Proof of Brocard's Theorem

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§ Brocard's Theorem

Let $ABCD$ is a cyclic quadrilateral inscribed in a circle with centre O. P, Q, R are the intersection points of (BA - CD), (DA - CB) *and* (AC - BD) *respectively. Then, point O is the orthocentre of ΔPQR. (In fact, P is the pole of QR, Q is the pole of PR, and R is the pole of PQ)*

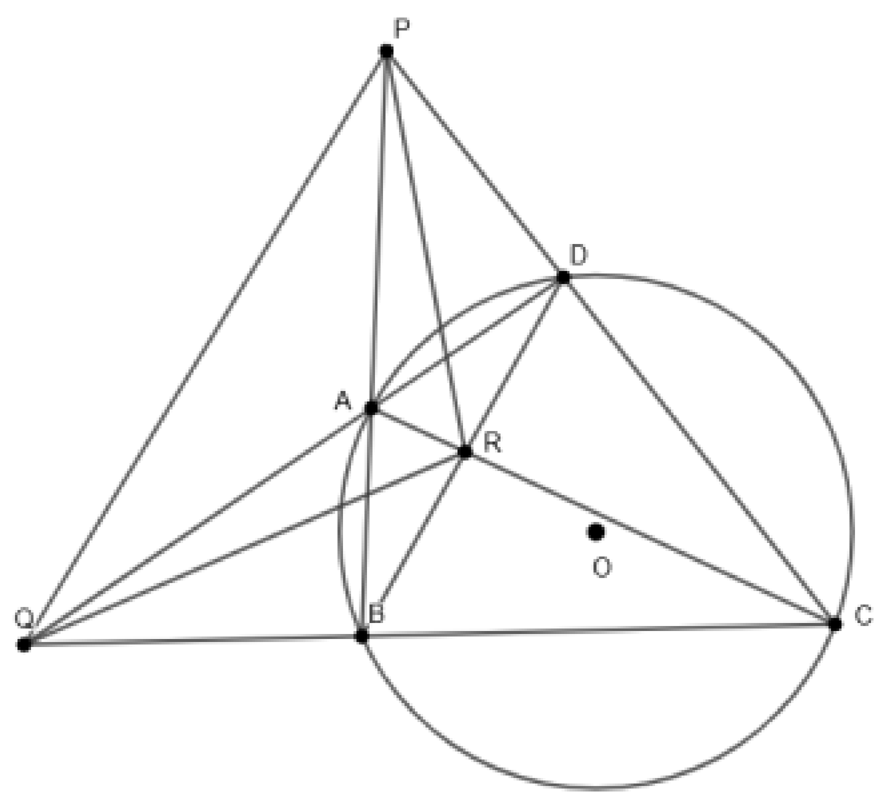

Figure 1 : O is the orthocentre of ΔPQR

§ Prerequisites

Brocard's theorem is a very powerful tool in synthetic as well as in projective geometry. Many of you know this theorem well but not so much familiar with the proof. So, here I am trying to give a complete proof of this theorem step by step. Some ideas of symmedians, projective geometry, perspectivity, harmonic bundles, poles and polars etc. will be needed.

§1 Symmedians

Symmedian is the reflection of median over the corresponding angle bisector of a triangle (isogonal of the median).

Lemma 1.1 : In $\Delta ABC, P$ be a point on BC, then $\frac{PB}{PC} = \frac{AB^2}{AC^2}$ if and only if AP is a symmedian.

(Just draw the median AM, use the fact $\angle BAP = \angle CAM, \angle BAM = \angle CAP$ and ratio lemma .)

Lemma 1.2 : If the tangents at B and C to circumcircle of ΔABC intersect at K then the line AK is a symmedian.

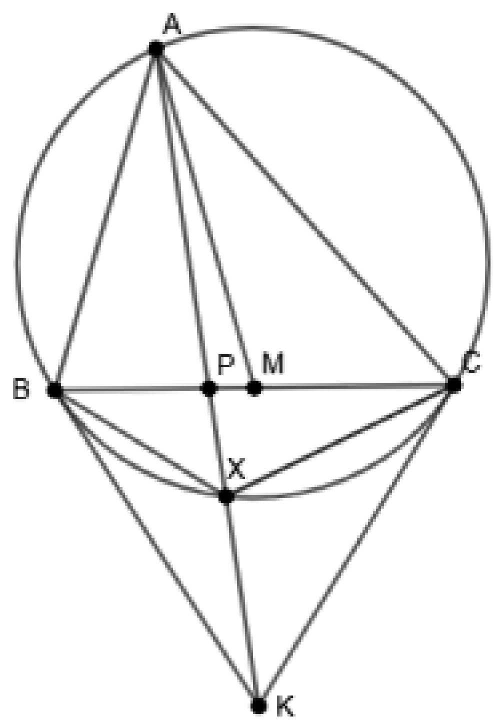

Figure 2 : The A -symmedian of ΔABC

Proof : Let P be the intersection of AK with BC. So by lemma 1.1, it is enough to show that $\frac{BP}{CP} = \frac{AB^2}{AC^2}$

In the above figure, we have $\frac{BP}{CP} = \frac{BK}{CK} \cdot \frac{sinBKP}{sinCKP}$

As , $BK = CK$ (tangent from same point to the circumcircle) , hence $\frac{BP}{CP} = \frac{sinBKP}{sinCKP}$

Applying Sine laws in ΔKAB and ΔKAC , we get

$$\frac{AB}{sinBKP} = \frac{AK}{sinABK} = \frac{AK}{sin(A+B)}$$

and

$$\frac{AC}{sinCKP} = \frac{AK}{sinACK} = \frac{AK}{sin(A+C)}$$

[KB and KC are tangents to (ABC) , so $\angle KBC = \angle KCB = \angle A$]

Hence,

$$\frac{sinBKP}{sinCKP} = \frac{AB}{AC} \cdot \frac{sin(A+C)}{sin(A+B)} = \frac{AB}{AC} \cdot \frac{sinB}{sinC} = \frac{AB^2}{AC^2}$$

So, we get $\dfrac{BP}{CP} = \dfrac{AB^2}{AC^2}$. Hence, AK is a symmedian of ΔABC .

Lemma 1.3 : In $\Delta ABC, AK$ is the A - symmedian of ΔABC with K on BC. Let AK meet (ABC) at X. Then $\dfrac{AB}{AC} = \dfrac{BX}{CX}$

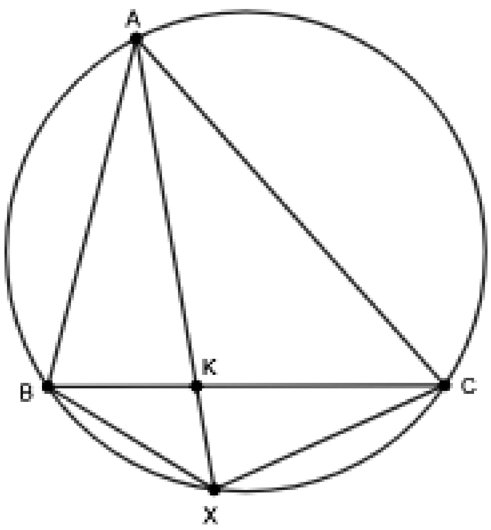

Figure 3 : $ABXC$ is called harmonic quadrilateral
(cyclic and product of opposite sides are equal)

Proof : By ratio lemma,

$$\frac{BX}{CX} \cdot \frac{sinBXA}{sinCXA} = \frac{BP}{CP}$$

Since , $\dfrac{BP}{CP} = \dfrac{AB^2}{BC^2}$ and $\angle BXA = \angle C, \angle CXA = \angle B$,

$$\frac{BX}{CX} = \frac{AB^2}{AC^2} \cdot \frac{sinB}{sinC} = \frac{AB^2}{AC^2} \cdot \frac{AC}{AB} = \frac{AB}{AC}$$

§2 Cross Ratios – Projective Geometry

For any given four collinear points A, B, X, Y, the cross ratio is

$$(A, B; X, Y) = \frac{XA}{XB} \div \frac{YA}{YB}$$

When four lines a, b, c, d are concurrent at some point P, then the cross ratio will be

$$(a, b; c, d) = \frac{sin\angle(c,a)}{sin\angle(c,b)} \div \frac{sin\angle(d,a)}{sin\angle(d,b)}$$

Where $\angle(x, y)$ is the angle between the lines x, y.

If A, B, X, Y are collinear points on lines a, b, x, y (respectively) concurrent at K,

$$K(A, B; X, Y) = (a, b; x, y)$$

$K(A, B; X, Y)$ is called a pencil of lines.

Lemma 2.1 : If $P(A, B; X, Y)$ is a pencil of lines and A, B, X, Y are collinear then

$$P(A, B; X, Y) = (A, B; X, Y)$$

(Just apply sine laws on the corresponding triangles)

Lemma 2.2 : If A, B, X, Y are concyclic and P is any point on the circumcircle, then

$$P(A, B; X, Y) = \pm \frac{XA}{XB} \div \frac{YA}{YB}$$

Here, $P(A, B; X, Y)$ does not depend on P .

If two lines s and t are given such that points $A, B, C, D)$ lie on s. Let P be a point and the intersection points of PA, PB, PC, PD with line t are A', B', C', D' respectively . Then,

$$P(A, B; C, D) = P(A', B'; C', D') = (A, B; C, D) = (A', B'; C', D')$$

This is called perspectivity at P . This is denoted by

$$(A, B; C, D) \overset{P}{=} (A', B', C', D')$$

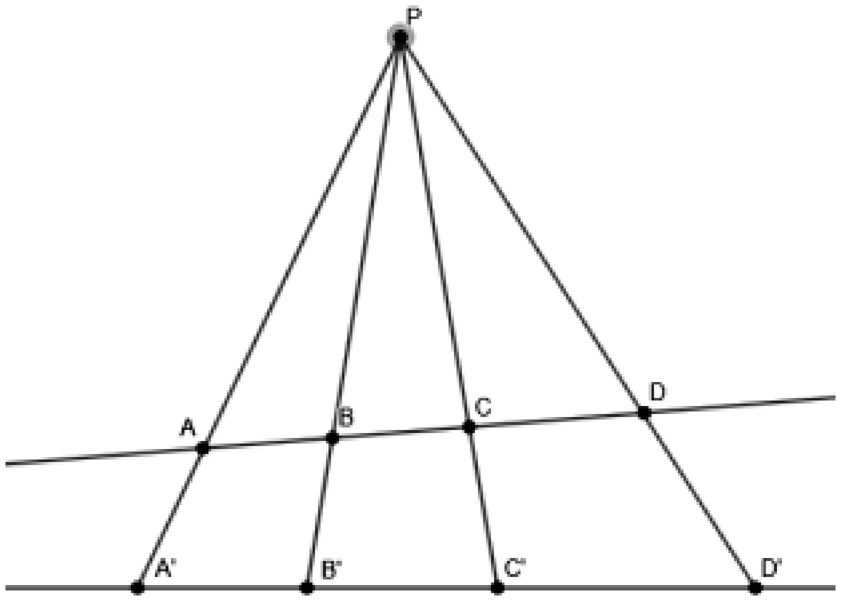

Figure 4 : Projecting (A, B, C, D) from s to t

This will be same even if s is a circle instead of a line ,that is P, A, B, C, D are concyclic. The cross ratio will be preserved.

§3 Harmonic Bundles

For four collinear points A, B, X, Y , if $(A, B; X, Y) = -1$ then, A, B, X, Y is called a harmonic bundle .
(The sign is negative as the direction is opposite)

Lemma 3.1 : Let Γ be a circle. P be a point outside it. Let PX and PY be tangents to Γ. If a line through P intersecs Γ at A and B and K be the intersection point of AB and XY. Then, $(A, B; K, P)$ is a harmonic bundle.

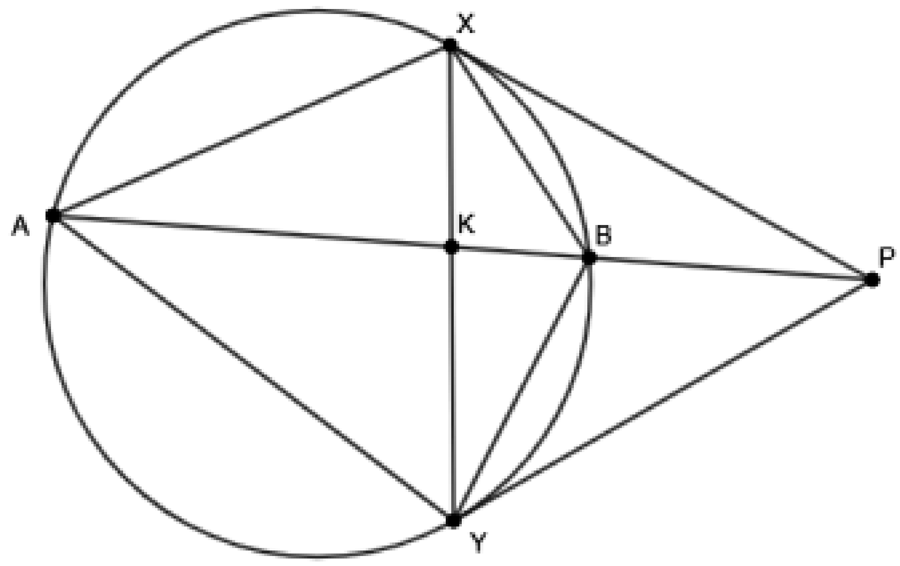

Figure 5 : $AXBY$ is a harmonic quadrilateral.

Proof: From lemma 1.3, we know that $\dfrac{AY}{BY} = \dfrac{AX}{BX}$.So, $ABXY$ is harmonic. That means, $(A, B; X, Y) = -1$

We can write,

$$(A, B; X, Y) \stackrel{X}{=} (A, B; K, P)$$

Because, we are projecting from the point X lying on the circle onto the line AB.
(As PX is tangent to Γ and if we bring a point M very very close to X, XM behaves as the tangent . So, XX is indeed PX.)

Lemma 3.2 : Let ABC be a triangle. AD, BE, CF are concurrent lines with D on BC, E on AC and F on AB. The line EF meets BC at X (may be point at infinity). Then $(B, C; X, D)$ is a harmonic bundle.

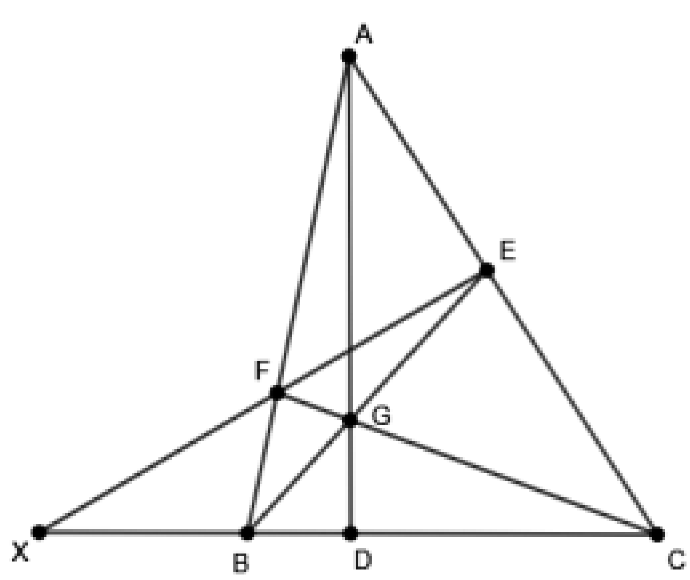

Figure 6 : $(B, C; X, D) = -1$
(Apply Ceva's theorem and Menelaus's theorem , then compare the ratios $\frac{BX}{CX}$ and $\frac{BD}{CD}$)

§4 Poles and Polars

Let Γ be a circle with centre O . P be a point on the plane. Let Q be the inverse of P with respect to Γ.(That is , O, Q, P are collinear and $OQ \cdot OP = radius^2$)
Then, the **Polar of point** P is the line passing through Q perpendicular to OP.

When P is Γ then its polar is the line (let's say l) through the two tangency points from P to Γ. Here, P is the **pole** of the line l.

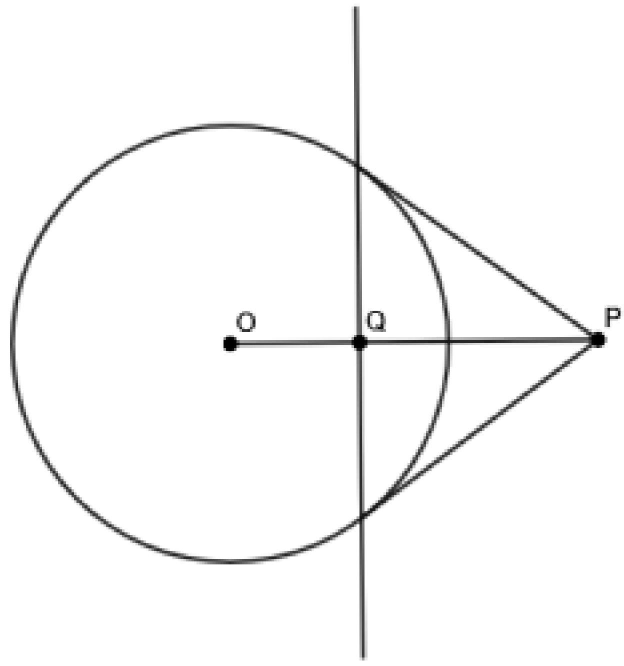

Figure 7 : The line throgh Q perpendicular to OP is the polar of P.

- **La Hire's Theorem**: A point X lies on the polar of a point Y if and only if Y lies on the polar of X. (Hint: Find similar triangles)

Lemma 4 : Let PQ be a line, points R, S lies on PQ. Then R lies on the polar of S if and only if $(P, Q; R, S) = -1$

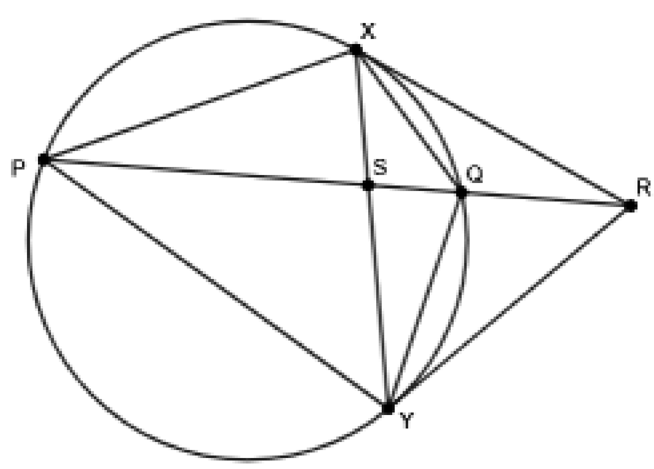

Figure 8 : $(P, Q; R, S) = -1$

Proof : Let Ω be a circle containing P, Q. Now we will consider the case when R is outside Ω (La Hire's). Draw tangents RX, RY to Ω. Let the intersection of XY and PQ is S' . From lemma 3.1, we get
$(P, Q; R, S') = -1$ $(P, Q; R, S') = (P, Q; S', R)$ because both are -1

So, S lies on the polar of R if and only if $(P, Q; R, S) = -1$
(Because , the harmonic conjugate of R with respect to PQ is unique , so $S' = S$)

Now the crucial part comes...

§ Proof of Brocard's Theorem

Statement : Let $ABCD$ be an cyclic quadrilateral inscribed in a circle with center O, and X, Y and Z are the intersection points of $(AB, CD), (BC, DA)$ and (AC, BD). Then ,

(i)X is the pole of YZ, Y is the pole of ZX and Z is the pole of XY.
(ii) O is the orthocenter of triangle XYZ.

(2nd result is just the consequence of the 1st)

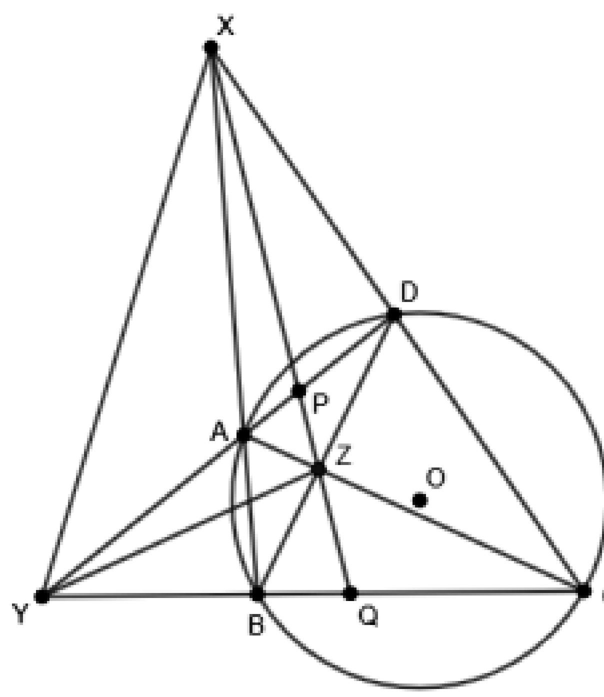

Figure 9 : O is the orthocentre of XYZ

Proof : Let the intersection of XZ with AD and BC are P and Q respectively. From lemma 3.2, we get $(B, C; Y, Q)$ is a harmonic bundle. Now,

$$-1 = (B, C; Y, Q) \overset{X}{=} (A, D; Y, P)$$

So, $(A, D; Y, P)$ is also harmonic. By lemma 4 , P and Q both lie on the polar of Y. As , the polar has to be a straight line, then the polar of Y is PQ, which is same as XZ.

Similarly, X is the pole of YZ and Z is the pole of XY. (ΔXYZ is called self-polar) From the definition of poles and polars, we get O is the orthocentre of ΔXYZ.