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Circles Inscribed in Segments

ANANDA BHADURI

As I was going through Evan Chen's *Euclidean Geometry for Mathematical Olympiads*, I came across this remarkable problem.

We are given that ω is a circle with centre O. AB is a chord of ω. If a circle is tangent to ω at P (internally) and tangent to AB at Q, prove that P,Q and the midpoint of the arc $\widehat{AB}$ not containing P are collinear.

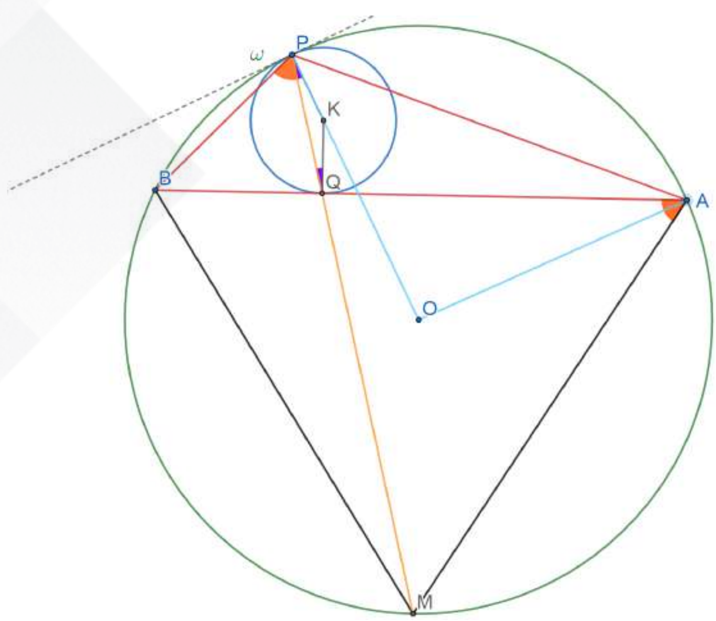

Figure 1

I start solving this problem by reasoning backwards. Once I have reduced it into a much more convenient form, I will present a proof.

Keywords: Circles, angles, chord, tangent, power of a point, orthogonal circles

Let PQ meet the arc $\overset{\frown}{AB}$ not containing P at M. We need to prove that M is the midpoint of the arc AB not containing P. That is, we need to prove that

$$MA = MB.$$

This reduces to proving that ∠MAB = ∠MBA. Now, we use the fact that angles in the same segment are equal. Since ∠MAB = ∠MPB and ∠MBA = ∠MPA, it is sufficient to prove that

$$\Leftrightarrow \angle MPB = \angle MPA \qquad \text{..........(1)}$$

We will prove (1) below

Proof:

Observe that O, P and K are collinear, where K is the centre of the inscribed circle, O is the centre of the outer circle and P is the point of contact.

Since the circle inscribed in the segment is tangent to AB at Q,

$$\angle KQA = KQB = 90^\circ \qquad \text{.......(2)}$$

We compute ∠MPA and show that it must be equal to ∠MPB.

∠MPA is the sum of the two angles ∠MPO and ∠OPA.

$$\begin{aligned}
\angle MPO \text{ is nothing but } &\angle KPQ \\
&= \angle KQP && (KP = KQ) \\
&= 90^\circ - \angle PQB && (\text{from } (2)) \\
&= 90^\circ - \angle AQM
\end{aligned}$$

$$\begin{aligned}
\angle OPA &= \frac{180^\circ - \angle POA}{2} && (\text{from the isosceles triangle POA}) \\
&= 90^\circ - \angle PMA && (\angle POA = 2\,\angle PMA)
\end{aligned}$$

Adding these two values,

$$\begin{aligned}
\angle MPA &= \angle MPO + \angle OPA \\
&= 180^\circ - (\angle AQM + \angle PMA) \\
&= \angle MAB \\
&= \angle MPB && (\text{angles in the same segment})
\end{aligned}$$

And this proves (1).

Now, we come to our first corollary:

The length of the tangent from M to any circle inscribed in segment AB is equal to MA.

We begin by using the fact that triangles MPA and MAQ are similar. This is because ∠MPA = ∠MAB = ∠MAQ (as proved earlier) and ∠PMA = ∠QMA.

Similarity yields the following ratio:

$$\frac{MQ}{MA} = \frac{MA}{MP}$$

This leads to:

$$MP \cdot MQ = MA^2$$

Observe that MP · MQ is the **power of point** M with respect to the inscribed circle. (The definitions of words written in bold can be found in the appendix.) But the power of a point of a point outside the circle is equal to the square of the length of the tangent from the point to the said circle.

If the length of the tangent from M to the inscribed circle is *t*. Then,

$$MA^2 = MP \cdot MQ = t^2$$

What does this result imply? *The length of the tangent from M to the circle is fixed by the segment AB and does not depend on the position of the circle within the segment.*

Now for our second corollary:

Let γ be the circle passing through A and B with centre M and radius MA or MB. *Let γ intersect the inscribed circle at X and Y and intersect PQ at I. Then,*

i. MX and MY are tangent to the inscribed circle.

ii. I is the incentre of triangle PAB.

The first part is the first corollary in disguise. In fact, the inscribed circle and γ are **orthogonal circles**.

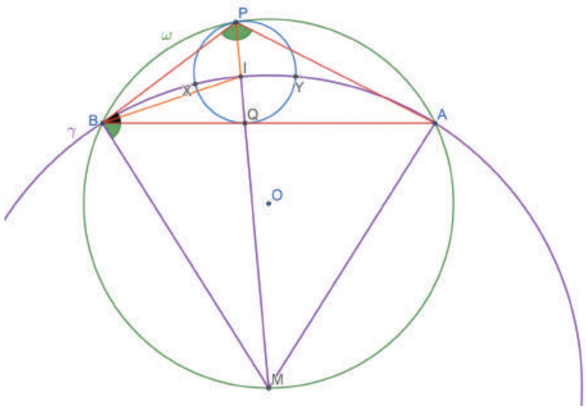

Figure 2

What about the second part? Since I lies on PM (which is the bisector of ∠APB), it is sufficient to prove that IB is the bisector of ∠PBA.

Since MB = MI,

$$\angle \mathbf{MBI} = \angle \mathbf{MIB} \qquad (2)$$

$$\begin{aligned}\angle MBI &= \angle MBA + \angle ABI\\ &= \angle MPA + \angle ABI \quad \text{(angles in the same segment)}\\ &= 1/2\angle APB + \angle ABI \quad \text{(MP bisects } \angle APB\text{)}\end{aligned}$$

$$\begin{aligned}\angle MIB &= \angle IPB + \angle IBP \quad \text{(exterior angle of a triangle)}\\ &= \angle MPB + \angle IBP\\ &= 1/2\angle APB + \angle IBP\end{aligned}$$

Substituting these values in (2),

$$1/2\angle APB + \angle ABI = 1/2\angle APB + \angle IBP$$

$$\Rightarrow \angle ABI = \angle IBP$$

Hence, IB bisects ∠PBA.

I would like to present a problem from the IMO 1992 shortlist which was proposed by Shailesh Shirali. This problem appeared in the article on Mathematical Olympiads in India in the November edition.

Problem 1

Two circles G_1 and G_2 are inscribed in a segment of circle G and touch each other externally at a point W. Let A be a point of intersection of a common internal tangent to G_1 and G_2 with the arc of the segment and let B and C be the endpoints of the chord. Prove that W is the in-centre of triangle ABC.

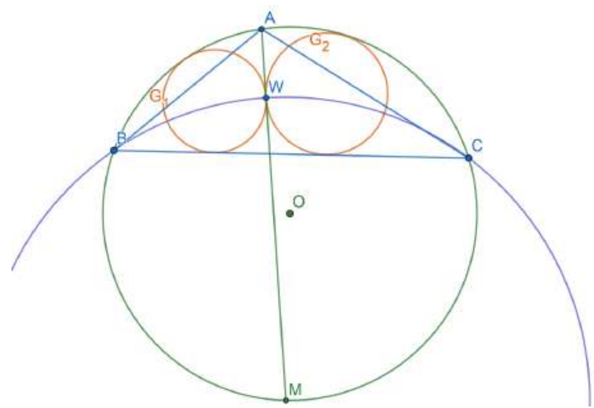

Figure 3

(*Note from the editor:* For those unfamiliar with the term 'internal tangent,' an internal tangent to two circles is a common tangent to the two circles such that the two circles lie on opposite sides of the tangent line. In the case of an external tangent, the two circles lie on the same side of the tangent line. Wikipedia uses the terms 'inner tangent' and 'outer tangent' but these are non-standard terms.)

We define point M to be the intersection of the common internal tangent of G_1 and G_2 and the circle G. The figure suggests that M is the midpoint of the arc $\overset{\frown}{BC}$ not containing A. To prove this, we will require this lemma:

The tangents from a point P to G_1 and G_2 are equal if and only if it lies on the common internal tangent to G_1 and G_2.

If P is on the common internal tangent, the tangent from P to both the circles is equal to the length PW. Therefore, we need to prove that the tangents from a point not on the common internal tangent to G_1 and G_2 cannot be equal.

Here is a brief sketch of the proof. The reader is expected to fill in the details.

- Let there be a point P' not on the common internal tangent such that the tangents from P' to both the circles are equal.
- Let P'W meet G_1 and G_2 at K and L respectively. Prove that K and L are distinct.
- Consider the powers of point P' (see appendix 1) with respect to both the circles. Prove that they must be unequal.
- Since the power of a point with respect to a circle is the square of the length of the tangent, we get a contradiction.

How can we use this lemma? According to the first corollary, the tangent from the midpoint of the arc $\widehat{BC}$ not containing A to circles G_1 and G_2 must be equal. Hence, the midpoint lies on the common internal tangent and is point M.

This also implies that the tangent from M to both the circles is equal to MW. But it is also equal to MA (from the first corollary).

Therefore, we can construct a circle centred at M with radius MW which passes through A and B. Now the figure of the problem resembles that of the second corollary. We can use the second part of the second corollary to complete the proof.

Appendix 1: Power of a point

There are many ways to define the power of a point. For a point P and a circle with centre O and radius r, the power of point P is the quantity $OP^2 - r^2$. If P is outside the circle, the power of point P is the length of the tangent from P to the circle squared. If a line through P intersects the circle at A and B, then the power of point P is also $PA \cdot PB$ (Note that the power of a point inside a circle is negative according to the first definition. But the third definition suggests that it is positive. The power of a point inside a circle is indeed negative and $PA \cdot PB$ is the *absolute value* of the power of a point.)

Here, we only prove that the three definitions are equal for a point outside a circle.

Suppose PT is the tangent from P to the circle.

It is obvious from the Pythagoras theorem that $PT^2 = OP^2 - r^2$

We have to prove that $PT^2 = PA \cdot PB$.

Let TC be a diameter of the circle.

This implies that ∠TAC is a right angle.

$$\Rightarrow \angle TCA = 90° - \angle ATC$$

But ∠CTP is also a right angle,

$$\Rightarrow \angle ATP = 90° - \angle ATC$$

$$\Rightarrow \angle ATP = \angle TCA = \angle TBA$$

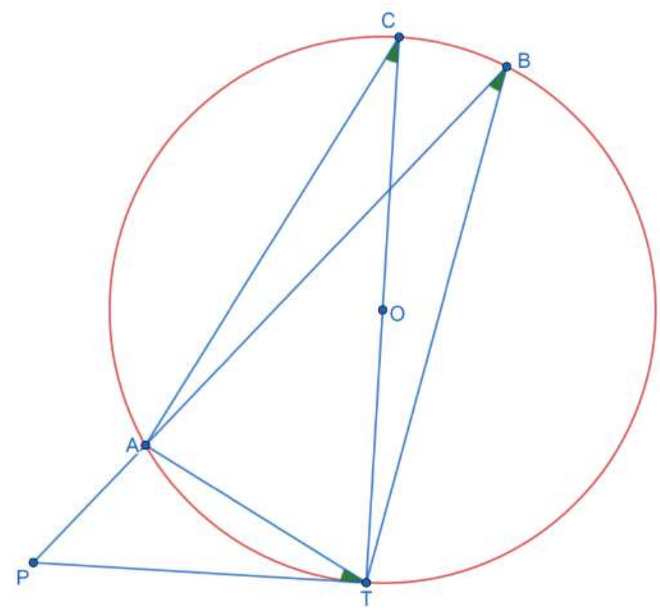

Figure 4

Also,

$$\angle APT = \angle BPT$$

The last two equations imply that triangles PAT and PBT are similar.

We get this ratio from similarity:

$$\frac{PA}{PT} = \frac{PT}{PB}$$

This gives us the required result.

Appendix 2: Orthogonal circles

Let Ω and ω be two circles intersecting with centres O_1 and O_2 respectively which intersect at A and B. If O_1A and O_1B are tangents to ω, the two circles are orthogonal. In fact, this implies that O_2A and O_2B are tangents to Ω. Orthogonal circles have lots of nice properties. We encourage the reader to find them. For example, one interesting feature is that the points O_1, O_2, A, B lie on a circle whose centre is the midpoint of segment O_1O_2.The proof is trivial and is left to the reader.

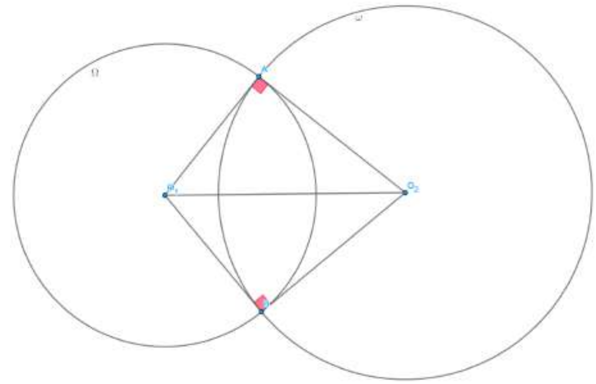

Figure 5

We conclude this appendix with a question: What is the power of point O_1 with respect to ω? What about that of O_2 with respect to Ω?

Acknowledgments

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ANANDA BHADURI is a class VIII student of Sarala Birla Gyan Jyoti, Guwahati. He is particularly interested in Euclidean Geometry. He is a voracious reader, who enjoys playing football and chess. He may be contacted at anandabhaduri25@gmail.com.