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"American" Olympiad Triangle Geometry Configurations

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Feb. 17 2021

Made in the USA

Contents

Chapter 0

Introduction

People say that config geo is too easy to train for, and that it's dying. Well, here's a nail in that coffin.

The handout is designed to be read from front to back, mostly because I wrote it generally from front to back. If you haven't seen many of these before it won't make sense back to front, especially because in each of the chapters, I carry point labelings over from section to section. However feel free to do whatever makes most sense to you. There are 100 problems at the end, from 0 to 99, and I added solutions to all of the ones in which the number 9 appears in the problem number (e.g. 9, 19,91), and I added hints to all of the ones in which have either an 8 or 9 in the problem number. I think this gives a good balance between not giving hints/solutions to all of the problems (because I'm lazy) and helping you out.

It goes without saying that I discovered none of these theorems myself, and only a few of them have proofs that I found solely by myself. This is more just a curation of what I think are important/interesting configurations in triangle geometry plus the proofs of these that I think are the best (or sometimes just my proofs, hehe). As part of that, I avoid doing any non-synthetic techniques, except for the Dumpty Point section, but there I only use it for it's own sake.

Note that the names Ex,Queue,Humpty,Dumpty,Sharkydevil,Iran Lemma,Midpoint of Altitudes Lemma, Median-Incircle Concurrency, Fact 5,and possibly others I may be forgetting are all olympiad colloquialisms. Why Point is something I made up to match these. A rule of thumb is that if you can't find an actual source that uses the names I use, you probably can't cite it on olympiad. Neither can you usually cite it as well known. This is a purpose of the handout: so that you see the proofs of what others on AoPS and other sites cite as well known, so you can recreate it on an olympiad.

The prerequisite knowledge necessary is EGMO, but if you don't know much of EGMO, even if you only know angle chasing so far, there should be some of stuff in here that you can understand(the "Nine-Point Circle and Poncelet Points" section is one of the more underrated topics, and for the purposes of this handout, is just angle chasing). I do go over configurations already in EGMO though, for completeness.

Also I'm calling this a handout, even though I used the document style book while creating this, because I thought it would be nicer with chapters and sections rather than sections and subsections as one would do in the document style article. Additionally in this handout I use a couple of names and notations that aren't necessarily standardized.

- I use $P_1 - P_2 - P_3 - \cdots$ to mean that the points $P_1, P_2, P_3, \cdots$ are collinear.
- I use $\measuredangle$ heavily, to mean the directed angle mod 180.
- I use $(P_1P_2P_3\cdots)$ to mean the circumcircle of $P_1, P_2, P_3, \cdots$.
- I use (P_1P_2) to mean the circle with diameter P_1P_2.
- I use degrees instead of radians, so instead of $\frac{\pi}{2}$ I use 90.
- I use $\sqrt{bc}$ invert to mean that if there is a $\triangle ABC$, to invert around A with radius $\sqrt{AB \cdot AC}$ and then reflect the image over the angle bisector of A. To be clear, in saying $\sqrt{bc}$ inversion, I also imply the reflection across the angle bisector.
- I use synthetic to exclude long trig and length bashes and to include inversions and especially projective (which plays a very large role in the handout).
- I use the word disgusting to mean the kinds of geo not covered in this handout.
- I use American Geo to mean ... umm ..., I really don't know. No one really does, but most agree that the problems in this handout are "American" Geo, even though there are only 16 official USA problems and 16 ELMO/ELMO SL Problems.

This handout is divided into an "Orthocenter" chapter and an "Incenter" chapter, mostly because in my head there's 2 main types of configs. However note that this is loose and I talk about other triangle centers occasionally (symmedian point as an example).

Finally, many thanks to AoPS user amar_04 for his suggestions. He has seen a lot of great problems and is really pro. All mistakes and typos in this handout are mine alone.

Chapter 1

Orthocenter Configurations

1.1 Orthocenter Itself

Let's first define what the orthocenter is:

Definition 1.1.1 (Orthocenter)**.** The **orthocenter** is the intersection of all of the altitudes of a triangle.

This exists by Miquel's theorem on the feet of the altitudes.

Theorem 1.1.1 (Orthocentric System)**.** In a triangle ABC, if H is the orthocenter of ABC, then A is the orthocenter of BHC, B is the orthocenter of AHC, and C is the orthocenter of AHB. These form an orthocentic system.

While this is quite obvious, it actually shows up quite a lot.

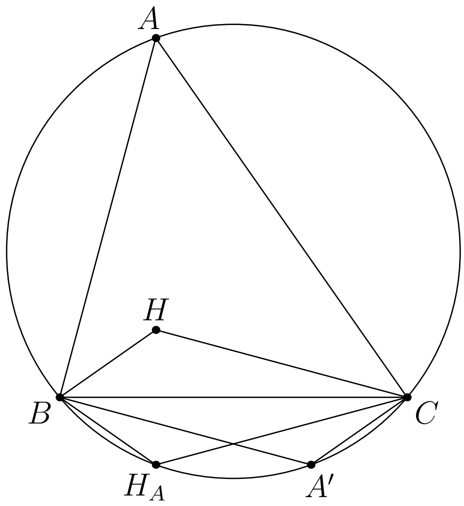
Theorem 1.1.2. The circumcenter and orthocenter are isogonal conjugates.

$\measuredangle BAH = 90 - \measuredangle B$, $\measuredangle OAC = 90 - \measuredangle B$.

If you know the lemma regarding isogonal conjugates, which we will prove later, the reflections of H across the sides of ABC have circumcenter O, and the reflections of O across the sides of ABC should have circumcenter H. We'll end up seeing both of those show up here:

Theorem 1.1.3 ("Orthocenter Reflections")**.** Let H_A be the intersection of $\overline{AH}$ and (ABC). Let A' be the antipode of A with respect to (ABC). $BHCH_A$ is a kite and $BHCA'$ is a parallelogram.

Time for the first diagram:



Behold, the triangle I'll use for the entire chapter.

This is just angle chasing. We see that $\measuredangle H_ABC = 90 - \measuredangle ACB = \measuredangle CBH$, and similarly $\measuredangle H_ACB = \measuredangle BCH$, so $BHCH_A$ is a kite.

$\measuredangle HCB = 90 - \measuredangle CBA = \measuredangle A'AC = \measuredangle A'BC$, and similarly $\measuredangle HBC = \measuredangle A'CB$, so $BHCA'$ is a parallelogram.

Since H_A is the reflection of H over $\overline{BC}$, this shows that $(H_AH_BH_C)$ has center O. This also means that (BHC) is the reflection of (ABC) over $\overline{BC}$. Thus the reflection of O over $\overline{BC}$, O_A, is R away from H, where R is the circumradius of (ABC). This means that $(O_AO_BO_C)$ is congruent to (ABC) and has center H, as desired.

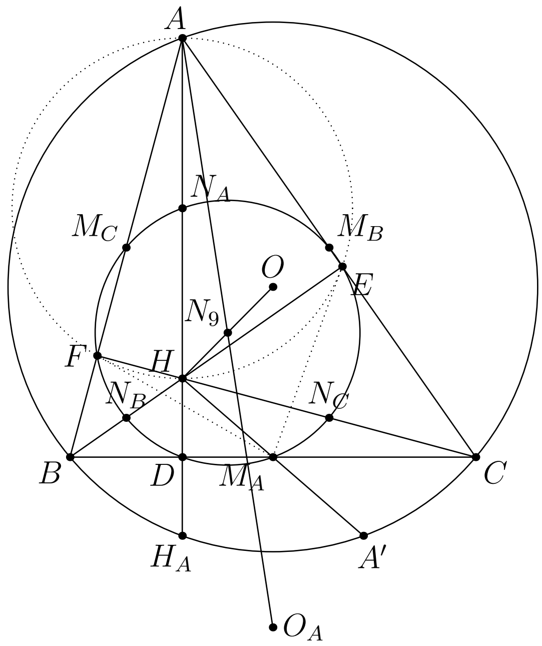
Definition 1.1.2 (Orthic Triangle)**.** The **orthic triangle** of $\triangle ABC$ is the triangle made by the feet of the altitudes of $\triangle ABC$.

Definition 1.1.3 (Euler Line)**.** The line through the circumcenter, centroid, and orthocenter.

This exists because the homothety with factor $\frac{-1}{2}$ centered at G takes H to the orthocenter of the medial triangle, or O.

1.2 Nine-Point Circle and Poncelet Points

Let $D = \overline{AH} \cap \overline{BC}$ and let M_A be the midpoint of BC. It's clear that D is the midpoint of HH_A and M_A is the midpoint of HA'. What happens if we dilate (ABC) by a factor of $\frac{1}{2}$ around H?



Hope you like the labels, they'll be a recurring theme for this section

We see that in addition to D and M_A, we get all the other altitude feet and other midpoints. That's 6 points already. The other three, N_A, N_B, N_C, represent the midpoints of AH, BH, CH, which we also get on this circle. The other points I've marked in this diagram are N_9, O, and O_A. O, as you can already guess, is the circumcenter of (ABC). N_9 is the center of the nine-point circle (hence N_9), and is the midpoint of OH, by the homothety. O_A is the reflection of O over $\overline{BC}$.

Definition 1.2.1 (Nine-Point Circle)**.** In a triangle ABC, the feet of the altitudes, the midpoints of the sides, and the midpoints between each of the vertices and the orthocenter H are concyclic, and they all lie on the **Nine-Point Circle**. In addition this circle is the circle resulting from a homothety with factor $\frac{1}{2}$ around H to the circumcircle of ABC. As such its center is the midpoint of the circumcenter and orthocenter of ABC.

Corollary 1.2.1. $\overline{M_AF}$ and $\overline{M_AE}$ are tangent to $(AEHF)$.

$(AEHF)$ exists because $\measuredangle AEH = 90 = \measuredangle AFH$. This circle has center N_A. $\measuredangle M_AFN_A = 90 = \measuredangle M_AEN_A$ because the homothety at H taking the nine-point circle to the circumcircle of ABC takes N_A to A and M_A to A'.

Corollary 1.2.2. N_AOM_AH is a parallelogram

We see that since M_AN_A has midpoint N_9, as well as HO, as desired.

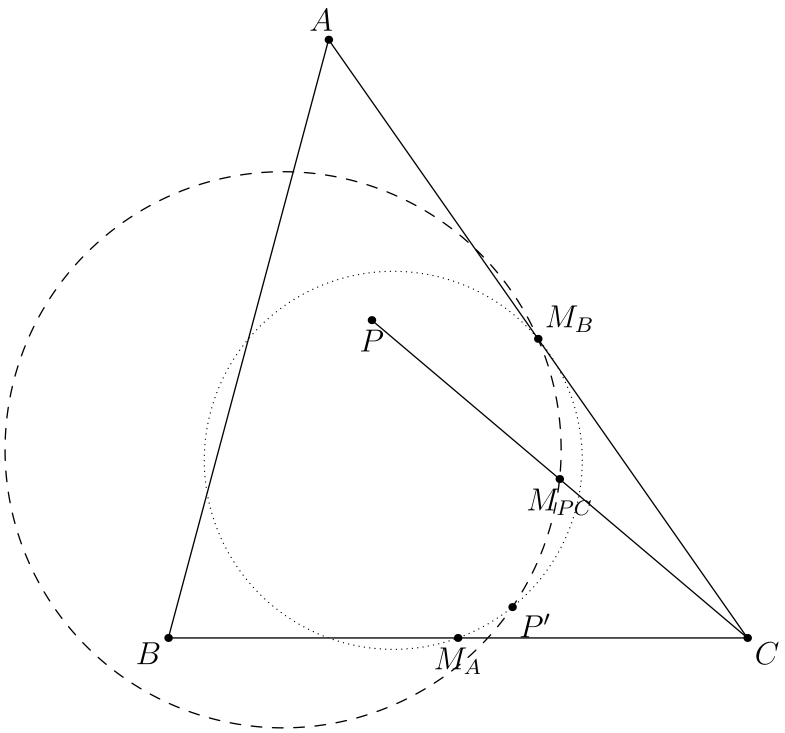
Corollary 1.2.3. N_9 is the midpoint of AO_A.

This is because of either parallelograms or a homothety of the nine-point circle at A with factor 2.

Corollary 1.2.4. Let H be the orthocenter of $\triangle ABC$. Then the Nine-Point Circles of AHB, BHC, CHA, and ABC are the same.

A natural question to ask, if you like asking questions that I just happen to want to answer, is what if we replace H with an arbitrary point P.

Definition 1.2.2 (Poncelet Point)**.** Let A, B, C, P be 4 points not in an orthocentric system. The 9-point circles of $\triangle ABC, \triangle BCP, \triangle CAP, \triangle ABP$ all meet at one point, the **Poncelet Point** of $ABCP$, P' for our purposes.



Guess what P is defined as (hint: Asymptote is a vector-graphics language)?

The plan here is to let P' be the intersection of the nine-point circle of $\triangle ABC$ (dotted) and $\triangle APC$ (dashed). Then we can show that P' is on the nine-point circle of $\triangle BPC$, which by symmetry will put it on the nine-point circle of $\triangle APB$. To do this, we can let M_{PC} be the midpoint of PC. Then $\measuredangle M_A P' M_{PC} = \measuredangle PCB$ suffices. This is true because

$$\measuredangle M_A P' M_{PC} = \measuredangle M_A P' M_B - \measuredangle M_{PC} P' M_B =$$

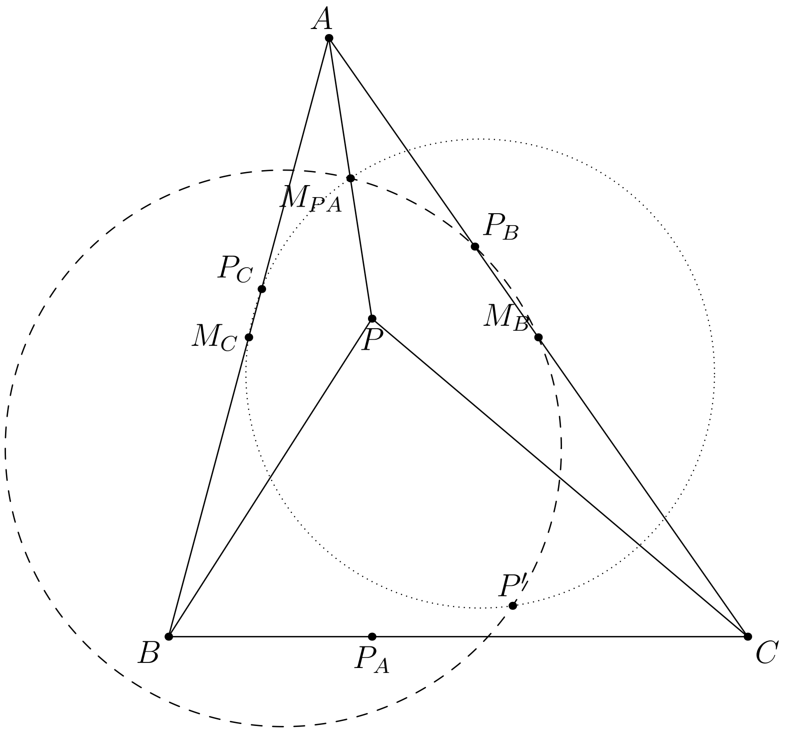
$$\measuredangle ACB - \measuredangle ACP =$$

$$\measuredangle PCB.$$

Corollary 1.2.5. If H is the orthocenter of ABC, then $ABCP$, $ABHP$, $BCHP$, and $ACHP$ have the same Poncelet Point.

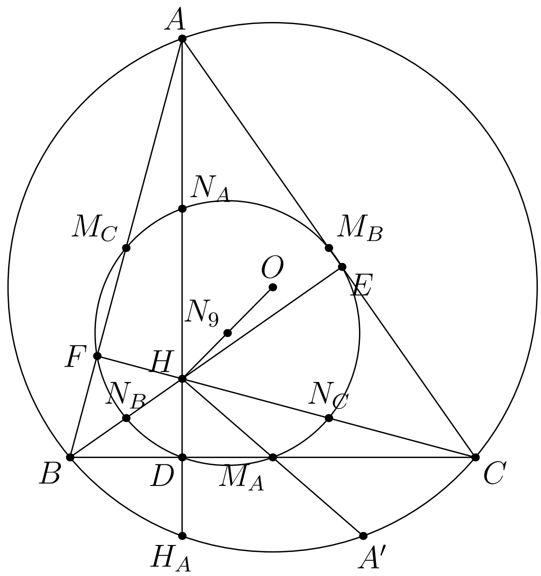
Theorem 1.2.1. The Poncelet point P' lies on the circumcircle of the pedal triangle of P.

This is one of those things that, if it looks true, it probably is because angle chase, but you don't actually want to prove.



$\triangle P_A P_B P_C$ is the pedal triangle of P

We can first calculate $\measuredangle P_B P' P_C$, and then show it equals $\measuredangle P_B P_A P_C$. Going back to this diagram:



If you're wondering, $\angle A = 50°, \angle B = 75°, \angle C = 55°$

we see that $\measuredangle DN_A M_A = \measuredangle BAC - 2\measuredangle BAD$ by undoing the homothety that created the nine-point circle in the first place. Using this,

$$\measuredangle P_B P' P_C =$$

$$\measuredangle P_B P' M_B + \measuredangle M_B P' M_{PA} + \measuredangle M_{PA} P' M_C + \measuredangle M_C P' P_{MC} =$$
$$\measuredangle APC - 2\measuredangle APP_B + \measuredangle PAC + \measuredangle BAP + 2\measuredangle APP_C - \measuredangle APB =$$
$$\measuredangle BPC + 2\measuredangle P_B P P_C + \measuredangle BAC.$$

We can now note that AP_BPP_C is cyclic, along with BP_CPP_A and CP_APP_B. This allows us to reduce the above even further, to just $\measuredangle BPC + \measuredangle P_BPP_C$. This note also allows us to calculate $\measuredangle P_BP_AP_C$:

$$\measuredangle P_BP_AP_C = \measuredangle P_BP_AP + \measuredangle PP_AP_C =$$
$$\measuredangle P_BCP + \measuredangle PBP_C =$$
$$\measuredangle BPP_C + \measuredangle P_BPC =$$
$$\measuredangle BPC + \measuredangle P_BPP_C$$

as desired.

While discussing the pedal triangle, it's important to state the following.

Theorem 1.2.2. Dilate $\triangle P_AP_BP_C$ around P with factor 2, to get $P'_AP'_BP'_C$. Then the center of the circle of these three new points is the isogonal conjugate of P.

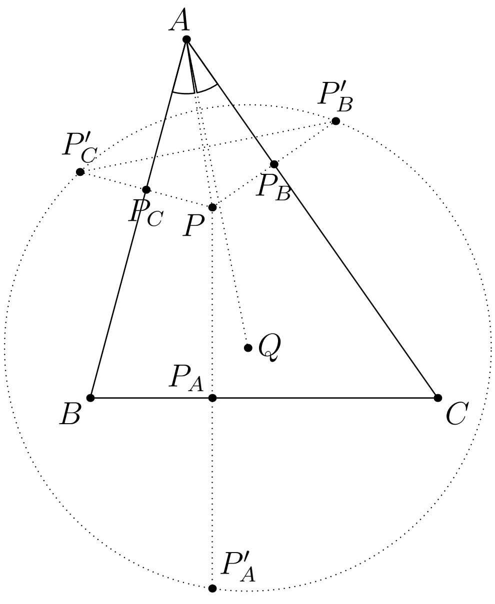
Clearly $AP'_B = AP = AP'_C$. Thus A is on the perpendicular bisector of $P'_BP'_C$. Let Q be the isogonal conjugate of P. Then

$$\measuredangle P'_CAQ = \measuredangle P'_CAB + \measuredangle BAQ =$$
$$\measuredangle BAP + \measuredangle PAC = \measuredangle BAC.$$

Similarly

$$\measuredangle QAP'_B = \measuredangle BAC,$$

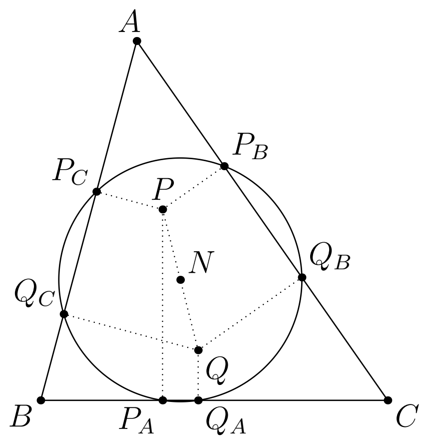
so Q is on the perpendicular bisector of $P'_BP'_C$ as desired.



Lots of dotted lines

Dilating this by a factor of $\frac{1}{2}$ around P gives that N, the midpoint of PQ, is the circumcenter of $\triangle P_A P_B P_C$. Using this same logic for Q, N is also the circumcenter of $\triangle Q_A Q_B Q_C$(which is the pedal triangle of Q). However, N is also equidistant from P_A and Q_A. Thus we have the following theorem.

Theorem 1.2.3 (Six Point Circle)**.** Let P and Q be isogonal conjugates in $\triangle ABC$. Then let $\triangle P_A P_B P_C$ be the pedal triangle of P, and let $\triangle Q_A Q_B Q_C$ be the pedal triangle of Q. Then $P_A, P_B, P_C, Q_A, Q_B, Q_C$ are all concyclic, in a six point circle with center N.

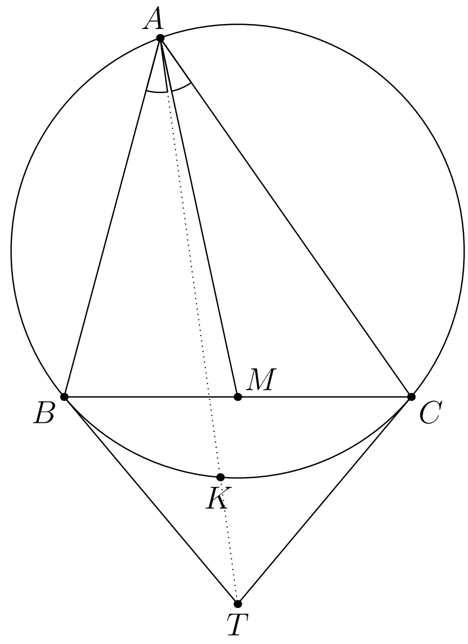


What if $P \equiv H$?

1.3 Symmedians

The symmedian, as you can probably guess, is symmetric to the median in some way. Specifically, it's the reflection of the median(here the midpoint of BC is M) over the angle bisector.

Definition 1.3.1 (Symmedian)**.** In a triangle ABC, the A-**Symmedian** is isogonal to the A-median.



I defined K in asymptote with it's equivalent in complex numbers, which we will see later.

Theorem 1.3.1 (Harmonic Quad)**.** $(A, K; B, C) = -1$

This is because
$$\frac{AB}{BK} = \frac{AM}{MC} = -\frac{AM}{MB} = -\frac{AC}{CK}$$
when using directed lengths. This condition is symmetric, and it is easily seen to work in reverse. This symmetry gives the next theorem.

Theorem 1.3.2. $\overline{KA}$ is a symmedian of $\triangle KBC$, $\overline{CB}$ is a symmedian of $\triangle ACK$, and $\overline{BC}$ is a symmedian of $\triangle KBA$.

This also lets us show that:

Theorem 1.3.3. Let T be the intersection of the tangents from B and C to (ABC). Then $A - K - T$.

If we let T_B be the intersection of the B tangent and $\overline{AK}$, and we define T_C similarly, then

$$(A, K; \overline{AK} \cap \overline{BC}, T_B) \overset{B}{=} (A, K; C, B) = -1 = (A, K; B, C) \overset{C}{=} (A, K; \overline{AK} \cap \overline{BC}, T_C).$$

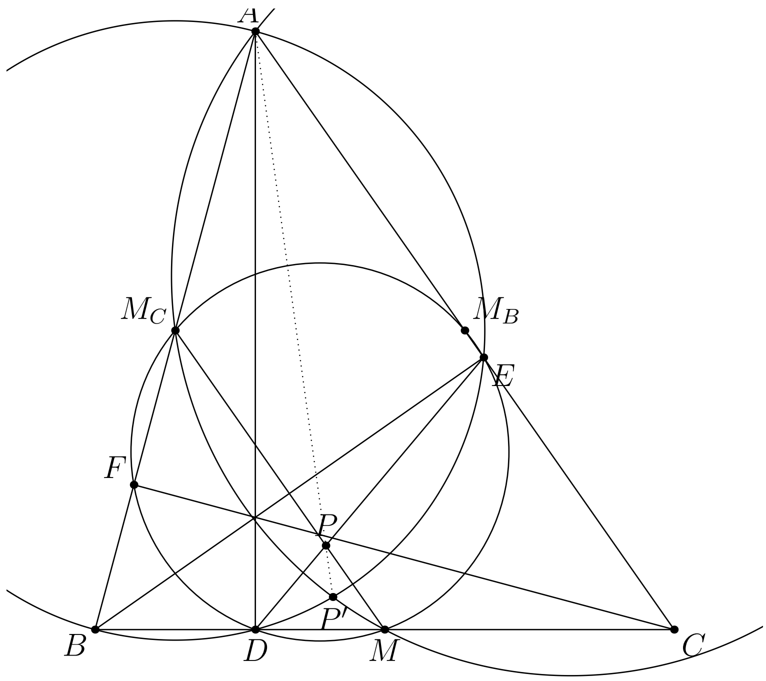
Thus $T_B \equiv T_C \equiv T$.

Bringing back in the orthic triangle, we have the following:

Theorem 1.3.4. If D is the foot of the altitude from A to $\overline{BC}$, and E and F are defined as such for B and C respectively. If M_B and M_C are the midpoints of AC and AB respectively, then

- $\overline{DE} \cap \overline{MM_C}$ is on the A-symmedian of $\triangle ABC$
- $\overline{DF} \cap \overline{MM_B}$ is on the A-symmedian of $\triangle ABC$

We can let $P = \overline{DE} \cap \overline{MM_C}$, and show it's on the symmedian, and the other bullet point follows by symmetry.



Yay I learned to clip a diagram in asymptote

We know that D, E, M, M_C are all on the nine-point circle of $\triangle ABC$. Thus if P' is the intersection of $(AEDB)$ and (AM_CM), then by PoP P is on $\overline{AP'}$. Thus it suffices to show that $\overline{AP'}$ is a symmedian.

$$\measuredangle BAP' = 2\measuredangle BAP' - \measuredangle BAP' =$$

$$\measuredangle BM_CP' - \measuredangle BAP' = \measuredangle BM_CM - \measuredangle P'M_CM - \measuredangle BAP' =$$

$$\measuredangle BAC - \measuredangle P'AM - \measuredangle BAP' = \measuredangle MAC$$

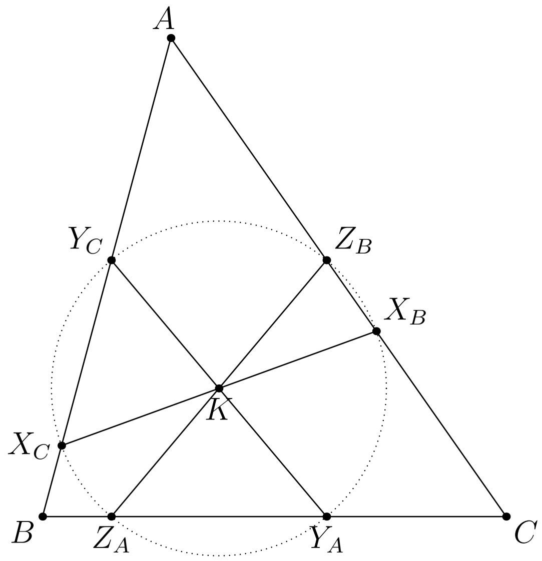
as desired.

We can now move on to the symmedian point, which I will also denote as K. I could have chosen L for Lemoine Point, but K seems more generally accepted. Note that when talking about the A-symmedian touchpoint above, I used K, because that's also generally accepted.

Definition 1.3.2 (Symmedian Point)**.** The **symmedian point** of ABC is the intersection of the 3 symmedians of ABC.

This exists because it's the isogonal conjugate of the centroid.

Theorem 1.3.5 (Second Lemoine Circle)**.** Let X_B be the intersection of the A-antiparallel of BC through K and $\overline{AC}$. Let X_C be analogous for $\overline{AB}$. Define Y_A, Y_C, Z_A, and Z_B similarly. $X_B, X_C, Y_A, Y_C, Z_A, Z_B$ are concyclic.



$\overline{X_BX_C}$ being antiparallel to BC means BX_CX_BC is cyclic

Reflect $\triangle AX_BX_C$ around the angle bisector of $\angle BAC$. Then K goes to a point on the A-median and $\overline{X_BX_C}$ is parallel to $\overline{BC}$. Thus K is the midpoint of X_BX_C. Since

$$\measuredangle Y_AZ_AK = \measuredangle BAC = \measuredangle KY_AZ_A,$$

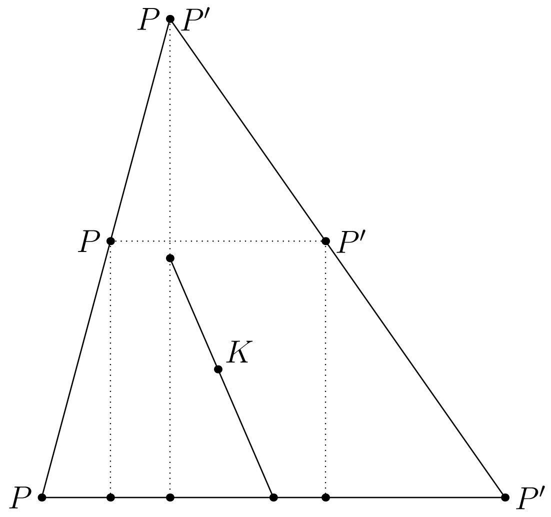
$KY_A = KZ_A$. Thus we have that $KX_B = KX_C = KY_C = KY_A = KZ_A = KZ_B$, so not only are these 6 points concyclic, the circle through them has center K.

Corollary 1.3.1. The A-symmedian bisects the antiparallels from A to BC. The converse is also true.

Theorem 1.3.6 (Schwatt Line)**.** The midpoint of the A-altitude, the midpoint of BC, and the symmedian point of ABC are concurrent.

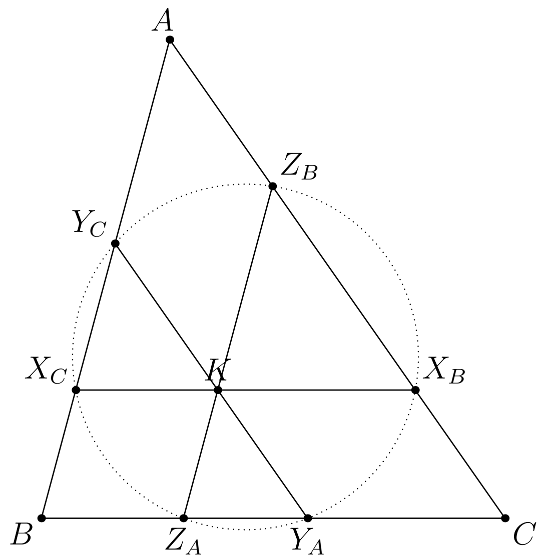
While this can be done with barycentrics or projective, it can be proved non-projectively in a very elegant way. The main claim is that this line is the locus of the centers of rectangles with 2 vertices on BC, one on AB, and one on AC. To prove this is a line, imagine varying a point P on $\overline{AB}$. Then the distance from $\overline{BC}$ to P varies linearly as P moves from A to B(linearly). Let P' be a point on $\overline{AC}$ such that $\overline{PP'}||\overline{AC}$. Then the distance of the midpoint of PP' to the A-altitude varies linearly as P varies linearly. Thus the locus of the centers of these rectangles is a line.

The midpoint of the A-altitude is obviously on this line, and so is the midpoint of BC (think of the degenerate rectangles created). K is on this line because of the rectangle $Z_AY_CZ_BY_A$ from above.



Everyone loves points that move, right?

Theorem 1.3.7 (First Lemoine Circle)**.** Let X_B be the intersection of the parallel to $\overline{BC}$ through K and $\overline{AC}$. Let X_C be the analogous intersection for $\overline{AB}$. Let Y_A, Y_C, Z_A, Z_B be analogous points. $X_B, X_C, Y_A, Y_C, Z_A, Z_B$ lie on a circle.



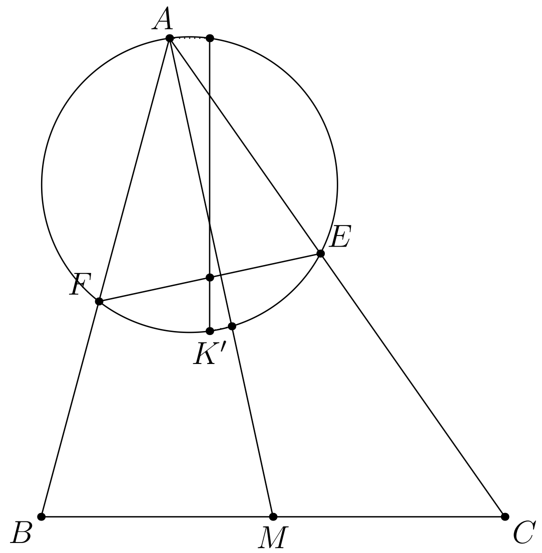
The symmedian point is $X(6)$ for ETC people

We see that AZ_BKY_C is a parallelogram, so $\overline{AK}$ bisects Y_CZ_B. This means that $\overline{Y_CZ_B}$ is antiparallel to $\overline{BC}$ (with respect to A), so $X_CY_CZ_BX_B$ is cyclic. By symmetry $X_CY_CZ_BX_BY_AZ_A$ is cyclic, as desired.

Theorem 1.3.8 (Lemoine's Pedal Triangle Theorem)**.** The symmedian point K is the only point in a $\triangle ABC$ that is the centroid of its pedal triangle.

Let K' be any point on the A-symmedian, and let E and F be the feet from K' to $\overline{AC}$ and $\overline{AB}$. Then we will show that the line through K' perpendicular to $\overline{BC}$ bisects EF, and the converse will also easily be seen to be true.

We have that $AEK'F$ is cyclic. $\measuredangle EFA = 90 - \measuredangle K'FE = 90 - \measuredangle K'AE = 90 - \measuredangle K'AC = 90 - \measuredangle BAM$, so $\overline{EF} \perp \overline{AM}$. Let ℓ be the line through K' perpendicular to $\overline{BC}$. Then $(F, E; \ell \cap \overline{EF}, P_\infty) \overset{K'}{=} (F, E; \ell \cap (AEF), \overline{AM} \cap (AEF)) \overset{A}{=} (B, C; P_\infty, M) = -1$ as desired.



For K', I chose a special point (that has a nice vector representation) on the A-symmedian that we will talk about later.

1.4 Ex Points,Queue Points, Humpty Points

As it doesn't make sense to talk about one of these sets of points without mentioning the other 2, I lumped them into one big section. This section has a lot of theorems that are relatively easy to prove, in contrast to other chapters which take more work to prove their theorems, but have less of them. First we can define the Ex Point. Assume that $\triangle DEF$ is the orthic triangle.

Definition 1.4.1 (Ex Points)**.** The A-**Ex Point** X_A is the intersection of $\overline{EF}$ and $\overline{BC}$.

This definition brings another theorem with it:

Theorem 1.4.1. $(X_A, D; B, C) = -1$

Thinking about the 9-point circle, we see yet another theorem:

Theorem 1.4.2 (Orthic Axis)**.** X_A, X_B, X_C lie on a line, called the orthic axis.

Since

$$X_AB \cdot X_AC = X_AD \cdot X_AM,$$

X_A is on the radical axis of the circumcircle of $\triangle ABC$ and the nine-point circle of $\triangle ABC$.

One more Ex-Point Theorem: Looking at $(BFEC)$, we have the perfect setup for Brocard.

Theorem 1.4.3. H is the orthocenter of AX_AM

Now here are the definitions of the Queue and Humpty Points, which will make it clear why I'm talking about them both together.

Definition 1.4.2 (Queue Points)**.** The A-**Queue Point**, Q_A, is the intersection of (AH) and (ABC).

Definition 1.4.3 (Humpty Points)**.** The A-**Humpty Point**, H_A, is the intersection of (HA) and (HBC).

You can probably see the connection. They're the same point, just in different triangles.

The Humpty Point is also called the HM point, but I like Humpty better.

Thus there are many properties that the Queue and Humpty points share. For example:

Theorem 1.4.4. Let M be the midpoint of BC. Then Q_A is on $\overline{HM}$.

Let A' be the antipode of A on (ABC). Then since $\measuredangle AQ_AH = 90 = \measuredangle AQ_AA'$, $Q_A - H - A'$. Since $H - M - A'$, $Q_A - H - M$. This leads to the analogous Humpty Point Theorem:

Theorem 1.4.5. Let M be the midpoint of BC. Then H_A is on $\overline{AM}$.

From now on I'm not going to state theorems for both points, because that's quite redundant.

Theorem 1.4.6. Let R and S be points on $\overline{BC}$ such that $(R, S; B, C) = -1$. A, H_A, R, S are concyclic.

This follows from the fact that

$$MA \cdot MH_A = ME^2 = MB \cdot MC = MR \cdot MS.$$

Corollary 1.4.1 (Appolonius Circle)**.** H_A lies on the A-Apollonius Circle

Theorem 1.4.7. Q_A is the center of spiral similarity from FE to BC.

This is purely because of the definition of Q_A, in terms of the cyclic quad $BFEC$.

Let W be the intersection of $\overline{AM}$ and (ABC). Let K be the intersection of the A-symmedian and (ABC).

Theorem 1.4.8. As H_A is on (BHC), naturally reflections over $\overline{BC}$ produce useful results.

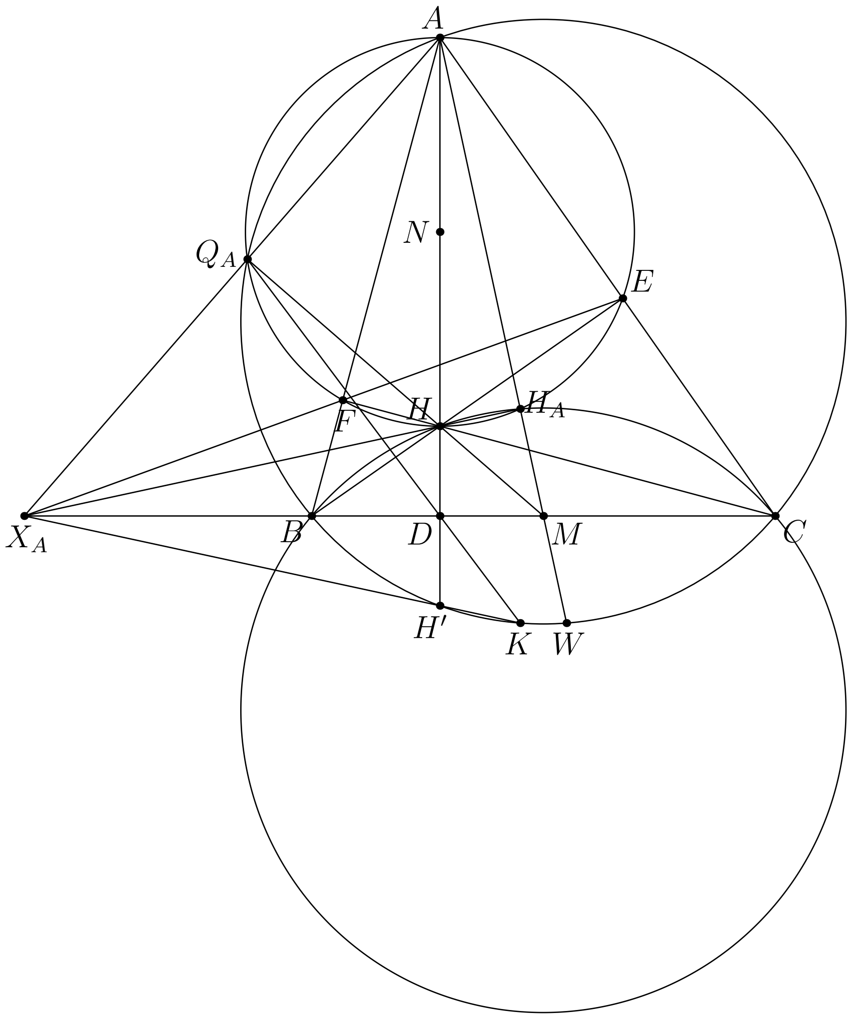
- W is the reflection of H_A over M
- K is the reflection of H_A over $\overline{BC}$.
- $X_A - H' - K$

The first thing follows from the fact that reflecting (BCH) over M gives (ABC), and the second thing follows from the first. The third follows from reflecting $\overline{X_AHH_A}$ across $\overline{BC}$.

Theorem 1.4.9. Let H' be the reflection of H over $\overline{BC}$. Then $-1 = (Q_A, H'; B, C)$ and $Q_A - D - K$

This is just because $(Q_A, H'; B, C) \stackrel{A}{=} (X_A, D; B, C) = -1$ and $-1 = (A, K; B, C) \stackrel{D}{=} (H', \overline{KD} \cap (ABC); B, C) = (H', Q_A; B, C)$.

Here's a diagram to hopefully recap some of the previous few theorems.



N is the midpoint of AH

With this we get lots of cyclic quads:

Theorem 1.4.10. All of these quadrilaterals are cyclic because of the many right angles produced, plus PoP in some cases:

- X_AQ_AHD
- $X_AQ_AH_AM$
- HH_AMD
- $DMKH'$
- $AMDQ_A$
- X_ADH_AA

We actually have even more cyclic quads using other methods.

Theorem 1.4.11. Although there aren't obvious right angles for these, they can be shown to be cyclic quite easily as well.

- $KH'EF$
- $Q_A N H_A D$
- $H' H H_A K$

$KH'EF$ is cyclic because

$$X_A F \cdot X_A E = X_A B \cdot X_A C = X_A H' \cdot X_A K.$$

$Q_A N H_A D$ is cyclic because it's the nine-point circle of $\triangle A X_A M$. $H' H H_A K$ is cyclic because it's an isosceles trapezoid.

Because of the definition of H_A, we have an important angle equality:

Theorem 1.4.12.

$$\measuredangle CBH_A = \measuredangle BAH_A$$

and

$$\measuredangle H_A CB = \measuredangle H_A AC$$

This is because

$$\measuredangle CBH_A = \measuredangle CHH_A =$$

$$\measuredangle CHA - \measuredangle H_A HA =$$

$$90 + \measuredangle HAH_A - \measuredangle CBA = \measuredangle BAH + \measuredangle HAH_A =$$

$$\measuredangle BAH_A,$$

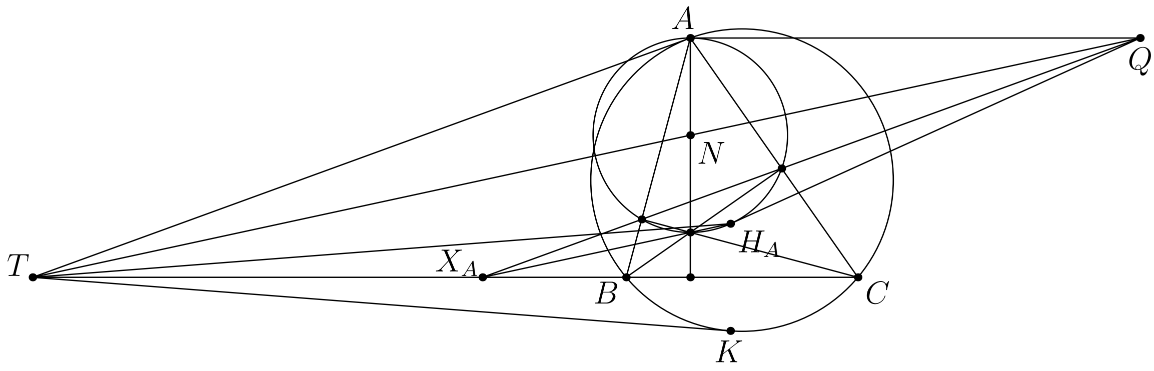
as desired (the other one is true by symmetry).

We also have a fairly often tested collinearity that's quite simple with knowledge of H_A.

Theorem 1.4.13. Let T be the intersection of $\overline{BC}$ and the tangent at A to (ABC). Let N be the midpoint of AH. Finally let Q be the point on $\overline{EF}$ such that $\overline{AQ}||\overline{BC}$. $T - N - Q$.

The hidden Humpty Point part is that this line is the perpendicular bisector of AH_A.

- Because $(A, K; B, C) = -1$, $TA = TK$. Since K and H_A are reflections across $\overline{BC}$, $TA = TK = TH_A$.
- Since A and H_A are on (AEF), which has center N, $NA = NH_A$.
- Since AH_A is the A-symmedian of $\triangle AEF$, $(A, H_A; E, F) = -1$. $\overline{AQ}$ is tangent to (AEF), so $AQ = AH_A$.



Maybe I shouldn't have chosen this triangle

This also means that this goes through a fourth point that shows up often enough:

Corollary 1.4.2. The midpoint of AH_A is on $\overline{TNQ}$.

Since H_A is on (BHC), a dilation around A with a scale factor of $\frac{1}{2}$ puts the midpoint of AH_A on the nine-point circle of $\triangle ABC$. Thus this point is usually classified as the intersection of $\overline{AM}$ and the nine-point circle of $\triangle ABC$.

1.5 Dumpty Points

Theorem 1.5.1. The Poncelet Point of A, B, C, H_A, where H_A is the A-Humpty Point, is the midpoint of AH_A.

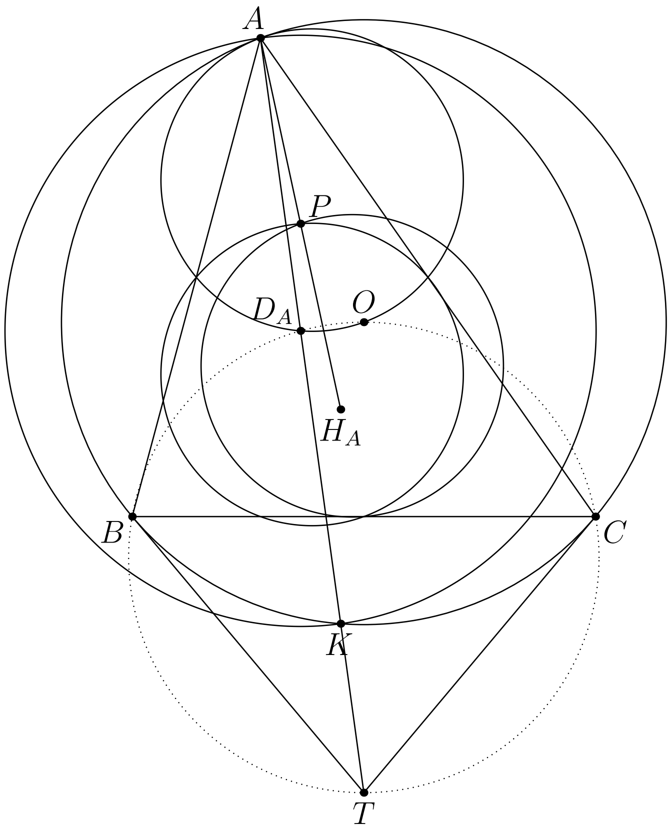
Dilate (ABC) around H_A with scale factor $\frac{1}{2}$. This circle contains the midpoint of BH_A, the midpoint of CH_A, and the midpoint of BC, so it is the nine-point circle of $\triangle BCH_A$. Thus the midpoint of AH_A, P, is on the nine-point circle of $\triangle BCH_A$. P is also on the nine-point circle of $\triangle ABC$, meaning it is the Poncelet Point of $ABCH_A$.

We now have that P is on the circumcircle of the pedal triangle of H_A. Dilate this circle around H_A by a factor of 2. We get that this new circle, ω, goes through A and K, where $\overline{AK}$ is a symmedian.

We can now define the Dumpty Point:

Definition 1.5.1 (Dumpty Point)**.** The A-**Dumpty Point**, D_A is the Isogonal Conjugate of the A-Humpty Point

D_A is the center of ω. However we also know that because H_A is on the A-median, D_A is on $\overline{AK}$. This means that D_A is on the midpoint of AK.



Let T be the intersection of the tangents to (ABC) at B and C

Because D_A is the midpoint of AK we have that D_A is on (AO), where O is the center of (ABC). Since $\measuredangle OD_AA = 90 = \measuredangle OD_AT$, D_A is on (BOC). This leads us to the classical definition of the Dumpty Point (none of this is classical by the way).

Definition 1.5.2 (Classical Dumpty Point Definition)**.** The A-**Dumpty Point** is the intersection of (AO) and (BOC).

From the fact that D_A is the isogonal conjugate to the Humpty Point, and in the Humpty Point

$$\measuredangle CBH_A = \measuredangle BAH_A$$

and

$$\measuredangle H_ACB = \measuredangle H_AAC,$$

we have that

$$\measuredangle D_ABA = \measuredangle D_AAC$$

and

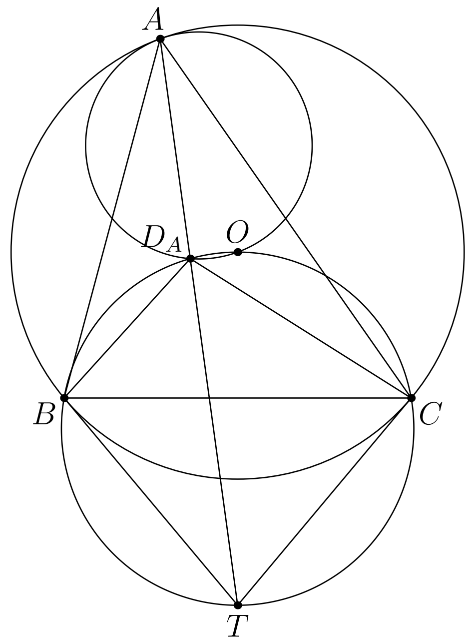
$$\measuredangle BAD_A = \measuredangle ACD_A.$$

This gives us the following result:

Theorem 1.5.2. D_A is the center of spiral similarity sending BA to AC (and the center of spiral similarity from CA to AB).

This is quite profound in that it's unclear what the center of spiral similarity would be in this scenario.

In a contest situation, it would likely be cumbersome to list out all of the above, plus results on the Humpty Point, just for these results. I used this proof because I thought it was quite nice and tied well with the previous sections. Here's another proof that this is the desired spiral center, starting with the assumption that D_A is the intersection of (AO) and (BOC).



A pure angle chase

$$\measuredangle D_ABA = \measuredangle BD_AA + \measuredangle D_AAB =$$

$$\measuredangle BD_AT + \measuredangle D_AAB = \measuredangle BCT + \measuredangle D_AAB =$$

$$\measuredangle BAC - \measuredangle BAD_A = \measuredangle D_AAC.$$

Similarly

$$\measuredangle BAD_A = \measuredangle ACD_A,$$

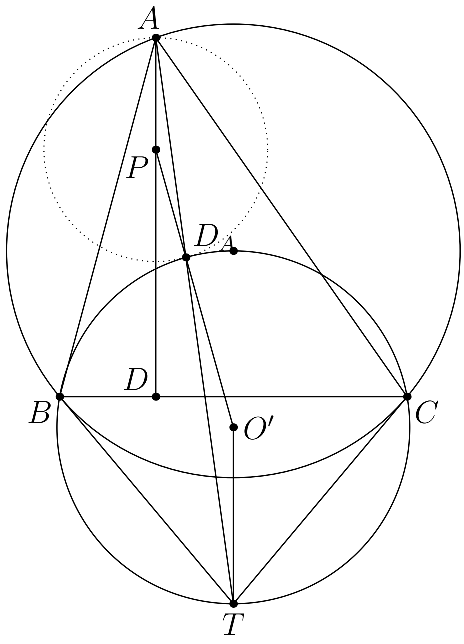
as desired. We can then state this next fact:

Theorem 1.5.3. Let D be the foot of the altitude from A to $\overline{BC}$ and let M_B and M_C denote the midpoints of AC and AB respectively. Then D_A is the D-Humpty Point of $\triangle DM_BM_C$.

This is because we can do a homothety at A with factor 2 and then reflect the image across $\overline{BC}$. D would go to A, M_B would go to C, M_C would go to B, and D_A would go to H_A.

There's actually another fact about D_A concerning the A-altitude:

Theorem 1.5.4. The circle through A and D_A tangent to (BOC) is centered on the A-altitude.



This diagram makes it quite obvious what's going on

If O' is the center of (BOC), then this is equivalent to proving that the perpendicular bisector of AD_A and $\overline{O'D_A}$ meet on $\overline{AD}$(where D is the foot of the altitude from A to $\overline{BC}$). If we let this intersection point be P, then

$$\measuredangle PAD_A = \measuredangle O'TD_A = \measuredangle TD_AO' = \measuredangle AD_AP$$

In bashing, the Dumpty and Humpty Points are suprisingly nice to work with.

Theorem 1.5.5. Assuming A, B, and C are on the unit circle, and represented as a, b, c:

- $D_A = \frac{a^2-bc}{2a-b-c}$
- $H_A = \frac{ab^2+ac^2-b^2c-bc^2}{ab+ac-2bc}$

The Dumpty Point definition follows straight from the spiral sim. The Humpty Point will take a little more work.

$$K = \overline{AD_A} \cap (ABC) = 2 \cdot \frac{a^2-bc}{2a-b-c} - a = \frac{ab+ac-2bc}{2a-b-c}.$$

This means that

$$W = \overline{AM} \cap (ABC) = \frac{bc}{\frac{ab+ac-2bc}{2a-b-c}} = \frac{2abc-b^2c-bc^2}{ab+ac-2bc}.$$

Reflecting this over the midpoint of BC, M, gives

$$H_A = b + c + \frac{b^2c+bc^2-2abc}{ab+ac-2bc} = \frac{ab^2+ac^2-b^2c-bc^2}{ab+ac-2bc}.$$

Now that we've cornered D_A and H_A in complex, and defined the symmedian intersection point in the process, we can move on to bary.

Theorem 1.5.6. In barycentric coordinates, if $A = (1, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 1)$, then

- $D_A = (b^2 + c^2 - a^2 : b^2 : c^2)$, or alternatively $D_A = (2S_A : b^2 : c^2)$
- $H_A = (a^2 : b^2 + c^2 - a^2 : b^2 + c^2 - a^2)$, or alternatively $H_A = (a^2 : 2S_A : 2S_A)$

We get that

$$K = (-a^2 : 2b^2 : 2c^2)$$

because

$$-a^2 \cdot 2b^2 \cdot 2c^2 - b^2 \cdot -a^2 \cdot 2c^2 - c^2 \cdot -a^2 \cdot 2b^2 = 0.$$

Thus $D_A = (2b^2 + 2c^2 - a^2 - a^2 : 2b^2 : 2c^2) = (b^2 + c^2 - a^2 : b^2 : c^2)$. Because H_A is its isogonal conjugate

$$H_A = (\frac{a^2}{2S_A} : 1 : 1) = (a^2 : 2S_A : 2S_A).$$

1.6 Why Points

I'm calling these Why Points because as of yet, they don't seem to have a name other than the clunky "ISL 2011 G4 Points".

Definition 1.6.1 (Why Point)**.** In $\triangle ABC$, let A' be the reflection of A over the perpendicular bisector of BC and let D be the foot of the altitude from A to $\overline{BC}$. Then the A-**Why Point**, Y_A, is the intersection of $\overline{A'D} \cap (ABC)$

The first theorem doesn't concern the Why Points, but it's important for later.

Theorem 1.6.1. Let G be the centroid of $\triangle ABC$. Then $D - G - A'$.

This is immediate from a homothety around G with a scale factor of $\frac{-1}{2}$. This takes (ABC) to the nine-point circle of $\triangle ABC$, and it takes A' to the intersection of $\overline{BC}$ and this nine-point circle that is not the midpoint of BC. This means that A' goes to D. This also means that $A'G = 2DG$.

If we let $W = \overline{AM} \cap (ABC)$, and M be the midpoint of BC, then we have

$$GA \cdot GW = GA' \cdot GY_A$$

or

$$GM \cdot GW = GD \cdot GY_A.$$

Thus we have our second property

Theorem 1.6.2. $DMWY_A$ is cyclic

Extend $\overline{Y_AW}$ to $\overline{BC}$ at X'_A. Then

$$X'_AD \cdot X'_AM = X'_AY_A \cdot X'_AW = X'_AB \cdot X'_AC,$$

so $X'_A \equiv X_A$.

Theorem 1.6.3. $X_A - Y_A - W$

Let H_A be the intersection of $\overline{AD}$ and (ABC) (not the A-Humpty Point). Then we have the following theorem:

Theorem 1.6.4. Y_A is the Humpty Point of $\triangle H_A D X_A$

The orthocenter of $\triangle H_A D X_A$ is D. Since $\measuredangle H_A Y_A D = 90$ (in definition of Humpty Point) and B, C, H_A, Y_A are cyclic, where B and C satisfy $(X_A, D; B, C) = -1$ (Theorem 1.4.6), Y_A is the desired Humpty Point.

Corollary 1.6.1. $\overline{H_A Y_A}$ bisects $X_A D$

Since H, the orthocenter of ABC, is the reflection of H_A over $\overline{X_A D}$, by the definition of the Humpty Point we get two more results.

Theorem 1.6.5. Let A^* be the antipode of A on (ABC). Then $-1 = (A, K; Y_A, A^*)$.

Let N be the midpoint of $X_A D$. Then

$$-1 = (D, X_A; N, P_\infty) \stackrel{H_A}{=} (A, K; Y_A, A^*).$$

Corollary 1.6.2. Let T be the intersection of the A-tangent to (ABC) and $\overline{BC}$. Then $T - Y_A - A^*$.

Theorem 1.6.6. Y_A is on (HDX_A), or the circle with diameter $X_A H$.

This circle is the reflection of $(H_A D X_A)$ over $\overline{DX_A}$, giving the result.

Corollary 1.6.3. Y_A is the spiral center taking $X_A H$ to DH_A

This is because $\measuredangle X_A Y_A H = 90 = \measuredangle D Y_A H_A$ and $\measuredangle Y_A X_A H = \measuredangle Y_A D H = \measuredangle Y_A D H_A$.

Looking at $\overline{AY_A}$ we see that this passes through a special point.

Theorem 1.6.7. $\overline{AY_A}$ goes through Z, the foot of the altitude from H to $\overline{EF}$.

This is because under an inversion around A with radius $\sqrt{AH \cdot AD}$, $(X_A Y_A D H)$ inverts to itself (note that $AH \cdot AD = AE \cdot AC = AF \cdot AB$). This means that Y_A goes to the point on $\overline{EF}$ that is on $(X_A Y_A D H)$, which is Z.

Theorem 1.6.8 (ISL 2011 G4 with better labels)**.** Let M_B and M_C be the midpoints of AC and AB respectively. Then $(Y_A M_B M_C)$ is tangent to (ABC).

Let P be the foot from O to $\overline{M_B M_C}$. Then by centroid homotheties P is on $\overline{DG}$, or $\overline{Y_A A'}$. We then have

$$\measuredangle M_C P Y_A = \measuredangle A A' Y_A = \measuredangle A B Y_A = M_C B Y_A$$

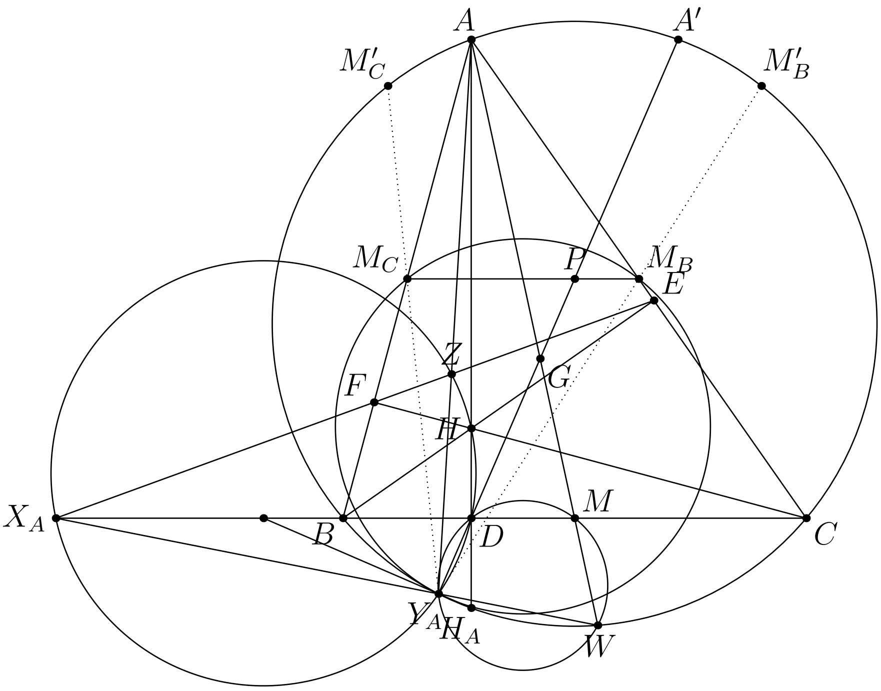
and similarly

$$\measuredangle M_B P Y_A = \measuredangle M_B C Y_A.$$

Thus $M_C B Y_A P$ and $M_B C Y_A P$ are cyclic. Since P is on the perpendicular bisector of BC,

$$\measuredangle M_C Y_A B = \measuredangle M_C P B = 90 - \measuredangle BPM - 90 - \measuredangle MPC = \measuredangle CPM_B = \measuredangle C Y_A M_B.$$

Letting M_B' and M_C' be the intersection of $\overline{Y_A M_B}$ and $\overline{Y_A M_C}$ respectively with (ABC). Then because $\overline{M_B' M_C'} || \overline{BC} || \overline{M_B M_C}$, a homothety centered at Y_A takes (ABC) to $(Y_A M_B M_C)$.



The dot between X_A and D is the midpoint of X_AD

1.7 Others

Let P be a point on the circumcircle of $\triangle ABC$.

Theorem 1.7.1 (Simson Line)**.** The feet of the altitudes from P to each of the sides of $\triangle ABC$ are collinear on the Simson Line.

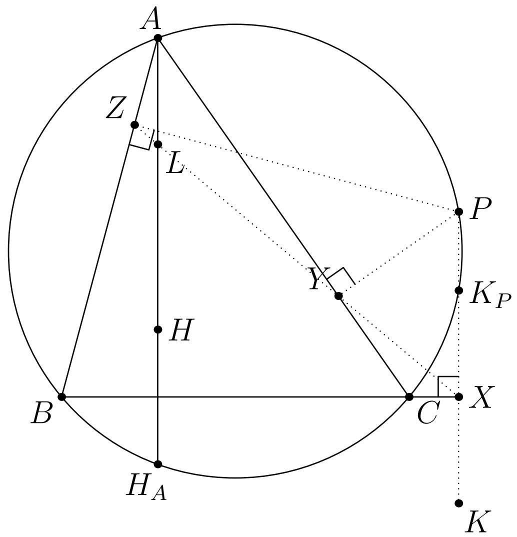
Let H is the orthocenter of ABC, let K be the orthocenter of PBC, let H_A be the reflection of H over $\overline{BC}$, let K_P be the reflection of K over $\overline{BC}$, let L be the intersection of the A-altitude and P-simson line of ABC, and let X be the foot of P to $\overline{BC}$.

Theorem 1.7.2. The following are all parallelograms:

- LAK_PX
- $LAXK$
- $APKH$
- $LPXH$

This leads to the corollary

Corollary 1.7.1. The P-simson line of $\triangle ABC$ bisects PH, where H is the orthocenter of $\triangle ABC$.



"All you need to do is construct a parallelogram!"-James Tao

- We have that $APYZ$, $BZPX$, and $CYPX$ are cyclic because of the 90-degree angles formed.

  $$\measuredangle PYZ = \measuredangle PAZ = \measuredangle PCB = \measuredangle PCX = \measuredangle PYX$$

  gives this collinearity.

- $$\measuredangle PK_PA = \measuredangle PCA = \measuredangle PXY = \measuredangle PXL$$

  proves that LAK_PX is a parallelogram.

- Since $K_PX = KX$, $LAXK$ is a parallelogram.

- Since APK_PH_A is an isosceles trapezoid, and reflecting K_PH_A over $\overline{BC}$ gives KH, $APKH$ is a parallelogram.

- Since $AH = PK$ and $AL = XK$, $LH = PX$, giving that $LPXH$ is a parallelogram.

We can actually say more than Corollary 1.7.1.

Corollary 1.7.2. The midpoint of PH is the Poncelet Point of $ABCP$.

Notice that the midpoint of PH is on the nine-point circle of $\triangle ABC$. Since it's also on the nine-point circles of $\triangle AHP$, $\triangle BHP$, $\triangle CHP$, it's the Poncelet Point of A, B, C, P, as desired. Notice that we also used this logic at the beginning of the Dumpty Point section, but with H, B, C, H_A.

Theorem 1.7.3 (Paralleogram Isogonality Lemma)**.** For a triangle ABC, let P be the a point such that $\measuredangle ACP = \measuredangle PBA$ and let Q be the point such that $BPCQ$ is a parallelogram. Then $\overline{AP}$ and $\overline{AQ}$ are isogonal.

Let R be the point such that $APBR$ is a parallelogram. Then $ACQR$ is also a parallelogram.

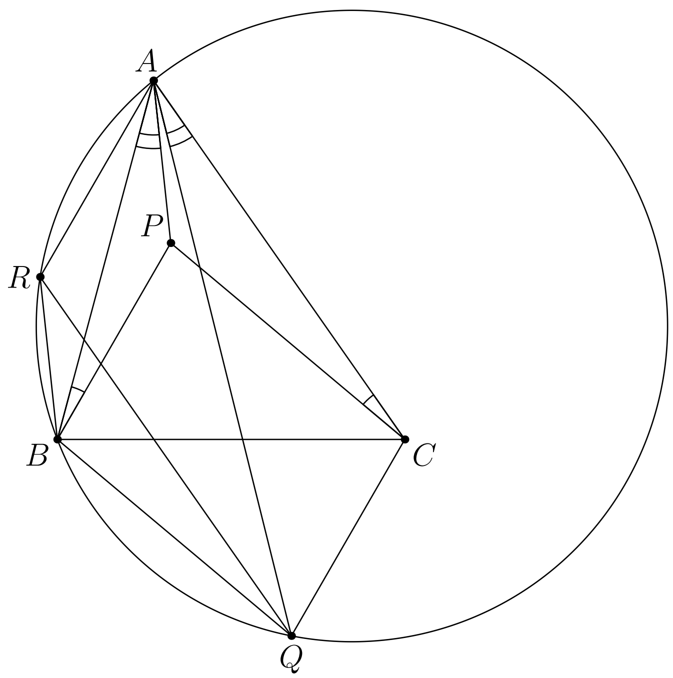
$$\measuredangle RQB = \measuredangle CQB - \measuredangle CQR =$$

$$\measuredangle BPC - \measuredangle RAC = \measuredangle BPC - \measuredangle BAC - \measuredangle RAB =$$

$$\measuredangle ACB + \measuredangle CBA - \measuredangle PCB - \measuredangle CBP - \measuredangle PBA =$$

$$\measuredangle ACP = \measuredangle PBA = \measuredangle RAB.$$

Thus $ARBQ$ is a cyclic quadrilateral, so $\measuredangle BAP = \measuredangle ABR = \measuredangle AQR = \measuredangle QAC$, as desired.



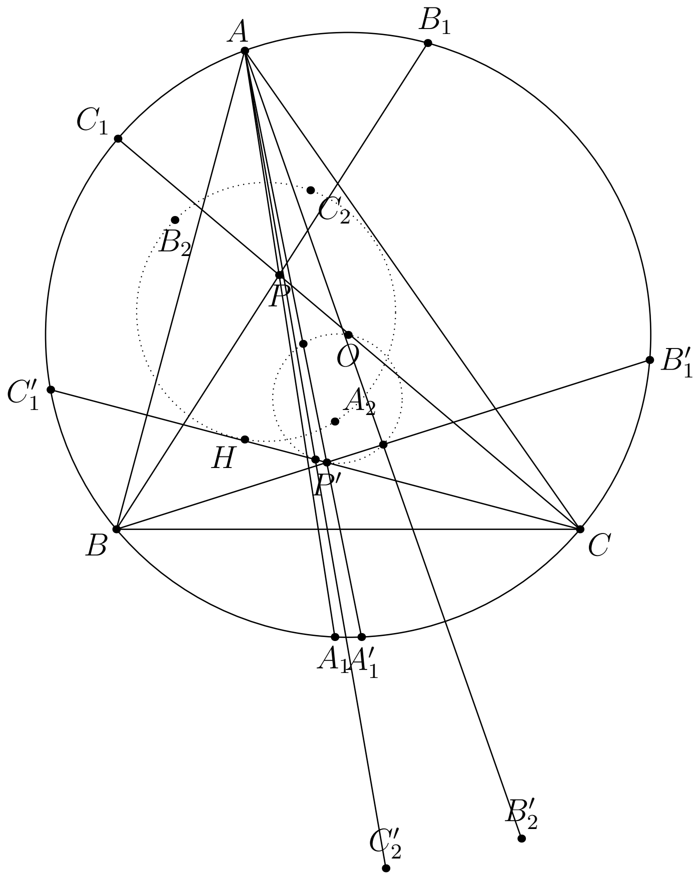
You could also prove this with DDIT but sadly EGMO doesn't talk about DDIT.

Now let P be an arbitrary point.

Theorem 1.7.4 (Hagge Circle)**.** Let $\overline{AP} \cap (ABC) = A_1$, and let A_2 be the reflection of A_1 across $\overline{BC}$. Define B_2 and C_2 similarly. Then H, A_2, B_2, C_2 are concyclic on the P-Hagge Circle.

Let P' be the isogonal conjugate of P. Then let A_1', B_1', C_1' be the inersection of $\overline{AP'}, \overline{BP'}, \overline{CP'}$ respectively with (ABC). Let B_2' and C_2' be the reflections of B_1' and C_1' across the midpoint of BC.

Since B_1' is clearly the reflection of B_2 across the midpoint of AC, $AB_1'CB_2$ is a parallelogram. Since CB_2BB_2' is also a parallelogram, $AB_1'B_2'B$ is a parallelogram. Thus the midpoint of AB_2' is the foot from O to $\overline{BB_1'}$, or the foot from O to $\overline{BP'}$. Thus the midpoints of AA_1', AB_2', AC_2' are on the circle with diameter OP'. A homothety at A with factor 2 plus a reflection across the midpoint of BC proves the theorem.



This is pretty much copy pasted from Telv Cohl

Corollary 1.7.3. The center of the Hagge Circle is the reflection of P' over the nine-point center of $\triangle ABC$.

Chapter 2

Incenter Configurations

2.1 Fact 5 and Excentral-Orthic Duality

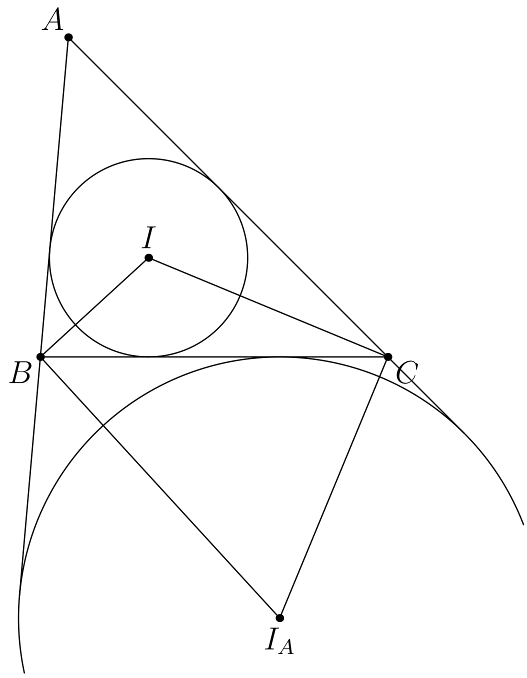
We can start with the definitions of the incenter, incircle, excenter, and excircle.

Definition 2.1.1 (Incircle)**.** The **incircle** of $\triangle ABC$ is the circle inside $\triangle ABC$ that is tangent to all of its sides.

Definition 2.1.2 (Excircle)**.** The A-**excircle** of $\triangle ABC$ is the circle tangent to all 3 lines $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ created by a homothety of the incircle around A.

Definition 2.1.3 (Incenter)**.** The **incenter** of $\triangle ABC$ is the center of the incircle.

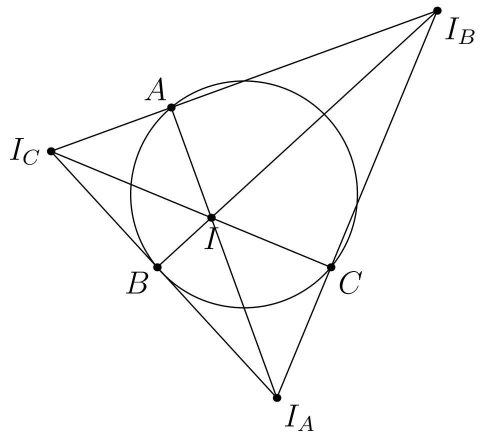
Definition 2.1.4 (Excenter)**.** The A-**excenter** of $\triangle ABC$ is the center of the A-excircle.



Hopefully you see what I mean

We see that I bisects the angles of $\triangle ABC$. We also get that $\overline{BI_A}$ and $\overline{CI_A}$ are the external angle bisectors of $\angle B$ and $\angle C$ respectively. We then get that because of this $\measuredangle I_ABI = 90 = \measuredangle I_ACI$.

Also because of the homothety at A from the incircle to the A-excircle we have that $A{-}I{-}I_A$. This leads to excentral-orthic duality.



What do you think is O(the center of (ABC)) in terms of $(I_AI_BI_C)$?

Theorem 2.1.1 (Excentral-Orthic Duality)**.** In $\triangle I_AI_BI_C$, I is its orthocenter and $\triangle ABC$ is its orthic triangle.

This leads to the so-called Fact 5:

Theorem 2.1.2 (Fact 5)**.** Let M_A be the midpoint of the arc BC on (ABC) not containing A. M_A is the midpoint of II_A, and thus the circumcenter of $(BICI_A)$.

If we remember that (ABC) is the nine-point circle of $\triangle I_AI_BI_C$, then this becomes obvious.

It is common for problems to be stated in terms of $\triangle ABC$, when it would make more sense for them to be stated in terms of $\triangle I_AI_BI_C$. Similarly it is common for problems to be stated in $\triangle ABC$ when it would be better stated in terms of its orthic triangle.

Especially concerning $\sqrt{bc}$ inversion, it is worth mentioning that $\triangle ABI \sim \triangle AI_AC$. This can be angle chased, or one can notice that since $\overline{BI}$ and $\overline{CI_A}$ meet at I_B and A, I_B, C, I and A, I_B, I_A, B are cyclic, A is the center of spiral similarity between BI and I_AC.

Theorem 2.1.3. $AI \cdot AI_A = AB \cdot AC$, so under $\sqrt{bc}$ inversion I goes to I_A.

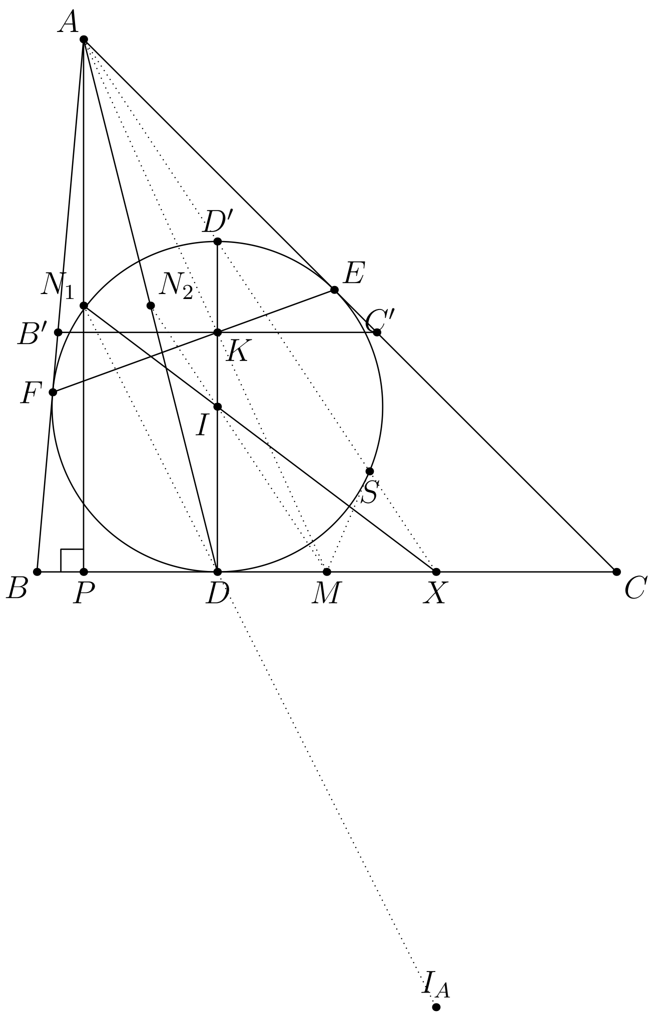
2.2 Incircle Homotheties

Let D be the tangency point of the incircle and $\overline{BC}$, and let X be the analogous point for the A-excircle. Then because the midpoint of I and I_A is on the perpendicular bisector of BC, the midpoint of D and X is M, the midpoint of BC.

Let D' be the antipode of D on the incircle. Let N_1 be the midpoint of the AP, where the foot of A to $\overline{BC}$ is P, and let N_2 be the midpoint of AD. Let E and F be the tangency points of the incircle with $\overline{AC}$ and $\overline{AB}$ respectively. Finally let K be the intersection of $\overline{DI}$ and $\overline{EF}$.

Theorem 2.2.1. The following collinearities hold:

- $A - D' - X$
- $N_2 - I - M$
- $N_1 - I - X$ - Midpoint of Altitude Lemma
- $N_1 - D - I_A$ - Midpoint of Altitude Lemma
- $A - K - M$ - Median-Incircle Concurrency



In this chapter we have $\angle A = 50$, $\angle B = 85$, and $\angle C = 45$.

For the first 4 collinearities,

- $A - D' - X$ because the homothety sending the incircle to the A-excircle sends D' to X.
- $N_2 - I - M$ because it's the D-midline of $\triangle ADX$.
- $N_1 - I - X$ because it's the X-median of $\triangle APX$.
- $N_1 - D - I_A$ because $\frac{PD}{PX} = \frac{DI}{XI_A}$ and $N_1 - I - X$.

Corollary 2.2.1. $\overline{AX}||\overline{IM}$ and $\overline{AD}||\overline{MI_A}$

The first one is by a homothety at D with factor 2 and the second one can result from a homothety at X with factor 2 (by homothety the bottom point of the excircle is collinear with A and D).

For the last collinearity, let B' and C' be the intersection of the parallel of $\overline{BC}$ through K and $\overline{AB}$ and $\overline{AC}$, respectively. Then $\measuredangle B'KI = \measuredangle B'FI = 90$ and $\measuredangle IKC' = \measuredangle IEC' = 90$, so $B'KFI$ and $IKEC'$ are cyclic.

We have

$$\measuredangle IB'A = \measuredangle IB'K + \measuredangle KB'A = \measuredangle IFE + \measuredangle CBA = \measuredangle IAC + \measuredangle CBA.$$

Similarly

$$\measuredangle IC'A = \measuredangle IAB + \measuredangle BCA.$$

Since these are equal, $AB'IC'$ is cyclic. Thus the homothety sending $(AB'IC)$ to (ABC) around A sends K to M, as desired.

If instead of I, we used I_A (and we used the A-excircle instead of the incircle), by a homothety around A, the last collinearity still holds.

While I used a homothety solution (because I named this chapter Incircle Homotheties), try finding the polar of K with respect to the incircle, and proving that K is on the A-median that way.

Let S be the intersection of $\overline{AX}$ and (DEF).

Theorem 2.2.2. $\overline{SM}$ is tangent to (DEF).

$\measuredangle DSX = \measuredangle DSD' = 90$, so M is the circumcenter of $\triangle DSX$, meaning that $MD = MS$, as desired.

Here are some definitions that, while not very related to incircle homotheties, are important to state:

Definition 2.2.1 (Gergonne Point)**.** The **Gergonne Point** is the intersection of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$.

This exists either by noting that the Gergonne Point is the symmedian point of $\triangle DEF$ or by using Ceva's theorem on $\triangle ABC$.

Definition 2.2.2 (Intouch Triangle)**.** The **Intouch Triangle** or **Contact Triangle** of $\triangle ABC$ is the pedal triangle of its incenter, I.

2.3 Feuerbach Point

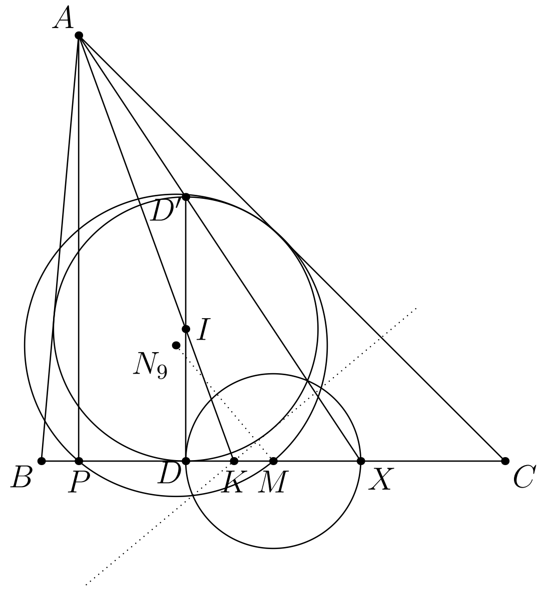
So far in this handout we've seen 1 special instance of a Poncelet Point; that of A, B, C, H_A, where H_A is the A-Humpty Point of $\triangle ABC$. Now we will see another special instance:

Definition 2.3.1 (Feuerbach Point)**.** The **Feuerbach Point** Fe is the Poncelet Point of A, B, C, I, where I is the incenter of $\triangle ABC$.

As the Poncelet Point of these 4 points, it would have to be one of the intersections of the nine-point circle of $\triangle ABC$ and the incircle of $\triangle ABC$, as the incircle is the pedal triangle of I. We can actually state something stronger about the intersections of these two circles, and the excircles.

Theorem 2.3.1. The nine-point circle of $\triangle ABC$ is tangent to its incircle and its 3 excircles.

It suffices to show that the nine-point circle is tangent to the incircle and the A-excircle. I'll only prove it for the incircle, as the A-excircle case is nearly identical.



A weird inversion

Let P be the foot of the altitude from A to $\overline{BC}$. Let $K = \overline{AI} \cap \overline{BC}$, and let N_9 be the center of the nine-point circle. Then since

$$(P, K; D, X) \stackrel{A}{=} (P_\infty, I; D, D') = -1,$$

we have

$$MP \cdot MJ = MD \cdot MX.$$

The trick is to now invert around (DX). The incircle is orthogonal to this circle, so it stays in place. The nine-point circle goes to a line through K perpendicular to $\overline{MN_9}$. It suffices to show that this line is tangent to the incircle.

Let the line through K tangent to the incircle be ℓ. Then

$$\measuredangle(\ell, \overline{BC}) = 2\measuredangle AKB = 2\measuredangle IAB + 2\measuredangle ABK = \measuredangle CAB + 2\measuredangle ABC.$$

Since this is equivalent to the angle made by the line perpendicular to $\overline{N_9M}$ and $\overline{BC}$, the nine-point circle is tangent to the incircle, as desired.

An interesting side note because of this is to realize that the nine-point circle is tangent to not 2, not 4, but 16 circles as a result of the above, because of the quadrality of the orthocentric system.

This proof also leads to this:

Corollary 2.3.1. Let D^* be the reflection of D over $\overline{AI}$. Then $\overline{MD^*}$ goes through Fe.

Theorem 2.3.2. Let M_{AI} denote the midpoint of A and I. Then $M_{AI} - D' - Fe$.

We will actually prove that $M_{BI} - E' - Fe$ and $M_{CI} - F' - Fe$, where M_{BI},M_{CI},E',F' are defined as expected. By symmetry $M_{AI} - D' - Fe$.

$$\measuredangle DE'F' = 90 - \measuredangle DEF = 90 - \measuredangle BDF = \measuredangle CBI.$$

Similarly

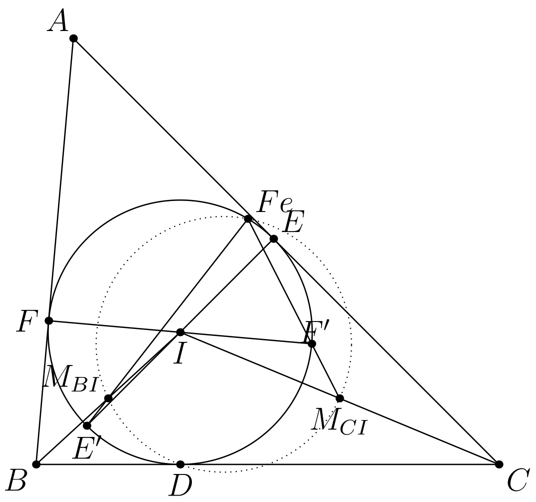
$$\measuredangle DF'E' = \measuredangle BCI.$$

Thus

$$\triangle DE'F' \sim \triangle IBC.$$

$\triangle DM_{BI}M_{CI}$ is similar to $\triangle IBC$ as well because a dilation around I with factor 2 sends $\triangle DM_{BI}M_{CI}$ to the reflection of $\triangle IBC$ across $\overline{BC}$.

Thus D is the center of spiral similarity between $M_{BI}M_{CI}$ and $E'F'$. $\overline{M_{BI}E'}$ and $\overline{M_{CI}F'}$ must meet on the intersection of $(DE'F')$ and $(DM_{BI}M_{CI})$. Since Fe is the Poncelet point of $ABCI$, $(DE'F')$ is the circumcircle of the pedal triangle of I, and $(DM_{BI}M_{CI})$ is the nine-point circle of $\triangle BIC$, Fe is this intersection point, as desired.



Poncelet Points are quite helpful, and I've only scratched the surface of the surface of them.

2.4 Sharkydevil

Definition 2.4.1 (Sharkydevil Point)**.** The A-Sharkydevil Point, S, is the intersection of $(AEIF)$ and (ABC).

If A' is the antipode of A on (ABC), then since $\measuredangle ASA' = \measuredangle ASI = 90$, $S - I - A'$. Invert around the incircle. A goes to the midpoint of EF, B goes to the midpoint of DF, and C goes to the midpoint of DE. Therefore (ABC) goes to the 9-point circle of DEF. $(AEIF)$ goes to $\overline{EF}$, so we have the following theorem.

Theorem 2.4.1. Let P be the foot of the altitude from D to $\overline{EF}$. Then $S-P-I-A'$.

Additionally by the definition of the Sharkydevil Point we have the following:

Theorem 2.4.2. S is the center of spiral similarity sending FE to BC.

This spiral similarity is actually quite interesting. It sends F to B, E to C, I to M_A(the midpoint of the arc BC of (ABC) not containing A), but where does it send P?

Theorem 2.4.3. The spiral similarity sending FE to BC sends P to D, or alternatively $S-D-M_A$.

Since $\overline{PD}||\overline{IM_A}$ and D is on $\overline{BC}$, this is true. Additionally, $\frac{BS}{SC}=\frac{BE}{CF}=\frac{BD}{CD}$, so $\overline{SD}$ bisects $\angle BSC$. This means that $S-D-M_A$.

Theorem 2.4.4. Let L be the midpoint of $\widehat{BAC}$. Then $\overline{LS}$, $\overline{EF}$, and $\overline{BC}$ concur.

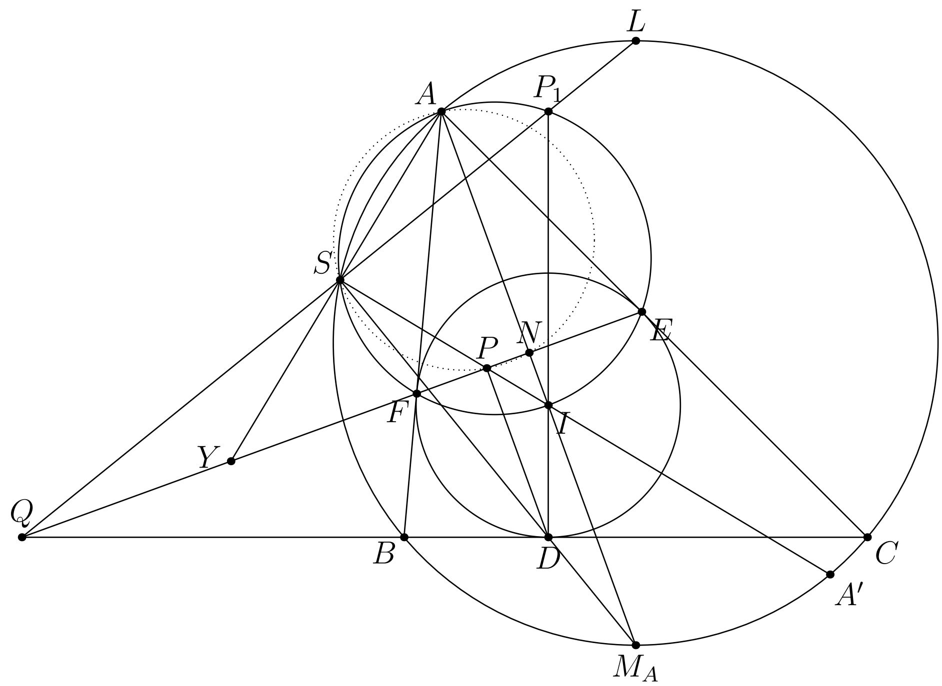
This is because $(\overline{LS}\cap\overline{BC},D;B,C)\stackrel{S}{=}(L,M_A;B,C)=-1$. We can call $\overline{LS}\cap\overline{BC}$ as Q. Let Y be the intersection of $\overline{AS}$ and $\overline{EF}$.

Theorem 2.4.5. If $P_1=\overline{DI}\cap(AEF)$, then $S-P_1-L$.

$\measuredangle SP_1D=\measuredangle SP_1I=\measuredangle SAI=\measuredangle SAM_A=\measuredangle SLM_A$, so $\triangle SDP_1$ and $\triangle SM_AL$ are homothetic, as desired.

Theorem 2.4.6. Y is the D-Ex Point of $\triangle DEF$.

This is simply because $(Y,P;E,F)\stackrel{S}{=}(A,I;E,F)=-1$. Alternatively if N is the midpoint of EF then $\measuredangle ASP=\measuredangle ASI=90-\measuredangle ANP$, so $ASPN$ is cyclic, and $YF\cdot YE=YS\cdot YA=YP\cdot YN$.



The best *shakes head back and forth vigorously* is yet *head continues shaking back and forth vigorously* to come.

Because of all of the right angles and PoP produced, there are tons of concyclic points. However I'll just show two rather useful examples:

Theorem 2.4.7. Let Q_D be the D-Queue Point and let K be the intersection between $\overline{AI}$ and $\overline{BC}$. Then A, S, Q_D, D, K all are concyclic.

$$YQ_D \cdot YD = YF \cdot YE = YS \cdot YA,$$

so A, S, Q_D, D are concyclic.

$$\measuredangle AKD = \measuredangle ACB + \measuredangle M_A AC = \measuredangle ACB + \measuredangle BCM_A = \measuredangle ACM_A = \measuredangle ASM_A = \measuredangle ASD$$

so A, S, D, K are concyclic too, as desired.

Theorem 2.4.8. (MDQ_D) and (ABC) have radical axis $\overline{EF}$.

Since

$$QD \cdot QM = QB \cdot QC$$

and

$$YA \cdot YS = YD \cdot YQ_D,$$

this is true.

The remaining theorems in this section are an entry into the very rich configuration in the next section.

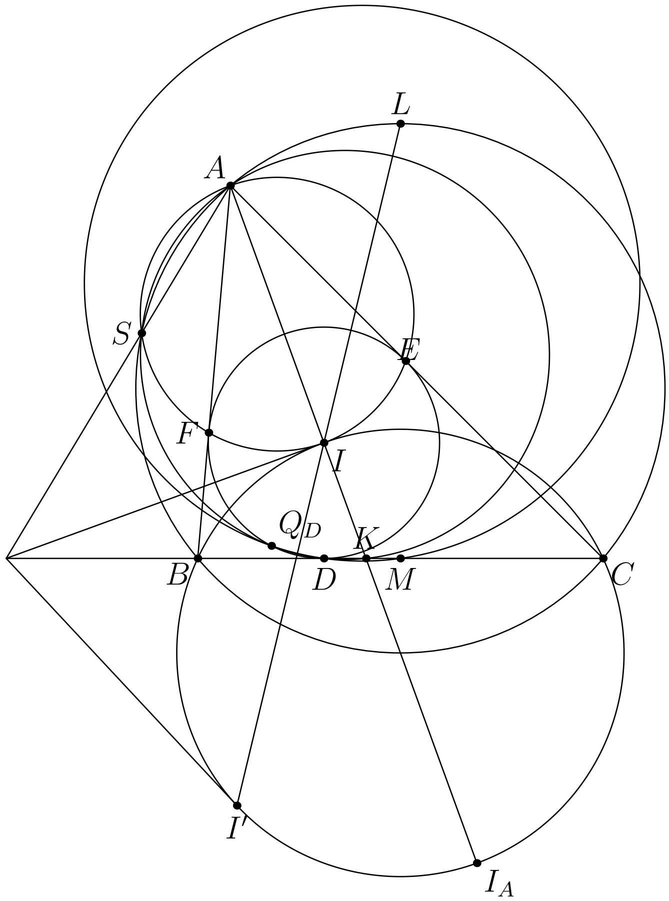
Theorem 2.4.9. Let I' be the intersection of $\overline{LI}$ and (BIC). Then the tangents to (BIC) at I, I' meet at the intersection of $\overline{AS}$ and $\overline{BC}$.

The first observation you hopefully had is that I said that the tangents to (BIC) at I and I' meet on $\overline{BC}$. This is because $\overline{LB}$ and $\overline{LC}$ are tangents to (BIC), so $(I, I'; B, C) = -1$.

Since both $(AEIF)$ and (BIC) are centered on $\overline{AM_A}$, the tangent to (BIC) at I is actually the line at I perpendicular to $\overline{AI}$, or the radical axis of $(AEIF)$ and (BIC). Radical axis on these two circles plus (ABC) gives that this tangent goes through $\overline{AS} \cap \overline{BC}$.

Theorem 2.4.10. Let $Z = \overline{AL} \cap \overline{BC}$. Then Z, A, I, D, I' are all concyclic.

Notice that $\measuredangle ZAI = \measuredangle ZDI = 90$. It suffices to show that $\measuredangle ZI'I = 90$. $\measuredangle I_A I'I = 90$, so it actually suffices to show that $Z - I' - I_A$. $-1 = (I', I; B, C) \stackrel{I_A}{=} (\overline{I'I_A} \cap \overline{BC}, K; B, C)$ as desired.

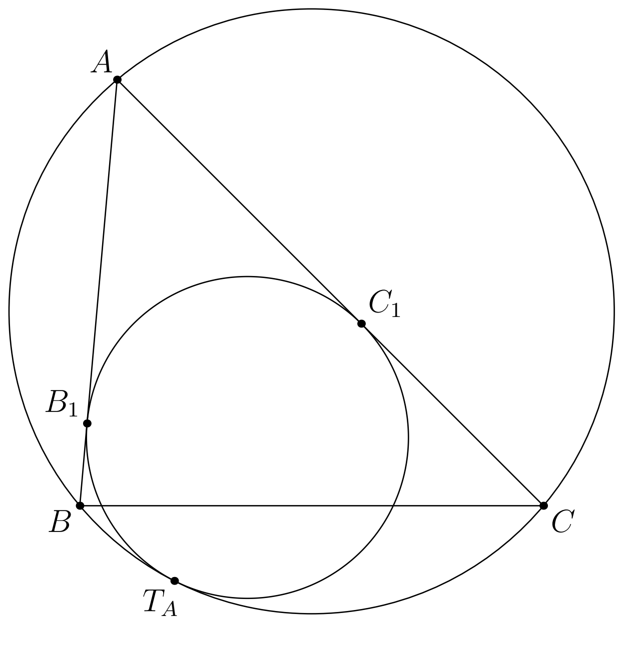


With Z in the diagram it was almost unreadable.

2.5 Mixtilinear Incircle

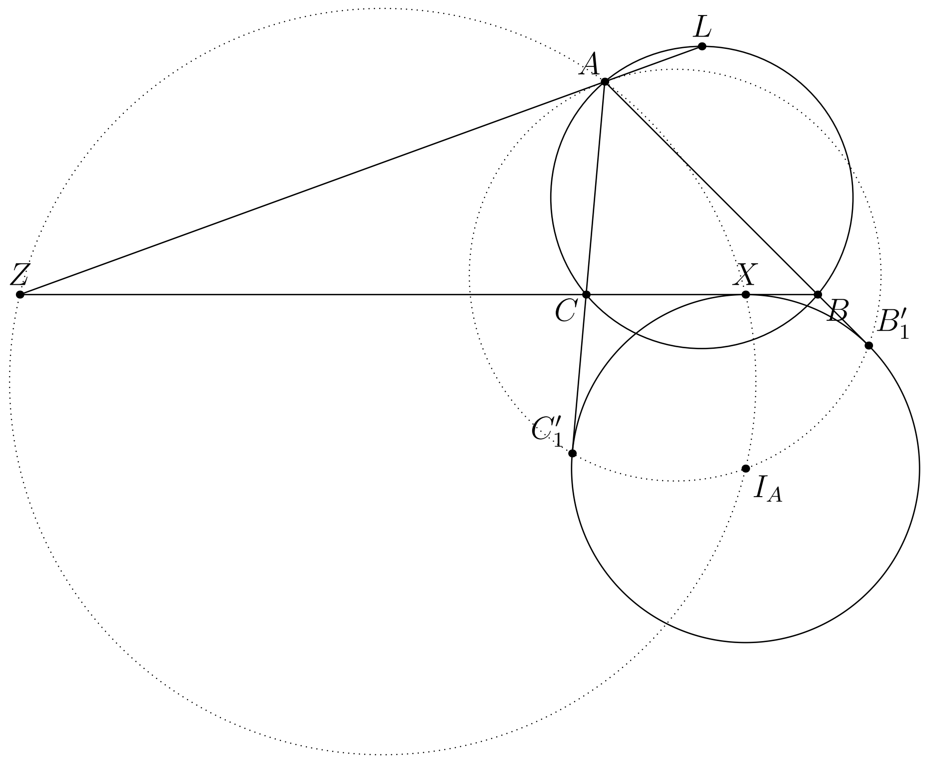
Let's start with a few definitions.

Definition 2.5.1 (Mixtilinear Incircle)**.** The A-**mixtilinear incircle** is tangent to $\overline{AB}$, $\overline{AC}$, and internally tangent to (ABC). The A-**mixtilinear touchpoint** T_A is the tangency point of the mixtilinear incircle and (ABC).



Let B_1 and C_1 be the tangency points of the A-mixtilinear incircle with $\overline{AB}$ and $\overline{AC}$ respectively.

This looks quite intractable at first sight. How would one deal with this weird tangency condition? However, $\sqrt{bc}$ inverting makes this the A-excircle, something we can handle more easily.



Get ready

Let B_1' and C_1' be the images of B_1 and C_1. We have that $\measuredangle AB_1'I_A = \measuredangle AC_1I_A' = 90$, so

$AB_1'I_AC_1'$ is cyclic. Additionally $\measuredangle ZAI_A = \measuredangle ZXI_A = 90$, so $ZAXI_A$ is cyclic. While these might seem quite arbitrary, $\sqrt{bc}$ inverting back we get these theorems.

Theorem 2.5.1. $\overline{AT_A}$ and $\overline{AX}$ are isogonal in $\triangle ABC$

Theorem 2.5.2. $\overline{L - I - T_A}$

Reflecting $\overline{AX}$ over the perpendicular bisector of BC we get another theorem.

Theorem 2.5.3. Let A' be the reflection of A over the perpendicular bisector of BC. Then $\overline{T_AD}$ passes through A'.

Theorem 2.5.4. $\overline{B_1C_1}$ is the line perpendicular to $\overline{AI}$ at I.

Theorem 2.5.5. Let M_C be the midpoint of the arc AB not containing C, and define M_B similarly. Then $T_A - B_1 - M_C$ and $T_A - C_1 - M_B$.

Since the tangents to B_1 and C_1 on $(T_AB_1C_1)$ are $\overline{AB}$ and $\overline{AC}$ respectively, and the tangents to M_C and M_B on $(T_AM_CM_B)$ are parallel to $\overline{AB}$ and $\overline{AC}$ respectively, the homothety at T sending the A-mixtilinear incircle to (ABC) sends B_1 to M_C and C_1 to M_B.

Let O_A be the center of the A-mixtilinear incircle.

Theorem 2.5.6. O_A is on (AB_1C_1).

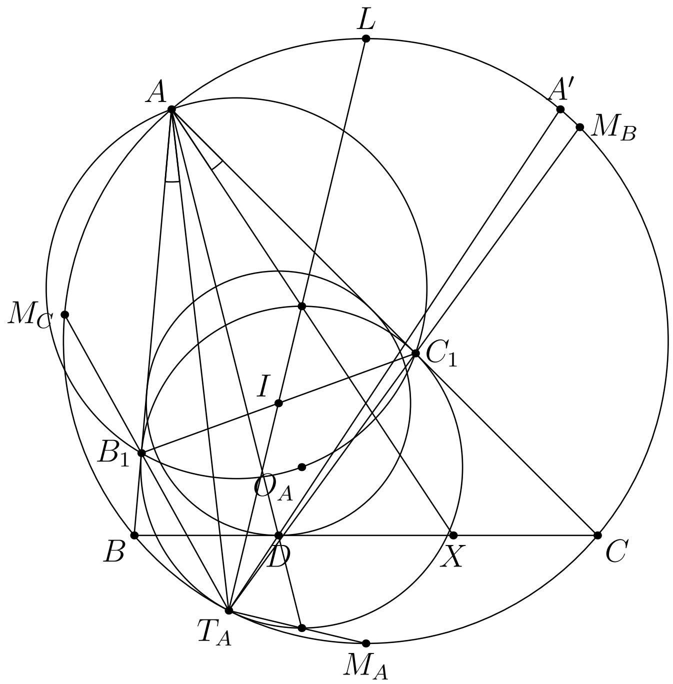
This is because $\measuredangle C_1O_AB_1 = 2\measuredangle C_1T_AB_1 = 2\measuredangle M_BAM_C = \measuredangle C_1AB_1$.

Theorem 2.5.7. An inversion about the circle centered at A with radius AI switches the A-mixtillinear incircle and the incircle.

This is because from similar triangles we have $\frac{AE}{AI} = \frac{AI}{AC_1}$.

Theorem 2.5.8. If M_A is the antipode of L on (ABC) then $\overline{AD}$ and $\overline{T_AM_A}$ meet on the "bottom point" of the A-mixtilinear incircle and $\overline{AX}$ and $\overline{T_AL}$ meet at the "top point" of the A-mixtilinear incircle.

A homothety sending the incircle to the A-mixtillinear circle at A means that the top and bottom points of the A-mixtilinear incircle are on $\overline{AX}$ and $\overline{AD}$ respectively. Similarly the homothety sending the A-mixtillinear incircle to (ABC) at T_A means that the top and bottom points of the A-mixtilinear incircle are on $\overline{T_AL}$ and $\overline{T_AM_A}$ respectively.



Already 8 theorems, and many more waiting

Realize that in the previous section, we already saw $\overline{B_1C_1}$. Additionally, note that M_A is the center of (BIC) and that $\measuredangle M_AT_AI = 90$. This means that $\overline{T_AM_A}$ is the perpendicular bisector of II'. Thus we have the following.

Theorem 2.5.9. $\overline{AS}$, $\overline{B_1C_1}$, $\overline{BC}$, and $\overline{T_AM_A}$ are concurrent.

Let T be the intersection of $\overline{T_AM_A}$ and $\overline{BC}$. Let R be the intersection of $\overline{LI}$ and $\overline{BC}$.

Theorem 2.5.10. R is the orthocenter of $\triangle TLM_A$.

This is because $\overline{LR}$ is perpendicular to $\overline{M_AT}$ and $\overline{TR}$ is perpendicular to $\overline{LM_A}$.

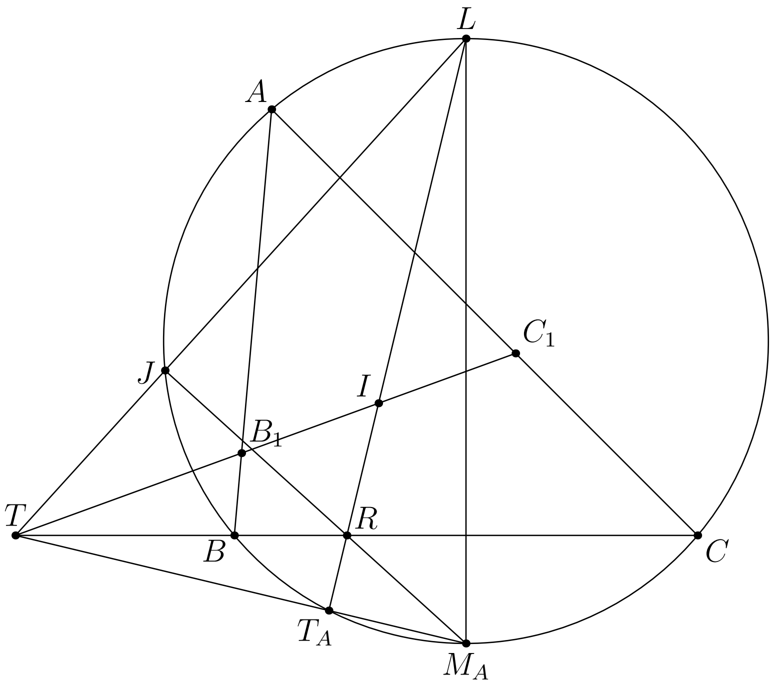
Let J be the intersection of $\overline{LT}$ and $\overline{M_AR}$.

Theorem 2.5.11. $(T_A, J; B, C) = -1$.

First of all $\measuredangle M_AJL = 90$, so J is on (ABC). From there we can use $(L, M_A; B, C) \stackrel{R}{=} (T_A, J, B, C)$.

Theorem 2.5.12. $(R, T; B, C) = -1$.

This is just from $(T_A, J; B, C) \stackrel{L}{=} (R, T; B, C)$.



A non-crowded diagram

Now let J' be the intersection of (AB_1C_1) and (ABC). J' is the center of spiral similarity from B_1C_1 to BC. It is also the center of spiral similarity from B_1I to BM. Thus $TJ'IM$ is cyclic. However $TJIM$ is cyclic, because J, I, M are all on the circle with diameter TM_A.

Theorem 2.5.13. A, C_1, B_1, J are all concyclic.

Theorem 2.5.14. Let D_A be the intersection of the altitude from D to $\overline{EF}$ and (DEF). Let Q_D be the D-Queue Point of $\triangle DEF$. $A - D_A - Q_D - T_A$.

It suffices to show that $A - D_A - T_A$ since $A - D_A - Q_D$ comes from $(Q_D, D_A; E, F) = -1$.The homothety at A sending (DEF) to $(T_AB_1C_1)$ plus the homothety sending $(T_AB_1C_1)$ to (ABC) sends D, E, F to M_A, M_B, M_C respectively, because the tangents at M_A, M_B, M_C to (ABC) are parallel to $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively, which are the tangents to (DEF) at D, E, F respectively. Since $M_AA \perp \overline{M_BM_C}$(consider that $\overline{M_BM_C}$ is the perpendicular bisector of AI) these two homotheties send D_A to A. Thus D_A is on $\overline{AT_A}$ as desired.

The next point I'll bring up in this section is E_O, the intersection of $\overline{AP}$ and (ABC). The reason this is named as such is because it showed up in TSTST 2020/2, and Evan Chen said he hadn't seen it before, which means that it isn't well known, because he's such an authority. It's not like he's had a decade of olympiad experience and he was a Gold Medalist at IMO, right. It's also not like I started olympiad a couple months ago, right. Thus the point is named E_O for Evan is Old. I know that Evan Chen created the whole notion of American Geo in the first place but ... yeah.

Theorem 2.5.15. $(S, E_O; B, C) = -1$

$(Y, P; E, F) \overset{A}{=} (S, E_O; B, C)$ gives this important fact.

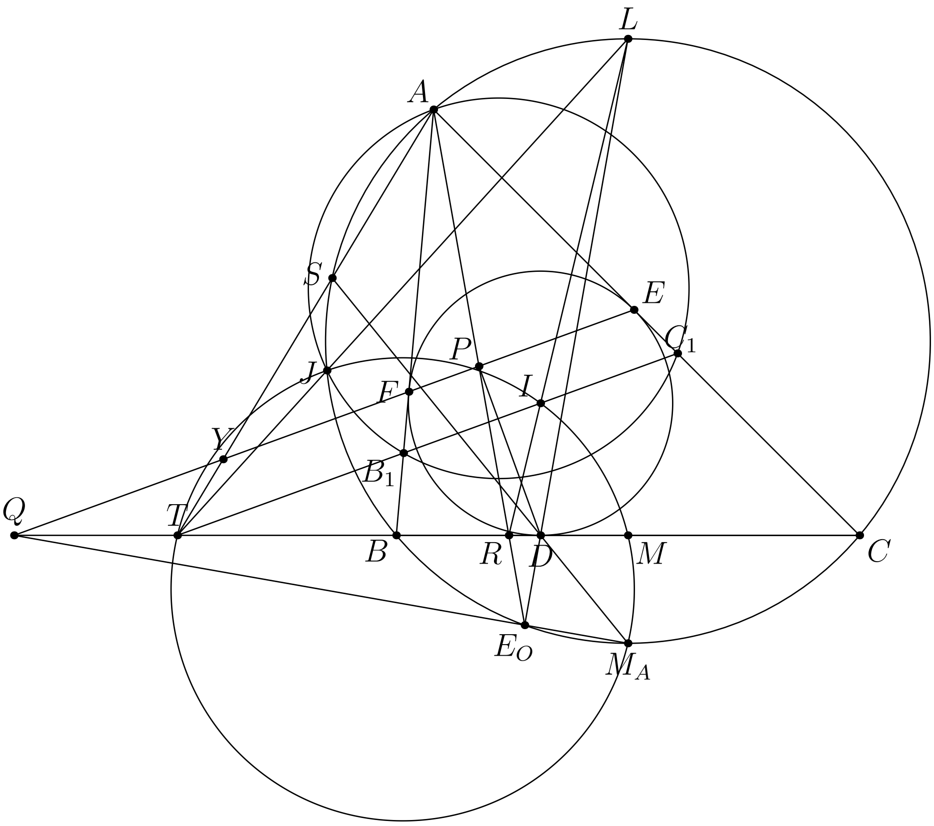
Theorem 2.5.16. Because of the previous theorem there are many collinearities worth mentioning.

- $A - P - R - E_O$
- $L - D - E_O$
- $Q - E_O - M_A(Q = \overline{EF} \cap \overline{BC})$

For the first collinearity, $-1 = (Y, P; E, F) \stackrel{A}{=} (T, \overline{AP} \cap \overline{BC}; B, C)$ gives $A - P - R$, or $A - P - R - E_O$, as desired.

For the second collinearity $-1 = (S, E_O; B, C) \stackrel{D}{=} (M_A, \overline{E_O D} \cap (ABC); B, C)$ gives $L - D - E_O$. Note that if one needs to just prove that $\overline{AP}$ and $\overline{ND}$ intersect on $\overline{BC}$, then looking at the spiral sim taking FE to BC plus an angle chase should suffice.

For the final collinearity $-1 = (S, E_O; B, C) \stackrel{M_A}{=} (\overline{M_A E_O}, D; B, C)$ gives $Q - D - E_O$.



Back to a semi-crowded diagram

Theorem 2.5.17. If H is the orthocenter of $\triangle DEF$ and H_D is the D-Humpty Point of $\triangle DEF$, then an inversion about the incircle sends T_A and E_O the midpoints of DH and DH_D respectively.

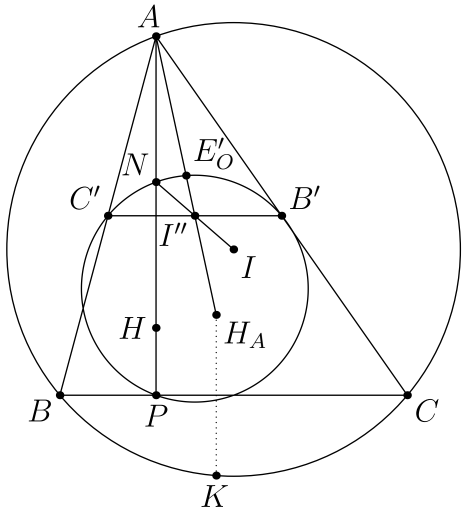
The first thing to note is that the inversion of (ABC) around the incircle goes to the nine-point circle of $\triangle DEF$

Let I' be the reflection of I over T_A, or the point on (BIC) such that $-1 = (I, I'; B, C)$. Then Let B' and C' be the midpoints of DF and DE respectively. I' inverts to the midpoint of B' and C'. Let N be the midpoint of DH. Since $NB'IC'$ is a parallelogram (dilate it by a factor of 2 around D), the midpoint of NI is the midpoint of $B'C'$, so T_A inverts to N, as desired.

Let P be the foot of D to $\overline{EF}$. Then E_O inverts to a point E_O' on the nine-point circle of $\triangle DEF$, or a point on $(DB'C')$, such that $-1 = (D, E_O'; B', C')$, because inversion preserves cross ratio. We can recast this in terms of $\triangle DEF$.

Let $\triangle ABC$ have orthocenter H, A-Humpty Point H_A, and let the midpoint of AH_A be E_O'. Let $\overline{AH} \cap \overline{BC} = P$, let the midpoint of AC be B', and let the midpoint of AB be C'. Then prove that $(P, E_O'; B', C') = -1$.

Dilate P, E_O', B', C' around A by a factor of 2 and reflect their images around $\overline{BC}$. P goes to A, E_O goes to K, B' goes to C, and C' goes to B, proving the recasting.



What does (IT_AE_O) invert to?

Corollary 2.5.1. E_O lies on $(ZAIDI')$

This is because $(ZAIDI')$ turns into the D-median of $\triangle DEF$ under an inversion about the incircle. In addition, $LZ \cdot LA = LD \cdot LE_O$ by shooting lemma also suffices, and is shorter on an olympiad.

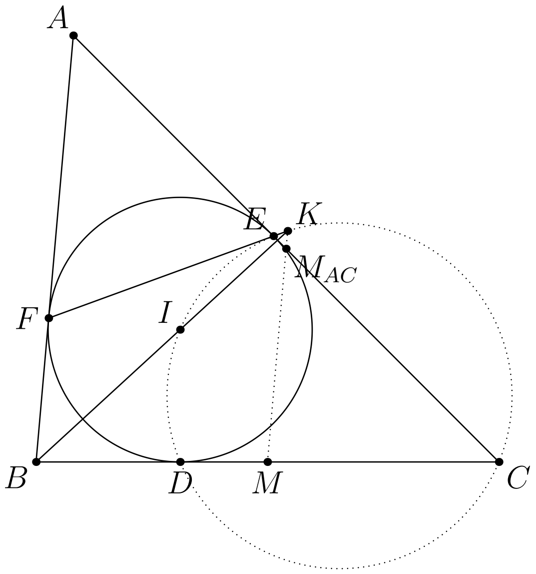
2.6 Iran Lemma

This chapter, while having only one theorem, is separated because I want to make it very clear that it is very useful.

The Iran Lemma, named after its use in the famous Iran TST 2009/9(a modified version appears as Problem 3.67), goes as follows:

Theorem 2.6.1 (Iran Lemma)**.** Let M and M_{AC} be the midpoints of $\overline{BC}$ and $\overline{AC}$ respectively. Then the following concur:

- $\overline{EF}$
- $\overline{MM_{AC}}$
- The C-altitude of $\triangle BIC$
- $\overline{BI}$



Let $K = \overline{EF} \cap \overline{BI}$

Since

$$\measuredangle EKI = \measuredangle IBF + \measuredangle EFB = \measuredangle IBA + 90 - \measuredangle FAI = \measuredangle ECI,$$

we have $EKIC$ cyclic. This means that

$$\measuredangle IKC = 90 = \measuredangle BKC,$$

proving that K is the foot of the altitude from C to $\overline{BI}$.

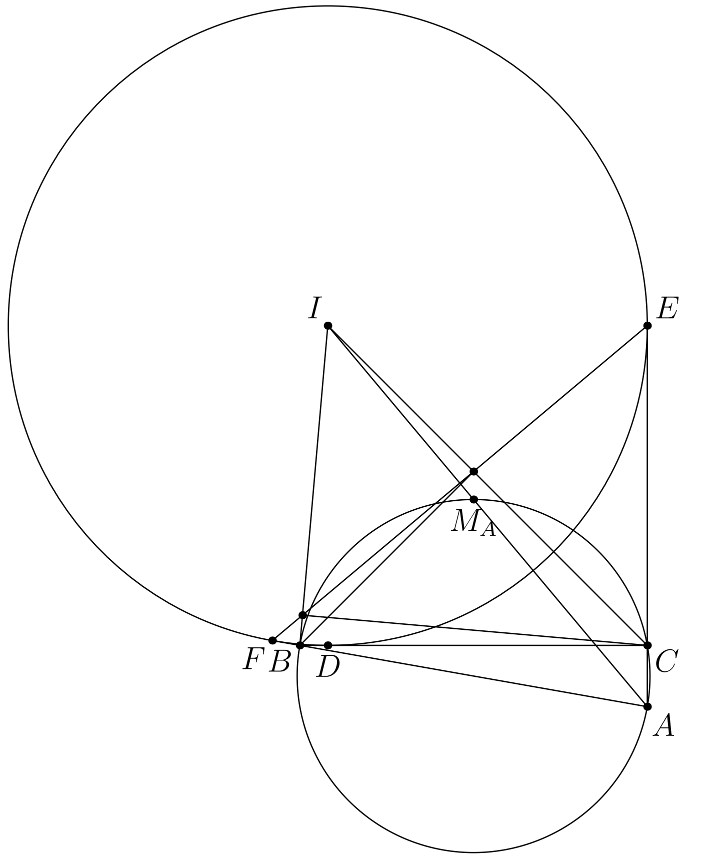
Since K is on (BC), and

$$\measuredangle CMM_B = 2\measuredangle CBI,$$

$\overline{MM_{AC}}$ goes through K.

It is helpful to think of this lemma as the concurrence of an intouch chord, midline, and angle bisector, all on different angles (here I used A,C, and B respectively). This also means that when looking at $\triangle BIC$, the foot from B to $\overline{CI}$ and the foot from C to $\overline{BI}$ both lie on $\overline{EF}$. This trick of looking at $\triangle BIC$ is helpful in some circumstances, especially when the orthocenter of it shows up. In addition, this theorem works exactly the same with I_A instead of I.

The above trick of recasting in terms of $\triangle BIC$ shows up extremely often, especially when you can define the points in such a way that A, E, F are not very important.



Ok I know here $\angle BIC$ is acute so I is an excenter of $\triangle ABC$ but the point still stands.

2.7 Others

When the line $\overline{OI}$ shows up, often, but not always, homotheties are present.

Theorem 2.7.1. $\triangle DEF \sim \triangle M_AM_BM_C \sim \triangle I_AI_BI_C$, and all of the pairwise centers of homothety are on $\overline{OI}$.

We already stated above that $\triangle DEF \sim \triangle M_AM_BM_C$ above, with the homothety that sends the incircle to the A-mixtilinear incircle plus the homothety that sends the A-mixtilinear incircle to (ABC). Since these two have circumcenters I and O respectively, the center of homothety between these two is on $\overline{OI}$.

We already have that $\triangle M_AM_BM_C \sim \triangle I_AI_BI_C$ by a homothety at I with factor 2. Since O is the nine-point center of $\triangle I_AI_BI_C$, the circumcenter of O is on $\overline{OI}$. Since the circumcenter of $\triangle DEF$ is I, and $\triangle DEF \sim \triangle M_AM_BM_C \sim \triangle I_AI_BI_C$, the center of homothety between $\triangle DEF$ and $\triangle I_AI_BI_C$ is on $\overline{OI}$.

Corollary 2.7.1. The Euler lines of $\triangle DEF$, $\triangle M_AM_BM_C$, and $\triangle I_AI_BI_C$ are all $\overline{OI}$.

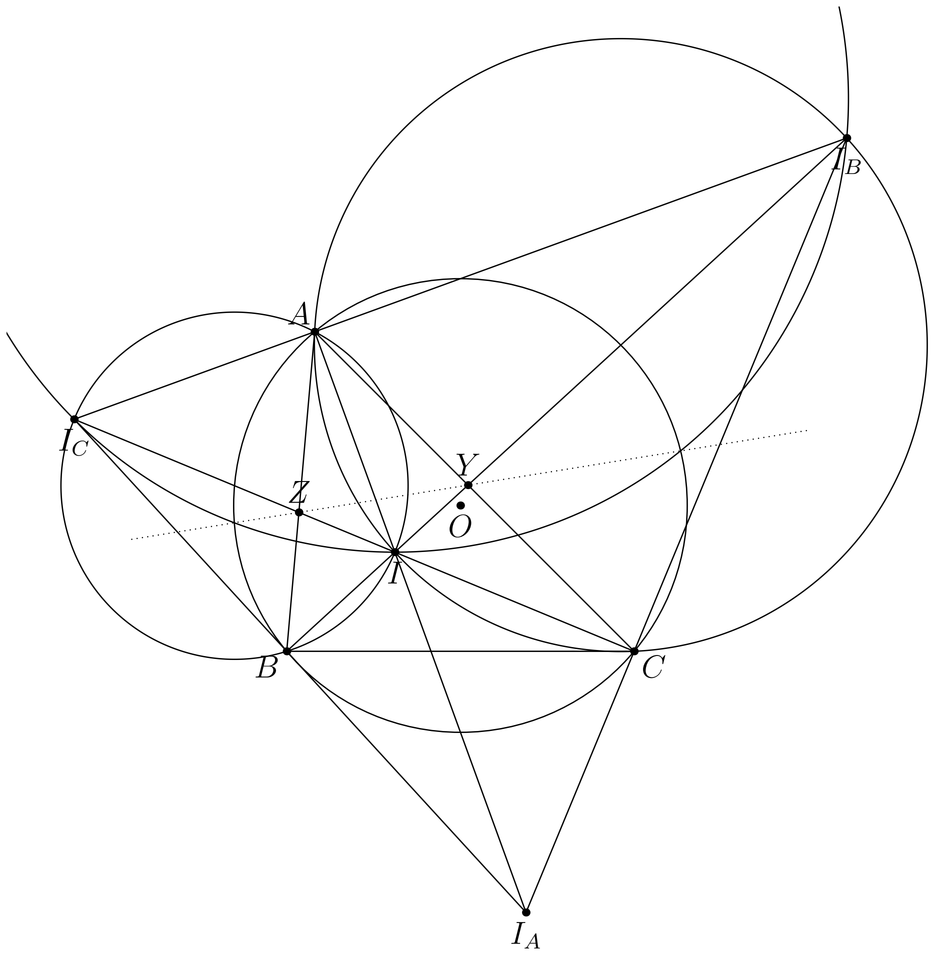
Corollary 2.7.2. The homothety center between $\triangle DEF$ and $\triangle M_AM_BM_C$ is on $\overline{AT_A}$(and by symmetry the intersection of $\overline{AT_A}$, $\overline{BT_B}$, and $\overline{CT_C}$ is this homothety center).

Monge on (ABC), the A-mixtilinear incircle, and the incircle means that the exsimilicenter of the incircle and (ABC), or the homothety center between $\triangle DEF$ and $\triangle M_AM_BM_C$, is on $\overline{AT_A}$.

Now that we've seen theorems on $\overline{OI}$, let's see one on $\overline{OI_A}$.

Theorem 2.7.2. Let $Y = \overline{BI} \cap \overline{AC}$ and $Z = \overline{CI} \cap \overline{AB}$. Then $\overline{OI_A} \perp \overline{YZ}$.

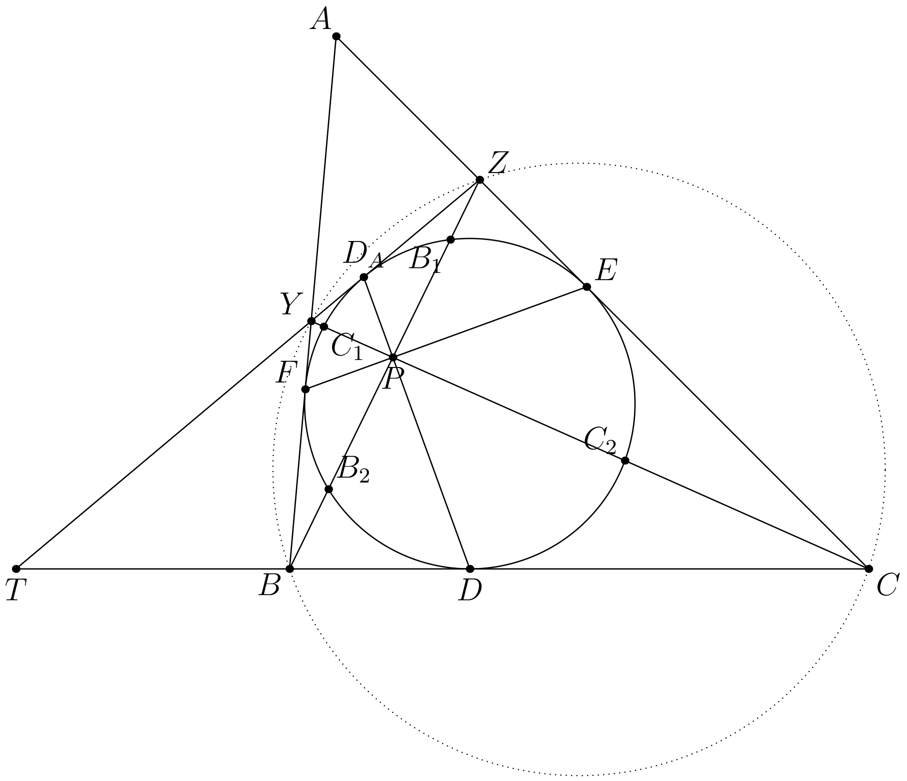
This follows from a lot of radical axis. Introducing I, I_B, I_C, Y is the radical center of (II_B), (II_BI_C), (ABC), and Z is the radical center of (II_C),(II_BI_C),(ABC). Thus $\overline{YZ}$ is the radical axis of (II_BI_C) and (ABC). The line through O and the center of (II_BI_C) is $\overline{OI_A}$ by Excentral-Orthic Duality.



Radical axis is quite powerful, especially as shown here

The next 3 theorems are about a tangent to the incircle.

Theorem 2.7.3. Let P be the foot from D to $\overline{EF}$ and let $Y = \overline{AB} \cap \overline{CP}$,$Z = \overline{AC} \cap \overline{BP}$. Then $\overline{YZ}$ is tangent to the incircle on $D_A = \overline{DP} \cap (DEF)$.



This makes $\overline{YZ}$ antiparallel to $\overline{BC}$.

$(B_1, B_2; D, E) \overset{P}{=} (B_2, B_1; D_A, F)$, so $\overline{ZD_A}$ is tangent to (DEF). Similarly $\overline{YD_A}$ is tangent to (DEF), so $\overline{YZ}$ is tangent to (DEF) at D_A.

Theorem 2.7.4. *$BCZY$ is cyclic.*

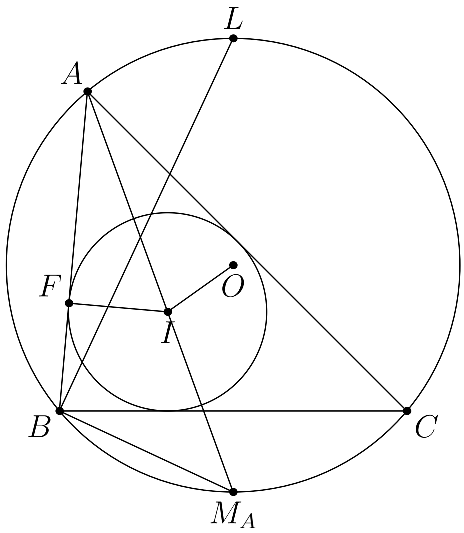
Since $\measuredangle YZC = 2\measuredangle D_ADE = 2\measuredangle FED = \measuredangle YBC$, our claim is proved.

Theorem 2.7.5. *$\overline{YZ}$ goes through T, or the point on $\overline{BC}$ such that $\measuredangle AIT = 90$.*

Since $\overline{BC}$ is the tangent to (DEF) at D and $\overline{YZ}$ is tangent to (DEF) at D_A, they meet at T by angle chasing. Additionally since $\overline{BZ}$ and $\overline{CY}$ meet at P and $\overline{AP} \cap \overline{BC}$ is R, $(T, R; B, C) = -1$, gives the conclusion.

Theorem 2.7.6 (Euler's Theorem)**.** *In a $\triangle ABC$, if O is the circumcenter, R is the circumradius, I is the incenter, and r is the inradius, then $OI^2 = R(R - 2r)$.*

Let L be the midpoint of $\widehat{BAC}$ and let M_A be the antipode of L on (ABC). Let F be the foot from I to $\overline{AB}$. Then $\frac{AI}{r} = \frac{AI}{IF} = \frac{LM_A}{BM_A} = \frac{2R}{IM_A}$, so $AI \cdot IM_A = 2Rr$. By Power of a Point we get that $R^2 - OI^2 = AI \cdot IM_A = 2Rr$, so $OI^2 = R(R - 2r)$. This, along with the existence of the Feuerbach tangency, is a proof that $R \geq 2r$.



This theorem has among the most original names in existence.

Corollary 2.7.3 (Poncelet's Porism for triangles and circles)**.** If Γ and ω are the circumcircle and incircle of $\triangle ABC$ respectively, then for any point P on Γ, there is a triangle with P as a vertex that also has Γ and ω as its circumcircle and incircle respectively.

If I is the incenter of $\triangle ABC$, then let $P' = \overline{PI} \cap \Gamma$. Let the circle centered at P' with radius $P'I$ intersect Γ at P_1 and P_2 respectively. Then $\triangle PP_1P_2$ has incenter I by Fact 5. By Euler's theorem the inradius of this triangle must be the same as the inradius of $\triangle ABC$, so ω is the incircle of $\triangle PP_1P_2$ as desired.

As hinted in the name, the full statement of Poncelet's Porism is much more general. However for our purposes this is good enough.

Chapter 3

Problems

"Why was the six afraid of the seven?"

"Because seven ate nine."

"Why was the student afraid of the six"

"Because P6s are hard"

"Fear not. They say to become a master you must do something 10000 times. If you speed your slow self up by 100, you should be a master by the end of these 100 problems."

While I have put these in roughly ascending order in terms of difficulty, arguing about problem difficulty is fun, as many have noted before. Basically the ordering, for the most part, is a piece of garbage.

Problem 3.0 (North Korea TST 2013/1)**.** The incircle of a non-isosceles triangle ABC with the center I touches the sides BC, CA, AB at A_1, B_1, C_1 respectively. The line AI meets the circumcircle of ABC at A_2. The line B_1C_1 meets the line BC at A_3 and the line A_2A_3 meets the circumcircle of ABC at $A_4 (\neq A_2)$. Define B_4, C_4 similarly. Prove that the lines AA_4, BB_4, CC_4 are concurrent.

Problem 3.1 (GOTEEM 3)**.** Let D be a point in the plane of $\triangle ABC$. Define D_A, D_B, D_C to be the reflections of D over BC, CA, AB, respectively. Prove that the circumcircles of $\triangle D_ABC$, $\triangle D_BCA$, $\triangle D_CAB$, $\triangle D_AD_BD_C$ concur at a point P. Moreover, prove that the midpoint of $\overline{DP}$ lies on the nine-point circle of $\triangle ABC$.

Problem 3.2 (USA TST 2011/1)**.** In an acute scalene triangle ABC, points D, E, F lie on sides BC, CA, AB, respectively, such that $AD \perp BC, BE \perp CA, CF \perp AB$. Altitudes AD, BE, CF meet at orthocenter H. Points P and Q lie on segment EF such that $AP \perp EF$ and $HQ \perp EF$. Lines DP and QH intersect at point R. Compute HQ/HR.

Problem 3.3 (USA TST 2008/7)**.** Let ABC be a triangle with G as its centroid. Let P be a variable point on segment BC. Points Q and R lie on sides AC and AB respectively, such that $PQ \parallel AB$ and $PR \parallel AC$. Prove that, as P varies along segment BC, the circumcircle of triangle AQR passes through a fixed point X such that $\angle BAG = \angle CAX$.

Problem 3.4 (USA TSTST 2017/1)**.** Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$ and that $\angle A \neq 90^\circ$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection of lines AQ and EF. Prove that $PR \perp OH$.

Problem 3.5 (USAJMO 2014/6). Let ABC be a triangle with incenter I, incircle γ, and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively and let E and F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U and V be the intersections of lines $\overline{EF}$ with $\overline{MN}$ and $\overline{MP}$ respectively, and let X be the midpoint of the arc BAC of Γ.

(a) Prove that I lies on $\overline{CV}$

(b) Prove that the line $\overline{XI}$ bisects the segment UV

.

Problem 3.6 (ISL 2016 G2). Let ABC be a triangle with circumcircle Γ and incenter I. Let M be the midpoint of BC. The points D, E, F are selected on sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a points X other than A. Prove that lines $\overline{XD}$ and $\overline{AM}$ meet on Γ.

Problem 3.7 (Korea 2020/2). H is the orthocenter of an acute triangle ABC, and let M be the midpoint of BC. Suppose (AH) meets AB and AC at D, E respectively. AH meets DE at P, and the line through H perpendicular to AH meets DM at Q. Prove that P, Q, B are collinear.

Problem 3.8 (USA TSTST 2011/4). Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

Hint: 20

Problem 3.9 (USAMO 2008/2). Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.

Hint: 86

Problem 3.10 (USA TST 2005/6). Let ABC be an acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

Problem 3.11 (AoPS Community). Given acute triangle ABC, $BE \perp AC, CF \perp AB$ at E, F, resp. M is midpoint of BC, N is intersection of EF and AM. $NK \perp BC$ at K. Prove that AK is symmedian of triangle ABC.

Problem 3.12 (USA TSTST 2015/2). Let ABC be a scalene triangle. Let K_a, L_a and M_a be the respective intersections with BC of the internal angle bisector, external angle bisector, and the median from A. The circumcircle of AK_aL_a intersects AM_a a second time at point X_a different from A. Define X_b and X_c analogously. Prove that the circumcenter of $X_aX_bX_c$ lies on the Euler line of ABC.

(The Euler line of ABC is the line passing through the circumcenter, centroid, and orthocenter of ABC.)

Problem 3.13 (STEMS 2019 CAT B P6). In triangle ABC, with circumcircle Γ, the incircle ω has center I and touches sides $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Point Q lies on $\overline{EF}$ and point R lies on ω such that $\overline{DQ} \perp \overline{EF}$ and D, Q, R are collinear. Ray AR meets ω again at P and Γ again at S. Ray AQ meets BC at T. Let M be the midpoint of BC and let O be the circumcenter of triangle MPD. Prove that O, T, I, S are collinear.

Problem 3.14 (Vietnam TST 2003/2). Given a triangle ABC. Let O be the circumcenter of this triangle ABC. Let H, K, L be the feet of the altitudes of triangle ABC from the vertices A, B, C, respectively. Denote by A_0, B_0, C_0 the midpoints of these altitudes AH, BK, CL, respectively. The incircle of triangle ABC has center I and touches the sides BC, CA, AB at the points D, E, F, respectively. Prove that the four lines A_0D, B_0E, C_0F and OI are concurrent. (When the point O concides with I, we consider the line OI as an arbitrary line passing through O.)

Problem 3.15 (Taiwan TST Round 2 2019 Day 1 P2). Let ABC be a scalene triangle,let I be its incenter and let Ω be its circumcircle. Let M be the midpoint of BC. The incircle ω touches CA, AB at E, F respectively. Suppose that the line EF intersects Ω at two points P, Q, and let R be the point on the circumcircle Γ of $\triangle MPQ$ such that MR is perpendicular to PQ. Prove that AR, Γ,and ω intersect at one point.

Problem 3.16 (Sun Yat-sen University bi-weekly problem). Let I be the incenter of $\triangle ABC$. D, E, F be the symmetric point of I wrt BC, CA, AB respectively. Prove that there exists a point P such that $AP \perp DP, BP \perp EP, CP \perp FP$.

Problem 3.17 (2013 ELMO SL G3). In $\triangle ABC$, a point D lies on line BC. The circumcircle of ABD meets AC at F (other than A), and the circumcircle of ADC meets AB at E (other than A). Prove that as D varies, the circumcircle of AEF always passes through a fixed point other than A, and that this point lies on the median from A to BC.

Problem 3.18 (Korea Winter Program Practice Test 2018/5). Let ΔABC be a triangle with circumcenter O and circumcircle w. Let S be the center of the circle which is tangent with AB, AC, and w (in the inside), and let the circle meet w at point K. Let the circle with diameter AS meet w at T. If M is the midpoint of BC, show that K, T, M, O are concyclic.

Hint: 38

Problem 3.19 (2019 ELMO SL G3). Let $\triangle ABC$ be an acute triangle with incenter I and circumcenter O. The incircle touches sides BC, CA, and AB at D, E, and F respectively, and A' is the reflection of A over O. The circumcircles of ABC and $A'EF$ meet at G, and the circumcircles of AMG and $A'EF$ meet at a point $H \neq G$, where M is the midpoint of EF. Prove that if GH and EF meet at T, then $DT \perp EF$.

Hint: 26

Problem 3.20 (2013 ELMO SL G2). Let ABC be a scalene triangle with circumcircle Γ, and let D,E,F be the points where its incircle meets BC, AC, AB respectively. Let the circumcircles of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$ meet Γ a second time at X, Y, Z respectively. Prove that the perpendiculars from A, B, C to AX, BY, CZ respectively are concurrent.

Problem 3.21 (USA TST 2014/1). Let ABC be an acute triangle, and let X be a variable interior point on the minor arc BC of its circumcircle. Let P and Q be the feet of the perpendiculars from X to lines CA and CB, respectively. Let R be the intersection of line PQ

and the perpendicular from B to AC. Let ℓ be the line through P parallel to XR. Prove that as X varies along minor arc BC, the line ℓ always passes through a fixed point. (Specifically: prove that there is a point F, determined by triangle ABC, such that no matter where X is on arc BC, line ℓ passes through F.)

Problem 3.22 (ELMO SL 2018/1). Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

Problem 3.23 (AoPS user math_pi_rate). Let X_A and Y_A be the A-intouch point and the foot of the A-internal angle bisector in a $\triangle ABC$. Define X_B, Y_B and X_C, Y_C analogously. Then prove that the radical center of $\odot AX_AY_A$, $\odot BX_BY_B$, $\odot CX_CY_C$ lies on $\overline{OI}$(O is the circumcenter and I is the incenter respectively of $\triangle ABC$).

Problem 3.24 (ELMO 2020/4). Let acute scalene triangle ABC have orthocenter H and altitude AD with D on side BC. Let M be the midpoint of side BC, and let D' be the reflection of D over M. Let P be a point on line $D'H$ such that lines AP and BC are parallel, and let the circumcircles of $\triangle AHP$ and $\triangle BHC$ meet again at $G \neq H$. Prove that $\angle MHG = 90°$.

Problem 3.25 (ELMO 2012/1). In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

Problem 3.26 (Japan 2017/3). Let ABC be an acute-angled triangle with the circumcenter O. Let D, E and F be the feet of the altitudes from A, B and C, respectively, and let M be the midpoint of BC. AD and EF meet at X, AO and BC meet at Y, and let Z be the midpoint of XY. Prove that A, Z, M are collinear.

Problem 3.27 (Sharygin 2015 Final Round Grade 10(penultimate grade) Problem 3). Let A_1, B_1 and C_1 be the midpoints of sides BC, CA and AB of triangle ABC, respectively. Points B_2 and C_2 are the midpoints of segments BA_1 and CA_1 respectively. Point B_3 is symmetric to C_1 wrt B, and C_3 is symmetric to B_1 wrt C. Prove that one of common points of circles BB_2B_3 and CC_2C_3 lies on the circumcircle of triangle ABC.

Problem 3.28 (ELMO 2018/4). Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90°$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Hint: 59

Problem 3.29 (ELMO 2017/2). Let ABC be a triangle with orthocenter H, and let M be the midpoint of $\overline{BC}$. Suppose that P and Q are distinct points on the circle with diameter $\overline{AH}$, different from A, such that M lies on line PQ. Prove that the orthocenter of $\triangle APQ$ lies on the circumcircle of $\triangle ABC$.

Hint: 87

Problem 3.30 (CJMO 2019/3)**.** Let I be the incenter of $\triangle ABC$, and M the midpoint of $\overline{BC}$. Let Ω be the nine-point circle of $\triangle BIC$. Suppose that $\overline{BC}$ intersects Ω at a point $D \neq M$. If Y is the intersection of $\overline{BC}$ and the A-intouch chord, and X is the projection of Y onto $\overline{AM}$, prove that X lies on Ω, and the intersection of the tangents to Ω at D and X intersect on the A-intouch chord of $\triangle ABC$.

Problem 3.31 (GOTEEM 1)**.** Let ABC be a scalene triangle. The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let G be a point on the incircle of $\triangle ABC$ such that $\angle AGD = 90°$. If lines DG and EF intersect at P, prove that AP is parallel to BC.

Problem 3.32 (GGG1/4)**.** Let ABC be an acute triangle, let ω be its incircle, and let M_A be the midpoint of minor arc $\hat{BC}$ on the circumcircle of ABC. Let ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at points D, E, F, respectively, and let H be the foot of the altitude from A to $\overline{BC}$. Denote by L the intersection of $\overleftrightarrow{M_AD}$ and $\overleftrightarrow{AH}$. Let I_E and I_F denote the E and F-excenters of triangle ELF, respectively.

Prove that I_E and I_F lie on ω.

Problem 3.33 (USAJMO 2016/1 generalization)**.** The triangle $\triangle ABC$ is inscribed in the circle ω. Let P be a variable point on the arc $\hat{BC}$ that does not contain A, and let I_B and I_C denote the incenters of triangles $\triangle ABP$ and $\triangle ACP$, respectively.

Prove that as P varies, the circumcircle of $\triangle PI_BI_C$ passes through a fixed point.

Problem 3.34 (ELMO 2010/6)**.** Let ABC be a triangle with circumcircle ω, incenter I, and A-excenter I_A. Let the incircle and the A-excircle hit BC at D and E, respectively, and let M be the midpoint of arc BC without A. Consider the circle tangent to BC at D and arc BAC at T. If TI intersects ω again at S, prove that SI_A and ME meet on ω.

Problem 3.35 (ELMO 2014/5)**.** Let ABC be a triangle with circumcenter O and orthocenter H. Let ω_1 and ω_2 denote the circumcircles of triangles BOC and BHC, respectively. Suppose the circle with diameter $\overline{AO}$ intersects ω_1 again at M, and line AM intersects ω_1 again at X. Similarly, suppose the circle with diameter $\overline{AH}$ intersects ω_2 again at N, and line AN intersects ω_2 again at Y. Prove that lines MN and XY are parallel.

Problem 3.36 (USA TST 2015/1)**.** Let ABC be a non-isosceles triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Denote by M the midpoint of $\overline{BC}$. Let Q be a point on the incircle such that $\angle AQD = 90°$. Let P be the point inside the triangle on line AI for which $MD = MP$. Prove that either $\angle PQE = 90°$ or $\angle PQF = 90°$.

Problem 3.37 (BMOSL 2018/2)**.** Let ABC be a triangle inscribed in circle Γ with center O. Let H be the orthocenter of triangle ABC and let K be the midpoint of OH. Tangent of Γ at B intersects the perpendicular bisector of AC at L. Tangent of Γ at C intersects the perpendicular bisector of AB at M. Prove that AK and LM are perpendicular.

Problem 3.38 (Folklore I think)**.** Given a triangle ABC with circumcenter O, let M be the midpoint of BC and D be the foot from A to $\overline{BC}$. $P = \overline{OD} \cap \overline{AM}$. Prove that $P = \overline{OD} \cap \overline{AM}$ lies on the radical axis of (BOC) and the nine-point circle of (ABC).

Hints: 35,92

Problem 3.39 (Vietnam 2019/6)**.** Given an acute triangle ABC and (O) be its circumcircle, and H is its orthocenter. Let M, N, P be midpoints of BC, CA, AB, respectively. D, E, F are the feet of the altitudes from A, B and C, respectively. Let K symmetry with H through BC. DE intersects MP at X, DF intersects MN at Y.

a XY intersects smaller arc BC of (O) at Z. Prove that K, Z, E, F are concyclic.

b KE, KF intersect (O) at S, T ($S, T \neq K$), respectively. Prove that BS, CT, XY are concurrent.

Hints: 32,14

Problem 3.40 (AoPS user tutubixu9198)**.** Let ABC be a triangle with circumcenter O and incenter I. Let (O_1) be a circle ex-tangent to (BOC) and tangent to AB, AC at M, N. Prove that MN passes through midpoint of AI.

Problem 3.41 (Sharygin Correspondence Round 2020/15)**.** Let H be the orthocenter of a nonisosceles triangle ABC. The bisector of angle BHC meets AB and AC at points P and Q respectively. The perpendiculars to AB and AC from P and Q meet at K. Prove that KH bisects the segment BC.

Problem 3.42 (GOTEEM 2)**.** Let ABC be an acute triangle with $AB \neq AC$, and let D, E, F be the feet of the altitudes from A, B, C, respectively. Let P be a point on DE such that $AP \perp AB$ and let Q be a point on DF such that $AQ \perp AC$. Lines PQ and BC intersect at T. If M is the midpoint of $\overline{BC}$, prove that $\angle MAT = 90°$.

Problem 3.43 (2012 ELMO SL G3)**.** ABC is a triangle with incenter I. The foot of the perpendicular from I to BC is D, and the foot of the perpendicular from I to AD is P. Prove that $\angle BPD = \angle DPC$.

Problem 3.44 (2017 ISL G4)**.** In triangle ABC, let ω be the excircle opposite to A. Let D, E and F be the points where ω is tangent to BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

Problem 3.45 (GGG3 6)**.** Let ABC be a scalene triangle with incenter I and circumcircle Ω; L is the midpoint of arc BAC and A' is diametrically opposite A on Ω. D is the foot of the perpendicular from I to $\overline{BC}$. $\overleftrightarrow{LI}$ meets $\overleftrightarrow{BC}$ and Ω at X and Y, respectively, and $\overleftrightarrow{LD}$ meets Ω again at Z. $\overleftrightarrow{XZ}$ meets $\overleftrightarrow{A'I}$ at T. $\overleftrightarrow{DI}$ meets the circle with diameter $\overline{AI}$ again at P. Show that the second intersection between $\overleftrightarrow{PT}$ and the circle with diameter $\overline{AI}$ lies on $\overline{AY}$.

Problem 3.46 (Iran TST 2012 Day 1 P2)**.** Consider ω, the circumcircle of a triangle ABC. D is the midpoint of arc BAC and I is the incenter of $\triangle ABC$. Let DI intersect BC at E and ω for the second time at F. Let P be a point on line AF such that PE is parallel to AI. Prove that PE is the angle bisector of angle BPC.

Problem 3.47 (Centroamerican Olympiad 2016/6)**.** Let $\triangle ABC$ be triangle with incenter I and circumcircle Γ. Let $M = BI \cap \Gamma$ and $N = CI \cap \Gamma$, the line parallel to MN through I cuts AB, AC in P and Q. Prove that the circumradius of $\odot(BNP)$ and $\odot(CMQ)$ are equal.

Problem 3.48 (2019 ELMO SL G1)**.** Let ABC be an acute triangle with orthocenter H and circumcircle Γ. Let BH intersect AC at E, and let CH intersect AB at F. Let AH intersect Γ again at $P \neq A$. Let PE intersect Γ again at $Q \neq P$. Prove that BQ bisects segment $\overline{EF}$.

Hints: 11,79

Problem 3.49 (Our one true god Wolfram Alpha)**.** Let $\triangle ABC$ have orthic triangle DEF. Prove that the Euler lines of $\triangle AEF, \triangle BDF, \triangle CDE$ are all concurrent.

Hint: 7,69,57

Problem 3.50 (Romania JBMO TST 2019/1.3)**.** Let ABC a triangle, I the incenter, D the contact point of the incircle with the side BC and E the foot of the bisector of the angle A. If M is the midpoint of the arc BC which contains the point A of the circumcircle of the triangle ABC and $\{F\} = DI \cap AM$, prove that MI passes through the midpoint of $[EF]$.

Problem 3.51 (MMOSL G3)**.** Let D, E, and F be the respective feet of the A, B, and C altitudes in $\triangle ABC$, and let M and N be the respecive midpoints of $\overline{AC}$ and $\overline{AB}$. Lines DF and DE intersect the line through A parallel to BC at X and Y, respectively. Lines MX and YN intersect at Z. Prove that the circumcircles of $\triangle EFZ$ and $\triangle XYZ$ are tangent.

Problem 3.52 (AoPS user MP8148)**.** In triangle ABC let the incenter be I. Suppose that the A-mixtilinear incircle ω touches (ABC) at T, and $Q = \overline{IT} \cap \overline{BC}$. Show that the homothety sending ω to (ABC) sends Q to the midpoint of $\overline{IQ}$.

Problem 3.53 (Myself (i3435))**.** Let ABC be a triangle with orthocenter H. Let $\overline{BH} \cap \overline{AC} = E$, and $\overline{CH} \cap \overline{AB} = F$. Let N be the midpoint of EF and let X be the point besides N on $\overline{EF}$ such that $HN = HX$. Let $\overline{AX}$ intersect (ABC) at Y. Let A' be the point on (ABC) such that $\overline{AA'} || \overline{BC}$ and let M be the midpoint of BC. Prove that $\overline{A'Y}$ bisects AM.

Problem 3.54 (Error: Not Found)**.** Let ABC be an acute scalene triangle with orthocenter H. Prove that the midpoint of AH lies on the incircle of ABC if and only if the incenter of ABC lies on the circle with diameter AH.

Problem 3.55 (IGO Advanced 2020/2)**.** Let $\triangle ABC$ be an acute-angled triangle with its incenter I. Suppose that N is the midpoint of the arc $\widehat{BAC}$ of the circumcircle of triangle $\triangle ABC$, and P is a point such that $ABPC$ is a parallelogram.Let Q be the reflection of A over N and R the projection of A on $\overline{QI}$. Show that the line $\overline{AI}$ is tangent to the circumcircle of triangle $\triangle PQR$.

Problem 3.56 (AoPS user jayme)**.** Let ABC be an acute triangle with orthocenter H, DEF as the orthic triangle, and X as the foot from A to $\overline{EF}$. He and Hf are the orthocenters of $\triangle HFD$, $\triangle HED$ respectively. Prove that $\overline{HeHf} \perp \overline{DX}$.

Problem 3.57 (Japan 2019/4)**.** Let ABC be a triangle with its inceter I, incircle w, and let M be a midpoint of the side BC. A line through the point A perpendicular to the line BC and a line through the point M perpendicular to the line AI meet at K. Show that a circle with line segment AK as the diameter touches w.

Problem 3.58 (Brazil 2013/6)**.** The incircle of triangle ABC touches sides BC, CA and AB at points D, E and F, respectively. Let P be the intersection of lines AD and BE. The reflections of P with respect to EF, FD and DE are X, Y and Z, respectively. Prove that lines AX, BY and CZ are concurrent at a point on line IO, where I and O are the incenter and circumcenter of triangle ABC.

Hints: 67,53

Problem 3.59 (ISL 2002 G7)**.** The incircle Ω of the acute-angled triangle ABC is tangent to its side BC at a point K. Let AD be an altitude of triangle ABC, and let M be the midpoint of the segment AD. If N is the common point of the circle Ω and the line KM (distinct from K), then prove that the incircle Ω and the circumcircle of triangle BCN are tangent to each other at the point N.

Hints: 72,12

Problem 3.60 (USA TSTST 2016/2)**.** Let ABC be a scalene triangle with orthocenter H and circumcenter O. Denote by M, N the midpoints of $\overline{AH}$, $\overline{BC}$. Suppose the circle γ with diameter $\overline{AH}$ meets the circumcircle of ABC at $G \neq A$, and meets line AN at a point $Q \neq A$. The tangent to γ at G meets line OM at P. Show that the circumcircles of $\triangle GNQ$ and $\triangle MBC$ intersect at a point T on $\overline{PN}$.

Problem 3.61 (CAMO 2020/3)**.** Let ABC be a triangle with incircle ω, and let ω touch $\overline{BC}, \overline{CA}, \overline{AB}$ at D, E, F respectively. Point M is the midpoint of EF, and T is the point on ω such that $\overline{DT}$ is a diameter of ω. Line $\overline{MT}$ meets the line through A parallel to $\overline{BC}$ at P and ω again at Q. Lines $\overline{DF}$ and $\overline{DE}$ intersect line AP at X and Y respectively. Prove that the circumcircles of $\triangle APQ$ and $\triangle DXY$ are tangent.

Problem 3.62 (AQGO 2020/6)**.**]Let ΔABC be a triangle with orthocenter H and $\overline{BH}$ meet $\overline{AC}$ at E and $\overline{CH}$ meet $\overline{AB}$ at F. Let $\overline{EF}$ intersect the line through A parallel to $\overline{BC}$ at X and the tangent to (ABC) at A intersect $\overline{BC}$ at Y. Let $\overline{XY}$ intersect AB at P and let XY meet AC at Q. Let O be the circumcenter of ΔAPQ and $\overline{AO}$ meet $\overline{BC}$ at T. Let V be the projection of H on $\overline{AT}$ and M be the midpoint of BC. Then prove that (BHC) and (TVM) are tangent to each other.

Problem 3.63 (Myself (i3435))**.** Let ABC be a triangle such that the foot of the altitude from A to $\overline{BC}$ is D, the foot of the altitude form B to $\overline{AC}$ is E, and the foot of the altitude from C to $\overline{AB}$ is F. Let $X = \overline{EF} \cap \overline{BC}$. Let the circle with diameter AD hit $\overline{DE}$ at E', $\overline{DF}$ at F', and (DEF) at the non-D point L. Let $\overline{GI} \cap \overline{EF} = K$. Then prove that X, D, L, K are concyclic.

Problem 3.64 (Romania TST 2009 Day 3 P3)**.** Let ABC be a non-isosceles triangle, in which X, Y, and Z are the tangency points of the incircle of center I with sides BC, CA and AB respectively. Denoting by O the circumcircle of $\triangle ABC$, line OI meets BC at a point D. The perpendicular dropped from X to YZ intersects AD at E. Prove that YZ is the perpendicular bisector of $[EX]$.

Problem 3.65 (2017 USAMO P3)**.** Let ABC be a scalene triangle with circumcircle Ω and incenter I. Ray AI meets $\overline{BC}$ at D and meets Ω again at M; the circle with diameter DM cuts Ω again at K. Lines MK and BC meet at S, and N is the midpoint of $\overline{IS}$. The circumcircles of $\triangle KID$ and $\triangle MAN$ intersect at points L_1 and L_2. Prove that Ω passes through L_1 or L_2.

Problem 3.66 (Taiwan TST 2015 Round 3 Quiz 3 P2)**.** In a scalene triangle ABC with incenter I, the incircle is tangent to sides CA and AB at points E and F. The tangents to the circumcircle of triangle AEF at E and F meet at S. Lines EF and BC intersect at T. Prove that the circle with diameter ST is orthogonal to the nine-point circle of triangle BIC.

Problem 3.67 (Modified Iran TST 2009/9)**.** Let $\triangle ABC$ have incenter I and contact triangle DEF. Let H be the orthocenter of $\triangle BIC$, and let P be the foot from D to $\overline{EF}$. If the midpoint of EF is N, the midpoint of DP is N^*, and Ge is the gergonne point of $\triangle ABC$, then prove that $N^* - Ge - N - H$.

Problem 3.68 (POGCHAMP 6)**.** Triangle ABC has incenter I, A-excenter I_A, and let D be the foot of I onto BC. M is the midpoint of arc $\widehat{BAC}$ in $\odot(ABC)$, MA intersects BC at E, and line I_AD intersects $\odot(AID)$ again at $F \neq D$. Let O be the circumcenter of $\triangle AMF$ and suppose the line through E perpendicular to OI meets MI_A at G. Find, with proof, the value of $\frac{I_AG}{MG}$.

Hints: 49,56

Problem 3.69 (AoPS user proglote)**.** Let T be the homothetic center of the intouch and excentral triangle of a triangle ABC. Prove that T is collinear with the centroid of $\triangle ABC$, G and the Gergonne Point of $\triangle ABC$, Ge.

Hints: 84,76,28

Problem 3.70 (2012 ELMO SL G7)**.** Let $\triangle ABC$ be an acute triangle with circumcenter O such that $AB < AC$, let Q be the intersection of the external bisector of $\angle A$ with BC, and let P be a point in the interior of $\triangle ABC$ such that $\triangle BPA$ is similar to $\triangle APC$. Show that $\angle QPA + \angle OQB = 90^\circ$.

Problem 3.71 (Fake USAMO 2020/3)**.** Let $\triangle ABC$ be a scalene triangle with circumcenter O, incenter I, and incircle ω. Let ω touch the sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at points D, E, and F respectively. Let T be the projection of D to $\overline{EF}$. The line AT intersects the circumcircle of $\triangle ABC$ again at point $X \neq A$. The circumcircles of $\triangle AEX$ and $\triangle AFX$ intersect ω again at points $P \neq E$ and $Q \neq F$ respectively. Prove that the lines EQ, FP, and OI are concurrent.

Problem 3.72 (Myself (i3435))**.** Let ABC be a triangle. Let E, F be the feet of the altitudes of $\triangle ABC$ from B, C respectively. Let M be the midpoint of BC and let N be the midpoint of EF. Let X be the point such that $\triangle XMN$ is similar and similarly oriented to $\triangle XFE$. Let Y be the point such that $\triangle YMN$ is similar and similarly oriented to $\triangle YEF$. Let $Z = \overline{FX} \cap \overline{EY}$ and let T be the intersection of the tangents to (ABC) at B and C. Prove $\overline{AZ}$ bisects MT.

Problem 3.73 (Myself(i3435))**.** In triangle ABC, let I be the incenter, and let the incircle hit the sides $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ at D, E, F respectively. Let S be the non-A intersection of (AEF) and (ABC), let $J = \overline{AI} \cap \overline{BC}$, and let N be the midpoint of EF. Let $\overline{SJ}$ and $\overline{DN}$ intersect at P. Prove P is on the 9-point circle of DEF.

Problem 3.74 (2015 Taiwan TST Round 3 Quiz 1 P2)**.** Let O be the circumcircle of the triangle ABC. Two circles O_1, O_2 are tangent to each of the circle O and the rays $\overrightarrow{AB}, \overrightarrow{AC}$, with O_1 interior to O, O_2 exterior to O. The common tangent of O, O_1 and the common tangent of O, O_2 intersect at the point X. Let M be the midpoint of the arc BC (not containing the point A) on the circle O, and the segment $\overline{AA'}$ be the diameter of O. Prove that X, M, and A' are collinear.

Problem 3.75 (Myself(i3435))**.** Let ABC be a triangle with circumcenter O, incenter I, and intouch triangle DEF. Let L be the midpoint of arc $\widehat{BAC}$, and let $\overline{LD}$ intersect (ABC) again at X. Let (EFX) meet (ABC) again at Y. Then let $\overline{AX} = \ell_A$ and $\overline{LY} = m_A$. Define ℓ_B, ℓ_C, m_B, m_C similarly. Prove that $\ell_A, \ell_B, \ell_C, m_A, m_B, m_C$, and $\overline{OI}$ all concur at one point.

Problem 3.76 (AoPS user MP8148). Let ABC be a triangle with incenter I, orthocenter H, and circumcenter O. Show that the line parallel through O parallel to $\overline{BC}$ and the polar of H with respect to (BIC) meet on $\overline{AI}$.

Problem 3.77 (SORY P6). Let $\triangle ABC$ be a triangle with incenter I. Let the incircle be tangent to the sides BC, CA, AB at D, E, F respectively. Let P be the foot of the perpendicular from D onto $\overline{EF}$. Assume that BP, CP intersect the sides AC, AB at Y, Z respectively. Finally let the rays IP, YZ meet the circumcircle of $\triangle ABC$ in R, X respectively.

Prove that the tangent from X to the incircle and RD intersect on the circumcircle of $\triangle ABC$.

Problem 3.78 (AoPS user beiqiang). Let acute triangle ABC be inscribed in a circle ω, and suppose the incircle of ABC touches side BC at N. Let ω' be a circle tangent to BC at N, and tangent to ω at T such that ω' is on the same side of BC as A. Let O be the center of ω' and I_a be the A-excenter. Let the midpoint of smaller arc BC be L, and A' be the A-antipode. Let TN and AO intersect at P.

Show that:

1. T is on NL and IA',
2. $\{AO, NI_a\}, \{BC, PI_a\}, \{ON, A'P\}$ are parallel,
3. LI_aPA' and ATI_aP are cyclic, and
4. AT, $A'L$ and I_aP are concurrent.

Hints: 88,8,44,85

Problem 3.79 (2017 ELMO SL G4). Let ABC be an acute triangle with incenter I and circumcircle ω. Suppose a circle ω_B is tangent to BA, BC, and internally tangent to ω at B_1, while a circle ω_C is tangent to CA, CB, and internally tangent to ω at C_1. If B_2, C_2 are the points opposite to B, C on ω, respectively, and X denotes the intersection of B_1C_2, B_2C_1, prove that $XA = XI$.

Hints: 81,58

Problem 3.80 (AoPS User MP8148). Let ABC be a triangle with incenter I, circumcenter O, and circumcircle Γ. Let M be the midpoint of the arc BC not containing A. Suppose the incircle is tangent to $\overline{BC}$ at D, and N is the midpoint of $\overline{IM}$. Denote Ω to be the circumcircle of $\triangle AON$. Show that

a Lines $\overline{OI}$ and $\overline{MD}$ meet on the radical axis of Γ and Ω.

b Lines $\overline{OI}$ and $\overline{ND}$ meet on Γ.

Hints: 36,22

Problem 3.81 (2014 ELMO SL G8). In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

Hints: 27,30

Problem 3.82 (USA TST 2016/2). Let ABC be a scalene triangle with circumcircle Ω, and suppose the incircle of ABC touches BC at D. The angle bisector of $\angle A$ meets BC and Ω at E and F. The circumcircle of $\triangle DEF$ intersects the A-excircle at S_1, S_2, and Ω at $T \neq F$. Prove that line AT passes through either S_1 or S_2.

Hints: 13,41

Problem 3.83 (EMMO Juniors 2016/5). Let $\triangle ABC$ be a triangle with circumcenter O and circumcircle Γ. The point X lies on Γ such that AX is the A- symmedian of triangle $\triangle ABC$. The line through X perpendicular to AX intersects AB, AC in F, E, respectively. Denote by γ the nine-point circle of triangle $\triangle AEF$, and let Γ and γ intersect again in $P \neq X$. Further, let the tangent to Γ at A meet the line BC in Y, and let Z be the antipode of A with respect to circle Γ. Prove that the points Y, P, Z are collinear.

Hints: 39,47

Problem 3.84 (All Russian 2013 Grade 11 (12 in American System) P8 (why am I putting Russia problems in an American Geo handout?)). Let ω be the incircle of $\triangle ABC$ and with center I. Let Γ be the circumcircle of the triangle BIC. Circles ω and Γ intersect at the points X and Y. Let Z be the intersection of the common tangents of the circles ω and Γ. Show that the circumcircle of the triangle XYZ is tangent to the circumcircle of $\triangle ABC$.

Hints: 2,71,73,21

Problem 3.85 (AoPS user MP8148). In scalene triangle ABC with $AB \neq AC$, I is the incenter, and O is the circumcenter. Let L be the midpoint of arc BAC, P be the point on $\overline{BC}$ such that $\overline{PI} \perp \overline{OI}$, and Q be the point on $\overline{AL}$ such that $\overline{QP} \perp \overline{LI}$.

If the incircle is tangent to $\overline{BC}$ at D and D' is the reflection of D over I, show that $QD' = QI$.

Hints: 80,94,23

Problem 3.86 (Mathematical Reflections O451). Let ABC be a triangle, Γ be its circumcircle, ω its incircle, and I the incenter. Let M be the midpoint of BC. The incircle ω is tangent to $\overline{AB}$ and $\overline{AC}$ at F and E respectively. Suppose $\overline{EF}$ meets Γ at distinct points P and Q. Let J denote the point on EF such that MJ is perpendicular on EF. Show that IJ and the radical axis of (MPQ) and (AJI) intersect on Γ.

Hint: 89

Problem 3.87 (AoPS Problem Making Contest 2016/7). Let $\triangle ABC$ be given, it's $A-$mixtilinear incircle, ω, and it's excenter I_A. Let H be the foot of altitude from A to BC, E midpoint of arc $\widehat{BAC}$ and denote by M and N, midpoints of BC and AH, respectively. Suposse that $MN \cap AE = \{P\}$ and that line I_AP meet ω at S and T in this order: $I_A - T - S - P$.

Prove that circumcircle of $\triangle BSC$ and ω are tangent to each other.

Hints: 68,63,60

Problem 3.88 (AoPS user tutubixu9198)**.** Let ABC be a triangle with circumcircle (O). The tangents to (O) at the vertices B and C meet at a point S. Let d be the internal bisector of the angle BAC. Let the perpendicular bisector of AB, AC intersect d at M, N respectively and $P = BM \cap CN$. Prove that A, I, S are collinear where I is the incenter of triangle PMN.

Hints: 61,3

Problem 3.89 (AoPS User Supercali)**.** In ΔABC, let F_e be the Feuerbach point, let I be the incentre, let H be orthocentre of ΔBIC and let A' be reflection of A in F_e. Then, prove that AH, IA' and BC are concurrent.

Hints: 6,50,48,77

Problem 3.90 (Iran TST Third Round 2020 Geometry P4)**.** Triangle ABC is given. Let O be it's circumcenter. Let I be the center of it's incircle.The external angle bisector of A meet BC at D. And I_A is the A-excenter . The point K is chosen on the line AI such that $AK = 2AI$ and A is closer to K than I. If the segment DF is the diameter of the circumcircle of triangle DKI_A, then prove $OF = 3OI$.

Hints: 43,66

Problem 3.91 (ELMO 2016/6)**.** Elmo is now learning olympiad geometry. In triangle ABC with $AB \neq AC$, let its incircle be tangent to sides BC, CA, and AB at D, E, and F, respectively. The internal angle bisector of $\angle BAC$ intersects lines DE and DF at X and Y, respectively. Let S and T be distinct points on side BC such that $\angle XSY = \angle XTY = 90°$. Finally, let γ be the circumcircle of $\triangle AST$.

a Help Elmo show that γ is tangent to the circumcircle of $\triangle ABC$.

b Help Elmo show that γ is tangent to the incircle of $\triangle ABC$.

Hints: 29,62,1

Problem 3.92 (AoPS Community)**.** In a triangle ABC, let $\triangle DEF$ be the intouch triangle. Let P be the foot from D to $\overline{EF}$, and let I and H be the incenter and orthocenter respectively of $\triangle ABC$. Prove that $\angle HPI$ is bisected by $\overline{PD}$.

Hints: 15,45,40,96

Problem 3.93 (2020 Taiwan TST Round 2 Mock IMO Day 6)**.** Let I, O, ω, Ω be the incenter, circumcenter, the incircle, and the circumcircle, respectively, of a scalene triangle ABC. The incircle ω is tangent to side BC at point D. Let S be the point on the circumcircle Ω such that AS, OI, BC are concurrent. Let H be the orthocenter of triangle BIC. Point T lies on Ω such that $\angle ATI$ is a right angle. Prove that the points D, T, H, S are concyclic.

Hints: 5,91,65,37,17

Problem 3.94 (Posted by AoPS user buratinogigle)**.** Let ABC be a triangle inscribed in circle (O). Tangent at A meets BC at X. Median AM meets (O) again at P. Q lies on ray MP such that $PQ = 2PM$. Choose the points R on line OX and D on segment BC such that $RD = RA = RQ$. Let S, T be on perpendicular bisector of AD such that $BS \perp BA$ and $CT \perp CA$. Prove that $ASDT$ is a rhombus.

Hints: 64,82,52,46,42

Problem 3.95 (Encyclopedia of Triangle Centers)**.** In $\triangle ABC$, let Be be the Bevan Point, or the reflection of the incenter over the circumcenter. If Be' is the antipode of Be on $(BCBe)$, then prove that $\overline{ABe'}$ goes through the homothety center between the intouch and excentral triangle.

Hints: 9,19,75

Problem 3.96 (Myself (i3435))**.** Let ABC be a triangle with circumcenter O and let the A-symmedian intersect $\overline{BC}$ at D. Let P be the foot from B to $\overline{AO}$ and let Q be the foot from C to $\overline{AO}$. Let A_1 be the intersection of $\overline{DP}$ and $\overline{AC}$ and let A_2 be the intersection of $\overline{DQ}$ and $\overline{AB}$. Define B_1, B_2, C_1, C_2 similarly. Prove that $A_1, A_2, B_1, B_2, C_1, C_2$ are all concyclic.

Hints: 16,95,31,70,25

Problem 3.97 (India TST???)**.** Let ABC be a triangle with circumcircle Γ and altitudes $\overline{AD}, \overline{BE}, \overline{CF}$ meeting at H. Let ω be the circumcircle of $\triangle DEF$. Point $S \neq A$ lies on Γ such that $DS = DA$. Line $\overline{AD}$ meets $\overline{EF}$ at Q, and meets ω at $L \neq D$. Point M is chosen such that DM is a diameter of ω. Point P lies on $\overline{EF}$ with $\overline{DP} \perp \overline{EF}$. Prove that lines SH, MQ, PL are concurrent.

Hints: 34,74,4,18,93

Problem 3.98 (USAMO 2016/3)**.** Let $\triangle ABC$ be an acute triangle, and let I_B, I_C, and O denote its B-excenter, C-excenter, and circumcenter, respectively. Points E and Y are selected on $\overline{AC}$ such that $\angle ABY = \angle CBY$ and $\overline{BE} \perp \overline{AC}$. Similarly, points F and Z are selected on $\overline{AB}$ such that $\angle ACZ = \angle BCZ$ and $\overline{CF} \perp \overline{AB}$.

Lines $\overleftrightarrow{I_BF}$ and $\overleftrightarrow{I_CE}$ meet at P. Prove that $\overline{PO}$ and $\overline{YZ}$ are perpendicular.

Hints: 90,54,33,83,51

Problem 3.99 (USA TSTST 2016/6)**.** Let ABC be a triangle with incenter I, and whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F, respectively. Let K be the foot of the altitude from D to $\overline{EF}$. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1 and C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1 and B_2. Prove that the radical axis of the circumcircles of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

Hints: 78,24,10,55

Chapter 4

Hints

1. For part b, prove that the incircle and $(XSYT)$ are orthogonal.
2. Using PoP, where should (XYZ) go through for this problem to be true.
3. Try projecting in a couple different ways. Involve (BOC).
4. Use projective to show that $\overline{MQ}$ and $\overline{PL}$ meet on ω. Use some dilations to prove that if S' is the point on ω such that $\overline{MQ}, \overline{PL}, \overline{S'H}$ are concurrent, then it suffices to show that $\overline{LS'} \perp \overline{MH}$.
5. Show that it suffices for A, I, S, H to be concyclic.
6. Project in multiple ways.
7. Notice that they intersect at on the nine-point circle. Try and guess where they intersect.
8. Show that $\overline{AO} || \overline{DI_a} \implies ATI_aP$ cyclic $\implies$ the rest.
9. Try drawing a line perpendicular to $\overline{ABe'}$.
10. Use PoP to show that the orthocenter of $\triangle BIC$ is on the desired radical axis.
11. Draw in the symmedian
12. Note that the tangents from T to Ω meet Ω at N and K.
13. Prove that $T = T_A$.
14. Pascal's
15. Prove that $\overline{PD}$ bisects $\angle BPC$
16. What circle is this?
17. $\sqrt{bc}$ invert (in $\triangle BIC$).
18. Let T_A be the D-mixtillinear touch-point of $\triangle DEF$. Then show that $\overline{DT_A}$ and $\overline{M_AS'}$ intersect on the tangent to ω at M by projective.
19. Draw (ABe).

20. Prove that $MNPQ$ is cyclic.
21. Appolonius Circle
22. For the first hint, try finding a certain 9-point circle. Then try finding a spiral center for part b.
23. Try inverting about the incircle for both claims in the second hint.
24. Invert (AIC) and (AIB) around the incircle to show that the midpoint of EF is on the radical axis.
25. Draw the segment between A and the intersection of the A-tangent to (ABC) and $\overline{BC}$. Prove that both lines in last hint trisect the segment.
26. This means that T is on the radical axis of (AMG) and $(A'EF)$. Is there a special point one of the circles must go through?
27. What does ω_A pass through on $\overline{BC}$.
28. Neither T or Ge go to each other. However what other point is on this line that homotheties from T and G send Ge to?
29. Let M be the midpoint of XY, or the center of $(XSYT)$. Prove that A, S, M, T are cyclic.
30. Once you locate X, prove that A, A', O, X are concyclic.
31. Project on $\overline{AO}$. Reduce the problem to showing that $\overline{AK} \cap (ABC) - Y_A - \overline{KA_1} \cap \overline{BC}$(assume this and then solve the problem).
32. What is the point Z?
33. Show that this line goes through the foot of I_A on $\overline{AC}$.
34. They meet on ω. Use excentral-orthic duality.
35. What should the midpoint of PX be? What is the line perpendicular to $\overline{PT}$ at this point?
36. Show that T_A is on (AON).
37. Rephrase the problem in terms of (BIC).
38. What is the relation between K and T?
39. What is P?
40. Parallelogram Isogonality Lemma
41. Reflect (DEF) over $\overline{EF}$.
42. What is the most reduced form of $AS = AT$ under the inversion.
43. Use excentral-orthic duality.

44. What is the antipode of D on ω'.
45. What points are on $\overline{PI}$
46. Since R is on $\overline{OX}$, what other point is on (ADQ)? What does (ADQ) invert to?
47. Invert around A.
48. Show that if D is the foot from I to $\overline{BC}$, then using projective show that $\overline{IA'}$ trisecting AD suffices.
49. Excentral-orthic duality
50. Remember the Iran Lemma and the Median-Incircle Concurrency to involve H.
51. Reflect I_A over each of the vertices of the A-extouch triangle.
52. Note that inversion preserves cross ratio. Prove that Q inverts to the symmedian point of $\triangle ABC$.
53. Look at the reflection condition as an equal angle condition.
54. Draw the line through I_C through the midpoint of BE.
55. Construct $\overline{HB}$ and $\overline{HC}$. What 4 points are collinear.
56. G is the centroid of $\triangle I_A I_B I_C$. What is a way to use (AMF) to tie X and G together?
57. If you have guessed the 4-points that have the Poncelet Point correctly, then the rest should be angle chasing.
58. Remember Corollary 2.7.1 of Theorem 2.7.2.
59. What should the midpoint of PX be? What is the line perpendicular to $\overline{PT}$ at this point?
60. If $K = \overline{AI} \cap \overline{BC}$, then $-1 = (A, K; I, I_A)$.
61. What is the P-excenter of $\triangle PMN$?
62. Try using Shooting lemma.
63. Remember that T' is the intersection of $\overline{AD}$ and $\overline{T_A M_A}$.
64. $\sqrt{bc}$ invert.
65. What might be a better way to deal with the awkward point Q?
66. Dilate around the orthocenter by a factor of $\frac{1}{2}$.
67. Where does $\overline{AX}$ intersect $\overline{EF}$? Use projective to handle the reflection condition?
68. Let R be the intersection of $\overline{EI}$ and $\overline{BC}$ and let T' be the "bottom point" of ω. Prove that $R - T' - I_A$, and the finish.
69. Think of Poncelet Points.

70. Draw the tangent at A to (ABC). Prove that $\overline{K_AK'}$ and $\overline{K_AY_A}$ interact with it in a special way.

71. Let $\triangle DEF$ be the intouch triangle, and let $Q = \overline{EF} \cap \overline{BC}$. Prove that $\overline{XY} \cap \overline{BC}$ is the midpoint of DQ.

72. Let the tangent to N to Ω meet $\overline{BC}$ at T. Use PoP.

73. Let S be the A-Sharkydevil Point. Show that (SXY) is tangent to (ABC), and it suffices that S, X, Y, Z concyclic.

74. Do Problem 3.34.

75. Remember radical axis.

76. 99% of the time, when G is present, homotheties are present. Additionally the presence of T suggests homotheties are present.

77. Use multiple homotheties to finish.

78. Do Problem 3.67, and show that the midpoint of EF is on this radical axis.

79. Pascal's (I know this is hint 14 but I'm lazy)

80. Show that Q is the center of some circle that goes through D' and I.

81. Pascal's (I know this is hint 14 but I'm lazy)

82. Where does Q go under $\sqrt{bc}$ inversion?

83. One of the triangles in the homothety is $\triangle II_BI_C$.

84. Because G is present, what technique should be used?

85. Finish the problem with projective.

86. What is F?

87. What is the antipode of A on (APQ)?

88. Prove that T is on $\overline{NL}$, and thus the A-Sharkydevil Point, using homothety.

89. Do Problem 3.15.

90. P must be on $\overline{OI_A}$. Show that P is the center of a homothety between two triangles.

91. Show that if $\overline{OI} \cap (BIC) = Q$, that it suffices for A, I, Q, H to be concyclic.

92. Draw in the tangential triangle of $\triangle ABC$.

93. Find a suitable rotation at T_A that proves that $\overline{LS'} \perp \overline{MH}$.

94. Show that P is on the perpendicular bisector of IT_A, and if A' is the reflection of I across A, show that $A'D'IT_A$ is cyclic.

95. Let A_1' be the point on $\overline{AC}$ such that $\overline{KA_1'} \perp \overline{AO}$, where K is the symmedian point of $\triangle ABC$.

96. Remember Theorem 2.7.4.

Chapter 5

References

Now I just have to say "Hey kids" and not put anything else here and this will officially be a Sam O'Nella Academy video.

Hey kids.

Chapter 6

Solutions

Problem 9

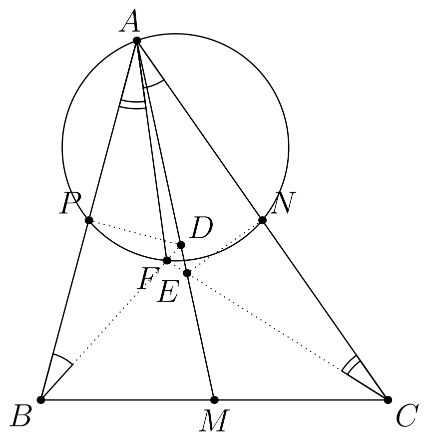
Define F' as the A-Dumpty-Point of $\triangle ABC$. Since F' is the center of spiral similarity from BA to AC,

$$\measuredangle ACF' = \measuredangle BAF' = \measuredangle MAC.$$

Similarly

$$\measuredangle ABF' = \measuredangle MAB,$$

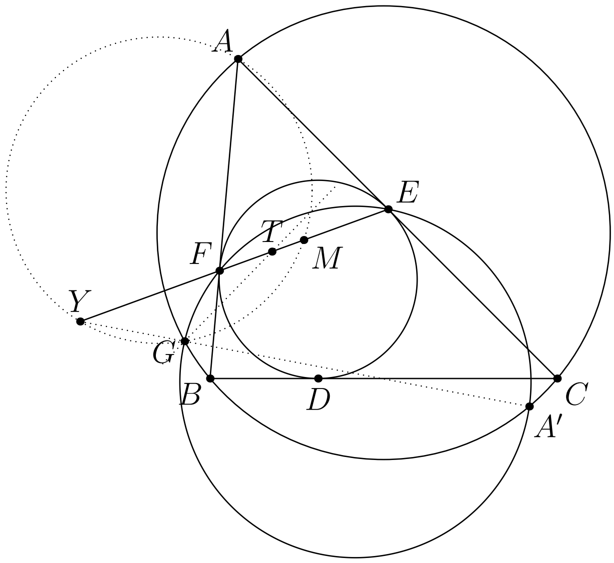
so $F' \equiv F$. Thus A, N, F, P are concyclic as desired.



At the top the anglemarks are for $\measuredangle BAF$ and $\measuredangle FAC$

Problem 19

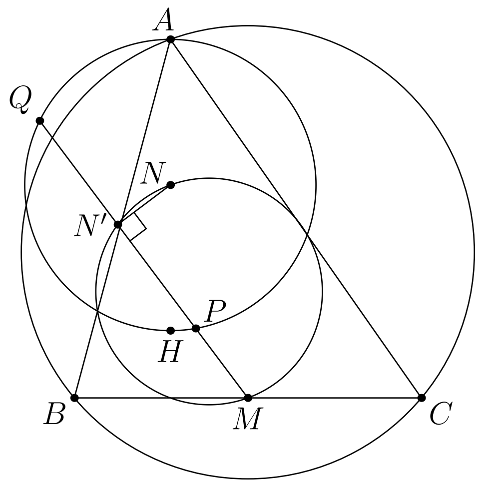
This is equivalent to T being on the radical axis of (AMG) and $(A'EF)$. Let Y be the D-Ex Point of $\triangle DEF$. Then it suffices to show that Y is on (AMG), because $MT \cdot YT = ET \cdot FT$. Since $\measuredangle AMY = 90$, $\measuredangle AGY$ should equal to 90. This is equivalent to $Y - G - A'$. This is true because the power of Y with respect to (ABC) and (DEF) is the same, so Y is on the radical axis of (ABC) and $(A'EF)$, as desired.



The important part is to recognize Y

Problem 29

Since H is the antipode of A on (AH), it suffices to show that the midpoint of PQ is on the dilation of (ABC) around H with a factor of $\frac{1}{2}$, or the nine-point circle of (ABC). Let the midpoint of AH be N and the midpoint of PQ be N'. Since $\measuredangle NN'M = 90$, this is true.



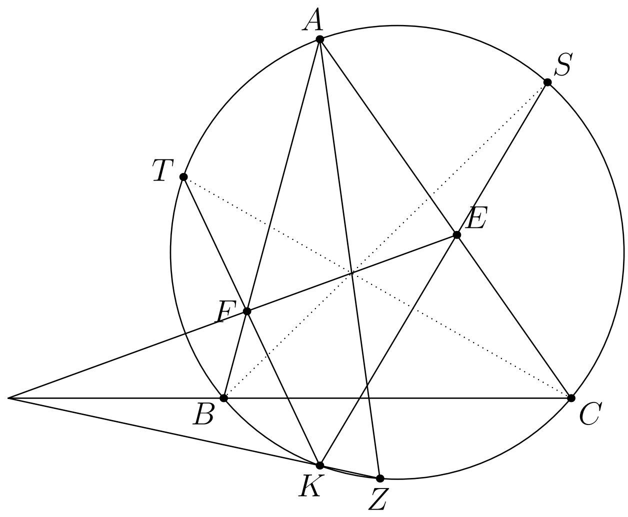
The trick is to look at $\triangle APQ$, then the rest falls into place.

Problem 39

We already know that Z is the intersection of the A-symmedian and (ABC) and that K, Z, E, F are already concyclic. The only difficulty is actually proving that BS, CT, XY are concurrent.

The important thing to notice is that BS and CT seem to intersect on $\overline{AZ} \cap \overline{EF}$. Pascal's on $BSKTCA$ proves that $\overline{BS}$ and $\overline{CT}$ meet on $\overline{EF}$ and Pascal's on $AZKSBC$ proves that

$\overline{BS}$ and $\overline{AZ}$ meet on $\overline{EF}$ (because $\overline{KZ}$ and $\overline{BC}$ meet on the A-Ex Point, which is on $\overline{EF}$) as desired.



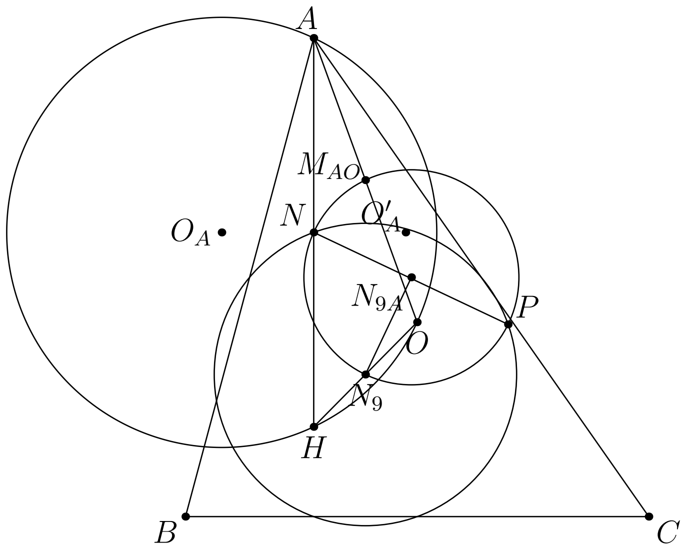
Although I never mention Pascal in the content portion of the handout, it's very important

Problem 49

Let P be the Poncelet Point of A, B, C, O(where O is the circumcenter) and let N be the midpoint of AH(where H is the orthocenter). Since $\overline{EF}$ is antiparallel to $\overline{BC}$, we want to show that $\measuredangle PNA = \measuredangle AOH$, and by symmetry this will be true for B and C. We know that P is on the nine-point circle of $\triangle AHO$, and thus the radical axis of the nine-point circles of $\triangle ABC$ and $\triangle AHO$.

Let N_9 and N_{9A} be the nine-point centers of these 2 respectively. Let M_{AO} be the midpoint of A and O. Then since $M_{AO}N_9 \perp \overline{BC}$, we want to show that $90 - \measuredangle PNA = \measuredangle M_{AO}N_9N_{9A} = 90 - \measuredangle AOH$. If O_A is the circumcenter of $\triangle AOH$ and O'_A is the reflection of O_A over N, then we get $\measuredangle M_{AO}N_9N_{9A} = \measuredangle AHO'_A = \measuredangle O_AHA = 90 - \measuredangle AOH$, as desired.

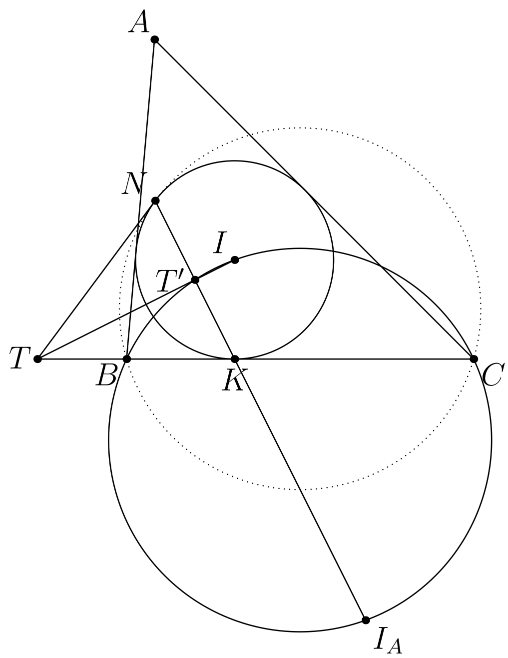
I originally guessed the Poncelet Point of A, B, C and the symmedian point K. While this is also right... you'd be better off reading this to find out why.



This is a nice wishful thinking problem: with the right guess of P, it's just an angle chase.

Problem 59

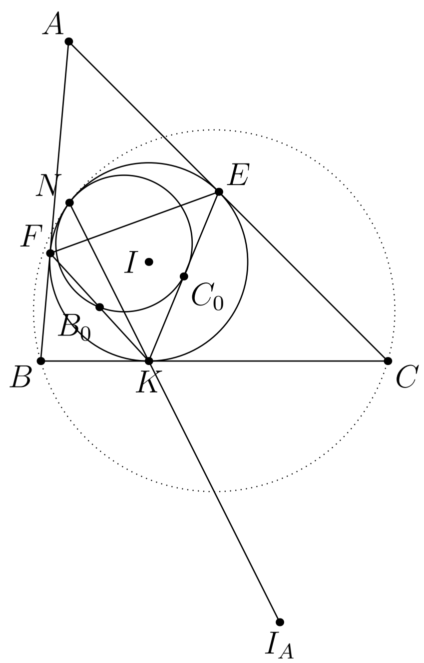
Let T be the point on $\overline{BC}$ such that TN is tangent to Ω. We have that the A-excenter I_A is on $\overline{KN}$. Let T' be the midpoint of KN. Then $TN^2 = TT' \cdot TI$. Since $\measuredangle I_AT'I = 90$, T' is on (BIC), so $TN^2 = TB \cdot TC$ as desired.



Clearly people in 2002 didn't know how to label correctly.

However there is another method, which is actually quite interesting. Let the intouch triangle be $\triangle KEF$. Note that the I_A-Why Point of the excentral triangle is on $\overline{KI_A}$. Additionally the homothety center between the excentral triangle and $\triangle KEF$ is on $\overline{KI_A}$, so dilating from the excentral triangle to $\triangle KEF$ means that N is the K-Why Point of $\triangle KEF$.

Let B_0 and C_0 be the midpoints of KF and KE respectively. Since (B_0C_0N) is tangent to Ω, an inversion around Ω means that (BCN) is tangent to Ω as well.

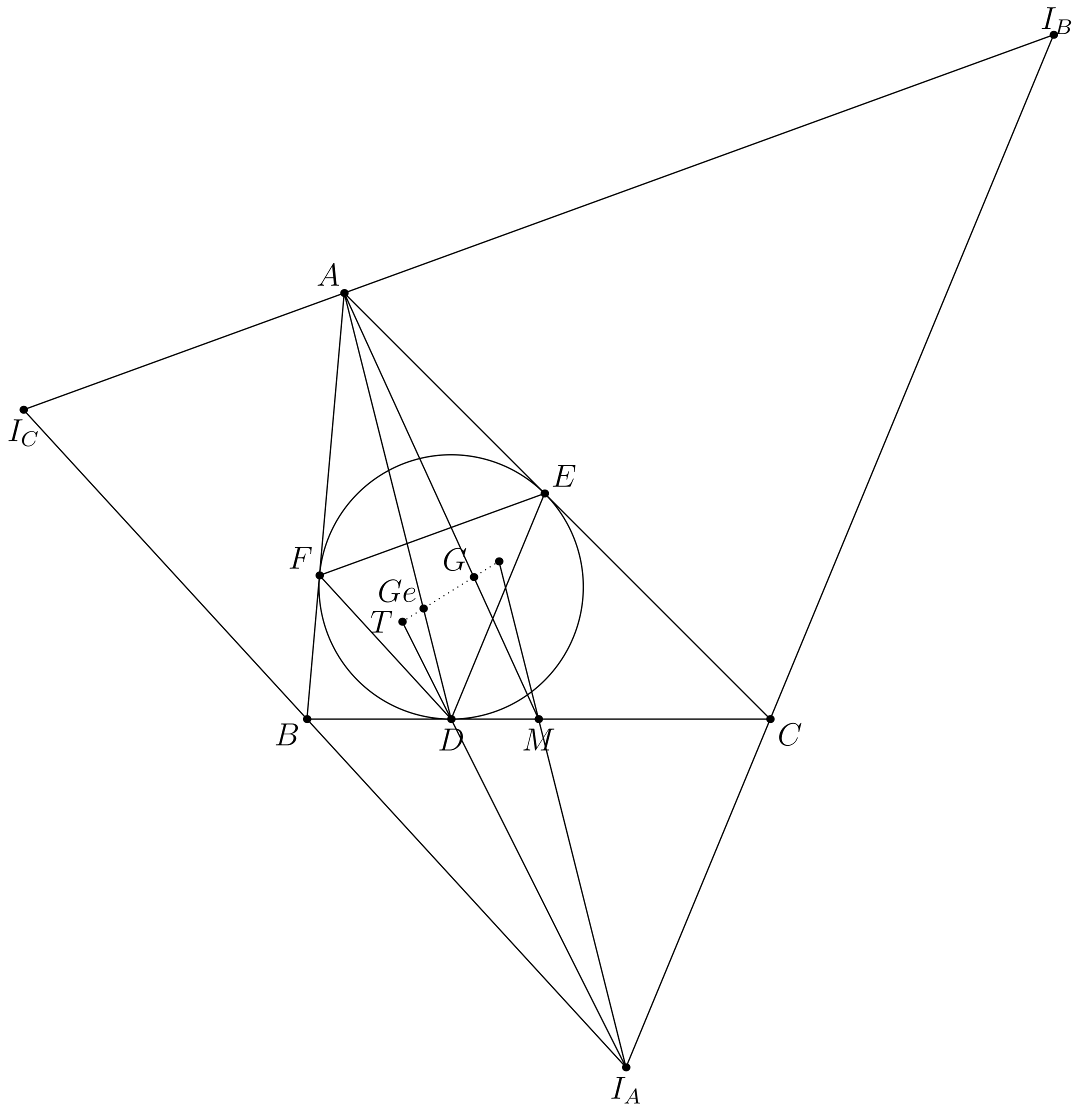


Note that I is the orthocenter of the excentral triangle and that K is the foot from I to $\overline{BC}$.

Problem 69

Let $\triangle DEF$ be the intouch triangle of $\triangle ABC$ and let I_A be the A-excenter of $\triangle ABC$. Let M be the midpoint of BC. $\overline{DA}$ is the D-symmedian of $\triangle DEF$ and $\overline{MI_A}$ is the I_A-symmedian of the excentral triangle. Thus T, Ge, and the symmedian point of the excentral triangle are collinear.

Because $\overline{AD}||\overline{MI_A}$, a homothety at G sending A to M also means that Ge,G, and the symmedian point of the excentral triangle are collinear, as desired.



This symmedian point is actually the Mittenpunkt of $\triangle ABC$, which is the isogonal conjugate of T.

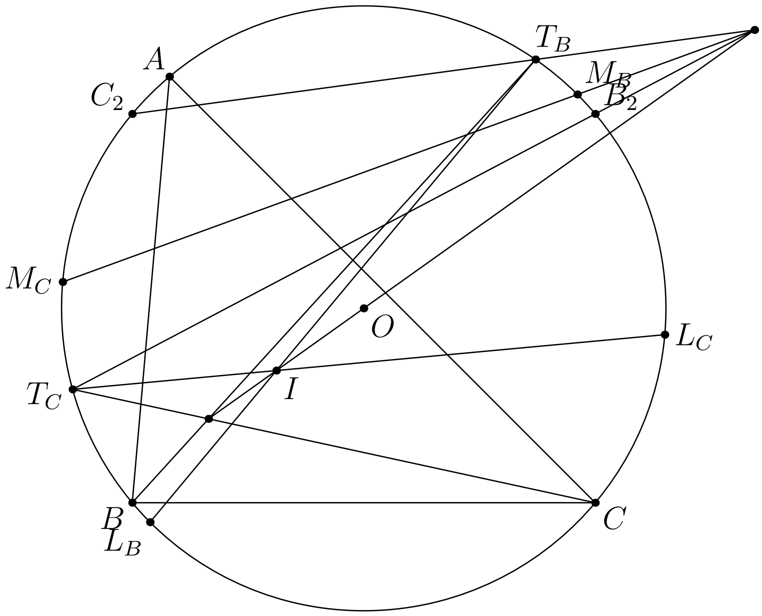
Problem 79

Rename B_1 and C_1 as T_B and T_C respectively. That was for no reason except to attain inner peace.

Define L_B and L_C as the midpoints of arcs $\widehat{ABC}$ and $\widehat{ACB}$ respectively, and define M_B and M_C as their antipodes, respectively. Then Pascal's on $BT_BC_2CT_CB_2$ gives that $\overline{T_BC_2}$ and $\overline{T_CB_2}$ meet on $\overline{OI}$, because $\overline{BB_2}$ and $\overline{CC_2}$ meet at O and $\overline{BT_B}$ and $\overline{CT_C}$ meet on $\overline{OI}$, which is because of Theorem 2.7.1 and Corollary 2.7.2.

Pascal's on $M_BM_CCC_2T_BL_B$ gives that $\overline{T_BC_2}$ and $\overline{M_BM_C}$ meet on $\overline{OI}$ because $\overline{L_BM_B}$ and $\overline{CC_2}$ meet on O and $\overline{M_CC}$ and $\overline{L_BT_B}$ meet on I.

Thus $\overline{T_BC_2}$ and $\overline{T_CB_2}$ meet on $\overline{M_BM_C}$, which is the perpendicular bisector of AI, as desired.

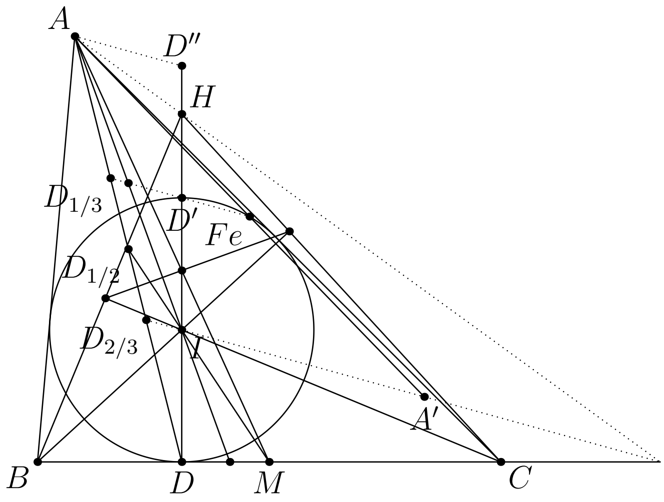


The especially interesting part about this problem is the concurrence with $\overline{OI}$

Problem 89

Let D' be the reflection of D over I, and let D'' be the reflection of I over D'. Let $D_{1/3}$ be the point between A and D such that $2AD_{1/3} = DD_{2/3}$, let $D_{1/2}$ be the point between A and D such that $AD_{1/2} = DD_{1/2}$, and let $D_{2/3}$ be the point between A and D such that $AD_{2/3} = 2DD_{2/3}$. A homothety at D with factor $\frac{3}{2}$ sends $D_{1/3}D'$ to AD'', and a homothety at I with factor $\frac{1}{2}$ means that $D_{1/3}D'$ bisects AI. Thus $D_{1/3} - D' - Fe$ and $D_{2/3} - I - A'$ by a homothety at A with factor 2.

By the Iran Lemma and Median-Extouch Lemma, if M is the midpoint of BC, then $(D, M; \overline{AI} \cap \overline{BC}, \overline{IA'} \cap \overline{BC}) \stackrel{I}{=} (D, D_{1/2}; A, D_{2/3}) = -1 = (D, \overline{AM} \cap \overline{DI}; I, H) \stackrel{A}{=} (D, M; \overline{AI} \cap \overline{BC}, \overline{AH} \cap \overline{BC})$, as desired.



I like the $(0, \frac{2}{3}; 1, \frac{1}{2}) = -1$ harmonic bundle a lot, I dunno about you.

Problem 90

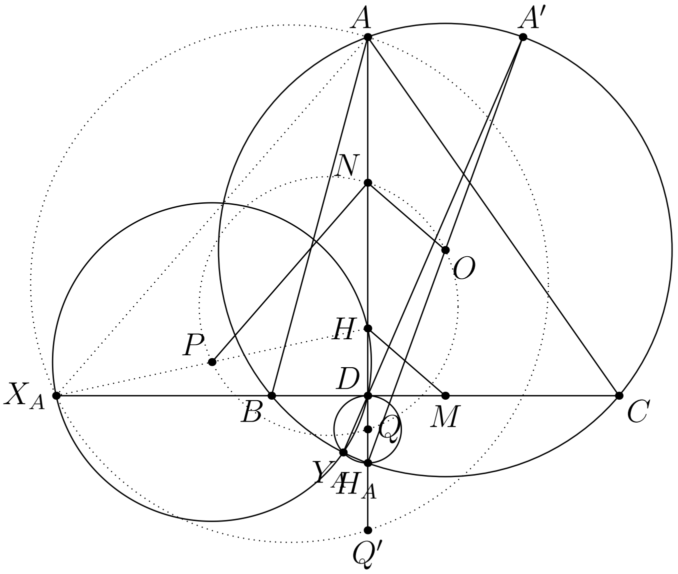
We can restate the problem as following through Excentral-Orthic Duality.

Let $\triangle ABC$ have orthocenter H and circumcenter O. Let X_A be the A-Ex Point and let L be the reflection of H over O. If D is the foot from A to $\overline{BC}$, H_A is the reflection of H over D, and Q' is the reflection of D over H_A, then prove that $\measuredangle X_AAL = 90 = \measuredangle X_AQ'L$.

Let N be the midpoint of AH, let P be the midpoint of X_AH, and let Q be the midpoint of HQ', or the midpoint of DH_A. It suffices to show that $\measuredangle PNO = 90 = \measuredangle PQO$.

If M is the midpoint of BC, then $\overline{HM}||\overline{NO}$. Since H is the orthocenter of $\triangle AX_AM$, $\overline{HM} \perp \overline{AX_A}$. Since $\overline{AX_A}||\overline{PN}$, $\measuredangle PNO = 90$.

Let A' be the reflection of A across the perpendicular bisector of BC. By a homothety centered at H_A $\overline{QO}||\overline{DA'}$. Additionally, P is the center of (XH) and Q is the center of (DH_A). If Y_A is the A-Why Point of $\triangle ABC$, then the radical axis of these two circles is $\overline{DY_A}$ or $\overline{DA'}$. Thus $\overline{PQ} \perp \overline{DA'}$, as desired.



Excentral-Orthic Duality seemed to be a theme in this test

Problem 91

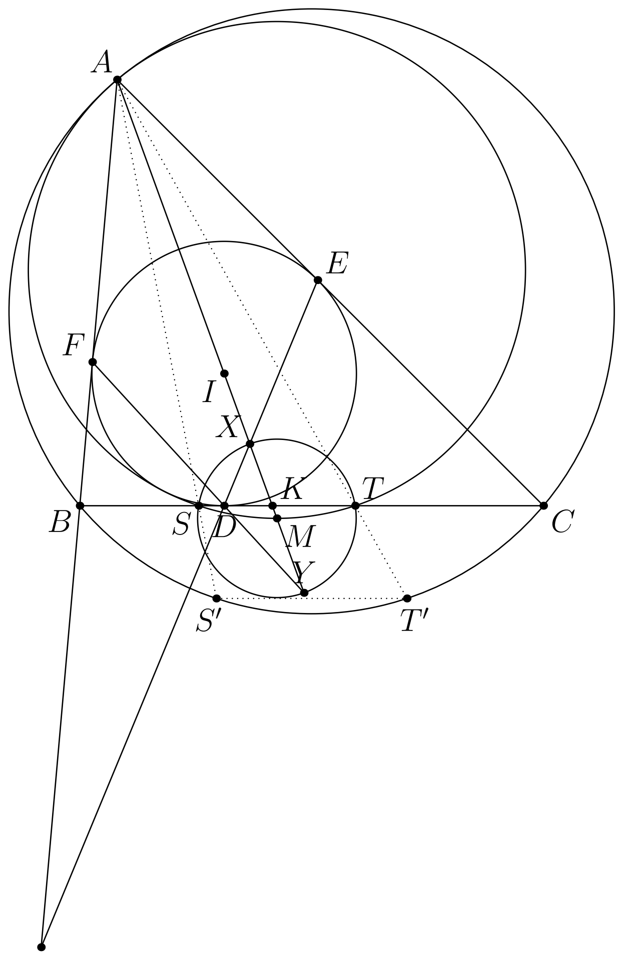
By the Iran Lemma X is on (IB) and Y is on (IC). Thus under an inversion about the incircle X goes to Y and Y goes to X, so (XY), or $(XSYT)$, inverts to itself. Similarly, and inversion around $(XSYT)$ inverts the incircle to itself.

Let K be the intersection of $\overline{XY}$ and $\overline{BC}$. Then $(A, K; X, Y) \overset{D}{=} (A, B; F, \overline{DE} \cap \overline{AB}) = -1$. Thus if M is the midpoint of XY, $MK \cdot MA = MX \cdot MY$. Under an inversion about $(XSYT)$, A goes to K, so (AST) goes to $\overline{BC}$, which is tangent to the incircle, proving part b. (You can similarly prove that (AST) is tangent to the A-excircle of $\triangle ABC$.)

From this inversion, we see that A, S, M, T are concyclic. Thus

$$\measuredangle SAM = \measuredangle STM = \measuredangle MST = \measuredangle MAT.$$

Letting S' and T' be the intersection of $\overline{AS}$ and $\overline{AT}$ with (ABC), $\overline{S'T'} || \overline{BC}$, so a dilation at A takes (ABC) to (AST). This proves part a, and finishes the problem.



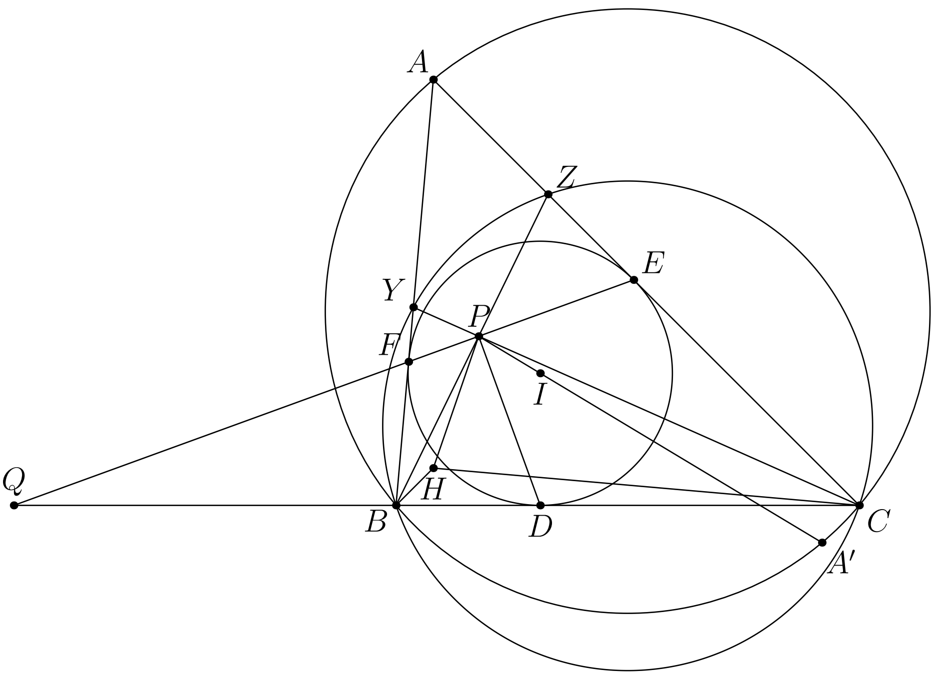
This is probably one of my favorite problem statements.

Problem 92

Let $Q = \overline{EF} \cap \overline{BC}$. Then $(Q, D; B, C) = -1$ and $\measuredangle QPD = 90$, so $\overline{PD}$ is an angle bisector of $\angle BPC$.

Let A' be the antipode of A on (ABC). Then $P - I - A'$ and $HBA'C$ is a parallelogram. Thus for $\overline{PH}$ and $\overline{PA'}$ to be isogonal in $\triangle PBC$, we need $\measuredangle PCH = \measuredangle HBP$, or $\measuredangle ACH - \measuredangle ACP = \measuredangle HBA - \measuredangle PBA$. Since $\measuredangle ABH = \measuredangle ACH = 90 - \measuredangle BAC$, it suffices to show that $\measuredangle ACP = \measuredangle PBA$.

Let $Y = \overline{CP} \cap \overline{AB}$ and let $Z = \overline{BP} \cap \overline{AC}$. Then $BCZY$ is cyclic. $\measuredangle ACP = \measuredangle ZYC = \measuredangle PBA$, as desired.

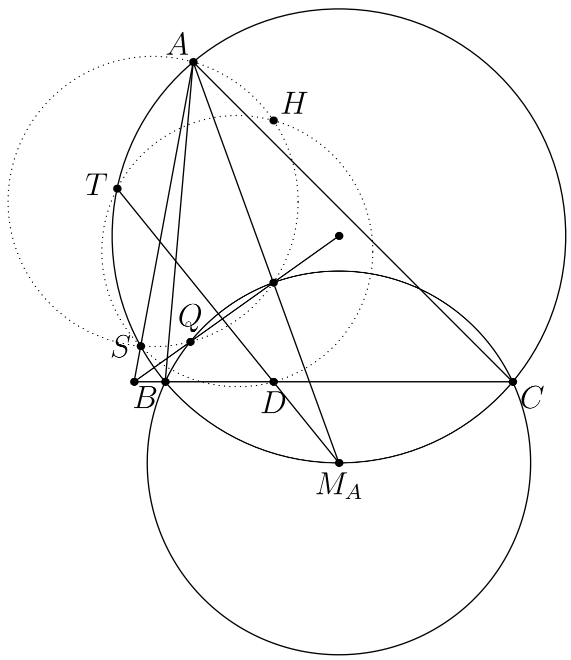


This problem nicely ties the orthocenter with the incenter.

Problem 93

Let M_A be the midpoint of $\widehat{BC}$ on (ABC) not containing A. Then $T - D - M_A$, so $\measuredangle IAS = \measuredangle M_A AS = \measuredangle M_A TS = \measuredangle DTS$. In addition $\measuredangle DHS = \measuredangle IHS$, so it suffices to show that A, I, S, H are concyclic.

Let $Q = \overline{OI} \cap (BIC)$. Then by radical axis it suffices to show that A, I, Q, H are concyclic. We can rephrase the problem in terms of $\triangle BIC$.

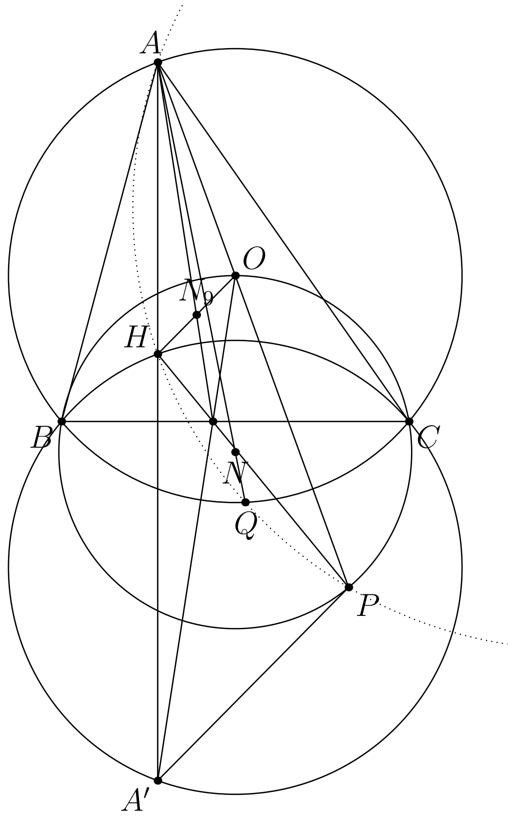


The definition of Q initially looks just as hopeless as the defintion of T, but it's actually better because Q is more well-defined in terms of $\triangle BIC$.

Let $\triangle ABC$ have orthocenter H and circumcenter O. Let N be the circumcenter of (BOC) and let $\overline{AN} \cap (ABC) = Q$. Let $\overline{AO} \cap (BOC) = P$. Prove that A, H, Q, P are concyclic.

$\sqrt{bc}$ invert. O goes to A', the reflection of A across $\overline{BC}$, so (BOC) goes to (BHC). If N_9 is the nine-point center of $\triangle ABC$, then $\overline{AN}$ goes to $\overline{AN_9}$. Thus the image of Q is $\overline{AN_9} \cap \overline{BC}$. The image of H is P, and the image of P is H. Thus we want to show that $\overline{HP}$ and $\overline{AN_9}$ concur on $\overline{BC}$.

Note that since H goes to P and O goes to A' under $\sqrt{bc}$ inversion, $\overline{HO} || \overline{A'P}$. We have that $\overline{OA'}$ is the reflection of $\overline{AN_9}$ over $\overline{BC}$, so we want to show that $\overline{OA'}$ and $\overline{HP}$ concur on $\overline{AN_9}$. Since N_9 is the midpoint of HO, by Ceva's on $\triangle AHO$, this is true.

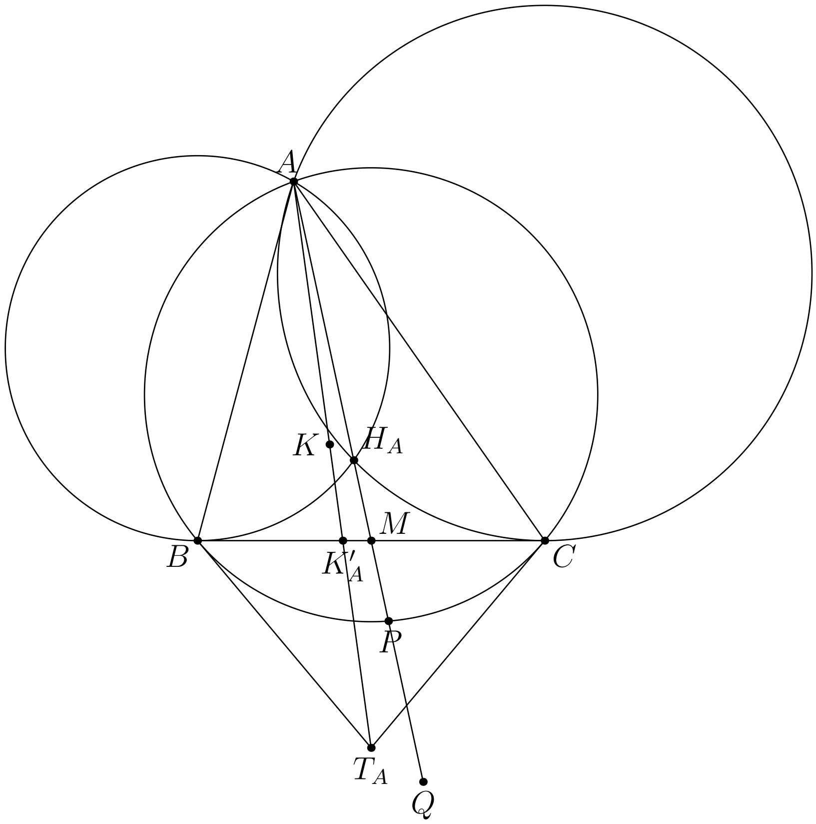


While not the fastest solution, I think that this is the most easily motivatable by the contents of the handout.

Problem 94

Let H_A be the A-Humpty Point. Then since $AM \cdot H_AM = CM^2$, (AH_AC) is tangent to $\overline{BC}$. Similarly (AH_AB) is tangent to $\overline{BC}$. $\sqrt{bc}$ inverting, we see that H_A maps to T_A, the intersection of the tangents to (ABC) at B, C.

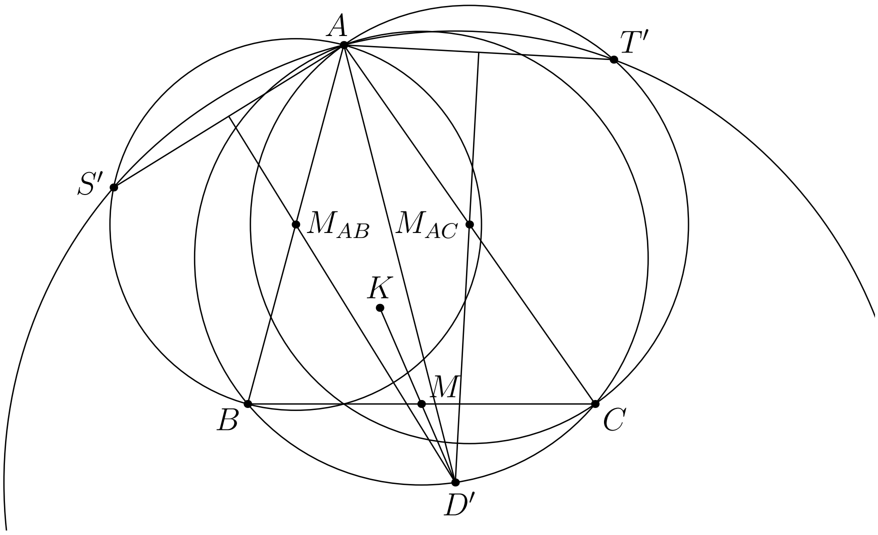
Let K be the symmedian point and let $K'_A = \overline{AK} \cap \overline{BC}$. Then $(A, K'_A; K, T_A) = -1$. Since $(P_\infty, P; Q, H_A) = -1$, K maps to Q under $\sqrt{bc}$ inversion.



This can be used to prove that $AK = \frac{bc\sqrt{2b^2+2c^2-a^2}}{a^2+b^2+c^2}$.

Let $K_A = \overline{AK} \cap (ABC)$. Then since $XK_A = XA$ and $OK_A = OA$, $RK_A = RA$, so K_A is on (ADQ). Under $\sqrt{bc}$ inversion, K_A goes to M and K goes to Q, so (ADQ) inverts to the A-Schwatt Line of $\triangle ABC$. D inverts to D', an intersection of this line with (ABC).

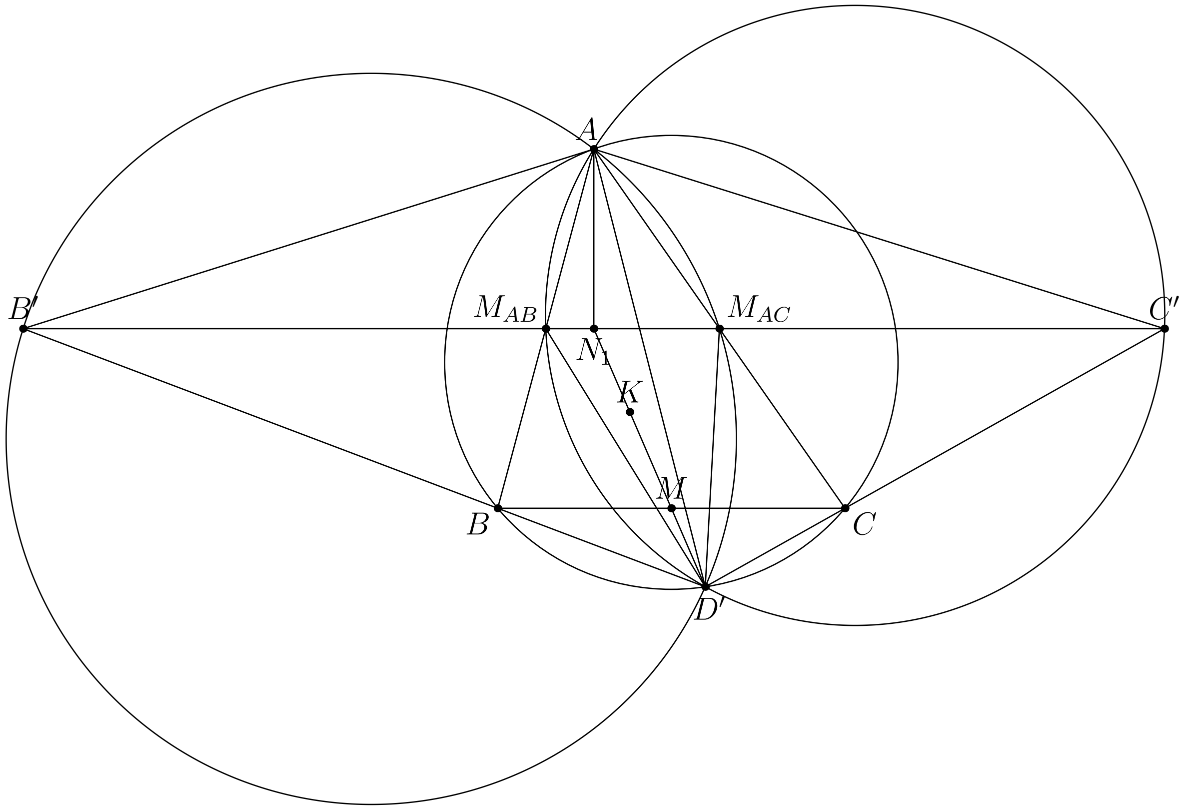
The circle centered at D' passing through A meets (AB) and (AC) at S' and T' respectively, which are the images of S and T under the $\sqrt{bc}$ inversion. Let M_{AC} and M_{AB} be the midpoints of AC and AB respectively. Then S' and T' are the reflections of A across $\overline{D'M_{AB}}$ and $\overline{D'M_{AC}}$ respectively. For $AS = AT$, A must be the same distance from $\overline{D'M_{AB}}$ and $\overline{D'M_{AC}}$. Thus our problem is reduced to $\measuredangle M_{AC}D'A = \measuredangle AD'M_{AB}$.



But why must D' be on the A-Schwatt Line?

Let $\overline{D'B}$ and $\overline{D'C}$ hit $\overline{M_{AB}M_{AC}}$ at B' and C' respectively. Let N_1 be the midpoint of the A-altitude. N_1 is on $\overline{KM}$ and $\overline{M_{AB}M_{AC}}$. Since $BM = MC$, $B'N_1 = N_1C'$, so $\measuredangle M_{AC}B'A = \measuredangle M_{AB}C'A$.

Since $\measuredangle AM_{AC}B' = \measuredangle ACB = \measuredangle AD'B = \measuredangle AD'B'$, B', A, M_{AC}, D' are cyclic. By symmetry C', A, M_{AB}, D' are cyclic, so $\measuredangle M_{AC}D'A = \measuredangle M_{AC}B'A = \measuredangle M_{AB}C'A = \measuredangle AC'M_{AB} = \measuredangle AD'M_{AB}$, as desired.



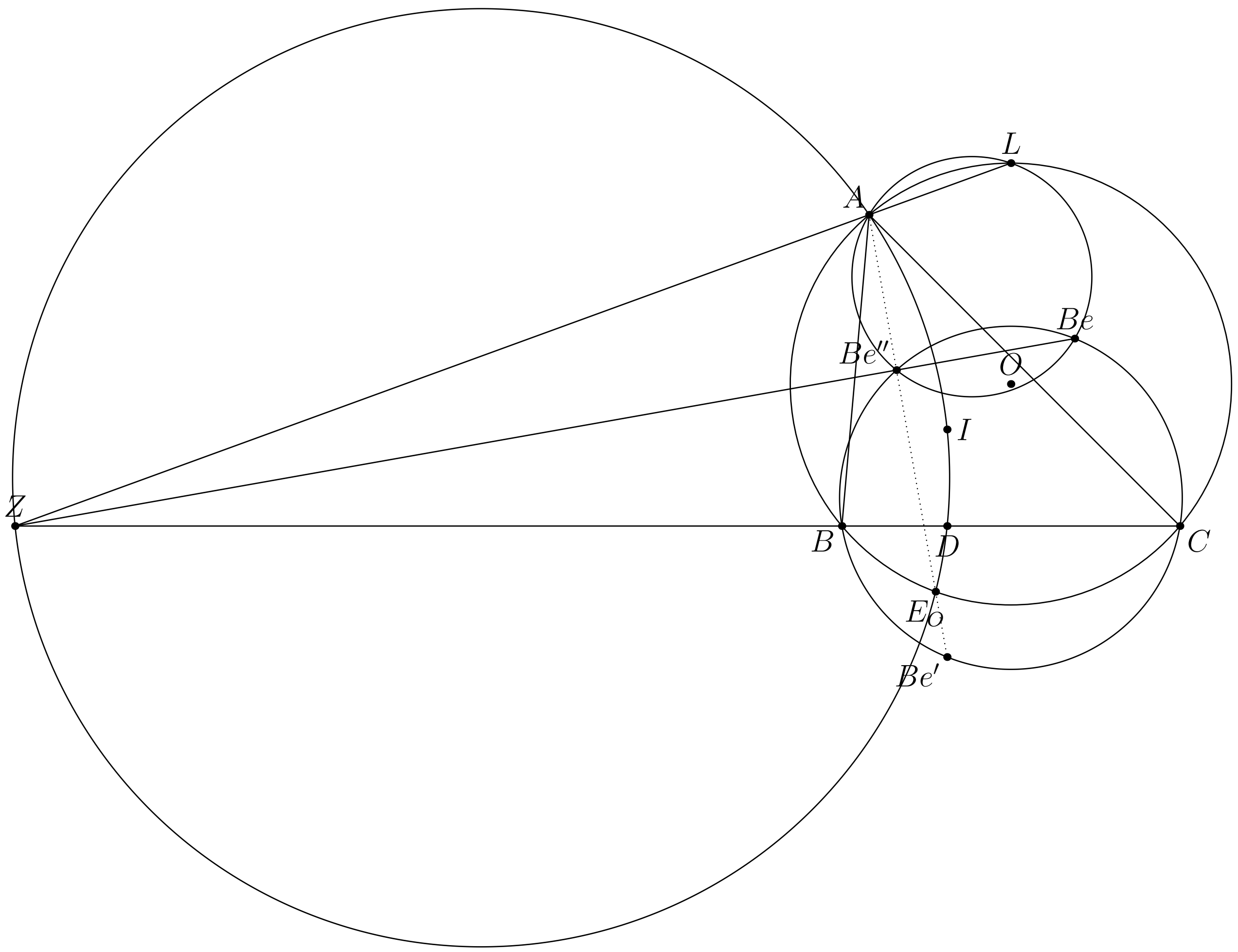
Note that the two possible points for D are the only ones on $\overline{BC}$ that make $ASDT$ a rhombus.

Problem 95

Let the intouch triangle be $\triangle DEF$ and let P be the foot from D to $\overline{EF}$. Then the homothety from $\triangle DEF$ to the excentral triangle sends P to A, so we want to show that Be' is on $\overline{AP}$, or $\overline{AE_O}$.

Let (ABe) and $(BCBe)$ meet again at Be''. Then $\measuredangle ABe''Be = \measuredangle BeB''Be = 90$, so $A - Be'' - Be'$ and $\overline{BeBe''} \perp \overline{ABe'}$. Let L be the midpoint of $\widehat{BAC}$. Since I and O are the orthocenter and nine-point center respectively of the excentral triangle, Be is the circumcenter of the excentral triangle. Since L is the midpoint of I_BI_C by excentral-orthic duality, L is on (ABe).

Radical axis on (ABe), (ABC), and $(BCBe)$ means that if $Z = \overline{AL} \cap \overline{BC}$, it suffices to show that $\overline{AX_{57}} \perp \overline{BeZ}$. $\overline{BeZ}$ is parallel to the line between O and the midpoint of ZI by a homothety around I. Since $\measuredangle ZAI = 90$, we want to show that $\overline{AE_O}$ is the radical axis of (ABC) and (ZI). However by Corollary 2.5.1, we are done.



Finally, a circle actually intersected at L and it's not just my eyes tricking me.

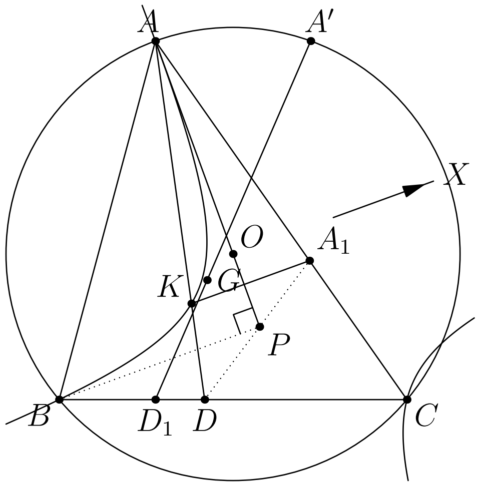
Problem 96

First I present the backwards version of how I created this, which I think is one of the shortest solutions for this problem.

Let A' be the reflection of A over the perpendicular bisector of BC, let G be the centroid of $\triangle ABC$, and let D_1 be the foot from A to $\overline{BC}$. Then $A' - G - D_1$.

Isogonally conjugate this line. A' goes to X, the point at infinity of the A-antiparallels of $\triangle ABC$, G goes to K, the symmedian point of $\triangle ABC$, and $\overline{AD_1}$ is isogonal to $\overline{AO}$, so the conic through A, B, C, K, X is tangent to $\overline{AO}$.

Pascal's on $AAKXBC$ gives that A_1 is the intersection of the A-antiparallel of $\triangle ABC$ through K and AC. We now finish by Second Lemoine Circle.



As you can probably tell the only two things I know about conics are Pascal's and isogonally conjugating a line gives a circumconic.

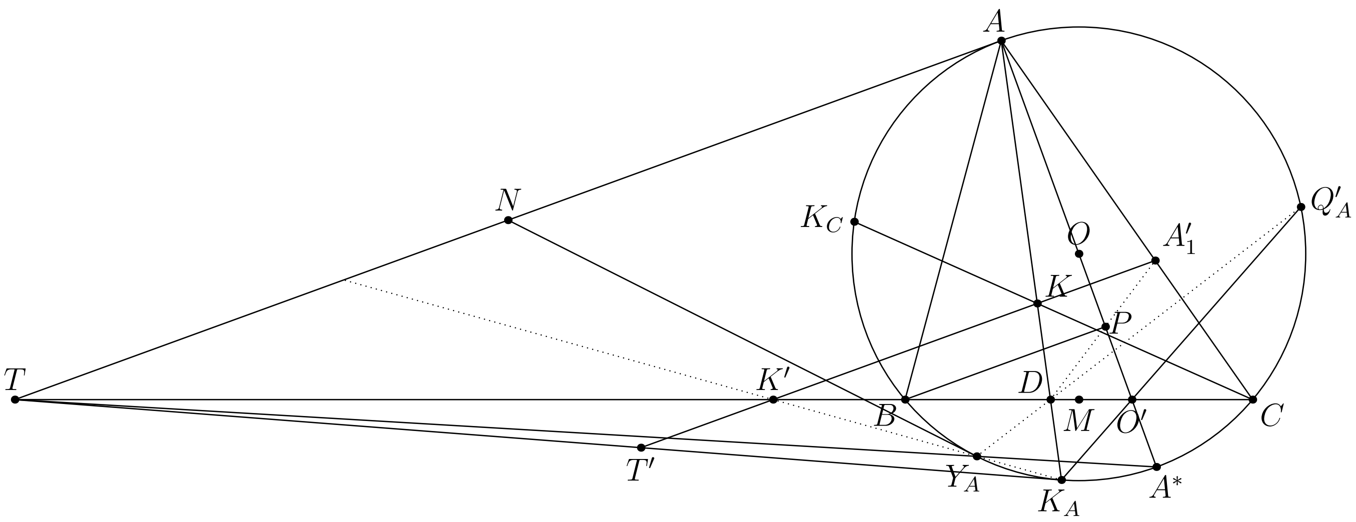
However, using a bunch of projections, there is another method, which the hints hopefully led you towards.

Let A_1' be the intersection of the A-antiparallel through K and $\overline{AC}$. We want to show that $D-P-A_1'$. Let K' be the intersection of $\overline{KA_1}$ and $\overline{BC}$, let $K_A=\overline{AK}\cap(ABC)$, and let Y_A be the A-Why Point. We first try to show that $K'-Y_A-K_A$

Let T be the intersection of the A-tangent to (ABC) and $\overline{BC}$. Then let N be the midpoint of AT and let A^* be the antipode of A on (ABC). $(A,A^*;Y_A,\overline{Y_AP_{\infty AT}})-1=(A,T;N,P_{\infty AT})\overset{Y_A}{=}(A,A^*;\overline{NY_A}\cap(ABC),\overline{Y_AP_{\infty AT}}\cap(ABC))$, so $\overline{NY_A}$ is tangent to (ABC) at Y_A. Thus projecting $-1=(Y_A,A^*;A,K_A)$ through Y_A to $\overline{AT}$ gives that $\overline{K_AY_A}$ trisects AT.

Let $T'=\overline{TK_A}\cap\overline{K'K_A}$. Then by homothety it suffices to show that K' trisects KT'. $(K,T';K',P_\infty)$ $\overset{T}{=}(K,K_A;D,A)$. Let $K_C=\overline{CK}\cap(ABC)$ and let M be the midpoint of BC. Then $(K,K_A;D,A)$ $\overset{C}{=}(K_C,K_A;B,A)=\frac{\frac{K_CB}{K_CA}}{\frac{K_AB}{K_AA}}=\frac{\frac{AC}{BC}}{\frac{MC}{AC}}=2$ as desired.

Now onto the main problem. Let $O'=\overline{AO}\cap\overline{BC}$ and let $Q_A'=\overline{K_AO'}\cap(ABC)$. Then $-1=(A,K_A;B,C)\overset{O'}{=}(A^*,Q_A';B,C)\overset{Y_A}{=}(T,\overline{Y_AQ_A'}\cap\overline{BC};B,C)$ so Y_A-D-Q_A'. $(A,P;\overline{K'K}\cap\overline{AO},O')\overset{P_\infty AT}{=}(T,B;K',O')\overset{K_A}{=}(K_A,B;Y_A,Q_A')\overset{D}{=}(A,C;Q_A',Y_A)\overset{K_A}{=}(D,C;O',K')=(C,D;K',O')\overset{A_1'}{=}(A,\overline{A_1'D}\cap\overline{AO};\overline{K'K}\cap\overline{AO},O')$ as desired.



Sometimes non-harmonic cross ratios help.

Problem 97

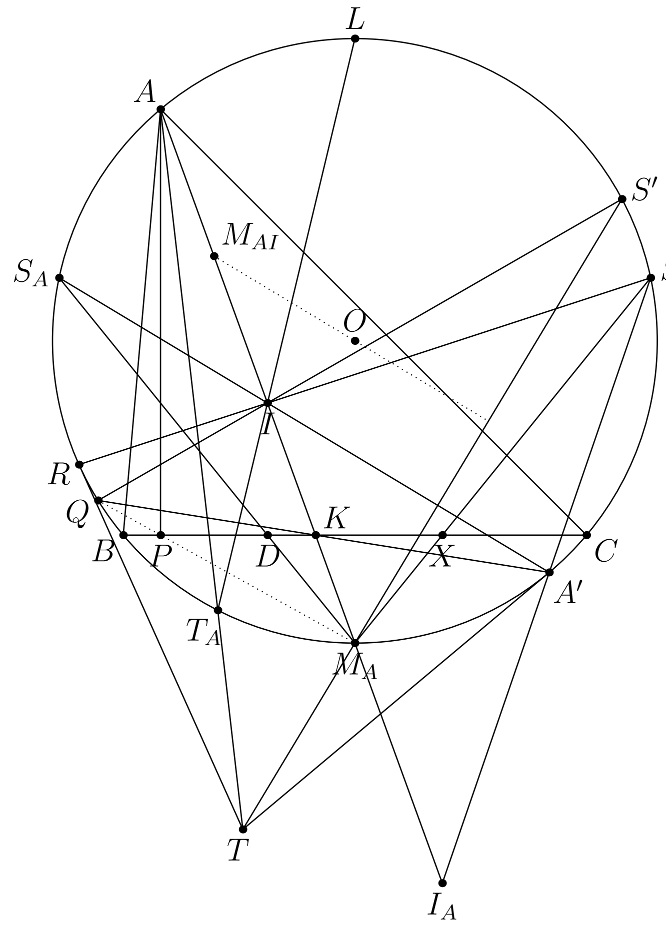
Using excentral-orthic duality, we have the following:

Let $\triangle ABC$ have incenter I, circumcenter O, and A-excenter I_A. Let M_A be the midpoint of the arc $\hat{BC}$ on (ABC) not containing A and let A' be the antipode of A on $\triangle ABC$. Let P be the foot of A on $\overline{BC}$ and let $K = \overline{AM_A} \cap \overline{BC}$. Then prove that if $\overline{A'K} \cap (ABC) = Q$, $M_A - P - Q$. Furthermore, if $\overline{IQ} \cap (ABC) = S'$ and the midpoint of AI is M_{AI}, then prove that $M_{AI}M_A = M_{AI}S'$.

Let D be the foot from I onto $\overline{BC}$ and let X be the foot from I_A onto $\overline{BC}$. Then by Problem 3.34 $\overline{A'I_A}$ and $\overline{MX}$ meet at S'_A, the reflection of the A-Sharkydevil Point S_A over the perpendicular bisector of BC. $-1 = (A, K; I, I_A) \stackrel{A'}{=} (A, Q; S_A, S'_A) \stackrel{M_A}{=} (K, \overline{M_AQ} \cap \overline{BC}; D, X)$ as desired.

We want to show that $\overline{M_{AI}O} \perp \overline{M_AS'}$, or that $\overline{IA'} \perp \overline{M_AS'}$. Let $\overline{S'_AI} \cap (ABC) = R$. The position of R isn't really that important, but if L is the antipode of M_A on (ABC) and T_A is the A-mixtillinear touchpoint on (ABC), then $(A, Q; S_A, S'_A) \stackrel{I}{=} (M_A, S'; A', R)$ and $(L, M_A; S_A, S'_A) \stackrel{I}{=} (T_A, A; A', R)$. Thus $\overline{AT_A}$ and $\overline{M_AS'}$ meet on the tangent to (ABC) at A', let's say at T.

Credits go to the AoPS user The_Turtle for this last part. I complex bashed this in terms of $\triangle M_AA'T_A$ but they found a better solution. Notice that $TA'M_AT_A \sim A'AIT_A$, since $\measuredangle TT_AA' = \measuredangle A'T_AA$, $\measuredangle T_AA'T = \measuredangle T_AAA'$, $\measuredangle T_AA'M_A = \measuredangle T_AAI$, and $\measuredangle M_AA'T = \measuredangle IAA'$. Since a 90 degree rotation plus a dilation at T_A brings these 2 to each other, $\overline{M_AS'} \perp \overline{IA'}$ as desired.



This was apparently Green Practice Test 3 P3 (out of 4) during MOP 2019. Do I need to say that this is copyrighted by MAA? I dunno, I never went to MOP. The Green MOP tests are copyrighted by MAA, as they were developed under contract as part of MOP.

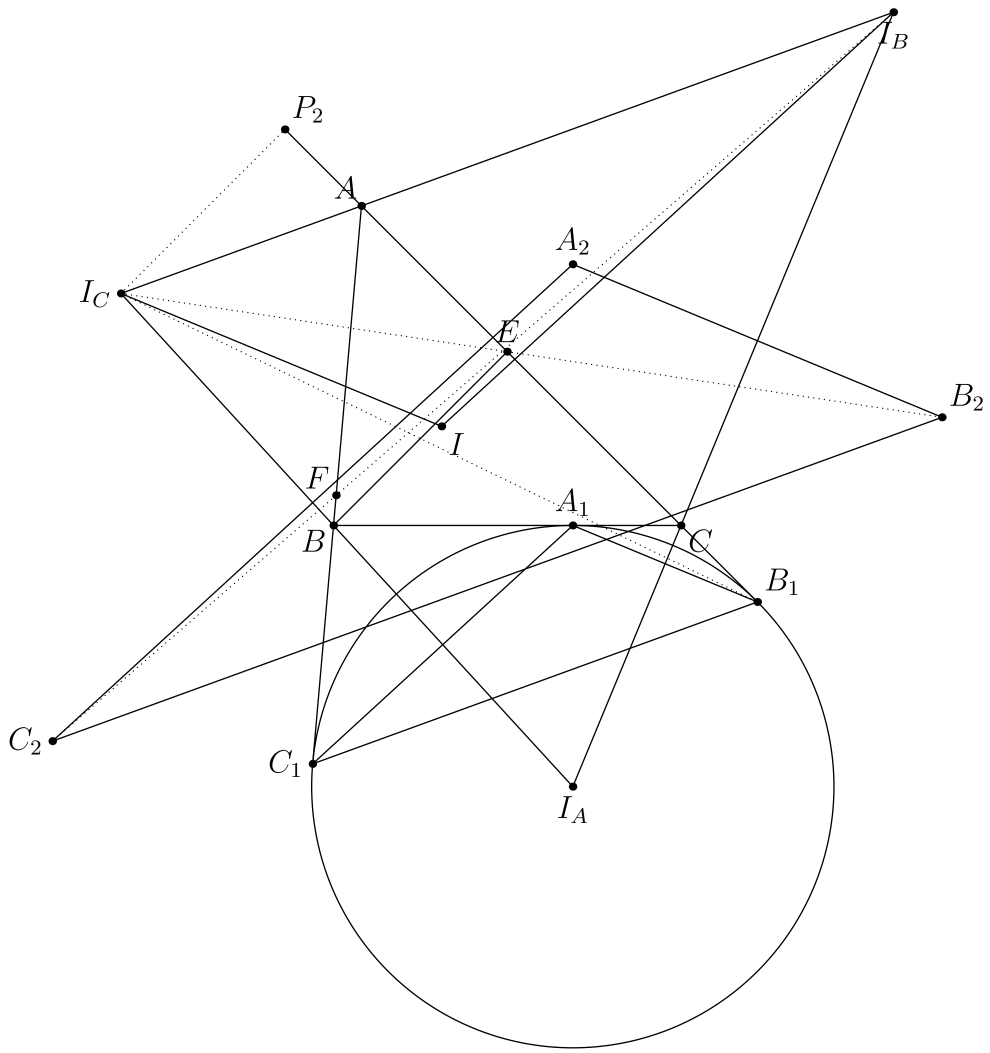
Problem 98

This solution is due to v_Enhance on the corresponding AoPS thread

Let I_A be the A-excenter and let $\triangle A_1B_1C_1$ be the A-extouch triangle (defined similarly as the intouch triangle). Let D be the foot from A to $\overline{BC}$ and let A_2, B_2, C_2 be the reflections of I_A across A_1, B_1, C_1.

Let $\overline{I_CB} \cap \overline{AC} = P_1$ and let P_2 be the foot from I_C to $\overline{AC}$. Then $-1 = (P_1, B; I_C, I_A) = (P_1, E; P_2, B_1) \stackrel{I_C}{=} (B, E; P_\infty, \overline{I_CB_1} \cap \overline{BE})$, showing that $\overline{I_CB_1}$ bisects BE. Thus $I_C - E - B_2$ and similarly $I_B - F - B_2$.

Since $\overline{A_1B_1} || \overline{II_C}$, $\overline{A_1C_1} || \overline{II_B}$, and $\overline{B_1C_1} || \overline{I_BI_C}$, $\triangle A_2B_2C_2 \sim \triangle II_CI_B$. The center of homothety between these is P. Thus P is on the line between I_A, which is the circumcenter of $\triangle A_2B_2C_2$, and the circumcenter of $\triangle II_CI_B$. Thus P is on $\overline{OI_A}$, which is perpendicular to $\overline{YZ}$, as desired.



Finding $\triangle A_2B_2C_2$ is why this is 45 Mohs

Problem 99

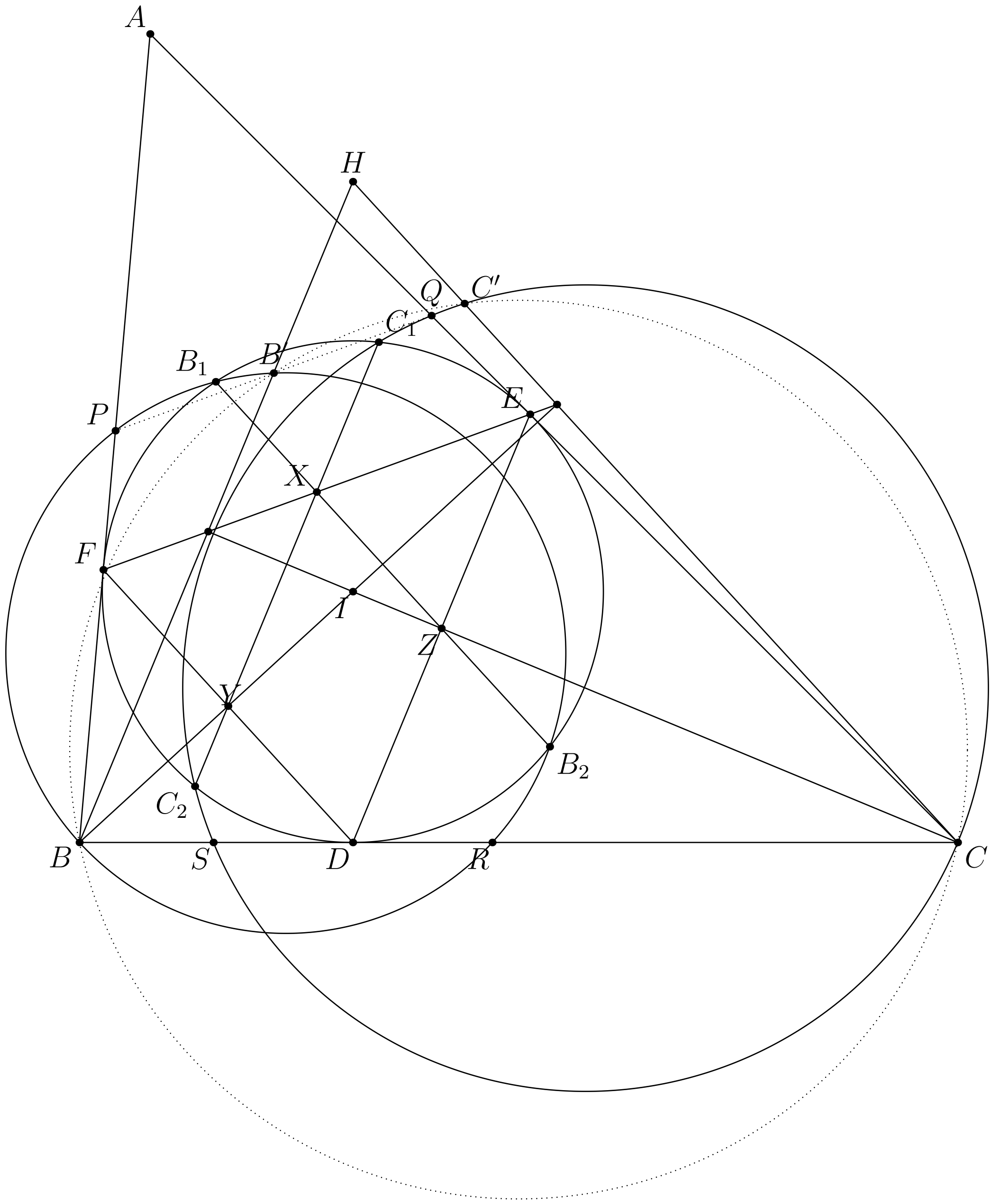
The first part of this problem is proving that X, the midpoint of EF, is on the desired radical axis. Let Y be the midpoint of DF and let Z be the midpoint of DE. Then since under an inversion about the incircle (AIC) inverts to $\overline{XZ}$ and (AIB) inverts to $\overline{XY}$, the radical center of (BB_1B_2), (CC_1C_2), and (DEF) is X.

The rest of the solution is due to anantmudgal09 and EulerMacaroni, on the corresponding AoPS thread).

Let $\overline{BA}, \overline{BC}$ meet ω_B at P, R respectively; define Q, S similarly. Let H be the orthocenter of $\triangle BIC$; $B' = \overline{HB} \cap \omega_B, C' = \overline{HC} \cap \omega_C$.

Note that $\measuredangle RPB' = \measuredangle RBB' = \measuredangle DFE$ and $\measuredangle SQC' = \measuredangle DEF$. Also, $\measuredangle(PR, QS) = \measuredangle EDF$ since $\overline{PR} \parallel \overline{DF}$ and $\overline{QS} \parallel \overline{DE}$. Observe that $\measuredangle RPB' + \measuredangle SQC' + \measuredangle(PR, QS) = 180^\circ$, hence P, Q, B', C' are collinear.

Consequently, B, C, B', C' are concyclic (say, on γ); indeed, $\measuredangle CC'Q = \measuredangle CSQ = \measuredangle CBB'$ as $\overline{SQ} \parallel \overline{BB'}$. Apply radical axis theorem to ω_B, ω_C and γ, to get that H lies on the radical axis of ω_B, ω_C, as desired.



Try solving this problem using the Gergonne Point instead of the orthocenter of $\triangle BIC$.

Problem 100

Got Lil' Wayne pumpin' on my iPod
Pumpin' on the subs in the back of my crew cab
Redneck rockin' like a rockstar
Sling a lil' mud off the back, we can do that
Friday night football, Saturday Last Call, Sunday Hallelujah
If you like it up loud and you're hillbilly proud
Then you know what I'm talking about

Let me hear you say, Truck Yeah
Wanna get it jacked up yeah
Let's crank it on up yeah
With a little bit of luck I can find me a girl with a Truck Yeah
We can love it on up yeah
'Til the sun comes up yeah
And if you think this life I love is a little too country
Truck Yeah

Our party in the club is a honky tonk downtown
Yeah that's where I like to hang out
Chillin' in the back room
Hangin' with my whole crew
Sippin' on a cold brew, hey now!
Got a mixed up playlist, DJ play this
Wanna hear a country song
If you like it up loud and you're hillbilly proud
Throw your hands up now, let me hear you shout

Truck Yeah
Wanna get it jacked up yeah
Let's crank it on up yeah
With a little bit of luck I can find me a girl with a Truck Yeah
We can love it on up yeah
'Til the sun comes up yeah
And if you think this life I love is a little too country
Truck Yeah

Rap or country, city farm
It don't matter who you are
Got a little fight, got a little love
Got a little redneck in your blood
Are you one of us?

Truck Yeah
Wanna get it jacked up yeah
Let's crank it on up yeah
With a little bit of luck I can find me a girl with a Truck Yeah
We can love it on up yeah
'Til the sun comes up yeah
And if you think this life I love is a little too country
You're right on the money
Truck Yeah!

This is a noncommerical use of the song "Truck Yeah". Watch the music video, it's even cringier.