Barycentric Coordinates

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1 Barycentric Coordinates: Definition

1.1 Definition

Consider placing masses of 2, 3, and 7 at vertices A, B, and C of a non-degenerate triangle. Letting D be the point of BC so that $BD/DC = 7/3$, we see that lever BC balances at fulcrum D, meaning the triangle ABC balances along cevian AD. Likewise, it balances along cevians BE and CF where $CE/EA = 2/7$ and $AF/FB = 3/2$. With a bit of physical intuition, these balancing lines should all pass through the center of mass of the system, and we define this point of concurrency P (the center of mass) to have barycentric coordinates $(2:3:7)$ with respect to triangle ABC. More generally,

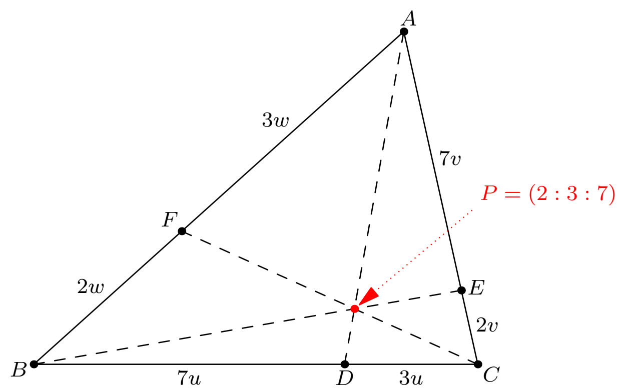

Figure 1: Barycentric coordinates definition

Barycentric Coordinates Definition. The point with coordinates $(x : y : z)$ is the center of mass of the system when masses of x, y, and z (which may be zero or even negative!) are placed at the vertices A, B and C of the reference triangle.

Equivalently, as explained above, we may say that

Definition′. The point $P = (x : y : z)$ is the point whose traces D, E, F satisfy $\frac{BD}{DC} = \frac{z}{y}$, $\frac{CE}{EA} = \frac{x}{z}$, and $\frac{AF}{FB} = \frac{y}{x}$, using signed ratios.

Using this formulation, we can give a quick proof of Ceva's theorem, which states that cevians AD, BE, and CF are concurrent if and only if the three ratios $\frac{BD}{DC} = r_a$, $\frac{CE}{EA} = r_b$, and $\frac{AF}{FB} = r_c$ have product 1. The point $P = (r_a r_b : 1 : r_a)$ can be seen to have traces D, E, and F', where $\frac{AF'}{F'B} = \frac{1}{r_a r_b}$. Cevians AD, BE, CF are concurrent if and only if $F = F'$, i.e. if and only if $r_c = \frac{1}{r_a r_b}$, QED. (The barycentric coordinates of P can be written more symmetrically as $P = (\sqrt[3]{r_a r_b^2} : \sqrt[3]{r_b r_c^2} : \sqrt[3]{r_c r_a^2})$.)

There is one more description of barycentric coordinates that is often much more useful that the previous two:

Definition″. The point $P = (x : y : z)$ is the point in the plane of triangle ABC so that the three (signed) areas $[PBC]$, $[PCA]$, and $[PAB]$ are in the ratio $x : y : z$ (which explains the chosen notation).

Indeed, we have

$$\frac{[PAB]}{[PCA]} = \frac{[PAD]}{[PDC]} = \frac{PD}{DC} = \frac{z}{y},$$

and likewise for $[PBC]$.

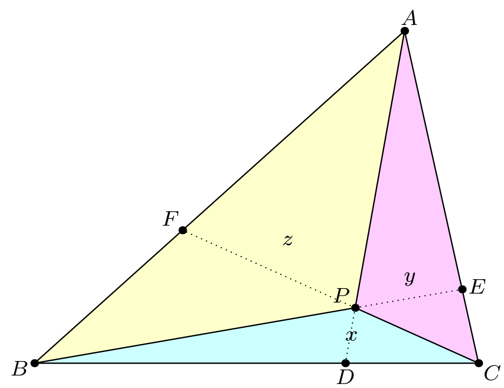

Figure 2: Barycentric coordinates through areas

Note that these coordinates are *homogeneous*. That is, the points $(x : y : z)$ and $(kx : ky : kz)$ for any nonzero constant k are the same. For this reason, it is often desirable to normalize so that the coordinates have sum 1. When P is in normalized form, we will use the following notation:

$$\underline{P} = (x : y : z) \text{ normalized} = \frac{1}{x+y+z}(x, y, z) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right).$$

(Note the commas instead of colons in the last two representations.) It can be verified that such a scaling is not possible, i.e. $x + y + z = 0$, if and only if AD, BE, CF are *parallel*, which means the point of concurrency P corresponds to a point at infinity in the projective plane.

1.2 Examples

Let's calculate the normalized barycentric coordinates for a few common triangle centers.

1.2.1 Centroid

The centroid G is the point of concurrency of the medians of $\triangle ABC$, i.e. the point of concurrency when D, E, F are taken as midpoints. As $\frac{BD}{DC} = \frac{CE}{EA} = \frac{AF}{FB} = 1$, the second definition above shows that

$$\underline{G} = (1 : 1 : 1) \text{ normalized} = \frac{1}{3}(1, 1, 1).$$

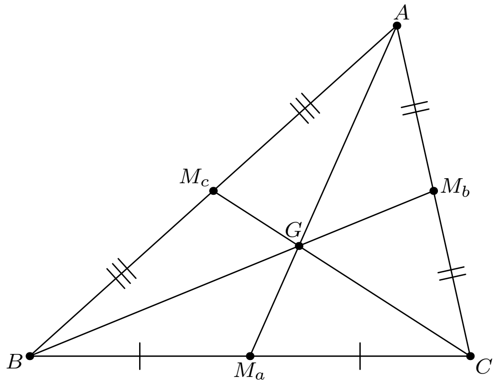

Figure 3: Centroid

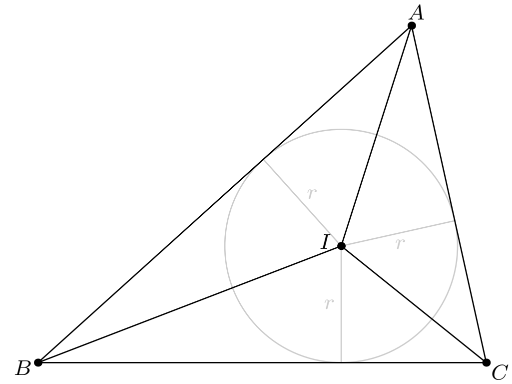

Figure 4: Incenter

1.2.2 Incenter

With I as the incenter of $\triangle ABC$ and r the inradius, we see that $[IBC] = \frac{1}{2}ar$, so we can write $I = (\frac{1}{2}ar : \frac{1}{2}br : \frac{1}{2}cr)$, or after normalizing,

$$\underline{I} = \frac{1}{2s}(a, b, c)$$

(where $s = \frac{a+b+c}{2}$ is the semiperimeter).

1.2.3 Spieker Center

If the sides of triangle ABC are traced with a uniform wire, the Spieker center S_p is the center of mass of the wire. As the wire BC has mass proportional to a and has center of mass M_a, the Spieker center it is the center of mass when weights of a, b, and c are placed at the midpoints M_a, M_b, M_c of BC, CA, AB respectively. This implies two important facts about this center. First, basing our barycentric coordinate system around the medial triangle $M_aM_bM_c$ gives S_p the coordinates $(a : b : c)$, and since $M_aM_bM_c \sim ABC$, we find that S_p is the incenter of the medial triangle. Secondly, the center of mass of the system with weights a, b, c at M_a, M_b, M_c is the same as the center of mass of the system which has weights of $\frac{b+c}{2}$, $\frac{c+a}{2}$, $\frac{a+b}{2}$ at A, B, C respectively, so the barycentric coordinates of S_p with respect to ABC are

$$\underline{S_p} = \frac{1}{4s}(b + c, c + a, a + b).$$

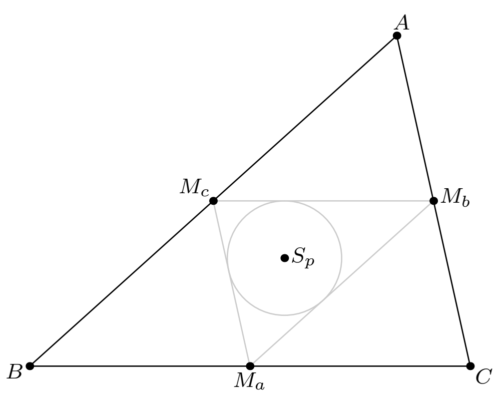

Figure 5: Speiker Center

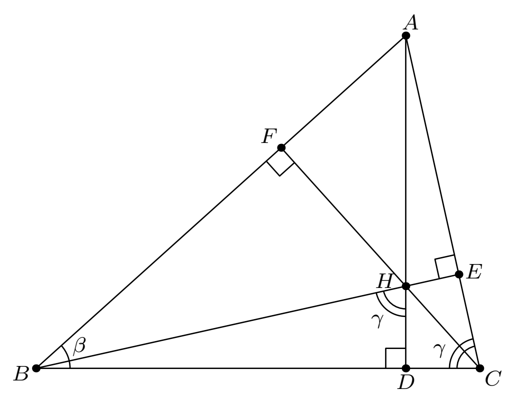

Figure 6: Orthocenter

1.2.4 Orthocenter

In Figure 6, notice that $\angle BHD = 90 - \angle DBH = \angle ECB = \gamma$, so

$$[HBC] = \frac{1}{2}aHD = \frac{1}{2}aAD\tan\gamma = \frac{1}{2}ac\cos\beta\tan\gamma = \frac{c}{2\sin\gamma}a\cos\beta\cos\gamma = Ra\cos\beta\cos\gamma.$$

Thus,

$$\underline{H} = (a\cos\beta\cos\gamma : b\cos\gamma\cos\alpha : c\cos\alpha\cos\beta) \text{ normalized.}$$

By the Law of Cosines, $a\cos\beta\cos\gamma = \frac{1}{4abc}(c^2+a^2-b^2)(a^2+b^2-c^2)$, so we also have

$$\underline{H} = \big((c^2+a^2-b^2)(a^2+b^2-c^2) : (a^2+b^2-c^2)(b^2+c^2-a^2) : (b^2+c^2-a^2)(c^2+a^2-b^2)\big) \text{ normalized.}$$

But what are the normalizing factors? The sum of the coordinates in the second expression can be expanded to

$$\sum_{\text{cyc}}(c^2+a^2-b^2)(a^2+b^2-c^2) = 2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4$$
$$= (a+b+c)(a+b-c)(b+c-a)(c+a-b) = 16(s)(s-a)(s-b)(s-c) = 16[ABC]^2 = 16r^2s^2,$$

so we find the two equivalent normalized forms

$$\underline{H} = \frac{1}{16r^2s^2}\big((c^2+a^2-b^2)(a^2+b^2-c^2),-,-\big) = \frac{R}{rs}(a\cos\beta\cos\gamma,-,-)$$

(we used the well-known formula $abc = 4srR$ to obtain the second expression from the first).

2 Collinearity

Barycentric coordinates can be used to detect when three points are on a line. Suppose we have two normalized points $\underline{P} = (x_1, y_1, z_1)$ and $\underline{Q} = (x_2, y_2, z_2)$, which means $x_1 = [PBC]/[ABC]$, etc. Let P_x, P_y, P_z be the projections from P to lines BC, CA, AB respectively, and likewise for Q. Since $[PBC] = \frac{1}{2}\cdot PP_x \cdot a$, we have $PP_x = 2[PBC]/a = \frac{2[ABC]}{a}\cdot x_1$, and likewise $QQ_x = \frac{2[ABC]}{a}\cdot x_2$.

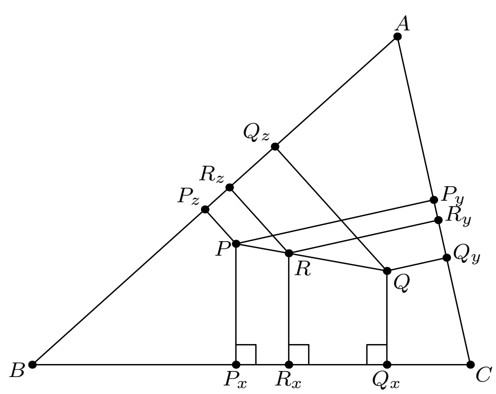

Figure 7: Collinearity condition

Consider the point $R = (x_3, y_3, z_3)$ on line BC so that $PR/PQ = k$ for some real number k, and let R_x be its projection onto BC. By right trapezoid PP_xQ_xQ, it can be seen that $RR_x = (1-k)\cdot PP_x + (k)\cdot QQ_x$, i.e. that

$x_3 = (1-k) \cdot x_1 + (k) \cdot x_2$. As the same holds for y_3 and z_3, we find that

$$\underline{R} = (1-k)\underline{P} + (k)\underline{Q},$$

so $\underline{R}$ is simply a weighted average of $\underline{P}$ and $\underline{Q}$. When homogeneity is taken into account (i.e. without assuming P and Q have been normalized), the criterion for collinearity becomes the following:

Collinearity Condition. Points $P = (x_1 : y_1 : z_1)$, $Q = (x_2 : y_2 : z_2)$, and $R = (x_3 : y_3 : z_3)$ are collinear if and only if the vectors (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) are linearly dependent, i.e. the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

is zero. Furthermore, if $\underline{P}$, $\underline{Q}$, and $\underline{R}$ are normalized, the value of k for which $\underline{R} = (1-k)\underline{P} + (k)\underline{Q}$ corresponds to the ratio $k = PR/PQ$.

2.1 Examples

2.1.1 Euler Line

The points H, G, and O lie on a line in that order with $OH = 3OG$. To see why, we need the normalized barycentric coordinates for O (the other two were calculated above in sections 1.2.1 and 1.2.4). The area of OBC is $\frac{1}{2}R^2 \sin 2\alpha = R^2 \sin\alpha\cos\alpha = \frac{1}{2}Ra\cos\alpha$, so (with a bit of work) we obtain the following equivalent normalized expressions:

$$\underline{O} = \frac{R}{2rs}\left(a\cos\alpha, -, -\right) = \frac{1}{4s^2r^2}\left(a^2(b^2+c^2-a^2), -, -\right).$$

We simply need to illustrate that $3\underline{G} = \underline{H} + 2\underline{O}$, i.e. that

$$1 = \frac{Ra}{rs} \cdot \cos\beta\cos\gamma + \frac{Ra}{rs} \cdot \cos\alpha.$$

Notice that $\cos\alpha + \cos\beta\cos\gamma = \cos\alpha + \cos(\beta+\gamma) + \sin\beta\sin\gamma = \frac{bc}{4R^2}$, so the right side of the previous equation equals $\frac{abc}{4srR} = 1$, as needed.

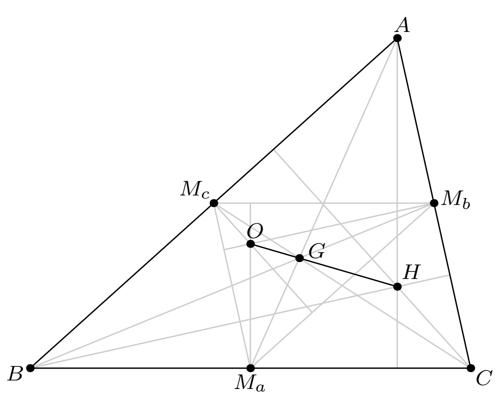

Figure 8: Euler line

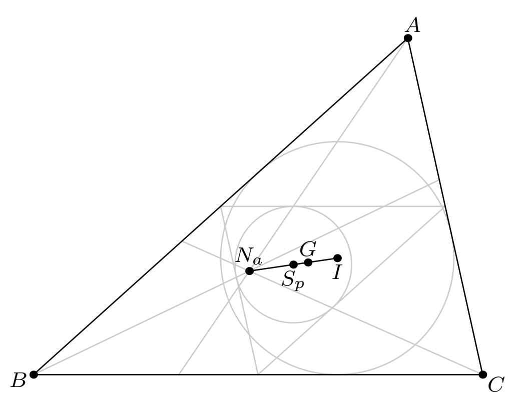

Figure 9: Nagel line

2.1.2 Nagel Line

The Nagel point N_a, Spieker center S_p, centroid G, and incenter I are collinear along the so-called Nagel Line in that order, with S_p and G respectively bisecting and trisecting segment N_aI. The Nagel point is defined as the point of concurrency of AD, BE, and CF where DEF is the extouch triangle, meaning the excircle opposite vertex A is tangent to BC at D, and similarly for E and F. Since $BD/DC = (s-c)/(s-b)$, and likewise for E and F, we obtain $\underline{N_a} = \frac{1}{s}(s-a, s-b, s-c)$. Now the statement above is easy to verify:

$$\frac{1}{2}\left(\underline{N_a} + \underline{I}\right) = \left(\frac{\frac{s-a}{s} + \frac{a}{2s}}{2}, -, -\right) = \left(\frac{b+c}{4s}, -, -\right) = \underline{S_p}$$

and

$$\frac{1}{3}\underline{N_a} + \frac{2}{3}\underline{I} = \left(\frac{s-a}{3s} + \frac{a}{3s}, -, -\right) = \left(\frac{1}{3}, -, -\right) = \underline{G}.$$

3 Area

Barycentric coordinates can also be used to calculate triangle areas, as follows:

Area Formula. For three points $\underline{P} = (x_1, y_1, z_1)$, $\underline{Q} = (x_2, y_2, z_2)$, $\underline{R} = (x_3, y_3, z_3)$ written in normalized barycentric coordinates with respect to triangle ABC, we have

$$\frac{[PQR]}{[ABC]} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

Proof. Choose a point O not in the plane of $\triangle ABC$, and set up a three dimensional coordinate system with $O = (0,0,0)$, $A = (1,0,0)$, $B = (0,1,0)$, and $C = (0,0,1)$ (note that this need not be an orthonormal frame!). It is not difficult to show that for a point $X = (x, y, z)$ with $x + y + z = 1$, i.e. in the plane of $\triangle ABC$, the coordinates (x, y, z) in the coordinate system correspond to its normalized barycentric coordinates $\underline{X}$. Letting $\mathcal{P}_{ABC}$ be the parallelepiped spanned by vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, and likewise for $\mathcal{P}_{PQR}$, the definition of determinant via volume gives

$$\frac{\mathrm{vol}(\mathcal{P}_{PQR})}{\mathrm{vol}(\mathcal{P}_{ABC})} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}.$$

We have $\mathrm{vol}(\mathcal{P}_{ABC}) = 6\,\mathrm{vol}(OABC) = 2[ABC] \cdot h$, where h is the length of the height from O to plane ABC, and likewise, $\mathrm{vol}(\mathcal{P}_{PQR}) = 2[PQR] \cdot h$. Thus, the ratio above equals

$$\frac{[PQR]}{[ABC]},$$

as claimed. □

3.1 Example: Triangle OIH

Problem 1. The area of triangle OIH is $\frac{1}{8r}(a-b)(b-c)(c-a)$.

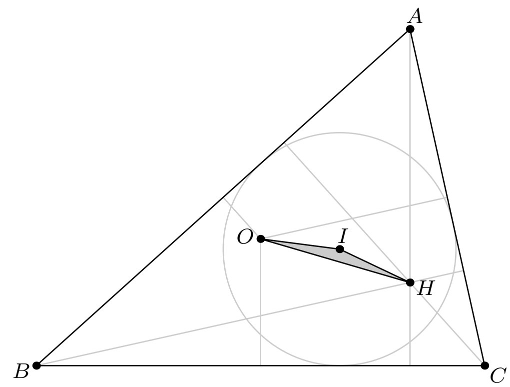

Figure 10: Triangle OIH

Solution. According to the previous formula, the area of triangle OIH is

$$[ABC] \cdot \frac{R}{2rs} \cdot \frac{1}{2s} \cdot \frac{R}{rs} \cdot \begin{vmatrix} a\cos\alpha & b\cos\beta & c\cos\gamma \\ a & b & c \\ a\cos\beta\cos\gamma & b\cos\gamma\cos\alpha & c\cos\alpha\cos\beta \end{vmatrix} = \frac{R^2abc}{4rs^2} \cdot \begin{vmatrix} \cos\alpha & \cos\beta & c\cos\gamma \\ 1 & 1 & 1 \\ \cos\beta\cos\gamma & \cos\gamma\cos\alpha & \cos\alpha\cos\beta \end{vmatrix}$$

$$= \frac{R^3}{s} \cdot (\cos\beta - \cos\alpha)(\cos\gamma - \cos\beta)(\cos\alpha - \cos\gamma).$$

Finally, using the formula

$$\cos\beta - \cos\alpha = \frac{c^2+a^2-b^2}{2ca} - \frac{b^2+c^2-a^2}{2bc} = \frac{(a-b)(a+b+c)(a+b-c)}{2abc} = \frac{(a-b)(s-c)}{2rR},$$

the above simplifies to

$$\frac{1}{8sr^3}(s-a)(s-b)(s-c)(a-b)(b-c)(c-a) = \frac{1}{8r}(a-b)(b-c)(c-a),$$

as claimed. (The last simplification is due to Heron's formula: $rs = \sqrt{s(s-a)(s-b)(s-c)}$.) □

4 Problems

Problem 2 (MOP 2006)**.** Triangle ABC is inscribed in circle ω. Point P lies on line BC such that line PA is tangent to ω. The bisector of $\angle APB$ meets segments AB and AC at D and E respectively. Segments BE and CD meet at Q. Given that line PQ passes through the center of ω, compute $\angle BAC$.

Solution. By similar triangles PBA and PAC, $\frac{PB}{PA} = \frac{PA}{PC} = \frac{c}{b}$, so $\frac{BD}{DA} = \frac{c}{b}$ and $\frac{AE}{EC} = \frac{c}{b}$. This is enough to identify $D = (c:b:0)$, $E = (b:0:c)$, and $Q = (bc:b^2:c^2)$. Point P, lying on line BC, has the form $P = (0:x:y)$ for some x and y, and since P, D, and E are collinear, we have $\begin{vmatrix} 0 & x & y \\ c & b & 0 \\ b & 0 & c \end{vmatrix} = 0$, i.e. $\frac{x}{y} = -\frac{b^2}{c^2}$. So $P = (0:b^2:-c^2)$.

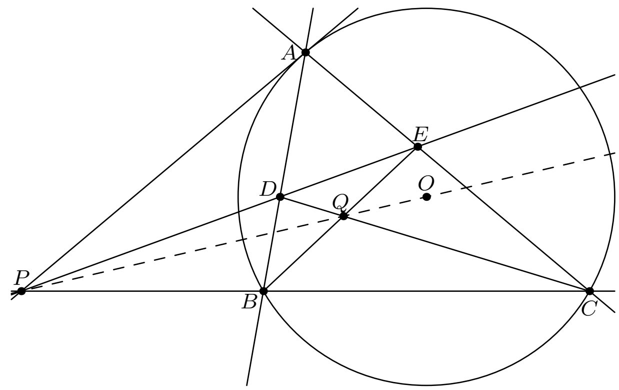

Figure 11: Problem 2

Finally, since P, Q, and $O = (a\cos\alpha : - : -)$ are collinear, we find that

$$\begin{vmatrix} 0 & b^2 & -c^2 \\ bc & b^2 & c^2 \\ a\cos\alpha & b\cos\beta & c\cos\gamma \end{vmatrix} = bc \cdot \begin{vmatrix} 0 & b & -c \\ bc & b & c \\ a\cos\alpha & \cos\beta & \cos\gamma \end{vmatrix} = 0$$

which simplifies to $2abc\cos\alpha = bc^2\cos\beta + b^2c\cos\gamma = bc(c\cos\beta + b\cos\gamma) = abc$, i.e. $\cos\alpha = \frac{1}{2}$ and $\alpha = 60°$. $\square$

Problem 3 (USAMO 2001 #2)**.** Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

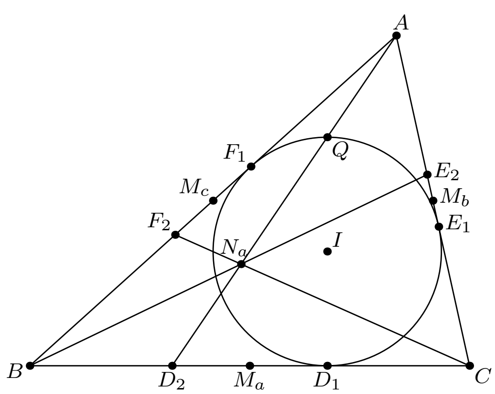

Figure 12: Problem 3

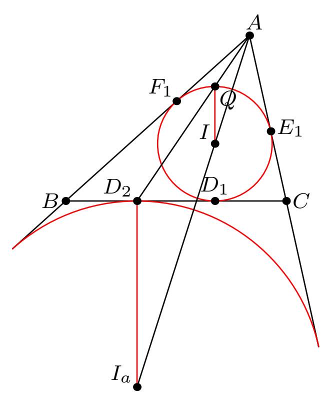

Figure 13: A, Q, D_1 collinearity

Solution. We can locate most of the points in the diagram: $I = \frac{1}{2s}(a,b,c)$, $D_1 = \frac{1}{a}(0, s-c, s-b)$, $D_2 = \frac{1}{a}(0, s-b, s-c)$ (the point of tangency of BC with A's excircle), $E_2 = \frac{1}{b}(s-a, 0, s-c)$, and $P = N_a = \frac{1}{s}(s-a, s-b, s-c)$ (this is the Nagel point). To find Q, we note that the homothecy at A taking A's excircle to the incircle must take D_2 to Q. This means the radius IQ is parallel to $I_aD_2 \parallel ID_1$, i.e. Q is diametrically opposite to D_1 along the incircle: $Q = 2I - D_1 = \left(\frac{a}{s}, \frac{b}{s} - \frac{s-c}{a}, \frac{c}{s} - \frac{s-b}{a}\right)$. Now, all we must show is $AQ = PD_2$.

While distances in general are not pretty in barycentric coordinates, we are saved by the fact that all four points are on a line. We offer two ways to finish. The first is to appeal to the 3D coordinate frame view developed in

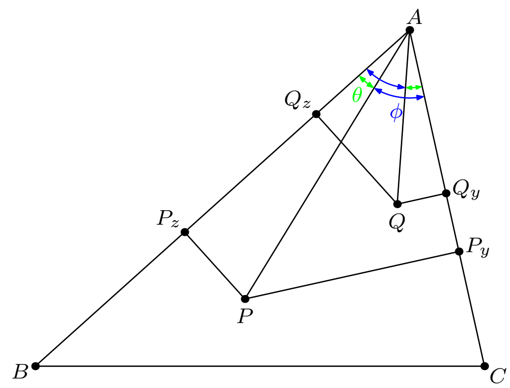

Figure 14: Isogonal conjugates

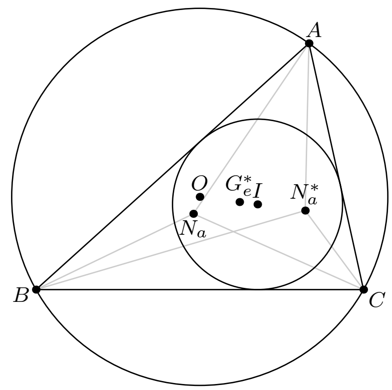

Figure 15: Problem 4

section 3 and claim it suffices to show that $\overrightarrow{AQ} = \overrightarrow{PD_2}$ (since these points are already in normalized form), i.e. that

$$\left(\frac{a}{s} - 1, \frac{b}{s} - \frac{s-c}{a}, \frac{c}{s} - \frac{s-b}{a}\right) = \left(-\frac{s-a}{s}, \frac{s-b}{a} - \frac{s-b}{s}, \frac{s-c}{a} - \frac{s-c}{s}\right).$$

This can be checked directly. The first coordinates are clearly equal; equality of the second coordinates boils down to $2s = a + b + c$; and likewise for the third coordinates. Another method is to show that $[AQB] = [PD_2B]$. This becomes

$$\begin{vmatrix} 1 & 0 & 0 \\ \frac{a}{s} & \frac{b}{s} - \frac{s-c}{a} & \frac{c}{s} - \frac{s-b}{a} \\ 0 & 1 & 0 \end{vmatrix} = \begin{vmatrix} \frac{s-a}{s} & \frac{s-b}{s} & \frac{s-c}{s} \\ 0 & \frac{s-b}{a} & \frac{s-c}{a} \\ 0 & 1 & 0 \end{vmatrix},$$

i.e. $\frac{s-b}{a} - \frac{c}{s} = -\frac{s-a}{s} \cdot \frac{s-c}{a}$. But again, this is just a rewriting of $a + b + c = s$. $\square$

Problem 4. Show that the isogonal conjugate of the Nagel point is the center of positive homothecy between the incircle and circumcircle. Likewise, the isogonal conjugate of the Gergonne point is the center of negative homothecy between the two circles.

Solution. To do this, first we need to calculate the isogonal conjugate P^* of a general point $P = (x : y : z)$. Let P_y and P_z be the projections from P to AC and AB respectively, and likewise for P_y^* and P_z^*. Setting $\angle BAP = \theta$ and $\angle PAC = \phi$, we have

$$\frac{[P^*CA]}{[P^*AB]} = \frac{b}{c} \cdot \frac{P^*P_y^*}{P^*P_z^*} = \frac{b}{c} \cdot \frac{\sin\theta}{\sin\phi} = \frac{b}{c} \cdot \frac{PP_z}{PP_y} = \frac{b^2}{c^2} \cdot \frac{[PAB]}{[PCA]} = \frac{b^2/y}{c^2/z},$$

so we find $P^* = \left(\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}\right)$.

Now, since $N_a = (s-a : s-b : s-c)$ normalized, the above gives

$$N_a^* = \left(\frac{a^2}{s-a} : - : -\right) \text{ normalized} = \left(a^2(s-b)(s-c) : - : -\right) \text{ normalized}.$$

Using the identity $\sin^2\frac{\alpha}{2} = \frac{(s-b)(s-c)}{bc}$, the above can be manipulated as follows:

$$\begin{aligned}
N_a^* &= \left(a^2 bc \sin^2\tfrac{\alpha}{2} : - : -\right) \text{ normalized} \\
&= (a(1-\cos\alpha) : - : -) \text{ normalized} \\
&= \big((a:b:c) - (a\cos\alpha : b\cos\beta : c\cos\gamma) \big) \text{ normalized} \\
&= \left(2s\cdot I - \frac{2rs}{R}\cdot O\right) \text{ normalized} \\
&= \left(R\cdot I - r\cdot O\right) \text{ normalized} \\
&= \frac{R}{R-r}\cdot I - \frac{r}{R-r}\cdot O.
\end{aligned}$$

This means that N_a^* is on line OI such that $N_a^* I / N_a^* O = r/R$, i.e. N_a^* is exactly the center of homothecy taking I to O with positive ratio R/r. Likewise, it may be calculated that $G_e^* = \frac{R}{R+r}\cdot I + \frac{r}{R+r}\cdot O$, i.e. G_e^* is the center of homothecy taking I to O with negative ratio $-R/r$. □