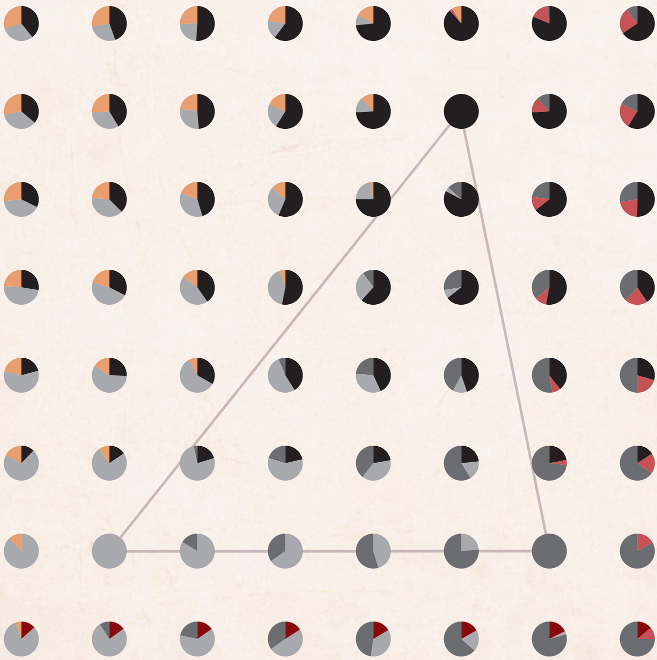


22 TRILINEAR COORDINATES

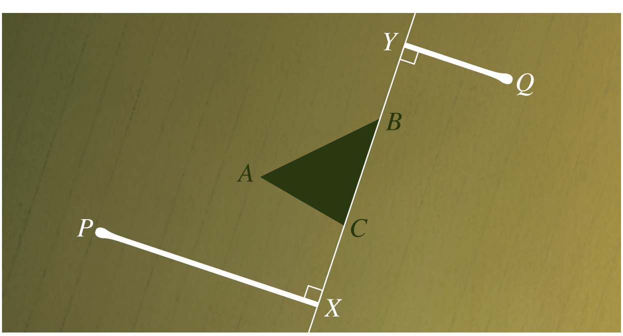
This is my last lesson under the heading of "Euclidean geometry". If you look back to the start, we have built a fairly impressive structure from modest beginnings. Throughout it all, I have aspired to a synthetic approach to the subject, which is to say that I have avoided attaching a coordinate system to the plane, with all the powerful analytic techniques that come by doing so. I feel that it is in the classical spirit of the subject to try to maintain this synthetic stance for as long as possible. But as we now move into the more modern development of the subject, it is time to shift positions. As a result, much of the rest of this work will take on a decidedly different flavor. With this lesson, I hope to capture the inflection point of that shift in stance, from the synthetic to the analytic.

Trilinear coordinates

In this lesson, we will look at trilinear coordinates, a coordinate system that is closely tied to the concurrence results of the last few lessons. Essentially, trilinear coordinates are defined by measuring signed distances from the sides of a given triangle.

DEF: THE SIGNED DISTANCE TO A SIDE OF A TRIANGLE
Given a side s of a triangle $\triangle ABC$ and a point P, let $|P,s|$ denote the (minimum) distance from P to the line containing s. Then define the signed distance from P to s as

$$[P,s] = \begin{cases} |P,s| & \text{if } P \text{ is on the same side of } s \text{ as the triangle} \\ -|P,s| & \text{if } P \text{ is on the opposite side of } s \text{ from the triangle} \end{cases}$$



$[P,BC] = |PX|$
$[Q,BC] = -|QY|$

From these signed distances, every triangle creates a kind of coordinate system in which a point P in the plane is assigned three coordinates

$$\alpha = [P,BC] \quad \beta = [P,AC] \quad \gamma = [P,AB].$$

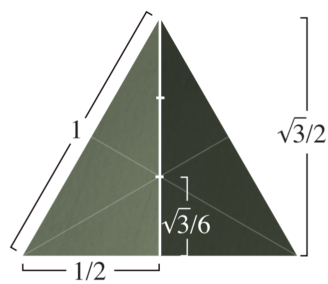
This information is consolidated into the notation $P = [\alpha : \beta : \gamma]$. There is an important thing to notice about this system of coordinates: while every point corresponds to a triple of real numbers, not every triple of real numbers corresponds to a point. For instance, when $\triangle ABC$ is equilateral with sides of length one, there is no point with coordinates $[2:2:2]$. Fortunately, there is a way around this limitation, via an equivalence relation.

AN EQUIVALENCE RELATION ON COORDINATES

Two sets of trilinear coordinates $[a:b:c]$ and $[d:b':c']$ are equivalent, written $[a:b:c] \sim [d':b':c']$, if there is a real number $k \neq 0$ so that

$$a' = ka \quad b' = kb \quad c' = kc.$$

Consider again that equilateral triangle $\triangle ABC$ with sides of length one. Okay, there is no point which is a distance of two from each side. But $[2:2:2]$ is equivalent to $[\sqrt{3}/6 : \sqrt{3}/6 : \sqrt{3}/6]$, and there is a point which is a distance of $\sqrt{3}/6$ from each side– the center of the triangle. That brings us to the definition of trilinear coordinates.



DEF: TRILINEAR COORDINATES

The trilinear coordinates of a point P with respect to a triangle $\triangle ABC$ is the equivalence class of triples $[k\alpha : k\beta : k\gamma]$ (with $k \neq 0$) where

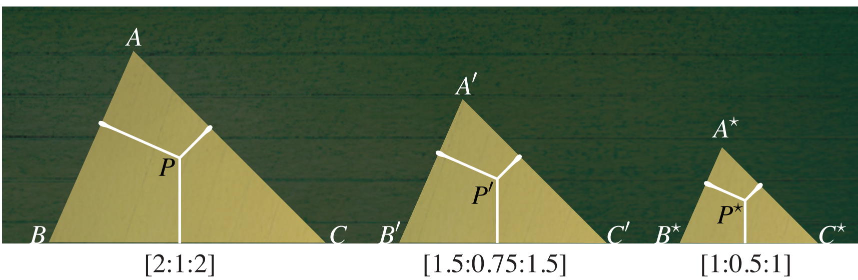
$$\alpha = [P,BC] \quad \beta = [P,AC] \quad \gamma = [P,AB].$$

The coordinates corresponding to the actual signed distances, when $k = 1$, are called the exact trilinear coordinates of P.

Because each coordinate is actually an equivalence class, there is an immediately useful relationship between trilinear coordinates in similar triangles. Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are similar, with a scaling constant k so that

$$|A'B'| = k|AB| \quad |B'C'| = k|BC| \quad |C'A'| = k|CA|.$$

Suppose that P and P' are points that are positioned similarly with respect to those triangles (so that $|A'P'| = k|AP|$, $|B'P'| = k|BP|$, and $|C'P'| = k|CP|$). Then the coordinates of P as determined by $\triangle ABC$ will be equivalent to the coordinates of P' as determined by $\triangle A'B'C'$.



Exact trilinear coordinates of similarly positioned points in similar triangles.

With that in mind, let's get back to the question of whether every *equivalence class* of triples of real numbers corresponds to a point. Straight out of the gate, the answer is no– the coordinates $[0:0:0]$ do not correspond to any point. As it turns out, that is the exception.

THM: THE RANGE OF THE TRILINEARS

Given a triangle $\triangle ABC$ and real numbers x, y, and z, not all zero, there is a point whose trilinear coordinates with respect to $\triangle ABC$ are $[x:y:z]$.

Proof. There are essentially two cases: one where all three of x, y, and z have the same sign, and one where they do not. I will look at the first case in detail. The second differs at just one crucial step, so I will leave the details of that case to you. In both cases, my approach is a constructive one, but it does take a rather indirect path. Instead of trying to find a point inside $\triangle ABC$ with the correct coordinates, I will start with a point P, and then build a new triangle $\triangle abc$ around it.

That new triangle will

1. be similar to the original $\triangle ABC$, and
2. be positioned so that the trilinear coordinates of P with respect to $\triangle abc$ are $[x : y : z]$.

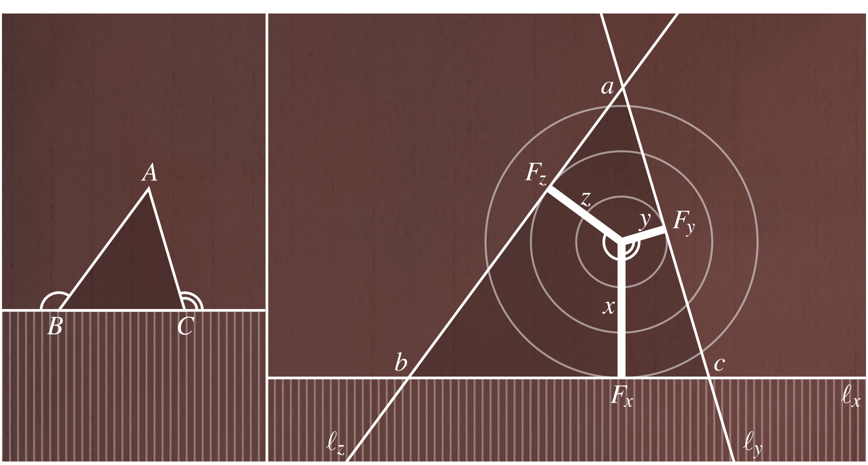
Then the similarly positioned point in $\triangle ABC$ will have to have those same coordinates relative to $\triangle ABC$.

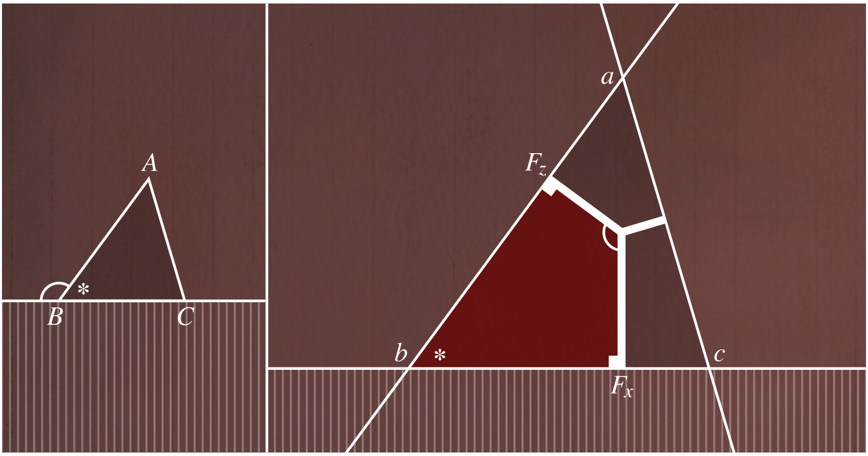
Case 1. $[+ : + : +] \sim [- : - : -]$
Consider the situation where all three numbers x, y, and z are greater than or equal to zero (of course, they cannot all be zero, since a point cannot be on all three sides of a triangle). This also handles the case where all three coordinates are negative, since $[x : y : z] \sim [-x : -y : -z]$. Mark a point F_x which is a distance x away from P. On opposite sides of the ray $PF_x \rightarrow$, draw out two more rays to form angles measuring $\pi - (\angle B)$ and $\pi - (\angle C)$. On the first ray, mark the point F_z which is a distance z from P. On the second, mark the point F_y which is a distance y from P. Let

ℓ_x be the line through F_x that is perpendicular to PF_x,
ℓ_y be the line through F_y that is perpendicular to PF_y,
ℓ_z be the line through F_z that is perpendicular to PF_z.

Label their points of intersection as

$$a = \ell_y \cap \ell_z \quad b = \ell_x \cap \ell_z \quad c = \ell_x \cap \ell_y.$$





Clearly, the trilinear coordinates of P relative to $\triangle abc$ are $[x:y:z]$. To see that $\triangle abc$ and $\triangle ABC$ are similar, let's compare their interior angles. The quadrilateral PF_xbF_z has right angles at vertex F_x and F_z and an angle measuring $\pi - (\angle B)$ at vertex P. Since the angle sum of a quadrilateral is 2π, that means $(\angle b) = (\angle B)$, so they are congruent. By a similar argument, $\angle c$ and $\angle C$ must be congruent. By A·A similarity, then, $\triangle ABC$ and $\triangle abc$ are similar.

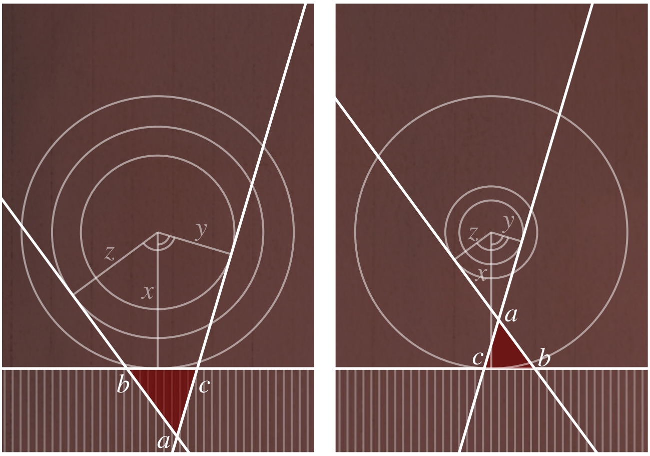
Case 2. $[+:-:-] \sim [-:+:+]$
Other than some letter shuffling, this also handles scenarios of the form $[-:+:-]$, $[+:-:+]$, $[-:-:+]$, and $[+:+:-]$. Use the same construction as in the previous case, but with one important change: in the previous construction, we needed

$$(\angle F_zPF_x) = \pi - (\angle B) \quad \& \quad (\angle F_yPF_x) = \pi - (\angle C).$$

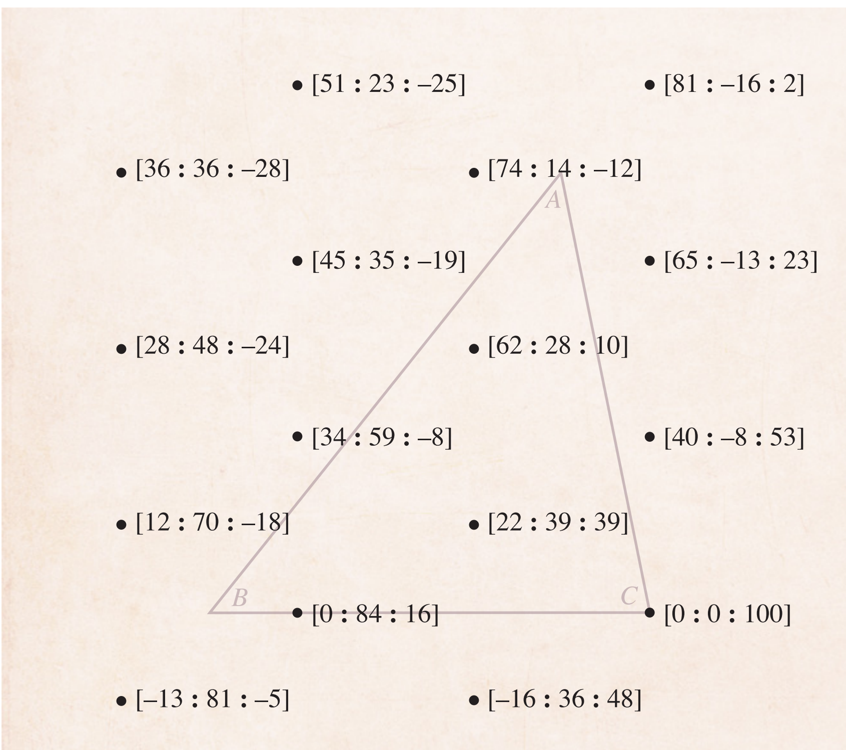
This time we are going to want

$$(\angle F_zPF_x) = (\angle B) \quad \& \quad (\angle F_yPF_x) = (\angle C).$$

The construction still forms a triangle $\triangle abc$ that is similar to $\triangle ABC$, but now P lies outside of it. Depending upon the location of a relative to the line ℓ_x, the signed distances from P to BC, AC, and AB, respectively are either x, y, and z, or $-x$, $-y$ and $-z$. Either way, since $[x:y:z]$ is equivalent to $[-x:-y:-z]$, P has the correct coordinates. $\square$



Case 2.
(l) exact trilinears with form [–:+:+]
(r) exact trilinears with form [+:–:–]



Trilinear coordinates of a few points, normalized so that the sum of the magnitudes of the coordinates is 100, and rounded to the nearest integer.

Trilinears of the classical centers

The classical triangle centers that we have studied in the last few lessons tend to have elegant trilinear coordinates. The rest of this lesson is dedicated to finding a few of them. The easiest of these, of course, is the incenter. Since it is equidistant from each of the three sides of the triangle, its trilinear coordinates are $[1:1:1]$. The others will require a little bit more work. These formulas are valid for all triangles, but if $\triangle ABC$ is obtuse, then one of its angles is obtuse, and thus far we have only really discussed the trigonometry of acute angles. For that reason, in these proofs I will restrict my attention to acute triangles. Of course, you have surely seen the unit circle extension of the trigonometric functions to all angle measures, so I encourage you to complete the proof by considering triangles that are not acute.

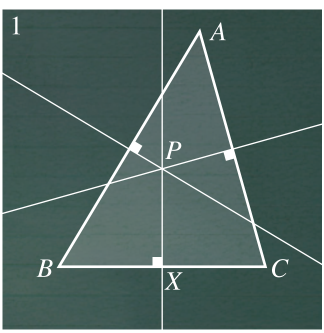
TRILINEARS OF THE CIRCUMCENTER

The trilinear coordinates of the circumcenter of $\triangle ABC$ are

$$[\cos A : \cos B : \cos C].$$

Proof. First the labels. Label the circumcenter P. Recall that the circumcenter is the intersection of the perpendicular bisectors of the three sides of the triangle. Let's take just one of those: the perpendicular bisector to BC. It intersects BC at its midpoint– call that point X. Now we can calculate the first exact trilinear coordinate in just a few steps, which I will justify below.

$$[P,BC] \underset{①}{=} |PX| \underset{②}{=} |PB|\cos(\angle BPX) \underset{③}{=} |PB|\cos(\angle BAC).$$

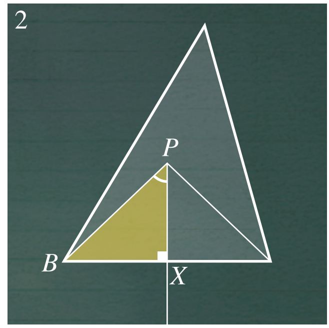


1. The minimum distance from P to BC is along the perpendicular– so $|P,BC| = |P,X|$. We have assumed that $\triangle ABC$ is acute. That places P inside the triangle, on the same side of BC as A, which means that the signed distance $[P,BC]$ is positive. Therefore

$$[P,BC] = |P,BC| = |PX|.$$

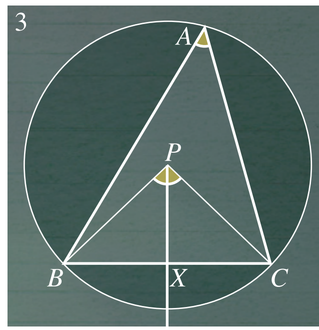
2. Look at $\angle BPX$ in the triangle $\triangle BPX$:

$$\cos(\angle BPX) = \frac{|PX|}{|PB|}$$
$$\implies |PX| = |PB|\cos(\angle BPX).$$



3. Segment PX splits $\triangle BPC$ into two pieces, $\triangle BPX$ and $\triangle CPX$, which are congruent by S·A·S. Thus PX evenly divides the *angle* $\angle BPC$ into two congruent pieces, and so

$$(\angle BPX) = \tfrac{1}{2}(\angle BPC).$$



Recall that the circumcenter is the center of the circle which passes through all three vertices A, B, and C. With respect to that circle, $\angle BAC$ is an inscribed angle, and $\angle BPC$ is the corresponding central angle. According to the Inscribed Angle Theorem,

$$(\angle BAC) = \tfrac{1}{2}(\angle BPC).$$

That means that $(\angle BPX) = (\angle BAC)$.

With that same argument we can find the signed distances to the other two sides as well.

$$[P,AC] = |PC|\cos(\angle ABC) \quad \& \quad [P,AB] = |PA|\cos(\angle BCA)$$

Gather that information together to get the exact trilinear coordinates of the circumcenter

$$P = [|PB|\cos(\angle A) : |PC|\cos(\angle B) : |PA|\cos(\angle C)].$$

Finally, observe that PA, PB, and PC are all the same length– they are radii of the circumcircle. Therefore, we can factor out that constant to get an equivalent representation

$$P = [\cos(\angle A) : \cos(\angle B) : \cos(\angle C)].$$

$\square$

TRILINEARS OF THE ORTHOCENTER

The trilinear coordinates of the orthocenter of $\triangle ABC$ are

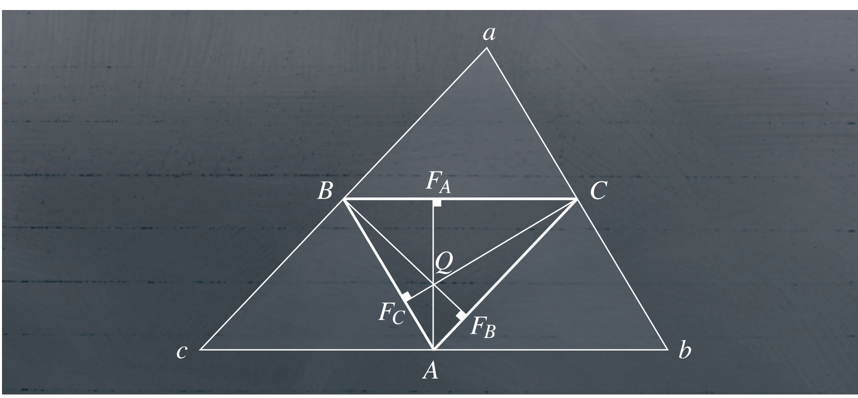
$$[\cos B \cos C : \cos A \cos C : \cos A \cos B].$$

Proof. Label the orthocenter Q. Recall that it is the intersection of the three altitudes of the triangle. Label the feet of those altitudes

F_A: the foot of the altitude through A,
F_B: the foot of the altitude through B, and
F_C: the foot of the altitude through C.

Now think back to the way we proved that the altitudes concur in lesson 19– it was by showing that they are the perpendicular bisectors of a larger triangle $\triangle abc$, where

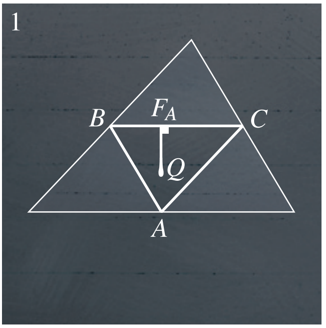
bc passed through A and was parallel to BC,
ac passed through B and was parallel to AC, and
ab passed through C and was parallel to AB.



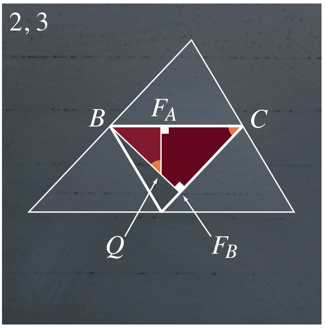
We are going to need that triangle again. Here is the essential calculation, with commentary explaining the steps below.

$$\begin{aligned}[Q,BC] &\overset{①}{=} |QF_A| \overset{②}{=} |QB|\cos(\angle F_AQB) \overset{③}{=} |QB|\cos(\angle C)\\ &\underset{④}{=} |Qa|\cos(\angle aQB)\cos(\angle C) \underset{⑤}{=} |Qa|\cos(\angle B)\cos(\angle C)\end{aligned}$$

1. The distance from Q to BC is measured along the perpendicular, so $|Q,BC| = |QF_A|$, but since we assumed our triangle is acute, Q will be inside $\triangle ABC$ and that means the signed distance $[Q,BC]$ is positive. So
$$[Q,BC] = |Q,BC| = |QF_A|.$$

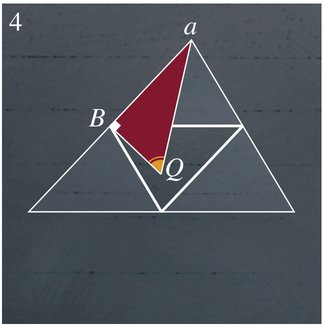


2. Look at the right triangle $\triangle F_AQB$. In it,
$$\cos(\angle F_AQB) = \frac{|QF_A|}{|QB|}$$
$$\implies |QF_A| = |QB|\cos(\angle F_AQB).$$

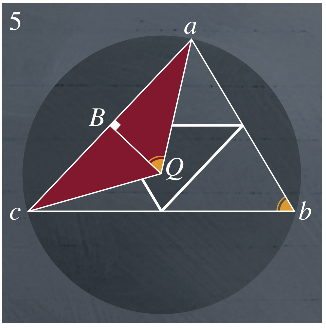


3. By A·A, $\triangle F_AQB \sim \triangle F_BCB$ (they share the angle at B and both have a right angle). Therefore
$$\angle F_AQB \simeq \angle F_BCB.$$

4. Look at the right triangle $\triangle aQB$. In it,
$$\cos(\angle aQB) = \frac{|QB|}{|Qa|}$$
$$\implies |QB| = |Qa|\cos(\angle aQB).$$



5. The orthocenter Q of $\triangle ABC$ is actually the circumcenter of the larger triangle $\triangle abc$. The angle $\angle abc$ is an inscribed angle in the circumcircle whose corresponding central angle is $\angle aQc$. By the Inscribed Angle Theorem, then,
$$(\angle abc) = \tfrac{1}{2}(\angle aQc).$$
The segment QB bisects $\angle aQc$ though, so
$$(\angle aQB) = \tfrac{1}{2}(\angle aQc).$$
That means $\angle aQB \simeq \angle abc$, which is, in turn congruent to $\angle B$ in the original triangle.



Through similar calculations,

$$[Q,AC] = |Qb|\cos(\angle A)\cos(\angle C)$$
$$[Q,AB] = |Qc|\cos(\angle A)\cos(\angle B).$$

That gives the exact trilinear coordinates for the orthocenter as

$$Q = [|Qa|\cos(\angle B)\cos(\angle C) : |Qb|\cos(\angle A)\cos(\angle C) : |Qc|\cos(\angle A)\cos(\angle B)]$$

Of course Qa, Qb and Qc are all the same length, though, since they are radii of the circumcircle of $\triangle abc$. Factoring out that constant gives an equivalent set of coordinates

$$Q = [\cos(\angle B)\cos(\angle C) : \cos(\angle A)\cos(\angle C) : \cos(\angle A)\cos(\angle B)].$$

□

TRILINEARS OF THE CENTROID

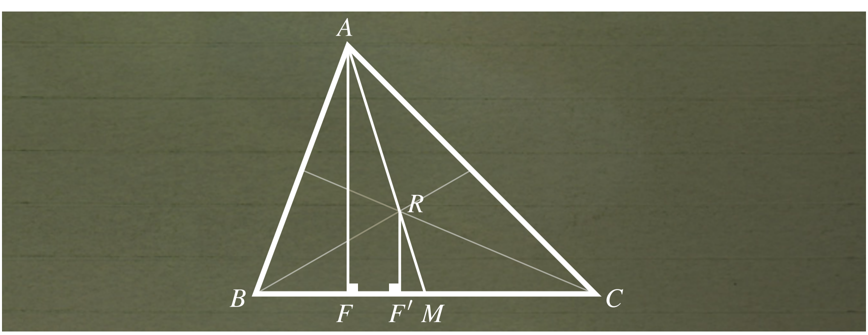
The trilinear coordinates of the centroid of $\triangle ABC$ are

$$[|AB|\cdot|AC| : |BA|\cdot|BC| : |CA|\cdot|CB|].$$

Proof. First the labels:

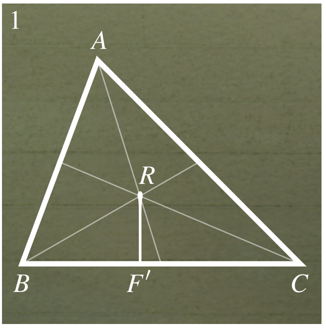
F: the foot of the altitude through A;
M: the midpoint of the side BC;
R: the centroid of $\triangle ABC$ (the intersection of the medians);
F': the foot of the perpendicular through R to the side BC.

In addition, just for convenience write $a = |BC|$, $b = |AC|$, and $c = |AB|$.

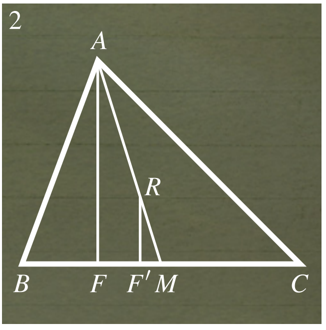


The last few results relied upon some essential property of the center in question– for the circumcenter it was the fact that it is equidistant from the three vertices; for the orthocenter, that it is the circumcenter of a larger triangle. This argument also draws upon such a property– that the centroid is located 2/3 of the way down a median from the vertex. Let's look at $[R,BC]$ which is one of the signed distances needed for the trilinear coordinates.

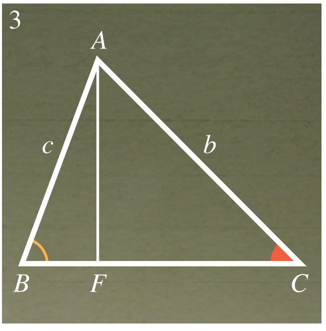
$$[R,BC] \underset{①}{=} |RF'| \underset{②}{=} \tfrac{1}{3}|AF| \underset{③}{=} \tfrac{1}{3}c\sin(\angle B) = \tfrac{1}{3}b\sin(\angle C)$$



1. Unlike the circumcenter and orthocenter, the median is always in the interior of the triangle, even when the triangle is right or obtuse. Therefore the signed distance $[R,BC]$ is the positive distance $|R,BC|$. Since RF' is the perpendicular to BC that passes through R, $|RF'|$ measures that distance.



2. This is the key step. Between the median AM and the parallel lines AF and RF' there are two triangles, $\triangle AFM$ and $\triangle RF'M$. These triangles are similar by A·A (they share the angle at M and the right angles at F and F' are congruent). Furthermore, because R is located 2/3 of the way down the median from the vertex, $|RM| = \frac{1}{3}|AM|$. The legs of those triangles must be in the same ratio, so $|RF'| = \frac{1}{3}|AF|$.



3. The goal is to relate $|AF|$ to the sides and angles of the original triangle, and we can now easily do that in two ways. In the right triangle $\triangle AFB$,

$$\sin(\angle B) = \frac{|AF|}{c} \implies |AF| = c\sin(\angle B),$$

and in the right triangle $\triangle AFC$,

$$\sin(\angle C) = \frac{|AF|}{b} \implies |AF| = b\sin(\angle C).$$

Similarly, we can calculate the distances to the other two sides as

$$[R,AC] = \tfrac{1}{3}a\sin(\angle C) = \tfrac{1}{3}c\sin(\angle A)$$
$$[R,AB] = \tfrac{1}{3}b\sin(\angle A) = \tfrac{1}{3}a\sin(\angle B)$$

and so the exact trilinear coordinates of the centroid can be written as

$$R = \left[\tfrac{1}{3}c\sin(\angle B) : \tfrac{1}{3}a\sin(\angle C) : \tfrac{1}{3}b\sin(\angle A)\right].$$

There is still a little more work to get to the more symmetric form presented in the theorem. Note from the calculation in step (3) above, that,

$$c\sin(\angle B) = b\sin(\angle C) \implies \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}$$

Likewise, the ratio $\sin(\angle A)/a$ also has that same value (this is the "law of sines"). Therefore we can multiply by the value $3b/\sin(\angle B)$ in the first coordinate, $3c/\sin(\angle C)$ in the second coordinate, and $3a/\sin(\angle A)$ in the third coordinate, and since they are all equal, the result is an equivalent set of trilinear coordinates for the centroid $R = [bc : ca : ab]$. □

To close out this lesson, and as well this section of the book, I want to make passing reference to another triangular coordinate system called barycentric coordinates. The trilinear coordinates that we have just studied put the incenter at the center of the triangle in the sense that it is the one point where are three coordinates are equal. With barycentric coordinates, that centermost point is the centroid. This is useful because if the triangle is a flat plate with a uniform density, then the centroid marks the location of the center of mass (the balance point). The barycentric coordinates of another point, then, give information about how to redistribute the mass of the plate so that *that point* is the balance point. Barycentric coordinates are usually presented in conjunction with the trilinear coordinates as the two are closely related. I am not going to do that though because I think we need to talk about area first, and area is still a ways away.

Exercises

1. (On the existence of similarly-positioned points) Suppose that $\triangle ABC$ and $\triangle A'B'C'$ are similar, with scaling constant k, so that

$$|AB| = k|AB| \quad |B'C'| = k|BC| \quad |C'A'| = k|CA|.$$

Given any point P, show that there exists a unique point P so that

$$[A'P'] = k[AP] \quad [B'P'] = k[BP] \quad [C'P'] = k[CP].$$

2. (On the uniqueness of trilinear coordinate representations) For a given triangle $\triangle ABC$, is it possible for two distinct points P and Q to have the same trilinear coordinates?

3. What are the trilinear coordinates of the three excenters of a triangle?

4. Show that the trilinear coordinates of the center of the nine-point circle of $\triangle ABC$ are

$$[\cos((\angle B) - (\angle C)) : \cos((\angle C) - (\angle A)) : \cos((\angle A) - (\angle B))].$$

This one is a little tricky, so here is a hint if you are not sure where to start. Suppose that $\angle B$ is larger than $\angle C$. Label

O: the center of the nine-point circle,
P: the circumcenter,
M: the midpoint of BC, and
X: the foot of the perpendicular from O to BC.

The key is to show that the angle $\angle POX$ is congruent to $\angle B$ and that $\angle POM$ is congruent to $\angle C$. That will mean $(\angle MOX) = (\angle B) - (\angle C)$.