Features

Complex …… NAPOLEON'S THEOREM …… Made Simple

Part 2

SHAILESH SHIRALI

In an earlier issue of At Right Angles, we had studied a gem of Euclidean geometry called Napoleon's Theorem, a result discovered in post-revolution France. We had offered proofs of the theorem that were computational in nature, based on trigonometry and complex numbers. We continue our study of the theorem in this article, and offer proofs that are more geometric in nature; they make extremely effective use of the geometry of rotations.

Napoleon's Theorem states the following. Let ABC be an arbitrary triangle. With the three sides of the triangle as bases, construct three equilateral triangles, each one outside $\triangle ABC$. Next, mark the centres P, Q, R of these three equilateral triangles. Napoleon's theorem asserts that $\triangle PQR$ is equilateral, irrespective of the shape of $\triangle ABC$. (See Figure 1.)

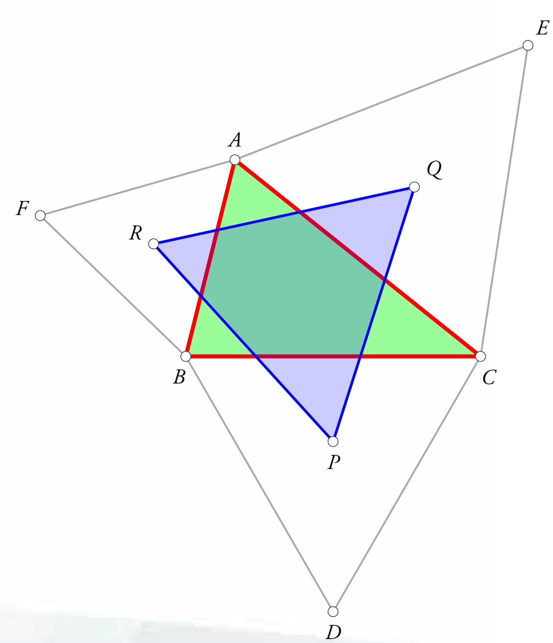


Figure 1

Keywords: Napoleon, equilateral triangle, rotation, parallelogram, basic proportionality theorem

In Part 1 of this article, we had considered computational proofs of Napoleon's theorem. In the trigonometric proof, we derived an expression for the length of one side of $\triangle PQR$ in terms of the sides and the angles of the $\triangle ABC$ (i.e., in terms of a, b, c, A, B, C). After going through the computations, we discovered that the resulting expression is symmetric in the parameters of the parent triangle. This fact suffices to prove that triangle PQR is equilateral.

Now we study an extremely elegant pure geometry proof of Napoleon's theorem; it makes very effective use of rotational geometry. In the literature, it is ascribed to an Irish mathematician, MacCool [1].

Before proceeding, we make a comment about rotations. Figure 2 shows a segment AB being subjected to two different rotations, both centred at a point O. The first one is through an angle of $+30^\circ$ (the positive sign tells us that the rotation is in a *counterclockwise direction*); it takes segment AB to segment A_1B_1. The second one is through an angle of -30° (the negative sign tells us that the rotation is in a *clockwise direction*); it takes segment AB to segment A_2B_2. Note that segments AB, A_1B_1 and A_2B_2 have equal length.

Now we get back to the proof of Napoleon's theorem. Consider a rotation through an angle of -30°, centred at B (see Figure 3; the rotation is in a clockwise direction). Our interest is in what this rotation 'does' to points R and P, i.e., where it takes these two points. Since $\measuredangle ABR = 30^\circ$ and $\measuredangle DBP = 30^\circ$, it follows that the image R_1 of R lies on side AB, and the image P_1 of P lies on side BD.

We argue as follows. The steps of the reasoning are laid out in itemised form at the right side of the diagram.

To see why $BR_1/BA = 1/\sqrt{3} = BP_1/BD$, you will first need to understand why $BR/BA = 1/\sqrt{3} = BP/BD$. But this follows from the basic geometry of an equilateral triangle. We leave the details for you to fill in.

From the fact that $BR_1/BA = BP_1/BD$, we deduce (using the basic proportionality theorem) that

$$R_1P_1 \parallel AD, \qquad \frac{R_1P_1}{AD} = \frac{1}{\sqrt{3}}. \tag{1}$$

Since $RP = R_1P_1$, it follows that:

$$\frac{RP}{AD} = \frac{1}{\sqrt{3}}. \tag{2}$$

In just the same way, we consider a rotation through an angle of $+30^\circ$, centred at C. Then, if the rotation takes Q and P to Q_2 and P_2, respectively, it follows that Q_2 lies on side AC, and P_2 lies on side CD; and arguing as earlier, we conclude that

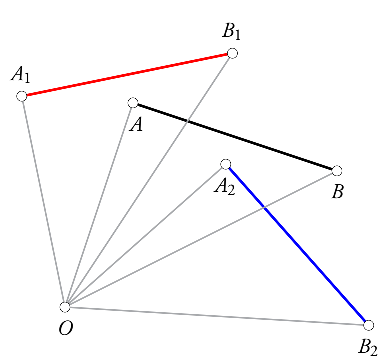
$$Q_2P_2 \parallel AD, \qquad \frac{Q_2P_2}{AD} = \frac{1}{\sqrt{3}}, \tag{3}$$

and

$$\frac{QP}{AD} = \frac{1}{\sqrt{3}}. \tag{4}$$

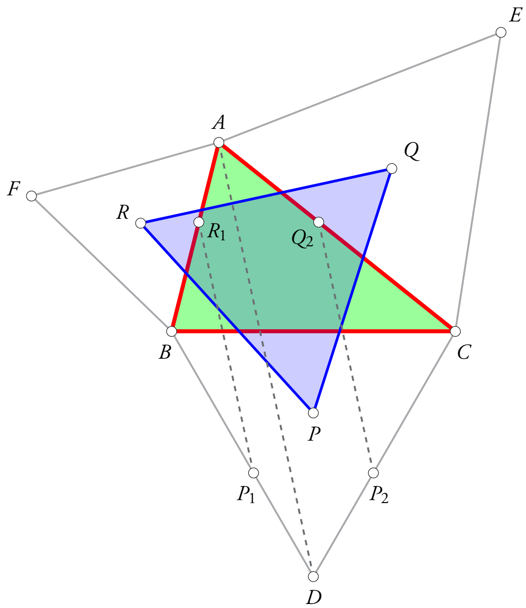
From (2) and (4), we conclude that $RP = QP$.

At this stage, we can proceed in two different ways. One way is to say that the same argument can be repeated for another pair of sides of $\triangle PQR$ and to conclude that equality therefore holds for the lengths of that pair of sides of $\triangle PQR$, and to



- $\measuredangle AOA_1 = +30^\circ = \measuredangle BOB_1$
- $\measuredangle AOA_2 = -30^\circ = \measuredangle BOB_2$
- Segments AB, A_1B_1 and A_2B_2 have equal length

Figure 2



(a) $R_1P_1 = RP$

(b) $BR_1 = BR$

(c) $BR_1/BA = 1/\sqrt{3}$

(d) $BP_1 = BP$

(e) $BP_1/BD = 1/\sqrt{3}$

(f) $BR_1/BA = BP_1/BD$

(g) $R_1P_1 \parallel AD$

(h) $R_1P_1/AD = 1/\sqrt{3}$

Figure 3

conclude from this that all three sides of the triangle have the same length. From this it follows that $\triangle PQR$ is equilateral. (We do not actually have to repeat all the steps of the argument. All we need to say is that since the argument worked for this particular pair of sides, it will also work for another pair of sides. Note that this is an appeal to symmetry.)

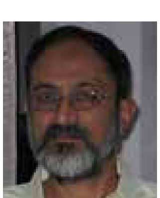
Another way is to say that $RP = QP$ and $\measuredangle RPQ = 60^\circ$; this is so because R_1P_1 is parallel to Q_2P_2, and we had obtained these two segments by rotations of segments RP and QP through 30° and 30° respectively, the first one through a rotation of -30° (i.e., 30° in a clockwise direction), and the second one through a rotation of $+30^\circ$ (i.e., 30° in an anticlockwise direction). So the two rotations are in *opposite directions*. After the two rotations, the resulting segments are parallel to each other, which means that *prior* to the rotations they must have been inclined at an angle of $(+30)^\circ - (-30)^\circ = 60^\circ$ to each other. This suffices to prove that $\triangle PQR$ is equilateral. $\square$

This proof is to be admired for its elegance and its compactness! It shows just how much can be accomplished using arguments belonging only to elementary geometry.

In Part 3 of this series, we will consider generalisations and further aspects of Napoleon's theorem.

References

1. M. R. F. Smyth, "MacCool's Proof of Napoleon's Theorem", Irish Math. Soc. Bulletin 59 (2007), 71–77, http://www.maths.tcd.ie/pub/ims/bull59/M5903.pdf



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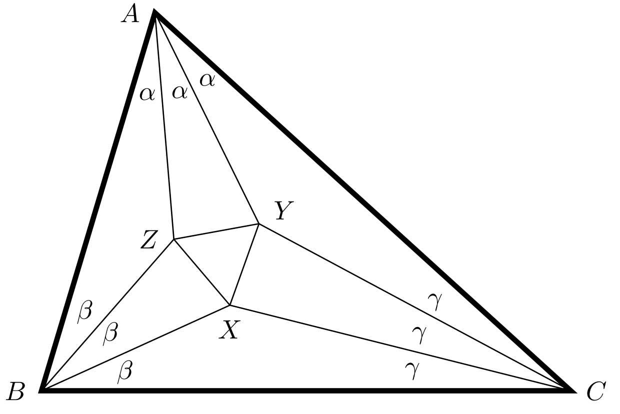
MacCool's Proof of Morley's Miracle

M. R. F. SMYTH

One of the most beautiful results in plane geometry is known as *Morley's Miracle* (1899). In essence it states that the triangle XYZ in the figure below is always equilateral. It features prominently on the front cover of the popular work [2] but is "still not as well-known as it deserves to be" [3]. The excellent web article [1] continues to track its development and also hosts a wide variety of proofs. None of the early proofs was easy but since 1990 elementary ones have emerged which are *backward* in the sense that they start from the equilateral triangle and eventually reconstruct the original. Finding a *direct* proof that matches them in brevity and simplicity has always been an elusive goal [3].



So I was amazed to find just such a proof in MacCool's notebooks and indeed it was so short that I nearly missed it. At first glance he seemed to be merely doodling, but moments later he had finished the proof and was working on something completely different.

Those readers who haven't heard of MacCool's notebooks may be surprised to learn that I am still less than halfway through the first one. Translation from the Ogham script is proving a long slow process and I am deeply indebted to one correspondent who reviewed and improved upon my original efforts, often spotting intricacies that I had overlooked. Although the gist of his arguments is always clear MacCool delights in recording only a minimum of information, and this particular proof was little more than a sketch decorated with jottings of line segments and angles. Like all the rest so far it is based solely on straight line geometry and similar triangles, but anyone interested in more advanced concepts may be pleased to know that diagrams containing circles begin to appear early in book two.

In his doodle the unit of measure is the perpendicular DX, and the lengths of BX and CX are s and s'. E and F are points on BC where $\angle BXE = \angle FXC = 60^o$. P and P' are where $BP = s$ and $CP' = s'$ and S is constructed so that $BS = s$ and $\angle SBX = 120^o$. This makes the four marked angles 60^o (even if ΔABC is obtuse). The rest of his construction is self-explanatory.
Now by (vi) and (vii) $2ST = 2SU + 2UT = s + 2\left(s - 2s^{-1}\right) = 3s - 4s^{-1}$ and $\Delta BQV \sim \Delta BDX$ yields $VQ = 1 - 4s^{-2}$ so $PQ = PV + VQ = 3 - 4s^{-2}$ thus

$$2ST = sPQ.$$

Then by (iv) and (v)

$$AY = \left(\frac{AC}{s'}\right)\left(\frac{PS}{ST}\right) = \frac{2AC.PS}{ss'PQ}.$$

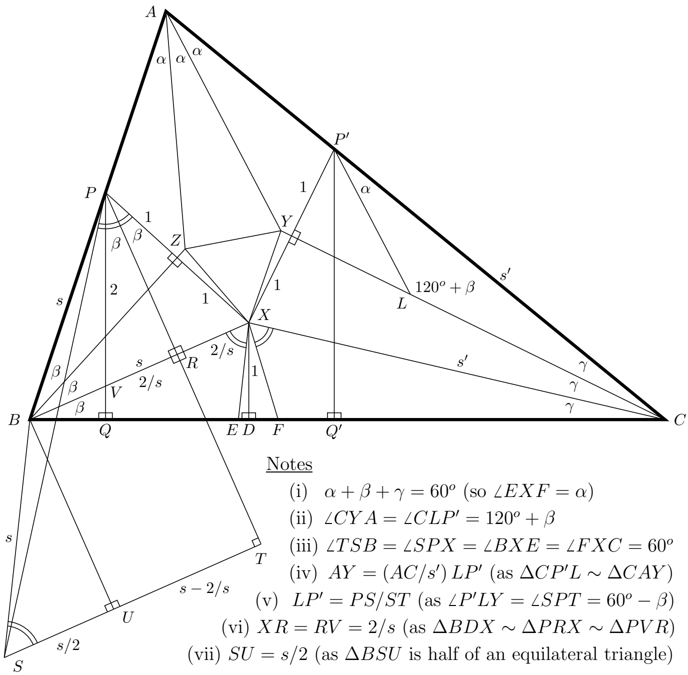
If W is the midpoint of PS then since ΔBSP is isosceles ΔBWP and ΔEDX will have identical angles, hence $\Delta BWP \sim \Delta EDX$ giving $XE = 2s/PS$. Therefore

$$XE.AY = \frac{4AC}{s'PQ}$$

and, by symmetry,

$$\frac{XE.AY}{XF.AZ} = \left(\frac{4AC}{s'PQ}\right)\left(\frac{sP'Q'}{4AB}\right) = \frac{sAC.P'Q'}{s'AB.PQ} = 1$$

because $PQ(AB/s) = P'Q'(AC/s')$ is the height of ΔABC. However this means $AZ : AY = XE : XF$ and as $\angle ZAY = \alpha = \angle EXF$



then $\Delta AZY \sim \Delta XEF$. Hence $\angle YZA = \angle FEX = 60^o + \beta$ and $\angle AYZ = \angle XFE = 60^o + \gamma$. Analogous arguments for ΔBXZ and ΔCYX show $\angle ZXB = 60^o + \gamma$, $\angle CXY = 60^o + \beta$ and $\angle BZX = \angle CYX = 60^o + \alpha$. All the angles in the doodle may now be deduced in terms of α, β, γ and it transpires that every angle of ΔXYZ is 60^o.

Here are some comments on the proof leading to a slight variation that may help to make it more intuitive. The underlying idea is to treat it as a series of left/right linkages. The results that $2ST = sPQ$ and $XE = 2s/PS$ are clearly "internal" to the left hand side. On the other hand LP' has a foot in each camp since it can be expressed both in terms of objects from the left PS/ST and objects from the right $s'AY/AC$. Equating these expressions gives a "cross-linkage" $XE.PQ = 4AC/(s'AY)$ and its companion $4AB/(sAZ) = XF.P'Q'$ which may then be combined to form the complicated looking quotient above. Even MacCool seems to have been shocked by the final devastating cross-linkage $PQ.AB/s = P'Q'.AC/s'$ which reduces this quotient to unity. After that the rest is plain sailing.

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[1] A. Bogomolny, http://www.cut-the-knot.org/triangle/Morley/index.shtml Morley's Miracle *from Interactive Mathematics Miscellany and Puzzles*

[2] H. S. M. Coxeter, and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, 1967.

[3] R. K. Guy, The Lighthouse Theorem, Morley and Malfatti — A budget of paradoxes, *Amer. Math. Monthly*, **114** (2007), 97–141.

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MacCool's Proof of Napoleon's Theorem

A sequel to The MacCool/West Point [1]

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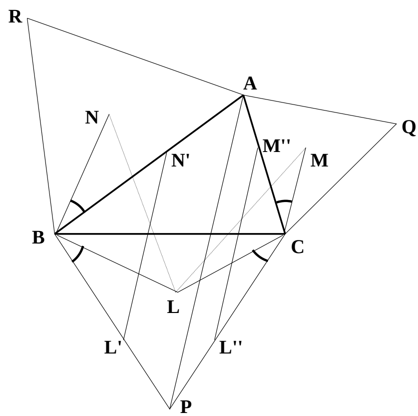
I came across this incredibly short proof in one of MacCool's notebooks. Napoleon's Theorem is one of the most often proved results in mathematics, but having scoured the World Wide Web at some length I have yet to find a proof that comes near to matching this particular one for either brevity or simplicity.

MacCool refers to equilateral triangles as *e-triangles* and he uses κ to denote the distance from a vertex of an e-triangle with unit side to its centroid. Naturally κ is a universal constant. He also treats anti-clockwise rotations as positive and clockwise rotations as negative.

Theorem 1. *If exterior e-triangles are erected on the sides of any triangle then their centroids form a fourth e-triangle.*



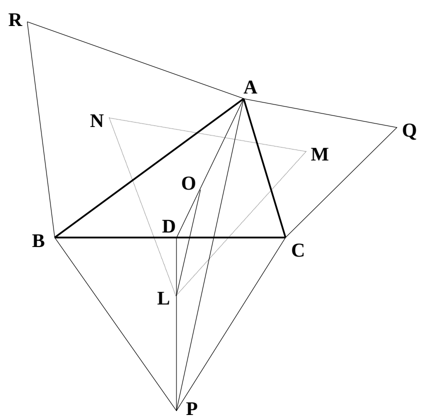
[1] Irish Math. Soc. Bulletin **57** (2006), 93–97

Proof. Let ABC be any triangle and construct the three exterior e-triangles with centroids L, M, N as shown. Rotate LN by $-30°$ about B to give $L'N'$ and LM by $+30°$ about C giving $L''M''$. Since all four marked angles are $30°$ it follows that L', N', L'', M'' will lie on BP, BA, CP, CA respectively and $\kappa = BL' : BP = BN' : BA = CL'' : CP = CM'' : CA$. Then by similarity $L'N' = \kappa AP = L''M''$ and $L'N' \parallel AP \parallel L''M''$ so $LN = LM$ and the angle between them is $30° + 30° = 60°$. Hence ΔLMN is an e-triangle. □

Theorem 1 is the classical Napoleon theorem. MacCool refers to the resultant e-triangle as the *outer triangle* to distinguish it from the *inner triangle* whose vertices are the centroids of the internally erected e-triangles.

The proof shows that each side of the outer triangle is equal to κAP. Since it could equally well have used BQ or CR instead this means $AP = BQ = CR$. The common length of these three lines is central to the next result. Also required is the fact that the centroid lies one third of the way along any median. This important property is easily deduced by observing that the medians of any triangle dissect it into six pieces of equal area.

Theorem 2. *The centroids of the outer triangle and the original triangle are coincident.*



Proof. Let D be the mid point of BC, O be the centroid of ΔABC, and L be the centroid of ΔBPC. Then $DA = 3DO$ and $DP = 3DL$ so ΔDLO and ΔDPA are similar, giving $AP \parallel OL$ and $AP = 3\,OL$.

Likewise $BQ = 3\,OM$ and $CR = 3\,ON$. Since $AP = BQ = CR$ the distances from O to the vertices of ΔLMN are equal. As ΔLMN is equilateral O must be its centroid. □

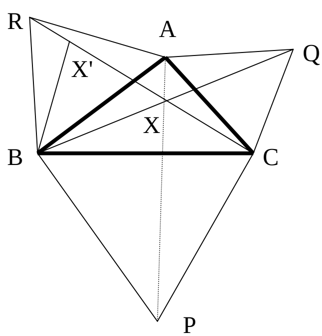
Next MacCool fixes ΔBPC and allows A to vary continuously throughout the plane. He notes that the proofs of these two theorems still apply whenever A drops below the level of BC, in effect making the angle at A reflexive and the angles at B and C negative. Essentially this is because the three e-triangles always retain their original orientation. For the orientation of an e-triangle to change under continuous deformation its area must first become zero which means that it must shrink to a point, but for the e-triangles in question this can only happen at B or C. So long as A avoids those two points no orientational changes to the e-triangles can occur.

However one subtle change does take place as A drops below BC in that the orientation of ΔABC itself changes. When that happens the e-triangles become internal rather than external. This has the following consequence.

Theorem 3. *The inner triangle is an e-triangle whose centroid coincides with the centroid of the original triangle.*

The next result gives an alternative proof that $AP = BQ = CR$. Only the "external" proof is given since the "internal" case is handled by exactly the same proof with the assumption that A lies below rather than above BC.

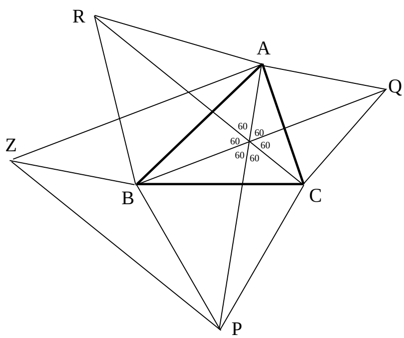
Theorem 4. *Suppose external (internal) e-triangles are erected on the sides of a given triangle. Then the three lines joining each vertex of the given triangle to the remote vertex of the opposite e-triangle are equal in length, concurrent, and cut one another at angles of* $60°$.



Proof. Let ΔABC be given and CBP, ACQ, BAR be the external e-triangles. Clearly ΔABQ is a $+60°$ rotation of ΔARC about A, ΔBCR is a $+60°$ rotation of ΔBPA about B and ΔCAP is a $+60°$ rotation of ΔCQB about C. It follows that $AP = BQ = CR$ and all angles of intersection are $60°$. To prove concurrency assume BQ and CR cut at X and construct BX' by rotating BX through $+60°$ about B as shown. Since $\angle BXR = 60°$ and $BX = BX'$ it follows that X' must lie on CR. However a rotation of the line $CX'R$ through $-60°$ about B will map $C \mapsto P$, $R \mapsto A$, and $X' \mapsto X$. Therefore A, X, and P are collinear which means that AP, BQ, CR must be concurrent. $\square$

MacCool next studies the areas of the various triangles. He uses (UVW) to denote the *algebraic* area of ΔUVW. In other words (UVW) is equal to the area of ΔUVW when the orientation of ΔUVW is positive, and minus that value whenever the orientation is negative.

Lemma 5. *In the diagram below BPC, ACQ, and ARB are e-triangles whose mean area is Ω, and Z is constructed so that $AZBQ$ is a parallelogram. Then AZP is also an e-triangle and $2(AZP) = 3\Omega + 3(ABC)$.*



Proof. As $AZBQ$ is a parallelogram $\angle ZAP$ is alternate to an angle of $60°$ so it too is $60°$. Also $AP = BQ = AZ$ so AZP must be an e-triangle. Clearly

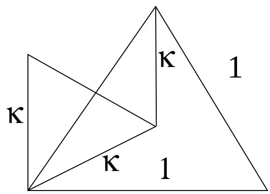
$$\begin{aligned}(AZP) &= (ABP) + (BZP) + (AZB) \text{ by tesselation}\\ &= (ABP) + (APC) + (ABQ)\end{aligned}$$

as $(APC) = (BZP)$ and $(ABQ) = (AZB)$.

Now $(BCR) = (ABP)$ and $(BCQ) = (APC)$ and $(ARC) = (ABQ)$ therefore

$$\begin{aligned} 2(AZP) &= (ABP)+(APC)+(ABQ)+(BCR)+(BCQ)+(ARC) \\ &= 3\Omega + 3(ABC). \end{aligned}$$ □

The diagram below shows two e-triangles, one with unit side and the other with side κ. Although I have found no evidence that MacCool was familiar with Pythagoras, he inferred from this diagram that $3\kappa^2 = 1$ and he deduced that the areas of the inner and outer triangles were one third the area of an e-triangle of side AP.



The area of the smaller equilateral triangle is clearly κ^2 that of the larger, from which it follows that κ must satisfy the equation :

$$3\kappa^2 = 1$$

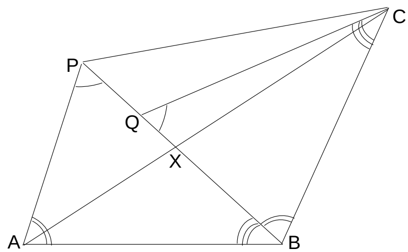
Theorem 6. *The mean area of the three e-triangles plus (minus) the area of the original triangle equals twice the area of the outer (inner) triangle.*

Proof. Let Δ be the area of the outer triangle. As explained on the previous page $(AZP) = 3\Delta$. Applying Lemma 5 now yields $2\Delta = \Omega+(ABC)$. Alternatively, if Δ is the area of the inner triangle this equation still holds, but there is a caveat. The orientations of ΔAZP and the inner triangle don't change as long as A avoids the point P where the latter shrinks to a point, but ΔABC has changed its orientation and so the value of (ABC) is now negative. Hence rewriting the equation in positive terms, $2\Delta = \Omega - (ACB)$. □

Corollary 7. *The area of the outer triangle is that of the inner triangle plus that of the original one.*

Finally MacCool presents a generalisation of Theorem 1.

Lemma 8. *Let A, B, C be non-collinear and X any point between A and C. Construct P and Q on BX such that $\angle PAB = \angle XBC$ and $\angle QCB = \angle XBA$. Then the triangles PAB and QBC are directly* *similar, moreover P and Q coincide if and only if $AX : XC = AB^2 : BC^2$.*



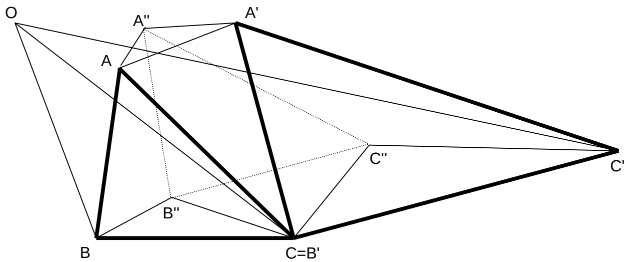
Proof. Clearly ΔPAB and ΔQBC are directly similar. Suppose $BC = \lambda AB$ and $XC = \mu AX$. Then $(QBC) = \lambda^2(PAB)$ whereas $(PBC) = \mu(PAB)$. If P and Q coincide then clearly $\mu = \lambda^2$. Conversely if $\mu = \lambda^2$ then $(PBC) = (QBC)$ so $(PQC) = 0$ which implies $P = Q$. $\square$

Note that if AB and BC have equal length then ΔPAB and ΔPBC are similar (but not directly similar) for all points P on the bisector of $\angle ABC$. Also the lines AB and BC (extended) divide the plane into four zones, and if a point O exists such that ΔOAB and ΔOBC are directly similar then O must lie in the zone that includes the line segment AC. This leads to a key result.

Corollary 9. *If the points A, B, C are non-collinear then there exists a unique point O such that the triangles OAB and OBC are directly similar.*

Theorem 10 (Generalised Napoleon)**.** *Let ABC and $A'B'C'$ be directly similar triangles with a common vertex $C = B'$. Suppose A'', B'', C'' are chosen such that the triangles $AA'A''$, $BB'B''$, $CC'C''$ are directly similar. Then so too are the triangles $A''B''C''$ and ABC.*

Proof. There are 3 separate cases. First if B' is midway between B and C' then $ABB'A'$ is a parallelogram and the result follows easily. Otherwise if B, B', C' are collinear take O to be the point where AA' cuts BB'. Then $\Delta A'B'C'$ is a dilation of ΔABC and it is clear that $\Delta A''B''C''$ may be obtained from ΔABC by a rotation



of $\angle AOA''(=\angle BOB''=\angle COC'')$ about O followed by a dilation of size OA''/OA. So once again the result holds. Finally if B, B', C' aren't collinear apply Corollary 9 to $\Delta BB'C'$ (aka BCC') giving the point O such that OBB' and OCC' are directly similar. Let $\theta=\angle BOB'=\angle COC'$ and $\lambda=OB'{:}OB=OC'{:}OC$. Let τ be the transformation that first rotates through the angle θ about O and then dilates by the scaling factor λ. Clearly τ preserves directly similar figures and maps $B\mapsto B', C\mapsto C'$ so as ABC and $A'B'C'$ are directly similar it must also map $A\mapsto A'$. Thus $\angle AOA'=\theta$ and $OA'{:}OA=\lambda$ from which it follows that $\Delta OAA'$ is directly similar to both $\Delta OBB'$ and $\Delta OCC'$. Then $OAA''A'$, $OBB''B'$, $OCC''C'$ are directly similar quadrilaterals so OAA'', OBB'', OCC'' are directly similar triangles. Thus $OA'':OA=OB'':OB=OC'':OC=\mu$ and $\angle AOA''=\angle BOB''=\angle COC''=\phi$ for some μ and ϕ. That means the quadrilateral $OA''B''C''$ may be obtained from $OABC$ by rotating it through ϕ about O and dilating the result by the scaling factor μ. Therefore $\Delta A''B''C''$ and ΔABC are directly similar. $\square$

The wheel has come full circle. To derive Napoleon's Theorem from this result take ΔABC to be equilateral and choose A'' so that $\Delta AA'A''$ is isosceles with base AA' and base angles of $30°$.

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MACCOOL'S SECOND PROOF OF MORLEY'S MIRACLE

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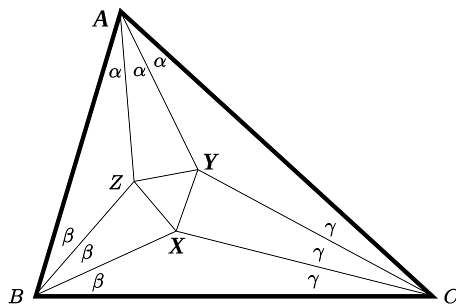
In memory of Kenneth Beales and Trevor West

Abstract. Here is a traditional proof of Morley's miracle that is unrivalled for brevity and simplicity. It stems from a sadly neglected mathematical gem published in 1914.

1. Introduction

That the triangle XYZ in the figure below is always equilateral is formally known as *Morley's trisector theorem* and informally as *Morley's miracle*. Its modern discovery dates back to 1899 and since then it has been proved many times by a wide variety of methods. The website [1] tracks developments and plays host to roughly twenty proofs including MacCool's original effort.



However as [3] explained, the proof there was based entirely on straight line geometry and similar triangles. It opined that MacCool's second notebook which was marked "Advanced" and contained diagrams of circles might hold an alternative proof. And so indeed it has proved, although it has taken me a very long time to decipher the Ogham. So whilst I have yet to find any evidence that MacCool was familiar with Pythagoras, the result that we know today as the inscribed angle theorem [Euclid: Book 3, Prop 22] does

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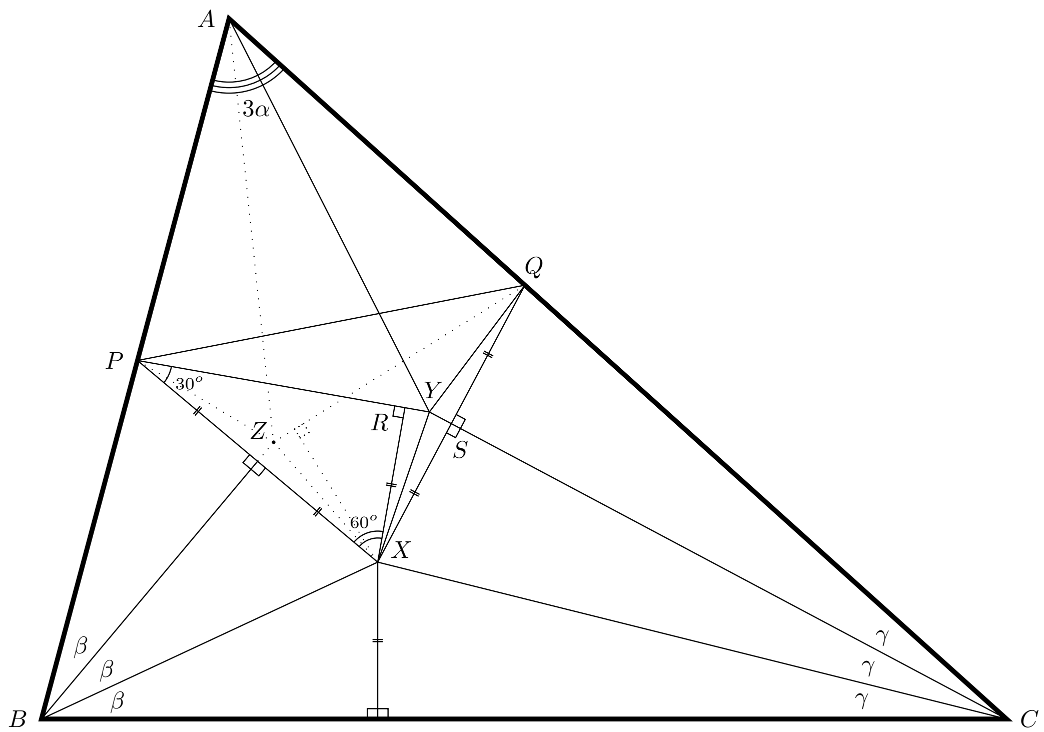
indeed appear in his "Advanced" notebook, and soon afterwards comes the second proof of Morley's theorem. This is even shorter and easier than his "Basic" one, and completely debunks the urban myth that all purely geometric proofs must necessarily be longer and more complex than the "backward" ones.

2. Proof

His proof runs as follows. In any triangle ABC let X be the Morley vertex adjacent to BC. First construct the points P and Q on AB and AC respectively such that $|BP| = |BX|$ and $|CQ| = |CX|$. Then construct the right-angled triangle PRX with hypotenuse PX and $\angle XPR = 30°$ as shown below. The six marked segments will all have equal length. Produce PR and the trisector CS to meet in Y. Note that the three right-angled triangles, ΔRXY and ΔSXY and ΔSQY, (which MacCool calls *wedges*) have equal hypotenuses and an equal (marked) side therefore they are congruent. Evidently $\alpha + \beta + \gamma = 60°$.



Now $\angle QXP = 360° - 2(90° - \beta) - 2(90° - \gamma) = 120° - 2\alpha$. So $\angle YXR = \angle SXY = \angle YQS = \frac{1}{2}(\angle QXP - 60°) = 30° - \alpha$. As ΔPQX is isosceles its base angles $\angle XPQ$ and $\angle PQX$ are both $30° + \alpha$ so $\angle YPQ = \alpha$ and $\angle PQY = (30° + \alpha) - (30° - \alpha) = 2\alpha$.

Therefore $\angle QYP = 180° - 3\alpha$. And now for the advanced bit. Finn spots this is supplementary to $\angle BAC$ making $APYQ$ a cyclic quadrilateral. Consequently $\angle YAQ = \angle YPQ = \alpha$ which fixes Y as the Morley vertex adjacent to AC. Next he performs a similar construction (shown in outline) starting from a right-angled triangle on hypotenuse QX and angles 30° and 60° at Q and X respectively. This generates three more wedges which are clearly congruent to the first three, plus the Morley vertex Z adjacent to AB. In particular $|XY| = |XZ|$ and $\angle YXZ = 60°$ from which he deduces that ΔXYZ is equilateral. ∎

3. Conclusion

After scanning numerous proofs the only "modern" one I've seen that is remotely like this is given in [2] and attributed to W. E. Philip. William Edward Philip was Third Wrangler at Cambridge in 1894, but despite many references to [2] in the literature the beauty of his proof seems to have been strangely overlooked. Indeed [2] also contains a version of Leon Bankoff's 1962 trigonometric proof, long regarded as the easiest non-backward approach to the theorem. As years passed without anyone finding a short, simple, non-backward geometric proof a mistaken belief has proliferated that no such proof exists. So, as the centenary of its publication approaches, the time seems ripe to call attention to [2] and bring it back centre stage.

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[3] M. R. F. Smyth: *MacCool's proof of Morley's Miracle*, Irish Math. Soc. Bulletin 63 (2009), 63–66

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Napoleon's Theorem, Shakespeare's Theorem, and Desargues's Theorem

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ABSTRACT

A very short and inevitably successful proof of Napoleon's theorem is given in terms of complex algebra. Another two related theorems, also concerned with centroids, are then proved by using an even simpler vector method. I eponymize these theorems to Shakespeare and Bacon in order to obey Stigler's law of eponymy. Finally, Desargues's theorem is related to the 3D monocular effect.

My aim in this brief note is to entertain and perchance to educate rather than to produce something new under the sun.

Consider a triangle ABC, labelled anticlockwise in the order A, B, C. (See Fig. 1.) Construct three equilateral triangles CBP_1, ACP_2, and BAP_3 external to ΔABC. Then the centroids G_1, G_2, G_3 of these three equilateral triangles form another equilateral triangle which I shall call the *derived* Napoleonic triangle. Someone who respected the Godfather, named the result "Napoleon's theorem" and therefore, by Stigler's Law of Eponymy (Stigler, 1980, "eponymy is always wrong", see also Good, 1985), we infer that it was more likely to have been discovered by Henry VIII. One proof was given in this journal recently by Boyd and Raychowdhury (1992).

It is intuitively obvious in advance that a short and automatic proof can be obtained by the machinery of complex algebra (if no mistakes are made!). This proof might not be too familiar though I would be surprised if it hasn't been discovered many times in the last hundred years so I give it with no claim to originality. I describe the "machinery" in three sentences, followed by the proof in four sentences.

A complex number z can be used to denote either a point in the Argand diagram (complex plane) or the vector $\vec{Oz}$ from the origin to that point, or, by the definition of a vector, any vector obtained from $\vec{Oz}$ by parallel displacement. Thus $\vec{AB}$ can be written as B − A (where B and A are complex numbers). For present purposes, O can be anywhere except "at infinity".

The line segment $\vec{CP}_1$, can be obtained by rotating the segment $\vec{AB}$ about C through an angle $\pi/3$ (anticlockwise); therefore $P_1 = C + \omega(B - C)$ where $\omega = \exp(\pi i/3)$. Therefore

$$G_1 = \frac{1}{3}[C + B + C + \omega(B - C)] = \frac{1}{3}[B(1 + \omega) + C(2 - \omega)]$$

and by "symmetry",

$$G_2 = \frac{1}{3}[C(1 + \omega) + A(2 - \omega)].$$

Therefore

$$\vec{G_1G_2} = G_2 - G_1 = \frac{1}{3}[A(2-\omega) - B(1+\omega) + C(2\omega - 1)].$$

Similarly,

$$\vec{G_2G_3} = \frac{1}{3}[A(2\omega - 1) + B(2 - \omega) - C(1 + \omega)$$

and this is seen to be $\omega^2 \; \vec{G_1G_2}$ (an anticlockwise rotation of $\vec{G_1G_2}$ by $2\pi/3$) by using the equations $\omega^3 = -1$ and $\omega^2 = \omega - 1$.

Napoleon's second theorem (for example, Coxeter and Greitzer, p. 64) states that a similar result is true if the original equilateral triangles are all reflected in their "bases" BC, CA and AB. (The "inner" Napoleon theorem.) This can be proved by essentially the same argument as above, as the reader can verify. (Just change the sign of ω.)

Coxeter and Greitzer (1967, p. 166) prove that the outer and inner Napoleon derived triangles have the same centroid. This again follows "automatically" from the "complex" method because

$$\frac{1}{3}(G_1 + G_2 + G_3) = \frac{1}{9}\{[B(1+\omega) + C(2-\omega)] + [C(1+\omega) + A(2-\omega)] + [A(1+\omega) + B(2-\omega)]\} = \frac{1}{3}(A + B + C)$$

so the centroid of the outer derived Napoleonic triangle is at the centroid of the original triangle ABC, and similarly so is the centroid of the inner derived Napoloeonic triangle.

For a short history of Napoleon's theorem and allied matters, with many references, see Wetzel (1992).

Sometimes the most elementary use of vectors, without even appealing to inner products or vector products, is more powerful than the use of complex numbers. This is because vectors are defined in any number of dimensions. (For the advanced use of vectors, in fact Grassmanians, in geometry, see Forder, 1941.) This (elementary) method is illustrated by the proof of the following result which I shall call "Shakespeare's theorem" so as to bridge the "two cultures" (the humanities and the sciences) while continuing to obey Stigler's law. It could also be called a shrinkage theorem.

Given n points $A_1, A_2, \ldots, A_n$ in d dimensions ($d = 1, 2, 3, \ldots$), let G_i denote the centroid of the ($n-1$) points obtained by omitting A_i. Then the construct consisting of $G_1, G_2, \ldots, G_n$ is similar (in the standard technical sense of Euclidean geometry) to that consisting of $A_1, A_2, \ldots, A_n$ and is scaled down by a factor n-1. The two constructs have the same centroid. Moreover, corresponding sides are "antiparallel", that is, parallel but oriented in opposite directions. The theorem is illustrated in Figure 1 for the case $n = 4$, $d = 2$. (The case $n = 3$ is familiar and the case $n = 2$ is trivial.) To visualize Figure 2 as a theorem about a tetrahedron ($n = 4$, $d = 3$) instead of a quadrilateral, it is better viewed with only one eye. That eye acts as a center of projection. With both eyes open one has more visual cues (conscious or subconscious) that a diagram is in the plane of the paper.

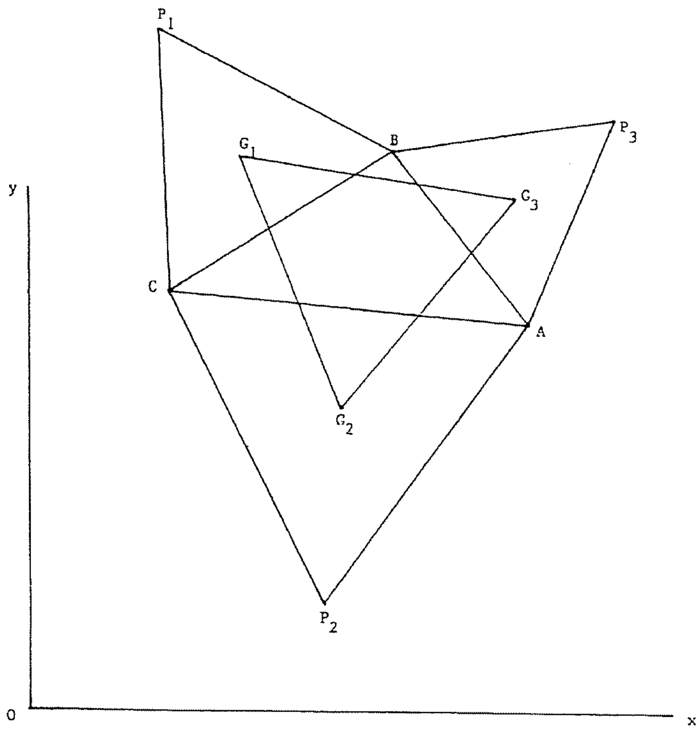

FIGURE 1. Napoleon's theorem.

Proof of Shakespeare's theorem. We have

$$G_1 = \frac{1}{n-1}(A_2 + A_3 + \ldots + A_n)$$

and

$$G_2 = \frac{1}{n-1}(A_1 + A_3 + A_4 + \ldots + A_n)$$

so

$$\overrightarrow{G_1G_2} = G_2 - G_1 = \frac{1}{n-1}(A_1 - A_2)$$

$$= \frac{1}{n-1}\,\overrightarrow{A_2A_1}$$

and similarly

$$\overrightarrow{G_iG_j} = \frac{1}{n-1}\,\overrightarrow{A_jA_i}$$

for all pairs i and j. This equation expresses the required "antiparallelism" and a scaling down of the original construct by a factor (n-1). The main part of the

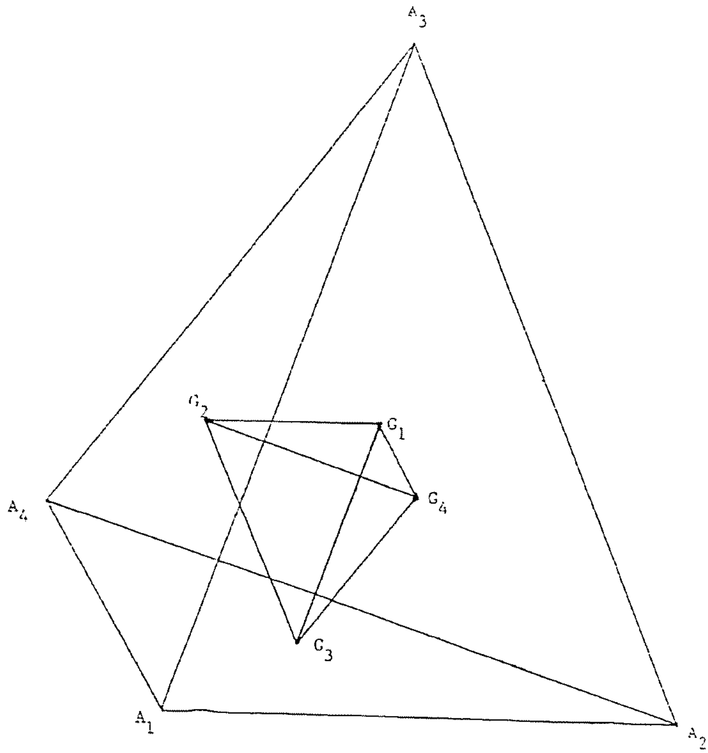

FIGURE 2. The shrunken quadrilateral or tetrahedron. If the theorem were used repeatedly, the successive tetrahedra would shrink down to a point, namely G.

theorem now follows from the fact that two figures (each connected and without "hinges") are congruent if one can be obtained from the other by a length-preserving transformation (see, for example, Coxeter and Greitzer, 1967, p. 80).

The proof that the two constructs have the same centroid is too simple to include here.

Bacon's Theorem. The lines $\overrightarrow{A_iG_i}$ ($i = 1, 2, ..., n$) *are concurrent.*

Proof. Denote by G the centroid of $A_1, A_2, ..., A_n$. We have

$$G = (A_1 + A_2 + ... + A_n)/n$$
$$= (1/n)A1 + (1 - 1/n)G_1 .$$

Because the sum of the two coefficients is 1, it follows that G lies on the line $\overrightarrow{A_1G_1}$ and similarly on the lines $\overrightarrow{A_iG_i}$ for all i. *A fortiori* these n lines are concurrent and we have brought home the Bacon.

Excuse for the eponymy. It has often been conjectured that "Shakespeare" was a pseudonym for Francis Bacon.

Statistical interpretation. Figure 2 and the allied discussion have an obvious interpretation (in a multivariate way) in terms of the Introduction of Efron (1982). The similarity (in the technical sense) of the shrunken structure to the original one makes it intuitively obvious that the Jackknife technique of "missing one out" cannot lead to an improved estimate of the population mean. But, as Efron's Introduction implies, the Jackknife (and Bootstrap) were not proposed in relation to the population mean.

"Projecting back" to one eye. Desargues's famous theorem in projective geometry states that if two triangles are in perspective from a point then they are also "in perspective from a line" (see, for example, Coxeter and Greitzer, 1967, p. 70). The theorem is thus expressible in terms of the incidence relations in only two dimensions, but its proof is not. When the axioms of incidence in three dimensions are assumed, the proof is simple because the two planes of the triangles meet in a line. See, for example, Baker (1943), pp. 22-23. This proof can be made especially intuitive if the diagram is viewed from one eye and is projected backwards into three dimensions. This visual technique of backward projection was used above to transform a quadrilateral into a tetrahedron. For another example, especially elegant, of viewing a two-dimensional diagram as if it were in three dimensions so as to give greater insight, see Honsberger (1976, page 19) who acknowledges Frank Bernhart.

The technique provides a bridge between geometry and artistic appreciation, as may very well have been noticed by Desargues (1591-1661) who was an architect as well as a geometer. For when a picture has strong perspective features, such as a road receding into the distance, it can be made to look especially three-dimensional when viewed with one eye closed. This is found to be true by a high fraction of male viewers. For a write-up of a sampling experiment on this topic see Good (1986). This "3D monocular effect" was noticeable by 28 out of the 37 subjects questioned, mostly male. Professor Dr. Anton Hajos of the University of Giessen, who is an expert on human visual perception, informed me in 1987 (i) that the 3D monocular effect is known, (ii) that it is less clear for photographs taken at a great distance, and (iii) that it is less noticeable by female subjects. He did not confirm that the effect is strongest for pictures containing a "lot of perspective". He cited Hofmann (1925, page 434 ff).

Only recently did I notice the relationship between this property of visual perception and the visual proof of Desargues's theorem although I became aware of the latter in about 1936 while attending a course on projective geometry given by F.P. White in Cambridge, England. My excuse for taking more than fifty years to notice the relationship is that unfortunately one tends to keep the two cultures in watertight compartments.

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S. Louridas and M. Rassias: Problem-Solving and Selected Topics in Euclidean Geometry: In the Spirit of the Mathematical Olympiads, Springer, 2013. ISBN:978-1-4614-7272-8, EUR 42.79, 235 pp.

REVIEWED BY JIM LEAHY

The success of the International Mathematical Olympiad (IMO) has helped to revive interest in Euclidean geometry and to halt somewhat its decline during the second half of the twentieth century. Consequently there is a constant trickle of new publications on the subject of which the book under review is one. Both authors have connections with the IMO. Sotirios E. Louridas has been a coach of the Greek Mathematical Olympiad team while Michael Th. Rassias is a winner of a silver medal at the IMO 2003 in Tokyo and holds a Master of Advanced Study from the University of Cambridge.

The book has six chapters with a foreword by Fields Medalist Michael H. Freedman. Chapter 1, Introduction, is short with a little history of geometry and containing Euclid's axioms and postulates. Chapter 2 deals with the basic concepts of logic and covers methods of proof including proof by analysis, by synthesis, proof by contradiction and proof by induction with examples. The one induction example is more a problem in number theory than geometry having the theorem of Pythagoras as a starting point. There is no other problem in the book that uses proof by induction. Chapter 3 covers geometrical transformations, viz. translations, symmetry, rotations, homothety and inversion. These are illustrated with examples and some theorems with proofs. The section on inversion will be found particularly useful to students and teachers as it gives several examples of its power in solving certain types of problems. Some of the later IMO type problems in the book also use inversion, something not common in many publications on Euclidean geometry.

Chapter 4 is a collection of thirty-eight theorems some of which are proved. The selection of theorems is excellent. Knowledge of these theorems together with the theorems of Euclid would go a long way towards solving many a geometrical problem. The proof of

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Feuerbach's theorem, Theorem 4.21 in the book, contains an error and the proof of Morley's theorem, Theorem 4.11, is not correct. In the latter case if the word 'isosceles', used twice, is replaced by 'equilateral' the proof would be correct but incomplete. Interestingly the construction at the beginning of this proof is similar to the construction used in MacCool's proof of Morley's theorem [1].

Chapter 5 offers sixty-five problems divided into three categories, problems with basic theory, problems with more advanced theory and geometrical inequalities. There is little difference between the first two categories and many of the problems are of IMO standard. The solutions follow in chapter 6 forming the main body of the work. Reading through the solutions is not easy. In some cases parts of the solutions seem to have been omitted and a good deal has been left to the reader usually without any comment from the authors. The statement of problem 6.2.25 p.169 is false as the wrong angles are designated as being equal. The solution uses the correct two angles but if you were attempting to solve the problem without consulting the solution, which you would expect a reader to do, your work would be in vain. The problems are restated before each solution in chapter 6 and the error is repeated. The solution of problem 6.2.22 p.164 is also incorrect since it would require the side of an inscribed pentagon to also be a tangent to the circumscribing circle! Obviously there are serveral misprints in this solution. On the other hand some of the proofs are quite innovative and the solution of problem 6.2.15 p.151 is an excellent example of the use of inversion.

There is an appendix on the Golden Section which is a reprinting of an article by Dirk Jan Struik in [2]. I fail to see the point of this as it is a popular article containing all the usual material of such articles which can be found in many publications and on the internet. Besides the Golden Section is not mentioned anywhere in the main text of the book. There is also a useful index of symbols used in the text, a subject index and a list of references, ninety-nine in all, including a reference to Wiles' paper on the solution of Fermat's Last Theorem! Since there are no references in the text, apart from acknowledging authors of problems, the references should rightly be called a bibliography.

To summarise, this is not a book showing how to solve problems in geometry except in the sense of learning from seeing problems solved. This is not a criticism as much can be learned in this way particularly if a solution has been attempted beforehand. The book is beautifully produced, the quotions at the head of each chapter adding to the publication. However the work is unfortunately marred by poor editing and proofreading. I counted over sixty errors, omissions, typos or misprints, mostly the latter, which does not make for easy reading. In addition some solutions have no diagram. Woody Guthrie, the American folk singer, once said that he liked books with errors as it made them more human. I doubt he ever read a mathematics text.

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