Fourier series — an example

Recall that the *Fourier series* for a 2π-periodic function f has the complex form

$$S(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}, \qquad c_n = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx. \tag{1}$$

The convergence theorem states that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, 2π-periodic, and piecewise C^1, then the series converges uniformly on $\mathbb{R}$, and $S(x) = f(x)$ for all $x \in \mathbb{R}$. Taking the limit $N \to \infty$ in the generalized Pythagorean theorem (14) and using the formula in the line before (15), we get

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx. \tag{2}$$

This result is called *Parseval's identity.*

Example. We can define $f(x)$ so that $f(x) = x^2$ for $x \in [-\pi, \pi]$ (note $f(-\pi) = f(\pi)$), and $f(x + 2\pi) = f(x)$ for all x. Then

$$c_n = (f, e_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 (\cos nx - i \sin nx)\, dx.$$

Since $x^2 \cos nx$ is even and $x^2 \sin nx$ is odd in x, the $x^2 \sin nx$ term integrates to zero and we find, after repeated integrations by parts, that for $n \neq 0$,

$$2\pi c_n = 2 \int_0^{\pi} x^2 \cos nx\, dx = 2 \left(x^2 \frac{\sin nx}{n} - 2x \frac{-\cos nx}{n^2} + 2 \frac{-\sin nx}{n^3} \right) \Bigg|_0^{\pi} = \frac{4\pi \cos n\pi}{n^2},$$

hence $c_n = 2(-1)^n / n^2$ for $n \neq 0$. The case $n = 0$ we have to treat separately:

$$c_0 = (f, e_0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2\, dx = \frac{1}{\pi} \int_0^{\pi} x^2\, dx = \frac{\pi^2}{3}.$$

Therefore, since $c_n = c_{-n}$, by the convergence theorem we have that for all $x \in [-\pi, \pi]$,

$$x^2 = c_0 + \sum_{n=1}^{\infty} c_n (e^{inx} + e^{-inx}) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx$$

Furthermore, Parseval's identity says that

$$|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + |c_{-n}|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x^2|^2\, dx,$$

which means

$$\frac{\pi^4}{9} + 2 \sum_{n=1}^{\infty} \frac{4}{n^4} = \frac{1}{\pi} \int_0^{\pi} x^4\, dx = \frac{1}{\pi} \frac{\pi^5}{5} = \frac{\pi^4}{5}.$$

Therefore (!)

$$\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \pi^4 \left(\frac{1}{5} - \frac{1}{9} \right) \frac{1}{8} = \frac{\pi^4}{90}.$$