Power of a Point and Radical Axis

Tovi Wen

NYC Math Team

§1 Power of a Point

This handout will cover the topic power of a point, and one of its more powerful uses in radical axes.

Definition (Power of a Point)

Given a circle ω with center O and radius r, and a point P, the **power** of P with respect to ω, which we will denote as (P, ω) is $OP^2 - r^2$. Note that

- If P is outside ω then (P, ω) is positive.
- If P is on ω then $(P, \omega) = 0$.
- If P is inside ω then (P, ω) is negative.

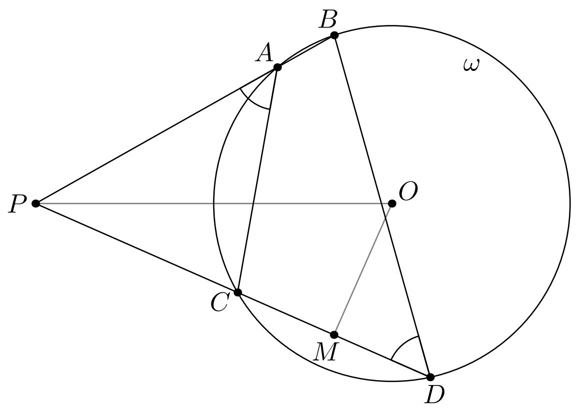
This definition does not feel particularly motivated, and therefore when taught at a more elementary level, it is often skipped and replaced by the following nice property.

Theorem

Let ω be a circle and let P be a point not on ω. If a line passing through P meets ω at distinct points A and B then

$$(P, \omega) = \begin{cases} PA \cdot PB & \text{if } P \text{ lies outside } \omega, \\ -PA \cdot PB & \text{if } P \text{ lies inside } \omega \end{cases}$$

Proof. It is not immediately obvious why the quantity $PA \cdot PB$ should be fixed for any line passing through P. Draw another chord of ω passing through P as shown.



Recall that opposite angles in a cyclic quadrilateral are supplementary. This gives $\angle PAC = \angle PDB$ so as $\angle APC \equiv \angle BPD$ is shared, we have $\triangle PAC \sim \triangle PDB$. In particular,

$$\frac{PA}{PC} = \frac{PD}{PB} \implies PA \cdot PB = PC \cdot PD.$$

We now show this quantity is equal to $OP^2 - r^2$. Let M be the midpoint of $\overline{CD}$ so that $OM \perp PM$. Then,

$$\begin{aligned} PC \cdot PD &= (PM - CM)(PM + CM) \\ &= PM^2 - CM^2 \\ &= (PO^2 - OM^2) - (OC^2 - OM^2) \\ &= PO^2 - OC^2 \end{aligned}$$

as desired. □

§2 Power of a Point Exercises

Problem 1 (2020 AMC12B)**.** In unit square $ABCD$, the inscribed circle ω intersects $\overline{CD}$ at M, and $\overline{AM}$ intersects ω at a point P different from M. What is AP?

Problem 2. Point P is chosen on the common chord of circles C_1 and C_2. Assume that P lies outside of both circles. Prove that the length of the tangent from P to C_1 is equal to the length of the tangent from P to C_2.

Problem 3. Let ω and γ be two circles intersecting at P and Q. Let their common external tangent touch ω at A and γ at B. Prove that $\overline{PQ}$ passes through the midpoint M of $\overline{AB}$.

Problem 4 (2019 AIME I)**.** In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

Problem 5. Let $\triangle ABC$ be equilateral, have side length 1, and have circumcircle ω. A chord of ω is trisected by $\overline{AB}$ and $\overline{AC}$. What is the length of this chord?

§3 What's the Radical Axis?

Definition (Radical Axis)

Given two non-concentric circles ω_1 and ω_2, there exists a line ℓ consisting of all points P for which $(P, \omega_1) = P(\omega_2)$. The line ℓ is known as the **radical axis** of ω_1 and ω_2.

This means if ω_1 has center O_1 and radius r_1, and ω_2 has center O_2 and radius r_2 then

$$PO_1^2 - r_1^2 = PO_2^2 - r_2^2.$$

Why does ℓ exist? is a natural question to ask. Here's a slightly non-rigorous proof.

Proof. Let P be a point such that $(P, \omega_1) = (P, \omega_2)$. Let D be the foot of the altitude from P to $\overline{O_1O_2}$.

Let $DO_1 = x_1$ and $DO_2 = x_2$. Set $C = x_1 + x_2$. By the Pythagroean Theorem, we have

$$PO_1^2 = PD^2 + x_1^2 \text{ and } PO_2^2 = PD^2 + x_2^2.$$

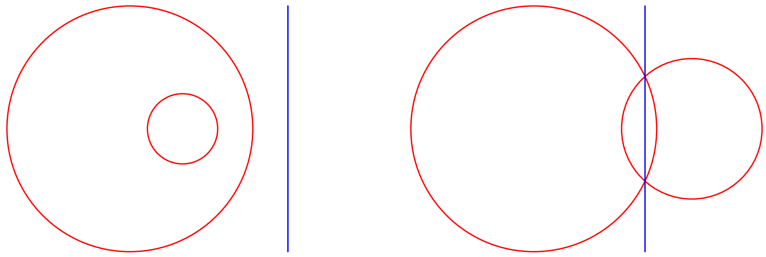
Combining and simplifying yields

$$PD^2 + x_1^2 - r_1^2 = PD^2 + x_2^2 - r_2^2 \implies x_1^2 - x_2^2 = r_1^2 - r_2^2 \implies x_1 - x_2 = \frac{r_1 - r_2}{C}.$$

As this last quantity is fixed, so are x_1 and x_2. This means that all P satisfying $(P, \omega_1) = (P, \omega_2)$ lie on the same line ℓ. The proof that all $P \in \ell$ work is a simple computation. □

Where did we use the fact that ω_1 and ω_2 were non-concentric in the above proof?

Pictured on the following page is an example of the radical axes of two circles. Note that when ω_1 and ω_2 intersect, their radical axis is simply their common chord.



§4 The Radical Axis Theorem

The following is the whole point of all of this and the reason for dedicating an entire talk to this topic.

Theorem (Radical Axis Theorem)

The pairwise radical axes of three non-concentric circles are concurrent. Note that this means the common chords of three pairwise intersecting circles are concurrent.

The proof is amazingly short.

Proof. Let the circles be $\omega_1, \omega_2, \omega_3$ and let the radical axes of (ω_1, ω_2) and (ω_2, ω_3) intersect at P. Then

$$(P, \omega_1) = (P, \omega_2) = (P, \omega_3)$$

so P lies on the radical axis of (ω_1, ω_3) also. □

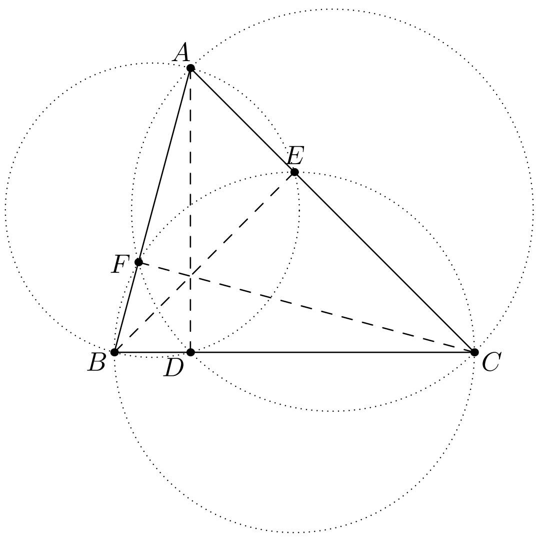
Remark. *Note that this proof is isomorphic to the proof of the existence of the circumcenter.*

If you ever want to prove that three strange lines are concurrent, the radical axis theorem will often by the best way to go.

Example 6 (Existence of the orthocenter)

Prove that the three altitudes in a triangle are concurrent.

Let $\triangle ABC$ be the triangle and let its altitudes be $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ with $D \in BC$, $E \in CA$, and $F \in AB$. Note that points E and F lie on the circle (BC) with diameter $\overline{BC}$ and similar results hold for (CA) and (AB).



But now, we recognize that line AD is the radical axis of (AB), (CA), line BE is the radical axis of (AB), (BC), and line CF is the radical axis of (BC), (CA) so by the Radical Axis theorem, AD, BE, CF concur as required.

§5 Degenerate Circles

Technically, a point is a circle of radius 0. One fascinating use of the radical axis theorem is when we apply it to a set of circles, some of which are just points.

Example 7 (Existence of the circumcenter)

Prove that the perpendicular bisectors of the sides of a triangle are concurrent.

Proof. Let $\triangle ABC$ be the triangle and view A as a circle ω_A with radius 0. Define ω_B and ω_C similarly. The perpendicular bisector of $\overline{BC}$ is just the radical axis of (ω_B, ω_C) so the three perpendicular bisectors concur at the radical center of $\omega_A, \omega_B, \omega_C$. □

Obviously the above example is silly as the use of radical axes is completely contrived but this is far from always true. Take a look at the following problem given on a real olympiad.

Example 8

Let ABC be a triangle with circumcenter O and P be a point. Let the tangent to the circumcircle of $\triangle BPC$ at P intersect BC at A'. Define points $B' \in CA$ and $C' \in AB$ similarly. Prove that points A', B', C' are collinear on a line perpendicular to OP.

The condition that $\overline{A'B'C'} \perp \overline{OP}$ leads us to believe that the line in question might be the radical axis of $\odot(ABC) \overset{\text{def}}{=} \Omega$ and some other circle. In fact, this is the circle ω with center P and radius 0. To see this, note that

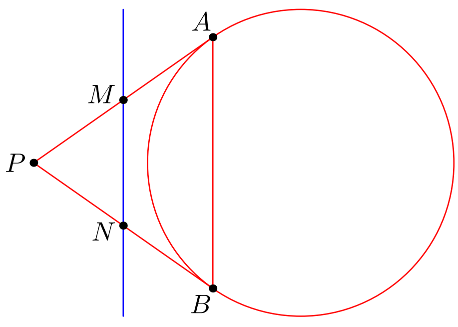
$$(A', \omega) = A'P^2 = A'B \cdot A'C = (A', \Omega)$$

so A' lies on the radical axis ℓ of ω and Ω. Similarly, we can prove $B', C' \in \ell$ so A', B', C' are collinear on the radical axis and we are done.

We've seen examples exploiting the power of a point definition of the radical axis. It is also helpful to explicitly define the radical axis of circle and a point outside it.

Lemma

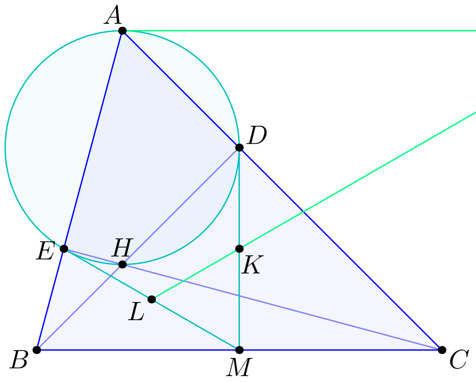
Let P be a point outside circle ω. The tangents to ω at P meet ω at distinct points A and B. Then the P-midline of $\triangle PAB$ is the radical axis of $\odot(P)$ and ω.



We'll present one last olympiad problem in full.

Example 9 (Iran TST 2011)

In acute triangle ABC angle B is greater than angle C. Let M is midpoint of BC. Let D and E are the feet of the altitude from C and B, respectively. Let K and L are midpoint of ME and MD, respectively. If KL intersect the line through A parallel to BC in T, prove that $TA = TM$.



This example is instructive as it highlights the following claim.

Claim. MD, ME, and the line through A parallel to BC are all tangent to $\odot(AEF)$.

Proof. Note that D, E lie on the circle with diameter $\overline{BC}$ and center M. Hence, $MD = ME$ and

$$\angle DME = 2\angle ABD = 180^\circ - 2\angle A$$

which gives $\angle MED = \angle MDE = \angle A$ so MD, ME are tangent. Moreover, the circle has diameter $\overline{AH}$ where H is the orthocenter, so since the line through A parallel to BC is perpendicular to AH, it must be tangent to $\odot(ADE)$ as desired. □

Now for the cool part. Notice that by the Lemma, $\overline{KL}$ is the radical axis of $\odot(ADE)$ and the circle at M with radius 0. In particular,

$$TA^2 = (T, \odot(ADE)) = (T, \odot(M)) = TM^2$$

so $TA = TM$ as desired.

§6 Radical Axis Exercises

Problem 1 (2017 AMC12B)**.** A circle has center $(-10, -4)$ and radius 13. Another circle has center $(3, 9)$ and radius $\sqrt{65}$. The line passing through the two points of intersection of the two circles has equation $x + y = c$. What is c?

Problem 2. Given two non-intersecting circles, can you construct their radical axis using a compass and a striaghtedge?

Problem 3. Let $\triangle ABC$ have orthocenter H. Points D and E lie on sides AB and AC, respectively. Prove that H lies on the radical axis of the circle with diameter $\overline{CD}$ and the circle with diameter $\overline{BE}$.

Problem 4 (USAJMO 2012)**.** Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic

§7 Challenging Problems

Here are some advanced problems for you to try. Be forewarned that these get pretty hard, and don't worry if you aren't able to solve any of them just yet.

Problem 1 (ISL 1995)**.** The incircle of triangle $\triangle ABC$ touches the sides BC, CA, AB at D, E, F respectively. X is a point inside triangle of $\triangle ABC$ such that the incircle of triangle $\triangle XBC$ touches BC at D, and touches CX and XB at Y and Z respectively. Show that E, F, Z, Y are concyclic.

Problem 2 (IMO 1995)**.** Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at X and Y. The line XY meets BC at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

Problem 3 (Orthic Axis)**.** Let $\triangle ABC$ have circumcenter O, orthocenter H, and alitudes AD, BE, CF. Let EF meet BC at X, let FD meet CA at Y, and let DE meet AB at Z. Prove that X, Y, Z are collinear on a line perpendicular to OH.

Problem 4 (IMO 2000)**.** Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

Problem 5 (2020 AIME I)**.** Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.

Problem 6 (2016 AIME I)**.** Circles ω_1 and ω_2 intersect at points X and Y. Line ℓ is tangent to ω_1 and ω_2 at A and B, respectively, with line AB closer to point X than to Y. Circle ω passes through A and B intersecting ω_1 again at $D \neq A$ and intersecting ω_2 again at $C \neq B$. The three points C, Y, D are collinear, $XC = 67$, $XY = 47$, and $XD = 37$. Find AB^2.

Problem 7 (Fake USAJMO 2020)**.** Let $\triangle ABC$ be a triangle. Points D, E, and F are placed on sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively such that $EF \parallel BC$. The line DE meets the circumcircle of $\triangle ADC$ again at $X \neq D$. Similarly, the line DF meets the circumcircle of $\triangle ADB$ again at $Y \neq D$. If D_1 is the reflection of D across the midpoint of $\overline{BC}$, prove that the four points D, D_1, X, and Y lie on a circle.

Problem 8 (Coaxality Lemma)**.** Circles $\omega_1, \omega_2, \omega_3$ all pass through points X and Y. If points P and Q lie on ω_3, show that

$$\frac{(P, \omega_1)}{(P, \omega_2)} = \frac{(Q, \omega_1)}{(Q, \omega_2)}.$$

Problem 9 (Russian Olympiad 2011)**.** The perimeter of triangle ABC is 4. Point X is marked at ray AB and point Y is marked at ray AC such that $AX = AY = 1$. Let BC intersect XY at point M. Prove that perimeter of either $\triangle ABM$ or $\triangle ACM$ is 2.

Problem 10 (PUMaC Finals 2017)**.** Triangle ABC has incenter I. The line through I perpendicular to AI meets the circumcircle of ABC at distinct points P and Q, where P and B are on the same side of AI. Let X be the point such that $PX \parallel CI$ and $QX \parallel BI$. Show that PB, QC, and IX intersect at a common point.