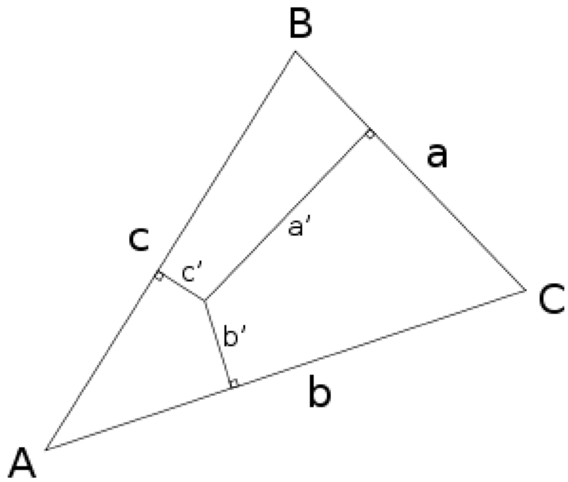
Trilinear coordinates



In geometry, the **trilinear coordinates** $x:y:z$ of a point relative to a given triangle describe the relative directed distances from the three sidelines of the triangle. Trilinear coordinates are an example of homogeneous coordinates. The ratio $x:y$ is the ratio of the perpendicular distances from the point to the sides (extended if necessary) opposite vertices A and B respectively; the ratio $y:z$ is the ratio of the perpendicular distances from the point to the sidelines opposite vertices B and C respectively; and likewise for $z:x$ and vertices C and A.

In the diagram at right, the trilinear coordinates of the indicated interior point are the actual distances (a', b', c'), or equivalently in ratio form, $ka':kb':kc'$ for any positive constant k. If a point is on a sideline of the reference triangle, its corresponding trilinear coordinate is 0. If an exterior point is on the opposite side of a sideline from the interior of the triangle, its trilinear coordinate associated with that sideline is negative. It is impossible for all three trilinear coordinates to be non-positive.

The name "trilinear coordinates" is sometimes abbreviated to "trilinears".

Notation

The ratio notation $x:y:z$ for trilinear coordinates is different from the ordered triple notation (a', b', c') for actual directed distances. Here each of x, y, and z has no meaning by itself; its ratio to one of the others *does* have meaning. Thus "comma notation" for trilinear coordinates should be avoided, because the notation (x, y, z), which means an ordered triple, does not allow, for example, $(x, y, z) = (2x, 2y, 2z)$, whereas the "colon notation" does allow $x : y : z = 2x : 2y : 2z$.

Examples

The trilinear coordinates of the incenter of a triangle ABC are 1 : 1 : 1; that is, the (directed) distances from the incenter to the sidelines BC, CA, AB are proportional to the actual distances denoted by (r, r, r), where r is the inradius of triangle ABC. Given side lengths a, b, c we have:

- $A = 1 : 0 : 0$
- $B = 0 : 1 : 0$
- $C = 0 : 0 : 1$
- incenter = 1 : 1 : 1
- centroid = $bc : ca : ab = 1/a : 1/b : 1/c = \csc A : \csc B : \csc C$.
- circumcenter = $\cos A : \cos B : \cos C$.
- orthocenter = $\sec A : \sec B : \sec C$.
- nine-point center = $\cos(B - C) : \cos(C - A) : \cos(A - B)$.
- symmedian point = $a : b : c = \sin A : \sin B : \sin C$.
- A-excenter = $-1 : 1 : 1$
- B-excenter = $1 : -1 : 1$
- C-excenter = $1 : 1 : -1$.

Note that, in general, the incenter is not the same as the centroid; the centroid has barycentric coordinates 1 : 1 : 1 (these being proportional to actual signed areas of the triangles *BGC*, *CGA*, *AGB*, where *G* = centroid.)

The midpoint of, for example, side *BC* has trilinear coordinates in actual sideline distances $(0, \frac{\Delta}{b}, \frac{\Delta}{c})$ for triangle area Δ, which in arbitrarily specified relative distances simplifies to $0 : ca : ab$. The coordinates in actual sideline distances of the foot of the altitude from *A* to *BC* are $(0, \frac{2\Delta}{a} \cos C, \frac{2\Delta}{a} \cos B)$, which in purely relative distances simplifies to $0 : \cos C : \cos B$.[1]:p. 96

Formulas

Collinearities and concurrencies

Trilinear coordinates enable many algebraic methods in triangle geometry. For example, three points

P = *p* : *q* : *r*
U = *u* : *v* : *w*
X = *x* : *y* : *z*

are collinear if and only if the determinant

$$D = \begin{vmatrix} p & q & r \\ u & v & w \\ x & y & z \end{vmatrix}$$

equals zero. Thus if *x:y:z* is a variable point, the equation of a line through the points *P* and *U* is *D* = 0.[1]:p. 23 From this, every straight line has a linear equation homogeneous in *x, y, z*. Every equation of the form *lx*+*my*+*nz* = 0 in real coefficients is a real straight line of finite points unless *l : m: n* is proportional to *a : b : c*, the side lengths, in which case we have the locus of points at infinity.[1]:p. 40

The dual of this proposition is that the lines

pα + qβ + rγ = 0
uα + vβ + wγ = 0,
xα + yβ + zγ = 0

concur in a point (α, β, γ) if and only if *D* = 0.[1]:p. 28

Also, if the actual directed distances are used when evaluating the determinant of *D*, then the area of triangle *PUX* is *KD*, where $K = abc/8\Delta^2$ (and where *Δ* is the area of triangle *ABC*, as above) if triangle *PUX* has the same orientation (clockwise or counterclockwise) as triangle *ABC*, and $K = -abc/8\Delta^2$ otherwise.

Parallel lines

Two lines with trilinear equations $lx + my + nz = 0$ and $l'x + m'y + n'z = 0$ are parallel if and only if[1]:p. 98,#xi

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ a & b & c \end{vmatrix} = 0,$$

where *a, b, c* are the side lengths.

Angle between two lines

The tangents of the angles between two lines with trilinear equations $lx + my + nz = 0$ and $l'x + m'y + n'z = 0$ are given by[1]:p.50

$$\pm \frac{(mn' - m'n)\sin A + (nl' - n'l)\sin B + (lm' - l'm)\sin C}{ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C}.$$

Perpendicular lines

Thus two lines with trilinear equations $lx + my + nz = 0$ and $l'x + m'y + n'z = 0$ are perpendicular if and only if

$$ll' + mm' + nn' - (mn' + m'n)\cos A - (nl' + n'l)\cos B - (lm' + l'm)\cos C = 0.$$

Altitude

The equation of the altitude from vertex *A* to side *BC* is[1]:p.98,#x

$$y\cos B - z\cos C = 0.$$

Line in terms of distances from vertices

The equation of a line with variable distances *p, q, r* from the vertices *A*, *B*, *C* whose opposite sides are *a, b, c* is[1]:p. 97,#viii

$$apx + bqy + crz = 0.$$

Actual-distance trilinear coordinates

The trilinears with the coordinate values *a', b', c'* being the actual perpendicular distances to the sides satisfy[1]:p. 11

$$aa' + bb' + cc' = 2\Delta$$

for triangle sides *a, b, c* and area Δ. This can be seen in the figure at the top of this article, with interior point *P* partitioning triangle *ABC* into three triangles *PBC*, *PCA*, and *PAB* with respective areas (1/2)*aa'* , (1/2)*bb'* , and (1/2)*cc'* .

Distance between two points

The distance *d* between two points with actual-distance trilinears $a_i : b_i : c_i$ is given by[1]:p. 46

$$d^2 \sin^2 C = (a_1 - a_2)^2 + (b_1 - b_2)^2 + 2(a_1 - a_2)(b_1 - b_2) \cos C$$

or in a more symmetric way

$$d^2 = \frac{abc}{4\Delta^2} \left(a(b_1 - b_2)(c_2 - c_1) + b(c_1 - c_2)(a_2 - a_1) + c(a_1 - a_2)(b_2 - b_1) \right).$$

Distance from a point to a line

The distance d from a point $a' : b' : c'$, in trilinear coordinates of actual distances, to a straight line $lx + my + nz = 0$ is[1]:p. 48

$$d = \frac{la' + mb' + nc'}{\sqrt{l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C}}.$$

Quadratic curves

The equation of a conic section in the variable trilinear point $x : y : z$ is[1]:p.118

$$rx^2 + sy^2 + tz^2 + 2uyz + 2vzx + 2wxy = 0.$$

It has no linear terms and no constant term.

The equation of a circle of radius r having center at actual-distance coordinates (a', b', c') is[1]:p.287

$$(x - a')^2 \sin 2A + (y - b')^2 \sin 2B + (z - c')^2 \sin 2C = 2r^2 \sin A \sin B \sin C.$$

Circumconics

The equation in trilinear coordinates x, y, z of any circumconic of a triangle is[1]:p. 192

$$lyz + mzx + nxy = 0.$$

If the parameters l, m, n respectively equal the side lengths a, b, c (or the sines of the angles opposite them) then the equation gives the circumcircle.[1]:p. 199

Each distinct circumconic has a center unique to itself. The equation in trilinear coordinates of the circumconic with center $x' : y' : z'$ is[1]:p. 203

$$yz(x' - y' - z') + zx(y' - z' - x') + xy(z' - x' - y') = 0.$$

Inconics

Every conic section inscribed in a triangle has an equation in trilinear coordinates:[1]:p. 208

$$l^2x^2 + m^2y^2 + n^2z^2 \pm 2mnyz \pm 2nlzx \pm 2lmxy = 0,$$

with exactly one or three of the unspecified signs being negative.

The equation of the incircle can be simplified to[1]:p. 210, p.214

$$\pm\sqrt{x}\cos\frac{A}{2}\pm\sqrt{y}\cos\frac{B}{2}\pm\sqrt{z}\cos\frac{C}{2}=0,$$

while the equation for, for example, the excircle adjacent to the side segment opposite vertex A can be written as[1]:p. 215

$$\pm\sqrt{-x}\cos\frac{A}{2}\pm\sqrt{y}\cos\frac{B}{2}\pm\sqrt{z}\cos\frac{C}{2}=0.$$

Cubic curves

Many cubic curves are easily represented using trilinear coordinates. For example, the pivotal self-isoconjugate cubic *Z(U,P)*, as the locus of a point X such that the P-isoconjugate of X is on the line UX is given by the determinant equation

$$\begin{vmatrix} x & y & z \\ qryz & rpzx & pqxy \\ u & v & w \end{vmatrix} = 0.$$

Among named cubics *Z(U,P)* are the following:

Thomson cubic: *Z(X(2),X(1))*, where *X(2)* = centroid, *X(1)* = incenter
Feuerbach cubic: *Z(X(5),X(1))*, where *X(5)* = Feuerbach point
Darboux cubic: *Z(X(20),X(1))*, where *X(20)* = De Longchamps point
Neuberg cubic: *Z(X(30),X(1))*, where *X(30)* = Euler infinity point.

Conversions

Between trilinear coordinates and distances from sidelines

For any choice of trilinear coordinates $x{:}y{:}z$ to locate a point, the actual distances of the point from the sidelines are given by $a' = kx$, $b' = ky$, $c' = kz$ where k can be determined by the formula $k=\frac{2\Delta}{ax+by+cz}$ in which a, b, c are the respective sidelengths BC, CA, AB, and Δ is the area of ABC.

Between barycentric and trilinear coordinates

A point with trilinear coordinates $x : y : z$ has barycentric coordinates $ax : by : cz$ where a, b, c are the sidelengths of the triangle. Conversely, a point with barycentrics $\alpha : \beta : \gamma$ has trilinear coordinates $\alpha/a : \beta/b : \gamma/c$.

Between Cartesian and trilinear coordinates

Given a reference triangle ABC, express the position of the vertex B in terms of an ordered pair of Cartesian coordinates and represent this algebraically as a vector $\underline{B}$, using vertex C as the origin. Similarly define the position vector of vertex A as $\underline{A}$. Then any point P associated with the reference triangle ABC can be defined in a Cartesian system as a vector $\underline{P} = k_1\underline{A} + k_2\underline{B}$. If this point P has trilinear coordinates $x : y : z$ then the conversion formula from the coefficients k_1 and k_2 in the Cartesian representation to the trilinear coordinates is, for side lengths a, b, c opposite vertices A, B, C,

$$x:y:z=\frac{k_1}{a}:\frac{k_2}{b}:\frac{1-k_1-k_2}{c},$$

and the conversion formula from the trilinear coordinates to the coefficients in the Cartesian representation is

$$k_1=\frac{ax}{ax+by+cz},\quad k_2=\frac{by}{ax+by+cz}.$$

More generally, if an arbitrary origin is chosen where the Cartesian coordinates of the vertices are known and represented by the vectors $\underline{A}$, $\underline{B}$ and $\underline{C}$ and if the point P has trilinear coordinates $x : y : z$, then the Cartesian coordinates of $\underline{P}$ are the weighted average of the Cartesian coordinates of these vertices using the barycentric coordinates ax, by and cz as the weights. Hence the conversion formula from the trilinear coordinates x, y, z to the vector of Cartesian coordinates $\underline{P}$ of the point is given by

$$\underline{P}=\frac{ax}{ax+by+cz}\underline{A}+\frac{by}{ax+by+cz}\underline{B}+\frac{cz}{ax+by+cz}\underline{C},$$

where the side lengths are $|\underline{C}-\underline{B}|=a$, $|\underline{A}-\underline{C}|=b$ and $|\underline{B}-\underline{A}|=c$.

See also

- Morley's trisector theorem#Morley's triangles, giving examples of numerous points expressed in trilinear coordinates
- Ternary plot
- Viviani's theorem

References

1. William Allen Whitworth (1866) Trilinear Coordinates and Other Methods of Analytical Geometry of Two Dimensions: an elementary treatise (http://ebooks.library.cornell.edu/cgi/t/text/text-idx?c=math;idno=01190002), link from Cornell University Historical Math Monographs.

External links

- Weisstein, Eric W. "Trilinear Coordinates" (https://mathworld.wolfram.com/TrilinearCoordinates.html). *MathWorld*.
- Encyclopedia of Triangle Centers - ETC (http://faculty.evansville.edu/ck6/encyclopedia/ETC.html) by Clark Kimberling; has trilinear coordinates (and barycentric) for more than 7000 triangle centers