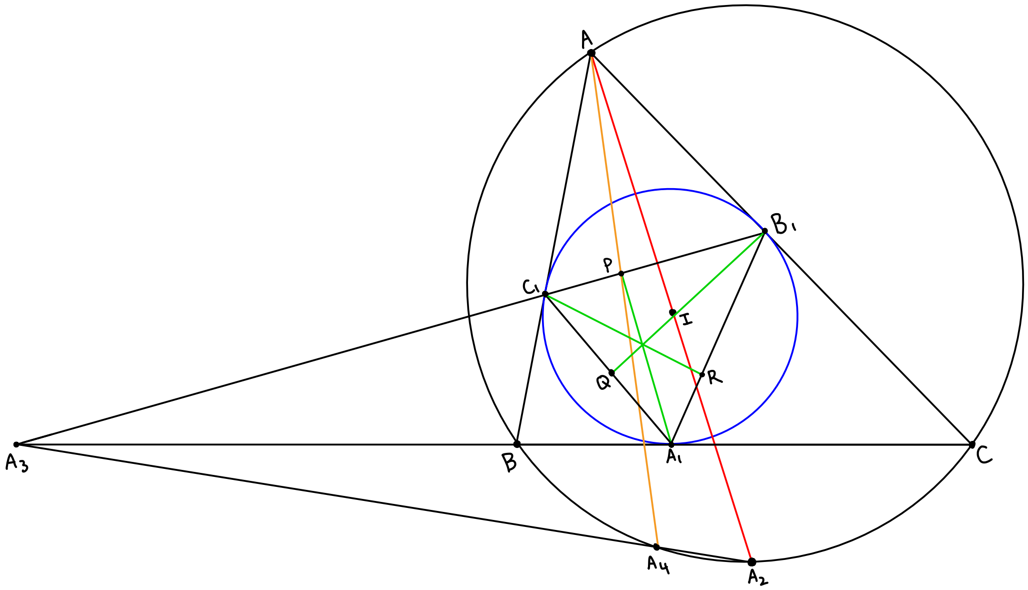
Murica Solutions

3.0 North Korea TST 2013/1

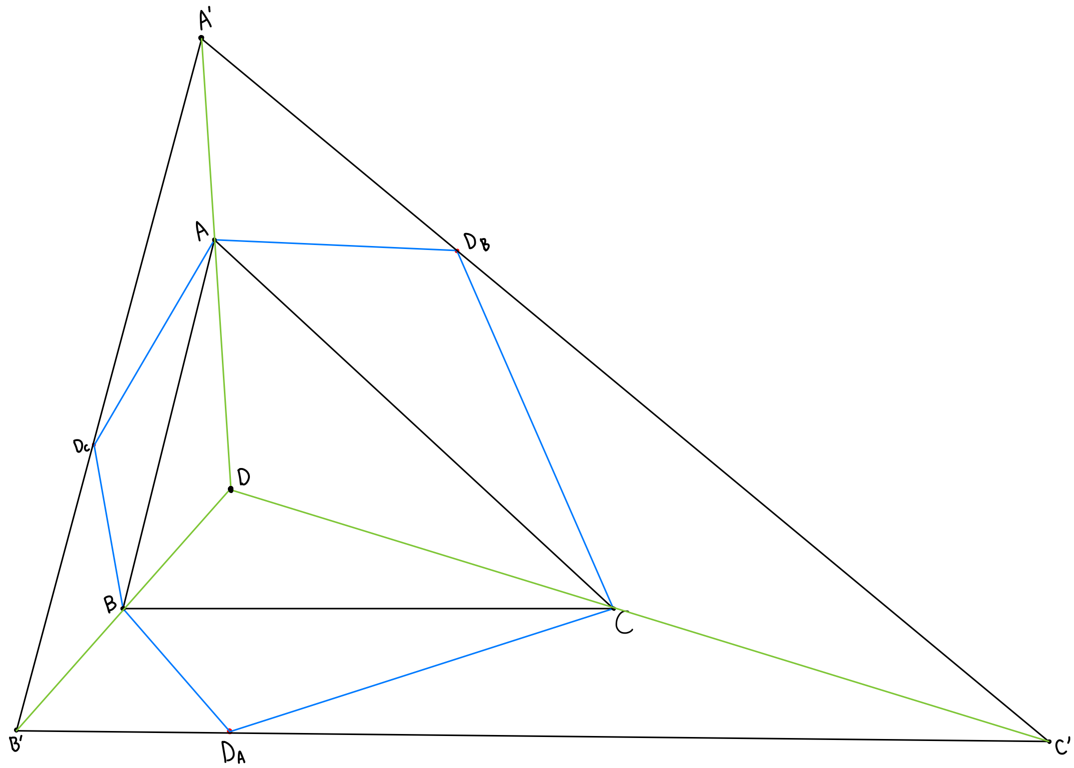
Problem 3.0 (North Korea TST 2013/1)**.** The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4 (\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.



We see that A_4 is the E_0 point, so we have $\overline{APA_4}$, where P is the foot from A_1 to B_1C_1. Since $A_1P \cap B_1Q \cap C_1R$ and $AA_1 \cap BB_1 \cap CC_1$, by Cevian Nest (3-23 in EGMO), we have $AP \cap BQ \cap CR$ as desired.

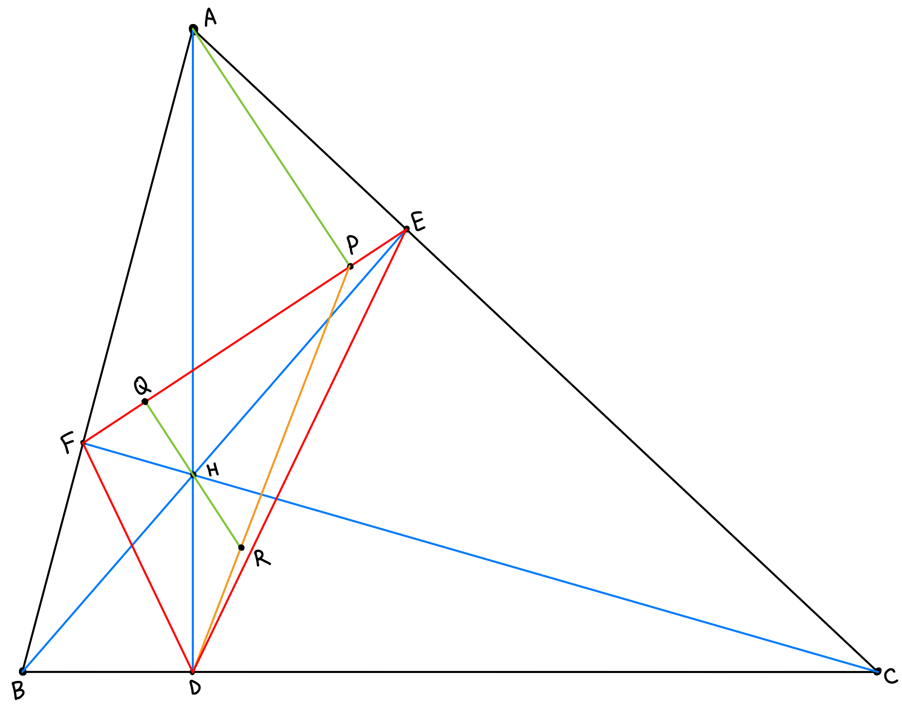
3.1 GOTEEM 3

Problem 3.1 (GOTEEM 3)**.** Let D be a point in the plane of $\triangle ABC$. Define D_A, D_B, D_C to be the reflections of D over BC, CA, AB, respectively. Prove that the circumcircles of $\triangle D_ABC$, $\triangle D_BCA$, $\triangle D_CAB$, $\triangle D_AD_BD_C$ concur at a point P. Moreover, prove that the midpoint of $\overline{DP}$ lies on the nine-point circle of $\triangle ABC$.



Take a 2x homothety. we see that (D_ABC), (D_BAC), (D_cAB) are the 9 point circles of $DB'C'$, $DA'C'$, $DA'B'$, so they intersect at the Poncelet Point. we see the 9 point circle of $A'B'C'$ also goes through this point P, so the midpoint of DP (homothety back to ABC) brings P onto the 9 point circle of ABC as desired.

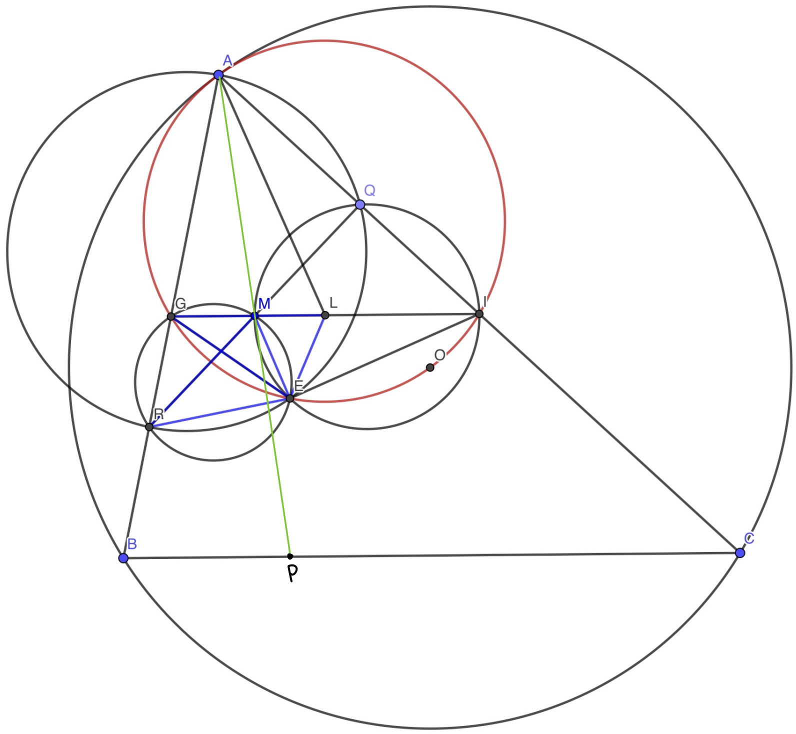
Problem 3.2 (USA TST 2011/1)**.** In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.



Note that H is the incenter of ΔDEF, and by a homothety from the incircle $\to$ excircle, P goes through the "top" point of the incircle of ΔDEF. Since Q is the "bottom" point, $HQ = HR$, as QR is a diameter.

3.3 USA TST 2008/7

Problem 3.3 (USA TST 2008/7). Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

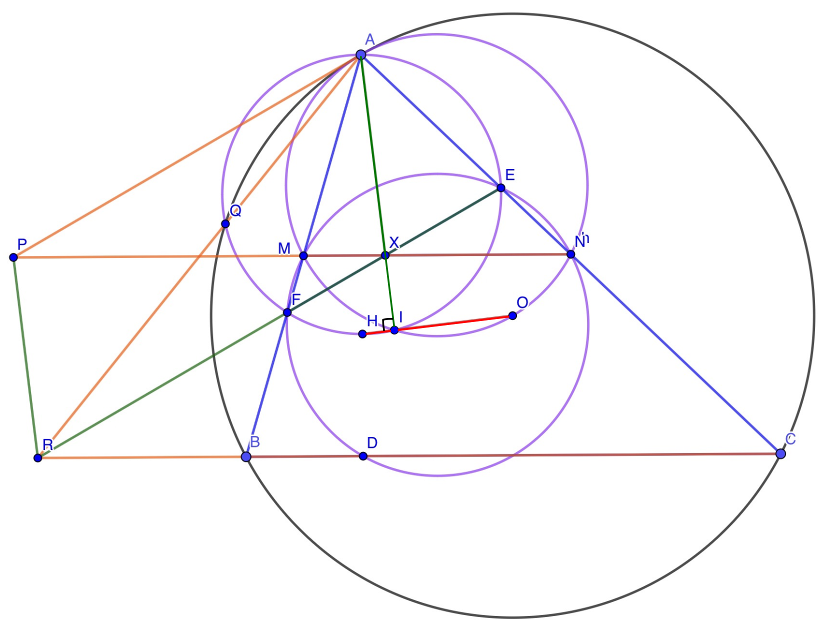


Let E be the A dumpty point. Let G, I be the midpoints of AB, AC. Let Q be a point on BC, and R = (AQE) ∩ AB. We show that there exist P st AQRP is a parallelogram (R is the unique point st there exists parallelogram APQR).

We have E is the center of spiral similarity mapping QR to GI. We show that if M = RQ ∩ GI, and GL = LI, then $\overline{QMR} \to \overline{GLI}$. We have ∠LGE = ∠MQE ∠GLE = ∠AIE = ∠QGE, since AE is the symmedian, so M is the midpoint of QP. Thus, the reflection of A over the midpoint of PQ lies on BC as desired.

3.4 USA TSTST 2017/1

Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90^\circ$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.



We have, by radical axis on (AEF), (BEFC), (ABC), that $AQ \cap EF \cap BC = R$

We show $APRX$ is a parallelogram. We have $AP \parallel RX$ since $\measuredangle PAB = \measuredangle ACB = \measuredangle AFE$, and $AP = RX$ since $\frac{EN}{AN} = \frac{EX}{AP}$ and $\frac{EX}{XR} = \frac{EN}{CN}$ and since $AN = CN$, $AP = XR$. Thus, $AX \parallel PR$.

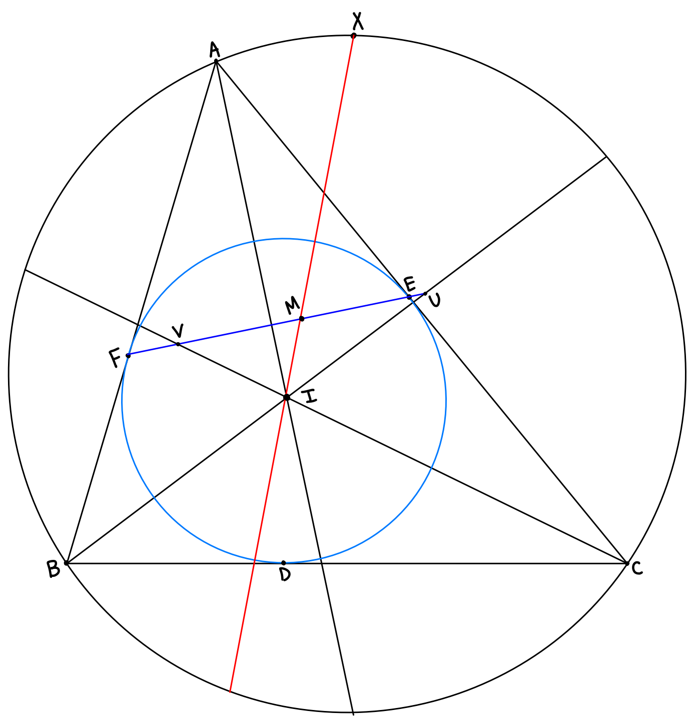
Consider (AH), (AO), (DEF). The radical center is $MN \cap EF = X$ and lies on the radical axis of (AO) and (AH), so $AX \perp OH$ as desired (X lies on AI where AI is the altitude from A to OH).

3.5 USAJMO 2014/6

Problem 3.5 (USAJMO 2014/6). Let ABC be a triangle with incenter I, incircle γ, and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively and let E and F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U and V be the intersections of lines $\overline{EF}$ with $\overline{MN}$ and $\overline{MP}$ respectively, and let X be the midpoint of the arc BAC of Γ.

(a) Prove that I lies on $\overline{CV}$

(b) Prove that the line $\overline{XI}$ bisects the segment UV

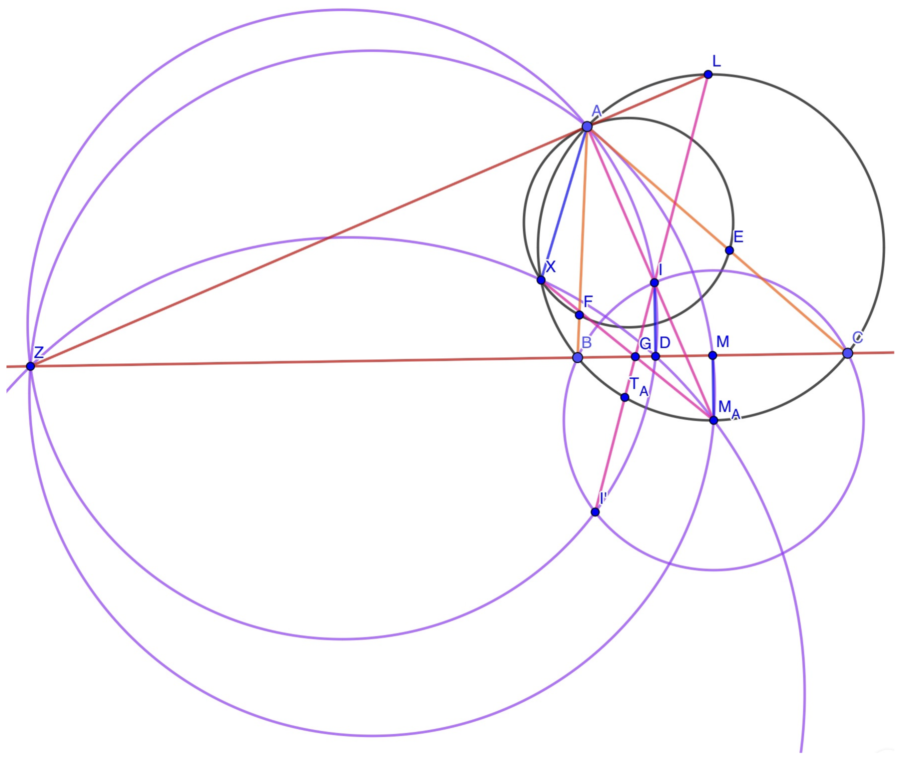


Part a follows from Iran Lemma.

Note that $UVBC$ is cyclic so $\Delta IUV \sim \Delta IBC$. Note that XB, XC are tangent to (IBC), so XI is a symmedian of ΔIBC and thus the median of ΔIUV.

3.6 ISL 2016 G2

Problem 3.6 (ISL 2016 G2)**.** Let ABC be a triangle with circumcircle Γ and incenter I. Let M be the midpoint of BC. The points D, E, F are selected on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a points X other than A. Prove that lines $\overline{XD}$ and $\overline{AM}$ meet on Γ.



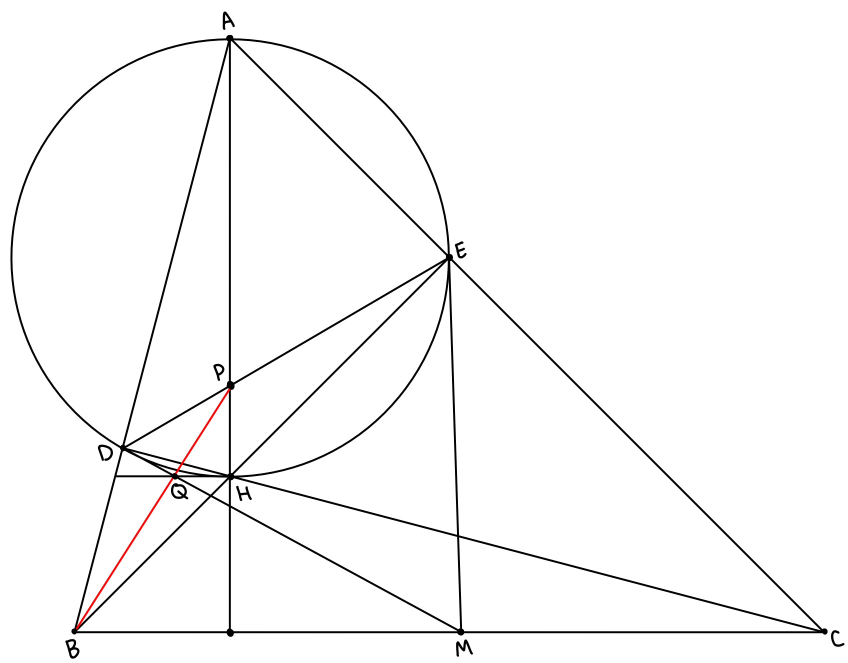
Note that $(ZAMM_A)$, $(ZAIDI')$ so

$ZG \cdot GD = IG \cdot GI' = BG \cdot GC = XG \cdot GM_A$ so (XDM_AZ), thus, we have $\measuredangle(AM, XD) = \angle AMZ - \angle XDZ = \angle AM_AZ - \angle XM_AZ = \angle AM_AX$

as desired.

3.7 Korea 2020/2

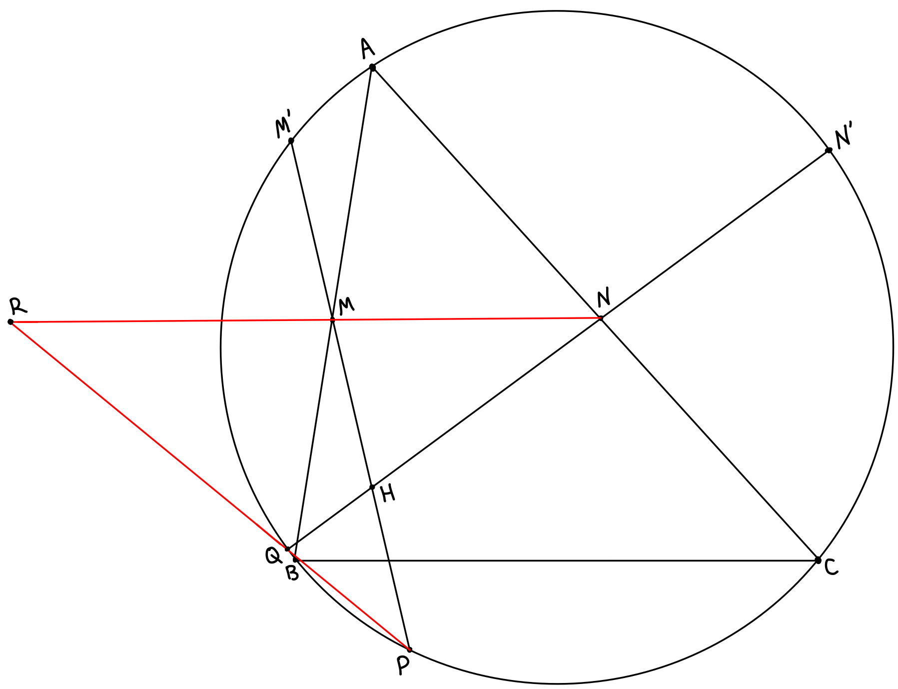
Problem 3.7 (Korea 2020/2). H is the orthocenter of an acute triangle ABC, and let M be the midpoint of BC. Suppose (AH) meets AB and AC at D, E respectively. AH meets DE at P, and the line through H perpendicular to AH meets DM at Q. Prove that P, Q, B are collinear.



Note that $DD \cap HH = Q$ and consider Pascals on $(DDEHHA)$ which gives the desired.

3.8 USA TSTST 2011/4

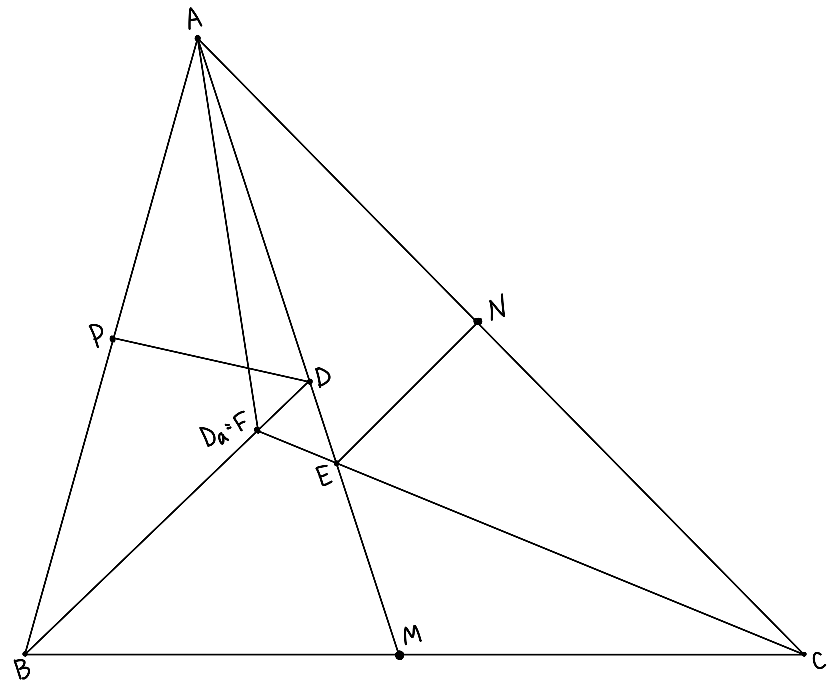
Problem 3.8 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.



We have $HM = MM'$ and $HN = NN'$, so since $HM \cdot HP = HN \cdot HQ$ so $HM' \cdot HP = HN' \cdot HQ$, we have $(MNPQ)$. Radical Axis on (AMN), $(MNQP)$, (ABC) gives $R \in AA$, as desired.

3.9 USAMO 2008/2

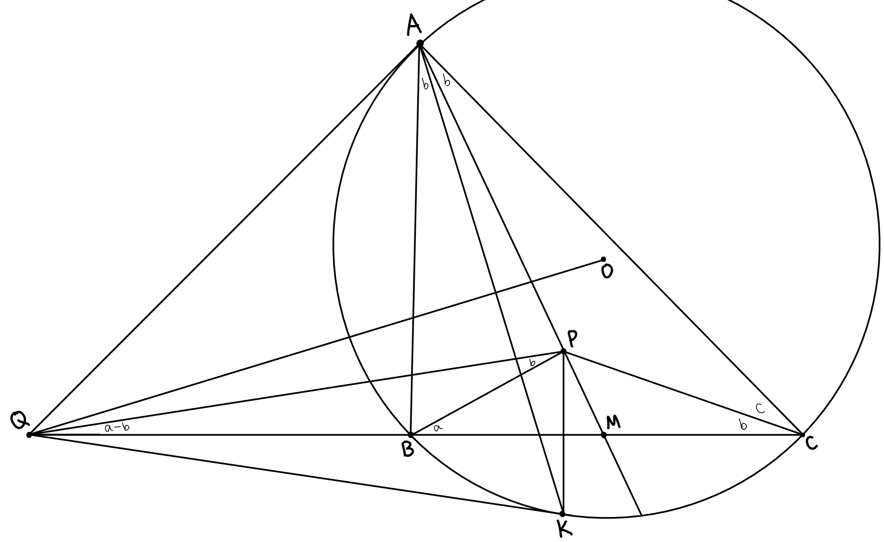
Problem 3.9 (USAMO 2008/2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.



We show the Dumpty Point satisfies these properties. The Dumpty point is the center of spiral similarity sending $AB \to AC$. We have $\angle BAD = \angle CAD_A = \angle ABD_A$, so F on BD, and similarly, D_A on CE. Thus, $F = D_A$ so $(ANFP)$.

3.10 USA TST 2005/6

Problem 3.10 (USA TST 2005/6). Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.



The angle conditions imply that P is the A-HM point, so since P is the reflection of K (K is the point such that $(AK;BC)=-1$) so $QP = QK = QA$ if $Q = AA \cap KK \cap BC$. Let $\angle PBC = a$ and $\angle PCB = b$, and $\angle PCA = c$.

Note that QP is tangent to (PBC) as $QA^2 = QP^2 = QB \cdot QC$.

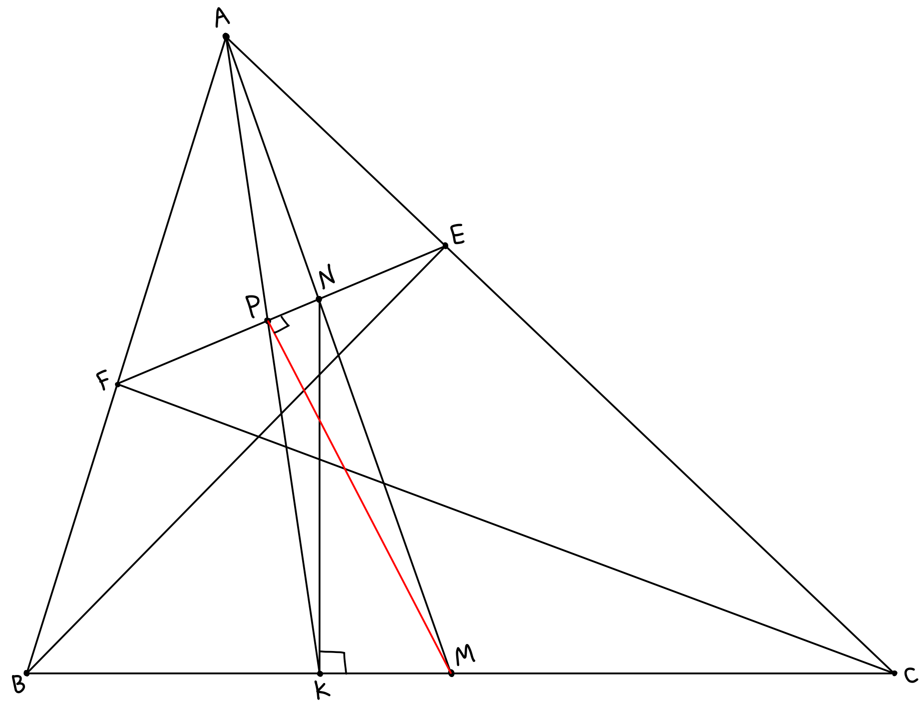
Thus, $\angle PQB = a-b$, $\angle AQP = 180 - 2a - 2b - 2c$.

We have $2\angle OQB = \angle AQK - 2\angle KQC = \angle AQK - 2\angle PQC$

$= (180 - 4b - 2c) - (2a - 2b) = 180 - 2a - 2b - 2c$ as desired.

3.11 AOPS Community, NK perp BC and AK symmedian

Problem 3.11 (AoPS Community). Given acute triangle ABC, $BE \perp AC, CF \perp AB$ at E, F, resp. M is midpoint of BC, N is intersection of EF and AM. $NK \perp BC$ at K. Prove that AK is symmedian of triangle ABC.

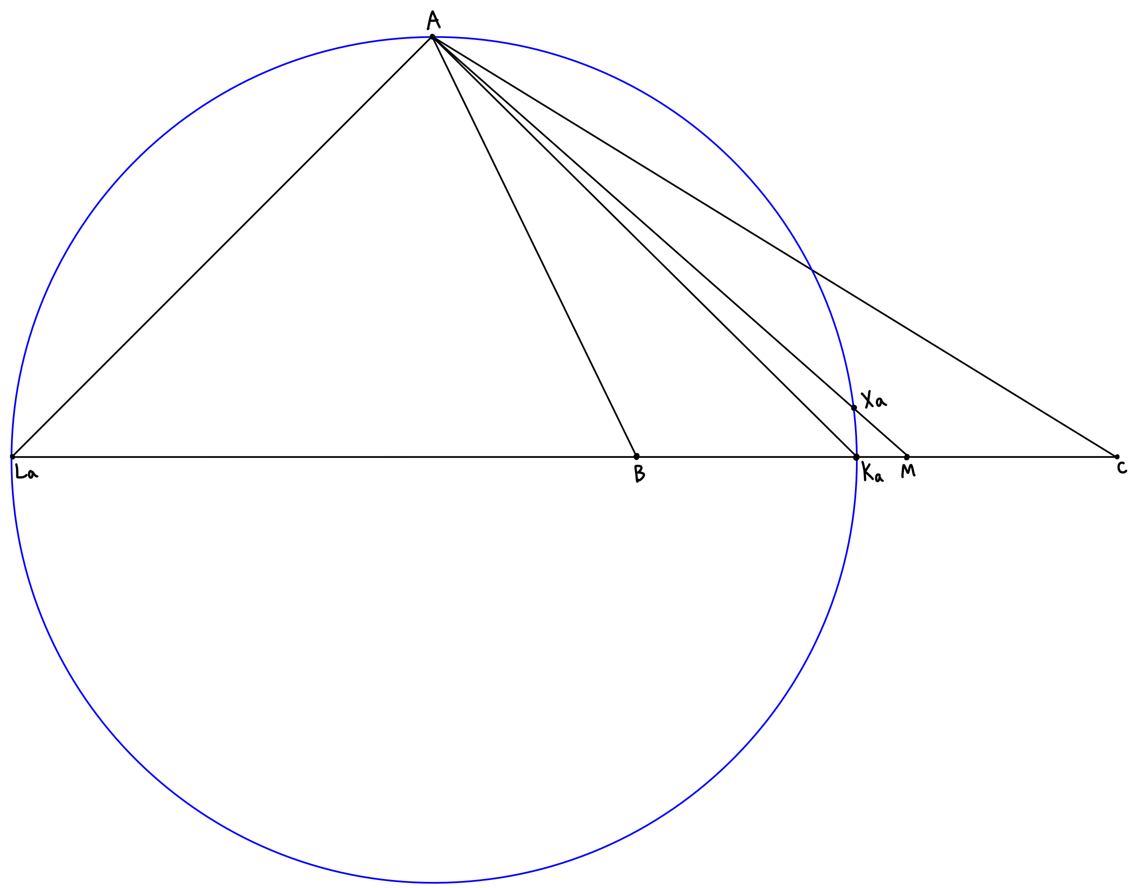


Consider the midpoint of EF, P. We have $MP \perp EF$, so $(MNPK)$, so $\angle NPK = \angle NMK = \angle APE$ by homothety, so $\overline{APK}$.
Thus, AK is a symmedian as AP is a median.

3.12 USA TSTST 2015/2

Problem 3.12 (USA TSTST 2015/2). Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

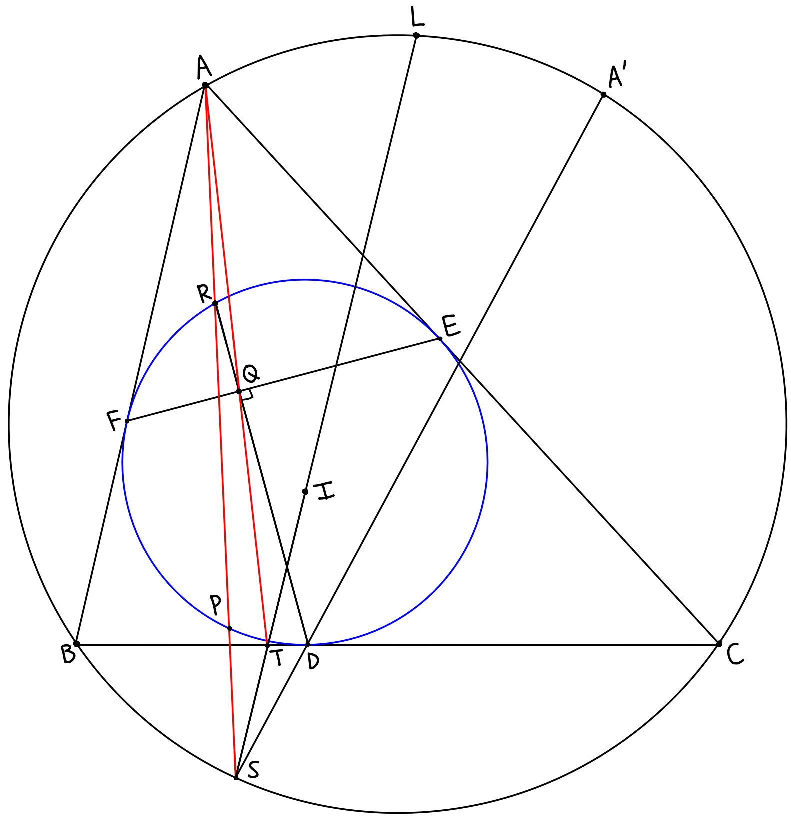
(The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)



We have $(L_a K_a; BC) = -1$ so $MB^2 = ML_a \cdot MK_a = MA \cdot MX_a$ so X_a is the A-HM point of ABC. We have $\angle HX_aM = 90^\circ$, so $\angle HX_aG = 90^\circ$. Thus, X_a lies on (HG), so the center of $(X_aX_bX_c)$ is on the Euler Line.

3.13 STEMS 2019 CAT B P6

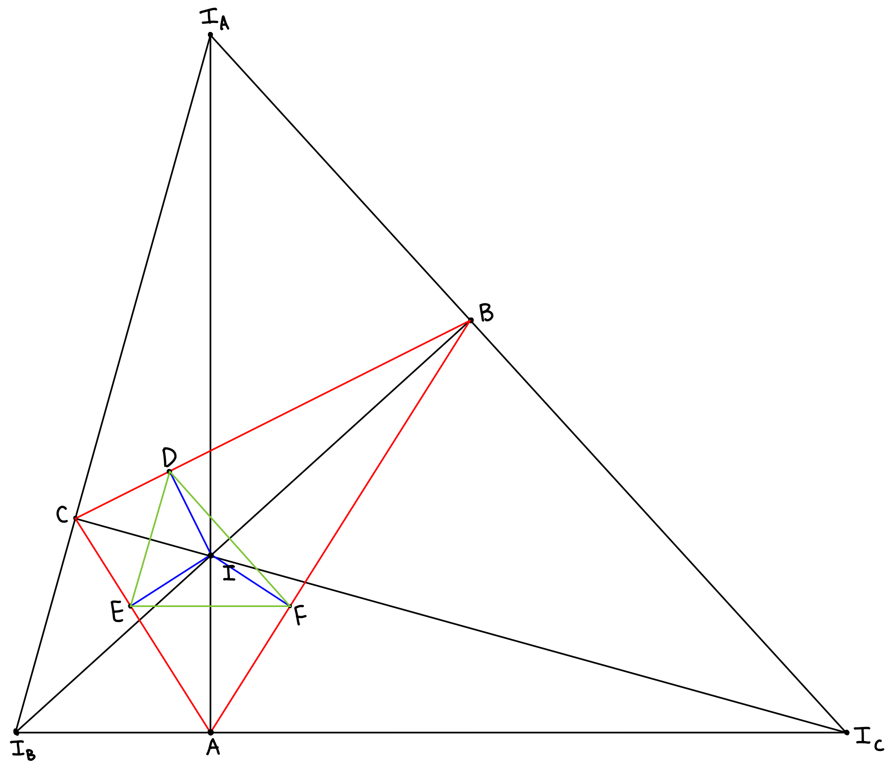
Problem 3.13 (STEMS 2019 CAT B P6)**.** In triangle ABC, with circumcircle Γ, the incircle ω has center I and touches sides $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Point Q lies on $\overline{EF}$ and point R lies on ω such that $\overline{DQ} \perp \overline{EF}$ and D, Q, R are collinear. Ray AR meets ω again at P and Γ again at S. Ray AQ meets BC at T. Let M be the midpoint of BC and let O be the circumcenter of triangle MPD. Prove that O, T, I, S are collinear.



By Configs, we have $\overline{ITS}$. We also have $\overline{SDA'}$ so IT is the perp bisector of PD, so $\overline{OTIS}$.

3.14 Vietnam TST 2003/2

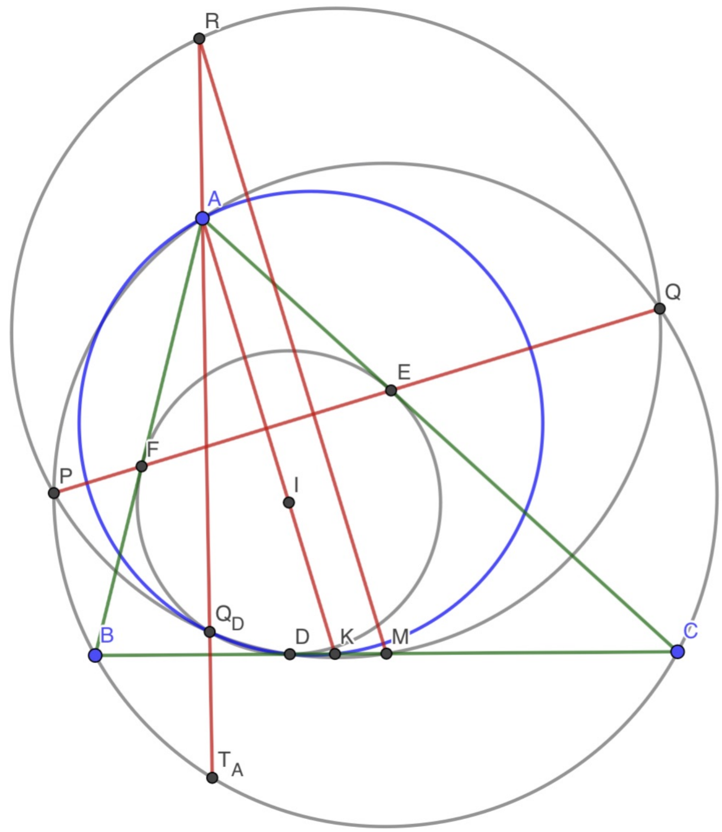
Problem 3.14 (Vietnam TST 2003/2)**.** Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)



By Midpoints of Altitudes, $\overline{A_0D} = \overline{I_AD}$. Note that $I_AI_BI_C$ and DEF are homothetic since $EF \parallel I_BI_C$, $DF \parallel I_AI_C$, $DE \parallel I_AI_B$. Note that the center of $I_AI_BI_C$, O, I are collinear (Euler line of $I_AI_BI_C$ as the circumcenter, orthocenter, 9-point center) as desired (center of homothety on $\overline{OI}$).

3.15 Taiwan TST Round 2 2019 Day 1 P2

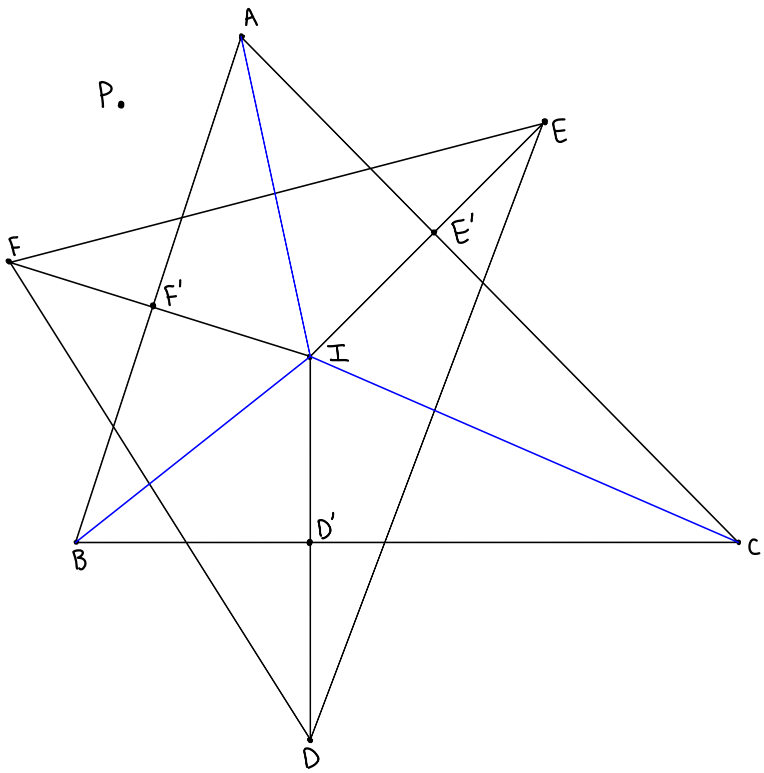
Problem 3.15 (Taiwan TST Round 2 2019 Day 1 P2)**.** Let ABC be a scalene triangle, let I be its incenter and let Ω be its circumcircle. Let M be the midpoint of BC. The incircle ω touches CA, AB at E, F respectively. Suppose that the line EF intersects Ω at two points P, Q, and let R be the point on the circumcircle Γ of $\triangle MPQ$ such that MR is perpendicular to PQ. Prove that AR, Γ, and ω intersect at one point.



Let Q_D be the D-Queue point of DEF. We have (MDQ_DPQ) since (MDQ_D) and (ABC) have rad axis EF. Note that $\measuredangle(AT_A, RM) = \measuredangle(AT_A\, AK)$ and (AQ_DDK) and (AQ_DDM), so $\overline{ARQ_D}$, as desired.

3.16 Sun Yat-sen University bi-weekly problem

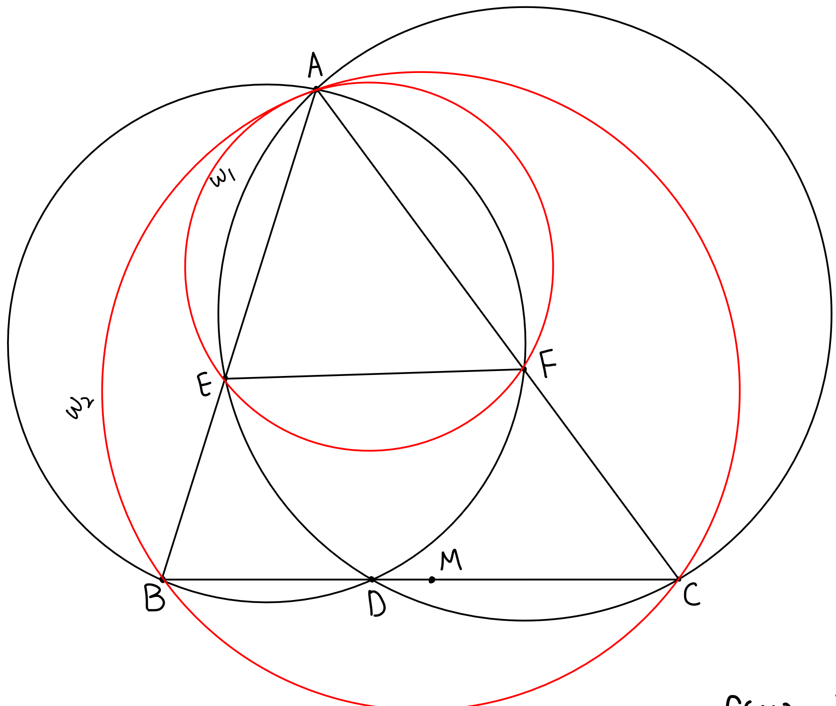
Problem 3.16 (Sun Yat-sen University bi-weekly problem)**.** Let I be the incenter of $\triangle ABC$. D, E, F be the symmetric point of I wrt BC, CA, AB respectively. Prove that there exists a point P such that $AP \perp DP, BP \perp EP, CP \perp FP$.



Taking a $\frac{1}{2}$ homothety at I, we see that (AFB), (AEC), (BCD) and (DEF) intersect (at $2P'-I$, where P' is the poncelet point of $ABCI$). Thus, $\angle APD = \angle APF + \angle FPD =$ $\angle FED + \angle ABF = \angle FED + \angle ABI = 90^\circ$, so P is our desired point.

3.17 2013 ELMO SL G3

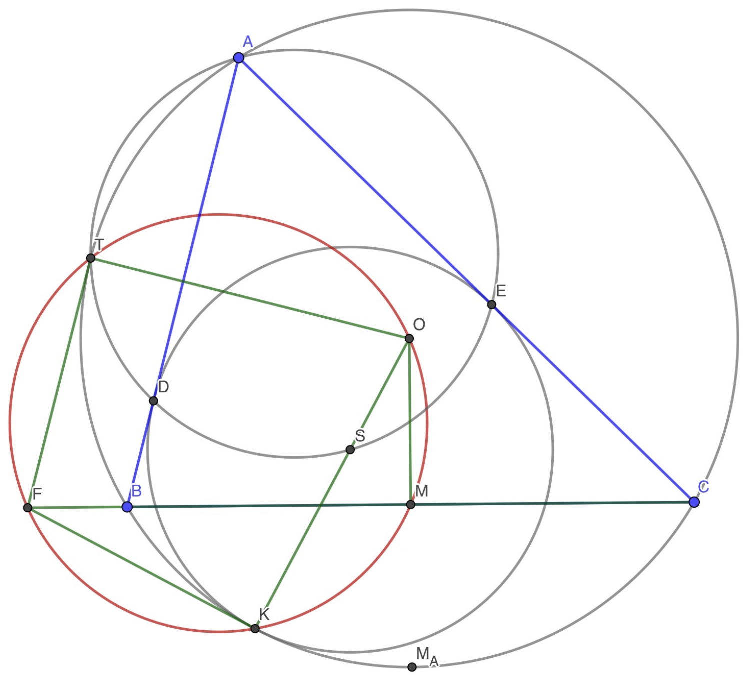
Problem 3.17 (2013 ELMO SL G3)**.** In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.



Consider $f(X) = \text{Pow}(w_1) - \text{Pow}(w_2)$. We have $f(M) = \dfrac{f(B)+f(C)}{2}$ and $f(B) = BE \cdot BA = BD \cdot BC$ and $f(C) = CD \cdot CB$, so $f(M) = \dfrac{BD \cdot BC + CD \cdot CB}{2} = \dfrac{BC^2}{2}$, and $f(M) = \text{pow}_M(w_1) + \dfrac{BC^2}{4} = \dfrac{BC^2}{2}$ so $\text{pow}_M(w_1) = \dfrac{BC^2}{4}$, and since $MH_A \cdot MA = MB^2 = \dfrac{BC^2}{4}$, $H_A \in (AEF)$, as desired.

3.18 Korea Winter Program Practice Test 2018/5

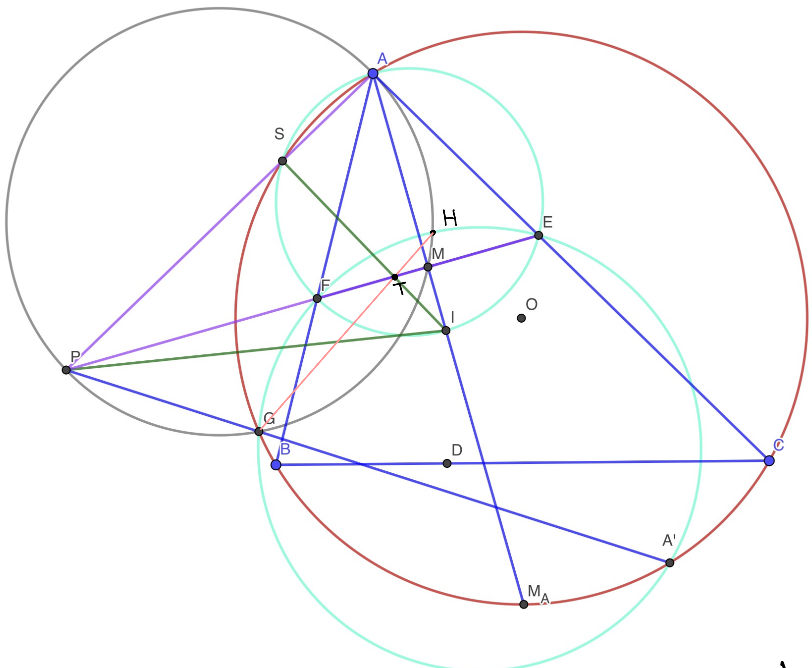
Problem 3.18 (Korea Winter Program Practice Test 2018/5)**.** Let ΔABC be a triangle with circumcenter O and circumcircle w. Let S be the center of the circle which is tangent with AB, AC, and w (in the inside), and let the circle meet w at point K. Let the circle with diameter AS meet w at T. If M is the midpoint of BC, show that K, T, M, O are concyclic.



By Mixti config, we know that $T = (ADES) \cap (ABC)$ and $(TK;BC) = -1$. Let $F = TT \cap KK$, so $(OTFK)$, and since $\angle OKF = \angle OMF$, $(OMTK)$.

3.19 2019 ELMO SL G3

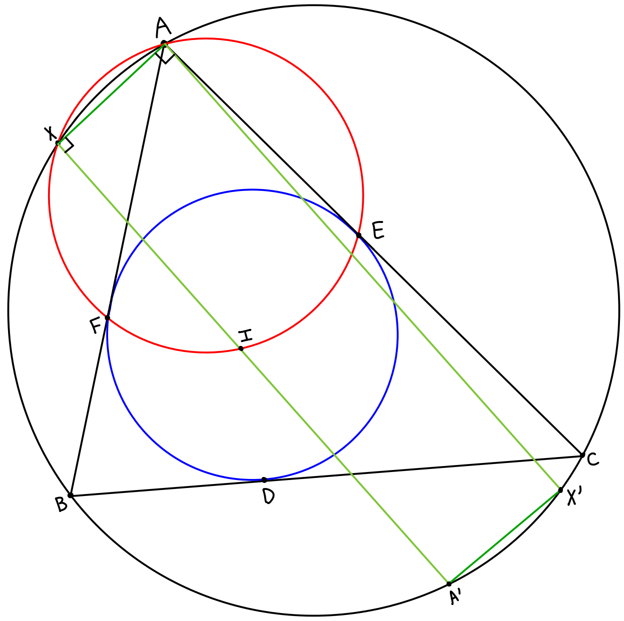
Problem 3.19 (2019 ELMO SL G3)**.** Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.



Let $P = AS \cap EF$, and $G' = PA' \cap (ABC)$. We have $PS \cdot PA = PF \cdot PE = PG' \cdot PA'$ so $G' = G$. Additionally, since $\angle AGA' = \angle AGP = \angle AMP = 90^\circ$, $(AMPG)$. By Radical Axis on $(AMPG)$, (AEF), $(A'EF)$, we have $SI \cap EF \cap GH = T$. However, since $T = SI \cap EF$, $DT \perp EF$.

3.20 2013 ELMO SL G2

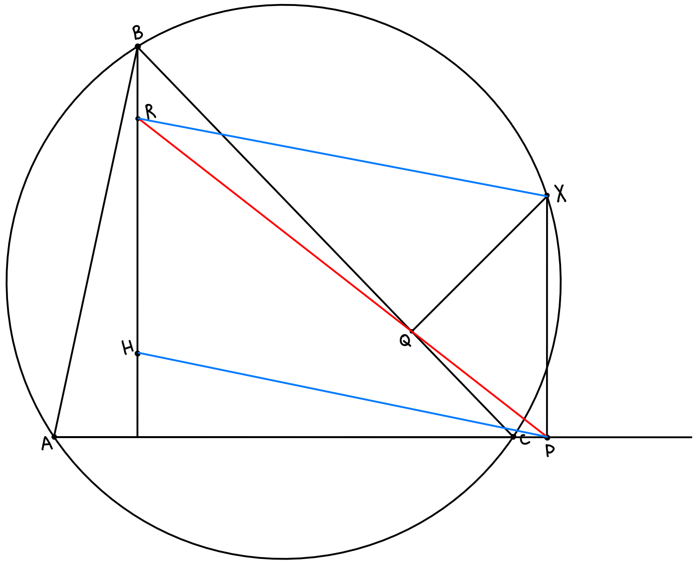
Problem 3.20 (2013 ELMO SL G2). Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.



We have that $\overline{XIA'}$, so AX' is a reflection of $A'X$ over O, so $2O-I$ lies on AX', BY', CZ'

3.21 USA TST 2014/1

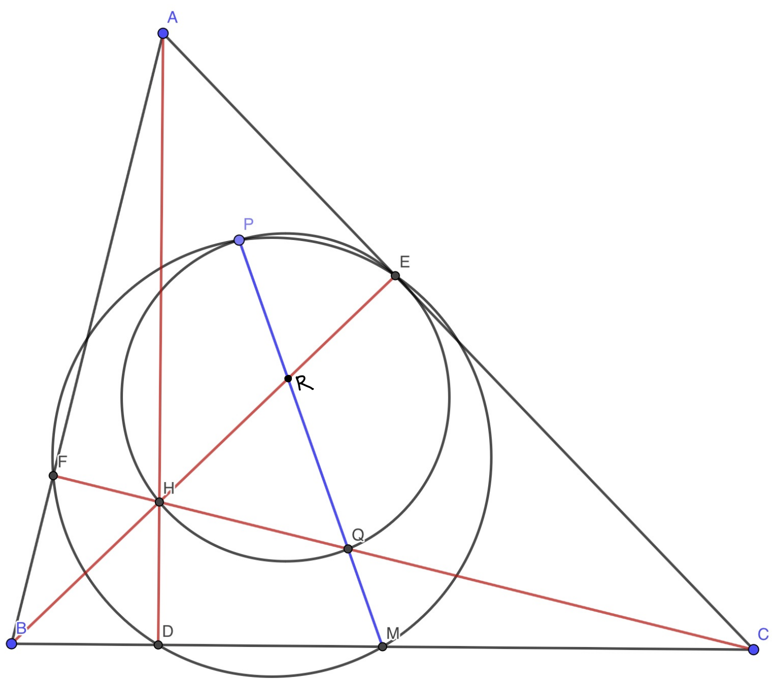
Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc BC, line ℓ passes through F.)



Note that PQ is the Simson line, so $XPHR$ is a parallelogram as desired ($H \in \ell$).

3.22 ELMO SL 2018/1

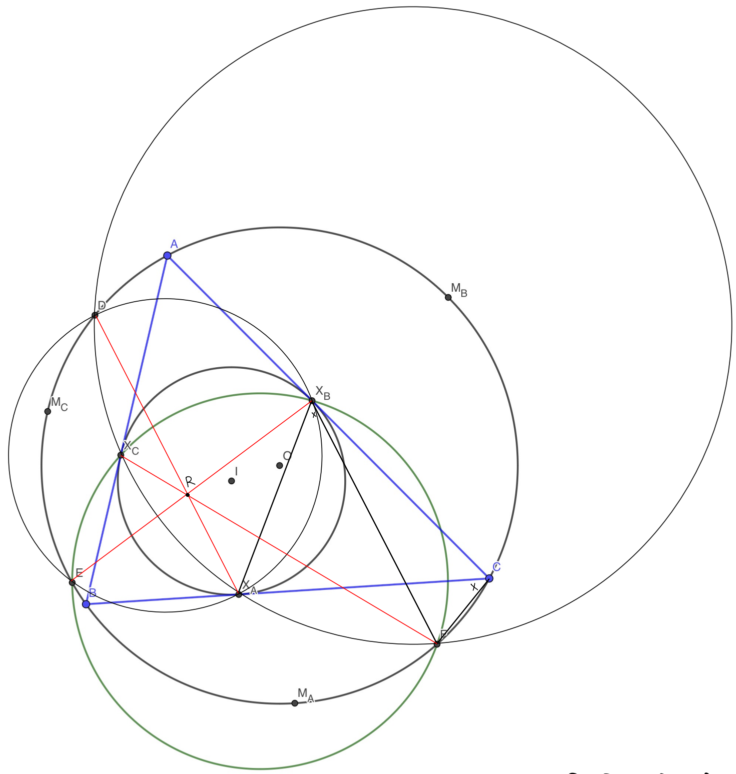
Problem 3.22 (ELMO SL 2018/1). Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.



let $Q = PM \cap CH$. we have
$\measuredangle EPM = \measuredangle EDM = \measuredangle EDB = \measuredangle CAB = \measuredangle EHC$ so $(PEHQ)$. Similarly,
$(FHRP)$ as desired (fixed point is M).

3.23 HSO math_pi_rate

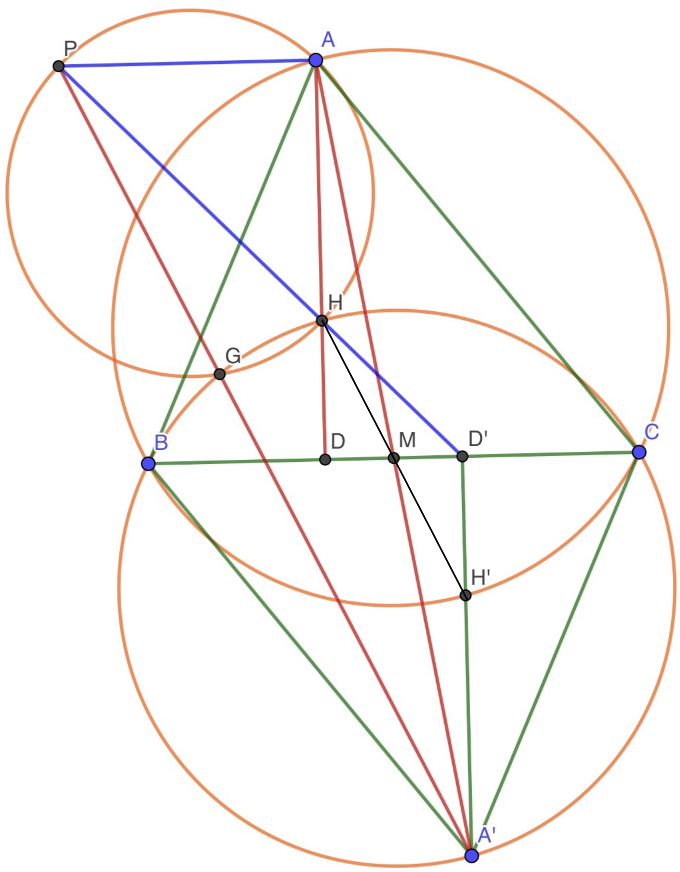
Problem 3.23 (AoPS user math_pi_rate)**.** Let X_A and Y_A be the A-intouch point and the foot of the A-internal angle bisector in a $\triangle ABC$. Define X_B, Y_B and X_C, Y_C analogously. Then prove that the radical center of $\odot AX_AY_A$, $\odot BX_BY_B$, $\odot CX_CY_C$ lies on $\overline{OI}$(O is the circumcenter and I is the incenter respectively of $\triangle ABC$).



Let $D = (AX_BX_C) \cap (ABC)$ and define E, F similarly. By configs, $(ADX_AY_A), (BEX_BY_B), (CFX_CY_C)$.
We have (X_CX_BEF) as $\angle X_CEF = \angle AEF - \angle AEX_C = (180 - C - x) - \left(\frac{A}{2} - \frac{B}{2}\right) = (A+B-x) - \left(\frac{A}{2} - \frac{C}{2}\right) = \frac{A}{2} + \frac{C}{2} + B - x$
and $\angle X_CX_BF = x + 90 - \frac{B}{2} = \frac{A}{2} + \frac{C}{2} + x$ so $\angle X_CEF + \angle X_CX_BF = 180^\circ$.
Thus, let $(X_CX_BEF), (X_AX_BDE), (X_AX_CDF)$ have radical center R.
Since $DR \cdot RX_A = ER \cdot RX_B = FR \cdot RX_C$, R is the radical center of $(ADX_A), (BEX_B), (CFX_C)$ as well. Additionally, since $\overline{DX_AM_A}$, $\overline{EX_BM_B}$, $\overline{FX_CM_C}$, we have $R = \overline{DM_A} \cap \overline{EM_B} \cap \overline{FM_C}$, so R is the center of homothety mapping $\triangle DEF \to \triangle M_AM_BM_C$, thus it lies on $\overline{OI}$ (centers of our triangles).

3.24 ELMO 2020/4

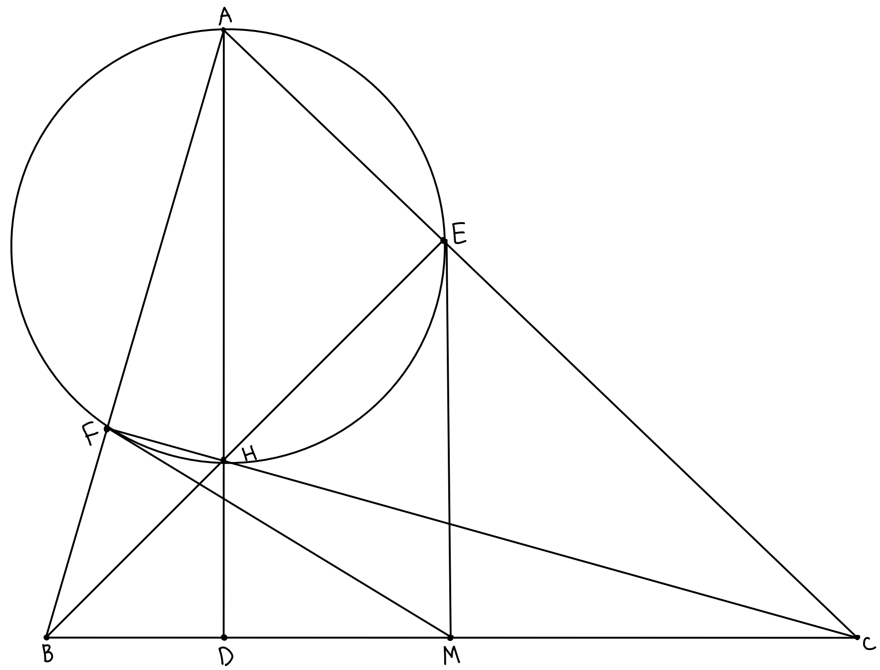
Problem 3.24 (ELMO 2020/4)**.** Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90°$.



we have $\overline{PGA'}$ as $\measuredangle PGH = \measuredangle HGA' = 90^\circ$. We also have $HH' \parallel PA'$ as $\frac{D'H}{HP} = \frac{HD}{AH} = \frac{DH'}{H'A'}$ so since $GH \perp AA'$, $HG \perp MH$ as desired.

3.25 ELMO 2012/1

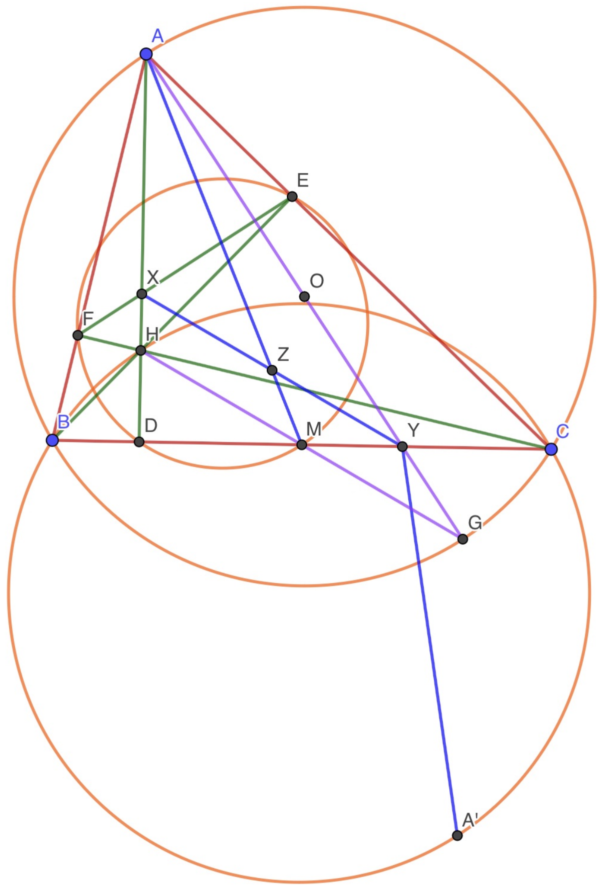
Problem 3.25 (ELMO 2012/1). In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.



Consider the radical center of w, w_1, w_2. We see that it is $EE \cap FF = M$ so M is on the radical axis of w_1, w_2, so the radical axis is MD. Thus, w_1 and w_2 meet again on BC.

3.26 Japan 2017/3

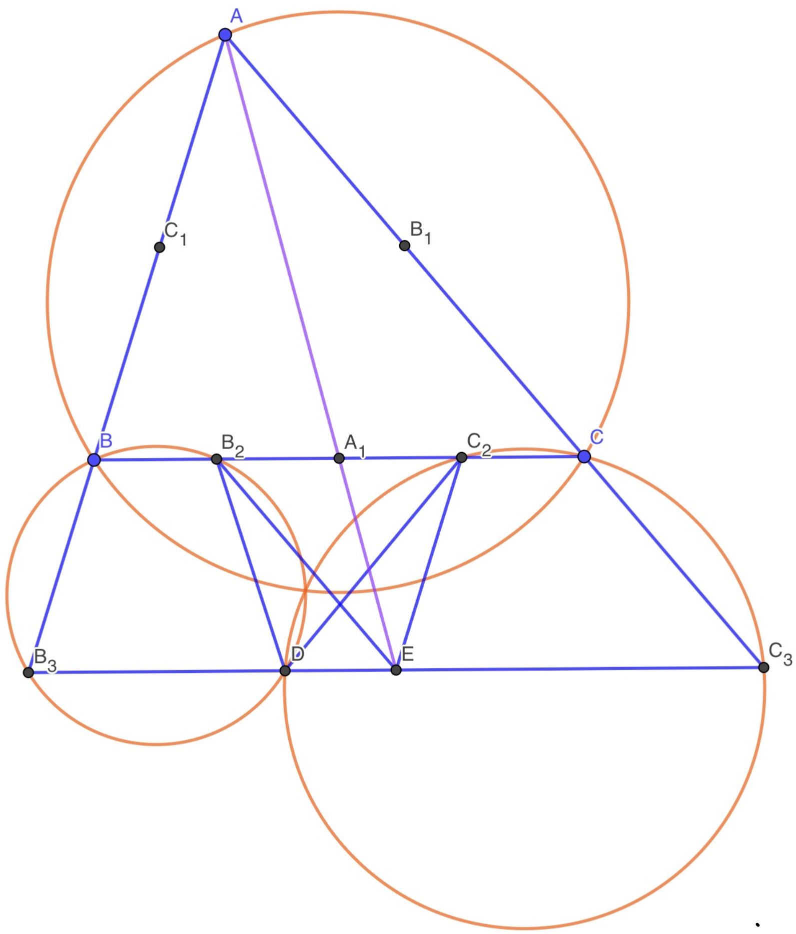
Problem 3.26 (Japan 2017/3)**.** Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the mid-point of XY. Prove that A, Z, M are collinear.



Notice that $\frac{AX}{AH} = \frac{AY}{AG}$ as $\square AEHF \sim \square ABGC$, so $XY \parallel HG$. Thus, $\overline{AZM}$, as Z is the midpoint of AY, and M is the midpoint of HG.

3.27 Sharygin 2015 Final Round Grade 10 (penultimate grade) Problem 3

Problem 3.27 (Sharygin 2015 Final Round Grade 10(penultimate grade) Problem 3). Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB of triangle ABC, respectively. Points B_2 and C_2 are the midpoints of segments BA_1 and CA_1 respectively. Point B_3 is symmetric to C_1 wrt B, and C_3 is symmetric to B_1 wrt C. Prove that one of common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

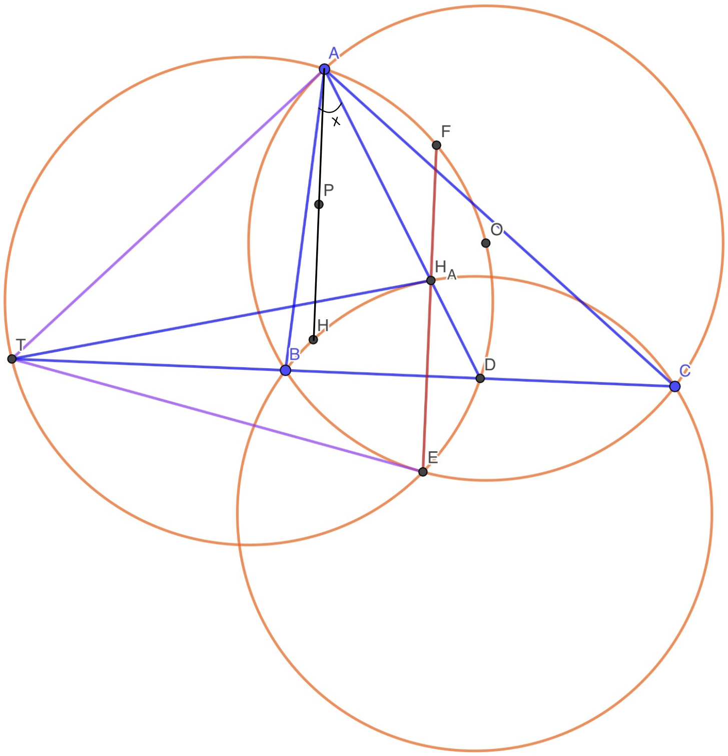


Let E be the reflection of A over A_1 with a $\frac{1}{2}$ homothety ($A_1E = \frac{1}{2}AA_1$), and let $D = \overline{B_3C_3E} \cap (EB_2C_2)$. Note that $\measuredangle B_2BB_3 = \measuredangle B_2BA = \measuredangle B_2C_2E = \measuredangle B_2DB_3$, so (BB_2B_3D) and (CC_2DC_3). By Miquel's, we have $(ABC) \cap (BB_2B_3) \cap (CC_2C_3)$ as desired.

3.28 ELMO 2018/4

Problem 3.28 (ELMO 2018/4)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90°$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

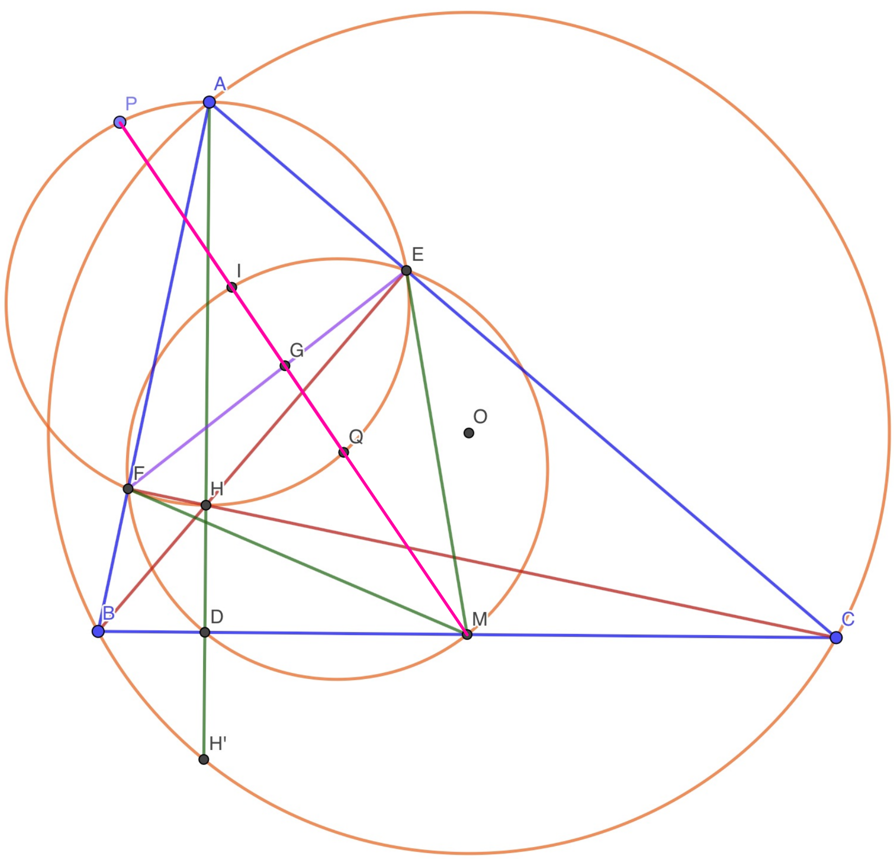
*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.



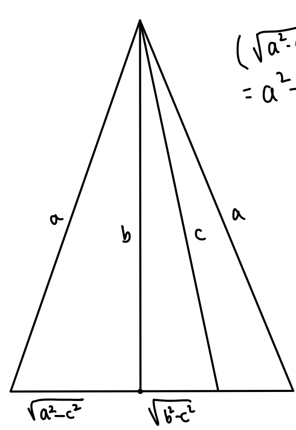
Let $F = H_AE \cap (ATO)$. We have $PH = PA = PH_A$. Let $\angle BAD = x$.
Thus, $\angle PAH_A = x - (90 - B) = x + B - 90$. We have
$\angle APH_A = 180 - (2x + 2B - 180) = 360 - 2x - 2B$. Thus, $\angle AFE = \angle ADE = 2\angle BDA$
$= 2(180 - x - B) = 360 - 2x - 2B$. Thus, APH_AF is a parallelogram.
We have $FA = FH_A$ so $\overline{TPF} \to F = X$ so the midpoint of $\overline{PX}$ is the midpoint $\overline{AH_A}$, and is on the 9 point circle.

3.29 ELMO 2017/2

Problem 3.29 (ELMO 2017/2). Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.



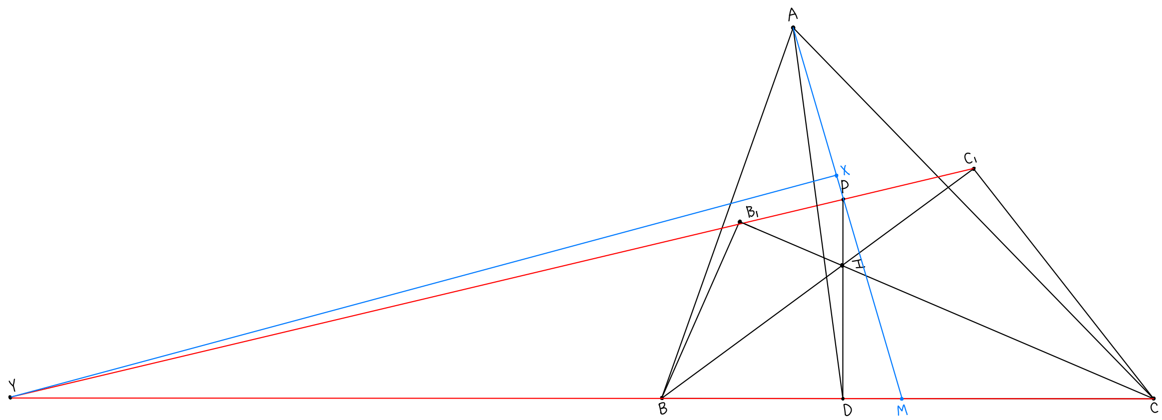
The orthocenter of APQ is the reflection of H over I. We prove that I lies on the 9 point circle, which implies our result. Note that $(MG;QP)=-1$, so $MG\cdot MI = MP\cdot MQ = ME^2$. Thus, $MG\cdot GI = MG\cdot(MI-MG) = ME^2-MG^2 = EG\cdot GF$, as desired (we have $(IEMF)$).



$$\left(\sqrt{a^2-c^2}+\sqrt{b^2-c^2}\right)\left(\sqrt{a^2-c^2}-\sqrt{b^2-c^2}\right)$$
$$=a^2-b^2$$

3.30 CJMO 2019/3

Problem 3.30 (CJMO 2019/3)**.** Let I be the incenter of $\triangle ABC$, and M the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X intersect on the A-intouch chord of $\triangle ABC$.



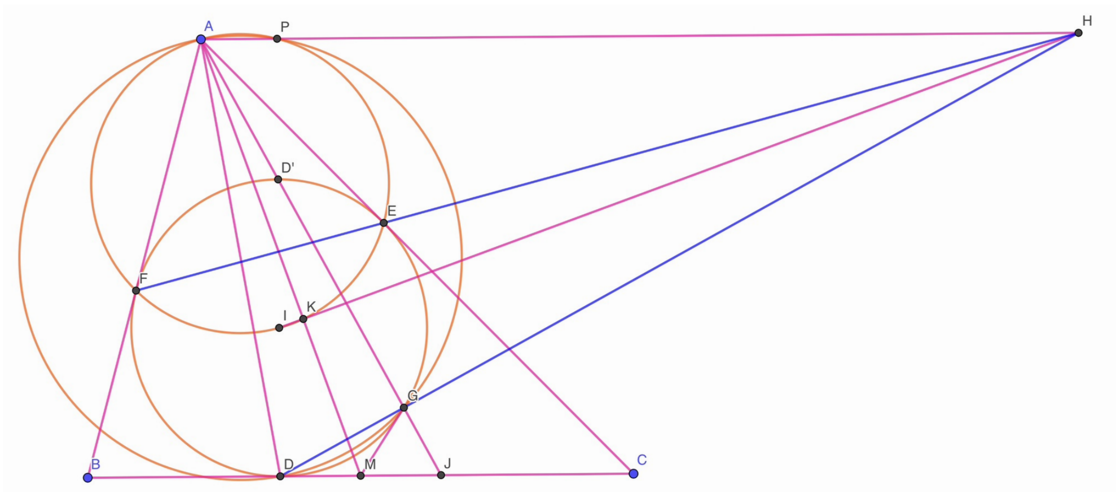
Iran lemma tells us that B_1C_1 is the intouch chord. Let $P = AM \cap B_1C_1 \cap DI$ (median incircle concurrency). We invert about (B_1C_1BC) and we have $\overline{B_1C_1} \leftrightarrow (B_1C_1M)$ (9 point of $\triangle IBC$), so $Y \leftrightarrow D$. We have $\angle MXY = \angle MY'X'$ so $\angle MDX' = 90^\circ$ so X' is on AM and ID so X' is P. Since $P \in B_1C_1$, $X \in (B_1C_1BC)$ as desired. Now, note that

$$(DX;B_1C_1) = (D'X';B_1'C_1') = (YP;B_1C_1) = (YD;B_1C_1) = -1$$

so $DD \cap XX \in \overline{B_1C_1}$

3.31 GOTEEM 1

Problem 3.31 (GOTEEM 1)**.** Let ABC be a scalene triangle. The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let G be a point on the incircle of $\triangle ABC$ such that $\angle AGD = 90°$. If lines DG and EF intersect at P, prove that AP is parallel to BC.

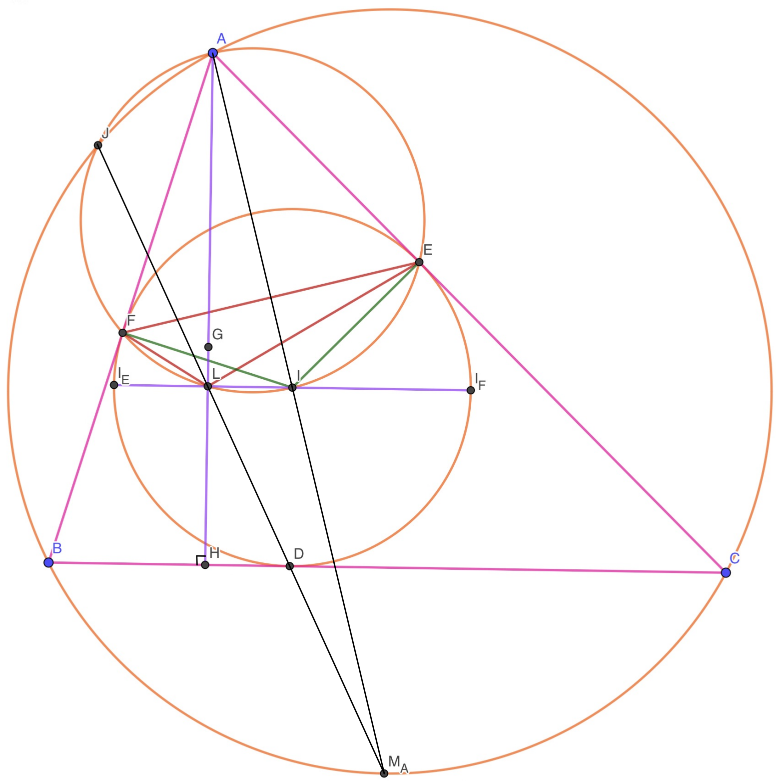


Let $P \in \overline{DI}$ and $AP \parallel BC$. We see that $G = AD' \cap (DEF)$, so $\measuredangle AGD = \measuredangle APD = 90°$ so $(PADG)$. We also have $\measuredangle API = \measuredangle AEI$ so $(APEIF)$. Thus, radical axis on (AEF), (DEF), (ADG) gives that $H \in \overline{AP}$ as desired.

3.32 GGG1/4

Problem 3.32 (GGG1/4)**.** Let ABC be an acute triangle, let ω be its incircle, and let M_A be the midpoint of minor arc $\hat{BC}$ on the circumcircle of ABC. Let ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at points D, E, F, respectively, and let H be the foot of the altitude from A to $\overline{BC}$. Denote by L the intersection of $\overleftrightarrow{M_AD}$ and $\overleftrightarrow{AH}$. Let I_E and I_F denote the E and F-excenters of triangle ELF, respectively.

Prove that I_E and I_F lie on ω.

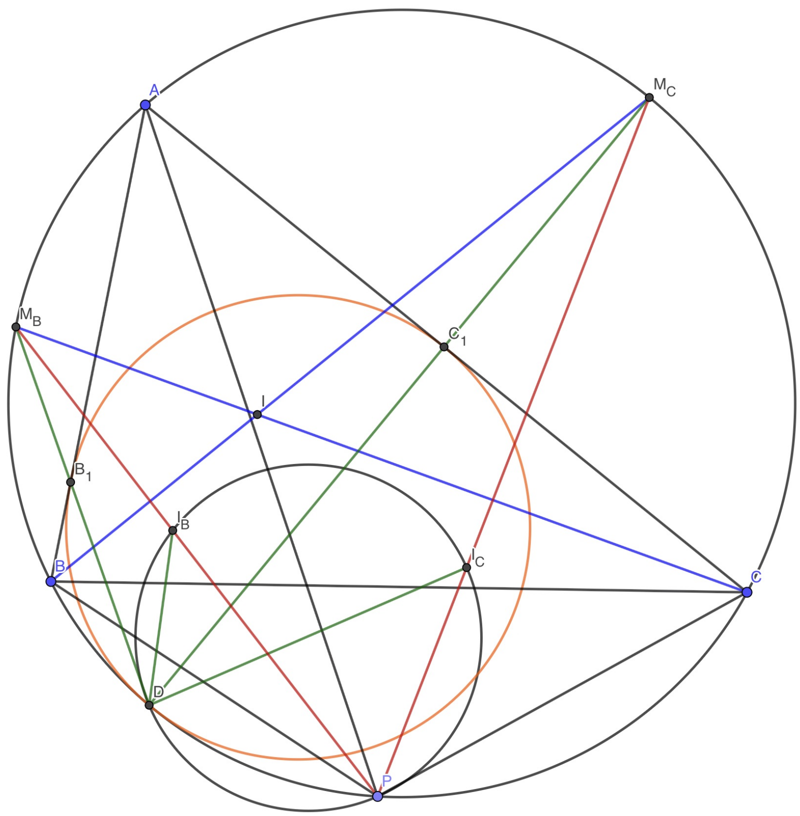


Let L be $\overline{M_AD} \cap (AEF)$. We have $\measuredangle JLI = \measuredangle JAM_A = \measuredangle JBM_A = \measuredangle JDC$, so $LI \parallel BC$, so $\overline{ALH}$. Since $AE = AF$, the incenter of EFL is on AL, so I_E and I_F are on the line through L parallel to BC. Consider (I_EI_FEF). We know the center must be on I_EI_F and the perpendicular bisector of EF. Thus, the center must be I, so $(I_EI_FEF) = \omega$ as desired.

3.33 USAJMO 2016/1 generalization

Problem 3.33 (USAJMO 2016/1 generalization)**.** The triangle $\triangle ABC$ is inscribed in the circle ω. Let P be a variable point on the arc $\hat{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of $\triangle PI_BI_C$ passes through a fixed point.



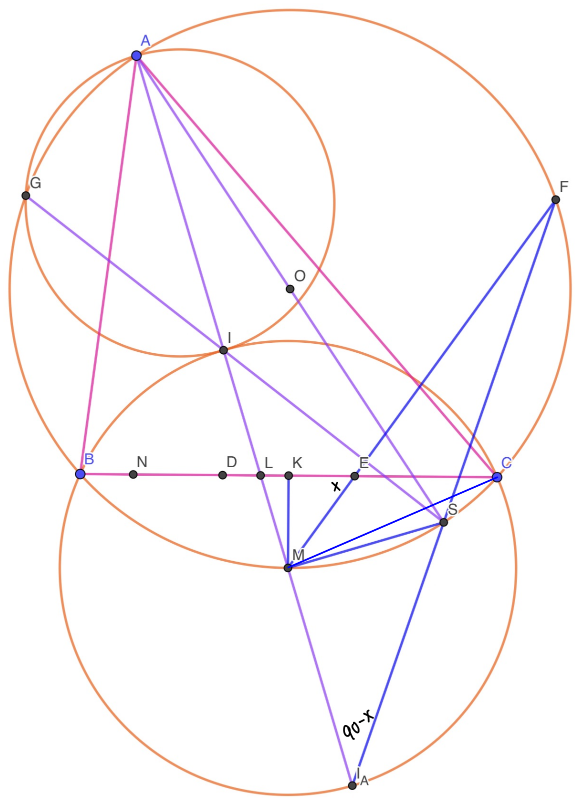
We show the mixtilinear touchpoint is the fixed point. We show that $\triangle M_B D I_B \sim \triangle M_C D I_C$, or $\frac{M_B D}{M_B I_B} = \frac{M_C D}{M_C I_C}$. Let $\frac{M_B D}{M_B B_1} = \frac{M_C D}{M_C C_1} = x$

Note that $M_B I_B = M_B A$ and $M_B B_1 \cdot M_B D = M_B A^2$, or $x(M_B D)^2 = M_B A^2$ or $M_B D = \frac{M_B A}{\sqrt{x}}$. Thus, $\frac{M_B D}{M_B I_B} = \frac{1}{\sqrt{x}} = \frac{M_C D}{M_C I_C}$

(by symmetry) as desired.

3.34 ELMO 2010/6

Problem 3.34 (ELMO 2010/6). Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.



By Configs, we know S is the antipode of A. We show that $\triangle MKE \sim \triangle MSI_A$. We show that $\frac{MK}{KE} = \frac{MI_A}{MS} = \frac{MC}{MS}$. Let $2R=1$.

We have $MS = \cos\left(C+\frac{A}{2}\right)$. We also have $MK = MC\sin\frac{A}{2}$ and $KE = \frac{b-c}{2}$.

Our expression simplifies into $\frac{\cos\left(C+\frac{A}{2}\right)}{MC} = \frac{\frac{b-c}{2}}{MC\sin\left(\frac{A}{2}\right)}$ or we wish to show $\cos\left(C+\frac{A}{2}\right)\sin\left(\frac{A}{2}\right) = \frac{b-c}{2}$.

We also have by LOS on $\triangle MCL$ that $\frac{\sin B}{CL} = \frac{\sin\left(\frac{A}{2}\right)}{ML}$ and we also have from $\triangle MLK$ that $\cos\left(C+\frac{A}{2}\right) = \frac{LK}{ML}$. Thus, our expression is now $\frac{LK}{ML}\cdot\frac{ML}{CL}\sin B = \frac{b-c}{2}$ or $\frac{LK}{CL} = \frac{b-c}{2}$.

We have $LK = \frac{ab}{b+c} - \frac{a}{2} = \frac{a(b-c)}{2(b+c)}$ and $CL = \frac{ab}{b+c}$ and $\frac{\frac{a(b-c)}{2(b+c)}}{\frac{ab}{b+c}}\cdot\frac{1}{b} = \frac{b-c}{2}$ is true.

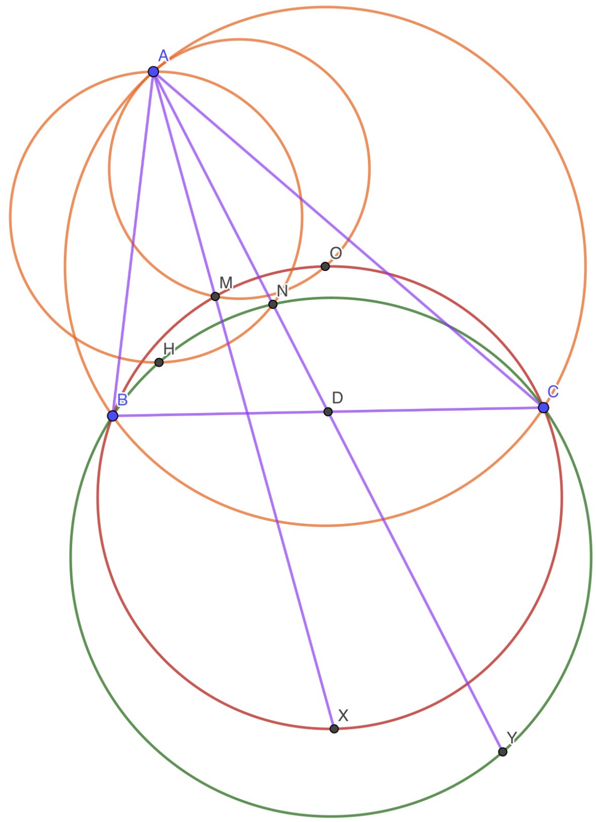
Now let $\angle MEK = x$.

We have $\angle(SI_A, ME) = \left(\left(B+\frac{A}{2}\right) - (90-x)\right) - x = B + \frac{A}{2} - 90 = \frac{B}{2} - \frac{C}{2} = \angle MAS$

as desired.

3.35 ELMO 2014/5

Problem 3.35 (ELMO 2014/5)**.** Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.



We have M is the dumpty point, N is the humpty point, so

$$M = \frac{a^2-bc}{2a-b-c}, \quad N = \frac{ab^2+ac^2-b^2c-bc^2}{ab+ac-2bc}, \quad X = \frac{2bc}{b+c}, \quad Y = b+c-a.$$

Thus, $M-N = \dfrac{(a-b)(a-c)(ab+ac-b^2-c^2)}{(2a-b-c)(ab+ac-2bc)}$ and $X-Y = \dfrac{(ab+ac-b^2-c^2)}{b+c}$

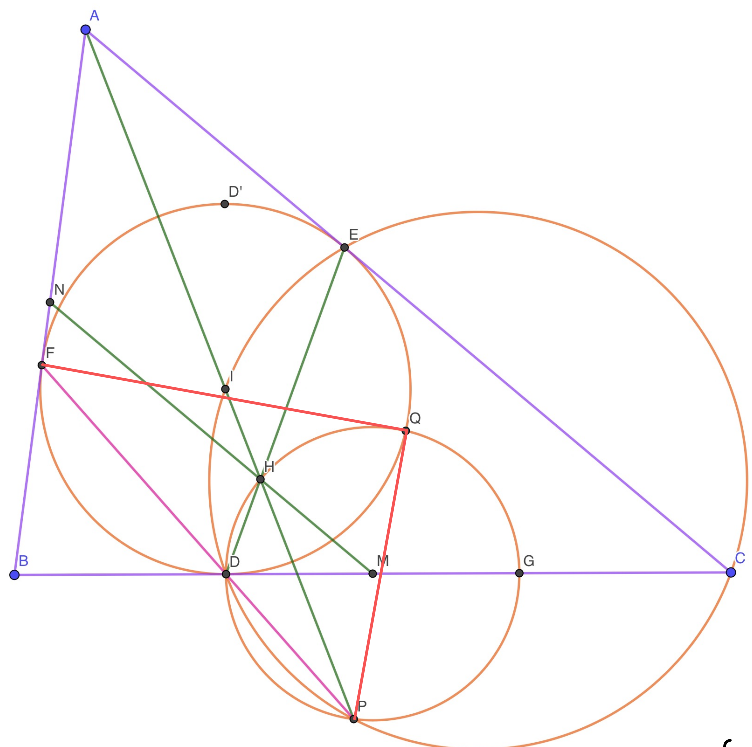
So, $\dfrac{M-N}{X-Y} = \dfrac{(a-b)(a-c)(b+c)}{(2a-b-c)(ab+ac-2bc)}$, and this quantity is real as

$$\overline{\frac{M-N}{X-Y}} = \frac{\left(\frac{(b-a)}{ab}\right)\left(\frac{(c-a)}{ac}\right)\left(\frac{b+c}{bc}\right)}{\left(\frac{2bc-bc-ac}{abc}\right)\left(\frac{c+b-a}{abc}\right)} = \frac{M-N}{X-Y}$$

as desired.

3.36 USA TST 2015/1

Problem 3.36 (USA TST 2015/1)**.** Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$.



Note that AI and FD intersect on $(CDIE)$ as $\angle(AI, FD) = \frac{\angle C}{2}$

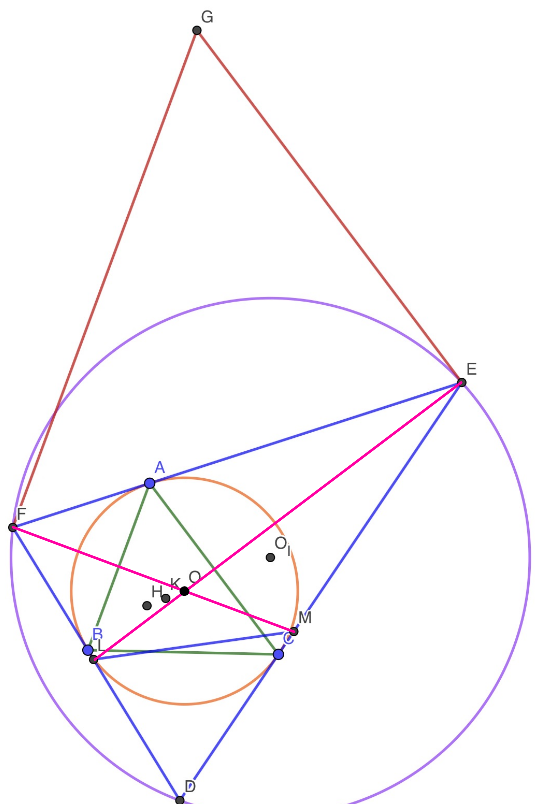
Let $P = AI \cap FD$. We invert about the incircle. We see that P inverts to H, where $H \in CD, (BDF), AI$. Thus, $MH \parallel CA$, by Iran Lemma as $H = ED \cap AI$, so $MD = MH$, as $CD = CE$. We also have $MD = MQ$ by our configs (MQ tangent to (DEF)). We also have $IH \cdot IP = ID^2$ due to inversion and $IQ^2 = IH \cdot IP$, so $(DHQP)$. We see that M must be the center as $MD = MQ = MH$. Thus, P is our desired point. We see that

$\measuredangle FQP = \measuredangle QFP + \measuredangle FPQ = \measuredangle QDM + \measuredangle FPQ = \measuredangle QPG + \measuredangle DPQ = 90°$

as desired.

3.37 BMO SL 2018/2

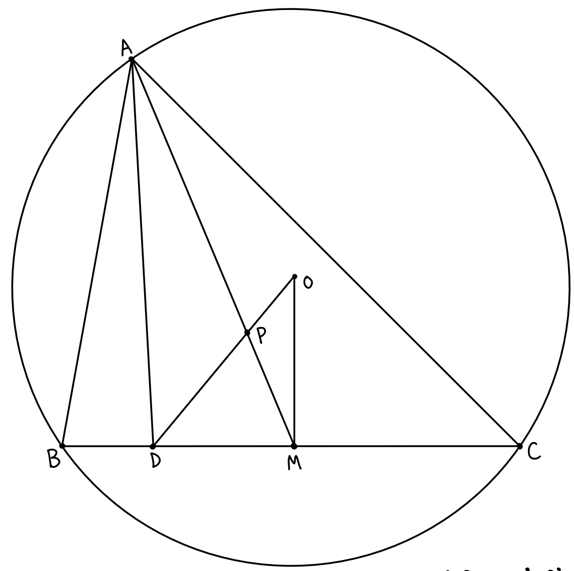
Problem 3.37 (BMOSL 2018/2)**.** Let ABC be a triangle inscribed in circle Γ with center O. Let H be the orthocenter of triangle ABC and let K be the midpoint of OH. Tangent of Γ at B intersects the perpendicular bisector of AC at L. Tangent of Γ at C intersects the perpendicular bisector of AB at M. Prove that AK and LM are perpendicular.



Consider the tangental triangle of ABC. We see that EL and FM are the perp bisectors of AC, AB, so FM and EL are actually angle bisectors. Thus, if G is the D-excenter, by configs, $O_1 G \perp LM$. We know that $\triangle ABC$ is homothetic to the triangle formed by the D, E, F excenters, so by the homothety, $AK \parallel O_1 G$ (O_1 is the 9 point center) so $AK \perp LM$

3.38 Folklore P=OD cap AM on radical axis of (BOC) and 9 point circle of (ABC)

Problem 3.38 (Folklore I think)**.** Given a triangle ABC with circumcenter O, let M be the midpoint of BC and D be the foot from A to $\overline{BC}$. $P = \overline{OD} \cap \overline{AM}$. Prove that $P = \overline{OD} \cap \overline{AM}$ lies on the radical axis of (BOC) and the nine-point circle of (ABC).



We use linearity of POAP. Consider $f(x) = \text{pow}_x(\text{9 point circle}) - \text{pow}_x((BOC))$.

First we compute $f(O)$. We have $OH^2 = 9R^2 - (a^2+b^2+c^2)$, so $\text{pow}_O(\text{9 point circle}) = OK^2 - \left(\frac{R}{2}\right)^2$

or $f(O) = 2R^2 - \left(\frac{a^2+b^2+c^2}{4}\right)$.

Now we compute $f(D)$. We have $\text{pow}_D((BOC)) = BD \cdot CD = \frac{AD^2}{\tan B \tan C}$. Thus $f(D) = \frac{AD^2}{\tan B \tan C}$

By linearity of POAP, $f(P) = \frac{OP \cdot f(D) + DP \cdot f(O)}{OP} = \frac{OM f(D) + AD f(O)}{OM + AD}$.

We show $OM f(D) + AD f(O) = 0$. WLOG let $R = \frac{1}{2}$, so $OM = \frac{1}{2}\cos A$.

$$AD\left(2R^2 - \frac{a^2+b^2+c^2}{4}\right) + \left(\tfrac{1}{2}\cos A\right)\left(\frac{AD^2}{\tan B \tan C}\right) = 0 \rightarrow \left(1 - \frac{a^2+b^2+c^2}{2}\right) + \frac{\cos A \cdot \frac{bc}{a}\sin A}{\tan B \tan C} = 0$$

$$\rightarrow \left(2 - (a^2+b^2+c^2)\right) + 2\cos A \cos B \cos C = 0$$

Since $A+B+C = 180$, $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$ (2013 AIME II)

Thus, $\left(2 - (a^2+b^2+c^2)\right) + \left(1 - \cos^2 A - \cos^2 B - \cos^2 C\right) = 0$

$$\rightarrow \left(-a^2 - b^2 - c^2\right) + \left(1 - \cos^2 A\right) + \left(1 - \cos^2 B\right) + \left(1 - \cos^2 C\right) = 0$$

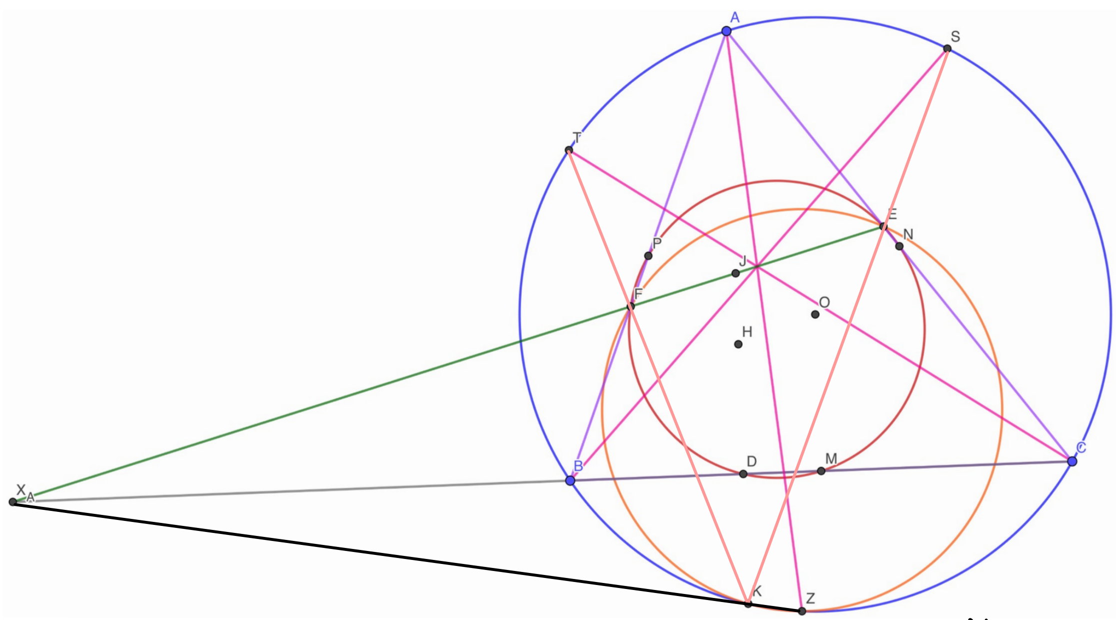
which is true as $\sin A = a$.

3.39 Vietnam 2019/6

Problem 3.39 (Vietnam 2019/6)**.** Given an acute triangle ABC and (O) be its circumcircle, and H is its orthocenter. Let M, N, P be midpoints of BC, CA, AB, respectively. D, E, F are the feet of the altitudes from A, B and C, respectively. Let K symmetry with H through BC. DE intersects MP at X, DF intersects MN at Y.

a XY intersects smaller arc BC of (O) at Z. Prove that K, Z, E, F are concyclic.

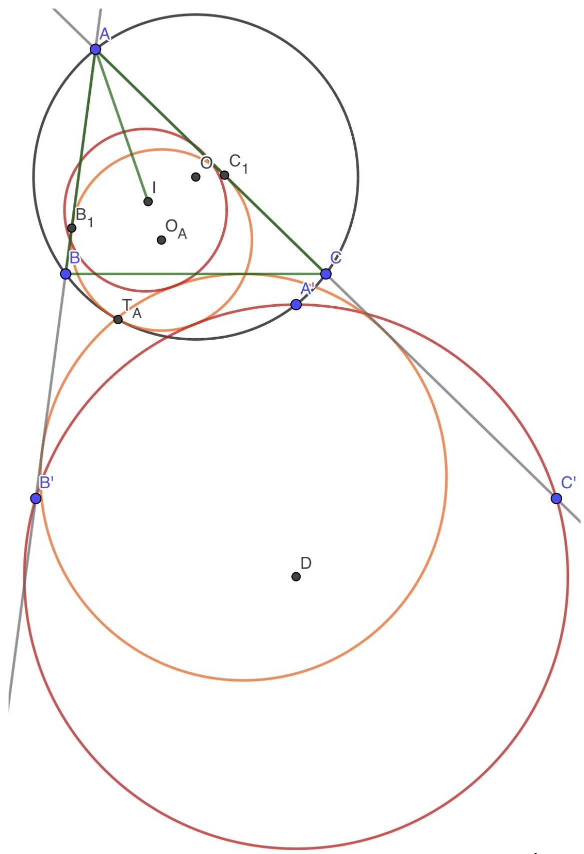
b KE, KF intersect (O) at S, T ($S, T \neq K$), respectively. Prove that BS, CT, XY are concurrent.



By Configs, we know that X,Y are on the A symmedian.
Thus, $\overline{X_AKZ}$ and $\overline{X_AEF}$ so $X_AE \cdot X_AF = X_AK \cdot X_AZ$ or $(KEFZ)$.
Additionally, Pascals on KTCBAZ and KSBCAZ
gives $AZ \cap SB \cap CT \cap EF$ as desired.

3.40 AOPS tutubixu9198 (BOC) is tangent to 1/2 homothety of mixtilinear incircle

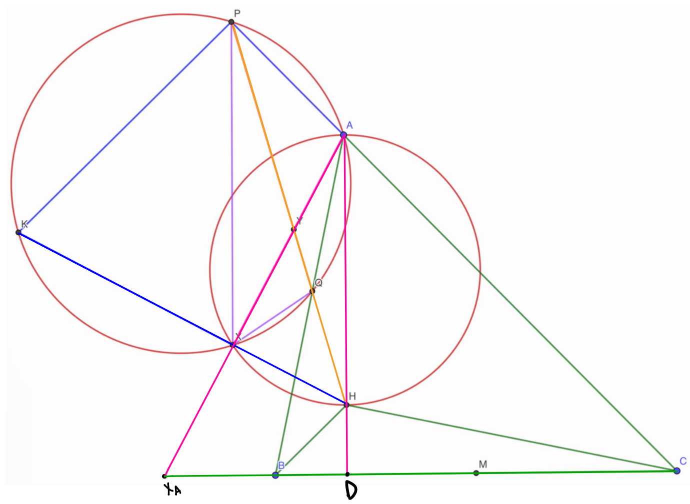
Problem 3.40 (AoPS user tutubixu9198). Let ABC be a triangle with circumcenter O and incenter I. Let (O_1) be a circle ex-tangent to (BOC) and tangent to AB, AC at M, N. Prove that MN passes through midpoint of AI.



First consider a 2x homothety of (O_1) to the mixtilinear incircle and a 2x homothety of (BOC). Now, consider a $\sqrt{bc}$ inversion about A, which sends our mixti incircle to the A-excircle and (BOC) to the 9 point circle. By Configs, we know the excircle is tangent to the 9 point circle as desired.

3. 41 Sharygin Correspondence Round 2020/15

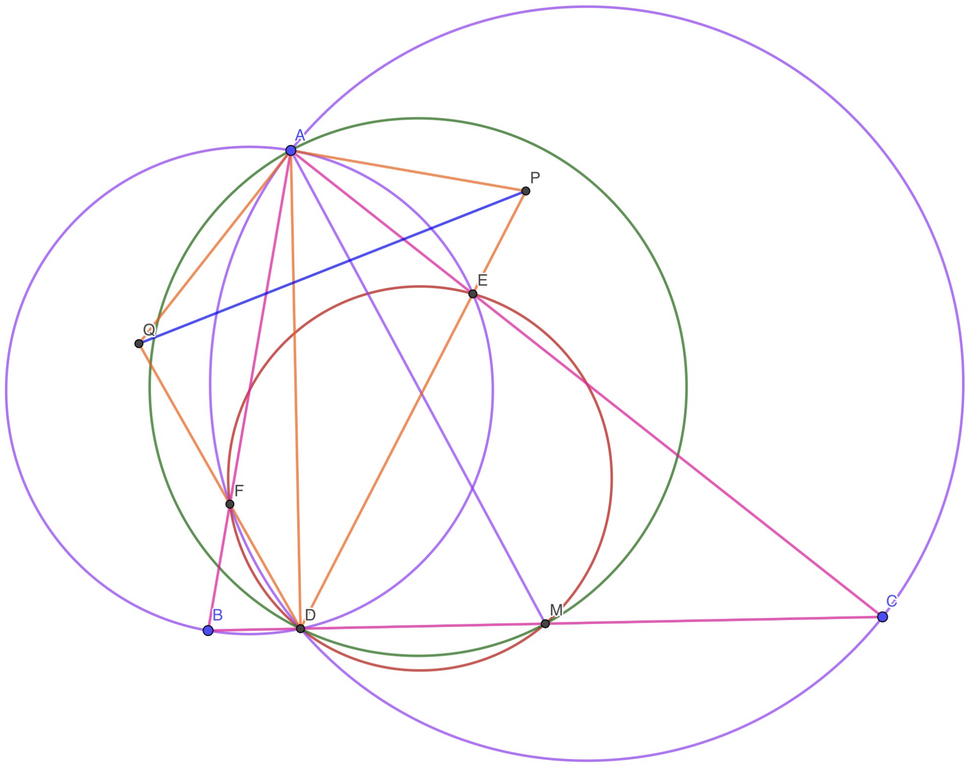
Problem 3.41 (Sharygin Correspondence Round 2020/15). Let H be the orthocenter of a nonisosceles triangle ABC. The bisector of angle BHC meets AB and AC at points P and Q respectively. The perpendiculars to AB and AC from P and Q meet at K. Prove that KH bisects the segment BC.



Let $X = (AK) \cap KH$. Note that $AP = AQ$, as $\measuredangle QBH = \measuredangle HCP$ and $\measuredangle QHB = \measuredangle CHP$. Thus, AX is an angle bisector, and KX is an exterior angle bisector so $(KA;PQ) \overset{X}{=} (PQ;YH) \overset{A}{=} (CB;DX_A) = -1$, Thus, X is the Queue Point and thus $\overline{XHM}$ by Configs.

3.42 GOTEEM 2

Problem 3.42 (GOTEEM 2)**.** Let ABC be an acute triangle with $AB \neq AC$, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P be a point on DE such that $AP \perp AB$ and let Q be a point on DF such that $AQ \perp AC$. Lines PQ and BC intersect at T. If M is the midpoint of $\overline{BC}$, prove that $\angle MAT = 90°$.

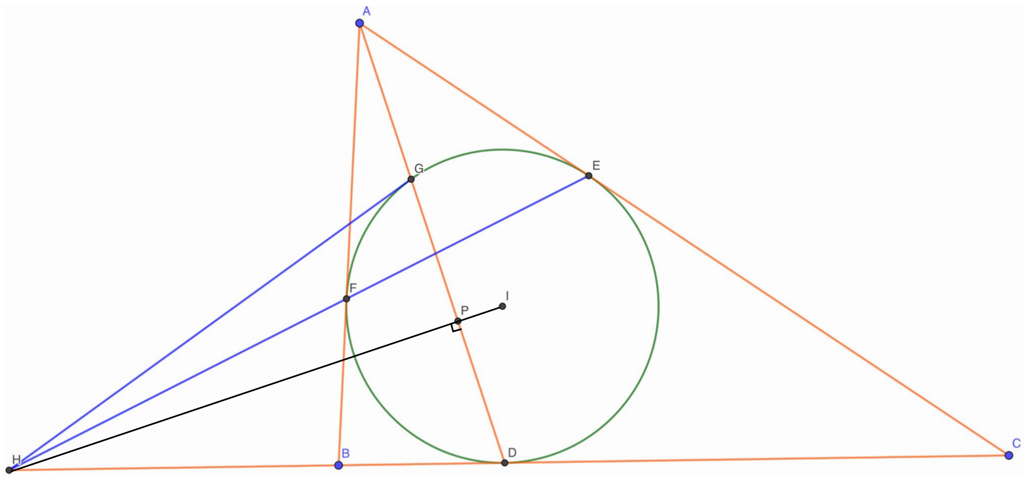


We have QA is tangent to (ADC) and PA is tangent to (ADB), so $QA^2 = QF \cdot QD$, $PA^2 = PE \cdot PD$.

Consider the radical center of (A), (DEF), (ADM). We see the radical axis of (A) and (DEF) is PQ, and (DEF) and (ADM) is BC, so the radical center is T. Thus, since (ADM) has diameter AM, $AT \perp AM$ as desired (radical axis of (A) and (ADM) is tangent through A, so $\angle MAT = 90°$).

3.43 2012 ELMO SL G3

Problem 3.43 (2012 ELMO SL G3)**.** ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

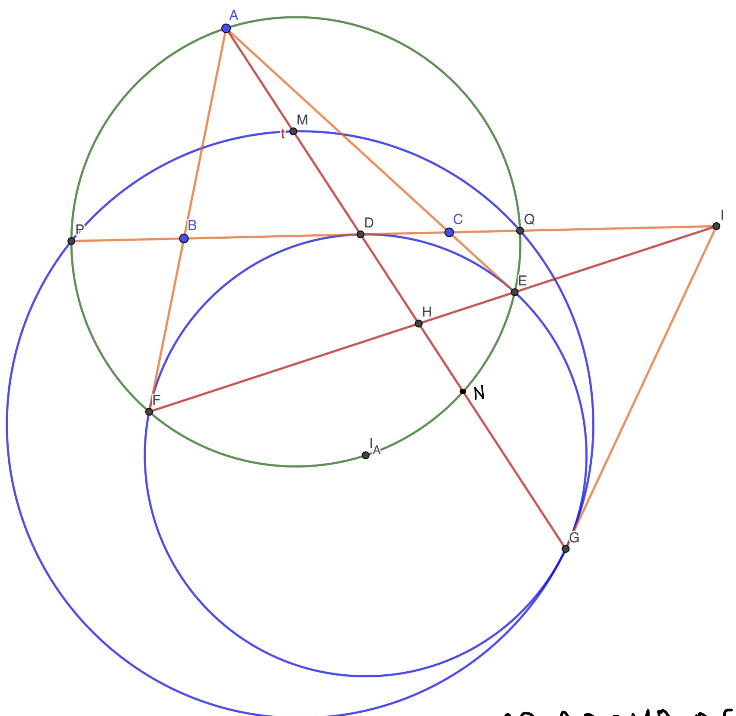


we have $(EF;GD) = -1$, so $H = DD \cap EF \cap GG \cap IP$ (as $H = GG \cap DD$)

so $(HD;BC) = -1 \rightarrow \angle BPD = \angle DPC$.

3.44 2017 ISL G4

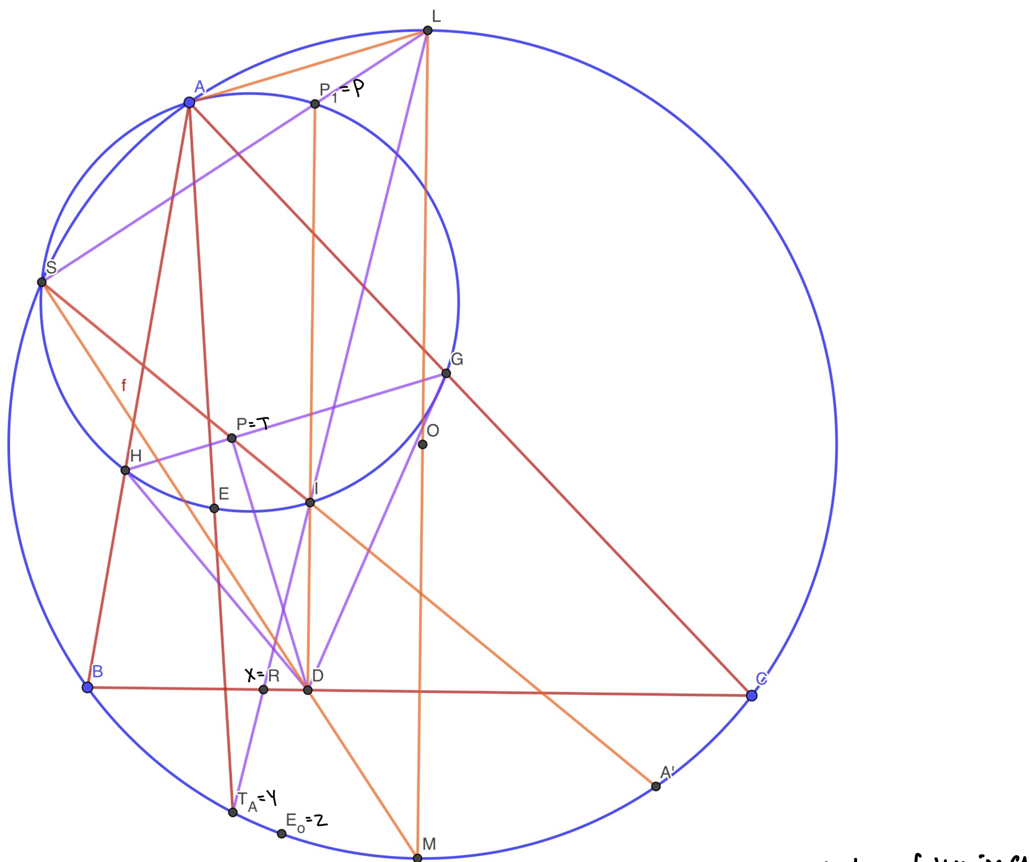
Problem 3.44 (2017 ISL G4). In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.



Let $N = (AEF) \cap AD$ we have $AD \cdot DN = PQ \cdot DQ = MD \cdot DG$ so $(MQGP)$.
Let $G = AD \cap (DEF)$. We have $(DG;EF) = -1$ so $DD \cap EF \cap GG = I$. Consider the radical center of $(AEF), (MPQ), (DEF)$, which is I. Thus, IG is the common tangent to (DEF) and (MPQ) as desired (meaning the two circles are tangent).

3.45 GGG3 6

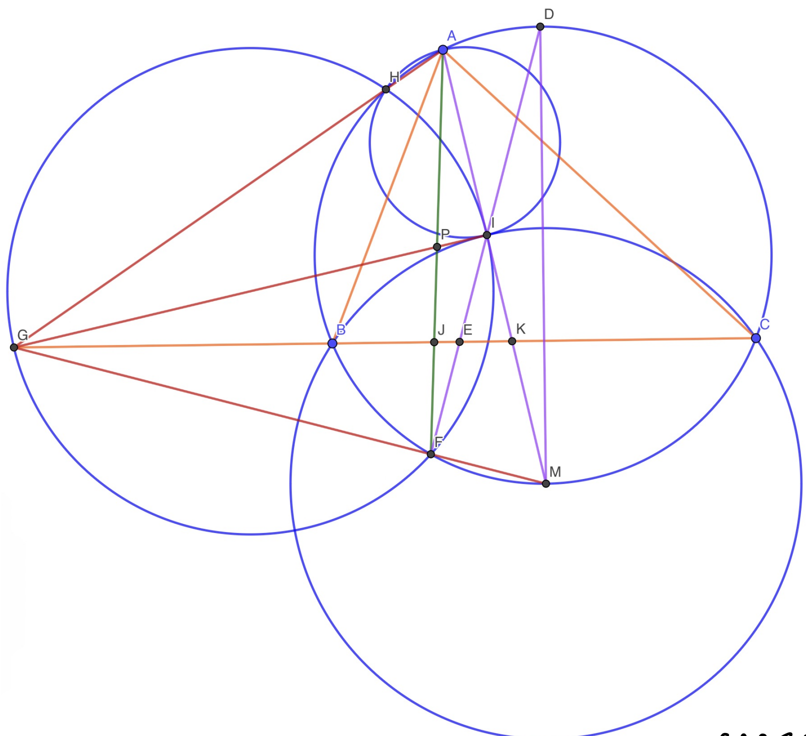
Problem 3.45 (GGG3 6)**.** Let ABC be a scalene triangle with incenter I and circumcircle Ω; L is the midpoint of arc BAC and A' is diametrically opposite A on Ω. D is the foot of the perpendicular from I to $\overline{BC}$. $\overleftrightarrow{LI}$ meets $\overleftrightarrow{BC}$ and Ω at X and Y, respectively, and $\overleftrightarrow{LD}$ meets Ω again at Z. $\overleftrightarrow{XZ}$ meets $\overleftrightarrow{A'I}$ at T. $\overleftrightarrow{DI}$ meets the circle with diameter $\overline{AI}$ again at P. Show that the second intersection between $\overleftrightarrow{PT}$ and the circle with diameter $\overline{AI}$ lies on $\overline{AY}$.



Replace the given variable names with our config ones, and the following collinearities all result from Configs. Let $E = AT_A \cap (AI)$, and we have $\angle SP_1E = \angle SAT_A = \angle SLT_A$ so $P_1E \parallel T_AL$. We show $P_1P \parallel LT_A$. We have $\frac{SP_1}{SL} = \frac{SD}{SM} = \frac{SP}{SI}$ as desired, so $\overline{P_1PE}$ as desired.

3.46 Iran TST 2012 Day 1 P2

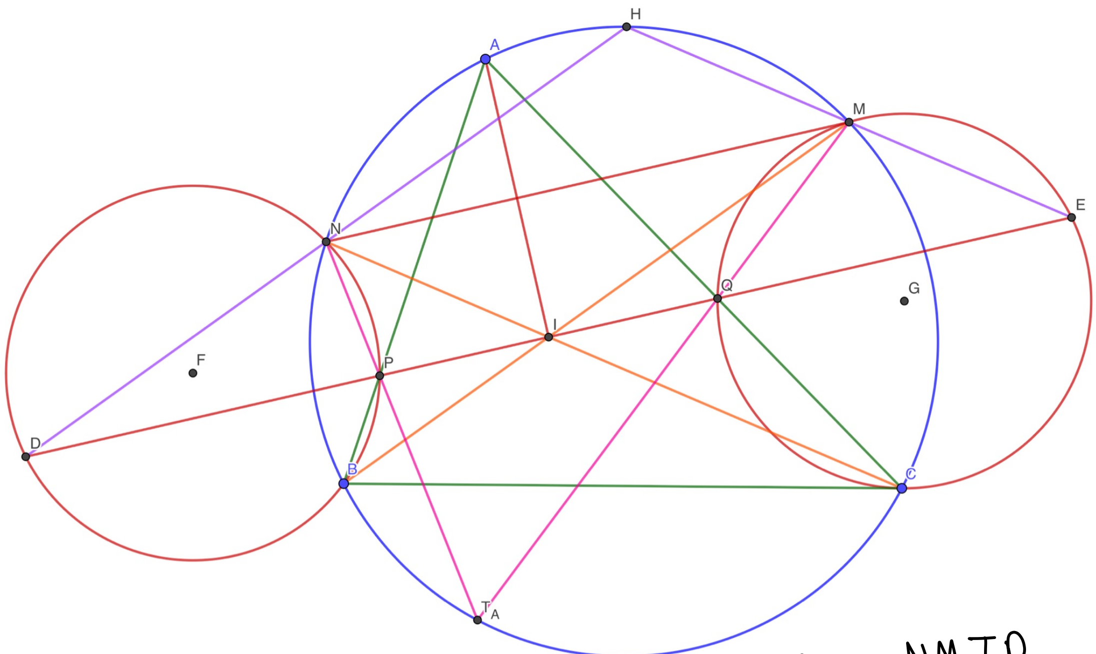
Problem 3.46 (Iran TST 2012 Day 1 P2)**.** Consider ω, the circumcircle of a triangle ABC. D is the midpoint of arc BAC and I is the incenter of $\triangle ABC$. Let DI intersect BC at E and ω for the second time at F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE is the angle bisector of angle BPC.



Let $G = MF \cap BC$. We have $\angle PEG + \angle EFG = 180^\circ$ so $(PEFG)$.
Thus, $EP \perp TP$ and since $TI \perp AM$, $\overline{TPI}$. Thus, $(GE;BC) \overset{F}{=} (MD;BC)$
$= -1$ so as $\angle GPE = 90^\circ$, $\angle BPE = \angle EPC$ as desired.

3.47 Centroamerican Olympiad 2016/6

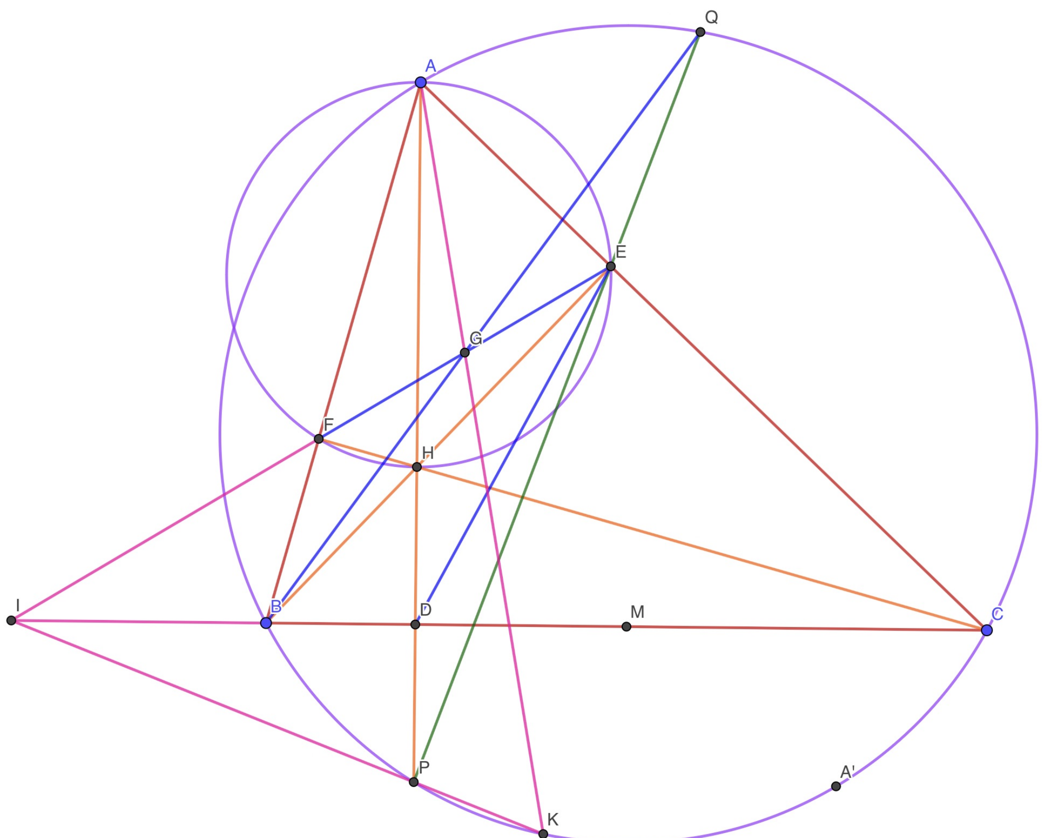
Problem 3.47 (Centroamerican Olympiad 2016/6). Let $\triangle ABC$ be triangle with incenter I and circumcircle Γ. Let $M = BI \cap \Gamma$ and $N = CI \cap \Gamma$, the line parallel to MN through I cuts AB, AC in P and Q. Prove that the circumradius of $\odot(BNP)$ and $\odot(CMQ)$ are equal.



We have $\angle NDP = \angle NBP = \angle NMB$ and $NM \parallel PQ$ so $NMID$ and $NMIE$ are parallelograms. Thus, $ID = NM = IE$ and since $IP = IQ$, $DP = QE$. Since AI is the perp bisector of AI, ND and ME intersect on (ABC). Thus, $\angle PND = \angle HNT_A = \angle HMT_A = \angle QME$, so (NPB) and (MQE) have the same radius

3.48 2019 ELMO SL G1

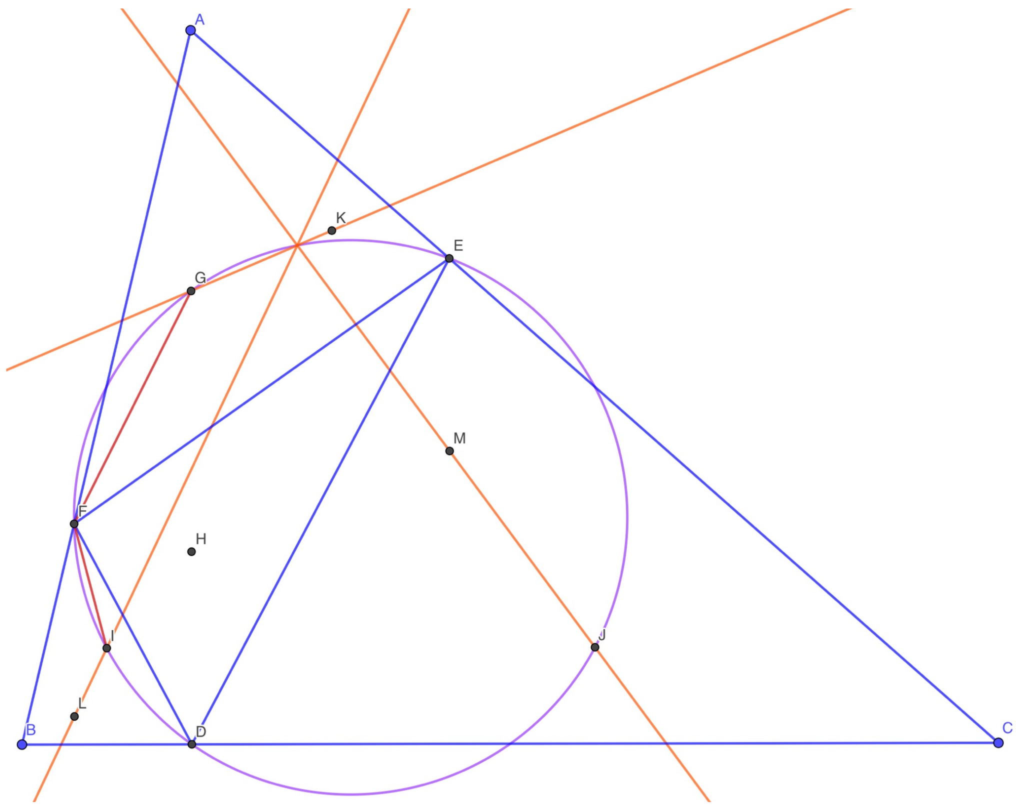
Problem 3.48 (2019 ELMO SL G1). Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.



Let K be the intersection of the A symmedian with (ABC). Thus, $PK \cap BC \cap EF = I$ by configs. Pascals on $CAKPQB$ gives that $BQ \cap AK$ is on $\overline{EI}$, thus, since AK bisects EF, so does BQ.

3.49 Wolfram Alpha Euler lines of AEF, BDF, CDE are concurrent

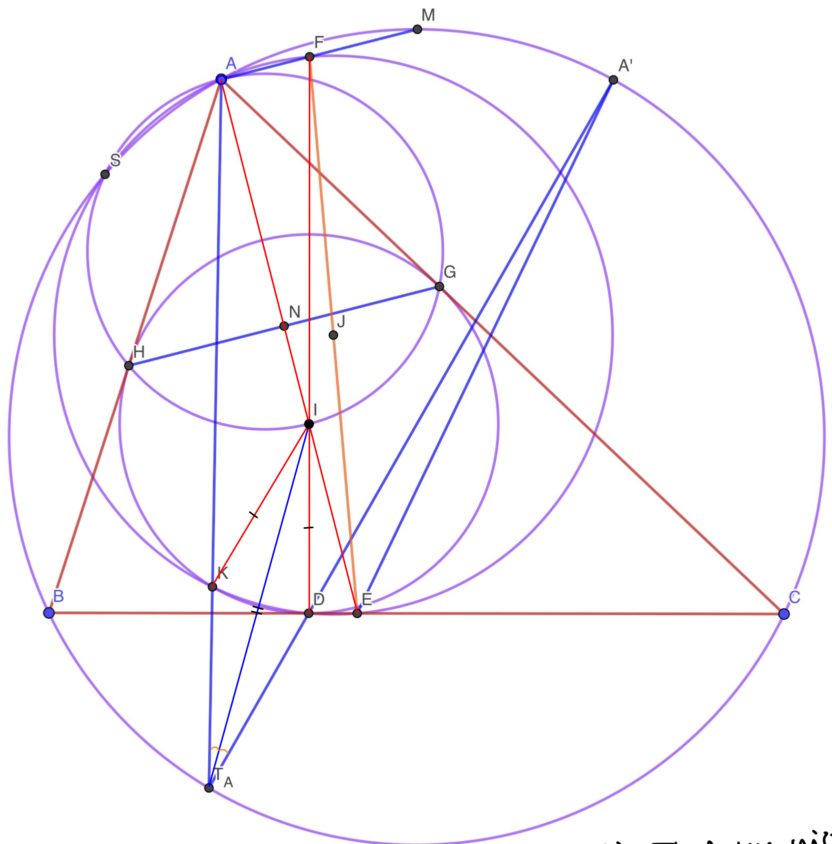
Problem 3.49 (Our one true god Wolfram Alpha). Let $\triangle ABC$ have orthic triangle DEF. Prove that the Euler lines of $\triangle AEF, \triangle BDF, \triangle CDE$ are all concurrent.



Note that the centes of AEF, BDF, CDE are on the 9 point circle as they are the midpoints of AH, BH, CH. Since $\triangle AEF$ and $\triangle BDF$ are similar, $\angle(GF, KG) = \angle(IF, LI)$ so $\angle(FI, GF) = \angle(LI, KG)$, so LI and KG intersect on the 9 point circle. Thus, LI, KG, JM all intersect on the 9 point circle as desired.

3.50 Romania JBMO TST 2019/1.3

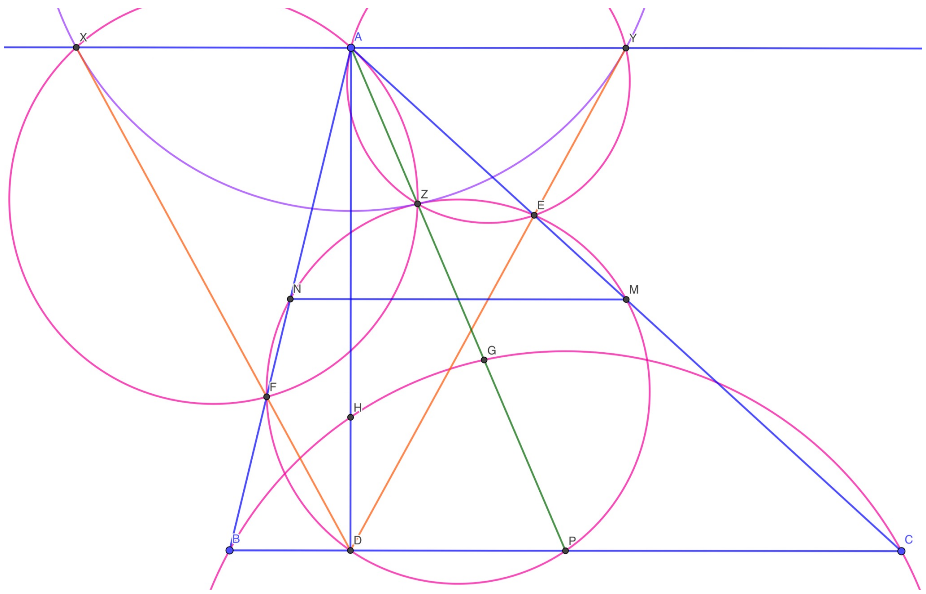
Problem 3.50 (Romania JBMO TST 2019/1.3). Let ABC a triangle, I the incenter, D the contact point of the incircle with the side BC and E the foot of the bisector of the angle A. If M is the midpoint of the arc BC which contains the point A of the circumcircle of the triangle ABC and $\{F\} = DI \cap AM$, prove that MI passes through the midpoint of $[EF]$.



Since $\angle MAI = 90^\circ$ and $FDE = 90^\circ$, we have $(DEFA)$. Thus, we wish to show that MI passes through the center of $(ADEFK)$. (The point K lies on our circle due to configs). However, note that $T_A I$ bisects $\angle AT_A I$. Since $IK = ID$ and $\angle IDT_A, \angle IKT_A > 90$, we have (by LoS), $\angle IDT_A = \angle IKT_A$ and $T_A K = T_A D$, $\overline{T_A I M}$ is the perpendicular bisector of KD, so MI passes through the center of $(AEFD)$ and thus EF.

3. 51 MMOSL G3

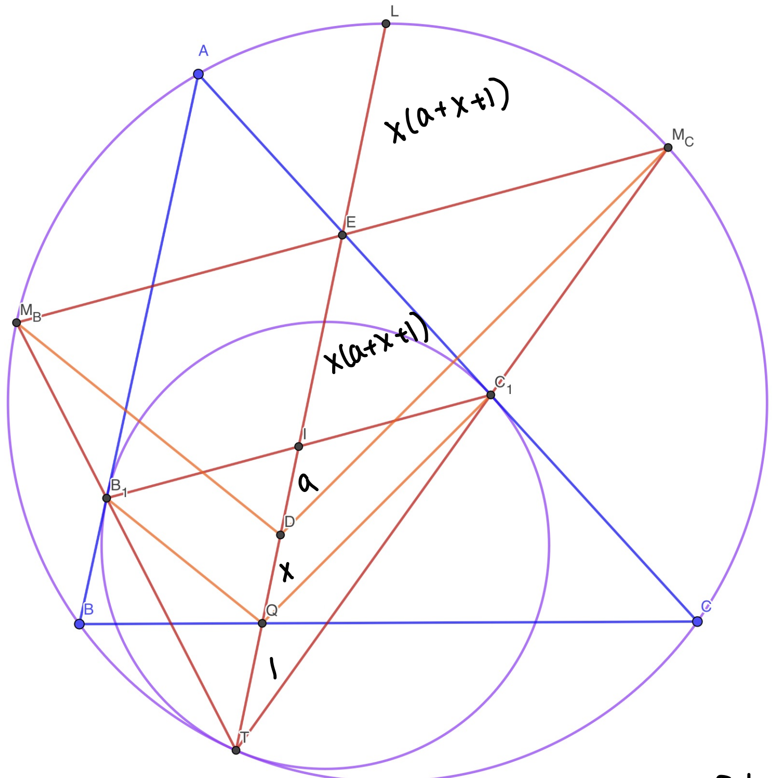
Problem 3.51 (MMOSL G3). Let D, E, and F be the respective feet of the A, B, and C altitudes in $\triangle ABC$, and let M and N be the respecive midpoints of $\overline{AC}$ and $\overline{AB}$. Lines DF and DE intersect the line through A parallel to BC at X and Y, respectively. Lines MX and YN intersect at Z. Prove that the circumcircles of $\triangle EFZ$ and $\triangle XYZ$ are tangent.



Let Z' be a $\frac{1}{2}$ homothety at A of G (the Humpty point). We have $\angle AZ'F = \angle FZ'P = \angle FDB = \angle AXF$ so $(AZ'FX)$ and similarly, $(AZ'EY)$. Thus, $\angle AZ'X + \angle AZ'M = \angle AFX + 180 - \angle MZ'P = 180$ so $\overline{XZ'M}$ and similarly $\overline{YZ'N}$ so $Z' = Z$ as desired. Thus, $(EFZ) = (MNZ)$, and since $MN \parallel XY$, a homothety at Z means (MNZ) and (XYZ) are tangent.

3.52 AoPS MP8148 mixtilinear homothety sends Q to the midpoint of IQ

Problem 3.52 (AoPS user MP8148). In triangle ABC let the incenter be I. Suppose that the A-mixtilinear incircle ω touches (ABC) at T, and $Q = \overline{IT} \cap \overline{BC}$. Show that the homothety sending ω to (ABC) sends Q to the midpoint of $\overline{IQ}$.



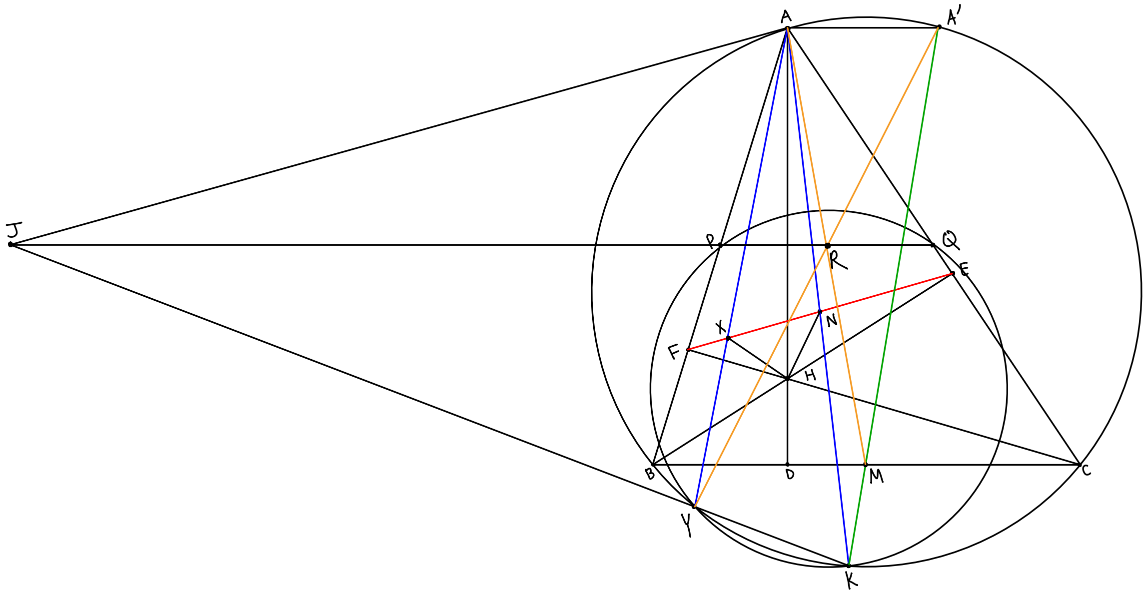
We let the factor of homothety be x. Since $EI = EL$, and $M_BM_C = LB$, we have $LE \cdot ET = \frac{1}{4} LQ \cdot LT = \frac{LB^2}{4}$

$4x(a+x+1)(x+1)(a+x+1) = (2ax+2x^2+3x+a)(2ax+2x^2+3x+a+1)$

$\hookrightarrow a^2+a-x^2-x=0 \rightarrow (a-x)(a+x+1)=0 \rightarrow a=x$ as desired.

3.53 AoPS i3435 A'Y bisects AM

Problem 3.53 (Myself (i3435)). Let ABC be a triangle with orthocenter H. Let $\overline{BH} \cap \overline{AC} = E$, and $\overline{CH} \cap \overline{AB} = F$. Let N be the midpoint of EF and let X be the point besides N on $\overline{EF}$ such that $HN = HX$. Let $\overline{AX}$ intersect (ABC) at Y. Let A' be the point on (ABC) such that $\overline{AA'} || \overline{BC}$ and let M be the midpoint of BC. Prove that $\overline{A'Y}$ bisects AM.



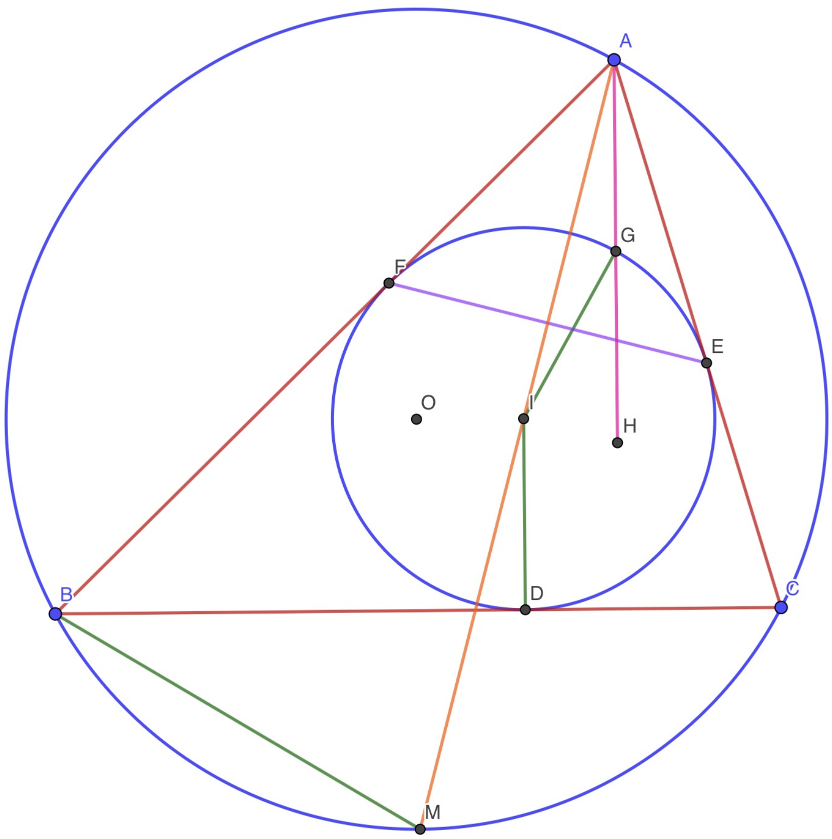
We see that AN intersects (ABC) at the point K such that $(AK;BC) = -1$. Consider a $\sqrt{AH \cdot AD}$ inversion. We see that $X \leftrightarrow Y$, $N \leftrightarrow K$, since $EF \leftrightarrow (ABC)$. We see that P, Q go to the $2\times$ homotheties of F, E about A. Let these points be F', E'. Note that H is on the perp bisectors of AF', AE' so H is the circumcenter of $AF'E'$, so H is on the perp bisector of F', E'. Thus, $(XYF'E')$ or $(PQKY)$.

Radical Axis on (APQ), $(PQKY)$, (ABC) gives $AA \cap PQ \cap KY = J$. Additionally, since $\overline{KMA'}$, we have $\measuredangle JYA = \measuredangle KYA = \measuredangle KA'A = \measuredangle AA'R = \measuredangle JRA$ so $(JYAR)$. We then have $\angle AYR = \angle AJR = \angle B - \angle C$ and $\angle AYA' = \angle ABA' = \angle B - \angle C$

so $\overline{ARY}$ as desired.

3.54 I on (AH) iff midpoint of H on incircle

Problem 3.54 (Error: Not Found). Let ABC be an acute scalene triangle with orthocenter H. Prove that the midpoint of AH lies on the incircle of ABC if and only if the incenter of ABC lies on the circle with diameter AH.



We show both conditions imply $AH = 2r$. Let $R = \frac{1}{2}$. If I on (AH), then $AM = \frac{AI}{AH}$, $AI = \frac{r}{\sin\left(\frac{A}{2}\right)}$, $IM = \sin\left(\frac{A}{2}\right) \to AH = \frac{AI}{AM} = \frac{\frac{r}{\sin\left(\frac{A}{2}\right)}}{\frac{r}{\sin\frac{A}{2}} + \sin\frac{A}{2}} \cdot \cos A \to \frac{r}{r + \sin^2\frac{A}{2}} = \frac{2r}{2r + 1 - \cos A} = \cos A \to \cos A = 2r$ $(AH = 2r)$.

If the midpoint of AH is on the incircle, then $\cos\left(\frac{A}{2} + C - 90\right) = AM = \frac{AI^2 + AG^2 - IG^2}{2AI \cdot IG}$. We have $AI = \frac{r}{\sin\frac{A}{2}}$, $AG = \frac{\cos A}{2}$, $IG = r$.

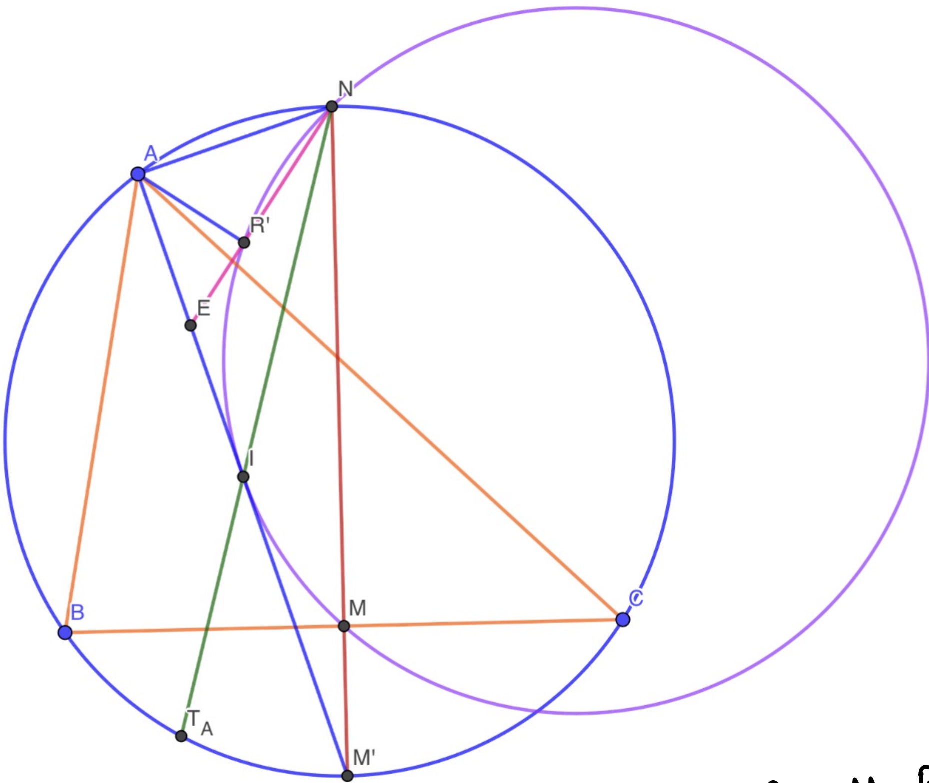
Thus, $\left(\frac{r}{\sin\frac{A}{2}} + \sin\frac{A}{2}\right) \cdot \frac{r}{\sin\frac{A}{2}} \cos A = \frac{r^2}{\sin^2\frac{A}{2}} + \frac{\cos^2 A}{4} - r^2$, or $\left(\frac{2r + 1 - \cos A}{1 - \cos A}\right) r \cos A = \frac{2r^2}{1 - \cos A} + \frac{\cos^2 A}{4} - r^2$

Simplifying, $r^2(\cos A - 1) + r\cos A(1 - \cos A) = \frac{\cos^2 A(1 - \cos A)}{4} \to r^2 - r\cos A + \frac{\cos^2 A}{4} = 0$ or $r = \frac{\cos A}{2} \to AH = 2r$.

Thus, the 2 conditions are equivalent as desired.

3.55 IGO Advanced 2020/2

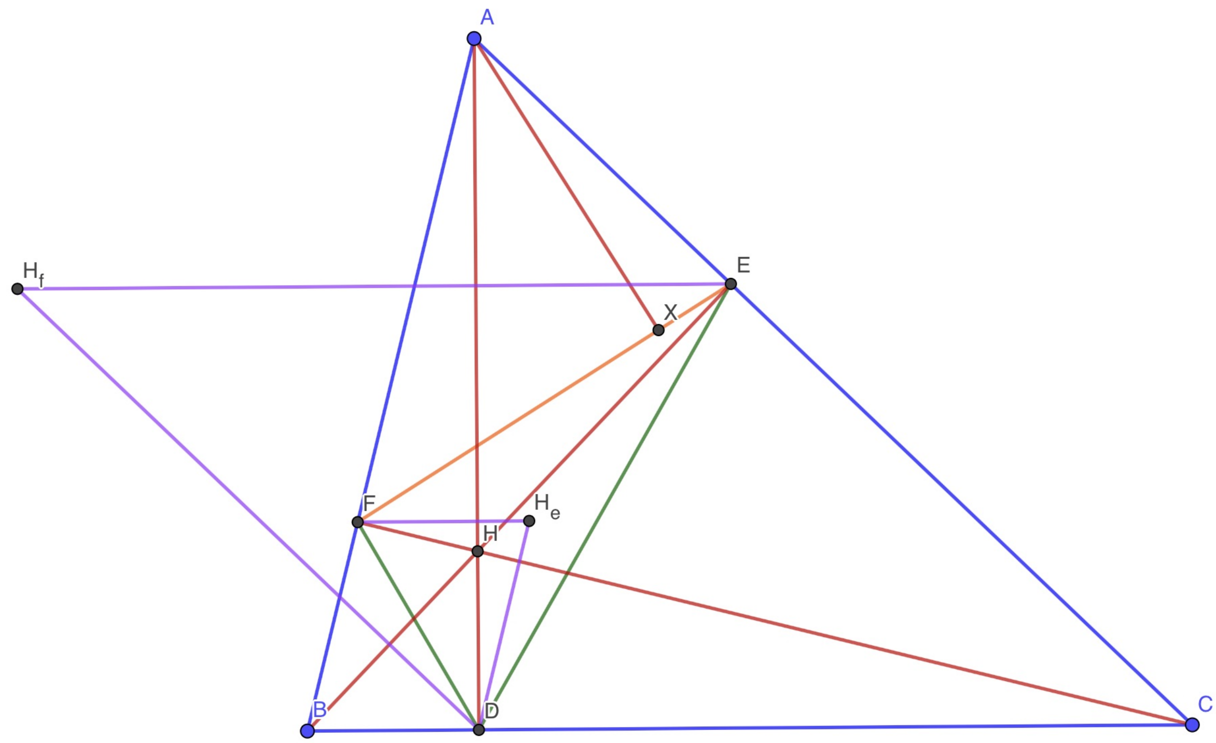
Problem 3.55 (IGO Advanced 2020/2)**.** Let $\triangle ABC$ be an acute-angled triangle with its incenter I. Suppose that N is the midpoint of the arc $\widehat{BAC}$ of the circumcircle of triangle $\triangle ABC$, and P is a point such that $ABPC$ is a parallelogram.Let Q be the reflection of A over N and R the projection of A on $\overline{QI}$. Show that the line $\overline{AI}$ is tangent to the circumcircle of triangle $\triangle PQR$.



Take a $\frac{1}{2}$ homothety at A. Then $P \to M$, $Q \to N$, $R \to$ proj of A onto EN, where E is the midpoint of AI. Thus, $ER' \cdot EN = EA^2 = EI^2$, so AM' is tangent to $(R'NI)$. Note that $IM'^2 = M'B^2 = M'M \cdot M'N$, so AM' is tangent to (INM). Thus, R', M are both on the unique circle through I, N tangent to AM', so we have $(MNR'I)$ tangent to $AI \to (PQR)$ tangent to AI by homothety.

3.56 AOPS user jayme HeHf perp DX

Problem 3.56 (AoPS user jayme). Let ABC be an acute triangle with orthocenter H, DEF as the orthic triangle, and X as the foot from A to $\overline{EF}$. He and Hf are the orthocenters of $\triangle HFD$, $\triangle HED$ respectively. Prove that $\overline{HeHf} \perp \overline{DX}$.



We have $D = \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right)$, $E = \frac{1}{2}\left(a+b+c-\frac{ac}{b}\right)$, $F = \frac{1}{2}\left(a+b+c-\frac{ab}{c}\right)$.

The altitude from A to EF is $X = \dfrac{(\bar{e}-\bar{f})(a) + \frac{e-f}{a} + \bar{e}f - e\bar{f}}{2(\bar{e}-\bar{f})}$.

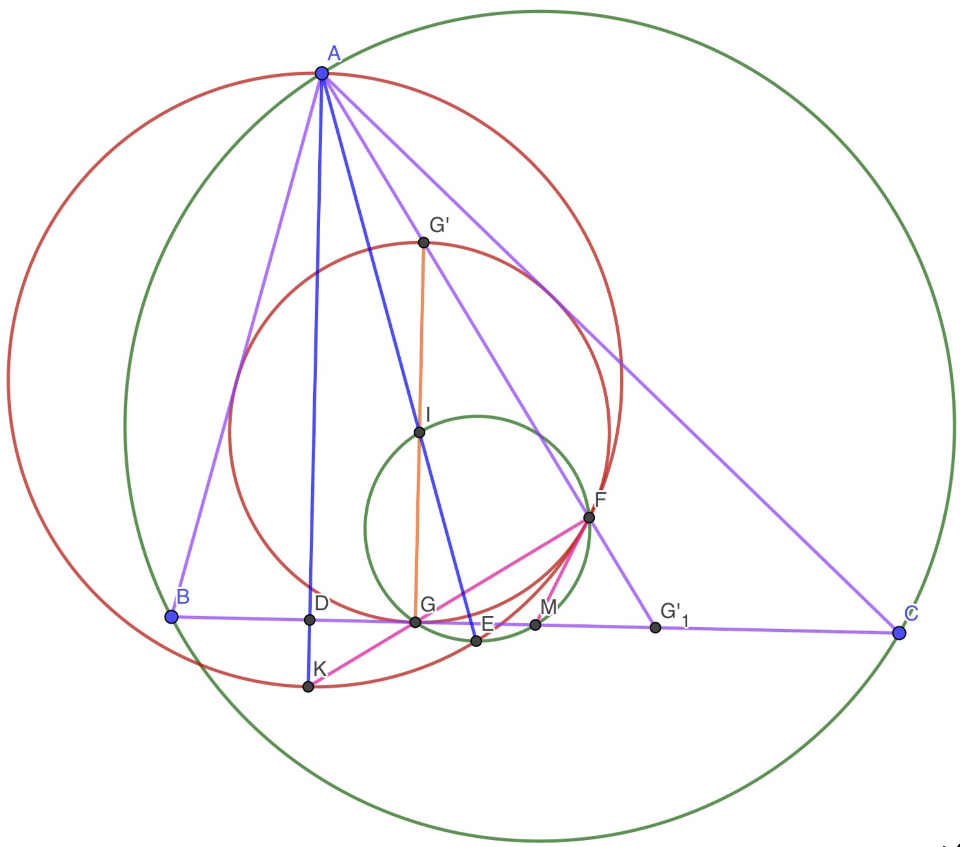
Simplifying, $X = \dfrac{a^2b + a^2c - ab^2 - ac^2 + b^2c + bc^2 + 2abc}{4bc}$, so $D - X = \dfrac{(a-b)(a-c)(ab+ac+2bc)}{4abc}$

We have $He - Hf = (f+b-d) - (d+e-c) = (c-b)\left(\dfrac{ab+ac+2bc}{2bc}\right)$.

Thus, $\dfrac{D-X}{He-Hf} = \dfrac{(a-b)(a-c)}{2a(c-b)}$, $\overline{\left(\dfrac{D-X}{He-Hf}\right)} = (c-b)\left(\dfrac{ab+ac+2bc}{2bc}\right)$ as desired.

3.57 Japan 2019/4

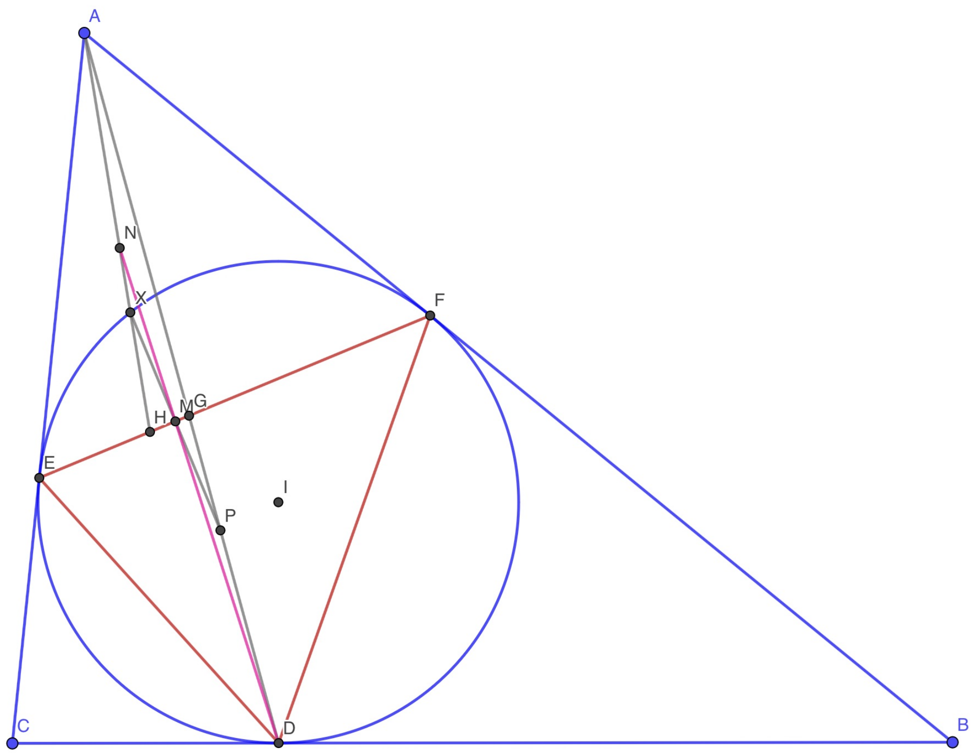
Problem 3.57 (Japan 2019/4). Let ABC be a triangle with its inceter I, incircle w, and let M be a midpoint of the side BC. A line through the point A perpendicular to the line BC and a line through the point M perpendicular to the line AI meet at K. Show that a circle with line segment AK as the diameter touches w.



Let G' be the antipode of G, and F be the intersection of AG' and the incircle.
Let K be $FG \cap AD$, so note $(FG'G)$ and (AFK) are tangent by homothety, and that $\angle AFK = 90^\circ$.
Now we just need to prove $MK \perp AI$. Let $E = (AK) \cap (IM) = (MGIF)$ (MG, MF are tangents to the incircle). Note
$\angle EIG = \angle KFE = \angle KAE \Rightarrow \overline{AIE}$, and since $\angle AEK = \angle IEM = \angle AEM = 90^\circ$, $\overline{MEK}$, so $MK \perp AI$ as desired.

3.58 Brazil 2013/6

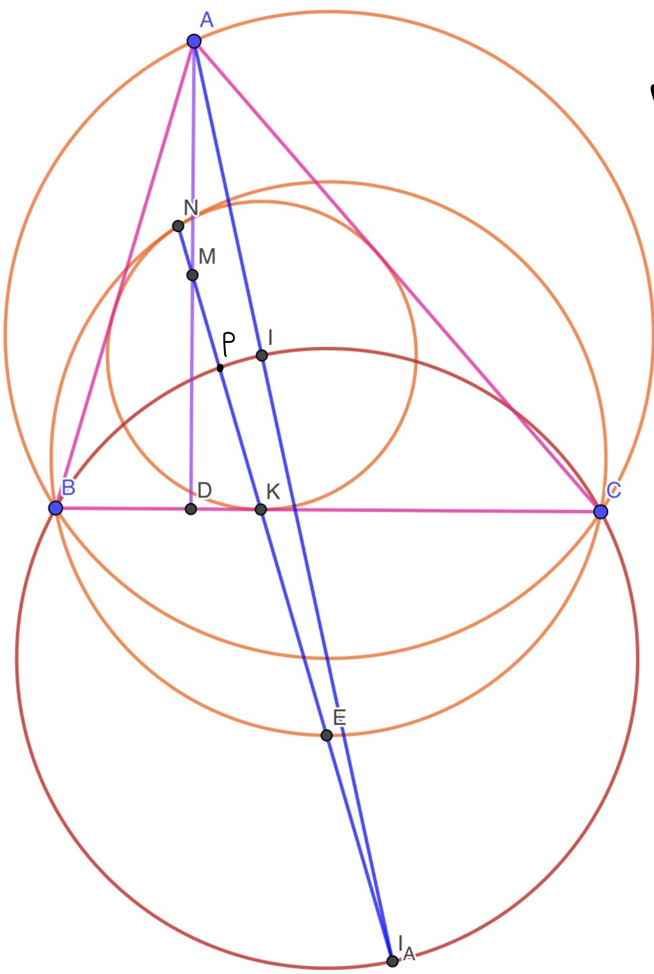
Problem 3.58 (Brazil 2013/6)**.** The incircle of triangle ABC touches sides BC, CA and AB at points D, E and F, respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of triangle ABC.



Note P is the Gergonne point, so $(AP;GD) \overset{M}{=} (AX;HN) = -1$, where $N = DM \cap AX$. Note $(XP, M\infty) \overset{D}{=} (XA;ND) = -1$, so $DH \parallel XP$, so H is the foot of the altitude from D to EF. The orthic triangle of DEF is homothetic to ABC which gives the desired.

3.59 ISL 2002 G7

Problem 3.59 (ISL 2002 G7). The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.



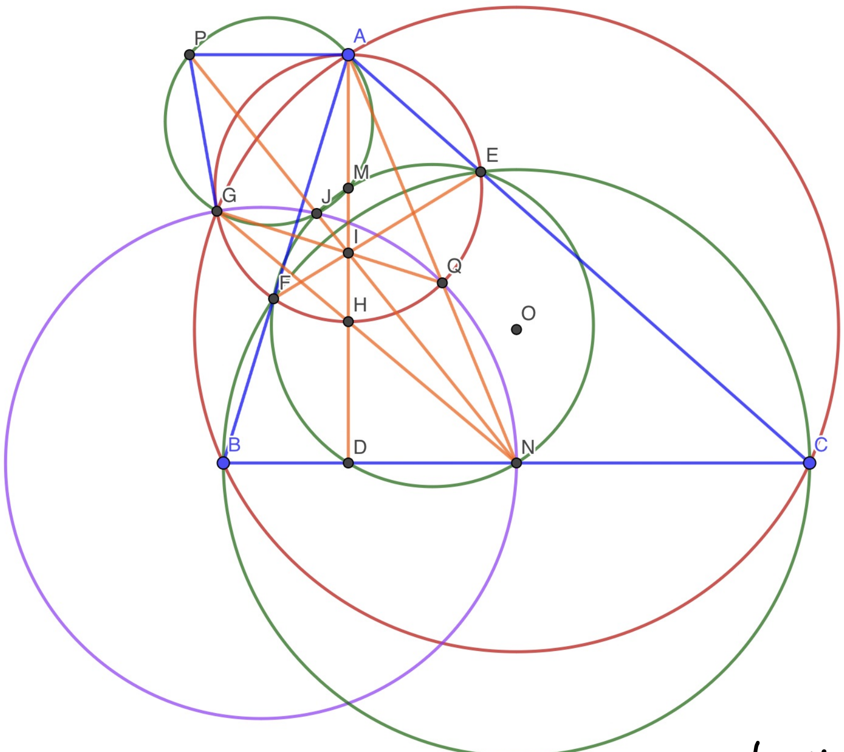
Let the midpoint of KI_A be E.

We have $(BNCE)$ since $NK \cdot KE = PK \cdot KI_A = BK \cdot KC$.

Note that $BE = EC$, since K, I_A are symmetric about the midpoint of BC. By shooting lemma, since our circle goes through N and is tangent to BC at K, we see it is also tangent to (NBC) as $\overline{NKE}$.

3.60 USA TSTST 2016/2

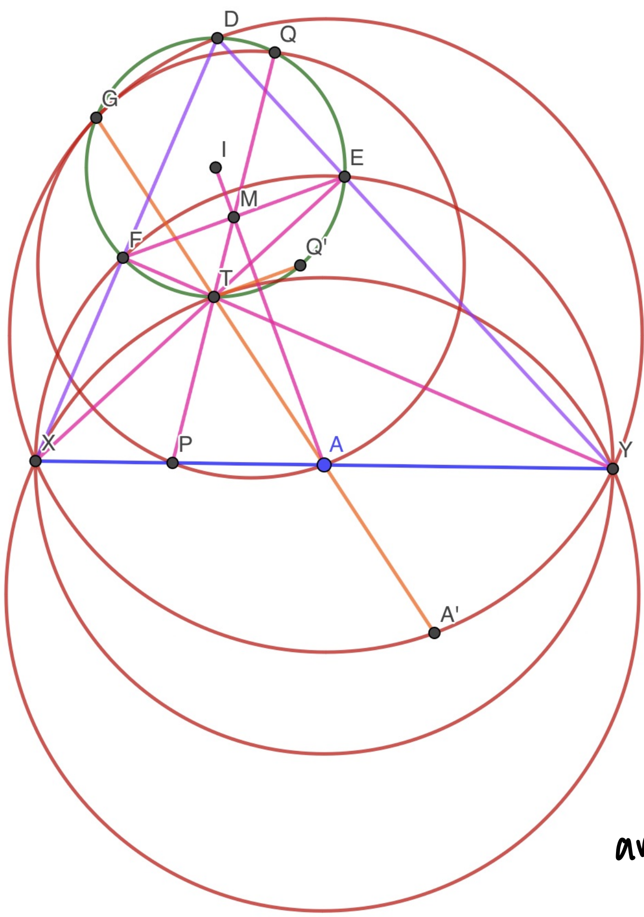
Problem 3.60 (USA TSTST 2016/2). Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.



We see that $P = AA \cap GG$. Let $I = EF \cap MD$, so $(AH;ID) = -1 \rightarrow$ since $\angle AGH = 90^\circ$,
$\angle IGH = \angle NGD = \angle DAN = \angle HAQ \rightarrow \overline{GIQ}$. Pascals on $AAQGGH$ gives $\overline{PIN}$.
Let $J = PN \cap (MEFN)$, so $FI \cdot IE = JI \cdot IN = GI \cdot IQ \rightarrow (GJQN)$.
We also have $(PAMJG)$ since $\angle MJN = \angle PJM = 90^\circ$. (since G is queue point)
Thus, radical axis on (PAG), (AEF), (MEF) gives $AG \cap MJ \cap EF \cap BC$, and by POAP, we see that $(MJBC)$, as desired.

3.61 CAMO 2020/3

Problem 3.61 (CAMO 2020/3). Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Point M is the midpoint of EF, and T is the point on ω such that $\overline{DT}$ is a diameter of ω. Line $\overline{MT}$ meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines $\overline{DF}$ and $\overline{DE}$ intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.



Let DXY be the reference triangle. Note that we have $AF = AE = AX = AY$, so $\angle XEY = \angle XFY = 90°$, so we have $T = YF \cap XE$.

G is the Queue point, so $\angle FQT = \angle FET = \angle FYP$ $\to (PYQF), (XPEQ)$. Thus,

$PT \cdot TQ = YT \cdot TF = AT \cdot TG \to (APGQ)$.

$\angle GTF = \angle ATY = \angle XTP = \angle QTE \to GQ \parallel FE$.

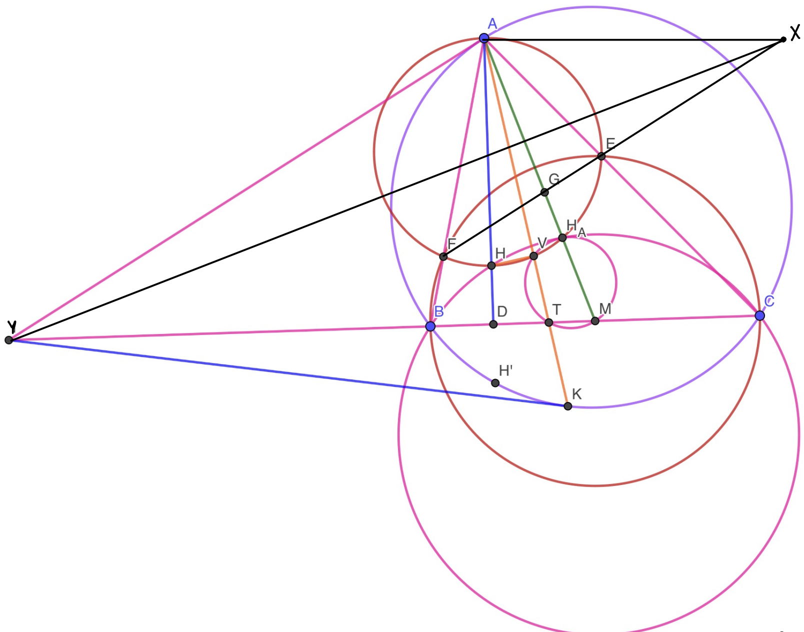
We have $AT = AQ'$, where $Q' = AQ \cap (DEF)$.

So, $TQ' \parallel EF$, so TQ' is tangent to (TXY), as $\angle Q'TY = \angle EFY = \angle TXY$.

Inverting about $(EFXY)$, $(DGXY) \to (TXY)$ and $(APQG) \to TQ'$, as desired.

3.62 AQGO 2020/6

Problem 3.62 (AQGO 2020/6).]Let ΔABC be a triangle with orthocenter H and $\overline{BH}$ meet $\overline{AC}$ at E and $\overline{CH}$ meet $\overline{AB}$ at F. Let $\overline{EF}$ intersect the line through A parallel to $\overline{BC}$ at X and the tangent to (ABC) at A intersect $\overline{BC}$ at Y. Let $\overline{XY}$ intersect AB at P and let XY meet AC at Q. Let O be the circumcenter of ΔAPQ and $\overline{AO}$ meet $\overline{BC}$ at T. Let V be the projection of H on $\overline{AT}$ and M be the midpoint of BC. Then prove that (BHC) and (TVM) are tangent to each other.

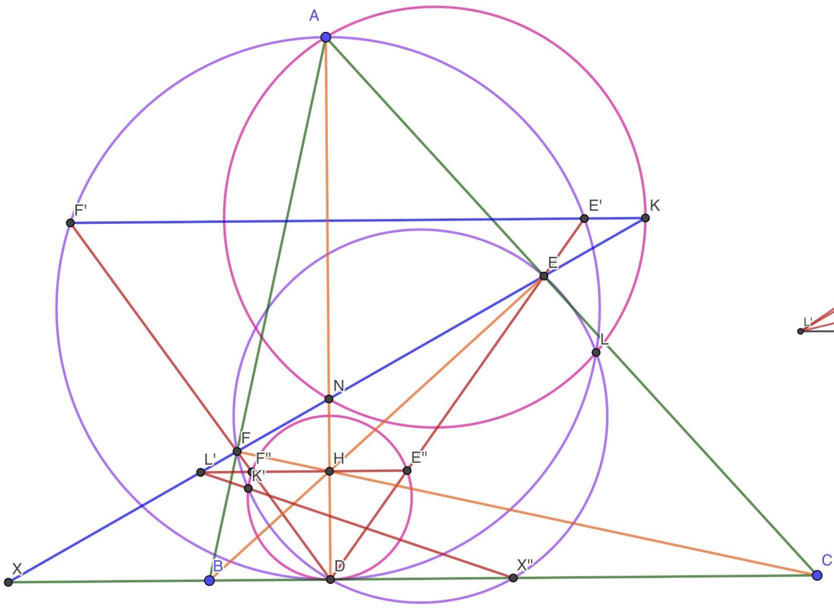


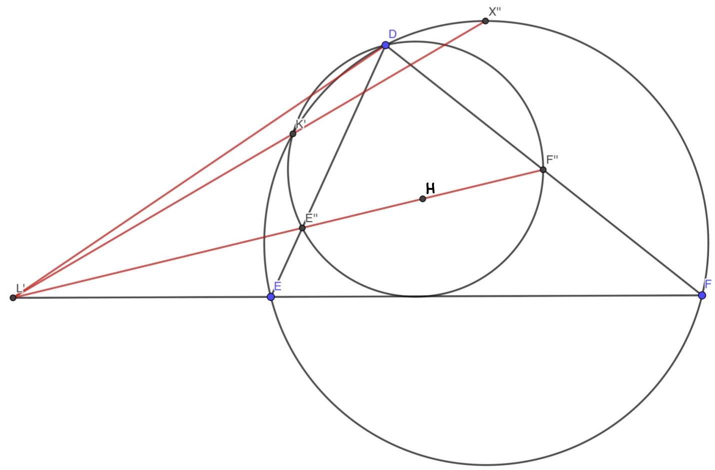
Theorem 1.4.13 gives AH_A has perp bisector XY, thus, $PQ \perp AM$, so AO is the symmedian of ABC. Note that $HDVT$, $HDMH_A$ are cyclic so by POAP, (VH_AMT). So we wish to show (H_AVTM) is tangent to (BH_AC) at H_A. Invert about $(BCEF)$. Note that $(YT;BC) = -1$, so Y and T swap ($Y = AA \cap KK$). So, $(H_AMT) \longleftrightarrow YA$, which is tangent to (BH_AC) as desired.

3.63 i3435 X, D, L, K cyclic

Problem 3.63 (Myself (i3435)). Let ABC be a triangle such that the foot of the altitude from A to $\overline{BC}$ is D, the foot of the altitude form B to $\overline{AC}$ is E, and the foot of the altitude from C to $\overline{AB}$ is F. Let $X = \overline{EF} \cap \overline{BC}$. Let the circle with diameter AD hit $\overline{DE}$ at E', $\overline{DF}$ at F', and (DEF) at the non-D point L. Let $\overline{GI} \cap \overline{EF} = K$. Then prove that X, D, L, K are concyclic.

$(E'F')$





Consider $\triangle DEF$ as our reference triangle and perform a $\sqrt{bc}$ inversion. Note that $DE' = DF'$. We have

$EF \leftrightarrow (DEF)$ so $X \leftrightarrow X''$ (midpoint of arc EF). We have $(AD) \leftrightarrow$ the line through H (let this line be ℓ)

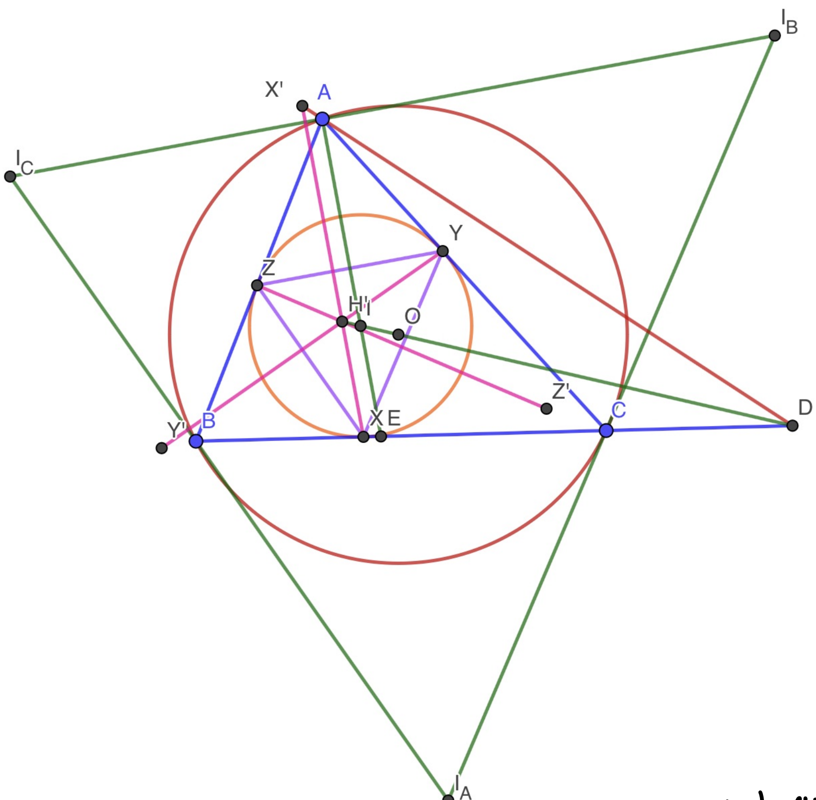
parallel to BC [since $\overleftrightarrow{BC} \leftrightarrow \overleftrightarrow{BC}$ and $DH \cdot DA = DE \cdot DF$]. Then, $E'' \leftrightarrow \ell \cap DE$ and $F'' \leftrightarrow \ell \cap DF$.

We have $K' = (DE''F'') \cap (DEF)$ and $L' = E''F'' \cap EF$ (K' and L' are the inverted K, L)

By configs, we have, $L'K'X''$ as desired

3.64 Romania TST 2009 Day 3 P3

Problem 3.64 (Romania TST 2009 Day 3 P3)**.** Let ABC be a non-isosceles triangle, in which X, Y, and Z are the tangency points of the incircle of center I with sides BC, CA and AB respectively. Denoting by O the circumcircle of $\triangle ABC$, line OI meets BC at a point D. The perpendicular dropped from X to YZ intersects AD at E. Prove that YZ is the perpendicular bisector of $[EX]$.



Let X' be the reflection of X over YZ. We show that $\frac{X'H'}{XH'} = \frac{AI}{IE}$ in order to show that $AX \cap H'I \cap XE = D$. We have $\frac{X'H'}{XH'} = \frac{(2\times \text{height of } XYZ) - XH'}{XH'} \cdot \frac{YZ}{YZ}$

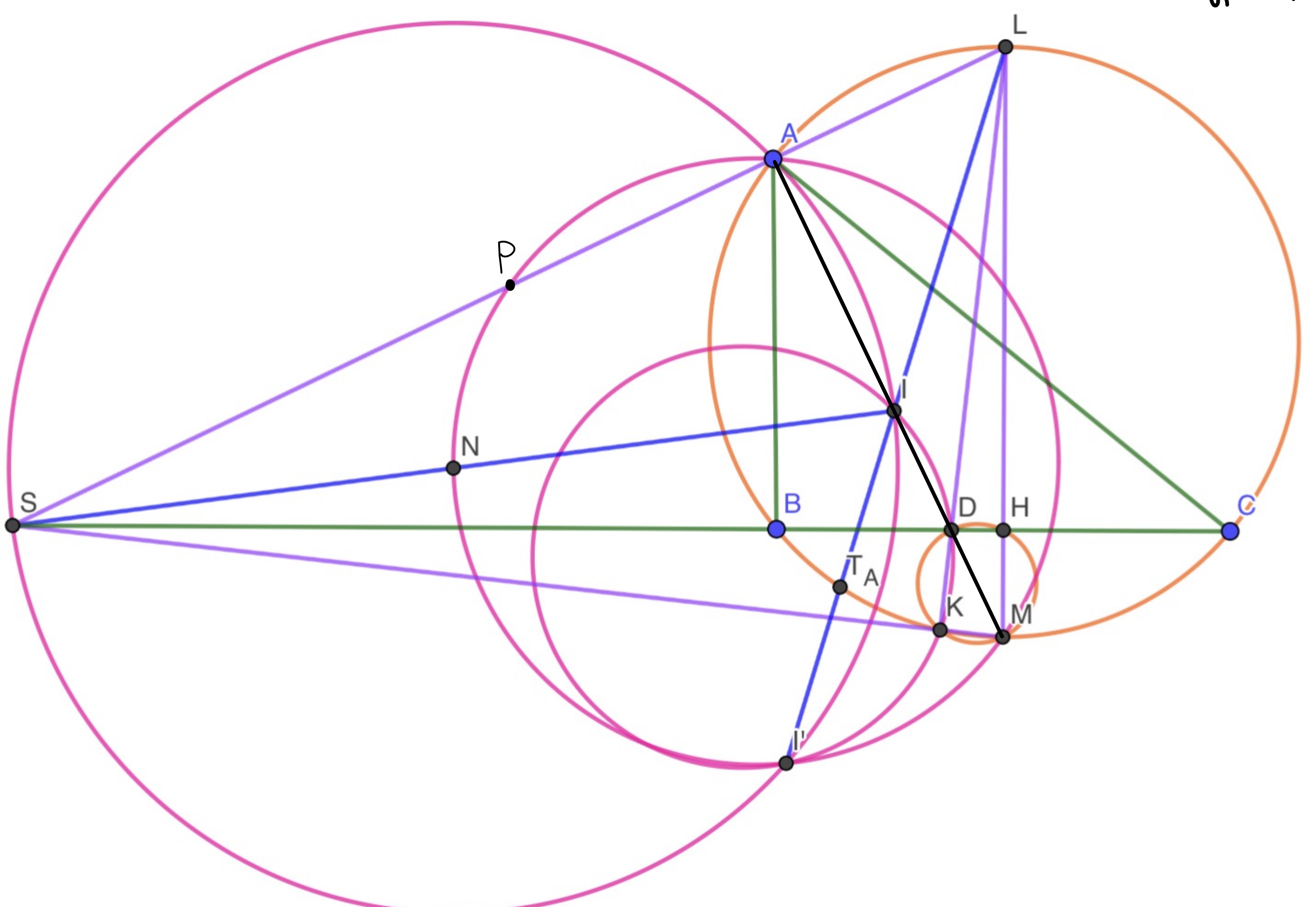
$$= \frac{(2\times\text{area}) - XH' \cdot YZ}{XH' \cdot YZ} = \frac{2r^2(\sin(2X)+\sin(2Y)+\sin(2Z))}{2r\sin X \, 2r\cos X} - 1 = \frac{\sin(2Y)+\sin(2Z)}{\sin(2X)}$$

And $\frac{AI}{IE} = \frac{\sin B + \sin C}{\sin A}$ (mass points) $= \frac{\sin(2Y)+\sin(2Z)}{\sin(2X)}$ as desired.

3.65 2017 USAMO P3 MAN and KID

Problem 3.65 (2017 USAMO P3)**.** Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter DM cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through L_1 or L_2.

the midpoint of IL_1 or I



we have LA ∩ DH ∩ KM = S, so

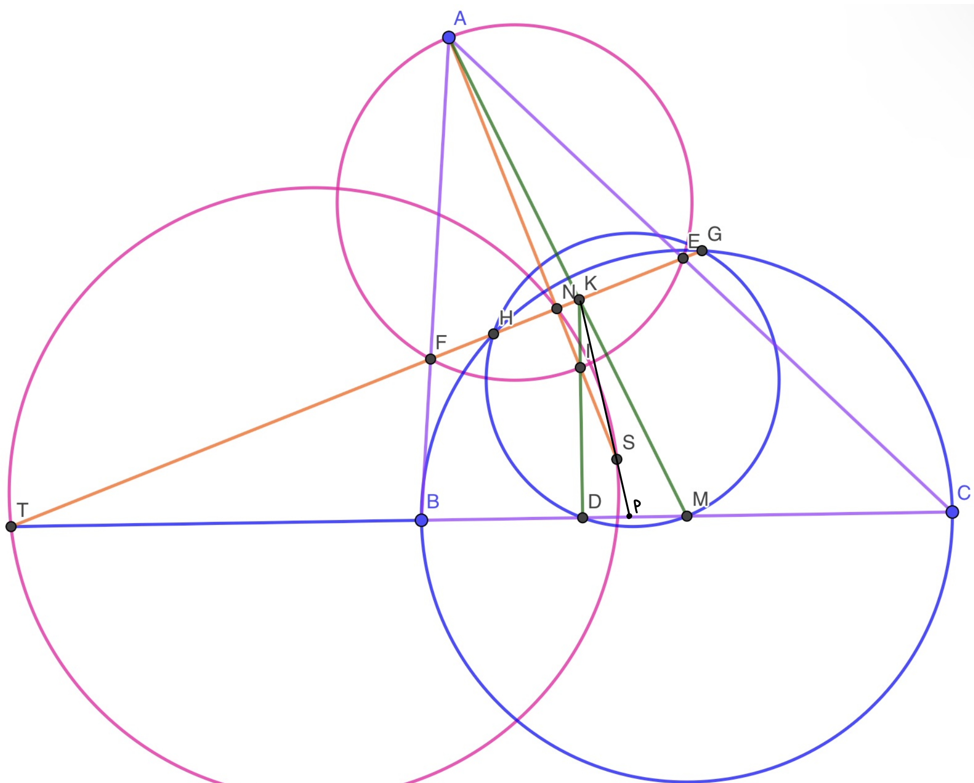
we have $\overline{LAS}$, since $K = LD \cap (ABC)$, since $MA \perp LA$, we have $LD \cdot LK = LI \cdot LI' = LB^2 \to (IDKI')$, by configs. $(H = LM \cap BC)$. Let $I' = (IBC) \cap AI$. We also have

we also let $\angle AMN = x$ and $\angle ALT_A = x \to \angle AIT_A = 90+x$ so $\angle ANI' = 180-2x \to \angle AI'N = x$.

Thus, $(MANI')$. Since T_A is the midpoint of II', we are done.

3.66 Taiwan TST 2015 Round 3 Quiz 3 P2

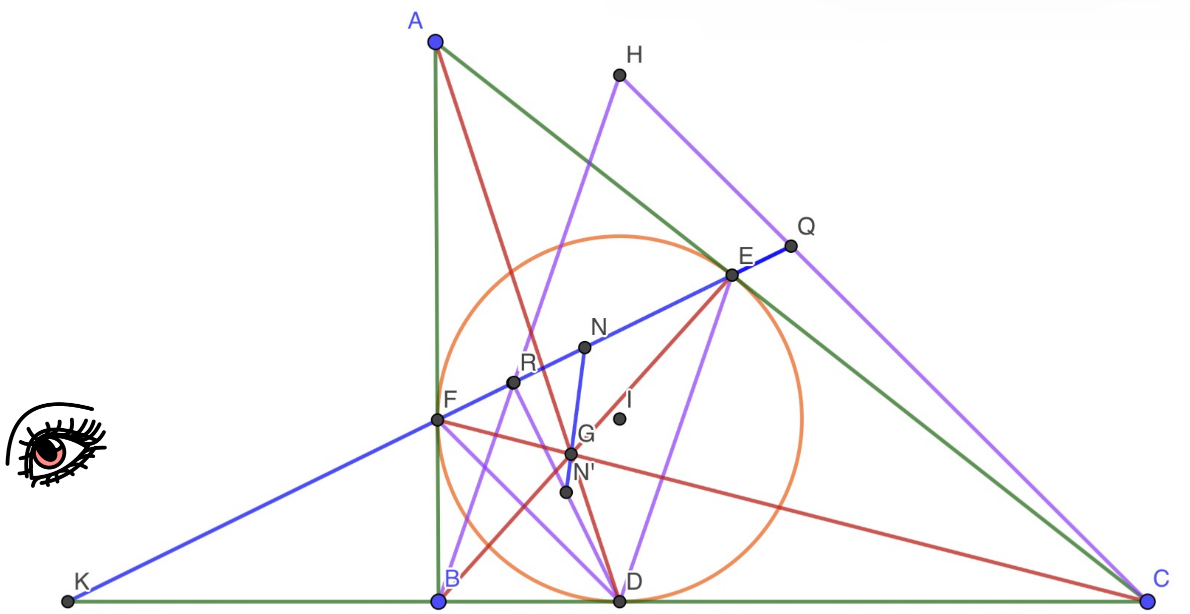
Problem 3.66 (Taiwan TST 2015 Round 3 Quiz 3 P2)**.** In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.



We have $(AI;NS) \overset{K}{=} (MD;T,P) = -1$. and $(TK;HG)$ (G, H are the feet from B, C to IC, BI) $\overset{I}{=} (TD;BC) = -1$. Thus, T is on the polar of S wrt $(DMGH)$, (since S is on KP). Thus, we know that the circle with diameter ST is orthogonal to the 9 point circle

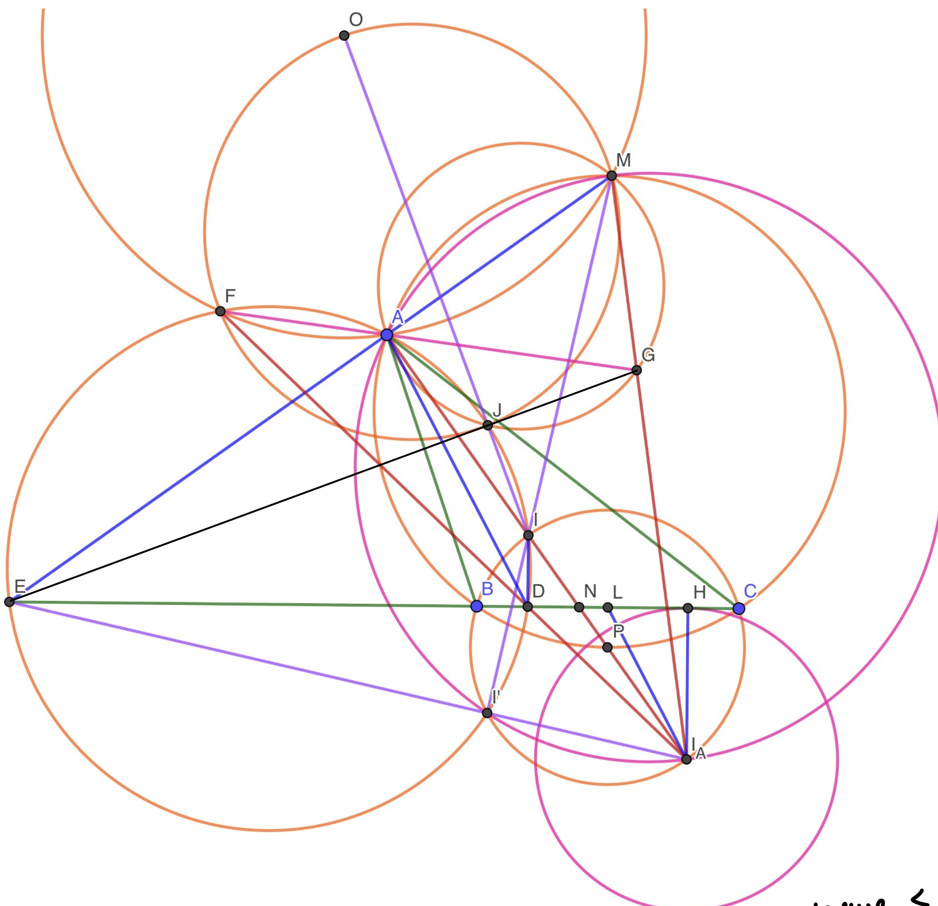
3.67 Modified Iran TST 2009/9

Problem 3.67 (Modified Iran TST 2009/9). Let $\triangle ABC$ have incenter I and contact triangle DEF. Let H be the orthocenter of $\triangle BIC$, and let P be the foot from D to $\overline{EF}$. If the midpoint of EF is N, the midpoint of DP is N^*, and Ge is the gergonne point of $\triangle ABC$, then prove that $N^* - Ge - N - H$.



Note by Iran lemma, $R = CI \cap EF$, $Q = BI \cap EF$ and are the feet of the altitudes. Thus, H is $BP \cap CQ$. Also, since I is the orthocenter of $\triangle HBC$, it is the incenter of $\triangle DPQ$. By, midpoints of altitudes, since $IN \perp PQ$, H lies on $N'N$. Now, since N, G, N' is the schwatt line of $\triangle DEF$, we know that $\overline{HNGN'}$ as desired.

Problem 3.68 (POGCHAMP 6). Triangle ABC has incenter I, A-excenter I_A, and let D be the foot of I onto BC. M is the midpoint of arc $\widehat{BAC}$ in $\odot(ABC)$, MA intersects BC at E, and line I_AD intersects $\odot(AID)$ again at $F \neq D$. Let O be the circumcenter of $\triangle AMF$ and suppose the line through E perpendicular to OI meets MI_A at G. Find, with proof, the value of $\frac{I_AG}{MG}$.



We have $J = OI \cap EG$, and $\angle IJE = 90^\circ$ so $J \in (AID)$. We have $\angle AJG = \angle AJE = \angle AI'E$ $= \angle AMI_A$, since $(MAI'I_A)$ (by radical axis). We let $\angle FAE = x$, so $\angle OJF = 90 - \angle FJE = 90 - x$, and $\angle FAM = 180 - x$, so $\angle FOM = 2x \to \angle OMF = \angle OFM = 90 - x$, so $(OMFJ)$. Thus, $\angle OJF = \angle OJM$ so $\angle FJE = \angle MJG = \angle FAE = \angle MAG \to \overline{FAG}$. This problem can now be reduced to a trig bash.

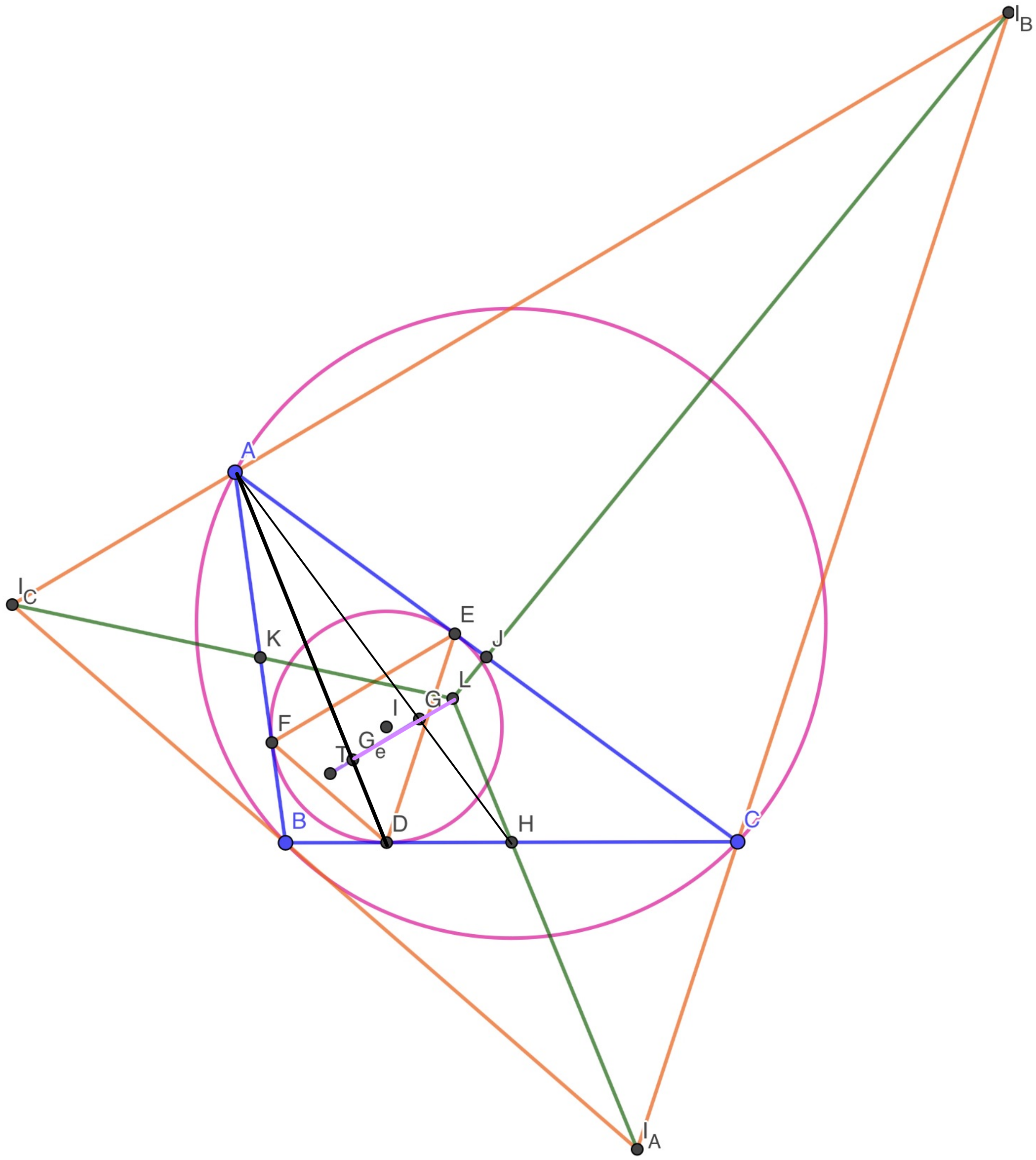
We have $\frac{GI_A}{GM} = \frac{AI_A \sin\angle GAI_A}{AM \sin\angle MAG} = \frac{AI_A \sin FAI_A}{AM \sin FAE} = \frac{AI_A \sin IDI_A}{AM \sin CDI_A} = \frac{AI_A}{AM} \tan\angle DI_AH$. We prove this is equal to 2.

We have $AI_A = \frac{R_A}{\sin(\frac{A}{2})}$, $\frac{DH}{HI} = \frac{c-b}{R_A}$, so $AI_A \tan\angle DI_AH = \frac{c-b}{\sin(\frac{A}{2})}$. We have $AM = \cos(B + \frac{A}{2}) = \frac{NL}{NP}$ (let $R = \frac{1}{2}$)

$\frac{NP}{\sin(\frac{A}{2})} = \frac{NC}{\sin B} \to NP = \frac{NC \sin(\frac{A}{2})}{b}$. Thus, $AM = \frac{NL \cdot b}{NC \sin(\frac{A}{2})} = \frac{\frac{c-b}{2}}{\sin(\frac{A}{2})}$, as desired.

3.69 proglote T (nomothetic center) collinear with G and Ge

Problem 3.69 (AoPS user proglote)**.** Let T be the homothetic center of the intouch and excentral triangle of a triangle ABC. Prove that T is collinear with the centroid of $\triangle ABC$, G and the Gergonne Point of $\triangle ABC$, Ge.



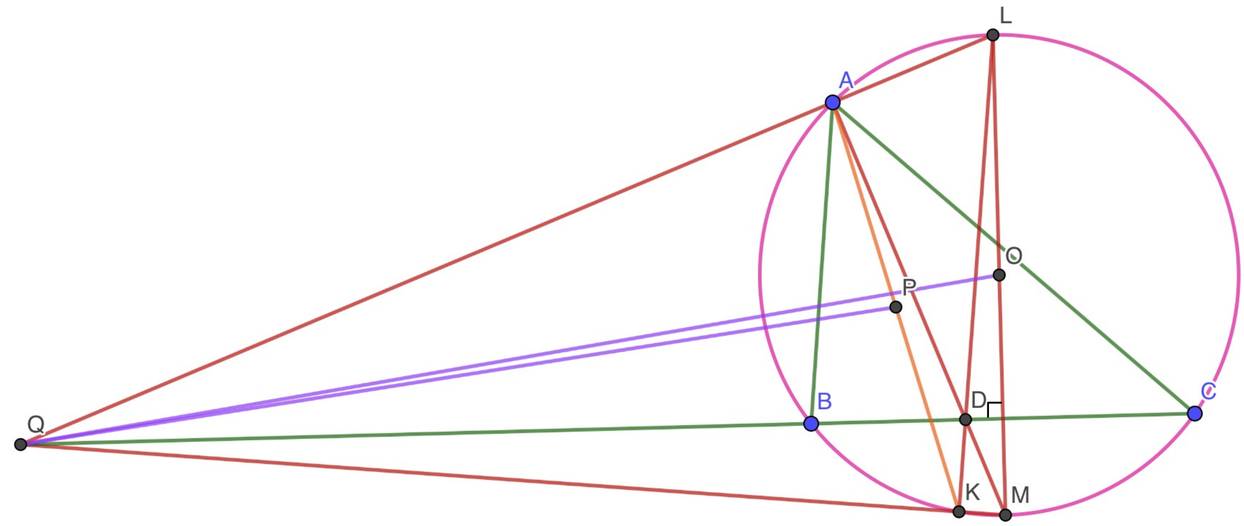
Notice that Ge is the symmedian point of DEF, and T is the symmedian point of $I_AI_BI_C$, so $\overline{GeLT}$.

Note G is the center of homothety that sends AD to HI_A, JI_B to BE, I_CK to CF, so it sends $CF \cap AD \cap BE = Ge$ to $I_CK \cap HI_A \cap JI_B$ to L, so

$\overline{GeGT} \to \overline{GeGLT}$.

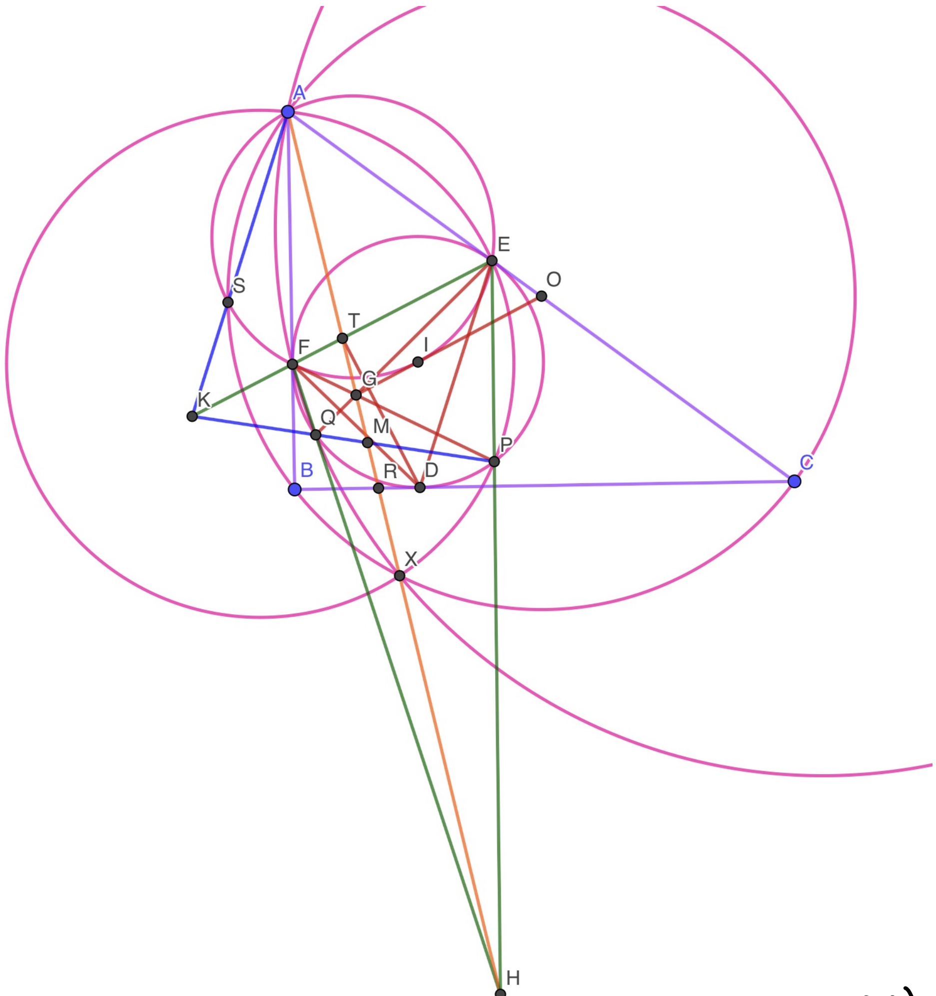
3.70 2012 ELMO SL G7

Problem 3.70 (2012 ELMO SL G7). Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90°$.



Note P is the Humpty Point, and let $K = AP \cap (ABC)$, so $(AK;BC) = -1$.
Additionally, by configs, $\overline{QKM}$, and $BC \cap AM \cap LK$, as if we let $D = LK \cap BC$,
$(AK;BC) \overset{D}{=} (ML;BC) = -1 \rightarrow \overline{ALM}$. Thus, Q is the orthocenter of $\triangle LDM$.
Thus, QP and QO are isogonal conjugates as $\triangle QAK \sim \triangle QML$.
Thus, $\angle QPA = \angle QOM$, and $\angle QOM + \angle OQB = 90°$ as desired.

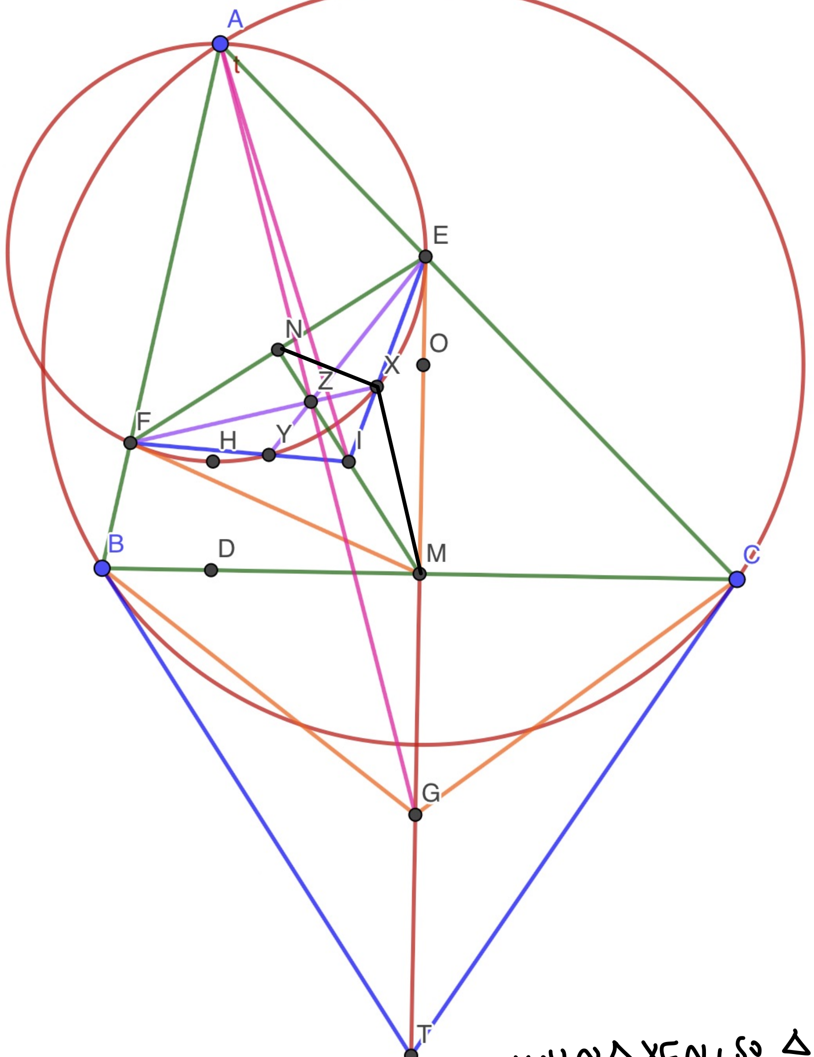
Problem 3.71 (Fake USAMO 2020/3). Let $\triangle ABC$ be a scalene triangle with circumcenter O, incenter I, and incircle ω. Let ω touch the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F respectively. Let T be the projection of D to $\overline{EF}$. The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. The circumcircles of $\triangle AEX$ and $\triangle AFX$ intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that the lines EQ, FP, and OI are concurrent.



Radical Axis gives us $FQ \cap EP \cap AX = H$ (or Pascals on FFQEEP). Now, notice that $(K = AS \cap EF)$ $(KT;FE) \stackrel{H}{=} (KM;QP) = -1$, so TM is the polar of K, or $KI \perp AR$ and KH is the polar of G wrt the incircle (Brocards on FEQP). However, notice that KH is the radical axis of the incircle + circumcircle since $KS \cdot KA = KF \cdot KE$ and $HQ \cdot HF = HX \cdot HA$, so $KH \perp OI$, and since $KH \perp IG$, $\overline{IOG}$.

3.72 i3435 AZ bisects MT

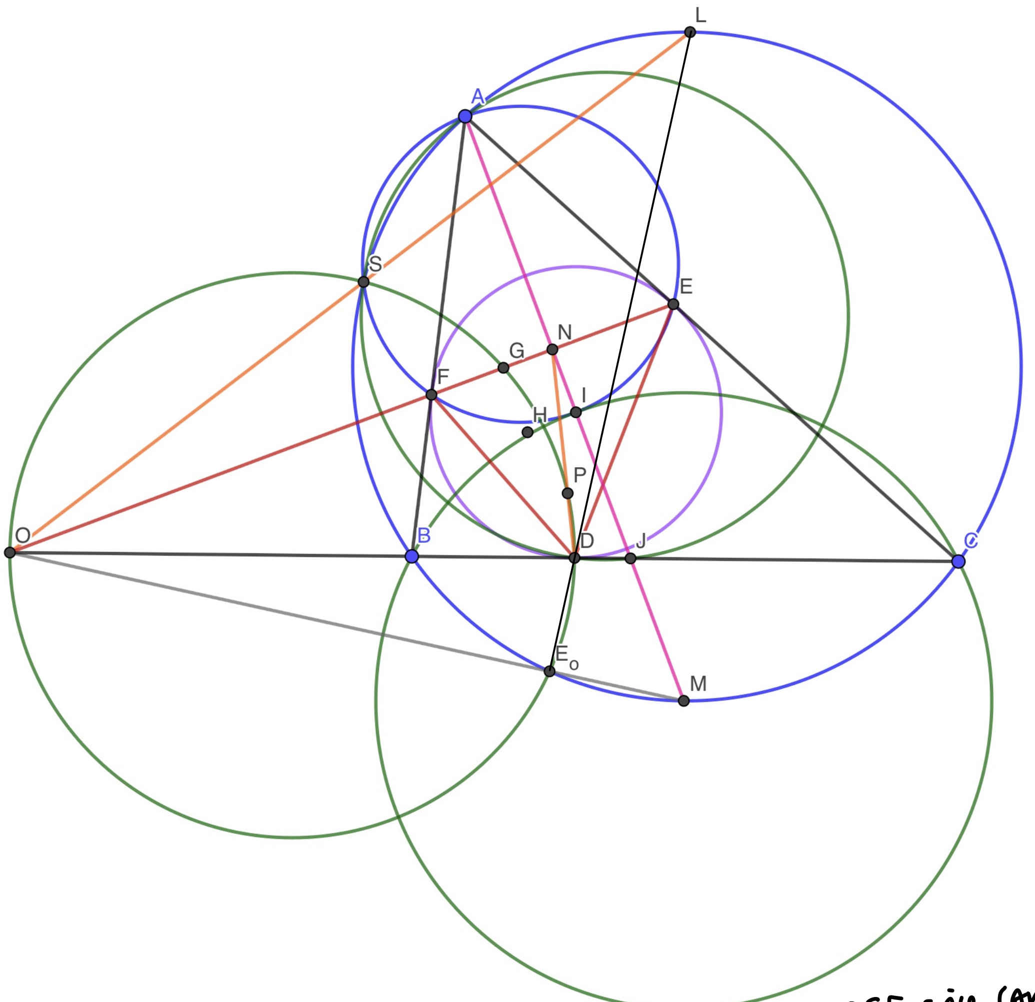
Problem 3.72 (Myself (i3435)). Let ABC be a triangle. Let E, F be the feet of the altitudes of $\triangle ABC$ from B, C respectively. Let M be the midpoint of BC and let N be the midpoint of EF. Let X be the point such that $\triangle XMN$ is similar and similarly oriented to $\triangle XFE$. Let Y be the point such that $\triangle YMN$ is similar and similarly oriented to $\triangle YEF$. Let $Z = \overline{FX} \cap \overline{EY}$ and let T be the intersection of the tangents to (ABC) at B and C. Prove $\overline{AZ}$ bisects MT.



Let I be the midpoint of MN. Note $\triangle XNI \sim \triangle XFN$, so $\triangle XNM \sim \triangle XEF$. (X is foot from N to EI). Similarly Y, the foot from N to FI satisfies these properties. By the second isogonality lemma, AZ and AI are isogonal conjugates. Notice AEMF is similar to ABTC, so AZ bisects MT since AI bisects MN.

3.73 i3435 P on 9 point of DEF

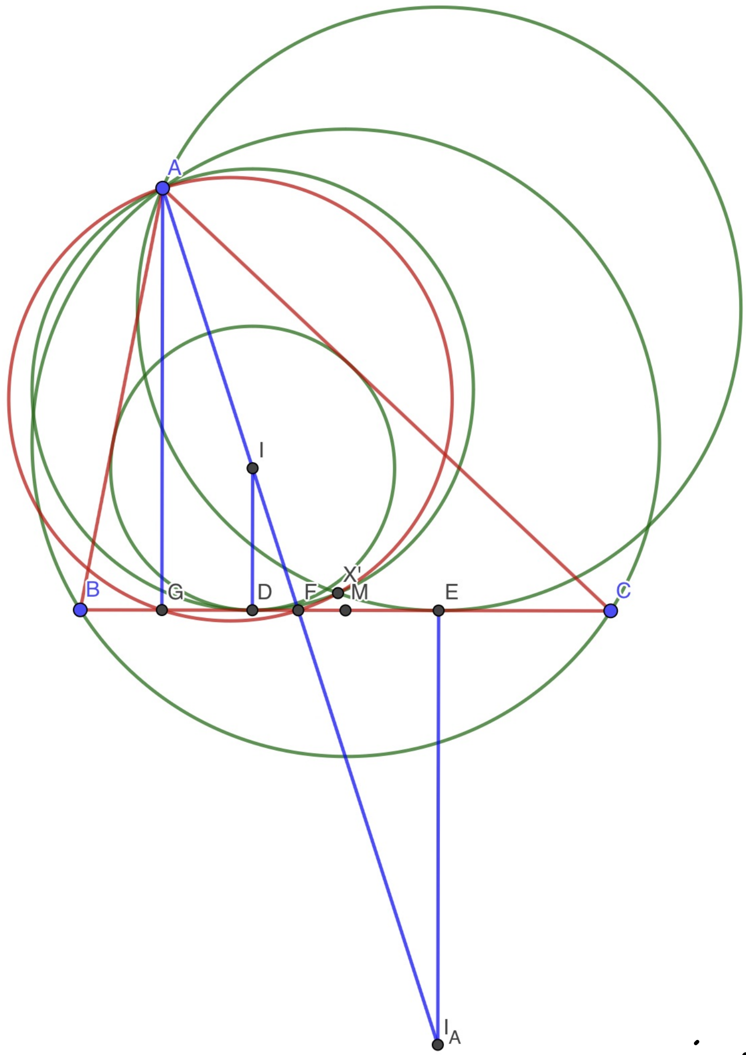
Problem 3.73 (Myself(i3435)). In triangle ABC, let I be the incenter, and let the incircle hit the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at D, E, F respectively. Let S be the non-A intersection of (AEF) and (ABC), let $J = \overline{AI} \cap \overline{BC}$, and let N be the midpoint of EF. Let $\overline{SJ}$ and $\overline{DN}$ intersect at P. Prove P is on the 9-point circle of DEF.



let P be the inverse of E_0 wrt (DEF) (lies on 9 point of DEF since (ABC) inverts to it). we have ($SGDE_0$), where $DG \perp EF$. If we let $O := LS \cap EF \cap BC$ then note (SOE_0D) by configs, so (SOE_0DG). Pascals on $PSDDOE_0$ gives that $SP \cap DO$ is on AM as desired.

3.74 2015 Taiwan TST Round3 Quiz 1 P2 X, M, A’ are collinear

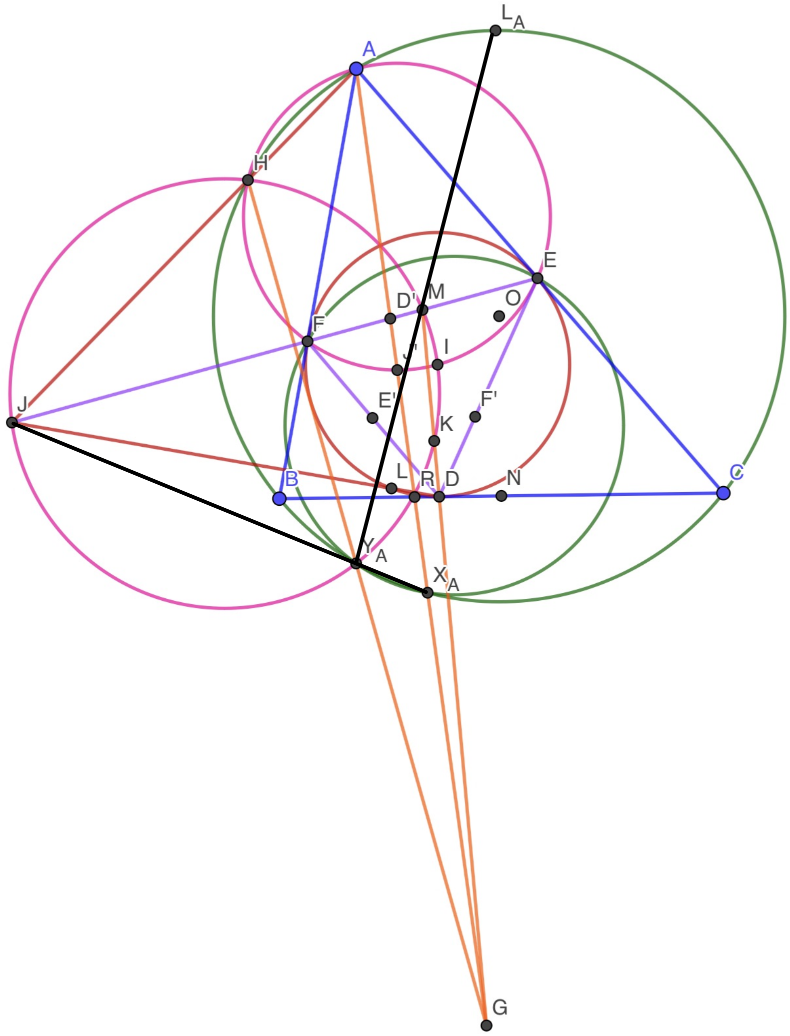
Problem 3.74 (2015 Taiwan TST Round 3 Quiz 1 P2). Let O be the circumcircle of the triangle ABC. Two circles O_1, O_2 are tangent to each of the circle O and the rays $\overrightarrow{AB}, \overrightarrow{AC}$, with O_1 interior to O, O_2 exterior to O. The common tangent of O, O_1 and the common tangent of O, O_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle O, and the segment $\overline{AA'}$ be the diameter of O. Prove that X, M, and A' are collinear.



Do a $\sqrt{bc}$ inversion. Note that O_1 and O_2 invert to the incircle + excircle, and the common tangents invert to ω_1, ω_2, which are tangent to BC at D, E (incircle + excircle touchpoints) through A. We need to show $G \leftrightarrow A'$, $X' \leftrightarrow X$, F, A are cyclic, or $MX' \cdot MA = MF \cdot MG = MD^2$. However, note that $(AF; I I_A) \overset{BC}{=} (GF; DE)$, so $MF \cdot MG = MD^2$ as desired.

3.75 i3435 concur on OI

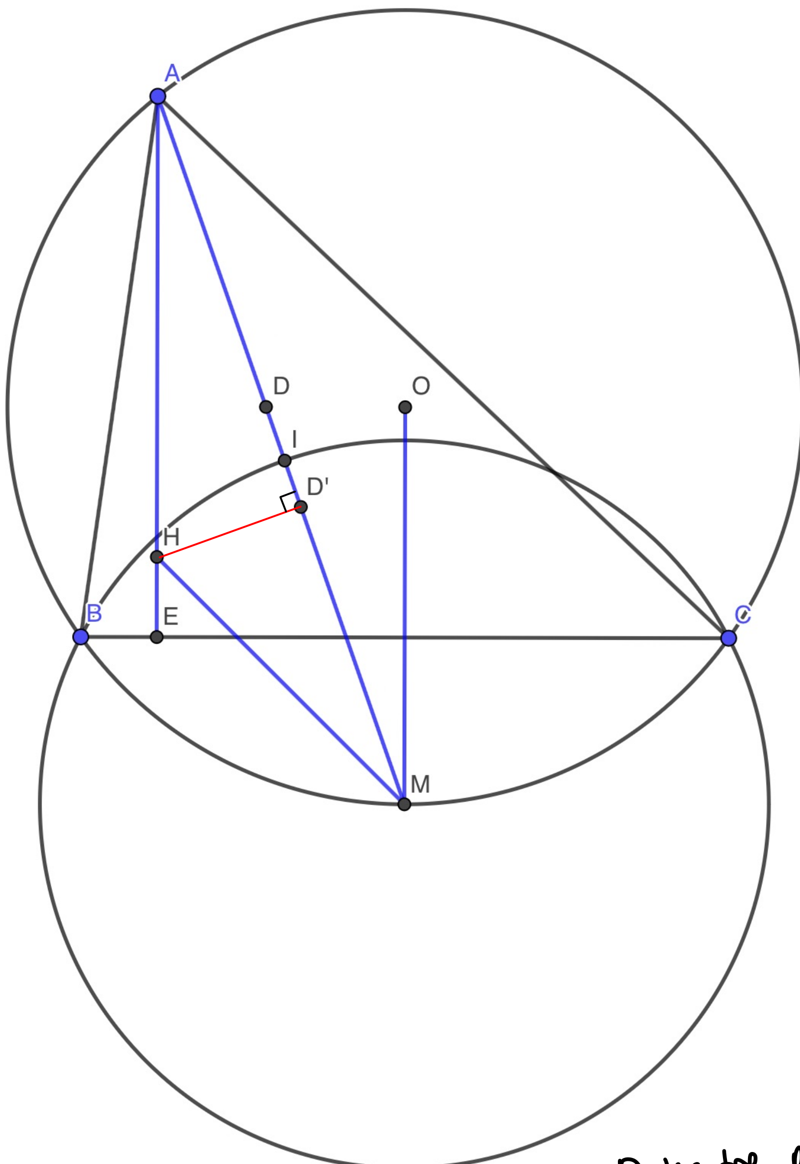
Problem 3.75 (Myself(i3435)). Let ABC be a triangle with circumcenter O, incenter I, and intouch triangle DEF. Let L be the midpoint of arc $\widehat{BAC}$, and let $\overline{LD}$ intersect (ABC) again at X. Let (EFX) meet (ABC) again at Y. Then let $\overline{AX} = \ell_A$ and $\overline{LY} = m_A$. Define ℓ_B, ℓ_C, m_B, m_C similarly. Prove that $\ell_A, \ell_B, \ell_C, m_A, m_B, m_C$, and $\overline{OI}$ all concur at one point.



Note K_A' is the E_0 point, so $\overline{ARK_A'}$. Note ABC is homothetic to $D'E'F'$, so AD', BE', CF' intersect at the center of homothety between the two triangles ($\ell_A \cap \ell_B \cap \ell_C$ on OI). Note $X_A \leftrightarrow K$, consider $(IHKM)$, the image of line AD' after inverting about the incircle, lies on our circle since (2.5.17), which lies on DM. Note $J = AH \cap EF \cap XY$ by radical axis, lies on our circle since $\angle IMJ = \angle IHJ = 90°$. We also have $(HARN)$ by our 2.5.16 config, so $\angle HMJ = \angle ANR = AHX_A = \angle HY_AJ$, so (Y_AKIM). Note by inversion, $(MAKX_A)$, so by rad axis on (ABC), (MAK), (MIK), we have $MK \cap AX_A \cap HY_A = G$. Finally, Pascals on $AM_AHY_AL_AX_A$ gives $\overline{L_AMY_A}$, and since $L_AL_BL_C$ is homothetic to the midpoint triangle of DEF we see $\ell_A, \ell_B, \ell_C, m_A, m_B, m_C$, OI intersect at the center of homothety between the 9 point circle of DEF and (ABC).

3.76 MP8148 polar or H WRT BIC

Problem 3.76 (AoPS user MP8148). Let ABC be a triangle with incenter I, orthocenter H, and circumcenter O. Show that the line parallel through O parallel to $\overline{BC}$ and the polar of H with respect to (BIC) meet on $\overline{AI}$.



Let D' be the foot from H to AI. Let $R = \frac{1}{2}$. Let D be the point on AI st $OD \parallel BC$.

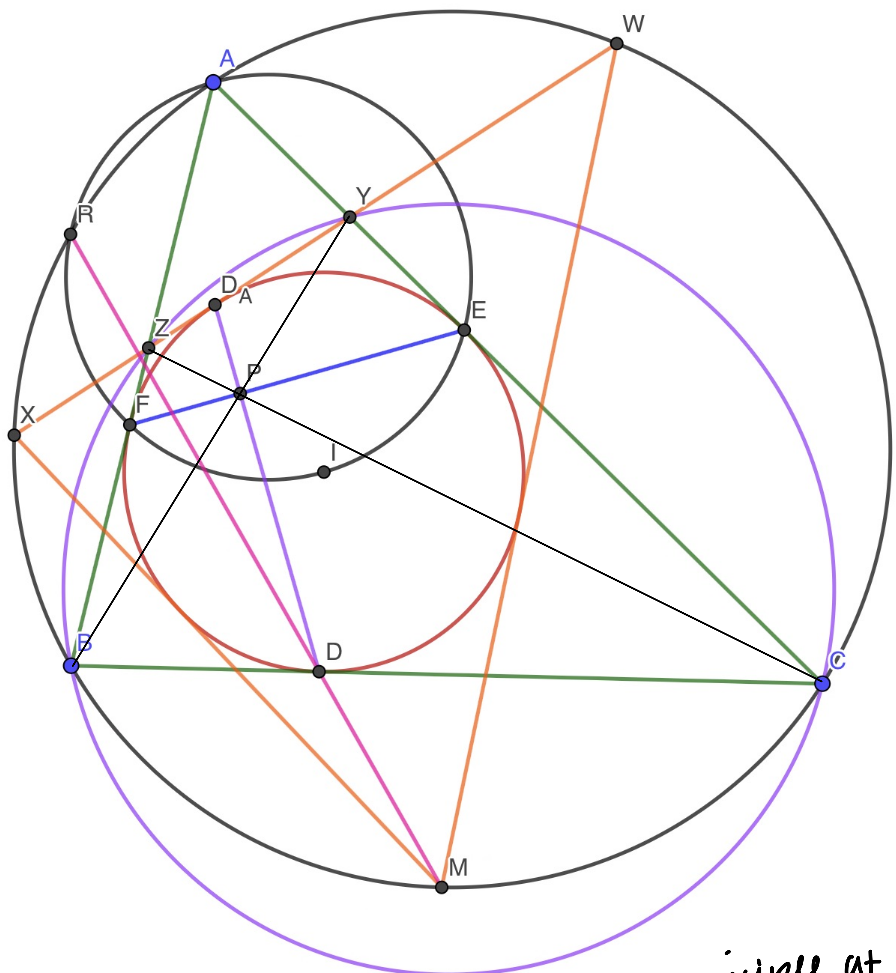
We show $MD \cdot MD' = MI^2$. We have $AH = \cos A$, so $AD' = \cos A \sin(C + \frac{A}{2})$. We have $D'M = AM - AD' = \sin(C + \frac{A}{2})(1 - \cos A)$. Note $\frac{OM}{MD} = \sin(C + \frac{A}{2})$, so $MD = \frac{1}{2\sin(C + \frac{A}{2})}$.

Thus, $MD \cdot MD' = \frac{1}{2}(1 - \cos A) = \sin^2(\frac{A}{2}) = MC^2 = MI^2$. Thus, HD' is the polar of D, so D lies on the polar of H.

3.77 SORY P6

Problem 3.77 (SORY P6). Let $\triangle ABC$ be a triangle with incenter I. Let the incircle be tangent to the sides BC, CA, AB at D, E, F respectively. Let P be the foot of the perpendicular from D onto $\overline{EF}$. Assume that BP, CP intersect the sides AC, AB at Y, Z respectively. Finally let the rays IP, YZ meet the circumcircle of $\triangle ABC$ in R, X respectively.

Prove that the tangent from X to the incircle and RD intersect on the circumcircle of $\triangle ABC$.



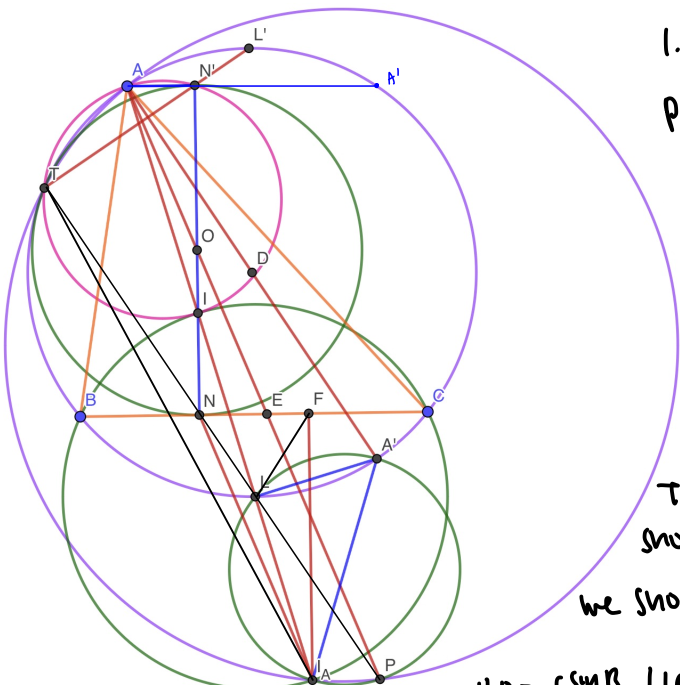
By theorem 2.73, we know XW is tangent to our incircle at D_A (or $DP \perp (DEF)$)
we wish to show XMW has (DEF) as the incircle. Note YZ is antiparallel to
BC, so A is the midpoint of arc ZW. Since (DEF) is tangent to XW and
$\overline{MIA}$, so MXW is the unique triangle with M as a vertex with circumcircle (ABC) and
incircle (DEF), so XM is tangent to (DEF) as desired.

3.78 Beiqiang

Problem 3.78 (AoPS user beiqiang). Let acute triangle ABC be inscribed in a circle ω, and suppose the incircle of ABC touches side BC at N. Let ω' be a circle tangent to BC at N, and tangent to ω at T such that ω' is on the same side of BC as A. Let O be the center of ω' and I_a be the A-excenter. Let the midpoint of smaller arc BC be L, and A' be the A-antipode. Let TN and AO intersect at P.

Show that:

1. T is on NL and IA',
2. $\{AO, NI_a\}, \{BC, PI_a\}, \{ON, A'P\}$ are parallel,
3. LI_aPA' and ATI_aP are cyclic, and
4. AT, $A'L$ and I_aP are concurrent.



1. We have T is the sharky devil point since $T = NL \cap (ABC)$, so $T \cap IA'$ as well.

We show $\triangle AN'O \sim \triangle NFI_A$.

Notice that if you construct the line through A tangent to our circle, it intersects at A', the reflection of A over the perp bisector of BC (shooting lemma), let the tangency point be N'. Thus, $ON' \perp AA'$ and $ON \perp BC$, so $\overline{NON'}$. Now, we can show $\triangle ANO \sim \triangle NFI_A$.

We show $\frac{N'O}{AN'} = \frac{I_AF}{NF}$. $AN' = (s-b) - c\cos B$, $NN' = c\sin B$, $N'O = \frac{c\sin B}{2}$ (let $R = \frac{1}{2}$), we wish to show $\frac{\frac{a+c-b}{2} - c\cos B}{\frac{c\sin B}{2}} = \frac{\frac{b-c}{2}}{\frac{s}{s-a}\cdot r}$.

We have $\frac{c\sin B}{2} = \frac{K}{a}$ ($K = [ABC]$) and $r = \frac{K}{s}$, so $\frac{(s-b)-c\cos B}{\frac{K}{a}} = \frac{\frac{b-c}{2}}{\frac{s}{s-a}\cdot\frac{K}{s}}$. We substitute $\cos B = \frac{a^2+c^2-b^2}{2ac}$ to get $\left((s-b) - c\left(\frac{a^2+c^2-b^2}{2ac}\right)\right)\cdot a = (s-a)\left(\frac{b-c}{2}\right)$ which is true.

Now $\angle N'AE = \angle OEN = \angle FNI_A$ so $AO \parallel NI_A$, $\angle PNI_A = \angle TI_A A = \angle TPA \to (TI_APA)$ (since $I_AL^2 = LB^2 = LN \cdot LT$). We have $\angle I_APT = \angle TCL = \angle TNB$, and from ELMO 2011/6, $\angle LFB = \angle LA'I_A$, so $\hookrightarrow \angle LNC = \angle LA'I_A = \angle NPI_A \to I_AP \parallel BC$

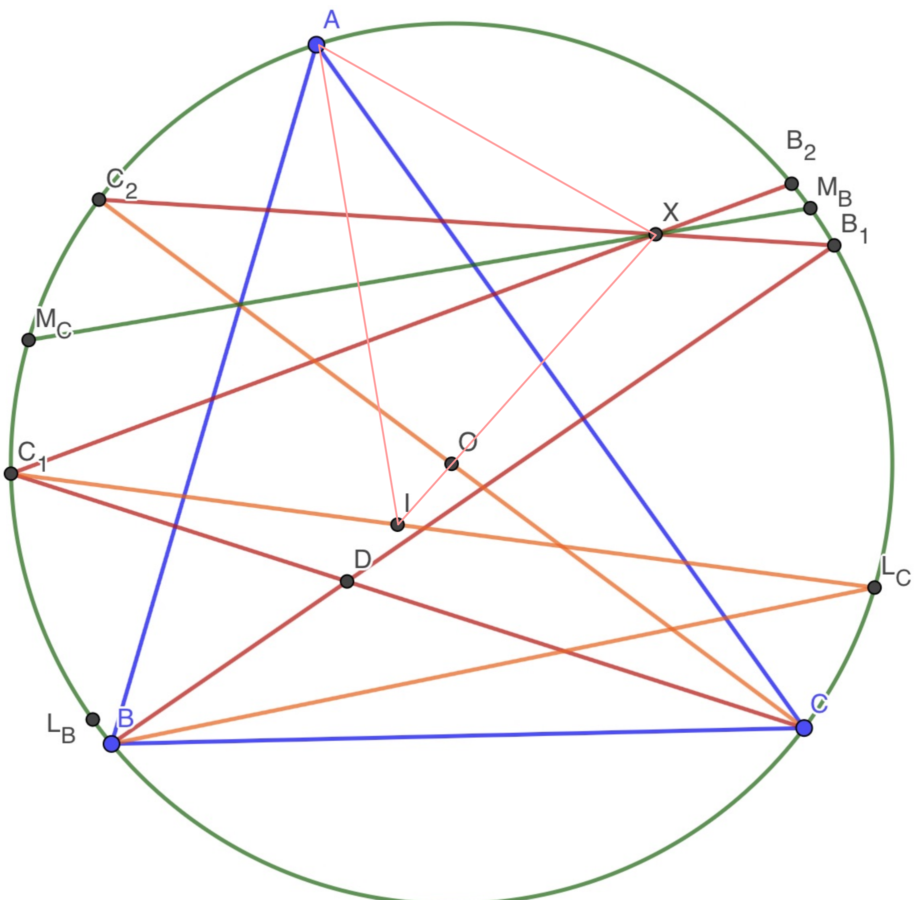
$\angle LPI_A = \angle LAT = \angle LCB = \angle LNB = \angle LNF = \angle LFN = \angle LAI_A \to (LA'I_AP)$.

$\angle API_A = \angle A'LI_A = 90^\circ \to A'P \perp I_AP$, so $ON \parallel A'P$.

Finally, radical axis on (ATI_AP), $(LA'I_AP)$, $(ATLA')$ gives $AT \cap A'L \cap I_AP$.

3.79 2017 ELMO SL G4

Problem 3.79 (2017 ELMO SL G4). Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points opposite to B, C on ω, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.



Consider pascals on $B_2C_1L_CM_CM_BB \to B_2C_1 \cap M_BM_C$ on OI
$C_2B_1L_BM_BM_CC \to B_1C_2 \cap B_BM_C$ on OI

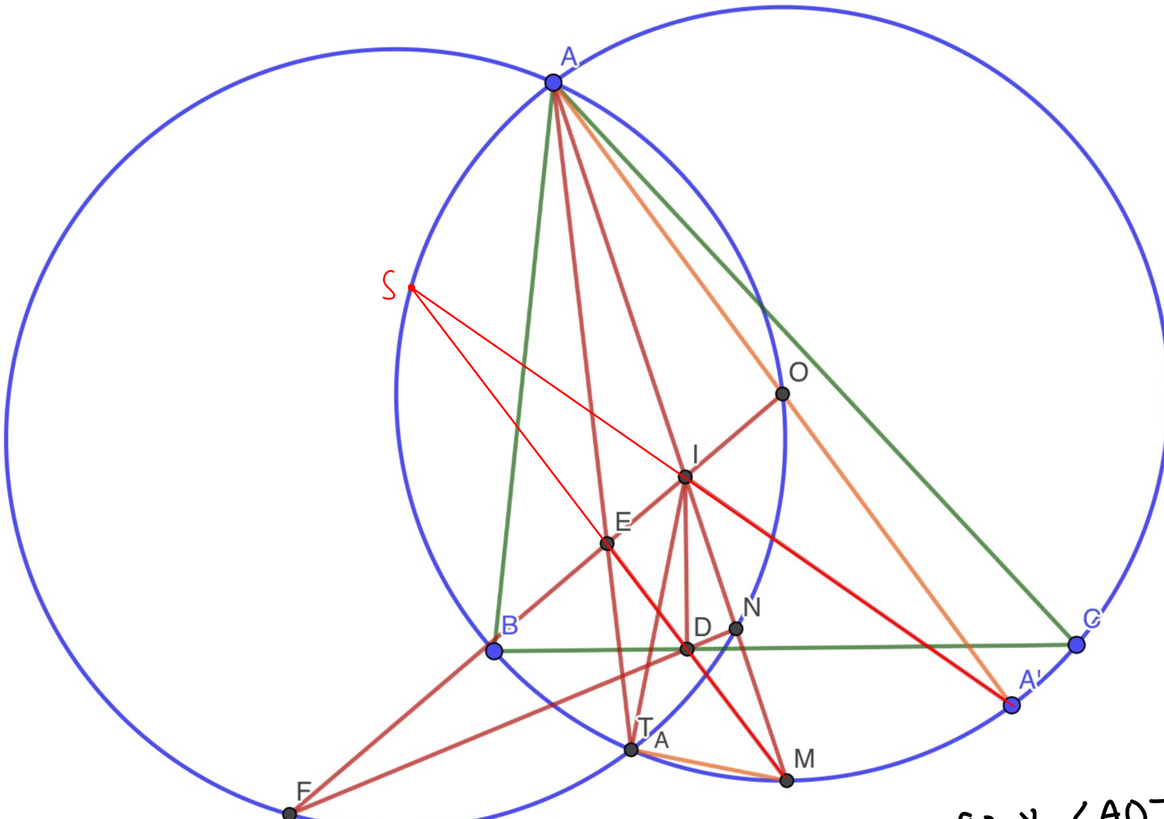
Thus, $M_BM_C \cap B_2C_1 \cap B_1C_2 = X$, so X lies on the perpendicular bisector of AI (M_BM_C) as desired.

3.80 MP8148 OI cap MD, ND

Problem 3.80 (AoPS User MP8148). Let ABC be a triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of the arc BC not containing A. Suppose the incircle is tangent to $\overline{BC}$ at D, and N is the midpoint of $\overline{IM}$. Denote Ω to be the circumcircle of $\triangle AON$. Show that

a Lines $\overline{OI}$ and $\overline{MD}$ meet on the radical axis of Γ and Ω.

b Lines $\overline{OI}$ and $\overline{ND}$ meet on Γ.

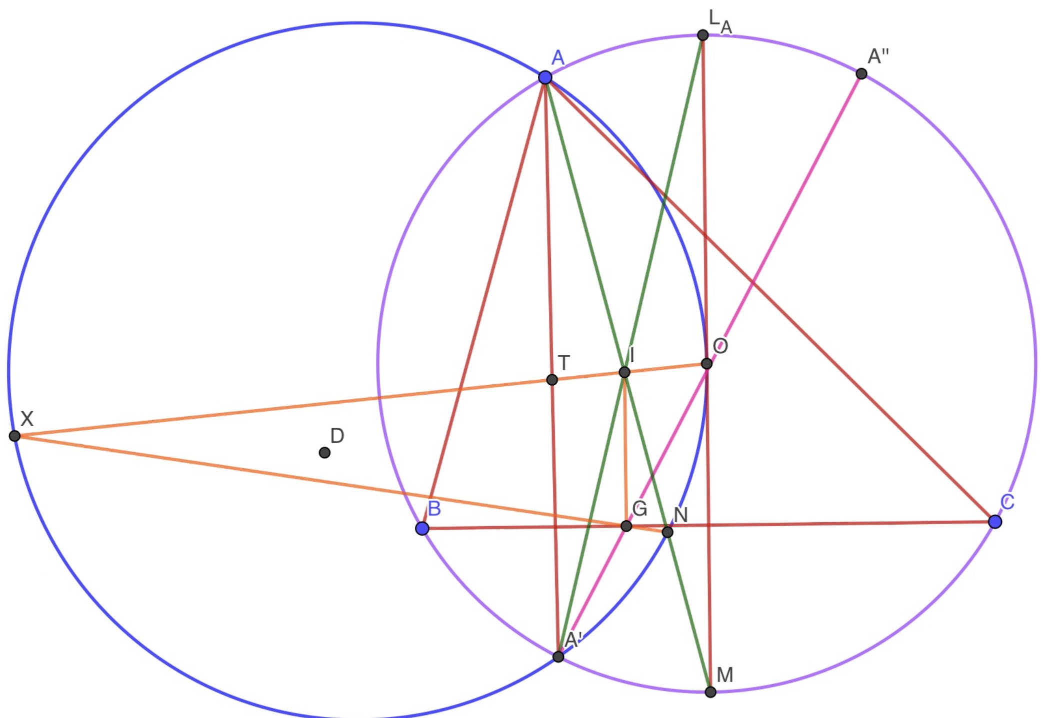


We have $\angle T_A N M = 2x$, $\angle A N T_A = 180-2x$, $\angle A M T_A = 90-x$, $\angle A O T_A = 180-2x$, so $(AONT_A)$. Note that $MD \cap OI$ on the center of homothety between the intouch triangle + the triangle composed of M_A, M_B, M_C, which also lies on AT_A.

Let $F = OI \cap ND$. Notice that $\triangle MDI \sim \triangle MIS \sim \triangle A'IA$, so $\triangle ION \sim \triangle AIO$ so $\angle AOI = \angle IND \rightarrow (FONA)$

3.81 2014 ELMO SL G8

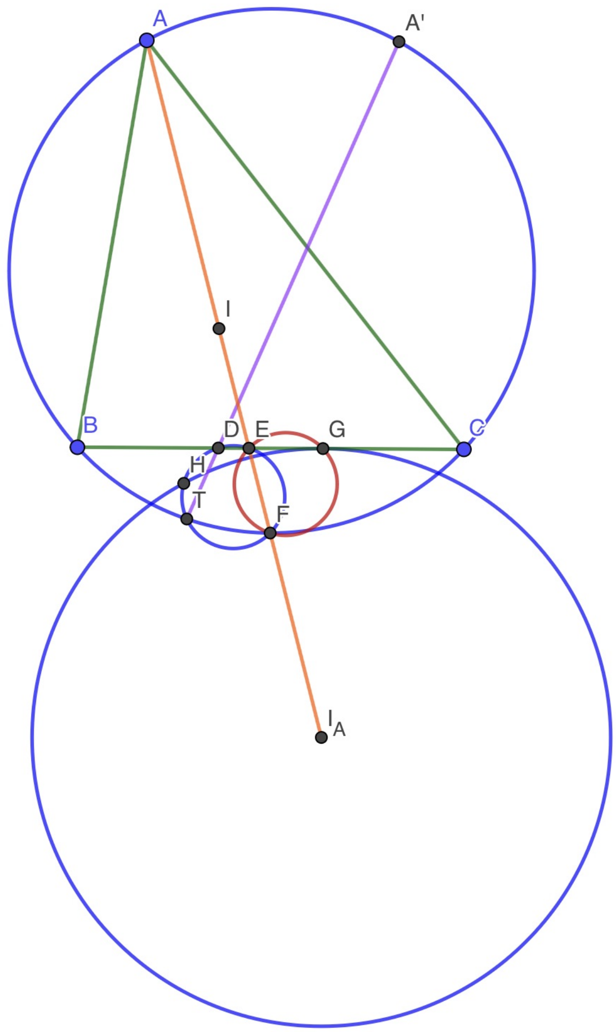
Problem 3.81 (2014 ELMO SL G8). In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.



The previous (3.80) problem tells us that the midpoint of IM is on (AOA'), and that $OI \cap NG$ is on (AOA'). Let $X = IO \cap NG$. We have $\triangle AOI \sim \triangle ING$ so $\angle OAN = \angle GIN = \angle IXN \rightarrow (XIG)$ is tangent to AN. We have $\angle IXN = \angle NAO = \angle AMO = \angle AA'L_A = \angle L_A A'G$, so $(AIXG)$. Note that $OI \cdot IX = AI \cdot IN = AI \cdot IM \cdot \frac{1}{2} = \frac{1}{2} Pow(I)$, a fixed value so X is a fixed point (symmetric wrt B, C) so we are done ($\angle AXO = \angle OXA'$ by fact 5).

3.82 USA TST 2016/2

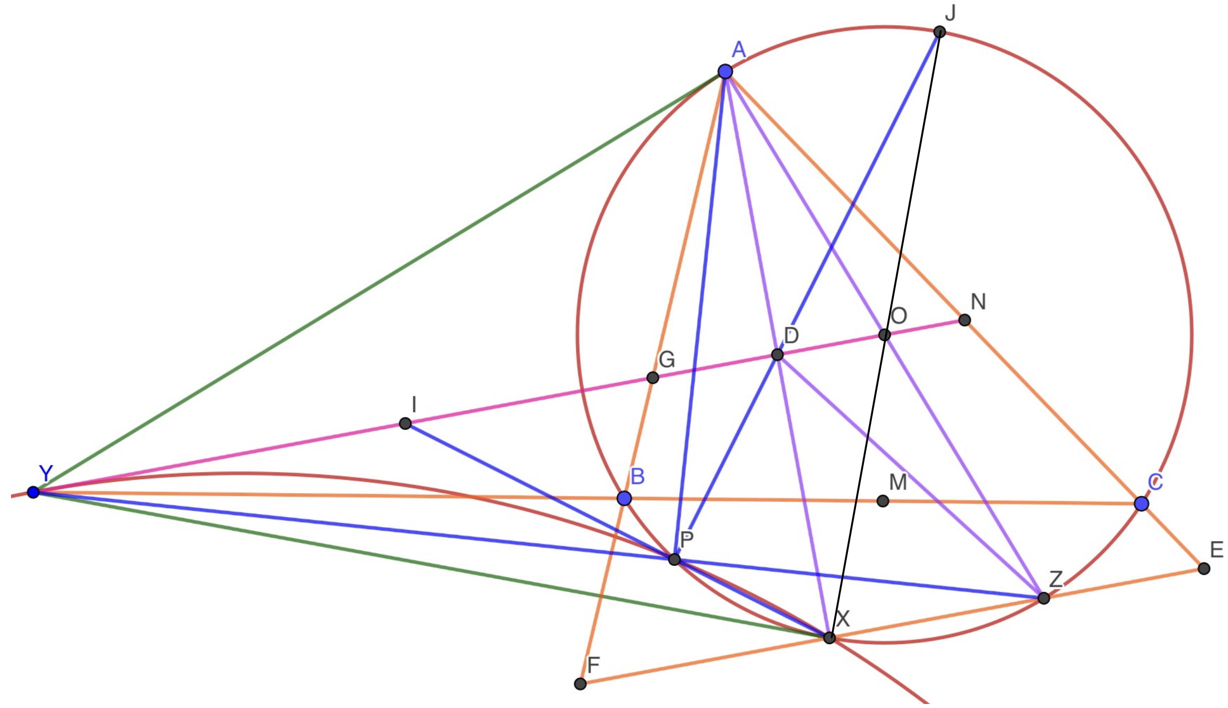
Problem 3.82 (USA TST 2016/2)**.** Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.



We have $\angle TFE = \angle AA'D = \angle A'DB = \angle EDT$, so T is the mixtilinear touchpoint ($AA' \parallel BC$). Let H be the reflection of G over EF. Then, $AH \cdot AT = AE \cdot AF$ ($\overline{AHT}$ since AG, AT are isog conjugates), so $H \in (EFD)$ and H on excircle.

3.83 EMMO Juniors 2016/5

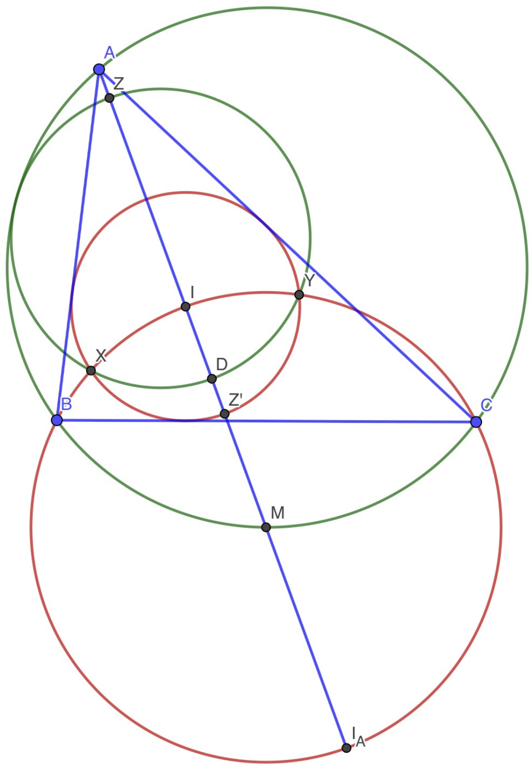
Problem 3.83 (EMMO Juniors 2016/5). Let $\triangle ABC$ be a triangle with circumcenter O and circumcircle Γ. The point X lies on Γ such that AX is the A- symmedian of triangle $\triangle ABC$. The line through X perpendicular to AX intersects AB, AC in F, E, respectively. Denote by γ the nine-point circle of triangle $\triangle AEF$, and let Γ and γ intersect again in $P \neq X$. Further, let the tangent to Γ at A meet the line BC in Y, and let Z be the antipode of A with respect to circle Γ. Prove that the points Y, P, Z are collinear.



We have $YO \parallel EF$ since $AX \perp EF$ and $AX \perp YO$. Let G, N be the midpoints of AF and AE, so we have $\overline{YGN}$. Let D be the A-Dumpty point, and $I = YD \cap PX$ (let $P = YZ \cap (ABC)$)). Note that $\angle IYP = \angle PZX = \angle PXY$ so $IY^2 = IP \cdot IX$. Note that Pascals on $PJXXAZ$ gives ($J = PD \cap (ABC)$) $JX \cap AZ = O$. Thus, $\angle DPX = \angle IDX = 90°$ so $ID^2 = IP \cdot IX$, so $YI = YD$. Note that $(YDGN) \overset{A}{=} (AXBC) = -1$, so by Harmonic Bundles, $IG \cdot IN = YI^2 = IP \cdot IX$ so $P \in (GNX)$ as desired.

3.84 All Russian 2013 Grade 11 P8

Problem 3.84 (All Russian 2013 Grade 11 (12 in American System) P8 (why am I putting Russia problems in an American Geo handout?)). Let ω be the incircle of $\triangle ABC$ and with center I. Let Γ be the circumcircle of the triangle BIC. Circles ω and Γ intersect at the points X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of $\triangle ABC$.

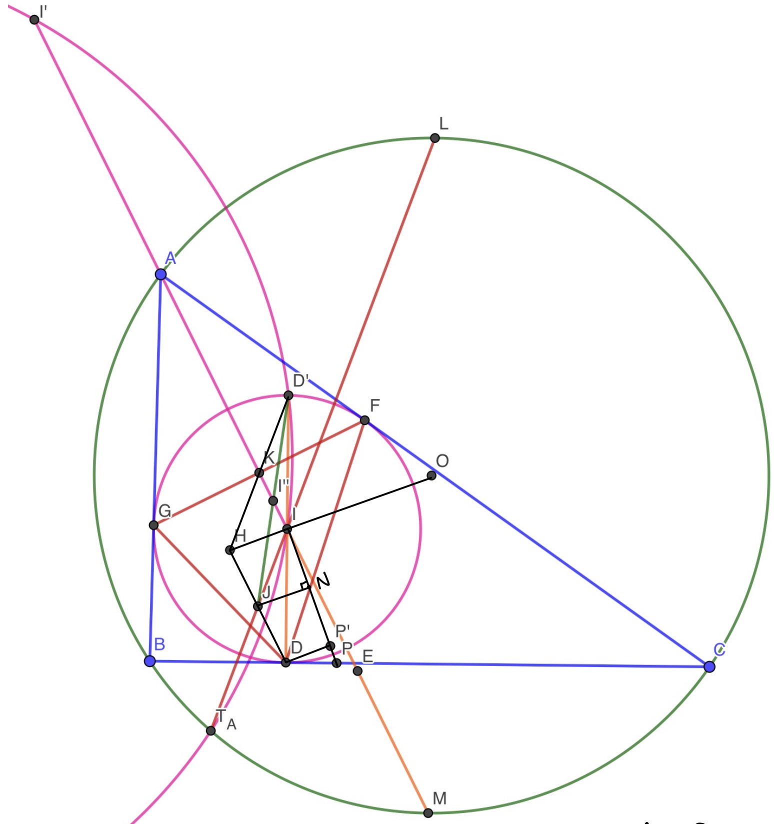


We show that (XYZ) inverts to incircle under the inversion about (BIC). Note $\frac{ZI}{ZI+IM} = \frac{r}{IM}$, let $IM = R$, so $ZI = \frac{Rr}{R-r}$. Thus, $MZ' \cdot MZ = (ZI+IM)(IM-r)$ $= \left(\frac{Rr}{R-r} + R\right)(R-r) = R^2$ so ($Z' = AM \cap$ incircle), we see $(XYZ) \leftrightarrow (XYZ')$ and $(ABC) \leftrightarrow BC$ as desired.

3.85 MP8148 QD'=QI

Problem 3.85 (AoPS user MP8148). In scalene triangle ABC with $AB \neq AC$, I is the in-center, and O is the circumcenter. Let L be the midpoint of arc BAC, P be the point on $\overline{BC}$ such that $\overline{PI} \perp \overline{OI}$, and Q be the point on $\overline{AL}$ such that $\overline{QP} \perp \overline{LI}$.

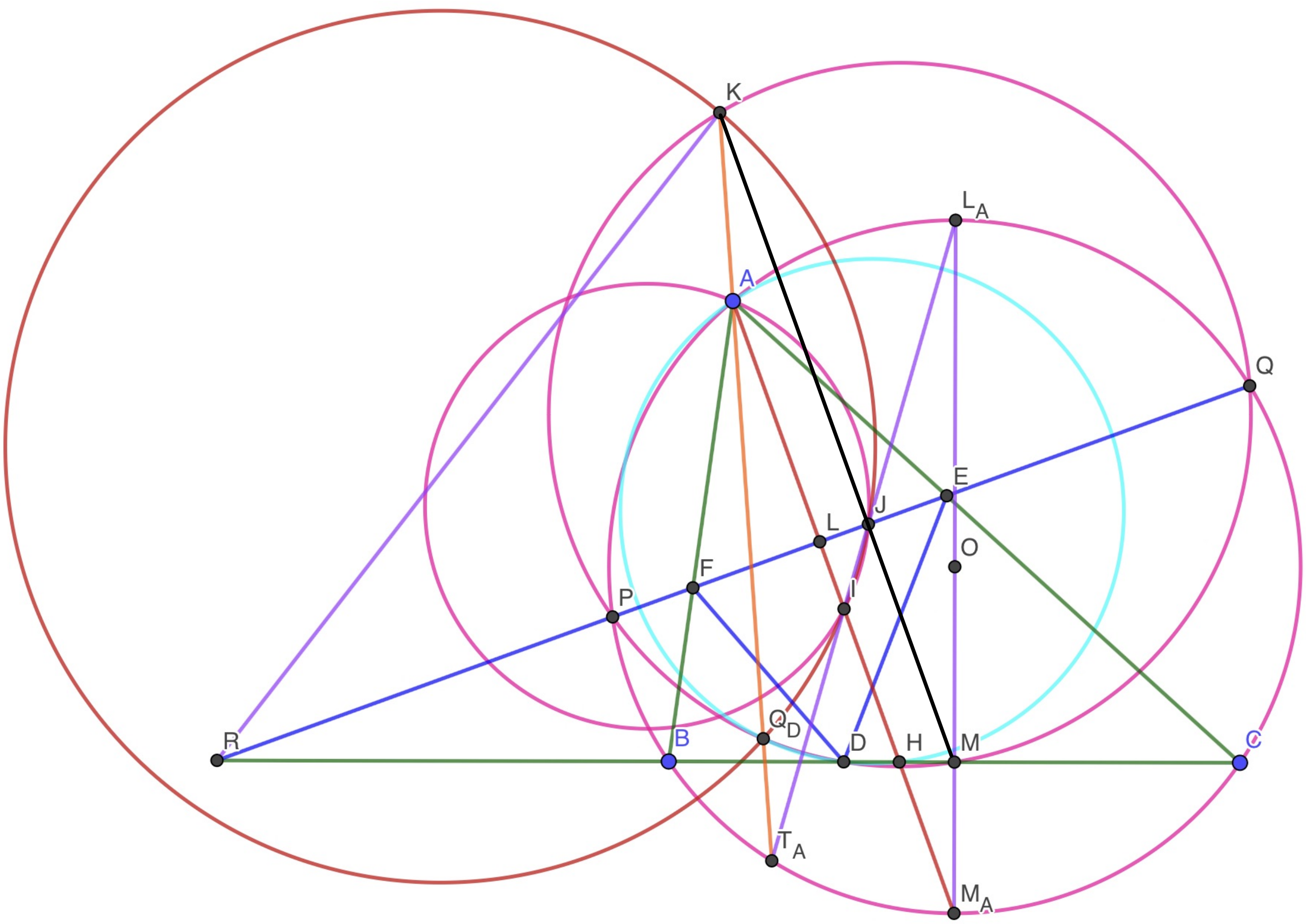
If the incircle is tangent to $\overline{BC}$ at D and D' is the reflection of D over I, show that $QD' = QI$.



Let I' be the reflection of I over A. Then, we want to show Q is the center of $(I'D'I)$. Consider an inversion about the incircle. If H is the orthocenter of DGF, we know T_A inverts to the midpoint of DH, where H is the orthocenter of GFD (note $\overline{HIO}$ as well). Then $IH \perp IP'$, and $DP' \perp IP'$ (where P' is the image of P under inversion). Thus, the projection of J onto IP' is N, where $IN = NP'$, as $HJ = JD$. Thus, $IJ = JP'$. Thus, $\angle JP'I = \angle PT_AI = \angle T_AIP$ so $IP = PT_A$, so Q is on the perpendicular bisector of IT_A, and since $\overline{AL} \perp IA$, it is also on the perpendicular bisector of II'. Now, note that $\overline{D'KH}$ and $\overline{D'ID}$, and I' inverts to the midpoint of IK, so we have $\overline{DI''J} \leftrightarrow (I''D'T_A)$, so Q is the center of this circle as desired.

3.86 Mathematical Reflections O451

Problem 3.86 (Mathematical Reflections O451)**.** Let ABC be a triangle, Γ be its circumcircle, ω its incircle, and I the incenter. Let M be the midpoint of BC. The incircle ω is tangent to $\overline{AB}$ and $\overline{AC}$ at F and E respectively. Suppose $\overline{EF}$ meets Γ at distinct points P and Q. Let J denote the point on EF such that MJ is perpendicular on EF. Show that IJ and the radical axis of (MPQ) and (AJI) intersect on Γ.

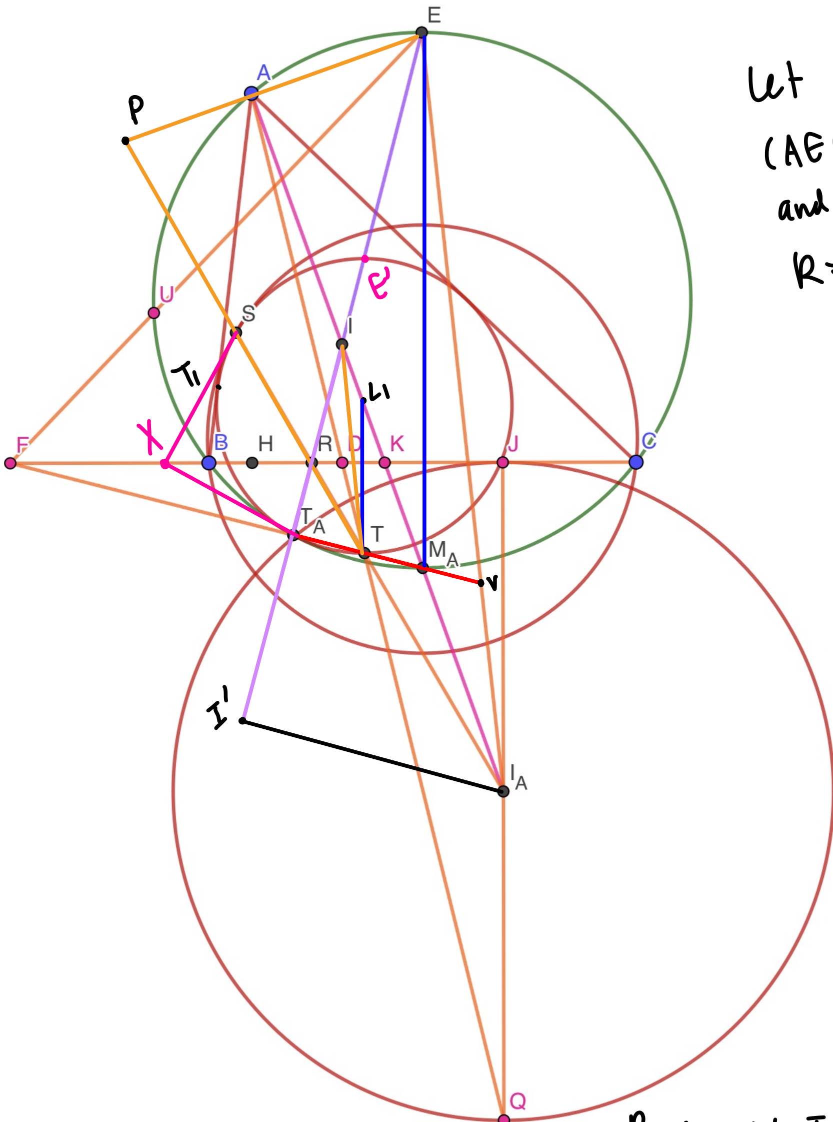


Let $J = M_AT_A \cap EF$, and we show $\frac{L_AM_A}{M_AI} = \frac{L_AM}{MJ}$. Note that $MR\cos\left(\frac{A}{2}+C\right) = MJ$. Since $(R\,D;B\,C) = -1$, we have $MR = \frac{\frac{a^2}{4}}{\frac{b-c}{2}} = \frac{a^2}{2(b-c)}$. We have $M_AI = M_AC = \sin\left(\frac{A}{2}\right)$ (let $R = \frac{1}{2}$) and $L_AM = MC \cdot \frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}$, and $L_AM_A = 1$. We wish to show $\frac{1}{\sin\left(\frac{A}{2}\right)} = \frac{MC\frac{\cos\left(\frac{A}{2}\right)}{\sin\left(\frac{A}{2}\right)}}{\frac{a^2}{2(b-c)}\cos\left(\frac{A}{2}+C\right)}$. Simplifying, we see this is true, so $MJ \perp EF$ (orig def of J). Now note by 3.15, we have $\cos\left(\frac{A}{2}+C\right) = \frac{HM}{HM_A} = \frac{HM\sin B}{HC\sin\left(\frac{A}{2}\right)} = \frac{b-c}{2b}\frac{\sin B}{\sin\left(\frac{A}{2}\right)} = \frac{b-c}{2\sin\left(\frac{A}{2}\right)}$. We have $KM \perp PQ$ so $\overline{KJM}$, and note $\angle MKQ_D = M_AAT_A = \angle T_AL_AM_A = T_AID = \angle T_AIQ_D$, so (Q_DIJK), so $T_AQ_D \cdot T_AK = T_AI \cdot T_AJ \rightarrow T_A$ on rad axis as desired.

3.87 AoPS Problem Making Contest 2016/7

Problem 3.87 (AoPS Problem Making Contest 2016/7). Let $\triangle ABC$ be given, it's $A-$mixtilinear incircle, ω, and it's excenter I_A. Let H be the foot of altitude from A to BC, E midpoint of arc $\widehat{BAC}$ and denote by M and N, midpoints of BC and AH, respectively. Suposse that $MN \cap AE = \{P\}$ and that line I_AP meet ω at S and T in this order: $I_A - T - S - P$.

Prove that circumcircle of $\triangle BSC$ and ω are tangent to each other.



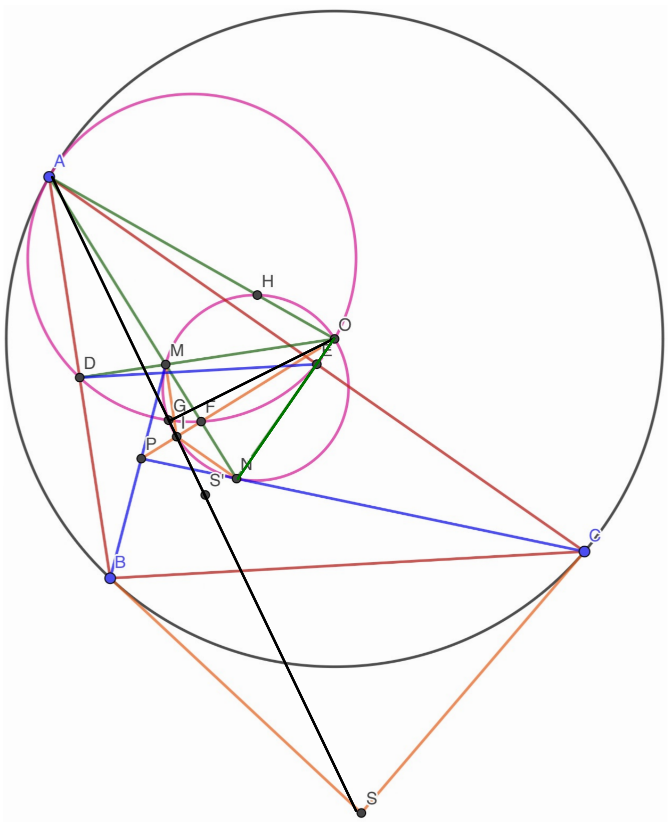
Let $L = AE \cap BC$. We have $(AEI'I_A)$ by radical axis + configs, and $I_AI'L \parallel M_AT_A$.

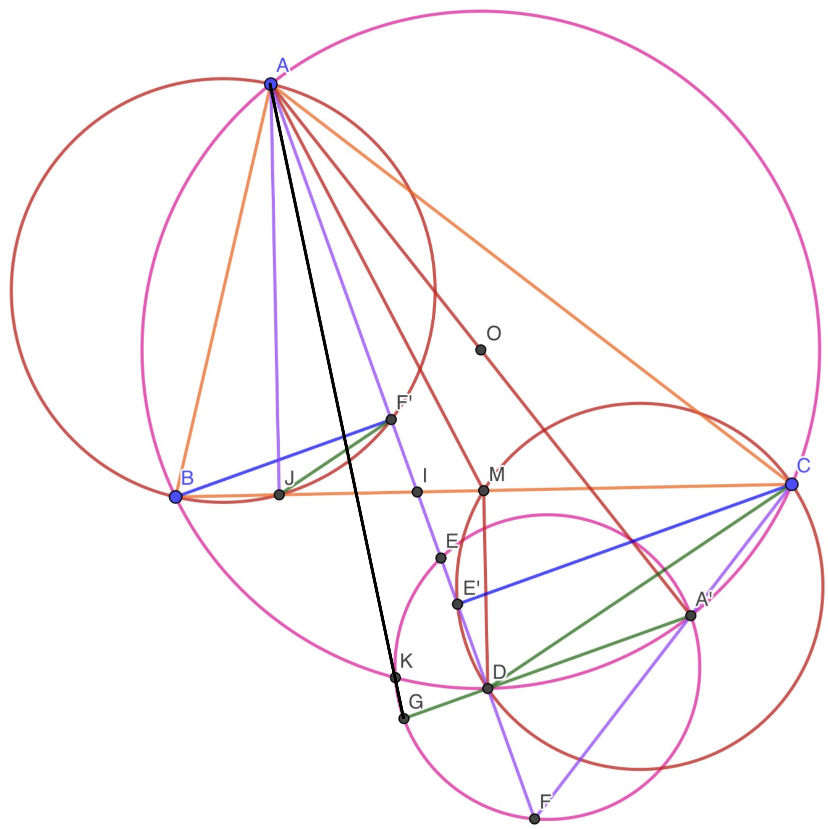
$R = BC \cap ET_A$.

Let $T = PI_A \cap T_AM_A$. We have $(AH;N\infty) \stackrel{M}{=} (LA;PE) \stackrel{R}{=} (KA;I_AI) = -1$ so $\overline{I_APR}$. If we let T' be the $M_AT_A \cap v$, then $L_1T' \parallel I_AE$. We also have $\frac{IL_1}{L_1T'} = \frac{IL_1}{L_1T_1} = \sin\angle IT_1L_1 = \sin\angle BAM_A$
$= \frac{M_AC}{M_AE} = \frac{M_AI_A}{M_AE}$. Thus, since $\angle IL_1T' = \angle I_AM_AE$, we have $\triangle IL_1T' \sim \triangle I_AM_AE$ so $IT' \parallel I_AE$ so since $IM_A = M_AI_A$, $T'M_A = M_AV'$, where $V' = T_AM_A \cap EI_A$. If we let $V = T_AT \cap EI_A$, then $(LA;PE) \stackrel{I_A}{=} (TV;M_A\infty)$, so $TM_A = M_AV \rightarrow T = T'$. Thus, T is the midpoint of the arc made by BC in ω. We have $(ST_A;BC) \stackrel{R}{=} (TE';BC) = -1$ so $SS \cap T_AT_A = X$.
We have $XT_A^2 = XB \cdot XC = XS^2$ as desired.

3.88 tutubixu

Problem 3.88 (AoPS user tutubixu9198)**.** Let ABC be a triangle with circumcircle (O). The tangents to (O) at the vertices B and C meet at a point S. Let d be the internal bisector of the angle BAC. Let the perpendicular bisector of AB, AC intersect d at M, N respectively and $P = BM \cap CN$. Prove that A, I, S are collinear where I is the incenter of triangle PMN.



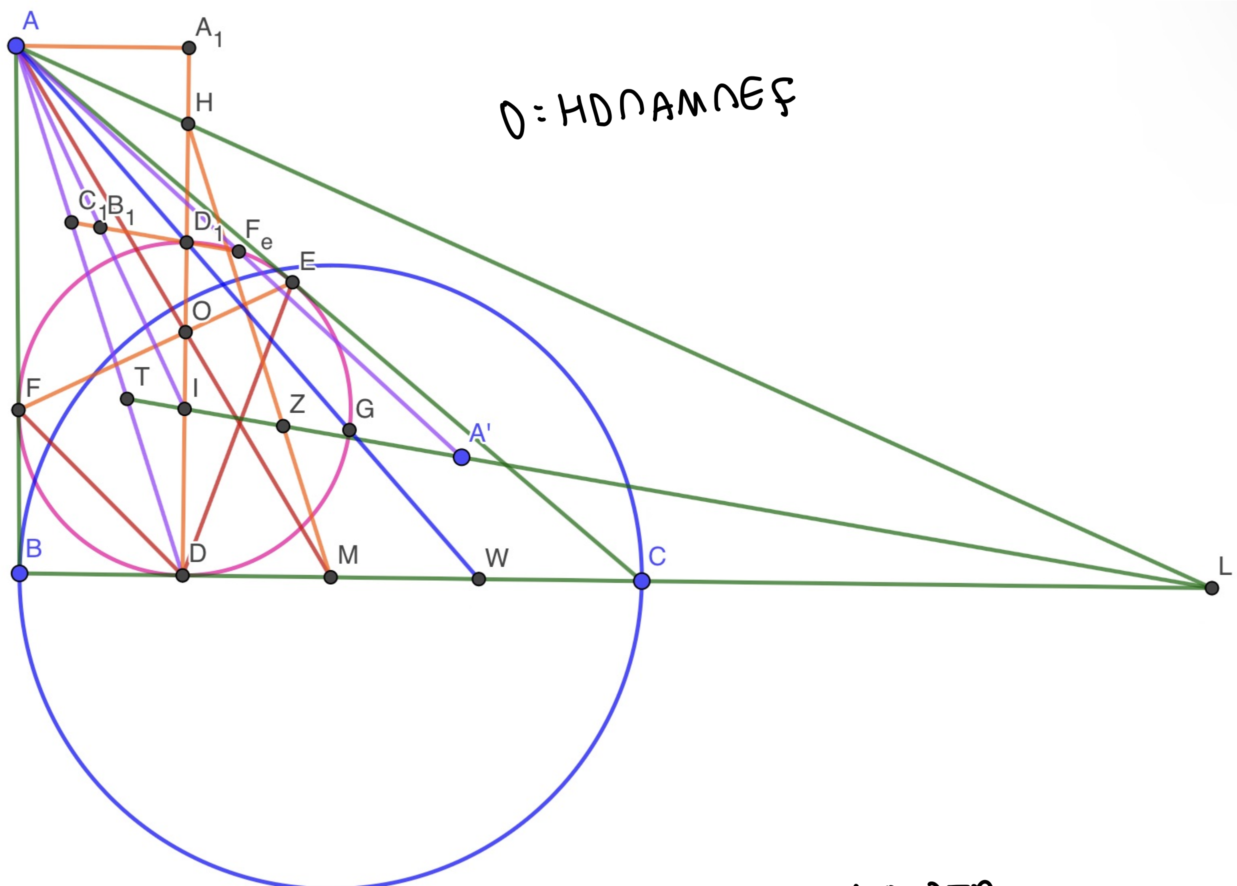


Note that $\angle PMN = \angle PNM$ and that OM, ON are the external angle bisectors of $\angle PMN, \angle PNM$, so $\angle IMO = \angle INO = 90°$. Choose $\triangle ADE$ as the reference triangle instead, and redraw the problem as follows:

Let ABC be a triangle. Let A' be the antipode of A, and let AD be the angle bisector. Then, let $F = AD \cap (A')$, $E = A'B \cap AD$. Then, let G be the antipode of A in (AEF). A $\sqrt{bc}$ inver[s]ion gives $A' \to J$, and $F \to (ABJ) \cap AD = F'$, the foot from B to AD, and E', the foot from C to AD.

Let $K = (ABC) \cap (AEF)$, or the foot from A' to AG. However, note that $(JF'ME')$, since $\triangle JBF' \sim \triangle E'DM$, so $\angle F'JM = \angle FE'M$, so $K \leftrightarrow M$, so $\overline{AKG}$ is the A-symmedian as desired

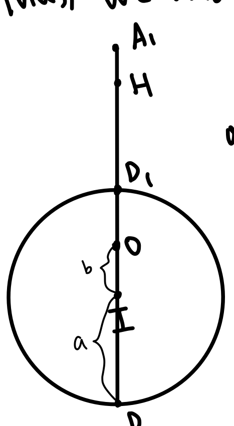
Problem 3.89 (AoPS User Supercali). In ΔABC, let F_e be the Feuerbach point, let I be the incentre, let H be orthocentre of ΔBIC and let A' be reflection of A in F_e. Then, prove that AH, IA' and BC are concurrent.



If AH intersects BC at L and $LI \cap AD = T$, we show $AT = 2TD$.

We have $\frac{TD}{HZ} = \frac{HI}{OI} \rightarrow TD = HZ \cdot \frac{HI}{ID} = AT \cdot \frac{OM}{AO} \cdot \frac{HI}{ID}$, as $\frac{AO}{OM} = \frac{AD}{HM} = \frac{AT}{HZ}$

Thus, we wish to show $\frac{OM}{AO} \cdot \frac{HI}{ID} = \frac{OD}{OA_1} \cdot \frac{HI}{ID} = 2$. Note that AA_1 is the



polar of O, so $IO \cdot IA_1 = ID^2$. We also have $(HI;OD)$, since I is the orthocenter of ΔHBC. Thus $OH = \frac{b(a+b)}{a-b}$, using cross ratios. Thus,

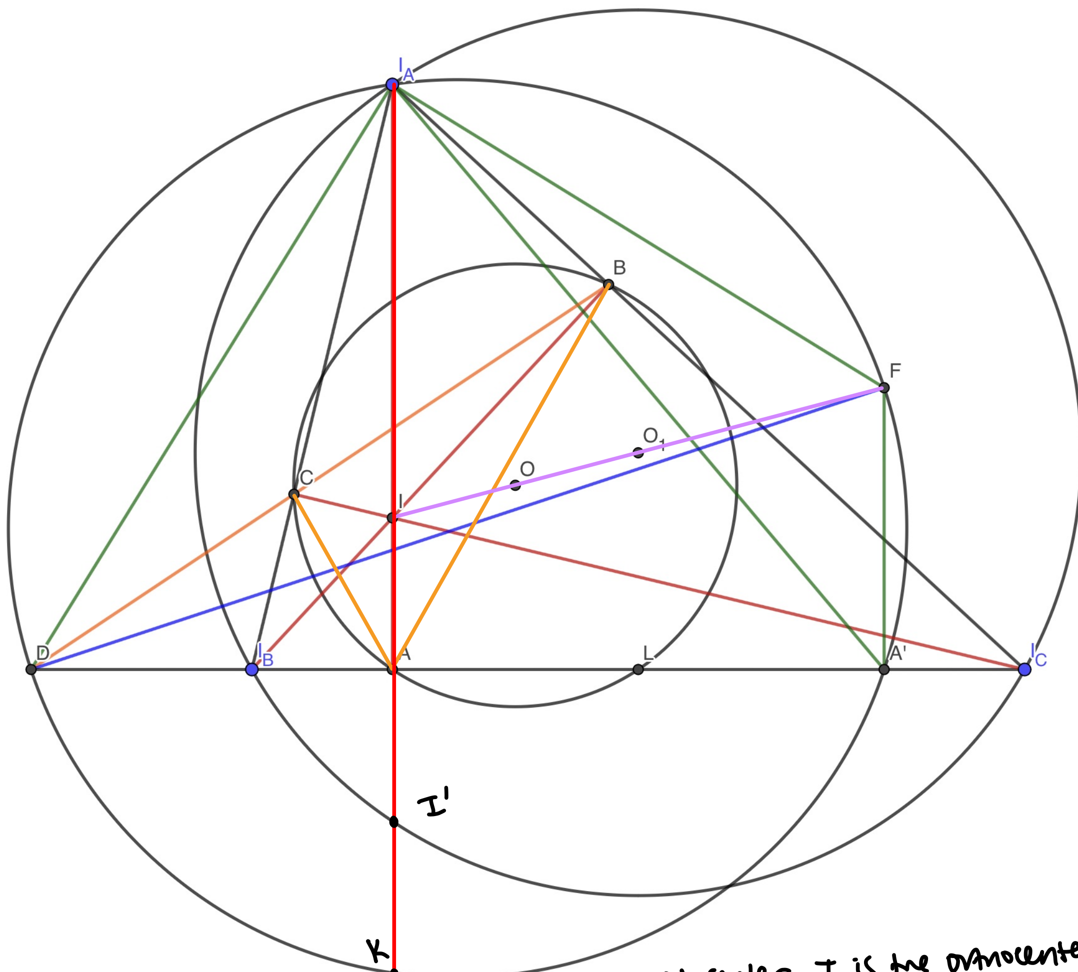
$OD = a+b$, $OA_1 = \frac{a^2}{b} - b = \frac{(a^2-b^2)}{b}$, $HI = \frac{b(a+b)}{a-b} + b = \frac{2ab}{a-b}$, $ID = a$, so we see this

is true.

To finish, notice by configs, $\overline{F_e D_1 G}$ where G is the midpoint of AT. Note $AC' = C'T$, and since $DI = ID_1$ and $DT = TG$, $F_e C_1 \parallel \overline{LIT}$, so since $AF_e = F_e A'$, $A' \in \overline{LIT}$ as desired.

3.90 Iran TST Third Round 2020 Geometry P4

Problem 3.90 (Iran TST Third Round 2020 Geometry P4). Triangle ABC is given. Let O be it's circumcenter. Let I be the center of it's incircle.The external angle bisector of A meet BC at D. And I_A is the A-excenter . The point K is chosen on the line AI such that $AK = 2AI$ and A is closer to K than I. If the segment DF is the diameter of the circumcircle of triangle DKI_A, then prove $OF = 3OI$.

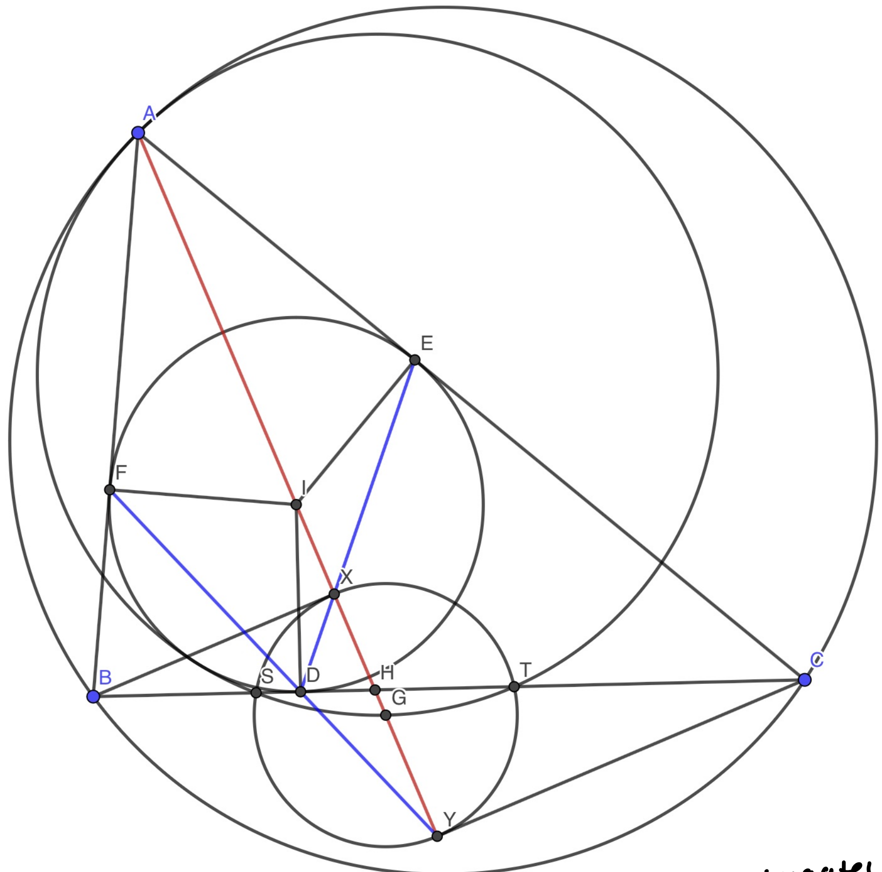


We work in terms of $I_A I_B I_C$. O is the 9 point center, I is the orthocenter.
Note that $(DA;I_BI_C) = -1$, so we can verify that $DA \cdot AL = I_BA \cdot AI_C$, so $(I'DLI_A)$ and
thus, (KDI_AA') where A' is the reflection of A over L. We can find that
$DI_B = \dfrac{ac\cos B}{b\cos C - c\cos B}$ using our harmonic bundle, and we let F be the reflection of I over
O_1, and prove $\angle DFA' = \angle DI_AA'$ by proving $\tan(DFA') = \tan(DI_AA')$. We have $AI = \cos B \cos C$,
$O_1L = \frac{1}{2}\cos A$, so $FA' = \cos A - \cos B\cos C$. This reduces to a trig bash ($\tan(\angle DI_AA')$
$= \tan(DI_AA + AI_AA')$) which is relatively clean.

Problem 3.91 (ELMO 2016/6)**.** Elmo is now learning olympiad geometry. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

a Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

b Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.

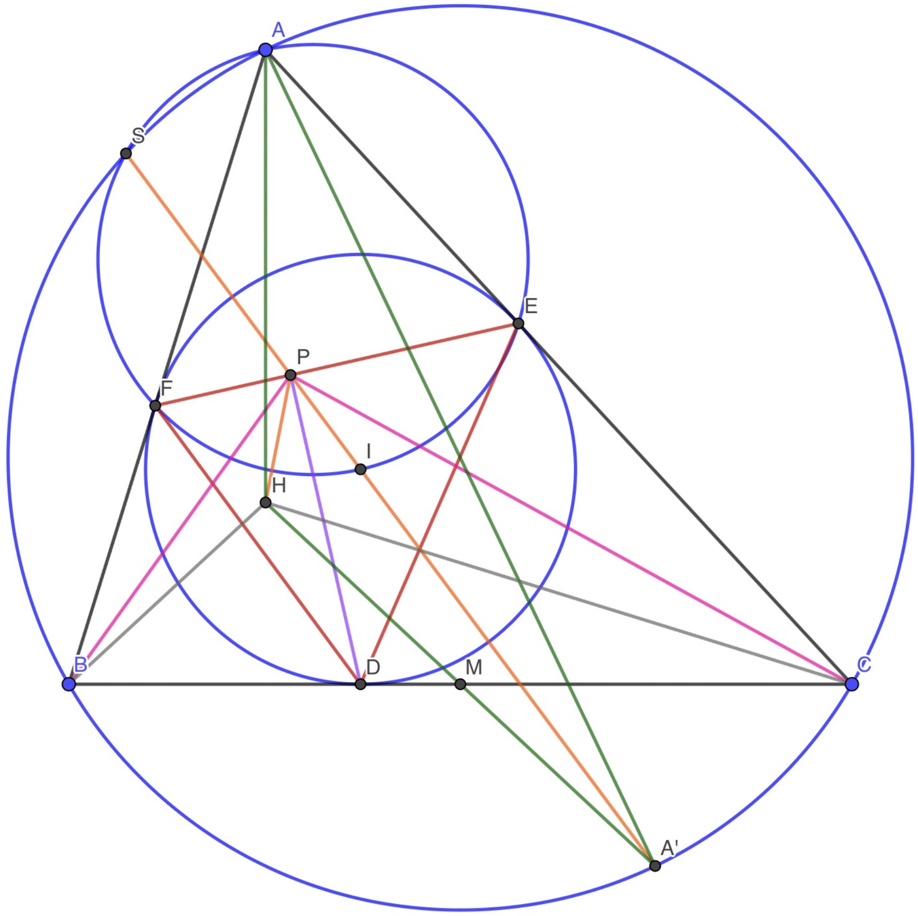


We have $\frac{BS}{SC} \cdot \frac{BT}{TC} = \frac{BX^2}{CY^2} = \frac{AB^2}{AC^2}$ so AS, AT are isogonal conjugates. Note that if $AS \cap (ABC) = S'$, $AT \cap (ABC) = T'$, then $S'T' \parallel ST$, so (AST) is homothetic to $(AS'T')$ at A (so they are tangent). Note $GS = GT$ so by fact 5, $(AGST)$. We will show (DEF) and (XY) are orthogonal, or $IX \cdot IY = r^2$. Let $R = \frac{1}{2}$, so we have $AX = c\cos\left(\frac{A}{2}\right)$, $AY = b\cos\left(\frac{A}{2}\right)$ $\rightarrow$ show $\left(c\cos\left(\frac{A}{2}\right) - \frac{r}{\sin\left(\frac{A}{2}\right)}\right)\left(b\cos\left(\frac{A}{2}\right) - \frac{r}{\sin\left(\frac{A}{2}\right)}\right) = r^2$

$\left(\frac{ac}{2} - r\right)\left(\frac{ab}{2} - r\right) = r^2\left(\sin\left(\frac{A}{2}\right)^2\right) \rightarrow r\left[\frac{ab+ac}{2}\right] - \frac{a^2bc}{4} = r^2\left(1 - \sin\left(\frac{A}{2}\right)^2\right) = r^2\left(\frac{1+\cos A}{2}\right)$. Plugging in $r = \frac{abc}{a+b+c}$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$, we see this is true. Thus, DEF inverts to itself under an (XY) inversion, and (AST) inverts to BC, as desired.

3.92 AoPS Community <HPI bisected by PD

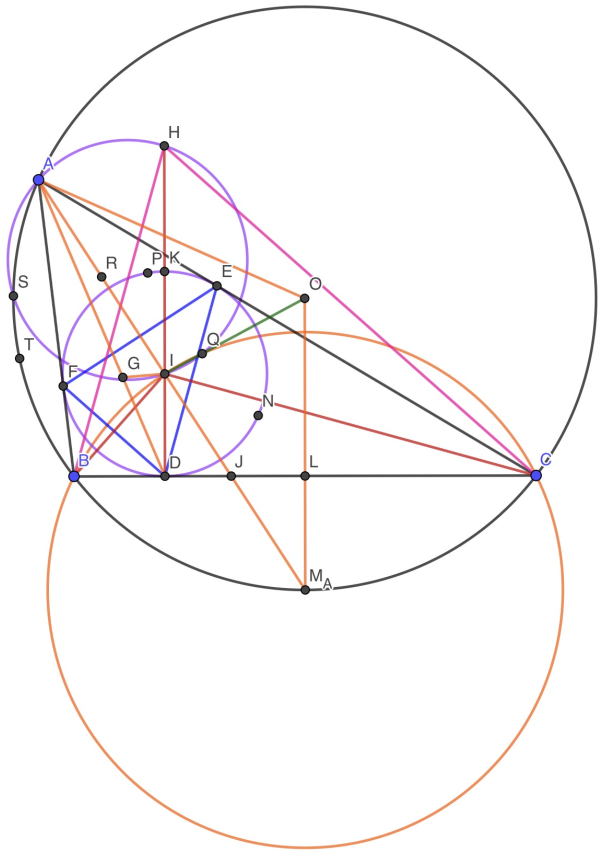
Problem 3.92 (AoPS Community). In a triangle ABC, let $\triangle DEF$ be the intouch triangle. Let P be the foot from D to $\overline{EF}$, and let I and H be the incenter and orthocenter respectively of $\triangle ABC$. Prove that $\angle HPI$ is bisected by $\overline{PD}$.

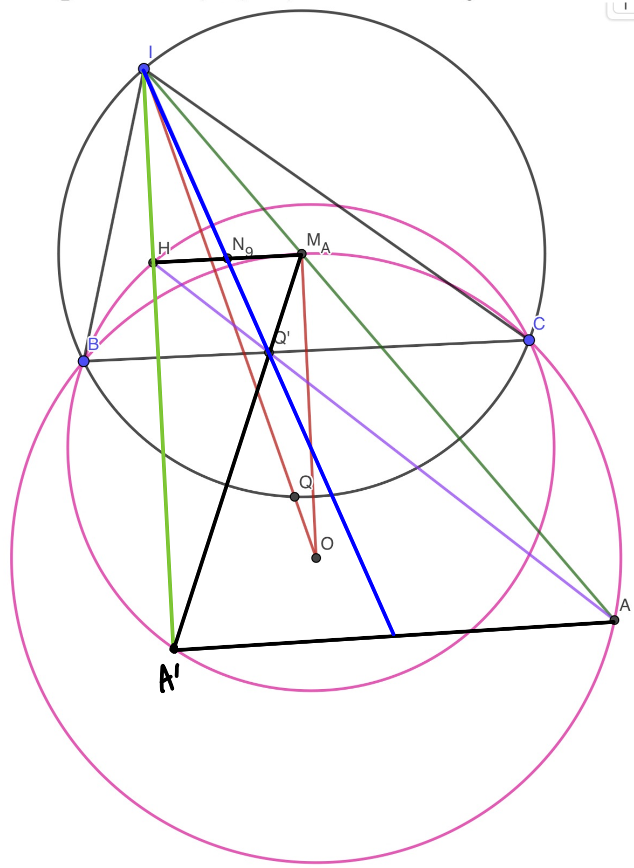


Let $X = EF \cap BC$. We have $(XD;BC)$ by 2012 ELMO SL G3, so as $\angle XPD$ is right, $\angle BPD = \angle DPC$. Additionally, note $\overline{SPIA'}$, and by isogonality lemma, since $BHCA'$ is a parallelogram, $\angle BPH = \angle CPA'$. Thus, PD bisects $\angle HPI$.

3.93 2020 Taiwan TST Round 2 Mock IMO Day 6

Problem 3.93 (2020 Taiwan TST Round 2 Mock IMO Day 6)**.** Let I, O, ω, Ω be the incenter, circumcenter, the incircle, and the circumcircle, respectively, of a scalene triangle ABC. The incircle ω is tangent to side BC at point D. Let S be the point on the circumcircle Ω such that AS, OI, BC are concurrent. Let H be the orthocenter of triangle BIC. Point T lies on Ω such that $\angle ATI$ is a right angle. Prove that the points D, T, H, S are concyclic.





so $\angle STD = \angle STM_A = \angle SAI$. Thus, we wish

Note that T is the Sharky-Devil point, Then, by Radical axis, $(AISQ)$.

to prove $\angle SAI = \angle SHI$, or $(AISH)$. Let $OI \cap (BIC) = Q$. Note that $\triangle AIB \sim$

Thus, we prove $(AIQH)$. Now, preform a $\sqrt{bc}$ inversion about BIC.

$\triangle CIH$, so $AI \cdot IH = IB \cdot IC$, so $A \leftrightarrow H$. We see that $M_A \leftrightarrow A'$, the reflection of I over

BC. Note that BOC and BHC swap, so if O_A is the center of BHC, then IO_A

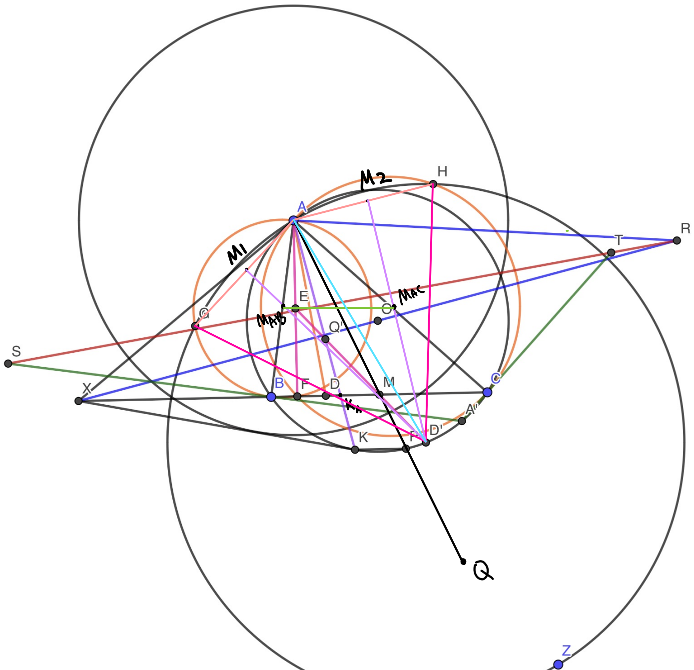
and IO are isogonal conjugates. Thus, $Q \leftrightarrow IN_9 \cap BC = Q'$. We wish to show

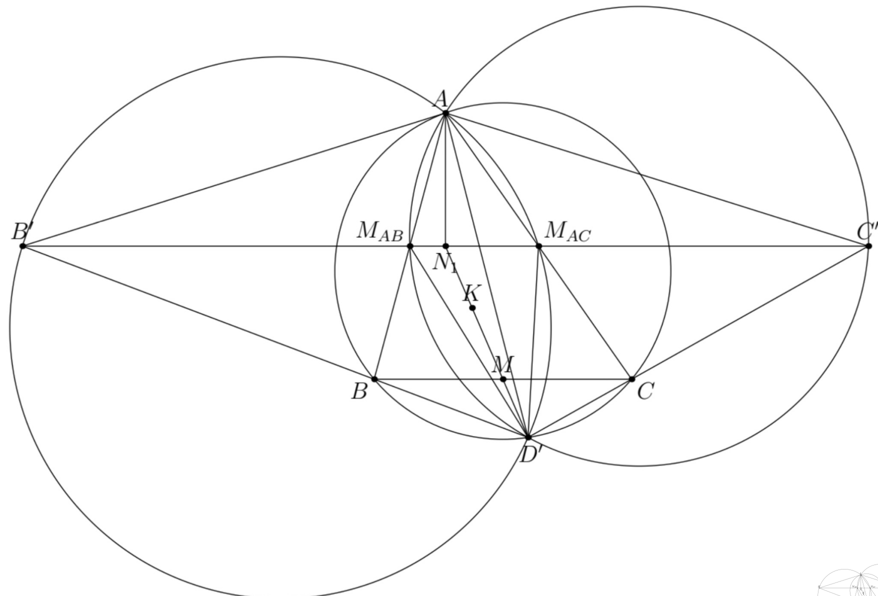
$HQ'A$. Note that M_AA' is the reflection of IN_9 over BC, and $IH \cdot IA = IM_A \cdot IA' \rightarrow$

$HM_A \parallel A'A$ $\rightarrow$ by Cevas, $A'M_A \cap HA \cap IN_9$, and thus, intersect on BC, as desired.

3.94 buratinogigle ASDT rhombus

Problem 3.94 (Posted by AoPS user buratinogigle)**.** Let ABC be a triangle inscribed in circle (O). Tangent at A meets BC at X. Median AM meets (O) again at P. Q lies on ray MP such that $PQ = 2PM$. Choose the points R on line OX and D on segment BC such that $RD = RA = RQ$. Let S, T be on perpendicular bisector of AD such that $BS \perp BA$ and $CT \perp CA$. Prove that $ASDT$ is a rhombus.

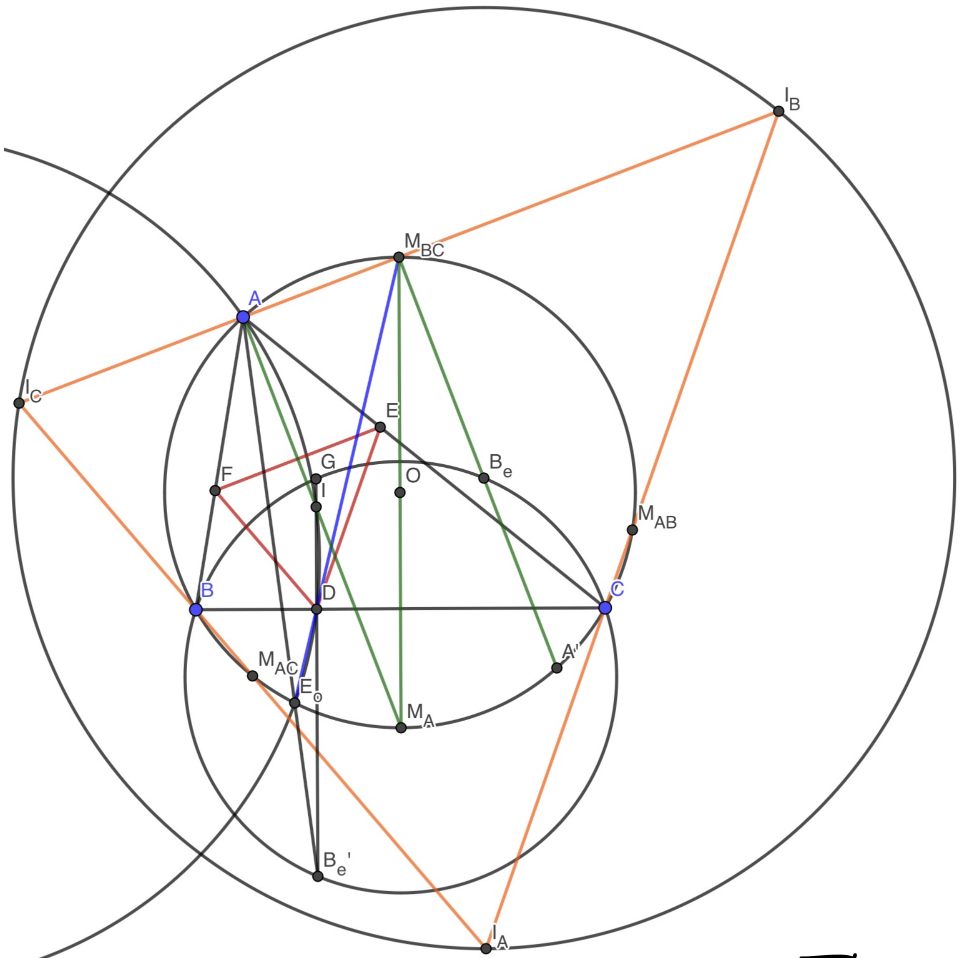




We show Q inverts to Q', the symmedian point. We have $K_A \leftrightarrow P$ (under $\sqrt{bc}$ invert), and $\overline{AK_A} \cdot \overline{a^2+b^2+c^2}$ and $AK_A \cdot AP = bc$, so we show $AK_A \cdot AP = AQ' \cdot AQ$ or $\frac{AQ'}{AK_A} = \frac{AP}{AQ}$, which is relatively simple via Stewarts + POAP. We have $K \in (ADQ)$ since $R \in OX$, so $(AKDQ) \leftrightarrow Q'M$, the Schwatt line of ABC. Let E be the midpoint of altitude AD. Then, $D' \leftrightarrow \overline{Q'EM} \cap (ABC)$. We see that BS, TC go to $(ABF), (CFA)$, and the perpendicular bisector of AD goes to a circle centered at D' through A. Now, we just need to show $AD'M_1 = \angle AD'M_2$ or $\angle AD'M_{AB} = \angle AD'M_{AC}$.

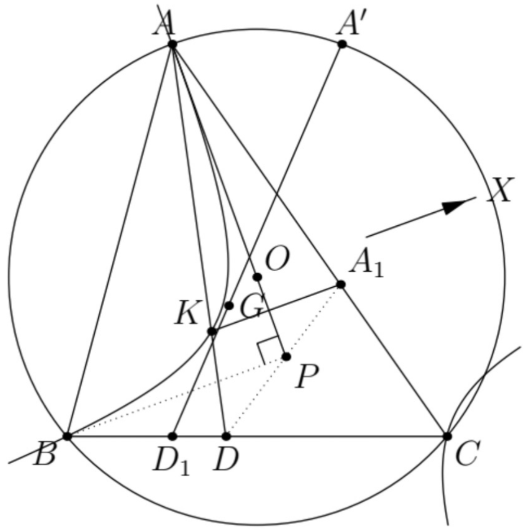
Let $D'B$ and $D'C$ intersect $M_{AB}M_{AC}$ at B', C'. We have $\angle D'AM_{AC} = \angle D'BC = \angle DB'C'$ so $(AM_AD'B')$, $(AM_{AB}D'C')$. We also have $D'MN_1$, so N_1 is the midpoint of $B'C'$, and since $AN_1 \perp B'C'$, $\angle AB'M_{AC} = \angle AC'M_B = \angle AD'M_{AB} = \angle AD'M_{AC}$, as desired.

Problem 3.95 (Encyclopedia of Triangle Centers)**.** In $\triangle ABC$, let Be be the Bevan Point, or the reflection of the incenter over the circumcenter. If Be' is the antipode of Be on $(BCBe)$, then prove that $\overline{ABe'}$ goes through the homothety center between the intouch and excentral triangle.



Since $\overline{M_A O M_{BC}}$, $\overline{AIM}$, $\overline{AOA'}$, $\overline{IOBe}$, then $\overline{A' Be M_{BC}}$, because the two lines are each others reflection over O. let $G = ID \cap (BCBe)$. Note $GBeCB$ is an isosceles trapezoid, since the feet from Be and G are symmetric wrt BC. Thus, $Be' \in ID$ ($Be' = \overline{GID} \cap (BeBC)$). Note that $M_{BC}A'IM_A$ is a parallelogram, so $M_AI = M_{BC}Be = M_{BC}G$, so $\angle M_{BC}GI = \angle GIM_A = \angle AID = \angle AE_oD$. Thus, $AE_o \cap ID = X$ has (XE_oGM_{BC}) is cyclic, so $E_oD \cdot DM = BD \cdot DC = GD \cdot DX$, so $X \in (BCBe)$, so $X = Be'$, as desired (AE_o passes through the altitude from D to EF and A, so the center of homothety lies on it).

Problem 3.96 (Myself (i3435)). Let ABC be a triangle with circumcenter O and let the A-symmedian intersect $\overline{BC}$ at D. Let P be the foot from B to $\overline{AO}$ and let Q be the foot from C to $\overline{AO}$. Let A_1 be the intersection of $\overline{DP}$ and $\overline{AC}$ and let A_2 be the intersection of $\overline{DQ}$ and $\overline{AB}$. Define B_1, B_2, C_1, C_2 similarly. Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ are all concyclic.



Let A' be the reflection of A over the perpendicular bisector of BC, let G be the centroid of $\triangle ABC$, and let D_1 be the foot from A to $\overline{BC}$. Then $A' - G - D_1$.

Isogonally conjugate this line. A' goes to X, the point at infinity of the A-antiparallels of $\triangle ABC$, G goes to K, the symmedian point of $\triangle ABC$, and $\overline{AD_1}$ is isogonal to $\overline{AO}$, so the conic through A, B, C, K, X is tangent to $\overline{AO}$.

Pascal's on $AAKXBC$ gives that A_1 is the intersection of the A-antiparallel of $\triangle ABC$ through K and AC. We now finish by Second Lemoine Circle.

However, using a bunch of projections, there is another method, which the hints hopefully led you towards.

Let A_1' be the intersection of the A-antiparallel through K and $\overline{AC}$. We want to show that $D - P - A_1'$. Let K' be the intersection of $\overline{KA_1}$ and $\overline{BC}$, let $K_A = \overline{AK} \cap (ABC)$, and let Y_A be the A-Why Point. We first try to show that $K' - Y_A - K_A$

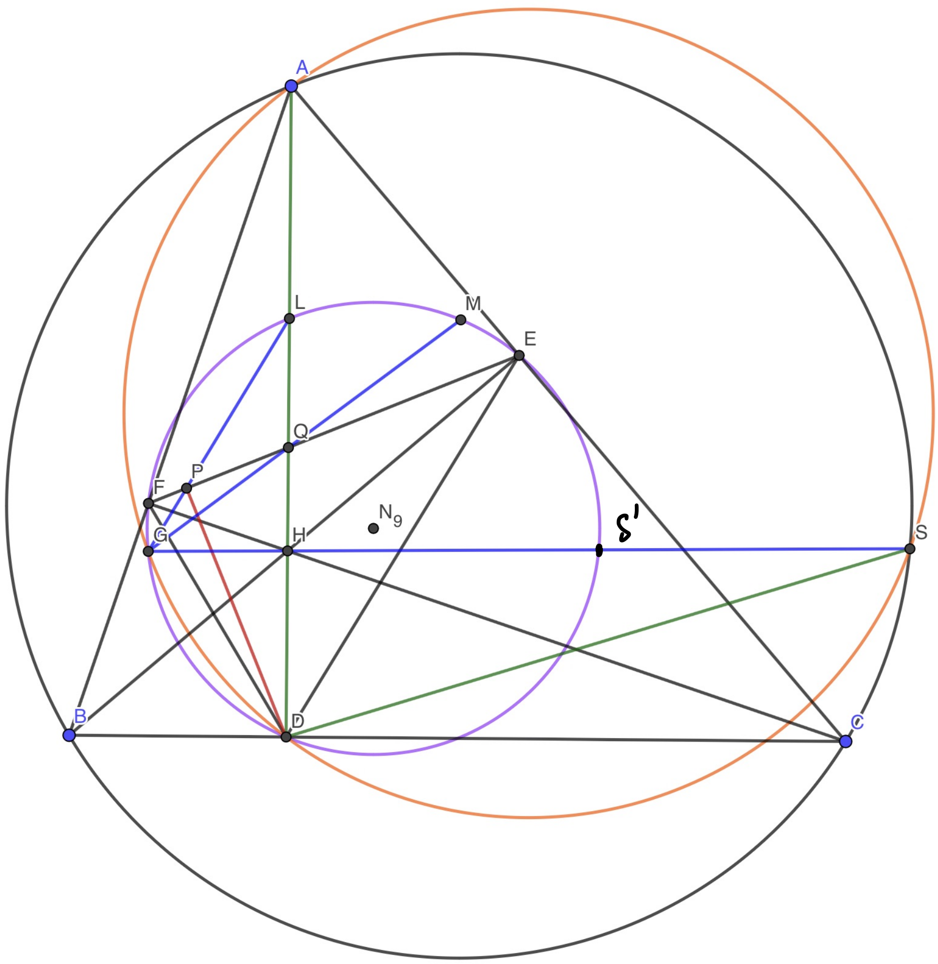
Let T be the intersection of the A-tangent to (ABC) and $\overline{BC}$. Then let N be the midpoint of AT and let A^* be the antipode of A on (ABC). $(A, A^*; Y_A, \overline{Y_A P_{\infty AT}}) - 1 = (A, T; N, P_{\infty AT}) \overset{Y_A}{=} (A, A^*; \overline{NY_A} \cap (ABC), \overline{Y_A P_{\infty AT}} \cap (ABC))$, so $\overline{NY_A}$ is tangent to (ABC) at Y_A. Thus projecting $-1 = (Y_A, A^*; A, K_A)$ through Y_A to $\overline{AT}$ gives that $\overline{K_A Y_A}$ trisects AT.

Let $T' = \overline{TK_A} \cap \overline{K'K_A}$. Then by homothety it suffices to show that K' trisects KT'. $(K, T'; K', P_\infty) \overset{T}{=} (K, K_A; D, A)$. Let $K_C = \overline{CK} \cap (ABC)$ and let M be the midpoint of BC. Then $(K, K_A; D, A) \overset{C}{=} (K_C, K_A; B, A) = \frac{\frac{K_C B}{K_C A}}{\frac{K_A B}{K_A A}} = \frac{\frac{AC}{BC}}{\frac{MC}{AC}} = 2$ as desired.

Now onto the main problem. Let $O' = \overline{AO} \cap \overline{BC}$ and let $Q_A' = \overline{K_A O'} \cap (ABC)$. Then $-1 = (A, K_A; B, C) \overset{O'}{=} (A^*, Q_A'; B, C) \overset{Y_A}{=} (T, \overline{Y_A Q_A'} \cap \overline{BC}; B, C)$ so $Y_A - D - Q_A'$. $(A, P; \overline{K'K} \cap \overline{AO}, O') \overset{P_\infty AT}{=} (T, B; K', O') \overset{K_A}{=} (K_A, B; Y_A, Q_A') \overset{D}{=} (A, C; Q_A', Y_A) \overset{K_A}{=} (D, C; O', K') = (C, D; K', O') \overset{A_1'}{=} (A, \overline{A_1' D} \cap \overline{AO}; \overline{K'K} \cap \overline{AO}, O')$ as desired.

3.97 India TST SH, MQ, PL concurrent

Problem 3.97 (India TST???). Let ABC be a triangle with circumcircle Γ and altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ meeting at H. Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Γ such that $DS = DA$. Line $\overline{AD}$ meets $\overline{EF}$ at Q, and meets ω at $L \neq D$. Point M is chosen such that DM is a diameter of ω. Point P lies on $\overline{EF}$ with $\overline{DP} \perp \overline{EF}$. Prove that lines SH, MQ, PL are concurrent.

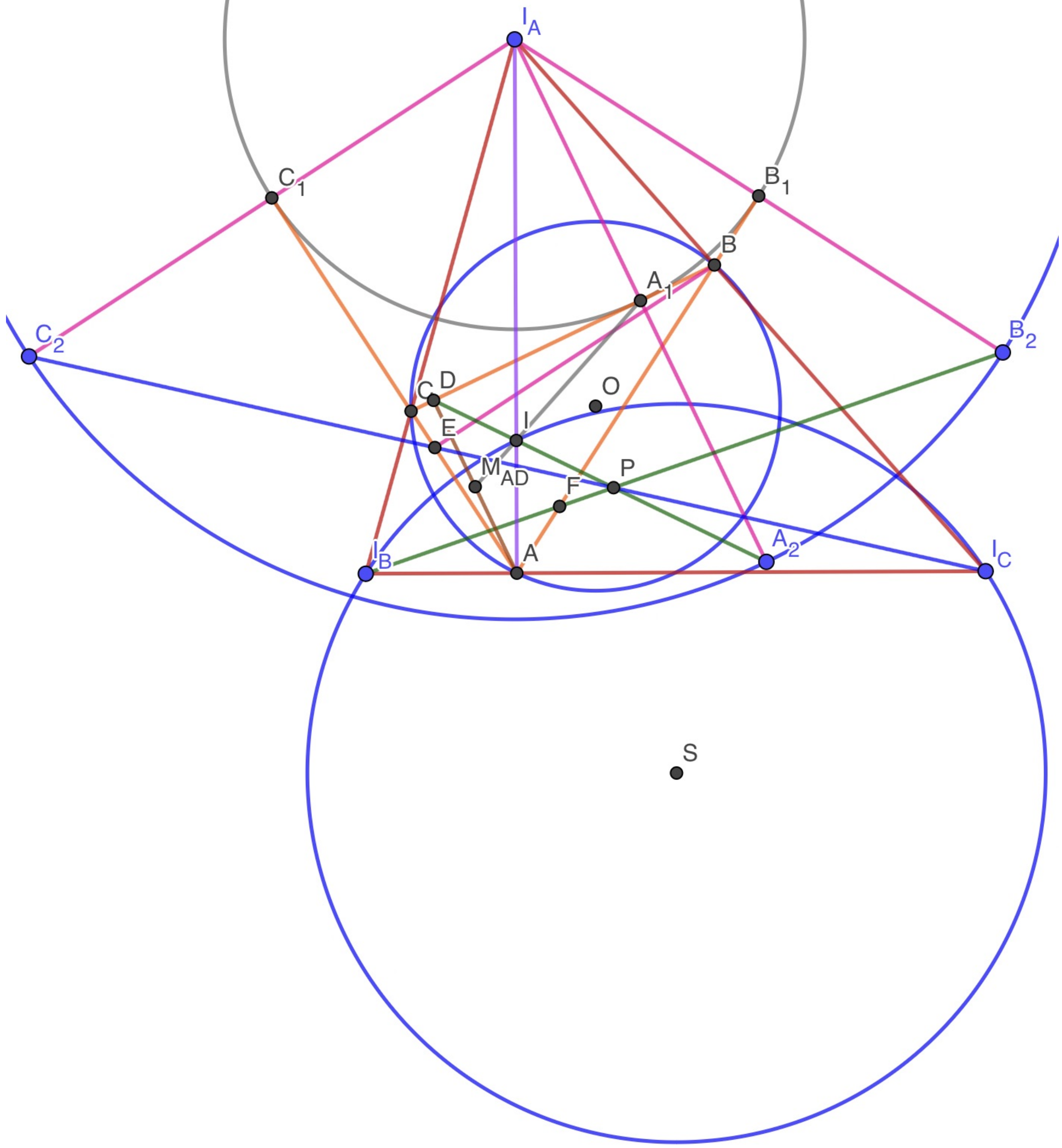


Let $G = LP \cap MQ$. Note that by a $\sqrt{bc}$ invert, $DP \cdot DM = DQ \cdot DL \to \frac{DP}{DL} = \frac{DQ}{DM}$ and since $\angle DPL = \angle LDM$, we have $\angle PLD = \angle QMD$. Thus, LP and QM intersect on (DEF). Now, note $(AH;QD) = -1$, because $\angle QEH = \angle HED$ and $\angle AEH = 90°$. Thus, $\angle AGQ = \angle QGH$. Let $GH \cap (DAG)$ be S, so $\angle SGD = \angle AGD \to DA = DS$. Now, since $GH \cdot HS' = DH \cdot HL$ and $GH \cdot HS = DH \cdot HA$, we have $HS = 2HS' \to S$ on (ABC) as desired.

3.98 USAMO 2016/3

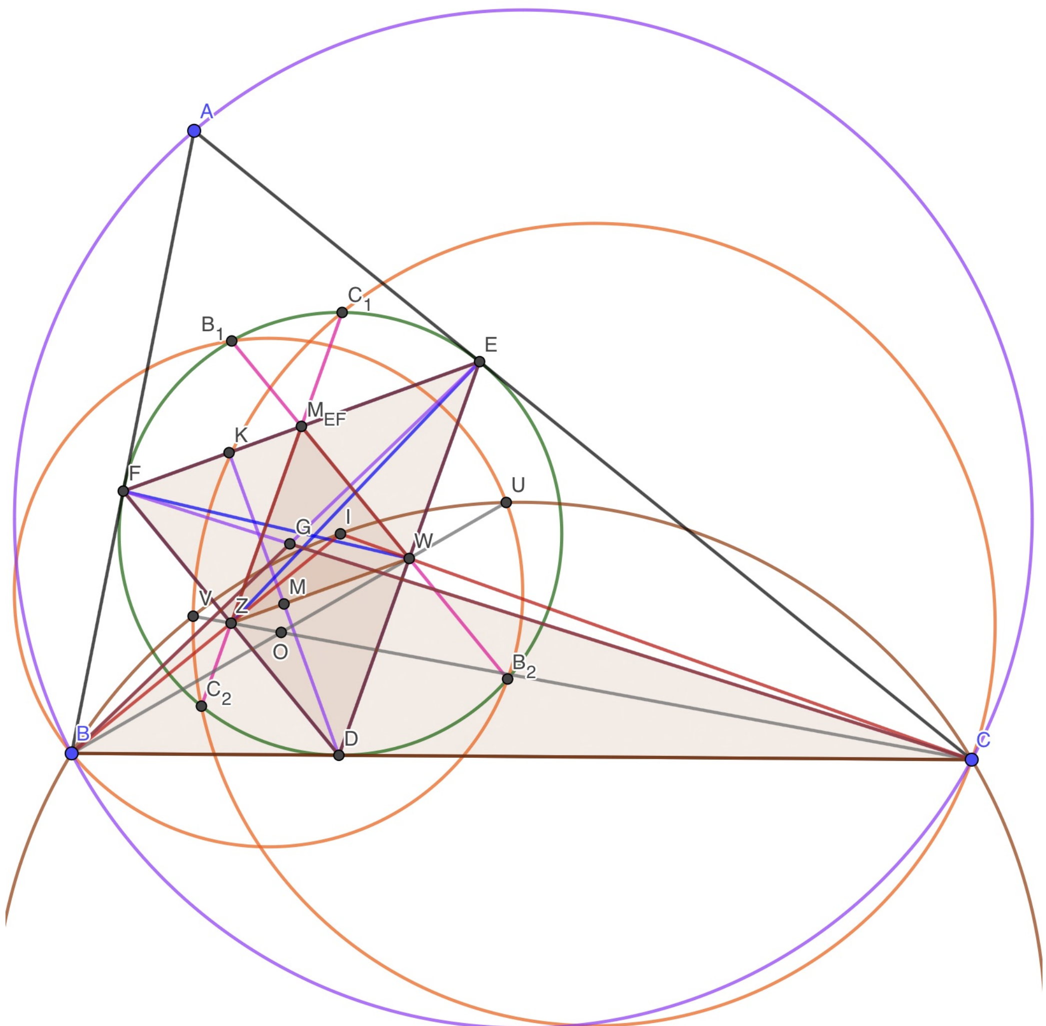
Problem 3.98 (USAMO 2016/3). Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_BF}$ and $\overleftrightarrow{I_CE}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.



We have by Theorem 2.72 that $SI_AO \perp YZ$, we will show P lies on $\overline{SI_A}$. Consider the excircle of ABC and let the tangency points be A_1, B_1, C_1. Now, let A_2, B_2, C_2 be the reflections of I_A over the tangency points. We show that P is the center of homothety between $(A_2B_2C_2)$ and (II_BI_C), which proves the desired. Notice that $\overline{M_{AD}IA_1}$ by midpoints of altitudes, and since $\overline{A_1II_A}$, we have $\overline{DIA_2}$. We know that $BE \cdot b = 2K$, and $C_1I_A = \frac{s}{s-a}r = \frac{s}{s-a} \cdot \frac{K}{s} = \frac{K}{s-a}$. Thus, $BE = \frac{2K}{b}$, $C_2I_A = \frac{2K}{s-a}$. We have $I_AB = \frac{s-c}{\cos(90-\frac{B}{2})}$ and $I_CB = \frac{s-a}{\cos(90-\frac{B}{2})}$, so $\frac{I_CB}{I_CI_A} = \frac{s-a}{b} = \frac{BE}{C_2I_A}$ so since $BE \parallel C_2I_A$, we have $\overline{C_2EI_C}$, and symmetrically, $\overline{B_2FI_B}$. However, note that $B_2C_2 \parallel I_BI_C$, so P is the center of homothety mapping $(A_2B_2C_2)$ to (II_BI_C) as desired (we also have $A_2B_2 \parallel II_B$, $A_2C_2 \parallel II_B$ and we're done by homothety).

Problem 3.99 (USA TSTST 2016/6). Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.



Radical axis of (AFIE), (AIB), (DEF) gives $\overline{B_1B_2M_{EF}}$. We see that $B_1B_2 \perp IM_B$ (M_B = midpoint of arc), so $B_1B_2 \parallel DF$, and thus, if we let M_{EF}, W, Z be the medial triangle then $\overline{B_1WB_2}$, $\overline{C_1ZC_2}$. We have that by Radical axis on (BB_1B_2), (BIC), (CC_1C_2C) that the radical center is on BW and CZ so $BW \cap CZ$ is on our radical axis. ($WB_2 \cdot WB_1 = VW \cdot WB$, and $\overline{VWB}$ by symmetry from before).

By Cevian Bundles on DEF, GBC, $M_{EF}ZW$ (DM_{EF}, EZ, FW concur and so do DG, FB, EC), so $DM_{EF}G \cap ZC \cap WB$, is on the radical axis. Thus $\overline{GM_{EF}}$ is our radical axis → Schwatz line, so M is also on the radical axis.