A Fresh Look at the "Hot Hand" Paradox

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We discuss the "hot hand" paradox within the framework of the backward Kolmogorov equation. We use this approach to understand the apparently paradoxical features of the statistics of fixed-length sequences of heads and tails upon repeated fair coin flips. In particular, we compute the average waiting time for the appearance of specific sequences. For sequences of length 2, the average time until the appearance of the sequence HH (heads-heads) equals 6, while the waiting time for the sequence HT (heads-tails) equals 4. These results require a few simple calculational steps by the Kolmogorov approach. We also give complete results for sequences of lengths 3, 4, and 5; the extension to longer sequences is straightforward (albeit more tedious). Finally, we compute the waiting times $T_{n\mathrm{H}}$ for an arbitrary length sequences of all heads and $T_{n(\mathrm{HT})}$ for the sequence of alternating heads and tails. For large n, $T_{2n\mathrm{H}} \sim 3T_{n(\mathrm{HT})}$.

I. INTRODUCTION

In a repeated flips of a fair coin, the outcomes H (heads) or T (tails) occur with 50% probability. Thus in a long string of N coin flips, the number of heads and tails, N_H and N_T, will be nearly equal, with $|N_H - N_T|$ of the order of $\sqrt{N}$. Given that H and T appear equiprobably, a naive expectation might be that the average frequencies of specific fixed-length sequences of H's and T's should be the same; that is, the sequence HHTH should occur with the same frequency as HTTH. As a corollary of this expectation, the waiting time before encountering either of these sequences should be the same. Surprisingly, this expectation is false.

The paradoxical nature of this so-called "hot hand" paradox has spawned considerable discussion and literature that has ultimately resolved this intriguing issue, see, e.g., Refs. [1–10]. However, the approaches given in these references are complicated and the simplicity of the mechanism that underlies the apparent paradox can be lost in calculational details. Here we give an alternative route to understand the hot hand paradox that is based on the backward Kolmogorov equations [11, 12]. This formulation has proved to be extremely useful in a variety of first-passage processes. We will use this approach to compute the waiting time for specific sequences of H's and T's of length up to 5, and it may be straightforwardly extended to longer sequences. We give an intuitive reason why different sequences of the same length do not occur with the same frequency. We also derive the waiting time for particularly simple sequences of arbitrary length, namely, the sequence of n consecutive H's and the sequence of n consecutive (HT)'s. We find that $T_{2n\mathrm{H}} \sim 3T_{n(\mathrm{HT})}$, so that $2n$ heads in a row is three times less frequent than n (HT)'s in a row.

The idea underlying the backward Kolmogorov equation is quite simple. Consider a Markov process that is currently in a particular state S. We want to compute the average time $T_{S\to F}$ until the process reaches a specified final state F. Suppose that there are two possible outcomes at each stage of the process that occur with equal probability. That is, from state S, the process transitions either to state S' or to S'', each with probability $\frac{1}{2}$. Suppose further that the time required for each transition equals 1. Since the Markov process has no memory, when either of the states S' or S'' are reached, the the process starts anew. Consequently, the hitting time from S is just the average of the hitting times starting from either S' or S'' plus the time spent in the transition itself. That is

$$T_{S\to F} = \tfrac{1}{2}(T_{S'\to F} + 1) + \tfrac{1}{2}(T_{S''\to F} + 1)\,. \tag{1}$$

We will use this basic equation to compute the waiting time for specific sequences of H's and T's of a given length as a result of repeated flips of a fair coin.

II. DOUBLET SEQUENCES

We start with the simplest example of length-2 sequences. The possible sequences are HH, HT, TH, and HH. Because the coin is fair, we obtain the same statistics by the substitution $H \leftrightarrow T$, so that the waiting time for the sequences TT and TH is the same as that for HH and HT. Consequently, we only need to consider the first two sequences. How long does one have to wait before encountering each of these sequences in a long string of fair coin flips?

Using Eq. (1) as our starting point, we first compute the waiting time T_{HH} to encounter an HH sequence. For this purpose, we introduce the auxiliary restricted times:

- A, the average waiting time for the sequence HH starting with an H.
- B, the average waiting time for the sequence HH starting with a T.

These two times obey the backward equations

$$\begin{aligned} A &= \tfrac{1}{2}\times 2 + \tfrac{1}{2}(1+B) \\ B &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+A)\,. \end{aligned} \tag{2a}$$

These two equations express the waiting times A and B as the average time to reach the desired final state after one coin flip, plus the time for a single coin flip itself. Thus in the equation for A, the first term accounts for the next coin flip being H (which occurs with probability $\frac{1}{2}$) after which the sequence HH has been generated. The factor 2 arises because it takes two coin flips to generate the sequence HH; the fact that the starting state is H does not count as time step. The second term accounts for the next coin flip being T. Again, the probability for this event is $\frac{1}{2}$. Once a T appears, the waiting time to generate an HH sequence is B by definition. Consequently, the factor $(1+B)$ accounts for the time spent in making a single coin flip plus the waiting time when the sequence string starts with T. Solving these two equations gives $A=5$, $B=7$. Since H and T appear equiprobably, on average, in a long series of fair coin flips, we have $T_{HH}=\frac{1}{2}(A+B)=6$.

For T_{HT}, we introduce

- A, the average waiting time for the sequence HT starting with H.
- B, the average waiting time for the sequence HT starting with T.

Using the same reasoning as given above, these two times obey the backward equations

$$\begin{aligned} A &= \tfrac{1}{2}\times 2 + \tfrac{1}{2}(1+A) \\ B &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B)\,. \end{aligned} \tag{2b}$$

The solution to (2b) are $(A,B)=(3,5)$. Again, because H and T appear equiprobably in a long series of coin flips, $T_{HT}=\frac{1}{2}(A+B)=4$.

Why are these two times different? The key lies in the second term of first lines of Eqs. (2a) and (2b). These terms account for a "mistake". For example, in Eq. (2a), if the next coin flip is T, one has to "start over" to generate HH. The soonest that the next HH can happen immediately after a T is after two more coin flips. In contrast, in Eq. (2b), if the next coin flip is H (again a mistake), the process "starts over". Now, however, the soonest that the next HT sequence can appear is after only one more coin flip.

III. TRIPLET SEQUENCES

Let's now generalize to triplet sequences. The $2^3=8$ distinct triplet sequences are HHH, HHT, HTH, and THH and their counterparts that are obtained by the substitution H↔T. By left/right symmetry, the triplets HHT and THH have identical statistics, so the only distinct sequences are HHH, HHT, and HTH.

We define $T_{\rm HHH}$ as the waiting time to encounter the sequence with three consecutive H's. To compute this time, we define the auxiliary restricted times:

- A, the waiting time for HHH when the current state is H;
- B, the waiting time for HHH when the current state is HH;
- C, the waiting time for HHH when the current state is T.

Following the same reasoning that led to Eqs. (2a), the above times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\ B &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+C) \\ C &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+C)\,. \end{aligned} \tag{3}$$

The first term in the equation for B merits explanation. From the state HH, the desired sequence HHH is obtained with probability $\frac{1}{2}$, while the time for this event is 2. Here the time is measured starting *before* the second H has been added to the sequence. The solution to (3) is $(A,B,C)=(13,9,15)$. Since the probability to find an H or a T are equal, the waiting time to encounter the sequence HHH is just the average of the times to find HHH when starting with an H or starting with a T. Thus $T_{\rm HHH}=\frac{1}{2}(A+C)=14$.

Similarly, let $T_{\rm HHT}$ be the waiting time to encounter the sequence HHT. Here, we introduce the auxiliary times:

- A, the waiting time for HHT when the current state is H;
- B, the waiting time for HHT when the current state is HH;
- C, the waiting time for HHT when the current state is T.

These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\ B &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+B) \\ C &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+C)\,, \end{aligned} \tag{4}$$

There is a subtlety in the second equation that needs explanation. If the initial state is HH, then after adding an H, the current state is still HH, so that that the second term involves B. This feature that the initial state consists of a subsequence of length greater than one plays an increasing role for longer sequences (see Appendices A and B). The solution to (4) is $(A, B, C) = (7, 3, 9)$. The waiting time to encounter the sequence HHT is the average of the times to find HHT after an H or after a T, which gives again $T_{\rm HHT} = \frac{1}{2}(A + C) = 8$.

Finally, let $T_{\rm HTH}$ be the waiting time to encounter the sequence HTH. We introduce the auxiliary times:

- A, the waiting time for HTH when the current state is H;
- B, the waiting time for HTH when the current state is HT;
- C, the waiting time for HHT when the current state is T.

These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B) \\ B &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+C) \\ C &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+C)\,, \end{aligned} \tag{5}$$

with solutions $(A, B, C) = (9, 7, 11)$. Thus the waiting time to encounter the sequence HTH is the average of the times to find HTH after an H or after a T, which gives $T_{\rm HTH} = \frac{1}{2}(A + C) = 10$.

To summarize, $T_{\rm HHH} = 14$, $T_{\rm HHT} = 8$, and $T_{\rm HTH} = 10$, in agreement with the times quoted in, for example, Ref. [6].

IV. QUARTET AND QUINTET SEQUENCES

The six distinct quartets are HHHH, HHHT, HHTH, HHTT, HTHT, and HTTH. To not inflict even more tedious algebra upon the casual reader, all the calculational details are given in Appendix A. Here we merely quote the waiting times in reverse time order:

$$T_{\rm 4H} = 30 \quad T_{\rm HTHT} = 20 \quad T_{\rm HHTH} = T_{\rm HTTH} = 18 \quad T_{\rm HHHT} = T_{\rm HHTT} = 16\,. \tag{6}$$

There are nine distinct quintets: HHHHH, HHHHT, HHHTH, HHTHH, HHHTT, HHTHT, HTHHT, HTHTH, and HTTHH. There are additional non-independent sequences that are obtained by either the interchange $H \leftrightarrow T$ or by reading the above sequences in reverse order. The waiting times are (see Appendix B for details):

$$T_{\rm 5H} = 62 \quad T_{\rm HTHTH} = 42 \quad T_{\rm HHTHH} = 38 \quad T_{\rm HTHHT} = 36 \quad T_{\rm HHHTH} = T_{\rm HTTHH} = 34 \quad T_{\rm HHHTH} = T_{\rm HHTHT} = 32\,. \tag{7}$$

All these results agree with those given in [6].

V. SIMPLE ARBITRARY LENGTH SEQUENCES

A. n Consecutive H's

While the calculational details for longer sequences are straightforward, they become progressively more tedious as the sequence length is increased. However, for the sequence of n consecutive H's, the equations for the restricted times are sufficiently systematic in character that they can be solved. To this end, we first define the following set of restricted times:

- A_k, the waiting time for nH when the current state consists of k consecutive H's;

- B, the waiting time for nH when the current state is T.

These times satisfy

$$\begin{aligned}
A_1 &= \tfrac{1}{2}(1+A_2)+\tfrac{1}{2}(1+B)\\
A_2 &= \tfrac{1}{2}(1+A_3)+\tfrac{1}{2}(1+B)\\
&\vdots\\
A_{n-2} &= \tfrac{1}{2}(1+A_{N-1})+\tfrac{1}{2}(1+B)\\
A_{n-1} &= \tfrac{1}{2}(2)+\tfrac{1}{2}(1+B)\\
B &= \tfrac{1}{2}(1+A_1)+\tfrac{1}{2}(1+B)\,.
\end{aligned} \tag{8a}$$

From the last equation, we have $B = 2 + A_1$, while from the penultimate equation we can replace the factor $\frac{1}{2}(1+B)$ everywhere with $A_{n-1} - 1$. Thus Eqs. (8a) become

$$\begin{aligned}
A_1 &= \tfrac{1}{2}(1+A_2)+A_{n-1}-1\\
A_2 &= \tfrac{1}{2}(1+A_3)+A_{n-1}-1\\
&\vdots\\
A_{n-3} &= \tfrac{1}{2}(1+A_{n-2})+A_{n-1}-1\\
A_{n-2} &= \tfrac{1}{2}(1+A_{n-1})+A_{n-1}-1
\end{aligned} \tag{8b}$$

Now the equation for A_{n-2} can be written in terms of A_{n-1} only. Similarly, the equation for A_{n-3} can be written in terms of A_{n-1} only. Continuing this procedure, we find

$$A_{n-k} = -\frac{2^{k-1}-1}{2^{k-1}} + \frac{2^k-1}{2^{k-1}}A_{n-1}\,. \tag{9a}$$

In particular

$$A_1 = -\frac{2^{n-2}-1}{2^{n-2}} + \frac{2^{n-1}-1}{2^{n-2}}A_{n-1}\,. \tag{9b}$$

From the original equation for A_{n-1}, we eliminate B in favor of A_1 and obtain $A_{n-1} = \frac{5}{2} + \frac{1}{2}A_1$. Using this in (9b), we solve for A_1 and find, after some straightforward steps, $A_1 = 2^{n+1} - 3$. Finally, $T_{\rm nH}$ is the average of the waiting times starting from H and starting from T. That is,

$$T_{\rm nH} = \frac{1}{2}(A_1+B) = 2^{n+1}-2\,. \tag{10}$$

B. n Consecutive (HT)'s

A similar calculation can be carried out for the sequence of n consecutive (HT)'s. Here, we first define the following set of restricted times:

- A_{2k-1}, the waiting time for n(HT) when the current state is $(k-1)$(HT)H;
- A_{2k}, the waiting time for n(HT) when the current state is k(HT);
- B, the waiting time for n(HT) when the current state is T.

These times satisfy

$$\begin{aligned} A_1 &= \tfrac{1}{2}(1+A_2) + \tfrac{1}{2}(1+A_1) \\ A_2 &= \tfrac{1}{2}(1+A_3) + \tfrac{1}{2}(1+B) \\ A_3 &= \tfrac{1}{2}(1+A_4) + \tfrac{1}{2}(1+A_1) \\ A_4 &= \tfrac{1}{2}(1+A_5) + \tfrac{1}{2}(1+B) \\ &\vdots \\ A_{2n-3} &= \tfrac{1}{2}(1+A_{2n-2}) + \tfrac{1}{2}(1+A_1) \\ A_{2n-2} &= \tfrac{1}{2}(1+A_{2n-1}) + \tfrac{1}{2}(1+B) \\ A_{2n-1} &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+A_1) \\ B &= \tfrac{1}{2}(1+A_1) + \tfrac{1}{2}(1+B)\,. \end{aligned} \tag{11}$$

From the last two equations, we obtain

$$\tfrac{1}{2}(1+A_1) = A_{2n-1} \qquad \tfrac{1}{2}(1+B) = A_{2n-1}\,. \tag{12}$$

We now eliminate A_1 and B from Eqs. (11). With this substitution, we can then recursively solve for A_{2n-2}, $A_{2n-3}, \ldots$ in terms of A_{2n-1} and obtain

$$A_{2n-k} = -\frac{S_k}{2^{k-1}} + \frac{2^k-1}{2^{k-1}}\,A_{2n-1}\,. \tag{13}$$

where

$$S_k = \sum_{j=0}^{(k-3)/2} 4^j\,.$$

We now use $k = 2n-1$ in (13) to obtain A_1 in terms of A_{2n-1}, and then use the first of (12) to eliminate A_{2n-1} in favor of A_1 and ultimately solve for A_1. After straightforward algebra, the final result is

$$A_1 = -2S_{2n-1} + 3(2\cdot 4^{n-1}-1) = -\tfrac{2}{3}(4^{n-1}-1) + 3(2\cdot 4^{n-1}-1)\,. \tag{14}$$

The waiting time $T_{n(\mathrm{HT})}$ is given by $\frac{1}{2}(A_1+B) = A_1+1$, and after some algebra, we obtain

$$T_{n(\mathrm{HT})} = \tfrac{4}{3}(4^n-1)\,. \tag{15}$$

It is intriguing to compare the times $T_{n\mathrm{H}}$ and $T_{n(\mathrm{HT})}$. The fair comparison is between $T_{2n\mathrm{H}}$ and $T_{n(\mathrm{HT})}$; i.e, between strings of the same length. Asymptotically, Eq. (10) gives $T_{2n\mathrm{H}} \sim 4^{n+1}$, while (15) gives $T_{n(\mathrm{HT})} \sim \frac{1}{3}\cdot 4^{n+1}$. One has to wait three times as long, on average, to encounter a sequence of $2n$ H's in a row compared to a sequence of n (HT)'s in a row.

VI. CONCLUDING COMMENTS

While most of the results given here are already known, the backward Kolmogorov approach provides a fresh and powerful perspective on how to calculate waiting times for specific sequences of H's and T's in a long string of repeated flips of a fair coin. Once one understands the idea that underlies the Kolmogorov approach, the computation of the waiting times for specific sequences is straightforward and direct. Another important aspect of this approach is that it is not limited to mean waiting times. This same method can be applied to compute any *functional* of the waiting time, such as higher moments or even the characteristic function, $\langle\exp(-sT)\rangle$.

A surprising conclusion of repeated fair coin flips is that the waiting times, or equivalently, the occurrence frequencies, for specific sequences of H's and T's of the same length are different. The effect is quite pronounced for a long string of $2n$ H's in a row compared to the string of n (HT)'s in row. For large n, one has to wait three times longer to encounter the former sequence compared to the latter.

Although our approach unambiguously illustrates the different waiting times/frequencies of fixed-length sequences, this seemingly paradoxical phenomenon simple requires careful thought to appreciate intuitively.

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Appendix A: Calculational Details for Quartet Sequences

The calculation T_{HHHH} was given in Sec. V A and we start with T_{HHHT}. To compute T_{HHHT}, we define A, B, C, and D as the waiting time for HHHT when the current state is H, HH, HHH, and T, respectively. These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+D) \\ B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+D) \\ C &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+C) \\ D &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\,, \end{aligned} \tag{A1}$$

whose solutions are $(A,B,C,D) = (15,11,3,17)$, from which $T_{\mathrm{HHHT}} = \frac{1}{2}(A+D) = 16$.

To compute T_{HHTH}, we define A, B, C, and D as the waiting time for HHTH when the current state is H, HH, HHT, and T, respectively. These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+D) \\ B &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\ C &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+D) \\ D &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\,, \end{aligned} \tag{A2}$$

whose solutions are $(A,B,C,D) = (17,13,11,19)$, from which $T_{\mathrm{HHTH}} = \frac{1}{2}(A+D) = 18$.

To compute T_{HHTT}, we define A, B, C, D as the waiting time for HHTT when the current state is H, HH, HHT, and T, respectively. These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+D) \\ B &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\ C &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+A) \\ D &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\,, \end{aligned} \tag{A3}$$

whose solutions are $(A,B,C,D) = (15,11,9,17)$, from which $T_{\mathrm{HHTT}} = \frac{1}{2}(A+D) = 16$.

To compute T_{HTHT}, we define A, B, C, D as the waiting time for HTHT when the current state is H, HT, HTH, and T, respectively. These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B) \\ B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+D) \\ C &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+A) \\ D &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\,, \end{aligned} \tag{A4}$$

whose solutions are $(A,B,C,D) = (19,17,11,21)$, from which $T_{\mathrm{HTHT}} = \frac{1}{2}(A+D) = 20$.

Finally, to compute T_{HTTH}, we define A, B, C, D as the waiting time for HTTH when the current state is H, HT, HTT, and T, respectively. These times satisfy

$$\begin{aligned} A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B) \\ B &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+C) \\ C &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+D) \\ D &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\,, \end{aligned} \tag{A5}$$

whose solutions are $(A,B,C,D) = (17,15,11,19)$, from which $T_{\mathrm{HTTH}} = \frac{1}{2}(A+D) = 18$.

Appendix B: Calculational Details for Quintet Sequences

Again, the calculation T_{HHHHH} was given in Sec. V A and we start with T_{HHHHT}. To compute T_{HHHHT}, we define A, B, C, D, E as the waiting time for HHHHT when the current state is H, HH, HHH, HHHH, and T, respectively.

These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+E) \\
B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+E) \\
C &= \tfrac{1}{2}(1+D) + \tfrac{1}{2}(1+E) \\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+D) \\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B1}$$

whose solutions are $(A, B, C, D, E) = (31, 27, 19, 3, 33)$ and we obtain $T_{\mathrm{HHHHT}} = \frac{1}{2}(A+E) = 32$.

To compute T_{HHHTH}, we define A, B, C, D, E as the waiting time for HHHTH when the current state is H, HH, HHH, HHHT, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+E) \\
B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+E) \\
C &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+D) \\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+E) \\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B2}$$

whose solutions are $(A, B, C, D, E) = (33, 29, 21, 19, 35)$ and we obtain $T_{\mathrm{HHHTH}} = \frac{1}{2}(A+E) = 34$.

To compute T_{HHTHH}, we define A, B, C, D, E as the waiting time for HHTHH when the current state is H, HH, HHT, HHTH, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+E) \\
B &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\
C &= \tfrac{1}{2}(1+D) + \tfrac{1}{2}(1+E) \\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+E) \\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B3}$$

whose solutions are $(A, B, C, D, E) = (37, 33, 31, 21, 39)$ and we obtain $T_{\mathrm{HHTHH}} = \frac{1}{2}(A+E) = 38$.

To compute T_{HHHTT}, we define A, B, C, D, E as the waiting time for HHHTT when the current state is H, HH, HHH, HHHT, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E) \\
B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+E) \\
C &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+D) \\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+A) \\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B4}$$

whose solutions are $(A, B, C, D, E) = (31, 27, 19, 17, 33)$ and we obtain $T_{\mathrm{HHHTT}} = \frac{1}{2}(A+E) = 32$.

To compute T_{HHTHT}, we define A, B, C, D, E as the waiting time for HHTHT when the current state is H, HH, HHT, HHTH, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+E) \\
B &= \tfrac{1}{2}(1+B) + \tfrac{1}{2}(1+C) \\
C &= \tfrac{1}{2}(1+D) + \tfrac{1}{2}(1+E) \\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+B) \\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B5}$$

whose solutions are $(A, B, C, D, E) = (31, 27, 25, 15, 33)$ and we obtain $T_{\mathrm{HHTHT}} = \frac{1}{2}(A+E) = 32$.

To compute T_{HTHHT}, we define A, B, C, D, E as the waiting time for HTHHT when the current state is H, HT, HTH, HTHH, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B)\\
B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+E)\\
C &= \tfrac{1}{2}(1+D) + \tfrac{1}{2}(1+B)\\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+A)\\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B6}$$

whose solutions are $(A, B, C, D, E) = (35, 33, 27, 19, 37)$ and we obtain $T_{\text{HTHHT}} = \frac{1}{2}(A + E) = 36$.

To compute T_{HTHTH}, we define A, B, C, D, E as the waiting time for HTHTH when the current state is H, HT, HTH, HTHT, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B)\\
B &= \tfrac{1}{2}(1+C) + \tfrac{1}{2}(1+E)\\
C &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+D)\\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+E)\\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B7}$$

whose solutions are $(A, B, C, D, E) = (41, 39, 33, 23, 43)$ and we obtain $T_{\text{HTHTH}} = \frac{1}{2}(A + E) = 42$.

To compute T_{HTTHH}, we define A, B, C, D, E as the waiting time for HTTHH when the current state is H, HT, HTT, HTTH, and T, respectively. These times satisfy

$$\begin{aligned}
A &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+B)\\
B &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+C)\\
C &= \tfrac{1}{2}(1+D) + \tfrac{1}{2}(1+E)\\
D &= \tfrac{1}{2}(2) + \tfrac{1}{2}(1+B)\\
E &= \tfrac{1}{2}(1+A) + \tfrac{1}{2}(1+E)
\end{aligned} \tag{B8}$$

whose solutions are $(A, B, C, D, E) = (33, 31, 27, 17, 35)$ and we obtain $T_{\text{HTTHH}} = \frac{1}{2}(A + E) = 34$.

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