

GAMBLER'S RUIN?

SOME ASPECTS OF COIN TOSSING

PORTER W. JOHNSON,* *Illinois Institute of Technology*

DAVID ATKINSON,** *University of Groningen*

Abstract

What is the average number of coin tosses needed before a particular sequence of heads and tails first turns up? This problem is solved in our paper, starting with doubles, a tail, followed by a head, turns up on average after only four tosses, while six tosses are needed for two successive heads. The method is extended to encompass the triples head-tail-tail and head-head-tail, but head-tail-head and head-head-head are surprisingly more recalcitrant. However, the general case is finally solved by using a new algorithm, even for relatively long strings. It is shown that the average number of tosses is always an even integer.

Keywords: probability, coin tossing, game theory, gambling

2010 Mathematics Subject Classification: Primary 60G40

Secondary 91A60;91A80

1. Do heads like tails?

The classic example of the gambler's fallacy in coin tossing is that a long run of heads is more likely to be succeeded by a tail than by yet another head. A variant is as follows. A gambler must make a stake of 5 euros. He can decide whether to bet on a head-head, or a tail-head outcome. In the first case the gambler wins n euros if n tosses are made until two successive heads first appear, and in the second case he wins

* Postal address: Illinois Institute of Technology, Chicago, IL 60616, U.S.A.

** Postal address: University of Groningen, 9712 GL Groningen, The Netherlands

n euros if n tosses are made until the sequence tail-head first appears. Does it make any difference which of the two wagers is favoured?

The gambler might reason that, assuming the coin to be fair, an equal number of heads and tails should come up in the long run, and therefore it is more likely that the sequence tail-head will come up than the sequence head-head. So he should bet on head-head, since it is likelier that more tosses will be needed than would be required for tail-head. Is the gambler wrong again? Are tail-head and head-head equally likely?

Notwithstanding any first intuition one might have, it turns out that the gambler is well-advised to bet on head-head, and not on tail-head. Indeed, on average he would win one euro per game with head-head, but lose one euro per game with tail-head. [1], [2]

In Section 2 we treat the problem of ‘doubles’, calculating the average number of tosses needed before a tail, followed by a head (TH) first comes up, and then the average number before two heads (HH) first come up. These averages are computed by first obtaining the probabilities that TH or HH first come up after m tosses, and then calculating the mean value of m . Since there is symmetry under an interchange of heads and tails, TH and HT yield the same average number of tosses, and similarly for TT and HH. In the HH case the probability turns out to be related to a Fibonacci number.

In Section 3 the probabilities associated with the doubles are recalculated by using recurrence relations; and this method is then applied to the ‘triples’ HTT and HHT. The corresponding calculations for HTH and HHH prove cumbersome using the technique of this section, and they are not therefore attempted.

The calculations in Sections 2 and 3 unexpectedly result in averages that are integers: why should the average number of tosses be a whole number? The answer to this question is given in Section 4, in which a new method is introduced that enables all the findings of the previous sections to be recovered, as well as providing the averages pertaining to the remaining triples. This technique makes it clear that all the averages must in fact be even integers. At the end of the paper a table is presented of the average numbers of tosses, not only for the doubles and triples, but also for the quadruples, quintuples and sextuples, all of which can be readily treated by the new method.

2. Doubles

We wish to calculate the probability that a tail first comes up in the $(m-1)$ st toss, followed immediately by a head in the m th toss. Representing a tail by 0 and a head by 1, and the sequence of tosses by a string of zeroes and ones, we see that this is the same as the probability that for the first time the $(m-1)$ st and m th positions are occupied by the sequence 01. Suppose that there are n tosses in all, so that we have 2^n distinct configurations of zeroes and ones, each being equiprobable. To calculate the probability of interest, we first consider the number of all the distinct configurations in which the sequence 01 is **absent**. If $n = 1$, so that there is only one toss, the sequence 01 is absent. For two tosses, $n = 2$, three of the four configurations are not 01. For $n = 3$, the number of configurations lacking 01 is four, and for $n = 4$ it is five. Table 1 enables one to see at a glance why, for general n , the number of configurations that do not contain 01 is $n + 1$. For to progress from n to $n + 1$ one can add a 0 to the right of each n -sequence without creating 01, and one can also add a 1 to the right of the configuration containing only ones. No other configuration tolerates the adjunction of 1, since all the rest have a 0 in the rightmost position.

| | | | |
|---|----|-----|------|
| 0 | 00 | 000 | 0000 |
| 1 | 10 | 100 | 1000 |
| | 11 | 110 | 1100 |
| | | 111 | 1110 |
| | | | 1111 |

Table 1: Sequences **not** containing 01

How many different sequences of length n are there, such that the $(m-1)$ st and m th positions are occupied by 01, and such that the first $m-2$ positions do not contain the sequence 01? We have just deduced that there will be $(m-2) + 1 = m-1$ such sequences in a string of length $m-2$. There are $n-m$ positions after the 01, and it is not important if 01 recurs, so there are 2^{n-m} distinct sequences here. Therefore the total number we are seeking is $(m-1)2^{n-m}$. Since the total number of distinct sequences is 2^n , we must normalize by this to obtain the probability, p_m , that 01 first

occurs in the $(m - 1)$ st and m th positions respectively:

$$p_m(01) = (m - 1) 2^{-m}$$

It is now straightforward to compute the average value of m , namely

$$N_{01} = \sum_{m=2}^{\infty} m p_m(01) = \sum_{m=2}^{\infty} m (m - 1) 2^{-m} = 4.$$

In the long run a tail, followed by a head, will come up after 4 tosses, so if our gambler were to put down his stake of 5 euros on the tail-head option, he would lose an average of one euro per game.

What difference does it make if we now require that a head comes up in the $(m - 1)$ st toss, followed immediately by another head in the m th toss? What is the probability that the $(m - 1)$ st and m th positions in a string of zeroes and ones are occupied for the first time by the sequence 11? We consider now the number of all the distinct configurations in which the sequence 11 is absent. If $n = 1$, with possibilities, 0 or 1, the sequence 11 is absent. For two tosses, $n = 2$, three of the four configurations are not 11. For $n = 3$, the number of configurations lacking 11 is five, and for $n = 4$ it is eight. These configurations are shown in Table 2, and evidently this array is quite different from that in Table 1.

| | | | |
|---|----|-----|------|
| 0 | 00 | 000 | 0000 |
| 1 | 10 | 100 | 1000 |
| | 01 | 010 | 0100 |
| | | 001 | 0010 |
| | | 101 | 1010 |
| | | | 0001 |
| | | | 1001 |
| | | | 0101 |

Table 2: Sequences **not** containing 11

This pattern yields a Fibonacci sequence, which is defined by the recursion

$$F_0 = 0 \qquad F_1 = 1 \qquad F_n = F_{n-1} + F_{n-2},$$

so $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, and so on. The number of configurations corresponding to n bits is F_{n+2} .

We should ask how many different sequences of length n there are, such that the $(m-1)$ st and m th positions are occupied for the first time by 11. We have just shown that there are F_m different sequences of length $m-2$ that lack 11. But this is not sufficient, for some of these F_m sequences end with 1, which is disallowed, since we want 11 to be in the $(m-1)$ st and m th positions, not the $(m-2)$ nd and $(m-1)$ st positions. So we must count only those of the F_m sequences that end in 0. There are precisely F_{m-1} of these sequences, as can be seen from Table 2. Indeed, they are the sequences of $m-3$ bits that lack 11, augmented by a 0 at the right end.

The total number of states in a sequence of n bits that have 11 for the first time in the $(m-1)$ st and m th positions is therefore $F_{m-1} 2^{n-m}$. On normalizing this we obtain the probability, $p_m(11)$, that 11 first occurs in the $(m-1)$ st and m th positions respectively:

$$p_m(11) = F_{m-1} 2^{-m}$$

The average value of m is therefore

$$N_{11} = \sum_{m=2}^{\infty} m p_m(11) = \sum_{m=2}^{\infty} m F_{m-1} 2^{-m} = 6. \quad (1)$$

The above sum was evaluated from the definition of Fibonacci numbers. In the long run a head, followed by another head, will come up after 6 tosses, so if our gambler were to put down his stake of 5 euros on the head-head option, he would gain an average of one euro per game.

3. Recursive solution

We now repeat the above calculations of N_{01} and N_{11} with a notation that leads directly to recursion relations. The new method will then be generalized to sequences involving 3 consecutive bits (triples). Let $K_m(01)$ be the number of different strings of zeroes and ones of length m , such that 01 occurs for the only time at the right-hand end of the string, as in the previous section. Similarly, $K_m(11)$ is the number of different strings of length m , such that 11 occurs for the only time at the right-hand end of that string.

Consider first 01, and define $M_m(01; j)$ to be the number of different strings of length m , such that 01 does **not** occur, and such that the rightmost bit is j ($j = 0$ or

1). We expect a recursion relation of the sort

$$M_m(01; j) = a M_{m-1}(01; 0) + b M_{m-1}(01; 1),$$

for $M_{m-1}(01; j)$ is, like $M_m(01; j)$, free of the sequence 01, and it is one bit shorter. Here a and b are 0 or 1, and we find

$$\begin{aligned} M_m(01; 0) &= M_{m-1}(01; 0) + M_{m-1}(01; 1) \\ M_m(01; 1) &= M_{m-1}(01; 1) \end{aligned} \tag{2}$$

Note that $M_{m-1}(01; 0)$ is missing in the second equation, i.e. $a = 0$, for we may not add a 1 at the end of a string of length $m - 1$ that terminates with a 0, because that would produce a string which terminates with 01, contrary to the definition of $M_m(01; 1)$. The second line of (2) says that $M_m(01; 1)$ is independent of m , and since $M_1(01; 1)$ is clearly one, we find $M_m(01; 1) = 1$, and the first line of (2) becomes

$$M_m(01; 0) = M_{m-1}(01; 0) + 1.$$

The solution is $M_m(01; 0) = m$, since $M_1(01; 0)$ is also clearly one. A string of length m that contains 01 only at its right-hand end can be thought of as a string of length $m - 1$ that does not contain 01, but which has 0 at its right-hand end, plus a 1 at the extreme right. This means that

$$K_m(01) = M_{m-1}(01; 0) = m - 1, \tag{3}$$

which agrees with our earlier result.

Let us now consider 11 and similarly define $M_m(11; j)$ to be the number of different strings of length m , such that 11 does not occur, and such that the rightmost bit is 0 or 1. The analogue of Eq.(2) is

$$\begin{aligned} M_m(11; 0) &= M_{m-1}(11; 0) + M_{m-1}(11; 1) \\ M_m(11; 1) &= M_{m-1}(11; 0) \end{aligned} \tag{4}$$

Here it is $M_{m-1}(11; 1)$ that is missing, for adding a 1 to the end of that sequence would produce a string containing 11. Replace m by $m - 1$ in the second line of (4) and substitute for $M_{m-1}(11; 1)$ in the first line of (4):

$$M_m(11; 0) = M_{m-1}(11; 0) + M_{m-2}(11; 0).$$

This is the Fibonacci recursion, and since $M_1(11;0) = 1$ and $M_2(11;0) = 2$, it follows that $M_m(11;0) = F_{m+1}$. Clearly,

$$K_m(11) = M_{m-1}(11;1) = M_{m-2}(11;0) = F_{m-1},$$

once more agreeing with the result of Section 2.

We now break new ground by considering triples. How many strings of length m are there for which 100 occurs only at the right-hand end? We call this number $K_m(100)$. Let $M_m(100;ij)$ be the number of strings of length m that do not contain the sequence 100, and which end with ij , i.e. either 00, 01, 10 or 11. The relevant recursion relations are now

$$\begin{aligned} M_m(100;00) &= M_{m-1}(100;00) \\ M_m(100;01) &= M_{m-1}(100;00) + M_{m-1}(100;10) \\ M_m(100;10) &= M_{m-1}(100;01) + M_{m-1}(100;11) \\ M_m(100;11) &= M_{m-1}(100;01) + M_{m-1}(100;11) \end{aligned} \tag{5}$$

The last three lines have two terms on the right, corresponding to the possibility of adding the relevant bit to the right of the string of length $m-1$. The first line lacks $M_{m-1}(100;10)$, because the adjunction of a 0 to the right would produce a string ending in 100. This line shows that $M_m(100;00)$ is independent of m , so $M_m(100;00) = M_2(100;00) = 1$. The second line of Eq.(5) is therefore

$$M_m(100;01) = 1 + M_{m-1}(100;10). \tag{6}$$

We next note that the right-hand sides of the third and fourth lines are identical, so $M_m(100;10) = M_m(100;11)$, and therefore

$$M_m(100;10) = M_{m-1}(100;01) + M_{m-1}(100;10). \tag{7}$$

Change m into $m-1$ in (6), and substitute the result into (7):

$$M_m(100;10) = 1 + M_{m-2}(100;10) + M_{m-1}(100;10). \tag{8}$$

It is clear that $1 + M_m(100;10)$ satisfies the Fibonacci recurrence relation. Since $M_3(100;10) = 2$, we conclude that $M_m(100;10) = F_{m+1} - 1$. Thus the number of

strings ending in 100 is

$$K_m(100) = M_{m-1}(100; 10) = F_m - 1, \quad (9)$$

so the average value of m is

$$N_{100} = \sum_{m=3}^{\infty} m [F_m - 1] 2^{-m} = 8. \quad (10)$$

We next consider the triple 110. Let $M_m(110; ij)$ be the number of strings of length m that do not contain the sequence 110, and which end with ij , i.e. either 00, 01, 10 or 11. The relevant recursion relations are now

$$\begin{aligned} M_m(110; 00) &= M_{m-1}(110; 00) + M_{m-1}(110; 10) \\ M_m(110; 01) &= M_{m-1}(110; 00) + M_{m-1}(110; 10) \\ M_m(110; 10) &= M_{m-1}(110; 01) \\ M_m(110; 11) &= M_{m-1}(110; 01) + M_{m-1}(110; 11) \end{aligned} \quad (11)$$

The third line lacks $M_{m-1}(110; 11)$, because the adjunction of a 0 to the right would produce a string containing 110. The right-hand sides of the first and second lines are identical, so $M_m(110; 00) = M_m(110; 01)$, and therefore

$$M_m(110; 01) = M_{m-1}(110; 01) + M_{m-2}(110; 01) \quad (12)$$

where the third line has also been used. This is the Fibonacci recursion again, and since $M_3(110; 01) = 2$ it follows that $M_m(110; 01) = M_m(110; 00) = F_m$, and also that $M_m(110; 10) = F_{m-1}$. The fourth line in Eq.(11) becomes $M_m(110; 11) = F_{m-1} + M_{m-1}(110; 11)$, and by iteration this yields

$$M_m(110; 11) = \sum_{p=1}^{m-1} F_p = F_{m+1} - 1 \quad (13)$$

Finally, the number of strings ending in 110 is

$$K_m(110) = M_{m-1}(100; 11) = F_m - 1,$$

From Eq.(9) we see that this is the same formula as the one we derived for the triple 100, so we also find $N_{110} = 8$.

The above method turns out to be inadequate for the calculation of N_{101} and N_{111} : an analogous treatment would involve more complicated numbers than those of Fibonacci, and so in the following section we examine a novel approach that turns out to be rather simpler.

4. All averages are even integers

An interesting feature of the average numbers, 4, 6 and 8 that we have calculated so far is that they are integral. The number of tosses in any one game is of course an integer; but why should their **average** be a whole number too? In this section we shall demonstrate that the average number of tosses needed before a given string,

$$T_m = (t_1 t_2 \dots t_m), \quad (14)$$

comes up for the first time is in general an integer, and moreover an **even** integer. Consider first a string of length n ,

$$S_n = (s_1 s_2 \dots s_n), \quad (15)$$

that does not contain T_m as a substring; and let σ_n be the number of such strings. A string of length $m + n$ that contains T_m just once, right at the end, has the form

$$(S_n T_m) = (s_1 s_2 \dots s_n t_1 t_2 \dots t_m). \quad (16)$$

Let τ_{m+n} be the number of such strings of length $m + n$ that terminate with T_m , but which do not contain any other instance of T_m .

To make a string of the type S_n from S_{n-1} one can add either $s_n = 0$ or $s_n = 1$ at the end, and thus there will often be twice as many strings of the type S_n as of the type S_{n-1} . However, if the addition of 0 or 1 to a string S_{n-1} results in the completion of the string T_m , the new string should not be counted as one of the tally σ_n , but rather as one of the tally τ_n , since it has become a string of length n , of which the last m bits constitute T_m . Thus

$$2\sigma_{n-1} = \sigma_n + \tau_n. \quad (17)$$

In the notation of Section 3, τ_{m+n} corresponds to $K_{m+n}(T_m)$, and σ_n corresponds to $M_n(T_m; 0) + M_n(T_m; 1)$.

If $m = 1$, $\sigma_1 = 1$ and $\tau_1 = 1$, but if $m \geq 2$, $\sigma_1 = 2$ and $\tau_1 = 0$, so in all cases (17) is true for $n = 1$ if we formally define $\sigma_0 = 1$. With this understanding, (17) is valid for $n = 1, 2, 3 \dots$

By iteration we find from Eq.(17) that

$$\sigma_n = -\tau_n - 2\tau_{n-1} - \dots - 2^{n-m}\tau_m + 2^{n-m+1}\sigma_{m-1}.$$

Since one cannot obtain T_m with only $m - 1$ bits, σ_{m-1} is equal to the total number of strings of length $m - 1$, namely 2^{m-1} , and thus

$$\sigma_n = 2^n \left(1 - \sum_{j=m}^n \frac{\tau_j}{2^j} \right) = 2^n \sum_{j=n+1}^{\infty} \frac{\tau_j}{2^j}, \quad (18)$$

where in the last step we have used the identity $\sum_{j=m}^{\infty} \tau_j / 2^j = 1$, which is proved in the Appendix — Eq.(30).

Dividing both sides of Eq.(18) by 2^n and summing, we find

$$\sum_{n=0}^{\infty} \frac{\sigma_n}{2^n} = \sum_{n=0}^{\infty} \sum_{j=n+1}^{\infty} \frac{\tau_j}{2^j}. \quad (19)$$

On changing the order of the summations, we can rewrite the right-hand side in the form

$$\sum_{j=0}^{\infty} \frac{\tau_j}{2^j} \sum_{n=0}^{j-1} 1 = \sum_{j=m}^{\infty} j \frac{\tau_j}{2^j}. \quad (20)$$

This is the average value of the lengths of the strings that terminate in T_m , for τ_j is the number of them with length j , while 2^{-j} is their probabilistic weight. The lower limit of the summation on the right-hand side has been changed to $j = m$, since $\tau_j = 0$ for $j < m$. In the notation of the previous sections, the sum (20) is N_{T_m} , so Eq.(19) becomes

$$N_{T_m} = \sum_{n=0}^{\infty} \frac{\sigma_n}{2^n}, \quad (21)$$

where we recall that $\sigma_0 = 1$.

Let us first consider the case in which $t_1 = 1$ and $t_j = 0$ for $2 \leq j \leq m$, so the string of length m has the form

$$T_m = (100 \dots 0).$$

A string of length $m + n$ that contains T_m just once, right at the end, has the form

$$(S_n T_m) = (s_1 s_2 \dots s_n 100 \dots 0). \quad (22)$$

The number of strings (22) is equal to the number of strings (15), there being only one way to make a string $S_n T_m$ from a string S_n , namely by adding T_m to its end, so

$$\sigma_n = \tau_{m+n} . \quad (23)$$

On substituting Eq.(23) into Eq.(21), we obtain

$$N_{100\dots 0} = \sum_{n=0}^{\infty} \frac{\tau_{m+n}}{2^n} = 2^m \sum_{j=m}^{\infty} \frac{\tau_j}{2^j} = 2^m ,$$

where Eq.(30) has been used.

The next case we shall treat is that in which $t_1 = 1$ for $1 \leq j \leq m$, so the string of length m contains only ones:

$$T_m = (111\dots 1)$$

Consider the string of length $m + n$ of the form

$$(S_n T_m) = (s_1 s_2 \dots s_n 111\dots 1) . \quad (24)$$

The number of such strings is equal to the number of strings (S_n), but this is not now equal to τ_{m+n} , since some of the strings S_n will end in 01, and when T_m is appended to those strings, they will belong to the tally τ_{m+n-1} . Similarly, some of the strings S_n will end in 011, and when T_m is appended to those strings, they will belong to the tally τ_{m+n-2} , and so on, up to the strings S_n that end in zero, followed by $m - 1$ ones. When T_m is appended to these strings, they will belong to the tally τ_{n+1} . So instead of Eq.(23), we have in the present case

$$\sigma_n = \sum_{j=1}^m \tau_{j+n} . \quad (25)$$

Substituting Eq.(25) into Eq.(21), we now find

$$N_{111\dots 1} = \sum_{n=0}^{\infty} 2^{-n} \sum_{j=1}^m \tau_{j+n} = \sum_{j=1}^m 2^j \sum_{i=m}^{\infty} \frac{\tau_i}{2^i} = \sum_{j=1}^m 2^j ,$$

where Eq.(30) has again been used.

$N_{100\dots 0}$ and $N_{111\dots 1}$ are two extreme cases. In general a string T_m of length m will generate a series

$$\sigma_n = \sum_{j=1}^m c_j \tau_{j+n} , \quad (26)$$

where always $c_m = 1$, but where some of the other coefficients c_j may be zero, the others being equal to one. In this general case we find

$$N_{T_m} = \sum_{n=0}^{\infty} 2^{-n} \sum_{j=1}^m c_j \tau_{j+n} = \sum_{j=1}^m c_j 2^j \sum_{i=m}^{\infty} \frac{\tau_i}{2^i} = \sum_{j=1}^m c_j 2^j, \quad (27)$$

once more with use of Eq.(30). This expression shows that N_{T_m} is always an even integer, being the sum of positive integral powers of two. The coefficient c_j is equal to zero if the string $(S_n T_m)$, truncated at order $n + j$, does not contain T_m for its last m bits; otherwise it is equal to one. For a given m , the following inequalities hold:

$$2^m \leq N_{T_m} \leq \sum_{j=1}^m 2^j = 2^{m+1} - 2,$$

with Eq.(23) and Eq.(25), respectively realizing the extreme possibilities.

By way of example, we shall show how to extract the coefficients c_j of Eq.(26) in a particular case in which $m = 5$, namely the string T_{10101} . On placing S_n before T_{10101} we obtain a string

$$(S_n T_5) = (s_1 s_2 \dots s_n 10101), \quad (28)$$

and the number of such strings is the same as the number of strings of the type S_n , namely σ_n . However, the strings (28) encompass not only those of length $5 + n$ that terminate with T_{10101} , but also strings of length $5 + n - 2 = 3 + n$, for which the first 10 of 10101 belongs to S_n , i.e. strings of the sort

$$(s_1 s_2 \dots s_{n-2} 1010101).$$

Similarly, (28) also includes strings of length $5 + n - 4 = 1 + n$ for which the first 1010 of 10101 belongs to S_n . No such string could have terminated at order $4 + n$, since the first four bits of 10101, namely 1010, are not consistent with the last four bits of the required termination; namely 0101. That is,

$$(s_1 s_2 \dots s_{n-1} s_n 10101),$$

cannot belong to the tally τ_{n+4} , irrespective of whether s_n is 0 or 1, so it must be part of the tally τ_{n+5} . For a similar reason, no such string could have terminated at order $2 + n$, since the $(2 + n)$ th bit would be 0, whereas the last bit of the termination should be 1. So we obtain

$$\sigma_n = \tau_{n+1} + \tau_{n+3} + \tau_{n+5},$$

and thus

$$N_{10101} = 2 + 2^3 + 2^5 = 42.$$

The results shown in Table 3 were obtained by an application of this method.

Appendix: Convergence and completeness

Divide both sides of Eq.(17) by 2^n and sum:

$$\sum_{n=m}^p \frac{\sigma_{n-1}}{2^{n-1}} = \sum_{n=m}^p \frac{\tau_n}{2^n} + \sum_{n=m}^p \frac{\sigma_n}{2^n},$$

and so

$$\sum_{n=m}^p \frac{\tau_n}{2^n} = \sum_{n=m-1}^{p-1} \frac{\sigma_n}{2^n} - \sum_{n=m}^p \frac{\sigma_n}{2^n} = \frac{\sigma_{m-1}}{2^{m-1}} - \frac{\sigma_p}{2^p}. \quad (29)$$

Now $\sigma_{m-1} = 2^{m-1}$, since all 2^{m-1} strings of length $m-1$ are of the type S_{m-1} . It follows that

$$\sum_{n=m}^p \frac{\tau_n}{2^n} \leq 1,$$

for all p , which means that the infinite series $\sum_{n=m}^{\infty} \tau_n/2^n$ is convergent, since all the terms are positive and the partial sums are uniformly bounded. This convergence means in particular that $\tau_n/2^n$ tends to 0 in the limit that n goes to infinity. However, from Eq.(26) we see that

$$\frac{\sigma_n}{2^n} = \sum_{j=1}^m c_j 2^j \frac{\tau_{j+n}}{2^{j+n}}$$

goes to zero as n tends to infinity at constant m . Therefore from Eq.(29) it follows that

$$\sum_{n=m}^{\infty} \frac{\tau_n}{2^n} = 1 - \lim_{p \rightarrow \infty} \frac{\sigma_p}{2^p} = 1. \quad (30)$$

Acknowledgement: We would like to thank Barteld Kooi for having brought this interesting problem to our notice.

References

- [1] FELLER, W. (1968). *An Introduction to Probability and its Application, Vol. I.*, 3rd edition. New York, Wiley.

- [2] GARDNER, M. (1988). “Nontransitive Paradoxes,” in *Time Travel and Other Mathematical Bewilderments*. New York: W. H. Freeman, pp. 64-66.

| Num bits | Average | Terminating String(s) |
|----------|---------|---|
| 2 | 4 | 10 |
| | 6 | 11 |
| 3 | 8 | 100, 110 |
| | 10 | 101 |
| | 14 | 111 |
| 4 | 16 | 1000, 1100, 1110 |
| | 18 | 1001, 1011, 1101 |
| | 20 | 1010 |
| | 30 | 1111 |
| 5 | 32 | 10000, 10100, 11000, 11010, 11100, 11110 |
| | 34 | 10001, 10011, 10111, 11001, 11101 |
| | 36 | 10010, 10110 |
| | 38 | 11011 |
| | 42 | 10101 |
| | 62 | 11111 |
| 6 | 64 | 100000, 101000, 101100, 110000, 110010, 110100, 111000, 111010, 111100, 111110 |
| | 66 | 100001, 100011, 100101, 100111, 101001, 101011, 101111, 110001, 110101, 111001, 111101 |
| | 68 | 100010, 100110, 101110 |
| | 70 | 110011, 110111, 111011 |
| | 72 | 100100, 110110 |
| | 74 | 101101 |
| | 84 | 101010 |
| | 126 | 111111 |

Average number of tosses for doubles, triples, quadruples, quintuples and sextuples

Table 3