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Shortlisted Problems with Solutions

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Shortlisted Problems with Solutions

Contents

Contributing Countries

Austria, Australia, Belgium, Bulgaria, Canada, Croatia, Czech Republic, Estonia, Finland, Greece, India, Indonesia, Iran, Japan, Korea (North), Korea (South), Lithuania, Luxembourg, Mexico, Moldova, Netherlands, New Zealand, Poland, Romania, Russia, Serbia, South Africa, Sweden, Thailand, Taiwan, Turkey, Ukraine, United Kingdom, United States of America

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Algebra

A1. Given a sequence $a_1, a_2, \ldots, a_n$ of real numbers. For each i $(1 \le i \le n)$ define

$$d_i = \max\{a_j : 1 \le j \le i\} - \min\{a_j : i \le j \le n\}$$

and let

$$d = \max\{d_i : 1 \le i \le n\}.$$

(a) Prove that for arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$,

$$\max\bigl\{|x_i - a_i| : 1 \le i \le n\bigr\} \ge \frac{d}{2}. \tag{1}$$

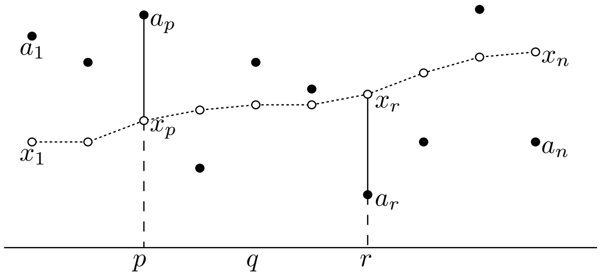
(b) Show that there exists a sequence $x_1 \le x_2 \le \ldots \le x_n$ of real numbers such that we have equality in (1).

(New Zealand)

Solution 1. (a) Let $1 \le p \le q \le r \le n$ be indices for which

$$d = d_q, \qquad a_p = \max\{a_j : 1 \le j \le q\}, \quad a_r = \min\{a_j : q \le j \le n\}$$

and thus $d = a_p - a_r$. (These indices are not necessarily unique.)



For arbitrary real numbers $x_1 \le x_2 \le \ldots \le x_n$, consider just the two quantities $|x_p - a_p|$ and $|x_r - a_r|$. Since

$$(a_p - x_p) + (x_r - a_r) = (a_p - a_r) + (x_r - x_p) \ge a_p - a_r = d,$$

we have either $a_p - x_p \ge \dfrac{d}{2}$ or $x_r - a_r \ge \dfrac{d}{2}$. Hence,

$$\max\{|x_i - a_i| : 1 \le i \le n\} \ge \max\bigl\{|x_p - a_p|, |x_r - a_r|\bigr\} \ge \max\{a_p - x_p, x_r - a_r\} \ge \frac{d}{2}.$$

(b) Define the sequence (x_k) as

$$x_1 = a_1 - \frac{d}{2}, \qquad x_k = \max\left\{x_{k-1},\ a_k - \frac{d}{2}\right\} \quad \text{for } 2 \le k \le n.$$

We show that we have equality in (1) for this sequence.

By the definition, sequence (x_k) is non-decreasing and $x_k - a_k \ge -\frac{d}{2}$ for all $1 \le k \le n$. Next we prove that

$$x_k - a_k \le \frac{d}{2} \quad \text{for all } 1 \le k \le n. \tag{2}$$

Consider an arbitrary index $1 \le k \le n$. Let $\ell \le k$ be the smallest index such that $x_k = x_\ell$. We have either $\ell = 1$, or $\ell \ge 2$ and $x_\ell > x_{\ell-1}$. In both cases,

$$x_k = x_\ell = a_\ell - \frac{d}{2}. \tag{3}$$

Since

$$a_\ell - a_k \le \max\{a_j : 1 \le j \le k\} - \min\{a_j :\ k \le j \le n\} = d_k \le d,$$

equality (3) implies

$$x_k - a_k = a_\ell - a_k - \frac{d}{2} \le d - \frac{d}{2} = \frac{d}{2}.$$

We obtained that $-\frac{d}{2} \le x_k - a_k \le \frac{d}{2}$ for all $1 \le k \le n$, so

$$\max\big\{|x_i - a_i| :\ 1 \le i \le n\big\} \le \frac{d}{2}.$$

We have equality because $|x_1 - a_1| = \frac{d}{2}$.

Solution 2. We present another construction of a sequence (x_i) for part (b).

For each $1 \le i \le n$, let

$$M_i = \max\{a_j : 1 \le j \le i\} \quad \text{and} \quad m_i = \min\{a_j :\ i \le j \le n\}.$$

For all $1 \le i < n$, we have

$$M_i = \max\{a_1, \ldots, a_i\} \le \max\{a_1, \ldots, a_i, a_{i+1}\} = M_{i+1}$$

and

$$m_i = \min\{a_i, a_{i+1}, \ldots, a_n\} \le \min\{a_{i+1}, \ldots, a_n\} = m_{i+1}.$$

Therefore sequences (M_i) and (m_i) are non-decreasing. Moreover, since a_i is listed in both definitions,

$$m_i \le a_i \le M_i.$$

To achieve equality in (1), set

$$x_i = \frac{M_i + m_i}{2}.$$

Since sequences (M_i) and (m_i) are non-decreasing, this sequence is non-decreasing as well.

From $d_i = M_i - m_i$ we obtain that

$$-\frac{d_i}{2} = \frac{m_i - M_i}{2} = x_i - M_i \le x_i - a_i \le x_i - m_i = \frac{M_i - m_i}{2} = \frac{d_i}{2}.$$

Therefore

$$\max\bigl\{|x_i - a_i| : 1 \le i \le n\bigr\} \le \max\left\{\frac{d_i}{2} : 1 \le i \le n\right\} = \frac{d}{2}.$$

Since the opposite inequality has been proved in part (a), we must have equality.

A2. Consider those functions $f\colon \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f\big(f(n)\big) - 1 \tag{1}$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

($\mathbb{N}$ denotes the set of all positive integers.)

(Bulgaria)

Answer. 1, 2, ..., 2008.

Solution. Suppose that a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies (1). For arbitrary positive integers $m > n$, by (1) we have

$$f(m) = f\big(n + (m-n)\big) \geq f(n) + f\big(f(m-n)\big) - 1 \geq f(n),$$

so f is nondecreasing.

Function $f \equiv 1$ is an obvious solution. To find other solutions, assume that $f \not\equiv 1$ and take the smallest $a \in \mathbb{N}$ such that $f(a) > 1$. Then $f(b) \geq f(a) > 1$ for all integer $b \geq a$.

Suppose that $f(n) > n$ for some $n \in \mathbb{N}$. Then we have

$$f\big(f(n)\big) = f\Big(\big(f(n)-n\big) + n\Big) \geq f\big(f(n)-n\big) + f\big(f(n)\big) - 1,$$

so $f\big(f(n)-n\big) \leq 1$ and hence $f(n)-n < a$. Then there exists a maximal value of the expression $f(n)-n$; denote this value by c, and let $f(k)-k = c \geq 1$. Applying the monotonicity together with (1), we get

$$\begin{aligned} 2k + c \geq f(2k) = f(k+k) &\geq f(k) + f\big(f(k)\big) - 1 \\ &\geq f(k) + f(k) - 1 = 2(k+c) - 1 = 2k + (2c-1), \end{aligned}$$

hence $c \leq 1$ and $f(n) \leq n+1$ for all $n \in \mathbb{N}$. In particular, $f(2007) \leq 2008$.

Now we present a family of examples showing that all values from 1 to 2008 can be realized. Let

$$f_j(n) = \max\{1, n+j-2007\} \quad \text{for } j = 1, 2, \dots, 2007; \qquad f_{2008}(n) = \begin{cases} n, & 2007 \nmid n, \\ n+1, & 2007 \mid n. \end{cases}$$

We show that these functions satisfy the condition (1) and clearly $f_j(2007) = j$.

To check the condition (1) for the function f_j ($j \leq 2007$), note first that f_j is nondecreasing and $f_j(n) \leq n$, hence $f_j\big(f_j(n)\big) \leq f_j(n) \leq n$ for all $n \in \mathbb{N}$. Now, if $f_j(m) = 1$, then the inequality (1) is clear since $f_j(m+n) \geq f_j(n) \geq f_j\big(f_j(n)\big) = f_j(m) + f_j\big(f_j(n)\big) - 1$. Otherwise,

$$f_j(m) + f_j\big(f_j(n)\big) - 1 \leq (m+j-2007) + n = (m+n) + j - 2007 = f_j(m+n).$$

In the case $j = 2008$, clearly $n+1 \geq f_{2008}(n) \geq n$ for all $n \in \mathbb{N}$; moreover, $n+1 \geq f_{2008}\big(f_{2008}(n)\big)$ as well. Actually, the latter is trivial if $f_{2008}(n) = n$; otherwise, $f_{2008}(n) = n+1$, which implies $2007 \nmid n+1$ and hence $n+1 = f_{2008}(n+1) = f_{2008}\big(f_{2008}(n)\big)$.

So, if $2007 \mid m+n$, then

$$f_{2008}(m+n) = m+n+1 = (m+1) + (n+1) - 1 \geq f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1.$$

Otherwise, $2007 \nmid m+n$, hence $2007 \nmid m$ or $2007 \nmid n$. In the former case we have $f_{2008}(m) = m$, while in the latter one $f_{2008}\big(f_{2008}(n)\big) = f_{2008}(n) = n$, providing

$$f_{2008}(m) + f_{2008}\big(f_{2008}(n)\big) - 1 \leq (m+n+1) - 1 = f_{2008}(m+n).$$

Comment. The examples above are not unique. The values $1, 2, \dots, 2008$ can be realized in several ways. Here we present other two constructions for $j \le 2007$, without proof:

$$g_j(n) = \begin{cases} 1, & n < 2007, \\ j, & n = 2007, \\ n, & n > 2007; \end{cases} \qquad h_j(n) = \max\left\{1, \left\lfloor \frac{jn}{2007} \right\rfloor\right\}.$$

Also the example for $j = 2008$ can be generalized. In particular, choosing a divisor $d > 1$ of 2007, one can set

$$f_{2008,d}(n) = \begin{cases} n, & d \nmid n, \\ n+1, & d \mid n. \end{cases}$$

A3. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}.$$

(Estonia)

Solution 1. For each real $t\in(0,1)$,

$$\frac{1+t^2}{1+t^4}=\frac{1}{t}-\frac{(1-t)(1-t^3)}{t(1+t^4)}<\frac{1}{t}.$$

Substituting $t=x^k$ and $t=y^k$,

$$0<\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}<\sum_{k=1}^{n}\frac{1}{x^k}=\frac{1-x^n}{x^n(1-x)}\quad\text{and}\quad 0<\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}<\sum_{k=1}^{n}\frac{1}{y^k}=\frac{1-y^n}{y^n(1-y)}.$$

Since $1-y^n=x^n$ and $1-x^n=y^n$,

$$\frac{1-x^n}{x^n(1-x)}=\frac{y^n}{x^n(1-x)},\qquad \frac{1-y^n}{y^n(1-y)}=\frac{x^n}{y^n(1-y)}$$

and therefore

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{y^n}{x^n(1-x)}\cdot\frac{x^n}{y^n(1-y)}=\frac{1}{(1-x)(1-y)}.$$

Solution 2. We prove

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}<\frac{0.7001}{(1-x)(1-y)}.\tag{1}$$

The idea is to estimate each term on the left-hand side with the same constant. To find the upper bound for the expression $\frac{1+x^{2k}}{1+x^{4k}}$, consider the function $f(t)=\frac{1+t}{1+t^2}$ in interval $(0,1)$. Since

$$f'(t)=\frac{1-2t-t^2}{(1+t^2)^2}=\frac{(\sqrt{2}+1+t)(\sqrt{2}-1-t)}{(1+t^2)^2},$$

the function increases in interval $(0,\sqrt{2}-1]$ and decreases in $[\sqrt{2}-1,1)$. Therefore the maximum is at point $t_0=\sqrt{2}-1$ and

$$f(t)=\frac{1+t}{1+t^2}\le f(t_0)=\frac{1+\sqrt{2}}{2}=\alpha.$$

Applying this to each term on the left-hand side of (1), we obtain

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)\le n\alpha\cdot n\alpha=(n\alpha)^2.\tag{2}$$

To estimate $(1-x)(1-y)$ on the right-hand side, consider the function

$$g(t)=\ln(1-t^{1/n})+\ln\bigl(1-(1-t)^{1/n}\bigr).$$

Substituting s for $1-t$, we have

$$-ng'(t) = \frac{t^{1/n-1}}{1-t^{1/n}} - \frac{s^{1/n-1}}{1-s^{1/n}} = \frac{1}{st}\left(\frac{(1-t)t^{1/n}}{1-t^{1/n}} - \frac{(1-s)s^{1/n}}{1-s^{1/n}}\right) = \frac{h(t)-h(s)}{st}.$$

The function

$$h(t) = t^{1/n}\frac{1-t}{1-t^{1/n}} = \sum_{i=1}^{n} t^{i/n}$$

is obviously increasing for $t \in (0,1)$, hence for these values of t we have

$$g'(t) > 0 \iff h(t) < h(s) \iff t < s = 1-t \iff t < \frac{1}{2}.$$

Then, the maximum of $g(t)$ in $(0,1)$ is attained at point $t_1 = 1/2$ and therefore

$$g(t) \le g\left(\frac{1}{2}\right) = 2\ln(1-2^{-1/n}), \qquad t \in (0,1).$$

Substituting $t = x^n$, we have $1-t = y^n$, $(1-x)(1-y) = \exp g(t)$ and hence

$$(1-x)(1-y) = \exp g(t) \le (1-2^{-1/n})^2. \tag{3}$$

Combining (2) and (3), we get

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) \le (\alpha n)^2 \cdot 1 \le (\alpha n)^2\frac{(1-2^{-1/n})^2}{(1-x)(1-y)} = \frac{\left(\alpha n(1-2^{-1/n})\right)^2}{(1-x)(1-y)}.$$

Applying the inequality $1-\exp(-t) < t$ for $t = \dfrac{\ln 2}{n}$, we obtain

$$\alpha n(1-2^{-1/n}) = \alpha n\left(1-\exp\left(-\frac{\ln 2}{n}\right)\right) < \alpha n \cdot \frac{\ln 2}{n} = \alpha\ln 2 = \frac{1+\sqrt{2}}{2}\ln 2.$$

Hence,

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right) < \frac{\left(\frac{1+\sqrt{2}}{2}\ln 2\right)^2}{(1-x)(1-y)}.$$

Comment. It is a natural idea to compare the sum $S_n(x) = \sum_{k=1}^{n}\dfrac{1+x^{2k}}{1+x^{4k}}$ with the integral $I_n(x) = \displaystyle\int_0^n \frac{1+x^{2t}}{1+x^{4t}}\mathrm{d}t$. Though computing the integral is quite standard, many difficulties arise. First, the integrand $\dfrac{1+x^{2k}}{1+x^{4k}}$ has an increasing segment and, depending on x, it can have a decreasing segment as well. So comparing $S_n(x)$ and $I_n(x)$ is not completely obvious. We can add a term to fix the estimate, e.g. $S_n \le I_n + (\alpha-1)$, but then the final result will be weak for the small values of n. Second, we have to minimize $(1-x)(1-y)I_n(x)I_n(y)$ which leads to very unpleasant computations.

However, by computer search we found that the maximum of $I_n(x)I_n(y)$ is at $x = y = 2^{-1/n}$, as well as the maximum of $S_n(x)S_n(y)$, and the latter is less. Hence, one can conjecture that the exact constant which can be put into the numerator on the right-hand side of (1) is

$$\left(\ln 2 \cdot \int_0^1 \frac{1+4^{-t}}{1+16^{-t}}\mathrm{d}t\right)^2 = \frac{1}{4}\left(\frac{1}{2}\ln\frac{17}{2} + \arctan 4 - \frac{\pi}{4}\right)^2 \approx 0.6484.$$

A4. Find all functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f\big(x+f(y)\big) = f(x+y) + f(y) \tag{1}$$

for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

(Thaliand)

Answer. $f(x) = 2x$.

Solution 1. First we show that $f(y) > y$ for all $y \in \mathbb{R}^+$. Functional equation (1) yields $f\big(x+f(y)\big) > f(x+y)$ and hence $f(y) \neq y$ immediately. If $f(y) < y$ for some y, then setting $x = y - f(y)$ we get

$$f(y) = f\Big(\big(y-f(y)\big)+f(y)\Big) = f\Big(\big(y-f(y)\big)+y\Big) + f(y) > f(y),$$

contradiction. Therefore $f(y) > y$ for all $y \in \mathbb{R}^+$.

For $x \in \mathbb{R}^+$ define $g(x) = f(x) - x$; then $f(x) = g(x) + x$ and, as we have seen, $g(x) > 0$. Transforming (1) for function $g(x)$ and setting $t = x + y$,

$$f\big(t+g(y)\big) = f(t) + f(y),$$

$$g\big(t+g(y)\big) + t + g(y) = \big(g(t)+t\big) + \big(g(y)+y\big)$$

and therefore

$$g\big(t+g(y)\big) = g(t) + y \qquad \text{for all } t > y > 0. \tag{2}$$

Next we prove that function $g(x)$ is injective. Suppose that $g(y_1) = g(y_2)$ for some numbers $y_1, y_2 \in \mathbb{R}^+$. Then by (2),

$$g(t) + y_1 = g\big(t+g(y_1)\big) = g\big(t+g(y_2)\big) = g(t) + y_2$$

for all $t > \max\{y_1, y_2\}$. Hence, $g(y_1) = g(y_2)$ is possible only if $y_1 = y_2$.

Now let u, v be arbitrary positive numbers and $t > u + v$. Applying (2) three times,

$$g\big(t+g(u)+g(v)\big) = g\big(t+g(u)\big) + v = g(t) + u + v = g\big(t+g(u+v)\big).$$

By the injective property we conclude that $t + g(u) + g(v) = t + g(u+v)$, hence

$$g(u) + g(v) = g(u+v). \tag{3}$$

Since function $g(v)$ is positive, equation (3) also shows that g is an increasing function.

Finally we prove that $g(x) = x$. Combining (2) and (3), we obtain

$$g(t) + y = g\big(t+g(y)\big) = g(t) + g\big(g(y)\big)$$

and hence

$$g\big(g(y)\big) = y.$$

Suppose that there exists an $x \in \mathbb{R}^+$ such that $g(x) \neq x$. By the monotonicity of g, if $x > g(x)$ then $g(x) > g\big(g(x)\big) = x$. Similarly, if $x < g(x)$ then $g(x) < g\big(g(x)\big) = x$. Both cases lead to contradiction, so there exists no such x.

We have proved that $g(x) = x$ and therefore $f(x) = g(x) + x = 2x$ for all $x \in \mathbb{R}^+$. This function indeed satisfies the functional equation (1).

Comment. It is well-known that the additive property (3) together with $g(x) \geq 0$ (for $x > 0$) imply $g(x) = cx$. So, after proving (3), it is sufficient to test functions $f(x) = (c+1)x$.

Solution 2. We prove that $f(y) > y$ and introduce function $g(x) = f(x) - x > 0$ in the same way as in Solution 1.

For arbitrary $t > y > 0$, substitute $x = t - y$ into (1) to obtain

$$f\big(t + g(y)\big) = f(t) + f(y)$$

which, by induction, implies

$$f\big(t + ng(y)\big) = f(t) + nf(y) \qquad \text{for all } t > y > 0,\ n \in \mathbb{N}. \tag{4}$$

Take two arbitrary positive reals y and z and a third fixed number $t > \max\{y, z\}$. For each positive integer k, let $\ell_k = \left\lfloor k\dfrac{g(y)}{g(z)} \right\rfloor$. Then $t + kg(y) - \ell_k g(z) \geq t > z$ and, applying (4) twice,

$$f\big(t + kg(y) - \ell_k g(z)\big) + \ell_k f(z) = f\big(t + kg(y)\big) = f(t) + kf(y),$$
$$0 < \frac{1}{k} f\big(t + kg(y) - \ell_k g(z)\big) = \frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z).$$

As $k \to \infty$ we get

$$0 \leq \lim_{k\to\infty} \left(\frac{f(t)}{k} + f(y) - \frac{\ell_k}{k} f(z) \right) = f(y) - \frac{g(y)}{g(z)} f(z) = f(y) - \frac{f(y) - y}{f(z) - z} f(z)$$

and therefore

$$\frac{f(y)}{y} \leq \frac{f(z)}{z}.$$

Exchanging variables y and z, we obtain the reverse inequality. Hence, $\dfrac{f(y)}{y} = \dfrac{f(z)}{z}$ for arbitrary y and z; so function $\dfrac{f(x)}{x}$ is constant, $f(x) = cx$.

Substituting back into (1), we find that $f(x) = cx$ is a solution if and only if $c = 2$. So the only solution for the problem is $f(x) = 2x$.

A5. Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that

$$a(m+n) \leq 2a(m) + 2a(n) \quad \text{for all } m, n \geq 1, \tag{1}$$

and

$$a(2^k) \leq \frac{1}{(k+1)^c} \quad \text{for all } k \geq 0. \tag{2}$$

Prove that the sequence $a(n)$ is bounded.

(Croatia)

Solution 1. For convenience, define $a(0) = 0$; then condition (1) persists for all pairs of nonnegative indices.

Lemma 1. For arbitrary nonnegative indices $n_1, \ldots, n_k$, we have

$$a\left(\sum_{i=1}^{k} n_i\right) \leq \sum_{i=1}^{k} 2^i a(n_i) \tag{3}$$

and

$$a\left(\sum_{i=1}^{k} n_i\right) \leq 2k \sum_{i=1}^{k} a(n_i). \tag{4}$$

Proof. Inequality (3) is proved by induction on k. The base case $k = 1$ is trivial, while the induction step is provided by

$$a\left(\sum_{i=1}^{k+1} n_i\right) = a\left(n_1 + \sum_{i=2}^{k+1} n_i\right) \leq 2a(n_1) + 2a\left(\sum_{i=1}^{k} n_{i+1}\right) \leq 2a(n_1) + 2\sum_{i=1}^{k} 2^i a(n_{i+1}) = \sum_{i=1}^{k+1} 2^i a(n_i).$$

To establish (4), first the inequality

$$a\left(\sum_{i=1}^{2^d} n_i\right) \leq 2^d \sum_{i=1}^{2^d} a(n_i)$$

can be proved by an obvious induction on d. Then, turning to (4), we find an integer d such that $2^{d-1} < k \leq 2^d$ to obtain

$$a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k} n_i + \sum_{i=k+1}^{2^d} 0\right) \leq 2^d \left(\sum_{i=1}^{k} a(n_i) + \sum_{i=k+1}^{2^d} a(0)\right) = 2^d \sum_{i=1}^{k} a(n_i) \leq 2k \sum_{i=1}^{k} a(n_i).$$

$\square$

Fix an increasing unbounded sequence $0 = M_0 < M_1 < M_2 < \ldots$ of real numbers; the exact values will be defined later. Let n be an arbitrary positive integer and write

$$n = \sum_{i=0}^{d} \varepsilon_i \cdot 2^i, \qquad \text{where } \varepsilon_i \in \{0, 1\}.$$

Set $\varepsilon_i = 0$ for $i > d$, and take some positive integer f such that $M_f > d$. Applying (3), we get

$$a(n) = a\left(\sum_{k=1}^{f} \sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right) \leq \sum_{k=1}^{f} 2^k a\left(\sum_{M_{k-1} \leq i < M_k} \varepsilon_i \cdot 2^i\right).$$

Note that there are less than $M_k - M_{k-1} + 1$ integers in interval $[M_{k-1}, M_k)$; hence, using (4) we have

$$\begin{aligned}
a(n) &\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1) \sum_{M_{k-1} \le i < M_k} \varepsilon_i \cdot a(2^i) \\
&\le \sum_{k=1}^{f} 2^k \cdot 2(M_k - M_{k-1} + 1)^2 \max_{M_{k-1} \le i < M_k} a(2^i) \\
&\le \sum_{k=1}^{f} 2^{k+1}(M_k + 1)^2 \cdot \frac{1}{(M_{k-1}+1)^c} = \sum_{k=1}^{f} \left(\frac{M_k + 1}{M_{k-1}+1}\right)^2 \frac{2^{k+1}}{(M_{k-1}+1)^{c-2}}.
\end{aligned}$$

Setting $M_k = 4^{k/(c-2)} - 1$, we obtain

$$a(n) \le \sum_{k=1}^{f} 4^{2/(c-2)} \frac{2^{k+1}}{(4^{(k-1)/(c-2)})^{c-2}} = 8 \cdot 4^{2/(c-2)} \sum_{k=1}^{f} \left(\frac{1}{2}\right)^k < 8 \cdot 4^{2/(c-2)},$$

and the sequence $a(n)$ is bounded.

Solution 2.

Lemma 2. Suppose that $s_1, \ldots, s_k$ are positive integers such that

$$\sum_{i=1}^{k} 2^{-s_i} \le 1.$$

Then for arbitrary positive integers $n_1, \ldots, n_k$ we have

$$a\left(\sum_{i=1}^{k} n_i\right) \le \sum_{i=1}^{k} 2^{s_i} a(n_i).$$

Proof. Apply an induction on k. The base cases are $k = 1$ (trivial) and $k = 2$ (follows from the condition (1)). Suppose that $k > 2$. We can assume that $s_1 \le s_2 \le \cdots \le s_k$. Note that

$$\sum_{i=1}^{k-1} 2^{-s_i} \le 1 - 2^{-s_{k-1}},$$

since the left-hand side is a fraction with the denominator $2^{s_{k-1}}$, and this fraction is less than 1. Define $s'_{k-1} = s_{k-1} - 1$ and $n'_{k-1} = n_{k-1} + n_k$; then we have

$$\sum_{i=1}^{k-2} 2^{-s_i} + 2^{-s'_{k-1}} \le (1 - 2 \cdot 2^{-s_{k-1}}) + 2^{1-s_{k-1}} = 1.$$

Now, the induction hypothesis can be applied to achieve

$$\begin{aligned}
a\left(\sum_{i=1}^{k} n_i\right) = a\left(\sum_{i=1}^{k-2} n_i + n'_{k-1}\right) &\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s'_{k-1}} a(n'_{k-1}) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}-1} \cdot 2\big(a(n_{k-1}) + a(n_k)\big) \\
&\le \sum_{i=1}^{k-2} 2^{s_i} a(n_i) + 2^{s_{k-1}} a(n_{k-1}) + 2^{s_k} a(n_k).
\end{aligned}$$

□

Let $q = c/2 > 1$. Take an arbitrary positive integer n and write

$$n = \sum_{i=1}^{k} 2^{u_i}, \qquad 0 \le u_1 < u_2 < \cdots < u_k.$$

Choose $s_i = \lfloor \log_2 (u_i + 1)^q \rfloor + d$ $(i = 1, \ldots, k)$ for some integer d. We have

$$\sum_{i=1}^{k} 2^{-s_i} = 2^{-d} \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor},$$

and we choose d in such a way that

$$\frac{1}{2} < \sum_{i=1}^{k} 2^{-s_i} \le 1.$$

In particular, this implies

$$2^d < 2 \sum_{i=1}^{k} 2^{-\lfloor \log_2 (u_i+1)^q \rfloor} < 4 \sum_{i=1}^{k} \frac{1}{(u_i+1)^q}.$$

Now, by Lemma 2 we obtain

$$\begin{aligned} a(n) = a\left(\sum_{i=1}^{k} 2^{u_i} \right) &\le \sum_{i=1}^{k} 2^{s_i} a(2^{u_i}) \le \sum_{i=1}^{k} 2^d (u_i+1)^q \cdot \frac{1}{(u_i+1)^{2q}} \\ &= 2^d \sum_{i=1}^{k} \frac{1}{(u_i+1)^q} < 4 \left(\sum_{i=1}^{k} \frac{1}{(u_i+1)^q} \right)^2, \end{aligned}$$

which is bounded since $q > 1$.

Comment 1. In fact, Lemma 2 (applied to the case $n_i = 2^{u_i}$ only) provides a sharp bound for any $a(n)$. Actually, let $b(k) = \dfrac{1}{(k+1)^c}$ and consider the sequence

$$a(n) = \min \left\{ \sum_{i=1}^{k} 2^{s_i} b(u_i) \;\middle|\; k \in \mathbb{N}, \quad \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad \sum_{i=1}^{k} 2^{u_i} = n \right\}. \tag{5}$$

We show that this sequence satisfies the conditions of the problem. Take two arbitrary indices m and n. Let

$$\begin{aligned} a(m) &= \sum_{i=1}^{k} 2^{s_i} b(u_i), \qquad & \sum_{i=1}^{k} 2^{-s_i} \le 1, \quad & \sum_{i=1}^{k} 2^{u_i} = m; \\ a(n) &= \sum_{i=1}^{l} 2^{r_i} b(w_i), \qquad & \sum_{i=1}^{l} 2^{-r_i} \le 1, \quad & \sum_{i=1}^{l} 2^{w_i} = n. \end{aligned}$$

Then we have

$$\sum_{i=1}^{k} 2^{-1-s_i} + \sum_{i=1}^{l} 2^{-1-r_i} \le \frac{1}{2} + \frac{1}{2} = 1, \qquad \sum_{i=1}^{k} 2^{u_i} + \sum_{i=1}^{l} 2^{w_i} = m + n,$$

so by (5) we obtain

$$a(n+m) \le \sum_{i=1}^{k} 2^{1+s_i} b(u_i) + \sum_{i=1}^{l} 2^{1+r_i} b(w_i) = 2a(m) + 2a(n).$$

Comment 2. The condition $c > 2$ is sharp; we show that the sequence (5) is not bounded if $c \le 2$.

First, we prove that for an arbitrary n the minimum in (5) is attained with a sequence (u_i) consisting of distinct numbers. To the contrary, assume that $u_{k-1} = u_k$. Replace u_{k-1} and u_k by a single number $u'_{k-1} = u_k + 1$, and s_{k-1} and s_k by $s'_{k-1} = \min\{s_{k-1}, s_k\}$. The modified sequences provide a better bound since

$$2^{s'_{k-1}} b(u'_{k-1}) = 2^{s'_{k-1}} b(u_k + 1) < 2^{s_{k-1}} b(u_{k-1}) + 2^{s_k} b(u_k)$$

(we used the fact that $b(k)$ is decreasing). This is impossible.

Hence, the claim is proved, and we can assume that the minimum is attained with $u_1 < \cdots < u_k$; then

$$n = \sum_{i=1}^{k} 2^{u_i}$$

is simply the binary representation of n. (In particular, it follows that $a(2^n) = b(n)$ for each n.)

Now we show that the sequence $\big(a(2^k - 1)\big)$ is not bounded. For some $s_1, \dots, s_k$ we have

$$a(2^k - 1) = a\left(\sum_{i=1}^{k} 2^{i-1}\right) = \sum_{i=1}^{k} 2^{s_i} b(i-1) = \sum_{i=1}^{k} \frac{2^{s_i}}{i^c}.$$

By the Cauchy–Schwarz inequality we get

$$a(2^k - 1) = a(2^k - 1) \cdot 1 \ge \left(\sum_{i=1}^{k} \frac{2^{s_i}}{i^c}\right)\left(\sum_{i=1}^{k} \frac{1}{2^{s_i}}\right) \ge \left(\sum_{i=1}^{k} \frac{1}{i^{c/2}}\right)^2,$$

which is unbounded.

For $c \le 2$, it is also possible to show a concrete counterexample. Actually, one can prove that the sequence

$$a\left(\sum_{i=1}^{k} 2^{u_i}\right) = \sum_{i=1}^{k} \frac{i}{(u_i+1)^2} \qquad (0 \le u_1 < \ldots < u_k)$$

satisfies (1) and (2) but is not bounded.

A6. Let $a_1, a_2, \ldots, a_{100}$ be nonnegative real numbers such that $a_1^2 + a_2^2 + \ldots + a_{100}^2 = 1$. Prove that

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_{100}^2 a_1 < \frac{12}{25}.$$

(Poland)

Solution. Let $S = \sum_{k=1}^{100} a_k^2 a_{k+1}$. (As usual, we consider the indices modulo 100, e.g. we set $a_{101} = a_1$ and $a_{102} = a_2$.)

Applying the Cauchy-Schwarz inequality to sequences (a_{k+1}) and $(a_k^2 + 2a_{k+1}a_{k+2})$, and then the AM-GM inequality to numbers a_{k+1}^2 and a_{k+2}^2,

$$\begin{aligned}(3S)^2 &= \left(\sum_{k=1}^{100} a_{k+1}(a_k^2 + 2a_{k+1}a_{k+2})\right)^2 \le \left(\sum_{k=1}^{100} a_{k+1}^2\right)\left(\sum_{k=1}^{100} (a_k^2 + 2a_{k+1}a_{k+2})^2\right) \qquad (1)\\ &= 1 \cdot \sum_{k=1}^{100} (a_k^2 + 2a_{k+1}a_{k+2})^2 = \sum_{k=1}^{100} (a_k^4 + 4a_k^2 a_{k+1}a_{k+2} + 4a_{k+1}^2 a_{k+2}^2)\\ &\le \sum_{k=1}^{100} \big(a_k^4 + 2a_k^2(a_{k+1}^2 + a_{k+2}^2) + 4a_{k+1}^2 a_{k+2}^2\big) = \sum_{k=1}^{100} \big(a_k^4 + 6a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2\big).\end{aligned}$$

Applying the trivial estimates

$$\sum_{k=1}^{100} (a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) \le \left(\sum_{k=1}^{100} a_k^2\right)^2 \quad \text{and} \quad \sum_{k=1}^{100} a_k^2 a_{k+1}^2 \le \left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right),$$

we obtain that

$$(3S)^2 \le \left(\sum_{k=1}^{100} a_k^2\right)^2 + 4\left(\sum_{i=1}^{50} a_{2i-1}^2\right)\left(\sum_{j=1}^{50} a_{2j}^2\right) \le 1 + \left(\sum_{i=1}^{50} a_{2i-1}^2 + \sum_{j=1}^{50} a_{2j}^2\right)^2 = 2,$$

hence

$$S \le \frac{\sqrt{2}}{3} \approx 0.4714 < \frac{12}{25} = 0.48.$$

Comment 1. By applying the Lagrange multiplier method, one can see that the maximum is attained at values of a_i satisfying

$$a_{k-1}^2 + 2a_k a_{k+1} = 2\lambda a_k \qquad (2)$$

for all $k = 1, 2, \ldots, 100$. Though this system of equations seems hard to solve, it can help to find the estimate above; it may suggest to have a closer look at the expression $a_{k-1}^2 a_k + 2a_k^2 a_{k+1}$.

Moreover, if the numbers $a_1, \ldots, a_{100}$ satisfy (2), we have equality in (1). (See also Comment 3.)

Comment 2. It is natural to ask what is the best constant c_n in the inequality

$$a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1 \le c_n \big(a_1^2 + a_2^2 + \ldots + a_n^2\big)^{3/2}. \qquad (3)$$

For $1 \le n \le 4$ one may prove $c_n = 1/\sqrt{n}$ which is achieved when $a_1 = a_2 = \ldots = a_n$. However, the situation changes completely if $n \ge 5$. In this case we do not know the exact value of c_n. By computer search it can be found that $c_n \approx 0.4514$ and it is realized for example if

$$a_1 \approx 0.5873, \quad a_2 \approx 0.6771, \quad a_3 \approx 0.4224, \quad a_4 \approx 0.1344, \quad a_5 \approx 0.0133$$

and $a_k \approx 0$ for $k \ge 6$. This example also proves that $c_n > 0.4513$.

Comment 3. The solution can be improved in several ways to give somewhat better bounds for c_n. Here we show a variant which proves $c_n < 0.4589$ for $n \geq 5$.

The value of c_n does not change if negative values are also allowed in (3). So the problem is equivalent to maximizing

$$f(a_1, a_2, \ldots, a_n) = a_1^2 a_2 + a_2^2 a_3 + \ldots + a_n^2 a_1$$

on the unit sphere $a_1^2 + a_2^2 + \ldots + a_n^2 = 1$ in $\mathbb{R}^n$. Since the unit sphere is compact, the function has a maximum and we can apply the Lagrange multiplier method; for each maximum point there exists a real number λ such that

$$a_{k-1}^2 + 2a_k a_{k+1} = \lambda \cdot 2a_k \qquad \text{for all } k = 1, 2, \ldots, n.$$

Then

$$3S = \sum_{k=1}^{n} \left(a_{k-1}^2 a_k + 2a_k^2 a_{k+1}\right) = \sum_{k=1}^{n} 2\lambda a_k^2 = 2\lambda$$

and therefore

$$a_{k-1}^2 + 2a_k a_{k+1} = 3Sa_k \qquad \text{for all } k = 1, 2, \ldots, n. \tag{4}$$

From (4) we can derive

$$9S^2 = \sum_{k=1}^{n} (3Sa_k)^2 = \sum_{k=1}^{n} \left(a_{k-1}^2 + 2a_k a_{k+1}\right)^2 = \sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + 4\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \tag{5}$$

and

$$3S^2 = \sum_{k=1}^{n} 3Sa_{k-1}^2 a_k = \sum_{k=1}^{n} a_{k-1}^2 \left(a_{k-1}^2 + 2a_k a_{k+1}\right) = \sum_{k=1}^{n} a_k^4 + 2\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2}. \tag{6}$$

Let p be a positive number. Combining (5) and (6) and applying the AM-GM inequality,

$$\begin{aligned}
(9+3p)S^2 &= (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + (4+2p)\sum_{k=1}^{n} a_k^2 a_{k+1} a_{k+2} \\
&\leq (1+p)\sum_{k=1}^{n} a_k^4 + 4\sum_{k=1}^{n} a_k^2 a_{k+1}^2 + \sum_{k=1}^{n}\left(2(1+p)a_k^2 a_{k+2}^2 + \frac{(2+p)^2}{2(1+p)} a_k^2 a_{k+1}^2\right) \\
&= (1+p)\sum_{k=1}^{n}(a_k^4 + 2a_k^2 a_{k+1}^2 + 2a_k^2 a_{k+2}^2) + \left(4 + \frac{(2+p)^2}{2(1+p)} - 2(1+p)\right)\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&\leq (1+p)\left(\sum_{k=1}^{n} a_k^2\right)^2 + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2 \\
&= (1+p) + \frac{8+4p-3p^2}{2(1+p)}\sum_{k=1}^{n} a_k^2 a_{k+1}^2.
\end{aligned}$$

Setting $p = \dfrac{2+2\sqrt{7}}{3}$ which is the positive root of $8 + 4p - 3p^2 = 0$, we obtain

$$S \leq \sqrt{\frac{1+p}{9+3p}} = \sqrt{\frac{5+2\sqrt{7}}{33+6\sqrt{7}}} \approx 0.458879.$$

A7. Let $n > 1$ be an integer. In the space, consider the set

$$S = \{(x, y, z) \mid x, y, z \in \{0, 1, \dots, n\},\ x + y + z > 0\}.$$

Find the smallest number of planes that jointly contain all $(n+1)^3 - 1$ points of S but none of them passes through the origin.

(Netherlands)

Answer. $3n$ planes.

Solution. It is easy to find $3n$ such planes. For example, planes $x = i$, $y = i$ or $z = i$ $(i = 1, 2, \dots, n)$ cover the set S but none of them contains the origin. Another such collection consists of all planes $x + y + z = k$ for $k = 1, 2, \dots, 3n$.

We show that $3n$ is the smallest possible number.

Lemma 1. Consider a nonzero polynomial $P(x_1, \dots, x_k)$ in k variables. Suppose that P vanishes at all points $(x_1, \dots, x_k)$ such that $x_1, \dots, x_k \in \{0, 1, \dots, n\}$ and $x_1 + \dots + x_k > 0$, while $P(0, 0, \dots, 0) \neq 0$. Then $\deg P \geq kn$.

Proof. We use induction on k. The base case $k = 0$ is clear since $P \neq 0$. Denote for clarity $y = x_k$.

Let $R(x_1, \dots, x_{k-1}, y)$ be the residue of P modulo $Q(y) = y(y-1)\dots(y-n)$. Polynomial $Q(y)$ vanishes at each $y = 0, 1, \dots, n$, hence $P(x_1, \dots, x_{k-1}, y) = R(x_1, \dots, x_{k-1}, y)$ for all $x_1, \dots, x_{k-1}, y \in \{0, 1, \dots, n\}$. Therefore, R also satisfies the condition of the Lemma; moreover, $\deg_y R \leq n$. Clearly, $\deg R \leq \deg P$, so it suffices to prove that $\deg R \geq nk$.

Now, expand polynomial R in the powers of y:

$$R(x_1, \dots, x_{k-1}, y) = R_n(x_1, \dots, x_{k-1})y^n + R_{n-1}(x_1, \dots, x_{k-1})y^{n-1} + \dots + R_0(x_1, \dots, x_{k-1}).$$

We show that polynomial $R_n(x_1, \dots, x_{k-1})$ satisfies the condition of the induction hypothesis.

Consider the polynomial $T(y) = R(0, \dots, 0, y)$ of degree $\leq n$. This polynomial has n roots $y = 1, \dots, n$; on the other hand, $T(y) \not\equiv 0$ since $T(0) \neq 0$. Hence $\deg T = n$, and its leading coefficient is $R_n(0, 0, \dots, 0) \neq 0$. In particular, in the case $k = 1$ we obtain that coefficient R_n is nonzero.

Similarly, take any numbers $a_1, \dots, a_{k-1} \in \{0, 1, \dots, n\}$ with $a_1 + \dots + a_{k-1} > 0$. Substituting $x_i = a_i$ into $R(x_1, \dots, x_{k-1}, y)$, we get a polynomial in y which vanishes at all points $y = 0, \dots, n$ and has degree $\leq n$. Therefore, this polynomial is null, hence $R_i(a_1, \dots, a_{k-1}) = 0$ for all $i = 0, 1, \dots, n$. In particular, $R_n(a_1, \dots, a_{k-1}) = 0$.

Thus, the polynomial $R_n(x_1, \dots, x_{k-1})$ satisfies the condition of the induction hypothesis. So, we have $\deg R_n \geq (k-1)n$ and $\deg P \geq \deg R \geq \deg R_n + n \geq kn$. □

Now we can finish the solution. Suppose that there are N planes covering all the points of S but not containing the origin. Let their equations be $a_i x + b_i y + c_i z + d_i = 0$. Consider the polynomial

$$P(x, y, z) = \prod_{i=1}^{N} (a_i x + b_i y + c_i z + d_i).$$

It has total degree N. This polynomial has the property that $P(x_0, y_0, z_0) = 0$ for any $(x_0, y_0, z_0) \in S$, while $P(0, 0, 0) \neq 0$. Hence by Lemma 1 we get $N = \deg P \geq 3n$, as desired.

Comment 1. There are many other collections of $3n$ planes covering the set S but not covering the origin.

Solution 2. We present a different proof of the main Lemma 1. Here we confine ourselves to the case $k = 3$, which is applied in the solution, and denote the variables by x, y and z. (The same proof works for the general statement as well.)

The following fact is known with various proofs; we provide one possible proof for the completeness.

Lemma 2. For arbitrary integers $0 \le m < n$ and for an arbitrary polynomial $P(x)$ of degree m,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k) = 0. \tag{1}$$

Proof. We use an induction on n. If $n = 1$, then $P(x)$ is a constant polynomial, hence $P(1) - P(0) = 0$, and the base is proved.

For the induction step, define $P_1(x) = P(x+1) - P(x)$. Then clearly $\deg P_1 = \deg P - 1 = m - 1 < n - 1$, hence by the induction hypothesis we get

$$\begin{aligned}
0 &= -\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P_1(k) = \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\big(P(k) - P(k+1)\big) \\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k) - \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k+1) \\
&= \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}P(k) + \sum_{k=1}^{n}(-1)^k\binom{n-1}{k-1}P(k) \\
&= P(0) + \sum_{k=1}^{n-1}(-1)^k\left(\binom{n-1}{k-1} + \binom{n-1}{k}\right)P(k) + (-1)^n P(n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}P(k). \quad \square
\end{aligned}$$

Now return to the proof of Lemma 1. Suppose, to the contrary, that $\deg P = N < 3n$. Consider the sum

$$\Sigma = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}\binom{n}{k}P(i,j,k).$$

The only nonzero term in this sum is $P(0,0,0)$ and its coefficient is $\binom{n}{0}^3 = 1$; therefore $\Sigma = P(0,0,0) \ne 0$.

On the other hand, if $P(x,y,z) = \sum_{\alpha+\beta+\gamma\le N} p_{\alpha,\beta,\gamma}x^\alpha y^\beta z^\gamma$, then

$$\begin{aligned}
\Sigma &= \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{i+j+k}\binom{n}{i}\binom{n}{j}\binom{n}{k}\sum_{\alpha+\beta+\gamma\le N} p_{\alpha,\beta,\gamma}i^\alpha j^\beta k^\gamma \\
&= \sum_{\alpha+\beta+\gamma\le N} p_{\alpha,\beta,\gamma}\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^\alpha\right)\left(\sum_{j=0}^{n}(-1)^j\binom{n}{j}j^\beta\right)\left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}k^\gamma\right).
\end{aligned}$$

Consider an arbitrary term in this sum. We claim that it is zero. Since $N < 3n$, one of three inequalities $\alpha < n$, $\beta < n$ or $\gamma < n$ is valid. For the convenience, suppose that $\alpha < n$. Applying Lemma 2 to polynomial x^α, we get $\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^\alpha = 0$, hence the term is zero as required.

This yields $\Sigma = 0$ which is a contradiction. Therefore, $\deg P \ge 3n$.

Comment 2. The proof does not depend on the concrete coefficients in Lemma 2. Instead of this Lemma, one can simply use the fact that there exist numbers $\alpha_0, \alpha_1, \ldots, \alpha_n$ ($\alpha_0 \neq 0$) such that

$$\sum_{k=0}^{n} \alpha_k k^m = 0 \qquad \text{for every } 0 \leq m < n.$$

This is a system of homogeneous linear equations in variables α_i. Since the number of equations is less than the number of variables, the only nontrivial thing is that there exists a solution with $\alpha_0 \neq 0$. It can be shown in various ways.

Combinatorics

C1. Let $n > 1$ be an integer. Find all sequences $a_1, a_2, \ldots, a_{n^2+n}$ satisfying the following conditions:

(a) $a_i \in \{0, 1\}$ for all $1 \le i \le n^2 + n$;

(b) $a_{i+1} + a_{i+2} + \ldots + a_{i+n} < a_{i+n+1} + a_{i+n+2} + \ldots + a_{i+2n}$ for all $0 \le i \le n^2 - n$.

(Serbia)

Answer. Such a sequence is unique. It can be defined as follows:

$$a_{u+vn} = \begin{cases} 0, & u + v \le n, \\ 1, & u + v \ge n + 1 \end{cases} \qquad \text{for all } 1 \le u \le n \text{ and } 0 \le v \le n. \tag{1}$$

The terms can be arranged into blocks of length n as

$$(\underbrace{0 \ldots 0}_{n})\,(\underbrace{0 \ldots 0}_{n-1}\,1)\,(\underbrace{0 \ldots 0}_{n-2}\,1\,1) \ldots (\underbrace{0 \ldots 0}_{n-v}\,\underbrace{1 \ldots 1}_{v}) \ldots (0\,\underbrace{1 \ldots 1}_{n-1})\,(\underbrace{1 \ldots 1}_{n}).$$

Solution 1. Consider a sequence (a_i) satisfying the conditions. For arbitrary integers $0 \le k \le l \le n^2 + n$ denote $S(k, l] = a_{k+1} + \cdots + a_l$. (If $k = l$ then $S(k, l] = 0$.) Then condition (b) can be rewritten as $S(i, i + n] < S(i + n, i + 2n]$ for all $0 \le i \le n^2 - n$. Notice that for $0 \le k \le l \le m \le n^2 + n$ we have $S(k, m] = S(k, l] + S(l, m]$.

By condition (b),

$$0 \le S(0, n] < S(n, 2n] < \cdots < S(n^2, n^2 + n] \le n.$$

We have only $n + 1$ distinct integers in the interval $[0, n]$; hence,

$$S\big(vn, (v + 1)n\big] = v \qquad \text{for all } 0 \le v \le n. \tag{2}$$

In particular, $S(0, n] = 0$ and $S(n^2, n^2 + n] = n$, therefore

$$a_1 = a_2 = \ldots = a_n = 0, \tag{3}$$

$$a_{n^2+1} = a_{n^2+2} = \ldots = a_{n^2+n} = 1. \tag{4}$$

Subdivide sequence (a_i) into $n+1$ blocks, each consisting of n consecutive terms, and number them from 0 to n. We show by induction on v that the vth blocks has the form

$$(\underbrace{0 \ldots 0}_{n-v}\,\underbrace{1 \ldots 1}_{v}).$$

The base case $v = 0$ is provided by (3).

Consider the vth block for $v > 0$. By (2), it contains some "ones". Let the first "one" in this block be at the uth position (that is, $a_{u+vn} = 1$). By the induction hypothesis, the $(v-1)$th and vth blocks of (a_i) have the form

$$(\underbrace{0\ \dots\ \mathrlap{\overbrace{}^{\sum = v}}0\ \dots\ 0}_{n-v+1}\,\underbrace{1\ \dots\ 1}_{v-1})\,(\underbrace{0\ \dots\ 0}_{u-1}\,1 * \dots *),$$

where each star can appear to be any binary digit. Observe that $u \le n - v + 1$, since the sum in this block is v. Then, the fragment of length n bracketed above has exactly $(v-1)+1$ ones, i. e. $S\big(u + (v-1)n, u + vn\big] = v$. Hence,

$$v = S\big(u + (v-1)n, u + vn\big] < S\big(u + vn, u + (v+1)n\big] < \dots < S\big(u + (n-1)n, u + n^2\big] \le n;$$

we have $n - v + 1$ distinct integers in the interval $[v, n]$, therefore $S(u + (t-1)n, u + tn] = t$ for each $t = v, \dots, n$.

Thus, the end of sequence (a_i) looks as following:

$$(\underbrace{\overbrace{0\ \dots\ 0}^{u \text{ zeroes}}\,\mathrlap{\overbrace{}^{\sum = v}}0\ \dots\ 0\,1\ \dots\ 1}_{\sum = v-1})\,(\underbrace{0\ \dots\ 0\,1\mathrlap{\overbrace{}^{\sum = v+1}}\,*\ \dots\ *}_{\sum = v})\,(\underbrace{*\ \dots\ *\,\overbrace{*\ \dots\ *}^{\dots}}_{\sum = v+1})\,\overbrace{\dots\,(\underbrace{1\ \dots\ 1}}^{\sum = n}\!\!\!\!\underbrace{\overbrace{1\ \dots\ 1}^{n-u \text{ ones}}}_{\sum = n})$$

(each bracketed fragment contains n terms). Computing in two ways the sum of all digits above, we obtain $n - u = v - 1$ and $u = n - v + 1$. Then, the first $n - v$ terms in the vth block are zeroes, and the next v terms are ones, due to the sum of all terms in this block. The statement is proved.

We are left to check that the sequence obtained satisfies the condition. Notice that $a_i \le a_{i+n}$ for all $1 \le i \le n^2$. Moreover, if $1 \le u \le n$ and $0 \le v \le n-1$, then $a_{u+vn} < a_{u+vn+n}$ exactly when $u + v = n$. In this case we have $u + vn = n + v(n-1)$.

Consider now an arbitrary index $0 \le i \le n^2 - n$. Clearly, there exists an integer v such that $n + v(n-1) \in [i+1, i+n]$. Then, applying the above inequalities we obtain that condition (b) is valid.

Solution 2. Similarly to Solution 1, we introduce the notation $S(k, l]$ and obtain (2), (3), and (4) in the same way. The sum of all elements of the sequence can be computed as

$$S(0, n^2 + n] = S(0, n] + S(n, 2n] + \ldots + S(n^2, n^2 + n] = 0 + 1 + \ldots + n.$$

For an arbitrary integer $0 \le u \le n$, consider the numbers

$$S(u, u+n] < S(u+n, u+2n] < \ldots < S\big(u + (n-1)n, u + n^2\big]. \tag{5}$$

They are n distinct integers from the $n+1$ possible values $0, 1, 2, \dots, n$. Denote by m the "missing" value which is not listed. We determine m from $S(0, n^2 + n]$. Write this sum as

$$S(0, n^2+n] = S(0, u] + S(u, u+n] + S(u+n, u+2n] + \ldots + S(u+(n-1)n, u+n^2] + S(u+n^2, n^2+n].$$

Since $a_1 = a_2 = \ldots = a_u = 0$ and $a_{u+n^2+1} = \ldots = a_{n^2+n} = 1$, we have $S(0, u] = 0$ and $S(u + n^2, n + n^2] = n - u$. Then

$$0 + 1 + \ldots + n = S(0, n^2 + n] = 0 + \big((0 + 1 + \ldots + n) - m\big) + (n - u),$$

so $m = n - u$.

Hence, the numbers listed in (5) are $0, 1, \dots, n-u-1$ and $n-u+1, \dots, n$, respectively, therefore

$$S\big(u+vn, u+(v+1)n\big] = \begin{cases} v, & v \le n-u-1, \\ v+1, & v \ge n-u \end{cases} \quad \text{for all } 0 \le u \le n,\ 0 \le v \le n-1. \tag{6}$$

Conditions (6), together with (3), provide a system of linear equations in variables a_i. Now we solve this system and show that the solution is unique and satisfies conditions (a) and (b).

First, observe that any solution of the system (3), (6) satisfies the condition (b). By the construction, equations (6) immediately imply (5). On the other hand, all inequalities mentioned in condition (b) are included into the chain (5) for some value of u.

Next, note that the system (3), (6) is redundant. The numbers $S\big(kn, (k+1)n\big]$, where $1 \le k \le n-1$, appear twice in (6). For $u = 0$ and $v = k$ we have $v \le n-u-1$, and (6) gives $S\big(kn, (k+1)n\big] = v = k$. For $u = n$ and $v = k-1$ we have $v \ge n-u$ and we obtain the same value, $S\big(kn, (k+1)n\big] = v+1 = k$. Therefore, deleting one equation from each redundant pair, we can make every sum $S(k, k+n]$ appear exactly once on the left-hand side in (6).

Now, from (3), (6), the sequence (a_i) can be reconstructed inductively by

$$a_1 = a_2 = \ldots = a_{n-1} = 0, \qquad a_{k+n} = S(k, k+n] - (a_{k+1} + a_{k+2} + \ldots + a_{k+n-1}) \quad (0 \le k \le n^2),$$

taking the values of $S(k, k+n]$ from (6). This means first that there exists at most one solution of our system. Conversely, the constructed sequence obviously satisfies all equations (3), (6) (the only missing equation is $a_n = 0$, which follows from $S(0, n] = 0$). Hence it satisfies condition (b), and we are left to check condition (a) only.

For arbitrary integers $1 \le u, t \le n$ we get

$$\begin{aligned} a_{u+tn} - a_{u+(t-1)n} &= S\big(u+(t-1)n, u+tn\big] - S\big((u-1)+(t-1)n, (u-1)+tn\big] \\ &= \begin{cases} (t-1)-(t-1) = 0, & t \le n-u, \\ t-(t-1) = 1, & t = n-u+1, \\ t-t = 0, & t \ge n-u+2. \end{cases} \end{aligned}$$

Since $a_u = 0$, we have

$$a_{u+vn} = a_{u+vn} - a_u = \sum_{t=1}^{v} (a_{u+tn} - a_{u+(t-1)n})$$

for all $1 \le u, v \le n$. If $v < n-u+1$ then all terms are 0 on the right-hand side. If $v \ge n-u+1$, then variable t attains the value $n-u+1$ once. Hence,

$$a_{u+vn} = \begin{cases} 0, & u+v \le n, \\ 1, & u+v \ge n+1, \end{cases}$$

according with (1). Note that the formula is valid for $v = 0$ as well.

Finally, we presented the direct formula for (a_i), and we have proved that it satisfies condition (a). So, the solution is complete.

C2. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

(Japan)

Solution 1. Call the directions of the sides of the square horizontal and vertical. A horizontal or vertical line, which intersects the interior of the square but does not intersect the interior of any rectangle, will be called a *splitting line*. A rectangle having no point on the boundary of the square will be called an *interior rectangle*.

Suppose, to the contrary, that there exists a dissection of the square into more than one rectangle, such that no interior rectangle and no splitting line appear. Consider such a dissection with the least possible number of rectangles. Notice that this number of rectangles is greater than 2, otherwise their common side provides a splitting line.

If there exist two rectangles having a common side, then we can replace them by their union (see Figure 1). The number of rectangles was greater than 2, so in a new dissection it is greater than 1. Clearly, in the new dissection, there is also no splitting line as well as no interior rectangle. This contradicts the choice of the original dissection.

Denote the initial square by $ABCD$, with A and B being respectively the lower left and lower right vertices. Consider those two rectangles a and b containing vertices A and B, respectively. (Note that $a \neq b$, otherwise its top side provides a splitting line.) We can assume that the height of a is not greater than that of b. Then consider the rectangle c neighboring to the lower right corner of a (it may happen that $c = b$). By aforementioned, the heights of a and c are distinct. Then two cases are possible.

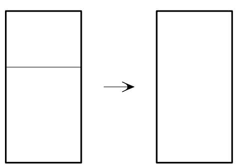


Figure 1

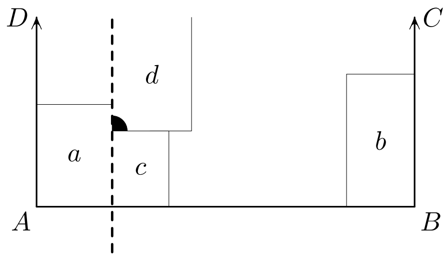


Figure 2

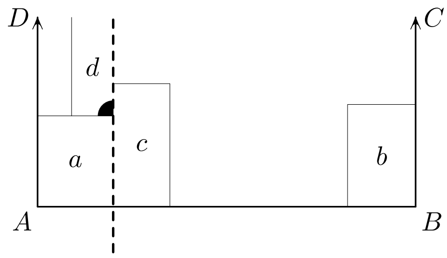


Figure 3

Case 1. The height of c is less than that of a. Consider the rectangle d which is adjacent to both a and c, i. e. the one containing the angle marked in Figure 2. This rectangle has no common point with BC (since a is not higher than b), as well as no common point with AB or with AD (obviously). Then d has a common point with CD, and its left side provides a splitting line. Contradiction.

Case 2. The height of c is greater than that of a. Analogously, consider the rectangle d containing the angle marked on Figure 3. It has no common point with AD (otherwise it has a common side with a), as well as no common point with AB or with BC (obviously). Then d has a common point with CD. Hence its right side provides a splitting line, and we get the contradiction again.

Solution 2. Again, we suppose the contrary. Consider an arbitrary counterexample. Then we know that each rectangle is attached to at least one side of the square. Observe that a rectangle cannot be attached to two opposite sides, otherwise one of its sides lies on a splitting line.

We say that two rectangles are *opposite* if they are attached to opposite sides of $ABCD$. We claim that there exist two opposite rectangles having a common point.

Consider the union L of all rectangles attached to the left. Assume, to the contrary, that L has no common point with the rectangles attached to the right. Take a polygonal line p connecting the top and the bottom sides of the square and passing close from the right to the boundary of L (see Figure 4). Then all its points belong to the rectangles attached either to the top or to the bottom. Moreover, the upper end-point of p belongs to a rectangle attached to the top, and the lower one belongs to an other rectangle attached to the bottom. Hence, there is a point on p where some rectangles attached to the top and to the bottom meet each other. So, there always exists a pair of neighboring opposite rectangles.

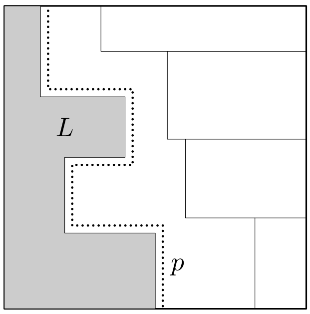


Figure 4

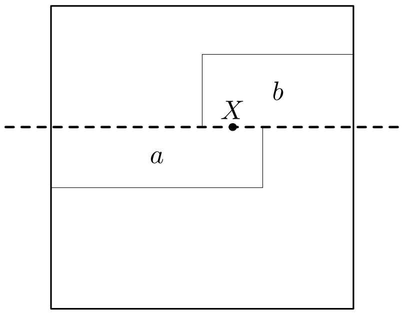


Figure 5

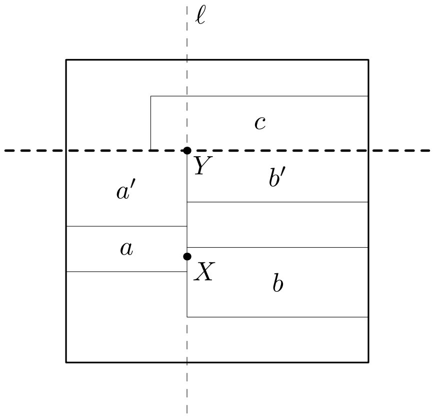


Figure 6

Now, take two opposite neighboring rectangles a and b. We can assume that a is attached to the left and b is attached to the right. Let X be their common point. If X belongs to their horizontal sides (in particular, X may appear to be a common vertex of a and b), then these sides provide a splitting line (see Figure 5). Otherwise, X lies on the vertical sides. Let ℓ be the line containing these sides.

Since ℓ is not a splitting line, it intersects the interior of some rectangle. Let c be such a rectangle, closest to X; we can assume that c lies above X. Let Y be the common point of ℓ and the bottom side of c (see Figure 6). Then Y is also a vertex of two rectangles lying below c.

So, let Y be the upper-right and upper-left corners of the rectangles a' and b', respectively. Then a' and b' are situated not lower than a and b, respectively (it may happen that $a = a'$ or $b = b'$). We claim that a' is attached to the left. If $a = a'$ then of course it is. If $a \neq a'$ then a' is above a, below c and to the left from b'. Hence, it can be attached to the left only.

Analogously, b' is attached to the right. Now, the top sides of these two rectangles pass through Y, hence they provide a splitting line again. This last contradiction completes the proof.

C3. Find all positive integers n, for which the numbers in the set $S=\{1,2,\ldots,n\}$ can be colored red and blue, with the following condition being satisfied: the set $S\times S\times S$ contains exactly 2007 ordered triples (x,y,z) such that (i) x,y,z are of the same color and (ii) $x+y+z$ is divisible by n.

(Netherlands)

Answer. $n=69$ and $n=84$.

Solution. Suppose that the numbers $1,2,\ldots,n$ are colored red and blue. Denote by R and B the sets of red and blue numbers, respectively; let $|R|=r$ and $|B|=b=n-r$. Call a triple $(x,y,z)\in S\times S\times S$ *monochromatic* if x,y,z have the same color, and *bichromatic* otherwise. Call a triple (x,y,z) *divisible* if $x+y+z$ is divisible by n. We claim that there are exactly r^2-rb+b^2 divisible monochromatic triples.

For any pair $(x,y)\in S\times S$ there exists a unique $z_{x,y}\in S$ such that the triple $(x,y,z_{x,y})$ is divisible; so there are exactly n^2 divisible triples. Furthermore, if a divisible triple (x,y,z) is bichromatic, then among x, y, z there are either one blue and two red numbers, or vice versa. In both cases, exactly one of the pairs (x,y), (y,z) and (z,x) belongs to the set $R\times B$. Assign such pair to the triple (x,y,z).

Conversely, consider any pair $(x,y)\in R\times B$, and denote $z=z_{x,y}$. Since $x\neq y$, the triples (x,y,z), (y,z,x) and (z,x,y) are distinct, and (x,y) is assigned to each of them. On the other hand, if (x,y) is assigned to some triple, then this triple is clearly one of those mentioned above. So each pair in $R\times B$ is assigned exactly three times.

Thus, the number of bichromatic divisible triples is three times the number of elements in $R\times B$, and the number of monochromatic ones is $n^2-3rb=(r+b)^2-3rb=r^2-rb+b^2$, as claimed.

So, to find all values of n for which the desired coloring is possible, we have to find all n, for which there exists a decomposition $n=r+b$ with $r^2-rb+b^2=2007$. Therefore, $9\mid r^2-rb+b^2=(r+b)^2-3rb$. From this it consequently follows that $3\mid r+b$, $3\mid rb$, and then $3\mid r$, $3\mid b$. Set $r=3s$, $b=3c$. We can assume that $s\geq c$. We have $s^2-sc+c^2=223$.

Furthermore,

$$892=4(s^2-sc+c^2)=(2c-s)^2+3s^2\geq 3s^2\geq 3s^2-3c(s-c)=3(s^2-sc+c^2)=669,$$

so $297\geq s^2\geq 223$ and $17\geq s\geq 15$. If $s=15$ then

$$c(15-c)=c(s-c)=s^2-(s^2-sc+c^2)=15^2-223=2$$

which is impossible for an integer c. In a similar way, if $s=16$ then $c(16-c)=33$, which is also impossible. Finally, if $s=17$ then $c(17-c)=66$, and the solutions are $c=6$ and $c=11$. Hence, $(r,b)=(51,18)$ or $(r,b)=(51,33)$, and the possible values of n are $n=51+18=69$ and $n=51+33=84$.

Comment. After the formula for the number of monochromatic divisible triples is found, the solution can be finished in various ways. The one presented is aimed to decrease the number of considered cases.

C4. Let $A_0 = (a_1, \dots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \dots, x_n)$ we construct a new sequence A_{k+1} in the following way.

1. We choose a partition $\{1, \dots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

$$\left|\sum_{i \in I} x_i - \sum_{j \in J} x_j\right|$$

attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily.

2. We set $A_{k+1} = (y_1, \dots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

Prove that for some k, the sequence A_k contains an element x such that $|x| \geq n/2$.

(Iran)

Solution.

Lemma. Suppose that all terms of the sequence $(x_1, \dots, x_n)$ satisfy the inequality $|x_i| < a$. Then there exists a partition $\{1, 2, \dots, n\} = I \cup J$ into two disjoint sets such that

$$\left|\sum_{i \in I} x_i - \sum_{j \in J} x_j\right| < a. \tag{1}$$

Proof. Apply an induction on n. The base case $n = 1$ is trivial. For the induction step, consider a sequence $(x_1, \dots, x_n)$ $(n > 1)$. By the induction hypothesis there exists a splitting $\{1, \dots, n-1\} = I' \cup J'$ such that

$$\left|\sum_{i \in I'} x_i - \sum_{j \in J'} x_j\right| < a.$$

For convenience, suppose that $\sum\limits_{i \in I'} x_i \geq \sum\limits_{j \in J'} x_j$. If $x_n \geq 0$ then choose $I = I'$, $J = J \cup \{n\}$; otherwise choose $I = I' \cup \{n\}$, $J = J'$. In both cases, we have $\sum\limits_{i \in I'} x_i - \sum\limits_{j \in J'} x_j \in [0, a)$ and $|x_n| \in [0, a)$; hence

$$\sum_{i \in I} x_i - \sum_{j \in J} x_j = \sum_{i \in I'} x_i - \sum_{j \in J'} x_j - |x_n| \in (-a, a),$$

as desired. $\square$

Let us turn now to the problem. To the contrary, assume that for all k, all the numbers in A_k lie in interval $(-n/2, n/2)$. Consider an arbitrary sequence $A_k = (b_1, \dots, b_n)$. To obtain the term b_i, we increased and decreased number a_i by one several times. Therefore $b_i - a_i$ is always an integer, and there are not more than n possible values for b_i. So, there are not more than n^n distinct possible sequences A_k, and hence two of the sequences $A_1, A_2, \dots, A_{n^n+1}$ should be identical, say $A_p = A_q$ for some $p < q$.

For any positive integer k, let S_k be the sum of squares of elements in A_k. Consider two consecutive sequences $A_k = (x_1, \dots, x_n)$ and $A_{k+1} = (y_1, \dots, y_n)$. Let $\{1, 2, \dots, n\} = I \cup J$ be the partition used in this step — that is, $y_i = x_i + 1$ for all $i \in I$ and $y_j = x_j - 1$ for all $j \in J$. Since the value of $\left|\sum\limits_{i \in I} x_i - \sum\limits_{j \in J} x_j\right|$ is the smallest possible, the Lemma implies that it is less than $n/2$. Then we have

$$S_{k+1} - S_k = \sum_{i \in I}\left((x_i+1)^2 - x_i^2\right) + \sum_{j \in J}\left((x_j-1)^2 - x_j^2\right) = n + 2\left(\sum_{i \in I} x_i - \sum_{j \in J} x_j\right) > n - 2 \cdot \frac{n}{2} = 0.$$

Thus we obtain $S_q > S_{q-1} > \dots > S_p$. This is impossible since $A_p = A_q$ and hence $S_p = S_q$.

C5. In the Cartesian coordinate plane define the strip $S_n = \{(x, y) \mid n \le x < n + 1\}$ for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

(Romania)

Solution. If S_n and S_{n+a} have the same color for some integer n, then we can choose the rectangle with vertices $(n, 0) \in S_n$, $(n, b) \in S_n$, $(n + a, 0) \in S_{n+a}$, and $(n + a, b) \in S_{n+a}$, and we are done. So it can be assumed that S_n and S_{n+a} have opposite colors for each n.

Similarly, it also can be assumed that S_n and S_{n+b} have opposite colors. Then, by induction on $|p| + |q|$, we obtain that for arbitrary integers p and q, strips S_n and $S_{n+pa+qb}$ have the same color if $p + q$ is even, and these two strips have opposite colors if $p + q$ is odd.

Let $d = \gcd(a, b)$, $a_1 = a/d$ and $b_1 = b/d$. Apply the result above for $p = b_1$ and $q = -a_1$. The strips S_0 and $S_{0+b_1a-a_1b}$ are identical and therefore they have the same color. Hence, $a_1 + b_1$ is even. By the construction, a_1 and b_1 are coprime, so this is possible only if both are odd.

Without loss of generality, we can assume $a > b$. Then $a_1 > b_1 \ge 1$, so $a_1 \ge 3$.

Choose integers k and ℓ such that $ka_1 - \ell b_1 = 1$ and therefore $ka - \ell b = d$. Since a_1 and b_1 are odd, $k + \ell$ is odd as well. Hence, for every integer n, strips S_n and $S_{n+ka-\ell b} = S_{n+d}$ have opposite colors. This also implies that the coloring is periodic with period $2d$, i.e. strips S_n and S_{n+2d} have the same color for every n.

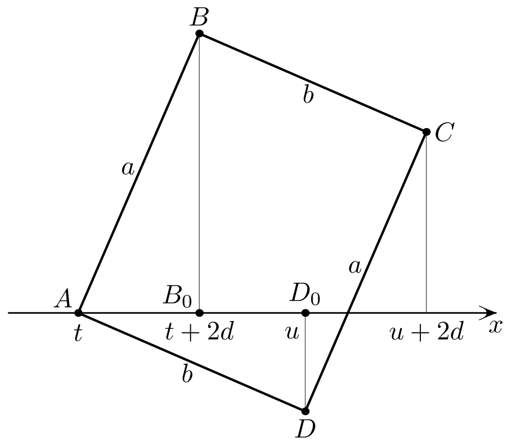


Figure 1

We will construct the desired rectangle $ABCD$ with $AB = CD = a$ and $BC = AD = b$ in a position such that vertex A lies on the x-axis, and the projection of side AB onto the x-axis is of length $2d$ (see Figure 1). This is possible since $a = a_1 d > 2d$. The coordinates of the vertices will have the forms

$$A = (t, 0), \quad B = (t + 2d, y_1), \quad C = (u + 2d, y_2), \quad D = (u, y_3).$$

Let $\varphi = \sqrt{a_1^2 - 4}$. By Pythagoras' theorem,

$$y_1 = BB_0 = \sqrt{a^2 - 4d^2} = d\sqrt{a_1^2 - 4} = d\varphi.$$

So, by the similar triangles ADD_0 and BAB_0, we have the constraint

$$u - t = AD_0 = \frac{AD}{AB} \cdot BB_0 = \frac{bd}{a}\varphi \tag{1}$$

for numbers t and u. Computing the numbers y_2 and y_3 is not required since they have no effect to the colors.

Observe that the number φ is irrational, because φ^2 is an integer, but φ is not: $a_1 > \varphi \geq \sqrt{a_1^2 - 2a_1 + 2} > a_1 - 1$.

By the periodicity, points A and B have the same color; similarly, points C and D have the same color. Furthermore, these colors depend only on the values of t and u. So it is sufficient to choose numbers t and u such that vertices A and D have the same color.

Let w be the largest positive integer such that there exist w consecutive strips $S_{n_0}, S_{n_0+1}, \ldots, S_{n_0+w-1}$ with the same color, say red. (Since S_{n_0+d} must be blue, we have $w \leq d$.) We will choose t from the interval $(n_0, n_0 + w)$.

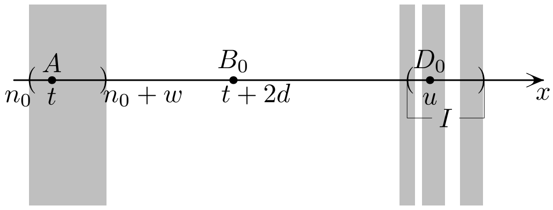


Figure 2

Consider the interval $I = \left(n_0 + \frac{bd}{a}\varphi, n_0 + \frac{bd}{a}\varphi + w\right)$ on the x-axis (see Figure 2). Its length is w, and the end-points are irrational. Therefore, this interval intersects $w + 1$ consecutive strips. Since at most w consecutive strips may have the same color, interval I must contain both red and blue points. Choose $u \in I$ such that the line $x = u$ is red and set $t = u - \frac{bd}{a}\varphi$, according to the constraint (1). Then $t \in (n_0, n_0 + w)$ and $A = (t, 0)$ is red as well as $D = (u, y_3)$.

Hence, variables u and t can be set such that they provide a rectangle with four red vertices.

Comment. The statement is false for squares, i.e. in the case $a = b$. If strips S_{2ka}, S_{2ka+1}, …, $S_{(2k+1)a-1}$ are red, and strips $S_{(2k+1)a}$, $S_{(2k+1)a+1}$, …, $S_{(2k+2)a-1}$ are blue for every integer k, then each square of size $a \times a$ has at least one red and at least one blue vertex as well.

C6. In a mathematical competition some competitors are friends; friendship is always mutual. Call a group of competitors a *clique* if each two of them are friends. The number of members in a clique is called its *size*.

It is known that the largest size of cliques is even. Prove that the competitors can be arranged in two rooms such that the largest size of cliques in one room is the same as the largest size of cliques in the other room.

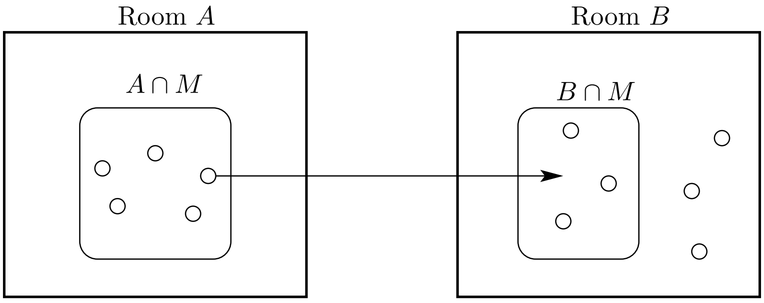
(Russia)

Solution. We present an algorithm to arrange the competitors. Let the two rooms be *Room A* and *Room B*. We start with an initial arrangement, and then we modify it several times by sending one person to the other room. At any state of the algorithm, A and B denote the sets of the competitors in the rooms, and $c(A)$ and $c(B)$ denote the largest sizes of cliques in the rooms, respectively.

Step 1. *Let M be one of the cliques of largest size, $|M| = 2m$. Send all members of M to Room A and all other competitors to Room B.*

Since M is a clique of the largest size, we have $c(A) = |M| \geq c(B)$.

Step 2. *While $c(A) > c(B)$, send one person from Room A to Room B.*



Note that $c(A) > c(B)$ implies that Room A is not empty.

In each step, $c(A)$ decreases by one and $c(B)$ increases by at most one. So at the end we have $c(A) \leq c(B) \leq c(A) + 1$.

We also have $c(A) = |A| \geq m$ at the end. Otherwise we would have at least $m+1$ members of M in Room B and at most $m-1$ in Room A, implying $c(B) - c(A) \geq (m+1) - (m-1) = 2$.

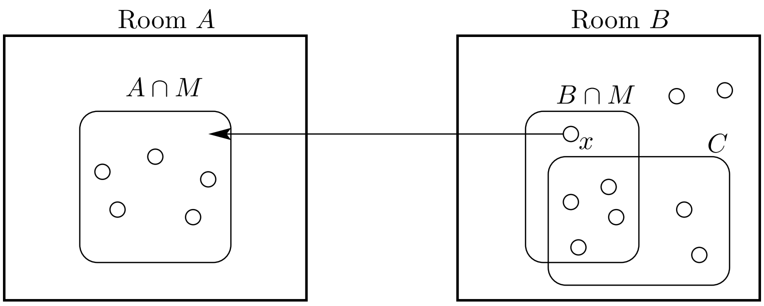
Step 3. *Let $k = c(A)$. If $c(B) = k$ then STOP.*

If we reached $c(A) = c(B) = k$ then we have found the desired arrangement.

In all other cases we have $c(B) = k + 1$.

From the estimate above we also know that $k = |A| = |A \cap M| \geq m$ and $|B \cap M| \leq m$.

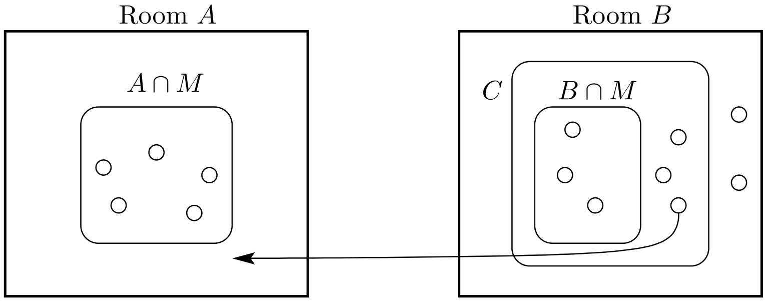
Step 4. *If there exists a competitor $x \in B \cap M$ and a clique $C \subset B$ such that $|C| = k + 1$ and $x \notin C$, then move x to Room A and STOP.*



After moving x back to Room A, we will have $k + 1$ members of M in Room A, thus $c(A) = k + 1$. Due to $x \notin C$, $c(B) = |C|$ is not decreased, and after this step we have $c(A) = c(B) = k + 1$.

If there is no such competitor x, then in Room B, all cliques of size $k+1$ contain $B \cap M$ as a subset.

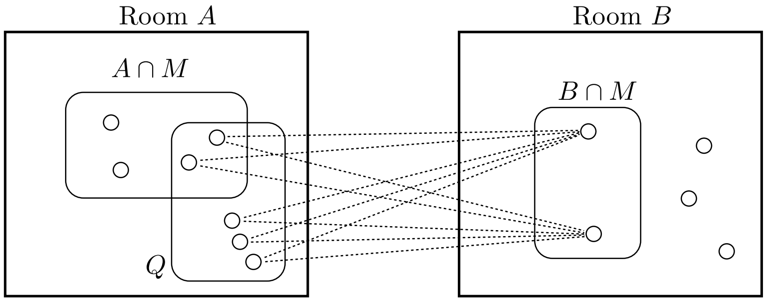
Step 5. *While $c(B) = k+1$, choose a clique $C \subset B$ such that $|C| = k+1$ and move one member of $C \setminus M$ to Room A.*



Note that $|C| = k+1 > m \geq |B \cap M|$, so $C \setminus M$ cannot be empty.

Every time we move a single person from Room B to Room A, so $c(B)$ decreases by at most 1. Hence, at the end of this loop we have $c(B) = k$.

In Room A we have the clique $A \cap M$ with size $|A \cap M| = k$ thus $c(A) \geq k$. We prove that there is no clique of larger size there. Let $Q \subset A$ be an arbitrary clique. We show that $|Q| \leq k$.



In Room A, and specially in set Q, there can be two types of competitors:

– Some members of M. Since M is a clique, they are friends with all members of $B \cap M$.

– Competitors which were moved to Room A in Step 5. Each of them has been in a clique with $B \cap M$ so they are also friends with all members of $B \cap M$.

Hence, all members of Q are friends with all members of $B \cap M$. Sets Q and $B \cap M$ are cliques themselves, so $Q \cup (B \cap M)$ is also a clique. Since M is a clique of the largest size,

$$|M| \geq |Q \cup (B \cap M)| = |Q| + |B \cap M| = |Q| + |M| - |A \cap M|,$$

therefore

$$|Q| \leq |A \cap M| = k.$$

Finally, after Step 5 we have $c(A) = c(B) = k$.

Comment. Obviously, the statement is false without the assumption that the largest clique size is even.

C7. Let $\alpha < \dfrac{3-\sqrt{5}}{2}$ be a positive real number. Prove that there exist positive integers n and $p > \alpha \cdot 2^n$ for which one can select $2p$ pairwise distinct subsets $S_1, \ldots, S_p, T_1, \ldots, T_p$ of the set $\{1, 2, \ldots, n\}$ such that $S_i \cap T_j \neq \varnothing$ for all $1 \leq i, j \leq p$.

(Austria)

Solution. Let k and m be positive integers (to be determined later) and set $n = km$. Decompose the set $\{1, 2, \ldots, n\}$ into k disjoint subsets, each of size m; denote these subsets by $A_1, \ldots, A_k$. Define the following families of sets:

$$\mathcal{S} = \big\{S \subset \{1, 2, \ldots, n\} : \ \forall i \ S \cap A_i \neq \varnothing\big\},$$
$$\mathcal{T}_1 = \big\{T \subset \{1, 2, \ldots, n\} : \ \exists i \ A_i \subset T\big\}, \qquad \mathcal{T} = \mathcal{T}_1 \setminus \mathcal{S}.$$

For each set $T \in \mathcal{T} \subset \mathcal{T}_1$, there exists an index $1 \leq i \leq k$ such that $A_i \subset T$. Then for all $S \in \mathcal{S}$, $S \cap T \supset S \cap A_i \neq \varnothing$. Hence, each $S \in \mathcal{S}$ and each $T \in \mathcal{T}$ have at least one common element.

Below we show that the numbers m and k can be chosen such that $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^n$. Then, choosing $p = \min\big\{|\mathcal{S}|, |\mathcal{T}|\big\}$, one can select the desired $2p$ sets $S_1, \ldots, S_p$ and $T_1, \ldots, T_p$ from families $\mathcal{S}$ and $\mathcal{T}$, respectively. Since families $\mathcal{S}$ and $\mathcal{T}$ are disjoint, sets S_i and T_j will be pairwise distinct.

To count the sets $S \in \mathcal{S}$, observe that each A_i has $2^m - 1$ nonempty subsets so we have $2^m - 1$ choices for $S \cap A_i$. These intersections uniquely determine set S, so

$$|\mathcal{S}| = (2^m - 1)^k. \tag{1}$$

Similarly, if a set $H \subset \{1, 2, \ldots, n\}$ does not contain a certain set A_i then we have $2^m - 1$ choices for $H \cap A_i$: all subsets of A_i, except A_i itself. Therefore, the complement of $\mathcal{T}_1$ contains $(2^m - 1)^k$ sets and

$$|\mathcal{T}_1| = 2^{km} - (2^m - 1)^k. \tag{2}$$

Next consider the family $\mathcal{S} \setminus \mathcal{T}_1$. If a set S intersects all A_i but does not contain any of them, then there exists $2^m - 2$ possible values for each $S \cap A_i$: all subsets of A_i except $\varnothing$ and A_i. Therefore the number of such sets S is $(2^m - 2)^k$, so

$$|\mathcal{S} \setminus \mathcal{T}_1| = (2^m - 2)^k. \tag{3}$$

From (1), (2), and (3) we obtain

$$|\mathcal{T}| = |\mathcal{T}_1| - |\mathcal{S} \cap \mathcal{T}_1| = |\mathcal{T}_1| - \big(|\mathcal{S}| - |\mathcal{S} \setminus \mathcal{T}_1|\big) = 2^{km} - 2(2^m - 1)^k + (2^m - 2)^k.$$

Let $\delta = \dfrac{3-\sqrt{5}}{2}$ and $k = k(m) = \big[2^m \log \frac{1}{\delta}\big]$. Then

$$\lim_{m\to\infty} \frac{|\mathcal{S}|}{2^{km}} = \lim_{m\to\infty} \left(1 - \frac{1}{2^m}\right)^k = \exp\left(-\lim_{m\to\infty} \frac{k}{2^m}\right) = \delta$$

and similarly

$$\lim_{m\to\infty} \frac{|\mathcal{T}|}{2^{km}} = 1 - 2\lim_{m\to\infty}\left(1 - \frac{1}{2^m}\right)^k + \lim_{m\to\infty}\left(1 - \frac{2}{2^m}\right)^k = 1 - 2\delta + \delta^2 = \delta.$$

Hence, if m is sufficiently large then $\dfrac{|\mathcal{S}|}{2^{mk}}$ and $\dfrac{|\mathcal{T}|}{2^{mk}}$ are greater than α (since $\alpha < \delta$). So $|\mathcal{S}|, |\mathcal{T}| > \alpha \cdot 2^{mk} = \alpha \cdot 2^n$.

Comment. It can be proved that the constant $\dfrac{3-\sqrt{5}}{2}$ is sharp. Actually, if $S_1, \ldots, S_p, T_1, \ldots, T_p$ are distinct subsets of $\{1, 2, \ldots, n\}$ such that each S_i intersects each T_j, then $p < \dfrac{3-\sqrt{5}}{2} \cdot 2^n$.

C8. Given a convex n-gon P in the plane. For every three vertices of P, consider the triangle determined by them. Call such a triangle *good* if all its sides are of unit length.

Prove that there are not more than $\frac{2}{3}n$ good triangles.

(Ukraine)

Solution. Consider all good triangles containing a certain vertex A. The other two vertices of any such triangle lie on the circle ω_A with unit radius and center A. Since P is convex, all these vertices lie on an arc of angle less than $180°$. Let L_AR_A be the shortest such arc, oriented clockwise (see Figure 1). Each of segments AL_A and AR_A belongs to a unique good triangle. We say that the good triangle with side AL_A is *assigned counterclockwise* to A, and the second one, with side AR_A, is *assigned clockwise* to A. In those cases when there is a single good triangle containing vertex A, this triangle is assigned to A twice.

There are at most two assignments to each vertex of the polygon. (Vertices which do not belong to any good triangle have no assignment.) So the number of assignments is at most $2n$.

Consider an arbitrary good triangle ABC, with vertices arranged clockwise. We prove that ABC is assigned to its vertices at least three times. Then, denoting the number of good triangles by t, we obtain that the number K of all assignments is at most $2n$, while it is not less than $3t$. Then $3t \le K \le 2n$, as required.

Actually, we prove that triangle ABC is assigned either counterclockwise to C or clockwise to B. Then, by the cyclic symmetry of the vertices, we obtain that triangle ABC is assigned either counterclockwise to A or clockwise to C, and either counterclockwise to B or clockwise to A, providing the claim.

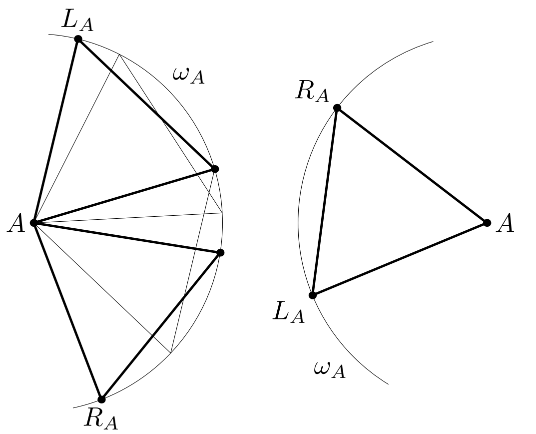


Figure 1

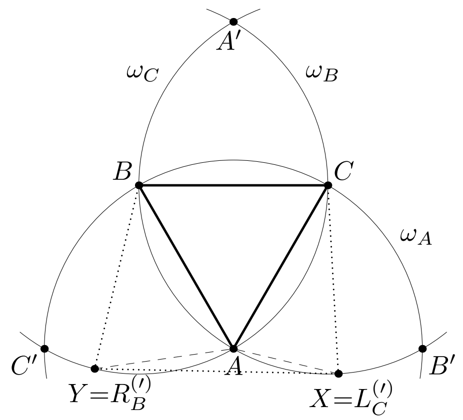


Figure 2

Assume, to the contrary, that $L_C \neq A$ and $R_B \neq A$. Denote by A', B', C' the intersection points of circles ω_A, ω_B and ω_C, distinct from A, B, C (see Figure 2). Let $CL_CL'_C$ be the good triangle containing CL_C. Observe that the angle of arc L_CA is less than $120°$. Then one of the points L_C and L'_C belongs to arc $B'A$ of ω_C; let this point be X. In the case when $L_C = B'$ and $L'_C = A$, choose $X = B'$.

Analogously, considering the good triangle BR'_BR_B which contains BR_B as an edge, we see that one of the points R_B and R'_B lies on arc AC' of ω_B. Denote this point by Y, $Y \neq A$. Then angles XAY, YAB, BAC and CAX (oriented clockwise) are not greater than $180°$. Hence, point A lies in quadrilateral $XYBC$ (either in its interior or on segment XY). This is impossible, since all these five points are vertices of P.

Hence, each good triangle has at least three assignments, and the statement is proved.

Comment 1. Considering a diameter AB of the polygon, one can prove that every good triangle containing either A or B has at least four assignments. This observation leads to $t \le \left\lfloor \frac{2}{3}(n-1) \right\rfloor$.

Comment 2. The result $t \leq \left\lfloor \frac{2}{3}(n-1) \right\rfloor$ is sharp. To construct a polygon with $n = 3k+1$ vertices and $t = 2k$ triangles, take a rhombus $AB_1C_1D_1$ with unit side length and $\angle B_1 = 60^\circ$. Then rotate it around A by small angles obtaining rhombi $AB_2C_2D_2, \ldots, AB_kC_kD_k$ (see Figure 3). The polygon $AB_1 \ldots B_kC_1 \ldots C_kD_1 \ldots D_k$ has $3k+1$ vertices and contains $2k$ good triangles.

The construction for $n = 3k$ and $n = 3k-1$ can be obtained by deleting vertices D_n and D_{n-1}.

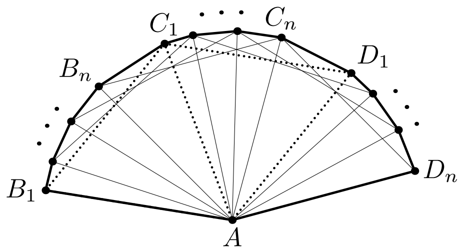


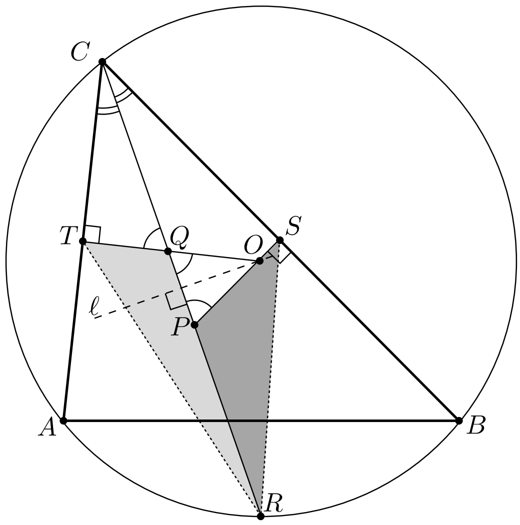
Figure 3

Geometry

G1. In triangle ABC, the angle bisector at vertex C intersects the circumcircle and the perpendicular bisectors of sides BC and CA at points R, P, and Q, respectively. The midpoints of BC and CA are S and T, respectively. Prove that triangles RQT and RPS have the same area.

(Czech Republic)

Solution 1. If $AC = BC$ then triangle ABC is isosceles, triangles RQT and RPS are symmetric about the bisector CR and the statement is trivial. If $AC \neq BC$ then it can be assumed without loss of generality that $AC < BC$.



Denote the circumcenter by O. The right triangles CTQ and CSP have equal angles at vertex C, so they are similar, $\angle CPS = \angle CQT = \angle OQP$ and

$$\frac{QT}{PS} = \frac{CQ}{CP}. \tag{1}$$

Let ℓ be the perpendicular bisector of chord CR; of course, ℓ passes through the circumcenter O. Due to the equal angles at P and Q, triangle OPQ is isosceles with $OP = OQ$. Then line ℓ is the axis of symmetry in this triangle as well. Therefore, points P and Q lie symmetrically on line segment CR,

$$RP = CQ \quad \text{and} \quad RQ = CP. \tag{2}$$

Triangles RQT and RPS have equal angles at vertices Q and P, respectively. Then

$$\frac{\text{area}(RQT)}{\text{area}(RPS)} = \frac{\frac{1}{2} \cdot RQ \cdot QT \cdot \sin \angle RQT}{\frac{1}{2} \cdot RP \cdot PS \cdot \sin \angle RPS} = \frac{RQ}{RP} \cdot \frac{QT}{PS}.$$

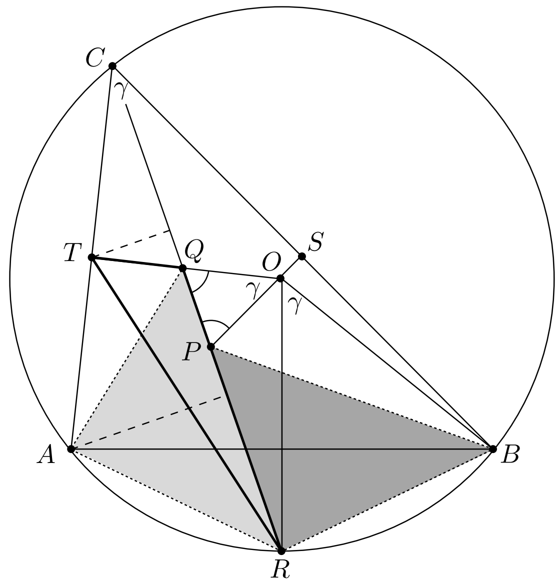
Substituting (1) and (2),

$$\frac{\text{area}(RQT)}{\text{area}(RPS)} = \frac{RQ}{RP} \cdot \frac{QT}{PS} = \frac{CP}{CQ} \cdot \frac{CQ}{CP} = 1.$$

Hence, $\text{area}(RQT) = \text{area}(RSP)$.

Solution 2. Assume again $AC < BC$. Denote the circumcenter by O, and let γ be the angle at C. Similarly to the first solution, from right triangles CTQ and CSP we obtain that $\angle OPQ = \angle OQP = 90^\circ - \frac{\gamma}{2}$. Then triangle OPQ is isosceles, $OP = OQ$ and moreover $\angle POQ = \gamma$.

As is well-known, point R is the midpoint of arc AB and $\angle ROA = \angle BOR = \gamma$.



Consider the rotation around point O by angle γ. This transform moves A to R, R to B and Q to P; hence triangles RQA and BPR are congruent and they have the same area.

Triangles RQT and RQA have RQ as a common side, so the ratio between their areas is

$$\frac{\text{area}(RQT)}{\text{area}(RQA)} = \frac{d(T, CR)}{d(A, CR)} = \frac{CT}{CA} = \frac{1}{2}.$$

($d(X, YZ)$ denotes the distance between point X and line YZ).

It can be obtained similarly that

$$\frac{\text{area}(RPS)}{\text{area}(BPR)} = \frac{CS}{CB} = \frac{1}{2}.$$

Now the proof can be completed as

$$\text{area}(RQT) = \frac{1}{2}\,\text{area}(RQA) = \frac{1}{2}\,\text{area}(BPR) = \text{area}(RPS).$$

G2. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90°$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

(Canada)

Solution 1. Let N be the midpoint of segment BT (see Figure 1). Line XN is the axis of symmetry in the isosceles triangle BXT, thus $\angle TNX = 90°$ and $\angle BXN = \angle NXT$. Moreover, in triangle BCT, line MN is the midline parallel to CT; hence $\angle CTM = \angle NMT$.

Due to the right angles at points M and N, these points lie on the circle with diameter XT. Therefore,

$$\angle MTB = \angle MTN = \angle MXN \qquad \text{and} \qquad \angle CTM = \angle NMT = \angle NXT = \angle BXN.$$

Hence

$$\angle MTB - \angle CTM = \angle MXN - \angle BXN = \angle MXB = \angle MAB$$

which does not depend on X.

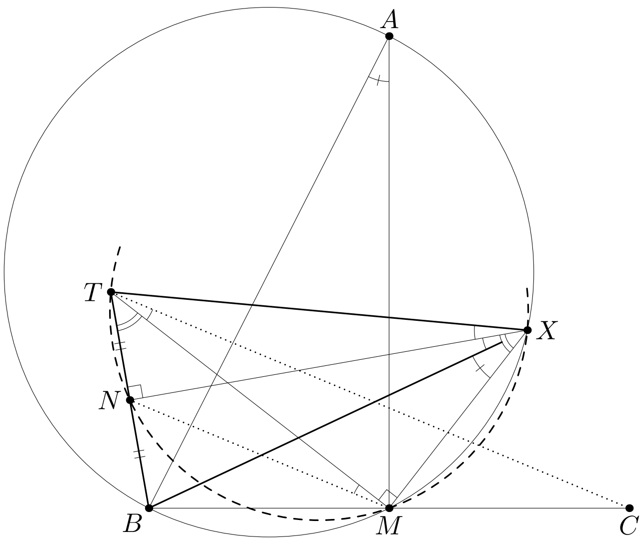


Figure 1

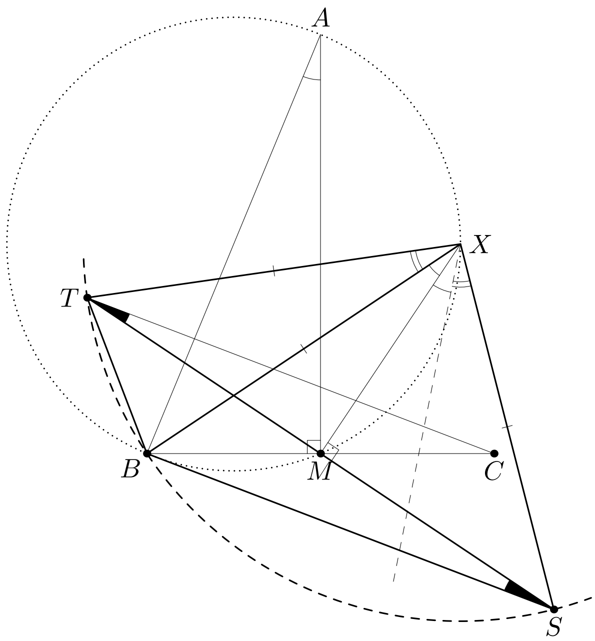


Figure 2

Solution 2. Let S be the reflection of point T over M (see Figure 2). Then XM is the perpendicular bisector of TS, hence $XB = XT = XS$, and X is the circumcenter of triangle BST. Moreover, $\angle BSM = \angle CTM$ since they are symmetrical about M. Then

$$\angle MTB - \angle CTM = \angle STB - \angle BST = \frac{\angle SXB - \angle BXT}{2}.$$

Observe that $\angle SXB = \angle SXT - \angle BXT = 2\angle MXT - \angle BXT$, so

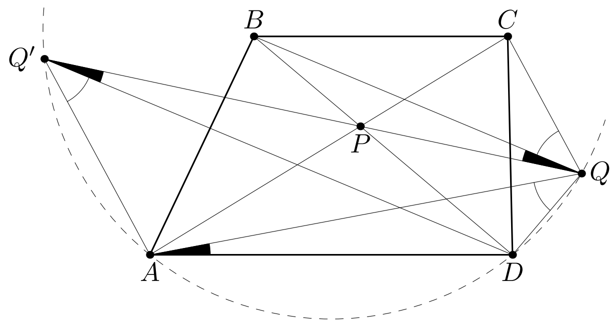
$$\angle MTB - \angle CTM = \frac{2\angle MXT - 2\angle BXT}{2} = \angle MXB = \angle MAB,$$

which is constant.

G3. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

(Ukraine)

Solution. Let $t = \dfrac{AD}{BC}$. Consider the homothety h with center P and scale $-t$. Triangles PDA and PBC are similar with ratio t, hence $h(B) = D$ and $h(C) = A$.



Let $Q' = h(Q)$ (see Figure 1). Then points Q, P and Q' are obviously collinear. Points Q and P lie on the same side of AD, as well as on the same side of BC; hence Q' and P are also on the same side of $h(BC) = AD$, and therefore Q and Q' are on the same side of AD. Moreover, points Q and C are on the same side of BD, while Q' and A are on the opposite side (see Figure above).

By the homothety, $\angle AQ'D = \angle CQB = \angle AQD$, hence quadrilateral $AQ'QD$ is cyclic. Then

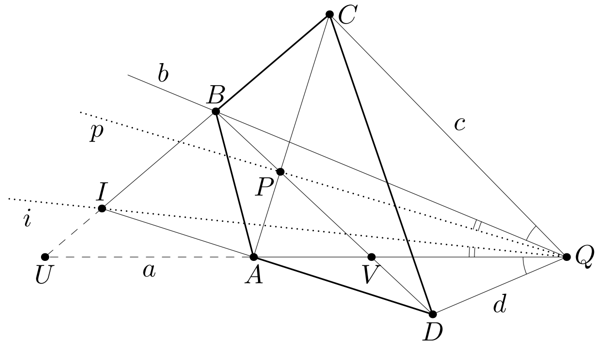
$$\angle DAQ = \angle DQ'Q = \angle DQ'P = \angle BQP$$

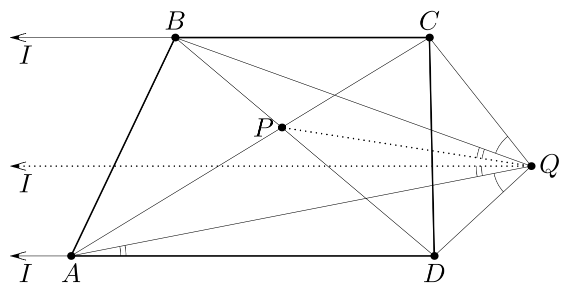
(the latter equality is valid by the homothety again).

Comment. The statement of the problem is a limit case of the following result.

In an arbitrary quadrilateral $ABCD$, let $P = AC \cap BD$, $I = AD \cap BC$, and let Q be an arbitrary point which is not collinear with any two of points A, B, C, D. Then $\angle AQD = \angle CQB$ if and only if $\angle BQP = \angle IQA$ (angles are oriented; see Figure below to the left).

In the special case of the trapezoid, I is an ideal point and $\angle DAQ = \angle IQA = \angle BQP$.





Let $a = QA$, $b = QB$, $c = QC$, $d = QD$, $i = QI$ and $p = QP$. Let line QA intersect lines BC and BD at points U and V, respectively. On lines BC and BD we have

$$(abci) = (UBCI) \qquad \text{and} \qquad (badp) = (abpd) = (VBPD).$$

Projecting from A, we get

$$(abci) = (UBCI) = (VBPD) = (badp).$$

Suppose that $\angle AQD = \angle CQB$. Let line p' be the reflection of line i about the bisector of angle AQB. Then by symmetry we have $(badp') = (abci) = (badp)$. Hence $p = p'$, as desired.

The converse statement can be proved analogously.

G4. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A, and let ℓ intersect segment DC and line BC at points F and G, respectively. Suppose that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

(Luxembourg)

Solution. If $CF = CG$, then $\angle FGC = \angle GFC$, hence $\angle GAB = \angle GFC = \angle FGC = \angle FAD$, and ℓ is a bisector.

Assume that $CF < GC$. Let EK and EL be the altitudes in the isosceles triangles ECF and EGC, respectively. Then in the right triangles EKF and ELC we have $EF = EC$ and

$$KF = \frac{CF}{2} < \frac{GC}{2} = LC,$$

so

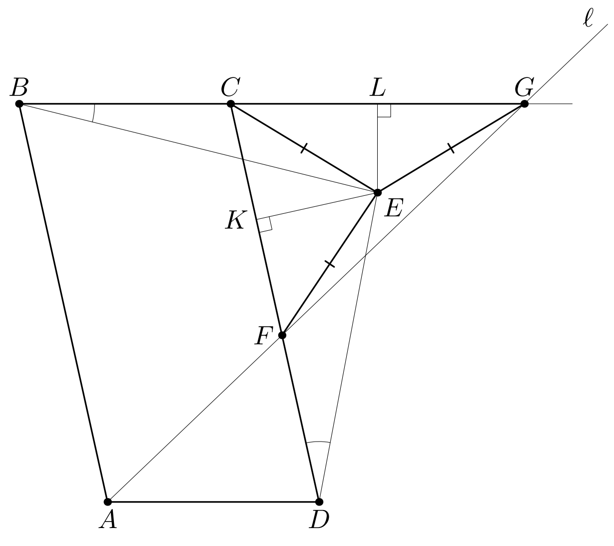
$$KE = \sqrt{EF^2 - KF^2} > \sqrt{EC^2 - LC^2} = LE.$$

Since quadrilateral $BCED$ is cyclic, we have $\angle EDC = \angle EBC$, so the right triangles BEL and DEK are similar. Then $KE > LE$ implies $DK > BL$, and hence

$$DF = DK - KF > BL - LC = BC = AD.$$

But triangles ADF and GCF are similar, so we have $1 > \dfrac{AD}{DF} = \dfrac{GC}{CF}$; this contradicts our assumption.

The case $CF > GC$ is completely similar. We consequently obtain the converse inequalities $KF > LC$, $KE < LE$, $DK < BL$, $DF < AD$, hence $1 < \dfrac{AD}{DF} = \dfrac{GC}{CF}$; a contradiction.



G5. Let ABC be a fixed triangle, and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Let P be a variable point on the circumcircle. Let lines PA_1, PB_1, PC_1 meet the circumcircle again at A', B', C' respectively. Assume that the points A, B, C, A', B', C' are distinct, and lines AA', BB', CC' form a triangle. Prove that the area of this triangle does not depend on P.

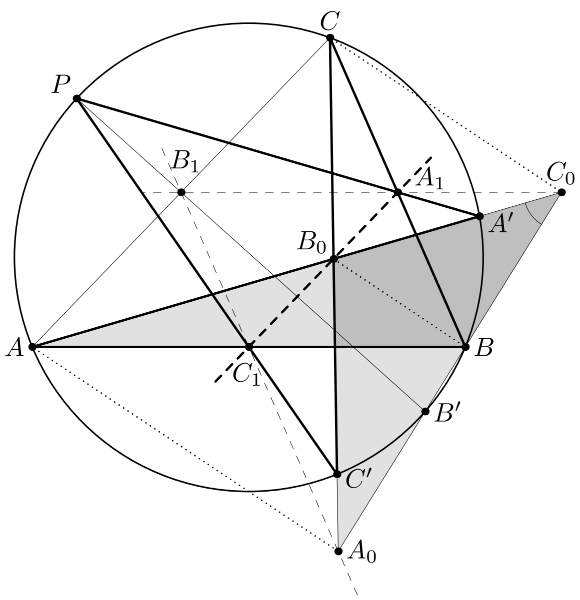
(United Kingdom)

Solution 1. Let A_0, B_0, C_0 be the points of intersection of the lines AA', BB' and CC' (see Figure). We claim that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$, hence it is constant.

Consider the inscribed hexagon $ABCC'PA'$. By Pascal's theorem, the points of intersection of its opposite sides (or of their extensions) are collinear. These points are $AB \cap C'P = C_1$, $BC \cap PA' = A_1$, $CC' \cap A'A = B_0$. So point B_0 lies on the midline A_1C_1 of triangle ABC. Analogously, points A_0 and C_0 lie on lines B_1C_1 and A_1B_1, respectively.

Lines AC and A_1C_1 are parallel, so triangles $B_0C_0A_1$ and AC_0B_1 are similar; hence we have

$$\frac{B_0C_0}{AC_0} = \frac{A_1C_0}{B_1C_0}.$$



Analogously, from $BC \parallel B_1C_1$ we obtain

$$\frac{A_1C_0}{B_1C_0} = \frac{BC_0}{A_0C_0}.$$

Combining these equalities, we get

$$\frac{B_0C_0}{AC_0} = \frac{BC_0}{A_0C_0},$$

or

$$A_0C_0 \cdot B_0C_0 = AC_0 \cdot BC_0.$$

Hence we have

$$\operatorname{area}(A_0B_0C_0) = \frac{1}{2}A_0C_0 \cdot B_0C_0 \sin\angle A_0C_0B_0 = \frac{1}{2}AC_0 \cdot BC_0 \sin\angle AC_0B = \operatorname{area}(ABC_0).$$

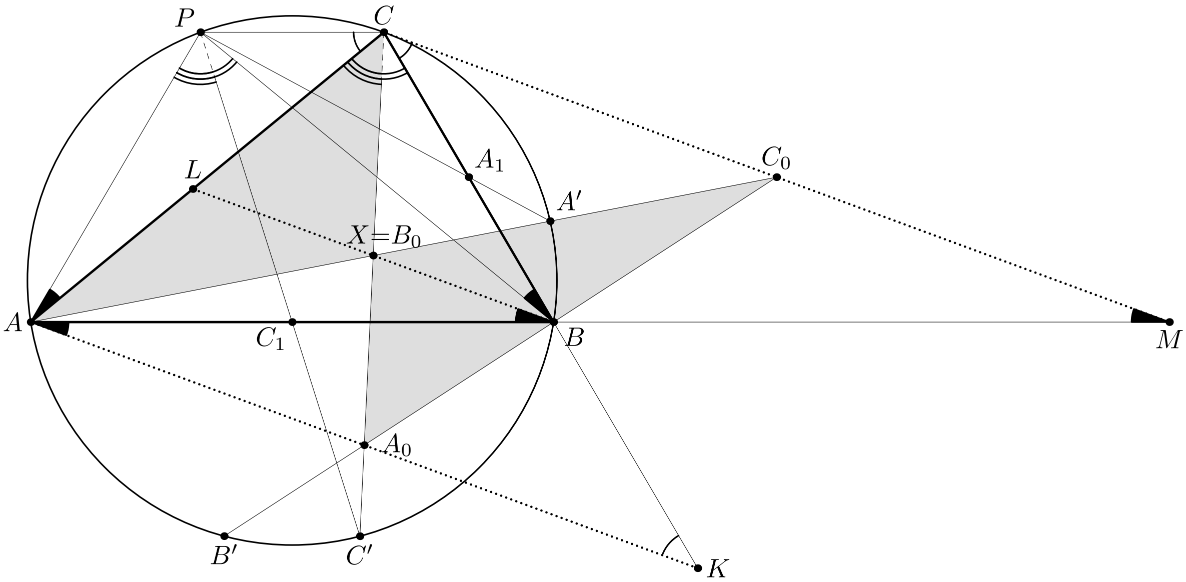
Since C_0 lies on the midline, we have $d(C_0, AB) = \frac{1}{2}d(C, AB)$ (we denote by $d(X, YZ)$ the distance between point X and line YZ). Then we obtain

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ABC_0) = \frac{1}{2}AB \cdot d(C_0, AB) = \frac{1}{4}AB \cdot d(C, AB) = \frac{1}{2}\operatorname{area}(ABC).$$

Solution 2. Again, we prove that $\operatorname{area}(A_0B_0C_0) = \frac{1}{2}\operatorname{area}(ABC)$.

We can assume that P lies on arc AC. Mark a point L on side AC such that $\angle CBL = \angle PBA$; then $\angle LBA = \angle CBA - \angle CBL = \angle CBA - \angle PBA = \angle CBP$. Note also that $\angle BAL = \angle BAC = \angle BPC$ and $\angle LCB = \angle APB$. Hence, triangles BAL and BPC are similar, and so are triangles LCB and APB.

Analogously, mark points K and M respectively on the extensions of sides CB and AB beyond point B, such that $\angle KAB = \angle CAP$ and $\angle BCM = \angle PCA$. For analogous reasons, $\angle KAC = \angle BAP$ and $\angle ACM = \angle PCB$. Hence $\triangle ABK \sim \triangle APC \sim \triangle MBC$, $\triangle ACK \sim \triangle APB$, and $\triangle MAC \sim \triangle BPC$. From these similarities, we have $\angle CMB = \angle KAB = \angle CAP$, while we have seen that $\angle CAP = \angle CBP = \angle LBA$. Hence, $AK \parallel BL \parallel CM$.



Let line CC' intersect BL at point X. Note that $\angle LCX = \angle ACC' = \angle APC' = \angle APC_1$, and PC_1 is a median in triangle APB. Since triangles APB and LCB are similar, CX is a median in triangle LCB, and X is a midpoint of BL. For the same reason, AA' passes through this midpoint, so $X = B_0$. Analogously, A_0 and C_0 are the midpoints of AK and CM.

Now, from $AA_0 \parallel CC_0$, we have

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(AC_0A_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACA_0) - \operatorname{area}(AB_0A_0) = \operatorname{area}(ACB_0).$$

Finally,

$$\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0) = \frac{1}{2}B_0L \cdot AC \sin ALB_0 = \frac{1}{4}BL \cdot AC \sin ALB = \frac{1}{2}\operatorname{area}(ABC).$$

Comment 1. The equality $\operatorname{area}(A_0B_0C_0) = \operatorname{area}(ACB_0)$ in Solution 2 does not need to be proved since the following fact is frequently known.

Suppose that the lines KL and MN are parallel, while the lines KM and LN intersect in a point E. Then $\operatorname{area}(KEN) = \operatorname{area}(MEL)$.

Comment 2. It follows immediately from both solutions that $AA_0 \parallel BB_0 \parallel CC_0$. These lines pass through an ideal point which is isogonally conjugate to P. It is known that they are parallel to the Simson line of point Q which is opposite to P on the circumcircle.

Comment 3. If $A = A'$, then one can define the line AA' to be the tangent to the circumcircle at point A. Then the statement of the problem is also valid in this case.

G6. Determine the smallest positive real number k with the following property.

Let $ABCD$ be a convex quadrilateral, and let points A_1, B_1, C_1 and D_1 lie on sides AB, BC, CD and DA, respectively. Consider the areas of triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1; let S be the sum of the two smallest ones, and let S_1 be the area of quadrilateral $A_1B_1C_1D_1$. Then we always have $kS_1 \geq S$.

(U.S.A.)

Answer. $k = 1$.

Solution. Throughout the solution, triangles AA_1D_1, BB_1A_1, CC_1B_1, and DD_1C_1 will be referred to as *border triangles*. We will denote by $[\mathcal{R}]$ the area of a region $\mathcal{R}$.

First, we show that $k \geq 1$. Consider a triangle ABC with unit area; let A_1, B_1, K be the midpoints of its sides AB, BC, AC, respectively. Choose a point D on the extension of BK, close to K. Take points C_1 and D_1 on sides CD and DA close to D (see Figure 1). We have $[BB_1A_1] = \frac{1}{4}$. Moreover, as $C_1, D_1, D \to K$, we get $[A_1B_1C_1D_1] \to [A_1B_1K] = \frac{1}{4}$, $[AA_1D_1] \to [AA_1K] = \frac{1}{4}$, $[CC_1B_1] \to [CKB_1] = \frac{1}{4}$ and $[DD_1C_1] \to 0$. Hence, the sum of the two smallest areas of border triangles tends to $\frac{1}{4}$, as well as $[A_1B_1C_1D_1]$; therefore, their ratio tends to 1, and $k \geq 1$.

We are left to prove that $k = 1$ satisfies the desired property.

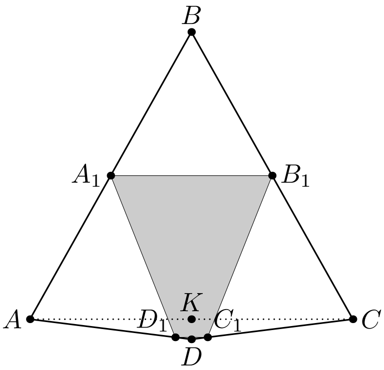


Figure 1

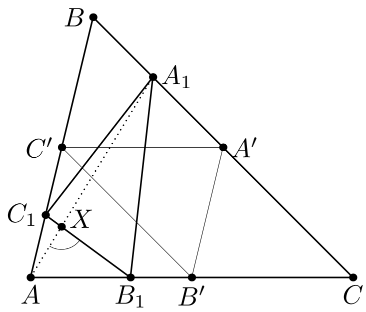


Figure 2

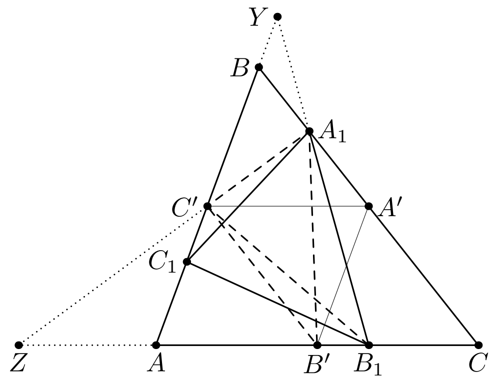


Figure 3

Lemma. Let points A_1, B_1, C_1 lie respectively on sides BC, CA, AB of a triangle ABC. Then $[A_1B_1C_1] \geq \min\{[AC_1B_1], [BA_1C_1], [CB_1A_1]\}$.

Proof. Let A', B', C' be the midpoints of sides BC, CA and AB, respectively.

Suppose that two of points A_1, B_1, C_1 lie in one of triangles $AC'B'$, $BA'C'$ and $CB'A'$ (for convenience, let points B_1 and C_1 lie in triangle $AC'B'$; see Figure 2). Let segments B_1C_1 and AA_1 intersect at point X. Then X also lies in triangle $AC'B'$. Hence $A_1X \geq AX$, and we have

$$\frac{[A_1B_1C_1]}{[AC_1B_1]} = \frac{\frac{1}{2}A_1X \cdot B_1C_1 \cdot \sin \angle A_1XC_1}{\frac{1}{2}AX \cdot B_1C_1 \cdot \sin \angle AXB_1} = \frac{A_1X}{AX} \geq 1,$$

as required.

Otherwise, each one of triangles $AC'B'$, $BA'C'$, $CB'A'$ contains exactly one of points A_1, B_1, C_1, and we can assume that $BA_1 < BA'$, $CB_1 < CB'$, $AC_1 < AC'$ (see Figure 3). Then lines B_1A_1 and AB intersect at a point Y on the extension of AB beyond point B, hence $\frac{[A_1B_1C_1]}{[A_1B_1C']} = \frac{C_1Y}{C'Y} > 1$; also, lines A_1C' and CA intersect at a point Z on the extension of CA beyond point A, hence $\frac{[A_1B_1C']}{[A_1B'C']} = \frac{B_1Z}{B'Z} > 1$. Finally, since $A_1A' \parallel B'C'$, we have $[A_1B_1C_1] > [A_1B_1C'] > [A_1B'C'] = [A'B'C'] = \frac{1}{4}[ABC]$.

Now, from $[A_1B_1C_1] + [AC_1B_1] + [BA_1C_1] + [CB_1A_1] = [ABC]$ we obtain that one of the remaining triangles AC_1B_1, BA_1C_1, CB_1A_1 has an area less than $\frac{1}{4}[ABC]$, so it is less than $[A_1B_1C_1]$. $\square$

Now we return to the problem. We say that triangle $A_1B_1C_1$ is *small* if $[A_1B_1C_1]$ is less than each of $[BB_1A_1]$ and $[CC_1B_1]$; otherwise this triangle is *big* (the similar notion is introduced for triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$). If both triangles $A_1B_1C_1$ and $C_1D_1A_1$ are big, then $[A_1B_1C_1]$ is not less than the area of some border triangle, and $[C_1D_1A_1]$ is not less than the area of another one; hence, $S_1 = [A_1B_1C_1] + [C_1D_1A_1] \geq S$. The same is valid for the pair of $B_1C_1D_1$ and $D_1A_1B_1$. So it is sufficient to prove that in one of these pairs both triangles are big.

Suppose the contrary. Then there is a small triangle in each pair. Without loss of generality, assume that triangles $A_1B_1C_1$ and $D_1A_1B_1$ are small. We can assume also that $[A_1B_1C_1] \leq [D_1A_1B_1]$. Note that in this case ray D_1C_1 intersects line BC.

Consider two cases.

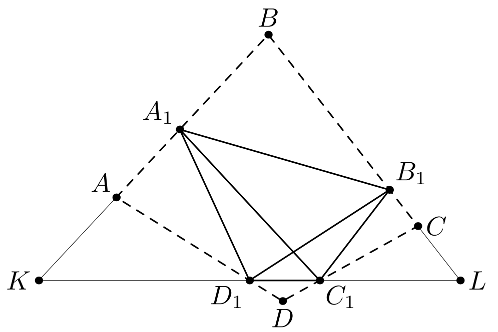


Figure 4

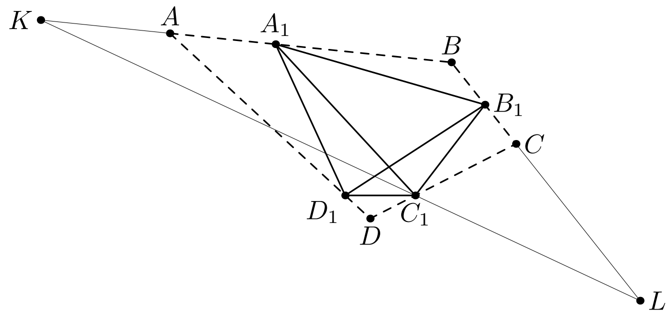


Figure 5

Case 1. Ray C_1D_1 intersects line AB at some point K. Let ray D_1C_1 intersect line BC at point L (see Figure 4). Then we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$, $[A_1B_1C_1] < [BB_1A_1]$ (both — since $[A_1B_1C_1]$ is small), and $[A_1B_1C_1] \leq [D_1A_1B_1] < [AA_1D_1] < [KA_1D_1] < [KA_1C_1]$ (since triangle $D_1A_1B_1$ is small). This contradicts the Lemma, applied for triangle $A_1B_1C_1$ inside LKB.

Case 2. Ray C_1D_1 does not intersect AB. Then choose a "sufficiently far" point K on ray BA such that $[KA_1C_1] > [A_1B_1C_1]$, and that ray KC_1 intersects line BC at some point L (see Figure 5). Since ray C_1D_1 does not intersect line AB, the points A and D_1 are on different sides of KL; then A and D are also on different sides, and C is on the same side as A and B. Then analogously we have $[A_1B_1C_1] < [CC_1B_1] < [LC_1B_1]$ and $[A_1B_1C_1] < [BB_1A_1]$ since triangle $A_1B_1C_1$ is small. This (together with $[A_1B_1C_1] < [KA_1C_1]$) contradicts the Lemma again.

G7. Given an acute triangle ABC with angles α, β and γ at vertices A, B and C, respectively, such that $\beta > \gamma$. Point I is the incenter, and R is the circumradius. Point D is the foot of the altitude from vertex A. Point K lies on line AD such that $AK = 2R$, and D separates A and K. Finally, lines DI and KI meet sides AC and BC at E and F, respectively.

Prove that if $IE = IF$ then $\beta \le 3\gamma$.

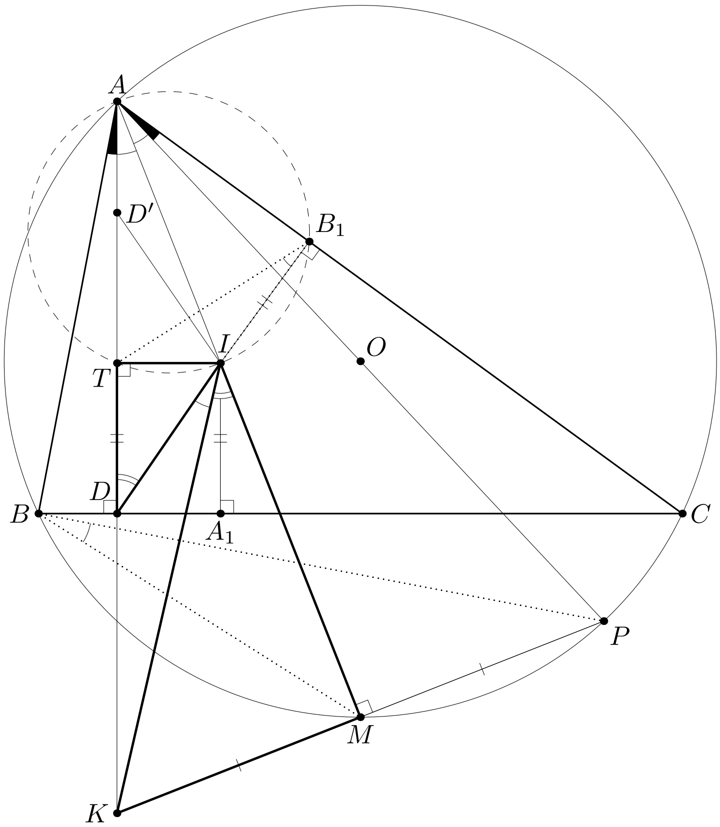
(Iran)

Solution 1. We first prove that

$$\angle KID = \frac{\beta - \gamma}{2} \tag{1}$$

even without the assumption that $IE = IF$. Then we will show that the statement of the problem is a consequence of this fact.

Denote the circumcenter by O. On the circumcircle, let P be the point opposite to A, and let the angle bisector AI intersect the circle again at M. Since $AK = AP = 2R$, triangle AKP is isosceles. It is known that $\angle BAD = \angle CAO$, hence $\angle DAI = \angle BAI - \angle BAD = \angle CAI - \angle CAO = \angle OAI$, and AM is the bisector line in triangle AKP. Therefore, points K and P are symmetrical about AM, and $\angle AMK = \angle AMP = 90°$. Thus, M is the midpoint of KP, and AM is the perpendicular bisector of KP.



Denote the perpendicular feet of incenter I on lines BC, AC, and AD by A_1, B_1, and T, respectively. Quadrilateral DA_1IT is a rectangle, hence $TD = IA_1 = IB_1$.

Due to the right angles at T and B_1, quadrilateral AB_1IT is cyclic. Hence $\angle B_1TI = \angle B_1AI = \angle CAM = \angle BAM = \angle BPM$ and $\angle IB_1T = \angle IAT = \angle MAK = \angle MAP = \angle MBP$. Therefore, triangles B_1TI and BPM are similar and $\dfrac{IT}{IB_1} = \dfrac{MP}{MB}$.

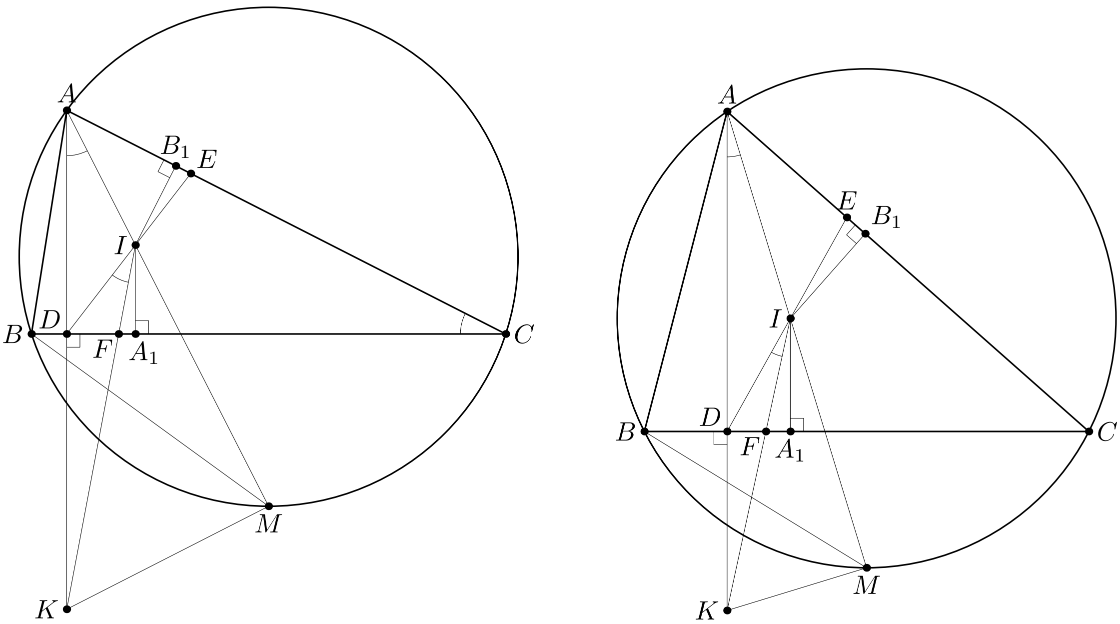
It is well-known that $MB = MC = MI$. Then right triangles ITD and KMI are also similar, because $\dfrac{IT}{TD}=\dfrac{IT}{IB_1}=\dfrac{MP}{MB}=\dfrac{KM}{MI}$. Hence, $\angle KIM=\angle IDT=\angle IDA$, and

$$\angle KID=\angle MID-\angle KIM=(\angle IAD+\angle IDA)-\angle IDA=\angle IAD.$$

Finally, from the right triangle ADB we can compute

$$\angle KID=\angle IAD=\angle IAB-\angle DAB=\frac{\alpha}{2}-(90^\circ-\beta)=\frac{\alpha}{2}-\frac{\alpha+\beta+\gamma}{2}+\beta=\frac{\beta-\gamma}{2}.$$

Now let us turn to the statement and suppose that $IE=IF$. Since $IA_1=IB_1$, the right triangles IEB_1 and IFA_1 are congruent and $\angle IEB_1=\angle IFA_1$. Since $\beta>\gamma$, A_1 lies in the interior of segment CD and F lies in the interior of A_1D. Hence, $\angle IFC$ is acute. Then two cases are possible depending on the order of points A, C, B_1 and E.



If point E lies between C and B_1 then $\angle IFC=\angle IEA$, hence quadrilateral $CEIF$ is cyclic and $\angle FCE=180^\circ-\angle EIF=\angle KID$. By (1), in this case we obtain $\angle FCE=\gamma=\angle KID=\dfrac{\beta-\gamma}{2}$ and $\beta=3\gamma$.

Otherwise, if point E lies between A and B_1, quadrilateral $CEIF$ is a deltoid such that $\angle IEC=\angle IFC<90^\circ$. Then we have $\angle FCE>180^\circ-\angle EIF=\angle KID$. Therefore, $\angle FCE=\gamma>\angle KID=\dfrac{\beta-\gamma}{2}$ and $\beta<3\gamma$.

Comment 1. In the case when quadrilateral $CEIF$ is a deltoid, one can prove the desired inequality without using (1). Actually, from $\angle IEC=\angle IFC<90^\circ$ it follows that $\angle ADI=90^\circ-\angle EDC<\angle AED-\angle EDC=\gamma$. Since the incircle lies inside triangle ABC, we have $AD>2r$ (here r is the inradius), which implies $DT<TA$ and $DI<AI$; hence $\dfrac{\beta-\gamma}{2}=\angle IAD<\angle ADI<\gamma$.

Solution 2. We give a different proof for (1). Then the solution can be finished in the same way as above.

Define points M and P again; it can be proved in the same way that AM is the perpendicular bisector of KP. Let J be the center of the excircle touching side BC. It is well-known that points B, C, I, J lie on a circle with center M; denote this circle by ω_1.

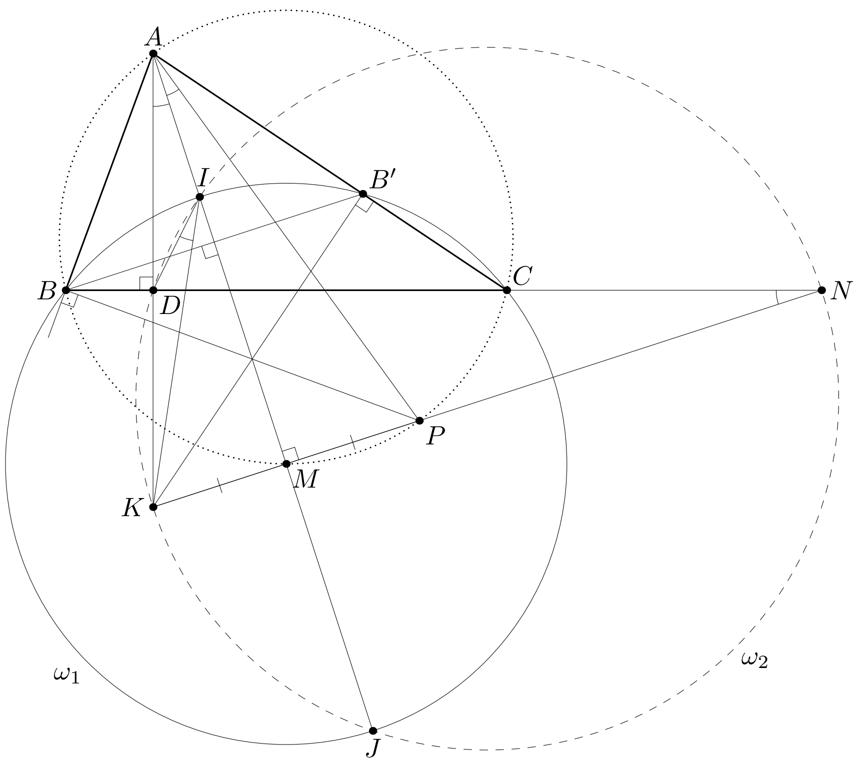
Let B' be the reflection of point B about the angle bisector AM. By the symmetry, B' is the second intersection point of circle ω_1 and line AC. Triangles PBA and $KB'A$ are symmetrical with respect to line AM, therefore $\angle KB'A = \angle PBA = 90°$. By the right angles at D and B', points K, D, B', C are concyclic and

$$AD \cdot AK = AB' \cdot AC.$$

From the cyclic quadrilateral $IJCB'$ we obtain $AB' \cdot AC = AI \cdot AJ$ as well, therefore

$$AD \cdot AK = AB' \cdot AC = AI \cdot AJ$$

and points I, J, K, D are also concyclic. Denote circle $IDKJ$ by ω_2.



Let N be the point on circle ω_2 which is opposite to K. Since $\angle NDK = 90° = \angle CDK$, point N lies on line BC. Point M, being the center of circle ω_1, is the midpoint of segment IJ, and KM is perpendicular to IJ. Therefore, line KM is the perpendicular bisector of IJ and hence it passes through N.

From the cyclic quadrilateral $IDKN$ we obtain

$$\angle KID = \angle KND = 90° - \angle DKN = 90° - \angle AKM = \angle MAK = \frac{\beta - \gamma}{2}.$$

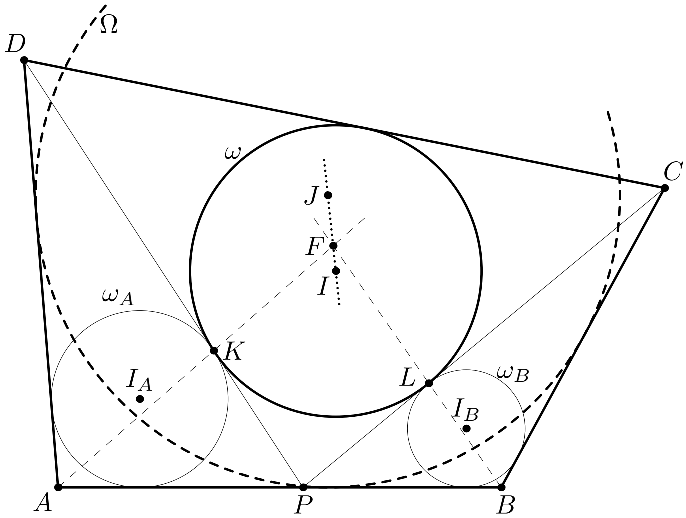
Comment 2. The main difficulty in the solution is finding (1). If someone can guess this fact, he or she can compute it in a relatively short way.

One possible way is finding and applying the relation $AI^2 = 2R(h_a - 2r)$, where $h_a = AD$ is the length of the altitude. Using this fact, one can see that triangles AKI and AID' are similar (here D' is the point symmetrical to D about T). Hence, $\angle MIK = \angle DD'I = \angle IDD'$. The proof can be finished as in Solution 1.

G8. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

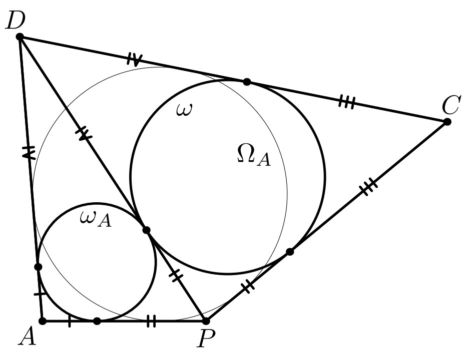
(Poland)

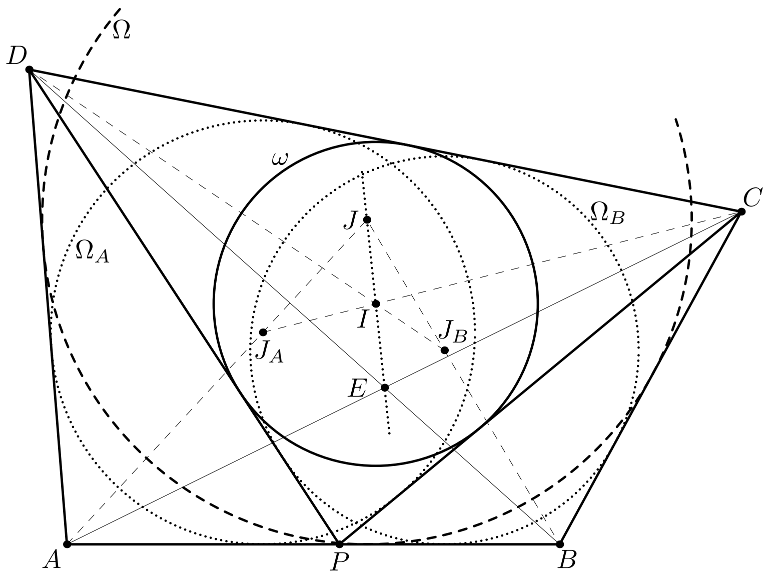
Solution. Let Ω be the circle tangent to segment AB and to rays AD and BC; let J be its center. We prove that points E and F lie on line IJ.



Denote the incircles of triangles ADP and BCP by ω_A and ω_B. Let h_1 be the homothety with a negative scale taking ω to Ω. Consider this homothety as the composition of two homotheties: one taking ω to ω_A (with a negative scale and center K), and another one taking ω_A to Ω (with a positive scale and center A). It is known that in such a case the three centers of homothety are collinear (this theorem is also referred to as the theorem on the three similitude centers). Hence, the center of h_1 lies on line AK. Analogously, it also lies on BL, so this center is F. Hence, F lies on the line of centers of ω and Ω, i. e. on IJ (if $I = J$, then $F = I$ as well, and the claim is obvious).

Consider quadrilateral $APCD$ and mark the equal segments of tangents to ω and ω_A (see the figure below to the left). Since circles ω and ω_A have a common point of tangency with PD, one can easily see that $AD + PC = AP + CD$. So, quadrilateral $APCD$ is circumscribed; analogously, circumscribed is also quadrilateral $BCDP$. Let Ω_A and Ω_B respectively be their incircles.





Consider the homothety h_2 with a positive scale taking ω to Ω. Consider h_2 as the composition of two homotheties: taking ω to Ω_A (with a positive scale and center C), and taking Ω_A to Ω (with a positive scale and center A), respectively. So the center of h_2 lies on line AC. By analogous reasons, it lies also on BD, hence this center is E. Thus, E also lies on the line of centers IJ, and the claim is proved.

Comment. In both main steps of the solution, there can be several different reasonings for the same claims. For instance, one can mostly use Desargues' theorem instead of the three homotheties theorem. Namely, if I_A and I_B are the centers of ω_A and ω_B, then lines I_AI_B, KL and AB are concurrent (by the theorem on three similitude centers applied to ω, ω_A and ω_B). Then Desargues' theorem, applied to triangles AI_AK and BI_BL, yields that the points $J = AI_A \cap BI_B$, $I = I_AK \cap I_BL$ and $F = AK \cap BL$ are collinear.

For the second step, let J_A and J_B be the centers of Ω_A and Ω_B. Then lines J_AJ_B, AB and CD are concurrent, since they appear to be the two common tangents and the line of centers of Ω_A and Ω_B. Applying Desargues' theorem to triangles AJ_AC and BJ_BD, we obtain that the points $J = AJ_A \cap BJ_B$, $I = CJ_A \cap DJ_B$ and $E = AC \cap BD$ are collinear.

Number Theory

N1. Find all pairs (k, n) of positive integers for which $7^k - 3^n$ divides $k^4 + n^2$.

(Austria)

Answer. $(2,4)$.

Solution. Suppose that a pair (k, n) satisfies the condition of the problem. Since $7^k - 3^n$ is even, $k^4 + n^2$ is also even, hence k and n have the same parity. If k and n are odd, then $k^4 + n^2 \equiv 1 + 1 = 2 \pmod 4$, while $7^k - 3^n \equiv 7 - 3 \equiv 0 \pmod 4$, so $k^4 + n^2$ cannot be divisible by $7^k - 3^n$. Hence, both k and n must be even.

Write $k = 2a$, $n = 2b$. Then $7^k - 3^n = 7^{2a} - 3^{2b} = \dfrac{7^a - 3^b}{2} \cdot 2(7^a + 3^b)$, and both factors are integers. So $2(7^a + 3^b) \mid 7^k - 3^n$ and $7^k - 3^n \mid k^4 + n^2 = 2(8a^4 + 2b^2)$, hence

$$7^a + 3^b \le 8a^4 + 2b^2. \tag{1}$$

We prove by induction that $8a^4 < 7^a$ for $a \ge 4$, $2b^2 < 3^b$ for $b \ge 1$ and $2b^2 + 9 \le 3^b$ for $b \ge 3$. In the initial cases $a = 4$, $b = 1$, $b = 2$ and $b = 3$ we have $8 \cdot 4^4 = 2048 < 7^4 = 2401$, $2 < 3$, $2 \cdot 2^2 = 8 < 3^2 = 9$ and $2 \cdot 3^2 + 9 = 3^3 = 27$, respectively.

If $8a^4 < 7^a$ $(a \ge 4)$ and $2b^2 + 9 \le 3^b$ $(b \ge 3)$, then

$$8(a+1)^4 = 8a^4 \left(\frac{a+1}{a}\right)^4 < 7^a \left(\frac{5}{4}\right)^4 = 7^a \frac{625}{256} < 7^{a+1} \qquad \text{and}$$
$$2(b+1)^2 + 9 < (2b^2 + 9)\left(\frac{b+1}{b}\right)^2 \le 3^b \left(\frac{4}{3}\right)^2 = 3^b \frac{16}{9} < 3^{b+1},$$

as desired.

For $a \ge 4$ we obtain $7^a + 3^b > 8a^4 + 2b^2$ and inequality (1) cannot hold. Hence $a \le 3$, and three cases are possible.

Case 1: $a = 1$. Then $k = 2$ and $8 + 2b^2 \ge 7 + 3^b$, thus $2b^2 + 1 \ge 3^b$. This is possible only if $b \le 2$. If $b = 1$ then $n = 2$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 2^2}{7^2 - 3^2} = \dfrac{1}{2}$, which is not an integer. If $b = 2$ then $n = 4$ and $\dfrac{k^4 + n^2}{7^k - 3^n} = \dfrac{2^4 + 4^2}{7^2 - 3^4} = -1$, so $(k, n) = (2, 4)$ is a solution.

Case 2: $a = 2$. Then $k = 4$ and $k^4 + n^2 = 256 + 4b^2 \ge |7^4 - 3^n| = |49 - 3^b| \cdot (49 + 3^b)$. The smallest value of the first factor is 22, attained at $b = 3$, so $128 + 2b^2 \ge 11(49 + 3^b)$, which is impossible since $3^b > 2b^2$.

Case 3: $a = 3$. Then $k = 6$ and $k^4 + n^2 = 1296 + 4b^2 \ge |7^6 - 3^n| = |343 - 3^b| \cdot (343 + 3^b)$. Analogously, $|343 - 3^b| \ge 100$ and we have $324 + b^2 \ge 25(343 + 3^b)$, which is impossible again.

We find that there exists a unique solution $(k, n) = (2, 4)$.

N2. Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

(Canada)

Solution. Let the prime factorization of b be $b = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $p_1, \dots, p_s$ are distinct primes. Our goal is to show that all exponents α_i are divisible by n, then we can set $A = p_1^{\alpha_1/n} \dots p_s^{\alpha_s/n}$.

Apply the condition for $k = b^2$. The number $b - a_k^n$ is divisible by b^2 and hence, for each $1 \leq i \leq s$, it is divisible by $p_i^{2\alpha_i} > p_i^{\alpha_i}$ as well. Therefore

$$a_k^n \equiv b \equiv 0 \pmod{p_i^{\alpha_i}}$$

and

$$a_k^n \equiv b \not\equiv 0 \pmod{p_i^{\alpha_i+1}},$$

which implies that the largest power of p_i dividing a_k^n is $p_i^{\alpha_i}$. Since a_k^n is a complete nth power, this implies that α_i is divisible by n.

Comment. If $n = 8$ and $b = 16$, then for each *prime* p there exists an integer a_p such that $b - a_p^n$ is divisible by p. Actually, the congruency $x^8 - 16 \equiv 0 \pmod{p}$ expands as

$$(x^2 - 2)(x^2 + 2)(x^2 - 2x + 2)(x^2 + 2x + 2) \equiv 0 \pmod{p}.$$

Hence, if -1 is a quadratic residue modulo p, then congruency $x^2 + 2x + 2 = (x+1)^2 + 1 \equiv 0$ has a solution. Otherwise, one of congruencies $x^2 \equiv 2$ and $x^2 \equiv -2$ has a solution.

Thus, the solution cannot work using only prime values of k.

N3. Let X be a set of 10 000 integers, none of them is divisible by 47. Prove that there exists a 2007-element subset Y of X such that $a-b+c-d+e$ is not divisible by 47 for any $a,b,c,d,e \in Y$.

(Netherlands)

Solution. Call a set M of integers *good* if $47 \nmid a-b+c-d+e$ for any $a,b,c,d,e \in M$.

Consider the set $J=\{-9,-7,-5,-3,-1,1,3,5,7,9\}$. We claim that J is good. Actually, for any $a,b,c,d,e \in J$ the number $a-b+c-d+e$ is odd and

$$-45=(-9)-9+(-9)-9+(-9) \le a-b+c-d+e \le 9-(-9)+9-(-9)+9=45.$$

But there is no odd number divisible by 47 between -45 and 45.

For any $k=1,\ldots,46$ consider the set

$$A_k=\{x \in X \mid \exists j \in J: \quad kx \equiv j \pmod{47}\}.$$

If A_k is not good, then $47 \mid a-b+c-d+e$ for some $a,b,c,d,e \in A_k$, hence $47 \mid ka-kb+kc-kd+ke$. But set J contains numbers with the same residues modulo 47, so J also is not good. This is a contradiction; therefore each A_k is a good subset of X.

Then it suffices to prove that there exists a number k such that $|A_k| \ge 2007$. Note that each $x \in X$ is contained in exactly 10 sets A_k. Then

$$\sum_{k=1}^{46} |A_k| = 10|X| = 100\,000,$$

hence for some value of k we have

$$|A_k| \ge \frac{100\,000}{46} > 2173 > 2007.$$

This completes the proof.

Comment. For the solution, it is essential to find a good set consisting of 10 different residues. Actually, consider a set X containing almost uniform distribution of the nonzero residues (i. e. each residue occurs 217 or 218 times). Let $Y \subset X$ be a good subset containing 2007 elements. Then the set K of all residues appearing in Y contains not less than 10 residues, and obviously this set is good.

On the other hand, there is no good set K consisting of 11 different residues. The Cauchy–Davenport theorem claims that for any sets A, B of residues modulo a prime p,

$$|A+B| \ge \min\{p, |A|+|B|-1\}.$$

Hence, if $|K| \ge 11$, then $|K+K| \ge 21$, $|K+K+K| \ge 31 > 47-|K+K|$, hence $|K+K+K+(-K)+(-K)|=47$, and $0 \equiv a+c+e-b-d \pmod{47}$ for some $a,b,c,d,e \in K$.

From the same reasoning, one can see that a good set K containing 10 residues should satisfy equalities $|K+K|=19=2|K|-1$ and $|K+K+K|=28=|K+K|+|K|-1$. It can be proved that in this case set K consists of 10 residues forming an arithmetic progression. As an easy consequence, one obtains that set K has the form aJ for some nonzero residue a.

N4. For every integer $k \geq 2$, prove that 2^{3k} divides the number

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} \tag{1}$$

but 2^{3k+1} does not.

(Poland)

Solution. We use the notation $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$ and $(2n)!! = 2 \cdot 4 \cdots (2n) = 2^n n!$ for any positive integer n. Observe that $(2n)! = (2n)!!\,(2n-1)!! = 2^n n!\,(2n-1)!!$.

For any positive integer n we have

$$\binom{4n}{2n} = \frac{(4n)!}{(2n)!^2} = \frac{2^{2n}(2n)!\,(4n-1)!!}{(2n)!^2} = \frac{2^{2n}}{(2n)!}(4n-1)!!,$$
$$\binom{2n}{n} = \frac{1}{(2n)!}\left(\frac{(2n)!}{n!}\right)^2 = \frac{1}{(2n)!}\left(2^n(2n-1)!!\right)^2 = \frac{2^{2n}}{(2n)!}(2n-1)!!^2.$$

Then expression (1) can be rewritten as follows:

$$\binom{2^{k+1}}{2^k} - \binom{2^k}{2^{k-1}} = \frac{2^{2^k}}{(2^k)!}(2^{k+1}-1)!! - \frac{2^{2^k}}{(2^k)!}(2^k-1)!!^2$$
$$= \frac{2^{2^k}(2^k-1)!!}{(2^k)!} \cdot \Big((2^k+1)(2^k+3)\dots(2^k+2^k-1) - (2^k-1)(2^k-3)\dots(2^k-2^k+1)\Big). \tag{2}$$

We compute the exponent of 2 in the prime decomposition of each factor (the first one is a rational number but not necessarily an integer; it is not important).

First, we show by induction on n that the exponent of 2 in $(2^n)!$ is $2^n - 1$. The base case $n = 1$ is trivial. Suppose that $(2^n)! = 2^{2^n-1}(2d+1)$ for some integer d. Then we have

$$(2^{n+1})! = 2^{2^n}(2^n)!\,(2^{n+1}-1)!! = 2^{2^n}2^{2^n-1} \cdot (2d+1)(2^{n+1}-1)!! = 2^{2^{n+1}-1} \cdot (2q+1)$$

for some integer q. This finishes the induction step.

Hence, the exponent of 2 in the first factor in (2) is $2^k - (2^k - 1) = 1$.

The second factor in (2) can be considered as the value of the polynomial

$$P(x) = (x+1)(x+3)\dots(x+2^k-1) - (x-1)(x-3)\dots(x-2^k+1). \tag{3}$$

at $x = 2^k$. Now we collect some information about $P(x)$.

Observe that $P(-x) = -P(x)$, since $k \geq 2$. So $P(x)$ is an odd function, and it has nonzero coefficients only at odd powers of x. Hence $P(x) = x^3Q(x) + cx$, where $Q(x)$ is a polynomial with integer coefficients.

Compute the exponent of 2 in c. We have

$$c = 2(2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{1}{2i-1} = (2^k-1)!!\sum_{i=1}^{2^{k-1}}\left(\frac{1}{2i-1} + \frac{1}{2^k-2i+1}\right)$$
$$= (2^k-1)!!\sum_{i=1}^{2^{k-1}} \frac{2^k}{(2i-1)(2^k-2i+1)} = 2^k\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} = 2^kS.$$

For any integer $i = 1, \ldots, 2^{k-1}$, denote by a_{2i-1} the residue inverse to $2i-1$ modulo 2^k. Clearly, when $2i-1$ runs through all odd residues, so does a_{2i-1}, hence

$$S = \sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)(2^k-2i+1)} \equiv -\sum_{i=1}^{2^{k-1}} \frac{(2^k-1)!!}{(2i-1)^2} \equiv -\sum_{i=1}^{2^{k-1}} (2^k-1)!!\, a_{2i-1}^2$$
$$= -(2^k-1)!! \sum_{i=1}^{2^{k-1}} (2i-1)^2 = -(2^k-1)!! \frac{2^{k-1}(2^{2k}-1)}{3} \pmod{2^k}.$$

Therefore, the exponent of 2 in S is $k-1$, so $c = 2^k S = 2^{2k-1}(2t+1)$ for some integer t.

Finally we obtain that

$$P(2^k) = 2^{3k}Q(2^k) + 2^k c = 2^{3k}Q(2^k) + 2^{3k-1}(2t+1),$$

which is divisible exactly by 2^{3k-1}. Thus, the exponent of 2 in (2) is $1 + (3k-1) = 3k$.

Comment. The fact that (1) is divisible by 2^{2k} is known; but it does not help in solving this problem.

N5. Find all surjective functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for every $m, n \in \mathbb{N}$ and every prime p, the number $f(m+n)$ is divisible by p if and only if $f(m) + f(n)$ is divisible by p.

($\mathbb{N}$ is the set of all positive integers.)

(Iran)

Answer. $f(n) = n$.

Solution. Suppose that function $f\colon \mathbb{N} \to \mathbb{N}$ satisfies the problem conditions.

Lemma. For any prime p and any $x, y \in \mathbb{N}$, we have $x \equiv y \pmod{p}$ if and only if $f(x) \equiv f(y) \pmod{p}$. Moreover, $p \mid f(x)$ if and only if $p \mid x$.

Proof. Consider an arbitrary prime p. Since f is surjective, there exists some $x \in \mathbb{N}$ such that $p \mid f(x)$. Let

$$d = \min\bigl\{x \in \mathbb{N} : p \mid f(x)\bigr\}.$$

By induction on k, we obtain that $p \mid f(kd)$ for all $k \in \mathbb{N}$. The base is true since $p \mid f(d)$. Moreover, if $p \mid f(kd)$ and $p \mid f(d)$ then, by the problem condition, $p \mid f(kd+d) = f\bigl((k+1)d\bigr)$ as required.

Suppose that there exists an $x \in \mathbb{N}$ such that $d \nmid x$ but $p \mid f(x)$. Let

$$y = \min\bigl\{x \in \mathbb{N} : d \nmid x,\ p \mid f(x)\bigr\}.$$

By the choice of d, we have $y > d$, and $y - d$ is a positive integer not divisible by d. Then $p \nmid f(y-d)$, while $p \mid f(d)$ and $p \mid f\bigl(d + (y-d)\bigr) = f(y)$. This contradicts the problem condition. Hence, there is no such x, and

$$p \mid f(x) \iff d \mid x. \tag{1}$$

Take arbitrary $x, y \in \mathbb{N}$ such that $x \equiv y \pmod{d}$. We have $p \mid f\bigl(x + (2xd - x)\bigr) = f(2xd)$; moreover, since $d \mid 2xd + (y-x) = y + (2xd - x)$, we get $p \mid f\bigl(y + (2xd - x)\bigr)$. Then by the problem condition $p \mid f(x) + f(2xd - x)$, $p \mid f(y) + f(2xd - x)$, and hence $f(x) \equiv -f(2xd - x) \equiv f(y) \pmod{p}$.

On the other hand, assume that $f(x) \equiv f(y) \pmod{p}$. Again we have $p \mid f(x) + f(2xd - x)$ which by our assumption implies that $p \mid f(x) + f(2xd - x) + \bigl(f(y) - f(x)\bigr) = f(y) + f(2xd - x)$. Hence by the problem condition $p \mid f\bigl(y + (2xd - x)\bigr)$. Using (1) we get $0 \equiv y + (2xd - x) \equiv y - x \pmod{d}$.

Thus, we have proved that

$$x \equiv y \pmod{d} \iff f(x) \equiv f(y) \pmod{p}. \tag{2}$$

We are left to show that $p = d$: in this case (1) and (2) provide the desired statements.

The numbers $1, 2, \ldots, d$ have distinct residues modulo d. By (2), numbers $f(1), f(2), \ldots, f(d)$ have distinct residues modulo p; hence there are at least d distinct residues, and $p \geq d$. On the other hand, by the surjectivity of f, there exist $x_1, \ldots, x_p \in \mathbb{N}$ such that $f(x_i) = i$ for any $i = 1, 2, \ldots, p$. By (2), all these x_i's have distinct residues modulo d. For the same reasons, $d \geq p$. Hence, $d = p$. $\square$

Now we prove that $f(n) = n$ by induction on n. If $n = 1$ then, by the Lemma, $p \nmid f(1)$ for any prime p, so $f(1) = 1$, and the base is established. Suppose that $n > 1$ and denote $k = f(n)$. Note that there exists a prime $q \mid n$, so by the Lemma $q \mid k$ and $k > 1$.

If $k > n$ then $k - n + 1 > 1$, and there exists a prime $p \mid k - n + 1$; we have $k \equiv n - 1 \pmod{p}$. By the induction hypothesis we have $f(n-1) = n - 1 \equiv k = f(n) \pmod{p}$. Now, by the Lemma we obtain $n - 1 \equiv n \pmod{p}$ which cannot be true.

Analogously, if $k < n$, then $f(k-1) = k-1$ by induction hypothesis. Moreover, $n-k+1 > 1$, so there exists a prime $p \mid n-k+1$ and $n \equiv k-1 \pmod p$. By the Lemma again, $k = f(n) \equiv f(k-1) = k-1 \pmod p$, which is also false. The only remaining case is $k = n$, so $f(n) = n$.

Finally, the function $f(n) = n$ obviously satisfies the condition.

N6. Let k be a positive integer. Prove that the number $(4k^2-1)^2$ has a positive divisor of the form $8kn-1$ if and only if k is even.

(United Kingdom)

Solution. The statement follows from the following fact.

Lemma. For arbitrary positive integers x and y, the number $4xy-1$ divides $(4x^2-1)^2$ if and only if $x=y$.

Proof. If $x=y$ then $4xy-1=4x^2-1$ obviously divides $(4x^2-1)^2$ so it is sufficient to consider the opposite direction.

Call a pair (x,y) of positive integers *bad* if $4xy-1$ divides $(4x^2-1)^2$ but $x\neq y$. In order to prove that bad pairs do not exist, we present two properties of them which provide an infinite descent.

Property (i). *If (x,y) is a bad pair and $x<y$ then there exists a positive integer $z<x$ such that (x,z) is also bad.*

Let $r=\dfrac{(4x^2-1)^2}{4xy-1}$. Then

$$r=-r\cdot(-1)\equiv -r(4xy-1)=-(4x^2-1)^2\equiv -1 \pmod{4x}$$

and $r=4xz-1$ with some positive integer z. From $x<y$ we obtain that

$$4xz-1=\frac{(4x^2-1)^2}{4xy-1}<4x^2-1$$

and therefore $z<x$. By the construction, the number $4xz-1$ is a divisor of $(4x^2-1)^2$ so (x,z) is a bad pair.

Property (ii). *If (x,y) is a bad pair then (y,x) is also bad.*

Since $1=1^2\equiv(4xy)^2 \pmod{4xy-1}$, we have

$$(4y^2-1)^2\equiv\left(4y^2-(4xy)^2\right)^2=16y^4(4x^2-1)^2\equiv 0 \pmod{4xy-1}.$$

Hence, the number $4xy-1$ divides $(4y^2-1)^2$ as well.

Now suppose that there exists at least one bad pair. Take a bad pair (x,y) such that $2x+y$ attains its smallest possible value. If $x<y$ then property (i) provides a bad pair (x,z) with $z<y$ and thus $2x+z<2x+y$. Otherwise, if $y<x$, property (ii) yields that pair (y,x) is also bad while $2y+x<2x+y$. Both cases contradict the assumption that $2x+y$ is minimal; the Lemma is proved. $\square$

To prove the problem statement, apply the Lemma for $x=k$ and $y=2n$; the number $8kn-1$ divides $(4k^2-1)^2$ if and only if $k=2n$. Hence, there is no such n if k is odd and $n=k/2$ is the only solution if k is even.

Comment. The constant 4 in the Lemma can be replaced with an arbitrary integer greater than 1: if $a>1$ and $axy-1$ divides $(ax^2-1)^2$ then $x=y$.

N7. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \dots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n)$ for all $1 \le i \le k$.

(India)

Solution 1. For arbitrary prime p and positive integer n, denote by $\operatorname{ord}_p(n)$ the exponent of p in n. Thus,

$$\nu_p(n) = \operatorname{ord}_p(n!) = \sum_{i=1}^{n} \operatorname{ord}_p(i).$$

Lemma. Let p be a prime number, q be a positive integer, k and r be positive integers such that $p^k > r$. Then $\nu_p(qp^k + r) = \nu_p(qp^k) + \nu_p(r)$.

Proof. We claim that $\operatorname{ord}_p(qp^k + i) = \operatorname{ord}_p(i)$ for all $0 < i < p^k$. Actually, if $d = \operatorname{ord}_p(i)$ then $d < k$, so $qp^k + i$ is divisible by p^d, but only the first term is divisible by p^{d+1}; hence the sum is not.

Using this claim, we obtain

$$\nu_p(qp^k + r) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=qp^k+1}^{qp^k+r} \operatorname{ord}_p(i) = \sum_{i=1}^{qp^k} \operatorname{ord}_p(i) + \sum_{i=1}^{r} \operatorname{ord}_p(i) = \nu_p(qp^k) + \nu_p(r). \quad \square$$

For any integer a, denote by $\overline{a}$ its residue modulo d. The addition of residues will also be performed modulo d, i.e. $\overline{a} + \overline{b} = \overline{a+b}$. For any positive integer n, let $f(n) = \big(f_1(n), \dots, f_k(n)\big)$, where $f_i(n) = \overline{\nu_{p_i}(n)}$.

Define the sequence $n_1 = 1$, $n_{\ell+1} = (p_1 p_2 \dots p_k)^{n_\ell}$. We claim that

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

for any $\ell_1 < \ell_2 < \ldots < \ell_m$. (The addition of k-tuples is componentwise.) The base case $m = 1$ is trivial.

Suppose that $m > 1$. By the construction of the sequence, $p_i^{n_{\ell_1}}$ divides $n_{\ell_2} + \ldots + n_{\ell_m}$; clearly, $p_i^{n_{\ell_1}} > n_{\ell_1}$ for all $1 \le i \le k$. Therefore the Lemma can be applied for $p = p_i$, $k = r = n_{\ell_1}$ and $qp^k = n_{\ell_2} + \ldots + n_{\ell_m}$ to obtain

$$f_i(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f_i(n_{\ell_1}) + f_i(n_{\ell_2} + \ldots + n_{\ell_m}) \qquad \text{for all } 1 \le i \le k,$$

and hence

$$f(n_{\ell_1} + n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2} + \ldots + n_{\ell_m}) = f(n_{\ell_1}) + f(n_{\ell_2}) + \ldots + f(n_{\ell_m})$$

by the induction hypothesis.

Now consider the values $f(n_1)$, $f(n_2)$, ... There exist finitely many possible values of f. Hence, there exists an infinite sequence of indices $\ell_1 < \ell_2 < \ldots$ such that $f(n_{\ell_1}) = f(n_{\ell_2}) = \ldots$ and thus

$$f(n_{\ell_{m+1}} + n_{\ell_{m+2}} + \ldots + n_{\ell_{m+d}}) = f(n_{\ell_{m+1}}) + \ldots + f(n_{\ell_{m+d}}) = d \cdot f(n_{\ell_1}) = (\overline{0}, \dots, \overline{0})$$

for all m. We have found infinitely many suitable numbers.

Solution 2. We use the same Lemma and definition of the function f.

Let $S=\{f(n): n\in\mathbb{N}\}$. Obviously, set S is finite. For every $s\in S$ choose the minimal n_s such that $f(n_s)=s$. Denote $N=\max\limits_{s\in S} n_s$. Moreover, let g be an integer such that $p_i^g>N$ for each $i=1,2,\ldots,k$. Let $P=(p_1p_2\ldots p_k)^g$.

We claim that

$$\bigl\{f(n)\mid n\in[mP,mP+N]\bigr\}=S \tag{1}$$

for every positive integer m. In particular, since $(\overline{0},\ldots,\overline{0})=f(1)\in S$, it follows that for an arbitrary m there exists $n\in[mP,mP+N]$ such that $f(n)=(\overline{0},\ldots,\overline{0})$. So there are infinitely many suitable numbers.

To prove (1), let $a_i=f_i(mP)$. Consider all numbers of the form $n_{m,s}=mP+n_s$ with $s=(s_1,\ldots,s_k)\in S$ (clearly, all $n_{m,s}$ belong to $[mP,mP+N]$). Since $n_s\le N<p_i^g$ and $p_i^g \mid mP$, we can apply the Lemma for the values $p=p_i$, $r=n_s$, $k=g$, $qp^k=mP$ to obtain

$$f_i(n_{m,s})=f_i(mP)+f_i(n_s)=a_i+s_i;$$

hence for distinct $s,t\in S$ we have $f(n_{m,s})\neq f(n_{m,t})$.

Thus, the function f attains at least $|S|$ distinct values in $[mP,mP+N]$. Since all these values belong to S, f should attain all possible values in $[mP,mP+N]$.

Comment. Both solutions can be extended to prove the following statements.

Claim 1. For any K there exist infinitely many n divisible by K, such that $d \mid \nu_{p_i}(n)$ for each i.

Claim 2. For any $s\in S$, there exist infinitely many $n\in\mathbb{N}$ such that $f(n)=s$.

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SHORT LIST

WITH SOLUTIONS

49^{th} International Mathematical Olympiad

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Algebra

A1. Find all functions $f\colon (0,\infty) \to (0,\infty)$ such that

$$\frac{f(p)^2 + f(q)^2}{f(r^2) + f(s^2)} = \frac{p^2 + q^2}{r^2 + s^2}$$

for all $p, q, r, s > 0$ with $pq = rs$.

Solution. Let f satisfy the given condition. Setting $p = q = r = s = 1$ yields $f(1)^2 = f(1)$ and hence $f(1) = 1$. Now take any $x > 0$ and set $p = x$, $q = 1$, $r = s = \sqrt{x}$ to obtain

$$\frac{f(x)^2 + 1}{2f(x)} = \frac{x^2 + 1}{2x}\,.$$

This recasts into

$$\begin{gathered} xf(x)^2 + x = x^2 f(x) + f(x), \\ \bigl(xf(x) - 1\bigr)\bigl(f(x) - x\bigr) = 0. \end{gathered}$$

And thus,

$$\text{for every } \ x > 0, \quad \text{either } \ f(x) = x \ \text{ or } \ f(x) = \frac{1}{x}\,. \tag{1}$$

Obviously, if

$$f(x) = x \quad \text{for all } x > 0 \qquad \text{or} \qquad f(x) = \frac{1}{x} \quad \text{for all } x > 0 \tag{2}$$

then the condition of the problem is satisfied. We show that actually these two functions are the only solutions.

So let us assume that there exists a function f satisfying the requirement, other than those in (2). Then $f(a) \neq a$ and $f(b) \neq 1/b$ for some $a, b > 0$. By (1), these values must be $f(a) = 1/a$, $f(b) = b$. Applying now the equation with $p = a$, $q = b$, $r = s = \sqrt{ab}$ we obtain $(a^{-2} + b^2)/2f(ab) = (a^2 + b^2)/2ab$; equivalently,

$$f(ab) = \frac{ab(a^{-2} + b^2)}{a^2 + b^2}\,. \tag{3}$$

We know however (see (1)) that $f(ab)$ must be either ab or $1/ab$. If $f(ab) = ab$ then by (3) $a^{-2} + b^2 = a^2 + b^2$, so that $a = 1$. But, as $f(1) = 1$, this contradicts the relation $f(a) \neq a$. Likewise, if $f(ab) = 1/ab$ then (3) gives $a^2b^2(a^{-2} + b^2) = a^2 + b^2$, whence $b = 1$, in contradiction to $f(b) \neq 1/b$. Thus indeed the functions listed in (2) are the only two solutions.

Comment. The equation has as many as four variables with only one constraint $pq = rs$, leaving three degrees of freedom and providing a lot of information. Various substitutions force various useful properties of the function searched. We sketch one more method to reach conclusion (1); certainly there are many others.

Noticing that $f(1) = 1$ and setting, first, $p = q = 1$, $r = \sqrt{x}$, $s = 1/\sqrt{x}$, and then $p = x$, $q = 1/x$, $r = s = 1$, we obtain two relations, holding for every $x > 0$,

$$f(x) + f\left(\frac{1}{x}\right) = x + \frac{1}{x} \qquad \text{and} \qquad f(x)^2 + f\left(\frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2}\,. \tag{4}$$

Squaring the first and subtracting the second gives $2f(x)f(1/x) = 2$. Subtracting this from the second relation of (4) leads to

$$\left(f(x) - f\left(\frac{1}{x}\right)\right)^2 = \left(x - \frac{1}{x}\right)^2 \qquad \text{or} \qquad f(x) - f\left(\frac{1}{x}\right) = \pm\left(x - \frac{1}{x}\right).$$

The last two alternatives combined with the first equation of (4) imply the two alternatives of (1).

A2. (a) Prove the inequality

$$\frac{x^2}{(x-1)^2} + \frac{y^2}{(y-1)^2} + \frac{z^2}{(z-1)^2} \geq 1$$

for real numbers $x, y, z \neq 1$ satisfying the condition $xyz = 1$.

(b) Show that there are infinitely many triples of rational numbers x, y, z for which this inequality turns into equality.

Solution 1. (a) We start with the substitution

$$\frac{x}{x-1} = a, \qquad \frac{y}{y-1} = b, \qquad \frac{z}{z-1} = c, \qquad \text{i.e.,} \qquad x = \frac{a}{a-1}, \qquad y = \frac{b}{b-1}, \qquad z = \frac{c}{c-1}\,.$$

The inequality to be proved reads $a^2 + b^2 + c^2 \geq 1$. The new variables are subject to the constraints $a, b, c \neq 1$ and the following one coming from the condition $xyz = 1$,

$$(a-1)(b-1)(c-1) = abc.$$

This is successively equivalent to

$$\begin{aligned} a+b+c-1 &= ab+bc+ca, \\ 2(a+b+c-1) &= (a+b+c)^2 - (a^2+b^2+c^2), \\ a^2+b^2+c^2-2 &= (a+b+c)^2 - 2(a+b+c), \\ a^2+b^2+c^2-1 &= (a+b+c-1)^2. \end{aligned}$$

Thus indeed $a^2 + b^2 + c^2 \geq 1$, as desired.

(b) From the equation $a^2 + b^2 + c^2 - 1 = (a+b+c-1)^2$ we see that the proposed inequality becomes an equality if and only if both sums $a^2 + b^2 + c^2$ and $a + b + c$ have value 1. The first of them is equal to $(a+b+c)^2 - 2(ab+bc+ca)$. So the instances of equality are described by the system of two equations

$$a+b+c = 1, \qquad\qquad ab+bc+ca = 0$$

plus the constraint $a, b, c \neq 1$. Elimination of c leads to $a^2 + ab + b^2 = a + b$, which we regard as a quadratic equation in b,

$$b^2 + (a-1)b + a(a-1) = 0,$$

with discriminant

$$\Delta = (a-1)^2 - 4a(a-1) = (1-a)(1+3a).$$

We are looking for rational triples (a, b, c); it will suffice to have a rational such that $1-a$ and $1+3a$ are both squares of rational numbers (then Δ will be so too). Set $a = k/m$. We want $m-k$ and $m+3k$ to be squares of integers. This is achieved for instance by taking $m = k^2 - k + 1$ (clearly nonzero); then $m - k = (k-1)^2$, $m + 3k = (k+1)^2$. Note that distinct integers k yield distinct values of $a = k/m$.

And thus, if k is any integer and $m = k^2 - k + 1$, $a = k/m$ then $\Delta = (k^2-1)^2/m^2$ and the quadratic equation has rational roots $b = (m - k \pm k^2 \mp 1)/(2m)$. Choose e.g. the larger root,

$$b = \frac{m-k+k^2-1}{2m} = \frac{m+(m-2)}{2m} = \frac{m-1}{m}.$$

Computing c from $a+b+c=1$ then gives $c=(1-k)/m$. The condition $a,b,c\neq 1$ eliminates only $k=0$ and $k=1$. Thus, as k varies over integers greater than 1, we obtain an infinite family of rational triples (a,b,c)—and coming back to the original variables ($x=a/(a-1)$ etc.)—an infinite family of rational triples (x,y,z) with the needed property. (A short calculation shows that the resulting triples are $x=-k/(k-1)^2$, $y=k-k^2$, $z=(k-1)/k^2$; but the proof was complete without listing them.)

Comment 1. There are many possible variations in handling the equation system $a^2+b^2+c^2=1$, $a+b+c=1$ $(a,b,c\neq 1)$ which of course describes a circle in the (a,b,c)-space (with three points excluded), and finding infinitely many rational points on it.

Also the initial substitution $x=a/(a-1)$ (etc.) can be successfully replaced by other similar substitutions, e.g. $x=1-1/\alpha$ (etc.); or $x=x'-1$ (etc.); or $1-yz=u$ (etc.)—eventually reducing the inequality to $(\cdots)^2\geq 0$, the expression in the parentheses depending on the actual substitution.

Depending on the method chosen, one arrives at various sequences of rational triples (x,y,z) as needed; let us produce just one more such example: $x=(2r-2)/(r+1)^2$, $y=(2r+2)/(r-1)^2$, $z=(r^2-1)/4$ where r can be any rational number different from 1 or -1.

Solution 2 (an outline)**.** (a) Without changing variables, just setting $z=1/xy$ and clearing fractions, the proposed inequality takes the form

$$(xy-1)^2\big(x^2(y-1)^2+y^2(x-1)^2\big)+(x-1)^2(y-1)^2\geq (x-1)^2(y-1)^2(xy-1)^2.$$

With the notation $p=x+y$, $q=xy$ this becomes, after lengthy routine manipulation and a lot of cancellation

$$q^4-6q^3+2pq^2+9q^2-6pq+p^2\geq 0.$$

It is not hard to notice that the expression on the left is just $(q^2-3q+p)^2$, hence nonnegative.

(Without introducing p and q, one is of course led with some more work to the same expression, just written in terms of x and y; but then it is not that easy to see that it is a square.)

(b) To have equality, one needs $q^2-3q+p=0$. Note that x and y are the roots of the quadratic trinomial (in a formal variable t): t^2-pt+q. When $q^2-3q+p=0$, the discriminant equals

$$\delta=p^2-4q=(3q-q^2)^2-4q=q(q-1)^2(q-4).$$

Now it suffices to have both q and $q-4$ squares of rational numbers (then $p=3q-q^2$ and $\sqrt{\delta}$ are also rational, and so are the roots of the trinomial). On setting $q=(n/m)^2=4+(l/m)^2$ the requirement becomes $4m^2+l^2=n^2$ (with l,m,n being integers). This is just the Pythagorean equation, known to have infinitely many integer solutions.

Comment 2. Part (a) alone might also be considered as a possible contest problem (in the category of easy problems).

A3. Let $S \subseteq \mathbb{R}$ be a set of real numbers. We say that a pair (f, g) of functions from S into S is a *Spanish Couple* on S, if they satisfy the following conditions:

(i) Both functions are strictly increasing, i.e. $f(x) < f(y)$ and $g(x) < g(y)$ for all $x, y \in S$ with $x < y$;

(ii) The inequality $f(g(g(x))) < g(f(x))$ holds for all $x \in S$.

Decide whether there exists a Spanish Couple

(a) on the set $S = \mathbb{N}$ of positive integers;

(b) on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Solution. We show that the answer is NO for part (a), and YES for part (b).

(a) Throughout the solution, we will use the notation $g_k(x) = \overbrace{g(g(\ldots g}^{k}(x)\ldots))$, including $g_0(x) = x$ as well.

Suppose that there exists a Spanish Couple (f, g) on the set $\mathbb{N}$. From property (i) we have $f(x) \geq x$ and $g(x) \geq x$ for all $x \in \mathbb{N}$.

We claim that $g_k(x) \leq f(x)$ for all $k \geq 0$ and all positive integers x. The proof is done by induction on k. We already have the base case $k = 0$ since $x \leq f(x)$. For the induction step from k to $k + 1$, apply the induction hypothesis on $g_2(x)$ instead of x, then apply (ii):

$$g(g_{k+1}(x)) = g_k\big(g_2(x)\big) \leq f\big(g_2(x)\big) < g(f(x)).$$

Since g is increasing, it follows that $g_{k+1}(x) < f(x)$. The claim is proven.

If $g(x) = x$ for all $x \in \mathbb{N}$ then $f(g(g(x))) = f(x) = g(f(x))$, and we have a contradiction with (ii). Therefore one can choose an $x_0 \in S$ for which $x_0 < g(x_0)$. Now consider the sequence $x_0, x_1, \ldots$ where $x_k = g_k(x_0)$. The sequence is increasing. Indeed, we have $x_0 < g(x_0) = x_1$, and $x_k < x_{k+1}$ implies $x_{k+1} = g(x_k) < g(x_{k+1}) = x_{k+2}$.

Hence, we obtain a strictly increasing sequence $x_0 < x_1 < \ldots$ of positive integers which on the other hand has an upper bound, namely $f(x_0)$. This cannot happen in the set $\mathbb{N}$ of positive integers, thus no Spanish Couple exists on $\mathbb{N}$.

(b) We present a Spanish Couple on the set $S = \{a - 1/b : \ a, b \in \mathbb{N}\}$.

Let

$$\begin{aligned} f(a - 1/b) &= a + 1 - 1/b, \\ g(a - 1/b) &= a - 1/(b + 3^a). \end{aligned}$$

These functions are clearly increasing. Condition (ii) holds, since

$$f(g(g(a - 1/b))) = (a + 1) - 1/(b + 2 \cdot 3^a) < (a + 1) - 1/(b + 3^{a+1}) = g(f(a - 1/b)).$$

Comment. Another example of a Spanish couple is $f(a - 1/b) = 3a - 1/b$, $g(a - 1/b) = a - 1/(a+b)$. More generally, postulating $f(a - 1/b) = h(a) - 1/b$, $g(a - 1/b) = a - 1/G(a, b)$ with h increasing and G increasing in both variables, we get that $f \circ g \circ g < g \circ f$ holds if $G\big(a, G(a, b)\big) < G\big(h(a), b\big)$. A search just among linear functions $h(a) = Ca$, $G(a, b) = Aa + Bb$ results in finding that any integers $A > 0$, $C > 2$ and $B = 1$ produce a Spanish couple (in the example above, $A = 1$, $C = 3$). The proposer's example results from taking $h(a) = a + 1$, $G(a, b) = 3^a + b$.

A4. For an integer m, denote by $t(m)$ the unique number in $\{1,2,3\}$ such that $m+t(m)$ is a multiple of 3. A function $f\colon \mathbb{Z}\to\mathbb{Z}$ satisfies $f(-1)=0$, $f(0)=1$, $f(1)=-1$ and

$$f(2^n+m)=f(2^n-t(m))-f(m) \quad \text{for all integers } m,n\geq 0 \text{ with } 2^n>m.$$

Prove that $f(3p)\geq 0$ holds for all integers $p\geq 0$.

Solution. The given conditions determine f uniquely on the positive integers. The signs of $f(1),f(2),\ldots$ seem to change quite erratically. However values of the form $f(2^n-t(m))$ are sufficient to compute directly any functional value. Indeed, let $n>0$ have base 2 representation $n=2^{a_0}+2^{a_1}+\cdots+2^{a_k}$, $a_0>a_1>\cdots>a_k\geq 0$, and let $n_j=2^{a_j}+2^{a_{j-1}}+\cdots+2^{a_k}$, $j=0,\ldots,k$. Repeated applications of the recurrence show that $f(n)$ is an alternating sum of the quantities $f(2^{a_j}-t(n_{j+1}))$ plus $(-1)^{k+1}$. (The exact formula is not needed for our proof.)

So we focus attention on the values $f(2^n-1)$, $f(2^n-2)$ and $f(2^n-3)$. Six cases arise; more specifically,

$$t(2^{2k}-3)=2,\ t(2^{2k}-2)=1,\ t(2^{2k}-1)=3,\ t(2^{2k+1}-3)=1,\ t(2^{2k+1}-2)=3,\ t(2^{2k+1}-1)=2.$$

Claim. For all integers $k\geq 0$ the following equalities hold:

$$\begin{array}{lll} f(2^{2k+1}-3)=0, & f(2^{2k+1}-2)=3^k, & f(2^{2k+1}-1)=-3^k, \\ f(2^{2k+2}-3)=-3^k, & f(2^{2k+2}-2)=-3^k, & f(2^{2k+2}-1)=2\cdot 3^k. \end{array}$$

Proof. By induction on k. The base $k=0$ comes down to checking that $f(2)=-1$ and $f(3)=2$; the given values $f(-1)=0$, $f(0)=1$, $f(1)=-1$ are also needed. Suppose the claim holds for $k-1$. For $f(2^{2k+1}-t(m))$, the recurrence formula and the induction hypothesis yield

$$\begin{aligned} f(2^{2k+1}-3)&=f(2^{2k}+(2^{2k}-3))=f(2^{2k}-2)-f(2^{2k}-3)=-3^{k-1}+3^{k-1}=0,\\ f(2^{2k+1}-2)&=f(2^{2k}+(2^{2k}-2))=f(2^{2k}-1)-f(2^{2k}-2)=2\cdot 3^{k-1}+3^{k-1}=3^k,\\ f(2^{2k+1}-1)&=f(2^{2k}+(2^{2k}-1))=f(2^{2k}-3)-f(2^{2k}-1)=-3^{k-1}-2\cdot 3^{k-1}=-3^k. \end{aligned}$$

For $f(2^{2k+2}-t(m))$ we use the three equalities just established:

$$\begin{aligned} f(2^{2k+2}-3)&=f(2^{2k+1}+(2^{2k+1}-3))=f(2^{2k+1}-1)-f(2^{2k+1}-3)=-3^k-0=-3^k,\\ f(2^{2k+2}-2)&=f(2^{2k+1}+(2^{2k+1}-2))=f(2^{2k+1}-3)-f(2^{2k}-2)=0-3^k=-3^k,\\ f(2^{2k+2}-1)&=f(2^{2k+1}+(2^{2k+1}-1))=f(2^{2k+1}-2)-f(2^{2k+1}-1)=3^k+3^k=2\cdot 3^k. \end{aligned}$$

The claim follows.

A closer look at the six cases shows that $f(2^n-t(m))\geq 3^{(n-1)/2}$ if $2^n-t(m)$ is divisible by 3, and $f(2^n-t(m))\leq 0$ otherwise. On the other hand, note that $2^n-t(m)$ is divisible by 3 if and only if 2^n+m is. Therefore, for all nonnegative integers m and n,

(i) $f(2^n-t(m))\geq 3^{(n-1)/2}$ if 2^n+m is divisible by 3;

(ii) $f(2^n-t(m))\leq 0$ if 2^n+m is not divisible by 3.

One more (direct) consequence of the claim is that $|f(2^n-t(m))|\leq \frac{2}{3}\cdot 3^{n/2}$ for all $m,n\geq 0$.

The last inequality enables us to find an upper bound for $|f(m)|$ for m less than a given power of 2. We prove by induction on n that $|f(m)|\leq 3^{n/2}$ holds true for all integers $m,n\geq 0$ with $2^n>m$.

The base $n = 0$ is clear as $f(0) = 1$. For the inductive step from n to $n+1$, let m and n satisfy $2^{n+1} > m$. If $m < 2^n$, we are done by the inductive hypothesis. If $m \geq 2^n$ then $m = 2^n + k$ where $2^n > k \geq 0$. Now, by $|f(2^n - t(k))| \leq \frac{2}{3} \cdot 3^{n/2}$ and the inductive assumption,

$$|f(m)| = |f(2^n - t(k)) - f(k)| \leq |f(2^n - t(k))| + |f(k)| \leq \frac{2}{3} \cdot 3^{n/2} + 3^{n/2} < 3^{(n+1)/2}.$$

The induction is complete.

We proceed to prove that $f(3p) \geq 0$ for all integers $p \geq 0$. Since $3p$ is not a power of 2, its binary expansion contains at least two summands. Hence one can write $3p = 2^a + 2^b + c$ where $a > b$ and $2^b > c \geq 0$. Applying the recurrence formula twice yields

$$f(3p) = f(2^a + 2^b + c) = f(2^a - t(2^b + c)) - f(2^b - t(c)) + f(c).$$

Since $2^a + 2^b + c$ is divisible by 3, we have $f(2^a - t(2^b + c)) \geq 3^{(a-1)/2}$ by (i). Since $2^b + c$ is not divisible by 3, we have $f(2^b - t(c)) \leq 0$ by (ii). Finally $|f(c)| \leq 3^{b/2}$ as $2^b > c \geq 0$, so that $f(c) \geq -3^{b/2}$. Therefore $f(3p) \geq 3^{(a-1)/2} - 3^{b/2}$ which is nonnegative because $a > b$.

A5. Let a, b, c, d be positive real numbers such that

$$abcd = 1 \quad \text{and} \quad a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}.$$

Prove that

$$a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}.$$

Solution. We show that if $abcd = 1$, the sum $a+b+c+d$ cannot exceed a certain weighted mean of the expressions $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ and $\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

By applying the AM-GM inequality to the numbers $\frac{a}{b}$, $\frac{a}{b}$, $\frac{b}{c}$ and $\frac{a}{d}$, we obtain

$$a = \sqrt[4]{\frac{a^4}{abcd}} = \sqrt[4]{\frac{a}{b}\cdot\frac{a}{b}\cdot\frac{b}{c}\cdot\frac{a}{d}} \le \frac{1}{4}\left(\frac{a}{b}+\frac{a}{b}+\frac{b}{c}+\frac{a}{d}\right).$$

Analogously,

$$b \le \frac{1}{4}\left(\frac{b}{c}+\frac{b}{c}+\frac{c}{d}+\frac{b}{a}\right), \quad c \le \frac{1}{4}\left(\frac{c}{d}+\frac{c}{d}+\frac{d}{a}+\frac{c}{b}\right) \quad \text{and} \quad d \le \frac{1}{4}\left(\frac{d}{a}+\frac{d}{a}+\frac{a}{b}+\frac{d}{c}\right).$$

Summing up these estimates yields

$$a+b+c+d \le \frac{3}{4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)+\frac{1}{4}\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right).$$

In particular, if $a+b+c+d > \frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}$ then $a+b+c+d < \frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}$.

Comment. The estimate in the above solution was obtained by applying the AM-GM inequality to each column of the 4×4 array

$$\begin{array}{cccc} a/b & b/c & c/d & d/a \\ a/b & b/c & c/d & d/a \\ b/c & c/d & d/a & a/b \\ a/d & b/a & c/b & d/c \end{array}$$

and adding up the resulting inequalities. The same table yields a stronger bound: If $a, b, c, d > 0$ and $abcd = 1$ then

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^3\left(\frac{b}{a}+\frac{c}{b}+\frac{d}{c}+\frac{a}{d}\right) \ge (a+b+c+d)^4.$$

It suffices to apply Hölder's inequality to the sequences in the four rows, with weights $1/4$:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)^{1/4}\left(\frac{b}{c}+\frac{c}{d}+\frac{d}{a}+\frac{a}{b}\right)^{1/4}\left(\frac{a}{d}+\frac{b}{a}+\frac{c}{b}+\frac{d}{c}\right)^{1/4}$$
$$\ge \left(\frac{aaba}{bbcd}\right)^{1/4}+\left(\frac{bbcb}{ccda}\right)^{1/4}+\left(\frac{ccdc}{ddab}\right)^{1/4}+\left(\frac{ddad}{aabc}\right)^{1/4} = a+b+c+d.$$

A6. Let $f\colon \mathbb{R} \to \mathbb{N}$ be a function which satisfies

$$f\left(x + \frac{1}{f(y)}\right) = f\left(y + \frac{1}{f(x)}\right) \qquad \text{for all } x, y \in \mathbb{R}. \tag{1}$$

Prove that there is a positive integer which is not a value of f.

Solution. Suppose that the statement is false and $f(\mathbb{R}) = \mathbb{N}$. We prove several properties of the function f in order to reach a contradiction.

To start with, observe that one can assume $f(0) = 1$. Indeed, let $a \in \mathbb{R}$ be such that $f(a) = 1$, and consider the function $g(x) = f(x + a)$. By substituting $x + a$ and $y + a$ for x and y in (1), we have

$$g\left(x + \frac{1}{g(y)}\right) = f\left(x + a + \frac{1}{f(y+a)}\right) = f\left(y + a + \frac{1}{f(x+a)}\right) = g\left(y + \frac{1}{g(x)}\right).$$

So g satisfies the functional equation (1), with the additional property $g(0) = 1$. Also, g and f have the same set of values: $g(\mathbb{R}) = f(\mathbb{R}) = \mathbb{N}$. Henceforth we assume $f(0) = 1$.

Claim 1. For an arbitrary fixed $c \in \mathbb{R}$ we have $\left\{ f\left(c + \frac{1}{n}\right) : n \in \mathbb{N} \right\} = \mathbb{N}$.

Proof. Equation (1) and $f(\mathbb{R}) = \mathbb{N}$ imply

$$f(\mathbb{R}) = \left\{ f\left(x + \frac{1}{f(c)}\right) : x \in \mathbb{R} \right\} = \left\{ f\left(c + \frac{1}{f(x)}\right) : x \in \mathbb{R} \right\} \subset \left\{ f\left(c + \frac{1}{n}\right) : n \in \mathbb{N} \right\} \subset f(\mathbb{R}).$$

The claim follows.

We will use Claim 1 in the special cases $c = 0$ and $c = 1/3$:

$$\left\{ f\left(\frac{1}{n}\right) : n \in \mathbb{N} \right\} = \left\{ f\left(\frac{1}{3} + \frac{1}{n}\right) : n \in \mathbb{N} \right\} = \mathbb{N}. \tag{2}$$

Claim 2. If $f(u) = f(v)$ for some $u, v \in \mathbb{R}$ then $f(u+q) = f(v+q)$ for all nonnegative rational q. Furthermore, if $f(q) = 1$ for some nonnegative rational q then $f(kq) = 1$ for all $k \in \mathbb{N}$.

Proof. For all $x \in \mathbb{R}$ we have by (1)

$$f\left(u + \frac{1}{f(x)}\right) = f\left(x + \frac{1}{f(u)}\right) = f\left(x + \frac{1}{f(v)}\right) = f\left(v + \frac{1}{f(x)}\right).$$

Since $f(x)$ attains all positive integer values, this yields $f(u + 1/n) = f(v + 1/n)$ for all $n \in \mathbb{N}$. Let $q = k/n$ be a positive rational number. Then k repetitions of the last step yield

$$f(u+q) = f\left(u + \frac{k}{n}\right) = f\left(v + \frac{k}{n}\right) = f(v+q).$$

Now let $f(q) = 1$ for some nonnegative rational q, and let $k \in \mathbb{N}$. As $f(0) = 1$, the previous conclusion yields successively $f(q) = f(2q), f(2q) = f(3q), \ldots, f\left((k-1)q\right) = f(kq)$, as needed.

Claim 3. The equality $f(q) = f(q+1)$ holds for all nonnegative rational q.

Proof. Let m be a positive integer such that $f(1/m) = 1$. Such an m exists by (2). Applying the second statement of Claim 2 with $q = 1/m$ and $k = m$ yields $f(1) = 1$.

Given that $f(0) = f(1) = 1$, the first statement of Claim 2 implies $f(q) = f(q+1)$ for all nonnegative rational q.

Claim 4. The equality $f\left(\dfrac{1}{n}\right) = n$ holds for every $n \in \mathbb{N}$.

Proof. For a nonnegative rational q we set $x = q$, $y = 0$ in (1) and use Claim 3 to obtain

$$f\left(\frac{1}{f(q)}\right) = f\left(q + \frac{1}{f(0)}\right) = f(q+1) = f(q).$$

By (2), for each $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f(1/k) = n$. Applying the last equation with $q = 1/k$, we have

$$n = f\left(\frac{1}{k}\right) = f\left(\frac{1}{f(1/k)}\right) = f\left(\frac{1}{n}\right).$$

Now we are ready to obtain a contradiction. Let $n \in \mathbb{N}$ be such that $f(1/3 + 1/n) = 1$. Such an n exists by (2). Let $1/3 + 1/n = s/t$, where $s, t \in \mathbb{N}$ are coprime. Observe that $t > 1$ as $1/3 + 1/n$ is not an integer. Choose $k, l \in \mathbb{N}$ so that that $ks - lt = 1$.

Because $f(0) = f(s/t) = 1$, Claim 2 implies $f(ks/t) = 1$. Now $f(ks/t) = f(1/t + l)$; on the other hand $f(1/t + l) = f(1/t)$ by l successive applications of Claim 3. Finally, $f(1/t) = t$ by Claim 4, leading to the impossible $t = 1$. The solution is complete.

A7. Prove that for any four positive real numbers a, b, c, d the inequality

$$\frac{(a-b)(a-c)}{a+b+c}+\frac{(b-c)(b-d)}{b+c+d}+\frac{(c-d)(c-a)}{c+d+a}+\frac{(d-a)(d-b)}{d+a+b} \geq 0$$

holds. Determine all cases of equality.

Solution 1. Denote the four terms by

$$A=\frac{(a-b)(a-c)}{a+b+c}, \quad B=\frac{(b-c)(b-d)}{b+c+d}, \quad C=\frac{(c-d)(c-a)}{c+d+a}, \quad D=\frac{(d-a)(d-b)}{d+a+b}.$$

The expression $2A$ splits into two summands as follows,

$$2A=A'+A'' \qquad \text{where} \qquad A'=\frac{(a-c)^2}{a+b+c}, \qquad A''=\frac{(a-c)(a-2b+c)}{a+b+c};$$

this is easily verified. We analogously represent $2B=B'+B''$, $2C=C'+C''$, $2B=D'+D''$ and examine each of the sums $A'+B'+C'+D'$ and $A''+B''+C''+D''$ separately.

Write $s=a+b+c+d$; the denominators become $s-d$, $s-a$, $s-b$, $s-c$. By the Cauchy-Schwarz inequality,

$$\begin{aligned}&\left(\frac{|a-c|}{\sqrt{s-d}}\cdot\sqrt{s-d}+\frac{|b-d|}{\sqrt{s-a}}\cdot\sqrt{s-a}+\frac{|c-a|}{\sqrt{s-b}}\cdot\sqrt{s-b}+\frac{|d-b|}{\sqrt{s-c}}\cdot\sqrt{s-c}\right)^2\\ &\qquad\leq\left(\frac{(a-c)^2}{s-d}+\frac{(b-d)^2}{s-a}+\frac{(c-a)^2}{s-b}+\frac{(d-b)^2}{s-c}\right)\big(4s-s\big)=3s\big(A'+B'+C'+D'\big).\end{aligned}$$

Hence

$$A'+B'+C'+D' \geq \frac{\big(2|a-c|+2|b-d|\big)^2}{3s} \geq \frac{16\cdot|a-c|\cdot|b-d|}{3s}. \tag{1}$$

Next we estimate the absolute value of the other sum. We couple A'' with C'' to obtain

$$\begin{aligned} A''+C'' &= \frac{(a-c)(a+c-2b)}{s-d}+\frac{(c-a)(c+a-2d)}{s-b}\\ &= \frac{(a-c)(a+c-2b)(s-b)+(c-a)(c+a-2d)(s-d)}{(s-d)(s-b)}\\ &= \frac{(a-c)\big(-2b(s-b)-b(a+c)+2d(s-d)+d(a+c)\big)}{s(a+c)+bd}\\ &= \frac{3(a-c)(d-b)(a+c)}{M}, \qquad \text{with} \qquad M=s(a+c)+bd. \end{aligned}$$

Hence by cyclic shift

$$B''+D''=\frac{3(b-d)(a-c)(b+d)}{N}, \qquad \text{with} \qquad N=s(b+d)+ca.$$

Thus

$$A''+B''+C''+D''=3(a-c)(b-d)\left(\frac{b+d}{N}-\frac{a+c}{M}\right)=\frac{3(a-c)(b-d)W}{MN} \tag{2}$$

where

$$W=(b+d)M-(a+c)N=bd(b+d)-ac(a+c). \tag{3}$$

Note that

$$MN > \big(ac(a+c) + bd(b+d)\big)s \geq |W| \cdot s. \tag{4}$$

Now (2) and (4) yield

$$|A'' + B'' + C'' + D''| \leq \frac{3 \cdot |a-c| \cdot |b-d|}{s}\,. \tag{5}$$

Combined with (1) this results in

$$\begin{aligned} 2(A+B+C+D) &= (A'+B'+C'+D') + (A''+B''+C''+D'') \\ &\geq \frac{16 \cdot |a-c| \cdot |b-d|}{3s} - \frac{3 \cdot |a-c| \cdot |b-d|}{s} = \frac{7 \cdot |a-c| \cdot |b-d|}{3(a+b+c+d)} \geq 0. \end{aligned}$$

This is the required inequality. From the last line we see that equality can be achieved only if either $a = c$ or $b = d$. Since we also need equality in (1), this implies that actually $a = c$ and $b = d$ must hold simultaneously, which is obviously also a sufficient condition.

Solution 2. We keep the notations A, B, C, D, s, and also M, N, W from the preceding solution; the definitions of M, N, W and relations (3), (4) in that solution did not depend on the foregoing considerations. Starting from

$$2A = \frac{(a-c)^2 + 3(a+c)(a-c)}{a+b+c} - 2a + 2c,$$

we get

$$\begin{aligned} 2(A+C) &= (a-c)^2 \left(\frac{1}{s-d} + \frac{1}{s-b}\right) + 3(a+c)(a-c)\left(\frac{1}{s-d} - \frac{1}{s-b}\right) \\ &= (a-c)^2\,\frac{2s-b-d}{M} + 3(a+c)(a-c) \cdot \frac{d-b}{M} = \frac{p(a-c)^2 - 3(a+c)(a-c)(b-d)}{M} \end{aligned}$$

where $p = 2s - b - d = s + a + c$. Similarly, writing $q = s + b + d$ we have

$$2(B+D) = \frac{q(b-d)^2 - 3(b+d)(b-d)(c-a)}{N}\,;$$

specific grouping of terms in the numerators has its aim. Note that $pq > 2s^2$. By adding the fractions expressing $2(A+C)$ and $2(B+D)$,

$$2(A+B+C+D) = \frac{p(a-c)^2}{M} + \frac{3(a-c)(b-d)W}{MN} + \frac{q(b-d)^2}{N}$$

with W defined by (3).

Substitution $x = (a-c)/M$, $y = (b-d)/N$ brings the required inequality to the form

$$2(A+B+C+D) = Mpx^2 + 3Wxy + Nqy^2 \geq 0. \tag{6}$$

It will be enough to verify that the discriminant $\Delta = 9W^2 - 4MNpq$ of the quadratic trinomial $Mpt^2 + 3Wt + Nq$ is negative; on setting $t = x/y$ one then gets (6). The first inequality in (4) together with $pq > 2s^2$ imply $4MNpq > 8s^3\big(ac(a+c) + bd(b+d)\big)$. Since

$$(a+c)s^3 > (a+c)^4 \geq 4ac(a+c)^2 \quad \text{and likewise} \quad (b+d)s^3 > 4bd(b+d)^2,$$

the estimate continues as follows,

$$4MNpq > 8\big(4(ac)^2(a+c)^2 + 4(bd)^2(b+d)^2\big) > 32\big(bd(b+d) - ac(a+c)\big)^2 = 32W^2 \geq 9W^2.$$

Thus indeed $\Delta < 0$. The desired inequality (6) hence results. It becomes an equality if and only if $x = y = 0$; equivalently, if and only if $a = c$ and simultaneously $b = d$.

Comment. The two solutions presented above do not differ significantly; large portions overlap. The properties of the number W turn out to be crucial in both approaches. The Cauchy-Schwarz inequality, applied in the first solution, is avoided in the second, which requires no knowledge beyond quadratic trinomials.

The estimates in the proof of $\Delta < 0$ in the second solution seem to be very wasteful. However, they come close to sharp when the terms in one of the pairs (a, c), (b, d) are equal and much bigger than those in the other pair.

In attempts to prove the inequality by just considering the six cases of arrangement of the numbers a, b, c, d on the real line, one soon discovers that the cases which create real trouble are precisely those in which a and c are both greater or both smaller than b and d.

Solution 3.

$$(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= \Big((a-b)(a+b+d)\Big)\Big((a-c)(a+c+d)\Big)(b+c+d) =$$
$$= (a^2+ad-b^2-bd)(a^2+ad-c^2-cd)(b+c+d) =$$
$$= \big(a^4+2a^3d-a^2b^2-a^2bd-a^2c^2-a^2cd+a^2d^2-ab^2d-abd^2-ac^2d-acd^2+b^2c^2+b^2cd+bc^2d+bcd^2\big)(b+c+d) =$$
$$= a^4b+a^4c+a^4d+(b^3c^2+a^2d^3)-a^2c^3+(2a^3d^2-b^3a^2+c^3b^2)+$$
$$+(b^3cd-c^3da-d^3ab)+(2a^3bd+c^3db-d^3ac)+(2a^3cd-b^3da+d^3bc)$$
$$+(-a^2b^2c+3b^2c^2d-2ac^2d^2)+(-2a^2b^2d+2bc^2d^2)+(-a^2bc^2-2a^2c^2d-2ab^2d^2+2b^2cd^2)+$$
$$+(-2a^2bcd-ab^2cd-abc^2d-2abcd^2)$$

Introducing the notation $S_{xyzw} = \sum\limits_{cyc} a^x b^y c^z d^w$, one can write

$$\sum_{cyc}(a-b)(a-c)(a+b+d)(a+c+d)(b+c+d) =$$
$$= S_{4100}+S_{4010}+S_{4001}+2S_{3200}-S_{3020}+2S_{3002}-S_{3110}+2S_{3101}+2S_{3011}-3S_{2120}-6S_{2111} =$$
$$+\left(S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120}\right)+$$
$$+\left(S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111}\right)+$$
$$+\frac{9}{16}\Big(S_{3200}-S_{2210}-S_{2201}+S_{3002}\Big)+\frac{23}{16}\Big(S_{3200}-2S_{3101}+S_{3002}\Big)+\frac{39}{8}\Big(S_{3101}-S_{2111}\Big),$$

where the expressions

$$S_{4100}+S_{4001}+\frac{1}{2}S_{3110}+\frac{1}{2}S_{3011}-3S_{2120} = \sum_{cyc}\left(a^4b+bc^4+\frac{1}{2}a^3bc+\frac{1}{2}abc^3-3a^2bc^2\right),$$

$$S_{4010}-S_{3020}-\frac{3}{2}S_{3110}+\frac{3}{2}S_{3011}+\frac{9}{16}S_{2210}+\frac{9}{16}S_{2201}-\frac{9}{8}S_{2111} = \sum_{cyc}a^2c\left(a-c-\frac{3}{4}b+\frac{3}{4}d\right)^2,$$

$$S_{3200}-S_{2210}-S_{2201}+S_{3002} = \sum_{cyc}b^2(a^3-a^2c-ac^2+c^3) = \sum_{cyc}b^2(a+c)(a-c)^2,$$

$$S_{3200}-2S_{3101}+S_{3002} = \sum_{cyc}a^3(b-d)^2 \qquad \text{and} \qquad S_{3101}-S_{2111} = \frac{1}{3}\sum_{cyc}bd(2a^3+c^3-3a^2c)$$

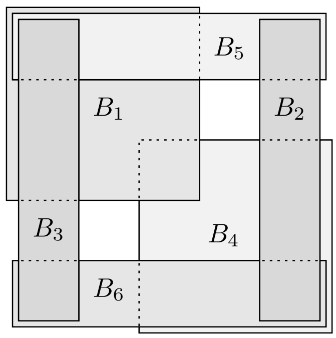
are all nonnegative.

Combinatorics

C1. In the plane we consider rectangles whose sides are parallel to the coordinate axes and have positive length. Such a rectangle will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

Solution. The maximum number of such boxes is 6. One example is shown in the figure.



Now we show that 6 is the maximum. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k and J_k be the projections of B_k onto the x- and y-axis, for $1 \le k \le n$.

If B_i and B_j intersect, with a common point (x, y), then $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$. So the intersections $I_i \cap I_j$ and $J_i \cap J_j$ are nonempty. Conversely, if $x \in I_i \cap I_j$ and $y \in J_i \cap J_j$ for some real numbers x, y, then (x, y) is a common point of B_i and B_j. Putting it around, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

For brevity we call two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_k and B_{k+1} do not intersect for each $k = 1, \ldots, n$. Hence (I_k, I_{k+1}) or (J_k, J_{k+1}) is a pair of disjoint intervals, $1 \le k \le n$. So there are at least n pairs of disjoint intervals among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1); (J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$.

Next, every two nonadjacent boxes intersect, hence their projections on both axes intersect, too. Then the claim below shows that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint, and the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$. Consequently $n \le 3 + 3 = 6$, as stated. Thus we are left with the claim and its justification.

Claim. Let $\Delta_1, \Delta_2, \ldots, \Delta_n$ be intervals on a straight line such that every two nonadjacent intervals intersect. Then Δ_k and Δ_{k+1} are disjoint for at most three values of $k = 1, \ldots, n$.

Proof. Denote $\Delta_k = [a_k, b_k]$, $1 \le k \le n$. Let $\alpha = \max(a_1, \ldots, a_n)$ be the rightmost among the left endpoints of $\Delta_1, \ldots, \Delta_n$, and let $\beta = \min(b_1, \ldots, b_n)$ be the leftmost among their right endpoints. Assume that $\alpha = a_2$ without loss of generality.

If $\alpha \le \beta$ then $a_i \le \alpha \le \beta \le b_i$ for all i. Every Δ_i contains α, and thus no disjoint pair (Δ_i, Δ_{i+1}) exists.

If $\beta < \alpha$ then $\beta = b_i$ for some i such that $a_i < b_i = \beta < \alpha = a_2 < b_2$, hence Δ_2 and Δ_i are disjoint. Now Δ_2 intersects all remaining intervals except possibly Δ_1 and Δ_3, so Δ_2 and Δ_i can be disjoint only if $i = 1$ or $i = 3$. Suppose by symmetry that $i = 3$; then $\beta = b_3$. Since each of the intervals $\Delta_4, \ldots, \Delta_n$ intersects Δ_2, we have $a_i \le \alpha \le b_i$ for $i = 4, \ldots, n$. Therefore $\alpha \in \Delta_4 \cap \ldots \cap \Delta_n$, in particular $\Delta_4 \cap \ldots \cap \Delta_n \neq \emptyset$. Similarly, $\Delta_5, \ldots, \Delta_n, \Delta_1$ all intersect Δ_3, so that $\Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1 \neq \emptyset$ as $\beta \in \Delta_5 \cap \ldots \cap \Delta_n \cap \Delta_1$. This leaves (Δ_1, Δ_2), (Δ_2, Δ_3) and (Δ_3, Δ_4) as the only candidates for disjoint interval pairs, as desired.

Comment. The problem is a two-dimensional version of the original proposal which is included below. The extreme shortage of easy and appropriate submissions forced the Problem Selection Committee to shortlist a simplified variant. The same one-dimensional Claim is used in both versions.

Original proposal. We consider parallelepipeds in three-dimensional space, with edges parallel to the coordinate axes and of positive length. Such a parallelepiped will be called a *box*. Two boxes *intersect* if they have a common point in their interior or on their boundary.

Find the largest n for which there exist n boxes $B_1, \ldots, B_n$ such that B_i and B_j intersect if and only if $i \not\equiv j \pm 1 \pmod{n}$.

The maximum number of such boxes is 9. Suppose that boxes $B_1, \ldots, B_n$ satisfy the condition. Let the closed intervals I_k, J_k and K_k be the projections of box B_k onto the x-, y- and z-axis, respectively, for $1 \le k \le n$. As before, B_i and B_j are disjoint if and only if their projections on at least one coordinate axis are disjoint.

We call again two boxes or intervals adjacent if their indices differ by 1 modulo n, and nonadjacent otherwise.

The adjacent boxes B_i and B_{i+1} do not intersect for each $i = 1, \ldots, n$. Hence at least one of the pairs (I_i, I_{i+1}), (J_i, J_{i+1}) and (K_i, K_{i+1}) is a pair of disjoint intervals. So there are at least n pairs of disjoint intervals among $(I_i, I_{i+1}), (J_i, J_{i+1}), (K_i, K_{i+1})$, $1 \le i \le n$.

Next, every two nonadjacent boxes intersect, hence their projections on the three axes intersect, too. Referring to the Claim in the solution of the two-dimensional version, we cocnclude that at most 3 pairs among $(I_1, I_2), \ldots, (I_{n-1}, I_n), (I_n, I_1)$ are disjoint; the same holds for $(J_1, J_2), \ldots, (J_{n-1}, J_n), (J_n, J_1)$ and $(K_1, K_2), \ldots, (K_{n-1}, K_n), (K_n, K_1)$. Consequently $n \le 3 + 3 + 3 = 9$, as stated.

For $n = 9$, the desired system of boxes exists. Consider the intervals in the following table:

i	I_i	J_i	K_i
1	$[1,4]$	$[1,6]$	$[3,6]$
2	$[5,6]$	$[1,6]$	$[1,6]$
3	$[1,2]$	$[1,6]$	$[1,6]$
4	$[3,6]$	$[1,4]$	$[1,6]$
5	$[1,6]$	$[5,6]$	$[1,6]$
6	$[1,6]$	$[1,2]$	$[1,6]$
7	$[1,6]$	$[3,6]$	$[1,4]$
8	$[1,6]$	$[1,6]$	$[5,6]$
9	$[1,6]$	$[1,6]$	$[1,2]$

We have $I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_4 = \emptyset$, $J_4 \cap J_5 = J_5 \cap J_6 = J_6 \cap J_7 = \emptyset$, and finally $K_7 \cap K_8 = K_8 \cap K_9 = K_9 \cap K_1 = \emptyset$. The intervals in each column intersect in all other cases. It follows that the boxes $B_i = I_i \times J_i \times K_i$, $i = 1, \ldots, 9$, have the stated property.

C2. For every positive integer n determine the number of permutations $(a_1, a_2, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ with the following property:

$$2(a_1 + a_2 + \cdots + a_k) \quad \text{is divisible by } k \quad \text{for } k = 1, 2, \dots, n.$$

Solution. For each n let F_n be the number of permutations of $\{1, 2, \dots, n\}$ with the required property; call them *nice*. For $n = 1, 2, 3$ every permutation is nice, so $F_1 = 1$, $F_2 = 2$, $F_3 = 6$.

Take an $n > 3$ and consider any nice permutation $(a_1, a_2, \dots, a_n)$ of $\{1, 2, \dots, n\}$. Then $n - 1$ must be a divisor of the number

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-1}) &= 2\big((1 + 2 + \cdots + n) - a_n\big) \\ &= n(n+1) - 2a_n = (n+2)(n-1) + (2 - 2a_n). \end{aligned}$$

So $2a_n - 2$ must be divisible by $n - 1$, hence equal to 0 or $n - 1$ or $2n - 2$. This means that

$$a_n = 1 \qquad \text{or} \qquad a_n = \frac{n+1}{2} \qquad \text{or} \qquad a_n = n.$$

Suppose that $a_n = (n+1)/2$. Since the permutation is nice, taking $k = n - 2$ we get that $n - 2$ has to be a divisor of

$$\begin{aligned} 2(a_1 + a_2 + \cdots + a_{n-2}) &= 2\big((1 + 2 + \cdots + n) - a_n - a_{n-1}\big) \\ &= n(n+1) - (n+1) - 2a_{n-1} = (n+2)(n-2) + (3 - 2a_{n-1}). \end{aligned}$$

So $2a_{n-1} - 3$ should be divisible by $n - 2$, hence equal to 0 or $n - 2$ or $2n - 4$. Obviously 0 and $2n - 4$ are excluded because $2a_{n-1} - 3$ is odd. The remaining possibility ($2a_{n-1} - 3 = n - 2$) leads to $a_{n-1} = (n+1)/2 = a_n$, which also cannot hold. This eliminates $(n+1)/2$ as a possible value of a_n. Consequently $a_n = 1$ or $a_n = n$.

If $a_n = n$ then $(a_1, a_2, \dots, a_{n-1})$ is a nice permutation of $\{1, 2, \dots, n-1\}$. There are F_{n-1} such permutations. Attaching n to any one of them at the end creates a nice permutation of $\{1, 2, \dots, n\}$.

If $a_n = 1$ then $(a_1-1, a_2-1, \dots, a_{n-1}-1)$ is a permutation of $\{1, 2, \dots, n-1\}$. It is also nice because the number

$$2\big((a_1-1) + \cdots + (a_k-1)\big) = 2(a_1 + \cdots + a_k) - 2k$$

is divisible by k, for any $k \le n - 1$. And again, any one of the F_{n-1} nice permutations $(b_1, b_2, \dots, b_{n-1})$ of $\{1, 2, \dots, n-1\}$ gives rise to a nice permutation of $\{1, 2, \dots, n\}$ whose last term is 1, namely $(b_1+1, b_2+1, \dots, b_{n-1}+1, 1)$.

The bijective correspondences established in both cases show that there are F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term 1 and also F_{n-1} nice permutations of $\{1, 2, \dots, n\}$ with the last term n. Hence follows the recurrence $F_n = 2F_{n-1}$. With the base value $F_3 = 6$ this gives the outcome formula $F_n = 3 \cdot 2^{n-2}$ for $n \ge 3$.

C3. In the coordinate plane consider the set S of all points with integer coordinates. For a positive integer k, two distinct points $A, B \in S$ will be called *k-friends* if there is a point $C \in S$ such that the area of the triangle ABC is equal to k. A set $T \subset S$ will be called a *k-clique* if every two points in T are k-friends. Find the least positive integer k for which there exists a k-clique with more than 200 elements.

Solution. To begin, let us describe those points $B \in S$ which are k-friends of the point $(0,0)$. By definition, $B = (u, v)$ satisfies this condition if and only if there is a point $C = (x, y) \in S$ such that $\frac{1}{2}|uy - vx| = k$. (This is a well-known formula expressing the area of triangle ABC when A is the origin.)

To say that there exist integers x, y for which $|uy - vx| = 2k$, is equivalent to saying that the greatest common divisor of u and v is also a divisor of $2k$. Summing up, a point $B = (u, v) \in S$ is a k-friend of $(0,0)$ if and only if $\gcd(u, v)$ divides $2k$.

Translation by a vector with integer coordinates does not affect k-friendship; if two points are k-friends, so are their translates. It follows that two points $A, B \in S$, $A = (s, t)$, $B = (u, v)$, are k-friends if and only if the point $(u - s, v - t)$ is a k-friend of $(0,0)$; i.e., if $\gcd(u - s, v - t)|2k$.

Let n be a positive integer which does not divide $2k$. We claim that a k-clique cannot have more than n^2 elements.

Indeed, all points $(x, y) \in S$ can be divided into n^2 classes determined by the remainders that x and y leave in division by n. If a set T has more than n^2 elements, some two points $A, B \in T$, $A = (s, t)$, $B = (u, v)$, necessarily fall into the same class. This means that $n|u - s$ and $n|v - t$. Hence $n|d$ where $d = \gcd(u - s, v - t)$. And since n does not divide $2k$, also d does not divide $2k$. Thus A and B are not k-friends and the set T is not a k-clique.

Now let $M(k)$ be the *least* positive integer which does not divide $2k$. Write $M(k) = m$ for the moment and consider the set T of all points (x, y) with $0 \le x, y < m$. There are m^2 of them. If $A = (s, t)$, $B = (u, v)$ are two distinct points in T then both differences $|u - s|$, $|v - t|$ are integers less than m and at least one of them is positive. By the definition of m, every positive integer less than m divides $2k$. Therefore $u - s$ (if nonzero) divides $2k$, and the same is true of $v - t$. So $2k$ is divisible by $\gcd(u - s, v - t)$, meaning that A, B are k-friends. Thus T is a k-clique.

It follows that the maximum size of a k-clique is $M(k)^2$, with $M(k)$ defined as above. We are looking for the minimum k such that $M(k)^2 > 200$.

By the definition of $M(k)$, $2k$ is divisible by the numbers $1, 2, \ldots, M(k)-1$, but not by $M(k)$ itself. If $M(k)^2 > 200$ then $M(k) \ge 15$. Trying to hit $M(k) = 15$ we get a contradiction immediately ($2k$ would have to be divisible by 3 and 5, but not by 15).

So let us try $M(k) = 16$. Then $2k$ is divisible by the numbers $1, 2, \ldots, 15$, hence also by their least common multiple L, but not by 16. And since L is not a multiple of 16, we infer that $k = L/2$ is the least k with $M(k) = 16$.

Finally, observe that if $M(k) \ge 17$ then $2k$ must be divisible by the least common multiple of $1, 2, \ldots, 16$, which is equal to $2L$. Then $2k \ge 2L$, yielding $k > L/2$.

In conclusion, the least k with the required property is equal to $L/2 = 180180$.

C4. Let n and k be fixed positive integers of the same parity, $k \geq n$. We are given $2n$ lamps numbered 1 through $2n$; each of them can be *on* or *off*. At the beginning all lamps are *off*. We consider sequences of k steps. At each step one of the lamps is switched (from *off* to *on* or from *on* to *off*).

Let N be the number of k-step sequences ending in the state: lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*.

Let M be the number of k-step sequences leading to the same state and not touching lamps $n+1, \ldots, 2n$ at all.

Find the ratio N/M.

Solution. A sequence of k switches ending in the state as described in the problem statement (lamps $1, \ldots, n$ *on*, lamps $n+1, \ldots, 2n$ *off*) will be called an *admissible process*. If, moreover, the process does not touch the lamps $n+1, \ldots, 2n$, it will be called *restricted*. So there are N admissible processes, among which M are restricted.

In every admissible process, restricted or not, each one of the lamps $1, \ldots, n$ goes from *off* to *on*, so it is switched an odd number of times; and each one of the lamps $n+1, \ldots, 2n$ goes from *off* to *off*, so it is switched an even number of times.

Notice that $M > 0$; i.e., restricted admissible processes do exist (it suffices to switch each one of the lamps $1, \ldots, n$ just once and then choose one of them and switch it $k - n$ times, which by hypothesis is an even number).

Consider any restricted admissible process $\mathbf{p}$. Take any lamp ℓ, $1 \leq \ell \leq n$, and suppose that it was switched k_ℓ times. As noticed, k_ℓ must be odd. Select arbitrarily an even number of these k_ℓ switches and replace each of them by the switch of lamp $n+\ell$. This can be done in $2^{k_\ell - 1}$ ways (because a k_ℓ-element set has $2^{k_\ell - 1}$ subsets of even cardinality). Notice that $k_1 + \cdots + k_n = k$.

These actions are independent, in the sense that the action involving lamp ℓ does not affect the action involving any other lamp. So there are $2^{k_1 - 1} \cdot 2^{k_2 - 1} \cdots 2^{k_n - 1} = 2^{k-n}$ ways of combining these actions. In any of these combinations, each one of the lamps $n+1, \ldots, 2n$ gets switched an even number of times and each one of the lamps $1, \ldots, n$ remains switched an odd number of times, so the final state is the same as that resulting from the original process $\mathbf{p}$.

This shows that every restricted admissible process $\mathbf{p}$ can be modified in 2^{k-n} ways, giving rise to 2^{k-n} distinct admissible processes (with all lamps allowed).

Now we show that every admissible process $\mathbf{q}$ can be achieved in that way. Indeed, it is enough to replace every switch of a lamp with a label $\ell > n$ that occurs in $\mathbf{q}$ by the switch of the corresponding lamp $\ell - n$; in the resulting process $\mathbf{p}$ the lamps $n+1, \ldots, 2n$ are not involved.

Switches of each lamp with a label $\ell > n$ had occurred in $\mathbf{q}$ an even number of times. So the performed replacements have affected each lamp with a label $\ell \leq n$ also an even number of times; hence in the overall effect the final state of each lamp has remained the same. This means that the resulting process $\mathbf{p}$ is admissible—and clearly restricted, as the lamps $n+1, \ldots, 2n$ are not involved in it any more.

If we now take process $\mathbf{p}$ and reverse all these replacements, then we obtain process $\mathbf{q}$. These reversed replacements are nothing else than the modifications described in the foregoing paragraphs.

Thus there is a one–to–(2^{k-n}) correspondence between the M restricted admissible processes and the total of N admissible processes. Therefore $N/M = 2^{k-n}$.

C5. Let $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ be a $(k+\ell)$-element set of real numbers contained in the interval $[0,1]$; k and ℓ are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left|\frac{1}{k}\sum_{x_i \in A} x_i - \frac{1}{\ell}\sum_{x_j \in S\setminus A} x_j\right| \le \frac{k+\ell}{2k\ell}.$$

Prove that the number of nice subsets is at least $\dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$.

Solution. For a k-element subset $A \subset S$, let $f(A) = \dfrac{1}{k}\displaystyle\sum_{x_i \in A} x_i - \dfrac{1}{\ell}\sum_{x_j \in S\setminus A} x_j$. Denote $\dfrac{k+\ell}{2k\ell} = d$. By definition a subset A is nice if $|f(A)| \le d$.

To each permutation $(y_1, y_2, \ldots, y_{k+\ell})$ of the set $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ we assign $k+\ell$ subsets of S with k elements each, namely $A_i = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, $i = 1, 2, \ldots, k+\ell$. Indices are taken modulo $k+\ell$ here and henceforth. In other words, if $y_1, y_2, \ldots, y_{k+\ell}$ are arranged around a circle in this order, the sets in question are all possible blocks of k consecutive elements.

Claim. At least two nice sets are assigned to every permutation of S.

Proof. Adjacent sets A_i and A_{i+1} differ only by the elements y_i and y_{i+k}, $i = 1, \ldots, k+\ell$. By the definition of f, and because $y_i, y_{i+k} \in [0,1]$,

$$|f(A_{i+1}) - f(A_i)| = \left|\left(\frac{1}{k} + \frac{1}{\ell}\right)(y_{i+k} - y_i)\right| \le \frac{1}{k} + \frac{1}{\ell} = 2d.$$

Each element $y_i \in S$ belongs to exactly k of the sets $A_1, \ldots, A_{k+\ell}$. Hence in k of the expressions $f(A_1), \ldots, f(A_{k+\ell})$ the coefficient of y_i is $1/k$; in the remaining ℓ expressions, its coefficient is $-1/\ell$. So the contribution of y_i to the sum of all $f(A_i)$ equals $k \cdot 1/k - \ell \cdot 1/\ell = 0$. Since this holds for all i, it follows that $f(A_1) + \cdots + f(A_{k+\ell}) = 0$.

If $f(A_p) = \min f(A_i)$, $f(A_q) = \max f(A_i)$, we obtain in particular $f(A_p) \le 0$, $f(A_q) \ge 0$. Let $p < q$ (the case $p > q$ is analogous; and the claim is true for $p = q$ as $f(A_i) = 0$ for all i).

We are ready to prove that at least two of the sets $A_1, \ldots, A_{k+\ell}$ are nice. The interval $[-d, d]$ has length $2d$, and we saw that adjacent numbers in the circular arrangement $f(A_1), \ldots, f(A_{k+\ell})$ differ by at most $2d$. Suppose that $f(A_p) < -d$ and $f(A_q) > d$. Then one of the numbers $f(A_{p+1}), \ldots, f(A_{q-1})$ lies in $[-d, d]$, and also one of the numbers $f(A_{q+1}), \ldots, f(A_{p-1})$ lies there. Consequently, one of the sets $A_{p+1}, \ldots, A_{q-1}$ is nice, as well as one of the sets $A_{q+1}, \ldots, A_{p-1}$. If $-d \le f(A_p)$ and $f(A_q) \le d$ then A_p and A_q are nice.

Let now $f(A_p) < -d$ and $f(A_q) \le d$. Then $f(A_p) + f(A_q) < 0$, and since $\sum f(A_i) = 0$, there is an $r \ne q$ such that $f(A_r) > 0$. We have $0 < f(A_r) \le f(A_q) \le d$, so the sets $f(A_r)$ and $f(A_q)$ are nice. The only case remaining, $-d \le f(A_p)$ and $d < f(A_q)$, is analogous.

Apply the claim to each of the $(k+\ell)!$ permutations of $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$. This gives at least $2(k+\ell)!$ nice sets, counted with repetitions: each nice set is counted as many times as there are permutations to which it is assigned.

On the other hand, each k-element set $A \subset S$ is assigned to exactly $(k+\ell)\,k!\,\ell!$ permutations. Indeed, such a permutation $(y_1, y_2, \ldots, y_{k+\ell})$ is determined by three independent choices: an index $i \in \{1, 2, \ldots, k+\ell\}$ such that $A = \{y_i, y_{i+1}, \ldots, y_{i+k-1}\}$, a permutation $(y_i, y_{i+1}, \ldots, y_{i+k-1})$ of the set A, and a permutation $(y_{i+k}, y_{i+k+1}, \ldots, y_{i-1})$ of the set $S \setminus A$.

In summary, there are at least $\dfrac{2(k+\ell)!}{(k+\ell)\,k!\,\ell!} = \dfrac{2}{k+\ell}\dbinom{k+\ell}{k}$ nice sets.

C6. For $n \geq 2$, let $S_1, S_2, \ldots, S_{2^n}$ be 2^n subsets of $A = \{1, 2, 3, \ldots, 2^{n+1}\}$ that satisfy the following property: There do not exist indices a and b with $a < b$ and elements $x, y, z \in A$ with $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Prove that at least one of the sets $S_1, S_2, \ldots, S_{2^n}$ contains no more than $4n$ elements.

Solution 1. We prove that there exists a set S_a with at most $3n + 1$ elements.

Given a $k \in \{1, \ldots, n\}$, we say that an element $z \in A$ is *k-good* to a set S_a if $z \in S_a$ and S_a contains two other elements x and y with $x < y < z$ such that $z - y < 2^k$ and $z - x \geq 2^k$. Also, $z \in A$ will be called *good* to S_a if z is k-good to S_a for some $k = 1, \ldots, n$.

We claim that each $z \in A$ can be k-good to at most one set S_a. Indeed, suppose on the contrary that z is k-good simultaneously to S_a and S_b, with $a < b$. Then there exist $y_a \in S_a$, $y_a < z$, and $x_b \in S_b$, $x_b < z$, such that $z - y_a < 2^k$ and $z - x_b \geq 2^k$. On the other hand, since $z \in S_a \cap S_b$, by the condition of the problem there is no element of S_a strictly between x_b and z. Hence $y_a \leq x_b$, implying $z - y_a \geq z - x_b$. However this contradicts $z - y_a < 2^k$ and $z - x_b \geq 2^k$. The claim follows.

As a consequence, a fixed $z \in A$ can be good to at most n of the given sets (no more than one of them for each $k = 1, \ldots, n$).

Furthermore, let $u_1 < u_2 < \cdots < u_m < \cdots < u_p$ be all elements of a fixed set S_a that are not good to S_a. We prove that $u_m - u_1 > 2(u_{m-1} - u_1)$ for all $m \geq 3$.

Indeed, assume that $u_m - u_1 \leq 2(u_{m-1} - u_1)$ holds for some $m \geq 3$. This inequality can be written as $2(u_m - u_{m-1}) \leq u_m - u_1$. Take the unique k such that $2^k \leq u_m - u_1 < 2^{k+1}$. Then $2(u_m - u_{m-1}) \leq u_m - u_1 < 2^{k+1}$ yields $u_m - u_{m-1} < 2^k$. However the elements $z = u_m$, $x = u_1$, $y = u_{m-1}$ of S_a then satisfy $z - y < 2^k$ and $z - x \geq 2^k$, so that $z = u_m$ is k-good to S_a.

Thus each term of the sequence $u_2 - u_1, u_3 - u_1, \ldots, u_p - u_1$ is more than twice the previous one. Hence $u_p - u_1 > 2^{p-1}(u_2 - u_1) \geq 2^{p-1}$. But $u_p \in \{1, 2, 3, \ldots, 2^{n+1}\}$, so that $u_p \leq 2^{n+1}$. This yields $p - 1 \leq n$, i. e. $p \leq n + 1$.

In other words, each set S_a contains at most $n + 1$ elements that are not good to it.

To summarize the conclusions, mark with red all elements in the sets S_a that are good to the respective set, and with blue the ones that are not good. Then the total number of red elements, counting multiplicities, is at most $n \cdot 2^{n+1}$ (each $z \in A$ can be marked red in at most n sets). The total number of blue elements is at most $(n + 1)2^n$ (each set S_a contains at most $n + 1$ blue elements). Therefore the sum of cardinalities of $S_1, S_2, \ldots, S_{2^n}$ does not exceed $(3n + 1)2^n$. By averaging, the smallest set has at most $3n + 1$ elements.

Solution 2. We show that one of the sets S_a has at most $2n + 1$ elements. In the sequel $|\cdot|$ denotes the cardinality of a (finite) set.

Claim. For $n \geq 2$, suppose that k subsets $S_1, \ldots, S_k$ of $\{1, 2, \ldots, 2^n\}$ (not necessarily different) satisfy the condition of the problem. Then

$$\sum_{i=1}^{k} (|S_i| - n) \leq (2n - 1)2^{n-2}.$$

Proof. Observe that if the sets S_i $(1 \leq i \leq k)$ satisfy the condition then so do their arbitrary subsets T_i $(1 \leq i \leq k)$. The condition also holds for the sets $t + S_i = \{t + x \mid x \in S_i\}$ where t is arbitrary.

Note also that a set may occur more than once among $S_1, \ldots, S_k$ only if its cardinality is less than 3, in which case its contribution to the sum $\sum_{i=1}^{k}(|S_i| - n)$ is nonpositive (as $n \geq 2$).

The proof is by induction on n. In the base case $n = 2$ we have subsets S_i of $\{1, 2, 3, 4\}$. Only the ones of cardinality 3 and 4 need to be considered by the remark above; each one of them occurs at most once among $S_1, \dots, S_k$. If $S_i = \{1,2,3,4\}$ for some i then no S_j is a 3-element subset in view of the condition, hence $\sum_{i=1}^{k}(|S_i|-2) \le 2$. By the condition again, it is impossible that $S_i = \{1,3,4\}$ and $S_j = \{2,3,4\}$ for some i, j. So if $|S_i| \le 3$ for all i then at most 3 summands $|S_i| - 2$ are positive, corresponding to 3-element subsets. This implies $\sum_{i=1}^{k}(|S_i|-2) \le 3$, therefore the conclusion is true for $n = 2$.

Suppose that the claim holds for some $n \ge 2$, and let the sets $S_1, \dots, S_k \subseteq \{1, 2, \dots, 2^{n+1}\}$ satisfy the given property. Denote $U_i = S_i \cap \{1, 2, \dots, 2^n\}$, $V_i = S_i \cap \{2^n + 1, \dots, 2^{n+1}\}$. Let

$$I = \{i \mid 1 \le i \le k,\ |U_i| \ne 0\}, \qquad J = \{1, \dots, k\} \setminus I.$$

The sets S_j with $j \in J$ are all contained in $\{2^n + 1, \dots, 2^{n+1}\}$, so the induction hypothesis applies to their translates $-2^n + S_j$ which have the same cardinalities. Consequently, this gives $\sum_{j\in J}(|S_j| - n) \le (2n-1)2^{n-2}$, so that

$$\sum_{j\in J}(|S_j| - (n+1)) \le \sum_{j\in J}(|S_j| - n) \le (2n-1)2^{n-2}. \tag{1}$$

For $i \in I$, denote by v_i the least element of V_i. Observe that if V_a and V_b intersect, with $a < b$, $a, b \in I$, then v_a is their unique common element. Indeed, let $z \in V_a \cap V_b \subseteq S_a \cap S_b$ and let m be the least element of S_b. Since $b \in I$, we have $m \le 2^n$. By the condition, there is no element of S_a strictly between $m \le 2^n$ and $z > 2^n$, which implies $z = v_a$.

It follows that if the element v_i is removed from each V_i, a family of pairwise disjoint sets $W_i = V_i \setminus \{v_i\}$ is obtained, $i \in I$ (we assume $W_i = \emptyset$ if $V_i = \emptyset$). As $W_i \subseteq \{2^n + 1, \dots, 2^{n+1}\}$ for all i, we infer that $\sum_{i\in I}|W_i| \le 2^n$. Therefore $\sum_{i\in I}(|V_i| - 1) \le \sum_{i\in I}|W_i| \le 2^n$.

On the other hand, the induction hypothesis applies directly to the sets U_i, $i \in I$, so that $\sum_{i\in \mathcal{I}}(|U_i| - n) \le (2n-1)2^{n-2}$. In summary,

$$\sum_{i\in I}(|S_i| - (n+1)) = \sum_{i\in I}(|U_i| - n) + \sum_{i\in I}(|V_i| - 1) \le (2n-1)2^{n-2} + 2^n. \tag{2}$$

The estimates (1) and (2) are sufficient to complete the inductive step:

$$\begin{aligned}\sum_{i=1}^{k}(|S_i| - (n+1)) &= \sum_{i\in I}(|S_i| - (n+1)) + \sum_{j\in J}(|S_j| - (n+1)) \\ &\le (2n-1)2^{n-2} + 2^n + (2n-1)2^{n-2} = (2n+1)2^{n-1}.\end{aligned}$$

Returning to the problem, consider $k = 2^n$ subsets $S_1, S_2, \dots, S_{2^n}$ of $\{1, 2, 3, \dots, 2^{n+1}\}$. If they satisfy the given condition, the claim implies $\sum_{i=1}^{2^n}(|S_i| - (n+1)) \le (2n+1)2^{n-1}$. By averaging again, we see that the smallest set has at most $2n + 1$ elements.

Comment. It can happen that each set S_i has cardinality at least $n + 1$. Here is an example by the proposer.

For $i = 1, \dots, 2^n$, let $S_i = \{i + 2^k \mid 0 \le k \le n\}$. Then $|S_i| = n + 1$ for all i. Suppose that there exist $a < b$ and $x < y < z$ such that $y, z \in S_a$ and $x, z \in S_b$. Hence $z = a + 2^k = b + 2^l$ for some $k > l$. Since $y \in S_a$ and $y < z$, we have $y \le a + 2^{k-1}$. So the element $x \in S_b$ satisfies

$$x < y \le a + 2^{k-1} = z - 2^{k-1} \le z - 2^l = b.$$

However the least element of S_b is $b + 1$, a contradiction.

Geometry

G1. In an acute-angled triangle ABC, point H is the orthocentre and A_0, B_0, C_0 are the midpoints of the sides BC, CA, AB, respectively. Consider three circles passing through H: ω_a around A_0, ω_b around B_0 and ω_c around C_0. The circle ω_a intersects the line BC at A_1 and A_2; ω_b intersects CA at B_1 and B_2; ω_c intersects AB at C_1 and C_2. Show that the points A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.

Solution 1. The perpendicular bisectors of the segments A_1A_2, B_1B_2, C_1C_2 are also the perpendicular bisectors of BC, CA, AB. So they meet at O, the circumcentre of ABC. Thus O is the only point that can possibly be the centre of the desired circle.

From the right triangle OA_0A_1 we get

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2. \tag{1}$$

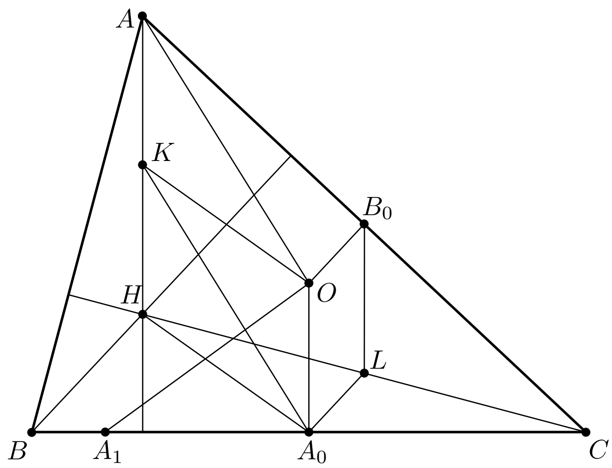
Let K be the midpoint of AH and let L be the midpoint of CH. Since A_0 and B_0 are the midpoints of BC and CA, we see that $A_0L \| BH$ and $B_0L \| AH$. Thus the segments A_0L and B_0L are perpendicular to AC and BC, hence parallel to OB_0 and OA_0, respectively. Consequently OA_0LB_0 is a parallelogram, so that OA_0 and B_0L are equal and parallel. Also, the midline B_0L of triangle AHC is equal and parallel to AK and KH.

It follows that AKA_0O and HA_0OK are parallelograms. The first one gives $A_0K = OA = R$, where R is the circumradius of ABC. From the second one we obtain

$$2(OA_0^2 + A_0H^2) = OH^2 + A_0K^2 = OH^2 + R^2. \tag{2}$$

(In a parallelogram, the sum of squares of the diagonals equals the sum of squares of the sides).

From (1) and (2) we get $OA_1^2 = (OH^2 + R^2)/2$. By symmetry, the same holds for the distances OA_2, OB_1, OB_2, OC_1 and OC_2. Thus A_1, A_2, B_1, B_2, C_1, C_2 all lie on a circle with centre at O and radius $(OH^2 + R^2)/2$.

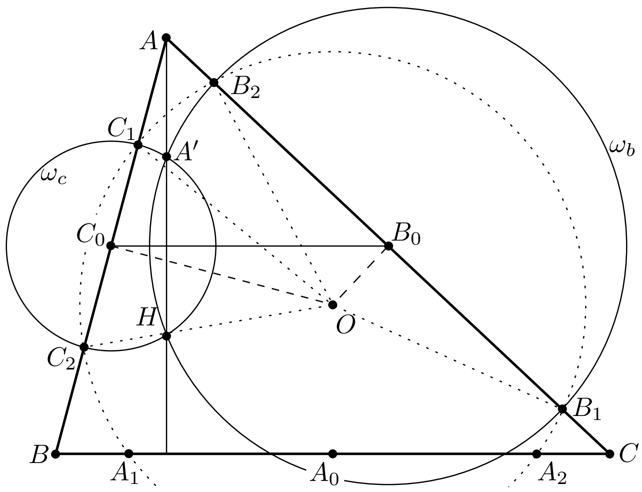


Solution 2. We are going to show again that the circumcentre O is equidistant from the six points in question.

Let A' be the second intersection point of ω_b and ω_c. The line B_0C_0, which is the line of centers of circles ω_b and ω_c, is a midline in triangle ABC, parallel to BC and perpendicular to the altitude AH. The points A' and H are symmetric with respect to the line of centers. Therefore A' lies on the line AH.

From the two circles ω_b and ω_c we obtain $AC_1 \cdot AC_2 = AA' \cdot AH = AB_1 \cdot AB_2$. So the quadrilateral $B_1B_2C_1C_2$ is cyclic. The perpendicular bisectors of the sides B_1B_2 and C_1C_2 meet at O. Hence O is the circumcentre of $B_1B_2C_1C_2$ and so $OB_1 = OB_2 = OC_1 = OC_2$.

Analogous arguments yield $OA_1 = OA_2 = OB_1 = OB_2$ and $OA_1 = OA_2 = OC_1 = OC_2$. Thus A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle centred at O.



Comment. The problem can be solved without much difficulty in many ways by calculation, using trigonometry, coordinate geometry or complex numbers. As an example we present a short proof using vectors.

Solution 3. Let again O and R be the circumcentre and circumradius. Consider the vectors

$$\overrightarrow{OA} = \mathbf{a}, \quad \overrightarrow{OB} = \mathbf{b}, \quad \overrightarrow{OC} = \mathbf{c}, \quad \text{where} \quad \mathbf{a}^2 = \mathbf{b}^2 = \mathbf{c}^2 = R^2.$$

It is well known that $\overrightarrow{OH} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. Accordingly,

$$\overrightarrow{A_0H} = \overrightarrow{OH} - \overrightarrow{OA_0} = (\mathbf{a} + \mathbf{b} + \mathbf{c}) - \frac{\mathbf{b} + \mathbf{c}}{2} = \frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2},$$

and

$$OA_1^2 = OA_0^2 + A_0A_1^2 = OA_0^2 + A_0H^2 = \left(\frac{\mathbf{b} + \mathbf{c}}{2}\right)^2 + \left(\frac{2\mathbf{a} + \mathbf{b} + \mathbf{c}}{2}\right)^2$$

$$= \frac{1}{4}(\mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) + \frac{1}{4}(4\mathbf{a}^2 + 4\mathbf{ab} + 4\mathbf{ac} + \mathbf{b}^2 + 2\mathbf{bc} + \mathbf{c}^2) = 2R^2 + (\mathbf{ab} + \mathbf{ac} + \mathbf{bc});$$

here $\mathbf{ab}$, $\mathbf{bc}$, etc. denote dot products of vectors. We get the same for the distances OA_2, OB_1, OB_2, OC_1 and OC_2.

G2. Given trapezoid $ABCD$ with parallel sides AB and CD, assume that there exist points E on line BC outside segment BC, and F inside segment AD, such that $\angle DAE = \angle CBF$. Denote by I the point of intersection of CD and EF, and by J the point of intersection of AB and EF. Let K be the midpoint of segment EF; assume it does not lie on line AB.

Prove that I belongs to the circumcircle of ABK if and only if K belongs to the circumcircle of CDJ.

Solution. Assume that the disposition of points is as in the diagram.

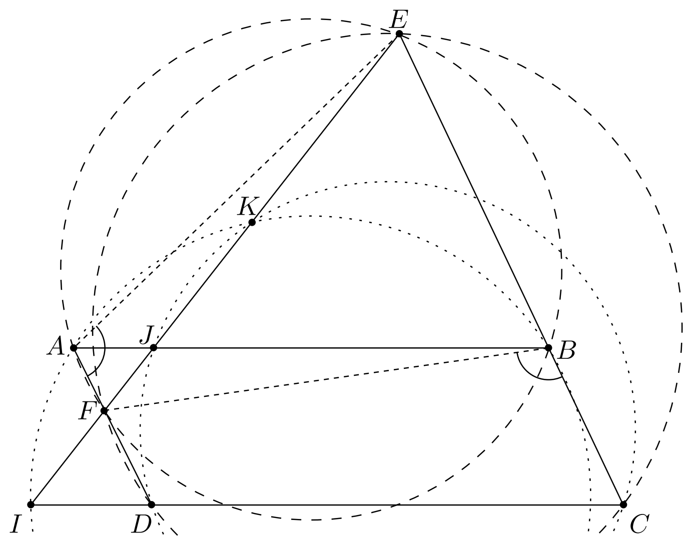
Since $\angle EBF = 180^\circ - \angle CBF = 180^\circ - \angle EAF$ by hypothesis, the quadrilateral $AEBF$ is cyclic. Hence $AJ \cdot JB = FJ \cdot JE$. In view of this equality, I belongs to the circumcircle of ABK if and only if $IJ \cdot JK = FJ \cdot JE$. Expressing $IJ = IF + FJ$, $JE = FE - FJ$, and $JK = \frac{1}{2}FE - FJ$, we find that I belongs to the circumcircle of ABK if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

Since $AEBF$ is cyclic and AB, CD are parallel, $\angle FEC = \angle FAB = 180^\circ - \angle CDF$. Then $CDFE$ is also cyclic, yielding $ID \cdot IC = IF \cdot IE$. It follows that K belongs to the circumcircle of CDJ if and only if $IJ \cdot IK = IF \cdot IE$. Expressing $IJ = IF + FJ$, $IK = IF + \frac{1}{2}FE$, and $IE = IF + FE$, we find that K is on the circumcircle of CDJ if and only if

$$FJ = \frac{IF \cdot FE}{2IF + FE}.$$

The conclusion follows.



Comment. While the figure shows B inside segment CE, it is possible that C is inside segment BE. Consequently, I would be inside segment EF and J outside segment EF. The position of point K on line EF with respect to points I, J may also vary.

Some case may require that an angle φ be replaced by $180^\circ - \varphi$, and in computing distances, a sum may need to become a difference. All these cases can be covered by the proposed solution if it is clearly stated that *signed* distances and angles are used.

G3. Let $ABCD$ be a convex quadrilateral and let P and Q be points in $ABCD$ such that $PQDA$ and $QPBC$ are cyclic quadrilaterals. Suppose that there exists a point E on the line segment PQ such that $\angle PAE = \angle QDE$ and $\angle PBE = \angle QCE$. Show that the quadrilateral $ABCD$ is cyclic.

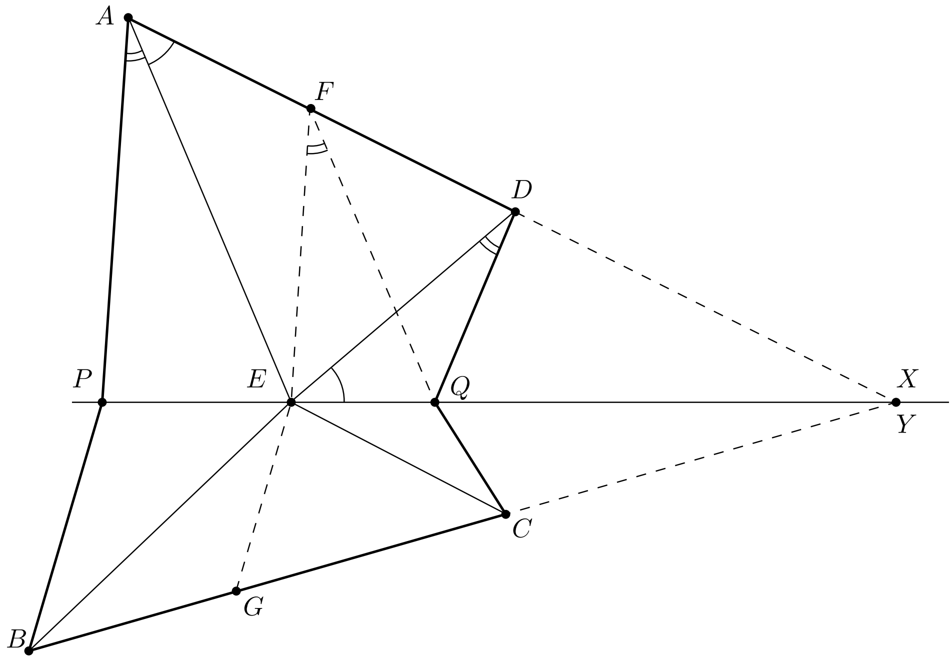
Solution 1. Let F be the point on the line AD such that $EF \| PA$. By hypothesis, the quadrilateral $PQDA$ is cyclic. So if F lies between A and D then $\angle EFD = \angle PAD = 180° - \angle EQD$; the points F and Q are on distinct sides of the line DE and we infer that $EFDQ$ is a cyclic quadrilateral. And if D lies between A and F then a similar argument shows that $\angle EFD = \angle EQD$; but now the points F and Q lie on the same side of DE, so that $EDFQ$ is a cyclic quadrilateral.

In either case we obtain the equality $\angle EFQ = \angle EDQ = \angle PAE$ which implies that $FQ \| AE$. So the triangles EFQ and PAE are either homothetic or parallel-congruent. More specifically, triangle EFQ is the image of PAE under the mapping f which carries the points P, E respectively to E, Q and is either a homothety or translation by a vector. Note that f is uniquely determined by these conditions and the position of the points P, E, Q alone.

Let now G be the point on the line BC such that $EG \| PB$. The same reasoning as above applies to points B, C in place of A, D, implying that the triangle EGQ is the image of PBE under the same mapping f. So f sends the four points A, P, B, E respectively to F, E, G, Q.

If $PE \neq QE$, so that f is a homothety with a centre X, then the lines AF, PE, BG—i.e. the lines AD, PQ, BC—are concurrent at X. And since $PQDA$ and $QPBC$ are cyclic quadrilaterals, the equalities $XA \cdot XD = XP \cdot XQ = XB \cdot XC$ hold, showing that the quadrilateral $ABCD$ is cyclic.

Finally, if $PE = QE$, so that f is a translation, then $AD \| PQ \| BC$. Thus $PQDA$ and $QPBC$ are isosceles trapezoids. Then also $ABCD$ is an isosceles trapezoid, hence a cyclic quadrilateral.



Solution 2. Here is another way to reach the conclusion that the lines AD, BC and PQ are either concurrent or parallel. From the cyclic quadrilateral $PQDA$ we get

$$\angle PAD = 180° - \angle PQD = \angle QDE + \angle QED = \angle PAE + \angle QED.$$

Hence $\angle QED = \angle PAD - \angle PAE = \angle EAD$. This in view of the tangent-chord theorem means that the circumcircle of triangle EAD is tangent to the line PQ at E. Analogously, the circumcircle of triangle EBC is tangent to PQ at E.

Suppose that the line AD intersects PQ at X. Since XE is tangent to the circle (EAD), $XE^2 = XA \cdot XD$. Also, $XA \cdot XD = XP \cdot XQ$ because P, Q, D, A lie on a circle. Therefore $XE^2 = XP \cdot XQ$.

It is not hard to see that this equation determines the position of the point X on the line PQ uniquely. Thus, if BC also cuts PQ, say at Y, then the analogous equation for Y yields $X = Y$, meaning that the three lines indeed concur. In this case, as well as in the case where $AD \| PQ \| BC$, the concluding argument is the same as in the first solution.

It remains to eliminate the possibility that e.g. AD meets PQ at X while $BC \| PQ$. Indeed, $QPBC$ would then be an isosceles trapezoid and the angle equality $\angle PBE = \angle QCE$ would force that E is the midpoint of PQ. So the length of XE, which is the geometric mean of the lengths of XP and XQ, should also be their arithmetic mean—impossible, as $XP \neq XQ$. The proof is now complete.

Comment. After reaching the conclusion that the circles (EDA) and (EBC) are tangent to PQ one may continue as follows. Denote the circles $(PQDA)$, (EDA), (EBC), $(QPBC)$ by ω_1, ω_2, ω_3, ω_4 respectively. Let ℓ_{ij} be the radical axis of the pair (ω_i, ω_j) for $i < j$. As is well-known, the lines ℓ_{12}, ℓ_{13}, ℓ_{23} concur, possibly at infinity (let this be the meaning of the word *concur* in this comment). So do the lines ℓ_{12}, ℓ_{14}, ℓ_{24}. Note however that ℓ_{23} and ℓ_{14} both coincide with the line PQ. Hence the pair ℓ_{12}, PQ is in both triples; thus the four lines ℓ_{12}, ℓ_{13}, ℓ_{24} and PQ are concurrent.

Similarly, ℓ_{13}, ℓ_{14}, ℓ_{34} concur, ℓ_{23}, ℓ_{24}, ℓ_{34} concur, and since $\ell_{14} = \ell_{23} = PQ$, the four lines ℓ_{13}, ℓ_{24}, ℓ_{34} and PQ are concurrent. The lines ℓ_{13} and ℓ_{24} are present in both quadruples, therefore all the lines ℓ_{ij} are concurrent. Hence the result.

G4. In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F are tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that the lines PE and QF intersect on the circumcircle of triangle AEF.

Solution 1. To approach the desired result we need some information about the slopes of the lines PE and QF; this information is provided by formulas (1) and (2) which we derive below.

The tangents BP and BQ to the two circles passing through A and F are equal, as $BP^2 = BA \cdot BF = BQ^2$. Consider the altitude AD of triangle ABC and its orthocentre H. From the cyclic quadrilaterals $CDFA$ and $CDHE$ we get $BA \cdot BF = BC \cdot BD = BE \cdot BH$. Thus $BP^2 = BE \cdot BH$, or $BP/BH = BE/BP$, implying that the triangles BPH and BEP are similar. Hence

$$\angle BPE = \angle BHP. \tag{1}$$

The point P lies between D and C; this follows from the equality $BP^2 = BC \cdot BD$. In view of this equality, and because $BP = BQ$,

$$DP \cdot DQ = (BP - BD) \cdot (BP + BD) = BP^2 - BD^2 = BD \cdot (BC - BD) = BD \cdot DC.$$

Also $AD \cdot DH = BD \cdot DC$, as is seen from the similar triangles BDH and ADC. Combining these equalities we obtain $AD \cdot DH = DP \cdot DQ$. Therefore $DH/DP = DQ/DA$, showing that the triangles HDP and QDA are similar. Hence $\angle HPD = \angle QAD$, which can be rewritten as $\angle BPH = \angle BAD + \angle BAQ$. And since BQ is tangent to the circumcircle of triangle FAQ,

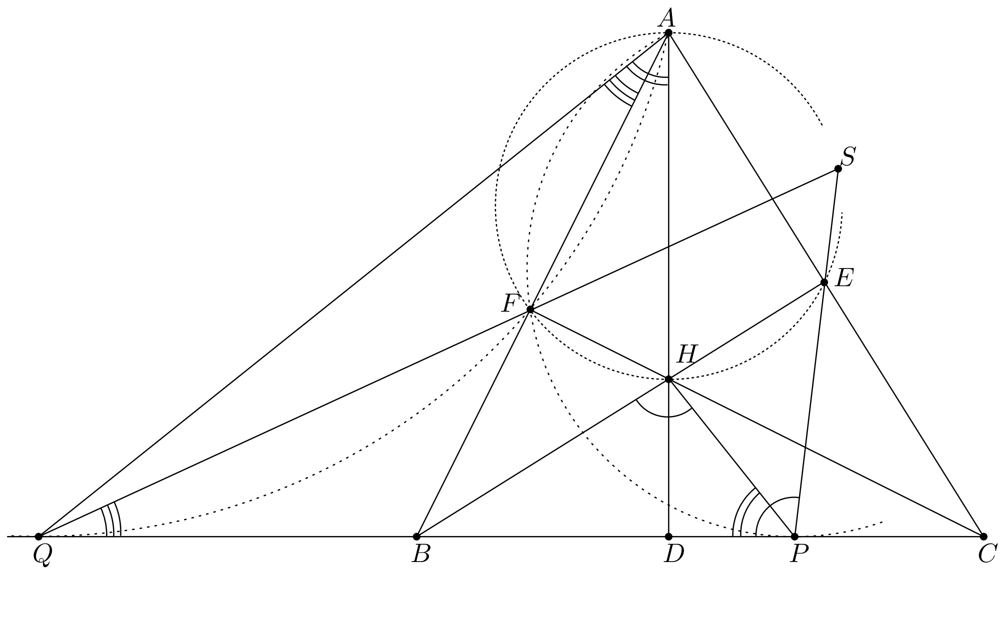
$$\angle BQF = \angle BAQ = \angle BPH - \angle BAD. \tag{2}$$

From (1) and (2) we deduce

$$\begin{aligned}\angle BPE + \angle BQF &= (\angle BHP + \angle BPH) - \angle BAD = (180^\circ - \angle PBH) - \angle BAD \\ &= (90^\circ + \angle BCA) - (90^\circ - \angle ABC) = \angle BCA + \angle ABC = 180^\circ - \angle CAB.\end{aligned}$$

Thus $\angle BPE + \angle BQF < 180^\circ$, which means that the rays PE and QF meet. Let S be the point of intersection. Then $\angle PSQ = 180^\circ - (\angle BPE + \angle BQF) = \angle CAB = \angle EAF$.

If S lies between P and E then $\angle PSQ = 180^\circ - \angle ESF$; and if E lies between P and S then $\angle PSQ = \angle ESF$. In either case the equality $\angle PSQ = \angle EAF$ which we have obtained means that S lies on the circumcircle of triangle AEF.



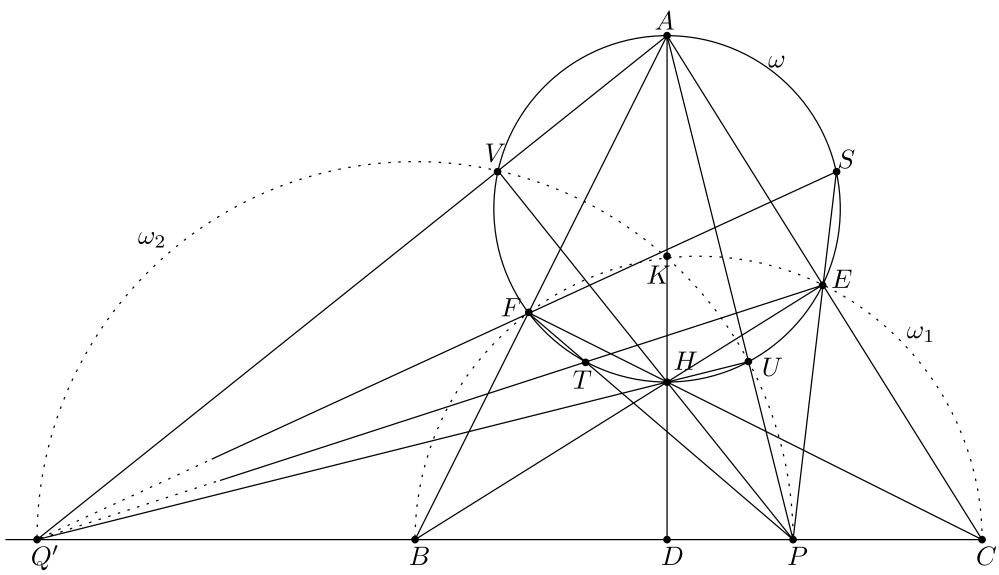
Solution 2. Let H be the orthocentre of triangle ABC and let ω be the circle with diameter AH, passing through E and F. Introduce the points of intersection of ω with the following lines emanating from P: $PA \cap \omega = \{A, U\}$, $PH \cap \omega = \{H, V\}$, $PE \cap \omega = \{E, S\}$. The altitudes of triangle AHP are contained in the lines AV, HU, BC, meeting at its orthocentre Q'.

By Pascal's theorem applied to the (tied) hexagon $AESFHV$, the points $AE \cap FH = C$, $ES \cap HV = P$ and $SF \cap VA$ are collinear, so FS passes through Q'.

Denote by ω_1 and ω_2 the circles with diameters BC and PQ', respectively. Let D be the foot of the altitude from A in triangle ABC. Suppose that AD meets the circles ω_1 and ω_2 at the respective points K and L.

Since H is the orthocentre of ABC, the triangles BDH and ADC are similar, and so $DA \cdot DH = DB \cdot DC = DK^2$; the last equality holds because BKC is a right triangle. Since H is the orthocentre also in triangle $AQ'P$, we analogously have $DL^2 = DA \cdot DH$. Therefore $DK = DL$ and $K = L$.

Also, $BD \cdot BC = BA \cdot BF$, from the similar triangles ABD, CBF. In the right triangle BKC we have $BK^2 = BD \cdot BC$. Hence, and because $BA \cdot BF = BP^2 = BQ^2$ (by the definition of P and Q in the problem statement), we obtain $BK = BP = BQ$. It follows that B is the centre of ω_2 and hence $Q' = Q$. So the lines PE and QF meet at the point S lying on the circumcircle of triangle AEF.



Comment 1. If T is the point defined by $PF \cap \omega = \{F, T\}$, Pascal's theorem for the hexagon $AFTEHV$ will analogously lead to the conclusion that the line ET goes through Q'. In other words, the lines PF and QE also concur on ω.

Comment 2. As is known from algebraic geometry, the points of the circle ω form a commutative groups with the operation defined as follows. Choose any point $0 \in \omega$ (to be the neutral element of the group) and a line ℓ exterior to the circle. For $X, Y \in \omega$, draw the line from the point $XY \cap \ell$ through 0 to its second intersection with ω and define this point to be $X + Y$.

In our solution we have chosen H to be the neutral element in this group and line BC to be ℓ. The fact that the lines AV, HU, ET, FS are concurrent can be deduced from the identities $A + A = 0$, $F = E + A$, $V = U + A = S + E = T + F$.

Comment 3. The problem was submitted in the following equivalent formulation:

Let BE and CF be altitudes of an acute triangle ABC. We choose P on the side BC and Q on the extension of CB beyond B such that $BQ^2 = BP^2 = BF \cdot AB$. If QF and PE intersect at S, prove that $ESAF$ is cyclic.

G5. Let k and n be integers with $0 \le k \le n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersection points of lines in L. Let O be a point in the plane not lying on any line of L.

A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least $\frac{1}{2}(k+1)(k+2)$ red points.

Solution. There are at least $\frac{1}{2}(k+1)(k+2)$ points in the intersection set I in view of the condition $n \ge k+2$.

For each point $P \in I$, define its *order* as the number of lines that intersect the open line segment OP. By definition, P is red if its order is at most k. Note that there is always at least one point $X \in I$ of order 0. Indeed, the lines in L divide the plane into regions, bounded or not, and O belongs to one of them. Clearly any corner of this region is a point of I with order 0.

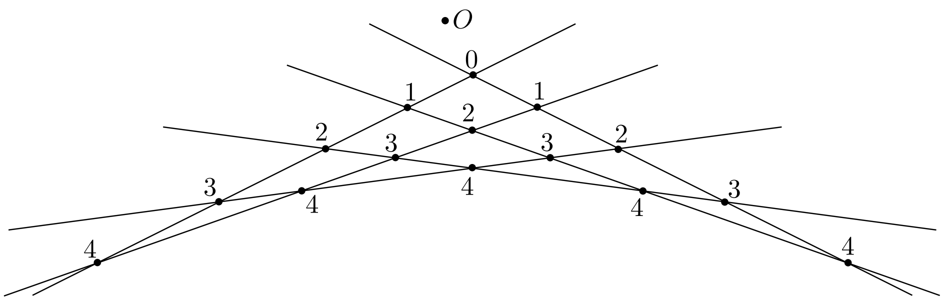
Claim. Suppose that two points $P, Q \in I$ lie on the same line of L, and no other line of L intersects the open line segment PQ. Then the orders of P and Q differ by at most 1.

Proof. Let P and Q have orders p and q, respectively, with $p \ge q$. Consider triangle OPQ. Now p equals the number of lines in L that intersect the interior of side OP. None of these lines intersects the interior of side PQ, and at most one can pass through Q. All remaining lines must intersect the interior of side OQ, implying that $q \ge p-1$. The conclusion follows.

We prove the main result by induction on k. The base $k=0$ is clear since there is a point of order 0 which is red. Assuming the statement true for $k-1$, we pass on to the inductive step. Select a point $P \in I$ of order 0, and consider one of the lines $\ell \in L$ that pass through P. There are $n-1$ intersection points on ℓ, one of which is P. Out of the remaining $n-2$ points, the k closest to P have orders not exceeding k by the Claim. It follows that there are at least $k+1$ red points on ℓ.

Let us now consider the situation with ℓ removed (together with all intersection points it contains). By hypothesis of induction, there are at least $\frac{1}{2}k(k+1)$ points of order not exceeding $k-1$ in the resulting configuration. Restoring ℓ back produces at most one new intersection point on each line segment joining any of these points to O, so their order is at most k in the original configuration. The total number of points with order not exceeding k is therefore at least $(k+1)+\frac{1}{2}k(k+1) = \frac{1}{2}(k+1)(k+2)$. This completes the proof.

Comment. The steps of the proof can be performed in reverse order to obtain a configuration of n lines such that equality holds simultaneously for all $0 \le k \le n-2$. Such a set of lines is illustrated in the Figure.



G6. There is given a convex quadrilateral $ABCD$. Prove that there exists a point P inside the quadrilateral such that

$$\angle PAB + \angle PDC = \angle PBC + \angle PAD = \angle PCD + \angle PBA = \angle PDA + \angle PCB = 90^\circ \qquad (1)$$

if and only if the diagonals AC and BD are perpendicular.

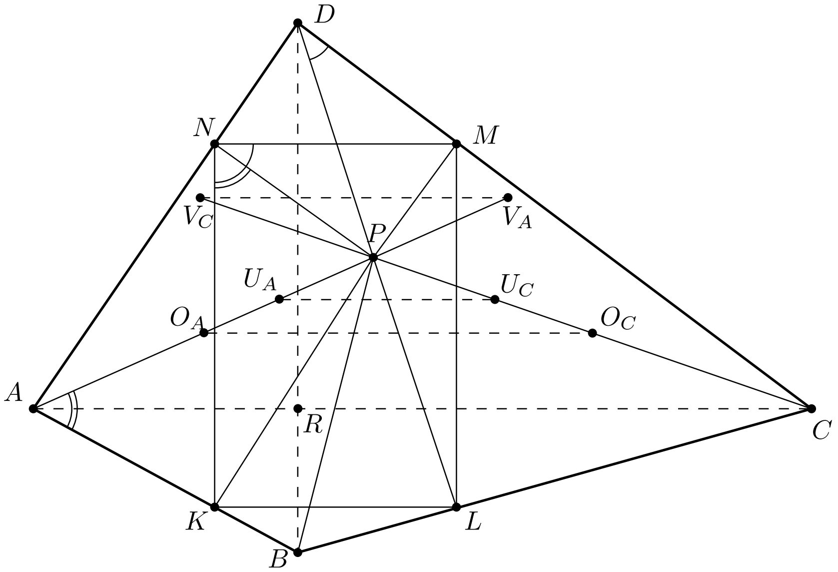
Solution 1. For a point P in $ABCD$ which satisfies (1), let K, L, M, N be the feet of perpendiculars from P to lines AB, BC, CD, DA, respectively. Note that K, L, M, N are interior to the sides as all angles in (1) are acute. The cyclic quadrilaterals $AKPN$ and $DNPM$ give

$$\angle PAB + \angle PDC = \angle PNK + \angle PNM = \angle KNM.$$

Analogously, $\angle PBC + \angle PAD = \angle LKN$ and $\angle PCD + \angle PBA = \angle MLK$. Hence the equalities (1) imply $\angle KNM = \angle LKN = \angle MLK = 90^\circ$, so that $KLMN$ is a rectangle. The converse also holds true, provided that K, L, M, N are interior to sides AB, BC, CD, DA.

(i) Suppose that there exists a point P in $ABCD$ such that $KLMN$ is a rectangle. We show that AC and BD are parallel to the respective sides of $KLMN$.

Let O_A and O_C be the circumcentres of the cyclic quadrilaterals $AKPN$ and $CMPL$. Line O_AO_C is the common perpendicular bisector of LM and KN, therefore O_AO_C is parallel to KL and MN. On the other hand, O_AO_C is the midline in the triangle ACP that is parallel to AC. Therefore the diagonal AC is parallel to the sides KL and MN of the rectangle. Likewise, BD is parallel to KN and LM. Hence AC and BD are perpendicular.



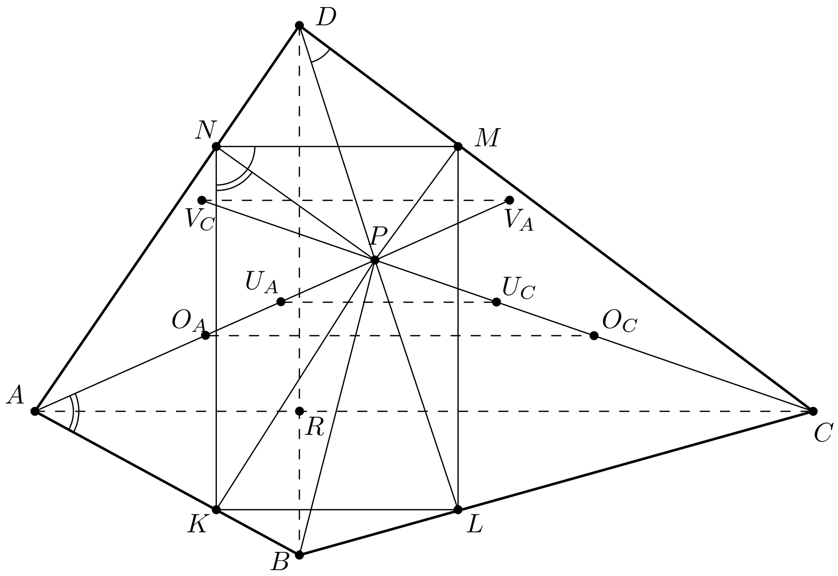
(ii) Suppose that AC and BD are perpendicular and meet at R. If $ABCD$ is a rhombus, P can be chosen to be its centre. So assume that $ABCD$ is not a rhombus, and let $BR < DR$ without loss of generality.

Denote by U_A and U_C the circumcentres of the triangles ABD and CDB, respectively. Let AV_A and CV_C be the diameters through A and C of the two circumcircles. Since AR is an altitude in triangle ADB, lines AC and AV_A are isogonal conjugates, i. e. $\angle DAV_A = \angle BAC$. Now $BR < DR$ implies that ray AU_A lies in $\angle DAC$. Similarly, ray CU_C lies in $\angle DCA$. Both diameters AV_A and CV_C intersect BD as the angles at B and D of both triangles are acute. Also U_AU_C is parallel to AC as it is the perpendicular bisector of BD. Hence V_AV_C is parallel to AC, too. We infer that AV_A and CV_C intersect at a point P inside triangle ACD, hence inside $ABCD$.

Construct points K, L, M, N, O_A and O_C in the same way as in the introduction. It follows from the previous paragraph that K, L, M, N are interior to the respective sides. Now O_AO_C is a midline in triangle ACP again. Therefore lines AC, O_AO_C and U_AU_C are parallel.

The cyclic quadrilateral $AKPN$ yields $\angle NKP = \angle NAP$. Since $\angle NAP = \angle DAU_A = \angle BAC$, as specified above, we obtain $\angle NKP = \angle BAC$. Because PK is perpendicular to AB, it follows that NK is perpendicular to AC, hence parallel to BD. Likewise, LM is parallel to BD.

Consider the two homotheties with centres A and C which transform triangles ABD and CDB into triangles AKN and CML, respectively. The images of points U_A and U_C are O_A and O_C, respectively. Since U_AU_C and O_AO_C are parallel to AC, the two ratios of homothety are the same, equal to $\lambda = AN/AD = AK/AB = AO_A/AU_A = CO_C/CU_C = CM/CD = CL/CB$. It is now straightforward that $DN/DA = DM/DC = BK/BA = BL/BC = 1-\lambda$. Hence KL and MN are parallel to AC, implying that $KLMN$ is a rectangle and completing the proof.



Solution 2. For a point P distinct from A, B, C, D, let circles (APD) and (BPC) intersect again at Q ($Q = P$ if the circles are tangent). Next, let circles (AQB) and (CQD) intersect again at R. We show that if P lies in $ABCD$ and satisfies (1) then AC and BD intersect at R and are perpendicular; the converse is also true. It is convenient to use directed angles. Let $\measuredangle(UV, XY)$ denote the angle of counterclockwise rotation that makes line UV parallel to line XY. Recall that four noncollinear points U, V, X, Y are concyclic if and only if $\measuredangle(UX, VX) = \measuredangle(UY, VY)$.

The definitions of points P, Q and R imply

$$\begin{aligned}
\measuredangle(AR, BR) &= \measuredangle(AQ, BQ) = \measuredangle(AQ, PQ) + \measuredangle(PQ, BQ) = \measuredangle(AD, PD) + \measuredangle(PC, BC),\\
\measuredangle(CR, DR) &= \measuredangle(CQ, DQ) = \measuredangle(CQ, PQ) + \measuredangle(PQ, DQ) = \measuredangle(CB, PB) + \measuredangle(PA, DA),\\
\measuredangle(BR, CR) &= \measuredangle(BR, RQ) + \measuredangle(RQ, CR) = \measuredangle(BA, AQ) + \measuredangle(DQ, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(AP, AQ) + \measuredangle(DQ, DP) + \measuredangle(DP, CD)\\
&= \measuredangle(BA, AP) + \measuredangle(DP, CD).
\end{aligned}$$

Observe that the whole construction is reversible. One may start with point R, define Q as the second intersection of circles (ARB) and (CRD), and then define P as the second intersection of circles (AQD) and (BQC). The equalities above will still hold true.

Assume in addition that P is interior to $ABCD$. Then

$$\measuredangle(AD,PD)=\angle PDA,\ \measuredangle(PC,BC)=\angle PCB,\ \measuredangle(CB,PB)=\angle PBC,\ \measuredangle(PA,DA)=\angle PAD,$$
$$\measuredangle(BA,AP)=\angle PAB,\ \measuredangle(DP,CD)=\angle PDC.$$

(i) Suppose that P lies in $ABCD$ and satisfies (1). Then $\measuredangle(AR,BR)=\angle PDA+\angle PCB=90^\circ$ and similarly $\measuredangle(BR,CR)=\measuredangle(CR,DR)=90^\circ$. It follows that R is the common point of lines AC and BD, and that these lines are perpendicular.
(ii) Suppose that AC and BD are perpendicular and intersect at R. We show that the point P defined by the reverse construction (starting with R and ending with P) lies in $ABCD$. This is enough to finish the solution, because then the angle equalities above will imply (1).

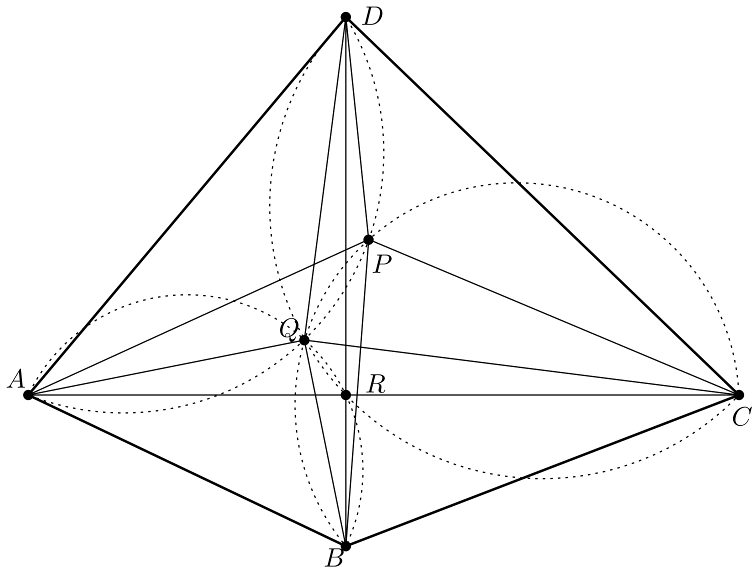
One can assume that Q, the second common point of circles (ABR) and (CDR), lies in $\angle ARD$. Then in fact Q lies in triangle ADR as angles AQR and DQR are obtuse. Hence $\angle AQD$ is obtuse, too, so that B and C are outside circle (ADQ) ($\angle ABD$ and $\angle ACD$ are acute).

Now $\angle CAB+\angle CDB=\angle BQR+\angle CQR=\angle CQB$ implies $\angle CAB<\angle CQB$ and $\angle CDB<\angle CQB$. Hence A and D are outside circle (BCQ). In conclusion, the second common point P of circles (ADQ) and (BCQ) lies on their arcs ADQ and BCQ.

We can assume that P lies in $\angle CQD$. Since

$$\angle QPC+\angle QPD=(180^\circ-\angle QBC)+(180^\circ-\angle QAD)=$$
$$=360^\circ-(\angle RBC+\angle QBR)-(\angle RAD-\angle QAR)=360^\circ-\angle RBC-\angle RAD>180^\circ,$$

point P lies in triangle CDQ, and hence in $ABCD$. The proof is complete.



G7. Let $ABCD$ be a convex quadrilateral with $AB \neq BC$. Denote by ω_1 and ω_2 the incircles of triangles ABC and ADC. Suppose that there exists a circle ω inscribed in angle ABC, tangent to the extensions of line segments AD and CD. Prove that the common external tangents of ω_1 and ω_2 intersect on ω.

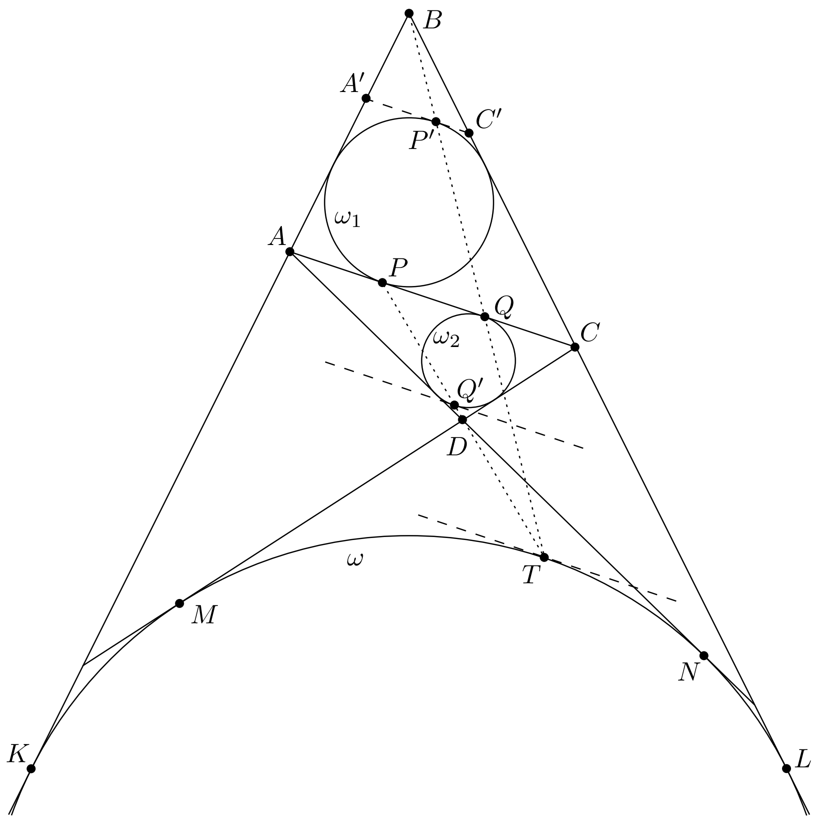
Solution. The proof below is based on two known facts.

Lemma 1. Given a convex quadrilateral $ABCD$, suppose that there exists a circle which is inscribed in angle ABC and tangent to the extensions of line segments AD and CD. Then $AB + AD = CB + CD$.

Proof. The circle in question is tangent to each of the lines AB, BC, CD, DA, and the respective points of tangency K, L, M, N are located as with circle ω in the figure. Then

$$AB + AD = (BK - AK) + (AN - DN), \quad CB + CD = (BL - CL) + (CM - DM).$$

Also $BK = BL$, $DN = DM$, $AK = AN$, $CL = CM$ by equalities of tangents. It follows that $AB + AD = CB + CD$.



For brevity, in the sequel we write "excircle AC" for the excircle of a triangle with side AC which is tangent to line segment AC and the extensions of the other two sides.

Lemma 2. The incircle of triangle ABC is tangent to its side AC at P. Let PP' be the diameter of the incircle through P, and let line BP' intersect AC at Q. Then Q is the point of tangency of side AC and excircle AC.

Proof. Let the tangent at P' to the incircle ω_1 meet BA and BC at A' and C'. Now ω_1 is the excircle $A'C'$ of triangle $A'BC'$, and it touches side $A'C'$ at P'. Since $A'C' \parallel AC$, the homothety with centre B and ratio BQ/BP' takes ω_1 to the excircle AC of triangle ABC. Because this homothety takes P' to Q, the lemma follows.

Recall also that if the incircle of a triangle touches its side AC at P, then the tangency point Q of the same side and excircle AC is the unique point on line segment AC such that $AP = CQ$.

We pass on to the main proof. Let ω_1 and ω_2 touch AC at P and Q, respectively; then $AP = (AC + AB - BC)/2$, $CQ = (CA + CD - AD)/2$. Since $AB - BC = CD - AD$ by Lemma 1, we obtain $AP = CQ$. It follows that in triangle ABC side AC and excircle AC are tangent at Q. Likewise, in triangle ADC side AC and excircle AC are tangent at P. Note that $P \neq Q$ as $AB \neq BC$.

Let PP' and QQ' be the diameters perpendicular to AC of ω_1 and ω_2, respectively. Then Lemma 2 shows that points B, P' and Q are collinear, and so are points D, Q' and P.

Consider the diameter of ω perpendicular to AC and denote by T its endpoint that is closer to AC. The homothety with centre B and ratio BT/BP' takes ω_1 to ω. Hence B, P' and T are collinear. Similarly, D, Q' and T are collinear since the homothety with centre D and ratio $-DT/DQ'$ takes ω_2 to ω.

We infer that points T, P' and Q are collinear, as well as T, Q' and P. Since $PP' \parallel QQ'$, line segments PP' and QQ' are then homothetic with centre T. The same holds true for circles ω_1 and ω_2 because they have PP' and QQ' as diameters. Moreover, it is immediate that T lies on the same side of line PP' as Q and Q', hence the ratio of homothety is positive. In particular ω_1 and ω_2 are not congruent.

In summary, T is the centre of a homothety with positive ratio that takes circle ω_1 to circle ω_2. This completes the solution, since the only point with the mentioned property is the intersection of the the common external tangents of ω_1 and ω_2.

Number Theory

N1. Let n be a positive integer and let p be a prime number. Prove that if a, b, c are integers (not necessarily positive) satisfying the equations

$$a^n + pb = b^n + pc = c^n + pa,$$

then $a = b = c$.

Solution 1. If two of a, b, c are equal, it is immediate that all the three are equal. So we may assume that $a \neq b \neq c \neq a$. Subtracting the equations we get $a^n - b^n = -p(b - c)$ and two cyclic copies of this equation, which upon multiplication yield

$$\frac{a^n - b^n}{a - b} \cdot \frac{b^n - c^n}{b - c} \cdot \frac{c^n - a^n}{c - a} = -p^3. \tag{1}$$

If n is odd then the differences $a^n - b^n$ and $a - b$ have the same sign and the product on the left is positive, while $-p^3$ is negative. So n must be even.

Let d be the greatest common divisor of the three differences $a - b$, $b - c$, $c - a$, so that $a - b = du$, $b - c = dv$, $c - a = dw$; $\gcd(u, v, w) = 1$, $u + v + w = 0$.

From $a^n - b^n = -p(b - c)$ we see that $(a - b)|p(b - c)$, i.e., $u|pv$; and cyclically $v|pw$, $w|pu$. As $\gcd(u, v, w) = 1$ and $u + v + w = 0$, at most one of u, v, w can be divisible by p. Supposing that the prime p does not divide any one of them, we get $u|v$, $v|w$, $w|u$, whence $|u| = |v| = |w| = 1$; but this quarrels with $u + v + w = 0$.

Thus p must divide exactly one of these numbers. Let e.g. $p|u$ and write $u = pu_1$. Now we obtain, similarly as before, $u_1|v$, $v|w$, $w|u_1$ so that $|u_1| = |v| = |w| = 1$. The equation $pu_1 + v + w = 0$ forces that the prime p must be even; i.e. $p = 2$. Hence $v + w = -2u_1 = \pm 2$, implying $v = w\ (= \pm 1)$ and $u = -2v$. Consequently $a - b = -2(b - c)$.

Knowing that n is even, say $n = 2k$, we rewrite the equation $a^n - b^n = -p(b - c)$ with $p = 2$ in the form

$$(a^k + b^k)(a^k - b^k) = -2(b - c) = a - b.$$

The second factor on the left is divisible by $a - b$, so the first factor $(a^k + b^k)$ must be ± 1. Then exactly one of a and b must be odd; yet $a - b = -2(b - c)$ is even. Contradiction ends the proof.

Solution 2. The beginning is as in the first solution. Assuming that a, b, c are not all equal, hence are all distinct, we derive equation (1) with the conclusion that n is even. Write $n = 2k$.

Suppose that p is odd. Then the integer

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + b^{n-1},$$

which is a factor in (1), must be odd as well. This sum of $n = 2k$ summands is odd only if a and b have different parities. The same conclusion holding for b, c and for c, a, we get that a, b, c, a alternate in their parities, which is clearly impossible.

Thus $p = 2$. The original system shows that a, b, c must be of the same parity. So we may divide (1) by p^3, i.e. 2^3, to obtain the following product of six integer factors:

$$\frac{a^k + b^k}{2} \cdot \frac{a^k - b^k}{a - b} \cdot \frac{b^k + c^k}{2} \cdot \frac{b^k - c^k}{b - c} \cdot \frac{c^k + a^k}{2} \cdot \frac{c^k - a^k}{c - a} = -1. \tag{2}$$

Each one of the factors must be equal to ± 1. In particular, $a^k + b^k = \pm 2$. If k is even, this becomes $a^k + b^k = 2$ and yields $|a| = |b| = 1$, whence $a^k - b^k = 0$, contradicting (2).

Let now k be odd. Then the sum $a^k + b^k$, with value ± 2, has $a + b$ as a factor. Since a and b are of the same parity, this means that $a + b = \pm 2$; and cyclically, $b + c = \pm 2$, $c + a = \pm 2$. In some two of these equations the signs must coincide, hence some two of a, b, c are equal. This is the desired contradiction.

Comment. Having arrived at the equation (1) one is tempted to write down all possible decompositions of $-p^3$ (cube of a prime) into a product of three integers. This leads to cumbersome examination of many cases, some of which are unpleasant to handle. One may do that just for $p = 2$, having earlier in some way eliminated odd primes from consideration.

However, the second solution shows that the condition of p being a prime is far too strong. What is actually being used in that solution, is that p is either a positive odd integer or $p = 2$.

N2. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers, $n \geq 3$. Prove that there exist distinct indices i and j such that $a_i + a_j$ does not divide any of the numbers $3a_1, 3a_2, \ldots, 3a_n$.

Solution. Without loss of generality, let $0 < a_1 < a_2 < \cdots < a_n$. One can also assume that $a_1, a_2, \ldots, a_n$ are coprime. Otherwise division by their greatest common divisor reduces the question to the new sequence whose terms are coprime integers.

Suppose that the claim is false. Then for each $i < n$ there exists a j such that $a_n + a_i$ divides $3a_j$. If $a_n + a_i$ is not divisible by 3 then $a_n + a_i$ divides a_j which is impossible as $0 < a_j \leq a_n < a_n + a_i$. Thus $a_n + a_i$ is a multiple of 3 for $i = 1, \ldots, n-1$, so that $a_1, a_2, \ldots, a_{n-1}$ are all congruent (to $-a_n$) modulo 3.

Now a_n is not divisible by 3 or else so would be all remaining a_i's, meaning that $a_1, a_2, \ldots, a_n$ are not coprime. Hence $a_n \equiv r \pmod 3$ where $r \in \{1, 2\}$, and $a_i \equiv 3 - r \pmod 3$ for all $i = 1, \ldots, n-1$.

Consider a sum $a_{n-1} + a_i$ where $1 \leq i \leq n-2$. There is at least one such sum as $n \geq 3$. Let j be an index such that $a_{n-1} + a_i$ divides $3a_j$. Observe that $a_{n-1} + a_i$ is not divisible by 3 since $a_{n-1} + a_i \equiv 2a_i \not\equiv 0 \pmod 3$. It follows that $a_{n-1} + a_i$ divides a_j, in particular $a_{n-1} + a_i \leq a_j$. Hence $a_{n-1} < a_j \leq a_n$, implying $j = n$. So a_n is divisible by all sums $a_{n-1} + a_i$, $1 \leq i \leq n-2$. In particular $a_{n-1} + a_i \leq a_n$ for $i = 1, \ldots, n-2$.

Let j be such that $a_n + a_{n-1}$ divides $3a_j$. If $j \leq n-2$ then $a_n + a_{n-1} \leq 3a_j < a_j + 2a_{n-1}$. This yields $a_n < a_{n-1} + a_j$; however $a_{n-1} + a_j \leq a_n$ for $j \leq n-2$. Therefore $j = n-1$ or $j = n$.

For $j = n-1$ we obtain $3a_{n-1} = k(a_n + a_{n-1})$ with k an integer, and it is straightforward that $k = 1$ ($k \leq 0$ and $k \geq 3$ contradict $0 < a_{n-1} < a_n$; $k = 2$ leads to $a_{n-1} = 2a_n > a_{n-1}$). Thus $3a_{n-1} = a_n + a_{n-1}$, i. e. $a_n = 2a_{n-1}$.

Similarly, if $j = n$ then $3a_n = k(a_n + a_{n-1})$ for some integer k, and only $k = 2$ is possible. Hence $a_n = 2a_{n-1}$ holds true in both cases remaining, $j = n-1$ and $j = n$.

Now $a_n = 2a_{n-1}$ implies that the sum $a_{n-1} + a_1$ is strictly between $a_n/2$ and a_n. But a_{n-1} and a_1 are distinct as $n \geq 3$, so it follows from the above that $a_{n-1} + a_1$ divides a_n. This provides the desired contradiction.

N3. Let $a_0, a_1, a_2, \ldots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all $n \geq 0$.

Solution. Since $a_i \geq \gcd(a_i, a_{i+1}) > a_{i-1}$, the sequence is strictly increasing. In particular $a_0 \geq 1$, $a_1 \geq 2$. For each $i \geq 1$ we also have $a_{i+1} - a_i \geq \gcd(a_i, a_{i+1}) > a_{i-1}$, and consequently $a_{i+1} \geq a_i + a_{i-1} + 1$. Hence $a_2 \geq 4$ and $a_3 \geq 7$. The equality $a_3 = 7$ would force equalities in the previous estimates, leading to $\gcd(a_2, a_3) = \gcd(4, 7) > a_1 = 2$, which is false. Thus $a_3 \geq 8$; the result is valid for $n = 0, 1, 2, 3$. These are the base cases for a proof by induction.

Take an $n \geq 3$ and assume that $a_i \geq 2^i$ for $i = 0, 1, \ldots, n$. We must show that $a_{n+1} \geq 2^{n+1}$. Let $\gcd(a_n, a_{n+1}) = d$. We know that $d > a_{n-1}$. The induction claim is reached immediately in the following cases:

$$\begin{array}{lll}
\text{if } a_{n+1} \geq 4d & \text{then} & a_{n+1} > 4a_{n-1} \geq 4 \cdot 2^{n-1} = 2^{n+1}\,; \\
\text{if } a_n \geq 3d & \text{then} & a_{n+1} \geq a_n + d \geq 4d > 4a_{n-1} \geq 4 \cdot 2^{n-1} = 2^{n+1}\,; \\
\text{if } a_n = d & \text{then} & a_{n+1} \geq a_n + d = 2a_n \geq 2 \cdot 2^n = 2^{n+1}.
\end{array}$$

The only remaining possibility is that $a_n = 2d$ and $a_{n+1} = 3d$, which we assume for the sequel. So $a_{n+1} = \frac{3}{2}a_n$.

Let now $\gcd(a_{n-1}, a_n) = d'$; then $d' > a_{n-2}$. Write $a_n = md'$ (m an integer). Keeping in mind that $d' \leq a_{n-1} < d$ and $a_n = 2d$, we get that $m \geq 3$. Also $a_{n-1} < d = \frac{1}{2}md'$, $a_{n+1} = \frac{3}{2}md'$. Again we single out the cases which imply the induction claim immediately:

$$\begin{array}{lll}
\text{if } m \geq 6 & \text{then} & a_{n+1} = \frac{3}{2}md' \geq 9d' > 9a_{n-2} \geq 9 \cdot 2^{n-2} > 2^{n+1}\,; \\
\text{if } 3 \leq m \leq 4 & \text{then} & a_{n-1} < \frac{1}{2} \cdot 4d', \text{ and hence } a_{n-1} = d', \\
 & & \qquad a_{n+1} = \frac{3}{2}ma_{n-1} \geq \frac{3}{2} \cdot 3a_{n-1} \geq \frac{9}{2} \cdot 2^{n-1} > 2^{n+1}.
\end{array}$$

So we are left with the case $m = 5$, which means that $a_n = 5d'$, $a_{n+1} = \frac{15}{2}d'$, $a_{n-1} < d = \frac{5}{2}d'$. The last relation implies that a_{n-1} is either d' or $2d'$. Anyway, $a_{n-1} | 2d'$.

The same pattern repeats once more. We denote $\gcd(a_{n-2}, a_{n-1}) = d''$; then $d'' > a_{n-3}$. Because d'' is a divisor of a_{n-1}, hence also of $2d'$, we may write $2d' = m'd''$ (m' an integer). Since $d'' \leq a_{n-2} < d'$, we get $m' \geq 3$. Also, $a_{n-2} < d' = \frac{1}{2}m'd''$, $a_{n+1} = \frac{15}{2}d' = \frac{15}{4}m'd''$. As before, we consider the cases:

$$\begin{array}{lll}
\text{if } m' \geq 5 & \text{then} & a_{n+1} = \frac{15}{4}m'd'' \geq \frac{75}{4}d'' > \frac{75}{4}a_{n-3} \geq \frac{75}{4} \cdot 2^{n-3} > 2^{n+1}\,; \\
\text{if } 3 \leq m' \leq 4 & \text{then} & a_{n-2} < \frac{1}{2} \cdot 4d'', \text{ and hence } a_{n-2} = d'', \\
 & & \qquad a_{n+1} = \frac{15}{4}m'a_{n-2} \geq \frac{15}{4} \cdot 3a_{n-2} \geq \frac{45}{4} \cdot 2^{n-2} > 2^{n+1}.
\end{array}$$

Both of them have produced the induction claim. But now there are no cases left. Induction is complete; the inequality $a_n \geq 2^n$ holds for all n.

N4. Let n be a positive integer. Show that the numbers

$$\binom{2^n-1}{0},\quad \binom{2^n-1}{1},\quad \binom{2^n-1}{2},\quad \ldots,\quad \binom{2^n-1}{2^{n-1}-1}$$

are congruent modulo 2^n to $1, 3, 5, \ldots, 2^n-1$ in some order.

Solution 1. It is well-known that all these numbers are odd. So the assertion that their remainders $\pmod{2^n}$ make up a permutation of $\{1, 3, \ldots, 2^n-1\}$ is equivalent just to saying that these remainders are all distinct. We begin by showing that

$$\binom{2^n-1}{2k}+\binom{2^n-1}{2k+1}\equiv 0 \pmod{2^n} \quad \text{and} \quad \binom{2^n-1}{2k}\equiv(-1)^k\binom{2^{n-1}-1}{k} \pmod{2^n}. \quad (1)$$

The first relation is immediate, as the sum on the left is equal to $\binom{2^n}{2k+1}=\frac{2^n}{2k+1}\binom{2^n-1}{2k}$, hence is divisible by 2^n. The second relation:

$$\binom{2^n-1}{2k}=\prod_{j=1}^{2k}\frac{2^n-j}{j}=\prod_{i=1}^{k}\frac{2^n-(2i-1)}{2i-1}\cdot\prod_{i=1}^{k}\frac{2^{n-1}-i}{i}\equiv(-1)^k\binom{2^{n-1}-1}{k} \pmod{2^n}.$$

This prepares ground for a proof of the required result by induction on n. The base case $n=1$ is obvious. Assume the assertion is true for $n-1$ and pass to n, denoting $a_k=\binom{2^{n-1}-1}{k}$, $b_m=\binom{2^n-1}{m}$. The induction hypothesis is that all the numbers a_k $(0\le k<2^{n-2})$ are distinct $\pmod{2^{n-1}}$; the claim is that all the numbers b_m $(0\le m<2^{n-1})$ are distinct $\pmod{2^n}$.

The congruence relations (1) are restated as

$$b_{2k}\equiv(-1)^k a_k\equiv -b_{2k+1} \pmod{2^n}. \quad (2)$$

Shifting the exponent in the first relation of (1) from n to $n-1$ we also have the congruence $a_{2i+1}\equiv -a_{2i} \pmod{2^{n-1}}$. We hence conclude:

If, for some $j, k<2^{n-2}$, $a_k\equiv -a_j \pmod{2^{n-1}}$, then $\{j,k\}=\{2i, 2i+1\}$ for some i. (3)

This is so because in the sequence $(a_k\colon k<2^{n-2})$ each term a_j is complemented to $0 \pmod{2^{n-1}}$ by only one other term a_k, according to the induction hypothesis.

From (2) we see that $b_{4i}\equiv a_{2i}$ and $b_{4i+3}\equiv a_{2i+1} \pmod{2^n}$. Let

$$M=\{m\colon 0\le m<2^{n-1},\ m\equiv 0 \text{ or } 3 \pmod 4\},\quad L=\{l\colon 0\le l<2^{n-1},\ l\equiv 1 \text{ or } 2 \pmod 4\}.$$

The last two congruences take on the unified form

$$b_m\equiv a_{\lfloor m/2\rfloor} \pmod{2^n} \quad \text{for all} \quad m\in M. \quad (4)$$

Thus all the numbers b_m for $m\in M$ are distinct $\pmod{2^n}$ because so are the numbers a_k (they are distinct $\pmod{2^{n-1}}$, hence also $\pmod{2^n}$).

Every $l\in L$ is paired with a unique $m\in M$ into a pair of the form $\{2k, 2k+1\}$. So (2) implies that also all the b_l for $l\in L$ are distinct $\pmod{2^n}$. It remains to eliminate the possibility that $b_m\equiv b_l \pmod{2^n}$ for some $m\in M$, $l\in L$.

Suppose that such a situation occurs. Let $m'\in M$ be such that $\{m', l\}$ is a pair of the form $\{2k, 2k+1\}$, so that (see (2)) $b_{m'}\equiv -b_l \pmod{2^n}$. Hence $b_{m'}\equiv -b_m \pmod{2^n}$. Since both m' and m are in M, we have by (4) $b_{m'}\equiv a_j$, $b_m\equiv a_k \pmod{2^n}$ for $j=\lfloor m'/2\rfloor$, $k=\lfloor m/2\rfloor$.

Then $a_j\equiv -a_k \pmod{2^n}$. Thus, according to (3), $j=2i$, $k=2i+1$ for some i (or vice versa). The equality $a_{2i+1}\equiv -a_{2i} \pmod{2^n}$ now means that $\binom{2^{n-1}-1}{2i}+\binom{2^{n-1}-1}{2i+1}\equiv 0 \pmod{2^n}$. However, the sum on the left is equal to $\binom{2^{n-1}}{2i+1}$. A number of this form cannot be divisible by 2^n. This is a contradiction which concludes the induction step and proves the result.

Solution 2. We again proceed by induction, writing for brevity $N = 2^{n-1}$ and keeping notation $a_k = \binom{N-1}{k}$, $b_m = \binom{2N-1}{m}$. Assume that the result holds for the sequence $(a_0, a_1, a_2, \dots, a_{N/2-1})$. In view of the symmetry $a_{N-1-k} = a_k$ this sequence is a permutation of $(a_0, a_2, a_4, \dots, a_{N-2})$. So the induction hypothesis says that this latter sequence, taken (mod N), is a permutation of $(1, 3, 5, \dots, N-1)$. Similarly, the induction claim is that $(b_0, b_2, b_4, \dots, b_{2N-2})$, taken (mod $2N$), is a permutation of $(1, 3, 5, \dots, 2N-1)$.

In place of the congruence relations (2) we now use the following ones,

$$b_{4i} \equiv a_{2i} \pmod N \qquad \text{and} \qquad b_{4i+2} \equiv b_{4i} + N \pmod{2N}. \tag{5}$$

Given this, the conclusion is immediate: the first formula of (5) together with the induction hypothesis tells us that $(b_0, b_4, b_8, \dots, b_{2N-4})$ (mod N) is a permutation of $(1, 3, 5, \dots, N-1)$. Then the second formula of (5) shows that $(b_2, b_6, b_{10}, \dots, b_{2N-2})$ (mod N) is exactly the same permutation; moreover, this formula distinguishes (mod $2N$) each b_{4i} from b_{4i+2}.

Consequently, these two sequences combined represent (mod $2N$) a permutation of the sequence $(1, 3, 5, \dots, N-1,\ N+1, N+3, N+5, \dots, N+N-1)$, and this is precisely the induction claim.

Now we prove formulas (5); we begin with the second one. Since $b_{m+1} = b_m \cdot \frac{2N-m-1}{m+1}$,

$$b_{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{2N-4i-2}{4i+2} = b_{4i} \cdot \frac{2N-4i-1}{4i+1} \cdot \frac{N-2i-1}{2i+1}.$$

The desired congruence $b_{4i+2} \equiv b_{4i} + N$ may be multiplied by the odd number $(4i+1)(2i+1)$, giving rise to a chain of successively equivalent congruences:

$$\begin{aligned}
b_{4i}(2N-4i-1)(N-2i-1) &\equiv (b_{4i}+N)(4i+1)(2i+1) &&\pmod{2N},\\
b_{4i}(2i+1-N) &\equiv (b_{4i}+N)(2i+1) &&\pmod{2N},\\
(b_{4i}+2i+1)N &\equiv 0 &&\pmod{2N};
\end{aligned}$$

and the last one is satisfied, as b_{4i} is odd. This settles the second relation in (5).

The first one is proved by induction on i. It holds for $i = 0$. Assume $b_{4i} \equiv a_{2i}$ (mod $2N$) and consider $i+1$:

$$b_{4i+4} = b_{4i+2} \cdot \frac{2N-4i-3}{4i+3} \cdot \frac{2N-4i-4}{4i+4}; \qquad a_{2i+2} = a_{2i} \cdot \frac{N-2i-1}{2i+1} \cdot \frac{N-2i-2}{2i+2}.$$

Both expressions have the fraction $\frac{N-2i-2}{2i+2}$ as the last factor. Since $2i+2 < N = 2^{n-1}$, this fraction reduces to ℓ/m with ℓ and m odd. In showing that $b_{4i+4} \equiv a_{2i+2}$ (mod $2N$), we may ignore this common factor ℓ/m. Clearing other odd denominators reduces the claim to

$$b_{4i+2}(2N-4i-3)(2i+1) \equiv a_{2i}(N-2i-1)(4i+3) \pmod{2N}.$$

By the inductive assumption (saying that $b_{4i} \equiv a_{2i}$ (mod $2N$)) and by the second relation of (5), this is equivalent to

$$(b_{4i}+N)(2i+1) \equiv b_{4i}(2i+1-N) \pmod{2N},$$

a congruence which we have already met in the preceding proof a few lines above. This completes induction (on i) and the proof of (5), hence also the whole solution.

Comment. One can avoid the words *congruent modulo* in the problem statement by rephrasing the assertion into: *Show that these numbers leave distinct remainders in division by* 2^n.

N5. For every $n \in \mathbb{N}$ let $d(n)$ denote the number of (positive) divisors of n. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ with the following properties:

(i) $d(f(x)) = x$ for all $x \in \mathbb{N}$;

(ii) $f(xy)$ divides $(x-1)y^{xy-1}f(x)$ for all $x, y \in \mathbb{N}$.

Solution. There is a unique solution: the function $f\colon \mathbb{N} \to \mathbb{N}$ defined by $f(1) = 1$ and

$$f(n) = p_1^{p_1^{a_1}-1} p_2^{p_2^{a_2}-1} \cdots p_k^{p_k^{a_k}-1} \text{ where } n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \text{ is the prime factorization of } n > 1. \quad (1)$$

Direct verification shows that this function meets the requirements.

Conversely, let $f\colon \mathbb{N} \to \mathbb{N}$ satisfy (i) and (ii). Applying (i) for $x = 1$ gives $d(f(1)) = 1$, so $f(1) = 1$. In the sequel we prove that (1) holds for all $n > 1$. Notice that $f(m) = f(n)$ implies $m = n$ in view of (i). The formula $d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1+1)\cdots(b_k+1)$ will be used throughout.

Let p be a prime. Since $d(f(p)) = p$, the formula just mentioned yields $f(p) = q^{p-1}$ for some prime q; in particular $f(2) = q^{2-1} = q$ is a prime. We prove that $f(p) = p^{p-1}$ for all primes p.

Suppose that p is odd and $f(p) = q^{p-1}$ for a prime q. Applying (ii) first with $x = 2$, $y = p$ and then with $x = p$, $y = 2$ shows that $f(2p)$ divides both $(2-1)p^{2p-1}f(2) = p^{2p-1}f(2)$ and $(p-1)2^{2p-1}f(p) = (p-1)2^{2p-1}q^{p-1}$. If $q \neq p$ then the odd prime p does not divide $(p-1)2^{2p-1}q^{p-1}$, hence the greatest common divisor of $p^{2p-1}f(2)$ and $(p-1)2^{2p-1}q^{p-1}$ is a divisor of $f(2)$. Thus $f(2p)$ divides $f(2)$ which is a prime. As $f(2p) > 1$, we obtain $f(2p) = f(2)$ which is impossible. So $q = p$, i. e. $f(p) = p^{p-1}$.

For $p = 2$ the same argument with $x = 2$, $y = 3$ and $x = 3$, $y = 2$ shows that $f(6)$ divides both $3^5 f(2)$ and $2^6 f(3) = 2^6 3^2$. If the prime $f(2)$ is odd then $f(6)$ divides $3^2 = 9$, so $f(6) \in \{1, 3, 9\}$. However then $6 = d(f(6)) \in \{d(1), d(3), d(9)\} = \{1, 2, 3\}$ which is false. In conclusion $f(2) = 2$.

Next, for each $n > 1$ the prime divisors of $f(n)$ are among the ones of n. Indeed, let p be the least prime divisor of n. Apply (ii) with $x = p$ and $y = n/p$ to obtain that $f(n)$ divides $(p-1)y^{n-1}f(p) = (p-1)y^{n-1}p^{p-1}$. Write $f(n) = \ell P$ where ℓ is coprime to n and P is a product of primes dividing n. Since ℓ divides $(p-1)y^{n-1}p^{p-1}$ and is coprime to $y^{n-1}p^{p-1}$, it divides $p-1$; hence $d(\ell) \leq \ell < p$. But (i) gives $n = d(f(n)) = d(\ell P)$, and $d(\ell P) = d(\ell)d(P)$ as ℓ and P are coprime. Therefore $d(\ell)$ is a divisor of n less than p, meaning that $\ell = 1$ and proving the claim.

Now (1) is immediate for prime powers. If p is a prime and $a \geq 1$, by the above the only prime factor of $f(p^a)$ is p (a prime factor does exist as $f(p^a) > 1$). So $f(p^a) = p^b$ for some $b \geq 1$, and (i) yields $p^a = d(f(p^a)) = d\left(p^b\right) = b + 1$. Hence $f(p^a) = p^{p^a-1}$, as needed.

Let us finally show that (1) is true for a general $n > 1$ with prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$. We saw that the prime factorization of $f(n)$ has the form $f(n) = p_1^{b_1} \cdots p_k^{b_k}$. For $i = 1, \ldots, k$, set $x = p_i^{a_i}$ and $y = n/x$ in (ii) to infer that $f(n)$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$. Hence $p_i^{b_i}$ divides $(p_i^{a_i} - 1)\, y^{n-1} f(p_i^{a_i})$, and because $p_i^{b_i}$ is coprime to $(p_i^{a_i} - 1)\, y^{n-1}$, it follows that $p_i^{b_i}$ divides $f(p_i^{a_i}) = p_i^{p_i^{a_i}-1}$. So $b_i \leq p_i^{a_i} - 1$ for all $i = 1, \ldots, k$. Combined with (i), these conclusions imply

$$p_1^{a_1} \cdots p_k^{a_k} = n = d(f(n)) = d\left(p_1^{b_1} \cdots p_k^{b_k}\right) = (b_1+1)\cdots(b_k+1) \leq p_1^{a_1} \cdots p_k^{a_k}.$$

Hence all inequalities $b_i \leq p_i^{a_i} - 1$ must be equalities, $i = 1, \ldots, k$, implying that (1) holds true. The proof is complete.

N6. Prove that there exist infinitely many positive integers n such that n^2+1 has a prime divisor greater than $2n+\sqrt{2n}$.

Solution. Let $p \equiv 1 \pmod 8$ be a prime. The congruence $x^2 \equiv -1 \pmod p$ has two solutions in $[1, p-1]$ whose sum is p. If n is the smaller one of them then p divides n^2+1 and $n \le (p-1)/2$. We show that $p > 2n + \sqrt{10n}$.

Let $n = (p-1)/2 - \ell$ where $\ell \ge 0$. Then $n^2 \equiv -1 \pmod p$ gives

$$\left(\frac{p-1}{2} - \ell\right)^2 \equiv -1 \pmod p \qquad \text{or} \qquad (2\ell+1)^2 + 4 \equiv 0 \pmod p.$$

Thus $(2\ell+1)^2+4 = rp$ for some $r \ge 0$. As $(2\ell+1)^2 \equiv 1 \equiv p \pmod 8$, we have $r \equiv 5 \pmod 8$, so that $r \ge 5$. Hence $(2\ell+1)^2+4 \ge 5p$, implying $\ell \ge \big(\sqrt{5p-4}-1\big)/2$. Set $\sqrt{5p-4} = u$ for clarity; then $\ell \ge (u-1)/2$. Therefore

$$n = \frac{p-1}{2} - \ell \le \frac{1}{2}\big(p-u\big).$$

Combined with $p = (u^2+4)/5$, this leads to $u^2 - 5u - 10n + 4 \ge 0$. Solving this quadratic inequality with respect to $u \ge 0$ gives $u \ge \big(5 + \sqrt{40n+9}\big)/2$. So the estimate $n \le \big(p-u\big)/2$ leads to

$$p \ge 2n + u \ge 2n + \frac{1}{2}\big(5 + \sqrt{40n+9}\big) > 2n + \sqrt{10n}.$$

Since there are infinitely many primes of the form $8k+1$, it follows easily that there are also infinitely many n with the stated property.

Comment. By considering the prime factorization of the product $\prod_{n=1}^{N}(n^2+1)$, it can be obtained that its greatest prime divisor is at least $cN\log N$. This could improve the statement as $p > n\log n$.

However, the proof applies some advanced information about the distribution of the primes of the form $4k+1$, which is inappropriate for high schools contests.

50th International Mathematical Olympiad

Bremen | Germany | 2009

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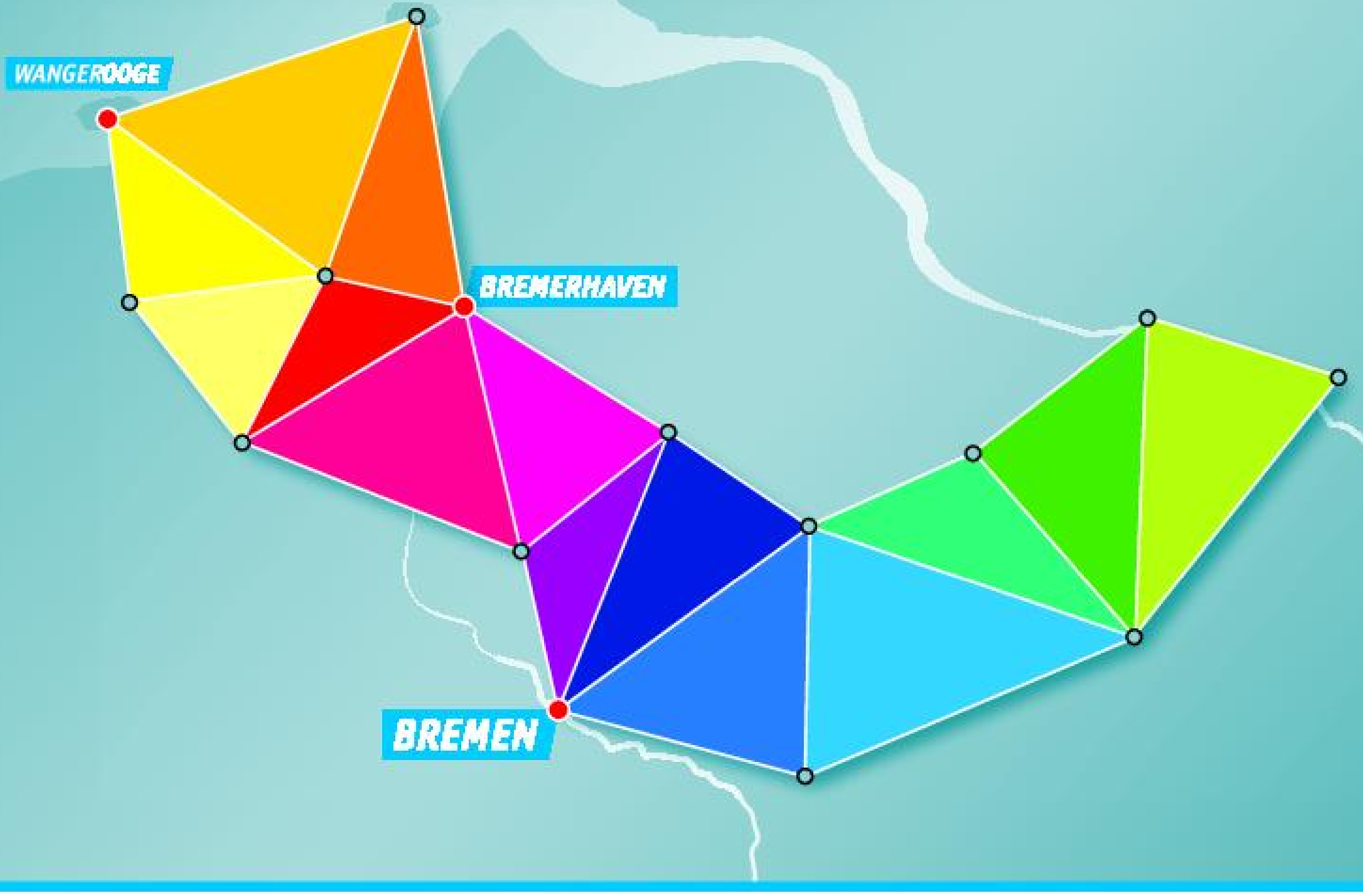
Problem Shortlist with solutions



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Problem Shortlist with Solutions

The Problem Selection Committee

The Note of Confidentiality **IMPORTANT**

We insistently ask everybody to consider the following IMO Regulations rule:

These Shortlist Problems have to be kept strictly confidential until IMO 2010.

The Problem Selection Committee

Konrad Engel, Karl Fegert, Andreas Felgenhauer, Hans-Dietrich Gronau, Roger Labahn, Bernd Mulansky, Jürgen Prestin, Christian Reiher, Peter Scholze, Eckard Specht, Robert Strich, Martin Welk

gratefully received

132 problem proposals submitted by 39 countries:

Algeria, Australia, Austria, Belarus, Belgium, Bulgaria, Colombia, Croatia, Czech Republic, El Salvador, Estonia, Finland, France, Greece, Hong Kong, Hungary, India, Ireland, Islamic Republic of Iran, Japan, Democratic People's Republic of Korea, Lithuania, Luxembourg, The former Yugoslav Republic of Macedonia, Mongolia, Netherlands, New Zealand, Pakistan, Peru, Poland, Romania, Russian Federation, Slovenia, South Africa, Taiwan, Turkey, Ukraine, United Kingdom, United States of America.

Layout: Roger Labahn with LaTeX & TeX
Drawings: Eckard Specht with `nicefig 2.0`

: The Problem Selection Committee

Contents

Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \leq b_2 \leq \ldots \leq b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \leq r_2 \leq \ldots \leq r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \leq w_2 \leq \ldots \leq w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = a + b + c$. Prove that

$$\frac{1}{(2a+b+c)^2} + \frac{1}{(2b+c+a)^2} + \frac{1}{(2c+a+b)^2} \leq \frac{3}{16}.$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y + f(x) - 1).$$

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab + bc + ca \leq 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}} + \sqrt{\frac{b^2+c^2}{b+c}} + \sqrt{\frac{c^2+a^2}{c+a}} + 3 \leq \sqrt{2}\left(\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}\right).$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f\left(x - f(y)\right) > yf(x) + x.$$

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \ldots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m, n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3, n)$ and $N(n, n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n, n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n + 2$.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \qquad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

C6 BGR (Bulgaria)

On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \ldots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \ldots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R - 1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle BAC$.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

G3 IRN (Islamic Republic of Iran)

Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \ldots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x+a)(x+b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \ldots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a - b$ divides $f(a) - f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n-1$.

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k - 2$ is divisible by 3.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n - 1)(b^n - 1)$ is not a perfect square.

Algebra

A1 CZE (Czech Republic)

Find the largest possible integer k, such that the following statement is true:

Let 2009 arbitrary non-degenerated triangles be given. In every triangle the three sides are colored, such that one is blue, one is red and one is white. Now, for every color separately, let us sort the lengths of the sides. We obtain

$$\begin{array}{rll} & b_1 \le b_2 \le \ldots \le b_{2009} & \text{the lengths of the blue sides,} \\ & r_1 \le r_2 \le \ldots \le r_{2009} & \text{the lengths of the red sides,} \\ \text{and} & w_1 \le w_2 \le \ldots \le w_{2009} & \text{the lengths of the white sides.} \end{array}$$

Then there exist k indices j such that we can form a non-degenerated triangle with side lengths b_j, r_j, w_j.

Solution. We will prove that the largest possible number k of indices satisfying the given condition is one.

Firstly we prove that b_{2009}, r_{2009}, w_{2009} are always lengths of the sides of a triangle. Without loss of generality we may assume that $w_{2009} \ge r_{2009} \ge b_{2009}$. We show that the inequality $b_{2009} + r_{2009} > w_{2009}$ holds. Evidently, there exists a triangle with side lengths w, b, r for the white, blue and red side, respectively, such that $w_{2009} = w$. By the conditions of the problem we have $b + r > w$, $b_{2009} \ge b$ and $r_{2009} \ge r$. From these inequalities it follows

$$b_{2009} + r_{2009} \ge b + r > w = w_{2009}.$$

Secondly we will describe a sequence of triangles for which w_j, b_j, r_j with $j < 2009$ are not the lengths of the sides of a triangle. Let us define the sequence Δ_j, $j = 1, 2, \ldots, 2009$, of triangles, where Δ_j has

a blue side of length $2j$,
a red side of length j for all $j \le 2008$ and 4018 for $j = 2009$,
and a white side of length $j+1$ for all $j \le 2007$, 4018 for $j = 2008$ and 1 for $j = 2009$.

Since

$$\begin{array}{rll} (j+1) + j > 2j & \ge j+1 > j, & \text{if } \ j \le 2007, \\ 2j + j > 4018 > 2j & > j, & \text{if } \ j = 2008, \\ 4018 + 1 > 2j & = 4018 > 1, & \text{if } \ j = 2009, \end{array}$$

such a sequence of triangles exists. Moreover, $w_j = j$, $r_j = j$ and $b_j = 2j$ for $1 \le j \le 2008$. Then

$$w_j + r_j = j + j = 2j = b_j,$$

i.e., b_j, r_j and w_j are not the lengths of the sides of a triangle for $1 \le j \le 2008$.

A2 EST (Estonia)

Let a, b, c be positive real numbers such that $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=a+b+c$. Prove that

$$\frac{1}{(2a+b+c)^2}+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\le\frac{3}{16}.$$

Solution 1. For positive real numbers x, y, z, from the arithmetic-geometric-mean inequality,

$$2x+y+z=(x+y)+(x+z)\ge 2\sqrt{(x+y)(x+z)},$$

we obtain

$$\frac{1}{(2x+y+z)^2}\le\frac{1}{4(x+y)(x+z)}.$$

Applying this to the left-hand side terms of the inequality to prove, we get

$$\begin{aligned}\frac{1}{(2a+b+c)^2}&+\frac{1}{(2b+c+a)^2}+\frac{1}{(2c+a+b)^2}\\ &\le\frac{1}{4(a+b)(a+c)}+\frac{1}{4(b+c)(b+a)}+\frac{1}{4(c+a)(c+b)}\\ &=\frac{(b+c)+(c+a)+(a+b)}{4(a+b)(b+c)(c+a)}=\frac{a+b+c}{2(a+b)(b+c)(c+a)}.\end{aligned}\tag{1}$$

A second application of the inequality of the arithmetic-geometric mean yields

$$a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\ge 6abc,$$

or, equivalently,

$$9(a+b)(b+c)(c+a)\ge 8(a+b+c)(ab+bc+ca).\tag{2}$$

The supposition $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=a+b+c$ can be written as

$$ab+bc+ca=abc(a+b+c).\tag{3}$$

Applying the arithmetic-geometric-mean inequality $x^2y^2+x^2z^2\ge 2x^2yz$ thrice, we get

$$a^2b^2+b^2c^2+c^2a^2\ge a^2bc+ab^2c+abc^2,$$

which is equivalent to

$$(ab+bc+ca)^2\ge 3abc(a+b+c).\tag{4}$$

Combining (1), (2), (3), and (4), we will finish the proof:

$$\begin{aligned}\frac{a+b+c}{2(a+b)(b+c)(c+a)} &= \frac{(a+b+c)(ab+bc+ca)}{2(a+b)(b+c)(c+a)} \cdot \frac{ab+bc+ca}{abc(a+b+c)} \cdot \frac{abc(a+b+c)}{(ab+bc+ca)^2} \\ &\le \frac{9}{2\cdot 8} \cdot 1 \cdot \frac{1}{3} = \frac{3}{16}.\end{aligned}$$

Solution 2. Equivalently, we prove the homogenized inequality

$$\frac{(a+b+c)^2}{(2a+b+c)^2} + \frac{(a+b+c)^2}{(a+2b+c)^2} + \frac{(a+b+c)^2}{(a+b+2c)^2} \le \frac{3}{16}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

for all positive real numbers a, b, c. Without loss of generality we choose $a+b+c=1$. Thus, the problem is equivalent to prove for all $a,b,c>0$, fulfilling this condition, the inequality

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} \le \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \tag{5}$$

Applying Jensen's inequality to the function $f(x) = \frac{x}{(1+x)^2}$, which is concave for $0 \le x \le 2$ and increasing for $0 \le x \le 1$, we obtain

$$\alpha\frac{a}{(1+a)^2} + \beta\frac{b}{(1+b)^2} + \gamma\frac{c}{(1+c)^2} \le (\alpha+\beta+\gamma)\frac{A}{(1+A)^2}, \quad \text{where} \quad A = \frac{\alpha a+\beta b+\gamma c}{\alpha+\beta+\gamma}.$$

Choosing $\alpha = \frac{1}{a}$, $\beta = \frac{1}{b}$, and $\gamma = \frac{1}{c}$, we can apply the harmonic-arithmetic-mean inequality

$$A = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \le \frac{a+b+c}{3} = \frac{1}{3} < 1.$$

Finally we prove (5):

$$\begin{aligned}\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} + \frac{1}{(1+c)^2} &\le \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{A}{(1+A)^2} \\ &\le \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\frac{\frac{1}{3}}{\left(1+\frac{1}{3}\right)^2} = \frac{3}{16}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).\end{aligned}$$

A3 FRA (France)

Determine all functions f from the set of positive integers into the set of positive integers such that for all x and y there exists a non degenerated triangle with sides of lengths

$$x, \quad f(y) \quad \text{and} \quad f(y+f(x)-1).$$

Solution. The identity function $f(x)=x$ is the only solution of the problem.

If $f(x)=x$ for all positive integers x, the given three lengths are x, $y=f(y)$ and $z=f(y+f(x)-1)=x+y-1$. Because of $x \geq 1$, $y \geq 1$ we have $z \geq \max\{x,y\} > |x-y|$ and $z < x+y$. From this it follows that a triangle with these side lengths exists and does not degenerate. We prove in several steps that there is no other solution.

Step 1. We show $f(1)=1$.
If we had $f(1)=1+m>1$ we would conclude $f(y)=f(y+m)$ for all y considering the triangle with the side lengths 1, $f(y)$ and $f(y+m)$. Thus, f would be m-periodic and, consequently, bounded. Let B be a bound, $f(x) \leq B$. If we choose $x > 2B$ we obtain the contradiction $x > 2B \geq f(y) + f(y+f(x)-1)$.

Step 2. For all positive integers z, we have $f(f(z))=z$.
Setting $x=z$ and $y=1$ this follows immediately from Step 1.

Step 3. For all integers $z \geq 1$, we have $f(z) \leq z$.
Let us show, that the contrary leads to a contradiction. Assume $w+1=f(z)>z$ for some z. From Step 1 we know that $w \geq z \geq 2$. Let $M=\max\{f(1), f(2), \ldots, f(w)\}$ be the largest value of f for the first w integers. First we show, that no positive integer t exists with

$$f(t) > \frac{z-1}{w} \cdot t + M, \tag{1}$$

otherwise we decompose the smallest value t as $t = wr+s$ where r is an integer and $1 \leq s \leq w$. Because of the definition of M, we have $t > w$. Setting $x=z$ and $y=t-w$ we get from the triangle inequality

$$z + f(t-w) > f((t-w)+f(z)-1) = f(t-w+w) = f(t).$$

Hence,

$$f(t-w) \geq f(t) - (z-1) > \frac{z-1}{w}(t-w) + M,$$

a contradiction to the minimality of t.

Therefore the inequality (1) fails for all $t \geq 1$, we have proven

$$f(t) \leq \frac{z-1}{w} \cdot t + M, \tag{2}$$

instead.

Now, using (2), we finish the proof of Step 3. Because of $z \leq w$ we have $\dfrac{z-1}{w} < 1$ and we can choose an integer t sufficiently large to fulfill the condition

$$\left(\frac{z-1}{w}\right)^2 t + \left(\frac{z-1}{w} + 1\right) M < t.$$

Applying (2) twice we get

$$f\left(f(t)\right) \leq \frac{z-1}{w} f(t) + M \leq \frac{z-1}{w}\left(\frac{z-1}{w} t + M\right) + M < t$$

in contradiction to Step 2, which proves Step 3.

Final step. Thus, following Step 2 and Step 3, we obtain

$$z = f(f(z)) \leq f(z) \leq z$$

and $f(z) = z$ for all positive integers z is proven.

A4 BLR (Belarus)

Let a, b, c be positive real numbers such that $ab+bc+ca \le 3abc$. Prove that

$$\sqrt{\frac{a^2+b^2}{a+b}}+\sqrt{\frac{b^2+c^2}{b+c}}+\sqrt{\frac{c^2+a^2}{c+a}}+3 \le \sqrt{2}\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right).$$

Solution. Starting with the terms of the right-hand side, the quadratic-arithmetic-mean inequality yields

$$\begin{aligned}\sqrt{2}\sqrt{a+b} &= 2\sqrt{\frac{ab}{a+b}}\sqrt{\frac{1}{2}\left(2+\frac{a^2+b^2}{ab}\right)}\\ &\ge 2\sqrt{\frac{ab}{a+b}}\cdot\frac{1}{2}\left(\sqrt{2}+\sqrt{\frac{a^2+b^2}{ab}}\right)=\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{a^2+b^2}{a+b}}\end{aligned}$$

and, analogously,

$$\sqrt{2}\sqrt{b+c} \ge \sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{b^2+c^2}{b+c}}, \qquad \sqrt{2}\sqrt{c+a} \ge \sqrt{\frac{2ca}{c+a}}+\sqrt{\frac{c^2+a^2}{c+a}}.$$

Applying the inequality between the arithmetic mean and the squared harmonic mean will finish the proof:

$$\sqrt{\frac{2ab}{a+b}}+\sqrt{\frac{2bc}{b+c}}+\sqrt{\frac{2ca}{c+a}} \ge 3\cdot\sqrt{\frac{3}{\sqrt{\frac{a+b}{2ab}}^2+\sqrt{\frac{b+c}{2bc}}^2+\sqrt{\frac{c+a}{2ca}}^2}}=3\cdot\sqrt{\frac{3abc}{ab+bc+ca}} \ge 3.$$

A5 BLR (Belarus)

Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x - f(y)) > yf(x) + x.$$

Solution 1. Assume that

$$f(x - f(y)) \leq yf(x) + x \quad \text{for all real } x, y. \tag{1}$$

Let $a = f(0)$. Setting $y = 0$ in (1) gives $f(x - a) \leq x$ for all real x and, equivalently,

$$f(y) \leq y + a \quad \text{for all real } y. \tag{2}$$

Setting $x = f(y)$ in (1) yields in view of (2)

$$a = f(0) \leq yf(f(y)) + f(y) \leq yf(f(y)) + y + a.$$

This implies $0 \leq y(f(f(y)) + 1)$ and thus

$$f(f(y)) \geq -1 \quad \text{for all } y > 0. \tag{3}$$

From (2) and (3) we obtain $-1 \leq f(f(y)) \leq f(y) + a$ for all $y > 0$, so

$$f(y) \geq -a - 1 \quad \text{for all } y > 0. \tag{4}$$

Now we show that

$$f(x) \leq 0 \quad \text{for all real } x. \tag{5}$$

Assume the contrary, i.e. there is some x such that $f(x) > 0$. Take any y such that

$$y < x - a \quad \text{and} \quad y < \frac{-a - x - 1}{f(x)}.$$

Then in view of (2)

$$x - f(y) \geq x - (y + a) > 0$$

and with (1) and (4) we obtain

$$yf(x) + x \geq f(x - f(y)) \geq -a - 1,$$

whence

$$y \geq \frac{-a - x - 1}{f(x)}$$

contrary to our choice of y. Thereby, we have established (5).

Setting $x = 0$ in (5) leads to $a = f(0) \leq 0$ and (2) then yields

$$f(x) \leq x \quad \text{for all real } x. \tag{6}$$

Now choose y such that $y > 0$ and $y > -f(-1) - 1$ and set $x = f(y) - 1$. From (1), (5) and

(6) we obtain

$$f(-1)=f(x-f(y))\le yf(x)+x=yf(f(y)-1)+f(y)-1\le y(f(y)-1)-1\le -y-1,$$

i.e. $y\le -f(-1)-1$, a contradiction to the choice of y.

Solution 2. Assume that

$$f(x-f(y))\le yf(x)+x \quad \text{for all real } x,y. \tag{7}$$

Let $a=f(0)$. Setting $y=0$ in (7) gives $f(x-a)\le x$ for all real x and, equivalently,

$$f(y)\le y+a \quad \text{for all real } y. \tag{8}$$

Now we show that

$$f(z)\ge 0 \quad \text{for all } z\ge 1. \tag{9}$$

Let $z\ge 1$ be fixed, set $b=f(z)$ and assume that $b<0$. Setting $x=w+b$ and $y=z$ in (7) gives

$$f(w)-zf(w+b)\le w+b \quad \text{for all real } w. \tag{10}$$

Applying (10) to $w, w+b, \ldots, w+(n-1)b$, where $n=1,2,\ldots$, leads to

$$\begin{aligned} f(w)-z^nf(w+nb) =& (f(w)-zf(w+b))+z\,(f(w+b)-zf(w+2b)) \\ &+\ \cdots\ +z^{n-1}\,(f(w+(n-1)b)-zf(w+nb)) \\ \le& (w+b)+z(w+2b)+\ \cdots\ +z^{n-1}(w+nb). \end{aligned}$$

From (8) we obtain

$$f(w+nb)\le w+nb+a$$

and, thus, we have for all positive integers n

$$f(w)\le (1+z+\cdots+z^{n-1}+z^n)w+(1+2z+\cdots+nz^{n-1}+nz^n)b+z^na. \tag{11}$$

With $w=0$ we get

$$a\le (1+2z+\cdots+nz^{n-1}+nz^n)b+az^n. \tag{12}$$

In view of the assumption $b<0$ we find some n such that

$$a>(nb+a)z^n \tag{13}$$

because the right hand side tends to $-\infty$ as $n\to\infty$. Now (12) and (13) give the desired contradiction and (9) is established. In addition, we have for $z=1$ the strict inequality

$$f(1)>0. \tag{14}$$

Indeed, assume that $f(1)=0$. Then setting $w=-1$ and $z=1$ in (11) leads to

$$f(-1)\le -(n+1)+a$$

which is false if n is sufficiently large.

To complete the proof we set $t=\min\{-a,-2/f(1)\}$. Setting $x=1$ and $y=t$ in (7) gives

$$f(1-f(t))\le tf(1)+1\le -2+1=-1. \tag{15}$$

On the other hand, by (8) and the choice of t we have $f(t) \leq t + a \leq 0$ and hence $1 - f(t) \geq 1$. The inequality (9) yields

$$f(1 - f(t)) \geq 0,$$

which contradicts (15).

A6 USA (United States of America)

Suppose that $s_1, s_2, s_3, \ldots$ is a strictly increasing sequence of positive integers such that the subsequences

$$s_{s_1},\ s_{s_2},\ s_{s_3},\ \ldots \qquad \text{and} \qquad s_{s_1+1},\ s_{s_2+1},\ s_{s_3+1},\ \ldots$$

are both arithmetic progressions. Prove that $s_1, s_2, s_3, \ldots$ is itself an arithmetic progression.

Solution 1. Let D be the common difference of the progression $s_{s_1}, s_{s_2}, \ldots$. Let for $n = 1, 2, \ldots$

$$d_n = s_{n+1} - s_n.$$

We have to prove that d_n is constant. First we show that the numbers d_n are bounded. Indeed, by supposition $d_n \geq 1$ for all n. Thus, we have for all n

$$d_n = s_{n+1} - s_n \leq d_{s_n} + d_{s_n+1} + \cdots + d_{s_{n+1}-1} = s_{s_{n+1}} - s_{s_n} = D.$$

The boundedness implies that there exist

$$m = \min\{d_n : n = 1, 2, \ldots\} \quad \text{and} \quad M = \max\{d_n : n = 1, 2, \ldots\}.$$

It suffices to show that $m = M$. Assume that $m < M$. Choose n such that $d_n = m$. Considering a telescoping sum of $m = d_n = s_{n+1} - s_n$ items not greater than M leads to

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+m} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+m-1} \leq mM \tag{1}$$

and equality holds if and only if all items of the sum are equal to M. Now choose n such that $d_n = M$. In the same way, considering a telescoping sum of M items not less than m we obtain

$$D = s_{s_{n+1}} - s_{s_n} = s_{s_n+M} - s_{s_n} = d_{s_n} + d_{s_n+1} + \cdots + d_{s_n+M-1} \geq Mm \tag{2}$$

and equality holds if and only if all items of the sum are equal to m. The inequalities (1) and (2) imply that $D = Mm$ and that

$$\begin{aligned} d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = M \quad &\text{if } d_n = m, \\ d_{s_n} = d_{s_n+1} = \cdots = d_{s_{n+1}-1} = m \quad &\text{if } d_n = M. \end{aligned}$$

Hence, $d_n = m$ implies $d_{s_n} = M$. Note that $s_n \geq s_1 + (n-1) \geq n$ for all n and moreover $s_n > n$ if $d_n = n$, because in the case $s_n = n$ we would have $m = d_n = d_{s_n} = M$ in contradiction to the assumption $m < M$. In the same way $d_n = M$ implies $d_{s_n} = m$ and $s_n > n$. Consequently, there is a strictly increasing sequence $n_1, n_2, \ldots$ such that

$$d_{s_{n_1}} = M, \quad d_{s_{n_2}} = m, \quad d_{s_{n_3}} = M, \quad d_{s_{n_4}} = m, \quad \ldots.$$

The sequence $d_{s_1}, d_{s_2}, \ldots$ is the sequence of pairwise differences of $s_{s_1+1}, s_{s_2+1}, \ldots$ and $s_{s_1}, s_{s_2}, \ldots$, hence also an arithmetic progression. Thus $m = M$.

Solution 2. Let the integers D and E be the common differences of the progressions $s_{s_1}, s_{s_2}, \ldots$ and $s_{s_1+1}, s_{s_2+1}, \ldots$, respectively. Let briefly $A = s_{s_1} - D$ and $B = s_{s_1+1} - E$. Then, for all positive integers n,

$$s_{s_n} = A + nD, \qquad s_{s_n+1} = B + nE.$$

Since the sequence $s_1, s_2, \ldots$ is strictly increasing, we have for all positive integers n

$$s_{s_n} < s_{s_n+1} \le s_{s_{n+1}},$$

which implies

$$A + nD < B + nE \le A + (n+1)D,$$

and thereby

$$0 < B - A + n(E - D) \le D,$$

which implies $D - E = 0$ and thus

$$0 \le B - A \le D. \tag{3}$$

Let $m = \min\{s_{n+1} - s_n : n = 1, 2, \ldots\}$. Then

$$B - A = (s_{s_1+1} - E) - (s_{s_1} - D) = s_{s_1+1} - s_{s_1} \ge m \tag{4}$$

and

$$D = A + (s_1 + 1)D - (A + s_1 D) = s_{s_{s_1+1}} - s_{s_{s_1}} = s_{B+D} - s_{A+D} \ge m(B - A). \tag{5}$$

From (3) we consider two cases.

Case 1. $B - A = D$.
Then, for each positive integer n, $s_{s_n+1} = B + nD = A + (n+1)D = s_{s_{n+1}}$, hence $s_{n+1} = s_n + 1$ and $s_1, s_2, \ldots$ is an arithmetic progression with common difference 1.

Case 2. $B - A < D$. Choose some positive integer N such that $s_{N+1} - s_N = m$. Then

$$\begin{aligned} m(A - B + D - 1) &= m((A + (N+1)D) - (B + ND + 1)) \\ &\le s_{A+(N+1)D} - s_{B+ND+1} = s_{s_{s_{N+1}}} - s_{s_{s_N}+1} \\ &= (A + s_{N+1}D) - (B + (s_N + 1)D) = (s_{N+1} - s_N)D + A - B - D \\ &= mD + A - B - D, \end{aligned}$$

i.e.,

$$(B - A - m) + (D - m(B - A)) \le 0. \tag{6}$$

The inequalities (4)-(6) imply that

$$B - A = m \quad \text{and} \quad D = m(B - A).$$

Assume that there is some positive integer n such that $s_{n+1} > s_n + m$. Then

$$m(m+1) \le m(s_{n+1} - s_n) \le s_{s_{n+1}} - s_{s_n} = (A + (n+1)D) - (A + nD)) = D = m(B - A) = m^2,$$

a contradiction. Hence $s_1, s_2, \ldots$ is an arithmetic progression with common difference m.

A7 JPN (Japan)

Find all functions f from the set of real numbers into the set of real numbers which satisfy for all real x, y the identity

$$f(xf(x+y)) = f(yf(x)) + x^2.$$

Solution 1. It is no hard to see that the two functions given by $f(x) = x$ and $f(x) = -x$ for all real x respectively solve the functional equation. In the sequel, we prove that there are no further solutions.

Let f be a function satisfying the given equation. It is clear that f cannot be a constant. Let us first show that $f(0) = 0$. Suppose that $f(0) \neq 0$. For any real t, substituting $(x, y) = (0, \frac{t}{f(0)})$ into the given functional equation, we obtain

$$f(0) = f(t), \tag{1}$$

contradicting the fact that f is not a constant function. Therefore, $f(0) = 0$. Next for any t, substituting $(x, y) = (t, 0)$ and $(x, y) = (t, -t)$ into the given equation, we get

$$f\left(tf(t)\right) = f(0) + t^2 = t^2,$$

and

$$f(tf(0)) = f(-tf(t)) + t^2,$$

respectively. Therefore, we conclude that

$$f(tf(t)) = t^2, \quad f(-tf(t)) = -t^2, \qquad \text{for every real } t. \tag{2}$$

Consequently, for every real v, there exists a real u, such that $f(u) = v$. We also see that if $f(t) = 0$, then $0 = f(tf(t)) = t^2$ so that $t = 0$, and thus 0 is the only real number satisfying $f(t) = 0$.

We next show that for any real number s,

$$f(-s) = -f(s). \tag{3}$$

This is clear if $f(s) = 0$. Suppose now $f(s) < 0$, then we can find a number t for which $f(s) = -t^2$. As $t \neq 0$ implies $f(t) \neq 0$, we can also find number a such that $af(t) = s$. Substituting $(x, y) = (t, a)$ into the given equation, we get

$$f(tf(t+a)) = f(af(t)) + t^2 = f(s) + t^2 = 0,$$

and therefore, $tf(t+a) = 0$, which implies $t + a = 0$, and hence $s = -tf(t)$. Consequently, $f(-s) = f(tf(t)) = t^2 = -(-t^2) = -f(s)$ holds in this case.

Finally, suppose $f(s) > 0$ holds. Then there exists a real number $t \neq 0$ for which $f(s) = t^2$. Choose a number a such that $tf(a) = s$. Substituting $(x, y) = (t, a-t)$ into the given equation, we get $f(s) = f(tf(a)) = f((a-t)f(t)) + t^2 = f((a-t)f(t)) + f(s)$. So we have $f((a-t)f(t)) = 0$, from which we conclude that $(a-t)f(t) = 0$. Since $f(t) \neq 0$, we get $a = t$ so that $s = tf(t)$ and thus we see $f(-s) = f(-tf(t)) = -t^2 = -f(s)$ holds in this case also. This observation finishes the proof of (3).

By substituting $(x, y) = (s, t)$, $(x, y) = (t, -s-t)$ and $(x, y) = (-s-t, s)$ into the given equation, we obtain

$$f(sf(s+t))) = f(tf(s)) + s^2,$$
$$f(tf(-s)) = f((-s-t)f(t)) + t^2,$$

and

$$f((-s-t)f(-t)) = f(sf(-s-t)) + (s+t)^2,$$

respectively. Using the fact that $f(-x) = -f(x)$ holds for all x to rewrite the second and the third equation, and rearranging the terms, we obtain

$$\begin{aligned} f(tf(s)) - f(sf(s+t)) &= -s^2, \\ f(tf(s)) - f((s+t)f(t)) &= -t^2, \\ f((s+t)f(t)) + f(sf(s+t)) &= (s+t)^2. \end{aligned}$$

Adding up these three equations now yields $2f(tf(s)) = 2ts$, and therefore, we conclude that $f(tf(s)) = ts$ holds for every pair of real numbers s, t. By fixing s so that $f(s) = 1$, we obtain $f(x) = sx$. In view of the given equation, we see that $s = \pm 1$. It is easy to check that both functions $f(x) = x$ and $f(x) = -x$ satisfy the given functional equation, so these are the desired solutions.

Solution 2. As in Solution 1 we obtain (1), (2) and (3).

Now we prove that f is injective. For this purpose, let us assume that $f(r) = f(s)$ for some $r \neq s$. Then, by (2)

$$r^2 = f(rf(r)) = f(rf(s)) = f((s-r)f(r)) + r^2,$$

where the last statement follows from the given functional equation with $x = r$ and $y = s - r$. Hence, $h = (s-r)f(r)$ satisfies $f(h) = 0$ which implies $h^2 = f(hf(h)) = f(0) = 0$, i.e., $h = 0$. Then, by $s \neq r$ we have $f(r) = 0$ which implies $r = 0$, and finally $f(s) = f(r) = f(0) = 0$. Analogously, it follows that $s = 0$ which gives the contradiction $r = s$.

To prove $|f(1)| = 1$ we apply (2) with $t = 1$ and also with $t = f(1)$ and obtain $f(f(1)) = 1$ and $(f(1))^2 = f(f(1) \cdot f(f(1))) = f(f(1)) = 1$.

Now we choose $\eta \in \{-1, 1\}$ with $f(1) = \eta$. Using that f is odd and the given equation with $x = 1, y = z$ (second equality) and with $x = -1, y = z + 2$ (fourth equality) we obtain

$$\begin{aligned} f(z) + 2\eta &= \eta(f(z\eta) + 2) = \eta(f(f(z+1)) + 1) = \eta(-f(-f(z+1)) + 1) \\ &= -\eta f((z+2)f(-1)) = -\eta f((z+2)(-\eta)) = \eta f((z+2)\eta) = f(z+2). \end{aligned} \tag{4}$$

Hence,

$$f(z + 2\eta) = \eta f(\eta z + 2) = \eta(f(\eta z) + 2\eta) = f(z) + 2.$$

Using this argument twice we obtain

$$f(z + 4\eta) = f(z + 2\eta) + 2 = f(z) + 4.$$

Substituting $z = 2f(x)$ we have

$$f(2f(x)) + 4 = f(2f(x) + 4\eta) = f(2f(x+2)),$$

where the last equality follows from (4). Applying the given functional equation we proceed to

$$f(2f(x+2)) = f(xf(2)) + 4 = f(2\eta x) + 4$$

where the last equality follows again from (4) with $z = 0$, i.e., $f(2) = 2\eta$. Finally, $f(2f(x)) = f(2\eta x)$ and by injectivity of f we get $2f(x) = 2\eta x$ and hence the two solutions.

Combinatorics

C1 NZL (New Zealand)

Consider 2009 cards, each having one gold side and one black side, lying in parallel on a long table. Initially all cards show their gold sides. Two players, standing by the same long side of the table, play a game with alternating moves. Each move consists of choosing a block of 50 consecutive cards, the leftmost of which is showing gold, and turning them all over, so those which showed gold now show black and vice versa. The last player who can make a legal move wins.

(a) Does the game necessarily end?

(b) Does there exist a winning strategy for the starting player?

Solution. (a) We interpret a card showing black as the digit 0 and a card showing gold as the digit 1. Thus each position of the 2009 cards, read from left to right, corresponds bijectively to a nonnegative integer written in binary notation of 2009 digits, where leading zeros are allowed. Each move decreases this integer, so the game must end.

(b) We show that there is no winning strategy for the starting player. We label the cards from right to left by $1, \ldots, 2009$ and consider the set S of cards with labels $50i$, $i = 1, 2, \ldots, 40$. Let g_n be the number of cards from S showing gold after n moves. Obviously, $g_0 = 40$. Moreover, $|g_n - g_{n+1}| = 1$ as long as the play goes on. Thus, after an odd number of moves, the non-starting player finds a card from S showing gold and hence can make a move. Consequently, this player always wins.

C2 ROU (Romania)

For any integer $n \geq 2$, let $N(n)$ be the maximal number of triples (a_i, b_i, c_i), $i = 1, \ldots, N(n)$, consisting of nonnegative integers a_i, b_i and c_i such that the following two conditions are satisfied:

(1) $a_i + b_i + c_i = n$ for all $i = 1, \ldots, N(n)$,

(2) If $i \neq j$, then $a_i \neq a_j, b_i \neq b_j$ and $c_i \neq c_j$.

Determine $N(n)$ for all $n \geq 2$.

Comment. The original problem was formulated for m-tuples instead for triples. The numbers $N(m,n)$ are then defined similarly to $N(n)$ in the case $m = 3$. The numbers $N(3,n)$ and $N(n,n)$ should be determined. The case $m = 3$ is the same as in the present problem. The upper bound for $N(n,n)$ can be proved by a simple generalization. The construction of a set of triples attaining the bound can be easily done by induction from n to $n+2$.

Solution. Let $n \geq 2$ be an integer and let $\{T_1, \ldots, T_N\}$ be any set of triples of nonnegative integers satisfying the conditions (1) and (2). Since the a-coordinates are pairwise distinct we have

$$\sum_{i=1}^{N} a_i \geq \sum_{i=1}^{N} (i-1) = \frac{N(N-1)}{2}.$$

Analogously,

$$\sum_{i=1}^{N} b_i \geq \frac{N(N-1)}{2} \quad \text{and} \quad \sum_{i=1}^{N} c_i \geq \frac{N(N-1)}{2}.$$

Summing these three inequalities and applying (1) yields

$$3\frac{N(N-1)}{2} \leq \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} b_i + \sum_{i=1}^{N} c_i = \sum_{i=1}^{N} (a_i + b_i + c_i) = nN,$$

hence $3\frac{N-1}{2} \leq n$ and, consequently,

$$N \leq \left\lfloor \frac{2n}{3} \right\rfloor + 1.$$

By constructing examples, we show that this upper bound can be attained, so $N(n) = \lfloor \frac{2n}{3} \rfloor + 1$.

We distinguish the cases $n = 3k-1$, $n = 3k$ and $n = 3k+1$ for $k \geq 1$ and present the extremal examples in form of a table.

$n = 3k-1$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k$		
a_i	b_i	c_i
0	$k+1$	$2k-2$
1	$k+2$	$2k-4$
$\vdots$	$\vdots$	$\vdots$
$k-1$	$2k$	0
k	0	$2k-1$
$k+1$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k-1$	$k-1$	1

$n = 3k$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k+1$		
a_i	b_i	c_i
0	k	$2k$
1	$k+1$	$2k-2$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	0
$k+1$	0	$2k-1$
$k+2$	1	$2k-3$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	1

$n = 3k+1$, $\lfloor \frac{2n}{3} \rfloor + 1 = 2k+1$		
a_i	b_i	c_i
0	k	$2k+1$
1	$k+1$	$2k-1$
$\vdots$	$\vdots$	$\vdots$
k	$2k$	1
$k+1$	0	$2k$
$k+2$	1	$2k-2$
$\vdots$	$\vdots$	$\vdots$
$2k$	$k-1$	2

It can be easily seen that the conditions (1) and (2) are satisfied and that we indeed have $\lfloor \frac{2n}{3} \rfloor + 1$ triples in each case.

Comment. A cute combinatorial model is given by an equilateral triangle, partitioned into n^2 congruent equilateral triangles by $n-1$ equidistant parallels to each of its three sides. Two chess-like bishops placed at any two vertices of the small triangles are said to *menace* one another if they lie on a same parallel. The problem is to determine the largest number of bishops that can be placed so that none menaces another. A bishop may be assigned three coordinates a, b, c, namely the numbers of sides of small triangles they are off each of the sides of the big triangle. It is readily seen that the sum of these coordinates is always n, therefore fulfilling the requirements.

C3 RUS (Russian Federation)

Let n be a positive integer. Given a sequence $\varepsilon_1, \ldots, \varepsilon_{n-1}$ with $\varepsilon_i = 0$ or $\varepsilon_i = 1$ for each $i = 1, \ldots, n-1$, the sequences $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ are constructed by the following rules:

$$a_0 = b_0 = 1, \quad a_1 = b_1 = 7,$$

$$a_{i+1} = \begin{cases} 2a_{i-1} + 3a_i, & \text{if } \varepsilon_i = 0, \\ 3a_{i-1} + a_i, & \text{if } \varepsilon_i = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1,$$

$$b_{i+1} = \begin{cases} 2b_{i-1} + 3b_i, & \text{if } \varepsilon_{n-i} = 0, \\ 3b_{i-1} + b_i, & \text{if } \varepsilon_{n-i} = 1, \end{cases} \quad \text{for each } i = 1, \ldots, n-1.$$

Prove that $a_n = b_n$.

Solution. For a binary word $w = \sigma_1 \ldots \sigma_n$ of length n and a letter $\sigma \in \{0, 1\}$ let $w\sigma = \sigma_1 \ldots \sigma_n \sigma$ and $\sigma w = \sigma \sigma_1 \ldots \sigma_n$. Moreover let $\overline{w} = \sigma_n \ldots \sigma_1$ and let $\emptyset$ be the empty word (of length 0 and with $\overline{\emptyset} = \emptyset$). Let (u, v) be a pair of two real numbers. For binary words w we define recursively the numbers $(u, v)^w$ as follows:

$$(u, v)^{\emptyset} = v, \quad (u, v)^0 = 2u + 3v, \quad (u, v)^1 = 3u + v,$$

$$(u, v)^{w\sigma\varepsilon} = \begin{cases} 2(u, v)^w + 3(u, v)^{w\sigma}, & \text{if } \varepsilon = 0, \\ 3(u, v)^w + (u, v)^{w\sigma}, & \text{if } \varepsilon = 1. \end{cases}$$

It easily follows by induction on the length of w that for all real numbers $u_1, v_1, u_2, v_2, \lambda_1$ and λ_2

$$(\lambda_1 u_1 + \lambda_2 u_2, \lambda_1 v_1 + \lambda_2 v_2)^w = \lambda_1 (u_1, v_1)^w + \lambda_2 (u_2, v_2)^w \tag{1}$$

and that for $\varepsilon \in \{0, 1\}$

$$(u, v)^{\varepsilon w} = (v, (u, v)^{\varepsilon})^w. \tag{2}$$

Obviously, for $n \geq 1$ and $w = \varepsilon_1 \ldots \varepsilon_{n-1}$, we have $a_n = (1, 7)^w$ and $b_n = (1, 7)^{\overline{w}}$. Thus it is sufficient to prove that

$$(1, 7)^w = (1, 7)^{\overline{w}} \tag{3}$$

for each binary word w. We proceed by induction on the length of w. The assertion is obvious if w has length 0 or 1. Now let $w\sigma\varepsilon$ be a binary word of length $n \geq 2$ and suppose that the assertion is true for all binary words of length at most $n - 1$.

Note that $(2, 1)^{\sigma} = 7 = (1, 7)^{\emptyset}$ for $\sigma \in \{0, 1\}$, $(1, 7)^0 = 23$, and $(1, 7)^1 = 10$.

First let $\varepsilon = 0$. Then in view of the induction hypothesis and the equalities (1) and (2), we obtain

$$\begin{aligned}(1, 7)^{w\sigma 0} &= 2(1, 7)^w + 3(1, 7)^{w\sigma} = 2(1, 7)^{\overline{w}} + 3(1, 7)^{\sigma\overline{w}} = 2(2, 1)^{\sigma\overline{w}} + 3(1, 7)^{\sigma\overline{w}} \\ &= (7, 23)^{\sigma\overline{w}} = (1, 7)^{0\sigma\overline{w}}.\end{aligned}$$

Now let $\varepsilon = 1$. Analogously, we obtain

$$(1,7)^{w\sigma 1} = 3(1,7)^{w} + (1,7)^{w\sigma} = 3(1,7)^{\overline{w}} + (1,7)^{\sigma\overline{w}} = 3(2,1)^{\sigma\overline{w}} + (1,7)^{\sigma\overline{w}}$$
$$= (7,10)^{\sigma\overline{w}} = (1,7)^{1\sigma\overline{w}}.$$

Thus the induction step is complete, (3) and hence also $a_n = b_n$ are proved.

Comment. The original solution uses the relation

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha}, \quad \alpha, \beta \in \{0,1\},$$

which can be proved by induction on the length of w. Then (3) also follows by induction on the length of w:

$$(1,7)^{\alpha\beta w} = ((1,7)^{w}, (1,7)^{\beta w})^{\alpha} = ((1,7)^{\overline{w}}, (1,7)^{\overline{w}\beta})^{\alpha} = (1,7)^{\overline{w}\beta\alpha}.$$

Here w may be the empty word.

C4 NLD (Netherlands)

For an integer $m \geq 1$, we consider partitions of a $2^m \times 2^m$ chessboard into rectangles consisting of cells of the chessboard, in which each of the 2^m cells along one diagonal forms a separate rectangle of side length 1. Determine the smallest possible sum of rectangle perimeters in such a partition.

Solution 1. For a $k \times k$ chessboard, we introduce in a standard way coordinates of the vertices of the cells and assume that the cell C_{ij} in row i and column j has vertices $(i-1, j-1), (i-1, j), (i, j-1), (i, j)$, where $i, j \in \{1, \ldots, k\}$. Without loss of generality assume that the cells C_{ii}, $i = 1, \ldots, k$, form a separate rectangle. Then we may consider the boards $B_k = \bigcup_{1 \leq i < j \leq k} C_{ij}$ below that diagonal and the congruent board $B'_k = \bigcup_{1 \leq j < i \leq k} C_{ij}$ above that diagonal separately because no rectangle can simultaneously cover cells from B_k and B'_k. We will show that for $k = 2^m$ the smallest total perimeter of a rectangular partition of B_k is $m2^{m+1}$. Then the overall answer to the problem is $2 \cdot m2^{m+1} + 4 \cdot 2^m = (m+1)2^{m+2}$.

First we inductively construct for $m \geq 1$ a partition of B_{2^m} with total perimeter $m2^{m+1}$. If $m = 0$, the board B_{2^m} is empty and the total perimeter is 0. For $m \geq 0$, the board $B_{2^{m+1}}$ consists of a $2^m \times 2^m$ square in the lower right corner with vertices $(2^m, 2^m), (2^m, 2^{m+1}), (2^{m+1}, 2^m)$, $(2^{m+1}, 2^{m+1})$ to which two boards congruent to B_{2^m} are glued along the left and the upper margin. The square together with the inductive partitions of these two boards yield a partition with total perimeter $4 \cdot 2^m + 2 \cdot m2^{m+1} = (m+1)2^{m+2}$ and the induction step is complete.

Let

$$D_k = 2k \log_2 k.$$

Note that $D_k = m2^{m+1}$ if $k = 2^m$. Now we show by induction on k that the total perimeter of a rectangular partition of B_k is at least D_k. The case $k = 1$ is trivial (see $m = 0$ from above). Let the assertion be true for all positive integers less than k. We investigate a fixed rectangular partition of B_k that attains the minimal total perimeter. Let R be the rectangle that covers the cell C_{1k} in the lower right corner. Let (i, j) be the upper left corner of R. First we show that $i = j$. Assume that $i < j$. Then the line from (i, j) to $(i+1, j)$ or from (i, j) to $(i, j-1)$ must belong to the boundary of some rectangle in the partition. Without loss of generality assume that this is the case for the line from (i, j) to $(i+1, j)$.

Case 1. No line from (i, l) to $(i+1, l)$ where $j < l < k$ belongs to the boundary of some rectangle of the partition.
Then there is some rectangle R' of the partition that has with R the common side from (i, j) to (i, k). If we join these two rectangles to one rectangle we get a partition with smaller total perimeter, a contradiction.

Case 2. There is some l such that $j < l < k$ and the line from (i, l) to $(i+1, l)$ belongs to the boundary of some rectangle of the partition.
Then we replace the upper side of R by the line $(i+1, j)$ to $(i+1, k)$ and for the rectangles whose lower side belongs to the line from (i, j) to (i, k) we shift the lower side upwards so that the new lower side belongs to the line from $(i+1, j)$ to $(i+1, k)$. In such a way we obtain a rectangular partition of B_k with smaller total perimeter, a contradiction.

Now the fact that the upper left corner of R has the coordinates (i, i) is established. Consequently, the partition consists of R, of rectangles of a partition of a board congruent to B_i and of rectangles of a partition of a board congruent to B_{k-i}. By the induction hypothesis, its total perimeter is at least

$$2(k-i)+2i+D_i+D_{k-i} \geq 2k+2i\log_2 i+2(k-i)\log_2(k-i). \tag{1}$$

Since the function $f(x)=2x\log_2 x$ is convex for $x>0$, Jensen's inequality immediately shows that the minimum of the right hand sight of (1) is attained for $i=k/2$. Hence the total perimeter of the optimal partition of B_k is at least $2k+2k/2\log_2 k/2+2(k/2)\log_2(k/2)=D_k$.

Solution 2. We start as in Solution 1 and present another proof that $m2^{m+1}$ is a lower bound for the total perimeter of a partition of B_{2^m} into n rectangles. Let briefly $M=2^m$. For $1\leq i\leq M$, let r_i denote the number of rectangles in the partition that cover some cell from row i and let c_j be the number of rectangles that cover some cell from column j. Note that the total perimeter p of all rectangles in the partition is

$$p=2\left(\sum_{i=1}^{M} r_i+\sum_{i=1}^{M} c_i\right).$$

No rectangle can simultaneously cover cells from row i and from column i since otherwise it would also cover the cell C_{ii}. We classify subsets S of rectangles of the partition as follows. We say that S is of type i, $1\leq i\leq M$, if S contains all r_i rectangles that cover some cell from row i, but none of the c_i rectangles that cover some cell from column i. Altogether there are $2^{n-r_i-c_i}$ subsets of type i. Now we show that no subset S can be simultaneously of type i and of type j if $i\neq j$. Assume the contrary and let without loss of generality $i<j$. The cell C_{ij} must be covered by some rectangle R. The subset S is of type i, hence R is contained in S. S is of type j, thus R does not belong to S, a contradiction. Since there are 2^n subsets of rectangles of the partition, we infer

$$2^n \geq \sum_{i=1}^{M} 2^{n-r_i-c_i}=2^n\sum_{i=1}^{M} 2^{-(r_i+c_i)}. \tag{2}$$

By applying Jensen's inequality to the convex function $f(x)=2^{-x}$ we derive

$$\frac{1}{M}\sum_{i=1}^{M} 2^{-(r_i+c_i)} \geq 2^{-\frac{1}{M}\sum_{i=1}^{M}(r_i+c_i)}=2^{-\frac{p}{2M}}. \tag{3}$$

From (2) and (3) we obtain

$$1\geq M2^{-\frac{p}{2M}}$$

and equivalently

$$p\geq m2^{m+1}.$$

C5 NLD (Netherlands)

Five identical empty buckets of 2-liter capacity stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Can the wicked Stepmother enforce a bucket overflow?

Solution 1. No, the Stepmother cannot enforce a bucket overflow and Cinderella can keep playing forever. Throughout we denote the five buckets by B_0, B_1, B_2, B_3, and B_4, where B_k is adjacent to bucket B_{k-1} and B_{k+1} $(k = 0,1,2,3,4)$ and all indices are taken modulo 5. Cinderella enforces that the following three conditions are satisfied at the beginning of every round:

(1) Two adjacent buckets (say B_1 and B_2) are empty.

(2) The two buckets standing next to these adjacent buckets (here B_0 and B_3) have total contents at most 1.

(3) The remaining bucket (here B_4) has contents at most 1.

These conditions clearly hold at the beginning of the first round, when all buckets are empty.

Assume that Cinderella manages to maintain them until the beginning of the r-th round ($r \geq 1$). Denote by x_k $(k = 0,1,2,3,4)$ the contents of bucket B_k at the beginning of this round and by y_k the corresponding contents after the Stepmother has distributed her liter of water in this round.

By the conditions, we can assume $x_1 = x_2 = 0$, $x_0 + x_3 \leq 1$ and $x_4 \leq 1$. Then, since the Stepmother adds one liter, we conclude $y_0+y_1+y_2+y_3 \leq 2$. This inequality implies $y_0+y_2 \leq 1$ or $y_1 + y_3 \leq 1$. For reasons of symmetry, we only consider the second case.

Then Cinderella empties buckets B_0 and B_4.

At the beginning of the next round B_0 and B_4 are empty (condition (1) is fulfilled), due to $y_1+y_3 \leq 1$ condition (2) is fulfilled and finally since $x_2 = 0$ we also must have $y_2 \leq 1$ (condition (3) is fulfilled).

Therefore, Cinderella can indeed manage to maintain the three conditions (1)–(3) also at the beginning of the $(r+1)$-th round. By induction, she thus manages to maintain them at the beginning of every round. In particular she manages to keep the contents of every single bucket at most 1 liter. Therefore, the buckets of 2-liter capacity will never overflow.

Solution 2. We prove that Cinderella can maintain the following two conditions and hence she can prevent the buckets from overflow:

(1′) Every two non-adjacent buckets contain a total of at most 1.

(2′) The total contents of all five buckets is at most $\frac{3}{2}$.

We use the same notations as in the first solution. The two conditions again clearly hold at the beginning. Assume that Cinderella maintained these two conditions until the beginning of the r-th round. A pair of non-neighboring buckets (B_i, B_{i+2}), $i = 0,1,2,3,4$ is called *critical* if $y_i + y_{i+2} > 1$. By condition (2′), after the Stepmother has distributed her water we have $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$. Therefore,

$$(y_0 + y_2) + (y_1 + y_3) + (y_2 + y_4) + (y_3 + y_0) + (y_4 + y_1) = 2(y_0 + y_1 + y_2 + y_3 + y_4) \le 5,$$

and hence there is a pair of non-neighboring buckets which is not critical, say (B_0, B_2). Now, if both of the pairs (B_3, B_0) and (B_2, B_4) are critical, we must have $y_1 < \frac{1}{2}$ and Cinderella can empty the buckets B_3 and B_4. This clearly leaves no critical pair of buckets and the total contents of all the buckets is then $y_1 + (y_0 + y_2) \le \frac{3}{2}$. Therefore, conditions (1′) and (2′) are fulfilled.

Now suppose that without loss of generality the pair (B_3, B_0) is not critical. If in this case $y_0 \le \frac{1}{2}$, then one of the inequalities $y_0 + y_1 + y_2 \le \frac{3}{2}$ and $y_0 + y_3 + y_4 \le \frac{3}{2}$ must hold. But then Cinderella can empty B_3 and B_4 or B_1 and B_2, respectively and clearly fulfill the conditions.

Finally consider the case $y_0 > \frac{1}{2}$. By $y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$, at least one of the pairs (B_1, B_3) and (B_2, B_4) is not critical. Without loss of generality let this be the pair (B_1, B_3). Since the pair (B_3, B_0) is not critical and $y_0 > \frac{1}{2}$, we must have $y_3 \le \frac{1}{2}$. But then, as before, Cinderella can maintain the two conditions at the beginning of the next round by either emptying B_1 and B_2 or B_4 and B_0.

Comments on GREEDY approaches. A natural approach for Cinderella would be a GREEDY strategy as for example: *Always remove as much water as possible from the system.* It is straightforward to prove that GREEDY can avoid buckets of capacity $\frac{5}{2}$ from overflowing: If before the Stepmothers move one has $x_0 + x_1 + x_2 + x_3 + x_4 \le \frac{3}{2}$ then after her move the inequality $Y = y_0 + y_1 + y_2 + y_3 + y_4 \le \frac{5}{2}$ holds. If now Cinderella removes the two adjacent buckets with maximum total contents she removes at least $\frac{2Y}{5}$ and thus the remaining buckets contain at most $\frac{3}{5} \cdot Y \le \frac{3}{2}$.

But GREEDY is in general not strong enough to settle this problem as can be seen in the following example:

- In an initial phase, the Stepmother brings all the buckets (after her move) to contents of at least $\frac{1}{2} - 2\epsilon$, where ϵ is an arbitrary small positive number. This can be done by always splitting the 1 liter she has to distribute so that all buckets have the same contents. After her r-th move the total contents of each of the buckets is then c_r with $c_1 = 1$ and $c_{r+1} = 1 + \frac{3}{5} \cdot c_r$ and hence $c_r = \frac{5}{2} - \frac{3}{2} \cdot \left(\frac{3}{5}\right)^{r-1}$. So the contents of each single bucket indeed approaches $\frac{1}{2}$ (from below). In particular, any two adjacent buckets have total contents strictly less than 1 which enables the Stepmother to always refill the buckets that Cinderella just emptied and then distribute the remaining water evenly over all buckets.
- After that phase GREEDY faces a situation like this $(\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon)$ and leaves a situation of the form $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, \frac{1}{2} - 2\epsilon, 0, 0)$.
- Then the Stepmother can add the amounts $(0, \frac{1}{4} + \epsilon, \epsilon, \frac{3}{4} - 2\epsilon, 0)$ to achieve a situation like this: $(y_0, y_1, y_2, y_3, y_4) = (\frac{1}{2} - 2\epsilon, \frac{3}{4} - \epsilon, \frac{1}{2} - \epsilon, \frac{3}{4} - 2\epsilon, 0)$.
- Now B_1 and B_2 are the adjacent buckets with the maximum total contents and thus GREEDY empties them to yield $(x_0, x_1, x_2, x_3, x_4) = (\frac{1}{2} - 2\epsilon, 0, 0, \frac{3}{4} - 2\epsilon, 0)$.
- Then the Stepmother adds $(\frac{5}{8}, 0, 0, \frac{3}{8}, 0)$, which yields $(\frac{9}{8} - 2\epsilon, 0, 0, \frac{9}{8} - 2\epsilon, 0)$.
- Now GREEDY can only empty one of the two nonempty buckets and in the next step the Stepmother adds her liter to the other bucket and brings it to $\frac{17}{8} - 2\epsilon$, i.e. an overflow.

A harder variant. Five identical empty buckets of capacity b stand at the vertices of a regular pentagon. Cinderella and her wicked Stepmother go through a sequence of rounds: At the beginning of every round, the Stepmother takes one liter of water from the nearby river and distributes it arbitrarily over the five buckets. Then Cinderella chooses a pair of neighboring buckets, empties them into the river, and puts them back. Then the next round begins. The Stepmother's goal is to make one of these buckets overflow. Cinderella's goal is to prevent this. Determine all bucket capacities b for which the Stepmother can enforce a bucket to overflow.

Solution to the harder variant. The answer is $b < 2$.

The previous proof shows that for all $b \geq 2$ the Stepmother cannot enforce overflowing. Now if $b < 2$, let R be a positive integer such that $b < 2 - 2^{1-R}$. In the first R rounds the Stepmother now ensures that at least one of the (nonadjacent) buckets B_1 and B_3 have contents of at least $1 - 2^{1-r}$ at the beginning of round r ($r = 1, 2, \ldots, R$). This is trivial for $r = 1$ and if it holds at the beginning of round r, she can fill the bucket which contains at least $1 - 2^{1-r}$ liters with another 2^{-r} liters and put the rest of her water – $1 - 2^{-r}$ liters – in the other bucket. As Cinderella now can remove the water of at most one of the two buckets, the other bucket carries its contents into the next round.

At the beginning of the R-th round there are $1 - 2^{1-R}$ liters in B_1 or B_3. The Stepmother puts the entire liter into that bucket and produces an overflow since $b < 2 - 2^{1-R}$.

C6 BGR (Bulgaria)

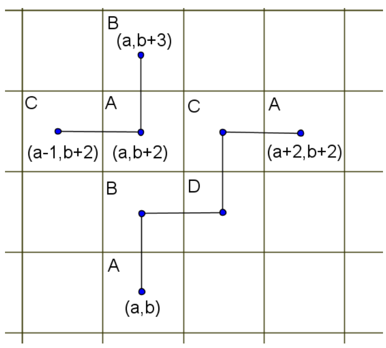
On a 999×999 board a *limp rook* can move in the following way: From any square it can move to any of its adjacent squares, i.e. a square having a common side with it, and every move must be a turn, i.e. the directions of any two consecutive moves must be perpendicular. A *non-intersecting route* of the limp rook consists of a sequence of pairwise different squares that the limp rook can visit in that order by an admissible sequence of moves. Such a non-intersecting route is called *cyclic*, if the limp rook can, after reaching the last square of the route, move directly to the first square of the route and start over.

How many squares does the longest possible cyclic, non-intersecting route of a limp rook visit?

Solution. The answer is $998^2 - 4 = 4 \cdot (499^2 - 1)$ squares.

First we show that this number is an upper bound for the number of cells a limp rook can visit. To do this we color the cells with four colors A, B, C and D in the following way: for $(i, j) \equiv (0, 0) \mod 2$ use A, for $(i, j) \equiv (0, 1) \mod 2$ use B, for $(i, j) \equiv (1, 0) \mod 2$ use C and for $(i, j) \equiv (1, 1) \mod 2$ use D. From an A-cell the rook has to move to a B-cell or a C-cell. In the first case, the order of the colors of the cells visited is given by $A, B, D, C, A, B, D, C, A, \ldots$, in the second case it is $A, C, D, B, A, C, D, B, A, \ldots$. Since the route is closed it must contain the same number of cells of each color. There are only 499^2 A-cells. In the following we will show that the rook cannot visit all the A-cells on its route and hence the maximum possible number of cells in a route is $4 \cdot (499^2 - 1)$.

Assume that the route passes through every single A-cell. Color the A-cells in black and white in a chessboard manner, i.e. color any two A-cells at distance 2 in different color. Since the number of A-cells is odd the rook cannot always alternate between visiting black and white A-cells along its route. Hence there are two A-cells of the same color which are four rook-steps apart that are visited directly one after the other. Let these two A-cells have row and column numbers (a, b) and $(a + 2, b + 2)$ respectively.



There is up to reflection only one way the rook can take from (a, b) to $(a + 2, b + 2)$. Let this way be $(a, b) \to (a, b + 1) \to (a + 1, b + 1) \to (a + 1, b + 2) \to (a + 2, b + 2)$. Also let without loss of generality the color of the cell $(a, b+1)$ be B (otherwise change the roles of columns and rows).

Now consider the A-cell $(a, b+2)$. The only way the rook can pass through it is via $(a-1, b+2) \to (a, b + 2) \to (a, b + 3)$ in this order, since according to our assumption after every A-cell the rook passes through a B-cell. Hence, to connect these two parts of the path, there must be a path connecting the cell $(a, b+3)$ and (a, b) and also a path connecting $(a+2, b+2)$ and $(a-1, b+2)$.

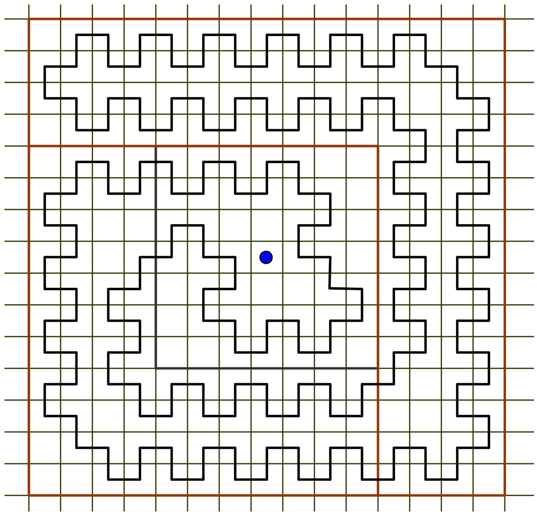
But these four cells are opposite vertices of a convex quadrilateral and the paths are outside of that quadrilateral and hence they must intersect. This is due to the following fact:

The path from (a, b) to $(a, b+3)$ together with the line segment joining these two cells form a closed loop that has one of the cells $(a-1, b+2)$ and $(a+2, b+2)$ in its inside and the other one on the outside. Thus the path between these two points must cross the previous path.

But an intersection is only possible if a cell is visited twice. This is a contradiction.

Hence the number of cells visited is at most $4 \cdot (499^2 - 1)$.

The following picture indicates a recursive construction for all $n \times n$-chessboards with $n \equiv 3 \mod 4$ which clearly yields a path that misses exactly one A-cell (marked with a dot, the center cell of the 15×15-chessboard) and hence, in the case of $n = 999$ crosses exactly $4 \cdot (499^2 - 1)$ cells.



C7 RUS (Russian Federation)

Variant 1. A grasshopper jumps along the real axis. He starts at point 0 and makes 2009 jumps to the right with lengths $1, 2, \ldots, 2009$ in an arbitrary order. Let M be a set of 2008 positive integers less than $1005 \cdot 2009$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Variant 2. Let n be a nonnegative integer. A grasshopper jumps along the real axis. He starts at point 0 and makes $n+1$ jumps to the right with pairwise different positive integral lengths $a_1, a_2, \ldots, a_{n+1}$ in an arbitrary order. Let M be a set of n positive integers in the interval $(0, s)$, where $s = a_1 + a_2 + \cdots + a_{n+1}$. Prove that the grasshopper can arrange his jumps in such a way that he never lands on a point from M.

Solution of Variant 1. We construct the set of landing points of the grasshopper.

Case 1. M does not contain numbers divisible by 2009.
We fix the numbers $2009k$ as landing points, $k = 1, 2, \ldots, 1005$. Consider the open intervals $I_k = (2009(k-1), 2009k)$, $k = 1, 2, \ldots, 1005$. We show that we can choose exactly one point outside of M as a landing point in 1004 of these intervals such that all lengths from 1 to 2009 are realized. Since there remains one interval without a chosen point, the length 2009 indeed will appear. Each interval has length 2009, hence a new landing point in an interval yields with a length d also the length $2009 - d$. Thus it is enough to implement only the lengths from $D = \{1, 2, \ldots, 1004\}$. We will do this in a greedy way. Let n_k, $k = 1, 2, \ldots, 1005$, be the number of elements of M that belong to the interval I_k. We order these numbers in a decreasing way, so let $p_1, p_2, \ldots, p_{1005}$ be a permutation of $\{1, 2, \ldots, 1005\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1005}}$. In I_{p_1} we do not choose a landing point. Assume that landing points have already been chosen in the intervals $I_{p_2}, \ldots, I_{p_m}$ and the lengths $d_2, \ldots, d_m$ from D are realized, $m = 1, \ldots, 1004$. We show that there is some $d \in D \setminus \{d_2, \ldots, d_m\}$ that can be implemented with a new landing point in $I_{p_{m+1}}$. Assume the contrary. Then the $1004 - (m-1)$ other lengths are obstructed by the $n_{p_{m+1}}$ points of M in $I_{p_{m+1}}$. Each length d can be realized by two landing points, namely $2009(p_{m+1} - 1) + d$ and $2009p_{m+1} - d$, hence

$$n_{p_{m+1}} \geq 2(1005 - m). \tag{1}$$

Moreover, since $|M| = 2008 = n_1 + \cdots + n_{1005}$,

$$2008 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}. \tag{2}$$

Consequently, by (1) and (2),

$$2008 \geq 2(m+1)(1005 - m).$$

The right hand side of the last inequality obviously attains its minimum for $m = 1004$ and this minimum value is greater than 2008, a contradiction.

Case 2. M does contain a number μ divisible by 2009.
By the pigeonhole principle there exists some $r \in \{1, \ldots, 2008\}$ such that M does not contain numbers with remainder r modulo 2009. We fix the numbers $2009(k-1) + r$ as landing points, $k = 1, 2, \ldots, 1005$. Moreover, $1005 \cdot 2009$ is a landing point. Consider the open intervals

$I_k = (2009(k-1)+r, 2009k+r)$, $k = 1, 2, \ldots, 1004$. Analogously to Case 1, it is enough to show that we can choose in 1003 of these intervals exactly one landing point outside of $M \setminus \{\mu\}$ such that each of the lengths of $D = \{1, 2, \ldots, 1004\} \setminus \{r\}$ are implemented. Note that r and $2009 - r$ are realized by the first and last jump and that choosing μ would realize these two differences again. Let n_k, $k = 1, 2, \ldots, 1004$, be the number of elements of $M \setminus \{\mu\}$ that belong to the interval I_k and $p_1, p_2, \ldots, p_{1004}$ be a permutation of $\{1, 2, \ldots, 1004\}$ such that $n_{p_1} \geq n_{p_2} \geq \cdots \geq n_{p_{1004}}$. With the same reasoning as in Case 1 we can verify that a greedy choice of the landing points in $I_{p_2}, I_{p_3}, \ldots, I_{p_{1004}}$ is possible. We only have to replace (1) by

$$n_{p_{m+1}} \geq 2(1004 - m)$$

(D has one element less) and (2) by

$$2007 \geq n_{p_1} + n_{p_2} + \cdots + n_{p_{m+1}} \geq (m+1)n_{p_{m+1}}.$$

Comment. The cardinality 2008 of M in the problem is the maximum possible value. For $M = \{1, 2, \ldots, 2009\}$, the grasshopper necessarily lands on a point from M.

Solution of Variant 2. First of all we remark that the statement in the problem implies a strengthening of itself: Instead of $|M| = n$ it is sufficient to suppose that $|M \cap (0, s - \overline{a}]| \leq n$, where $\overline{a} = \min\{a_1, a_2, \ldots, a_{n+1}\}$. This fact will be used in the proof.

We prove the statement by induction on n. The case $n = 0$ is obvious. Let $n > 0$ and let the assertion be true for all nonnegative integers less than n. Moreover let $a_1, a_2, \ldots, a_{n+1}, s$ and M be given as in the problem. Without loss of generality we may assume that $a_{n+1} < a_n < \cdots < a_2 < a_1$. Set

$$T_k = \sum_{i=1}^{k} a_i \quad \text{for } k = 0, 1, \ldots, n+1.$$

Note that $0 = T_0 < T_1 < \cdots < T_{n+1} = s$. We will make use of the induction hypothesis as follows:

Claim 1. It suffices to show that for some $m \in \{1, 2, \ldots, n+1\}$ the grasshopper is able to do at least m jumps without landing on a point of M and, in addition, after these m jumps he has jumped over at least m points of M.

Proof. Note that $m = n+1$ is impossible by $|M| = n$. Now set $n' = n - m$. Then $0 \leq n' < n$. The remaining $n' + 1$ jumps can be carried out without landing on one of the remaining at most n' forbidden points by the induction hypothesis together with a shift of the origin. This proves the claim.

An integer $k \in \{1, 2, \ldots, n+1\}$ is called *smooth*, if the grasshopper is able to do k jumps with the lengths $a_1, a_2, \ldots, a_k$ in such a way that he never lands on a point of M except for the very last jump, when he may land on a point of M.

Obviously, 1 is smooth. Thus there is a largest number k^*, such that all the numbers $1, 2, \ldots, k^*$ are smooth. If $k^* = n+1$, the proof is complete. In the following let $k^* \leq n$.

Claim 2. We have

$$T_{k^*} \in M \quad \text{and} \quad |M \cap (0, T_{k^*})| \geq k^*. \tag{3}$$

Proof. In the case $T_{k^*} \notin M$ any sequence of jumps that verifies the smoothness of k^* can be extended by appending a_{k^*+1}, which is a contradiction to the maximality of k^*. Therefore we have $T_{k^*} \in M$. If $|M \cap (0, T_{k^*})| < k^*$, there exists an $l \in \{1, 2, \ldots, k^*\}$ with $T_{k^*+1} - a_l \notin M$. By the induction hypothesis with $k^* - 1$ instead of n, the grasshopper is able to reach $T_{k^*+1} - a_l$

with k^* jumps of lengths from $\{a_1, a_2, \ldots, a_{k^*+1}\} \setminus \{a_l\}$ without landing on any point of M. Therefore k^*+1 is also smooth, which is a contradiction to the maximality of k^*. Thus Claim 2 is proved.

Now, by Claim 2, there exists a smallest integer $\overline{k} \in \{1, 2, \ldots, k^*\}$ with

$$T_{\overline{k}} \in M \quad \text{and} \quad |M \cap (0, T_{\overline{k}})| \geq \overline{k}.$$

Claim 3. It is sufficient to consider the case

$$|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k} - 1. \tag{4}$$

Proof. If $\overline{k} = 1$, then (4) is clearly satisfied. In the following let $\overline{k} > 1$. If $T_{\overline{k}-1} \in M$, then (4) follows immediately by the minimality of $\overline{k}$. If $T_{\overline{k}-1} \notin M$, by the smoothness of $\overline{k} - 1$, we obtain a situation as in Claim 1 with $m = \overline{k} - 1$ provided that $|M \cap (0, T_{\overline{k}-1}]| \geq \overline{k} - 1$. Hence, we may even restrict ourselves to $|M \cap (0, T_{\overline{k}-1}]| \leq \overline{k} - 2$ in this case and Claim 3 is proved.

Choose an integer $v \geq 0$ with $|M \cap (0, T_{\overline{k}})| = \overline{k} + v$. Let $r_1 > r_2 > \cdots > r_l$ be exactly those indices r from $\{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ for which $T_{\overline{k}} + a_r \notin M$. Then

$$n = |M| = |M \cap (0, T_{\overline{k}})| + 1 + |M \cap (T_{\overline{k}}, s)| \geq \overline{k} + v + 1 + (n + 1 - \overline{k} - l)$$

and consequently $l \geq v + 2$. Note that

$$T_{\overline{k}} + a_{r_1} - a_1 < T_{\overline{k}} + a_{r_1} - a_2 < \cdots < T_{\overline{k}} + a_{r_1} - a_{\overline{k}} < T_{\overline{k}} + a_{r_2} - a_{\overline{k}} < \cdots < T_{\overline{k}} + a_{r_{v+2}} - a_{\overline{k}} < T_{\overline{k}}$$

and that this are $\overline{k} + v + 1$ numbers from $(0, T_{\overline{k}})$. Therefore we find some $r \in \{\overline{k}+1, \overline{k}+2, \ldots, n+1\}$ and some $s \in \{1, 2, \ldots, \overline{k}\}$ with $T_{\overline{k}} + a_r \notin M$ and $T_{\overline{k}} + a_r - a_s \notin M$. Consider the set of jump lengths $B = \{a_1, a_2, \ldots, a_{\overline{k}}, a_r\} \setminus \{a_s\}$. We have

$$\sum_{x \in B} x = T_{\overline{k}} + a_r - a_s$$

and

$$T_{\overline{k}} + a_r - a_s - \min(B) = T_{\overline{k}} - a_s \leq T_{\overline{k}-1}.$$

By (4) and the strengthening, mentioned at the very beginning with $\overline{k} - 1$ instead of n, the grasshopper is able to reach $T_{\overline{k}} + a_r - a_s$ by $\overline{k}$ jumps with lengths from B without landing on any point of M. From there he is able to jump to $T_{\overline{k}} + a_r$ and therefore we reach a situation as in Claim 1 with $m = \overline{k} + 1$, which completes the proof.

C8 AUT (Austria)

For any integer $n \geq 2$, we compute the integer $h(n)$ by applying the following procedure to its decimal representation. Let r be the rightmost digit of n.

(1) If $r = 0$, then the decimal representation of $h(n)$ results from the decimal representation of n by removing this rightmost digit 0.

(2) If $1 \leq r \leq 9$ we split the decimal representation of n into a maximal right part R that solely consists of digits not less than r and into a left part L that either is empty or ends with a digit strictly smaller than r. Then the decimal representation of $h(n)$ consists of the decimal representation of L, followed by two copies of the decimal representation of $R-1$. For instance, for the number $n = 17{,}151{,}345{,}543$, we will have $L = 17{,}151$, $R = 345{,}543$ and $h(n) = 17{,}151{,}345{,}542{,}345{,}542$.

Prove that, starting with an arbitrary integer $n \geq 2$, iterated application of h produces the integer 1 after finitely many steps.

Solution 1. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. We recursively define ten functions $f_0, \ldots, f_9$ that map some strings into integers for $k = 9, 8, \ldots, 1, 0$. The function f_9 is only defined on strings x (including the empty string ε) that entirely consist of nines. If x consists of m nines, then $f_9(x) = m + 1$, $m = 0, 1, \ldots$. For $k \leq 8$, the domain of $f_k(x)$ is the set of all strings consisting only of digits that are $\geq k$. We write x in the form $x_0 k x_1 k x_2 k \ldots x_{m-1} k x_m$ where the strings x_s only consist of digits $\geq k+1$. Note that some of these strings might equal the empty string ε and that $m = 0$ is possible, i.e. the digit k does not appear in x. Then we define

$$f_k(x) = \sum_{s=0}^{m} 4^{f_{k+1}(x_s)}.$$

We will use the following obvious fact:

Fact 1. If x does not contain digits smaller than k, then $f_i(x) = 4^{f_{i+1}(x)}$ for all $i = 0, \ldots, k-1$. In particular, $f_i(\varepsilon) = 4^{9-i}$ for all $i = 0, 1, \ldots, 9$.

Moreover, by induction on $k = 9, 8, \ldots, 0$ it follows easily:

Fact 2. If the nonempty string x does not contain digits smaller than k, then $f_i(x) > f_i(\varepsilon)$ for all $i = 0, \ldots, k$.

We will show the essential fact:

Fact 3. $f_0(n) > f_0(h(n))$.

Then the empty string will necessarily be reached after a finite number of applications of h. But starting from a string without leading zeros, ε can only be reached via the strings $1 \to 00 \to 0 \to \varepsilon$. Hence also the number 1 will appear after a finite number of applications of h.

Proof of Fact 3. If the last digit r of n is 0, then we write $n = x_0 0 \ldots 0 x_{m-1} 0 \varepsilon$ where the x_i do not contain the digit 0. Then $h(n) = x_0 0 \ldots 0 x_{m-1}$ and $f_0(n) - f_0(h(n)) = f_0(\varepsilon) > 0$.

So let the last digit r of n be at least 1. Let $L = yk$ and $R = zr$ be the corresponding left and right parts where y is some string, $k \leq r-1$ and the string z consists only of digits not less than r. Then $n = ykzr$ and $h(n) = ykz(r-1)z(r-1)$. Let $d(y)$ be the smallest digit of y. We consider two cases which do not exclude each other.

Case 1. $d(y) \geq k$.
Then

$$f_k(n) - f_k(h(n)) = f_k(zr) - f_k(z(r-1)z(r-1)).$$

In view of Fact 1 this difference is positive if and only if

$$f_{r-1}(zr) - f_{r-1}(z(r-1)z(r-1)) > 0.$$

We have, using Fact 2,

$$f_{r-1}(zr) = 4^{f_r(zr)} = 4^{f_r(z)+4^{f_{r+1}(\varepsilon)}} \geq 4 \cdot 4^{f_r(z)} > 4^{f_r(z)} + 4^{f_r(z)} + 4^{f_r(\varepsilon)} = f_{r-1}(z(r-1)z(r-1)).$$

Here we use the additional definition $f_{10}(\varepsilon) = 0$ if $r = 9$. Consequently, $f_k(n) - f_k(h(n)) > 0$ and according to Fact 1, $f_0(n) - f_0(h(n)) > 0$.

Case 2. $d(y) \leq k$.
We prove by induction on $d(y) = k, k-1, \dots, 0$ that $f_i(n) - f_i(h(n)) > 0$ for all $i = 0, \dots, d(y)$. By Fact 1, it suffices to do so for $i = d(y)$. The initialization $d(y) = k$ was already treated in Case 1. Let $t = d(y) < k$. Write y in the form utv where v does not contain digits $\leq t$. Then, in view of the induction hypothesis,

$$f_t(n) - f_t(h(n)) = f_t(vkzr) - f_t(vkz(r-1)z(r-1)) = 4^{f_{t+1}(vkzr)} - 4^{f_{t+1}(vkz(r-1)z(r-1))} > 0.$$

Thus the inequality $f_{d(y)}(n) - f_{d(y)}(h(n)) > 0$ is established and from Fact 1 it follows that $f_0(n) - f_0(h(n)) > 0$.

Solution 2. We identify integers $n \geq 2$ with the digit-strings, briefly strings, of their decimal representation and extend the definition of h to all non-empty strings with digits from 0 to 9. Moreover, let us define that the empty string, ε, is being mapped to the empty string. In the following all functions map the set of strings into the set of strings. For two functions f and g let $g \circ f$ be defined by $(g \circ f)(x) = g(f(x))$ for all strings x and let, for non-negative integers n, f^n denote the n-fold application of f. For any string x let $s(x)$ be the smallest digit of x, and for the empty string let $s(\varepsilon) = \infty$. We define nine functions $g_1, \dots, g_9$ as follows: Let $k \in \{1, \dots, 9\}$ and let x be a string. If $x = \varepsilon$ then $g_k(x) = \varepsilon$. Otherwise, write x in the form $x = yzr$ where y is either the empty string or ends with a digit smaller than k, $s(z) \geq k$ and r is the rightmost digit of x. Then $g_k(x) = zr$.

Lemma 1. We have $g_k \circ h = g_k \circ h \circ g_k$ for all $k = 1, \dots, 9$.

Proof of Lemma 1. Let $x = yzr$ be as in the definition of g_k. If $y = \varepsilon$, then $g_k(x) = x$, whence

$$g_k(h(x)) = g_k(h(g_k(x))). \tag{1}$$

So let $y \neq \varepsilon$.

Case 1. z contains a digit smaller than r.
Let $z = uav$ where $a < r$ and $s(v) \geq r$. Then

$$h(x) = \begin{cases} yuav & \text{if } r = 0, \\ yuav(r-1)v(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = h(uavr) = \begin{cases} uav & \text{if } r = 0, \\ uav(r-1)v(r-1) & \text{if } r > 0. \end{cases}$$

Since y ends with a digit smaller than k, (1) is obviously true.

Case 2. z does not contain a digit smaller than r.
Let $y = uv$ where u is either the empty string or ends with a digit smaller than r and $s(v) \geq r$. We have

$$h(x) = \begin{cases} uvz & \text{if } r = 0, \\ uvz(r-1)vz(r-1) & \text{if } r > 0 \end{cases}$$

and

$$h(g_k(x)) = h(zr) = \begin{cases} z & \text{if } r = 0, \\ z(r-1)z(r-1) & \text{if } r > 0. \end{cases}$$

Recall that y and hence v ends with a digit smaller than k, but all digits of v are at least r. Now if $r > k$, then $v = \varepsilon$, whence the terminal digit of u is smaller than k, which entails

$$g_k(h(x)) = z(r-1)z(r-1) = g_k(h(g_k(x))).$$

If $r \leq k$, then

$$g_k(h(x)) = z(r-1) = g_k(h(g_k(x))) \ ,$$

so that in both cases (1) is true. Thus Lemma 1 is proved.

Lemma 2. Let $k \in \{1, \ldots, 9\}$, let x be a non-empty string and let n be a positive integer. If $h^n(x) = \varepsilon$ then $(g_k \circ h)^n(x) = \varepsilon$.

Proof of Lemma 2. We proceed by induction on n. If $n = 1$ we have

$$\varepsilon = h(x) = g_k(h(x)) = (g_k \circ h)(x).$$

Now consider the step from $n-1$ to n where $n \geq 2$. Let $h^n(x) = \varepsilon$ and let $y = h(x)$. Then $h^{n-1}(y) = \varepsilon$ and by the induction hypothesis $(g_k \circ h)^{n-1}(y) = \varepsilon$. In view of Lemma 1,

$$\begin{aligned} \varepsilon = (g_k \circ h)^{n-2}((g_k \circ h)(y)) &= (g_k \circ h)^{n-2}(g_k(h(y)) \\ &= (g_k \circ h)^{n-2}(g_k(h(g_k(y))) = (g_k \circ h)^{n-2}(g_k(h(g_k(h(x)))) = (g_k \circ h)^n(x). \end{aligned}$$

Thus the induction step is complete and Lemma 2 is proved.

We say that the non-empty string x *terminates* if $h^n(x) = \varepsilon$ for some non-negative integer n.

Lemma 3. Let $x = yzr$ where $s(y) \geq k$, $s(z) \geq k$, y ends with the digit k and z is possibly empty. If y and zr terminate then also x terminates.

Proof of Lemma 3. Suppose that y and zr terminate. We proceed by induction on k. Let $k = 0$. Obviously, $h(yw) = yh(w)$ for any non-empty string w. Let $h^n(zr) = \epsilon$. It follows easily by induction on m that $h^m(yzr) = yh^m(zr)$ for $m = 1, \ldots, n$. Consequently, $h^n(yzr) = y$. Since y terminates, also $x = yzr$ terminates.

Now let the assertion be true for all nonnegative integers less than k and let us prove it for k where $k \geq 1$. It turns out that it is sufficient to prove that $yg_k(h(zr))$ terminates. Indeed:

Case 1. $r = 0$.
Then $h(yzr) = yz = yg_k(h(zr))$.

Case 2. $0 < r \leq k$.
We have $h(zr) = z(r-1)z(r-1)$ and $g_k(h(zr)) = z(r-1)$. Then $h(yzr) = yz(r-1)yz(r-$

$1) = yg_k(h(zr))yg_k(h(zr))$ and we may apply the induction hypothesis to see that if $yg_kh(zr))$ terminates, then $h(yzr)$ terminates.

Case 3. $r > k$.
Then $h(yzr) = yh(zr) = yg_k(h(zr))$.

Note that $yg_k(h(zr))$ has the form $yz'r'$ where $s(z') \geq k$. By the same arguments it is sufficient to prove that $yg_k(h(z'r')) = y(g_k \circ h)^2(zr)$ terminates and, by induction, that $y(g_k \circ h)^m(zr)$ terminates for some positive integer m. In view of Lemma 2 there is some m such that $(g_k \circ h)^m(zr) = \epsilon$, so $x = yzr$ terminates if y terminates. Thus Lemma 3 is proved.

Now assume that there is some string x that does not terminate. We choose x minimal. If $x \geq 10$, we can write x in the form $x = yzr$ of Lemma 3 and by this lemma x terminates since y and zr are smaller than x. If $x \leq 9$, then $h(x) = (x-1)(x-1)$ and $h(x)$ terminates again by Lemma 3 and the minimal choice of x.

Solution 3. We commence by introducing some terminology. Instead of integers, we will consider the set S of all strings consisting of the digits $0, 1, \ldots, 9$, including the empty string ϵ. If $(a_1, a_2, \ldots, a_n)$ is a nonempty string, we let $\rho(a) = a_n$ denote the *terminal digit of* a and $\lambda(a)$ be the string with the last digit removed. We also define $\lambda(\epsilon) = \epsilon$ and denote the set of non-negative integers by $\mathbb{N}_0$.

Now let $k \in \{0, 1, 2, \ldots, 9\}$ denote any digit. We define a function $f_k : S \longrightarrow S$ on the set of strings: First, if the terminal digit of n belongs to $\{0, 1, \ldots, k\}$, then $f_k(n)$ is obtained from n by deleting this terminal digit, i.e $f_k(n) = \lambda(n)$. Secondly, if the terminal digit of n belongs to $\{k+1, \ldots, 9\}$, then $f_k(n)$ is obtained from n by the process described in the problem. We also define $f_k(\epsilon) = \epsilon$. Note that up to the definition for integers $n \leq 1$, the function f_0 coincides with the function h in the problem, through interpreting integers as digit strings. The argument will be roughly as follows. We begin by introducing a straightforward generalization of our claim about f_0. Then it will be easy to see that f_9 has all these stronger properties, which means that is suffices to show for $k \in \{0, 1, \ldots, 8\}$ that f_k possesses these properties provided that f_{k+1} does.

We continue to use k to denote any digit. The operation f_k is said to be *separating*, if the followings holds: Whenever a is an initial segment of b, there is some $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. The following two notions only apply to the case where f_k is indeed separating, otherwise they remain undefined. For every $a \in S$ we denote the least $N \in \mathbb{N}_0$ for which $f_k^N(a) = \epsilon$ occurs by $g_k(a)$ (because ϵ is an initial segment of a, such an N exists if f_k is separating). If for every two strings a and b such that a is a terminal segment of b one has $g_k(a) \leq g_k(b)$, we say that f_k is *coherent*. In case that f_k is separating and coherent we call the digit k *seductive*.

As $f_9(a) = \lambda(a)$ for all a, it is obvious that 9 is seductive. Hence in order to show that 0 is seductive, which clearly implies the statement of the problem, it suffices to take any $k \in \{0, 1, \ldots, 8\}$ such that $k+1$ is seductive and to prove that k has to be seductive as well. Note that in doing so, we have the function g_{k+1} at our disposal. We have to establish two things and we begin with

Step 1. f_k is separating.

Before embarking on the proof of this, we record a useful observation which is easily proved by induction on M.

Claim 1. For any strings A, B and any positive integer M such that $f_k^{M-1}(B) \neq \epsilon$, we have

$$f_k^M(AkB) = Akf_k^M(B).$$

Now we call a pair (a, b) of strings *wicked* provided that a is an initial segment of b, but there is no $N \in \mathbb{N}_0$ such that $f_k^N(b) = a$. We need to show that there are none, so assume that there were such pairs. Choose a wicked pair (a, b) for which $g_{k+1}(b)$ attains its minimal possible value. Obviously $b \neq \epsilon$ for any wicked pair (a, b). Let z denote the terminal digit of b. Observe that $a \neq b$, which means that a is also an initial segment of $\lambda(b)$. To facilitate the construction of the eventual contradiction, we prove

Claim 2. There cannot be an $N \in \mathbb{N}_0$ such that

$$f_k^N(b) = \lambda(b).$$

Proof of Claim 2. For suppose that such an N existed. Because $g_{k+1}(\lambda(b)) < g_{k+1}(b)$ in view of the coherency of f_{k+1}, the pair $(a, \lambda(b))$ is not wicked. But then there is some N' for which $f_k^{N'}(\lambda(b)) = a$ which entails $f_k^{N+N'}(b) = a$, contradiction. Hence Claim 2 is proved.

It follows that $z \leq k$ is impossible, for otherwise $N = 1$ violated Claim 2.

Also $z > k+1$ is impossible: Set $B = f_k(b)$. Then also $f_{k+1}(b) = B$, but $g_{k+1}(B) < g_{k+1}(b)$ and a is an initial segment of B. Thus the pair (a, B) is not wicked. Hence there is some $N \in \mathbb{N}_0$ with $a = f_k^N(B)$, which, however, entails $a = f_k^{N+1}(b)$.

We are left with the case $z = k+1$. Let L denote the left part and $R = R^*(k+1)$ the right part of b. Then we have symbolically

$$f_k(b) = LR^*kR^*k\ ,\ f_k^2(b) = LR^*kR^* \quad \text{and} \quad f_{k+1}(b) = LR^*.$$

Using that R^* is a terminal segment of LR^* and the coherency of f_{k+1}, we infer

$$g_{k+1}(R^*) \leq g_{k+1}(LR^*) < g_{k+1}(b).$$

Hence the pair (ϵ, R^*) is not wicked, so there is some minimal $M \in \mathbb{N}_0$ with $f_k^M(R^*) = \epsilon$ and by Claim 1 it follows that $f_k^{2+M}(b) = LR^*k$. Finally, we infer that $\lambda(b) = LR^* = f_k(LR^*k) = f_k^{3+M}(b)$, which yields a contradiction to Claim 2.

This final contradiction establishes that f_k is indeed separating.

Step 2. f_k is coherent.

To prepare the proof of this, we introduce some further pieces of terminology. A nonempty string $(a_1, a_2, \dots, a_n)$ is called a *hypostasis*, if $a_n < a_i$ for all $i = 1, \dots, n-1$. Reading an arbitrary string a backwards, we easily find a, possibly empty, sequence $(A_1, A_2, \dots, A_m)$ of hypostases such that $\rho(A_1) \leq \rho(A_2) \leq \cdots \leq \rho(A_m)$ and, symbolically, $a = A_1A_2 \dots A_m$. The latter sequence is referred to as the *decomposition* of a. So, for instance, $(20, 0, 9)$ is the decomposition of 2009 and the string 50 is a hypostasis. Next we explain when we say about two strings a and b that a is *injectible* into b. The definition is by induction on the length of b. Let $(B_1, B_2, \dots, B_n)$ be the decomposition of b into hypostases. Then a is injectible into b if for the decomposition $(A_1, A_2, \dots, A_m)$ of a there is a strictly increasing function $H : \{1, 2, \dots, m\} \longrightarrow \{1, 2, \dots, n\}$ satisfying

$\rho(A_i) = \rho(B_{H(i)})$ for all $i = 1, \dots, m$;

$\lambda(A_i)$ is injectible into $\lambda(B_{H(i)})$ for all $i = 1, \ldots, m$.

If one can choose H with $H(m) = n$, then we say that a is *strongly injectible* into b. Obviously, if a is a terminal segment of b, then a is strongly injectible into b.

Claim 3. If a and b are two nonempty strings such that a is strongly injectible into b, then $\lambda(a)$ is injectible into $\lambda(b)$.

Proof of Claim 3. Let $(B_1, B_2, \ldots, B_n)$ be the decomposition of b and let $(A_1, A_2, \ldots, A_m)$ be the decomposition of a. Take a function H exemplifying that a is strongly injectible into b. Let $(C_1, C_2, \ldots, C_r)$ be the decomposition of $\lambda(A_m)$ and let $(D_1, D_2, \ldots, D_s)$ be the decomposition of $\lambda(B_n)$. Choose a strictly increasing $H' : \{1, 2, \ldots, r\} \longrightarrow \{1, 2, \ldots s\}$ witnessing that $\lambda(A_m)$ is injectible into $\lambda(B_n)$. Clearly, $(A_1, A_2, \ldots, A_{m-1}, C_1, C_2, \ldots, C_r)$ is the decomposition of $\lambda(a)$ and $(B_1, B_2, \ldots, B_{n-1}, D_1, D_2, \ldots, D_s)$ is the decomposition of $\lambda(b)$. Then the function $H'' : \{1, 2, \ldots, m+r-1\} \longrightarrow \{1, 2, \ldots, n+s-1\}$ given by $H''(i) = H(i)$ for $i = 1, 2, \ldots, m-1$ and $H''(m-1+i) = n-1+H'(i)$ for $i = 1, 2, \ldots, r$ exemplifies that $\lambda(a)$ is injectible into $\lambda(b)$, which finishes the proof of the claim.

A pair (a, b) of strings is called *aggressive* if a is injectible into b and nevertheless $g_k(a) > g_k(b)$. Observe that if f_k was incoherent, which we shall assume from now on, then such pairs existed. Now among all aggressive pairs we choose one, say (a, b), for which $g_k(b)$ attains its least possible value. Obviously $f_k(a)$ cannot be injectible into $f_k(b)$, for otherwise the pair $(f_k(a), f_k(b))$ was aggressive and contradicted our choice of (a, b). Let $(A_1, A_2, \ldots, A_m)$ and $(B_1, B_2, \ldots, B_n)$ be the decompositions of a and b and take a function $H : \{1, 2, \ldots, m\} \longrightarrow \{1, 2, \ldots, n\}$ exemplifying that a is indeed injectible into b. If we had $H(m) < n$, then a was also injectible into the number b' whose decomposition is $(B_1, B_2, \ldots, B_{n-1})$ and by separativity of f_k we obtained $g_k(b') < g_k(b)$, whence the pair (a, b') was also aggressive, contrary to the minimality condition imposed on b. Therefore a is strongly injectible into b. In particular, a and b have a common terminal digit, say z. If we had $z \leq k$, then $f_k(a) = \lambda(a)$ and $f_k(b) = \lambda(b)$, so that by Claim 3, $f_k(a)$ was injectible into $f_k(b)$, which is a contradiction. Hence, $z \geq k+1$.

Now let r be the minimal element of $\{1, 2, \ldots, m\}$ for which $\rho(A_r) = z$. Then the maximal right part of a consisting of digits $\geq z$ is equal to R_a, the string whose decomposition is $(A_r, A_{r+1}, \ldots, A_m)$. Then $R_a - 1$ is a hypostasis and $(A_1, \ldots, A_{r-1}, R_a - 1, R_a - 1)$ is the decomposition of $f_k(a)$. Defining s and R_b in a similar fashion with respect to b, we see that $(B_1, \ldots, B_{s-1}, R_b - 1, R_b - 1)$ is the decomposition of $f_k(b)$. The definition of injectibility then easily entails that R_a is strongly injectible into R_b. It follows from Claim 3 that $\lambda(R_a) = \lambda(R_a - 1)$ is injectible into $\lambda(R_b) = \lambda(R_b - 1)$, whence the function $H' : \{1, 2, \ldots, r+1\} \longrightarrow \{1, 2, \ldots, s+1\}$, given by $H'(i) = H(i)$ for $i = 1, 2, \ldots, r-1$, $H'(r) = s$ and $H'(r+1) = s+1$ exemplifies that $f_k(a)$ is injectible into $f_k(b)$, which yields a contradiction as before.

This shows that aggressive pairs cannot exist, whence f_k is indeed coherent, which finishes the proof of the seductivity of k, whereby the problem is finally solved.

Geometry

G1 BEL (Belgium)

Let ABC be a triangle with $AB = AC$. The angle bisectors of A and B meet the sides BC and AC in D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45^\circ$. Find all possible values of $\angle BAC$.

Solution 1. Answer: $\angle BAC = 60^\circ$ or $\angle BAC = 90^\circ$ are possible values and the only possible values.

Let I be the incenter of triangle ABC, then K lies on the line CI. Let F be the point, where the incircle of triangle ABC touches the side AC; then the segments IF and ID have the same length and are perpendicular to AC and BC, respectively.

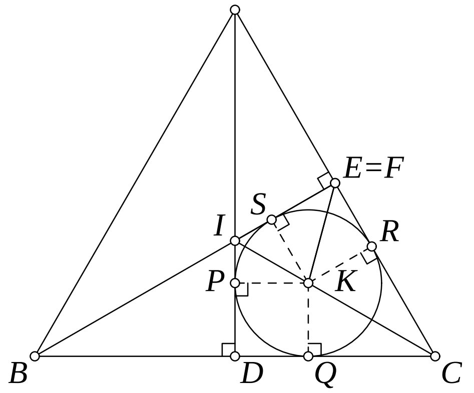


Figure 1

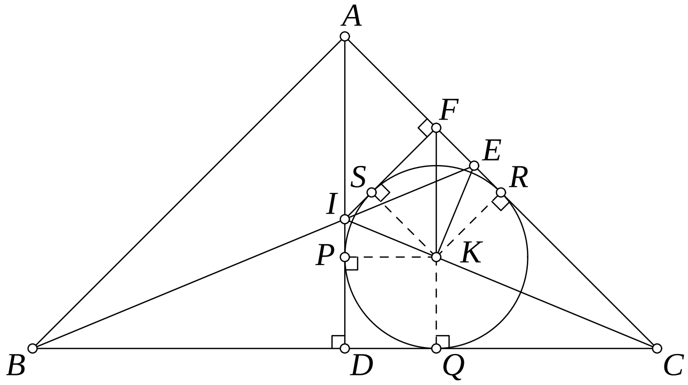


Figure 2

Let P, Q and R be the points where the incircle of triangle ADC touches the sides AD, DC and CA, respectively. Since K and I lie on the angle bisector of $\angle ACD$, the segments ID and IF are symmetric with respect to the line IC. Hence there is a point S on IF where the incircle of triangle ADC touches the segment IF. Then segments KP, KQ, KR and KS all have the same length and are perpendicular to AD, DC, CA and IF, respectively. So – regardless of the value of $\angle BEK$ – the quadrilateral $KRFS$ is a square and $\angle SFK = \angle KFC = 45^\circ$.

Consider the case $\angle BAC = 60^\circ$ (see Figure 1). Then triangle ABC is equilateral. Furthermore we have $F = E$, hence $\angle BEK = \angle IFK = \angle SEK = 45^\circ$. So 60° is a possible value for $\angle BAC$.

Now consider the case $\angle BAC = 90^\circ$ (see Figure 2). Then $\angle CBA = \angle ACB = 45^\circ$. Furthermore, $\angle KIE = \frac{1}{2}\angle CBA + \frac{1}{2}\angle ACB = 45^\circ$, $\angle AEB = 180^\circ - 90^\circ - 22.5^\circ = 67.5^\circ$ and $\angle EIA = \angle BID = 180^\circ - 90^\circ - 22.5^\circ = 67.5^\circ$. Hence triangle IEA is isosceles and a reflection of the bisector of $\angle IAE$ takes I to E and K to itself. So triangle IKE is symmetric with respect to this axis, i.e. $\angle KIE = \angle IEK = \angle BEK = 45^\circ$. So 90° is a possible value for $\angle BAC$, too.

If, on the other hand, $\angle BEK = 45^\circ$ then $\angle BEK = \angle IEK = \angle IFK = 45^\circ$. Then

- either $F = E$, which makes the angle bisector BI be an altitude, i.e., which makes triangle ABC isosceles with base AC and hence equilateral and so $\angle BAC = 60^\circ$,
- or E lies between F and C, which makes the points K, E, F and I concyclic, so $45^\circ = \angle KFC = \angle KFE = \angle KIE = \angle CBI + \angle ICB = 2 \cdot \angle ICB = 90^\circ - \frac{1}{2}\angle BAC$, and so $\angle BAC = 90^\circ$,

- or F lies between E and C, then again, K, E, F and I are concyclic, so $45^\circ = \angle KFC = 180^\circ - \angle KFE = \angle KIE$, which yields the same result $\angle BAC = 90^\circ$. (However, for $\angle BAC = 90^\circ$ E lies, in fact, between F and C, see Figure 2. So this case does not occur.)

This proves 90° and 60° to be the only possible values for $\angle BAC$.

Solution 2. Denote angles at A, B and C as usual by α, β and γ. Since triangle ABC is isosceles, we have $\beta = \gamma = 90^\circ - \frac{\alpha}{2} < 90^\circ$, so $\angle ECK = 45^\circ - \frac{\alpha}{4} = \angle KCD$. Since K is the incenter of triangle ADC, we have $\angle CDK = \angle KDA = 45^\circ$; furthermore $\angle DIC = 45^\circ + \frac{\alpha}{4}$. Now, if $\angle BEK = 45^\circ$, easy calculations within triangles BCE and KCE yield

$$\begin{aligned}
\angle KEC &= 180^\circ - \tfrac{\beta}{2} - 45^\circ - \beta = 135^\circ - \tfrac{3}{2}\beta = \tfrac{3}{2}(90^\circ - \beta) = \tfrac{3}{4}\alpha,\\
\angle IKE &= \tfrac{3}{4}\alpha + 45^\circ - \tfrac{\alpha}{4} = 45^\circ + \tfrac{\alpha}{2}.
\end{aligned}$$

So in triangles ICE, IKE, IDK and IDC we have (see Figure 3)

$$\frac{IC}{IE} = \frac{\sin\angle IEC}{\sin\angle ECI} = \frac{\sin(45^\circ + \frac{3}{4}\alpha)}{\sin(45^\circ - \frac{\alpha}{4})}, \qquad \frac{IE}{IK} = \frac{\sin\angle EKI}{\sin\angle IEK} = \frac{\sin(45^\circ + \frac{\alpha}{2})}{\sin 45^\circ},$$

$$\frac{IK}{ID} = \frac{\sin\angle KDI}{\sin\angle IKD} = \frac{\sin 45^\circ}{\sin(90^\circ - \frac{\alpha}{4})}, \qquad \frac{ID}{IC} = \frac{\sin\angle ICD}{\sin\angle CDI} = \frac{\sin(45^\circ - \frac{\alpha}{4})}{\sin 90^\circ}.$$

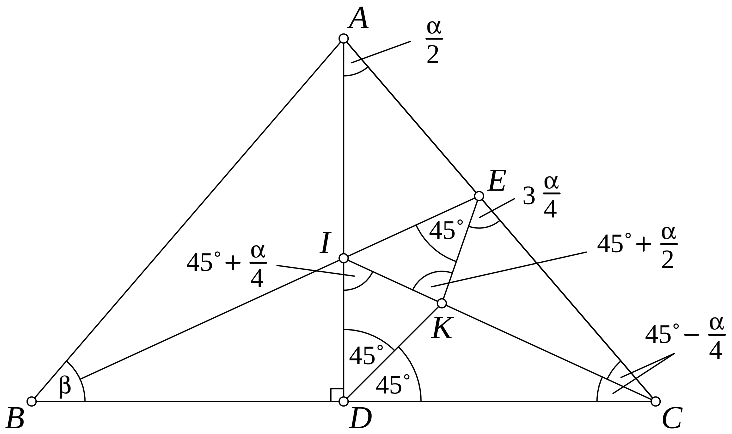


Figure 3

Multiplication of these four equations yields

$$1 = \frac{\sin(45^\circ + \frac{3}{4}\alpha)\sin(45^\circ + \frac{\alpha}{2})}{\sin(90^\circ - \frac{\alpha}{4})}.$$

But, since

$$\begin{aligned}
\sin\left(90^\circ - \tfrac{\alpha}{4}\right) &= \cos\tfrac{\alpha}{4} = \cos\left((45^\circ + \tfrac{3}{4}\alpha) - (45^\circ + \tfrac{\alpha}{2})\right)\\
&= \cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) + \sin\left(45^\circ + \tfrac{3}{4}\alpha\right)\sin\left(45^\circ + \tfrac{\alpha}{2}\right),
\end{aligned}$$

this is equivalent to

$$\sin(45^\circ + \tfrac{3}{4}\alpha)\sin(45^\circ + \tfrac{\alpha}{2}) = \cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) + \sin\left(45^\circ + \tfrac{3}{4}\alpha\right)\sin\left(45^\circ + \tfrac{\alpha}{2}\right)$$

and finally

$$\cos\left(45^\circ + \tfrac{3}{4}\alpha\right)\cos\left(45^\circ + \tfrac{\alpha}{2}\right) = 0.$$

But this means $\cos\left(45^\circ + \frac{3}{4}\alpha\right) = 0$, hence $45^\circ + \frac{3}{4}\alpha = 90^\circ$, i.e. $\alpha = 60^\circ$ or $\cos\left(45^\circ + \frac{\alpha}{2}\right) = 0$, hence $45^\circ + \frac{\alpha}{2} = 90^\circ$, i.e. $\alpha = 90^\circ$. So these values are the only two possible values for α.

On the other hand, both $\alpha = 90^\circ$ and $\alpha = 60^\circ$ yield $\angle BEK = 45^\circ$, this was shown in Solution 1.

G2 RUS (Russian Federation)

Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. The circle k passes through the midpoints of the segments BP, CQ, and PQ. Prove that if the line PQ is tangent to circle k then $OP = OQ$.

Solution 1. Let K, L, M, B', C' be the midpoints of BP, CQ, PQ, CA, and AB, respectively (see Figure 1). Since $CA \parallel LM$, we have $\angle LMP = \angle QPA$. Since k touches the segment PQ at M, we find $\angle LMP = \angle LKM$. Thus $\angle QPA = \angle LKM$. Similarly it follows from $AB \parallel MK$ that $\angle PQA = \angle KLM$. Therefore, triangles APQ and MKL are similar, hence

$$\frac{AP}{AQ} = \frac{MK}{ML} = \frac{\frac{QB}{2}}{\frac{PC}{2}} = \frac{QB}{PC}. \tag{1}$$

Now (1) is equivalent to $AP \cdot PC = AQ \cdot QB$ which means that the power of points P and Q with respect to the circumcircle of $\triangle ABC$ are equal, hence $OP = OQ$.

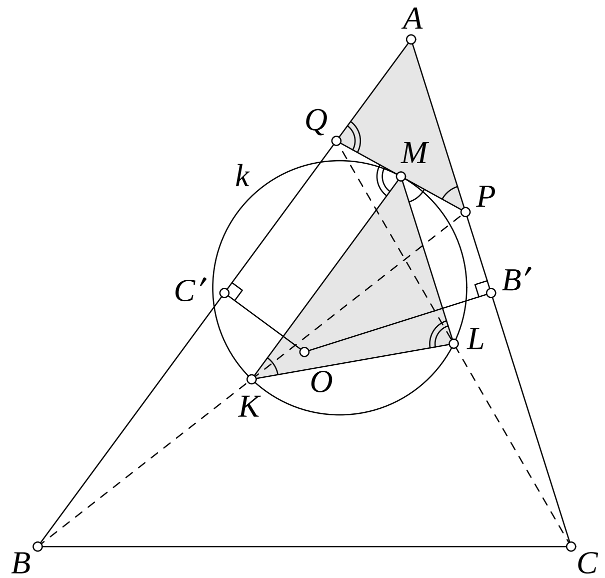


Figure 1

Comment. The last argument can also be established by the following calculation:

$$\begin{aligned} OP^2 - OQ^2 &= OB'^2 + B'P^2 - OC'^2 - C'Q^2 \\ &= (OA^2 - AB'^2) + B'P^2 - (OA^2 - AC'^2) - C'Q^2 \\ &= (AC'^2 - C'Q^2) - (AB'^2 - B'P^2) \\ &= (AC' - C'Q)(AC' + C'Q) - (AB' - B'P)(AB' + B'P) \\ &= AQ \cdot QB - AP \cdot PC. \end{aligned}$$

With (1), we conclude $OP^2 - OQ^2 = 0$, as desired.

Solution 2. Again, denote by K, L, M the midpoints of segments BP, CQ, and PQ, respectively. Let O, S, T be the circumcenters of triangles ABC, KLM, and APQ, respectively (see Figure 2). Note that MK and LM are the midlines in triangles BPQ and CPQ, respectively, so $\overrightarrow{MK} = \frac{1}{2}\overrightarrow{QB}$ and $\overrightarrow{ML} = \frac{1}{2}\overrightarrow{PC}$. Denote by $\mathrm{pr}_l(\overrightarrow{v})$ the projection of vector $\overrightarrow{v}$ onto line l. Then $\mathrm{pr}_{AB}(\overrightarrow{OT}) = \mathrm{pr}_{AB}(\overrightarrow{OA} - \overrightarrow{TA}) = \frac{1}{2}\overrightarrow{BA} - \frac{1}{2}\overrightarrow{QA} = \frac{1}{2}\overrightarrow{BQ} = \overrightarrow{KM}$ and $\mathrm{pr}_{AB}(\overrightarrow{SM}) = \mathrm{pr}_{MK}(\overrightarrow{SM}) = \frac{1}{2}\overrightarrow{KM} = \frac{1}{2}\mathrm{pr}_{AB}(\overrightarrow{OT})$. Analogously we get $\mathrm{pr}_{CA}(\overrightarrow{SM}) = \frac{1}{2}\mathrm{pr}_{CA}(\overrightarrow{OT})$. Since AB and CA are not parallel, this implies that $\overrightarrow{SM} = \frac{1}{2}\overrightarrow{OT}$.

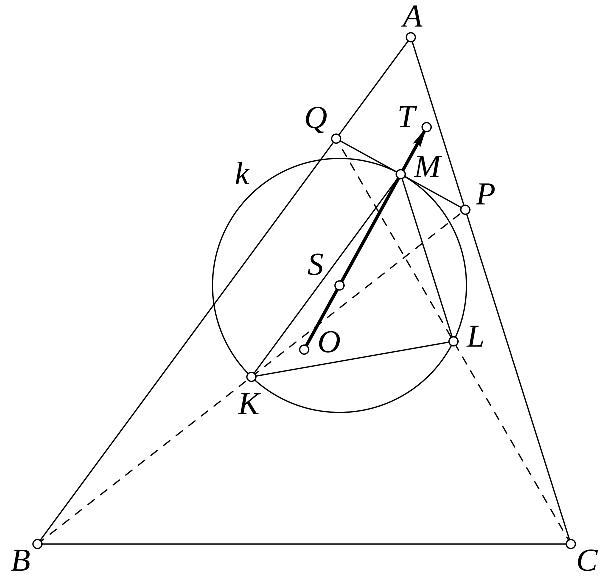


Figure 2

Now, since the circle k touches PQ at M, we get $SM \perp PQ$, hence $OT \perp PQ$. Since T is equidistant from P and Q, the line OT is a perpendicular bisector of segment PQ, and hence O is equidistant from P and Q which finishes the proof.

G3 IRN (Islamic Republic of Iran)

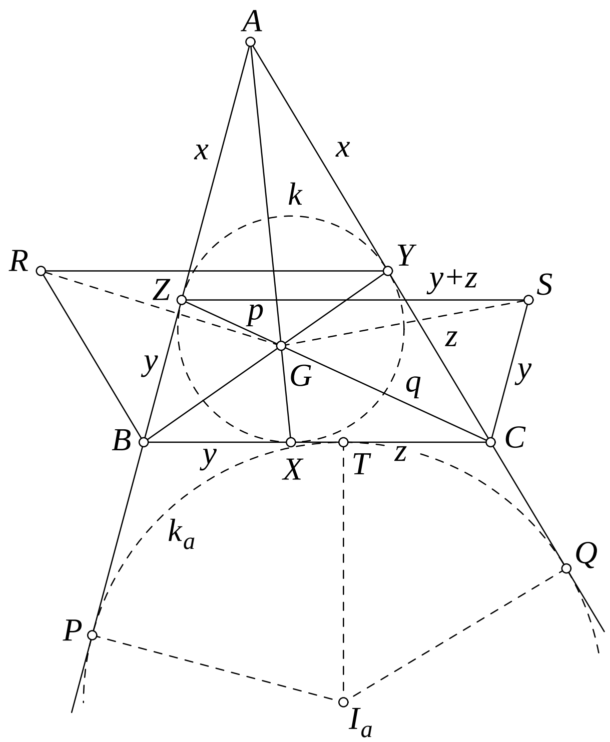
Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelograms.

Prove that $GR = GS$.

Solution 1. Denote by k the incircle and by k_a the excircle opposite to A of triangle ABC. Let k and k_a touch the side BC at the points X and T, respectively, let k_a touch the lines AB and AC at the points P and Q, respectively. We use several times the fact that opposing sides of a parallelogram are of equal length, that points of contact of the excircle and incircle to a side of a triangle lie symmetric with respect to the midpoint of this side and that segments on two tangents to a circle defined by the points of contact and their point of intersection have the same length. So we conclude

$$ZP = ZB + BP = XB + BT = BX + CX = ZS \quad \text{and}$$

$$CQ = CT = BX = BZ = CS.$$



So for each of the points Z, C, their distances to S equal the length of a tangent segment from this point to k_a. It is well-known, that all points with this property lie on the line ZC, which is the radical axis of S and k_a. Similar arguments yield that BY is the radical axis of R and k_a. So the point of intersection of ZC and BY, which is G by definition, is the radical center of R, S and k_a, from which the claim $GR = GS$ follows immediately.

Solution 2. Denote $x = AZ = AY$, $y = BZ = BX$, $z = CX = CY$, $p = ZG$, $q = GC$. Several lengthy calculations (Menelaos' theorem in triangle AZC, law of Cosines in triangles ABC and AZC and Stewart's theorem in triangle ZCS) give four equations for p, q, $\cos\alpha$

and GS in terms of x, y, and z that can be resolved for GS. The result is symmetric in y and z, so $GR = GS$. More in detail this means:

The line BY intersects the sides of triangle AZC, so MENELAOS' theorem yields $\frac{p}{q} \cdot \frac{z}{x} \cdot \frac{x+y}{y} = 1$, hence

$$\frac{p}{q} = \frac{xy}{yz + zx}. \tag{1}$$

Since we only want to show that the term for GS is symmetric in y and z, we abbreviate terms that are symmetric in y and z by capital letters, starting with $N = xy + yz + zx$. So (1) implies

$$\frac{p}{p+q} = \frac{xy}{xy + yz + zx} = \frac{xy}{N} \quad \text{and} \quad \frac{q}{p+q} = \frac{yz + zx}{xy + yz + zx} = \frac{yz + zx}{N}. \tag{2}$$

Now the law of Cosines in triangle ABC yields

$$\cos\alpha = \frac{(x+y)^2 + (x+z)^2 - (y+z)^2}{2(x+y)(x+z)} = \frac{2x^2 + 2xy + 2xz - 2yz}{2(x+y)(x+z)} = 1 - \frac{2yz}{(x+y)(x+z)}.$$

We use this result to apply the law of Cosines in triangle AZC:

$$\begin{aligned}(p+q)^2 &= x^2 + (x+z)^2 - 2x(x+z)\cos\alpha \\ &= x^2 + (x+z)^2 - 2x(x+z) \cdot \left(1 - \frac{2yz}{(x+y)(x+z)}\right) \\ &= z^2 + \frac{4xyz}{x+y}. \end{aligned} \tag{3}$$

Now in triangle ZCS the segment GS is a cevian, so with STEWART's theorem we have $py^2 + q(y+z)^2 = (p+q)(GS^2 + pq)$, hence

$$GS^2 = \frac{p}{p+q} \cdot y^2 + \frac{q}{p+q} \cdot (y+z)^2 - \frac{p}{p+q} \cdot \frac{q}{p+q} \cdot (p+q)^2.$$

Replacing the p's and q's herein by (2) and (3) yields

$$\begin{aligned} GS^2 &= \frac{xy}{N}y^2 + \frac{yz + zx}{N}(y+z)^2 - \frac{xy}{N} \cdot \frac{yz + zx}{N} \cdot \left(z^2 + \frac{4xyz}{x+y}\right) \\ &= \frac{xy^3}{N} + \underbrace{\frac{yz(y+z)^2}{N}}_{M_1} + \frac{zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} - \underbrace{\frac{4x^2y^2z^2}{N^2}}_{M_2} \\ &= \frac{xy^3 + zx(y+z)^2}{N} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \underbrace{\frac{x(y^3 + y^2z + yz^2 + z^3)}{N}}_{M_3} + \frac{xyz^2N}{N^2} - \frac{xyz^3(x+y)}{N^2} + M_1 - M_2 \\ &= \frac{x^2y^2z^2 + xy^2z^3 + x^2yz^3 - x^2yz^3 - xy^2z^3}{N^2} + M_1 - M_2 + M_3 \\ &= \frac{x^2y^2z^2}{N^2} + M_1 - M_2 + M_3, \end{aligned}$$

a term that is symmetric in y and z, indeed.

Comment. G is known as GERGONNE's point of $\triangle ABC$.

G4 UNK (United Kingdom)

Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G, and H.

Solution 1. It suffices to show that $\angle HEF = \angle HGE$ (see Figure 1), since in circle EGH the angle over the chord EH at G equals the angle between the tangent at E and EH.

First, $\angle BAD = 180° - \angle DCB = \angle FCD$. Since triangles FAB and FCD have also a common interior angle at F, they are similar.

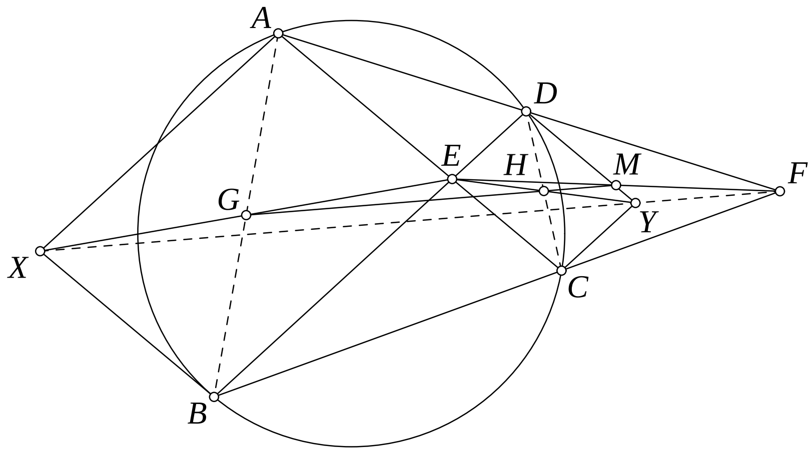


Figure 1

Denote by $\mathcal{T}$ the transformation consisting of a reflection at the bisector of $\angle DFC$ followed by a dilation with center F and factor of $\frac{FA}{FC}$. Then $\mathcal{T}$ maps F to F, C to A, D to B, and H to G. To see this, note that $\triangle FCA \sim \triangle FDB$, so $\frac{FA}{FC} = \frac{FB}{FD}$. Moreover, as $\angle ADB = \angle ACB$, the image of the line DE under $\mathcal{T}$ is parallel to AC (and passes through B) and similarly the image of CE is parallel to DB and passes through A. Hence E is mapped to the point X which is the fourth vertex of the parallelogram $BEAX$. Thus, in particular $\angle HEF = \angle FXG$.

As G is the midpoint of the diagonal AB of the parallelogram $BEAX$, it is also the midpoint of EX. In particular, E, G, X are collinear, and $EX = 2 \cdot EG$.

Denote by Y the fourth vertex of the parallelogram $DECY$. By an analogous reasoning as before, it follows that $\mathcal{T}$ maps Y to E, thus E, H, Y are collinear with $EY = 2 \cdot EH$. Therefore, by the intercept theorem, $HG \parallel XY$.

From the construction of $\mathcal{T}$ it is clear that the lines FX and FE are symmetric with respect to the bisector of $\angle DFC$, as are FY and FE. Thus, F, X, Y are collinear, which together with $HG \parallel XY$ implies $\angle FXE = \angle HGE$. This completes the proof.

Solution 2. We use the following

Lemma (Gauß). Let $ABCD$ be a quadrilateral. Let AB and CD intersect at P, and BC and DA intersect at Q. Then the midpoints K, L, M of AC, BD, and PQ, respectively, are collinear.

Proof: Let us consider the points Z that fulfill the equation

$$(ABZ) + (CDZ) = (BCZ) + (DAZ), \tag{1}$$

where (RST) denotes the oriented area of the triangle RST (see Figure 2).

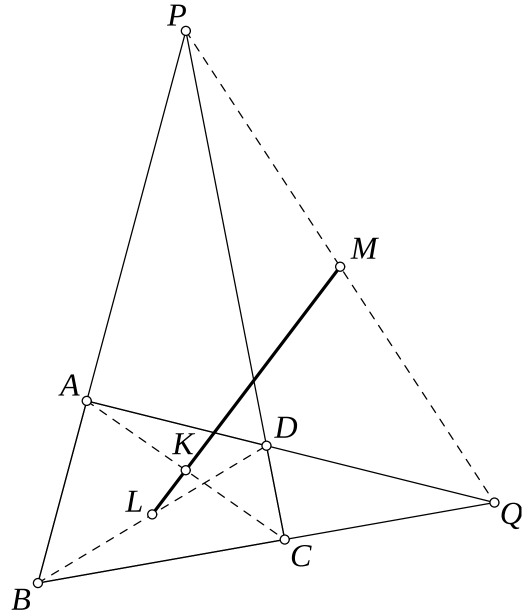


Figure 2

As (1) is linear in Z, it can either characterize a line, or be contradictory, or be trivially fulfilled for all Z in the plane. If (1) was fulfilled for all Z, then it would hold for $Z = A$, $Z = B$, which gives $(CDA) = (BCA)$, $(CDB) = (DAB)$, respectively, i.e. the diagonals of $ABCD$ would bisect each other, thus $ABCD$ would be a parallelogram. This contradicts the hypothesis that AD and BC intersect. Since E, F, G fulfill (1), it is the equation of a line which completes the proof of the lemma.

Now consider the parallelograms $EAXB$ and $ECYD$ (see Figure 1). Then G, H are the midpoints of EX, EY, respectively. Let M be the midpoint of EF. By applying the Lemma to the (re-entrant) quadrilateral $ADBC$, it is evident that G, H, and M are collinear. A dilation by a factor of 2 with center E shows that X, Y, F are collinear. Since $AX \parallel DE$ and $BX \parallel CE$, we have pairwise equal interior angles in the quadrilaterals $FDEC$ and $FBXA$. Since we have also $\angle EBA = \angle DCA = \angle CDY$, the quadrilaterals are similar. Thus, $\angle FXA = \angle CEF$.

Clearly the parallelograms $ECYD$ and $EBXA$ are similar, too, thus $\angle EXA = \angle CEY$. Consequently, $\angle FXE = \angle FXA - \angle EXA = \angle CEF - \angle CEY = \angle YEF$. By the converse of the tangent-chord angle theorem EF is tangent to the circle XEY. A dilation by a factor of $\frac{1}{2}$ completes the proof.

Solution 3. As in Solution 2, G, H, M are proven to be collinear. It suffices to show that $ME^2 = MG \cdot MH$. If $\boldsymbol{p} = \overrightarrow{OP}$ denotes the vector from circumcenter O to point P, the claim becomes

$$\left(\frac{\boldsymbol{e}-\boldsymbol{f}}{2}\right)^2 = \left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right)\left(\frac{\boldsymbol{e}+\boldsymbol{f}}{2} - \frac{\boldsymbol{c}+\boldsymbol{d}}{2}\right),$$

or equivalently

$$4\,\boldsymbol{e}\boldsymbol{f} - (\boldsymbol{e}+\boldsymbol{f})(\boldsymbol{a}+\boldsymbol{b}+\boldsymbol{c}+\boldsymbol{d}) + (\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{c}+\boldsymbol{d}) = 0. \tag{2}$$

With R as the circumradius of $ABCD$, we obtain for the powers $\mathcal{P}(E)$ and $\mathcal{P}(F)$ of E and F, respectively, with respect to the circumcircle

$$\mathcal{P}(E) = (\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = (\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2,$$
$$\mathcal{P}(F) = (\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = (\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2,$$

hence

$$(\boldsymbol{e}-\boldsymbol{a})(\boldsymbol{e}-\boldsymbol{c}) = \boldsymbol{e}^2 - R^2, \tag{3}$$
$$(\boldsymbol{e}-\boldsymbol{b})(\boldsymbol{e}-\boldsymbol{d}) = \boldsymbol{e}^2 - R^2, \tag{4}$$
$$(\boldsymbol{f}-\boldsymbol{a})(\boldsymbol{f}-\boldsymbol{d}) = \boldsymbol{f}^2 - R^2, \tag{5}$$
$$(\boldsymbol{f}-\boldsymbol{b})(\boldsymbol{f}-\boldsymbol{c}) = \boldsymbol{f}^2 - R^2. \tag{6}$$

Since F lies on the polar to E with respect to the circumcircle, we have

$$4\,\boldsymbol{e}\boldsymbol{f} = 4R^2. \tag{7}$$

Adding up (3) to (7) yields (2), as desired.

G5 POL (Poland)

Let P be a polygon that is convex and symmetric to some point O. Prove that for some parallelogram R satisfying $P \subset R$ we have

$$\frac{|R|}{|P|} \leq \sqrt{2}$$

where $|R|$ and $|P|$ denote the area of the sets R and P, respectively.

Solution 1. We will construct two parallelograms R_1 and R_3, each of them containing P, and prove that at least one of the inequalities $|R_1| \leq \sqrt{2}\,|P|$ and $|R_3| \leq \sqrt{2}\,|P|$ holds (see Figure 1).

First we will construct a parallelogram $R_1 \supseteq P$ with the property that the midpoints of the sides of R_1 are points of the boundary of P.

Choose two points A and B of P such that the triangle OAB has maximal area. Let a be the line through A parallel to OB and b the line through B parallel to OA. Let A', B', a' and b' be the points or lines, that are symmetric to A, B, a and b, respectively, with respect to O. Now let R_1 be the parallelogram defined by a, b, a' and b'.

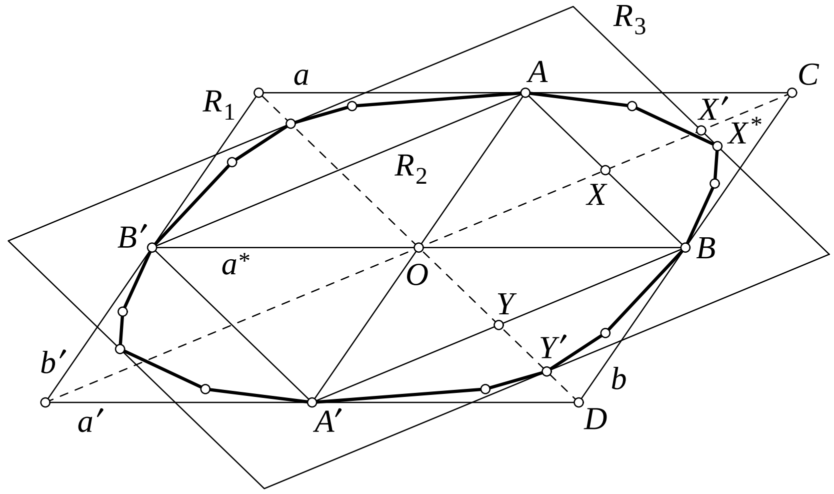


Figure 1

Obviously, A and B are located on the boundary of the polygon P, and A, B, A' and B' are midpoints of the sides of R_1. We note that $P \subseteq R_1$. Otherwise, there would be a point $Z \in P$ but $Z \notin R_1$, i.e., one of the lines a, b, a' or b' were between O and Z. If it is a, we have $|OZB| > |OAB|$, which is contradictory to the choice of A and B. If it is one of the lines b, a' or b' almost identical arguments lead to a similar contradiction.

Let R_2 be the parallelogram $ABA'B'$. Since A and B are points of P, segment $AB \subset P$ and so $R_2 \subset R_1$. Since A, B, A' and B' are midpoints of the sides of R_1, an easy argument yields

$$|R_1| = 2 \cdot |R_2|. \tag{1}$$

Let R_3 be the smallest parallelogram enclosing P defined by lines parallel to AB and BA'. Obviously $R_2 \subset R_3$ and every side of R_3 contains at least one point of the boundary of P. Denote by C the intersection point of a and b, by X the intersection point of AB and OC, and by X' the intersection point of XC and the boundary of R_3. In a similar way denote by D

the intersection point of b and a', by Y the intersection point of $A'B$ and OD, and by Y' the intersection point of YD and the boundary of R_3.

Note that $OC = 2 \cdot OX$ and $OD = 2 \cdot OY$, so there exist real numbers x and y with $1 \le x, y \le 2$ and $OX' = x \cdot OX$ and $OY' = y \cdot OY$. Corresponding sides of R_3 and R_2 are parallel which yields

$$|R_3| = xy \cdot |R_2|. \tag{2}$$

The side of R_3 containing X' contains at least one point X^* of P; due to the convexity of P we have $AX^*B \subset P$. Since this side of the parallelogram R_3 is parallel to AB we have $|AX^*B| = |AX'B|$, so $|OAX'B|$ does not exceed the area of P confined to the sector defined by the rays OB and OA. In a similar way we conclude that $|OB'Y'A'|$ does not exceed the area of P confined to the sector defined by the rays OB and OA'. Putting things together we have $|OAX'B| = x \cdot |OAB|$, $|OBDA'| = y \cdot |OBA'|$. Since $|OAB| = |OBA'|$, we conclude that $|P| \ge 2 \cdot |AX'BY'A'| = 2 \cdot (x \cdot |OAB| + y \cdot |OBA'|) = 4 \cdot \frac{x+y}{2} \cdot |OAB| = \frac{x+y}{2} \cdot R_2$; this is in short

$$\frac{x+y}{2} \cdot |R_2| \le |P|. \tag{3}$$

Since all numbers concerned are positive, we can combine (1)–(3). Using the arithmetic-geometric-mean inequality we obtain

$$|R_1| \cdot |R_3| = 2 \cdot |R_2| \cdot xy \cdot |R_2| \le 2 \cdot |R_2|^2 \left(\frac{x+y}{2}\right)^2 \le 2 \cdot |P|^2.$$

This implies immediately the desired result $|R_1| \le \sqrt{2} \cdot |P|$ or $|R_3| \le \sqrt{2} \cdot |P|$.

Solution 2. We construct the parallelograms R_1, R_2 and R_3 in the same way as in Solution 1 and will show that $\frac{|R_1|}{|P|} \le \sqrt{2}$ or $\frac{|R_3|}{|P|} \le \sqrt{2}$.

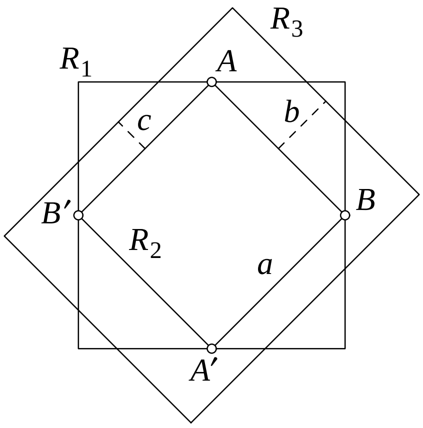


Figure 2

Recall that affine one-to-one maps of the plane preserve the ratio of areas of subsets of the plane. On the other hand, every parallelogram can be transformed with an affine map onto a square. It follows that without loss of generality we may assume that R_1 is a square (see Figure 2).

Then R_2, whose vertices are the midpoints of the sides of R_1, is a square too, and R_3, whose sides are parallel to the diagonals of R_1, is a rectangle.

Let $a > 0$, $b \ge 0$ and $c \ge 0$ be the distances introduced in Figure 2. Then $|R_1| = 2a^2$ and

$|R_3| = (a+2b)(a+2c)$.

Points A, A', B and B' are in the convex polygon P. Hence the square $ABA'B'$ is a subset of P. Moreover, each of the sides of the rectangle R_3 contains a point of P, otherwise R_3 would not be minimal. It follows that

$$|P| \geq a^2 + 2 \cdot \frac{ab}{2} + 2 \cdot \frac{ac}{2} = a(a+b+c).$$

Now assume that both $\frac{|R_1|}{|P|} > \sqrt{2}$ and $\frac{|R_3|}{|P|} > \sqrt{2}$, then

$$2a^2 = |R_1| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c)$$

and

$$(a+2b)(a+2c) = |R_3| > \sqrt{2} \cdot |P| \geq \sqrt{2} \cdot a(a+b+c).$$

All numbers concerned are positive, so after multiplying these inequalities we get

$$2a^2(a+2b)(a+2c) > 2a^2(a+b+c)^2.$$

But the arithmetic-geometric-mean inequality implies the contradictory result

$$2a^2(a+2b)(a+2c) \leq 2a^2 \left(\frac{(a+2b)+(a+2c)}{2} \right)^2 = 2a^2(a+b+c)^2.$$

Hence $\frac{|R_1|}{|P|} \leq \sqrt{2}$ or $\frac{|R_3|}{|P|} \leq \sqrt{2}$, as desired.

G6 UKR (Ukraine)

Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are the circumcenters and points H_1 and H_2 are the orthocenters of triangles ABP and DCP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the line H_1H_2 are concurrent.

Solution 1. We keep triangle ABP fixed and move the line CD parallel to itself uniformly, i.e. linearly dependent on a single parameter λ (see Figure 1). Then the points C and D also move uniformly. Hence, the points O_2, H_2 and E_2 move uniformly, too. Therefore also the perpendicular from E_2 on AB moves uniformly. Obviously, the points O_1, H_1, E_1 and the perpendicular from E_1 on CD do not move at all. Hence, the intersection point S of these two perpendiculars moves uniformly. Since H_1 does not move, while H_2 and S move uniformly along parallel lines (both are perpendicular to CD), it is sufficient to prove their collinearity for two different positions of CD.

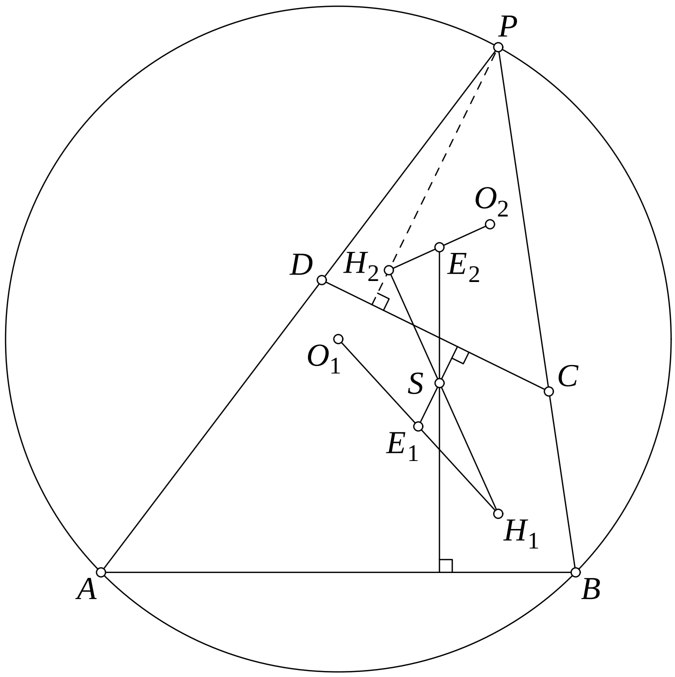


Figure 1

Let CD pass through either point A or point B. Note that by hypothesis these two cases are different. We will consider the case $A \in CD$, i.e. $A = D$. So we have to show that the perpendiculars from E_1 on AC and from E_2 on AB intersect on the altitude AH of triangle ABC (see Figure 2).

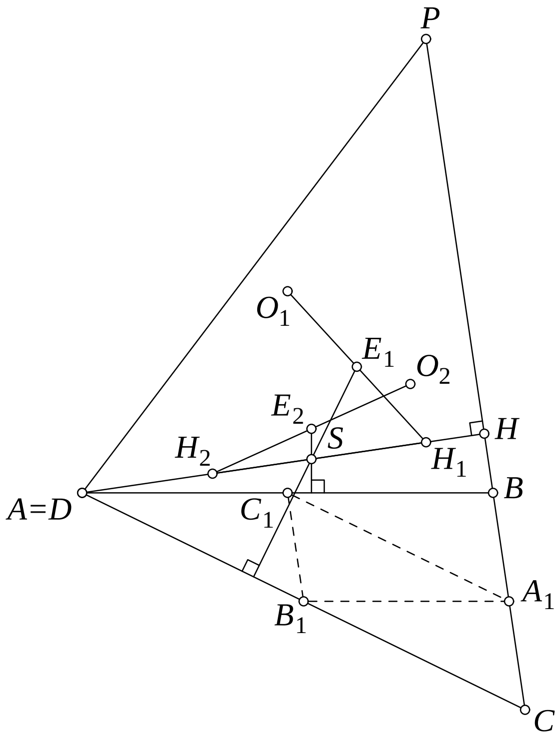


Figure 2

To this end, we consider the midpoints A_1, B_1, C_1 of BC, CA, AB, respectively. As E_1 is the center of FEUERBACH's circle (nine-point circle) of $\triangle ABP$, we have $E_1C_1 = E_1H$. Similarly, $E_2B_1 = E_2H$. Note further that a point X lies on the perpendicular from E_1 on A_1C_1 if and only if

$$XC_1^2 - XA_1^2 = E_1C_1^2 - E_1A_1^2.$$

Similarly, the perpendicular from E_2 on A_1B_1 is characterized by

$$XA_1^2 - XB_1^2 = E_2A_1^2 - E_2B_1^2.$$

The line H_1H_2, which is perpendicular to B_1C_1 and contains A, is given by

$$XB_1^2 - XC_1^2 = AB_1^2 - AC_1^2.$$

The three lines are concurrent if and only if

$$\begin{aligned}0 &= XC_1^2 - XA_1^2 + XA_1^2 - XB_1^2 + XB_1^2 - XC_1^2 \\ &= E_1C_1^2 - E_1A_1^2 + E_2A_1^2 - E_2B_1^2 + AB_1^2 - AC_1^2 \\ &= -E_1A_1^2 + E_2A_1^2 + E_1H^2 - E_2H^2 + AB_1^2 - AC_1^2,\end{aligned}$$

i.e. it suffices to show that

$$E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2 = \frac{AC^2 - AB^2}{4}.$$

We have

$$\frac{AC^2 - AB^2}{4} = \frac{HC^2 - HB^2}{4} = \frac{(HC + HB)(HC - HB)}{4} = \frac{HA_1 \cdot BC}{2}.$$

Let F_1, F_2 be the projections of E_1, E_2 on BC. Obviously, these are the midpoints of HP_1,

HP_2, where P_1, P_2 are the midpoints of PB and PC respectively. Then

$$\begin{aligned}
&E_1A_1^2 - E_2A_1^2 - E_1H^2 + E_2H^2\\
&= F_1A_1^2 - F_1H^2 - F_2A_1^2 + F_2H^2\\
&= (F_1A_1 - F_1H)(F_1A_1 + F_1H) - (F_2A_1 - F_2H)(F_2A_1 + F_2H)\\
&= A_1H \cdot (A_1P_1 - A_1P_2)\\
&= \frac{A_1H \cdot BC}{2}\\
&= \frac{AC^2 - AB^2}{4},
\end{aligned}$$

which proves the claim.

Solution 2. Let the perpendicular from E_1 on CD meet PH_1 at X, and the perpendicular from E_2 on AB meet PH_2 at Y (see Figure 3). Let φ be the intersection angle of AB and CD. Denote by M, N the midpoints of PH_1, PH_2 respectively.

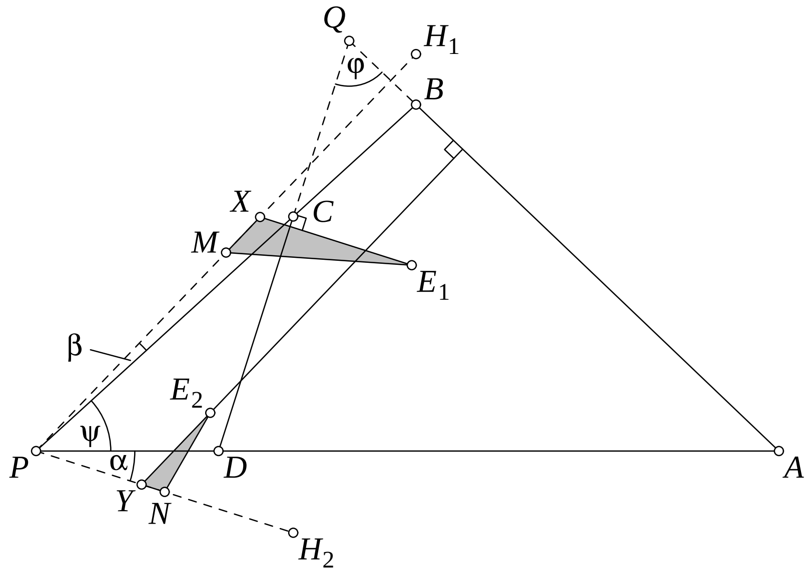


Figure 3

We will prove now that triangles E_1XM and E_2YN have equal angles at E_1, E_2, and supplementary angles at X, Y.

In the following, angles are understood as oriented, and equalities of angles modulo 180°.

Let $\alpha = \angle H_2PD$, $\psi = \angle DPC$, $\beta = \angle CPH_1$. Then $\alpha + \psi + \beta = \varphi$, $\angle E_1XH_1 = \angle H_2YE_2 = \varphi$, thus $\angle MXE_1 + \angle NYE_2 = 180°$.

By considering the FEUERBACH circle of $\triangle ABP$ whose center is E_1 and which goes through M, we have $\angle E_1MH_1 = \psi + 2\beta$. Analogous considerations with the FEUERBACH circle of $\triangle DCP$ yield $\angle H_2NE_2 = \psi + 2\alpha$. Hence indeed $\angle XE_1M = \varphi - (\psi + 2\beta) = (\psi + 2\alpha) - \varphi = \angle YE_2N$.

It follows now that

$$\frac{XM}{ME_1} = \frac{YN}{NE_2}\ .$$

Furthermore, ME_1 is half the circumradius of $\triangle ABP$, while PH_1 is the distance of P to the orthocenter of that triangle, which is twice the circumradius times the cosine of ψ. Together with analogous reasoning for $\triangle DCP$ we have

$$\frac{ME_1}{PH_1} = \frac{1}{4\cos\psi} = \frac{NE_2}{PH_2} .$$

By multiplication,

$$\frac{XM}{PH_1} = \frac{YN}{PH_2},$$

and therefore

$$\frac{PX}{XH_1} = \frac{H_2Y}{YP}.$$

Let E_1X, E_2Y meet H_1H_2 in R, S respectively.

Applying the intercept theorem to the parallels E_1X, PH_2 and center H_1 gives

$$\frac{H_2R}{RH_1} = \frac{PX}{XH_1},$$

while with parallels E_2Y, PH_1 and center H_2 we obtain

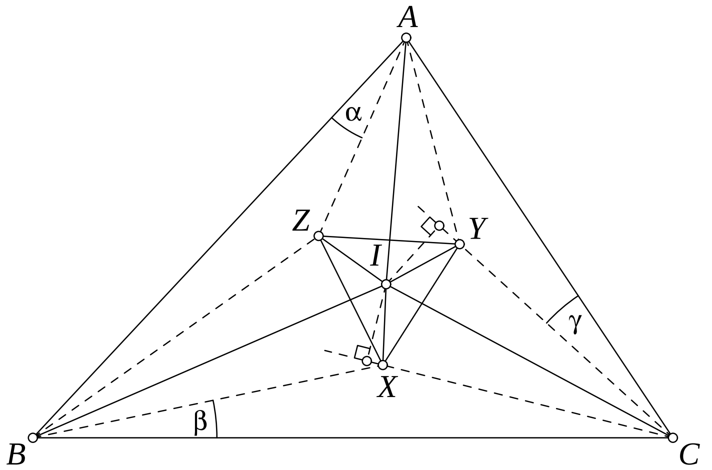
$$\frac{H_2S}{SH_1} = \frac{H_2Y}{YP}.$$

Combination of the last three equalities yields that R and S coincide.

G7 IRN (Islamic Republic of Iran)

Let ABC be a triangle with incenter I and let X, Y and Z be the incenters of the triangles BIC, CIA and AIB, respectively. Let the triangle XYZ be equilateral. Prove that ABC is equilateral too.

Solution. AZ, AI and AY divide $\angle BAC$ into four equal angles; denote them by α. In the same way we have four equal angles β at B and four equal angles γ at C. Obviously $\alpha+\beta+\gamma=\frac{180^\circ}{4}=45^\circ$; and $0^\circ<\alpha,\beta,\gamma<45^\circ$.



Easy calculations in various triangles yield $\angle BIC=180^\circ-2\beta-2\gamma=180^\circ-(90^\circ-2\alpha)=90^\circ+2\alpha$, hence (for X is the incenter of triangle BCI, so IX bisects $\angle BIC$) we have $\angle XIC=\angle BIX=\frac{1}{2}\angle BIC=45^\circ+\alpha$ and with similar aguments $\angle CIY=\angle YIA=45^\circ+\beta$ and $\angle AIZ=\angle ZIB=45^\circ+\gamma$. Furthermore, we have $\angle XIY=\angle XIC+\angle CIY=(45^\circ+\alpha)+(45^\circ+\beta)=135^\circ-\gamma$, $\angle YIZ=135^\circ-\alpha$, and $\angle ZIX=135^\circ-\beta$.

Now we calculate the lengths of IX, IY and IZ in terms of α, β and γ. The perpendicular from I on CX has length $IX\cdot\sin\angle CXI=IX\cdot\sin(90^\circ+\beta)=IX\cdot\cos\beta$. But CI bisects $\angle YCX$, so the perpendicular from I on CY has the same length, and we conclude

$$IX\cdot\cos\beta=IY\cdot\cos\alpha.$$

To make calculations easier we choose a length unit that makes $IX=\cos\alpha$. Then $IY=\cos\beta$ and with similar arguments $IZ=\cos\gamma$.

Since XYZ is equilateral we have $ZX=ZY$. The law of Cosines in triangles XYI, YZI yields

$$\begin{aligned}
&ZX^2=ZY^2\\
\Longrightarrow\quad &IZ^2+IX^2-2\cdot IZ\cdot IX\cdot\cos\angle ZIX=IZ^2+IY^2-2\cdot IZ\cdot IY\cdot\cos\angle YIZ\\
\Longrightarrow\quad &IX^2-IY^2=2\cdot IZ\cdot(IX\cdot\cos\angle ZIX-IY\cdot\cos\angle YIZ)\\
\Longrightarrow\quad &\underbrace{\cos^2\alpha-\cos^2\beta}_{\text{L.H.S.}}=\underbrace{2\cdot\cos\gamma\cdot(\cos\alpha\cdot\cos(135^\circ-\beta)-\cos\beta\cdot\cos(135^\circ-\alpha))}_{\text{R.H.S.}}.
\end{aligned}$$

A transformation of the left-hand side (L.H.S.) yields

$$\begin{aligned}
\text{L.H.S.}&=\cos^2\alpha\cdot\left(\sin^2\beta+\cos^2\beta\right)-\cos^2\beta\cdot\left(\sin^2\alpha+\cos^2\alpha\right)\\
&=\cos^2\alpha\cdot\sin^2\beta-\cos^2\beta\cdot\sin^2\alpha
\end{aligned}$$

$$= (\cos\alpha \cdot \sin\beta + \cos\beta \cdot \sin\alpha) \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha)$$
$$= \sin(\beta + \alpha) \cdot \sin(\beta - \alpha) = \sin(45^\circ - \gamma) \cdot \sin(\beta - \alpha)$$

whereas a transformation of the right-hand side (R.H.S.) leads to

$$\begin{aligned}
\text{R.H.S.} &= 2 \cdot \cos\gamma \cdot (\cos\alpha \cdot (-\cos(45^\circ + \beta)) - \cos\beta \cdot (-\cos(45^\circ + \alpha))) \\
&= 2 \cdot \frac{\sqrt{2}}{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot (\sin\beta - \cos\beta) + \cos\beta \cdot (\cos\alpha - \sin\alpha)) \\
&= \sqrt{2} \cdot \cos\gamma \cdot (\cos\alpha \cdot \sin\beta - \cos\beta \cdot \sin\alpha) \\
&= \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta - \alpha).
\end{aligned}$$

Equating L.H.S. and R.H.S. we obtain

$$\begin{aligned}
& \sin(45^\circ - \gamma) \cdot \sin(\beta - \alpha) = \sqrt{2} \cdot \cos\gamma \cdot \sin(\beta - \alpha) \\
\Longrightarrow \quad & \sin(\beta - \alpha) \cdot \left(\sqrt{2} \cdot \cos\gamma - \sin(45^\circ - \gamma)\right) = 0 \\
\Longrightarrow \quad & \alpha = \beta \quad \text{or} \quad \sqrt{2} \cdot \cos\gamma = \sin(45^\circ - \gamma).
\end{aligned}$$

But $\gamma < 45^\circ$; so $\sqrt{2} \cdot \cos\gamma > \cos\gamma > \cos 45^\circ = \sin 45^\circ > \sin(45^\circ - \gamma)$. This leaves $\alpha = \beta$. With similar reasoning we have $\alpha = \gamma$, which means triangle ABC must be equilateral.

G8 BGR (Bulgaria)

Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2, and I_3 the incenters of $\triangle ABM$, $\triangle MNC$, and $\triangle NDA$, respectively. Show that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

Solution 1. Let k_1, k_2 and k_3 be the incircles of triangles ABM, MNC, and NDA, respectively (see Figure 1). We shall show that the tangent h from C to k_1 which is different from CB is also tangent to k_3.

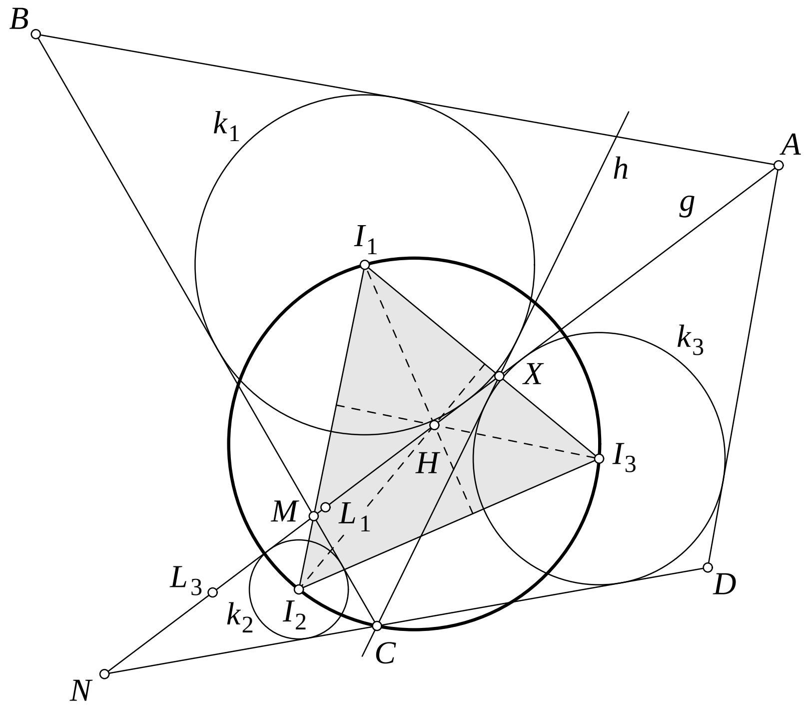


Figure 1

To this end, let X denote the point of intersection of g and h. Then $ABCX$ and $ABCD$ are circumscribed quadrilaterals, whence

$$CD - CX = (AB + CD) - (AB + CX) = (BC + AD) - (BC + AX) = AD - AX,$$

i.e.

$$AX + CD = CX + AD$$

which in turn reveals that the quadrilateral $AXCD$ is also circumscribed. Thus h touches indeed the circle k_3.

Moreover, we find that $\angle I_3CI_1 = \angle I_3CX + \angle XCI_1 = \frac{1}{2}(\angle DCX + \angle XCB) = \frac{1}{2}\angle DCB = \frac{1}{2}(180^\circ - \angle MCN) = 180^\circ - \angle MI_2N = \angle I_3I_2I_1$, from which we conclude that C, I_1, I_2, I_3 are concyclic.

Let now L_1 and L_3 be the reflection points of C with respect to the lines I_2I_3 and I_1I_2 respectively. Since I_1I_2 is the angle bisector of $\angle NMC$, it follows that L_3 lies on g. By analogous reasoning, L_1 lies on g.

Let H be the orthocenter of $\triangle I_1I_2I_3$. We have $\angle I_2L_3I_1 = \angle I_1CI_2 = \angle I_1I_3I_2 = 180^\circ - \angle I_1HI_2$, which entails that the quadrilateral $I_2HI_1L_3$ is cyclic. Analogously, $I_3HL_1I_2$ is cyclic.

Then, working with oriented angles modulo 180°, we have

$$\angle L_3HI_2 = \angle L_3I_1I_2 = \angle I_2I_1C = \angle I_2I_3C = \angle L_1I_3I_2 = \angle L_1HI_2,$$

whence L_1, L_3, and H are collinear. By $L_1 \neq L_3$, the claim follows.

Comment. The last part of the argument essentially reproves the following fact: The Simson line of a point P lying on the circumcircle of a triangle ABC with respect to that triangle bisects the line segment connecting P with the orthocenter of ABC.

Solution 2. We start by proving that C, I_1, I_2, and I_3 are concyclic.

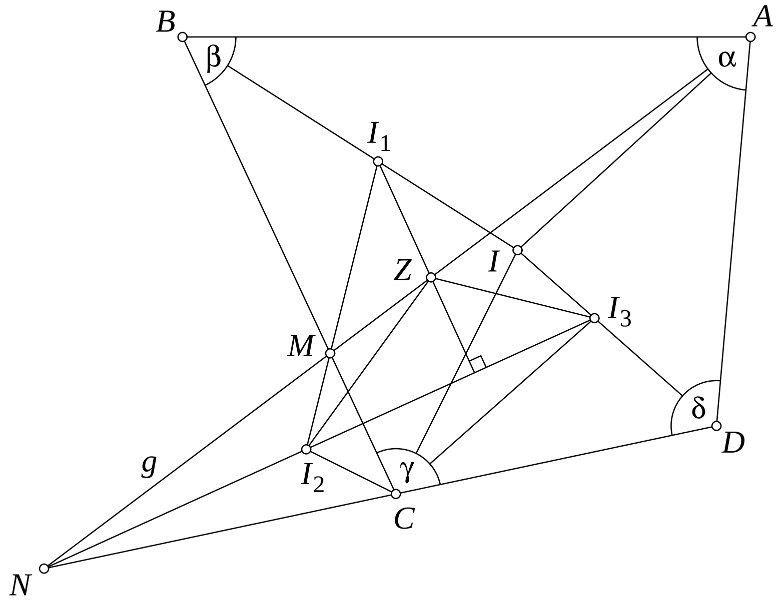


Figure 2

To this end, notice first that I_2, M, I_1 are collinear, as are N, I_2, I_3 (see Figure 2). Denote by α, β, γ, δ the internal angles of $ABCD$. By considerations in triangle CMN, it follows that $\angle I_3I_2I_1 = \frac{\gamma}{2}$. We will show that $\angle I_3CI_1 = \frac{\gamma}{2}$, too. Denote by I the incenter of $ABCD$. Clearly, $I_1 \in BI$, $I_3 \in DI$, $\angle I_1AI_3 = \frac{\alpha}{2}$.

Using the abbreviation $[X, YZ]$ for the distance from point X to the line YZ, we have because of $\angle BAI_1 = \angle IAI_3$ and $\angle I_1AI = \angle I_3AD$ that

$$\frac{[I_1, AB]}{[I_1, AI]} = \frac{[I_3, AI]}{[I_3, AD]}\,.$$

Furthermore, consideration of the angle sums in AIB, BIC, CID and DIA implies $\angle AIB + \angle CID = \angle BIC + \angle DIA = 180°$, from which we see

$$\frac{[I_1, AI]}{[I_3, CI]} = \frac{I_1I}{I_3I} = \frac{[I_1, CI]}{[I_3, AI]}\,.$$

Because of $[I_1, AB] = [I_1, BC]$, $[I_3, AD] = [I_3, CD]$, multiplication yields

$$\frac{[I_1, BC]}{[I_3, CI]} = \frac{[I_1, CI]}{[I_3, CD]}\,.$$

By $\angle DCI = \angle ICB = \gamma/2$ it follows that $\angle I_1CB = \angle I_3CI$ which concludes the proof of the above statement.

Let the perpendicular from I_1 on I_2I_3 intersect g at Z. Then $\angle MI_1Z = 90^\circ - \angle I_3I_2I_1 = 90^\circ - \gamma/2 = \angle MCI_2$. Since we have also $\angle ZMI_1 = \angle I_2MC$, triangles MZI_1 and MI_2C are similar. From this one easily proves that also MI_2Z and MCI_1 are similar. Because C, I_1, I_2, and I_3 are concyclic, $\angle MZI_2 = \angle MI_1C = \angle NI_3C$, thus NI_2Z and NCI_3 are similar, hence NCI_2 and NI_3Z are similar. We conclude $\angle ZI_3I_2 = \angle I_2CN = 90^\circ - \gamma/2$, hence $I_1I_2 \perp ZI_3$. This completes the proof.

Number Theory

N1 AUS (Australia)

A social club has n members. They have the membership numbers $1, 2, \ldots, n$, respectively. From time to time members send presents to other members, including items they have already received as presents from other members. In order to avoid the embarrassing situation that a member might receive a present that he or she has sent to other members, the club adds the following rule to its statutes at one of its annual general meetings:

"A member with membership number a is permitted to send a present to a member with membership number b if and only if $a(b-1)$ is a multiple of n."

Prove that, if each member follows this rule, none will receive a present from another member that he or she has already sent to other members.

Alternative formulation: Let G be a directed graph with n vertices $v_1, v_2, \ldots, v_n$, such that there is an edge going from v_a to v_b if and only if a and b are distinct and $a(b-1)$ is a multiple of n. Prove that this graph does not contain a directed cycle.

Solution 1. Suppose there is an edge from v_i to v_j. Then $i(j-1) = ij - i = kn$ for some integer k, which implies $i = ij - kn$. If $\gcd(i, n) = d$ and $\gcd(j, n) = e$, then e divides $ij - kn = i$ and thus e also divides d. Hence, if there is an edge from v_i to v_j, then $\gcd(j, n) \mid \gcd(i, n)$.

If there is a cycle in G, say $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$, then we have

$$\gcd(i_1, n) \mid \gcd(i_r, n) \mid \gcd(i_{r-1}, n) \mid \ldots \mid \gcd(i_2, n) \mid \gcd(i_1, n),$$

which implies that all these greatest common divisors must be equal, say be equal to t.

Now we pick any of the i_k, without loss of generality let it be i_1. Then $i_r(i_1 - 1)$ is a multiple of n and hence also (by dividing by t), $i_1 - 1$ is a multiple of $\frac{n}{t}$. Since i_1 and $i_1 - 1$ are relatively prime, also t and $\frac{n}{t}$ are relatively prime. So, by the Chinese remainder theorem, the value of i_1 is uniquely determined modulo $n = t \cdot \frac{n}{t}$ by the value of t. But, as i_1 was chosen arbitrarily among the i_k, this implies that all the i_k have to be equal, a contradiction.

Solution 2. If a, b, c are integers such that $ab - a$ and $bc - b$ are multiples of n, then also $ac - a = a(bc - b) + (ab - a) - (ab - a)c$ is a multiple of n. This implies that if there is an edge from v_a to v_b and an edge from v_b to v_c, then there also must be an edge from v_a to v_c. Therefore, if there are any cycles at all, the smallest cycle must have length 2. But suppose the vertices v_a and v_b form such a cycle, i. e., $ab - a$ and $ab - b$ are both multiples of n. Then $a - b$ is also a multiple of n, which can only happen if $a = b$, which is impossible.

Solution 3. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Then $i_1(i_2 - 1)$ is a multiple of n, i. e., $i_1 \equiv i_1 i_2 \bmod n$. Continuing in this manner, we get $i_1 \equiv i_1 i_2 \equiv i_1 i_2 i_3 \equiv i_1 i_2 i_3 \ldots i_r \bmod n$. But the same holds for all i_k, i. e., $i_k \equiv i_1 i_2 i_3 \ldots i_r \bmod n$. Hence $i_1 \equiv i_2 \equiv \cdots \equiv i_r \bmod n$, which means $i_1 = i_2 = \cdots = i_r$, a contradiction.

Solution 4. Let $n = k$ be the smallest value of n for which the corresponding graph has a cycle. We show that k is a prime power.
If k is not a prime power, it can be written as a product $k = de$ of relatively prime integers greater than 1. Reducing all the numbers modulo d yields a single vertex or a cycle in the corresponding graph on d vertices, because if $a(b-1) \equiv 0 \bmod k$ then this equation also holds modulo d. But since the graph on d vertices has no cycles, by the minimality of k, we must have that all the indices of the cycle are congruent modulo d. The same holds modulo e and hence also modulo $k = de$. But then all the indices are equal, which is a contradiction.
Thus k must be a prime power $k = p^m$. There are no edges ending at v_k, so v_k is not contained in any cycle. All edges not starting at v_k end at a vertex belonging to a non-multiple of p, and all edges starting at a non-multiple of p must end at v_1. But there is no edge starting at v_1. Hence there is no cycle.

Solution 5. Suppose there was a cycle $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_r} \to v_{i_1}$. Let $q = p^m$ be a prime power dividing n. We claim that either $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 0 \bmod q$ or $i_1 \equiv i_2 \equiv \cdots \equiv i_r \equiv 1 \bmod q$.

Suppose that there is an i_s not divisible by q. Then, as $i_s(i_{s+1} - 1)$ is a multiple of q, $i_{s+1} \equiv 1 \bmod p$. Similarly, we conclude $i_{s+2} \equiv 1 \bmod p$ and so on. So none of the labels is divisible by p, but since $i_s(i_{s+1} - 1)$ is a multiple of $q = p^m$ for all s, all i_{s+1} are congruent to 1 modulo q. This proves the claim.

Now, as all the labels are congruent modulo all the prime powers dividing n, they must all be equal by the Chinese remainder theorem. This is a contradiction.

N2 PER (Peru)

A positive integer N is called *balanced*, if $N = 1$ or if N can be written as a product of an even number of not necessarily distinct primes. Given positive integers a and b, consider the polynomial P defined by $P(x) = (x + a)(x + b)$.

(a) Prove that there exist distinct positive integers a and b such that all the numbers $P(1), P(2), \ldots, P(50)$ are balanced.

(b) Prove that if $P(n)$ is balanced for all positive integers n, then $a = b$.

Solution. Define a function f on the set of positive integers by $f(n) = 0$ if n is balanced and $f(n) = 1$ otherwise. Clearly, $f(nm) \equiv f(n) + f(m) \mod 2$ for all positive integers n, m.

(a) Now for each positive integer n consider the binary sequence $(f(n+1), f(n+2), \ldots, f(n+50))$. As there are only 2^{50} different such sequences there are two different positive integers a and b such that

$$(f(a+1), f(a+2), \ldots, f(a+50)) = (f(b+1), f(b+2), \ldots, f(b+50)).$$

But this implies that for the polynomial $P(x) = (x+a)(x+b)$ all the numbers $P(1)$, $P(2)$, $\ldots$, $P(50)$ are balanced, since for all $1 \leq k \leq 50$ we have $f(P(k)) \equiv f(a+k) + f(b+k) \equiv 2f(a+k) \equiv 0 \mod 2$.

(b) Now suppose $P(n)$ is balanced for all positive integers n and $a < b$. Set $n = k(b-a) - a$ for sufficiently large k, such that n is positive. Then $P(n) = k(k+1)(b-a)^2$, and this number can only be balanced, if $f(k) = f(k+1)$ holds. Thus, the sequence $f(k)$ must become constant for sufficiently large k. But this is not possible, as for every prime p we have $f(p) = 1$ and for every square t^2 we have $f(t^2) = 0$.

Hence $a = b$.

Comment. Given a positive integer k, a computer search for the pairs of positive integers (a, b), for which $P(1), P(2), \ldots, P(k)$ are all balanced yields the following results with minimal sum $a + b$ and $a < b$:

k	3	4	5	10	20
(a,b)	$(2,4)$	$(6,11)$	$(8,14)$	$(20,34)$	$(1751,3121)$

Therefore, trying to find a and b in part (a) of the problem cannot be done by elementary calculations.

N3 EST (Estonia)

Let f be a non-constant function from the set of positive integers into the set of positive integers, such that $a-b$ divides $f(a)-f(b)$ for all distinct positive integers a, b. Prove that there exist infinitely many primes p such that p divides $f(c)$ for some positive integer c.

Solution 1. Denote by $v_p(a)$ the exponent of the prime p in the prime decomposition of a.

Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value produced of f.

There are infinitely many positive integers a such that $v_{p_i}(a) > v_{p_i}(f(1))$ for all $i = 1, 2, \ldots, m$, e.g. $a = (p_1 p_2 \ldots p_m)^\alpha$ with α sufficiently large. Pick any such a. The condition of the problem then yields $a \mid (f(a+1) - f(1))$. Assume $f(a+1) \neq f(1)$. Then we must have $v_{p_i}(f(a+1)) \neq v_{p_i}(f(1))$ for at least one i. This yields $v_{p_i}(f(a+1) - f(1)) = \min\{v_{p_i}(f(a+1)), v_{p_i}(f(1))\} \leq v_{p_1}(f(1)) < v_{p_i}(a)$. But this contradicts the fact that $a \mid (f(a+1) - f(1))$.

Hence we must have $f(a+1) = f(1)$ for all such a.

Now, for any positive integer b and all such a, we have $(a+1-b) \mid (f(a+1) - f(b))$, i.e., $(a+1-b) \mid (f(1) - f(b))$. Since this is true for infinitely many positive integers a we must have $f(b) = f(1)$. Hence f is a constant function, a contradiction. Therefore, our initial assumption was false and there are indeed infinitely many primes p dividing $f(c)$ for some positive integer c.

Solution 2. Assume that there are only finitely many primes $p_1, p_2, \ldots, p_m$ that divide some function value of f. Since f is not identically 1, we must have $m \geq 1$.

Then there exist non-negative integers $\alpha_1, \ldots, \alpha_m$ such that

$$f(1) = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}.$$

We can pick a positive integer r such that $f(r) \neq f(1)$. Let

$$M = 1 + p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r).$$

Then for all $i \in \{1, \ldots, m\}$ we have that $p_i^{\alpha_i+1}$ divides $M-1$ and hence by the condition of the problem also $f(M) - f(1)$. This implies that $f(M)$ is divisible by $p_i^{\alpha_i}$ but not by $p_i^{\alpha_i+1}$ for all i and therefore $f(M) = f(1)$.

Hence

$$\begin{aligned}
M - r &> p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} \cdot (f(r) + r) - r \\
&\geq p_1^{\alpha_1+1} p_2^{\alpha_2+1} \ldots p_m^{\alpha_m+1} + (f(r) + r) - r \\
&> p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m} + f(r) \\
&\geq |f(M) - f(r)|.
\end{aligned}$$

But since $M - r$ divides $f(M) - f(r)$ this can only be true if $f(r) = f(M) = f(1)$, which contradicts the choice of r.

Comment. In the case that f is a polynomial with integer coefficients the result is well-known, see e.g. W. Schwarz, Einführung in die Methoden der Primzahltheorie, 1969.

N4 PRK (Democratic People's Republic of Korea)

Find all positive integers n such that there exists a sequence of positive integers $a_1, a_2, \ldots, a_n$ satisfying

$$a_{k+1} = \frac{a_k^2 + 1}{a_{k-1} + 1} - 1$$

for every k with $2 \leq k \leq n - 1$.

Solution 1. Such a sequence exists for $n = 1, 2, 3, 4$ and no other n. Since the existence of such a sequence for some n implies the existence of such a sequence for all smaller n, it suffices to prove that $n = 5$ is not possible and $n = 4$ is possible.

Assume first that for $n = 5$ there exists a sequence of positive integers $a_1, a_2, \ldots, a_5$ satisfying the conditions

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Assume a_1 is odd, then a_2 has to be odd as well and as then $a_2^2 + 1 \equiv 2 \mod 4$, a_3 has to be even. But this is a contradiction, since then the even number $a_2 + 1$ cannot divide the odd number $a_3^2 + 1$.

Hence a_1 is even.

If a_2 is odd, $a_3^2 + 1$ is even (as a multiple of $a_2 + 1$) and hence a_3 is odd, too. Similarly we must have a_4 odd as well. But then $a_3^2 + 1$ is a product of two even numbers $(a_2 + 1)(a_4 + 1)$ and thus is divisible by 4, which is a contradiction as for odd a_3 we have $a_3^2 + 1 \equiv 2 \mod 4$.

Hence a_2 is even. Furthermore $a_3 + 1$ divides the odd number $a_2^2 + 1$ and so a_3 is even. Similarly, a_4 and a_5 are even as well.

Now set $x = a_2$ and $y = a_3$. From the given condition we get $(x+1) | (y^2+1)$ and $(y+1) | (x^2+1)$. We will prove that there is no pair of positive even numbers (x, y) satisfying these two conditions, thus yielding a contradiction to the assumption.

Assume there exists a pair (x_0, y_0) of positive even numbers satisfying the two conditions $(x_0 + 1) | (y_0^2 + 1)$ and $(y_0 + 1) | (x_0^2 + 1)$.

Then one has $(x_0 + 1) | (y_0^2 + 1 + x_0^2 - 1)$, i.e., $(x_0 + 1) | (x_0^2 + y_0^2)$, and similarly $(y_0 + 1) | (x_0^2 + y_0^2)$. Any common divisor d of $x_0 + 1$ and $y_0 + 1$ must hence also divide the number $(x_0^2 + 1) + (y_0^2 + 1) - (x_0^2 + y_0^2) = 2$. But as $x_0 + 1$ and $y_0 + 1$ are both odd, we must have $d = 1$. Thus $x_0 + 1$ and $y_0 + 1$ are relatively prime and therefore there exists a positive integer k such that

$$k(x + 1)(y + 1) = x^2 + y^2$$

has the solution (x_0, y_0). We will show that the latter equation has no solution (x, y) in positive even numbers.

Assume there is a solution. Pick the solution (x_1, y_1) with the smallest sum $x_1 + y_1$ and assume $x_1 \geq y_1$. Then x_1 is a solution to the quadratic equation

$$x^2 - k(y_1 + 1)x + y_1^2 - k(y_1 + 1) = 0.$$

Let x_2 be the second solution, which by VIETA's theorem fulfills $x_1 + x_2 = k(y_1 + 1)$ and $x_1 x_2 = y_1^2 - k(y_1 + 1)$. If $x_2 = 0$, the second equation implies $y_1^2 = k(y_1 + 1)$, which is impossible, as $y_1 + 1 > 1$ cannot divide the relatively prime number y_1^2. Therefore $x_2 \neq 0$.

Also we get $(x_1 + 1)(x_2 + 1) = x_1 x_2 + x_1 + x_2 + 1 = y_1^2 + 1$ which is odd, and hence x_2 must be even and positive. Also we have $x_2 + 1 = \frac{y_1^2+1}{x_1+1} \leq \frac{y_1^2+1}{y_1+1} \leq y_1 \leq x_1$. But this means that the pair (x', y') with $x' = y_1$ and $y' = x_2$ is another solution of $k(x + 1)(y + 1) = x^2 + y^2$ in even positive numbers with $x' + y' < x_1 + y_1$, a contradiction.

Therefore we must have $n \leq 4$.

When $n = 4$, a possible example of a sequence is $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$.

Solution 2. It is easy to check that for $n = 4$ the sequence $a_1 = 4$, $a_2 = 33$, $a_3 = 217$ and $a_4 = 1384$ is possible.

Now assume there is a sequence with $n \geq 5$. Then we have in particular

$$\begin{aligned} a_2^2 + 1 &= (a_1 + 1)(a_3 + 1), \\ a_3^2 + 1 &= (a_2 + 1)(a_4 + 1), \\ a_4^2 + 1 &= (a_3 + 1)(a_5 + 1). \end{aligned}$$

Also assume without loss of generality that among all such quintuples $(a_1, a_2, a_3, a_4, a_5)$ we have chosen one with minimal a_1.

One shows quickly the following fact:

> If three positive integers x, y, z fulfill $y^2 + 1 = (x + 1)(z + 1)$ and if y is even, then x and z are even as well and either $x < y < z$ or $z < y < x$ holds. (1)

Indeed, the first part is obvious and from $x < y$ we conclude

$$z + 1 = \frac{y^2 + 1}{x + 1} \geq \frac{y^2 + 1}{y} > y,$$

and similarly in the other case.

Now, if a_3 was odd, then $(a_2 + 1)(a_4 + 1) = a_3^2 + 1 \equiv 2 \mod 4$ would imply that one of a_2 or a_4 is even, this contradicts (1). Thus a_3 and hence also a_1, a_2, a_4 and a_5 are even. According to (1), one has $a_1 < a_2 < a_3 < a_4 < a_5$ or $a_1 > a_2 > a_3 > a_4 > a_5$ but due to the minimality of a_1 the first series of inequalities must hold.

Consider the identity

$$(a_3+1)(a_1+a_3) = a_3^2-1+(a_1+1)(a_3+1) = a_2^2+a_3^2 = a_2^2-1+(a_2+1)(a_4+1) = (a_2+1)(a_2+a_4).$$

Any common divisor of the two odd numbers $a_2 + 1$ and $a_3 + 1$ must also divide $(a_2 + 1)(a_4 + 1) - (a_3 + 1)(a_3 - 1) = 2$, so these numbers are relatively prime. Hence the last identity shows that $a_1 + a_3$ must be a multiple of $a_2 + 1$, i.e. there is an integer k such that

$$a_1 + a_3 = k(a_2 + 1). \tag{2}$$

Now set $a_0 = k(a_1 + 1) - a_2$. This is an integer and we have

$$\begin{aligned} (a_0 + 1)(a_2 + 1) &= k(a_1 + 1)(a_2 + 1) - (a_2 - 1)(a_2 + 1) \\ &= (a_1 + 1)(a_1 + a_3) - (a_1 + 1)(a_3 + 1) + 2 \\ &= (a_1 + 1)(a_1 - 1) + 2 = a_1^2 + 1. \end{aligned}$$

Thus $a_0 \geq 0$. If $a_0 > 0$, then by (1) we would have $a_0 < a_1 < a_2$ and then the quintuple $(a_0, a_1, a_2, a_3, a_4)$ would contradict the minimality of a_1.

Hence $a_0 = 0$, implying $a_2 = a_1^2$. But also $a_2 = k(a_1 + 1)$, which finally contradicts the fact that $a_1 + 1 > 1$ is relatively prime to a_1^2 and thus cannot be a divisior of this number.

Hence $n \geq 5$ is not possible.

Comment 1. Finding the example for $n = 4$ is not trivial and requires a tedious calculation, but it can be reduced to checking a few cases. The equations $(a_1 + 1)(a_3 + 1) = a_2^2 + 1$ and $(a_2 + 1)(a_4 + 1) = a_3^2 + 1$ imply, as seen in the proof, that a_1 is even and a_2, a_3, a_4 are odd. The case $a_1 = 2$ yields $a_2^2 \equiv -1 \mod 3$ which is impossible. Hence $a_1 = 4$ is the smallest possibility. In this case $a_2^2 \equiv -1 \mod 5$ and a_2 is odd, which implies $a_2 \equiv 3$ or $a_2 \equiv 7 \mod 10$. Hence we have to start checking $a_2 = 7, 13, 17, 23, 27, 33$ and in the last case we succeed.

Comment 2. The choice of $a_0 = k(a_1 + 1) - a_2$ in the second solution appears more natural if one considers that by the previous calculations one has $a_1 = k(a_2+1) - a_3$ and $a_2 = k(a_3+1) - a_4$.

Alternatively, one can solve the equation (2) for a_3 and use $a_2^2 + 1 = (a_1 + 1)(a_3 + 1)$ to get $a_2^2 - k(a_1 + 1)a_2 + a_1^2 - k(a_1 + 1) = 0$. Now a_0 is the second solution to this quadratic equation in a_2 (Vieta jumping).

N5 HUN (Hungary)

Let $P(x)$ be a non-constant polynomial with integer coefficients. Prove that there is no function T from the set of integers into the set of integers such that the number of integers x with $T^n(x) = x$ is equal to $P(n)$ for every $n \geq 1$, where T^n denotes the n-fold application of T.

Solution 1. Assume there is a polynomial P of degree at least 1 with the desired property for a given function T. Let $A(n)$ denote the set of all $x \in \mathbb{Z}$ such that $T^n(x) = x$ and let $B(n)$ denote the set of all $x \in \mathbb{Z}$ for which $T^n(x) = x$ and $T^k(x) \neq x$ for all $1 \leq k < n$. Both sets are finite under the assumption made. For each $x \in A(n)$ there is a smallest $k \geq 1$ such that $T^k(x) = x$, i.e., $x \in B(k)$. Let $d = \gcd(k, n)$. There are positive integers r, s such that $rk - sn = d$ and hence $x = T^{rk}(x) = T^{sn+d}(x) = T^d(T^{sn}(x)) = T^d(x)$. The minimality of k implies $d = k$, i.e., $k|n$. On the other hand one clearly has $B(k) \subset A(n)$ if $k|n$ and thus we have $A(n) = \bigcup_{d|n} B(d)$ as a disjoint union and hence

$$|A(n)| = \sum_{d|n} |B(d)|.$$

Furthermore, for every $x \in B(n)$ the elements $x, T^1(x), T^2(x), \ldots, T^{n-1}(x)$ are n distinct elements of $B(n)$. The fact that they are in $A(n)$ is obvious. If for some $k < n$ and some $0 \leq i < n$ we had $T^k(T^i(x)) = T^i(x)$, i.e. $T^{k+i}(x) = T^i(x)$, that would imply $x = T^n(x) = T^{n-i}(T^i(x)) = T^{n-i}(T^{k+i}(x)) = T^k(T^n(x)) = T^k(x)$ contradicting the minimality of n. Thus $T^i(x) \in B(n)$ and $T^i(x) \neq T^j(x)$ for $0 \leq i < j \leq n-1$.

So indeed, T permutes the elements of $B(n)$ in (disjoint) cycles of length n and in particular one has $n \big| |B(n)|$.

Now let $P(x) = \sum_{i=0}^{k} a_i x^i$, $a_i \in \mathbb{Z}$, $k \geq 1$, $a_k \neq 0$ and suppose that $|A(n)| = P(n)$ for all $n \geq 1$. Let p be any prime. Then

$$p^2 \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_1(p^2 - p) + a_2(p^4 - p^2) + \ldots$$

Hence $p|a_1$ and since this is true for all primes we must have $a_1 = 0$.

Now consider any two different primes p and q. Since $a_1 = 0$ we have that

$$|A(p^2q)| - |A(pq)| = a_2(p^4q^2 - p^2q^2) + a_3(p^6q^3 - p^3q^3) + \ldots$$

is a multiple of p^2q. But we also have

$$p^2q \big| |B(p^2q)| = |A(p^2q)| - |A(pq)| - |B(p^2)|.$$

This implies

$$p^2q \big| |B(p^2)| = |A(p^2)| - |A(p)| = a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k).$$

Since this is true for every prime q we must have $a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k) = 0$ for every prime p. Since this expression is a polynomial in p of degree $2k$ (because $a_k \neq 0$) this is a contradiction, as such a polynomial can have at most $2k$ zeros.

Comment. The last contradiction can also be reached via

$$a_k = \lim_{p\to\infty} \frac{1}{p^{2k}} \big(a_2(p^4 - p^2) + a_3(p^6 - p^3) + \cdots + a_k(p^{2k} - p^k)\big) = 0.$$

Solution 2. As in the first solution define $A(n)$ and $B(n)$ and assume that a polynomial P with the required property exists. This again implies that $|A(n)|$ and $|B(n)|$ is finite for all positive integers n and that

$$P(n) = |A(n)| = \sum_{d|n} |B(d)| \qquad \text{and} \qquad n \big| |B(n)|.$$

Now, for any two distinct primes p and q, we have

$$P(0) \equiv P(pq) \equiv |B(1)| + |B(p)| + |B(q)| + |B(pq)| \equiv |B(1)| + |B(p)| \mod q.$$

Thus, for any fixed p, the expression $P(0) - |B(1)| - |B(p)|$ is divisible by arbitrarily large primes q which means that $P(0) = |B(1)| + |B(p)| = P(p)$ for any prime p. This implies that the polynomial P is constant, a contradiction.

N6 TUR (Turkey)

Let k be a positive integer. Show that if there exists a sequence $a_0, a_1, \ldots$ of integers satisfying the condition

$$a_n = \frac{a_{n-1} + n^k}{n} \quad \text{for all } n \geq 1,$$

then $k-2$ is divisible by 3.

Solution 1. *Part A.* For each positive integer k, there exists a polynomial P_k of degree $k-1$ with integer coefficients, i.e., $P_k \in \mathbb{Z}[x]$, and an integer q_k such that the polynomial identity

$$xP_k(x) = x^k + P_k(x-1) + q_k \tag{I_k}$$

is satisfied. To prove this, for fixed k we write

$$P_k(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$$

and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ and the number q_k successively. Obviously, we have $b_{k-1} = 1$. For $m = k-1, k-2, \ldots, 1$, comparing the coefficients of x^m in the identity (I_k) results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$, and finally $q_k = -P_k(-1)$.

Part B. Let k be a positive integer, and let $a_0, a_1, \ldots$ be a sequence of real numbers satisfying the recursion given in the problem. This recursion can be written as

$$a_n - P_k(n) = \frac{a_{n-1} - P_k(n-1)}{n} - \frac{q_k}{n} \quad \text{for all } n \geq 1,$$

which by induction gives

$$a_n - P_k(n) = \frac{a_0 - P_k(0)}{n!} - q_k \sum_{i=0}^{n-1} \frac{i!}{n!} \quad \text{for all } n \geq 1.$$

Therefore, the numbers a_n are integers for all $n \geq 1$ only if

$$a_0 = P_k(0) \quad \text{and} \quad q_k = 0.$$

Part C. Multiplying the identity (I_k) by $x^2 + x$ and subtracting the identities (I_{k+1}), (I_{k+2}) and $q_k x^2 = q_k x^2$ therefrom, we obtain

$$xT_k(x) = T_k(x-1) + 2x\big(P_k(x-1) + q_k\big) - (q_{k+2} + q_{k+1} + q_k),$$

where the polynomials $T_k \in \mathbb{Z}[x]$ are defined by $T_k(x) = (x^2+x)P_k(x) - P_{k+1}(x) - P_{k+2}(x) - q_k x$. Thus

$$xT_k(x) \equiv T_k(x-1) + q_{k+2} + q_{k+1} + q_k \bmod 2, \quad k = 1, 2, \ldots$$

Comparing the degrees, we easily see that this is only possible if T_k is the zero polynomial modulo 2, and

$$q_{k+2} \equiv q_{k+1} + q_k \bmod 2 \quad \text{for } k = 1, 2, \ldots$$

Since $q_1 = -1$ and $q_2 = 0$, these congruences finish the proof.

Solution 2. *Part A and B.* Let k be a positive integer, and suppose there is a sequence $a_0, a_1, \ldots$ as required. We prove: There exists a polynomial $P \in \mathbb{Z}[x]$, i. e., with integer coefficients, such that $a_n = P(n)$, $n = 0, 1, \ldots$, and $xP(x) = x^k + P(x-1)$.
To prove this, we write $P(x) = b_{k-1}x^{k-1} + \cdots + b_1 x + b_0$ and determine the coefficients $b_{k-1}, b_{k-2}, \ldots, b_0$ successively such that

$$xP(x) - x^k - P(x-1) = q,$$

where $q = q_k$ is an integer. Comparing the coefficients of x^m results in an expression of b_{m-1} as an integer linear combination of $b_{k-1}, \ldots, b_m$.
Defining $c_n = a_n - P(n)$, we get

$$\begin{aligned} P(n) + c_n &= \frac{P(n-1) + c_{n-1} + n^k}{n}, \quad \text{i. e.,} \\ q + nc_n &= c_{n-1}, \end{aligned}$$

hence

$$c_n = \frac{c_0}{n!} - q \cdot \frac{0! + 1! + \cdots + (n-1)!}{n!}.$$

We conclude $\lim_{n\to\infty} c_n = 0$, which, using $c_n \in \mathbb{Z}$, implies $c_n = 0$ for sufficiently large n. Therefore, we get $q = 0$ and $c_n = 0$, $n = 0, 1, \ldots$.

Part C. Suppose that $q = q_k = 0$, i. e. $xP(x) = x^k + P(x-1)$. To consider this identity for arguments $x \in \mathbb{F}_4$, we write $\mathbb{F}_4 = \{0, 1, \alpha, \alpha+1\}$. Then we get

$$\begin{aligned} \alpha P_k(\alpha) &= \alpha^k + P_k(\alpha+1) \quad \text{and} \\ (\alpha+1)P_k(\alpha+1) &= (\alpha+1)^k + P_k(\alpha), \end{aligned}$$

hence

$$\begin{aligned} P_k(\alpha) &= 1 \cdot P_k(\alpha) = (\alpha+1)\alpha P_k(\alpha) \\ &= (\alpha+1)P_k(\alpha+1) + (\alpha+1)\alpha^k \\ &= P_k(\alpha) + (\alpha+1)^k + (\alpha+1)\alpha^k. \end{aligned}$$

Now, $(\alpha+1)^{k-1} = \alpha^k$ implies $k \equiv 2 \bmod 3$.

Comment 1. For $k = 2$, the sequence given by $a_n = n+1$, $n = 0, 1, \ldots$, satisfies the conditions of the problem.

Comment 2. The first few polynomials P_k and integers q_k are

$$\begin{aligned}
&P_1(x) = 1, \quad q_1 = -1,\\
&P_2(x) = x + 1, \quad q_2 = 0,\\
&P_3(x) = x^2 + x - 1, \quad q_3 = 1,\\
&P_4(x) = x^3 + x^2 - 2x - 1, \quad q_4 = -1,\\
&P_5(x) = x^4 + x^3 - 3x^2 + 5, \quad q_5 = -2,\\
&P_6(x) = x^5 + x^4 - 4x^3 + 2x^2 + 10x - 5, \quad q_6 = 9,\\
&q_7 = -9, \quad q_8 = -50, \quad q_9 = 267, \quad q_{10} = -413, \quad q_{11} = -2180.
\end{aligned}$$

A lookup in the On-Line Encyclopedia of Integer Sequences (A000587) reveals that the sequence $q_1, -q_2, q_3, -q_4, q_5, \ldots$ is known as Uppuluri-Carpenter numbers. The result that $q_k = 0$ implies $k \equiv 2 \bmod 3$ is contained in
Murty, Summer: On the p-adic series $\sum_{n=0}^{\infty} n^k \cdot n!$. CRM Proc. and Lecture Notes 36, 2004.
As shown by Alexander (Non-Vanishing of Uppuluri-Carpenter Numbers, Preprint 2006), Uppuluri-Carpenter numbers are zero at most twice.

Comment 3. The numbers q_k can be written in terms of the Stirling numbers of the second kind. To show this, we fix the notation such that

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x-1)\cdots(x-k+1)\\
&+ S_{k-1,k-2}x(x-1)\cdots(x-k+2)\\
&+ \cdots + S_{k-1,0}x,
\end{aligned} \tag{$*$}$$

e. g., $S_{2,2} = 1$, $S_{2,1} = 3$, $S_{2,0} = 1$, and we define

$$\Omega_k = S_{k-1,k-1} - S_{k-1,k-2} + - \cdots .$$

Replacing x by $-x$ in $(*)$ results in

$$\begin{aligned}
x^k =& S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&- S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&+ - \cdots \pm S_{k-1,0}x.
\end{aligned}$$

Defining

$$\begin{aligned}
P(x) =& S_{k-1,k-1}(x+1)\cdots(x+k-1)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2})(x+1)\cdots(x+k-2)\\
&+ (S_{k-1,k-1} - S_{k-1,k-2} + S_{k-1,k-3})(x+1)\cdots(x+k-3)\\
&+ \cdots + \Omega_k,
\end{aligned}$$

we obtain

$$\begin{aligned}
xP(x) - P(x-1) &= S_{k-1,k-1}x(x+1)\cdots(x+k-1)\\
&\quad - S_{k-1,k-2}x(x+1)\cdots(x+k-2)\\
&\quad + - \cdots \pm S_{k-1,0}x - \Omega_k\\
&= x^k - \Omega_k,
\end{aligned}$$

hence $q_k = -\Omega_k$.

N7 MNG (Mongolia)

Let a and b be distinct integers greater than 1. Prove that there exists a positive integer n such that $(a^n-1)(b^n-1)$ is not a perfect square.

Solution 1. At first we notice that

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \left(1-\tfrac{1}{2}\cdot\alpha-\tfrac{1}{8}\cdot\alpha^2-\cdots\right)\left(1-\tfrac{1}{2}\cdot\beta-\tfrac{1}{8}\cdot\beta^2-\cdots\right)$$
$$= \sum_{k,\ell\geq 0} c_{k,\ell}\cdot\alpha^k\beta^\ell \quad \text{for all } \alpha,\beta\in(0,1), \tag{1}$$

where $c_{0,0}=1$ and $c_{k,\ell}$ are certain coefficients.

For an indirect proof, we suppose that $x_n=\sqrt{(a^n-1)(b^n-1)}\in\mathbb{Z}$ for all positive integers n. Replacing a by a^2 and b by b^2 if necessary, we may assume that a and b are perfect squares, hence $\sqrt{ab}$ is an integer.

At first we shall assume that $a^\mu\neq b^\nu$ for all positive integers μ,ν. We have

$$x_n = (\sqrt{ab})^n\left(1-\frac{1}{a^n}\right)^{\frac{1}{2}}\left(1-\frac{1}{b^n}\right)^{\frac{1}{2}} = \sum_{k,\ell\geq 0} c_{k,\ell}\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n. \tag{2}$$

Choosing k_0 and ℓ_0 such that $a^{k_0}>\sqrt{ab}$, $b^{\ell_0}>\sqrt{ab}$, we define the polynomial

$$P(x)=\prod_{k=0,\ell=0}^{k_0-1,\ell_0-1}(a^kb^\ell x-\sqrt{ab}) =: \sum_{i=0}^{k_0\cdot\ell_0} d_ix^i$$

with integer coefficients d_i. By our assumption, the zeros

$$\frac{\sqrt{ab}}{a^kb^\ell},\quad k=0,\ldots,k_0-1,\quad \ell=0,\ldots,\ell_0-1,$$

of P are pairwise distinct.

Furthermore, we consider the integer sequence

$$y_n=\sum_{i=0}^{k_0\cdot\ell_0} d_ix_{n+i},\quad n=1,2,\ldots \tag{3}$$

By the theory of linear recursions, we obtain

$$y_n=\sum_{\substack{k,\ell\geq 0\\ k\geq k_0 \text{ or } \ell\geq \ell_0}} e_{k,\ell}\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n,\quad n=1,2,\ldots, \tag{4}$$

with real numbers $e_{k,\ell}$. We have

$$|y_n|\leq \sum_{\substack{k,\ell\geq 0\\ k\geq k_0 \text{ or } \ell\geq \ell_0}} |e_{k,\ell}|\left(\frac{\sqrt{ab}}{a^kb^\ell}\right)^n =: M_n.$$

Because the series in (4) is obtained by a finite linear combination of the absolutely convergent series (1), we conclude that in particular $M_1 < \infty$. Since

$$\frac{\sqrt{ab}}{a^k b^\ell} \leq \lambda := \max\left\{\frac{\sqrt{ab}}{a^{k_0}}, \frac{\sqrt{ab}}{b^{\ell_0}}\right\} \quad \text{for all } k, \ell \geq 0 \text{ such that } k \geq k_0 \text{ or } \ell \geq \ell_0,$$

we get the estimates $M_{n+1} \leq \lambda M_n$, $n = 1, 2, \ldots$ Our choice of k_0 and ℓ_0 ensures $\lambda < 1$, which implies $M_n \to 0$ and consequently $y_n \to 0$ as $n \to \infty$. It follows that $y_n = 0$ for all sufficiently large n.

So, equation (3) reduces to $\sum_{i=0}^{k_0 \cdot \ell_0} d_i x_{n+i} = 0$.

Using the theory of linear recursions again, for sufficiently large n we have

$$x_n = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} f_{k,\ell} \left(\frac{\sqrt{ab}}{a^k b^\ell}\right)^n$$

for certain real numbers $f_{k,\ell}$.

Comparing with (2), we see that $f_{k,\ell} = c_{k,\ell}$ for all $k, \ell \geq 0$ with $k < k_0$, $\ell < \ell_0$, and $c_{k,\ell} = 0$ if $k \geq k_0$ or $\ell \geq \ell_0$, since we assumed that $a^\mu \neq b^\nu$ for all positive integers μ, ν.

In view of (1), this means

$$(1-\alpha)^{\frac{1}{2}}(1-\beta)^{\frac{1}{2}} = \sum_{k=0,\ell=0}^{k_0-1,\ell_0-1} c_{k,\ell} \cdot \alpha^k \beta^\ell \tag{5}$$

for all real numbers $\alpha, \beta \in (0,1)$. We choose $k^* < k_0$ maximal such that there is some i with $c_{k^*,i} \neq 0$. Squaring (5) and comparing coefficients of $\alpha^{2k^*}\beta^{2i^*}$, where i^* is maximal with $c_{k^*,i^*} \neq 0$, we see that $k^* = 0$. This means that the right hand side of (5) is independent of α, which is clearly impossible.

We are left with the case that $a^\mu = b^\nu$ for some positive integers μ and ν. We may assume that μ and ν are relatively prime. Then there is some positive integer c such that $a = c^\nu$ and $b = c^\mu$. Now starting with the expansion (2), i. e.,

$$x_n = \sum_{j \geq 0} g_j \left(\frac{\sqrt{c^{\mu+\nu}}}{c^j}\right)^n$$

for certain coefficients g_j, and repeating the arguments above, we see that $g_j = 0$ for sufficiently large j, say $j > j_0$. But this means that

$$(1-x^\mu)^{\frac{1}{2}}(1-x^\nu)^{\frac{1}{2}} = \sum_{j=0}^{j_0} g_j x^j$$

for all real numbers $x \in (0,1)$. Squaring, we see that

$$(1-x^\mu)(1-x^\nu)$$

is the square of a polynomial in x. In particular, all its zeros are of order at least 2, which implies $\mu = \nu$ by looking at roots of unity. So we obtain $\mu = \nu = 1$, i. e., $a = b$, a contradiction.

Solution 2. We set $a^2 = A$, $b^2 = B$, and $z_n = \sqrt{(A^n - 1)(B^n - 1)}$. Let us assume that z_n is an integer for $n = 1, 2, \ldots$ Without loss of generality, we may suppose that $b < a$. We determine an integer $k \geq 2$ such that $b^{k-1} \leq a < b^k$, and define a sequence $\gamma_1, \gamma_2, \ldots$ of rational numbers such that

$$2\gamma_1 = 1 \qquad \text{and} \qquad 2\gamma_{n+1} = \sum_{i=1}^{n} \gamma_i \gamma_{n-i} \quad \text{for} \quad n = 1, 2, \ldots$$

It could easily be shown that $\gamma_n = \frac{1 \cdot 1 \cdot 3 \ldots (2n-3)}{2 \cdot 4 \cdot 6 \ldots 2n}$, for instance by reading Vandermondes convolution as an equation between polynomials, but we shall have no use for this fact.

Using Landaus O–Notation in the usual way, we have

$$\begin{aligned}
&\left\{(ab)^n - \gamma_1 \left(\frac{a}{b}\right)^n - \gamma_2 \left(\frac{a}{b^3}\right)^n - \cdots - \gamma_k \left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n\right\}^2 \\
&= A^n B^n - 2\gamma_1 A^n - \sum_{i=2}^{k} \left(2\gamma_i - \sum_{j=1}^{i-1} \gamma_j \gamma_{i-j}\right) \left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n + O\left(B^n\right) \\
&= A^n B^n - A^n + O\left(B^n\right),
\end{aligned}$$

whence

$$z_n = (ab)^n - \gamma_1 \left(\frac{a}{b}\right)^n - \gamma_2 \left(\frac{a}{b^3}\right)^n - \cdots - \gamma_k \left(\frac{a}{b^{2k-1}}\right)^n + O\left(\frac{b}{a}\right)^n.$$

Now choose rational numbers $r_1, r_2, \ldots, r_{k+1}$ such that

$$(x - ab) \cdot (x - \tfrac{a}{b}) \ldots (x - \tfrac{a}{b^{2k-1}}) = x^{k+1} - r_1 x^k + - \cdots \pm r_{k+1},$$

and then a natural number M for which $Mr_1, Mr_2, \ldots Mr_{k+1}$ are integers. For known reasons,

$$M(z_{n+k+1} - r_1 z_{n+k} + - \cdots \pm r_{k+1} z_n) = O\left(\frac{b}{a}\right)^n$$

for all $n \in \mathbb{N}$ and thus there is a natural number N which is so large, that

$$z_{n+k+1} = r_1 z_{n+k} - r_2 z_{n+k-1} + - \cdots \mp r_{k+1} z_n$$

holds for all $n \geqslant N$. Now the theory of linear recursions reveals that there are some rational numbers $\delta_0, \delta_1, \delta_2, \ldots, \delta_k$ such that

$$z_n = \delta_0 (ab)^n - \delta_1 \left(\frac{a}{b}\right)^n - \delta_2 \left(\frac{a}{b^3}\right)^n - \cdots - \delta_k \left(\frac{a}{b^{2k-1}}\right)^n$$

for sufficiently large n, where $\delta_0 > 0$ as $z_n > 0$. As before, one obtains

$$\begin{aligned}
&A^n B^n - A^n - B^n + 1 = z_n^2 \\
&= \left\{\delta_0 (ab)^n - \delta_1 \left(\frac{a}{b}\right)^n - \delta_2 \left(\frac{a}{b^3}\right)^n - \cdots - \delta_k \left(\frac{a}{b^{2k-1}}\right)^n\right\}^2 \\
&= \delta_0^2 A^n B^n - 2\delta_0 \delta_1 A^n - \sum_{i=2}^{i=k} \left(2\delta_0 \delta_i - \sum_{j=1}^{j=i-1} \delta_j \delta_{i-j}\right) \left(\frac{A}{B^{i-1}}\right)^n + O\left(\frac{A}{B^k}\right)^n.
\end{aligned}$$

Easy asymptotic calculations yield $\delta_0 = 1$, $\delta_1 = \frac{1}{2}$, $\delta_i = \frac{1}{2} \sum_{j=1}^{j=i-1} \delta_j \delta_{i-j}$ for $i = 2, 3, \ldots, k-2$, and then $a = b^{k-1}$. It follows that $k > 2$ and there is some $P \in \mathbb{Q}[X]$ for which $(X-1)(X^{k-1}-1) = P(X)^2$. But this cannot occur, for instance as $X^{k-1} - 1$ has no double zeros. Thus our assumption that z_n was an integer for $n = 1, 2, \dots$ turned out to be wrong, which solves the problem.

Original formulation of the problem. a, b are positive integers such that $a \cdot b$ is not a square of an integer. Prove that there exists a (infinitely many) positive integer n such that $(a^n - 1)(b^n - 1)$ is not a square of an integer.

Solution. *Lemma.* Let c be a positive integer, which is not a perfect square. Then there exists an odd prime p such that c is not a quadratic residue modulo p.
Proof. Denoting the square-free part of c by c', we have the equality $\left(\frac{c'}{p}\right) = \left(\frac{c}{p}\right)$ of the corresponding Legendre symbols. Suppose that $c' = q_1 \cdots q_m$, where $q_1 < \cdots < q_m$ are primes. Then we have

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right).$$

Case 1. Let q_1 be odd. We choose a quadratic nonresidue r_1 modulo q_1 and quadratic residues r_i modulo q_i for $i = 2, \dots, m$. By the Chinese remainder theorem and the Dirichlet theorem, there exists a (infinitely many) prime p such that

$$\begin{aligned} p &\equiv r_1 \bmod q_1, \\ p &\equiv r_2 \bmod q_2, \\ &\vdots \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 1 \bmod 4. \end{aligned}$$

By our choice of the residues, we obtain

$$\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = \begin{cases} -1, & i = 1, \\ 1, & i = 2, \dots, m. \end{cases}$$

The congruence $p \equiv 1 \bmod 4$ implies that $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 1, \dots, m$, by the law of quadratic reciprocity. Thus

$$\left(\frac{c'}{p}\right) = \left(\frac{q_1}{p}\right) \cdots \left(\frac{q_m}{p}\right) = -1.$$

Case 2. Suppose $q_1 = 2$. We choose quadratic residues r_i modulo q_i for $i = 2, \dots, m$. Again, by the Chinese remainder theorem and the Dirichlet theorem, there exists a prime p such that

$$\begin{aligned} p &\equiv r_2 \bmod q_2, \\ &\vdots \quad \vdots \\ p &\equiv r_m \bmod q_m, \\ p &\equiv 5 \bmod 8. \end{aligned}$$

By the choice of the residues, we obtain $\left(\frac{p}{q_i}\right) = \left(\frac{r_i}{q_i}\right) = 1$ for $i = 2, \dots, m$. Since $p \equiv 1 \bmod 4$ we have $\left(\frac{q_i}{p}\right) = \left(\frac{p}{q_i}\right)$, $i = 2, \dots, m$, by the law of quadratic reciprocity. The congruence $p \equiv 5 \bmod 8$

implies that $\left(\frac{2}{p}\right)=-1$. Thus

$$\left(\frac{c'}{p}\right)=\left(\frac{2}{p}\right)\left(\frac{q_2}{p}\right)\cdots\left(\frac{q_m}{p}\right)=-1,$$

and the lemma is proved.

Applying the lemma for $c=a\cdot b$, we find an odd prime p such that

$$\left(\frac{ab}{p}\right)=\left(\frac{a}{p}\right)\cdot\left(\frac{b}{p}\right)=-1.$$

This implies either

$$a^{\frac{p-1}{2}}\equiv 1 \bmod p,\quad b^{\frac{p-1}{2}}\equiv -1 \bmod p,\qquad \text{or}\qquad a^{\frac{p-1}{2}}\equiv -1 \bmod p,\quad b^{\frac{p-1}{2}}\equiv 1 \bmod p.$$

Without loss of generality, suppose that $a^{\frac{p-1}{2}}\equiv 1 \bmod p$ and $b^{\frac{p-1}{2}}\equiv -1 \bmod p$. The second congruence implies that $b^{\frac{p-1}{2}}-1$ is not divisible by p. Hence, if the exponent $\nu_p(a^{\frac{p-1}{2}}-1)$ of p in the prime decomposition of $(a^{\frac{p-1}{2}}-1)$ is odd, then $(a^{\frac{p-1}{2}}-1)(b^{\frac{p-1}{2}}-1)$ is not a perfect square. If $\nu_p(a^{\frac{p-1}{2}}-1)$ is even, then $\nu_p(a^{\frac{p-1}{2}p}-1)$ is odd by the well-known power lifting property

$$\nu_p\left(a^{\frac{p-1}{2}p}-1\right)=\nu_p\left(a^{\frac{p-1}{2}}-1\right)+1.$$

In this case, $(a^{\frac{p-1}{2}p}-1)(b^{\frac{p-1}{2}p}-1)$ is not a perfect square.

Comment 1. In 1998, the following problem appeared in Crux Mathematicorum:
Problem 2344. Find all positive integers N that are quadratic residues modulo all primes greater than N.
The published solution (Crux Mathematicorum, 25(1999)4) is the same as the proof of the lemma given above, see also `http://www.mathlinks.ro/viewtopic.php?t=150495`.

Comment 2. There is also an elementary proof of the lemma. We cite Theorem 3 of Chapter 5 and its proof from the book
Ireland, Rosen: A Classical Introduction to Modern Number Theory, Springer 1982.

Theorem. Let a be a nonsquare integer. Then there are infinitely many primes p for which a is a quadratic nonresidue.

Proof. It is easily seen that we may assume that a is square-free. Let $a=2^e q_1 q_2\cdots q_n$, where q_i are distinct odd primes and $e=0$ or 1. The case $a=2$ has to be dealt with separately. We shall assume to begin with that $n\geq 1$, i.e., that a is divisible by an odd prime.

Let $\ell_1,\ell_2,\ldots,\ell_k$ be a finite set of odd primes not including any q_i. Let s be any quadratic nonresidue $\bmod q_n$, and find a simultaneous solution to the congruences

$$\begin{aligned}
x &\equiv 1 \bmod \ell_i, \quad i=1,\ldots,k,\\
x &\equiv 1 \bmod 8,\\
x &\equiv 1 \bmod q_i, \quad i=1,\ldots,n-1,\\
x &\equiv s \bmod q_n.
\end{aligned}$$

Call the solution b. b is odd. Suppose that $b=p_1p_2\cdots p_m$ is its prime decomposition. Since

$b \equiv 1 \bmod 8$ we have $\left(\frac{2}{b}\right) = 1$ and $\left(\frac{q_i}{b}\right) = \left(\frac{b}{q_i}\right)$ by a result on Jacobi symbols. Thus

$$\left(\frac{a}{b}\right) = \left(\frac{2}{b}\right)^e \left(\frac{q_1}{b}\right) \cdots \left(\frac{q_{n-1}}{b}\right)\left(\frac{q_n}{b}\right) = \left(\frac{b}{q_1}\right) \cdots \left(\frac{b}{q_{n-1}}\right)\left(\frac{b}{q_n}\right) = \left(\frac{1}{q_1}\right) \cdots \left(\frac{1}{q_{n-1}}\right)\left(\frac{s}{q_n}\right) = -1.$$

On the other hand, by the definition of $\left(\frac{a}{b}\right)$, we have $\left(\frac{a}{b}\right) = \left(\frac{a}{p_1}\right)\left(\frac{a}{p_2}\right) \cdots \left(\frac{a}{p_m}\right)$. It follows that $\left(\frac{a}{p_i}\right) = -1$ for some i.

Notice that ℓ_j does not divide b. Thus $p_i \notin \{\ell_1, \ell_2, \ldots, \ell_k\}$.

To summarize, if a is a nonsquare, divisible by an odd prime, we have found a prime p, outside of a given finite set of primes $\{2, \ell_1, \ell_2, \ldots, \ell_k\}$, such that $\left(\frac{a}{p}\right) = -1$. This proves the theorem in this case.

It remains to consider the case $a = 2$. Let $\ell_1, \ell_2, \ldots, \ell_k$ be a finite set of primes, excluding 3, for which $\left(\frac{2}{\ell_i}\right) = -1$. Let $b = 8\ell_1\ell_2 \cdots \ell_k + 3$. b is not divisible by 3 or any ℓ_i. Since $b \equiv 3 \bmod 8$ we have $\left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}} = -1$. Suppose that $b = p_1 p_2 \cdots p_m$ is the prime decomposition of b. Then, as before, we see that $\left(\frac{2}{p_i}\right) = -1$ for some i. $p_i \notin \{3, \ell_1, \ell_2, \ldots, \ell_k\}$. This proves the theorem for $a = 2$.

This proof has also been posted to `mathlinks`, see `http://www.mathlinks.ro/viewtopic.php?t=150495` again.

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Shortlisted Problems with Solutions

Contents

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The Shortlisted Problems should be kept strictly confidential until IMO 2011.

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Algebra

A1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f([x]y) = f(x)[f(y)]. \tag{1}$$

holds for all $x, y \in \mathbb{R}$. Here, by $[x]$ we denote the greatest integer not exceeding x.

(France)

Answer. $f(x) = \text{const} = C$, where $C = 0$ or $1 \le C < 2$.

Solution 1. First, setting $x = 0$ in (1) we get

$$f(0) = f(0)[f(y)] \tag{2}$$

for all $y \in \mathbb{R}$. Now, two cases are possible.

Case 1. Assume that $f(0) \neq 0$. Then from (2) we conclude that $[f(y)] = 1$ for all $y \in \mathbb{R}$. Therefore, equation (1) becomes $f([x]y) = f(x)$, and substituting $y = 0$ we have $f(x) = f(0) = C \neq 0$. Finally, from $[f(y)] = 1 = [C]$ we obtain that $1 \le C < 2$.

Case 2. Now we have $f(0) = 0$. Here we consider two subcases.

Subcase 2a. Suppose that there exists $0 < \alpha < 1$ such that $f(\alpha) \neq 0$. Then setting $x = \alpha$ in (1) we obtain $0 = f(0) = f(\alpha)[f(y)]$ for all $y \in \mathbb{R}$. Hence, $[f(y)] = 0$ for all $y \in \mathbb{R}$. Finally, substituting $x = 1$ in (1) provides $f(y) = 0$ for all $y \in \mathbb{R}$, thus contradicting the condition $f(\alpha) \neq 0$.

Subcase 2b. Conversely, we have $f(\alpha) = 0$ for all $0 \le \alpha < 1$. Consider any real z; there exists an integer N such that $\alpha = \dfrac{z}{N} \in [0, 1)$ (one may set $N = [z] + 1$ if $z \ge 0$ and $N = [z] - 1$ otherwise). Now, from (1) we get $f(z) = f([N]\alpha) = f(N)[f(\alpha)] = 0$ for all $z \in \mathbb{R}$.

Finally, a straightforward check shows that all the obtained functions satisfy (1).

Solution 2. Assume that $[f(y)] = 0$ for some y; then the substitution $x = 1$ provides $f(y) = f(1)[f(y)] = 0$. Hence, if $[f(y)] = 0$ for all y, then $f(y) = 0$ for all y. This function obviously satisfies the problem conditions.

So we are left to consider the case when $[f(a)] \neq 0$ for some a. Then we have

$$f([x]a) = f(x)[f(a)], \qquad \text{or} \qquad f(x) = \frac{f([x]a)}{[f(a)]}. \tag{3}$$

This means that $f(x_1) = f(x_2)$ whenever $[x_1] = [x_2]$, hence $f(x) = f([x])$, and we may assume that a is an integer.

Now we have

$$f(a) = f\left(2a \cdot \tfrac{1}{2}\right) = f(2a)\left[f\left(\tfrac{1}{2}\right)\right] = f(2a)[f(0)];$$

this implies $[f(0)] \neq 0$, so we may even assume that $a = 0$. Therefore equation (3) provides

$$f(x) = \frac{f(0)}{[f(0)]} = C \neq 0$$

for each x. Now, condition (1) becomes equivalent to the equation $C = C[C]$ which holds exactly when $[C] = 1$.

A2. Let the real numbers a, b, c, d satisfy the relations $a+b+c+d=6$ and $a^2+b^2+c^2+d^2=12$. Prove that
$$36 \leq 4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) \leq 48.$$

(Ukraine)

Solution 1. Observe that
$$\begin{aligned} 4(a^3+b^3+c^3+d^3)-(a^4+b^4+c^4+d^4) &= -\left((a-1)^4+(b-1)^4+(c-1)^4+(d-1)^4\right) \\ &\quad +6(a^2+b^2+c^2+d^2)-4(a+b+c+d)+4 \\ &= -\left((a-1)^4+(b-1)^4+(c-1)^4+(d-1)^4\right)+52. \end{aligned}$$

Now, introducing $x=a-1$, $y=b-1$, $z=c-1$, $t=d-1$, we need to prove the inequalities
$$16 \geq x^4+y^4+z^4+t^4 \geq 4,$$
under the constraint
$$x^2+y^2+z^2+t^2=(a^2+b^2+c^2+d^2)-2(a+b+c+d)+4=4 \tag{1}$$
(we will not use the value of $x+y+z+t$ though it can be found).

Now the rightmost inequality in (1) follows from the power mean inequality:
$$x^4+y^4+z^4+t^4 \geq \frac{(x^2+y^2+z^2+t^2)^2}{4}=4.$$
For the other one, expanding the brackets we note that
$$(x^2+y^2+z^2+t^2)^2=(x^4+y^4+z^4+t^4)+q,$$
where q is a nonnegative number, so
$$x^4+y^4+z^4+t^4 \leq (x^2+y^2+z^2+t^2)^2=16,$$
and we are done.

Comment 1. The estimates are sharp; the lower and upper bounds are attained at $(3,1,1,1)$ and $(0,2,2,2)$, respectively.

Comment 2. After the change of variables, one can finish the solution in several different ways. The latter estimate, for instance, it can be performed by moving the variables – since we need only the second of the two shifted conditions.

Solution 2. First, we claim that $0 \leq a, b, c, d \leq 3$. Actually, we have
$$a+b+c=6-d, \quad a^2+b^2+c^2=12-d^2,$$
hence the power mean inequality
$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3}$$
rewrites as
$$12-d^2 \geq \frac{(6-d)^2}{3} \quad \iff \quad 2d(d-3) \leq 0,$$

which implies the desired inequalities for d; since the conditions are symmetric, we also have the same estimate for the other variables.

Now, to prove the rightmost inequality, we use the obvious inequality $x^2(x-2)^2 \geq 0$ for each real x; this inequality rewrites as $4x^3 - x^4 \leq 4x^2$. It follows that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \leq 4(a^2 + b^2 + c^2 + d^2) = 48,$$

as desired.

Now we prove the leftmost inequality in an analogous way. For each $x \in [0,3]$, we have $(x+1)(x-1)^2(x-3) \leq 0$ which is equivalent to $4x^3 - x^4 \geq 2x^2 + 4x - 3$. This implies that

$$(4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) \geq 2(a^2 + b^2 + c^2 + d^2) + 4(a+b+c+d) - 12 = 36,$$

as desired.

Comment. It is easy to guess the extremal points $(0,2,2,2)$ and $(3,1,1,1)$ for this inequality. This provides a method of finding the polynomials used in Solution 2. Namely, these polynomials should have the form $x^4 - 4x^3 + ax^2 + bx + c$; moreover, the former polynomial should have roots at 2 (with an even multiplicity) and 0, while the latter should have roots at 1 (with an even multiplicity) and 3. These conditions determine the polynomials uniquely.

Solution 3. First, expanding $48 = 4(a^2 + b^2 + c^2 + d^2)$ and applying the AM–GM inequality, we have

$$\begin{aligned} a^4 + b^4 + c^4 + d^4 + 48 &= (a^4 + 4a^2) + (b^4 + 4b^2) + (c^4 + 4c^2) + (d^4 + 4d^2) \\ &\geq 2\left(\sqrt{a^4 \cdot 4a^2} + \sqrt{b^4 \cdot 4b^2} + \sqrt{c^4 \cdot 4c^2} + \sqrt{d^4 \cdot 4d^2}\right) \\ &= 4(|a^3| + |b^3| + |c^3| + |d^3|) \geq 4(a^3 + b^3 + c^3 + d^3), \end{aligned}$$

which establishes the rightmost inequality.

To prove the leftmost inequality, we first show that $a, b, c, d \in [0,3]$ as in the previous solution. Moreover, we can assume that $0 \leq a \leq b \leq c \leq d$. Then we have $a + b \leq b + c \leq \frac{2}{3}(b + c + d) \leq \frac{2}{3} \cdot 6 = 4$.

Next, we show that $4b - b^2 \leq 4c - c^2$. Actually, this inequality rewrites as $(c-b)(b+c-4) \leq 0$, which follows from the previous estimate. The inequality $4a - a^2 \leq 4b - b^2$ can be proved analogously.

Further, the inequalities $a \leq b \leq c$ together with $4a - a^2 \leq 4b - b^2 \leq 4c - c^2$ allow us to apply the Chebyshev inequality obtaining

$$\begin{aligned} a^2(4a - a^2) + b^2(4b - b^2) + c^2(4c - c^2) &\geq \frac{1}{3}(a^2 + b^2 + c^2)\left(4(a+b+c) - (a^2 + b^2 + c^2)\right) \\ &= \frac{(12 - d^2)(4(6-d) - (12 - d^2))}{3}. \end{aligned}$$

This implies that

$$\begin{aligned} (4a^3 - a^4) + (4b^3 - b^4) + (4c^3 - c^4) + (4d^3 - d^4) &\geq \frac{(12 - d^2)(d^2 - 4d + 12)}{3} + 4d^3 - d^4 \\ = \frac{144 - 48d + 16d^3 - 4d^4}{3} &= 36 + \frac{4}{3}(3-d)(d-1)(d^2-3). \end{aligned} \tag{2}$$

Finally, we have $d^2 \geq \frac{1}{4}(a^2 + b^2 + c^2 + d^2) = 3$ (which implies $d > 1$); so, the expression $\frac{4}{3}(3-d)(d-1)(d^2-3)$ in the right-hand part of (2) is nonnegative, and the desired inequality is proved.

Comment. The rightmost inequality is easier than the leftmost one. In particular, Solutions 2 and 3 show that only the condition $a^2 + b^2 + c^2 + d^2 = 12$ is needed for the former one.

A3. Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1$, $x_{102} = x_2$). Find the maximal possible value of the sum

$$S = \sum_{i=1}^{100} x_i x_{i+2}.$$

(Russia)

Answer. $\dfrac{25}{2}$.

Solution 1. Let $x_{2i} = 0$, $x_{2i-1} = \frac{1}{2}$ for all $i = 1, \ldots, 50$. Then we have $S = 50 \cdot \left(\frac{1}{2}\right)^2 = \frac{25}{2}$. So, we are left to show that $S \leq \frac{25}{2}$ for all values of x_i's satisfying the problem conditions.

Consider any $1 \leq i \leq 50$. By the problem condition, we get $x_{2i-1} \leq 1 - x_{2i} - x_{2i+1}$ and $x_{2i+2} \leq 1 - x_{2i} - x_{2i+1}$. Hence by the AM–GM inequality we get

$$\begin{aligned} x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2} &\leq (1 - x_{2i} - x_{2i+1})x_{2i+1} + x_{2i}(1 - x_{2i} - x_{2i+1}) \\ &= (x_{2i} + x_{2i+1})(1 - x_{2i} - x_{2i+1}) \leq \left(\frac{(x_{2i} + x_{2i+1}) + (1 - x_{2i} - x_{2i+1})}{2}\right)^2 = \frac{1}{4}. \end{aligned}$$

Summing up these inequalities for $i = 1, 2, \ldots, 50$, we get the desired inequality

$$\sum_{i=1}^{50}(x_{2i-1}x_{2i+1} + x_{2i}x_{2i+2}) \leq 50 \cdot \frac{1}{4} = \frac{25}{2}.$$

Comment. This solution shows that a bit more general fact holds. Namely, consider $2n$ nonnegative numbers $x_1, \ldots, x_{2n}$ in a row (with no cyclic notation) and suppose that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, 2, \ldots, 2n-2$. Then $\sum\limits_{i=1}^{2n-2} x_i x_{i+2} \leq \dfrac{n-1}{4}$.

The proof is the same as above, though if might be easier to find it (for instance, applying induction). The original estimate can be obtained from this version by considering the sequence $x_1, x_2, \ldots, x_{100}, x_1, x_2$.

Solution 2. We present another proof of the estimate. From the problem condition, we get

$$\begin{aligned} S = \sum_{i=1}^{100} x_i x_{i+2} \leq \sum_{i=1}^{100} x_i(1 - x_i - x_{i+1}) &= \sum_{i=1}^{100} x_i - \sum_{i=1}^{100} x_i^2 - \sum_{i=1}^{100} x_i x_{i+1} \\ &= \sum_{i=1}^{100} x_i - \frac{1}{2}\sum_{i=1}^{100}(x_i + x_{i+1})^2. \end{aligned}$$

By the AM–QM inequality, we have $\sum(x_i + x_{i+1})^2 \geq \frac{1}{100}\left(\sum(x_i + x_{i+1})\right)^2$, so

$$\begin{aligned} S \leq \sum_{i=1}^{100} x_i - \frac{1}{200}\left(\sum_{i=1}^{100}(x_i + x_{i+1})\right)^2 &= \sum_{i=1}^{100} x_i - \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)^2 \\ &= \frac{2}{100}\left(\sum_{i=1}^{100} x_i\right)\left(\frac{100}{2} - \sum_{i=1}^{100} x_i\right). \end{aligned}$$

And finally, by the AM–GM inequality

$$S \leq \frac{2}{100} \cdot \left(\frac{1}{2}\left(\sum_{i=1}^{100} x_i + \frac{100}{2} - \sum_{i=1}^{100} x_i\right)\right)^2 = \frac{2}{100} \cdot \left(\frac{100}{4}\right)^2 = \frac{25}{2}.$$

Comment. These solutions are not as easy as they may seem at the first sight. There are two different optimal configurations in which the variables have different values, and not all of sums of three consecutive numbers equal 1. Although it is easy to find the value $\frac{25}{2}$, the estimates must be done with care to preserve equality in the optimal configurations.

A4. A sequence $x_1, x_2, \dots$ is defined by $x_1 = 1$ and $x_{2k} = -x_k$, $x_{2k-1} = (-1)^{k+1}x_k$ for all $k \geq 1$. Prove that $x_1 + x_2 + \dots + x_n \geq 0$ for all $n \geq 1$.

(Austria)

Solution 1. We start with some observations. First, from the definition of x_i it follows that for each positive integer k we have

$$x_{4k-3} = x_{2k-1} = -x_{4k-2} \quad \text{and} \quad x_{4k-1} = x_{4k} = -x_{2k} = x_k. \tag{1}$$

Hence, denoting $S_n = \sum_{i=1}^n x_i$, we have

$$S_{4k} = \sum_{i=1}^{k}\big((x_{4k-3} + x_{4k-2}) + (x_{4k-1} + x_{4k})\big) = \sum_{i=1}^{k}(0 + 2x_k) = 2S_k, \tag{2}$$

$$S_{4k+2} = S_{4k} + (x_{4k+1} + x_{4k+2}) = S_{4k}. \tag{3}$$

Observe also that $S_n = \sum_{i=1}^n x_i \equiv \sum_{i=1}^n 1 = n \pmod 2$.

Now we prove by induction on k that $S_i \geq 0$ for all $i \leq 4k$. The base case is valid since $x_1 = x_3 = x_4 = 1$, $x_2 = -1$. For the induction step, assume that $S_i \geq 0$ for all $i \leq 4k$. Using the relations (1)–(3), we obtain

$$S_{4k+4} = 2S_{k+1} \geq 0, \quad S_{4k+2} = S_{4k} \geq 0, \quad S_{4k+3} = S_{4k+2} + x_{4k+3} = \frac{S_{4k+2} + S_{4k+4}}{2} \geq 0.$$

So, we are left to prove that $S_{4k+1} \geq 0$. If k is odd, then $S_{4k} = 2S_k \geq 0$; since k is odd, S_k is odd as well, so we have $S_{4k} \geq 2$ and hence $S_{4k+1} = S_{4k} + x_{4k+1} \geq 1$.

Conversely, if k is even, then we have $x_{4k+1} = x_{2k+1} = x_{k+1}$, hence $S_{4k+1} = S_{4k} + x_{4k+1} = 2S_k + x_{k+1} = S_k + S_{k+1} \geq 0$. The step is proved.

Solution 2. We will use the notation of S_n and the relations (1)–(3) from the previous solution.

Assume the contrary and consider the minimal n such that $S_{n+1} < 0$; surely $n \geq 1$, and from $S_n \geq 0$ we get $S_n = 0$, $x_{n+1} = -1$. Hence, we are especially interested in the set $M = \{n : S_n = 0\}$; our aim is to prove that $x_{n+1} = 1$ whenever $n \in M$ thus coming to a contradiction.

For this purpose, we first describe the set M inductively. We claim that (i) M consists only of even numbers, (ii) $2 \in M$, and (iii) for every even $n \geq 4$ we have $n \in M \iff [n/4] \in M$. Actually, (i) holds since $S_n \equiv n \pmod 2$, (ii) is straightforward, while (iii) follows from the relations $S_{4k+2} = S_{4k} = 2S_k$.

Now, we are left to prove that $x_{n+1} = 1$ if $n \in M$. We use the induction on n. The base case is $n = 2$, that is, the minimal element of M; here we have $x_3 = 1$, as desired.

For the induction step, consider some $4 \leq n \in M$ and let $m = [n/4] \in M$; then m is even, and $x_{m+1} = 1$ by the induction hypothesis. We prove that $x_{n+1} = x_{m+1} = 1$. If $n = 4m$ then we have $x_{n+1} = x_{2m+1} = x_{m+1}$ since m is even; otherwise, $n = 4m+2$, and $x_{n+1} = -x_{2m+2} = x_{m+1}$, as desired. The proof is complete.

Comment. Using the inductive definition of set M, one can describe it explicitly. Namely, M consists exactly of all positive integers not containing digits 1 and 3 in their 4-base representation.

A5. Denote by $\mathbb{Q}^+$ the set of all positive rational numbers. Determine all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ which satisfy the following equation for all $x, y \in \mathbb{Q}^+$:

$$f\big(f(x)^2 y\big) = x^3 f(xy). \tag{1}$$

(Switzerland)

Answer. The only such function is $f(x) = \dfrac{1}{x}$.

Solution. By substituting $y = 1$, we get

$$f\big(f(x)^2\big) = x^3 f(x). \tag{2}$$

Then, whenever $f(x) = f(y)$, we have

$$x^3 = \frac{f\big(f(x)^2\big)}{f(x)} = \frac{f\big(f(y)^2\big)}{f(y)} = y^3$$

which implies $x = y$, so the function f is injective.

Now replace x by xy in (2), and apply (1) twice, second time to $\big(y, f(x)^2\big)$ instead of (x, y):

$$f\big(f(xy)^2\big) = (xy)^3 f(xy) = y^3 f\big(f(x)^2 y\big) = f\big(f(x)^2 f(y)^2\big).$$

Since f is injective, we get

$$f(xy)^2 = f(x)^2 f(y)^2,$$
$$f(xy) = f(x) f(y).$$

Therefore, f is multiplicative. This also implies $f(1) = 1$ and $f(x^n) = f(x)^n$ for all integers n.

Then the function equation (1) can be re-written as

$$f\big(f(x)\big)^2 f(y) = x^3 f(x) f(y),$$
$$f\big(f(x)\big) = \sqrt{x^3 f(x)}. \tag{3}$$

Let $g(x) = x f(x)$. Then, by (3), we have

$$g\big(g(x)\big) = g\big(x f(x)\big) = x f(x) \cdot f\big(x f(x)\big) = x f(x)^2 f\big(f(x)\big) =$$
$$= x f(x)^2 \sqrt{x^3 f(x)} = \big(x f(x)\big)^{5/2} = \big(g(x)\big)^{5/2},$$

and, by induction,

$$\underbrace{g\Big(g\big(\dots g}_{n+1}(x)\dots\big)\Big) = \big(g(x)\big)^{(5/2)^n} \tag{4}$$

for every positive integer n.

Consider (4) for a fixed x. The left-hand side is always rational, so $\big(g(x)\big)^{(5/2)^n}$ must be rational for every n. We show that this is possible only if $g(x) = 1$. Suppose that $g(x) \neq 1$, and let the prime factorization of $g(x)$ be $g(x) = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ where $p_1, \dots, p_k$ are distinct primes and $\alpha_1, \dots, \alpha_k$ are nonzero integers. Then the unique prime factorization of (4) is

$$\underbrace{g\Big(g\big(\dots g}_{n+1}(x)\dots\big)\Big) = \big(g(x)\big)^{(5/2)^n} = p_1^{(5/2)^n \alpha_1} \cdots p_k^{(5/2)^n \alpha_k}$$

where the exponents should be integers. But this is not true for large values of n, for example $(\frac{5}{2})^n\alpha_1$ cannot be a integer number when $2^n \not| \alpha_1$. Therefore, $g(x) \neq 1$ is impossible.

Hence, $g(x) = 1$ and thus $f(x) = \dfrac{1}{x}$ for all x.

The function $f(x) = \dfrac{1}{x}$ satisfies the equation (1):

$$f(f(x)^2y) = \frac{1}{f(x)^2y} = \frac{1}{\left(\frac{1}{x}\right)^2y} = \frac{x^3}{xy} = x^3f(xy).$$

Comment. Among $\mathbb{R}^+ \to \mathbb{R}^+$ functions, $f(x) = \dfrac{1}{x}$ is not the only solution. Another solution is $f_1(x) = x^{3/2}$. Using transfinite tools, infinitely many other solutions can be constructed.

A6. Suppose that f and g are two functions defined on the set of positive integers and taking positive integer values. Suppose also that the equations $f(g(n)) = f(n) + 1$ and $g(f(n)) = g(n) + 1$ hold for all positive integers. Prove that $f(n) = g(n)$ for all positive integer n.

(Germany)

Solution 1. Throughout the solution, by $\mathbb{N}$ we denote the set of all positive integers. For any function $h\colon \mathbb{N} \to \mathbb{N}$ and for any positive integer k, define $h^k(x) = \underbrace{h\big(h\big(\ldots h}_{k}(x)\ldots\big)\big)$ (in particular, $h^0(x) = x$).

Observe that $f\big(g^k(x)\big) = f\big(g^{k-1}(x)\big) + 1 = \cdots = f(x) + k$ for any positive integer k, and similarly $g\big(f^k(x)\big) = g(x) + k$. Now let a and b are the minimal values attained by f and g, respectively; say $f(n_f) = a$, $g(n_g) = b$. Then we have $f\big(g^k(n_f)\big) = a + k$, $g\big(f^k(n_g)\big) = b + k$, so the function f attains all values from the set $N_f = \{a, a+1, \ldots\}$, while g attains all the values from the set $N_g = \{b, b+1, \ldots\}$.

Next, note that $f(x) = f(y)$ implies $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$; surely, the converse implication also holds. Now, we say that x and y are *similar* (and write $x \sim y$) if $f(x) = f(y)$ (equivalently, $g(x) = g(y)$). For every $x \in \mathbb{N}$, we define $[x] = \{y \in \mathbb{N} : x \sim y\}$; surely, $y_1 \sim y_2$ for all $y_1, y_2 \in [x]$, so $[x] = [y]$ whenever $y \in [x]$.

Now we investigate the structure of the sets $[x]$.

Claim 1. Suppose that $f(x) \sim f(y)$; then $x \sim y$, that is, $f(x) = f(y)$. Consequently, each class $[x]$ contains at most one element from N_f, as well as at most one element from N_g.

Proof. If $f(x) \sim f(y)$, then we have $g(x) = g\big(f(x)\big) - 1 = g\big(f(y)\big) - 1 = g(y)$, so $x \sim y$. The second statement follows now from the sets of values of f and g. $\square$

Next, we clarify which classes do not contain large elements.

Claim 2. For any $x \in \mathbb{N}$, we have $[x] \subseteq \{1, 2, \ldots, b-1\}$ if and only if $f(x) = a$. Analogously, $[x] \subseteq \{1, 2, \ldots, a-1\}$ if and only if $g(x) = b$.

Proof. We will prove that $[x] \nsubseteq \{1, 2, \ldots, b-1\} \iff f(x) > a$; the proof of the second statement is similar.

Note that $f(x) > a$ implies that there exists some y satisfying $f(y) = f(x) - 1$, so $f\big(g(y)\big) = f(y) + 1 = f(x)$, and hence $x \sim g(y) \geq b$. Conversely, if $b \leq c \sim x$ then $c = g(y)$ for some $y \in \mathbb{N}$, which in turn follows $f(x) = f\big(g(y)\big) = f(y) + 1 \geq a + 1$, and hence $f(x) > a$. $\square$

Claim 2 implies that there exists exactly one class contained in $\{1, \ldots, a-1\}$ (that is, the class $[n_g]$), as well as exactly one class contained in $\{1, \ldots, b-1\}$ (the class $[n_f]$). Assume for a moment that $a \leq b$; then $[n_g]$ is contained in $\{1, \ldots, b-1\}$ as well, hence it coincides with $[n_g]$. So, we get that

$$f(x) = a \iff g(x) = b \iff x \sim n_f \sim n_g. \tag{1}$$

Claim 3. $a = b$.

Proof. By Claim 2, we have $[a] \neq [n_f]$, so $[a]$ should contain some element $a' \geq b$ by Claim 2 again. If $a \neq a'$, then $[a]$ contains two elements $\geq a$ which is impossible by Claim 1. Therefore, $a = a' \geq b$. Similarly, $b \geq a$. $\square$

Now we are ready to prove the problem statement. First, we establish the following

Claim 4. For every integer $d \geq 0$, $f^{d+1}(n_f) = g^{d+1}(n_f) = a + d$.

Proof. Induction on d. For $d = 0$, the statement follows from (1) and Claim 3. Next, for $d > 1$ from the induction hypothesis we have $f^{d+1}(n_f) = f\big(f^d(n_f)\big) = f\big(g^d(n_f)\big) = f(n_f) + d = a + d$. The equality $g^{d+1}(n_f) = a + d$ is analogous. $\square$

Finally, for each $x \in \mathbb{N}$, we have $f(x) = a + d$ for some $d \geq 0$, so $f(x) = f\big(g^d(n_f)\big)$ and hence $x \sim g^d(n_f)$. It follows that $g(x) = g\big(g^d(n_f)\big) = g^{d+1}(n_f) = a + d = f(x)$ by Claim 4.

Solution 2. We start with the same observations, introducing the relation $\sim$ and proving Claim 1 from the previous solution.

Note that $f(a) > a$ since otherwise we have $f(a) = a$ and hence $g(a) = g\big(f(a)\big) = g(a) + 1$, which is false.

Claim $2'$. $a = b$.

Proof. We can assume that $a \leq b$. Since $f(a) \geq a + 1$, there exists some $x \in \mathbb{N}$ such that $f(a) = f(x) + 1$, which is equivalent to $f(a) = f\big(g(x)\big)$ and $a \sim g(x)$. Since $g(x) \geq b \geq a$, by Claim 1 we have $a = g(x) \geq b$, which together with $a \leq b$ proves the Claim. $\square$

Now, almost the same method allows to find the values $f(a)$ and $g(a)$.

Claim $3'$. $f(a) = g(a) = a + 1$.

Proof. Assume the contrary; then $f(a) \geq a + 2$, hence there exist some $x, y \in \mathbb{N}$ such that $f(x) = f(a) - 2$ and $f(y) = g(x)$ (as $g(x) \geq a = b$). Now we get $f(a) = f(x) + 2 = f\big(g^2(x)\big)$, so $a \sim g^2(x) \geq a$, and by Claim 1 we get $a = g^2(x) = g\big(f(y)\big) = 1 + g(y) \geq 1 + a$; this is impossible. The equality $g(a) = a + 1$ is similar.

Now, we are prepared for the proof of the problem statement. First, we prove it for $n \geq a$.

Claim $4'$. For each integer $x \geq a$, we have $f(x) = g(x) = x + 1$.

Proof. Induction on x. The base case $x = a$ is provided by Claim $3'$, while the induction step follows from $f(x + 1) = f\big(g(x)\big) = f(x) + 1 = (x + 1) + 1$ and the similar computation for $g(x + 1)$.

Finally, for an arbitrary $n \in \mathbb{N}$ we have $g(n) \geq a$, so by Claim $4'$ we have $f(n) + 1 = f\big(g(n)\big) = g(n) + 1$, hence $f(n) = g(n)$.

Comment. It is not hard now to describe all the functions $f : \mathbb{N} \to \mathbb{N}$ satisfying the property $f(f(n)) = f(n) + 1$. For each such function, there exists $n_0 \in \mathbb{N}$ such that $f(n) = n + 1$ for all $n \geq n_0$, while for each $n < n_0$, $f(n)$ is an arbitrary number greater than of equal to n_0 (these numbers may be different for different $n < n_0$).

A7. Let $a_1, \ldots, a_r$ be positive real numbers. For $n > r$, we inductively define

$$a_n = \max_{1\le k\le n-1}(a_k + a_{n-k}). \tag{1}$$

Prove that there exist positive integers $\ell \le r$ and N such that $a_n = a_{n-\ell} + a_\ell$ for all $n \ge N$.

(Iran)

Solution 1. First, from the problem conditions we have that each a_n $(n > r)$ can be expressed as $a_n = a_{j_1} + a_{j_2}$ with $j_1, j_2 < n$, $j_1 + j_2 = n$. If, say, $j_1 > r$ then we can proceed in the same way with a_{j_1}, and so on. Finally, we represent a_n in a form

$$a_n = a_{i_1} + \cdots + a_{i_k}, \tag{2}$$

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n. \tag{3}$$

Moreover, if a_{i_1} and a_{i_2} are the numbers in (2) obtained on the last step, then $i_1 + i_2 > r$. Hence we can adjust (3) as

$$1 \le i_j \le r, \quad i_1 + \cdots + i_k = n, \quad i_1 + i_2 > r. \tag{4}$$

On the other hand, suppose that the indices $i_1, \ldots, i_k$ satisfy the conditions (4). Then, denoting $s_j = i_1 + \cdots + i_j$, from (1) we have

$$a_n = a_{s_k} \ge a_{s_{k-1}} + a_{i_k} \ge a_{s_{k-2}} + a_{i_{k-1}} + a_{i_k} \ge \cdots \ge a_{i_1} + \cdots + a_{i_k}.$$

Summarizing these observations we get the following

Claim. For every $n > r$, we have

$$a_n = \max\{a_{i_1} + \cdots + a_{i_k} : \text{the collection } (i_1, \ldots, i_k) \text{ satisfies (4)}\}. \qquad \square$$

Now we denote

$$s = \max_{1\le i\le r} \frac{a_i}{i}$$

and fix some index $\ell \le r$ such that $s = \dfrac{a_\ell}{\ell}$.

Consider some $n \ge r^2\ell + 2r$ and choose an expansion of a_n in the form (2), (4). Then we have $n = i_1 + \cdots + i_k \le rk$, so $k \ge n/r \ge r\ell + 2$. Suppose that none of the numbers $i_3, \ldots, i_k$ equals ℓ. Then by the pigeonhole principle there is an index $1 \le j \le r$ which appears among $i_3, \ldots, i_k$ at least ℓ times, and surely $j \ne \ell$. Let us delete these ℓ occurrences of j from $(i_1, \ldots, i_k)$, and add j occurrences of ℓ instead, obtaining a sequence $(i_1, i_2, i'_3, \ldots, i'_{k'})$ also satisfying (4). By Claim, we have

$$a_{i_1} + \cdots + a_{i_k} = a_n \ge a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}},$$

or, after removing the coinciding terms, $\ell a_j \ge j a_\ell$, so $\dfrac{a_\ell}{\ell} \le \dfrac{a_j}{j}$. By the definition of ℓ, this means that $\ell a_j = j a_\ell$, hence

$$a_n = a_{i_1} + a_{i_2} + a_{i'_3} + \cdots + a_{i'_{k'}}.$$

Thus, for every $n \ge r^2\ell + 2r$ we have found a representation of the form (2), (4) with $i_j = \ell$ for some $j \ge 3$. Rearranging the indices we may assume that $i_k = \ell$.

Finally, observe that in this representation, the indices $(i_1, \ldots, i_{k-1})$ satisfy the conditions (4) with n replaced by $n - \ell$. Thus, from the Claim we get

$$a_{n-\ell} + a_\ell \ge (a_{i_1} + \cdots + a_{i_{k-1}}) + a_\ell = a_n,$$

which by (1) implies

$$a_n = a_{n-\ell} + a_\ell \qquad \text{for each } n \ge r^2\ell + 2r,$$

as desired.

Solution 2. As in the previous solution, we involve the expansion (2), (3), and we fix some index $1 \le \ell \le r$ such that
$$\frac{a_\ell}{\ell} = s = \max_{1\le i\le r} \frac{a_i}{i}.$$
Now, we introduce the sequence (b_n) as $b_n = a_n - sn$; then $b_\ell = 0$.

We prove by induction on n that $b_n \le 0$, and (b_n) satisfies the same recurrence relation as (a_n). The base cases $n \le r$ follow from the definition of s. Now, for $n > r$ from the induction hypothesis we have

$$b_n = \max_{1\le k\le n-1}(a_k + a_{n-k}) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k} + ns) - ns = \max_{1\le k\le n-1}(b_k + b_{n-k}) \le 0,$$

as required.

Now, if $b_k = 0$ for all $1 \le k \le r$, then $b_n = 0$ for all n, hence $a_n = sn$, and the statement is trivial. Otherwise, define

$$M = \max_{1\le i\le r} |b_i|, \quad \varepsilon = \min\{|b_i| : 1 \le i \le r,\ b_i < 0\}.$$

Then for $n > r$ we obtain

$$b_n = \max_{1\le k\le n-1}(b_k + b_{n-k}) \ge b_\ell + b_{n-\ell} = b_{n-\ell},$$

so
$$0 \ge b_n \ge b_{n-\ell} \ge b_{n-2\ell} \ge \cdots \ge -M.$$

Thus, in view of the expansion (2), (3) applied to the sequence (b_n), we get that each b_n is contained in a set

$$T = \{b_{i_1} + b_{i_2} + \cdots + b_{i_k} :\ i_1, \ldots, i_k \le r\} \cap [-M, 0]$$

We claim that this set is finite. Actually, for any $x \in T$, let $x = b_{i_1} + \cdots + b_{i_k}$ $(i_1, \ldots, i_k \le r)$. Then among b_{i_j}'s there are at most $\frac{M}{\varepsilon}$ nonzero terms (otherwise $x < \frac{M}{\varepsilon} \cdot (-\varepsilon) < -M$). Thus x can be expressed in the same way with $k \le \frac{M}{\varepsilon}$, and there is only a finite number of such sums.

Finally, for every $t = 1, 2, \ldots, \ell$ we get that the sequence

$$b_{r+t}, b_{r+t+\ell}, b_{r+t+2\ell}, \ldots$$

is non-decreasing and attains the finite number of values; therefore it is constant from some index. Thus, the sequence (b_n) is periodic with period ℓ from some index N, which means that

$$b_n = b_{n-\ell} = b_{n-\ell} + b_\ell \qquad \text{for all } n > N + \ell,$$

and hence

$$a_n = b_n + ns = (b_{n-\ell} + (n-\ell)s) + (b_\ell + \ell s) = a_{n-\ell} + a_\ell \qquad \text{for all } n > N + \ell,$$

as desired.

A8. Given six positive numbers a, b, c, d, e, f such that $a < b < c < d < e < f$. Let $a+c+e = S$ and $b + d + f = T$. Prove that

$$2ST > \sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}. \tag{1}$$

(South Korea)

Solution 1. We define also $\sigma = ac + ce + ae$, $\tau = bd + bf + df$. The idea of the solution is to interpret (1) as a natural inequality on the roots of an appropriate polynomial.

Actually, consider the polynomial

$$\begin{aligned} P(x) &= (b+d+f)(x-a)(x-c)(x-e) + (a+c+e)(x-b)(x-d)(x-f) \\ &= T(x^3 - Sx^2 + \sigma x - ace) + S(x^3 - Tx^2 + \tau x - bdf). \end{aligned} \tag{2}$$

Surely, P is cubic with leading coefficient $S + T > 0$. Moreover, we have

$$\begin{aligned} P(a) &= S(a-b)(a-d)(a-f) < 0, \qquad & P(c) &= S(c-b)(c-d)(c-f) > 0, \\ P(e) &= S(e-b)(e-d)(e-f) < 0, \qquad & P(f) &= T(f-a)(f-c)(f-e) > 0. \end{aligned}$$

Hence, each of the intervals (a, c), (c, e), (e, f) contains at least one root of $P(x)$. Since there are at most three roots at all, we obtain that there is exactly one root in each interval (denote them by $\alpha \in (a, c)$, $\beta \in (c, e)$, $\gamma \in (e, f)$). Moreover, the polynomial P can be factorized as

$$P(x) = (T+S)(x-\alpha)(x-\beta)(x-\gamma). \tag{3}$$

Equating the coefficients in the two representations (2) and (3) of $P(x)$ provides

$$\alpha + \beta + \gamma = \frac{2TS}{T+S}, \qquad \alpha\beta + \alpha\gamma + \beta\gamma = \frac{S\tau + T\sigma}{T+S}.$$

Now, since the numbers α, β, γ are distinct, we have

$$0 < (\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 = 2(\alpha+\beta+\gamma)^2 - 6(\alpha\beta+\alpha\gamma+\beta\gamma),$$

which implies

$$\frac{4S^2T^2}{(T+S)^2} = (\alpha+\beta+\gamma)^2 > 3(\alpha\beta+\alpha\gamma+\beta\gamma) = \frac{3(S\tau+T\sigma)}{T+S},$$

or

$$4S^2T^2 > 3(T+S)(T\sigma+S\tau),$$

which is exactly what we need.

Comment 1. In fact, one can locate the roots of $P(x)$ more narrowly: they should lie in the intervals (a, b), (c, d), (e, f).

Surely, if we change all inequality signs in the problem statement to non-strict ones, the (non-strict) inequality will also hold by continuity. One can also find when the equality is achieved. This happens in that case when $P(x)$ is a perfect cube, which immediately implies that $b = c = d = e (= \alpha = \beta = \gamma)$, together with the additional condition that $P''(b) = 0$. Algebraically,

$$\begin{aligned} 6(T+S)b - 4TS = 0 \qquad &\iff \qquad 3b(a+4b+f) = 2(a+2b)(2b+f) \\ &\iff \qquad f = \frac{b(4b-a)}{2a+b} = b\left(1 + \frac{3(b-a)}{2a+b}\right) > b. \end{aligned}$$

This means that for every pair of numbers a, b such that $0 < a < b$, there exists $f > b$ such that the point (a, b, b, b, b, f) is a point of equality.

Solution 2. Let

$$U=\frac{1}{2}\big((e-a)^2+(c-a)^2+(e-c)^2\big)=S^2-3(ac+ae+ce)$$

and

$$V=\frac{1}{2}\big((f-b)^2+(f-d)^2+(d-b)^2\big)=T^2-3(bd+bf+df).$$

Then

$$\begin{aligned}(\text{L.H.S.})^2-(\text{R.H.S.})^2&=(2ST)^2-(S+T)\big(S\cdot 3(bd+bf+df)+T\cdot 3(ac+ae+ce)\big)=\\ &=4S^2T^2-(S+T)\big(S(T^2-V)+T(S^2-U)\big)=(S+T)(SV+TU)-ST(T-S)^2,\end{aligned}$$

and the statement is equivalent with

$$(S+T)(SV+TU)>ST(T-S)^2. \tag{4}$$

By the Cauchy-Schwarz inequality,

$$(S+T)(TU+SV)\geq\big(\sqrt{S\cdot TU}+\sqrt{T\cdot SV}\big)^2=ST\big(\sqrt{U}+\sqrt{V}\big)^2. \tag{5}$$

Estimate the quantities $\sqrt{U}$ and $\sqrt{V}$ by the QM–AM inequality with the positive terms $(e-c)^2$ and $(d-b)^2$ being omitted:

$$\begin{aligned}\sqrt{U}+\sqrt{V}&>\sqrt{\frac{(e-a)^2+(c-a)^2}{2}}+\sqrt{\frac{(f-b)^2+(f-d)^2}{2}}\\ &>\frac{(e-a)+(c-a)}{2}+\frac{(f-b)+(f-d)}{2}=\left(f-\frac{d}{2}-\frac{b}{2}\right)+\left(\frac{e}{2}+\frac{c}{2}-a\right)\\ &=(T-S)+\frac{3}{2}(e-d)+\frac{3}{2}(c-b)>T-S.\end{aligned} \tag{6}$$

The estimates (5) and (6) prove (4) and hence the statement.

Solution 3. We keep using the notations σ and τ from Solution 1. Moreover, let $s=c+e$. Note that

$$(c-b)(c-d)+(e-f)(e-d)+(e-f)(c-b)<0,$$

since each summand is negative. This rewrites as

$$\begin{gathered}(bd+bf+df)-(ac+ce+ae)<(c+e)(b+d+f-a-c-e),\ \text{ or}\\ \tau-\sigma<s(T-S).\end{gathered} \tag{7}$$

Then we have

$$\begin{aligned}S\tau+T\sigma&=S(\tau-\sigma)+(S+T)\sigma<Ss(T-S)+(S+T)(ce+as)\\ &\leq Ss(T-S)+(S+T)\left(\frac{s^2}{4}+(S-s)s\right)=s\left(2ST-\frac{3}{4}(S+T)s\right).\end{aligned}$$

Using this inequality together with the AM–GM inequality we get

$$\begin{aligned}\sqrt{\frac{3}{4}(S+T)(S\tau+T\sigma)}&<\sqrt{\frac{3}{4}(S+T)s\left(2ST-\frac{3}{4}(S+T)s\right)}\\ &\leq\frac{\frac{3}{4}(S+T)s+2ST-\frac{3}{4}(S+T)s}{2}=ST.\end{aligned}$$

Hence,

$$2ST>\sqrt{3(S+T)\big(S(bd+bf+df)+T(ac+ae+ce)\big)}.$$

Comment 2. The expression (7) can be found by considering the sum of the roots of the quadratic polynomial $q(x) = (x-b)(x-d)(x-f) - (x-a)(x-c)(x-e)$.

Solution 4. We introduce the expressions σ and τ as in the previous solutions. The idea of the solution is to change the values of variables $a, \dots, f$ keeping the left-hand side unchanged and increasing the right-hand side; it will lead to a simpler inequality which can be proved in a direct way.

Namely, we change the variables (i) keeping the (non-strict) inequalities $a \le b \le c \le d \le e \le f$; (ii) keeping the values of sums S and T unchanged; and finally (iii) increasing the values of σ and τ. Then the left-hand side of (1) remains unchanged, while the right-hand side increases. Hence, the inequality (1) (and even a non-strict version of (1)) for the changed values would imply the same (strict) inequality for the original values.

First, we find the sufficient conditions for (ii) and (iii) to be satisfied.

Lemma. Let $x, y, z > 0$; denote $U(x,y,z) = x+y+z$, $\upsilon(x,y,z) = xy+xz+yz$. Suppose that $x'+y' = x+y$ but $|x-y| \ge |x'-y'|$; then we have $U(x',y',z) = U(x,y,z)$ and $\upsilon(x',y',z) \ge \upsilon(x,y,z)$ with equality achieved only when $|x-y| = |x'-y'|$.

Proof. The first equality is obvious. For the second, we have

$$\begin{aligned}\upsilon(x',y',z) = z(x'+y') + x'y' &= z(x'+y') + \frac{(x'+y')^2 - (x'-y')^2}{4} \\ &\ge z(x+y) + \frac{(x+y)^2 - (x-y)^2}{4} = \upsilon(x,y,z),\end{aligned}$$

with the equality achieved only for $(x'-y')^2 = (x-y)^2 \iff |x'-y'| = |x-y|$, as desired. $\square$

Now, we apply Lemma several times making the following changes. For each change, we denote the new values by the same letters to avoid cumbersome notations.

1. Let $k = \dfrac{d-c}{2}$. Replace (b,c,d,e) by $(b+k, c+k, d-k, e-k)$. After the change we have $a < b < c = d < e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by Lemma.

2. Let $\ell = \dfrac{e-d}{2}$. Replace (c,d,e,f) by $(c+\ell, d+\ell, e-\ell, f-\ell)$. After the change we have $a < b < c = d = e < f$, the values of S, T remain unchanged, but σ, τ strictly increase by the Lemma.

3. Finally, let $m = \dfrac{c-b}{3}$. Replace (a,b,c,d,e,f) by $(a+2m, b+2m, c-m, d-m, e-m, f-m)$. After the change, we have $a < b = c = d = e < f$ and S, T are unchanged. To check (iii), we observe that our change can be considered as a composition of two changes: $(a,b,c,d) \to (a+m, b+m, c-m, d-m)$ and $(a,b,e,f) \to (a+m, b+m, e-m, f-m)$. It is easy to see that each of these two consecutive changes satisfy the conditions of the Lemma, hence the values of σ and τ increase.

Finally, we come to the situation when $a < b = c = d = e < f$, and we need to prove the inequality

$$\begin{aligned}2(a+2b)(2b+f) &\ge \sqrt{3(a+4b+f)\big((a+2b)(b^2+2bf) + (2b+f)(2ab+b^2)\big)} \\ &= \sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f) + (2b+f)(2a+b)\big)}. \qquad (8)\end{aligned}$$

Now, observe that

$$2\cdot 2(a+2b)(2b+f) = 3b(a+4b+f) + \big((a+2b)(b+2f) + (2a+b)(2b+f)\big).$$

Hence (4) rewrites as

$$3b(a+4b+f)+\big((a+2b)(b+2f)+(2a+b)(2b+f)\big)$$
$$\geq 2\sqrt{3b(a+4b+f)\cdot\big((a+2b)(b+2f)+(2b+f)(2a+b)\big)},$$

which is simply the AM–GM inequality.

Comment 3. Here, we also can find all the cases of equality. Actually, it is easy to see that if some two numbers among b, c, d, e are distinct then one can use Lemma to increase the right-hand side of (1). Further, if $b=c=d=e$, then we need equality in (4); this means that we apply AM–GM to equal numbers, that is,

$$3b(a+4b+f)=(a+2b)(b+2f)+(2a+b)(2b+f),$$

which leads to the same equality as in Comment 1.

Combinatorics

C1. In a concert, 20 singers will perform. For each singer, there is a (possibly empty) set of other singers such that he wishes to perform later than all the singers from that set. Can it happen that there are exactly 2010 orders of the singers such that all their wishes are satisfied?

(Austria)

Answer. Yes, such an example exists.

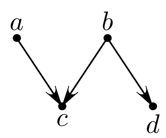
Solution. We say that an order of singers is *good* if it satisfied all their wishes. Next, we say that a number N is *realizable* by k singers (or k-*realizable*) if for some set of wishes of these singers there are exactly N good orders. Thus, we have to prove that a number 2010 is 20-realizable.

We start with the following simple

Lemma. Suppose that numbers n_1, n_2 are realizable by respectively k_1 and k_2 singers. Then the number $n_1 n_2$ is $(k_1 + k_2)$-realizable.

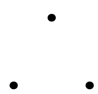
Proof. Let the singers $A_1, \ldots, A_{k_1}$ (with some wishes among them) realize n_1, and the singers B_1, $\ldots$, B_{k_2} (with some wishes among them) realize n_2. Add to each singer B_i the wish to perform later than all the singers A_j. Then, each good order of the obtained set of singers has the form $(A_{i_1}, \ldots, A_{i_{k_1}}, B_{j_1}, \ldots, B_{j_{k_2}})$, where $(A_{i_1}, \ldots, A_{i_{k_1}})$ is a good order for A_i's and $(B_{j_1}, \ldots, B_{j_{k_2}})$ is a good order for B_j's. Conversely, each order of this form is obviously good. Hence, the number of good orders is $n_1 n_2$. $\square$

In view of Lemma, we show how to construct sets of singers containing 4, 3 and 13 singers and realizing the numbers 5, 6 and 67, respectively. Thus the number $2010 = 6 \cdot 5 \cdot 67$ will be realizable by $4 + 3 + 13 = 20$ singers. These companies of singers are shown in Figs. 1–3; the wishes are denoted by arrows, and the number of good orders for each Figure stands below in the brackets.



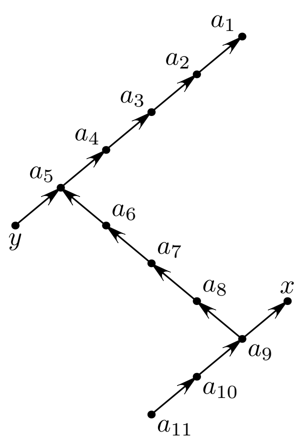
(5)

Fig. 1



(3)

Fig. 2



(67)

Fig. 3

For Fig. 1, there are exactly 5 good orders (a, b, c, d), (a, b, d, c), (b, a, c, d), (b, a, d, c), (b, d, a, c). For Fig. 2, each of 6 orders is good since there are no wishes.

Finally, for Fig. 3, the order of $a_1, \dots, a_{11}$ is fixed; in this line, singer x can stand before each of a_i $(i \le 9)$, and singer y can stand after each of a_j $(j \ge 5)$, thus resulting in $9 \cdot 7 = 63$ cases. Further, the positions of x and y in this line determine the whole order uniquely unless both of them come between the same pair (a_i, a_{i+1}) (thus $5 \le i \le 8$); in the latter cases, there are two orders instead of 1 due to the order of x and y. Hence, the total number of good orders is $63 + 4 = 67$, as desired.

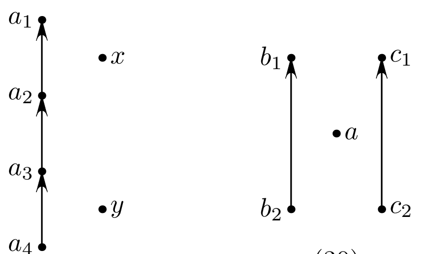
Comment. The number 20 in the problem statement is not sharp and is put there to respect the original formulation. So, if necessary, the difficulty level of this problem may be adjusted by replacing 20 by a smaller number. Here we present some improvements of the example leading to a smaller number of singers.

Surely, each example with < 20 singers can be filled with some "super-stars" who should perform at the very end in a fixed order. Hence each of these improvements provides a different solution for the problem. Moreover, the large variety of ideas standing behind these examples allows to suggest that there are many other examples.

1. Instead of building the examples realizing 5 and 6, it is more economic to make an example realizing 30; it may seem even simpler. Two possible examples consisting of 5 and 6 singers are shown in Fig. 4; hence the number 20 can be decreased to 19 or 18.

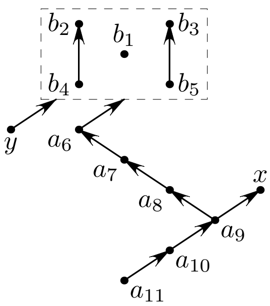
For Fig. 4a, the order of $a_1, \dots, a_4$ is fixed, there are 5 ways to add x into this order, and there are 6 ways to add y into the resulting order of $a_1, \dots, a_4, x$. Hence there are $5 \cdot 6 = 30$ good orders.

On Fig. 4b, for 5 singers a, b_1, b_2, c_1, c_2 there are $5! = 120$ orders at all. Obviously, exactly one half of them satisfies the wish $b_1 \leftarrow b_2$, and exactly one half of these orders satisfies another wish $c_1 \leftarrow c_2$; hence, there are exactly $5!/4 = 30$ good orders.



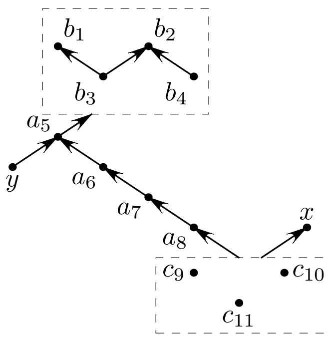
(30) a)   (30) b)

Fig. 4



(2010)

Fig. 5



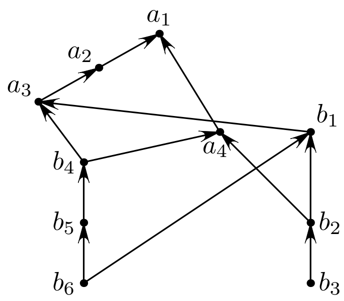
(2010)

Fig. 6

2. One can merge the examples for 30 and 67 shown in Figs. 4b and 3 in a smarter way, obtaining a set of 13 singers representing 2010. This example is shown in Fig. 5; an arrow from/to group $\{b_1, \dots, b_5\}$ means that there exists such arrow from each member of this group.

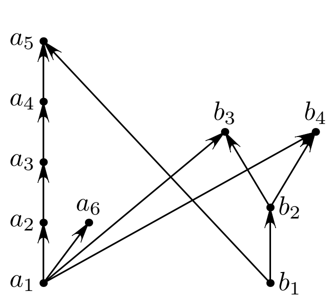
Here, as in Fig. 4b, one can see that there are exactly 30 orders of $b_1, \dots, b_5, a_6, \dots, a_{11}$ satisfying all their wishes among themselves. Moreover, one can prove in the same way as for Fig. 3 that each of these orders can be complemented by x and y in exactly 67 ways, hence obtaining $30 \cdot 67 = 2010$ good orders at all.

Analogously, one can merge the examples in Figs. 1–3 to represent 2010 by 13 singers as is shown in Fig. 6.



(67)

Fig. 7



(2010)

Fig. 8

3. Finally, we will present two other improvements; the proofs are left to the reader. The graph in Fig. 7 shows how 10 singers can represent 67. Moreover, even a company of 10 singers representing 2010 can be found; this company is shown in Fig. 8.

C2. On some planet, there are 2^N countries ($N \geq 4$). Each country has a flag N units wide and one unit high composed of N fields of size 1×1, each field being either yellow or blue. No two countries have the same flag.

We say that a set of N flags is *diverse* if these flags can be arranged into an $N \times N$ square so that all N fields on its main diagonal will have the same color. Determine the smallest positive integer M such that among any M distinct flags, there exist N flags forming a diverse set.

(Croatia)

Answer. $M = 2^{N-2} + 1$.

Solution. When speaking about the diagonal of a square, we will always mean the main diagonal.

Let M_N be the smallest positive integer satisfying the problem condition. First, we show that $M_N > 2^{N-2}$. Consider the collection of all 2^{N-2} flags having yellow first squares and blue second ones. Obviously, both colors appear on the diagonal of each $N \times N$ square formed by these flags.

We are left to show that $M_N \leq 2^{N-2} + 1$, thus obtaining the desired answer. We start with establishing this statement for $N = 4$.

Suppose that we have 5 flags of length 4. We decompose each flag into two parts of 2 squares each; thereby, we denote each flag as LR, where the 2×1 flags $L, R \in \mathcal{S} = \{\mathrm{BB}, \mathrm{BY}, \mathrm{YB}, \mathrm{YY}\}$ are its left and right parts, respectively. First, we make two easy observations on the flags 2×1 which can be checked manually.

(i) For each $A \in \mathcal{S}$, there exists only one 2×1 flag $C \in \mathcal{S}$ (possibly $C = A$) such that A and C cannot form a 2×2 square with monochrome diagonal (for part BB, that is YY, and for BY that is YB).

(ii) Let $A_1, A_2, A_3 \in \mathcal{S}$ be three distinct elements; then two of them can form a 2×2 square with yellow diagonal, and two of them can form a 2×2 square with blue diagonal (for all parts but BB, a pair (BY, YB) fits for both statements, while for all parts but BY, these pairs are (YB, YY) and (BB, YB)).

Now, let ℓ and r be the numbers of distinct left and right parts of our 5 flags, respectively. The total number of flags is $5 \leq r\ell$, hence one of the factors (say, r) should be at least 3. On the other hand, $\ell, r \leq 4$, so there are two flags with coinciding right part; let them be L_1R_1 and L_2R_1 ($L_1 \neq L_2$).

Next, since $r \geq 3$, there exist some flags L_3R_3 and L_4R_4 such that R_1, R_3, R_4 are distinct. Let $L'R'$ be the remaining flag. By (i), one of the pairs (L', L_1) and (L', L_2) can form a 2×2 square with monochrome diagonal; we can assume that L', L_2 form a square with a blue diagonal. Finally, the right parts of two of the flags L_1R_1, L_3R_3, L_4R_4 can also form a 2×2 square with a blue diagonal by (ii). Putting these 2×2 squares on the diagonal of a 4×4 square, we find a desired arrangement of four flags.

We are ready to prove the problem statement by induction on N; actually, above we have proved the base case $N = 4$. For the induction step, assume that $N > 4$, consider any $2^{N-2} + 1$ flags of length N, and arrange them into a large flag of size $(2^{N-2} + 1) \times N$. This flag contains a non-monochrome column since the flags are distinct; we may assume that this column is the first one. By the pigeonhole principle, this column contains at least $\left\lceil \frac{2^{N-2}+1}{2} \right\rceil = 2^{N-3} + 1$ squares of one color (say, blue). We call the flags with a blue first square *good*.

Consider all the good flags and remove the first square from each of them. We obtain at least $2^{N-3} + 1 \geq M_{N-1}$ flags of length $N - 1$; by the induction hypothesis, $N - 1$ of them can form a square Q with the monochrome diagonal. Now, returning the removed squares, we obtain a rectangle $(N-1) \times N$, and our aim is to supplement it on the top by one more flag.

If Q has a yellow diagonal, then we can take each flag with a yellow first square (it exists by a choice of the first column; moreover, it is not used in Q). Conversely, if the diagonal of Q is blue then we can take any of the $\geq 2^{N-3}+1-(N-1)>0$ remaining good flags. So, in both cases we get a desired $N \times N$ square.

Solution 2. We present a different proof of the estimate $M_N \leq 2^{N-2}+1$. We do not use the induction, involving Hall's lemma on matchings instead.

Consider arbitrary $2^{N-2}+1$ distinct flags and arrange them into a large $(2^{N-2}+1) \times N$ flag. Construct two bipartite graphs $G_{\mathrm{y}} = (V \cup V', E_{\mathrm{y}})$ and $G_{\mathrm{b}} = (V \cup V', E_{\mathrm{b}})$ with the common set of vertices as follows. Let V and V' be the set of columns and the set of flags under consideration, respectively. Next, let the edge (c, f) appear in E_{y} if the intersection of column c and flag f is yellow, and $(c, f) \in E_{\mathrm{b}}$ otherwise. Then we have to prove exactly that one of the graphs G_{y} and G_{b} contains a matching with all the vertices of V involved.

Assume that these matchings do not exist. By Hall's lemma, it means that there exist two sets of columns $S_{\mathrm{y}}, S_{\mathrm{b}} \subset V$ such that $|E_{\mathrm{y}}(S_{\mathrm{y}})| \leq |S_{\mathrm{y}}|-1$ and $|E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{b}}|-1$ (in the left-hand sides, $E_{\mathrm{y}}(S_{\mathrm{y}})$ and $E_{\mathrm{b}}(S_{\mathrm{b}})$ denote respectively the sets of all vertices connected to S_{y} and S_{b} in the corresponding graphs). Our aim is to prove that this is impossible. Note that $S_{\mathrm{y}}, S_{\mathrm{b}} \neq V$ since $N \leq 2^{N-2}+1$.

First, suppose that $S_{\mathrm{y}} \cap S_{\mathrm{b}} \neq \varnothing$, so there exists some $c \in S_{\mathrm{y}} \cap S_{\mathrm{b}}$. Note that each flag is connected to c either in G_{y} or in G_{b}, hence $E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}}) = V'$. Hence we have $2^{N-2}+1 = |V'| \leq |E_{\mathrm{y}}(S_{\mathrm{y}})| + |E_{\mathrm{b}}(S_{\mathrm{b}})| \leq |S_{\mathrm{y}}| + |S_{\mathrm{b}}| - 2 \leq 2N-4$; this is impossible for $N \geq 4$.

So, we have $S_{\mathrm{y}} \cap S_{\mathrm{b}} = \varnothing$. Let $y = |S_{\mathrm{y}}|$, $b = |S_{\mathrm{b}}|$. From the construction of our graph, we have that all the flags in the set $V'' = V' \backslash \big(E_{\mathrm{y}}(S_{\mathrm{y}}) \cup E_{\mathrm{b}}(S_{\mathrm{b}})\big)$ have blue squares in the columns of S_{y} and yellow squares in the columns of S_{b}. Hence the only undetermined positions in these flags are the remaining $N-y-b$ ones, so $2^{N-y-b} \geq |V''| \geq |V'| - |E_{\mathrm{y}}(S_{\mathrm{y}})| - |E_{\mathrm{b}}(S_{\mathrm{b}})| \geq 2^{N-2}+1-(y-1)-(b-1)$, or, denoting $c = y+b$, $2^{N-c}+c > 2^{N-2}+2$. This is impossible since $N \geq c \geq 2$.

C3. 2500 chess kings have to be placed on a 100×100 chessboard so that

(i) no king can capture any other one (i.e. no two kings are placed in two squares sharing a common vertex);

(ii) each row and each column contains exactly 25 kings.

Find the number of such arrangements. (Two arrangements differing by rotation or symmetry are supposed to be different.)

(Russia)

Answer. There are two such arrangements.

Solution. Suppose that we have an arrangement satisfying the problem conditions. Divide the board into 2×2 pieces; we call these pieces *blocks*. Each block can contain not more than one king (otherwise these two kings would attack each other); hence, by the pigeonhole principle each block must contain exactly one king.

Now assign to each block a letter T or B if a king is placed in its top or bottom half, respectively. Similarly, assign to each block a letter L or R if a king stands in its left or right half. So we define *T-blocks*, *B-blocks*, *L-blocks*, and *R-blocks*. We also combine the letters; we call a block *a TL-block* if it is simultaneously T-block and L-block. Similarly we define *TR-blocks*, *BL-blocks*, and *BR-blocks*. The arrangement of blocks determines uniquely the arrangement of kings; so in the rest of the solution we consider the 50×50 system of blocks (see Fig. 1). We identify the blocks by their coordinate pairs; the pair (i, j), where $1 \le i, j \le 50$, refers to the jth block in the ith row (or the ith block in the jth column). The upper-left block is $(1, 1)$.

The system of blocks has the following properties..

(i′) If (i, j) is a B-block then $(i + 1, j)$ is a B-block: otherwise the kings in these two blocks can take each other. Similarly: if (i, j) is a T-block then $(i - 1, j)$ is a T-block; if (i, j) is an L-block then $(i, j - 1)$ is an L-block; if (i, j) is an R-block then $(i, j + 1)$ is an R-block.

(ii′) Each column contains exactly 25 L-blocks and 25 R-blocks, and each row contains exactly 25 T-blocks and 25 B-blocks. In particular, the total number of L-blocks (or R-blocks, or T-blocks, or B-blocks) is equal to $25 \cdot 50 = 1250$.

Consider any B-block of the form $(1, j)$. By (i′), all blocks in the jth column are B-blocks; so we call such a column *B-column*. By (ii′), we have 25 B-blocks in the first row, so we obtain 25 B-columns. These 25 B-columns contain 1250 B-blocks, hence all blocks in the remaining columns are T-blocks, and we obtain 25 *T-columns*. Similarly, there are exactly 25 *L-rows* and exactly 25 *R-rows*.

Now consider an arbitrary pair of a T-column and a neighboring B-column (columns with numbers j and $j + 1$).

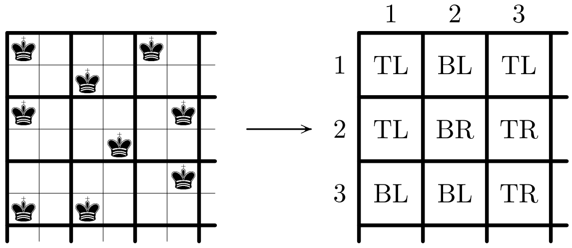


Fig. 1

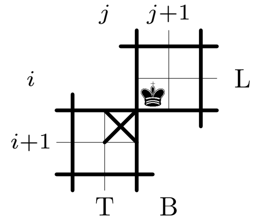


Fig. 2

Case 1. Suppose that the jth column is a T-column, and the $(j + 1)$th column is a B-column. Consider some index i such that the ith row is an L-row; then $(i, j + 1)$ is a BL-block. Therefore, $(i + 1, j)$ cannot be a TR-block (see Fig. 2), hence $(i + 1, j)$ is a TL-block, thus the

$(i+1)$th row is an L-row. Now, choosing the ith row to be the topmost L-row, we successively obtain that all rows from the ith to the 50th are L-rows. Since we have exactly 25 L-rows, it follows that the rows from the 1st to the 25th are R-rows, and the rows from the 26th to the 50th are L-rows.

Now consider the neighboring R-row and L-row (that are the rows with numbers 25 and 26). Replacing in the previous reasoning rows by columns and vice versa, the columns from the 1st to the 25th are T-columns, and the columns from the 26th to the 50th are B-columns. So we have a unique arrangement of blocks that leads to the arrangement of kings satisfying the condition of the problem (see Fig. 3).

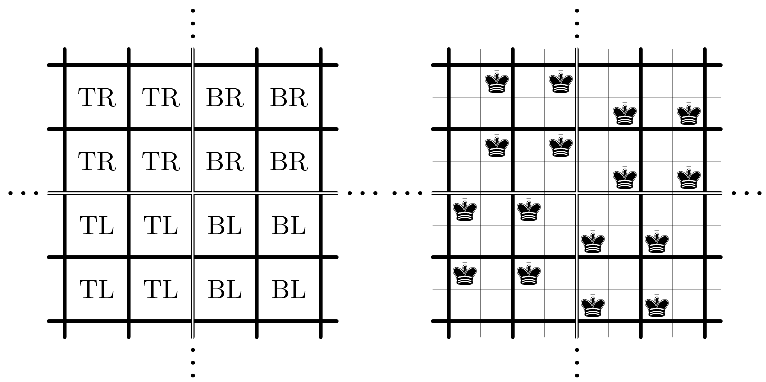


Fig. 3

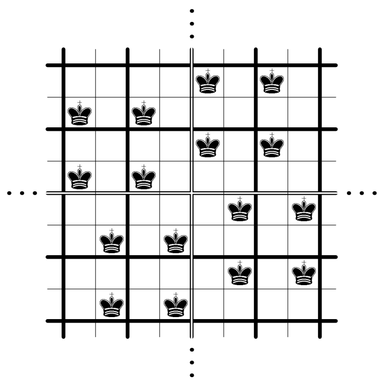


Fig. 4

Case 2. Suppose that the jth column is a B-column, and the $(j+1)$th column is a T-column. Repeating the arguments from Case 1, we obtain that the rows from the 1st to the 25th are L-rows (and all other rows are R-rows), the columns from the 1st to the 25th are B-columns (and all other columns are T-columns), so we find exactly one more arrangement of kings (see Fig. 4).

C4. Six stacks $S_1, \dots, S_6$ of coins are standing in a row. In the beginning every stack contains a single coin. There are two types of allowed moves:

Move 1: If stack S_k with $1 \le k \le 5$ contains at least one coin, you may remove one coin from S_k and add two coins to S_{k+1}.

Move 2: If stack S_k with $1 \le k \le 4$ contains at least one coin, then you may remove one coin from S_k and exchange stacks S_{k+1} and S_{k+2}.

Decide whether it is possible to achieve by a sequence of such moves that the first five stacks are empty, whereas the sixth stack S_6 contains exactly $2010^{2010^{2010}}$ coins.

C4′. Same as Problem C4, but the constant $2010^{2010^{2010}}$ is replaced by 2010^{2010}.

(Netherlands)

Answer. Yes (in both variants of the problem). There exists such a sequence of moves.

Solution. Denote by $(a_1, a_2, \dots, a_n) \to (a'_1, a'_2, \dots, a'_n)$ the following: if some consecutive stacks contain $a_1, \dots, a_n$ coins, then it is possible to perform several allowed moves such that the stacks contain $a'_1, \dots, a'_n$ coins respectively, whereas the contents of the other stacks remain unchanged.

Let $A = 2010^{2010}$ or $A = 2010^{2010^{2010}}$, respectively. Our goal is to show that

$$(1, 1, 1, 1, 1, 1) \to (0, 0, 0, 0, 0, A).$$

First we prove two auxiliary observations.

Lemma 1. $(a, 0, 0) \to (0, 2^a, 0)$ for every $a \ge 1$.

Proof. We prove by induction that $(a, 0, 0) \to (a-k, 2^k, 0)$ for every $1 \le k \le a$. For $k = 1$, apply Move 1 to the first stack:

$$(a, 0, 0) \to (a-1, 2, 0) = (a-1, 2^1, 0).$$

Now assume that $k < a$ and the statement holds for some $k < a$. Starting from $(a-k, 2^k, 0)$, apply Move 1 to the middle stack 2^k times, until it becomes empty. Then apply Move 2 to the first stack:

$$(a-k, 2^k, 0) \to (a-k, 2^k-1, 2) \to \dots \to (a-k, 0, 2^{k+1}) \to (a-k-1, 2^{k+1}, 0).$$

Hence,

$$(a, 0, 0) \to (a-k, 2^k, 0) \to (a-k-1, 2^{k+1}, 0). \qquad \square$$

Lemma 2. For every positive integer n, let $P_n = \underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n}$ (e.g. $P_3 = 2^{2^2} = 16$). Then $(a, 0, 0, 0) \to (0, P_a, 0, 0)$ for every $a \ge 1$.

Proof. Similarly to Lemma 1, we prove that $(a, 0, 0, 0) \to (a-k, P_k, 0, 0)$ for every $1 \le k \le a$.

For $k = 1$, apply Move 1 to the first stack:

$$(a, 0, 0, 0) \to (a-1, 2, 0, 0) = (a-1, P_1, 0, 0).$$

Now assume that the lemma holds for some $k < a$. Starting from $(a-k, P_k, 0, 0)$, apply Lemma 1, then apply Move 1 to the first stack:

$$(a-k, P_k, 0, 0) \to (a-k, 0, 2^{P_k}, 0) = (a-k, 0, P_{k+1}, 0) \to (a-k-1, P_{k+1}, 0, 0).$$

Therefore,

$$(a, 0, 0, 0) \to (a-k, P_k, 0, 0) \to (a-k-1, P_{k+1}, 0, 0). \qquad \square$$

Now we prove the statement of the problem.

First apply Move 1 to stack 5, then apply Move 2 to stacks S_4, S_3, S_2 and S_1 in this order. Then apply Lemma 2 twice:

$$(1,1,1,1,1,1) \to (1,1,1,1,0,3) \to (1,1,1,0,3,0) \to (1,1,0,3,0,0) \to (1,0,3,0,0,0) \to \\ \to (0,3,0,0,0,0) \to (0,0,P_3,0,0,0) = (0,0,16,0,0,0) \to (0,0,0,P_{16},0,0).$$

We already have more than A coins in stack S_4, since

$$A \le 2010^{2010^{2010}} < (2^{11})^{2010^{2010}} = 2^{11\cdot 2010^{2010}} < 2^{2010^{2011}} < 2^{(2^{11})^{2011}} = 2^{2^{11\cdot 2011}} < 2^{2^{2^{15}}} < P_{16}.$$

To decrease the number of coins in stack S_4, apply Move 2 to this stack repeatedly until its size decreases to $A/4$. (In every step, we remove a coin from S_4 and exchange the empty stacks S_5 and S_6.)

$$(0,0,0,P_{16},0,0) \to (0,0,0,P_{16}-1,0,0) \to (0,0,0,P_{16}-2,0,0) \to \\ \to \cdots \to (0,0,0,A/4,0,0).$$

Finally, apply Move 1 repeatedly to empty stacks S_4 and S_5:

$$(0,0,0,A/4,0,0) \to \cdots \to (0,0,0,0,A/2,0) \to \cdots \to (0,0,0,0,0,A).$$

Comment 1. Starting with only 4 stack, it is not hard to check manually that we can achieve at most 28 coins in the last position. However, around 5 and 6 stacks the maximal number of coins explodes. With 5 stacks it is possible to achieve more than $2^{2^{14}}$ coins. With 6 stacks the maximum is greater than $P_{P_{2^{14}}}$.

It is not hard to show that the numbers 2010^{2010} and $2010^{2010^{2010}}$ in the problem can be replaced by any nonnegative integer up to $P_{P_{2^{14}}}$.

Comment 2. The simpler variant C4′ of the problem can be solved without Lemma 2:

$$(1,1,1,1,1,1) \to (0,3,1,1,1,1) \to (0,1,5,1,1,1) \to (0,1,1,9,1,1) \to \\ \to (0,1,1,1,17,1) \to (0,1,1,1,0,35) \to (0,1,1,0,35,0) \to (0,1,0,35,0,0) \to \\ \to (0,0,35,0,0,0) \to (0,0,1,2^{34},0,0) \to (0,0,1,0,2^{2^{34}},0) \to (0,0,0,2^{2^{34}},0,0) \\ \to (0,0,0,2^{2^{34}}-1,0,0) \to \ldots \to (0,0,0,A/4,0,0) \to (0,0,0,0,A/2,0) \to (0,0,0,0,0,A).$$

For this reason, the PSC suggests to consider the problem C4 as well. Problem C4 requires more invention and technical care. On the other hand, the problem statement in C4′ hides the fact that the resulting amount of coins can be such incredibly huge and leaves this discovery to the students.

C5. $n \geq 4$ players participated in a tennis tournament. Any two players have played exactly one game, and there was no tie game. We call a company of four players *bad* if one player was defeated by the other three players, and each of these three players won a game and lost another game among themselves. Suppose that there is no bad company in this tournament. Let w_i and ℓ_i be respectively the number of wins and losses of the ith player. Prove that

$$\sum_{i=1}^{n}(w_i - \ell_i)^3 \geq 0. \tag{1}$$

(South Korea)

Solution. For any tournament T satisfying the problem condition, denote by $S(T)$ sum under consideration, namely

$$S(T) = \sum_{i=1}^{n}(w_i - \ell_i)^3.$$

First, we show that the statement holds if a tournament T has only 4 players. Actually, let $A = (a_1, a_2, a_3, a_4)$ be the number of wins of the players; we may assume that $a_1 \geq a_2 \geq a_3 \geq a_4$. We have $a_1 + a_2 + a_3 + a_4 = \binom{4}{2} = 6$, hence $a_4 \leq 1$. If $a_4 = 0$, then we cannot have $a_1 = a_2 = a_3 = 2$, otherwise the company of all players is bad. Hence we should have $A = (3, 2, 1, 0)$, and $S(T) = 3^3 + 1^3 + (-1)^3 + (-3)^3 = 0$. On the other hand, if $a_4 = 1$, then only two possibilities, $A = (3, 1, 1, 1)$ and $A = (2, 2, 1, 1)$ can take place. In the former case we have $S(T) = 3^3 + 3 \cdot (-2)^3 > 0$, while in the latter one $S(T) = 1^3 + 1^3 + (-1)^3 + (-1)^3 = 0$, as desired.

Now we turn to the general problem. Consider a tournament T with no bad companies and enumerate the players by the numbers from 1 to n. For every 4 players i_1, i_2, i_3, i_4 consider a "sub-tournament" $T_{i_1 i_2 i_3 i_4}$ consisting of only these players and the games which they performed with each other. By the abovementioned, we have $S(T_{i_1 i_2 i_3 i_4}) \geq 0$. Our aim is to prove that

$$S(T) = \sum_{i_1, i_2, i_3, i_4} S(T_{i_1 i_2 i_3 i_4}), \tag{2}$$

where the sum is taken over all 4-tuples of distinct numbers from the set $\{1, \dots, n\}$. This way the problem statement will be established.

We interpret the number $(w_i - \ell_i)^3$ as following. For $i \neq j$, let $\varepsilon_{ij} = 1$ if the ith player wins against the jth one, and $\varepsilon_{ij} = -1$ otherwise. Then

$$(w_i - \ell_i)^3 = \left(\sum_{j \neq i} \varepsilon_{ij}\right)^3 = \sum_{j_1, j_2, j_3 \neq i} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}.$$

Hence,

$$S(T) = \sum_{i \notin \{j_1, j_2, j_3\}} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}.$$

To simplify this expression, consider all the terms in this sum where two indices are equal. If, for instance, $j_1 = j_2$, then the term contains $\varepsilon_{ij_1}^2 = 1$, so we can replace this term by ε_{ij_3}. Make such replacements for each such term; obviously, after this change each term of the form ε_{ij_3} will appear $P(T)$ times, hence

$$S(T) = \sum_{|\{i, j_1, j_2, j_3\}| = 4} \varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3} + P(T)\sum_{i \neq j} \varepsilon_{ij} = S_1(T) + P(T)S_2(T).$$

We show that $S_2(T)=0$ and hence $S(T)=S_1(T)$ for each tournament. Actually, note that $\varepsilon_{ij}=-\varepsilon_{ji}$, and the whole sum can be split into such pairs. Since the sum in each pair is 0, so is $S_2(T)$.

Thus the desired equality (2) rewrites as

$$S_1(T)=\sum_{i_1,i_2,i_3,i_4} S_1(T_{i_1i_2i_3i_4}). \tag{3}$$

Now, if all the numbers j_1,j_2,j_3 are distinct, then the set $\{i,j_1,j_2,j_3\}$ is contained in exactly one 4-tuple, hence the term $\varepsilon_{ij_1}\varepsilon_{ij_2}\varepsilon_{ij_3}$ appears in the right-hand part of (3) exactly once, as well as in the left-hand part. Clearly, there are no other terms in both parts, so the equality is established.

Solution 2. Similarly to the first solution, we call the subsets of players as *companies*, and the k-element subsets will be called as k*-companies*.

In any company of the players, call a player *the local champion of the company* if he defeated all other members of the company. Similarly, if a player lost all his games against the others in the company then call him *the local loser of the company*. Obviously every company has at most one local champion and at most one local loser. By the condition of the problem, whenever a 4-company has a local loser, then this company has a local champion as well.

Suppose that k is some positive integer, and let us count all cases when a player is the local champion of some k-company. The ith player won against w_i other player. To be the local champion of a k-company, he must be a member of the company, and the other $k-1$ members must be chosen from those whom he defeated. Therefore, the ith player is the local champion of $\binom{w_i}{k-1}$ k-companies. Hence, the total number of local champions of all k-companies is $\sum_{i=1}^{n}\binom{w_i}{k-1}$.

Similarly, the total number of local losers of the k-companies is $\sum_{i=1}^{n}\binom{\ell_i}{k-1}$.

Now apply this for $k=2,3$ and 4.

Since every game has a winner and a loser, we have $\sum_{i=1}^{n} w_i=\sum_{i=1}^{n}\ell_i=\binom{n}{2}$, and hence

$$\sum_{i=1}^{n}\bigl(w_i-\ell_i\bigr)=0. \tag{4}$$

In every 3-company, either the players defeated one another in a cycle or the company has both a local champion and a local loser. Therefore, the total number of local champions and local losers in the 3-companies is the same, $\sum_{i=1}^{n}\binom{w_i}{2}=\sum_{i=1}^{n}\binom{\ell_i}{2}$. So we have

$$\sum_{i=1}^{n}\left(\binom{w_i}{2}-\binom{\ell_i}{2}\right)=0. \tag{5}$$

In every 4-company, by the problem's condition, the number of local losers is less than or equal to the number of local champions. Then the same holds for the total numbers of local champions and local losers in all 4-companies, so $\sum_{i=1}^{n}\binom{w_i}{3} \geq \sum_{i=1}^{n}\binom{\ell_i}{3}$. Hence,

$$\sum_{i=1}^{n}\left(\binom{w_i}{3} - \binom{\ell_i}{3}\right) \geq 0. \tag{6}$$

Now we establish the problem statement (1) as a linear combination of (4), (5) and (6). It is easy check that

$$(x-y)^3 = 24\left(\binom{x}{3} - \binom{y}{3}\right) + 24\left(\binom{x}{2} - \binom{y}{2}\right) - \big(3(x+y)^2 - 4\big)(x-y).$$

Apply this identity to $x = w_1$ and $y = \ell_i$. Since every player played $n-1$ games, we have $w_i + \ell_i = n-1$, and thus

$$(w_i - \ell_i)^3 = 24\left(\binom{w_i}{3} - \binom{\ell_i}{3}\right) + 24\left(\binom{w_i}{2} - \binom{\ell_i}{2}\right) - \big(3(n-1)^2 - 4\big)\Big(w_i - \ell_i\Big).$$

Then

$$\sum_{i=1}^{n}(w_i - \ell_i)^3 = 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{3} - \binom{\ell_i}{3}\right)}_{\geq 0} + 24\underbrace{\sum_{i=1}^{n}\left(\binom{w_i}{2} - \binom{\ell_i}{2}\right)}_{0} - \big(3(n-1)^2 - 4\big)\underbrace{\sum_{i=1}^{n}\Big(w_i - \ell_i\Big)}_{0} \geq 0.$$

C6. Given a positive integer k and other two integers $b > w > 1$. There are two strings of pearls, a string of b black pearls and a string of w white pearls. The *length* of a string is the number of pearls on it.

One cuts these strings in some steps by the following rules. In each step:

(i) The strings are ordered by their lengths in a non-increasing order. If there are some strings of equal lengths, then the white ones precede the black ones. Then k first ones (if they consist of more than one pearl) are chosen; if there are less than k strings longer than 1, then one chooses all of them.

(ii) Next, one cuts each chosen string into two parts differing in length by at most one.

(For instance, if there are strings of 5, 4, 4, 2 black pearls, strings of 8, 4, 3 white pearls and $k = 4$, then the strings of 8 white, 5 black, 4 white and 4 black pearls are cut into the parts $(4, 4)$, $(3, 2)$, $(2, 2)$ and $(2, 2)$, respectively.)

The process stops immediately after the step when a first isolated white pearl appears. Prove that at this stage, there will still exist a string of at least two black pearls.

(Canada)

Solution 1. Denote the situation after the ith step by A_i; hence A_0 is the initial situation, and $A_{i-1} \to A_i$ is the ith step. We call a string containing m pearls an m-string; it is an *m-w-string* or a *m-b-string* if it is white or black, respectively.

We continue the process until every string consists of a single pearl. We will focus on three moments of the process: (a) the first stage A_s when the first 1-string (no matter black or white) appears; (b) the first stage A_t where the total number of strings is greater than k (if such moment does not appear then we put $t = \infty$); and (c) the first stage A_f when all black pearls are isolated. It is sufficient to prove that in A_{f-1} (or earlier), a 1-w-string appears.

We start with some easy properties of the situations under consideration. Obviously, we have $s \leq f$. Moreover, all b-strings from A_{f-1} become single pearls in the fth step, hence all of them are 1- or 2-b-strings.

Next, observe that in each step $A_i \to A_{i+1}$ with $i \leq t-1$, all (>1)-strings were cut since there are not more than k strings at all; if, in addition, $i < s$, then there were no 1-string, so all the strings were cut in this step.

Now, let B_i and b_i be the lengths of the longest and the shortest b-strings in A_i, and let W_i and w_i be the same for w-strings. We show by induction on $i \leq \min\{s, t\}$ that (i) the situation A_i contains exactly 2^i black and 2^i white strings, (ii) $B_i \geq W_i$, and (iii) $b_i \geq w_i$. The base case $i = 0$ is obvious. For the induction step, if $i \leq \min\{s, t\}$ then in the ith step, each string is cut, thus the claim (i) follows from the induction hypothesis; next, we have $B_i = \lceil B_{i-1}/2 \rceil \geq \lceil W_{i-1}/2 \rceil = W_i$ and $b_i = \lfloor b_{i-1}/2 \rfloor \geq \lfloor w_{i-1}/2 \rfloor = w_i$, thus establishing (ii) and (iii).

For the numbers s, t, f, two cases are possible.

Case 1. Suppose that $s \leq t$ or $f \leq t + 1$ (and hence $s \leq t + 1$); in particular, this is true when $t = \infty$. Then in A_{s-1} we have $B_{s-1} \geq W_{s-1}$, $b_{s-1} \geq w_{s-1} > 1$ as $s - 1 \leq \min\{s, t\}$. Now, if $s = f$, then in A_{s-1}, there is no 1-w-string as well as no (>2)-b-string. That is, $2 = B_{s-1} \geq W_{s-1} \geq b_{s-1} \geq w_{s-1} > 1$, hence all these numbers equal 2. This means that in A_{s-1}, all strings contain 2 pearls, and there are 2^{s-1} black and 2^{s-1} white strings, which means $b = 2 \cdot 2^{s-1} = w$. This contradicts the problem conditions.

Hence we have $s \leq f - 1$ and thus $s \leq t$. Therefore, in the sth step each string is cut into two parts. Now, if a 1-b-string appears in this step, then from $w_{s-1} \leq b_{s-1}$ we see that a

1-w-string appears as well; so, in each case in the sth step a 1-w-string appears, while not all black pearls become single, as desired.

Case 2. Now assume that $t+1 \le s$ and $t+2 \le f$. Then in A_t we have exactly 2^t white and 2^t black strings, all being larger than 1, and $2^{t+1} > k \ge 2^t$ (the latter holds since 2^t is the total number of strings in A_{t-1}). Now, in the $(t+1)$st step, exactly k strings are cut, not more than 2^t of them being black; so the number of w-strings in A_{t+1} is at least $2^t + (k-2^t) = k$. Since the number of w-strings does not decrease in our process, in A_{f-1} we have at least k white strings as well.

Finally, in A_{f-1}, all b-strings are not larger than 2, and at least one 2-b-string is cut in the fth step. Therefore, at most $k-1$ white strings are cut in this step, hence there exists a w-string $\mathcal{W}$ which is not cut in the fth step. On the other hand, since a 2-b-string is cut, all ($\ge$2)-w-strings should also be cut in the fth step; hence $\mathcal{W}$ should be a single pearl. This is exactly what we needed.

Comment. In this solution, we used the condition $b \ne w$ only to avoid the case $b = w = 2^t$. Hence, if a number $b = w$ is not a power of 2, then the problem statement is also valid.

Solution 2. We use the same notations as introduced in the first paragraph of the previous solution. We claim that at every stage, there exist a u-b-string and a v-w-string such that either

(i) $u > v \ge 1$, or

(ii) $2 \le u \le v < 2u$, and there also exist $k-1$ of ($>v/2$)-strings other than considered above.

First, we notice that this statement implies the problem statement. Actually, in both cases (i) and (ii) we have $u > 1$, so at each stage there exists a ($\ge$2)-b-string, and for the last stage it is exactly what we need.

Now, we prove the claim by induction on the number of the stage. Obviously, for A_0 the condition (i) holds since $b > w$. Further, we suppose that the statement holds for A_i, and prove it for A_{i+1}. Two cases are possible.

Case 1. Assume that in A_i, there are a u-b-string and a v-w-string with $u > v$. We can assume that v is the length of the shortest w-string in A_i; since we are not at the final stage, we have $v \ge 2$. Now, in the $(i+1)$st step, two subcases may occur.

Subcase 1a. Suppose that either no u-b-string is cut, or both some u-b-string and some v-w-string are cut. Then in A_{i+1}, we have either a u-b-string and a ($\le v$)-w-string (and (i) is valid), or we have a $\lceil u/2 \rceil$-b-string and a $\lfloor v/2 \rfloor$-w-string. In the latter case, from $u > v$ we get $\lceil u/2 \rceil > \lfloor v/2 \rfloor$, and (i) is valid again.

Subcase 1b. Now, some u-b-string is cut, and no v-w-string is cut (and hence all the strings which are cut are longer than v). If $u' = \lceil u/2 \rceil > v$, then the condition (i) is satisfied since we have a u'-b-string and a v-w-string in A_{i+1}. Otherwise, notice that the inequality $u > v \ge 2$ implies $u' \ge 2$. Furthermore, besides a fixed u-b-string, other $k-1$ of ($\ge v+1$)-strings should be cut in the $(i+1)$st step, hence providing at least $k-1$ of ($\ge \lceil (v+1)/2 \rceil$)-strings, and $\lceil (v+1)/2 \rceil > v/2$. So, we can put $v' = v$, and we have $u' \le v < u \le 2u'$, so the condition (ii) holds for A_{i+1}.

Case 2. Conversely, assume that in A_i there exist a u-b-string, a v-w-string ($2 \le u \le v < 2u$) and a set S of $k-1$ other strings larger than $v/2$ (and hence larger than 1). In the $(i+1)$st step, three subcases may occur.

Subcase 2a. Suppose that some u-b-string is not cut, and some v-w-string is cut. The latter one results in a $\lfloor v/2 \rfloor$-w-string, we have $v' = \lfloor v/2 \rfloor < u$, and the condition (i) is valid.

Subcase 2b. Next, suppose that no v-w-string is cut (and therefore no u-b-string is cut as $u \leq v$). Then all k strings which are cut have the length $> v$, so each one results in a $(>v/2)$-string. Hence in A_{i+1}, there exist $k \geq k-1$ of $(>v/2)$-strings other than the considered u- and v-strings, and the condition (ii) is satisfied.

Subcase 2c. In the remaining case, all u-b-strings are cut. This means that all $(\geq u)$-strings are cut as well, hence our v-w-string is cut. Therefore in A_{i+1} there exists a $\lceil u/2 \rceil$-b-string together with a $\lfloor v/2 \rfloor$-w-string. Now, if $u' = \lceil u/2 \rceil > \lfloor v/2 \rfloor = v'$ then the condition (i) is fulfilled. Otherwise, we have $u' \leq v' < u \leq 2u'$. In this case, we show that $u' \geq 2$. If, to the contrary, $u' = 1$ (and hence $u = 2$), then all black and white (≥ 2)-strings should be cut in the $(i+1)$st step, and among these strings there are at least a u-b-string, a v-w-string, and $k-1$ strings in S ($k+1$ strings altogether). This is impossible.

Hence, we get $2 \leq u' \leq v' < 2u'$. To reach (ii), it remains to check that in A_{i+1}, there exists a set S' of $k-1$ other strings larger than $v'/2$. These will be exactly the strings obtained from the elements of S. Namely, each $s \in S$ was either cut in the $(i+1)$st step, or not. In the former case, let us include into S' the largest of the strings obtained from s; otherwise we include s itself into S'. All $k-1$ strings in S' are greater than $v/2 \geq v'$, as desired.

C7. Let $P_1, \ldots, P_s$ be arithmetic progressions of integers, the following conditions being satisfied:

(i) each integer belongs to at least one of them;

(ii) each progression contains a number which does not belong to other progressions.

Denote by n the least common multiple of steps of these progressions; let $n = p_1^{\alpha_1} \ldots p_k^{\alpha_k}$ be its prime factorization. Prove that

$$s \geq 1 + \sum_{i=1}^{k} \alpha_i (p_i - 1).$$

(Germany)

Solution 1. First, we prove the key lemma, and then we show how to apply it to finish the solution.

Let $n_1, \ldots, n_k$ be positive integers. By an $n_1 \times n_2 \times \cdots \times n_k$ *grid* we mean the set $N = \{(a_1, \ldots, a_k) : a_i \in \mathbb{Z},\ 0 \leq a_i \leq n_i - 1\}$; the elements of N will be referred to as *points*. In this grid, we define *a subgrid* as a subset of the form

$$L = \{(b_1, \ldots, b_k) \in N : b_{i_1} = x_{i_1}, \ldots, b_{i_t} = x_{i_t}\}, \tag{1}$$

where $I = \{i_1, \ldots, i_t\}$ is an arbitrary nonempty set of indices, and $x_{i_j} \in [0, n_{i_j} - 1]$ $(1 \leq j \leq t)$ are fixed integer numbers. Further, we say that a subgrid (1) is *orthogonal* to the ith coordinate axis if $i \in I$, and that it is *parallel* to the ith coordinate axis otherwise.

Lemma. Assume that the grid N is covered by subgrids $L_1, L_2, \ldots, L_s$ (this means $N = \bigcup_{i=1}^{s} L_i$) so that

(ii′) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Then

$$s \geq 1 + \sum_{i=1}^{k} (n_i - 1).$$

Proof. Assume to the contrary that $s \leq \sum_i (n_i - 1) = s'$. Our aim is to find a point that is not covered by $L_1, \ldots, L_s$.

The idea of the proof is the following. Imagine that we expand each subgrid to some maximal subgrid so that for the ith axis, there will be at most $n_i - 1$ maximal subgrids orthogonal to this axis. Then the desired point can be found easily: its ith coordinate should be that not covered by the maximal subgrids orthogonal to the ith axis. Surely, the conditions for existence of such expansion are provided by Hall's lemma on matchings. So, we will follow this direction, although we will apply Hall's lemma to some subgraph instead of the whole graph.

Construct a bipartite graph $G = (V \cup V', E)$ as follows. Let $V = \{L_1, \ldots, L_s\}$, and let $V' = \{v_{ij} : 1 \leq i \leq s,\ 1 \leq j \leq n_i - 1\}$ be some set of s' elements. Further, let the edge (L_m, v_{ij}) appear iff L_m is orthogonal to the ith axis.

For each subset $W \subset V$, denote

$$f(W) = \{v \in V' : (L, v) \in E \text{ for some } L \in W\}.$$

Notice that $f(V) = V'$ by (iii).

Now, consider the set $W \subset V$ containing the maximal number of elements such that $|W| > |f(W)|$; if there is no such set then we set $W = \varnothing$. Denote $W' = f(W)$, $U = V \backslash W$, $U' = V' \backslash W'$.

By our assumption and the Lemma condition, $|f(V)| = |V'| \geq |V|$, hence $W \neq V$ and $U \neq \varnothing$. Permuting the coordinates, we can assume that $U' = \{v_{ij} : 1 \leq i \leq \ell\}$, $W' = \{v_{ij} : \ell{+}1 \leq i \leq k\}$.

Consider the induced subgraph G' of G on the vertices $U \cup U'$. We claim that for every $X \subset U$, we get $|f(X) \cap U'| \geq |X|$ (so G' satisfies the conditions of Hall's lemma). Actually, we have $|W| \geq |f(W)|$, so if $|X| > |f(X) \cap U'|$ for some $X \subset U$, then we have

$$|W \cup X| = |W| + |X| > |f(W)| + |f(X) \cap U'| = |f(W) \cup (f(X) \cap U')| = |f(W \cup X)|.$$

This contradicts the maximality of $|W|$.

Thus, applying Hall's lemma, we can assign to each $L \in U$ some vertex $v_{ij} \in U'$ so that to distinct elements of U, distinct vertices of U' are assigned. In this situation, we say that $L \in U$ corresponds to the ith axis, and write $g(L) = i$. Since there are $n_i - 1$ vertices of the form v_{ij}, we get that for each $1 \leq i \leq \ell$, not more than $n_i - 1$ subgrids correspond to the ith axis.

Finally, we are ready to present the desired point. Since $W \neq V$, there exists a point $b = (b_1, b_2, \ldots, b_k) \in N \backslash (\cup_{L \in W} L)$. On the other hand, for every $1 \leq i \leq \ell$, consider any subgrid $L \in U$ with $g(L) = i$. This means exactly that L is orthogonal to the ith axis, and hence all its elements have the same ith coordinate c_L. Since there are at most $n_i - 1$ such subgrids, there exists a number $0 \leq a_i \leq n_i - 1$ which is not contained in a set $\{c_L : g(L) = i\}$. Choose such number for every $1 \leq i \leq \ell$. Now we claim that point $a = (a_1, \ldots, a_\ell, b_{\ell+1}, \ldots, b_k)$ is not covered, hence contradicting the Lemma condition.

Surely, point a cannot lie in some $L \in U$, since all the points in L have $g(L)$th coordinate $c_L \neq a_{g(L)}$. On the other hand, suppose that $a \in L$ for some $L \in W$; recall that $b \notin L$. But the points a and b differ only at first ℓ coordinates, so L should be orthogonal to at least one of the first ℓ axes, and hence our graph contains some edge (L, v_{ij}) for $i \leq \ell$. It contradicts the definition of W'. The Lemma is proved. $\square$

Now we turn to the problem. Let d_j be the step of the progression P_j. Note that since $n = \text{l.c.m.}(d_1, \ldots, d_s)$, for each $1 \leq i \leq k$ there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$. We assume that $n > 1$; otherwise the problem statement is trivial.

For each $0 \leq m \leq n-1$ and $1 \leq i \leq k$, let m_i be the residue of m modulo $p_i^{\alpha_i}$, and let $m_i = \overline{r_{i\alpha_i} \ldots r_{i1}}$ be the base p_i representation of m_i (possibly, with some leading zeroes). Now, we put into correspondence to m the sequence $r(m) = (r_{11}, \ldots, r_{1\alpha_1}, r_{21}, \ldots, r_{k\alpha_k})$. Hence $r(m)$ lies in a $\underbrace{p_1 \times \cdots \times p_1}_{\alpha_1 \text{ times}} \times \cdots \times \underbrace{p_k \times \cdots \times p_k}_{\alpha_k \text{ times}}$ grid N.

Surely, if $r(m) = r(m')$ then $p_i^{\alpha_i} \mid m_i - m_i'$, which follows $p_i^{\alpha_i} \mid m - m'$ for all $1 \leq i \leq k$; consequently, $n \mid m - m'$. So, when m runs over the set $\{0, \ldots, n-1\}$, the sequences $r(m)$ do not repeat; since $|N| = n$, this means that r is a bijection between $\{0, \ldots, n-1\}$ and N. Now we will show that for each $1 \leq i \leq s$, the set $L_i = \{r(m) : m \in P_i\}$ is a subgrid, and that for each axis there exists a subgrid orthogonal to this axis. Obviously, these subgrids cover N, and the condition (ii′) follows directly from (ii). Hence the Lemma provides exactly the estimate we need.

Consider some $1 \leq j \leq s$ and let $d_j = p_1^{\gamma_1} \ldots p_k^{\gamma_k}$. Consider some $q \in P_j$ and let $r(q) = (r_{11}, \ldots, r_{k\alpha_k})$. Then for an arbitrary q', setting $r(q') = (r'_{11}, \ldots, r'_{k\alpha_k})$ we have

$$q' \in P_j \quad \iff \quad p_i^{\gamma_i} \mid q - q' \text{ for each } 1 \leq i \leq k \quad \iff \quad r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i.$$

Hence $L_j = \{(r'_{11}, \ldots, r'_{k\alpha_k}) \in N : r_{i,t} = r'_{i,t} \text{ for all } t \leq \gamma_i\}$ which means that L_j is a subgrid containing $r(q)$. Moreover, in $L_{j(i)}$, all the coordinates corresponding to p_i are fixed, so it is orthogonal to all of their axes, as desired.

Comment 1. The estimate in the problem is sharp for every n. One of the possible examples is the following one. For each $1 \le i \le k$, $0 \le j \le \alpha_i - 1$, $1 \le k \le p-1$, let

$$P_{i,j,k} = kp_i^j + p_i^{j+1}\mathbb{Z},$$

and add the progression $P_0 = n\mathbb{Z}$. One can easily check that this set satisfies all the problem conditions. There also exist other examples.

On the other hand, the estimate can be adjusted in the following sense. For every $1 \le i \le k$, let $0 = \alpha_{i0}, \alpha_{i1}, \dots, \alpha_{ih_i}$ be all the numbers of the form $\mathrm{ord}_{p_i}(d_j)$ in an increasing order (we delete the repeating occurences of a number, and add a number $0 = \alpha_{i0}$ if it does not occur). Then, repeating the arguments from the solution one can obtain that

$$s \ge 1 + \sum_{i=1}^{k}\sum_{j=1}^{h_i}(p^{\alpha_j - \alpha_{j-1}} - 1).$$

Note that $p^\alpha - 1 \ge \alpha(p-1)$, and the equality is achieved only for $\alpha = 1$. Hence, for reaching the minimal number of the progressions, one should have $\alpha_{i,j} = j$ for all i, j. In other words, for each $1 \le j \le \alpha_i$, there should be an index t such that $\mathrm{ord}_{p_i}(d_t) = j$.

Solution 2. We start with introducing some notation. For positive integer r, we denote $[r] = \{1, 2, \dots, r\}$. Next, we say that a set of progressions $\mathcal{P} = \{P_1, \dots, P_s\}$ *cover* $\mathbb{Z}$ if each integer belongs to some of them; we say that this covering is *minimal* if no proper subset of $\mathcal{P}$ covers $\mathbb{Z}$. Obviously, each covering contains a minimal subcovering.

Next, for a minimal covering $\{P_1, \dots, P_s\}$ and for every $1 \le i \le s$, let d_i be the step of progression P_i, and h_i be some number which is contained in P_i but in none of the other progressions. We assume that $n > 1$, otherwise the problem is trivial. This implies $d_i > 1$, otherwise the progression P_i covers all the numbers, and $n = 1$.

We will prove a more general statement, namely the following

Claim. Assume that the progressions $P_1, \dots, P_s$ and number $n = p_1^{\alpha_1} \dots p_k^{\alpha_k} > 1$ are chosen as in the problem statement. Moreover, choose some nonempty set of indices $I = \{i_1, \dots, i_t\} \subseteq [k]$ and some positive integer $\beta_i \le \alpha_i$ for every $i \in I$. Consider the set of indices

$$T = \left\{ j : 1 \le j \le s, \text{and } p_i^{\alpha_i - \beta_i + 1} \mid d_j \text{ for some } i \in I \right\}.$$

Then

$$|T| \ge 1 + \sum_{i \in I} \beta_i(p_i - 1). \tag{2}$$

Observe that the Claim for $I = [k]$ and $\beta_i = \alpha_i$ implies the problem statement, since the left-hand side in (2) is not greater than s. Hence, it suffices to prove the Claim.

1. First, we prove the Claim assuming that all d_j's are prime numbers. If for some $1 \le i \le k$ we have at least p_i progressions with the step p_i, then they do not intersect and hence cover all the integers; it means that there are no other progressions, and $n = p_i$; the Claim is trivial in this case.

Now assume that for every $1 \le i \le k$, there are not more than $p_i - 1$ progressions with step p_i; each such progression covers the numbers with a fixed residue modulo p_i, therefore there exists a residue $q_i \mod p_i$ which is not touched by these progressions. By the Chinese Remainder Theorem, there exists a number q such that $q \equiv q_i \pmod{p_i}$ for all $1 \le i \le k$; this number cannot be covered by any progression with step p_i, hence it is not covered at all. A contradiction.

2. Now, we assume that the general Claim is not valid, and hence we consider a counterexample $\{P_1, \dots, P_s\}$ for the Claim; we can choose it to be minimal in the following sense:

• the number n is minimal possible among all the counterexamples;

• the sum $\sum_i d_i$ is minimal possible among all the counterexamples having the chosen value of n.

As was mentioned above, not all numbers d_i are primes; hence we can assume that d_1 is composite, say $p_1 \mid d_1$ and $d_1' = \frac{d_1}{p_1} > 1$. Consider a progression P_1' having the step d_1', and containing P_1. We will focus on two coverings constructed as follows.

(i) Surely, the progressions $P_1', P_2, \dots, P_s$ cover $\mathbb{Z}$, though this covering in not necessarily minimal. So, choose some minimal subcovering $\mathcal{P}'$ in it; surely $P_1' \in \mathcal{P}'$ since h_1 is not covered by $P_2, \dots, P_s$, so we may assume that $\mathcal{P}' = \{P_1', P_2, \dots, P_{s'}\}$ for some $s' \le s$. Furthermore, the period of the covering $\mathcal{P}'$ can appear to be less than n; so we denote this period by

$$n' = p_1^{\alpha_1-\sigma_1} \dots p_k^{\alpha_k-\sigma_k} = \text{l.c.m.}\big(d_1', d_2, \dots, d_{s'}\big).$$

Observe that for each $P_j \notin \mathcal{P}'$, we have $h_j \in P_1'$, otherwise h_j would not be covered by $\mathcal{P}$.

(ii) On the other hand, each nonempty set of the form $R_i = P_i \cap P_1'$ $(1 \le i \le s)$ is also a progression with a step $r_i = \text{l.c.m.}(d_i, d_1')$, and such sets cover P_1'. Scaling these progressions with the ratio $1/d_1'$, we obtain the progressions Q_i with steps $q_i = r_i/d_1'$ which cover $\mathbb{Z}$. Now we choose a minimal subcovering $\mathcal{Q}$ of this covering; again we should have $Q_1 \in \mathcal{Q}$ by the reasons of h_1. Now, denote the period of $\mathcal{Q}$ by

$$n'' = \text{l.c.m.}\{q_i : Q_i \in \mathcal{Q}\} = \frac{\text{l.c.m.}\{r_i : Q_i \in \mathcal{Q}\}}{d_1'} = \frac{p_1^{\gamma_1} \dots p_k^{\gamma_k}}{d_1'}.$$

Note that if $h_j \in P_1'$, then the image of h_j under the scaling can be covered by Q_j only; so, in this case we have $Q_j \in \mathcal{Q}$.

Our aim is to find the desired number of progressions in coverings $\mathcal{P}$ and $\mathcal{Q}$. First, we have $n \ge n'$, and the sum of the steps in $\mathcal{P}'$ is less than that in $\mathcal{P}$; hence the Claim is valid for $\mathcal{P}'$. We apply it to the set of indices $I' = \{i \in I : \beta_i > \sigma_i\}$ and the exponents $\beta_i' = \beta_i - \sigma_i$; hence the set under consideration is

$$T' = \Big\{j : 1 \le j \le s', \text{and } p_i^{(\alpha_i-\sigma_i)-\beta_i'+1} = p_i^{\alpha_i-\beta_i+1} \mid d_j \text{ for some } i \in I'\Big\} \subseteq T \cap [s'],$$

and we obtain that

$$|T \cap [s']| \ge |T'| \ge 1 + \sum_{i\in I'} (\beta_i - \sigma_i)(p_i - 1) = 1 + \sum_{i \in I} (\beta_i - \sigma_i)_+ (p_i - 1),$$

where $(x)_+ = \max\{x, 0\}$; the latter equality holds as for $i \notin I'$ we have $\beta_i \le \sigma_i$.

Observe that $x = (x-y)_+ + \min\{x, y\}$ for all x, y. So, if we find at least

$$G = \sum_{i\in I} \min\{\beta_i, \sigma_i\}(p_i - 1)$$

indices in $T \cap \{s'+1, \dots, s\}$, then we would have

$$|T| = |T \cap [s']| + |T \cap \{s'+1, \dots, s\}| \ge 1 + \sum_{i\in I} \big((\beta_i - \sigma_i)_+ + \min\{\beta_i, \sigma_i\}\big)(p_i - 1) = 1 + \sum_{i\in I} \beta_i(p_i - 1),$$

thus leading to a contradiction with the choice of $\mathcal{P}$. We will find those indices among the indices of progressions in $\mathcal{Q}$.

3. Now denote $I'' = \{i \in I : \sigma_i > 0\}$ and consider some $i \in I''$; then $p_i^{\alpha_i} \nmid n'$. On the other hand, there exists an index $j(i)$ such that $p_i^{\alpha_i} \mid d_{j(i)}$; this means that $d_{j(i)} \nmid n'$ and hence $P_{j(i)}$ cannot appear in $\mathcal{P}'$, so $j(i) > s'$. Moreover, we have observed before that in this case $h_{j(i)} \in P_1'$, hence $Q_{j(i)} \in \mathcal{Q}$. This means that $q_{j(i)} \mid n''$, therefore $\gamma_i = \alpha_i$ for each $i \in I''$ (recall here that $q_i = r_i/d_1'$ and hence $d_{j(i)} \mid r_{j(i)} \mid d_1' n''$).

Let $d_1' = p_1^{\tau_1} \dots p_k^{\tau_k}$. Then $n'' = p_1^{\gamma_1-\tau_1} \dots p_k^{\gamma_i-\tau_i}$. Now, if $i \in I''$, then for every β the condition $p_i^{(\gamma_i-\tau_i)-\beta+1} \mid q_j$ is equivalent to $p_i^{\alpha_i-\beta+1} \mid r_j$.

Note that $n'' \le n/d_1' < n$, hence we can apply the Claim to the covering $\mathcal{Q}$. We perform this with the set of indices I'' and the exponents $\beta_i'' = \min\{\beta_i, \sigma_i\} > 0$. So, the set under consideration is

$$\begin{aligned} T'' &= \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{(\gamma_i-\tau_i)-\min\{\beta_i,\sigma_i\}+1} \mid q_j \text{ for some } i \in I'' \right\} \\ &= \left\{ j : Q_j \in \mathcal{Q}, \text{and } p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j \text{ for some } i \in I'' \right\}, \end{aligned}$$

and we obtain $|T''| \ge 1 + G$. Finally, we claim that $T'' \subseteq T \cap \big(\{1\} \cup \{s'+1, \dots, s\}\big)$; then we will obtain $|T \cap \{s'+1, \dots, s\}| \ge G$, which is exactly what we need.

To prove this, consider any $j \in T''$. Observe first that $\alpha_i - \min\{\beta_i, \sigma_i\} + 1 > \alpha_i - \sigma_i \ge \tau_i$, hence from $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid r_j = \text{l.c.m.}(d_1', d_j)$ we have $p_i^{\alpha_i-\min\{\beta_i,\sigma_i\}+1} \mid d_j$, which means that $j \in T$. Next, the exponent of p_i in d_j is greater than that in n', which means that $P_j \notin \mathcal{P}'$. This may appear only if $j = 1$ or $j > s'$, as desired. This completes the proof.

Comment 2. A grid analogue of the Claim is also valid. It reads as following.

Claim. Assume that the grid N is covered by subgrids $L_1, L_2, \dots, L_s$ so that

(ii$'$) each subgrid contains a point which is not covered by other subgrids;

(iii) for each coordinate axis, there exists a subgrid L_i orthogonal to this axis.

Choose some set of indices $I = \{i_1, \dots, i_t\} \subset [k]$, and consider the set of indices

$$T = \{j : 1 \le j \le s, \text{and } L_j \text{ is orthogonal to the } i\text{th axis for some } i \in I\}.$$

Then

$$|T| \ge 1 + \sum_{i \in I} (n_i - 1).$$

This Claim may be proved almost in the same way as in Solution 1.

Geometry

G1. Let ABC be an acute triangle with D, E, F the feet of the altitudes lying on BC, CA, AB respectively. One of the intersection points of the line EF and the circumcircle is P. The lines BP and DF meet at point Q. Prove that $AP = AQ$.

(United Kingdom)

Solution 1. The line EF intersects the circumcircle at two points. Depending on the choice of P, there are two different cases to consider.

Case 1: The point P lies on the ray EF (see Fig. 1).

Let $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The quadrilaterals $BCEF$ and $CAFD$ are cyclic due to the right angles at D, E and F. So,

$$\begin{aligned}\angle BDF &= 180^\circ - \angle FDC = \angle CAF = \alpha,\\ \angle AFE &= 180^\circ - \angle EFB = \angle BCE = \gamma,\\ \angle DFB &= 180^\circ - \angle AFD = \angle DCA = \gamma.\end{aligned}$$

Since P lies on the arc AB of the circumcircle, $\angle PBA < \angle BCA = \gamma$. Hence, we have

$$\angle PBD + \angle BDF = \angle PBA + \angle ABD + \angle BDF < \gamma + \beta + \alpha = 180^\circ,$$

and the point Q must lie on the extensions of BP and DF beyond the points P and F, respectively.

From the cyclic quadrilateral $APBC$ we get

$$\angle QPA = 180^\circ - \angle APB = \angle BCA = \gamma = \angle DFB = \angle QFA.$$

Hence, the quadrilateral $AQPF$ is cyclic. Then $\angle AQP = 180^\circ - \angle PFA = \angle AFE = \gamma$.

We obtained that $\angle AQP = \angle QPA = \gamma$, so the triangle AQP is isosceles, $AP = AQ$.

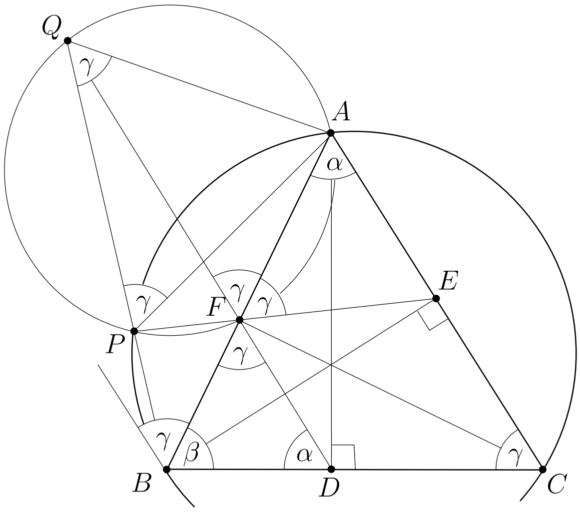


Fig. 1

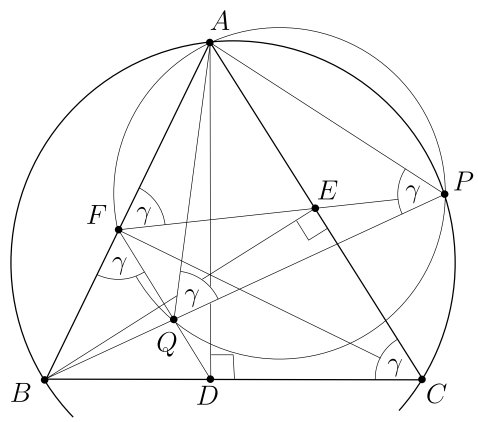


Fig. 2

Case 2: The point P lies on the ray FE (see Fig. 2). In this case the point Q lies inside the segment FD.

Similarly to the first case, we have

$$\angle QPA = \angle BCA = \gamma = \angle DFB = 180^\circ - \angle AFQ.$$

Hence, the quadrilateral $AFQP$ is cyclic.

Then $\angle AQP = \angle AFP = \angle AFE = \gamma = \angle QPA$. The triangle AQP is isosceles again, $\angle AQP = \angle QPA$ and thus $AP = AQ$.

Comment. Using signed angles, the two possible configurations can be handled simultaneously, without investigating the possible locations of P and Q.

Solution 2. For arbitrary points X, Y on the circumcircle, denote by $\widehat{XY}$ the central angle of the arc XY.

Let P and P' be the two points where the line EF meets the circumcircle; let P lie on the arc AB and let P' lie on the arc CA. Let BP and BP' meet the line DF and Q and Q', respectively (see Fig. 3). We will prove that $AP = AP' = AQ = AQ'$.

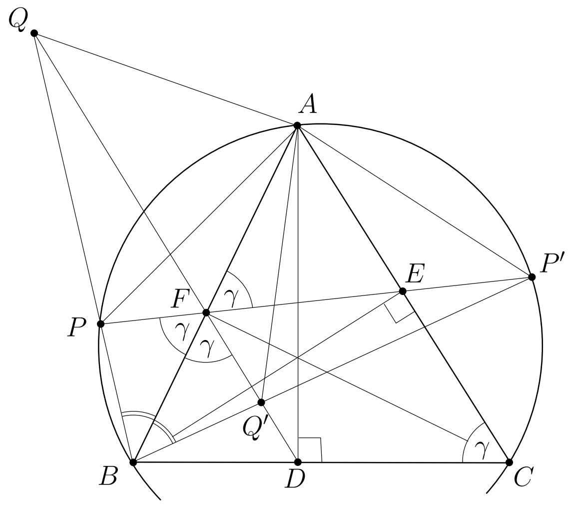


Fig. 3

Like in the first solution, we have $\angle AFE = \angle BFP = \angle DFB = \angle BCA = \gamma$ from the cyclic quadrilaterals $BCEF$ and $CAFD$.

By $\widehat{PB} + \widehat{P'A} = 2\angle AFP' = 2\gamma = 2\angle BCA = \widehat{AP} + \widehat{PB}$, we have

$$\widehat{AP} = \widehat{P'A}, \quad \angle PBA = \angle ABP' \qquad \text{and} \quad AP = AP'. \tag{1}$$

Due to $\widehat{AP} = \widehat{P'A}$, the lines BP and BQ' are symmetrical about line AB.

Similarly, by $\angle BFP = \angle Q'FB$, the lines FP and FQ' are symmetrical about AB. It follows that also the points P and P' are symmetrical to Q' and Q, respectively. Therefore,

$$AP = AQ' \quad \text{and} \quad AP' = AQ. \tag{2}$$

The relations (1) and (2) together prove $AP = AP' = AQ = AQ'$.

G2. Point P lies inside triangle ABC. Lines AP, BP, CP meet the circumcircle of ABC again at points K, L, M, respectively. The tangent to the circumcircle at C meets line AB at S. Prove that $SC = SP$ if and only if $MK = ML$.

(Poland)

Solution 1. We assume that $CA > CB$, so point S lies on the ray AB.

From the similar triangles $\triangle PKM \sim \triangle PCA$ and $\triangle PLM \sim \triangle PCB$ we get $\dfrac{PM}{KM} = \dfrac{PA}{CA}$ and $\dfrac{LM}{PM} = \dfrac{CB}{PB}$. Multiplying these two equalities, we get

$$\frac{LM}{KM} = \frac{CB}{CA} \cdot \frac{PA}{PB}.$$

Hence, the relation $MK = ML$ is equivalent to $\dfrac{CB}{CA} = \dfrac{PB}{PA}$.

Denote by E the foot of the bisector of angle B in triangle ABC. Recall that the locus of points X for which $\dfrac{XA}{XB} = \dfrac{CA}{CB}$ is the Apollonius circle Ω with the center Q on the line AB, and this circle passes through C and E. Hence, we have $MK = ML$ if and only if P lies on Ω, that is $QP = QC$.

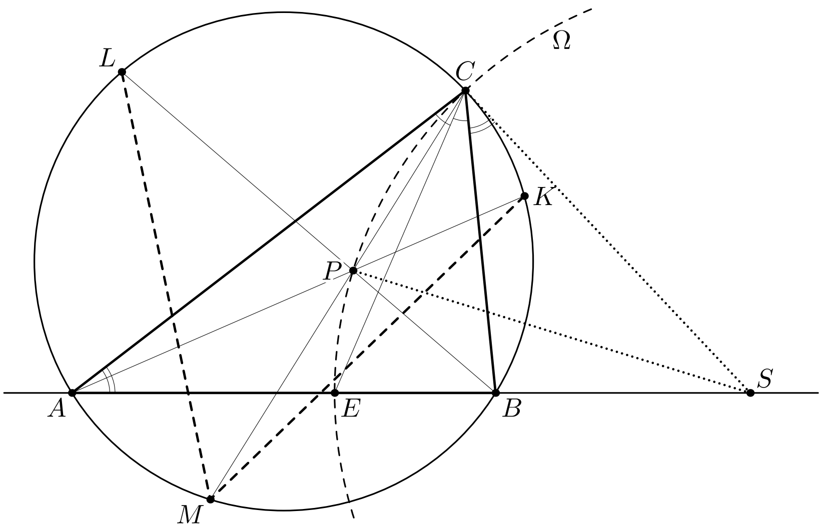


Fig. 1

Now we prove that $S = Q$, thus establishing the problem statement. We have $\angle CES = \angle CAE + \angle ACE = \angle BCS + \angle ECB = \angle ECS$, so $SC = SE$. Hence, the point S lies on AB as well as on the perpendicular bisector of CE and therefore coincides with Q.

Solution 2. As in the previous solution, we assume that S lies on the ray AB.

1. Let P be an arbitrary point inside both the circumcircle ω of the triangle ABC and the angle ASC, the points K, L, M defined as in the problem. We claim that $SP = SC$ implies $MK = ML$.

Let E and F be the points of intersection of the line SP with ω, point E lying on the segment SP (see Fig. 2).

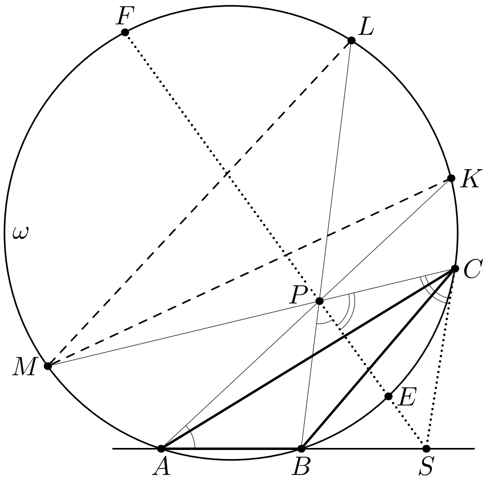


Fig. 2

We have $SP^2 = SC^2 = SA \cdot SB$, so $\dfrac{SP}{SB} = \dfrac{SA}{SP}$, and hence $\triangle PSA \sim \triangle BSP$. Then $\angle BPS = \angle SAP$. Since $2\angle BPS = \wideparen{BE} + \wideparen{LF}$ and $2\angle SAP = \wideparen{BE} + \wideparen{EK}$ we have

$$\wideparen{LF} = \wideparen{EK}. \tag{1}$$

On the other hand, from $\angle SPC = \angle SCP$ we have $\wideparen{EC} + \wideparen{MF} = \wideparen{EC} + \wideparen{EM}$, or

$$\wideparen{MF} = \wideparen{EM}. \tag{2}$$

From (1) and (2) we get $\wideparen{MFL} = \wideparen{MF} + \wideparen{FL} = \wideparen{ME} + \wideparen{EK} = \wideparen{MEK}$ and hence $MK = ML$. The claim is proved.

2. We are left to prove the converse. So, assume that $MK = ML$, and introduce the points E and F as above. We have $SC^2 = SE \cdot SF$; hence, there exists a point P' lying on the segment EF such that $SP' = SC$ (see Fig. 3).

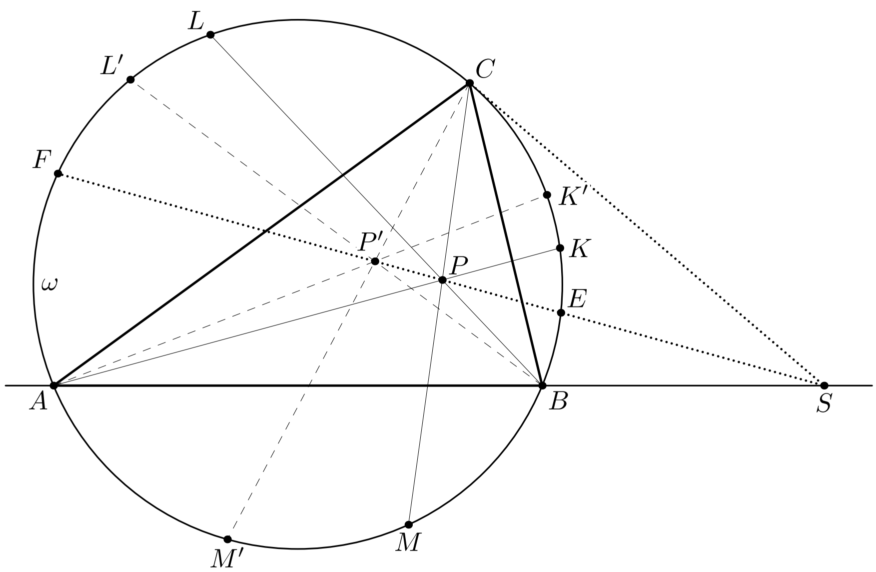


Fig. 3

Assume that $P \neq P'$. Let the lines AP', BP', CP' meet ω again at points K', L', M' respectively. Now, if P' lies on the segment PF then by the first part of the solution we have $\widehat{M'FL'} = \widehat{M'EK'}$. On the other hand, we have $\widehat{MFL} > \widehat{M'FL'} = \widehat{M'EK'} > \widehat{MEK}$, therefore $\widehat{MFL} > \widehat{MEK}$ which contradicts $MK = ML$.

Similarly, if point P' lies on the segment EP then we get $\widehat{MFL} < \widehat{MEK}$ which is impossible. Therefore, the points P and P' coincide and hence $SP = SP' = SC$.

Solution 3. We present a different proof of the converse direction, that is, $MK = ML \Rightarrow SP = SC$. As in the previous solutions we assume that $CA > CB$, and the line SP meets ω at E and F.

From $ML = MK$ we get $\widehat{MEK} = \widehat{MFL}$. Now we claim that $\widehat{ME} = \widehat{MF}$ and $\widehat{EK} = \widehat{FL}$.

To the contrary, suppose first that $\widehat{ME} > \widehat{MF}$; then $\widehat{EK} = \widehat{MEK} - \widehat{ME} < \widehat{MFL} - \widehat{MF} = \widehat{FL}$. Now, the inequality $\widehat{ME} > \widehat{MF}$ implies $2\angle SCM = \widehat{EC} + \widehat{ME} > \widehat{EC} + \widehat{MF} = 2\angle SPC$ and hence $SP > SC$. On the other hand, the inequality $\widehat{EK} < \widehat{FL}$ implies $2\angle SPK = \widehat{EK} + \widehat{AF} < \widehat{FL} + \widehat{AF} = 2\angle ABL$, hence

$$\angle SPA = 180^\circ - \angle SPK > 180^\circ - \angle ABL = \angle SBP.$$

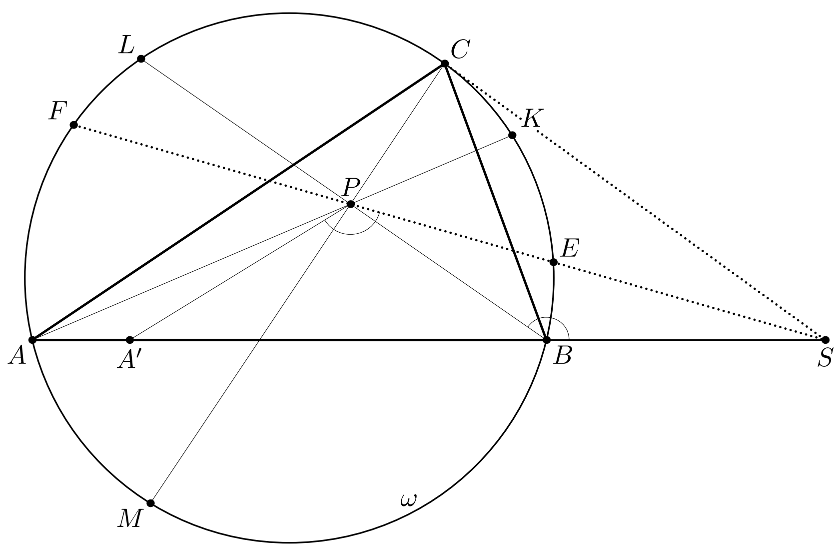


Fig. 4

Consider the point A' on the ray SA for which $\angle SPA' = \angle SBP$; in our case, this point lies on the segment SA (see Fig. 4). Then $\triangle SBP \sim \triangle SPA'$ and $SP^2 = SB \cdot SA' < SB \cdot SA = SC^2$. Therefore, $SP < SC$ which contradicts $SP > SC$.

Similarly, one can prove that the inequality $\widehat{ME} < \widehat{MF}$ is also impossible. So, we get $\widehat{ME} = \widehat{MF}$ and therefore $2\angle SCM = \widehat{EC} + \widehat{ME} = \widehat{EC} + \widehat{MF} = 2\angle SPC$, which implies $SC = SP$.

G3. Let $A_1A_2\dots A_n$ be a convex polygon. Point P inside this polygon is chosen so that its projections $P_1, \dots, P_n$ onto lines $A_1A_2, \dots, A_nA_1$ respectively lie on the sides of the polygon. Prove that for arbitrary points $X_1, \dots, X_n$ on sides $A_1A_2, \dots, A_nA_1$ respectively,

$$\max\left\{\frac{X_1X_2}{P_1P_2}, \dots, \frac{X_nX_1}{P_nP_1}\right\} \geq 1.$$

(Armenia)

Solution 1. Denote $P_{n+1} = P_1$, $X_{n+1} = X_1$, $A_{n+1} = A_1$.

Lemma. Let point Q lies inside $A_1A_2\dots A_n$. Then it is contained in at least one of the circumcircles of triangles $X_1A_2X_2, \dots, X_nA_1X_1$.

Proof. If Q lies in one of the triangles $X_1A_2X_2, \dots, X_nA_1X_1$, the claim is obvious. Otherwise Q lies inside the polygon $X_1X_2\dots X_n$ (see Fig. 1). Then we have

$$\begin{aligned}(\angle X_1A_2X_2 + \angle X_1QX_2) + \dots + (\angle X_nA_1X_1 + \angle X_nQX_1)\\ = (\angle X_1A_1X_2 + \dots + \angle X_nA_1X_1) + (\angle X_1QX_2 + \dots + \angle X_nQX_1) = (n-2)\pi + 2\pi = n\pi,\end{aligned}$$

hence there exists an index i such that $\angle X_iA_{i+1}X_{i+1} + \angle X_iQX_{i+1} \geq \frac{\pi n}{n} = \pi$. Since the quadrilateral $QX_iA_{i+1}X_{i+1}$ is convex, this means exactly that Q is contained the circumcircle of $\triangle X_iA_{i+1}X_{i+1}$, as desired. $\square$

Now we turn to the solution. Applying lemma, we get that P lies inside the circumcircle of triangle $X_iA_{i+1}X_{i+1}$ for some i. Consider the circumcircles ω and Ω of triangles $P_iA_{i+1}P_{i+1}$ and $X_iA_{i+1}X_{i+1}$ respectively (see Fig. 2); let r and R be their radii. Then we get $2r = A_{i+1}P \leq 2R$ (since P lies inside Ω), hence

$$P_iP_{i+1} = 2r\sin\angle P_iA_{i+1}P_{i+1} \leq 2R\sin\angle X_iA_{i+1}X_{i+1} = X_iX_{i+1},$$

QED.

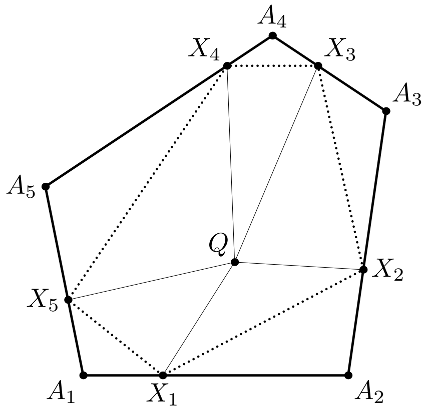


Fig. 1

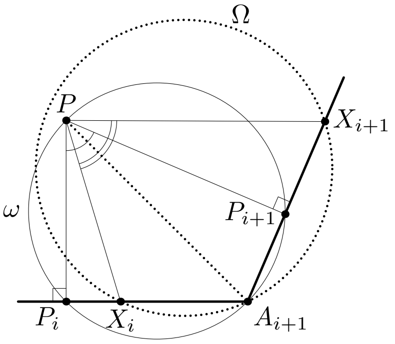


Fig. 2

Solution 2. As in Solution 1, we assume that all indices of points are considered modulo n.

We will prove a bit stronger inequality, namely

$$\max\left\{\frac{X_1X_2}{P_1P_2}\cos\alpha_1, \dots, \frac{X_nX_1}{P_nP_1}\cos\alpha_n\right\} \geq 1,$$

where α_i $(1 \leq i \leq n)$ is the angle between lines X_iX_{i+1} and P_iP_{i+1}. We denote $\beta_i = \angle A_iP_iP_{i-1}$ and $\gamma_i = \angle A_{i+1}P_iP_{i+1}$ for all $1 \leq i \leq n$.

Suppose that for some $1 \leq i \leq n$, point X_i lies on the segment A_iP_i, while point X_{i+1} lies on the segment $P_{i+1}A_{i+2}$. Then the projection of the segment X_iX_{i+1} onto the line P_iP_{i+1} contains segment P_iP_{i+1}, since γ_i and β_{i+1} are acute angles (see Fig. 3). Therefore, $X_iX_{i+1}\cos\alpha_i \geq P_iP_{i+1}$, and in this case the statement is proved.

So, the only case left is when point X_i lies on segment P_iA_{i+1} for all $1 \leq i \leq n$ (the case when each X_i lies on segment A_iP_i is completely analogous).

Now, assume to the contrary that the inequality

$$X_iX_{i+1}\cos\alpha_i < P_iP_{i+1} \tag{1}$$

holds for every $1 \leq i \leq n$. Let Y_i and Y'_{i+1} be the projections of X_i and X_{i+1} onto P_iP_{i+1}. Then inequality (1) means exactly that $Y_iY'_{i+1} < P_iP_{i+1}$, or $P_iY_i > P_{i+1}Y'_{i+1}$ (again since γ_i and β_{i+1} are acute; see Fig. 4). Hence, we have

$$X_iP_i\cos\gamma_i > X_{i+1}P_{i+1}\cos\beta_{i+1}, \qquad 1 \leq i \leq n.$$

Multiplying these inequalities, we get

$$\cos\gamma_1\cos\gamma_2\cdots\cos\gamma_n > \cos\beta_1\cos\beta_2\cdots\cos\beta_n. \tag{2}$$

On the other hand, the sines theorem applied to triangle PP_iP_{i+1} provides

$$\frac{PP_i}{PP_{i+1}} = \frac{\sin\left(\frac{\pi}{2}-\beta_{i+1}\right)}{\sin\left(\frac{\pi}{2}-\gamma_i\right)} = \frac{\cos\beta_{i+1}}{\cos\gamma_i}.$$

Multiplying these equalities we get

$$1 = \frac{\cos\beta_2}{\cos\gamma_1}\cdot\frac{\cos\beta_3}{\cos\gamma_2}\cdots\frac{\cos\beta_1}{\cos\gamma_n}$$

which contradicts (2).

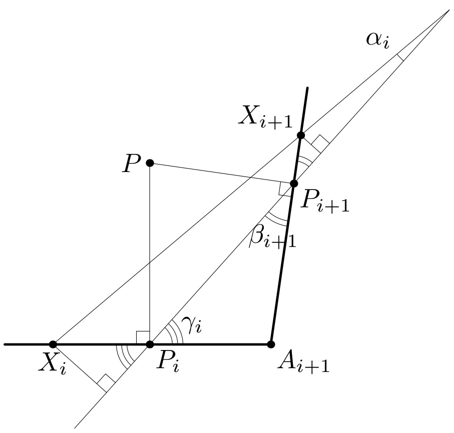


Fig. 3

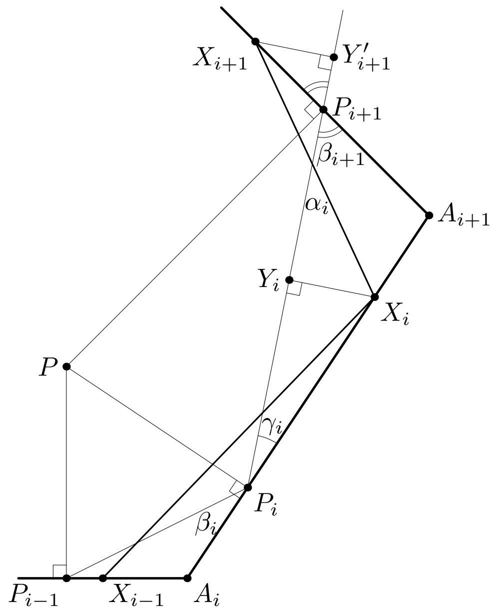


Fig. 4

G4. Let I be the incenter of a triangle ABC and Γ be its circumcircle. Let the line AI intersect Γ at a point $D \neq A$. Let F and E be points on side BC and arc BDC respectively such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

(Hong Kong)

Solution 1. Let X be the second point of intersection of line EI with Γ, and L be the foot of the bisector of angle BAC. Let G' and T be the points of intersection of segment DX with lines IF and AF, respectively. We are to prove that $G = G'$, or $IG' = G'F$. By the Menelaus theorem applied to triangle AIF and line DX, it means that we need the relation

$$1 = \frac{G'F}{IG'} = \frac{TF}{AT} \cdot \frac{AD}{ID}, \qquad \text{or} \quad \frac{TF}{AT} = \frac{ID}{AD}.$$

Let the line AF intersect Γ at point $K \neq A$ (see Fig. 1); since $\angle BAK = \angle CAE$ we have $\widehat{BK} = \widehat{CE}$, hence $KE \parallel BC$. Notice that $\angle IAT = \angle DAK = \angle EAD = \angle EXD = \angle IXT$, so the points I, A, X, T are concyclic. Hence we have $\angle ITA = \angle IXA = \angle EXA = \angle EKA$, so $IT \parallel KE \parallel BC$. Therefore we obtain $\dfrac{TF}{AT} = \dfrac{IL}{AI}$.

Since CI is the bisector of $\angle ACL$, we get $\dfrac{IL}{AI} = \dfrac{CL}{AC}$. Furthermore, $\angle DCL = \angle DCB = \angle DAB = \angle CAD = \frac{1}{2}\angle BAC$, hence the triangles DCL and DAC are similar; therefore we get $\dfrac{CL}{AC} = \dfrac{DC}{AD}$. Finally, it is known that the midpoint D of arc BC is equidistant from points I, B, C, hence $\dfrac{DC}{AD} = \dfrac{ID}{AD}$.

Summarizing all these equalities, we get

$$\frac{TF}{AT} = \frac{IL}{AI} = \frac{CL}{AC} = \frac{DC}{AD} = \frac{ID}{AD},$$

as desired.

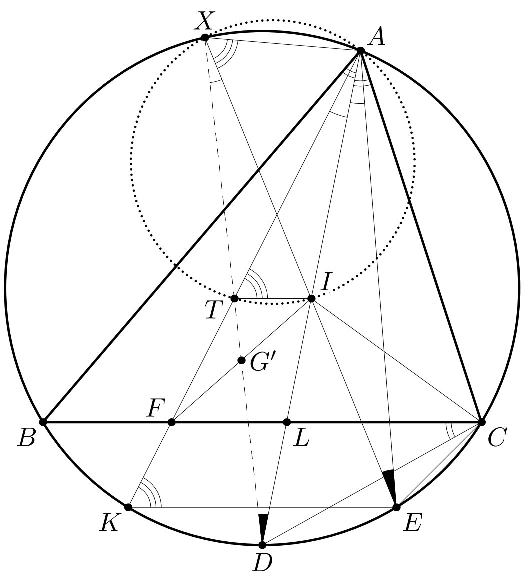


Fig. 1

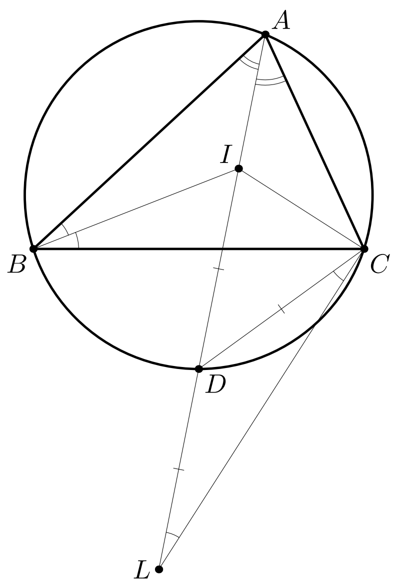


Fig. 2

Comment. The equality $\dfrac{AI}{IL} = \dfrac{AD}{DI}$ is known and can be obtained in many different ways. For instance, one can consider the inversion with center D and radius $DC = DI$. This inversion takes $\widehat{BAC}$ to the segment BC, so point A goes to L. Hence $\dfrac{IL}{DI} = \dfrac{AI}{AD}$, which is the desired equality.

Solution 2. As in the previous solution, we introduce the points X, T and K and note that it suffice to prove the equality

$$\frac{TF}{AT} = \frac{DI}{AD} \quad \Longleftrightarrow \quad \frac{TF + AT}{AT} = \frac{DI + AD}{AD} \quad \Longleftrightarrow \quad \frac{AT}{AD} = \frac{AF}{DI + AD}.$$

Since $\angle FAD = \angle EAI$ and $\angle TDA = \angle XDA = \angle XEA = \angle IEA$, we get that the triangles ATD and AIE are similar, therefore $\dfrac{AT}{AD} = \dfrac{AI}{AE}$.

Next, we also use the relation $DB = DC = DI$. Let J be the point on the extension of segment AD over point D such that $DJ = DI = DC$ (see Fig. 2). Then $\angle DJC = \angle JCD = \frac{1}{2}(\pi - \angle JDC) = \frac{1}{2}\angle ADC = \frac{1}{2}\angle ABC = \angle ABI$. Moreover, $\angle BAI = \angle JAC$, hence triangles ABI and AJC are similar, so $\dfrac{AB}{AJ} = \dfrac{AI}{AC}$, or $AB \cdot AC = AJ \cdot AI = (DI + AD) \cdot AI$.

On the other hand, we get $\angle ABF = \angle ABC = \angle AEC$ and $\angle BAF = \angle CAE$, so triangles ABF and AEC are also similar, which implies $\dfrac{AF}{AC} = \dfrac{AB}{AE}$, or $AB \cdot AC = AF \cdot AE$.

Summarizing we get

$$(DI + AD) \cdot AI = AB \cdot AC = AF \cdot AE \quad \Rightarrow \quad \frac{AI}{AE} = \frac{AF}{AD + DI} \quad \Rightarrow \quad \frac{AT}{AD} = \frac{AF}{AD + DI},$$

as desired.

Comment. In fact, point J is an excenter of triangle ABC.

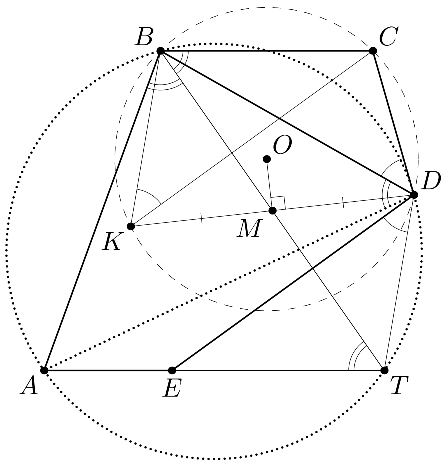
G5. Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90^\circ$, prove that $2\angle BDA = \angle CDE$.

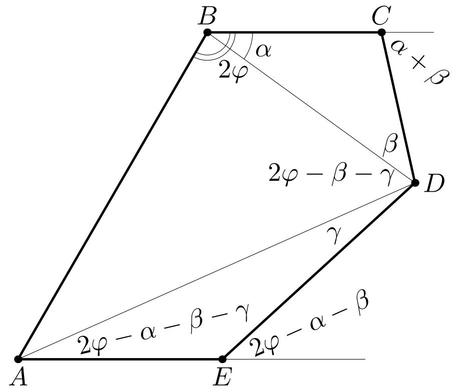
(Ukraine)

Solution 1. Choose point T on ray AE such that $AT = AB$; then from $AE \parallel BC$ we have $\angle CBT = \angle ATB = \angle ABT$, so BT is the bisector of $\angle ABC$. On the other hand, we have $ET = AT - AE = AB - AE = BC$, hence quadrilateral $BCTE$ is a parallelogram, and the midpoint M of its diagonal CE is also the midpoint of the other diagonal BT.

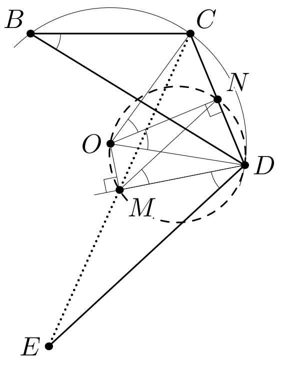
Next, let point K be symmetrical to D with respect to M. Then OM is the perpendicular bisector of segment DK, and hence $OD = OK$, which means that point K lies on the circumcircle of triangle BCD. Hence we have $\angle BDC = \angle BKC$. On the other hand, the angles BKC and TDE are symmetrical with respect to M, so $\angle TDE = \angle BKC = \angle BDC$.

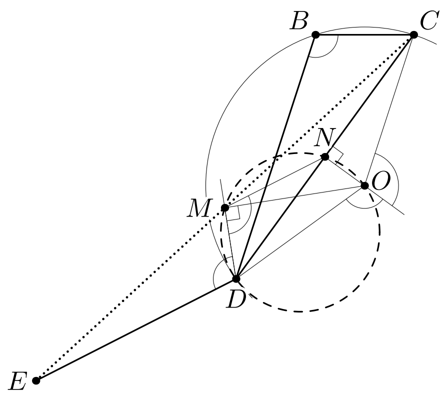
Therefore, $\angle BDT = \angle BDE + \angle EDT = \angle BDE + \angle BDC = \angle CDE = \angle ABC = 180^\circ - \angle BAT$. This means that the points A, B, D, T are concyclic, and hence $\angle ADB = \angle ATB = \frac{1}{2}\angle ABC = \frac{1}{2}\angle CDE$, as desired.





Solution 2. Let$\angle CBD = \alpha$, $\angle BDC = \beta$, $\angle ADE = \gamma$, and $\angle ABC = \angle CDE = 2\varphi$. Then we have $\angle ADB = 2\varphi - \beta - \gamma$, $\angle BCD = 180^\circ - \alpha - \beta$, $\angle AED = 360^\circ - \angle BCD - \angle CDE = 180^\circ - 2\varphi + \alpha + \beta$, and finally $\angle DAE = 180^\circ - \angle ADE - \angle AED = 2\varphi - \alpha - \beta - \gamma$.





Let N be the midpoint of CD; then $\angle DNO = 90^\circ = \angle DMO$, hence points M, N lie on the circle with diameter OD. Now, if points O and M lie on the same side of CD, we have $\angle DMN = \angle DON = \frac{1}{2}\angle DOC = \alpha$; in the other case, we have $\angle DMN = 180^\circ - \angle DON = \alpha$;

so, in both cases $\angle DMN = \alpha$ (see Figures). Next, since MN is a midline in triangle CDE, we have $\angle MDE = \angle DMN = \alpha$ and $\angle NDM = 2\varphi - \alpha$.

Now we apply the sine rule to the triangles ABD, ADE (twice), BCD and MND obtaining

$$\frac{AB}{AD} = \frac{\sin(2\varphi - \beta - \gamma)}{\sin(2\varphi - \alpha)}, \quad \frac{AE}{AD} = \frac{\sin\gamma}{\sin(2\varphi - \alpha - \beta)}, \quad \frac{DE}{AD} = \frac{\sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha - \beta)},$$
$$\frac{BC}{CD} = \frac{\sin\beta}{\sin\alpha}, \quad \frac{CD}{DE} = \frac{CD/2}{DE/2} = \frac{ND}{NM} = \frac{\sin\alpha}{\sin(2\varphi - \alpha)},$$

which implies

$$\frac{BC}{AD} = \frac{BC}{CD} \cdot \frac{CD}{DE} \cdot \frac{DE}{AD} = \frac{\sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma)}{\sin(2\varphi - \alpha) \cdot \sin(2\varphi - \alpha - \beta)}.$$

Hence, the condition $AB = AE + BC$, or equivalently $\dfrac{AB}{AD} = \dfrac{AE + BC}{AD}$, after multiplying by the common denominator rewrites as

$$\begin{aligned}
&\sin(2\varphi - \alpha - \beta) \cdot \sin(2\varphi - \beta - \gamma) = \sin\gamma \cdot \sin(2\varphi - \alpha) + \sin\beta \cdot \sin(2\varphi - \alpha - \beta - \gamma) \\
\iff\ & \cos(\gamma - \alpha) - \cos(4\varphi - 2\beta - \alpha - \gamma) = \cos(2\varphi - \alpha - 2\beta - \gamma) - \cos(2\varphi + \gamma - \alpha) \\
\iff\ & \cos(\gamma - \alpha) + \cos(2\varphi + \gamma - \alpha) = \cos(2\varphi - \alpha - 2\beta - \gamma) + \cos(4\varphi - 2\beta - \alpha - \gamma) \\
\iff\ & \cos\varphi \cdot \cos(\varphi + \gamma - \alpha) = \cos\varphi \cdot \cos(3\varphi - 2\beta - \alpha - \gamma) \\
\iff\ & \cos\varphi \cdot \big(\cos(\varphi + \gamma - \alpha) - \cos(3\varphi - 2\beta - \alpha - \gamma)\big) = 0 \\
\iff\ & \cos\varphi \cdot \sin(2\varphi - \beta - \alpha) \cdot \sin(\varphi - \beta - \gamma) = 0.
\end{aligned}$$

Since $2\varphi - \beta - \alpha = 180^\circ - \angle AED < 180^\circ$ and $\varphi = \frac{1}{2}\angle ABC < 90^\circ$, it follows that $\varphi = \beta + \gamma$, hence $\angle BDA = 2\varphi - \beta - \gamma = \varphi = \frac{1}{2}\angle CDE$, as desired.

G6. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

G6′. The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Prove that if the incenter of triangle ABC lies outside triangle XYZ, then one of the angles of triangle ABC is greater than $120°$.

(Bulgaria)

Solution 1 for G6. We will prove a stronger fact; namely, we will show that the incenter I of triangle ABC lies inside the incircle of triangle XYZ (and hence surely inside triangle XYZ itself). We denote by $d(U, VW)$ the distance between point U and line VW.

Denote by O the incenter of $\triangle XYZ$ and by r, r' and R' the inradii of triangles ABC, XYZ and the circumradius of XYZ, respectively. Then we have $R' = 2r'$, and the desired inequality is $OI \leq r'$. We assume that $O \neq I$; otherwise the claim is trivial.

Let the incircle of $\triangle ABC$ touch its sides BC, AC, AB at points A_1, B_1, C_1 respectively. The lines IA_1, IB_1, IC_1 cut the plane into 6 acute angles, each one containing one of the points A_1, B_1, C_1 on its border. We may assume that O lies in an angle defined by lines IA_1, IC_1 and containing point C_1 (see Fig. 1). Let A' and C' be the projections of O onto lines IA_1 and IC_1, respectively.

Since $OX = R'$, we have $d(O, BC) \leq R'$. Since $OA' \parallel BC$, it follows that $d(A', BC) = A'I + r \leq R'$, or $A'I \leq R' - r$. On the other hand, the incircle of $\triangle XYZ$ lies inside $\triangle ABC$, hence $d(O, AB) \geq r'$, and analogously we get $d(O, AB) = C'C_1 = r - IC' \geq r'$, or $IC' \leq r - r'$.

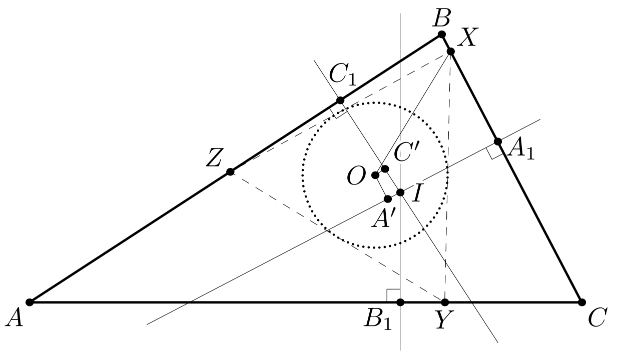


Fig. 1

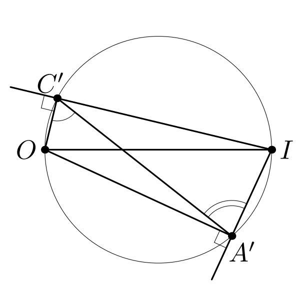


Fig. 2

Finally, the quadrilateral $IA'OC'$ is circumscribed due to the right angles at A' and C' (see Fig. 2). On its circumcircle, we have $\overparen{A'OC'} = 2\angle A'IC' < 180° = \overparen{OC'I}$, hence $180° \geq \overparen{IC'} > \overparen{A'O}$. This means that $IC' > A'O$. Finally, we have $OI \leq IA' + A'O < IA' + IC' \leq (R' - r) + (r - r') = R' - r' = r'$, as desired.

Solution 2 for G6. Assume the contrary. Then the incenter I should lie in one of triangles AYZ, BXZ, CXY — assume that it lies in $\triangle AYZ$. Let the incircle ω of $\triangle ABC$ touch sides BC, AC at point A_1, B_1 respectively. Without loss of generality, assume that point A_1 lies on segment CX. In this case we will show that $\angle C > 90°$ thus leading to a contradiction.

Note that ω intersects each of the segments XY and YZ at two points; let U, U' and V, V' be the points of intersection of ω with XY and YZ, respectively ($UY > U'Y$, $VY > V'Y$; see Figs. 3 and 4). Note that $60° = \angle XYZ = \frac{1}{2}(\overparen{UV} - \overparen{U'V'}) \leq \frac{1}{2}\overparen{UV}$, hence $\overparen{UV} \geq 120°$.

On the other hand, since I lies in $\triangle AYZ$, we get $\overset{\frown}{VUV'} < 180^\circ$, hence $\overset{\frown}{UA_1U'} \leq \overset{\frown}{UA_1V'} < 180^\circ - \overset{\frown}{UV} \leq 60^\circ$.

Now, two cases are possible due to the order of points Y, B_1 on segment AC.

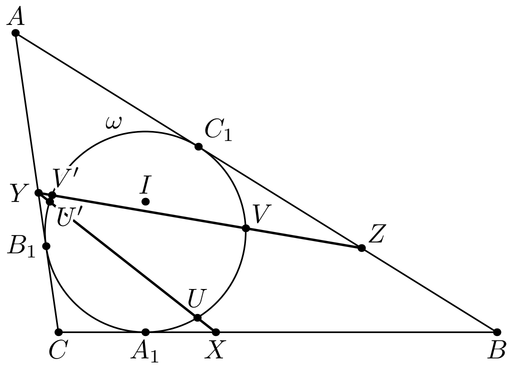


Fig. 3

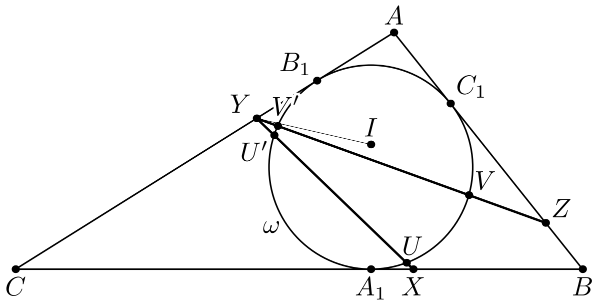


Fig. 4

Case 1. Let point Y lie on the segment AB_1 (see Fig. 3). Then we have $\angle YXC = \frac{1}{2}\left(\overset{\frown}{A_1U'} - \overset{\frown}{A_1U}\right) \leq \frac{1}{2}\overset{\frown}{UA_1U'} < 30^\circ$; analogously, we get $\angle XYC \leq \frac{1}{2}\overset{\frown}{UA_1U'} < 30^\circ$. Therefore, $\angle YCX = 180^\circ - \angle YXC - \angle XYC > 120^\circ$, as desired.

Case 2. Now let point Y lie on the segment CB_1 (see Fig. 4). Analogously, we obtain $\angle YXC < 30^\circ$. Next, $\angle IYX > \angle ZYX = 60^\circ$, but $\angle IYX < \angle IYB_1$, since YB_1 is a tangent and YX is a secant line to circle ω from point Y. Hence, we get $120^\circ < \angle IYB_1 + \angle IYX = \angle B_1YX = \angle YXC + \angle YCX < 30^\circ + \angle YCX$, hence $\angle YCX > 120^\circ - 30^\circ = 90^\circ$, as desired.

Comment. In the same way, one can prove a more general

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, and α is the least angle of $\triangle XYZ$. Then one of the angles of triangle ABC is greater than $3\alpha - 90^\circ$.

Solution for G6′. Assume the contrary. As in Solution 2, we assume that the incenter I of $\triangle ABC$ lies in $\triangle AYZ$, and the tangency point A_1 of ω and BC lies on segment CX. Surely, $\angle YZA \leq 180^\circ - \angle YZX = 120^\circ$, hence points I and Y lie on one side of the perpendicular bisector to XY; therefore $IX > IY$. Moreover, ω intersects segment XY at two points, and therefore the projection M of I onto XY lies on the segment XY. In this case, we will prove that $\angle C > 120^\circ$.

Let YK, YL be two tangents from point Y to ω (points K and A_1 lie on one side of XY; if Y lies on ω, we say $K = L = Y$); one of the points K and L is in fact a tangency point B_1 of ω and AC. From symmetry, we have $\angle YIK = \angle YIL$. On the other hand, since $IX > IY$, we get $XM < XY$ which implies $\angle A_1XY < \angle KYX$.

Next, we have $\angle MIY = 90^\circ - \angle IYX < 90^\circ - \angle ZYX = 30^\circ$. Since $IA_1 \perp A_1X$, $IM \perp XY$, $IK \perp YK$ we get $\angle MIA_1 = \angle A_1XY < \angle KYX = \angle MIK$. Finally, we get

$$\begin{aligned}\angle A_1IK < \angle A_1IL &= (\angle A_1IM + \angle MIK) + (\angle KIY + \angle YIL)\\ &< 2\angle MIK + 2\angle KIY = 2\angle MIY < 60^\circ.\end{aligned}$$

Hence, $\angle A_1IB_1 < 60^\circ$, and therefore $\angle ACB = 180^\circ - \angle A_1IB_1 > 120^\circ$, as desired.

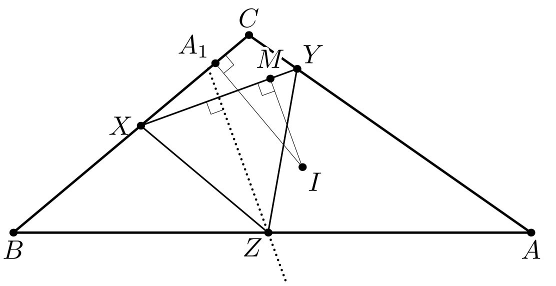


Fig. 5

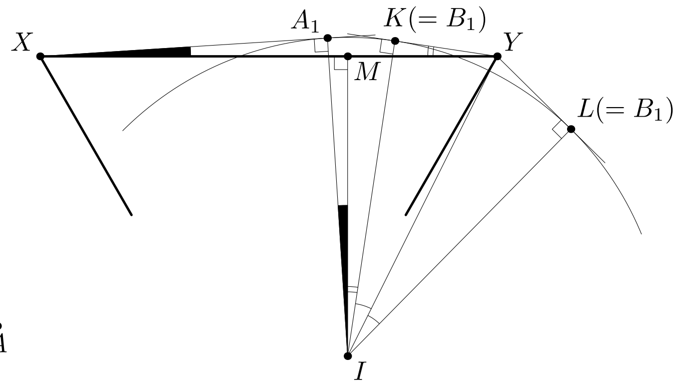


Fig. 6

Comment 1. The estimate claimed in G6$'$ is sharp. Actually, if $\angle BAC > 120^\circ$, one can consider an equilateral triangle XYZ with $Z = A$, $Y \in AC$, $X \in BC$ (such triangle exists since $\angle ACB < 60^\circ$). It intersects with the angle bisector of $\angle BAC$ only at point A, hence it does not contain I.

Comment 2. As in the previous solution, there is a generalization for an arbitrary triangle XYZ, but here we need some additional condition. The statement reads as follows.

Claim. Let the vertices X, Y, Z of a triangle XYZ lie respectively on the sides BC, CA, AB of a triangle ABC. Suppose that the incenter of triangle ABC lies outside triangle XYZ, α is the least angle of $\triangle XYZ$, and all sides of triangle XYZ are greater than $2r \cot \alpha$, where r is the inradius of $\triangle ABC$. Then one of the angles of triangle ABC is greater than 2α.

The additional condition is needed to verify that $XM > YM$ since it cannot be shown in the original way. Actually, we have $\angle MYI > \alpha$, $IM < r$, hence $YM < r \cot \alpha$. Now, if we have $XY = XM + YM > 2r \cot \alpha$, then surely $XM > YM$.

On the other hand, this additional condition follows easily from the conditions of the original problem. Actually, if $I \in \triangle AYZ$, then the diameter of ω parallel to YZ is contained in $\triangle AYZ$ and is thus shorter than YZ. Hence $YZ > 2r > 2r \cot 60^\circ$.

G7. Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below).

Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{k\ell}V_{i\ell}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{i\ell}$, $V_{kj}V_{k\ell}$ and arcs $V_{ij}V_{kj}$ and $V_{i\ell}V_{k\ell}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs.

Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}$, $\widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}$, $\widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.

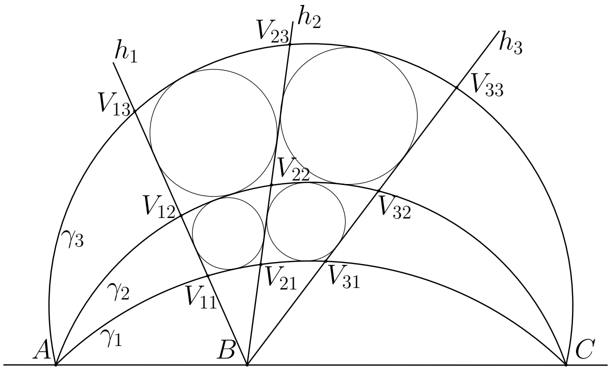


Fig. 1

(Hungary)

Solution. Denote by O_i and R_i the center and the radius of γ_i, respectively. Denote also by H the half-plane defined by AC which contains the whole configuration. For every point P in the half-plane H, denote by $d(P)$ the distance between P and line AC. Furthermore, for any $r > 0$, denote by $\Omega(P, r)$ the circle with center P and radius r.

Lemma 1. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are tangent to h_i and h_j.

(a) The locus of the centers of these circles is the angle bisector β_{ij} between h_i and h_j.

(b) There is a constant u_{ij} such that $r = u_{ij} \cdot d(P)$ for all such circles.

Proof. Part (a) is obvious. To prove part (b), notice that the circles which are tangent to h_i and h_j are homothetic with the common homothety center B (see Fig. 2). Then part (b) also becomes trivial. □

Lemma 2. For every $1 \le i < j \le 3$, consider those circles $\Omega(P, r)$ in the half-plane H which are externally tangent to γ_i and internally tangent to γ_j.

(a) The locus of the centers of these circles is an ellipse arc ε_{ij} with end-points A and C.

(b) There is a constant v_{ij} such that $r = v_{ij} \cdot d(P)$ for all such circles.

Proof. (a) Notice that the circle $\Omega(P, r)$ is externally tangent to γ_i and internally tangent to γ_j if and only if $O_iP = R_i + r$ and $O_j = R_j - r$. Therefore, for each such circle we have

$$O_iP + O_jP = O_iA + O_jA = O_iC + O_jC = R_i + R_j.$$

Such points lie on an ellipse with foci O_i and O_j; the diameter of this ellipse is $R_i + R_j$, and it passes through the points A and C. Let ε_{ij} be that arc AC of the ellipse which runs inside the half plane H (see Fig. 3.)

This ellipse arc lies between the arcs γ_i and γ_j. Therefore, if some point P lies on ε_{ij}, then $O_iP > R_i$ and $O_jP < R_j$. Now, we choose $r = O_iP - R_i = R_j - O_jP > 0$; then the

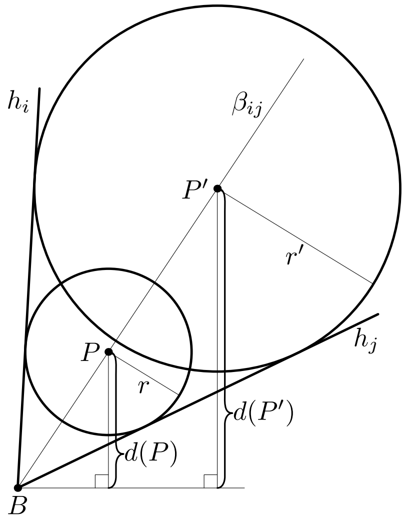


Fig. 2

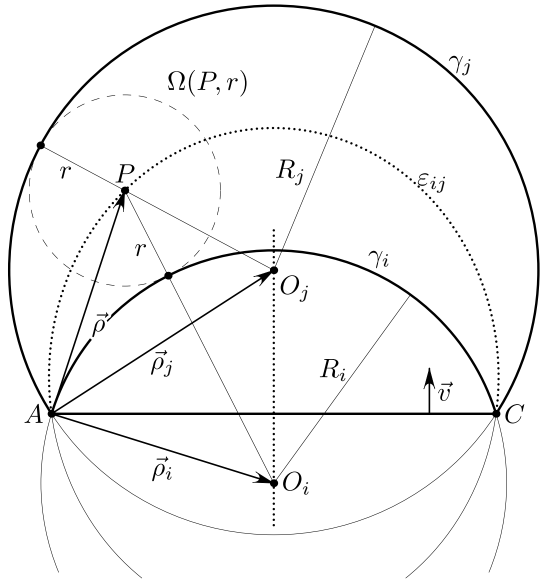


Fig. 3

circle $\Omega(P,r)$ touches γ_i externally and touches γ_j internally, so P belongs to the locus under investigation.

(b) Let $\vec{\rho} = \overrightarrow{AP}$, $\vec{\rho}_i = \overrightarrow{AO_i}$, and $\vec{\rho}_j = \overrightarrow{AO_j}$; let $d_{ij} = O_iO_j$, and let $\vec{v}$ be a unit vector orthogonal to AC and directed toward H. Then we have $|\vec{\rho}_i| = R_i$, $|\vec{\rho}_j| = R_j$, $|\overrightarrow{O_iP}| = |\vec{\rho} - \vec{\rho}_i| = R_i + r$, $|\overrightarrow{O_jP}| = |\vec{\rho} - \vec{\rho}_j| = R_j - r$, hence

$$
\begin{gathered}
(\vec{\rho} - \vec{\rho}_i)^2 - (\vec{\rho} - \vec{\rho}_j)^2 = (R_i + r)^2 - (R_j - r)^2, \\
(\vec{\rho}_i^{\,2} - \vec{\rho}_j^{\,2}) + 2\vec{\rho} \cdot (\vec{\rho}_j - \vec{\rho}_i) = (R_i^2 - R_j^2) + 2r(R_i + R_j), \\
d_{ij} \cdot d(P) = d_{ij} \vec{v} \cdot \vec{\rho} = (\vec{\rho}_j - \vec{\rho}_i) \cdot \vec{\rho} = r(R_i + R_j).
\end{gathered}
$$

Therefore,

$$
r = \frac{d_{ij}}{R_i + R_j} \cdot d(P),
$$

and the value $v_{ij} = \dfrac{d_{ij}}{R_i + R_j}$ does not depend on P. $\square$

Lemma 3. The curved quadrilateral $\mathcal{Q}_{ij} = V_{i,j}\overset{\frown}{V_{i+1,j}}V_{i+1,j+1}\overset{\frown}{V_{i,j+1}}$ is circumscribed if and only if $u_{i,i+1} = v_{j,j+1}$.

Proof. First suppose that the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed and $\Omega(P,r)$ is its inscribed circle. By Lemma 1 and Lemma 2 we have $r = u_{i,i+1} \cdot d(P)$ and $r = v_{j,j+1} \cdot d(P)$ as well. Hence, $u_{i,i+1} = v_{j,j+1}$.

To prove the opposite direction, suppose $u_{i,i+1} = v_{j,j+1}$. Let P be the intersection of the angle bisector $\beta_{i,i+1}$ and the ellipse arc $\varepsilon_{j,j+1}$. Choose $r = u_{i,i+1} \cdot d(P) = v_{j,j+1} \cdot d(P)$. Then the circle $\Omega(P,r)$ is tangent to the half lines h_i and h_{i+1} by Lemma 1, and it is tangent to the arcs γ_j and γ_{j+1} by Lemma 2. Hence, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed. $\square$

By Lemma 3, the statement of the problem can be reformulated to an obvious fact: If the equalities $u_{12} = v_{12}$, $u_{12} = v_{23}$, and $u_{23} = v_{12}$ hold, then $u_{23} = v_{23}$ holds as well.

Comment 1. Lemma 2(b) (together with the easy Lemma 1(b)) is the key tool in this solution. If one finds this fact, then the solution can be finished in many ways. That is, one can find a circle touching three of h_2, h_3, γ_2, and γ_3, and then prove that it is tangent to the fourth one in either synthetic or analytical way. Both approaches can be successful.

Here we present some discussion about this key Lemma.

1. In the solution above we chose an analytic proof for Lemma 2(b) because we expect that most students will use coordinates or vectors to examine the locus of the centers, and these approaches are less case-sensitive.

Here we outline a synthetic proof. We consider only the case when P does not lie in the line O_iO_j. The other case can be obtained as a limit case, or computed in a direct way.

Let S be the internal homothety center between the circles of γ_i and γ_j, lying on O_iO_j; this point does not depend on P. Let U and V be the points of tangency of circle $\sigma = \Omega(P, r)$ with γ_i and γ_j, respectively (then $r = PU = PV$); in other words, points U and V are the intersection points of rays O_iP, O_jP with arcs γ_i, γ_j respectively (see Fig. 4).

Due to the theorem on three homothety centers (or just to the Menelaus theorem applied to triangle O_iO_jP), the points U, V and S are collinear. Let T be the intersection point of line AC and the common tangent to σ and γ_i at U; then T is the radical center of σ, γ_i and γ_j, hence TV is the common tangent to σ and γ_j.

Let Q be the projection of P onto the line AC. By the right angles, the points U, V and Q lie on the circle with diameter PT. From this fact and the equality $PU = PV$ we get $\angle UQP = \angle UVP = \angle VUP = \angle SUO_i$. Since $O_iS \parallel PQ$, we have $\angle SO_iU = \angle QPU$. Hence, the triangles SO_iU and UPQ are similar and thus $\dfrac{r}{d(P)} = \dfrac{PU}{PQ} = \dfrac{O_iS}{O_iU} = \dfrac{O_iS}{R_i}$; the last expression is constant since S is a constant point. ☐

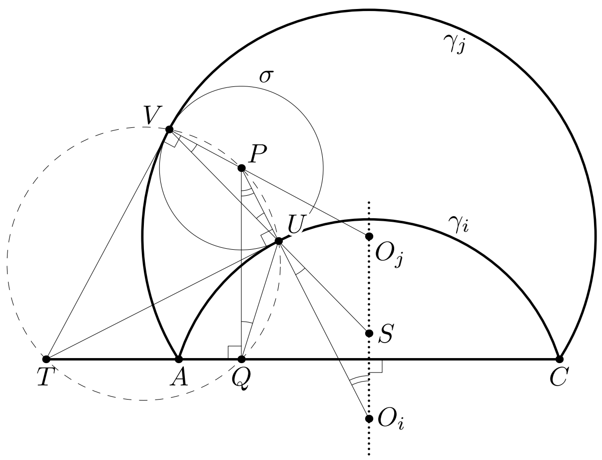


Fig. 4

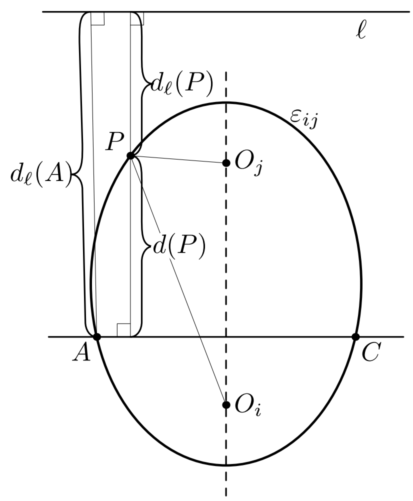


Fig. 5

2. Using some known facts about conics, the same statement can be proved in a very short way. Denote by ℓ the directrix of ellipse of ε_{ij} related to the focus O_j; since ε_{ij} is symmetrical about O_iO_j, we have $\ell \parallel AC$. Recall that for each point $P \in \varepsilon_{ij}$, we have $PO_j = \epsilon \cdot d_\ell(P)$, where $d_\ell(P)$ is the distance from P to ℓ, and ϵ is the eccentricity of ε_{ij} (see Fig. 5).

Now we have

$$r = R_j - (R_j - r) = AO_j - PO_j = \epsilon\big(d_\ell(A) - d_\ell(P)\big) = \epsilon\big(d(P) - d(A)\big) = \epsilon \cdot d(P),$$

and ϵ does not depend on P. ☐

Comment 2. One can find a spatial interpretations of the problem and the solution.

For every point (x,y) and radius $r>0$, represent the circle $\Omega\big((x,y),r\big)$ by the point (x,y,r) in space. This point is the apex of the cone with base circle $\Omega\big((x,y),r\big)$ and height r. According to Lemma 1, the circles which are tangent to h_i and h_j correspond to the points of a half line β'_{ij}, starting at B.

Now we translate Lemma 2. Take some $1\le i<j\le 3$, and consider those circles which are internally tangent to γ_j. It is easy to see that the locus of the points which represent these circles is a subset of a cone, containing γ_j. Similarly, the circles which are externally tangent to γ_i correspond to the points on the extension of another cone, which has its apex on the opposite side of the base plane Π. (See Fig. 6; for this illustration, the z-coordinates were multiplied by 2.)

The two cones are symmetric to each other (they have the same aperture, and their axes are parallel). As is well-known, it follows that the common points of the two cones are co-planar. So the intersection of the two cones is a a conic section — which is an ellipse, according to Lemma 2(a). The points which represent the circles touching γ_i and γ_j is an ellipse arc ε'_{ij} with end-points A and C.

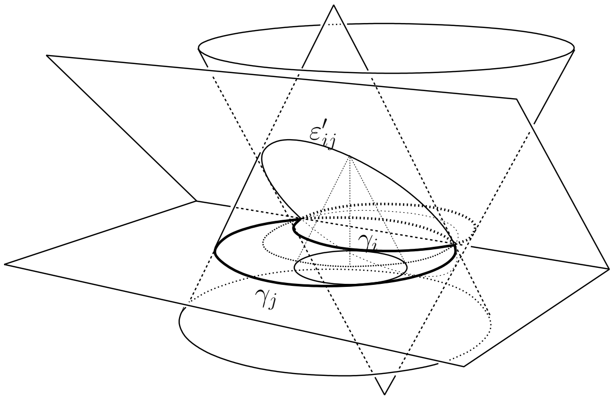


Fig. 6

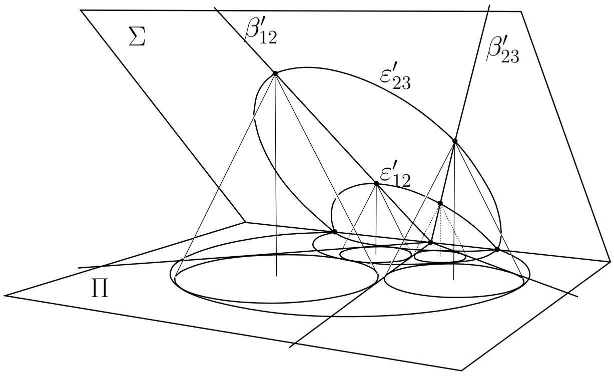
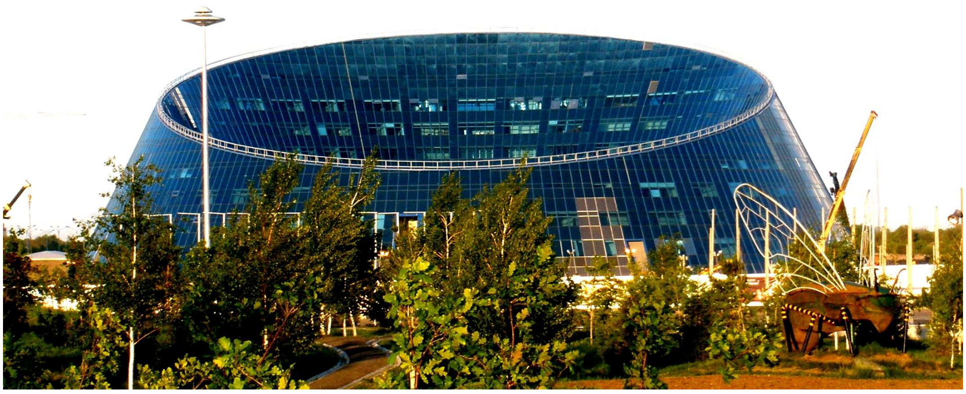


Fig. 7

Thus, the curved quadrilateral $\mathcal{Q}_{ij}$ is circumscribed if and only if $\beta'_{i,i+1}$ and $\varepsilon'_{j,j+1}$ intersect, i.e. if they are coplanar. If three of the four curved quadrilaterals are circumscribed, it means that ε'_{12}, ε'_{23}, β'_{12} and β'_{23} lie in the same plane Σ, and the fourth intersection comes to existence, too (see Fig. 7).



A connection between mathematics and real life:
the Palace of Creativity “Shabyt” (“Inspiration”) in Astana

Number Theory

N1. Find the least positive integer n for which there exists a set $\{s_1, s_2, \dots, s_n\}$ consisting of n distinct positive integers such that

$$\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\dots\left(1-\frac{1}{s_n}\right)=\frac{51}{2010}.$$

N1′. Same as Problem N1, but the constant $\frac{51}{2010}$ is replaced by $\frac{42}{2010}$.

(Canada)

Answer for Problem N1. $n = 39$.

Solution for Problem N1. Suppose that for some n there exist the desired numbers; we may assume that $s_1 < s_2 < \cdots < s_n$. Surely $s_1 > 1$ since otherwise $1-\frac{1}{s_1}=0$. So we have $2 \leq s_1 \leq s_2 - 1 \leq \cdots \leq s_n-(n-1)$, hence $s_i \geq i+1$ for each $i = 1, \dots, n$. Therefore

$$\begin{aligned}\frac{51}{2010}&=\left(1-\frac{1}{s_1}\right)\left(1-\frac{1}{s_2}\right)\dots\left(1-\frac{1}{s_n}\right)\\ &\geq\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\dots\left(1-\frac{1}{n+1}\right)=\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n}{n+1}=\frac{1}{n+1},\end{aligned}$$

which implies

$$n+1\geq\frac{2010}{51}=\frac{670}{17}>39,$$

so $n \geq 39$.

Now we are left to show that $n = 39$ fits. Consider the set $\{2, 3, \dots, 33, 35, 36, \dots, 40, 67\}$ which contains exactly 39 numbers. We have

$$\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{32}{33}\cdot\frac{34}{35}\cdots\frac{39}{40}\cdot\frac{66}{67}=\frac{1}{33}\cdot\frac{34}{40}\cdot\frac{66}{67}=\frac{17}{670}=\frac{51}{2010}, \tag{1}$$

hence for $n = 39$ there exists a desired example.

Comment. One can show that the example (1) is unique.

Answer for Problem N1′. $n = 48$.

Solution for Problem N1′. Suppose that for some n there exist the desired numbers. In the same way we obtain that $s_i \geq i+1$. Moreover, since the denominator of the fraction $\frac{42}{2010}=\frac{7}{335}$ is divisible by 67, some of s_i's should be divisible by 67, so $s_n \geq s_i \geq 67$. This means that

$$\frac{42}{2010}\geq\frac{1}{2}\cdot\frac{2}{3}\cdots\frac{n-1}{n}\cdot\left(1-\frac{1}{67}\right)=\frac{66}{67n},$$

which implies

$$n \geq \frac{2010 \cdot 66}{42 \cdot 67} = \frac{330}{7} > 47,$$

so $n \geq 48$.

Now we are left to show that $n = 48$ fits. Consider the set $\{2, 3, \dots, 33, 36, 37, \dots, 50, 67\}$ which contains exactly 48 numbers. We have

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{32}{33} \cdot \frac{35}{36} \cdots \frac{49}{50} \cdot \frac{66}{67} = \frac{1}{33} \cdot \frac{35}{50} \cdot \frac{66}{67} = \frac{7}{335} = \frac{42}{2010},$$

hence for $n = 48$ there exists a desired example.

Comment 1. In this version of the problem, the estimate needs one more step, hence it is a bit harder. On the other hand, the example in this version is not unique. Another example is

$$\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{46}{47} \cdot \frac{66}{67} \cdot \frac{329}{330} = \frac{1}{67} \cdot \frac{66}{330} \cdot \frac{329}{47} = \frac{7}{67 \cdot 5} = \frac{42}{2010}.$$

Comment 2. N1$'$ was the Proposer's formulation of the problem. We propose N1 according to the number of current IMO.

N2. Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr). \tag{1}$$

(Australia)

Answer. $(6,3)$, $(9,3)$, $(9,5)$, $(54,5)$.

Solution. For fixed values of n, the equation (1) is a simple quadratic equation in m. For $n \le 5$ the solutions are listed in the following table.

case	equation	discriminant	integer roots
$n = 0$	$m^2 - m + 2 = 0$	-7	none
$n = 1$	$m^2 - 3m + 6 = 0$	-15	none
$n = 2$	$m^2 - 7m + 18 = 0$	-23	none
$n = 3$	$m^2 - 15m + 54 = 0$	9	$m = 6$ and $m = 9$
$n = 4$	$m^2 - 31m + 162 = 0$	313	none
$n = 5$	$m^2 - 63m + 486 = 0$	$2025 = 45^2$	$m = 9$ and $m = 54$

We prove that there is no solution for $n \ge 6$.

Suppose that (m, n) satisfies (1) and $n \ge 6$. Since $m \mid 2 \cdot 3^n = m\bigl(2^{n+1} - 1\bigr) - m^2$, we have $m = 3^p$ with some $0 \le p \le n$ or $m = 2 \cdot 3^q$ with some $0 \le q \le n$.

In the first case, let $q = n - p$; then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 3^p + 2 \cdot 3^q.$$

In the second case let $p = n - q$. Then

$$2^{n+1} - 1 = m + \frac{2 \cdot 3^n}{m} = 2 \cdot 3^q + 3^p.$$

Hence, in both cases we need to find the nonnegative integer solutions of

$$3^p + 2 \cdot 3^q = 2^{n+1} - 1, \quad p + q = n. \tag{2}$$

Next, we prove bounds for p, q. From (2) we get

$$3^p < 2^{n+1} = 8^{\frac{n+1}{3}} < 9^{\frac{n+1}{3}} = 3^{\frac{2(n+1)}{3}}$$

and

$$2 \cdot 3^q < 2^{n+1} = 2 \cdot 8^{\frac{n}{3}} < 2 \cdot 9^{\frac{n}{3}} = 2 \cdot 3^{\frac{2n}{3}} < 2 \cdot 3^{\frac{2(n+1)}{3}},$$

so $p, q < \frac{2(n+1)}{3}$. Combining these inequalities with $p + q = n$, we obtain

$$\frac{n-2}{3} < p, q < \frac{2(n+1)}{3}. \tag{3}$$

Now let $h = \min(p, q)$. By (3) we have $h > \frac{n-2}{3}$; in particular, we have $h > 1$. On the left-hand side of (2), both terms are divisible by 3^h, therefore $9 \mid 3^h \mid 2^{n+1} - 1$. It is easy check that $\mathrm{ord}_9(2) = 6$, so $9 \mid 2^{n+1} - 1$ if and only if $6 \mid n + 1$. Therefore, $n + 1 = 6r$ for some positive integer r, and we can write

$$2^{n+1} - 1 = 4^{3r} - 1 = (4^{2r} + 4^r + 1)(2^r - 1)(2^r + 1). \tag{4}$$

Notice that the factor $4^{2r} + 4^r + 1 = (4^r - 1)^2 + 3 \cdot 4^r$ is divisible by 3, but it is never divisible by 9. The other two factors in (4), $2^r - 1$ and $2^r + 1$ are coprime: both are odd and their difference is 2. Since the whole product is divisible by 3^h, we have either $3^{h-1} \mid 2^r - 1$ or $3^{h-1} \mid 2^r + 1$. In any case, we have $3^{h-1} \leq 2^r + 1$. Then

$$3^{h-1} \leq 2^r + 1 \leq 3^r = 3^{\frac{n+1}{6}},$$
$$\frac{n-2}{3} - 1 < h - 1 \leq \frac{n+1}{6},$$
$$n < 11.$$

But this is impossible since we assumed $n \geq 6$, and we proved $6 \mid n + 1$.

N3. Find the smallest number n such that there exist polynomials $f_1, f_2, \dots, f_n$ with rational coefficients satisfying

$$x^2+7=f_1(x)^2+f_2(x)^2+\cdots+f_n(x)^2.$$

(Poland)

Answer. The smallest n is 5.

Solution 1. The equality $x^2+7=x^2+2^2+1^2+1^2+1^2$ shows that $n\le 5$. It remains to show that x^2+7 is not a sum of four (or less) squares of polynomials with rational coefficients.

Suppose by way of contradiction that $x^2+7=f_1(x)^2+f_2(x)^2+f_3(x)^2+f_4(x)^2$, where the coefficients of polynomials f_1, f_2, f_3 and f_4 are rational (some of these polynomials may be zero).

Clearly, the degrees of f_1, f_2, f_3 and f_4 are at most 1. Thus $f_i(x)=a_ix+b_i$ for $i=1,2,3,4$ and some rationals a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4. It follows that $x^2+7=\sum_{i=1}^4(a_ix+b_i)^2$ and hence

$$\sum_{i=1}^4 a_i^2=1, \quad \sum_{i=1}^4 a_ib_i=0, \quad \sum_{i=1}^4 b_i^2=7. \tag{1}$$

Let $p_i=a_i+b_i$ and $q_i=a_i-b_i$ for $i=1,2,3,4$. Then

$$\begin{aligned} \sum_{i=1}^4 p_i^2 &= \sum_{i=1}^4 a_i^2+2\sum_{i=1}^4 a_ib_i+\sum_{i=1}^4 b_i^2=8,\\ \sum_{i=1}^4 q_i^2 &= \sum_{i=1}^4 a_i^2-2\sum_{i=1}^4 a_ib_i+\sum_{i=1}^4 b_i^2=8\\ \text{and}\quad \sum_{i=1}^4 p_iq_i &= \sum_{i=1}^4 a_i^2-\sum_{i=1}^4 b_i^2=-6, \end{aligned}$$

which means that there exist a solution in integers x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4 and $m>0$ of the system of equations

$$\text{(i)}\ \sum_{i=1}^4 x_i^2=8m^2, \qquad \text{(ii)}\ \sum_{i=1}^4 y_i^2=8m^2, \qquad \text{(iii)}\ \sum_{i=1}^4 x_iy_i=-6m^2.$$

We will show that such a solution does not exist.

Assume the contrary and consider a solution with minimal m. Note that if an integer x is odd then $x^2\equiv 1 \pmod 8$. Otherwise (i.e., if x is even) we have $x^2\equiv 0 \pmod 8$ or $x^2\equiv 4 \pmod 8$. Hence, by (i), we get that x_1, x_2, x_3 and x_4 are even. Similarly, by (ii), we get that y_1, y_2, y_3 and y_4 are even. Thus the LHS of (iii) is divisible by 4 and m is also even. It follows that $(\frac{x_1}{2}, \frac{y_1}{2}, \frac{x_2}{2}, \frac{y_2}{2}, \frac{x_3}{2}, \frac{y_3}{2}, \frac{x_4}{2}, \frac{y_4}{2}, \frac{m}{2})$ is a solution of the system of equations (i), (ii) and (iii), which contradicts the minimality of m.

Solution 2. We prove that $n\le 4$ is impossible. Define the numbers a_i, b_i for $i=1,2,3,4$ as in the previous solution.

By Euler's identity we have

$$\begin{aligned}(a_1^2+a_2^2+a_3^2+a_4^2)(b_1^2+b_2^2+b_3^2+b_4^2) =&(a_1b_1+a_2b_2+a_3b_3+a_4b_4)^2+(a_1b_2-a_2b_1+a_3b_4-a_4b_3)^2\\ &+(a_1b_3-a_3b_1+a_4b_2-a_2b_4)^2+(a_1b_4-a_4b_1+a_2b_3-a_3b_2)^2.\end{aligned}$$

So, using the relations (1) from the Solution 1 we get that

$$7 = \left(\frac{m_1}{m}\right)^2 + \left(\frac{m_2}{m}\right)^2 + \left(\frac{m_3}{m}\right)^2, \tag{2}$$

where

$$\begin{aligned}
\frac{m_1}{m} &= a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3, \\
\frac{m_2}{m} &= a_1b_3 - a_3b_1 + a_4b_2 - a_2b_4, \\
\frac{m_3}{m} &= a_1b_4 - a_4b_1 + a_2b_3 - a_3b_2
\end{aligned}$$

and $m_1, m_2, m_3 \in \mathbb{Z}, m \in \mathbb{N}$.

Let m be a minimum positive integer number for which (2) holds. Then

$$8m^2 = m_1^2 + m_2^2 + m_3^2 + m^2.$$

As in the previous solution, we get that m_1, m_2, m_3, m are all even numbers. Then $\left(\frac{m_1}{2}, \frac{m_2}{2}, \frac{m_3}{2}, \frac{m}{2}\right)$ is also a solution of (2) which contradicts the minimality of m. So, we have $n \geq 5$. The example with $n = 5$ is already shown in Solution 1.

N4. Let a, b be integers, and let $P(x) = ax^3 + bx$. For any positive integer n we say that the pair (a, b) is *n-good* if $n \mid P(m) - P(k)$ implies $n \mid m - k$ for all integers m, k. We say that (a, b) is *very good* if (a, b) is n-good for infinitely many positive integers n.

(a) Find a pair (a, b) which is 51-good, but not very good.

(b) Show that all 2010-good pairs are very good.

(Turkey)

Solution. (a) We show that the pair $(1, -51^2)$ is good but not very good. Let $P(x) = x^3 - 51^2 x$. Since $P(51) = P(0)$, the pair $(1, -51^2)$ is not n-good for any positive integer that does not divide 51. Therefore, $(1, -51^2)$ is not very good.

On the other hand, if $P(m) \equiv P(k) \pmod{51}$, then $m^3 \equiv k^3 \pmod{51}$. By Fermat's theorem, from this we obtain

$$m \equiv m^3 \equiv k^3 \equiv k \pmod{3} \quad \text{and} \quad m \equiv m^{33} \equiv k^{33} \equiv k \pmod{17}.$$

Hence we have $m \equiv k \pmod{51}$. Therefore $(1, -51^2)$ is 51-good.

(b) We will show that if a pair (a, b) is 2010-good then (a, b) is 67^i-good for all positive integer i.

Claim 1. If (a, b) is 2010-good then (a, b) is 67-good.

Proof. Assume that $P(m) = P(k) \pmod{67}$. Since 67 and 30 are coprime, there exist integers m' and k' such that $k' \equiv k \pmod{67}$, $k' \equiv 0 \pmod{30}$, and $m' \equiv m \pmod{67}$, $m' \equiv 0 \pmod{30}$. Then we have $P(m') \equiv P(0) \equiv P(k') \pmod{30}$ and $P(m') \equiv P(m) \equiv P(k) \equiv P(k') \pmod{67}$, hence $P(m') \equiv P(k') \pmod{2010}$. This implies $m' \equiv k' \pmod{2010}$ as (a, b) is 2010-good. It follows that $m \equiv m' \equiv k' \equiv k \pmod{67}$. Therefore, (a, b) is 67-good. $\square$

Claim 2. If (a, b) is 67-good then $67 \mid a$.

Proof. Suppose that $67 \nmid a$. Consider the sets $\{at^2 \pmod{67} : 0 \le t \le 33\}$ and $\{-3as^2 - b \mod 67 : 0 \le s \le 33\}$. Since $a \not\equiv 0 \pmod{67}$, each of these sets has 34 elements. Hence they have at least one element in common. If $at^2 \equiv -3as^2 - b \pmod{67}$ then for $m = t \pm s$, $k = \mp 2s$ we have

$$\begin{aligned} P(m) - P(k) = a(m^3 - k^3) + b(m - k) &= (m - k)\big(a(m^2 + mk + k^2) + b\big) \\ &= (t \pm 3s)(at^2 + 3as^2 + b) \equiv 0 \pmod{67}. \end{aligned}$$

Since (a, b) is 67-good, we must have $m \equiv k \pmod{67}$ in both cases, that is, $t \equiv 3s \pmod{67}$ and $t \equiv -3s \pmod{67}$. This means $t \equiv s \equiv 0 \pmod{67}$ and $b \equiv -3as^2 - at^2 \equiv 0 \pmod{67}$. But then $67 \mid P(7) - P(2) = 67 \cdot 5a + 5b$ and $67 \nmid 7 - 2$, contradicting that (a, b) is 67-good. $\square$

Claim 3. If (a, b) is 2010-good then (a, b) is 67^i-good all $i \ge 1$.

Proof. By Claim 2, we have $67 \mid a$. If $67 \mid b$, then $P(x) \equiv P(0) \pmod{67}$ for all x, contradicting that (a, b) is 67-good. Hence, $67 \nmid b$.

Suppose that $67^i \mid P(m) - P(k) = (m - k)\big(a(m^2 + mk + k^2) + b\big)$. Since $67 \mid a$ and $67 \nmid b$, the second factor $a(m^2 + mk + k^2) + b$ is coprime to 67 and hence $67^i \mid m - k$. Therefore, (a, b) is 67^i-good. $\square$

Comment 1. In the proof of Claim 2, the following reasoning can also be used. Since 3 is not a quadratic residue modulo 67, either $au^2 \equiv -b \pmod{67}$ or $3av^2 \equiv -b \pmod{67}$ has a solution. The settings $(m, k) = (u, 0)$ in the first case and $(m, k) = (v, -2v)$ in the second case lead to $b \equiv 0 \pmod{67}$.

Comment 2. The pair $(67, 30)$ is n-good if and only if $n = d \cdot 67^i$, where $d \mid 30$ and $i \ge 0$. It shows that in part (b), one should deal with the large powers of 67 to reach the solution. The key property of number 67 is that it has the form $3k + 1$, so there exists a nontrivial cubic root of unity modulo 67.

N5. Let $\mathbb{N}$ be the set of all positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that the number $\big(f(m)+n\big)\big(m+f(n)\big)$ is a square for all $m, n \in \mathbb{N}$.

(U.S.A.)

Answer. All functions of the form $f(n) = n + c$, where $c \in \mathbb{N} \cup \{0\}$.

Solution. First, it is clear that all functions of the form $f(n) = n + c$ with a constant nonnegative integer c satisfy the problem conditions since $\big(f(m)+n\big)\big(f(n)+m\big) = (n+m+c)^2$ is a square.

We are left to prove that there are no other functions. We start with the following

Lemma. Suppose that $p \mid f(k) - f(\ell)$ for some prime p and positive integers k, ℓ. Then $p \mid k - \ell$.

Proof. Suppose first that $p^2 \mid f(k) - f(\ell)$, so $f(\ell) = f(k) + p^2 a$ for some integer a. Take some positive integer $D > \max\{f(k), f(\ell)\}$ which is not divisible by p and set $n = pD - f(k)$. Then the positive numbers $n + f(k) = pD$ and $n + f(\ell) = pD + \big(f(\ell) - f(k)\big) = p(D + pa)$ are both divisible by p but not by p^2. Now, applying the problem conditions, we get that both the numbers $\big(f(k)+n\big)\big(f(n)+k\big)$ and $\big(f(\ell)+n\big)\big(f(n)+\ell\big)$ are squares divisible by p (and thus by p^2); this means that the multipliers $f(n)+k$ and $f(n)+\ell$ are also divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$ as well.

On the other hand, if $f(k) - f(\ell)$ is divisible by p but not by p^2, then choose the same number D and set $n = p^3 D - f(k)$. Then the positive numbers $f(k) + n = p^3 D$ and $f(\ell) + n = p^3 D + \big(f(\ell) - f(k)\big)$ are respectively divisible by p^3 (but not by p^4) and by p (but not by p^2). Hence in analogous way we obtain that the numbers $f(n)+k$ and $f(n)+\ell$ are divisible by p, therefore $p \mid \big(f(n)+k\big) - \big(f(n)+\ell\big) = k - \ell$. $\square$

We turn to the problem. First, suppose that $f(k) = f(\ell)$ for some $k, \ell \in \mathbb{N}$. Then by Lemma we have that $k - \ell$ is divisible by every prime number, so $k - \ell = 0$, or $k = \ell$. Therefore, the function f is injective.

Next, consider the numbers $f(k)$ and $f(k+1)$. Since the number $(k+1) - k = 1$ has no prime divisors, by Lemma the same holds for $f(k+1) - f(k)$; thus $|f(k+1) - f(k)| = 1$.

Now, let $f(2) - f(1) = q$, $|q| = 1$. Then we prove by induction that $f(n) = f(1) + q(n-1)$. The base for $n = 1, 2$ holds by the definition of q. For the step, if $n > 1$ we have $f(n+1) = f(n) \pm q = f(1) + q(n-1) \pm q$. Since $f(n) \neq f(n-2) = f(1) + q(n-2)$, we get $f(n) = f(1) + qn$, as desired.

Finally, we have $f(n) = f(1) + q(n-1)$. Then q cannot be -1 since otherwise for $n \geq f(1)+1$ we have $f(n) \leq 0$ which is impossible. Hence $q = 1$ and $f(n) = (f(1) - 1) + n$ for each $n \in \mathbb{N}$, and $f(1) - 1 \geq 0$, as desired.

N6. The rows and columns of a $2^n \times 2^n$ table are numbered from 0 to $2^n - 1$. The cells of the table have been colored with the following property being satisfied: for each $0 \le i, j \le 2^n - 1$, the jth cell in the ith row and the $(i + j)$th cell in the jth row have the same color. (The indices of the cells in a row are considered modulo 2^n.)

Prove that the maximal possible number of colors is 2^n.

(Iran)

Solution. Throughout the solution we denote the cells of the table by coordinate pairs; (i, j) refers to the jth cell in the ith row.

Consider the directed graph, whose vertices are the cells of the board, and the edges are the arrows $(i, j) \to (j, i + j)$ for all $0 \le i, j \le 2^n - 1$. From each vertex (i, j), exactly one edge passes (to $(j, i + j \mod 2^n)$); conversely, to each cell (j, k) exactly one edge is directed (from the cell $(k - j \mod 2^n, j)$). Hence, the graph splits into cycles.

Now, in any coloring considered, the vertices of each cycle should have the same color by the problem condition. On the other hand, if each cycle has its own color, the obtained coloring obviously satisfies the problem conditions. Thus, the maximal possible number of colors is the same as the number of cycles, and we have to prove that this number is 2^n.

Next, consider any cycle $(i_1, j_1), (i_2, j_2), \dots$; we will describe it in other terms. Define a sequence $(a_0, a_1, \dots)$ by the relations $a_0 = i_1$, $a_1 = j_1$, $a_{n+1} = a_n + a_{n-1}$ for all $n \ge 1$ (we say that such a sequence is *a Fibonacci-type sequence*). Then an obvious induction shows that $i_k \equiv a_{k-1} \pmod{2^n}$, $j_k \equiv a_k \pmod{2^n}$. Hence we need to investigate the behavior of Fibonacci-type sequences modulo 2^n.

Denote by $F_0, F_1, \dots$ the Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \ge 0$. We also set $F_{-1} = 1$ according to the recurrence relation.

For every positive integer m, denote by $\nu(m)$ the exponent of 2 in the prime factorization of m, i.e. for which $2^{\nu(m)} \mid m$ but $2^{\nu(m)+1} \nmid m$.

Lemma 1. For every Fibonacci-type sequence $a_0, a_1, a_2, \dots$, and every $k \ge 0$, we have $a_k = F_{k-1}a_0 + F_k a_1$.

Proof. Apply induction on k. The base cases $k = 0, 1$ are trivial. For the step, from the induction hypothesis we get

$$a_{k+1} = a_k + a_{k-1} = (F_{k-1}a_0 + F_k a_1) + (F_{k-2}a_0 + F_{k-1}a_1) = F_k a_0 + F_{k+1}a_1. \qquad \square$$

Lemma 2. For every $m \ge 3$,

(a) we have $\nu(F_{3\cdot 2^{m-2}}) = m$;

(b) $d = 3 \cdot 2^{m-2}$ is the least positive index for which $2^m \mid F_d$;

(c) $F_{3\cdot 2^{m-2}+1} \equiv 1 + 2^{m-1} \pmod{2^m}$.

Proof. Apply induction on m. In the base case $m = 3$ we have $\nu(F_{3\cdot 2^{m-2}}) = F_6 = 8$, so $\nu(F_{3\cdot 2^{m-2}}) = \nu(8) = 3$, the preceding Fibonacci-numbers are not divisible by 8, and indeed $F_{3\cdot 2^{m-2}+1} = F_7 = 13 \equiv 1 + 4 \pmod 8$.

Now suppose that $m > 3$ and let $k = 3 \cdot 2^{m-3}$. By applying Lemma 1 to the Fibonacci-type sequence $F_k, F_{k+1}, \dots$ we get

$$F_{2k} = F_{k-1}F_k + F_k F_{k+1} = (F_{k+1} - F_k)F_k + F_{k+1}F_k = 2F_{k+1}F_k - F_k^2,$$
$$F_{2k+1} = F_k \cdot F_k + F_{k+1} \cdot F_{k+1} = F_k^2 + F_{k+1}^2.$$

By the induction hypothesis, $\nu(F_k) = m - 1$, and F_{k+1} is odd. Therefore we get $\nu(F_k^2) = 2(m-1) > (m-1) + 1 = \nu(2F_kF_{k+1})$, which implies $\nu(F_{2k}) = m$, establishing statement (a).

Moreover, since $F_{k+1} = 1 + 2^{m-2} + a2^{m-1}$ for some integer a, we get

$$F_{2k+1} = F_k^2 + F_{k+1}^2 \equiv 0 + (1 + 2^{m-2} + a2^{m-1})^2 \equiv 1 + 2^{m-1} \pmod{2^m},$$

as desired in statement (c).

We are left to prove that $2^m \nmid F_\ell$ for $\ell < 2k$. Assume the contrary. Since $2^{m-1} \mid F_\ell$, from the induction hypothesis it follows that $\ell > k$. But then we have $F_\ell = F_{k-1}F_{\ell-k} + F_k F_{\ell-k+1}$, where the second summand is divisible by 2^{m-1} but the first one is not (since F_{k-1} is odd and $\ell - k < k$). Hence the sum is not divisible even by 2^{m-1}. A contradiction. □

Now, for every pair of integers $(a, b) \neq (0, 0)$, let $\mu(a, b) = \min\{\nu(a), \nu(b)\}$. By an obvious induction, for every Fibonacci-type sequence $A = (a_0, a_1, \dots)$ we have $\mu(a_0, a_1) = \mu(a_1, a_2) = \dots$; denote this common value by $\mu(A)$. Also denote by $p_n(A)$ the period of this sequence modulo 2^n, that is, the least $p > 0$ such that $a_{k+p} \equiv a_k \pmod{2^n}$ for all $k \geq 0$.

Lemma 3. Let $A = (a_0, a_1, \dots)$ be a Fibonacci-type sequence such that $\mu(A) = k < n$. Then $p_n(A) = 3 \cdot 2^{n-1-k}$.

Proof. First, we note that the sequence $(a_0, a_1, \dots)$ has period p modulo 2^n if and only if the sequence $(a_0/2^k, a_1/2^k, \dots)$ has period p modulo 2^{n-k}. Hence, passing to this sequence we can assume that $k = 0$.

We prove the statement by induction on n. It is easy to see that for $n = 1, 2$ the claim is true; actually, each Fibonacci-type sequence A with $\mu(A) = 0$ behaves as $0, 1, 1, 0, 1, 1, \dots$ modulo 2, and as $0, 1, 1, 2, 3, 1, 0, 1, 1, 2, 3, 1, \dots$ modulo 4 (all pairs of residues from which at least one is odd appear as a pair of consecutive terms in this sequence).

Now suppose that $n \geq 3$ and consider an arbitrary Fibonacci-type sequence $A = (a_0, a_1, \dots)$ with $\mu(A) = 0$. Obviously we should have $p_{n-1}(A) \mid p_n(A)$, or, using the induction hypothesis, $s = 3 \cdot 2^{n-2} \mid p_n(A)$. Next, we may suppose that a_0 is even; hence a_1 is odd, and $a_0 = 2b_0$, $a_1 = 2b_1 + 1$ for some integers b_0, b_1.

Consider the Fibonacci-type sequence $B = (b_0, b_1, \dots)$ starting with (b_0, b_1). Since $a_0 = 2b_0 + F_0$, $a_1 = 2b_1 + F_1$, by an easy induction we get $a_k = 2b_k + F_k$ for all $k \geq 0$. By the induction hypothesis, we have $p_{n-1}(B) \mid s$, hence the sequence $(2b_0, 2b_1, \dots)$ is s-periodic modulo 2^n. On the other hand, by Lemma 2 we have $F_{s+1} \equiv 1 + 2^{n-1} \pmod{2^n}$, $F_{2s} \equiv 0 \pmod{2^n}$, $F_{2s+1} \equiv 1 \pmod{2^n}$, hence

$$\begin{aligned} a_{s+1} = 2b_{s+1} + F_{s+1} &\equiv 2b_1 + 1 + 2^{n-1} \not\equiv 2b_1 + 1 = a_1 \pmod{2^n}, \\ a_{2s} = 2b_{2s} + F_{2s} &\equiv 2b_0 + 0 = a_0 \pmod{2^n}, \\ a_{2s+1} = 2b_{2s+1} + F_{2s+1} &\equiv 2b_1 + 1 = a_1 \pmod{2^n}. \end{aligned}$$

The first line means that A is not s-periodic, while the other two provide that $a_{2s} \equiv a_0$, $a_{2s+1} \equiv a_1$ and hence $a_{2s+t} \equiv a_t$ for all $t \geq 0$. Hence $s \mid p_n(A) \mid 2s$ and $p_n(A) \neq s$, which means that $p_n(A) = 2s$, as desired. □

Finally, Lemma 3 provides a straightforward method of counting the number of cycles. Actually, take any number $0 \leq k \leq n-1$ and consider all the cells (i, j) with $\mu(i, j) = k$. The total number of such cells is $2^{2(n-k)} - 2^{2(n-k-1)} = 3 \cdot 2^{2n-2k-2}$. On the other hand, they are split into cycles, and by Lemma 3 the length of each cycle is $3 \cdot 2^{n-1-k}$. Hence the number of cycles consisting of these cells is exactly $\dfrac{3 \cdot 2^{2n-2k-2}}{3 \cdot 2^{n-1-k}} = 2^{n-k-1}$. Finally, there is only one cell $(0, 0)$ which is not mentioned in the previous computation, and it forms a separate cycle. So the total number of cycles is

$$1 + \sum_{k=0}^{n-1} 2^{n-1-k} = 1 + (1 + 2 + 4 + \dots + 2^{n-1}) = 2^n.$$

Comment. We outline a different proof for the essential part of Lemma 3. That is, we assume that $k = 0$ and show that in this case the period of (a_i) modulo 2^n coincides with the period of the Fibonacci numbers modulo 2^n; then the proof can be finished by the arguments from Lemma 2..

Note that p is a (not necessarily minimal) period of the sequence (a_i) modulo 2^n if and only if we have $a_0 \equiv a_p \pmod{2^n}$, $a_1 \equiv a_{p+1} \pmod{2^n}$, that is,

$$\begin{aligned} a_0 &\equiv a_p \equiv F_{p-1}a_0 + F_p a_1 = F_p(a_1 - a_0) + F_{p+1}a_0 \pmod{2^n}, \\ a_1 &\equiv a_{p+1} = F_p a_0 + F_{p+1}a_1 \pmod{2^n}. \end{aligned} \tag{1}$$

Now, If p is a period of (F_i) then we have $F_p \equiv F_0 = 0 \pmod{2^n}$ and $F_{p+1} \equiv F_1 = 1 \pmod{2^n}$, which by (1) implies that p is a period of (a_i) as well.

Conversely, suppose that p is a period of (a_i). Combining the relations of (1) we get

$$\begin{aligned} 0 = a_1 \cdot a_0 - a_0 \cdot a_1 &\equiv a_1\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) - a_0(F_p a_0 + F_{p+1}a_1) \\ &= F_p(a_1^2 - a_1a_0 - a_0^2) \pmod{2^n}, \\ a_1^2 - a_1a_0 - a_0^2 = (a_1 - a_0)a_1 - a_0 \cdot a_0 &\equiv (a_1 - a_0)(F_p a_0 + F_{p+1}a_1) - a_0\big(F_p(a_1 - a_0) + F_{p+1}a_0\big) \\ &= F_{p+1}(a_1^2 - a_1a_0 - a_0^2) \pmod{2^n}. \end{aligned}$$

Since at least one of the numbers a_0, a_1 is odd, the number $a_1^2 - a_1a_0 - a_0^2$ is odd as well. Therefore the previous relations are equivalent with $F_p \equiv 0 \pmod{2^n}$ and $F_{p+1} \equiv 1 \pmod{2^n}$, which means exactly that p is a period of $(F_0, F_1, \dots)$ modulo 2^n.

So, the sets of periods of (a_i) and (F_i) coincide, and hence the minimal periods coincide as well.

52nd International Mathematical Olympiad

12 – 24 July 2011

Amsterdam

The Netherlands

Problem Shortlist with Solutions

IMO2011 Amsterdam

International Mathematical Olympiad Amsterdam 2011

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Problem shortlist with solutions

IMPORTANT

IMO regulation: these shortlist problems have to be kept strictly confidential until IMO 2012.

The problem selection committee

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Armenia, Australia, Austria, Belarus, Belgium, Bosnia and Herzegovina, Brazil, Bulgaria, Canada, Colombia, Cyprus, Denmark, Estonia, Finland, France, Germany, Greece, Hong Kong, Hungary, India, Islamic Republic of Iran, Ireland, Israel, Japan, Kazakhstan, Republic of Korea, Luxembourg, Malaysia, Mexico, Mongolia, Montenegro, Pakistan, Poland, Romania, Russian Federation, Saudi Arabia, Serbia, Slovakia, Slovenia, Sweden, Taiwan, Thailand, Turkey, Ukraine, United Kingdom, United States of America

Algebra

A1

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

A2

A2

Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$

A3

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

A4

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

A5

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n+1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

A6

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x))$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

A7

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}.$$

Combinatorics

C1

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \ldots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

C2

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \leq k \leq 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

C3

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

C4

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

C5

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn $90°$ clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

C6

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

C7

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Geometry

G1

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

G2

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

G3

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

G4

G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

G5

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

G6

G6

Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

G7

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

G8

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

Number Theory

N1

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

N2

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

N3

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

N4

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a) - t(n), \quad t(n+a+1) - t(n+1), \quad \ldots, \quad t(n+2a-1) - t(n+a-1)$$

are divisible by 4.

N5

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

N6

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)} - 1$ divides $3^{P(n)} - 1$. Prove that $Q(x)$ is a constant polynomial.

N7

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

N8

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

A1

For any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers with sum $s_A = a_1 + a_2 + a_3 + a_4$, let p_A denote the number of pairs (i, j) with $1 \le i < j \le 4$ for which $a_i + a_j$ divides s_A. Among all sets of four distinct positive integers, determine those sets A for which p_A is maximal.

Answer. The sets A for which p_A is maximal are the sets the form $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer. For all these sets p_A is 4.

Solution. Firstly, we will prove that the maximum value of p_A is at most 4. Without loss of generality, we may assume that $a_1 < a_2 < a_3 < a_4$. We observe that for each pair of indices (i, j) with $1 \le i < j \le 4$, the sum $a_i + a_j$ divides s_A if and only if $a_i + a_j$ divides $s_A - (a_i + a_j) = a_k + a_l$, where k and l are the other two indices. Since there are 6 distinct pairs, we have to prove that at least two of them do not satisfy the previous condition. We claim that two such pairs are (a_2, a_4) and (a_3, a_4). Indeed, note that $a_2 + a_4 > a_1 + a_3$ and $a_3 + a_4 > a_1 + a_2$. Hence $a_2 + a_4$ and $a_3 + a_4$ do not divide s_A. This proves $p_A \le 4$.

Now suppose $p_A = 4$. By the previous argument we have

$$\begin{array}{lcl} a_1 + a_4 \mid a_2 + a_3 & \text{and} & a_2 + a_3 \mid a_1 + a_4, \\ a_1 + a_2 \mid a_3 + a_4 & \text{and} & a_3 + a_4 \nmid a_1 + a_2, \\ a_1 + a_3 \mid a_2 + a_4 & \text{and} & a_2 + a_4 \nmid a_1 + a_3. \end{array}$$

Hence, there exist positive integers m and n with $m > n \ge 2$ such that

$$\begin{cases} a_1 + a_4 = a_2 + a_3 \\ m(a_1 + a_2) = a_3 + a_4 \\ n(a_1 + a_3) = a_2 + a_4. \end{cases}$$

Adding up the first equation and the third one, we get $n(a_1 + a_3) = 2a_2 + a_3 - a_1$. If $n \ge 3$, then $n(a_1 + a_3) > 3a_3 > 2a_2 + a_3 > 2a_2 + a_3 - a_1$. This is a contradiction. Therefore $n = 2$. If we multiply by 2 the sum of the first equation and the third one, we obtain

$$6a_1 + 2a_3 = 4a_2,$$

while the sum of the first one and the second one is

$$(m + 1)a_1 + (m - 1)a_2 = 2a_3.$$

Adding up the last two equations we get

$$(m + 7)a_1 = (5 - m)a_2.$$

It follows that $5 - m \geq 1$, because the left-hand side of the last equation and a_2 are positive. Since we have $m > n = 2$, the integer m can be equal only to either 3 or 4. Substituting $(3, 2)$ and $(4, 2)$ for (m, n) and solving the previous system of equations, we find the families of solutions $\{d, 5d, 7d, 11d\}$ and $\{d, 11d, 19d, 29d\}$, where d is any positive integer.

A2

Determine all sequences $(x_1, x_2, \dots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \dots + 2011x_{2011}^n = a^{n+1} + 1.$$

Answer. The only sequence that satisfies the condition is

$$(x_1, \dots, x_{2011}) = (1, k, \dots, k) \quad \text{with } k = 2 + 3 + \dots + 2011 = 2023065.$$

Solution. Throughout this solution, the set of positive integers will be denoted by $\mathbb{Z}_+$.

Put $k = 2 + 3 + \dots + 2011 = 2023065$. We have

$$1^n + 2k^n + \cdots 2011k^n = 1 + k \cdot k^n = k^{n+1} + 1$$

for all n, so $(1, k, \dots, k)$ is a valid sequence. We shall prove that it is the only one.

Let a valid sequence $(x_1, \dots, x_{2011})$ be given. For each $n \in \mathbb{Z}_+$ we have some $y_n \in \mathbb{Z}_+$ with

$$x_1^n + 2x_2^n + \dots + 2011x_{2011}^n = y_n^{n+1} + 1.$$

Note that $x_1^n + 2x_2^n + \dots + 2011x_{2011}^n < (x_1 + 2x_2 + \dots + 2011x_{2011})^{n+1}$, which implies that the sequence (y_n) is bounded. In particular, there is some $y \in \mathbb{Z}_+$ with $y_n = y$ for infinitely many n.

Let m be the maximum of all the x_i. Grouping terms with equal x_i together, the sum $x_1^n + 2x_2^n + \dots + 2011x_{2011}^n$ can be written as

$$x_1^n + 2x_2^n + \dots + x_{2011}^n = a_m m^n + a_{m-1}(m-1)^n + \dots + a_1$$

with $a_i \geq 0$ for all i and $a_1 + \dots + a_m = 1 + 2 + \dots + 2011$. So there exist arbitrarily large values of n, for which

$$a_m m^n + \dots + a_1 - 1 - y \cdot y^n = 0. \tag{1}$$

The following lemma will help us to determine the a_i and y:

Lemma. Let integers $b_1, \dots, b_N$ be given and assume that there are arbitrarily large positive integers n with $b_1 + b_2 2^n + \dots + b_N N^n = 0$. Then $b_i = 0$ for all i.

Proof. Suppose that not all b_i are zero. We may assume without loss of generality that $b_N \neq 0$.

Dividing through by N^n gives

$$|b_N| = \left| b_{N-1}\left(\frac{N-1}{N}\right)^n + \cdots + b_1\left(\frac{1}{N}\right)^n \right| \leq (|b_{N-1}| + \cdots + |b_1|)\left(\frac{N-1}{N}\right)^n.$$

The expression $\left(\frac{N-1}{N}\right)^n$ can be made arbitrarily small for n large enough, contradicting the assumption that b_N be non-zero. $\square$

We obviously have $y > 1$. Applying the lemma to (1) we see that $a_m = y = m$, $a_1 = 1$, and all the other a_i are zero. This implies $(x_1, \ldots, x_{2011}) = (1, m, \ldots, m)$. But we also have $1 + m = a_1 + \cdots + a_m = 1 + \cdots + 2011 = 1 + k$ so $m = k$, which is what we wanted to show.

A3

Determine all pairs (f, g) of functions from the set of real numbers to itself that satisfy

$$g(f(x+y)) = f(x) + (2x+y)g(y)$$

for all real numbers x and y.

Answer. Either both f and g vanish identically, or there exists a real number C such that $f(x) = x^2 + C$ and $g(x) = x$ for all real numbers x.

Solution. Clearly all these pairs of functions satisfy the functional equation in question, so it suffices to verify that there cannot be any further ones. Substituting $-2x$ for y in the given functional equation we obtain

$$g(f(-x)) = f(x). \tag{1}$$

Using this equation for $-x-y$ in place of x we obtain

$$f(-x-y) = g(f(x+y)) = f(x) + (2x+y)g(y). \tag{2}$$

Now for any two real numbers a and b, setting $x = -b$ and $y = a+b$ we get

$$f(-a) = f(-b) + (a-b)g(a+b).$$

If c denotes another arbitrary real number we have similarly

$$f(-b) = f(-c) + (b-c)g(b+c)$$

as well as

$$f(-c) = f(-a) + (c-a)g(c+a).$$

Adding all these equations up, we obtain

$$\big((a+c)-(b+c)\big)g(a+b) + \big((a+b)-(a+c)\big)g(b+c) + \big((b+c)-(a+b)\big)g(a+c) = 0.$$

Now given any three real numbers x, y, and z one may determine three reals a, b, and c such that $x = b+c$, $y = c+a$, and $z = a+b$, so that we get

$$(y-x)g(z) + (z-y)g(x) + (x-z)g(y) = 0.$$

This implies that the three points $(x, g(x))$, $(y, g(y))$, and $(z, g(z))$ from the graph of g are collinear. Hence that graph is a line, i.e., g is either a constant or a linear function.

Let us write $g(x) = Ax + B$, where A and B are two real numbers. Substituting $(0, -y)$ for (x, y) in (2) and denoting $C = f(0)$, we have $f(y) = Ay^2 - By + C$. Now, comparing the coefficients of x^2 in (1) we see that $A^2 = A$, so $A = 0$ or $A = 1$.

If $A = 0$, then (1) becomes $B = -Bx + C$ and thus $B = C = 0$, which provides the first of the two solutions mentioned above.

Now suppose $A = 1$. Then (1) becomes $x^2 - Bx + C + B = x^2 - Bx + C$, so $B = 0$. Thus, $g(x) = x$ and $f(x) = x^2 + C$, which is the second solution from above.

Comment. Another way to show that $g(x)$ is either a constant or a linear function is the following. If we interchange x and y in the given functional equation and subtract this new equation from the given one, we obtain

$$f(x) - f(y) = (2y + x)g(x) - (2x + y)g(y).$$

Substituting $(x, 0)$, $(1, x)$, and $(0, 1)$ for (x, y), we get

$$\begin{aligned} f(x) - f(0) &= xg(x) - 2xg(0), \\ f(1) - f(x) &= (2x + 1)g(1) - (x + 2)g(x), \\ f(0) - f(1) &= 2g(0) - g(1). \end{aligned}$$

Taking the sum of these three equations and dividing by 2, we obtain

$$g(x) = x\big(g(1) - g(0)\big) + g(0).$$

This proves that $g(x)$ is either a constant of a linear function.

A4

Determine all pairs (f, g) of functions from the set of positive integers to itself that satisfy

$$f^{g(n)+1}(n) + g^{f(n)}(n) = f(n+1) - g(n+1) + 1$$

for every positive integer n. Here, $f^k(n)$ means $\underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

Answer. The only pair (f, g) of functions that satisfies the equation is given by $f(n) = n$ and $g(n) = 1$ for all n.

Solution. The given relation implies

$$f\left(f^{g(n)}(n)\right) < f(n+1) \quad \text{for all } n, \tag{1}$$

which will turn out to be sufficient to determine f.

Let $y_1 < y_2 < \ldots$ be all the values attained by f (this sequence might be either finite or infinite). We will prove that for every positive n the function f attains at least n values, and we have (i)$_n$: $f(x) = y_n$ if and only if $x = n$, and (ii)$_n$: $y_n = n$. The proof will follow the scheme

$$\text{(i)}_1,\ \text{(ii)}_1,\ \text{(i)}_2,\ \text{(ii)}_2,\ \ldots,\ \text{(i)}_n,\ \text{(ii)}_n,\ \ldots \tag{2}$$

To start, consider any x such that $f(x) = y_1$. If $x > 1$, then (1) reads $f\big(f^{g(x-1)}(x-1)\big) < y_1$, contradicting the minimality of y_1. So we have that $f(x) = y_1$ is equivalent to $x = 1$, establishing (i)$_1$.

Next, assume that for some n statement (i)$_n$ is established, as well as all the previous statements in (2). Note that these statements imply that for all $k \geq 1$ and $a < n$ we have $f^k(x) = a$ if and only if $x = a$.

Now, each value y_i with $1 \leq i \leq n$ is attained at the unique integer i, so y_{n+1} exists. Choose an arbitrary x such that $f(x) = y_{n+1}$; we necessarily have $x > n$. Substituting $x - 1$ into (1) we have $f\big(f^{g(x-1)}(x-1)\big) < y_{n+1}$, which implies

$$f^{g(x-1)}(x-1) \in \{1, \ldots, n\} \tag{3}$$

Set $b = f^{g(x-1)}(x-1)$. If $b < n$ then we would have $x - 1 = b$ which contradicts $x > n$. So $b = n$, and hence $y_n = n$, which proves (ii)$_n$. Next, from (i)$_n$ we now get $f(k) = n \iff k = n$, so removing all the iterations of f in (3) we obtain $x - 1 = b = n$, which proves (i)$_{n+1}$.

So, all the statements in (2) are valid and hence $f(n) = n$ for all n. The given relation between f and g now reads $n + g^n(n) = n + 1 - g(n+1) + 1$ or $g^n(n) + g(n+1) = 2$, from which it immediately follows that we have $g(n) = 1$ for all n.

Comment. Several variations of the above solution are possible. For instance, one may first prove by induction that the smallest n values of f are exactly $f(1) < \cdots < f(n)$ and proceed as follows. We certainly have $f(n) \geq n$ for all n. If there is an n with $f(n) > n$, then $f(x) > x$ for all $x \geq n$. From this we conclude $f^{g(n)+1}(n) > f^{g(n)}(n) > \cdots > f(n)$. But we also have $f^{g(n)+1} < f(n+1)$. Having squeezed in a function value between $f(n)$ and $f(n+1)$, we arrive at a contradiction.

In any case, the inequality (1) plays an essential rôle.

A5

Prove that for every positive integer n, the set $\{2, 3, 4, \ldots, 3n + 1\}$ can be partitioned into n triples in such a way that the numbers from each triple are the lengths of the sides of some obtuse triangle.

Solution. Throughout the solution, we denote by $[a, b]$ the set $\{a, a + 1, \ldots, b\}$. We say that $\{a, b, c\}$ is an *obtuse triple* if a, b, c are the sides of some obtuse triangle.

We prove by induction on n that there exists a partition of $[2, 3n + 1]$ into n obtuse triples A_i $(2 \leq i \leq n + 1)$ having the form $A_i = \{i, a_i, b_i\}$. For the base case $n = 1$, one can simply set $A_2 = \{2, 3, 4\}$. For the induction step, we need the following simple lemma.

Lemma. Suppose that the numbers $a < b < c$ form an obtuse triple, and let x be any positive number. Then the triple $\{a, b + x, c + x\}$ is also obtuse.

Proof. The numbers $a < b + x < c + x$ are the sides of a triangle because $(c + x) - (b + x) = c - b < a$. This triangle is obtuse since $(c+x)^2 - (b+x)^2 = (c-b)(c+b+2x) > (c-b)(c+b) > a^2$. $\square$

Now we turn to the induction step. Let $n > 1$ and put $t = \lfloor n/2 \rfloor < n$. By the induction hypothesis, there exists a partition of the set $[2, 3t + 1]$ into t obtuse triples $A'_i = \{i, a'_i, b'_i\}$ $(i \in [2, t + 1])$. For the same values of i, define $A_i = \{i, a'_i + (n - t), b'_i + (n - t)\}$. The constructed triples are obviously disjoint, and they are obtuse by the lemma. Moreover, we have

$$\bigcup_{i=2}^{t+1} A_i = [2, t + 1] \cup [n + 2, n + 2t + 1].$$

Next, for each $i \in [t + 2, n + 1]$, define $A_i = \{i, n + t + i, 2n + i\}$. All these sets are disjoint, and

$$\bigcup_{i=t+2}^{n+1} A_i = [t + 2, n + 1] \cup [n + 2t + 2, 2n + t + 1] \cup [2n + t + 2, 3n + 1],$$

so

$$\bigcup_{i=2}^{n+1} A_i = [2, 3n + 1].$$

Thus, we are left to prove that the triple A_i is obtuse for each $i \in [t + 2, n + 1]$.

Since $(2n + i) - (n + t + i) = n - t < t + 2 \leq i$, the elements of A_i are the sides of a triangle. Next, we have

$$(2n + i)^2 - (n + t + i)^2 = (n - t)(3n + t + 2i) \geq \frac{n}{2} \cdot (3n + 3(t + 1) + 1) > \frac{n}{2} \cdot \frac{9n}{2} \geq (n + 1)^2 \geq i^2,$$

so this triangle is obtuse. The proof is completed.

A6

Let f be a function from the set of real numbers to itself that satisfies

$$f(x+y) \leq yf(x) + f(f(x)) \tag{1}$$

for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Solution 1. Substituting $y = t - x$, we rewrite (1) as

$$f(t) \leq tf(x) - xf(x) + f(f(x)). \tag{2}$$

Consider now some real numbers a, b and use (2) with $t = f(a)$, $x = b$ as well as with $t = f(b)$, $x = a$. We get

$$\begin{aligned} f(f(a)) - f(f(b)) &\leq f(a)f(b) - bf(b), \\ f(f(b)) - f(f(a)) &\leq f(a)f(b) - af(a). \end{aligned}$$

Adding these two inequalities yields

$$2f(a)f(b) \geq af(a) + bf(b).$$

Now, substitute $b = 2f(a)$ to obtain $2f(a)f(b) \geq af(a) + 2f(a)f(b)$, or $af(a) \leq 0$. So, we get

$$f(a) \geq 0 \quad \text{for all } a < 0. \tag{3}$$

Now suppose $f(x) > 0$ for some real number x. From (2) we immediately get that for every $t < \dfrac{xf(x) - f(f(x))}{f(x)}$ we have $f(t) < 0$. This contradicts (3); therefore

$$f(x) \leq 0 \quad \text{for all real } x, \tag{4}$$

and by (3) again we get $f(x) = 0$ for all $x < 0$.

We are left to find $f(0)$. Setting $t = x < 0$ in (2) we get

$$0 \leq 0 - 0 + f(0),$$

so $f(0) \geq 0$. Combining this with (4) we obtain $f(0) = 0$.

Solution 2. We will also use the condition of the problem in form (2). For clarity we divide the argument into four steps.

Step 1. We begin by proving that f attains nonpositive values only. Assume that there exist some real number z with $f(z) > 0$. Substituting $x = z$ into (2) and setting $A = f(z)$, $B = -zf(z) - f(f(z))$ we get $f(t) \le At + B$ for all real t. Hence, if for any positive real number t we substitute $x = -t$, $y = t$ into (1), we get

$$\begin{aligned} f(0) &\le tf(-t) + f(f(-t)) \le t(-At+B) + Af(-t) + B \\ &\le -t(At-B) + A(-At+B) + B = -At^2 - (A^2 - B)t + (A+1)B. \end{aligned}$$

But surely this cannot be true if we take t to be large enough. This contradiction proves that we have indeed $f(x) \le 0$ for all real numbers x. Note that for this reason (1) entails

$$f(x+y) \le yf(x) \tag{5}$$

for all real numbers x and y.

Step 2. We proceed by proving that f has at least one zero. If $f(0) = 0$, we are done. Otherwise, in view of Step 1 we get $f(0) < 0$. Observe that (5) tells us now $f(y) \le yf(0)$ for all real numbers y. Thus we can specify a positive real number a that is so large that $f(a)^2 > -f(0)$. Put $b = f(a)$ and substitute $x = b$ and $y = -b$ into (5); we learn $-b^2 < f(0) \le -bf(b)$, i.e. $b < f(b)$. Now we apply (2) to $x = b$ and $t = f(b)$, which yields

$$f(f(b)) \le \big(f(b) - b\big)f(b) + f(f(b)),$$

i.e. $f(b) \ge 0$. So in view of Step 1, b is a zero of f.

Step 3. Next we show that if $f(a) = 0$ and $b < a$, then $f(b) = 0$ as well. To see this, we just substitute $x = b$ and $y = a - b$ into (5), thus getting $f(b) \ge 0$, which suffices by Step 1.

Step 4. By Step 3, the solution of the problem is reduced to showing $f(0) = 0$. Pick any zero r of f and substitute $x = r$ and $y = -1$ into (1). Because of $f(r) = f(r-1) = 0$ this gives $f(0) \ge 0$ and hence $f(0) = 0$ by Step 1 again.

Comment 1. Both of these solutions also show $f(x) \le 0$ for all real numbers x. As one can see from Solution 1, this task gets much easier if one already knows that f takes nonnegative values for sufficiently small arguments. Another way of arriving at this statement, suggested by the proposer, is as follows:

Put $a = f(0)$ and substitute $x = 0$ into (1). This gives $f(y) \le ay + f(a)$ for all real numbers y. Thus if for any real number x we plug $y = a - x$ into (1), we obtain

$$f(a) \le (a-x)f(x) + f(f(x)) \le (a-x)f(x) + af(x) + f(a)$$

and hence $0 \le (2a - x)f(x)$. In particular, if $x < 2a$, then $f(x) \ge 0$.

Having reached this point, one may proceed almost exactly as in the first solution to deduce $f(x) \le 0$ for all x. Afterwards the problem can be solved in a few lines as shown in steps 3 and 4 of the second solution.

Comment 2. The original problem also contained the question whether a nonzero function satisfying the problem condition exists. Here we present a family of such functions.

Notice first that if $g : (0, \infty) \longrightarrow [0, \infty)$ denotes any function such that

$$g(x + y) \geq yg(x) \tag{6}$$

for all positive real numbers x and y, then the function f given by

$$f(x) = \begin{cases} -g(x) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{7}$$

automatically satisfies (1). Indeed, we have $f(x) \leq 0$ and hence also $f(f(x)) = 0$ for all real numbers x. So (1) reduces to (5); moreover, this inequality is nontrivial only if x and y are positive. In this last case it is provided by (6).

Now it is not hard to come up with a nonzero function g obeying (6). E.g. $g(z) = Ce^z$ (where C is a positive constant) fits since the inequality $e^y > y$ holds for all (positive) real numbers y. One may also consider the function $g(z) = e^z - 1$; in this case, we even have that f is continuous.

A7

Let a, b, and c be positive real numbers satisfying $\min(a+b, b+c, c+a) > \sqrt{2}$ and $a^2+b^2+c^2 = 3$. Prove that

$$\frac{a}{(b+c-a)^2} + \frac{b}{(c+a-b)^2} + \frac{c}{(a+b-c)^2} \geq \frac{3}{(abc)^2}. \qquad (1)$$

Throughout both solutions, we denote the sums of the form $f(a,b,c) + f(b,c,a) + f(c,a,b)$ by $\sum f(a,b,c)$.

Solution 1. The condition $b + c > \sqrt{2}$ implies $b^2 + c^2 > 1$, so $a^2 = 3 - (b^2 + c^2) < 2$, i.e. $a < \sqrt{2} < b + c$. Hence we have $b + c - a > 0$, and also $c + a - b > 0$ and $a + b - c > 0$ for similar reasons.

We will use the variant of HÖLDER's inequality

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_1^{p+1}}{y_1^p} + \ldots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1 + x_2 + \ldots + x_n)^{p+1}}{(y_1 + y_2 + \ldots + y_n)^p},$$

which holds for all positive real numbers $p, x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$. Applying it to the left-hand side of (1) with $p = 2$ and $n = 3$, we get

$$\sum \frac{a}{(b+c-a)^2} = \sum \frac{(a^2)^3}{a^5(b+c-a)^2} \geq \frac{(a^2+b^2+c^2)^3}{\left(\sum a^{5/2}(b+c-a)\right)^2} = \frac{27}{\left(\sum a^{5/2}(b+c-a)\right)^2}. \qquad (2)$$

To estimate the denominator of the right-hand part, we use an instance of SCHUR's inequality, namely

$$\sum a^{3/2}(a-b)(a-c) \geq 0,$$

which can be rewritten as

$$\sum a^{5/2}(b+c-a) \leq abc(\sqrt{a} + \sqrt{b} + \sqrt{c}).$$

Moreover, by the inequality between the arithmetic mean and the fourth power mean we also have

$$\left(\frac{\sqrt{a} + \sqrt{b} + \sqrt{c}}{3}\right)^4 \leq \frac{a^2+b^2+c^2}{3} = 1,$$

i.e., $\sqrt{a} + \sqrt{b} + \sqrt{c} \leq 3$. Hence, (2) yields

$$\sum \frac{a}{(b+c-a)^2} \geq \frac{27}{\left(abc(\sqrt{a} + \sqrt{b} + \sqrt{c})\right)^2} \geq \frac{3}{a^2b^2c^2},$$

thus solving the problem.

Comment. In this solution, one may also start from the following version of HÖLDER's inequality

$$\left(\sum_{i=1}^{n} a_i^3\right)\left(\sum_{i=1}^{n} b_i^3\right)\left(\sum_{i=1}^{n} c_i^3\right) \geq \left(\sum_{i=1}^{n} a_i b_i c_i\right)^3$$

applied as

$$\sum \frac{a}{(b+c-a)^2} \cdot \sum a^3(b+c-a) \cdot \sum a^2(b+c-a) \geq 27.$$

After doing that, one only needs the slightly better known instances

$$\sum a^3(b+c-a) \leq (a+b+c)abc \qquad \text{and} \qquad \sum a^2(b+c-a) \leq 3abc$$

of SCHUR's Inequality.

Solution 2. As in Solution 1, we mention that all the numbers $b+c-a$, $a+c-b$, $a+b-c$ are positive. We will use only this restriction and the condition

$$a^5 + b^5 + c^5 \geq 3, \tag{3}$$

which is weaker than the given one. Due to the symmetry we may assume that $a \geq b \geq c$.

In view of (3), it suffices to prove the inequality

$$\sum \frac{a^3b^2c^2}{(b+c-a)^2} \geq \sum a^5,$$

or, moving all the terms into the left-hand part,

$$\sum \frac{a^3}{(b+c-a)^2}\left((bc)^2 - (a(b+c-a))^2\right) \geq 0. \tag{4}$$

Note that the signs of the expressions $(yz)^2 - (x(y+z-x))^2$ and $yz - x(y+z-x) = (x-y)(x-z)$ are the same for every positive x, y, z satisfying the triangle inequality. So the terms in (4) corresponding to a and c are nonnegative, and hence it is sufficient to prove that the sum of the terms corresponding to a and b is nonnegative. Equivalently, we need the relation

$$\frac{a^3}{(b+c-a)^2}(a-b)(a-c)(bc+a(b+c-a)) \geq \frac{b^3}{(a+c-b)^2}(a-b)(b-c)(ac+b(a+c-b)).$$

Obviously, we have

$$a^3 \geq b^3 \geq 0, \quad 0 < b+c-a \leq a+c-b, \quad \text{and} \quad a-c \geq b-c \geq 0,$$

hence it suffices to prove that

$$\frac{ab+ac+bc-a^2}{b+c-a} \geq \frac{ab+ac+bc-b^2}{c+a-b}.$$

Since all the denominators are positive, it is equivalent to

$$(c+a-b)(ab+ac+bc-a^2)-(ab+ac+bc-b^2)(b+c-a)\geq 0,$$

or

$$(a-b)(2ab-a^2-b^2+ac+bc)\geq 0.$$

Since $a\geq b$, the last inequality follows from

$$c(a+b)>(a-b)^2$$

which holds since $c>a-b\geq 0$ and $a+b>a-b\geq 0$.

C1

Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \dots, 2^{n-1}$. In a sequence of n moves we place all weights on the balance. In the first move we choose a weight and put it on the left pan. In each of the following moves we choose one of the remaining weights and we add it either to the left or to the right pan. Compute the number of ways in which we can perform these n moves in such a way that the right pan is never heavier than the left pan.

Answer. The number $f(n)$ of ways of placing the n weights is equal to the product of all odd positive integers less than or equal to $2n-1$, i.e. $f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)$.

Solution 1. Assume $n \geq 2$. We claim

$$f(n) = (2n-1)f(n-1). \tag{1}$$

Firstly, note that after the first move the left pan is always at least 1 heavier than the right one. Hence, any valid way of placing the n weights on the scale gives rise, by not considering weight 1, to a valid way of placing the weights $2, 2^2, \dots, 2^{n-1}$.

If we divide the weight of each weight by 2, the answer does not change. So these $n-1$ weights can be placed on the scale in $f(n-1)$ valid ways. Now we look at weight 1. If it is put on the scale in the first move, then it has to be placed on the left side, otherwise it can be placed either on the left or on the right side, because after the first move the difference between the weights on the left pan and the weights on the right pan is at least 2. Hence, there are exactly $2n-1$ different ways of inserting weight 1 in each of the $f(n-1)$ valid sequences for the $n-1$ weights in order to get a valid sequence for the n weights. This proves the claim.

Since $f(1) = 1$, by induction we obtain for all positive integers n

$$f(n) = (2n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1).$$

Comment 1. The word "compute" in the statement of the problem is probably too vague. An alternative but more artificial question might ask for the smallest n for which the number of valid ways is divisible by 2011. In this case the answer would be 1006.

Comment 2. It is useful to remark that the answer is the same for any set of weights where each weight is heavier than the sum of the lighter ones. Indeed, in such cases the given condition is equivalent to asking that during the process the heaviest weight on the balance is always on the left pan.

Comment 3. Instead of considering the lightest weight, one may also consider the last weight put on the balance. If this weight is 2^{n-1} then it should be put on the left pan. Otherwise it may be put on any pan; the inequality would not be violated since at this moment the heaviest weight is already put onto the left pan. In view of the previous comment, in each of these $2n-1$ cases the number of ways to place the previous weights is exactly $f(n-1)$, which yields (1).

Solution 2. We present a different way of obtaining (1). Set $f(0)=1$. Firstly, we find a recurrent formula for $f(n)$.

Assume $n \geq 1$. Suppose that weight 2^{n-1} is placed on the balance in the i-th move with $1 \leq i \leq n$. This weight has to be put on the left pan. For the previous moves we have $\binom{n-1}{i-1}$ choices of the weights and from Comment 2 there are $f(i-1)$ valid ways of placing them on the balance. For later moves there is no restriction on the way in which the weights are to be put on the pans. Therefore, all $(n-i)!2^{n-i}$ ways are possible. This gives

$$f(n)=\sum_{i=1}^{n}\binom{n-1}{i-1}f(i-1)(n-i)!2^{n-i}=\sum_{i=1}^{n}\frac{(n-1)!f(i-1)2^{n-i}}{(i-1)!}. \tag{2}$$

Now we are ready to prove (1). Using $n-1$ instead of n in (2) we get

$$f(n-1)=\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}.$$

Hence, again from (2) we get

$$\begin{aligned} f(n) &= 2(n-1)\sum_{i=1}^{n-1}\frac{(n-2)!f(i-1)2^{n-1-i}}{(i-1)!}+f(n-1) \\ &= (2n-2)f(n-1)+f(n-1)=(2n-1)f(n-1), \end{aligned}$$

QED.

Comment. There exist different ways of obtaining the formula (2). Here we show one of them.

Suppose that in the first move we use weight 2^{n-i+1}. Then the lighter $n-i$ weights may be put on the balance at any moment and on either pan. This gives $2^{n-i}\cdot(n-1)!/(i-1)!$ choices for the moves (moments and choices of pan) with the lighter weights. The remaining $i-1$ moves give a valid sequence for the $i-1$ heavier weights and this is the only requirement for these moves, so there are $f(i-1)$ such sequences. Summing over all $i=1,2,\ldots,n$ we again come to (2).

C2

Suppose that 1000 students are standing in a circle. Prove that there exists an integer k with $100 \le k \le 300$ such that in this circle there exists a contiguous group of $2k$ students, for which the first half contains the same number of girls as the second half.

Solution. Number the students consecutively from 1 to 1000. Let $a_i = 1$ if the ith student is a girl, and $a_i = 0$ otherwise. We expand this notion for all integers i by setting $a_{i+1000} = a_{i-1000} = a_i$. Next, let

$$S_k(i) = a_i + a_{i+1} + \cdots + a_{i+k-1}.$$

Now the statement of the problem can be reformulated as follows:

There exist an integer k with $100 \le k \le 300$ and an index i such that $S_k(i) = S_k(i+k)$.

Assume now that this statement is false. Choose an index i such that $S_{100}(i)$ attains the maximal possible value. In particular, we have $S_{100}(i-100) - S_{100}(i) < 0$ and $S_{100}(i) - S_{100}(i+100) > 0$, for if we had an equality, then the statement would hold. This means that the function $S(j) - S(j+100)$ changes sign somewhere on the segment $[i-100, i]$, so there exists some index $j \in [i-100, i-1]$ such that

$$S_{100}(j) \le S_{100}(j+100) - 1, \quad \text{but} \quad S_{100}(j+1) \ge S_{100}(j+101) + 1. \tag{1}$$

Subtracting the first inequality from the second one, we get $a_{j+100} - a_j \ge a_{j+200} - a_{j+100} + 2$, so

$$a_j = 0, \quad a_{j+100} = 1, \quad a_{j+200} = 0.$$

Substituting this into the inequalities of (1), we also obtain $S_{99}(j+1) \le S_{99}(j+101) \le S_{99}(j+1)$, which implies

$$S_{99}(j+1) = S_{99}(j+101). \tag{2}$$

Now let k and ℓ be the least positive integers such that $a_{j-k} = 1$ and $a_{j+200+\ell} = 1$. By symmetry, we may assume that $k \ge \ell$. If $k \ge 200$ then we have $a_j = a_{j-1} = \cdots = a_{j-199} = 0$, so $S_{100}(j-199) = S_{100}(j-99) = 0$, which contradicts the initial assumption. Hence $\ell \le k \le 199$.

Finally, we have

$$S_{100+\ell}(j-\ell+1) = (a_{j-\ell+1} + \cdots + a_j) + S_{99}(j+1) + a_{j+100} = S_{99}(j+1) + 1,$$
$$S_{100+\ell}(j+101) = S_{99}(j+101) + (a_{j+200} + \cdots + a_{j+200+\ell-1}) + a_{j+200+\ell} = S_{99}(j+101) + 1.$$

Comparing with (2) we get $S_{100+\ell}(j-\ell+1) = S_{100+\ell}(j+101)$ and $100 + \ell \le 299$, which again contradicts our assumption.

Comment. It may be seen from the solution that the number 300 from the problem statement can be replaced by 299. Here we consider some improvements of this result. Namely, we investigate which interval can be put instead of $[100, 300]$ in order to keep the problem statement valid.

First of all, the two examples

$$\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{167},\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{167},\underbrace{1,1,\dots,1}_{167},\underbrace{0,0,\dots,0}_{165}$$

and

$$\underbrace{1,1,\dots,1}_{249},\underbrace{0,0,\dots,0}_{251},\underbrace{1,1,\dots,1}_{249},\underbrace{0,0,\dots,0}_{251}$$

show that the interval can be changed neither to $[84, 248]$ nor to $[126, 374]$.

On the other hand, we claim that this interval can be changed to $[125, 250]$. Note that this statement is invariant under replacing all 1's by 0's and vice versa. Assume, to the contrary, that there is no admissible $k \in [125, 250]$. The arguments from the solution easily yield the following lemma.

Lemma. Under our assumption, suppose that for some indices $i < j$ we have $S_{125}(i) \le S_{125}(i+125)$ but $S_{125}(j) \ge S_{125}(j+125)$. Then there exists some $t \in [i, j-1]$ such that $a_t = a_{t-1} = \dots = a_{t-125} = 0$ and $a_{t+250} = a_{t+251} = \dots = a_{t+375} = 0$. □

Let us call a segment $[i, j]$ of indices a *crowd*, if (a) $a_i = a_{i+1} = \dots = a_j$, but $a_{i-1} \ne a_i \ne a_{j+1}$, and (b) $j - i \ge 125$. Now, using the lemma, one can get in the same way as in the solution that there exists some crowd. Take all the crowds in the circle, and enumerate them in cyclic order as $A_1, \dots, A_d$. We also assume always that $A_{s+d} = A_{s-d} = A_s$.

Consider one of the crowds, say A_1. We have $A_1 = [i, i+t]$ with $125 \le t \le 248$ (if $t \ge 249$, then $a_i = a_{i+1} = \dots = a_{i+249}$ and therefore $S_{125}(i) = S_{125}(i+125)$, which contradicts our assumption). We may assume that $a_i = 1$. Then we have $S_{125}(i+t-249) \le 125 = S_{125}(i+t-124)$ and $S_{125}(i) = 125 \ge S_{125}(i+125)$, so by the lemma there exists some index $j \in [i+t-249, i-1]$ such that the segments $[j-125, j]$ and $[j+250, j+375]$ are contained in some crowds.

Let us fix such j and denote the segment $[j+1, j+249]$ by B_1. Clearly, $A_1 \subseteq B_1$. Moreover, B_1 cannot contain any crowd other than A_1 since $|B_1| = 249 < 2 \cdot 126$. Hence it is clear that $j \in A_d$ and $j + 250 \in A_2$. In particular, this means that the genders of A_d and A_2 are different from that of A_1.

Performing this procedure for every crowd A_s, we find segments $B_s = [j_s + 1, j_s + 249]$ such that $|B_s| = 249$, $A_s \subseteq B_s$, and $j_s \in A_{s-1}$, $j_s + 250 \in A_{s+1}$. So, B_s covers the whole segment between A_{s-1} and A_{s+1}, hence the sets $B_1, \dots, B_d$ cover some 1000 consecutive indices. This implies $249d \ge 1000$, and $d \ge 5$. Moreover, the gender of A_i is alternating, so d is even; therefore $d \ge 6$.

Consider now three segments $A_1 = [i_1, i_1']$, $B_2 = [j_2 + 1, j_2 + 249]$, $A_3 = [i_3, i_3']$. By construction, we have $[j_2 - 125, j_2] \subseteq A_1$ and $[j_2 + 250, j_2 + 375] \subseteq A_3$, whence $i_1 \le j_2 - 125$, $i_3' \ge j_2 + 375$. Therefore $i_3' - i_1 \ge 500$. Analogously, if $A_4 = [i_4, i_4']$, $A_6 = [i_6, i_6']$ then $i_6' - i_4 \ge 500$. But from $d \ge 6$ we get $i_1 < i_3' < i_4 < i_6' < i_1 + 1000$, so $1000 > (i_3' - i_1) + (i_6' - i_4) \ge 500 + 500$. This final contradiction shows that our claim holds.

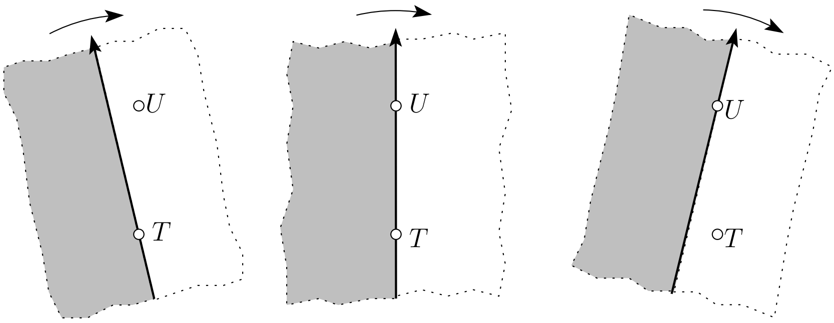
One may even show that the interval in the statement of the problem may be replaced by $[125, 249]$ (both these numbers cannot be improved due to the examples above). But a proof of this fact is a bit messy, and we do not present it here.

C3

Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. By a *windmill* we mean a process as follows. Start with a line ℓ going through a point $P \in \mathcal{S}$. Rotate ℓ clockwise around the *pivot* P until the line contains another point Q of $\mathcal{S}$. The point Q now takes over as the new pivot. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

Show that for a suitable $P \in \mathcal{S}$ and a suitable starting line ℓ containing P, the resulting windmill will visit each point of $\mathcal{S}$ as a pivot infinitely often.

Solution. Give the rotating line an orientation and distinguish its sides as the *oranje side* and the *blue side*. Notice that whenever the pivot changes from some point T to another point U, after the change, T is on the same side as U was before. Therefore, the number of elements of $\mathcal{S}$ on the oranje side and the number of those on the blue side remain the same throughout the whole process (except for those moments when the line contains two points).



First consider the case that $|\mathcal{S}| = 2n+1$ is odd. We claim that through any point $T \in \mathcal{S}$, there is a line that has n points on each side. To see this, choose an oriented line through T containing no other point of $\mathcal{S}$ and suppose that it has $n+r$ points on its oranje side. If $r=0$ then we have established the claim, so we may assume that $r \neq 0$. As the line rotates through 180° around T, the number of points of $\mathcal{S}$ on its oranje side changes by 1 whenever the line passes through a point; after 180°, the number of points on the oranje side is $n-r$. Therefore there is an intermediate stage at which the oranje side, and thus also the blue side, contains n points.

Now select the point P arbitrarily, and choose a line through P that has n points of $\mathcal{S}$ on each side to be the initial state of the windmill. We will show that during a rotation over 180°, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and select a line ℓ through T that separates $\mathcal{S}$ into equal halves. The point T is the unique point of $\mathcal{S}$ through which a line in this direction can separate the points of $\mathcal{S}$ into equal halves (parallel translation would disturb the balance). Therefore, when the windmill line is parallel to ℓ, it must be ℓ itself, and so pass through T.

Next suppose that $|\mathcal{S}| = 2n$. Similarly to the odd case, for every $T \in \mathcal{S}$ there is an oriented line through T with $n-1$ points on its oranje side and n points on its blue side. Select such an oriented line through an arbitrary P to be the initial state of the windmill.

We will now show that during a rotation over $360°$, the line of the windmill visits each point of $\mathcal{S}$ as a pivot. To see this, select any point T of $\mathcal{S}$ and an oriented line ℓ through T that separates $\mathcal{S}$ into two subsets with $n-1$ points on its oranje and n points on its blue side. Again, parallel translation would change the numbers of points on the two sides, so when the windmill line is parallel to ℓ with the same orientation, the windmill line must pass through T.

Comment. One may shorten this solution in the following way.

Suppose that $|\mathcal{S}| = 2n+1$. Consider any line ℓ that separates $\mathcal{S}$ into equal halves; this line is unique given its direction and contains some point $T \in \mathcal{S}$. Consider the windmill starting from this line. When the line has made a rotation of $180°$, it returns to the same location but the oranje side becomes blue and vice versa. So, for each point there should have been a moment when it appeared as pivot, as this is the only way for a point to pass from on side to the other.

Now suppose that $|\mathcal{S}| = 2n$. Consider a line having $n-1$ and n points on the two sides; it contains some point T. Consider the windmill starting from this line. After having made a rotation of $180°$, the windmill line contains some different point R, and each point different from T and R has changed the color of its side. So, the windmill should have passed through all the points.

C4

Determine the greatest positive integer k that satisfies the following property: The set of positive integers can be partitioned into k subsets $A_1, A_2, \ldots, A_k$ such that for all integers $n \geq 15$ and all $i \in \{1, 2, \ldots, k\}$ there exist two distinct elements of A_i whose sum is n.

Answer. The greatest such number k is 3.

Solution 1. There are various examples showing that $k = 3$ does indeed have the property under consideration. E.g. one can take

$$A_1 = \{1, 2, 3\} \cup \{3m \mid m \geq 4\},$$
$$A_2 = \{4, 5, 6\} \cup \{3m - 1 \mid m \geq 4\},$$
$$A_3 = \{7, 8, 9\} \cup \{3m - 2 \mid m \geq 4\}.$$

To check that this partition fits, we notice first that the sums of two distinct elements of A_i obviously represent all numbers $n \geq 1 + 12 = 13$ for $i = 1$, all numbers $n \geq 4 + 11 = 15$ for $i = 2$, and all numbers $n \geq 7 + 10 = 17$ for $i = 3$. So, we are left to find representations of the numbers 15 and 16 as sums of two distinct elements of A_3. These are $15 = 7+8$ and $16 = 7+9$.

Let us now suppose that for some $k \geq 4$ there exist sets $A_1, A_2, \ldots, A_k$ satisfying the given property. Obviously, the sets $A_1, A_2, A_3, A_4 \cup \cdots \cup A_k$ also satisfy the same property, so one may assume $k = 4$.

Put $B_i = A_i \cap \{1, 2, \ldots, 23\}$ for $i = 1, 2, 3, 4$. Now for any index i each of the ten numbers $15, 16, \ldots, 24$ can be written as sum of two distinct elements of B_i. Therefore this set needs to contain at least five elements. As we also have $|B_1| + |B_2| + |B_3| + |B_4| = 23$, there has to be some index j for which $|B_j| = 5$. Let $B_j = \{x_1, x_2, x_3, x_4, x_5\}$. Finally, now the sums of two distinct elements of A_j representing the numbers $15, 16, \ldots, 24$ should be exactly all the pairwise sums of the elements of B_j. Calculating the sum of these numbers in two different ways, we reach

$$4(x_1 + x_2 + x_3 + x_4 + x_5) = 15 + 16 + \ldots + 24 = 195.$$

Thus the number 195 should be divisible by 4, which is false. This contradiction completes our solution.

Comment. There are several variation of the proof that k should not exceed 3. E.g., one may consider the sets $C_i = A_i \cap \{1, 2, \ldots, 19\}$ for $i = 1, 2, 3, 4$. As in the previous solution one can show that for some index j one has $|C_j| = 4$, and the six pairwise sums of the elements of C_j should represent all numbers $15, 16, \ldots, 20$. Let $C_j = \{y_1, y_2, y_3, y_4\}$ with $y_1 < y_2 < y_3 < y_4$. It is not hard to deduce

$C_j = \{7, 8, 9, 11\}$, so in particular we have $1 \notin C_j$. Hence it is impossible to represent 21 as sum of two distinct elements of A_j, which completes our argument.

Solution 2. Again we only prove that $k \leq 3$. Assume that $A_1, A_2, \ldots, A_k$ is a partition satisfying the given property. We construct a graph $\mathcal{G}$ on the set $V = \{1, 2, \ldots, 18\}$ of *vertices* as follows. For each $i \in \{1, 2, \ldots, k\}$ and each $d \in \{15, 16, 17, 19\}$ we choose one pair of distinct elements $a, b \in A_i$ with $a + b = d$, and we draw an *edge* in the i^{th} color connecting a with b. By hypothesis, $\mathcal{G}$ has exactly 4 edges of each color.

Claim. The graph $\mathcal{G}$ contains at most one circuit.

Proof. Note that all the connected components of $\mathcal{G}$ are monochromatic and hence contain at most four edges. Thus also all circuits of $\mathcal{G}$ are monochromatic and have length at most four. Moreover, each component contains at most one circuit since otherwise it should contain at least five edges.

Suppose that there is a 4-cycle in $\mathcal{G}$, say with vertices a, b, c, and d in order. Then $\{a + b, b + c, c + d, d + a\} = \{15, 16, 17, 19\}$. Taking sums we get $2(a + b + c + d) = 15 + 16 + 17 + 19$ which is impossible for parity reasons. Thus all circuits of $\mathcal{G}$ are triangles.

Now if the vertices a, b, and c form such a triangle, then by a similar reasoning the set $\{a + b, b + c, c + a\}$ coincides with either $\{15, 16, 17\}$, or $\{15, 16, 19\}$, or $\{16, 17, 19\}$, or $\{15, 17, 19\}$. The last of these alternatives can be excluded for parity reasons again, whilst in the first three cases the set $\{a, b, c\}$ appears to be either $\{7, 8, 9\}$, or $\{6, 9, 10\}$, or $\{7, 9, 10\}$, respectively. Thus, a component containing a circuit should contain 9 as a vertex. Therefore there is at most one such component and hence at most one circuit. $\square$

By now we know that $\mathcal{G}$ is a graph with $4k$ edges, at least k components and at most one circuit. Consequently, $\mathcal{G}$ must have at least $4k + k - 1$ vertices. Thus $5k - 1 \leq 18$, and $k \leq 3$.

C5

Let m be a positive integer and consider a checkerboard consisting of m by m unit squares. At the midpoints of some of these unit squares there is an ant. At time 0, each ant starts moving with speed 1 parallel to some edge of the checkerboard. When two ants moving in opposite directions meet, they both turn 90° clockwise and continue moving with speed 1. When more than two ants meet, or when two ants moving in perpendicular directions meet, the ants continue moving in the same direction as before they met. When an ant reaches one of the edges of the checkerboard, it falls off and will not re-appear.

Considering all possible starting positions, determine the latest possible moment at which the last ant falls off the checkerboard or prove that such a moment does not necessarily exist.

Antswer. The latest possible moment for the last ant to fall off is $\frac{3m}{2} - 1$.

Solution. For $m = 1$ the answer is clearly correct, so assume $m > 1$. In the sequel, the word *collision* will be used to denote meeting of exactly two ants, moving in opposite directions.

If at the beginning we place an ant on the southwest corner square facing east and an ant on the southeast corner square facing west, then they will meet in the middle of the bottom row at time $\frac{m-1}{2}$. After the collision, the ant that moves to the north will stay on the board for another $m - \frac{1}{2}$ time units and thus we have established an example in which the last ant falls off at time $\frac{m-1}{2} + m - \frac{1}{2} = \frac{3m}{2} - 1$. So, we are left to prove that this is the latest possible moment.

Consider any collision of two ants a and a'. Let us change the rule for this collision, and enforce these two ants to turn anticlockwise. Then the succeeding behavior of all the ants does not change; the only difference is that a and a' swap their positions. These arguments may be applied to any collision separately, so we may assume that at any collision, either both ants rotate clockwise or both of them rotate anticlockwise by our own choice.

For instance, we may assume that there are only two types of ants, depending on their initial direction: *NE-ants*, which move only north or east, and *SW-ants*, moving only south and west. Then we immediately obtain that all ants will have fallen off the board after $2m - 1$ time units. However, we can get a better bound by considering the last moment at which a given ant collides with another ant.

Choose a coordinate system such that the corners of the checkerboard are $(0, 0)$, $(m, 0)$, (m, m) and $(0, m)$. At time t, there will be no NE-ants in the region $\{(x, y) : x + y < t + 1\}$ and no SW-ants in the region $\{(x, y) : x + y > 2m - t - 1\}$. So if two ants collide at (x, y) at time t, we have

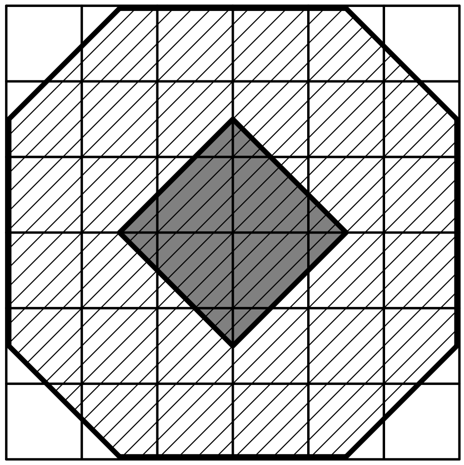
$$t + 1 \le x + y \le 2m - t - 1. \tag{1}$$

Analogously, we may change the rules so that each ant would move either alternatingly north and west, or alternatingly south and east. By doing so, we find that apart from (1) we also have $|x-y| \leq m-t-1$ for each collision at point (x, y) and time t.

To visualize this, put

$$B(t)=\left\{(x, y) \in[0, m]^{2}: t+1 \leq x+y \leq 2 m-t-1 \text { and }|x-y| \leq m-t-1\right\} .$$

An ant can thus only collide with another ant at time t if it happens to be in the region $B(t)$. The following figure displays $B(t)$ for $t=\frac{1}{2}$ and $t=\frac{7}{2}$ in the case $m=6$:



Now suppose that an NE-ant has its last collision at time t and that it does so at the point (x, y) (if the ant does not collide at all, it will fall off the board within $m-\frac{1}{2}<\frac{3 m}{2}-1$ time units, so this case can be ignored). Then we have $(x, y) \in B(t)$ and thus $x+y \geq t+1$ and $x-y \geq-(m-t-1)$. So we get

$$x \geq \frac{(t+1)-(m-t-1)}{2}=t+1-\frac{m}{2} .$$

By symmetry we also have $y \geq t+1-\frac{m}{2}$, and hence $\min \{x, y\} \geq t+1-\frac{m}{2}$. After this collision, the ant will move directly to an edge, which will take at most $m-\min \{x, y\}$ units of time. In sum, the total amount of time the ant stays on the board is at most

$$t+(m-\min \{x, y\}) \leq t+m-\left(t+1-\frac{m}{2}\right)=\frac{3 m}{2}-1 .$$

By symmetry, the same bound holds for SW-ants as well.

C6

Let n be a positive integer and let $W = \ldots x_{-1}x_0x_1x_2\ldots$ be an infinite periodic word consisting of the letters a and b. Suppose that the minimal period N of W is greater than 2^n.

A finite nonempty word U is said to *appear* in W if there exist indices $k \leq \ell$ such that $U = x_kx_{k+1}\ldots x_\ell$. A finite word U is called *ubiquitous* if the four words Ua, Ub, aU, and bU all appear in W. Prove that there are at least n ubiquitous finite nonempty words.

Solution. Throughout the solution, all the words are nonempty. For any word R of length m, we call the number of indices $i \in \{1,2,\ldots,N\}$ for which R coincides with the subword $x_{i+1}x_{i+2}\ldots x_{i+m}$ of W the *multiplicity* of R and denote it by $\mu(R)$. Thus a word R appears in W if and only if $\mu(R) > 0$. Since each occurrence of a word in W is both succeeded by either the letter a or the letter b and similarly preceded by one of those two letters, we have

$$\mu(R) = \mu(Ra) + \mu(Rb) = \mu(aR) + \mu(bR) \tag{1}$$

for all words R.

We claim that the condition that N is in fact the minimal period of W guarantees that each word of length N has multiplicity 1 or 0 depending on whether it appears or not. Indeed, if the words $x_{i+1}x_{i+2}\ldots x_{i+N}$ and $x_{j+1}\ldots x_{j+N}$ are equal for some $1 \leq i < j \leq N$, then we have $x_{i+a} = x_{j+a}$ for every integer a, and hence $j - i$ is also a period.

Moreover, since $N > 2^n$, at least one of the two words a and b has a multiplicity that is strictly larger than 2^{n-1}.

For each $k = 0, 1, \ldots, n-1$, let U_k be a subword of W whose multiplicity is strictly larger than 2^k and whose length is maximal subject to this property. Note that such a word exists in view of the two observations made in the two previous paragraphs.

Fix some index $k \in \{0,1,\ldots,n-1\}$. Since the word U_kb is longer than U_k, its multiplicity can be at most 2^k, so in particular $\mu(U_kb) < \mu(U_k)$. Therefore, the word U_ka has to appear by (1). For a similar reason, the words U_kb, aU_k, and bU_k have to appear as well. Hence, the word U_k is ubiquitous. Moreover, if the multiplicity of U_k were strictly greater than 2^{k+1}, then by (1) at least one of the two words U_ka and U_kb would have multiplicity greater than 2^k and would thus violate the maximality condition imposed on U_k.

So we have $\mu(U_0) \leq 2 < \mu(U_1) \leq 4 < \ldots \leq 2^{n-1} < \mu(U_{n-1})$, which implies in particular that the words $U_0, U_1, \ldots, U_{n-1}$ have to be distinct. As they have been proved to be ubiquitous as well, the problem is solved.

Comment 1. There is an easy construction for obtaining ubiquitous words from appearing words whose multiplicity is at least two. Starting with any such word U we may simply extend one of its occurrences in W forwards and backwards as long as its multiplicity remains fixed, thus arriving at a word that one might call the *ubiquitous prolongation* $p(U)$ of U.

There are several variants of the argument in the second half of the solution using the concept of prolongation. For instance, one may just take all ubiquitous words $U_1, U_2, \ldots, U_\ell$ ordered by increasing multiplicity and then prove for $i \in \{1, 2, \ldots, \ell\}$ that $\mu(U_i) \leq 2^i$. Indeed, assume that i is a minimal counterexample to this statement; then by the arguments similar to those presented above, the ubiquitous prolongation of one of the words $U_i a$, $U_i b$, aU_i or bU_i violates the definition of U_i.

Now the multiplicity of one of the two letters a and b is strictly greater than 2^{n-1}, so passing to ubiquitous prolongations once more we obtain $2^{n-1} < \mu(U_\ell) \leq 2^\ell$, which entails $\ell \geq n$, as needed.

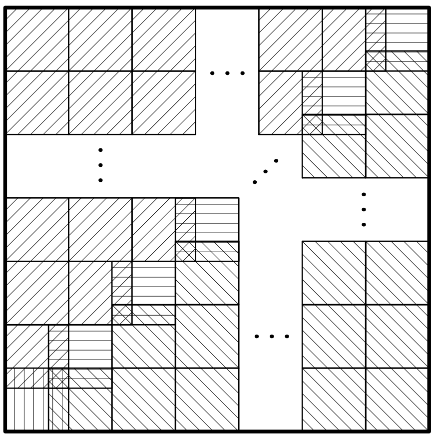
Comment 2. The bound n for the number of ubiquitous subwords in the problem statement is not optimal, but it is close to an optimal one in the following sense. There is a universal constant $C > 0$ such that for each positive integer n there exists an infinite periodic word W whose minimal period is greater than 2^n but for which there exist fewer than Cn ubiquitous words.

C7

On a square table of 2011 by 2011 cells we place a finite number of napkins that each cover a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?

Answer. $2011^2 - \big((52^2 - 35^2) \cdot 39 - 17^2\big) = 4044121 - 57392 = 3986729.$

Solution 1. Let $m = 39$, then $2011 = 52m - 17$. We begin with an example showing that there can exist 3986729 cells carrying the same positive number.



To describe it, we number the columns from the left to the right and the rows from the bottom to the top by $1, 2, \ldots, 2011$. We will denote each napkin by the coordinates of its lower-left cell. There are four kinds of napkins: first, we take all napkins $(52i + 36, 52j + 1)$ with $0 \le j \le i \le m - 2$; second, we use all napkins $(52i + 1, 52j + 36)$ with $0 \le i \le j \le m - 2$; third, we use all napkins $(52i + 36, 52i + 36)$ with $0 \le i \le m - 2$; and finally the napkin $(1, 1)$. Different groups of napkins are shown by different types of hatchings in the picture.

Now except for those squares that carry two or more different hatchings, all squares have the number 1 written into them. The number of these exceptional cells is easily computed to be $(52^2 - 35^2)m - 17^2 = 57392$.

We are left to prove that 3986729 is an upper bound for the number of cells containing the same number. Consider any configuration of napkins and any positive integer M. Suppose there are g cells with a number different from M. Then it suffices to show $g \ge 57392$. Throughout the solution, a *line* will mean either a row or a column.

Consider any line ℓ. Let $a_1, \ldots, a_{52m-17}$ be the numbers written into its consecutive cells. For $i = 1, 2, \ldots, 52$, let $s_i = \sum_{t \equiv i \pmod{52}} a_t$. Note that $s_1, \ldots, s_{35}$ have m terms each, while $s_{36}, \ldots, s_{52}$ have $m-1$ terms each. Every napkin intersecting ℓ contributes exactly 1 to each s_i;

hence the number s of all those napkins satisfies $s_1 = \cdots = s_{52} = s$. Call the line ℓ *rich* if $s > (m-1)M$ and *poor* otherwise.

Suppose now that ℓ is rich. Then in each of the sums $s_{36}, \ldots, s_{52}$ there exists a term greater than M; consider all these terms and call the corresponding cells the *rich bad cells* for this line. So, each rich line contains at least 17 cells that are bad for this line.

If, on the other hand, ℓ is poor, then certainly $s < mM$ so in each of the sums $s_1, \ldots, s_{35}$ there exists a term less than M; consider all these terms and call the corresponding cells the *poor bad cells* for this line. So, each poor line contains at least 35 cells that are bad for this line.

Let us call all indices congruent to $1, 2, \ldots,$ or 35 modulo 52 *small*, and all other indices, i.e. those congruent to $36, 37, \ldots,$ or 52 modulo 52, *big*. Recall that we have numbered the columns from the left to the right and the rows from the bottom to the top using the numbers $1, 2, \ldots, 52m - 17$; we say that a line is *big* or *small* depending on whether its index is big or small. By definition, all rich bad cells for the rows belong to the big columns, while the poor ones belong to the small columns, and vice versa.

In each line, we put a *strawberry* on each cell that is bad for this line. In addition, for each **small rich** line we put an *extra* strawberry on each of its (rich) bad cells. A cell gets the strawberries from its row and its column independently.

Notice now that a cell with a strawberry on it contains a number different from M. If this cell gets a strawberry by the extra rule, then it contains a number greater than M. Moreover, it is either in a small row and in a big column, or vice versa. Suppose that it is in a small row, then it is not bad for its column. So it has not more than two strawberries in this case. On the other hand, if the extra rule is not applied to some cell, then it also has not more than two strawberries. So, the total number N of strawberries is at most $2g$.

We shall now estimate N in a different way. For each of the $2 \cdot 35m$ small lines, we have introduced at least 34 strawberries if it is rich and at least 35 strawberries if it is poor, so at least 34 strawberries in any case. Similarly, for each of the $2 \cdot 17(m-1)$ big lines, we put at least $\min(17, 35) = 17$ strawberries. Summing over all lines we obtain

$$2g \geq N \geq 2(35m \cdot 34 + 17(m-1) \cdot 17) = 2(1479m - 289) = 2 \cdot 57392,$$

as desired.

Comment. The same reasoning applies also if we replace 52 by R and 2011 by $Rm - H$, where m, R, and H are integers with $m, R \geq 1$ and $0 \leq H \leq \frac{1}{3}R$. More detailed information is provided after the next solution.

Solution 2. We present a different proof of the estimate which is the hard part of the problem. Let $S = 35$, $H = 17$, $m = 39$; so the table size is $2011 = Sm + H(m-1)$, and the napkin size is $52 = S + H$. Fix any positive integer M and call a cell *vicious* if it contains a number distinct from M. We will prove that there are at least $H^2(m-1)+2SHm$ vicious cells.

Firstly, we introduce some terminology. As in the previous solution, we number rows and columns and we use the same notions of *small* and *big* indices and lines; so, an index is small if it is congruent to one of the numbers $1,2,\dots,S$ modulo $(S+H)$. The numbers $1,2,\dots,S+H$ will be known as *residues*. For two residues i and j, we say that a cell is of *type* (i,j) if the index of its row is congruent to i and the index of its column to j modulo $(S+H)$. The number of vicious cells of this type is denoted by v_{ij}.

Let s, s' be two variables ranging over small residues and let h, h' be two variables ranging over big residues. A cell is said to be of *class* A, B, C, or D if its type is of shape (s,s'), (s,h), (h,s), or (h,h'), respectively. The numbers of vicious cells belonging to these classes are denoted in this order by a, b, c, and d. Observe that each cell belongs to exactly one class.

Claim 1. We have

$$m \le \frac{a}{S^2}+\frac{b+c}{2SH}. \tag{1}$$

Proof. Consider an arbitrary small row r. Denote the numbers of vicious cells on r belonging to the classes A and B by α and β, respectively. As in the previous solution, we obtain that $\alpha \ge S$ or $\beta \ge H$. So in each case we have $\frac{\alpha}{S}+\frac{\beta}{H}\ge 1$.

Performing this argument separately for each small row and adding up all the obtained inequalities, we get $\frac{a}{S}+\frac{b}{H}\ge mS$. Interchanging rows and columns we similarly get $\frac{a}{S}+\frac{c}{H}\ge mS$. Summing these inequalities and dividing by $2S$ we get what we have claimed. $\square$

Claim 2. Fix two small residue s, s' and two big residues h, h'. Then $2m-1\le v_{ss'}+v_{sh'}+v_{hh'}$.

Proof. Each napkin covers exactly one cell of type (s,s'). Removing all napkins covering a vicious cell of this type, we get another collection of napkins, which covers each cell of type (s,s') either 0 or M times depending on whether the cell is vicious or not. Hence $(m^2-v_{ss'})M$ napkins are left and throughout the proof of Claim 2 we will consider only these remaining napkins. Now, using a red pen, write in each cell the number of napkins covering it. Notice that a cell containing a red number greater than M is surely vicious.

We call two cells *neighbors* if they can be simultaneously covered by some napkin. So, each cell of type (h,h') has not more than four neighbors of type (s,s'), while each cell of type (s,h') has not more than two neighbors of each of the types (s,s') and (h,h'). Therefore, each red number at a cell of type (h,h') does not exceed $4M$, while each red number at a cell of type (s,h') does not exceed $2M$.

Let x, y, and z be the numbers of cells of type (h,h') whose red number belongs to $(M,2M]$, $(2M,3M]$, and $(3M,4M]$, respectively. All these cells are vicious, hence $x+y+z\le v_{hh'}$. The red numbers appearing in cells of type (h,h') clearly sum up to $(m^2-v_{ss'})M$. Bounding each of these numbers by a multiple of M we get

$$(m^2-v_{ss'})M \le \big((m-1)^2-(x+y+z)\big)M+2xM+3yM+4zM,$$

i.e.

$$2m-1 \le v_{ss'} + x + 2y + 3z \le v_{ss'} + v_{hh'} + y + 2z.$$

So, to prove the claim it suffices to prove that $y + 2z \le v_{sh'}$.

For a cell δ of type (h, h') and a cell β of type (s, h') we say that δ *forces* β if there are more than M napkins covering both of them. Since each red number in a cell of type (s, h') does not exceed $2M$, it cannot be forced by more than one cell.

On the other hand, if a red number in a (h, h')-cell belongs to $(2M, 3M]$, then it forces at least one of its neighbors of type (s, h') (since the sum of red numbers in their cells is greater than $2M$). Analogously, an (h, h')-cell with the red number in $(3M, 4M]$ forces both its neighbors of type (s, h'), since their red numbers do not exceed $2M$. Therefore there are at least $y + 2z$ forced cells and clearly all of them are vicious, as desired. □

Claim 3. We have

$$2m-1 \le \frac{a}{S^2} + \frac{b+c}{2SH} + \frac{d}{H^2}. \tag{2}$$

Proof. Averaging the previous result over all S^2H^2 possibilities for the quadruple (s, s', h, h'), we get $2m-1 \le \frac{a}{S^2} + \frac{b}{SH} + \frac{d}{H^2}$. Due to the symmetry between rows and columns, the same estimate holds with b replaced by c. Averaging these two inequalities we arrive at our claim. □

Now let us multiply (2) by H^2, multiply (1) by $(2SH - H^2)$ and add them; we get

$$H^2(2m-1)+(2SH-H^2)m \le a\cdot\frac{H^2+2SH-H^2}{S^2}+(b+c)\frac{H^2+2SH-H^2}{2SH}+d = a\cdot\frac{2H}{S}+b+c+d.$$

The left-hand side is exactly $H^2(m-1) + 2SHm$, while the right-hand side does not exceed $a + b + c + d$ since $2H \le S$. Hence we come to the desired inequality.

Comment 1. Claim 2 is the key difference between the two solutions, because it allows to get rid of the notions of rich and poor cells. However, one may prove it by the "strawberry method" as well. It suffices to put a strawberry on each cell which is bad for an s-row, and a strawberry on each cell which is bad for an h'-column. Then each cell would contain not more than one strawberry.

Comment 2. Both solutions above work if the residue of the table size T modulo the napkin size R is at least $\frac{2}{3}R$, or equivalently if $T = Sm + H(m-1)$ and $R = S + H$ for some positive integers S, H, m such that $S \ge 2H$. Here we discuss all other possible combinations.

Case 1. If $2H \ge S \ge H/2$, then the sharp bound for the number of vicious cells is $mS^2 + (m-1)H^2$; it can be obtained by the same methods as in any of the solutions. To obtain an example showing that the bound is sharp, one may simply remove the napkins of the third kind from the example in Solution 1 (with an obvious change in the numbers).

Case 2. If $2S \le H$, the situation is more difficult. If $(S+H)^2 > 2H^2$, then the answer and the example are the same as in the previous case; otherwise the answer is $(2m-1)S^2 + 2SH(m-1)$, and the example is provided simply by $(m-1)^2$ nonintersecting napkins.

Now we sketch the proof of both estimates for Case 2. We introduce a more appropriate notation based on that from Solution 2. Denote by a_- and a_+ the number of cells of class A that contain the number which is strictly less than M and strictly greater than M, respectively. The numbers $b_\pm$, $c_\pm$, and $d_\pm$ are defined in a similar way. One may notice that the proofs of Claim 1 and Claims 2, 3 lead in fact to the inequalities

$$m-1 \leq \frac{b_- + c_-}{2SH} + \frac{d_+}{H^2} \quad \text{and} \quad 2m-1 \leq \frac{a}{S^2} + \frac{b_+ + c_+}{2SH} + \frac{d_+}{H^2}$$

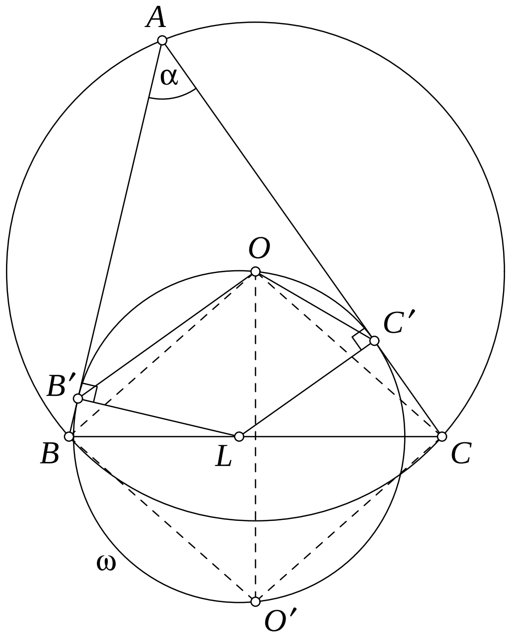
(to obtain the first one, one needs to look at the big lines instead of the small ones). Combining these inequalities, one may obtain the desired estimates.

These estimates can also be proved in some different ways, e.g. without distinguishing rich and poor cells.

G1

Let ABC be an acute triangle. Let ω be a circle whose center L lies on the side BC. Suppose that ω is tangent to AB at B' and to AC at C'. Suppose also that the circumcenter O of the triangle ABC lies on the shorter arc $B'C'$ of ω. Prove that the circumcircle of ABC and ω meet at two points.

Solution. The point B', being the perpendicular foot of L, is an interior point of side AB. Analogously, C' lies in the interior of AC. The point O is located inside the triangle $AB'C'$, hence $\angle COB < \angle C'OB'$.



Let $\alpha = \angle CAB$. The angles $\angle CAB$ and $\angle C'OB'$ are inscribed into the two circles with centers O and L, respectively, so $\angle COB = 2\angle CAB = 2\alpha$ and $2\angle C'OB' = 360^\circ - \angle C'LB'$. From the kite $AB'LC'$ we have $\angle C'LB' = 180^\circ - \angle C'AB' = 180^\circ - \alpha$. Combining these, we get

$$2\alpha = \angle COB < \angle C'OB' = \frac{360^\circ - \angle C'LB'}{2} = \frac{360^\circ - (180^\circ - \alpha)}{2} = 90^\circ + \frac{\alpha}{2},$$

so

$$\alpha < 60^\circ.$$

Let O' be the reflection of O in the line BC. In the quadrilateral $ABO'C$ we have

$$\angle CO'B + \angle CAB = \angle COB + \angle CAB = 2\alpha + \alpha < 180^\circ,$$

so the point O' is outside the circle ABC. Hence, O and O' are two points of ω such that one of them lies inside the circumcircle, while the other one is located outside. Therefore, the two circles intersect.

Comment. There are different ways of reducing the statement of the problem to the case $\alpha < 60^\circ$. E.g., since the point O lies in the interior of the isosceles triangle $AB'C'$, we have $OA < AB'$. So, if $AB' \leq 2LB'$ then $OA < 2LO$, which means that ω intersects the circumcircle of ABC. Hence the only interesting case is $AB' > 2LB'$, and this condition implies $\angle CAB = 2\angle B'AL < 2 \cdot 30^\circ = 60^\circ$.

G2

Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcenter and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

Solution 1. Let M be the point of intersection of the diagonals A_1A_3 and A_2A_4. On each diagonal choose a direction and let x, y, z, and w be the signed distances from M to the points A_1, A_2, A_3, and A_4, respectively.

Let ω_1 be the circumcircle of the triangle $A_2A_3A_4$ and let B_1 be the second intersection point of ω_1 and A_1A_3 (thus, $B_1 = A_3$ if and only if A_1A_3 is tangent to ω_1). Since the expression $O_1A_1^2 - r_1^2$ is the power of the point A_1 with respect to ω_1, we get

$$O_1A_1^2 - r_1^2 = A_1B_1 \cdot A_1A_3.$$

On the other hand, from the equality $MB_1 \cdot MA_3 = MA_2 \cdot MA_4$ we obtain $MB_1 = yw/z$. Hence, we have

$$O_1A_1^2 - r_1^2 = \left(\frac{yw}{z} - x\right)(z - x) = \frac{z - x}{z}(yw - xz).$$

Substituting the analogous expressions into the sought sum we get

$$\sum_{i=1}^{4} \frac{1}{O_iA_i^2 - r_i^2} = \frac{1}{yw - xz}\left(\frac{z}{z - x} - \frac{w}{w - y} + \frac{x}{x - z} - \frac{y}{y - w}\right) = 0,$$

as desired.

Comment. One might reformulate the problem by assuming that the quadrilateral $A_1A_2A_3A_4$ is convex. This should not really change the difficulty, but proofs that distinguish several cases may become shorter.

Solution 2. Introduce a Cartesian coordinate system in the plane. Every circle has an equation of the form $p(x, y) = x^2 + y^2 + l(x, y) = 0$, where $l(x, y)$ is a polynomial of degree at most 1. For any point $A = (x_A, y_A)$ we have $p(x_A, y_A) = d^2 - r^2$, where d is the distance from A to the center of the circle and r is the radius of the circle.

For each i in $\{1, 2, 3, 4\}$ let $p_i(x, y) = x^2 + y^2 + l_i(x, y) = 0$ be the equation of the circle with center O_i and radius r_i and let d_i be the distance from A_i to O_i. Consider the equation

$$\sum_{i=1}^{4} \frac{p_i(x, y)}{d_i^2 - r_i^2} = 1. \tag{1}$$

Since the coordinates of the points A_1, A_2, A_3, and A_4 satisfy (1) but these four points do not lie on a circle or on an line, equation (1) defines neither a circle, nor a line. Hence, the equation is an identity and the coefficient of the quadratic term $x^2 + y^2$ also has to be zero, i.e.

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = 0.$$

Comment. Using the determinant form of the equation of the circle through three given points, the same solution can be formulated as follows.

For $i = 1, 2, 3, 4$ let (u_i, v_i) be the coordinates of A_i and define

$$\Delta = \begin{vmatrix} u_1^2 + v_1^2 & u_1 & v_1 & 1 \\ u_2^2 + v_2^2 & u_2 & v_2 & 1 \\ u_3^2 + v_3^2 & u_3 & v_3 & 1 \\ u_4^2 + v_4^2 & u_4 & v_4 & 1 \end{vmatrix} \qquad \text{and} \qquad \Delta_i = \begin{vmatrix} u_{i+1} & v_{i+1} & 1 \\ u_{i+2} & v_{i+2} & 1 \\ u_{i+3} & v_{i+3} & 1 \end{vmatrix},$$

where $i+1$, $i+2$, and $i+3$ have to be read modulo 4 as integers in the set $\{1, 2, 3, 4\}$.

Expanding $\begin{vmatrix} u_1 & v_1 & 1 & 1 \\ u_2 & v_2 & 1 & 1 \\ u_3 & v_3 & 1 & 1 \\ u_4 & v_4 & 1 & 1 \end{vmatrix} = 0$ along the third column, we get $\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 = 0$.

The circle through A_{i+1}, A_{i+2}, and A_{i+3} is given by the equation

$$\frac{1}{\Delta_i} \begin{vmatrix} x^2 + y^2 & x & y & 1 \\ u_{i+1}^2 + v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2 + v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2 + v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = 0. \tag{2}$$

On the left-hand side, the coefficient of $x^2 + y^2$ is equal to 1. Substituting (u_i, v_i) for (x, y) in (2) we obtain the power of point A_i with respect to the circle through A_{i+1}, A_{i+2}, and A_{i+3}:

$$d_i^2 - r_i^2 = \frac{1}{\Delta_i} \begin{vmatrix} u_i^2 + v_i^2 & u_i & v_i & 1 \\ u_{i+1}^2 + v_{i+1}^2 & u_{i+1} & v_{i+1} & 1 \\ u_{i+2}^2 + v_{i+2}^2 & u_{i+2} & v_{i+2} & 1 \\ u_{i+3}^2 + v_{i+3}^2 & u_{i+3} & v_{i+3} & 1 \end{vmatrix} = (-1)^{i+1} \frac{\Delta}{\Delta_i}.$$

Thus, we have

$$\sum_{i=1}^{4} \frac{1}{d_i^2 - r_i^2} = \frac{\Delta_1 - \Delta_2 + \Delta_3 - \Delta_4}{\Delta} = 0.$$

G3

Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC, and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA, and AB. Prove that the midpoint of the segment EF lies on the line through the two intersection points of ω_E and ω_F.

Solution. Denote by P, Q, R, and S the projections of E on the lines DA, AB, BC, and CD respectively. The points P and Q lie on the circle with diameter AE, so $\angle QPE = \angle QAE$; analogously, $\angle QRE = \angle QBE$. So $\angle QPE + \angle QRE = \angle QAE + \angle QBE = 90^\circ$. By similar reasons, we have $\angle SPE + \angle SRE = 90^\circ$, hence we get $\angle QPS + \angle QRS = 90^\circ + 90^\circ = 180^\circ$, and the quadrilateral $PQRS$ is inscribed in ω_E. Analogously, all four projections of F onto the sides of $ABCD$ lie on ω_F.

Denote by K the meeting point of the lines AD and BC. Due to the arguments above, there is no loss of generality in assuming that A lies on segment DK. Suppose that $\angle CKD \geq 90^\circ$; then the circle with diameter CD covers the whole quadrilateral $ABCD$, so the points E, F cannot lie inside this quadrilateral. Hence our assumption is wrong. Therefore, the lines EP and BC intersect at some point P', while the lines ER and AD intersect at some point R'.

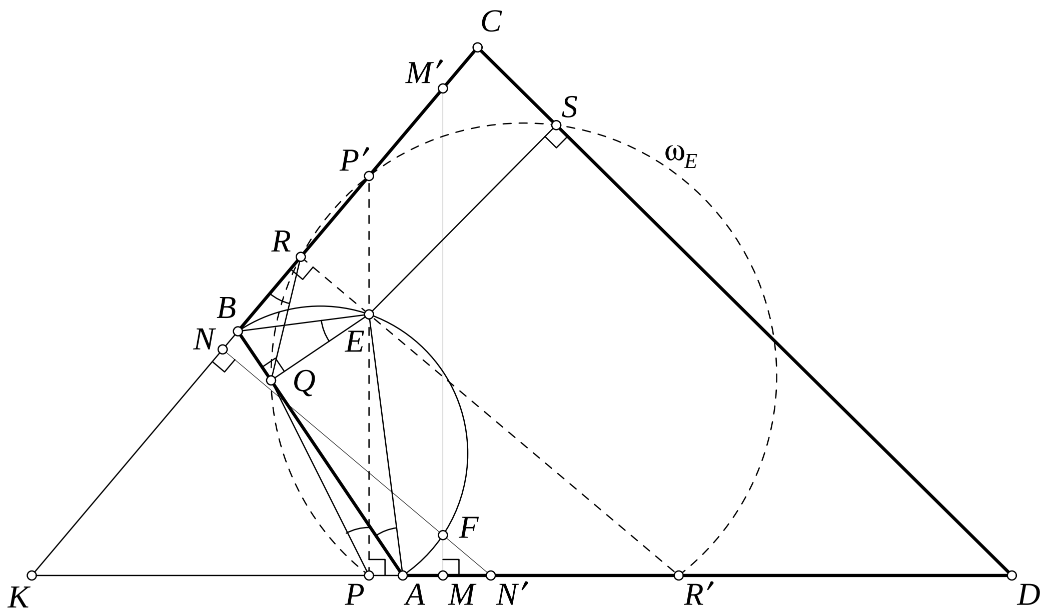


Figure 1

We claim that the points P' and R' also belong to ω_E. Since the points R, E, Q, B are concyclic, $\angle QRK = \angle QEB = 90^\circ - \angle QBE = \angle QAE = \angle QPE$. So $\angle QRK = \angle QPP'$, which means that the point P' lies on ω_E. Analogously, R' also lies on ω_E.

In the same manner, denote by M and N the projections of F on the lines AD and BC

respectively, and let $M' = FM \cap BC$, $N' = FN \cap AD$. By the same arguments, we obtain that the points M' and N' belong to ω_F.

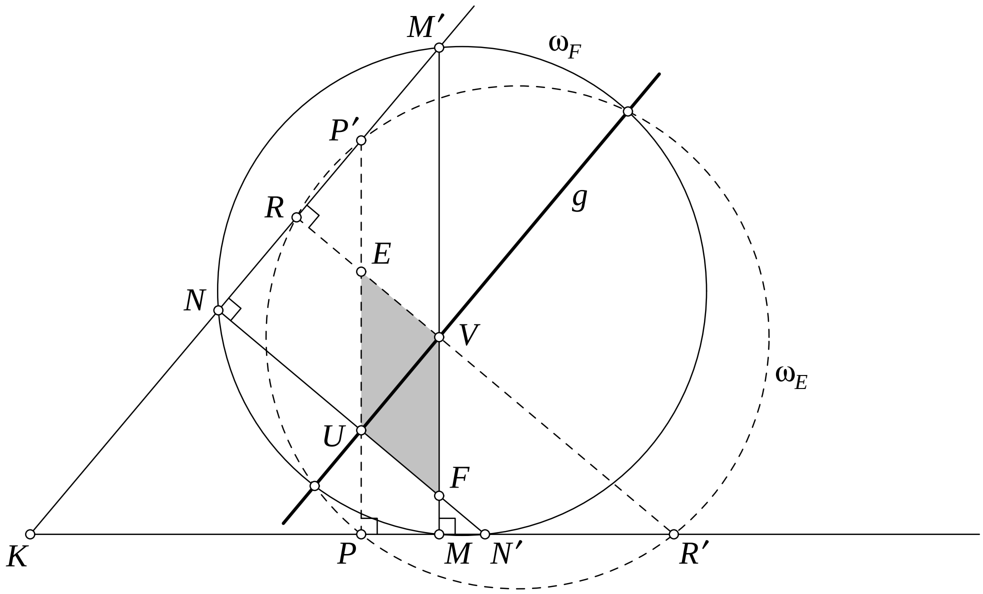


Figure 2

Now we concentrate on Figure 2, where all unnecessary details are removed. Let $U = NN' \cap PP'$, $V = MM' \cap RR'$. Due to the right angles at N and P, the points N, N', P, P' are concyclic, so $UN \cdot UN' = UP \cdot UP'$ which means that U belongs to the radical axis g of the circles ω_E and ω_F. Analogously, V also belongs to g.

Finally, since $EUFV$ is a parallelogram, the radical axis UV of ω_E and ω_F bisects EF.

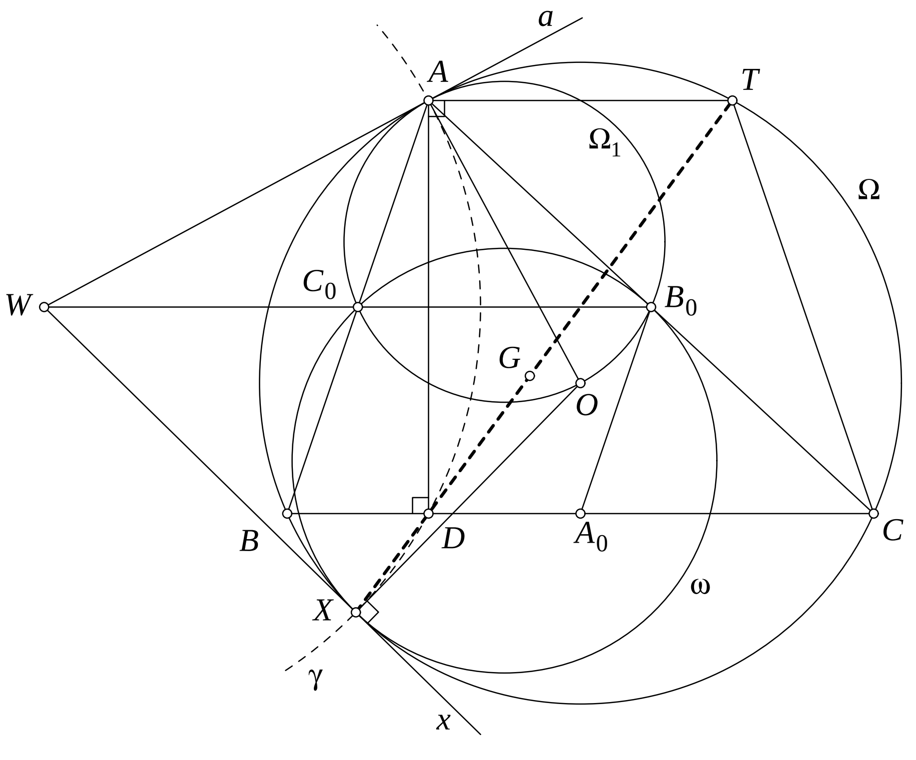
G4

Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A, and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G, and X are collinear.

Solution 1. If $AB = AC$, then the statement is trivial. So without loss of generality we may assume $AB < AC$. Denote the tangents to Ω at points A and X by a and x, respectively.

Let Ω_1 be the circumcircle of triangle AB_0C_0. The circles Ω and Ω_1 are homothetic with center A, so they are tangent at A, and a is their radical axis. Now, the lines a, x, and B_0C_0 are the three radical axes of the circles Ω, Ω_1, and ω. Since $a \nparallel B_0C_0$, these three lines are concurrent at some point W.

The points A and D are symmetric with respect to the line B_0C_0; hence $WX = WA = WD$. This means that W is the center of the circumcircle γ of triangle ADX. Moreover, we have $\angle WAO = \angle WXO = 90°$, where O denotes the center of Ω. Hence $\angle AWX + \angle AOX = 180°$.



Denote by T the second intersection point of Ω and the line DX. Note that O belongs to Ω_1. Using the circles γ and Ω, we find $\angle DAT = \angle ADX - \angle ATD = \frac{1}{2}(360° - \angle AWX) - \frac{1}{2}\angle AOX = 180° - \frac{1}{2}(\angle AWX + \angle AOX) = 90°$. So, $AD \perp AT$, and hence $AT \parallel BC$. Thus, $ATCB$ is an isosceles trapezoid inscribed in Ω.

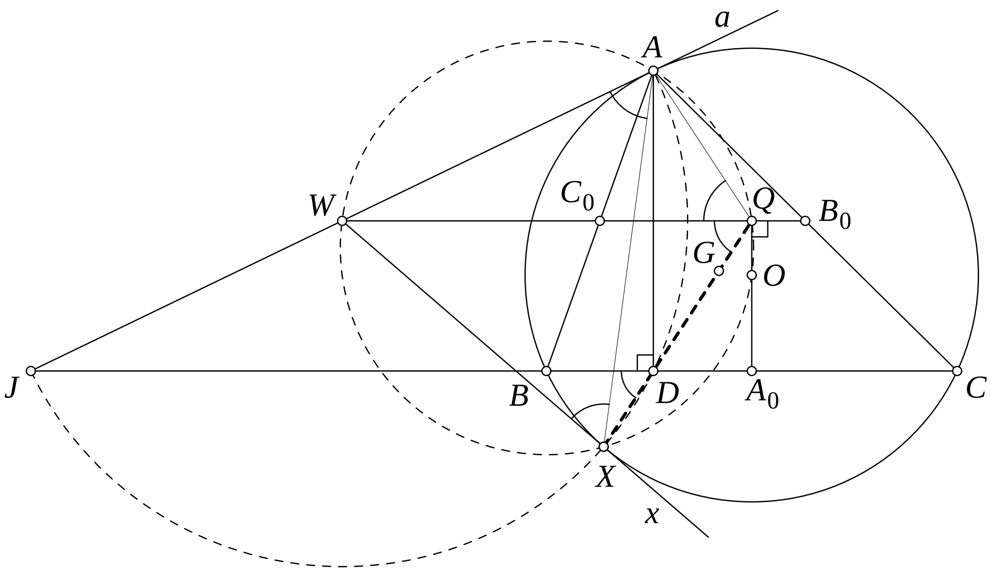
Denote by A_0 the midpoint of BC, and consider the image of $ATCB$ under the homothety h with center G and factor $-\frac{1}{2}$. We have $h(A) = A_0$, $h(B) = B_0$, and $h(C) = C_0$. From the symmetry about B_0C_0, we have $\angle TCB = \angle CBA = \angle B_0C_0A = \angle DC_0B_0$. Using $AT \parallel DA_0$, we conclude $h(T) = D$. Hence the points D, G, and T are collinear, and X lies on the same line.

Solution 2. We define the points A_0, O, and W as in the previous solution and we concentrate on the case $AB < AC$. Let Q be the perpendicular projection of A_0 on B_0C_0.

Since $\angle WAO = \angle WQO = \angle OXW = 90°$, the five points A, W, X, O, and Q lie on a common circle. Furthermore, the reflections with respect to B_0C_0 and OW map A to D and X, respectively. For these reasons, we have

$$\angle WQD = \angle AQW = \angle AXW = \angle WAX = \angle WQX.$$

Thus the three points Q, D, and X lie on a common line, say ℓ.



To complete the argument, we note that the homothety centered at G sending the triangle ABC to the triangle $A_0B_0C_0$ maps the altitude AD to the altitude A_0Q. Therefore it maps D to Q, so the points D, G, and Q are collinear. Hence G lies on ℓ as well.

Comment. There are various other ways to prove the collinearity of Q, D, and X obtained in the middle part of Solution 2. Introduce for instance the point J where the lines AW and BC intersect. Then the four points A, D, X, and J lie at the same distance from W, so the quadrilateral $ADXJ$ is cyclic. In combination with the fact that $AWXQ$ is cyclic as well, this implies

$$\angle JDX = \angle JAX = \angle WAX = \angle WQX.$$

Since $BC \parallel WQ$, it follows that Q, D, and X are indeed collinear.

G5

Let ABC be a triangle with incenter I and circumcircle ω. Let D and E be the second intersection points of ω with the lines AI and BI, respectively. The chord DE meets AC at a point F, and BC at a point G. Let P be the intersection point of the line through F parallel to AD and the line through G parallel to BE. Suppose that the tangents to ω at A and at B meet at a point K. Prove that the three lines AE, BD, and KP are either parallel or concurrent.

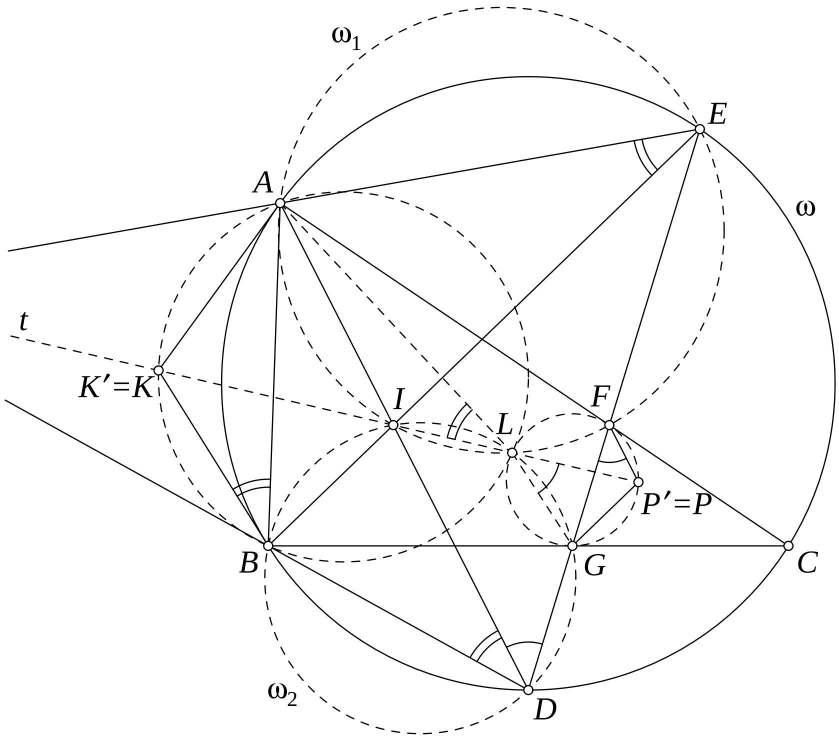
Solution 1. Since

$$\angle IAF = \angle DAC = \angle BAD = \angle BED = \angle IEF$$

the quadrilateral $AIFE$ is cyclic. Denote its circumcircle by ω_1. Similarly, the quadrilateral $BDGI$ is cyclic; denote its circumcircle by ω_2.

The line AE is the radical axis of ω and ω_1, and the line BD is the radical axis of ω and ω_2. Let t be the radical axis of ω_1 and ω_2. These three lines meet at the radical center of the three circles, or they are parallel to each other. We will show that t is in fact the line PK.

Let L be the second intersection point of ω_1 and ω_2, so $t = IL$. (If the two circles are tangent to each other then $L = I$ and t is the common tangent.)



Let the line t meet the circumcircles of the triangles ABL and FGL at $K' \neq L$ and $P' \neq L$, respectively. Using oriented angles we have

$$\angle(AB, BK') = \angle(AL, LK') = \angle(AL, LI) = \angle(AE, EI) = \angle(AE, EB) = \angle(AB, BK),$$

so $BK' \parallel BK$. Similarly we have $AK' \parallel AK$, and therefore $K' = K$. Next, we have

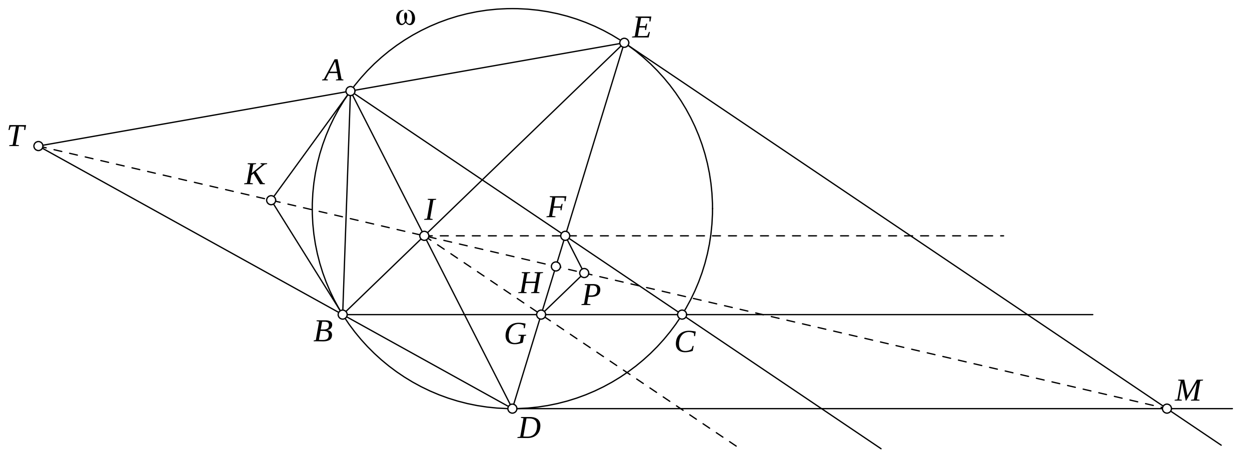
$$\angle(P'F, FG) = \angle(P'L, LG) = \angle(IL, LG) = \angle(ID, DG) = \angle(AD, DE) = \angle(PF, FG),$$

hence $P'F \parallel PF$ and similarly $P'G \parallel PG$. Therefore $P' = P$. This means that t passes through K and P, which finishes the proof.

Solution 2. Let M be the intersection point of the tangents to ω at D and E, and let the lines AE and BD meet at T; if AE and BD are parallel, then let T be their common ideal point. It is well-known that the points K and M lie on the line TI (as a consequence of Pascal's theorem, applied to the inscribed degenerate hexagons $AADBBE$ and $ADDBEE$).

The lines AD and BE are the angle bisectors of the angles $\angle CAB$ and $\angle ABC$, respectively, so D and E are the midpoints of the arcs BC and CA of the circle ω, respectively. Hence, DM is parallel to BC and EM is parallel to AC.

Apply Pascal's theorem to the degenerate hexagon $CADDEB$. By the theorem, the points $CA \cap DE = F$, $AD \cap EB = I$ and the common ideal point of lines DM and BC are collinear, therefore FI is parallel to BC and DM. Analogously, the line GI is parallel to AC and EM.



Now consider the homothety with scale factor $-\frac{FG}{ED}$ which takes E to G and D to F. Since the triangles EDM and GFI have parallel sides, the homothety takes M to I. Similarly, since the triangles DEI and FGP have parallel sides, the homothety takes I to P. Hence, the points M, I, P and the homothety center H must lie on the same line. Therefore, the point P also lies on the line $TKIM$.

Comment. One may prove that $IF \parallel BC$ and $IG \parallel AC$ in a more elementary way. Since $\angle ADE = \angle EDC$ and $\angle DEB = \angle CED$, the points I and C are symmetric about DE. Moreover, since the arcs AE and EC are equal and the arcs CD and DB are equal, we have $\angle CFG = \angle FGC$, so the triangle CFG is isosceles. Hence, the quadrilateral $IFCG$ is a rhombus.

G6

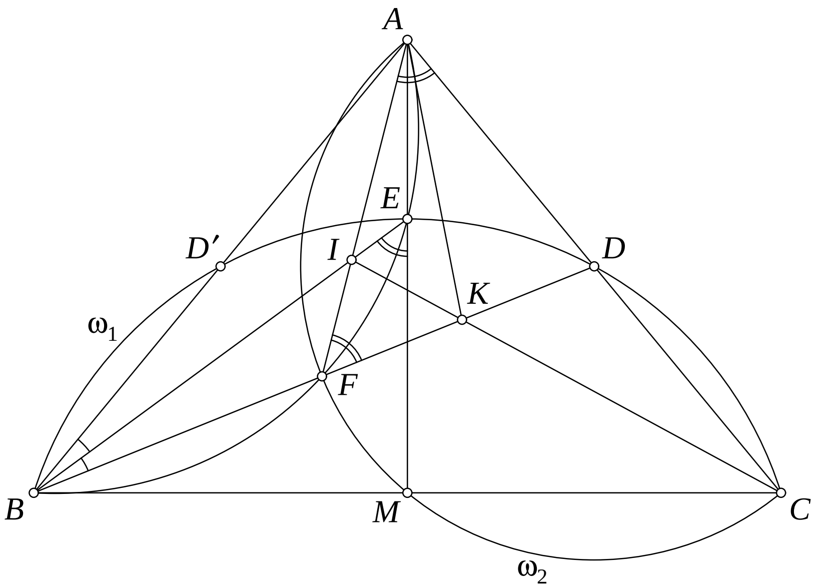
Let ABC be a triangle with $AB = AC$, and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B, and C in a point E inside the triangle ABC. The line BD intersects the circle through A, E, and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incenter of triangle KAB.

Solution 1. Let D' be the midpoint of the segment AB, and let M be the midpoint of BC. By symmetry at line AM, the point D' has to lie on the circle BCD. Since the arcs $D'E$ and ED of that circle are equal, we have $\angle ABI = \angle D'BE = \angle EBD = IBK$, so I lies on the angle bisector of $\angle ABK$. For this reason it suffices to prove in the sequel that the ray AI bisects the angle $\angle BAK$.

From

$$\angle DFA = 180^\circ - \angle BFA = 180^\circ - \angle BEA = \angle MEB = \frac{1}{2}\angle CEB = \frac{1}{2}\angle CDB$$

we derive $\angle DFA = \angle DAF$ so the triangle AFD is isosceles with $AD = DF$.



Applying MENELAUS's theorem to the triangle ADF with respect to the line CIK, and applying the angle bisector theorem to the triangle ABF, we infer

$$1 = \frac{AC}{CD} \cdot \frac{DK}{KF} \cdot \frac{FI}{IA} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{AB} = 2 \cdot \frac{DK}{KF} \cdot \frac{BF}{2 \cdot AD} = \frac{DK}{KF} \cdot \frac{BF}{AD}$$

and therefore

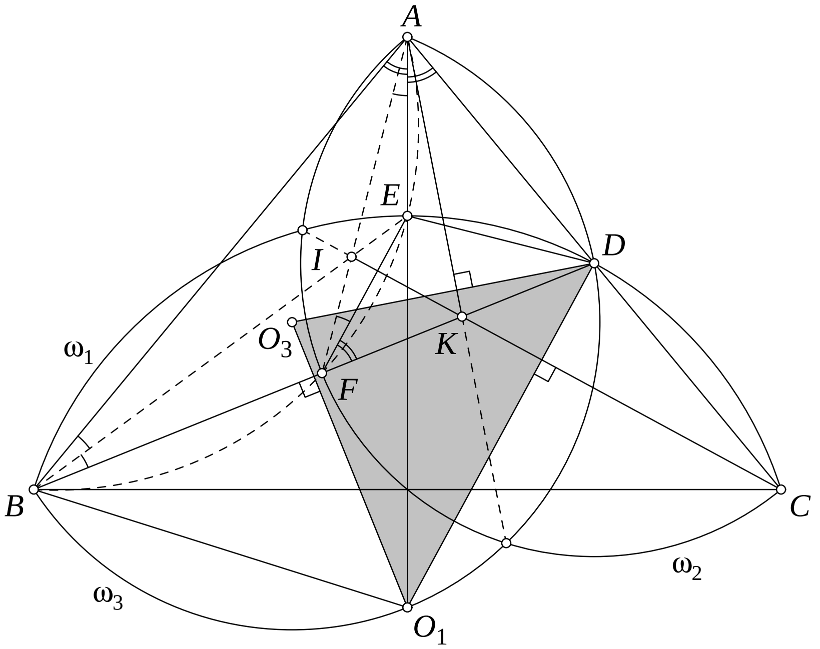
$$\frac{BD}{AD} = \frac{BF + FD}{AD} = \frac{BF}{AD} + 1 = \frac{KF}{DK} + 1 = \frac{DF}{DK} = \frac{AD}{DK}.$$

It follows that the triangles ADK and BDA are similar, hence $\angle DAK = \angle ABD$. Then

$$\angle IAB = \angle AFD - \angle ABD = \angle DAF - \angle DAK = \angle KAI$$

shows that the point K is indeed lying on the angle bisector of $\angle BAK$.

Solution 2. It can be shown in the same way as in the first solution that I lies on the angle bisector of $\angle ABK$. Here we restrict ourselves to proving that KI bisects $\angle AKB$.



Denote the circumcircle of triangle BCD and its center by ω_1 and by O_1, respectively. Since the quadrilateral $ABFE$ is cyclic, we have $\angle DFE = \angle BAE = \angle DAE$. By the same reason, we have $\angle EAF = \angle EBF = \angle ABE = \angle AFE$. Therefore $\angle DAF = \angle DFA$, and hence $DF = DA = DC$. So triangle AFC is inscribed in a circle ω_2 with center D.

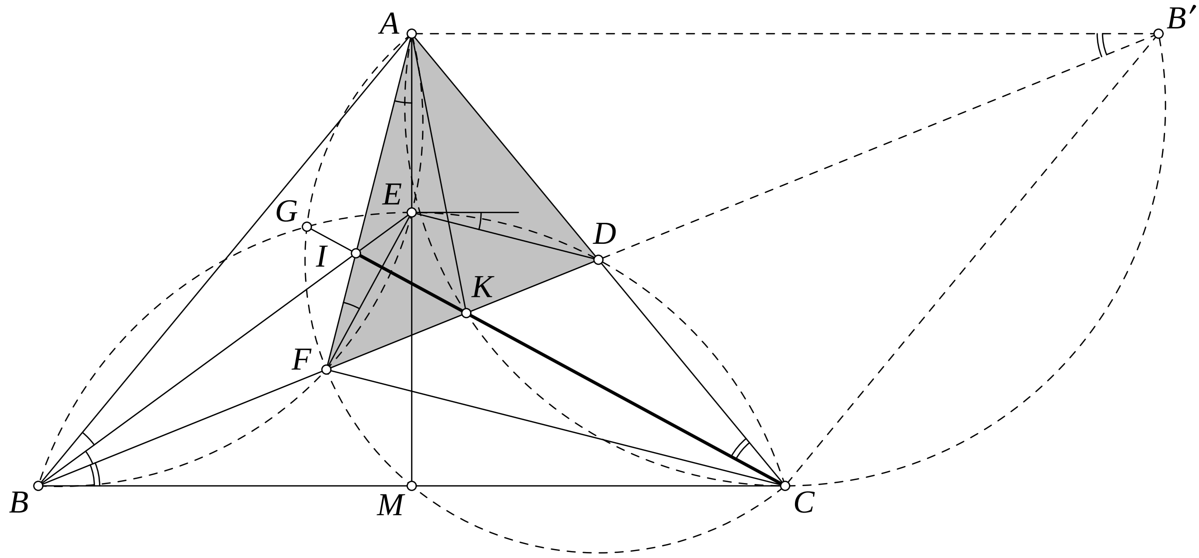
Denote the circumcircle of triangle ABD by ω_3, and let its center be O_3. Since the arcs BE and EC of circle ω_1 are equal, and the triangles ADE and FDE are congruent, we have $\angle AO_1B = 2\angle BDE = \angle BDA$, so O_1 lies on ω_3. Hence $\angle O_3O_1D = \angle O_3DO_1$.

The line BD is the radical axis of ω_1 and ω_3. Point C belongs to the radical axis of ω_1 and ω_2, and I also belongs to it since $AI \cdot IF = BI \cdot IE$. Hence $K = BD \cap CI$ is the radical center of ω_1, ω_2, and ω_3, and AK is the radical axis of ω_2 and ω_3. Now, the radical axes AK, BK and IK are perpendicular to the central lines O_3D, O_3O_1 and O_1D, respectively. By $\angle O_3O_1D = \angle O_3DO_1$, we get that KI is the angle bisector of $\angle AKB$.

Solution 3. Again, let M be the midpoint of BC. As in the previous solutions, we can deduce $\angle ABI = \angle IBK$. We show that the point I lies on the angle bisector of $\angle KAB$.

Let G be the intersection point of the circles AFC and BCD, different from C. The lines

CG, AF, and BE are the radical axes of the three circles $AGFC$, CDB, and $ABFE$, so $I = AF \cap BE$ is the radical center of the three circles and CG also passes through I.



The angle between line DE and the tangent to the circle BCD at E is equal to $\angle EBD = \angle EAF = \angle ABE = \angle AFE$. As the tangent at E is perpendicular to AM, the line DE is perpendicular to AF. The triangle AFE is isosceles, so DE is the perpendicular bisector of AF and thus $AD = DF$. Hence, the point D is the center of the circle AFC, and this circle passes through M as well since $\angle AMC = 90°$.

Let B' be the reflection of B in the point D, so $ABCB'$ is a parallelogram. Since $DC = DG$ we have $\angle GCD = \angle DBC = \angle KB'A$. Hence, the quadrilateral $AKCB'$ is cyclic and thus $\angle CAK = \angle CB'K = \angle ABD = 2\angle MAI$. Then

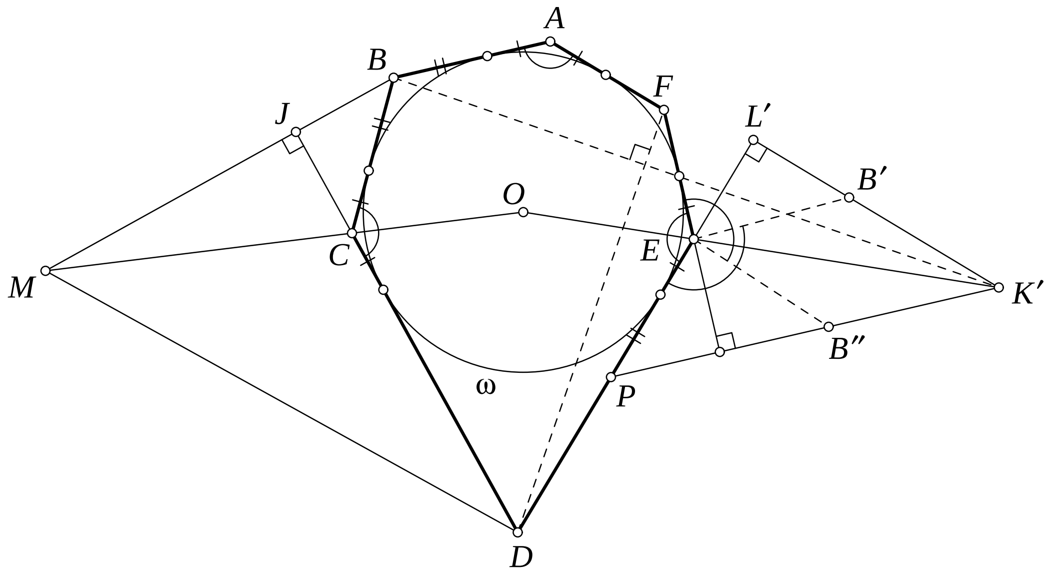
$$\angle IAB = \angle MAB - \angle MAI = \frac{1}{2}\angle CAB - \frac{1}{2}\angle CAK = \frac{1}{2}\angle KAB$$

and therefore AI is the angle bisector of $\angle KAB$.

G7

Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with center O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

Solution 1. Since ω and the circumcircle of triangle ACE are concentric, the tangents from A, C, and E to ω have equal lengths; that means that $AB = BC$, $CD = DE$, and $EF = FA$. Moreover, we have $\angle BCD = \angle DEF = \angle FAB$.



Consider the rotation around point D mapping C to E; let B' and L' be the images of the points B and J, respectively, under this rotation. Then one has $DJ = DL'$ and $B'L' \perp DE$; moreover, the triangles $B'ED$ and BCD are congruent. Since $\angle DEO < 90°$, the lines EO and $B'L'$ intersect at some point K'. We intend to prove that $K'B \perp DF$; this would imply $K = K'$, therefore $L = L'$, which proves the problem statement.

Analogously, consider the rotation around F mapping A to E; let B'' be the image of B under this rotation. Then the triangles FAB and FEB'' are congruent. We have $EB'' = AB = BC = EB'$ and $\angle FEB'' = \angle FAB = \angle BCD = \angle DEB'$, so the points B' and B'' are symmetrical with respect to the angle bisector EO of $\angle DEF$. So, from $K'B' \perp DE$ we get $K'B'' \perp EF$.

From these two relations we obtain

$$K'D^2 - K'E^2 = B'D^2 - B'E^2 \quad \text{and} \quad K'E^2 - K'F^2 = B''E^2 - B''F^2.$$

Adding these equalities and taking into account that $B'E = B''E$ we obtain

$$K'D^2 - K'F^2 = B'D^2 - B''F^2 = BD^2 - BF^2,$$

which means exactly that $K'B \perp DF$.

Comment. There are several variations of this solution. For instance, let us consider the intersection point M of the lines BJ and OC. Define the point K' as the reflection of M in the line DO. Then one has

$$DK'^2 - DB^2 = DM^2 - DB^2 = CM^2 - CB^2.$$

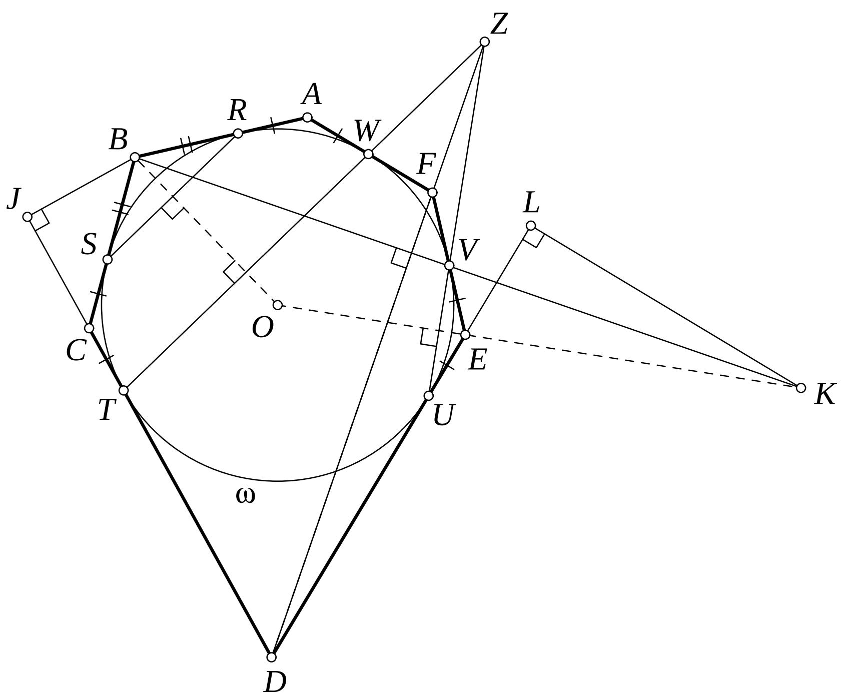
Next, consider the rotation around O which maps CM to EK'. Let P be the image of B under this rotation; so P lies on ED. Then $EF \perp K'P$, so

$$CM^2 - CB^2 = EK'^2 - EP^2 = FK'^2 - FP^2 = FK'^2 - FB^2,$$

since the triangles FEP and FAB are congruent.

Solution 2. Let us denote the points of tangency of AB, BC, CD, DE, EF, and FA to ω by R, S, T, U, V, and W, respectively. As in the previous solution, we mention that $AR = AW = CS = CT = EU = EV$.

The reflection in the line BO maps R to S, therefore A to C and thus also W to T. Hence, both lines RS and WT are perpendicular to OB, therefore they are parallel. On the other hand, the lines UV and WT are not parallel, since otherwise the hexagon $ABCDEF$ is symmetric with respect to the line BO and the lines defining the point K coincide, which contradicts the conditions of the problem. Therefore we can consider the intersection point Z of UV and WT.



Next, we recall a well-known fact that the points D, F, Z are collinear. Actually, D is the pole of the line UT, F is the pole of VW, and $Z = TW \cap UV$; so all these points belong to the polar line of $TU \cap VW$.

Now, we put O into the origin, and identify each point (say X) with the vector $\overrightarrow{OX}$. So, from now on all the products of points refer to the scalar products of the corresponding vectors.

Since $OK \perp UZ$ and $OB \perp TZ$, we have $K \cdot (Z - U) = 0 = B \cdot (Z - T)$. Next, the condition $BK \perp DZ$ can be written as $K \cdot (D - Z) = B \cdot (D - Z)$. Adding these two equalities we get

$$K \cdot (D - U) = B \cdot (D - T).$$

By symmetry, we have $D \cdot (D - U) = D \cdot (D - T)$. Subtracting this from the previous equation, we obtain $(K - D) \cdot (D - U) = (B - D) \cdot (D - T)$ and rewrite it in vector form as

$$\overrightarrow{DK} \cdot \overrightarrow{UD} = \overrightarrow{DB} \cdot \overrightarrow{TD}.$$

Finally, projecting the vectors $\overrightarrow{DK}$ and $\overrightarrow{DB}$ onto the lines UD and TD respectively, we can rewrite this equality in terms of segment lengths as $DL \cdot UD = DJ \cdot TD$, thus $DL = DJ$.

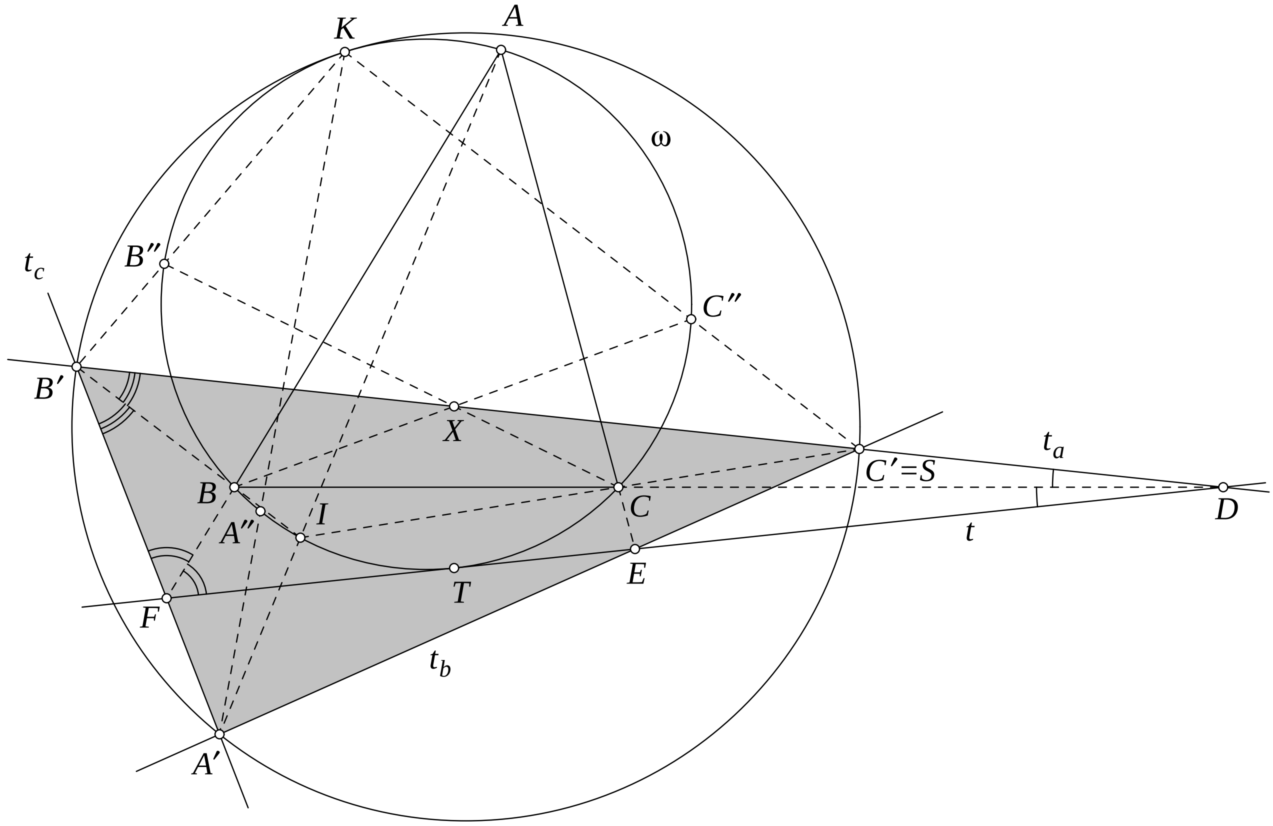
Comment. The collinearity of Z, F, and D may be shown in various more elementary ways. For instance, applying the sine theorem to the triangles DTZ and DUZ, one gets $\dfrac{\sin \angle DZT}{\sin \angle DZU} = \dfrac{\sin \angle DTZ}{\sin \angle DUZ}$; analogously, $\dfrac{\sin \angle FZW}{\sin \angle FZV} = \dfrac{\sin \angle FWZ}{\sin \angle FVZ}$. The right-hand sides are equal, hence so are the left-hand sides, which implies the collinearity of the points D, F, and Z.

There also exist purely synthetic proofs of this fact. E.g., let Q be the point of intersection of the circumcircles of the triangles ZTV and ZWU different from Z. Then QZ is the bisector of $\angle VQW$ since $\angle VQZ = \angle VTZ = \angle VUW = \angle ZQW$. Moreover, all these angles are equal to $\frac{1}{2}\angle VOW$, so $\angle VQW = \angle VOW$, hence the quadrilateral $VWOQ$ is cyclic. On the other hand, the points O, V, W lie on the circle with diameter OF due to the right angles; so Q also belongs to this circle. Since $FV = FW$, QF is also the bisector of $\angle VQW$, so F lies on QZ. Analogously, D lies on the same line.

G8

Let ABC be an acute triangle with circumcircle ω. Let t be a tangent line to ω. Let t_a, t_b, and t_c be the lines obtained by reflecting t in the lines BC, CA, and AB, respectively. Show that the circumcircle of the triangle determined by the lines t_a, t_b, and t_c is tangent to the circle ω.

To avoid a large case distinction, we will use the notion of *oriented angles*. Namely, for two lines ℓ and m, we denote by $\angle(\ell, m)$ the angle by which one may rotate ℓ anticlockwise to obtain a line parallel to m. Thus, all oriented angles are considered modulo $180°$.



Solution 1. Denote by T the point of tangency of t and ω. Let $A' = t_b \cap t_c$, $B' = t_a \cap t_c$, $C' = t_a \cap t_b$. Introduce the point A'' on ω such that $TA = AA''$ ($A'' \neq T$ unless TA is a diameter). Define the points B'' and C'' in a similar way.

Since the points C and B are the midpoints of arcs TC'' and TB'', respectively, we have

$$\begin{aligned}\angle(t, B''C'') &= \angle(t, TC'') + \angle(TC'', B''C'') = 2\angle(t, TC) + 2\angle(TC'', BC'')\\ &= 2\big(\angle(t, TC) + \angle(TC, BC)\big) = 2\angle(t, BC) = \angle(t, t_a).\end{aligned}$$

It follows that t_a and $B''C''$ are parallel. Similarly, $t_b \parallel A''C''$ and $t_c \parallel A''B''$. Thus, either the triangles $A'B'C'$ and $A''B''C''$ are homothetic, or they are translates of each other. Now we will prove that they are in fact homothetic, and that the center K of the homothety belongs to ω. It would then follow that their circumcircles are also homothetic with respect to K and are therefore tangent at this point, as desired.

We need the two following claims.

Claim 1. The point of intersection X of the lines $B''C$ and BC'' lies on t_a.

Proof. Actually, the points X and T are symmetric about the line BC, since the lines CT and CB'' are symmetric about this line, as are the lines BT and BC''. $\square$

Claim 2. The point of intersection I of the lines BB' and CC' lies on the circle ω.

Proof. We consider the case that t is not parallel to the sides of ABC; the other cases may be regarded as limit cases. Let $D = t \cap BC$, $E = t \cap AC$, and $F = t \cap AB$.

Due to symmetry, the line DB is one of the angle bisectors of the lines $B'D$ and FD; analogously, the line FB is one of the angle bisectors of the lines $B'F$ and DF. So B is either the incenter or one of the excenters of the triangle $B'DF$. In any case we have $\angle(BD, DF) + \angle(DF, FB) + \angle(B'B, B'D) = 90^\circ$, so

$$\angle(B'B, B'C') = \angle(B'B, B'D) = 90^\circ - \angle(BC, DF) - \angle(DF, BA) = 90^\circ - \angle(BC, AB).$$

Analogously, we get $\angle(C'C, B'C') = 90^\circ - \angle(BC, AC)$. Hence,

$$\angle(BI, CI) = \angle(B'B, B'C') + \angle(B'C', C'C) = \angle(BC, AC) - \angle(BC, AB) = \angle(AB, AC),$$

which means exactly that the points A, B, I, C are concyclic. $\square$

Now we can complete the proof. Let K be the second intersection point of $B'B''$ and ω. Applying Pascal's theorem to hexagon $KB''CIBC''$ we get that the points $B' = KB'' \cap IB$ and $X = B''C \cap BC''$ are collinear with the intersection point S of CI and $C''K$. So $S = CI \cap B'X = C'$, and the points C', C'', K are collinear. Thus K is the intersection point of $B'B''$ and $C'C''$ which implies that K is the center of the homothety mapping $A'B'C'$ to $A''B''C''$, and it belongs to ω.

Solution 2. Define the points T, A', B', and C' in the same way as in the previous solution. Let X, Y, and Z be the symmetric images of T about the lines BC, CA, and AB, respectively. Note that the projections of T on these lines form a Simson line of T with respect to ABC, therefore the points X, Y, Z are also collinear. Moreover, we have $X \in B'C'$, $Y \in C'A'$, $Z \in A'B'$.

Denote $\alpha = \angle(t, TC) = \angle(BT, BC)$. Using the symmetry in the lines AC and BC, we get

$$\angle(BC, BX) = \angle(BT, BC) = \alpha \quad \text{and} \quad \angle(XC, XC') = \angle(t, TC) = \angle(YC, YC') = \alpha.$$

Since $\angle(XC, XC') = \angle(YC, YC')$, the points X, Y, C, C' lie on some circle ω_c. Define the circles ω_a and ω_b analogously. Let ω' be the circumcircle of triangle $A'B'C'$.

Now, applying MIQUEL's theorem to the four lines $A'B'$, $A'C'$, $B'C'$, and XY, we obtain that the circles ω', ω_a, ω_b, ω_c intersect at some point K. We will show that K lies on ω, and that the tangent lines to ω and ω' at this point coincide; this implies the problem statement.

Due to symmetry, we have $XB = TB = ZB$, so the point B is the midpoint of one of the arcs XZ of circle ω_b. Therefore $\angle(KB, KX) = \angle(XZ, XB)$. Analogously, $\angle(KX, KC) = \angle(XC, XY)$. Adding these equalities and using the symmetry in the line BC we get

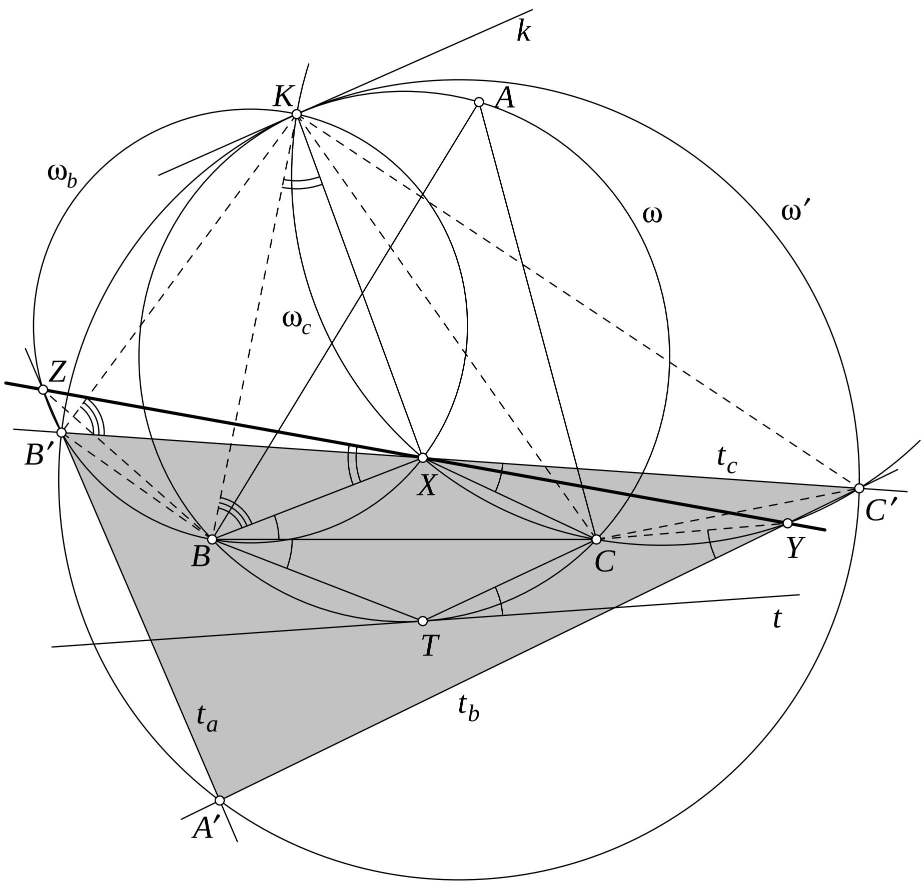
$$\angle(KB, KC) = \angle(XZ, XB) + \angle(XC, XZ) = \angle(XC, XB) = \angle(TB, TC).$$

Therefore, K lies on ω.

Next, let k be the tangent line to ω at K. We have

$$\begin{aligned}\angle(k, KC') &= \angle(k, KC) + \angle(KC, KC') = \angle(KB, BC) + \angle(XC, XC') \\ &= \big(\angle(KB, BX) - \angle(BC, BX)\big) + \alpha = \angle(KB', B'X) - \alpha + \alpha = \angle(KB', B'C'),\end{aligned}$$

which means exactly that k is tangent to ω'.



Comment. There exist various solutions combining the ideas from the two solutions presented above. For instance, one may define the point X as the reflection of T with respect to the line BC, and then introduce the point K as the second intersection point of the circumcircles of $BB'X$ and $CC'X$. Using the fact that BB' and CC' are the bisectors of $\angle(A'B', B'C')$ and $\angle(A'C', B'C')$ one can show successively that $K \in \omega$, $K \in \omega'$, and that the tangents to ω and ω' at K coincide.

N1

For any integer $d > 0$, let $f(d)$ be the smallest positive integer that has exactly d positive divisors (so for example we have $f(1) = 1$, $f(5) = 16$, and $f(6) = 12$). Prove that for every integer $k \geq 0$ the number $f(2^k)$ divides $f(2^{k+1})$.

Solution 1. For any positive integer n, let $d(n)$ be the number of positive divisors of n. Let $n = \prod_p p^{a(p)}$ be the prime factorization of n where p ranges over the prime numbers, the integers $a(p)$ are nonnegative and all but finitely many $a(p)$ are zero. Then we have $d(n) = \prod_p (a(p)+1)$. Thus, $d(n)$ is a power of 2 if and only if for every prime p there is a nonnegative integer $b(p)$ with $a(p) = 2^{b(p)} - 1 = 1 + 2 + 2^2 + \cdots + 2^{b(p)-1}$. We then have

$$n = \prod_p \prod_{i=0}^{b(p)-1} p^{2^i}, \qquad \text{and} \quad d(n) = 2^k \quad \text{with} \quad k = \sum_p b(p).$$

Let $\mathcal{S}$ be the set of all numbers of the form p^{2^r} with p prime and r a nonnegative integer. Then we deduce that $d(n)$ is a power of 2 if and only if n is the product of the elements of some finite subset $\mathcal{T}$ of $\mathcal{S}$ that satisfies the following condition: for all $t \in \mathcal{T}$ and $s \in \mathcal{S}$ with $s \mid t$ we have $s \in \mathcal{T}$. Moreover, if $d(n) = 2^k$ then the corresponding set $\mathcal{T}$ has k elements.

Note that the set $\mathcal{T}_k$ consisting of the smallest k elements from $\mathcal{S}$ obviously satisfies the condition above. Thus, given k, the smallest n with $d(n) = 2^k$ is the product of the elements of $\mathcal{T}_k$. This n is $f(2^k)$. Since obviously $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, it follows that $f(2^k) \mid f(2^{k+1})$.

Solution 2. This is an alternative to the second part of the Solution 1. Suppose k is a nonnegative integer. From the first part of Solution 1 we see that $f(2^k) = \prod_p p^{a(p)}$ with $a(p) = 2^{b(p)} - 1$ and $\sum_p b(p) = k$. We now claim that for any two distinct primes p, q with $b(q) > 0$ we have

$$m = p^{2^{b(p)}} > q^{2^{b(q)-1}} = \ell. \tag{1}$$

To see this, note first that ℓ divides $f(2^k)$. With the first part of Solution 1 one can see that the integer $n = f(2^k)m/\ell$ also satisfies $d(n) = 2^k$. By the definition of $f(2^k)$ this implies that $n \geq f(2^k)$ so $m \geq \ell$. Since $p \neq q$ the inequality (1) follows.

Let the prime factorization of $f(2^{k+1})$ be given by $f(2^{k+1}) = \prod_p p^{r(p)}$ with $r(p) = 2^{s(p)} - 1$. Since we have $\sum_p s(p) = k + 1 > k = \sum_p b(p)$ there is a prime p with $s(p) > b(p)$. For any prime $q \neq p$ with $b(q) > 0$ we apply inequality (1) twice and get

$$q^{2^{s(q)}} > p^{2^{s(p)-1}} \geq p^{2^{b(p)}} > q^{2^{b(q)-1}},$$

which implies $s(q) \geq b(q)$. It follows that $s(q) \geq b(q)$ for all primes q, so $f(2^k) \mid f(2^{k+1})$.

N2

Consider a polynomial $P(x) = (x + d_1)(x + d_2) \cdot \ldots \cdot (x + d_9)$, where $d_1, d_2, \ldots, d_9$ are nine distinct integers. Prove that there exists an integer N such that for all integers $x \geq N$ the number $P(x)$ is divisible by a prime number greater than 20.

Solution 1. Note that the statement of the problem is invariant under translations of x; hence without loss of generality we may suppose that the numbers $d_1, d_2, \ldots, d_9$ are positive.

The key observation is that there are only eight primes below 20, while $P(x)$ involves more than eight factors.

We shall prove that $N = d^8$ satisfies the desired property, where $d = \max\{d_1, d_2, \ldots, d_9\}$. Suppose for the sake of contradiction that there is some integer $x \geq N$ such that $P(x)$ is composed of primes below 20 only. Then for every index $i \in \{1, 2, \ldots, 9\}$ the number $x + d_i$ can be expressed as product of powers of the first 8 primes.

Since $x + d_i > x \geq d^8$ there is some prime power $f_i > d$ that divides $x + d_i$. Invoking the pigeonhole principle we see that there are two distinct indices i and j such that f_i and f_j are powers of the same prime number. For reasons of symmetry, we may suppose that $f_i \leq f_j$. Now both of the numbers $x + d_i$ and $x + d_j$ are divisible by f_i and hence so is their difference $d_i - d_j$. But as

$$0 < |d_i - d_j| \leq \max(d_i, d_j) \leq d < f_i,$$

this is impossible. Thereby the problem is solved.

Solution 2. Observe that for each index $i \in \{1, 2, \ldots, 9\}$ the product

$$D_i = \prod_{1 \leq j \leq 9, j \neq i} |d_i - d_j|$$

is positive. We claim that $N = \max\{D_1 - d_1, D_2 - d_2, \ldots, D_9 - d_9\} + 1$ satisfies the statement of the problem. Suppose there exists an integer $x \geq N$ such that all primes dividing $P(x)$ are smaller than 20. For each index i we reduce the fraction $(x + d_i)/D_i$ to lowest terms. Since $x + d_i > D_i$ the numerator of the fraction we thereby get cannot be 1, and hence it has to be divisible by some prime number $p_i < 20$.

By the pigeonhole principle, there are a prime number p and two distinct indices i and j such that $p_i = p_j = p$. Let p^{α_i} and p^{α_j} be the greatest powers of p dividing $x + d_i$ and $x + d_j$, respectively. Due to symmetry we may suppose $\alpha_i \leq \alpha_j$. But now p^{α_i} divides $d_i - d_j$ and hence also D_i, which means that all occurrences of p in the numerator of the fraction $(x + d_i)/D_i$ cancel out, contrary to the choice of $p = p_i$. This contradiction proves our claim.

Solution 3. Given a nonzero integer N as well as a prime number p we write $v_p(N)$ for the exponent with which p occurs in the prime factorization of $|N|$.

Evidently, if the statement of the problem were not true, then there would exist an infinite sequence (x_n) of positive integers tending to infinity such that for each $n \in \mathbb{Z}_+$ the integer $P(x_n)$ is not divisible by any prime number > 20. Observe that the numbers $-d_1, -d_2, \ldots, -d_9$ do not appear in this sequence.

Now clearly there exists a prime $p_1 < 20$ for which the sequence $v_{p_1}(x_n + d_1)$ is not bounded; thinning out the sequence (x_n) if necessary we may even suppose that

$$v_{p_1}(x_n + d_1) \longrightarrow \infty.$$

Repeating this argument eight more times we may similarly choose primes $p_2, \ldots, p_9 < 20$ and suppose that our sequence (x_n) has been thinned out to such an extent that $v_{p_i}(x_n + d_i) \longrightarrow \infty$ holds for $i = 2, \ldots, 9$ as well. In view of the pigeonhole principle, there are distinct indices i and j as well as a prime $p < 20$ such that $p_i = p_j = p$. Setting $k = v_p(d_i - d_j)$ there now has to be some n for which both $v_p(x_n + d_i)$ and $v_p(x_n + d_j)$ are greater than k. But now the numbers $x_n + d_i$ and $x_n + d_j$ are divisible by p^{k+1} whilst their difference $d_i - d_j$ is not – a contradiction.

Comment. This problem is supposed to be a relatively easy one, so one might consider adding the hypothesis that the numbers $d_1, d_2, \ldots, d_9$ be positive. Then certain merely technical issues are not going to arise while the main ideas required to solve the problems remain the same.

N3

Let $n \geq 1$ be an odd integer. Determine all functions f from the set of integers to itself such that for all integers x and y the difference $f(x) - f(y)$ divides $x^n - y^n$.

Answer. All functions f of the form $f(x) = \varepsilon x^d + c$, where ε is in $\{1, -1\}$, the integer d is a positive divisor of n, and c is an integer.

Solution. Obviously, all functions in the answer satisfy the condition of the problem. We will show that there are no other functions satisfying that condition.

Let f be a function satisfying the given condition. For each integer n, the function g defined by $g(x) = f(x) + n$ also satisfies the same condition. Therefore, by subtracting $f(0)$ from $f(x)$ we may assume that $f(0) = 0$.

For any prime p, the condition on f with $(x, y) = (p, 0)$ states that $f(p)$ divides p^n. Since the set of primes is infinite, there exist integers d and ε with $0 \leq d \leq n$ and $\varepsilon \in \{1, -1\}$ such that for infinitely many primes p we have $f(p) = \varepsilon p^d$. Denote the set of these primes by P. Since a function g satisfies the given condition if and only if $-g$ satisfies the same condition, we may suppose $\varepsilon = 1$.

The case $d = 0$ is easily ruled out, because 0 does not divide any nonzero integer. Suppose $d \geq 1$ and write n as $md + r$, where m and r are integers such that $m \geq 1$ and $0 \leq r \leq d - 1$. Let x be an arbitrary integer. For each prime p in P, the difference $f(p) - f(x)$ divides $p^n - x^n$. Using the equality $f(p) = p^d$, we get

$$p^n - x^n = p^r (p^d)^m - x^n \equiv p^r f(x)^m - x^n \equiv 0 \pmod{p^d - f(x)}$$

Since we have $r < d$, for large enough primes $p \in P$ we obtain

$$|p^r f(x)^m - x^n| < p^d - f(x).$$

Hence $p^r f(x)^m - x^n$ has to be zero. This implies $r = 0$ and $x^n = (x^d)^m = f(x)^m$. Since m is odd, we obtain $f(x) = x^d$.

Comment. If n is an even positive integer, then the functions f of the form

$$f(x) = \begin{cases} x^d + c \text{ for some integers,} \\ -x^d + c \text{ for the rest of integers,} \end{cases}$$

where d is a positive divisor of $n/2$ and c is an integer, also satisfy the condition of the problem. Together with the functions in the answer, they are all functions that satisfy the condition when n is even.

N4

For each positive integer k, let $t(k)$ be the largest odd divisor of k. Determine all positive integers a for which there exists a positive integer n such that all the differences

$$t(n+a)-t(n), \quad t(n+a+1)-t(n+1), \quad \ldots, \quad t(n+2a-1)-t(n+a-1)$$

are divisible by 4.

Answer. $a = 1$, 3, or 5.

Solution. A pair (a, n) satisfying the condition of the problem will be called a *winning pair*. It is straightforward to check that the pairs $(1,1)$, $(3,1)$, and $(5,4)$ are winning pairs.

Now suppose that a is a positive integer not equal to 1, 3, and 5. We will show that there are no winning pairs (a, n) by distinguishing three cases.

Case 1: a is even. In this case we have $a = 2^\alpha d$ for some positive integer α and some odd d. Since $a \geq 2^\alpha$, for each positive integer n there exists an $i \in \{0, 1, \ldots, a-1\}$ such that $n+i = 2^{\alpha-1}e$, where e is some odd integer. Then we have $t(n+i) = t(2^{\alpha-1}e) = e$ and

$$t(n+a+i) = t(2^\alpha d + 2^{\alpha-1}e) = 2d + e \equiv e + 2 \pmod{4}.$$

So we get $t(n+i) - t(n+a+i) \equiv 2 \pmod{4}$, and (a, n) is not a winning pair.

Case 2: a is odd and $a > 8$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, a-5\}$ such that $n+i = 2d$ for some odd d. We get

$$t(n+i) = d \not\equiv d+2 = t(n+i+4) \pmod{4}$$

and

$$t(n+a+i) = n+a+i \equiv n+a+i+4 = t(n+a+i+4) \pmod{4}.$$

Therefore, the integers $t(n+a+i) - t(n+i)$ and $t(n+a+i+4) - t(n+i+4)$ cannot be both divisible by 4, and therefore there are no winning pairs in this case.

Case 3: $a = 7$. For each positive integer n, there exists an $i \in \{0, 1, \ldots, 6\}$ such that $n+i$ is either of the form $8k+3$ or of the form $8k+6$, where k is a nonnegative integer. But we have

$$t(8k+3) \equiv 3 \not\equiv 1 \equiv 4k+5 = t(8k+3+7) \pmod{4}$$

and

$$t(8k+6) = 4k+3 \equiv 3 \not\equiv 1 \equiv t(8k+6+7) \pmod{4}.$$

Hence, there are no winning pairs of the form $(7, n)$.

N5

Let f be a function from the set of integers to the set of positive integers. Suppose that for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m-n)$. Prove that for all integers m, n with $f(m) \leq f(n)$ the number $f(n)$ is divisible by $f(m)$.

Solution 1. Suppose that x and y are two integers with $f(x) < f(y)$. We will show that $f(x) \mid f(y)$. By taking $m = x$ and $n = y$ we see that

$$f(x-y) \mid |f(x) - f(y)| = f(y) - f(x) > 0,$$

so $f(x-y) \leq f(y) - f(x) < f(y)$. Hence the number $d = f(x) - f(x-y)$ satisfies

$$-f(y) < -f(x-y) < d < f(x) < f(y).$$

Taking $m = x$ and $n = x - y$ we see that $f(y) \mid d$, so we deduce $d = 0$, or in other words $f(x) = f(x-y)$. Taking $m = x$ and $n = y$ we see that $f(x) = f(x-y) \mid f(x) - f(y)$, which implies $f(x) \mid f(y)$.

Solution 2. We split the solution into a sequence of claims; in each claim, the letters m and n denote arbitrary integers.

Claim 1. $f(n) \mid f(mn)$.

Proof. Since trivially $f(n) \mid f(1 \cdot n)$ and $f(n) \mid f((k+1)n) - f(kn)$ for all integers k, this is easily seen by using induction on m in both directions. □

Claim 2. $f(n) \mid f(0)$ and $f(n) = f(-n)$.

Proof. The first part follows by plugging $m = 0$ into Claim 1. Using Claim 1 twice with $m = -1$, we get $f(n) \mid f(-n) \mid f(n)$, from which the second part follows. □

From Claim 1, we get $f(1) \mid f(n)$ for all integers n, so $f(1)$ is the minimal value attained by f. Next, from Claim 2, the function f can attain only a finite number of values since all these values divide $f(0)$.

Now we prove the statement of the problem by induction on the number N_f of values attained by f. In the base case $N_f \leq 2$, we either have $f(0) \neq f(1)$, in which case these two numbers are the only values attained by f and the statement is clear, or we have $f(0) = f(1)$, in which case we have $f(1) \mid f(n) \mid f(0)$ for all integers n, so f is constant and the statement is obvious again.

For the induction step, assume that $N_f \geq 3$, and let a be the least positive integer with $f(a) > f(1)$. Note that such a number exists due to the symmetry of f obtained in Claim 2.

Claim 3. $f(n) \neq f(1)$ if and only if $a \mid n$.

Proof. Since $f(1) = \cdots = f(a-1) < f(a)$, the claim follows from the fact that

$$f(n) = f(1) \iff f(n+a) = f(1).$$

So it suffices to prove this fact.

Assume that $f(n) = f(1)$. Then $f(n+a) \mid f(a) - f(-n) = f(a) - f(n) > 0$, so $f(n+a) \leq f(a) - f(n) < f(a)$; in particular the difference $f(n+a) - f(n)$ is stricly smaller than $f(a)$. Furthermore, this difference is divisible by $f(a)$ and nonnegative since $f(n) = f(1)$ is the least value attained by f. So we have $f(n+a) - f(n) = 0$, as desired. For the converse direction we only need to remark that $f(n+a) = f(1)$ entails $f(-n-a) = f(1)$, and hence $f(n) = f(-n) = f(1)$ by the forward implication. □

We return to the induction step. So let us take two arbitrary integers m and n with $f(m) \leq f(n)$. If $a \nmid m$, then we have $f(m) = f(1) \mid f(n)$. On the other hand, suppose that $a \mid m$; then by Claim 3 $a \mid n$ as well. Now define the function $g(x) = f(ax)$. Clearly, g satisfies the conditions of the problem, but $N_g < N_f - 1$, since g does not attain $f(1)$. Hence, by the induction hypothesis, $f(m) = g(m/a) \mid g(n/a) = f(n)$, as desired.

Comment. After the fact that f attains a finite number of values has been established, there are several ways of finishing the solution. For instance, let $f(0) = b_1 > b_2 > \cdots > b_k$ be all these values. One may show (essentially in the same way as in Claim 3) that the set $S_i = \{n : f(n) \geq b_i\}$ consists exactly of all numbers divisible by some integer $a_i \geq 0$. One obviously has $a_i \mid a_{i-1}$, which implies $f(a_i) \mid f(a_{i-1})$ by Claim 1. So, $b_k \mid b_{k-1} \mid \cdots \mid b_1$, thus proving the problem statement.

Moreover, now it is easy to describe all functions satisfying the conditions of the problem. Namely, all these functions can be constructed as follows. Consider a sequence of nonnegative integers $a_1, a_2, \ldots, a_k$ and another sequence of positive integers $b_1, b_2, \ldots, b_k$ such that $|a_k| = 1$, $a_i \neq a_j$ and $b_i \neq b_j$ for all $1 \leq i < j \leq k$, and $a_i \mid a_{i-1}$ and $b_i \mid b_{i-1}$ for all $i = 2, \ldots, k$. Then one may introduce the function

$$f(n) = b_{i(n)}, \qquad \text{where} \quad i(n) = \min\{i : a_i \mid n\}.$$

These are all the functions which satisfy the conditions of the problem.

N6

Let $P(x)$ and $Q(x)$ be two polynomials with integer coefficients such that no nonconstant polynomial with rational coefficients divides both $P(x)$ and $Q(x)$. Suppose that for every positive integer n the integers $P(n)$ and $Q(n)$ are positive, and $2^{Q(n)}-1$ divides $3^{P(n)}-1$. Prove that $Q(x)$ is a constant polynomial.

Solution. First we show that there exists an integer d such that for all positive integers n we have $\gcd\big(P(n),Q(n)\big) \leq d$.

Since $P(x)$ and $Q(x)$ are coprime (over the polynomials with rational coefficients), Euclid's algorithm provides some polynomials $R_0(x), S_0(x)$ with rational coefficients such that $P(x)R_0(x)-Q(x)S_0(x)=1$. Multiplying by a suitable positive integer d, we obtain polynomials $R(x)=d\cdot R_0(x)$ and $S(x)=d\cdot S_0(x)$ with integer coefficients for which $P(x)R(x)-Q(x)S(x)=d$. Then we have $\gcd\big(P(n),Q(n)\big)\leq d$ for any integer n.

To prove the problem statement, suppose that $Q(x)$ is not constant. Then the sequence $Q(n)$ is not bounded and we can choose a positive integer m for which

$$M=2^{Q(m)}-1\geq 3^{\max\{P(1),P(2),\ldots,P(d)\}}. \tag{1}$$

Since $M=2^{Q(n)}-1 \mid 3^{P(n)}-1$, we have $2,3 \nmid M$. Let a and b be the multiplicative orders of 2 and 3 modulo M, respectively. Obviously, $a=Q(m)$ since the lower powers of 2 do not reach M. Since M divides $3^{P(m)}-1$, we have $b \mid P(m)$. Then $\gcd(a,b)\leq\gcd\big(P(m),Q(m)\big)\leq d$. Since the expression $ax-by$ attains all integer values divisible by $\gcd(a,b)$ when x and y run over all nonnegative integer values, there exist some nonnegative integers x,y such that $1\leq m+ax-by\leq d$.

By $Q(m+ax)\equiv Q(m) \pmod a$ we have

$$2^{Q(m+ax)}\equiv 2^{Q(m)}\equiv 1 \pmod M$$

and therefore

$$M \mid 2^{Q(m+ax)}-1 \mid 3^{P(m+ax)}-1.$$

Then, by $P(m+ax-by)\equiv P(m+ax) \pmod b$ we have

$$3^{P(m+ax-by)}\equiv 3^{P(m+ax)}\equiv 1 \pmod M.$$

Since $P(m+ax-by)>0$ this implies $M\leq 3^{P(m+ax-by)}-1$. But $P(m+ax-by)$ is listed among $P(1),P(2),\ldots,P(d)$, so

$$M<3^{P(m+ax-by)}\leq 3^{\max\{P(1),P(2),\ldots,P(d)\}}$$

which contradicts (1).

Comment. We present another variant of the solution above.

Denote the degree of P by k and its leading coefficient by p. Consider any positive integer n and let $a = Q(n)$. Again, denote by b the multiplicative order of 3 modulo $2^a - 1$. Since $2^a - 1 \mid 3^{P(n)} - 1$, we have $b \mid P(n)$. Moreover, since $2^{Q(n+at)} - 1 \mid 3^{P(n+at)} - 1$ and $a = Q(n) \mid Q(n + at)$ for each positive integer t, we have $2^a - 1 \mid 3^{P(n+at)} - 1$, hence $b \mid P(n + at)$ as well.

Therefore, b divides $\gcd\{P(n + at) : t \geq 0\}$; hence it also divides the number

$$\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} P(n + ai) = p \cdot k! \cdot a^k.$$

Finally, we get $b \mid \gcd\bigl(P(n), k! \cdot p \cdot Q(n)^k\bigr)$, which is bounded by the same arguments as in the beginning of the solution. So $3^b - 1$ is bounded, and hence $2^{Q(n)} - 1$ is bounded as well.

N7

Let p be an odd prime number. For every integer a, define the number

$$S_a = \frac{a}{1} + \frac{a^2}{2} + \cdots + \frac{a^{p-1}}{p-1}.$$

Let m and n be integers such that

$$S_3 + S_4 - 3S_2 = \frac{m}{n}.$$

Prove that p divides m.

Solution 1. For rational numbers p_1/q_1 and p_2/q_2 with the denominators q_1, q_2 not divisible by p, we write $p_1/q_1 \equiv p_2/q_2 \pmod p$ if the numerator $p_1q_2 - p_2q_1$ of their difference is divisible by p.

We start with finding an explicit formula for the residue of S_a modulo p. Note first that for every $k = 1, \ldots, p-1$ the number $\binom{p}{k}$ is divisible by p, and

$$\frac{1}{p}\binom{p}{k} = \frac{(p-1)(p-2)\cdots(p-k+1)}{k!} \equiv \frac{(-1)\cdot(-2)\cdots(-k+1)}{k!} = \frac{(-1)^{k-1}}{k} \pmod p$$

Therefore, we have

$$S_a = -\sum_{k=1}^{p-1} \frac{(-a)^k(-1)^{k-1}}{k} \equiv -\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} \pmod p.$$

The number on the right-hand side is integer. Using the binomial formula we express it as

$$-\sum_{k=1}^{p-1}(-a)^k \cdot \frac{1}{p}\binom{p}{k} = -\frac{1}{p}\left(-1 - (-a)^p + \sum_{k=0}^{p}(-a)^k\binom{p}{k}\right) = \frac{(a-1)^p - a^p + 1}{p}$$

since p is odd. So, we have

$$S_a \equiv \frac{(a-1)^p - a^p + 1}{p} \pmod p.$$

Finally, using the obtained formula we get

$$\begin{aligned} S_3 + S_4 - 3S_2 &\equiv \frac{(2^p - 3^p + 1) + (3^p - 4^p + 1) - 3(1^p - 2^p + 1)}{p} \\ &= \frac{4\cdot 2^p - 4^p - 4}{p} = -\frac{(2^p-2)^2}{p} \pmod p. \end{aligned}$$

By Fermat's theorem, $p \mid 2^p - 2$, so $p^2 \mid (2^p-2)^2$ and hence $S_3 + S_4 - 3S_2 \equiv 0 \pmod p$.

Solution 2. One may solve the problem without finding an explicit formula for S_a. It is enough to find the following property.

Lemma. For every integer a, we have $S_{a+1} \equiv S_{-a} \pmod{p}$.

Proof. We expand S_{a+1} using the binomial formula as

$$S_{a+1} = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k} \binom{k}{j} a^j = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \sum_{j=1}^{k} a^j \cdot \frac{1}{k} \binom{k}{j} \right) = \sum_{k=1}^{p-1} \frac{1}{k} + \sum_{j=1}^{p-1} a^j \sum_{k=j}^{p-1} \frac{1}{k} \binom{k}{j} a^k.$$

Note that $\frac{1}{k} + \frac{1}{p-k} = \frac{p}{k(p-k)} \equiv 0 \pmod{p}$ for all $1 \le k \le p-1$; hence the first sum vanishes modulo p. For the second sum, we use the relation $\frac{1}{k}\binom{k}{j} = \frac{1}{j}\binom{k-1}{j-1}$ to obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j}{j} \sum_{k=1}^{p-1} \binom{k-1}{j-1} \pmod{p}.$$

Finally, from the relation

$$\sum_{k=1}^{p-1} \binom{k-1}{j-1} = \binom{p-1}{j} = \frac{(p-1)(p-2)\dots(p-j)}{j!} \equiv (-1)^j \pmod{p}$$

we obtain

$$S_{a+1} \equiv \sum_{j=1}^{p-1} \frac{a^j(-1)^j}{j!} = S_{-a}. \qquad \square$$

Now we turn to the problem. Using the lemma we get

$$S_3 - 3S_2 \equiv S_{-2} - 3S_2 = \sum_{\substack{1 \le k \le p-1 \\ k \text{ is even}}} \frac{-2 \cdot 2^k}{k} + \sum_{\substack{1 \le k \le p-1 \\ k \text{ is odd}}} \frac{-4 \cdot 2^k}{k} \pmod{p}. \tag{1}$$

The first sum in (1) expands as

$$\sum_{\ell=1}^{(p-1)/2} \frac{-2 \cdot 2^{2\ell}}{2\ell} = -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell}.$$

Next, using Fermat's theorem, we expand the second sum in (1) as

$$-\sum_{\ell=1}^{(p-1)/2} \frac{2^{2\ell+1}}{2\ell-1} \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{2^{p+2\ell}}{p+2\ell-1} = -\sum_{m=(p+1)/2}^{p-1} \frac{2 \cdot 4^m}{2m} = -\sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} \pmod{p}$$

(here we set $m = \ell + \frac{p-1}{2}$). Hence,

$$S_3 - 3S_2 \equiv -\sum_{\ell=1}^{(p-1)/2} \frac{4^\ell}{\ell} - \sum_{m=(p+1)/2}^{p-1} \frac{4^m}{m} = -S_4 \pmod{p}.$$

N8

Let k be a positive integer and set $n = 2^k + 1$. Prove that n is a prime number if and only if the following holds: there is a permutation $a_1, \ldots, a_{n-1}$ of the numbers $1, 2, \ldots, n-1$ and a sequence of integers $g_1, g_2, \ldots, g_{n-1}$ such that n divides $g_i^{a_i} - a_{i+1}$ for every $i \in \{1, 2, \ldots, n-1\}$, where we set $a_n = a_1$.

Solution. Let $N = \{1, 2, \ldots, n-1\}$. For $a, b \in N$, we say that b *follows* a if there exists an integer g such that $b \equiv g^a \pmod{n}$ and denote this property as $a \to b$. This way we have a directed graph with N as set of vertices. If $a_1, \ldots, a_{n-1}$ is a permutation of $1, 2, \ldots, n-1$ such that $a_1 \to a_2 \to \ldots \to a_{n-1} \to a_1$ then this is a *Hamiltonian cycle* in the graph.

Step I. First consider the case when n is composite. Let $n = p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ be its prime factorization. All primes p_i are odd.

Suppose that $\alpha_i > 1$ for some i. For all integers a, g with $a \geq 2$, we have $g^a \not\equiv p_i \pmod{p_i^2}$, because g^a is either divisible by p_i^2 or it is not divisible by p_i. It follows that in any Hamiltonian cycle p_i comes immediately after 1. The same argument shows that $2p_i$ also should come immediately after 1, which is impossible. Hence, there is no Hamiltonian cycle in the graph.

Now suppose that n is square-free. We have $n = p_1 p_2 \ldots p_s > 9$ and $s \geq 2$. Assume that there exists a Hamiltonian cycle. There are $\frac{n-1}{2}$ even numbers in this cycle, and each number which follows one of them should be a quadratic residue modulo n. So, there should be at least $\frac{n-1}{2}$ nonzero quadratic residues modulo n. On the other hand, for each p_i there exist exactly $\frac{p_i+1}{2}$ quadratic residues modulo p_i; by the Chinese Remainder Theorem, the number of quadratic residues modulo n is exactly $\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2}$, including 0. Then we have a contradiction by

$$\frac{p_1+1}{2} \cdot \frac{p_2+1}{2} \cdot \ldots \cdot \frac{p_s+1}{2} \leq \frac{2p_1}{3} \cdot \frac{2p_2}{3} \cdot \ldots \cdot \frac{2p_s}{3} = \left(\frac{2}{3}\right)^s n \leq \frac{4n}{9} < \frac{n-1}{2}.$$

This proves the "if"-part of the problem.

Step II. Now suppose that n is prime. For any $a \in N$, denote by $\nu_2(a)$ the exponent of 2 in the prime factorization of a, and let $\mu(a) = \max\{t \in [0, k] \mid 2^t \to a\}$.

Lemma. For any $a, b \in N$, we have $a \to b$ if and only if $\nu_2(a) \leq \mu(b)$.

Proof. Let $\ell = \nu_2(a)$ and $m = \mu(b)$.

Suppose $\ell \leq m$. Since b follows 2^m, there exists some g_0 such that $b \equiv g_0^{2^m} \pmod{n}$. By $\gcd(a, n-1) = 2^\ell$ there exist some integers p and q such that $pa - q(n-1) = 2^\ell$. Choosing $g = g_0^{2^{m-\ell}p}$ we have $g^a = g_0^{2^{m-\ell}pa} = g_0^{2^m + 2^{m-\ell}q(n-1)} \equiv g_0^{2^m} \equiv b \pmod{n}$ by FERMAT's theorem. Hence, $a \to b$.

To prove the reverse statement, suppose that $a \to b$, so $b \equiv g^a \pmod{n}$ with some g. Then $b \equiv (g^{a/2^\ell})^{2^\ell}$, and therefore $2^\ell \to b$. By the definition of $\mu(b)$, we have $\mu(b) \geq \ell$. The lemma is proved. □

Now for every i with $0 \le i \le k$, let

$$A_i = \{a \in N \mid \nu_2(a) = i\},$$
$$B_i = \{a \in N \mid \mu(a) = i\},$$
$$\text{and } C_i = \{a \in N \mid \mu(a) \ge i\} = B_i \cup B_{i+1} \cup \ldots \cup B_k.$$

We claim that $|A_i| = |B_i|$ for all $0 \le i \le k$. Obviously we have $|A_i| = 2^{k-i-1}$ for all $i = 0, \ldots, k-1$, and $|A_k| = 1$. Now we determine $|C_i|$. We have $|C_0| = n - 1$ and by FERMAT's theorem we also have $C_k = \{1\}$, so $|C_k| = 1$. Next, notice that $C_{i+1} = \{x^2 \mod n \mid x \in C_i\}$. For every $a \in N$, the relation $x^2 \equiv a \pmod{n}$ has at most two solutions in N. Therefore we have $2|C_{i+1}| \le |C_i|$, with the equality achieved only if for every $y \in C_{i+1}$, there exist distinct elements $x, x' \in C_i$ such that $x^2 \equiv x'^2 \equiv y \pmod{n}$ (this implies $x + x' = n$). Now, since $2^k|C_k| = |C_0|$, we obtain that this equality should be achieved in each step. Hence $|C_i| = 2^{k-i}$ for $0 \le i \le k$, and therefore $|B_i| = 2^{k-i-1}$ for $0 \le i \le k-1$ and $|B_k| = 1$.

From the previous arguments we can see that for each $z \in C_i$ $(0 \le i < k)$ the equation $x^2 \equiv z^2 \pmod{n}$ has two solutions in C_i, so we have $n - z \in C_i$. Hence, for each $i = 0, 1, \ldots, k-1$, exactly half of the elements of C_i are odd. The same statement is valid for $B_i = C_i \setminus C_{i+1}$ for $0 \le i \le k-2$. In particular, each such B_i contains an odd number. Note that $B_k = \{1\}$ also contains an odd number, and $B_{k-1} = \{2^k\}$ since C_{k-1} consists of the two square roots of 1 modulo n.

Step III. Now we construct a Hamiltonian cycle in the graph. First, for each i with $0 \le i \le k$, connect the elements of A_i to the elements of B_i by means of an arbitrary bijection. After performing this for every i, we obtain a subgraph with all vertices having in-degree 1 and out-degree 1, so the subgraph is a disjoint union of cycles. If there is a unique cycle, we are done. Otherwise, we modify the subgraph in such a way that the previous property is preserved and the number of cycles decreases; after a finite number of steps we arrive at a single cycle.

For every cycle C, let $\lambda(C) = \min_{c \in C} \nu_2(c)$. Consider a cycle C for which $\lambda(C)$ is maximal. If $\lambda(C) = 0$, then for any other cycle C' we have $\lambda(C') = 0$. Take two arbitrary vertices $a \in C$ and $a' \in C'$ such that $\nu_2(a) = \nu_2(a') = 0$; let their direct successors be b and b', respectively. Then we can unify C and C' to a single cycle by replacing the edges $a \to b$ and $a' \to b'$ by $a \to b'$ and $a' \to b$.

Now suppose that $\lambda = \lambda(C) \ge 1$; let $a \in C \cap A_\lambda$. If there exists some $a' \in A_\lambda \setminus C$, then a' lies in another cycle C' and we can merge the two cycles in exactly the same way as above. So, the only remaining case is $A_\lambda \subset C$. Since the edges from A_λ lead to B_λ, we get also $B_\lambda \subset C$. If $\lambda \ne k-1$ then B_λ contains an odd number; this contradicts the assumption $\lambda(C) > 0$. Finally, if $\lambda = k-1$, then C contains 2^{k-1} which is the only element of A_{k-1}. Since $B_{k-1} = \{2^k\} = A_k$ and $B_k = \{1\}$, the cycle C contains the path $2^{k-1} \to 2^k \to 1$ and it contains an odd number again. This completes the proof of the "only if"-part of the problem.

Comment 1. The lemma and the fact $|A_i| = |B_i|$ together show that for every edge $a \to b$ of the Hamiltonian cycle, $\nu_2(a) = \mu(b)$ must hold. After this observation, the Hamiltonian cycle can be built in many ways. For instance, it is possible to select edges from A_i to B_i for $i = k, k-1, \ldots, 1$ in such a way that they form disjoint paths; at the end all these paths will have odd endpoints. In the final step, the paths can be closed to form a unique cycle.

Comment 2. Step II is an easy consequence of some basic facts about the multiplicative group modulo the prime $n = 2^k + 1$. The Lemma follows by noting that this group has order 2^k, so the a-th powers are exactly the $2^{\nu_2(a)}$-th powers. Using the existence of a primitive root g modulo n one sees that the map from $\{1, 2, \ldots, n-1\}$ to itself that sends a to $g^a \bmod n$ is a bijection that sends A_i to B_i for each $i \in \{0, \ldots, k\}$.

Shortlisted Problems with Solutions

53rd International Mathematical Olympiad

Mar del Plata, Argentina 2012

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2013

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2012 thank the following 40 countries for contributing 136 problem proposals:

Australia, Austria, Belarus, Belgium, Bulgaria, Canada, Cyprus, Czech Republic, Denmark, Estonia, Finland, France, Germany, Greece, Hong Kong, India, Iran, Ireland, Israel, Japan, Kazakhstan, Luxembourg, Malaysia, Montenegro, Netherlands, Norway, Pakistan, Romania, Russia, Serbia, Slovakia, Slovenia, South Africa, South Korea, Sweden, Thailand, Ukraine, United Kingdom, United States of America, Uzbekistan

Problem Selection Committee

Martín Avendaño
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Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

Here $X + Y$ denotes the set $\{x + y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

A3. Let $a_2, \ldots, a_n$ be $n - 1$ positive real numbers, where $n \geq 3$, such that $a_2a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2(1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

A5. Find all functions $f \colon \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y) \quad \text{for all } x, y \in \mathbb{R}$$

and $f(-1) \neq 0$.

A6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that $f^{2k}(n) = n + k$, and let k_n be the smallest such k. Prove that the sequence $k_1, k_2, \ldots$ is unbounded.

A7. We say that a function $f : \mathbb{R}^k \to \mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1, \ldots, x_k) = \max_{i=1,\ldots,m} \min_{j=1,\ldots,n} P_{i,j}(x_1, \ldots, x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \leq x \leq N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \geq 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \geq 1.99^k$ such that Ben cannot guarantee a win.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Geometry

G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

G5. Let ABC be a triangle with $\angle BCA = 90°$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Number Theory

N1. Call admissible a set A of integers that has the following property:

$$\text{If } x, y \in A \text{ (possibly } x = y\text{) then } x^2 + kxy + y^2 \in A \text{ for every integer } k.$$

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

N2. Find all triples (x, y, z) of positive integers such that $x \le y \le z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

N3. Determine all integers $m \ge 2$ such that every n with $\frac{m}{3} \le n \le \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

N4. An integer a is called friendly if the equation $(m^2 + n)(n^2 + m) = a(m - n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1, 2, \ldots, 2012\}$.

b) Decide whether $a = 2$ is friendly.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

N6. Let x and y be positive integers. If $x^{2^n} - 1$ is divisible by $2^n y + 1$ for every positive integer n, prove that $x = 1$.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Algebra

A1. Find all the functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a)$$

for all integers a, b, c satisfying $a + b + c = 0$.

Solution. The substitution $a = b = c = 0$ gives $3f(0)^2 = 6f(0)^2$, hence

$$f(0) = 0. \tag{1}$$

The substitution $b = -a$ and $c = 0$ gives $((f(a) - f(-a))^2 = 0$. Hence f is an even function:

$$f(a) = f(-a) \quad \text{for all } a \in \mathbb{Z}. \tag{2}$$

Now set $b = a$ and $c = -2a$ to obtain $2f(a)^2 + f(2a)^2 = 2f(a)^2 + 4f(a)f(2a)$. Hence

$$f(2a) = 0 \ \text{ or } \ f(2a) = 4f(a) \quad \text{for all } a \in \mathbb{Z}. \tag{3}$$

If $f(r) = 0$ for some $r \geq 1$ then the substitution $b = r$ and $c = -a-r$ gives $(f(a+r)-f(a))^2 = 0$. So f is periodic with period r, i. e.

$$f(a + r) = f(a) \quad \text{for all } a \in \mathbb{Z}.$$

In particular, if $f(1) = 0$ then f is constant, thus $f(a) = 0$ for all $a \in \mathbb{Z}$. This function clearly satisfies the functional equation. For the rest of the analysis, we assume $f(1) = k \neq 0$.

By (3) we have $f(2) = 0$ or $f(2) = 4k$. If $f(2) = 0$ then f is periodic of period 2, thus $f(\text{even}) = 0$ and $f(\text{odd}) = k$. This function is a solution for every k. We postpone the verification; for the sequel assume $f(2) = 4k \neq 0$.

By (3) again, we have $f(4) = 0$ or $f(4) = 16k$. In the first case f is periodic of period 4, and $f(3) = f(-1) = f(1) = k$, so we have $f(4n) = 0$, $f(4n+1) = f(4n+3) = k$, and $f(4n+2) = 4k$ for all $n \in \mathbb{Z}$. This function is a solution too, which we justify later. For the rest of the analysis, we assume $f(4) = 16k \neq 0$.

We show now that $f(3) = 9k$. In order to do so, we need two substitutions:

$$a = 1, b = 2, c = -3 \Longrightarrow f(3)^2 - 10kf(3) + 9k^2 = 0 \Longrightarrow f(3) \in \{k, 9k\},$$
$$a = 1, b = 3, c = -4 \Longrightarrow f(3)^2 - 34kf(3) + 225k^2 = 0 \Longrightarrow f(3) \in \{9k, 25k\}.$$

Therefore $f(3) = 9k$, as claimed. Now we prove inductively that the only remaining function is $f(x) = kx^2$, $x \in \mathbb{Z}$. We proved this for $x = 0, 1, 2, 3, 4$. Assume that $n \geq 4$ and that $f(x) = kx^2$ holds for all integers $x \in [0, n]$. Then the substitutions $a = n$, $b = 1$, $c = -n-1$ and $a = n-1$, $b = 2$, $c = -n-1$ lead respectively to

$$f(n + 1) \in \{k(n + 1)^2, k(n - 1)^2\} \qquad \text{and} \qquad f(n + 1) \in \{k(n + 1)^2, k(n - 3)^2\}.$$

Since $k(n - 1)^2 \neq k(n - 3)^2$ for $n \neq 2$, the only possibility is $f(n + 1) = k(n + 1)^2$. This completes the induction, so $f(x) = kx^2$ for all $x \geq 0$. The same expression is valid for negative values of x since f is even. To verify that $f(x) = kx^2$ is actually a solution, we need to check the identity $a^4 + b^4 + (a + b)^4 = 2a^2b^2 + 2a^2(a + b)^2 + 2b^2(a + b)^2$, which follows directly by expanding both sides.

Therefore the only possible solutions of the functional equation are the constant function $f_1(x) = 0$ and the following functions:

$$f_2(x) = kx^2 \qquad f_3(x) = \begin{cases} 0 & x \text{ even} \\ k & x \text{ odd} \end{cases} \qquad f_4(x) = \begin{cases} 0 & x \equiv 0 \pmod 4 \\ k & x \equiv 1 \pmod 2 \\ 4k & x \equiv 2 \pmod 4 \end{cases}$$

for any non-zero integer k. The verification that they are indeed solutions was done for the first two. For f_3 note that if $a + b + c = 0$ then either a, b, c are all even, in which case $f(a) = f(b) = f(c) = 0$, or one of them is even and the other two are odd, so both sides of the equation equal $2k^2$. For f_4 we use similar parity considerations and the symmetry of the equation, which reduces the verification to the triples $(0, k, k)$, $(4k, k, k)$, $(0, 0, 0)$, $(0, 4k, 4k)$. They all satisfy the equation.

Comment. We used several times the same fact: For any $a, b \in \mathbb{Z}$ the functional equation is a quadratic equation in $f(a+b)$ whose coefficients depend on $f(a)$ and $f(b)$:

$$f(a+b)^2 - 2(f(a) + f(b))f(a+b) + (f(a) - f(b))^2 = 0.$$

Its discriminant is $16f(a)f(b)$. Since this value has to be non-negative for any $a, b \in \mathbb{Z}$, we conclude that either f or $-f$ is always non-negative. Also, if f is a solution of the functional equation, then $-f$ is also a solution. Therefore we can assume $f(x) \geq 0$ for all $x \in \mathbb{Z}$. Now, the two solutions of the quadratic equation are

$$f(a+b) \in \left\{ \left(\sqrt{f(a)} + \sqrt{f(b)}\right)^2, \left(\sqrt{f(a)} - \sqrt{f(b)}\right)^2 \right\} \quad \text{for all } a, b \in \mathbb{Z}.$$

The computation of $f(3)$ from $f(1)$, $f(2)$ and $f(4)$ that we did above follows immediately by setting $(a, b) = (1, 2)$ and $(a, b) = (1, -4)$. The inductive step, where $f(n+1)$ is derived from $f(n)$, $f(n-1)$, $f(2)$ and $f(1)$, follows immediately using $(a, b) = (n, 1)$ and $(a, b) = (n-1, 2)$.

A2. Let $\mathbb{Z}$ and $\mathbb{Q}$ be the sets of integers and rationals respectively.

a) Does there exist a partition of $\mathbb{Z}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

b) Does there exist a partition of $\mathbb{Q}$ into three non-empty subsets A, B, C such that the sets $A + B$, $B + C$, $C + A$ are disjoint?

Here $X + Y$ denotes the set $\{x + y \mid x \in X, y \in Y\}$, for $X, Y \subseteq \mathbb{Z}$ and $X, Y \subseteq \mathbb{Q}$.

Solution 1. a) The residue classes modulo 3 yield such a partition:

$$A = \{3k \mid k \in \mathbb{Z}\}, \quad B = \{3k + 1 \mid k \in \mathbb{Z}\}, \quad C = \{3k + 2 \mid k \in \mathbb{Z}\}.$$

b) The answer is *no*. Suppose that $\mathbb{Q}$ can be partitioned into non-empty subsets A, B, C as stated. Note that for all $a \in A$, $b \in B$, $c \in C$ one has

$$a + b - c \in C, \quad b + c - a \in A, \quad c + a - b \in B. \tag{1}$$

Indeed $a+b-c \notin A$ as $(A+B)\cap(A+C) = \emptyset$, and similarly $a+b-c \notin B$, hence $a+b-c \in C$. The other two relations follow by symmetry. Hence $A+B \subset C+C$, $B+C \subset A+A$, $C+A \subset B+B$.

The opposite inclusions also hold. Let $a, a' \in A$ and $b \in B$, $c \in C$ be arbitrary. By (1) $a' + c - b \in B$, and since $a \in A$, $c \in C$, we use (1) again to obtain

$$a + a' - b = a + (a' + c - b) - c \in C.$$

So $A + A \subset B + C$ and likewise $B + B \subset C + A$, $C + C \subset A + B$. In summary

$$B + C = A + A, \qquad C + A = B + B, \qquad A + B = C + C.$$

Furthermore suppose that $0 \in A$ without loss of generality. Then $B = \{0\} + B \subset A + B$ and $C = \{0\} + C \subset A + C$. So, since $B + C$ is disjoint with $A + B$ and $A + C$, it is also disjoint with B and C. Hence $B + C$ is contained in $\mathbb{Z} \setminus (B \cup C) = A$. Because $B + C = A + A$, we obtain $A + A \subset A$. On the other hand $A = \{0\} + A \subset A + A$, implying $A = A + A = B + C$.

Therefore $A + B + C = A + A + A = A$, and now $B + B = C + A$ and $C + C = A + B$ yield $B + B + B = A + B + C = A$, $C + C + C = A + B + C = A$. In particular if $r \in \mathbb{Q} = A \cup B \cup C$ is arbitrary then $3r \in A$.

However such a conclusion is impossible. Take any $b \in B$ ($B \neq \emptyset$) and let $r = b/3 \in \mathbb{Q}$. Then $b = 3r \in A$ which is a contradiction.

Solution 2. We prove that the example for $\mathbb{Z}$ from the first solution is unique, and then use this fact to solve part b).

Let $\mathbb{Z} = A \cup B \cup C$ be a partition of $\mathbb{Z}$ with $A, B, C \neq \emptyset$ and $A + B$, $B + C$, $C + A$ disjoint. We need the relations (1) which clearly hold for $\mathbb{Z}$. Fix two consecutive integers from different sets, say $b \in B$ and $c = b+1 \in C$. For every $a \in A$ we have, in view of (1), $a - 1 = a + b - c \in C$ and $a + 1 = a + c - b \in B$. So every $a \in A$ is preceded by a number from C and followed by a number from B.

In particular there are pairs of the form $c, c + 1$ with $c \in C$, $c + 1 \in A$. For such a pair and any $b \in B$ analogous reasoning shows that each $b \in B$ is preceded by a number from A and followed by a number from C. There are also pairs $b, b - 1$ with $b \in B$, $b - 1 \in A$. We use them in a similar way to prove that each $c \in C$ is preceded by a number from B and followed by a number from A.

By putting the observations together we infer that A, B, C are the three congruence classes modulo 3. Observe that all multiples of 3 are in the set of the partition that contains 0.

Now we turn to part b). Suppose that there is a partition of $\mathbb{Q}$ with the given properties. Choose three rationals $r_i = p_i/q_i$ from the three sets A, B, C, $i = 1, 2, 3$, and set $N = 3q_1q_2q_3$.

Let $S \subset \mathbb{Q}$ be the set of fractions with denominators N (irreducible or not). It is obtained through multiplication of every integer by the constant $1/N$, hence closed under sums and differences. Moreover, if we identify each $k \in \mathbb{Z}$ with $k/N \in S$ then S is essentially the set $\mathbb{Z}$ with respect to addition. The numbers r_i belong to S because

$$r_1 = \frac{3p_1q_2q_3}{N}, \qquad r_2 = \frac{3p_2q_3q_1}{N}, \qquad r_3 = \frac{3p_3q_1q_2}{N}.$$

The partition $\mathbb{Q} = A \cup B \cup C$ of $\mathbb{Q}$ induces a partition $S = A' \cup B' \cup C'$ of S, with $A' = A \cap S$, $B' = B \cap S$, $C' = C \cap S$. Clearly $A' + B'$, $B' + C'$, $C' + A'$ are disjoint, so this partition has the properties we consider.

By the uniqueness of the example for $\mathbb{Z}$ the sets A', B', C' are the congruence classes modulo 3, multiplied by $1/N$. Also all multiples of $3/N$ are in the same set, A', B' or C'. This holds for r_1, r_2, r_3 in particular as they are all multiples of $3/N$. However r_1, r_2, r_3 are in different sets A', B', C' since they were chosen from different sets A, B, C. The contradiction ends the proof.

Comment. The uniqueness of the example for $\mathbb{Z}$ can also be deduced from the argument in the first solution.

A3. Let $a_2, \ldots, a_n$ be $n-1$ positive real numbers, where $n \geq 3$, such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Solution. The substitution $a_2 = \dfrac{x_2}{x_1}$, $a_3 = \dfrac{x_3}{x_2}$, $\ldots$, $a_n = \dfrac{x_1}{x_{n-1}}$ transforms the original problem into the inequality

$$(x_1+x_2)^2(x_2+x_3)^3 \cdots (x_{n-1}+x_1)^n > n^n x_1^2 x_2^3 \cdots x_{n-1}^n \qquad (*)$$

for all $x_1, \ldots, x_{n-1} > 0$. To prove this, we use the AM-GM inequality for each factor of the left-hand side as follows:

$$\begin{array}{rcll}
(x_1+x_2)^2 & & & \geq 2^2 x_1 x_2 \\
(x_2+x_3)^3 & = & \left(2\left(\frac{x_2}{2}\right)+x_3\right)^3 & \geq 3^3\left(\frac{x_2}{2}\right)^2 x_3 \\
(x_3+x_4)^4 & = & \left(3\left(\frac{x_3}{3}\right)+x_4\right)^4 & \geq 4^4\left(\frac{x_3}{3}\right)^3 x_4 \\
\vdots & & \vdots & \vdots \\
(x_{n-1}+x_1)^n & = & \left((n-1)\left(\frac{x_{n-1}}{n-1}\right)+x_1\right)^n & \geq n^n\left(\frac{x_{n-1}}{n-1}\right)^{n-1} x_1.
\end{array}$$

Multiplying these inequalities together gives (*), with inequality sign $\geq$ instead of $>$. However for the equality to occur it is necessary that $x_1 = x_2$, $x_2 = 2x_3$, $\ldots$, $x_{n-1} = (n-1)x_1$, implying $x_1 = (n-1)!x_1$. This is impossible since $x_1 > 0$ and $n \geq 3$. Therefore the inequality is strict.

Comment. One can avoid the substitution $a_i = x_i/x_{i-1}$. Apply the weighted AM-GM inequality to each factor $(1+a_k)^k$, with the same weights like above, to obtain

$$(1+a_k)^k = \left((k-1)\frac{1}{k-1}+a_k\right)^k \geq \frac{k^k}{(k-1)^{k-1}}a_k.$$

Multiplying all these inequalities together gives

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n \geq n^n a_2 a_3 \cdots a_n = n^n.$$

The same argument as in the proof above shows that the equality cannot be attained.

A4. Let f and g be two nonzero polynomials with integer coefficients and $\deg f > \deg g$. Suppose that for infinitely many primes p the polynomial $pf + g$ has a rational root. Prove that f has a rational root.

Solution 1. Since $\deg f > \deg g$, we have $|g(x)/f(x)| < 1$ for sufficiently large x; more precisely, there is a real number R such that $|g(x)/f(x)| < 1$ for all x with $|x| > R$. Then for all such x and all primes p we have

$$\big|pf(x) + g(x)\big| \geq \big|f(x)\big| \left(p - \frac{|g(x)|}{|f(x)|}\right) > 0.$$

Hence all real roots of the polynomials $pf + g$ lie in the interval $[-R, R]$.

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ where $n > m$, $a_n \neq 0$ and $b_m \neq 0$. Upon replacing $f(x)$ and $g(x)$ by $a_n^{n-1} f(x/a_n)$ and $a_n^{n-1} g(x/a_n)$ respectively, we reduce the problem to the case $a_n = 1$. In other words one can assume that f is monic. Then the leading coefficient of $pf + g$ is p, and if $r = u/v$ is a rational root of $pf + g$ with $(u, v) = 1$ and $v > 0$, then either $v = 1$ or $v = p$.

First consider the case when $v = 1$ infinitely many times. If $v = 1$ then $|u| \leq R$, so there are only finitely many possibilities for the integer u. Therefore there exist distinct primes p and q for which we have the same value of u. Then the polynomials $pf + g$ and $qf + g$ share this root, implying $f(u) = g(u) = 0$. So in this case f and g have an integer root in common.

Now suppose that $v = p$ infinitely many times. By comparing the exponent of p in the denominators of $pf(u/p)$ and $g(u/p)$ we get $m = n - 1$ and $pf(u/p) + g(u/p) = 0$ reduces to an equation of the form

$$\left(u^n + a_{n-1}pu^{n-1} + \ldots + a_0 p^n\right) + \left(b_{n-1}u^{n-1} + b_{n-2}pu^{n-2} + \ldots + b_0 p^{n-1}\right) = 0.$$

The equation above implies that $u^n + b_{n-1}u^{n-1}$ is divisible by p and hence, since $(u, p) = 1$, we have $u + b_{n-1} = pk$ with some integer k. On the other hand all roots of $pf + g$ lie in the interval $[-R, R]$, so that

$$\frac{|pk - b_{n-1}|}{p} = \frac{|u|}{p} < R,$$

$$|k| < R + \frac{|b_{n-1}|}{p} < R + |b_{n-1}|.$$

Therefore the integer k can attain only finitely many values. Hence there exists an integer k such that the number $\frac{pk - b_{n-1}}{p} = k - \frac{b_{n-1}}{p}$ is a root of $pf + g$ for infinitely many primes p. For these primes we have

$$f\left(k - b_{n-1}\frac{1}{p}\right) + \frac{1}{p} g\left(k - b_{n-1}\frac{1}{p}\right) = 0.$$

So the equation

$$f\left(k - b_{n-1}x\right) + xg\left(k - b_{n-1}x\right) = 0 \tag{1}$$

has infinitely many solutions of the form $x = 1/p$. Since the left-hand side is a polynomial, this implies that (1) is a polynomial identity, so it holds for all real x. In particular, by substituting $x = 0$ in (1) we get $f(k) = 0$. Thus the integer k is a root of f.

In summary the monic polynomial f obtained after the initial reduction always has an integer root. Therefore the original polynomial f has a rational root.

Solution 2. Analogously to the first solution, there exists a real number R such that the complex roots of all polynomials of the form $pf+g$ lie in the disk $|z| \le R$.

For each prime p such that $pf+g$ has a rational root, by GAUSS' lemma $pf+g$ is the product of two integer polynomials, one with degree 1 and the other with degree $\deg f - 1$. Since p is a prime, the leading coefficient of one of these factors divides the leading coefficient of f. Denote that factor by h_p.

By narrowing the set of the primes used we can assume that all polynomials h_p have the same degree and the same leading coefficient. Their complex roots lie in the disk $|z| \le R$, hence VIETA's formulae imply that all coefficients of all polynomials h_p form a bounded set. Since these coefficients are integers, there are only finitely many possible polynomials h_p. Hence there is a polynomial h such that $h_p = h$ for infinitely many primes p.

Finally, if p and q are distinct primes with $h_p = h_q = h$ then h divides $(p-q)f$. Since $\deg h = 1$ or $\deg h = \deg f - 1$, in both cases f has a rational root.

Comment. Clearly the polynomial h is a common factor of f and g. If $\deg h = 1$ then f and g share a rational root. Otherwise $\deg h = \deg f - 1$ forces $\deg g = \deg f - 1$ and g divides f over the rationals.

Solution 3. Like in the first solution, there is a real number R such that the real roots of all polynomials of the form $pf+g$ lie in the interval $[-R, R]$.

Let $p_1 < p_2 < \cdots$ be an infinite sequence of primes so that for every index k the polynomial $p_k f + g$ has a rational root r_k. The sequence $r_1, r_2, \dots$ is bounded, so it has a convergent subsequence $r_{k_1}, r_{k_2}, \dots$. Now replace the sequences $(p_1, p_2, \dots)$ and $(r_1, r_2, \dots)$ by $(p_{k_1}, p_{k_2}, \dots)$ and $(r_{k_1}, r_{k_2}, \dots)$; after this we can assume that the sequence $r_1, r_2, \dots$ is convergent. Let $\alpha = \lim_{k\to\infty} r_k$. We show that α is a rational root of f.

Over the interval $[-R, R]$, the polynomial g is bounded, $|g(x)| \le M$ with some fixed M. Therefore

$$|f(r_k)| = \left| f(r_k) - \frac{p_k f(r_k) + g(r_k)}{p_k} \right| = \frac{|g(r_k)|}{p_k} \le \frac{M}{p_k} \to 0,$$

and

$$f(\alpha) = f\Big(\lim_{k\to\infty} r_k\Big) = \lim_{k\to\infty} f(r_k) = 0.$$

So α is a root of f indeed.

Now let u_k, v_k be relative prime integers for which $r_k = \frac{u_k}{v_k}$. Let a be the leading coefficient of f, let $b = f(0)$ and $c = g(0)$ be the constant terms of f and g, respectively. The leading coefficient of the polynomial $p_k f + g$ is $p_k a$, its constant term is $p_k b + c$. So v_k divides $p_k a$ and u_k divides $p_k b + c$. Let $p_k b + c = u_k e_k$ (if $p_k b + c = u_k = 0$ then let $e_k = 1$).

We prove that α is rational by using the following fact. *Let (p_n) and (q_n) be sequences of integers such that the sequence (p_n/q_n) converges. If (p_n) or (q_n) is bounded then* $\lim(p_n/q_n)$ *is rational.*

Case 1: There is an infinite subsequence (k_n) of indices such that v_{k_n} divides a. Then (v_{k_n}) is bounded, so $\alpha = \lim_{n\to\infty}(u_{k_n}/v_{k_n})$ is rational.

Case 2: There is an infinite subsequence (k_n) of indices such that v_{k_n} does not divide a. For such indices we have $v_{k_n} = p_{k_n} d_{k_n}$ where d_{k_n} is a divisor of a. Then

$$\alpha = \lim_{n\to\infty} \frac{u_{k_n}}{v_{k_n}} = \lim_{n\to\infty} \frac{p_{k_n} b + c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}} + \lim_{n\to\infty} \frac{c}{p_{k_n} d_{k_n} e_{k_n}} = \lim_{n\to\infty} \frac{b}{d_{k_n} e_{k_n}}.$$

Because the numerator b in the last limit is bounded, α is rational.

A5. Find all functions $f\colon \mathbb{R}\to\mathbb{R}$ that satisfy the conditions

$$f(1+xy)-f(x+y)=f(x)f(y)\quad \text{for all } x,y\in\mathbb{R}$$

and $f(-1)\neq 0$.

Solution. The only solution is the function $f(x)=x-1$, $x\in\mathbb{R}$.

We set $g(x)=f(x)+1$ and show that $g(x)=x$ for all real x. The conditions take the form

$$g(1+xy)-g(x+y)=\big(g(x)-1\big)\big(g(y)-1\big)\quad \text{for all } x,y\in\mathbb{R} \text{ and } g(-1)\neq 1. \tag{1}$$

Denote $C=g(-1)-1\neq 0$. Setting $y=-1$ in (1) gives

$$g(1-x)-g(x-1)=C(g(x)-1). \tag{2}$$

Set $x=1$ in (2) to obtain $C(g(1)-1)=0$. Hence $g(1)=1$ as $C\neq 0$. Now plugging in $x=0$ and $x=2$ yields $g(0)=0$ and $g(2)=2$ respectively.

We pass on to the key observations

$$g(x)+g(2-x)=2\quad \text{for all } x\in\mathbb{R}, \tag{3}$$
$$g(x+2)-g(x)=2\quad \text{for all } x\in\mathbb{R}. \tag{4}$$

Replace x by $1-x$ in (2), then change x to $-x$ in the resulting equation. We obtain the relations $g(x)-g(-x)=C(g(1-x)-1)$, $g(-x)-g(x)=C(g(1+x)-1)$. Then adding them up leads to $C(g(1-x)+g(1+x)-2)=0$. Thus $C\neq 0$ implies (3).

Let u,v be such that $u+v=1$. Apply (1) to the pairs (u,v) and $(2-u,2-v)$:

$$g(1+uv)-g(1)=\big(g(u)-1\big)\big(g(v)-1\big),\quad g(3+uv)-g(3)=\big(g(2-u)-1\big)\big(g(2-v)-1\big).$$

Observe that the last two equations have equal right-hand sides by (3). Hence $u+v=1$ implies

$$g(uv+3)-g(uv+1)=g(3)-g(1).$$

Each $x\leq 5/4$ is expressible in the form $x=uv+1$ with $u+v=1$ (the quadratic function $t^2-t+(x-1)$ has real roots for $x\leq 5/4$). Hence $g(x+2)-g(x)=g(3)-g(1)$ whenever $x\leq 5/4$. Because $g(x)=x$ holds for $x=0,1,2$, setting $x=0$ yields $g(3)=3$. This proves (4) for $x\leq 5/4$. If $x>5/4$ then $-x<5/4$ and so $g(2-x)-g(-x)=2$ by the above. On the other hand (3) gives $g(x)=2-g(2-x)$, $g(x+2)=2-g(-x)$, so that $g(x+2)-g(x)=g(2-x)-g(-x)=2$. Thus (4) is true for all $x\in\mathbb{R}$.

Now replace x by $-x$ in (3) to obtain $g(-x)+g(2+x)=2$. In view of (4) this leads to $g(x)+g(-x)=0$, i. e. $g(-x)=-g(x)$ for all x. Taking this into account, we apply (1) to the pairs $(-x,y)$ and $(x,-y)$:

$$g(1-xy)-g(-x+y)=\big(g(x)+1\big)\big(1-g(y)\big),\quad g(1-xy)-g(x-y)=\big(1-g(x)\big)\big(g(y)+1\big).$$

Adding up yields $g(1-xy)=1-g(x)g(y)$. Then $g(1+xy)=1+g(x)g(y)$ by (3). Now the original equation (1) takes the form $g(x+y)=g(x)+g(y)$. Hence g is additive.

By additvity $g(1+xy)=g(1)+g(xy)=1+g(xy)$; since $g(1+xy)=1+g(x)g(y)$ was shown above, we also have $g(xy)=g(x)g(y)$ (g is multiplicative). In particular $y=x$ gives $g(x^2)=g(x)^2\geq 0$ for all x, meaning that $g(x)\geq 0$ for $x\geq 0$. Since g is additive and bounded from below on $[0,+\infty)$, it is linear; more exactly $g(x)=g(1)x=x$ for all $x\in\mathbb{R}$.

In summary $f(x)=x-1$, $x\in\mathbb{R}$. It is straightforward that this function satisfies the requirements.

Comment. There are functions that satisfy the given equation but vanish at -1, for instance the constant function 0 and $f(x)=x^2-1$, $x\in\mathbb{R}$.

A6. Let $f:\mathbb{N}\to\mathbb{N}$ be a function, and let f^m be f applied m times. Suppose that for every $n\in\mathbb{N}$ there exists a $k\in\mathbb{N}$ such that $f^{2k}(n)=n+k$, and let k_n be the smallest such k. Prove that the sequence $k_1,k_2,\dots$ is unbounded.

Solution. We restrict attention to the set

$$S=\{1,f(1),f^2(1),\dots\}.$$

Observe that S is unbounded because for every number n in S there exists a $k>0$ such that $f^{2k}(n)=n+k$ is in S. Clearly f maps S into itself; moreover f is injective on S. Indeed if $f^i(1)=f^j(1)$ with $i\neq j$ then the values $f^m(1)$ start repeating periodically from some point on, and S would be finite.

Define $g:S\to S$ by $g(n)=f^{2k_n}(n)=n+k_n$. We prove that g is injective too. Suppose that $g(a)=g(b)$ with $a<b$. Then $a+k_a=f^{2k_a}(a)=f^{2k_b}(b)=b+k_b$ implies $k_a>k_b$. So, since f is injective on S, we obtain

$$f^{2(k_a-k_b)}(a)=b=a+(k_a-k_b).$$

However this contradicts the minimality of k_a as $0<k_a-k_b<k_a$.

Let T be the set of elements of S that are not of the form $g(n)$ with $n\in S$. Note that $1\in T$ by $g(n)>n$ for $n\in S$, so T is non-empty. For each $t\in T$ denote $C_t=\{t,g(t),g^2(t),\dots\}$; call C_t the chain starting at t. Observe that distinct chains are disjoint because g is injective. Each $n\in S\backslash T$ has the form $n=g(n')$ with $n'<n$, $n'\in S$. Repeated applications of the same observation show that $n\in C_t$ for some $t\in T$, i. e. S is the disjoint union of the chains C_t.

If $f^n(1)$ is in the chain C_t starting at $t=f^{n_t}(1)$ then $n=n_t+2a_1+\dots+2a_j$ with

$$f^n(1)=g^j(f^{n_t}(1))=f^{2a_j}(f^{2a_{j-1}}(\cdots f^{2a_1}(f^{n_t}(1))))=f^{n_t}(1)+a_1+\dots+a_j.$$

Hence

$$f^n(1)=f^{n_t}(1)+\frac{n-n_t}{2}=t+\frac{n-n_t}{2}. \tag{1}$$

Now we show that T is infinite. We argue by contradiction. Suppose that there are only finitely many chains $C_{t_1},\dots,C_{t_r}$, starting at $t_1<\dots<t_r$. Fix N. If $f^n(1)$ with $1\le n\le N$ is in C_t then $f^n(1)=t+\frac{n-n_t}{2}\le t_r+\frac{N}{2}$ by (1). But then the $N+1$ distinct natural numbers $1,f(1),\dots,f^N(1)$ are all less than $t_r+\frac{N}{2}$ and hence $N+1\le t_r+\frac{N}{2}$. This is a contradiction if N is sufficiently large, and hence T is infinite.

To complete the argument, choose any k in $\mathbb{N}$ and consider the $k+1$ chains starting at the first $k+1$ numbers in T. Let t be the greatest one among these numbers. Then each of the chains in question contains a number not exceeding t, and at least one of them does not contain any number among $t+1,\dots,t+k$. So there is a number n in this chain such that $g(n)-n>k$, i. e. $k_n>k$. In conclusion $k_1,k_2,\dots$ is unbounded.

A7. We say that a function $f:\mathbb{R}^k\to\mathbb{R}$ is a metapolynomial if, for some positive integers m and n, it can be represented in the form

$$f(x_1,\ldots,x_k)=\max_{i=1,\ldots,m}\min_{j=1,\ldots,n}P_{i,j}(x_1,\ldots,x_k)$$

where $P_{i,j}$ are multivariate polynomials. Prove that the product of two metapolynomials is also a metapolynomial.

Solution. We use the notation $f(x)=f(x_1,\ldots,x_k)$ for $x=(x_1,\ldots,x_k)$ and $[m]=\{1,2,\ldots,m\}$. Observe that if a metapolynomial $f(x)$ admits a representation like the one in the statement for certain positive integers m and n, then they can be replaced by any $m'\geq m$ and $n'\geq n$. For instance, if we want to replace m by $m+1$ then it is enough to define $P_{m+1,j}(x)=P_{m,j}(x)$ and note that repeating elements of a set do not change its maximum nor its minimum. So one can assume that any two metapolynomials are defined with the same m and n. We reserve letters P and Q for polynomials, so every function called $P,P_{i,j},Q,Q_{i,j},\ldots$ is a polynomial function.

We start with a lemma that is useful to change expressions of the form $\min\max f_{i,j}$ to ones of the form $\max\min g_{i,j}$.

Lemma. Let $\{a_{i,j}\}$ be real numbers, for all $i\in[m]$ and $j\in[n]$. Then

$$\min_{i\in[m]}\max_{j\in[n]}a_{i,j}=\max_{j_1,\ldots,j_m\in[n]}\min_{i\in[m]}a_{i,j_i},$$

where the max in the right-hand side is over all vectors $(j_1,\ldots,j_m)$ with $j_1,\ldots,j_m\in[n]$.

Proof. We can assume for all i that $a_{i,n}=\max\{a_{i,1},\ldots,a_{i,n}\}$ and $a_{m,n}=\min\{a_{1,n},\ldots,a_{m,n}\}$. The left-hand side is $=a_{m,n}$ and hence we need to prove the same for the right-hand side. If $(j_1,j_2,\ldots,j_m)=(n,n,\ldots,n)$ then $\min\{a_{1,j_1},\ldots,a_{m,j_m}\}=\min\{a_{1,n},\ldots,a_{m,n}\}=a_{m,n}$ which implies that the right-hand side is $\geq a_{m,n}$. It remains to prove the opposite inequality and this is equivalent to $\min\{a_{1,j_1},\ldots,a_{m,j_m}\}\leq a_{m,n}$ for all possible $(j_1,j_2,\ldots,j_m)$. This is true because $\min\{a_{1,j_1},\ldots,a_{m,j_m}\}\leq a_{m,j_m}\leq a_{m,n}$. $\square$

We need to show that the family $\mathcal{M}$ of metapolynomials is closed under multiplication, but it turns out easier to prove more: that it is also closed under addition, maxima and minima.

First we prove the assertions about the maxima and the minima. If $f_1,\ldots,f_r$ are metapolynomials, assume them defined with the same m and n. Then

$$f=\max\{f_1,\ldots,f_r\}=\max\{\max_{i\in[m]}\min_{j\in[n]}P^1_{i,j},\ldots,\max_{i\in[m]}\min_{j\in[n]}P^r_{i,j}\}=\max_{s\in[r],i\in[m]}\min_{j\in[n]}P^s_{i,j}.$$

It follows that $f=\max\{f_1,\ldots,f_r\}$ is a metapolynomial. The same argument works for the minima, but first we have to replace *min max* by *max min*, and this is done via the lemma.

Another property we need is that if $f=\max\min P_{i,j}$ is a metapolynomial then so is $-f$. Indeed, $-f=\min(-\min P_{i,j})=\min\max P_{i,j}$.

To prove $\mathcal{M}$ is closed under addition let $f=\max\min P_{i,j}$ and $g=\max\min Q_{i,j}$. Then

$$f(x)+g(x)=\max_{i\in[m]}\min_{j\in[n]}P_{i,j}(x)+\max_{i\in[m]}\min_{j\in[n]}Q_{i,j}(x)$$

$$=\max_{i_1,i_2\in[m]}(\min_{j\in[n]}P_{i_1,j}(x)+\min_{j\in[n]}Q_{i_2,j}(x))=\max_{i_1,i_2\in[m]}\min_{j_1,j_2\in[n]}\big(P_{i_1,j_1}(x)+Q_{i_2,j_2}(x)\big),$$

and hence $f(x)+g(x)$ is a metapolynomial.

We proved that $\mathcal{M}$ is closed under sums, maxima and minima, in particular any function that can be expressed by sums, max, min, polynomials or even metapolynomials is in $\mathcal{M}$.

We would like to proceed with multiplication along the same lines like with addition, but there is an essential difference. In general the product of the maxima of two sets is not equal to the maximum of the product of the sets. We need to deal with the fact that $a < b$ and $c < d$ do not imply $ac < bd$. However this is true for $a, b, c, d \geq 0$.

In view of this we decompose each function $f(x)$ into its positive part $f^+(x) = \max\{f(x), 0\}$ and its negative part $f^-(x) = \max\{0, -f(x)\}$. Note that $f = f^+ - f^-$ and $f^+, f^- \in \mathcal{M}$ if $f \in \mathcal{M}$. The whole problem reduces to the claim that if f and g are metapolynomials with $f, g \geq 0$ then fg it is also a metapolynomial.

Assuming this claim, consider arbitrary $f, g \in \mathcal{M}$. We have

$$fg = (f^+ - f^-)(g^+ - g^-) = f^+g^+ - f^+g^- - f^-g^+ + f^-g^-,$$

and hence $fg \in \mathcal{M}$. Indeed, $\mathcal{M}$ is closed under addition, also $f^+g^+, f^+g^-, f^-g^+, f^-g^- \in \mathcal{M}$ because $f^+, f^-, g^+, g^- \geq 0$.

It remains to prove the claim. In this case $f, g \geq 0$, and one can try to repeat the argument for the sum. More precisely, let $f = \max \min P_{ij} \geq 0$ and $g = \max \min Q_{ij} \geq 0$. Then

$$fg = \max \min P_{i,j} \cdot \max \min Q_{i,j} = \max \min P^+_{i,j} \cdot \max \min Q^+_{i,j} = \max \min P^+_{i_1,j_1} \cdot Q^+_{i_2,j_2}.$$

Hence it suffices to check that $P^+Q^+ \in \mathcal{M}$ for any pair of polynomials P and Q. This reduces to the identity

$$u^+v^+ = \max\{0, \min\{uv, u, v\}, \min\{uv, uv^2, u^2v\}, \min\{uv, u, u^2v\}, \min\{uv, uv^2, v\}\},$$

with u replaced by $P(x)$ and v replaced by $Q(x)$. The formula is proved by a case-by-case analysis. If $u \leq 0$ or $v \leq 0$ then both sides equal 0. In case $u, v \geq 0$, the right-hand side is clearly $\leq uv$. To prove the opposite inequality we use that uv equals

$$\begin{array}{ll} \min\{uv, u, v\} & \text{if } 0 \leq u, v \leq 1, \\ \min\{uv, uv^2, u^2v\} & \text{if } 1 \leq u, v, \\ \min\{uv, u, u^2v\} & \text{if } 0 \leq v \leq 1 \leq u, \\ \min\{uv, uv^2, v\} & \text{if } 0 \leq u \leq 1 \leq v. \end{array}$$

Comment. The case $k = 1$ is simpler and can be solved by proving that a function $f : \mathbb{R} \to \mathbb{R}$ is a metapolynomial if and only if it is a piecewise polinomial (and continuos) function.

It is enough to prove that all such functions are metapolynomials, and this easily reduces to the following case. Given a polynomial $P(x)$ with $P(0) = 0$, the function f defined by $f(x) = P(x)$ for $x \geq 0$ and 0 otherwise is a metapolynomial. For this last claim, it suffices to prove that $(x^+)^n$ is a metapolynomial, and this follows from the formula $(x^+)^n = \max\{0, \min\{x^{n-1}, x^n\}, \min\{x^n, x^{n+1}\}\}$.

Combinatorics

C1. Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y+1, x)$ or $(x-1, x)$. Prove that she can perform only finitely many such iterations.

Solution 1. Note first that the allowed operation does not change the maximum M of the initial sequence. Let $a_1, a_2, \ldots, a_n$ be the numbers obtained at some point of the process. Consider the sum

$$S = a_1 + 2a_2 + \cdots + na_n.$$

We claim that S increases by a positive integer amount with every operation. Let the operation replace the pair (a_i, a_{i+1}) by a pair (c, a_i), where $a_i > a_{i+1}$ and $c = a_{i+1}+1$ or $c = a_i - 1$. Then the new and the old value of S differ by $d = (ic + (i+1)a_i) - (ia_i + (i+1)a_{i+1}) = a_i - a_{i+1} + i(c - a_{i+1})$. The integer d is positive since $a_i - a_{i+1} \geq 1$ and $c - a_{i+1} \geq 0$.

On the other hand $S \leq (1+2+\cdots+n)M$ as $a_i \leq M$ for all $i = 1, \ldots, n$. Since S increases by at least 1 at each step and never exceeds the constant $(1+2+\cdots+n)M$, the process stops after a finite number of iterations.

Solution 2. Like in the first solution note that the operations do not change the maximum M of the initial sequence. Now consider the reverse lexicographical order for n-tuples of integers. We say that $(x_1, \ldots, x_n) < (y_1, \ldots, y_n)$ if $x_n < y_n$, or if $x_n = y_n$ and $x_{n-1} < y_{n-1}$, or if $x_n = y_n$, $x_{n-1} = y_{n-1}$ and $x_{n-2} < y_{n-2}$, etc. Each iteration creates a sequence that is greater than the previous one with respect to this order, and no sequence occurs twice during the process. On the other hand there are finitely many possible sequences because their terms are always positive integers not exceeding M. Hence the process cannot continue forever.

Solution 3. Let the current numbers be $a_1, a_2, \ldots, a_n$. Define the *score* s_i of a_i as the number of a_j's that are less than a_i. Call the sequence $s_1, s_2, \ldots, s_n$ the score sequence of $a_1, a_2, \ldots, a_n$.

Let us say that a sequence $x_1, \ldots, x_n$ dominates a sequence $y_1, \ldots, y_n$ if the first index i with $x_i \neq y_i$ is such that $x_i < y_i$. We show that after each operation the new score sequence dominates the old one. Score sequences do not repeat, and there are finitely many possibilities for them, no more than $(n-1)^n$. Hence the process will terminate.

Consider an operation that replaces (x, y) by (a, x), with $a = y+1$ or $a = x-1$. Suppose that x was originally at position i. For each $j < i$ the score s_j does not increase with the change because $y \leq a$ and $x \leq x$. If s_j decreases for some $j < i$ then the new score sequence dominates the old one. Assume that s_j stays the same for all $j < i$ and consider s_i. Since $x > y$ and $y \leq a \leq x$, we see that s_i decreases by at least 1. This concludes the proof.

Comment. All three proofs work if x and y are not necessarily adjacent, and if the pair (x, y) is replaced by any pair (a, x), with a an integer satisfying $y \leq a \leq x$. There is nothing special about the "weights" $1, 2, \ldots, n$ in the definition of $S = \sum_{i=1}^{n} ia_i$ from the first solution. For any sequence $w_1 < w_2 < \cdots < w_n$ of positive integers, the sum $\sum_{i=1}^{n} w_i a_i$ increases by at least 1 with each operation.

Consider the same problem, but letting Alice replace the pair (x, y) by (a, x), where a is any positive integer less than x. The same conclusion holds in this version, i. e. the process stops eventually. The solution using the reverse lexicographical order works without any change. The first solution would require a special set of weights like $w_i = M^i$ for $i = 1, \ldots, n$.

Comment. The first and the second solutions provide upper bounds for the number of possible operations, respectively of order Mn^2 and M^n where M is the maximum of the original sequence. The upper bound $(n-1)^n$ in the third solution does not depend on M.

C2. Let $n \geq 1$ be an integer. What is the maximum number of disjoint pairs of elements of the set $\{1, 2, \ldots, n\}$ such that the sums of the different pairs are different integers not exceeding n?

Solution. Consider x such pairs in $\{1, 2, \ldots, n\}$. The sum S of the $2x$ numbers in them is at least $1+2+\cdots+2x$ since the pairs are disjoint. On the other hand $S \leq n+(n-1)+\cdots+(n-x+1)$ because the sums of the pairs are different and do not exceed n. This gives the inequality

$$\frac{2x(2x+1)}{2} \leq nx - \frac{x(x-1)}{2},$$

which leads to $x \leq \frac{2n-1}{5}$. Hence there are at most $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs with the given properties.

We show a construction with exactly $\left\lfloor \frac{2n-1}{5} \right\rfloor$ pairs. First consider the case $n = 5k+3$ with $k \geq 0$, where $\left\lfloor \frac{2n-1}{5} \right\rfloor = 2k+1$. The pairs are displayed in the following table.

Pairs	$3k+1$ 2	$3k$ 4	$\cdots$ $\cdots$	$2k+2$ $2k$	$4k+2$ 1	$4k+1$ 3	$\cdots$ $\cdots$	$3k+3$ $2k-1$	$3k+2$ $2k+1$
Sums	$3k+3$	$3k+4$	$\cdots$	$4k+2$	$4k+3$	$4k+4$	$\cdots$	$5k+2$	$5k+3$

The $2k+1$ pairs involve all numbers from 1 to $4k+2$; their sums are all numbers from $3k+3$ to $5k+3$. The same construction works for $n = 5k+4$ and $n = 5k+5$ with $k \geq 0$. In these cases the required number $\left\lfloor \frac{2n-1}{5} \right\rfloor$ of pairs equals $2k+1$ again, and the numbers in the table do not exceed $5k+3$. In the case $n = 5k+2$ with $k \geq 0$ one needs only $2k$ pairs. They can be obtained by ignoring the last column of the table (thus removing $5k+3$). Finally, $2k$ pairs are also needed for the case $n = 5k+1$ with $k \geq 0$. Now it suffices to ignore the last column of the table and then subtract 1 from each number in the first row.

Comment. The construction above is not unique. For instance, the following table shows another set of $2k+1$ pairs for the cases $n = 5k+3$, $n = 5k+4$, and $n = 5k+5$.

Pairs	1 $4k+1$	2 $4k-1$	$\cdots$ $\cdots$	k $2k+3$	$k+1$ $4k+2$	$k+2$ $4k$	$\cdots$ $\cdots$	$2k+1$ $2k+2$
Sums	$4k+2$	$4k+1$	$\cdots$	$3k+3$	$5k+3$	$5k+2$	$\cdots$	$4k+3$

The table for the case $n = 5k+2$ would be the same, with the pair $(k+1, 4k+2)$ removed. For the case $n = 5k+1$ remove the last column and subtract 2 from each number in the second row.

C3. In a 999×999 square table some cells are white and the remaining ones are red. Let T be the number of triples (C_1, C_2, C_3) of cells, the first two in the same row and the last two in the same column, with C_1 and C_3 white and C_2 red. Find the maximum value T can attain.

Solution. We prove that in an $n \times n$ square table there are at most $\frac{4n^4}{27}$ such triples.

Let row i and column j contain a_i and b_j white cells respectively, and let R be the set of red cells. For every red cell (i, j) there are $a_i b_j$ admissible triples (C_1, C_2, C_3) with $C_2 = (i, j)$, therefore

$$T = \sum_{(i,j)\in R} a_i b_j.$$

We use the inequality $2ab \le a^2 + b^2$ to obtain

$$T \le \frac{1}{2} \sum_{(i,j)\in R} (a_i^2 + b_j^2) = \frac{1}{2}\sum_{i=1}^{n}(n - a_i)a_i^2 + \frac{1}{2}\sum_{j=1}^{n}(n - b_j)b_j^2.$$

This is because there are $n - a_i$ red cells in row i and $n - b_j$ red cells in column j. Now we maximize the right-hand side.

By the AM-GM inequality we have

$$(n - x)x^2 = \frac{1}{2}(2n - 2x) \cdot x \cdot x \le \frac{1}{2}\left(\frac{2n}{3}\right)^3 = \frac{4n^3}{27},$$

with equality if and only if $x = \frac{2n}{3}$. By putting everything together, we get

$$T \le \frac{n}{2}\frac{4n^3}{27} + \frac{n}{2}\frac{4n^3}{27} = \frac{4n^4}{27}.$$

If $n = 999$ then any coloring of the square table with $x = \frac{2n}{3} = 666$ white cells in each row and column attains the maximum as all inequalities in the previous argument become equalities. For example color a cell (i, j) white if $i - j \equiv 1, 2, \ldots, 666 \pmod{999}$, and red otherwise.

Therefore the maximum value T can attain is $T = \frac{4 \cdot 999^4}{27}$.

Comment. One can obtain a better preliminary estimate with the Cauchy-Schwarz inequality:

$$T = \sum_{(i,j)\in R} a_i b_j \le \left(\sum_{(i,j)\in R} a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{(i,j)\in R} b_j^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n}(n - a_i)a_i^2\right)^{\frac{1}{2}} \cdot \left(\sum_{j=1}^{n}(n - b_j)b_j^2\right)^{\frac{1}{2}}.$$

It can be used to reach the same conclusion.

C4. Players A and B play a game with $N \geq 2012$ coins and 2012 boxes arranged around a circle. Initially A distributes the coins among the boxes so that there is at least 1 coin in each box. Then the two of them make moves in the order $B, A, B, A, \ldots$ by the following rules:

- On every move of his B passes 1 coin from every box to an adjacent box.
- On every move of hers A chooses several coins that were *not* involved in B's previous move and are in different boxes. She passes every chosen coin to an adjacent box.

Player A's goal is to ensure at least 1 coin in each box after every move of hers, regardless of how B plays and how many moves are made. Find the least N that enables her to succeed.

Solution. We argue for a general $n \geq 7$ instead of 2012 and prove that the required minimum N is $2n-2$. For $n = 2012$ this gives $N_{\min} = 4022$.

a) If $N = 2n-2$ player A can achieve her goal. Let her start the game with a *regular* distribution: $n-2$ boxes with 2 coins and 2 boxes with 1 coin. Call the boxes of the two kinds *red* and *white* respectively. We claim that on her first move A can achieve a regular distribution again, regardless of B's first move M. She acts according as the following situation S occurs after M or not: *The initial distribution contains a red box R with 2 white neighbors, and R receives no coins from them on move M.*

Suppose that S does not occur. Exactly one of the coins c_1 and c_2 in a given red box X is involved in M, say c_1. If M passes c_1 to the right neighbor of X, let A pass c_2 to its left neighbor, and vice versa. By doing so with all red boxes A performs a legal move M'. Thus M and M' combined move the 2 coins of every red box in opposite directions. Hence after M and M' are complete each neighbor of a red box X contains exactly 1 coin that was initially in X. So each box with a red neighbor is non-empty after M'. If initially there is a box X with 2 white neighbors (X is red and unique) then X receives a coin from at least one of them on move M since S does not occur. Such a coin is not involved in M', so X is also non-empty after M'. Furthermore each box Y has given away its initial content after M and M'. A red neighbor of Y adds 1 coin to it; a white neighbor adds at most 1 coin because it is not involved in M'. Hence each box contains 1 or 2 coins after M'. Because $N = 2n-2$, such a distribution is regular.

Now let S occur after move M. Then A leaves untouched the exceptional red box R. With all remaining red boxes she proceeds like in the previous case, thus making a legal move M''. Box R receives no coins from its neighbors on either move, so there is 1 coin in it after M''. Like above M and M'' combined pass exactly 1 coin from every red box different from R to each of its neighbors. Every box except R has a red neighbor different from R, hence all boxes are non-empty after M''. Next, each box Y except R loses its initial content after M and M''. A red neighbor of Y adds at most 1 coin to it; a white neighbor also adds at most 1 coin as it does not participate in M''. Thus each box has 1 or 2 coins after M'', and the obtained distribution is regular.

Player A can apply the described strategy indefinitely, so $N = 2n-2$ enables her to succeed.

b) For $N \leq 2n-3$ player B can achieve an empty box after some move of A. Let α be a set of ℓ consecutive boxes containing a total of $N(\alpha)$ coins. We call α an *arc* if $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Note that $\ell \geq 2$ by the last condition. Moreover if both extremes of α are non-empty boxes then $N(\alpha) \geq 2$, so that $N(\alpha) \leq 2\ell - 3$ implies $\ell \geq 3$. Observe also that if an extreme X of α has more than 1 coin then ignoring X yields a shorter arc. It follows that every arc contains an arc whose extremes have at most 1 coin each.

Given a clockwise labeling $1, 2, \ldots, n$ of the boxes, suppose that boxes $1, 2, \ldots, \ell$ form an arc α, with $\ell \leq n-2$ and $N(\alpha) \leq 2\ell - 3$. Suppose also that all $n \geq 7$ boxes are non-empty. Then B can move so that an arc α' with $N(\alpha') < N(\alpha)$ will appear after any response of A.

One may assume exactly 1 coin in boxes 1 and ℓ by a previous remark. Let B pass 1 coin in counterclockwise direction from box 1 and box n, and in clockwise direction from each remaining box. This leaves $N(\alpha)-2$ coins in the boxes of α. In addition, due to $3 \leq \ell \leq n-2$, box ℓ has exactly 1 coin c, the one received from box $\ell-1$.

Let player A's next move M pass $k \leq 2$ coins to boxes $1, 2, \ldots, \ell$ from the remaining ones. Only boxes 1 and ℓ can receive such coins, at most 1 each. If $k < 2$ then after move M boxes $1, 2, \ldots, \ell$ form an arc α' with $N(\alpha') < N(\alpha)$. If $k = 2$ then M adds a coin to box ℓ. Also M does not move coin c from ℓ because c is involved in the previous move of B. In summary boxes $1, 2, \ldots, \ell$ contain $N(\alpha)$ coins like before, so they form an arc. However there are 2 coins now in the extreme ℓ of the arc. Ignore ℓ to obtain a shorter arc α' with $N(\alpha') < N(\alpha)$.

Consider any initial distribution without empty boxes. Since $N \leq 2n-3$, there are at least 3 boxes in it with exactly 1 coin. It follows from $n \geq 7$ that some 2 of them are the extremes of an arc α. Hence B can make the move described above, which leads to an arc α' with $N(\alpha') < N(\alpha)$ after A's response. If all boxes in the new distribution are non-empty he can repeat the same, and so on. Because $N(\alpha)$ cannot decrease indefinitely, an empty box will occur after some move of A.

C5. The columns and the rows of a $3n \times 3n$ square board are numbered $1, 2, \ldots, 3n$. Every square (x, y) with $1 \leq x, y \leq 3n$ is colored asparagus, byzantium or citrine according as the modulo 3 remainder of $x + y$ is 0, 1 or 2 respectively. One token colored asparagus, byzantium or citrine is placed on each square, so that there are $3n^2$ tokens of each color.

Suppose that one can permute the tokens so that each token is moved to a distance of at most d from its original position, each asparagus token replaces a byzantium token, each byzantium token replaces a citrine token, and each citrine token replaces an asparagus token. Prove that it is possible to permute the tokens so that each token is moved to a distance of at most $d + 2$ from its original position, and each square contains a token with the same color as the square.

Solution. Without loss of generality it suffices to prove that the A-tokens can be moved to distinct A-squares in such a way that each A-token is moved to a distance at most $d + 2$ from its original place. This means we need a perfect matching between the $3n^2$ A-squares and the $3n^2$ A-tokens such that the distance in each pair of the matching is at most $d + 2$.

To find the matching, we construct a bipartite graph. The A-squares will be the vertices in one class of the graph; the vertices in the other class will be the A-tokens.

Split the board into 3×1 horizontal triminos; then each trimino contains exactly one A-square. Take a permutation π of the tokens which moves A-tokens to B-tokens, B-tokens to C-tokens, and C-tokens to A-tokens, in each case to a distance at most d. For each A-square S, and for each A-token T, connect S and T by an edge if T, $\pi(T)$ or $\pi^{-1}(T)$ is on the trimino containing S. We allow multiple edges; it is even possible that the same square and the same token are connected with three edges. Obviously the lengths of the edges in the graph do not exceed $d + 2$. By length of an edge we mean the distance between the A-square and the A-token it connects.

Each A-token T is connected with the three A-squares whose triminos contain T, $\pi(T)$ and $\pi^{-1}(T)$. Therefore in the graph all tokens are of degree 3. We show that the same is true for the A-squares. Let S be an arbitrary A-square, and let T_1, T_2, T_3 be the three tokens on the trimino containing S. For $i = 1, 2, 3$, if T_i is an A-token, then S is connected with T_i; if T_i is a B-token then S is connected with $\pi^{-1}(T_i)$; finally, if T_i is a C-token then S is connected with $\pi(T_i)$. Hence in the graph the A-squares also are of degree 3.

Since the A-squares are of degree 3, from every set $\mathcal{S}$ of A-squares exactly $3|\mathcal{S}|$ edges start. These edges end in at least $|\mathcal{S}|$ tokens because the A-tokens also are of degree 3. Hence every set $\mathcal{S}$ of A-squares has at least $|\mathcal{S}|$ neighbors among the A-tokens.

Therefore, by HALL's marriage theorem, the graph contains a perfect matching between the two vertex classes. So there is a perfect matching between the A-squares and A-tokens with edges no longer than $d + 2$. It follows that the tokens can be permuted as specified in the problem statement.

Comment 1. In the original problem proposal the board was infinite and there were only two colors. Having n colors for some positive integer n was an option; we chose $n = 3$. Moreover, we changed the board to a finite one to avoid dealing with infinite graphs (although Hall's theorem works in the infinite case as well).

With only two colors Hall's theorem is not needed. In this case we split the board into 2×1 dominos, and in the resulting graph all vertices are of degree 2. The graph consists of disjoint cycles with even length and infinite paths, so the existence of the matching is trivial.

Having more than three colors would make the problem statement more complicated, because we need a matching between every two color classes of tokens. However, this would not mean a significant increase in difficulty.

Comment 2. According to Wikipedia, the color *asparagus* (hexadecimal code `#87A96B`) is a tone of green that is named after the vegetable. Crayola created this color in 1993 as one of the 16 to be named in the Name The Color Contest. *Byzantium* (`#702963`) is a dark tone of purple. Its first recorded use as a color name in English was in 1926. *Citrine* (`#E4D00A`) is variously described as yellow, greenish-yellow, brownish-yellow or orange. The first known use of citrine as a color name in English was in the 14th century.

C6. Let k and n be fixed positive integers. In the liar's guessing game, Amy chooses integers x and N with $1 \le x \le N$. She tells Ben what N is, but not what x is. Ben may then repeatedly ask Amy whether $x \in S$ for arbitrary sets S of integers. Amy will always answer with *yes* or *no*, but she might lie. The only restriction is that she can lie at most k times in a row. After he has asked as many questions as he wants, Ben must specify a set of at most n positive integers. If x is in this set he wins; otherwise, he loses. Prove that:

a) If $n \ge 2^k$ then Ben can always win.

b) For sufficiently large k there exist $n \ge 1.99^k$ such that Ben cannot guarantee a win.

Solution. Consider an answer $A \in \{yes, no\}$ to a question of the kind "Is x in the set S?" We say that A is inconsistent with a number i if $A = yes$ and $i \notin S$, or if $A = no$ and $i \in S$. Observe that an answer inconsistent with the target number x is a lie.

a) Suppose that Ben has determined a set T of size m that contains x. This is true initially with $m = N$ and $T = \{1, 2, \dots, N\}$. For $m > 2^k$ we show how Ben can find a number $y \in T$ that is different from x. By performing this step repeatedly he can reduce T to be of size $2^k \le n$ and thus win.

Since only the size $m > 2^k$ of T is relevant, assume that $T = \{0, 1, \dots, 2^k, \dots, m-1\}$. Ben begins by asking repeatedly whether x is 2^k. If Amy answers *no* $k+1$ times in a row, one of these answers is truthful, and so $x \ne 2^k$. Otherwise Ben stops asking about 2^k at the first answer *yes*. He then asks, for each $i = 1, \dots, k$, if the binary representation of x has a 0 in the ith digit. Regardless of what the k answers are, they are all inconsistent with a certain number $y \in \{0, 1, \dots, 2^k - 1\}$. The preceding answer *yes* about 2^k is also inconsistent with y. Hence $y \ne x$. Otherwise the last $k+1$ answers are not truthful, which is impossible.

Either way, Ben finds a number in T that is different from x, and the claim is proven.

b) We prove that if $1 < \lambda < 2$ and $n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1$ then Ben cannot guarantee a win. To complete the proof, then it suffices to take λ such that $1.99 < \lambda < 2$ and k large enough so that

$$n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1 \ge 1.99^k.$$

Consider the following strategy for Amy. First she chooses $N = n+1$ and $x \in \{1, 2, \dots, n+1\}$ arbitrarily. After every answer of hers Amy determines, for each $i = 1, 2, \dots, n+1$, the number m_i of consecutive answers she has given by that point that are inconsistent with i. To decide on her next answer, she then uses the quantity

$$\phi = \sum_{i=1}^{n+1} \lambda^{m_i}.$$

No matter what Ben's next question is, Amy chooses the answer which minimizes ϕ.

We claim that with this strategy ϕ will always stay less than λ^{k+1}. Consequently no exponent m_i in ϕ will ever exceed k, hence Amy will never give more than k consecutive answers inconsistent with some i. In particular this applies to the target number x, so she will never lie more than k times in a row. Thus, given the claim, Amy's strategy is legal. Since the strategy does not depend on x in any way, Ben can make no deductions about x, and therefore he cannot guarantee a win.

It remains to show that $\phi < \lambda^{k+1}$ at all times. Initially each m_i is 0, so this condition holds in the beginning due to $1 < \lambda < 2$ and $n = \lfloor (2-\lambda)\lambda^{k+1} \rfloor - 1$. Suppose that $\phi < \lambda^{k+1}$ at some point, and Ben has just asked if $x \in S$ for some set S. According as Amy answers *yes* or *no*, the new value of ϕ becomes

$$\phi_1 = \sum_{i \in S} 1 + \sum_{i \notin S} \lambda^{m_i+1} \qquad \text{or} \qquad \phi_2 = \sum_{i \in S} \lambda^{m_i+1} + \sum_{i \notin S} 1.$$

Since Amy chooses the option minimizing ϕ, the new ϕ will equal $\min(\phi_1, \phi_2)$. Now we have

$$\min(\phi_1, \phi_2) \le \frac{1}{2}(\phi_1 + \phi_2) = \frac{1}{2}\left(\sum_{i \in S}\bigl(1 + \lambda^{m_i+1}\bigr) + \sum_{i \notin S}\bigl(\lambda^{m_i+1} + 1\bigr)\right) = \frac{1}{2}(\lambda\phi + n + 1).$$

Because $\phi < \lambda^{k+1}$, the assumptions $\lambda < 2$ and $n = \left\lfloor (2-\lambda)\lambda^{k+1} \right\rfloor - 1$ lead to

$$\min(\phi_1, \phi_2) < \frac{1}{2}(\lambda^{k+2} + (2-\lambda)\lambda^{k+1}) = \lambda^{k+1}.$$

The claim follows, which completes the solution.

Comment. Given a fixed k, let $f(k)$ denote the minimum value of n for which Ben can guarantee a victory. The problem asks for a proof that for large k

$$1.99^k \le f(k) \le 2^k.$$

A computer search shows that $f(k) = 2, 3, 4, 7, 11, 17$ for $k = 1, 2, 3, 4, 5, 6$.

C7. There are given 2^{500} points on a circle labeled $1, 2, \ldots, 2^{500}$ in some order. Prove that one can choose 100 pairwise disjoint chords joining some of these points so that the 100 sums of the pairs of numbers at the endpoints of the chosen chords are equal.

Solution. The proof is based on the following general fact.

Lemma. In a graph G each vertex v has degree d_v. Then G contains an independent set S of vertices such that $|S| \geq f(G)$ where

$$f(G) = \sum_{v \in G} \frac{1}{d_v + 1}.$$

Proof. Induction on $n = |G|$. The base $n = 1$ is clear. For the inductive step choose a vertex v_0 in G of minimum degree d. Delete v_0 and all of its neighbors $v_1, \ldots, v_d$ and also all edges with endpoints $v_0, v_1, \ldots, v_d$. This gives a new graph G'. By the inductive assumption G' contains an independent set S' of vertices such that $|S'| \geq f(G')$. Since no vertex in S' is a neighbor of v_0 in G, the set $S = S' \cup \{v_0\}$ is independent in G.

Let d'_v be the degree of a vertex v in G'. Clearly $d'_v \leq d_v$ for every such vertex v, and also $d_{v_i} \geq d$ for all $i = 0, 1, \ldots, d$ by the minimal choice of v_0. Therefore

$$f(G') = \sum_{v \in G'} \frac{1}{d'_v + 1} \geq \sum_{v \in G'} \frac{1}{d_v + 1} = f(G) - \sum_{i=0}^{d} \frac{1}{d_{v_i} + 1} \geq f(G) - \frac{d+1}{d+1} = f(G) - 1.$$

Hence $|S| = |S'| + 1 \geq f(G') + 1 \geq f(G)$, and the induction is complete. □

We pass on to our problem. For clarity denote $n = 2^{499}$ and draw all chords determined by the given $2n$ points. Color each chord with one of the colors $3, 4, \ldots, 4n - 1$ according to the sum of the numbers at its endpoints. Chords with a common endpoint have different colors. For each color c consider the following graph G_c. Its vertices are the chords of color c, and two chords are neighbors in G_c if they intersect. Let $f(G_c)$ have the same meaning as in the lemma for all graphs G_c.

Every chord ℓ divides the circle into two arcs, and one of them contains $m(\ell) \leq n - 1$ given points. (In particular $m(\ell) = 0$ if ℓ joins two consecutive points.) For each $i = 0, 1, \ldots, n - 2$ there are $2n$ chords ℓ with $m(\ell) = i$. Such a chord has degree at most i in the respective graph. Indeed let $A_1, \ldots, A_i$ be all points on either arc determined by a chord ℓ with $m(\ell) = i$ and color c. Every A_j is an endpoint of at most 1 chord colored c, $j = 1, \ldots, i$. Hence at most i chords of color c intersect ℓ.

It follows that for each $i = 0, 1, \ldots, n - 2$ the $2n$ chords ℓ with $m(\ell) = i$ contribute at least $\frac{2n}{i+1}$ to the sum $\sum_c f(G_c)$. Summation over $i = 0, 1, \ldots, n - 2$ gives

$$\sum_c f(G_c) \geq 2n \sum_{i=1}^{n-1} \frac{1}{i}.$$

Because there are $4n - 3$ colors in all, averaging yields a color c such that

$$f(G_c) \geq \frac{2n}{4n-3} \sum_{i=1}^{n-1} \frac{1}{i} > \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}.$$

By the lemma there are at least $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i}$ pairwise disjoint chords of color c, i. e. with the same sum c of the pairs of numbers at their endpoints. It remains to show that $\frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i} \geq 100$ for $n = 2^{499}$. Indeed we have

$$\sum_{i=1}^{n-1} \frac{1}{i} > \sum_{i=1}^{2^{400}} \frac{1}{i} = 1 + \sum_{k=1}^{400} \sum_{i=2^{k-1}+1}^{2^k} \frac{1}{i} > 1 + \sum_{k=1}^{400} \frac{2^{k-1}}{2^k} = 201 > 200.$$

This completes the solution.

Geometry

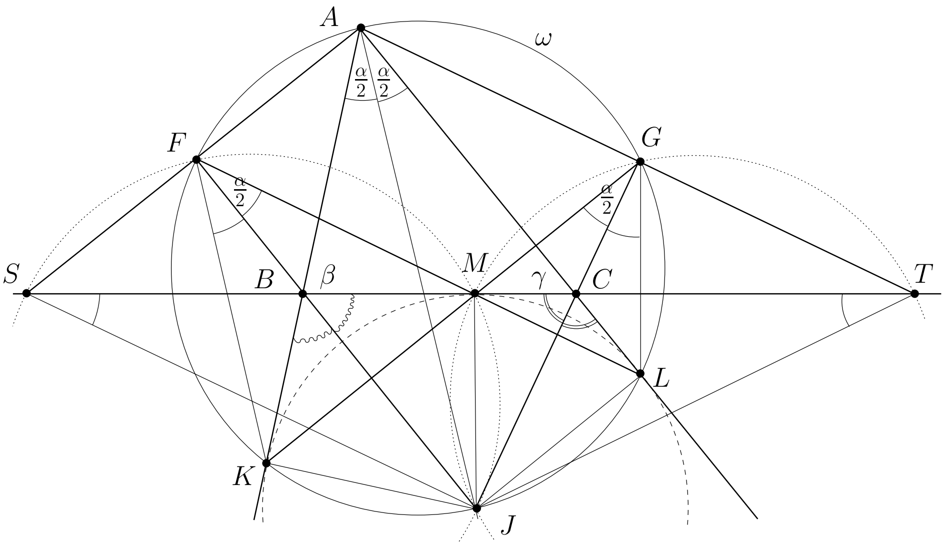
G1. In the triangle ABC the point J is the center of the excircle opposite to A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Solution. Let $\alpha = \angle CAB$, $\beta = \angle ABC$ and $\gamma = \angle BCA$. The line AJ is the bisector of $\angle CAB$, so $\angle JAK = \angle JAL = \frac{\alpha}{2}$. By $\angle AKJ = \angle ALJ = 90^\circ$ the points K and L lie on the circle ω with diameter AJ.

The triangle KBM is isosceles as BK and BM are tangents to the excircle. Since BJ is the bisector of $\angle KBM$, we have $\angle MBJ = 90^\circ - \frac{\beta}{2}$ and $\angle BMK = \frac{\beta}{2}$. Likewise $\angle MCJ = 90^\circ - \frac{\gamma}{2}$ and $\angle CML = \frac{\gamma}{2}$. Also $\angle BMF = \angle CML$, therefore

$$\angle LFJ = \angle MBJ - \angle BMF = \left(90^\circ - \frac{\beta}{2}\right) - \frac{\gamma}{2} = \frac{\alpha}{2} = \angle LAJ.$$

Hence F lies on the circle ω. (By the angle computation, F and A are on the same side of BC.) Analogously, G also lies on ω. Since AJ is a diameter of ω, we obtain $\angle AFJ = \angle AGJ = 90^\circ$.



The lines AB and BC are symmetric with respect to the external bisector BF. Because $AF \perp BF$ and $KM \perp BF$, the segments SM and AK are symmetric with respect to BF, hence $SM = AK$. By symmetry $TM = AL$. Since AK and AL are equal as tangents to the excircle, it follows that $SM = TM$, and the proof is complete.

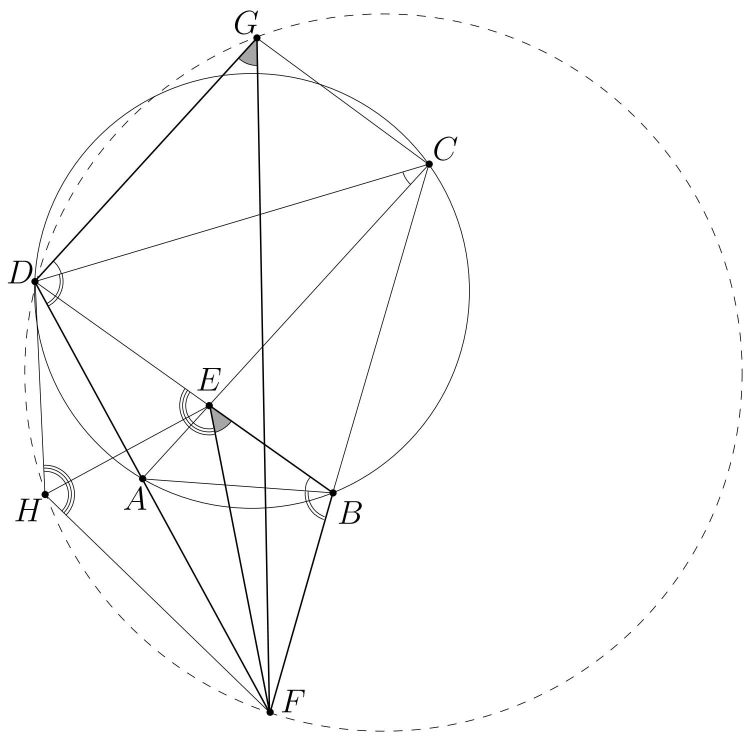
Comment. After discovering the circle $AFKJLG$, there are many other ways to complete the solution. For instance, from the cyclic quadrilaterals $JMFS$ and $JMGT$ one can find $\angle TSJ = \angle STJ = \frac{\alpha}{2}$. Another possibility is to use the fact that the lines AS and GM are parallel (both are perpendicular to the external angle bisector BJ), so $\frac{MS}{MT} = \frac{AG}{GT} = 1$.

G2. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

Solution. We show first that the triangles FDG and FBE are similar. Since $ABCD$ is cyclic, the triangles EAB and EDC are similar, as well as FAB and FCD. The parallelogram $ECGD$ yields $GD = EC$ and $\angle CDG = \angle DCE$; also $\angle DCE = \angle DCA = \angle DBA$ by inscribed angles. Therefore

$$\angle FDG = \angle FDC + \angle CDG = \angle FBA + \angle ABD = \angle FBE,$$
$$\frac{GD}{EB} = \frac{CE}{EB} = \frac{CD}{AB} = \frac{FD}{FB}.$$

It follows that FDG and FBE are similar, and so $\angle FGD = \angle FEB$.



Since H is the reflection of E with respect to FD, we conclude that

$$\angle FHD = \angle FED = 180^\circ - \angle FEB = 180^\circ - \angle FGD.$$

This proves that D, H, F, G are concyclic.

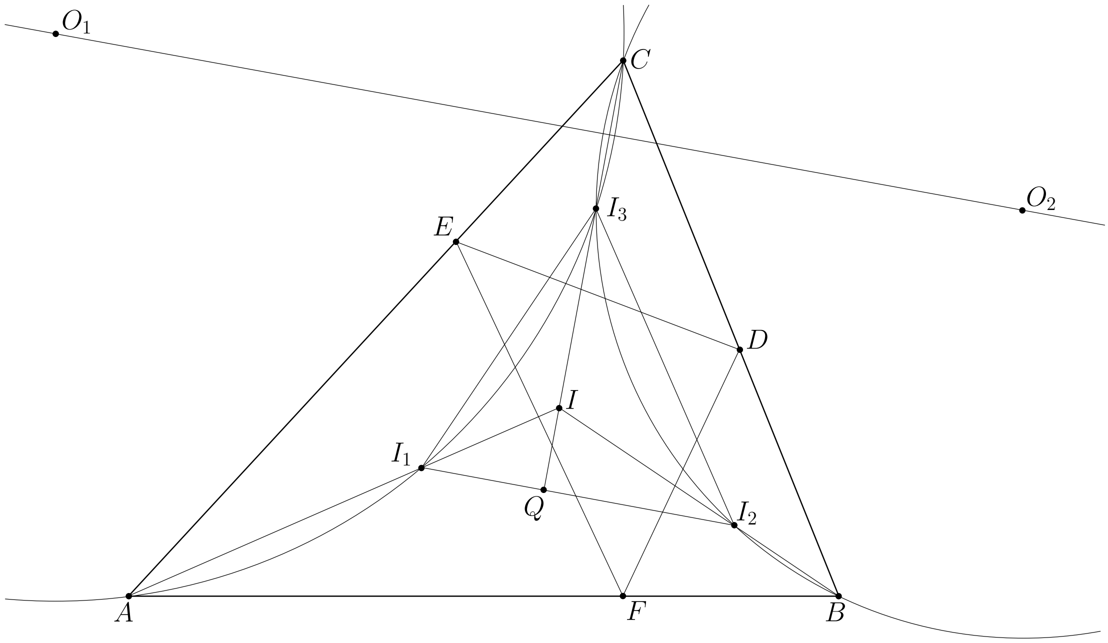
Comment. Points E and G are always in the half-plane determined by the line FD that contains B and C, but H is always in the other half-plane. In particular, $DHFG$ is cyclic if and only if $\angle FHD + \angle FGD = 180^\circ$.

G3. In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

Solution. Let $\angle CAB = \alpha$, $\angle ABC = \beta$, $\angle BCA = \gamma$. We start by showing that A, B, I_1 and I_2 are concyclic. Since AI_1 and BI_2 bisect $\angle CAB$ and $\angle ABC$, their extensions beyond I_1 and I_2 meet at the incenter I of the triangle. The points E and F are on the circle with diameter BC, so $\angle AEF = \angle ABC$ and $\angle AFE = \angle ACB$. Hence the triangles AEF and ABC are similar with ratio of similitude $\frac{AE}{AB} = \cos\alpha$. Because I_1 and I are their incenters, we obtain $I_1A = IA\cos\alpha$ and $II_1 = IA - I_1A = 2IA\sin^2\frac{\alpha}{2}$. By symmetry $II_2 = 2IB\sin^2\frac{\beta}{2}$. The law of sines in the triangle ABI gives $IA\sin\frac{\alpha}{2} = IB\sin\frac{\beta}{2}$. Hence

$$II_1 \cdot IA = 2\left(IA\sin\tfrac{\alpha}{2}\right)^2 = 2\left(IB\sin\tfrac{\beta}{2}\right)^2 = II_2 \cdot IB.$$

Therefore A, B, I_1 and I_2 are concyclic, as claimed.



In addition $II_1 \cdot IA = II_2 \cdot IB$ implies that I has the same power with respect to the circles (ACI_1), (BCI_2) and (ABI_1I_2). Then CI is the radical axis of (ACI_1) and (BCI_2); in particular CI is perpendicular to the line of centers O_1O_2.

Now it suffices to prove that $CI \perp I_1I_2$. Let CI meet I_1I_2 at Q, then it is enough to check that $\angle II_1Q + \angle I_1IQ = 90°$. Since $\angle I_1IQ$ is external for the triangle ACI, we have

$$\angle II_1Q + \angle I_1IQ = \angle II_1Q + (\angle ACI + \angle CAI) = \angle II_1I_2 + \angle ACI + \angle CAI.$$

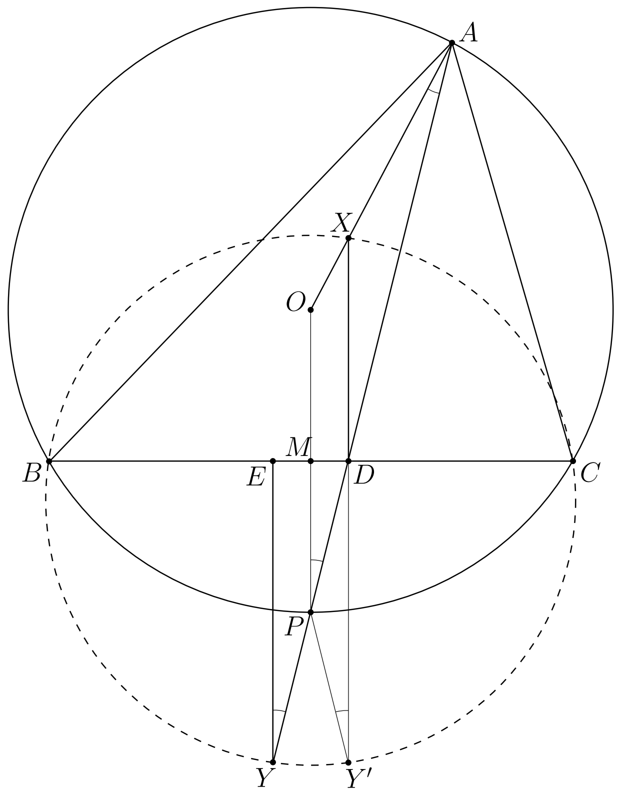
It remains to note that $\angle II_1I_2 = \frac{\beta}{2}$ from the cyclic quadrilateral ABI_1I_2, and $\angle ACI = \frac{\gamma}{2}$, $\angle CAI = \frac{\alpha}{2}$. Therefore $\angle II_1Q + \angle I_1IQ = \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = 90°$, completing the proof.

Comment. It follows from the first part of the solution that the common point $I_3 \neq C$ of the circles (ACI_1) and (BCI_2) is the incenter of the triangle CDE.

G4. Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

Solution. The bisector of $\angle BAC$ and the perpendicular bisector of BC meet at P, the midpoint of the minor arc $\widehat{BC}$ (they are different lines as $AB \neq AC$). In particular OP is perpendicular to BC and intersects it at M, the midpoint of BC.

Denote by Y' the reflexion of Y with respect to OP. Since $\angle BYC = \angle BY'C$, it suffices to prove that $BXCY'$ is cyclic.



We have

$$\angle XAP = \angle OPA = \angle EYP.$$

The first equality holds because $OA = OP$, and the second one because EY and OP are both perpendicular to BC and hence parallel. But $\{Y, Y'\}$ and $\{E, D\}$ are pairs of symmetric points with respect to OP, it follows that $\angle EYP = \angle DY'P$ and hence

$$\angle XAP = \angle DY'P = \angle XY'P.$$

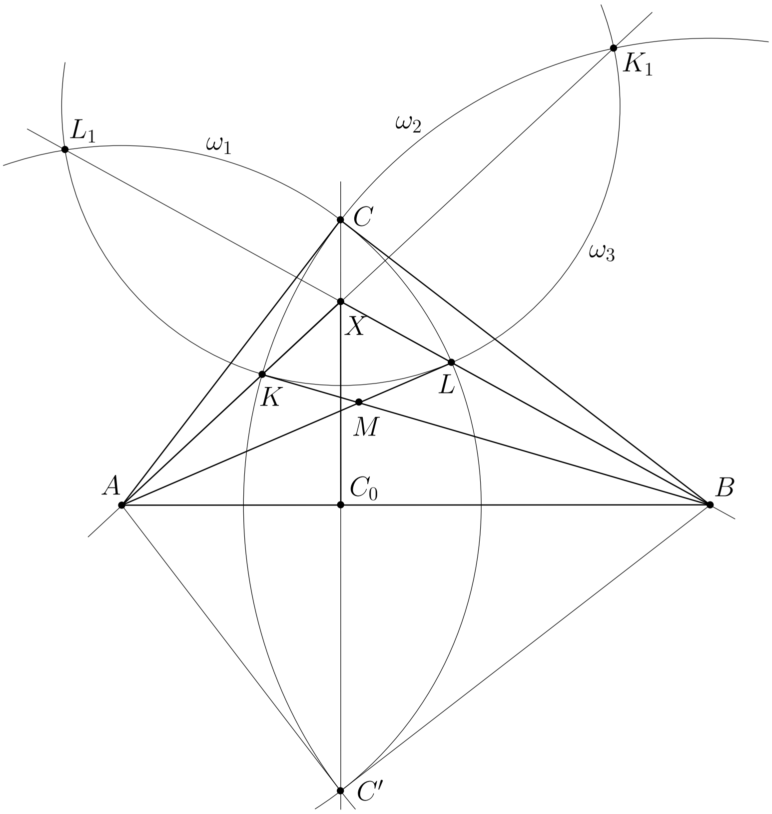
The last equation implies that $XAY'P$ is cyclic. By the powers of D with respect to the circles $(XAY'P)$ and $(ABPC)$ we obtain

$$XD \cdot DY' = AD \cdot DP = BD \cdot DC.$$

It follows that $BXCY'$ is cyclic, as desired.

G5. Let ABC be a triangle with $\angle BCA = 90°$, and let C_0 be the foot of the altitude from C. Choose a point X in the interior of the segment CC_0, and let K, L be the points on the segments AX, BX for which $BK = BC$ and $AL = AC$ respectively. Denote by M the intersection of AL and BK. Show that $MK = ML$.

Solution. Let C' be the reflection of C in the line AB, and let ω_1 and ω_2 be the circles with centers A and B, passing through L and K respectively. Since $AC' = AC = AL$ and $BC' = BC = BK$, both ω_1 and ω_2 pass through C and C'. By $\angle BCA = 90°$, AC is tangent to ω_2 at C, and BC is tangent to ω_1 at C. Let $K_1 \neq K$ be the second intersection of AX and ω_2, and let $L_1 \neq L$ be the second intersection of BX and ω_1.



By the powers of X with respect to ω_2 and ω_1,

$$XK \cdot XK_1 = XC \cdot XC' = XL \cdot XL_1,$$

so the points K_1, L, K, L_1 lie on a circle ω_3.

The power of A with respect to ω_2 gives

$$AL^2 = AC^2 = AK \cdot AK_1,$$

indicating that AL is tangent to ω_3 at L. Analogously, BK is tangent to ω_3 at K. Hence MK and ML are the two tangents from M to ω_3 and therefore $MK = ML$.

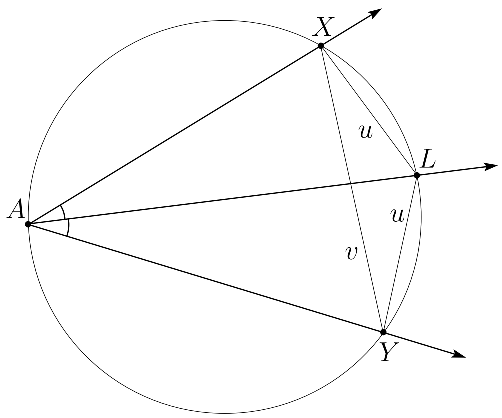
G6. Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

Solution. By MIQUEL's theorem the circles $(AEF) = \omega_A$, $(BFD) = \omega_B$ and $(CDE) = \omega_C$ have a common point, for arbitrary points D, E and F on BC, CA and AB. So ω_A passes through the common point $P \neq D$ of ω_B and ω_C.

Let ω_A, ω_B and ω_C meet the bisectors AI, BI and CI at $A \neq A'$, $B \neq B'$ and $C \neq C'$ respectively. The key observation is that A', B' and C' do not depend on the particular choice of D, E and F, provided that $BD + BF = CA$, $CD + CE = AB$ and $AE + AF = BC$ hold true (the last equality follows from the other two). For a proof we need the following fact.

Lemma. Given is an angle with vertex A and measure α. A circle ω through A intersects the angle bisector at L and sides of the angle at X and Y. Then $AX + AY = 2AL \cos \frac{\alpha}{2}$.

Proof. Note that L is the midpoint of arc $\widehat{XLY}$ in ω and set $XL = YL = u$, $XY = v$. By PTOLEMY's theorem $AX \cdot YL + AY \cdot XL = AL \cdot XY$, which rewrites as $(AX + AY)u = AL \cdot v$. Since $\angle LXY = \frac{\alpha}{2}$ and $\angle XLY = 180° - \alpha$, we have $v = 2\cos\frac{\alpha}{2} u$ by the law of sines, and the claim follows. $\square$



Apply the lemma to $\angle BAC = \alpha$ and the circle $\omega = \omega_A$, which intersects AI at A'. This gives $2AA' \cos\frac{\alpha}{2} = AE + AF = BC$; by symmetry analogous relations hold for BB' and CC'. It follows that A', B' and C' are independent of the choice of D, E and F, as stated.

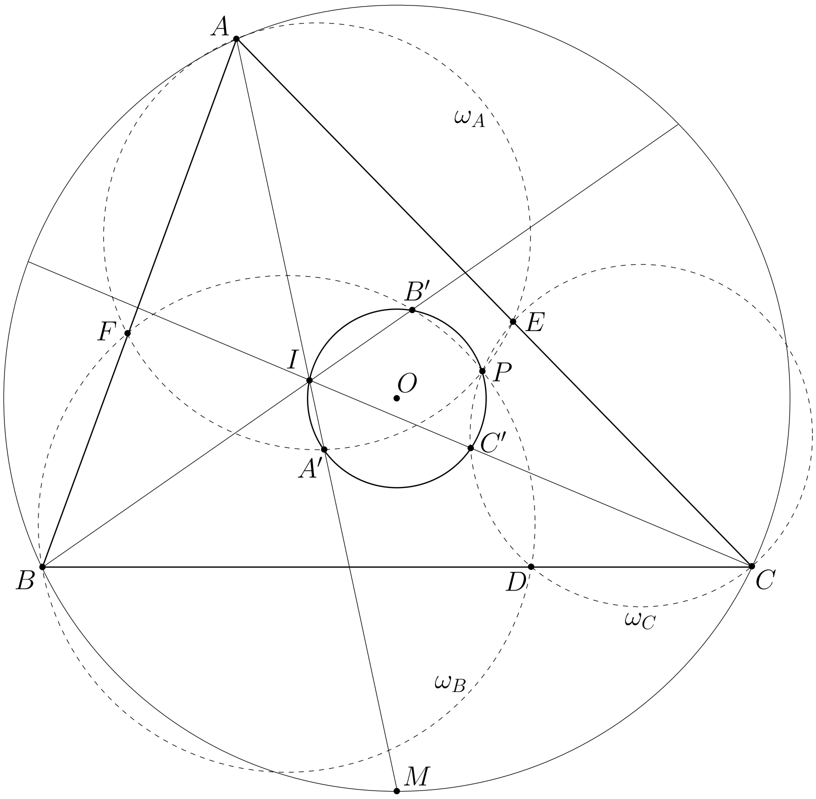
We use the lemma two more times with $\angle BAC = \alpha$. Let ω be the circle with diameter AI. Then X and Y are the tangency points of the incircle of ABC with AB and AC, and hence $AX = AY = \frac{1}{2}(AB + AC - BC)$. So the lemma yields $2AI \cos\frac{\alpha}{2} = AB + AC - BC$. Next, if ω is the circumcircle of ABC and AI intersects ω at $M \neq A$ then $\{X, Y\} = \{B, C\}$, and so $2AM \cos\frac{\alpha}{2} = AB + AC$ by the lemma. To summarize,

$$2AA' \cos\tfrac{\alpha}{2} = BC, \qquad 2AI \cos\tfrac{\alpha}{2} = AB + AC - BC, \qquad 2AM \cos\tfrac{\alpha}{2} = AB + AC. \qquad (*)$$

These equalities imply $AA' + AI = AM$, hence the segments AM and IA' have a common midpoint. It follows that I and A' are equidistant from the circumcenter O. By symmetry $OI = OA' = OB' = OC'$, so I, A', B', C' are on a circle centered at O.

To prove $OP = OI$, now it suffices to show that I, A', B', C' and P are concyclic. Clearly one can assume $P \neq I, A', B', C'$.

We use oriented angles to avoid heavy case distinction. The oriented angle between the lines l and m is denoted by $\angle(l, m)$. We have $\angle(l, m) = -\angle(m, l)$ and $\angle(l, m) + \angle(m, n) = \angle(l, n)$ for arbitrary lines l, m and n. Four distinct non-collinear points U, V, X, Y are concyclic if and only if $\angle(UX, VX) = \angle(UY, VY)$.



Suppose for the moment that A', B', P, I are distinct and noncollinear; then it is enough to check the equality $\angle(A'P, B'P) = \angle(A'I, B'I)$. Because A, F, P, A' are on the circle ω_A, we have $\angle(A'P, FP) = \angle(A'A, FA) = \angle(A'I, AB)$. Likewise $\angle(B'P, FP) = \angle(B'I, AB)$. Therefore

$$\angle(A'P, B'P) = \angle(A'P, FP) + \angle(FP, B'P) = \angle(A'I, AB) - \angle(B'I, AB) = \angle(A'I, B'I).$$

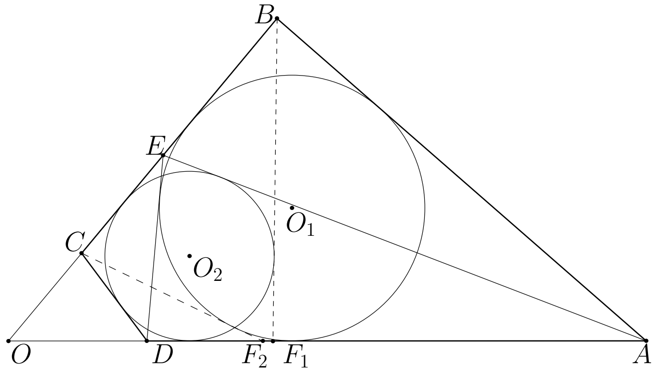
Here we assumed that $P \neq F$. If $P = F$ then $P \neq D, E$ and the conclusion follows similarly (use $\angle(A'F, B'F) = \angle(A'F, EF) + \angle(EF, DF) + \angle(DF, B'F)$ and inscribed angles in $\omega_A, \omega_B, \omega_C$).

There is no loss of generality in assuming A', B', P, I distinct and noncollinear. If ABC is an equilateral triangle then the equalities (*) imply that A', B', C', I, O and P coincide, so $OP = OI$. Otherwise at most one of A', B', C' coincides with I. If say $C' = I$ then $OI \perp CI$ by the previous reasoning. It follows that $A', B' \neq I$ and hence $A' \neq B'$. Finally A', B' and I are noncollinear because I, A', B', C' are concyclic.

Comment. The proposer remarks that the locus γ of the points P is an arc of the circle $(A'B'C'I)$. The reflection I' of I in O belongs to γ; it is obtained by choosing D, E and F to be the tangency points of the three excircles with their respective sides. The rest of the circle $(A'B'C'I)$, except I, can be included in γ by letting D, E and F vary on the extensions of the sides and assuming signed lengths. For instance if B is between C and D then the length BD must be taken with a negative sign. The incenter I corresponds to the limit case where D tends to infinity.

G7. Let $ABCD$ be a convex quadrilateral with non-parallel sides BC and AD. Assume that there is a point E on the side BC such that the quadrilaterals $ABED$ and $AECD$ are circumscribed. Prove that there is a point F on the side AD such that the quadrilaterals $ABCF$ and $BCDF$ are circumscribed if and only if AB is parallel to CD.

Solution. Let ω_1 and ω_2 be the incircles and O_1 and O_2 the incenters of the quadrilaterals $ABED$ and $AECD$ respectively. A point F with the stated property exists only if ω_1 and ω_2 are also the incircles of the quadrilaterals $ABCF$ and $BCDF$.



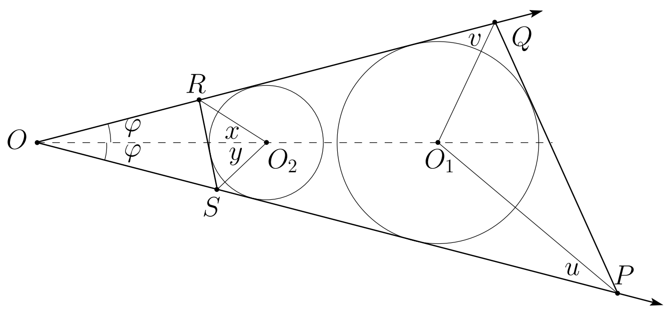
Let the tangents from B to ω_2 and from C to ω_1 (other than BC) meet AD at F_1 and F_2 respectively. We need to prove that $F_1 = F_2$ if and only if $AB \parallel CD$.

Lemma. The circles ω_1 and ω_2 with centers O_1 and O_2 are inscribed in an angle with vertex O. The points P, S on one side of the angle and Q, R on the other side are such that ω_1 is the incircle of the triangle PQO, and ω_2 is the excircle of the triangle RSO opposite to O. Denote $p = OO_1 \cdot OO_2$. Then exactly one of the following relations holds:

$$OP \cdot OR < p < OQ \cdot OS, \qquad OP \cdot OR > p > OQ \cdot OS, \qquad OP \cdot OR = p = OQ \cdot OS.$$

Proof. Denote $\angle OPO_1 = u$, $\angle OQO_1 = v$, $\angle OO_2R = x$, $\angle OO_2S = y$, $\angle POQ = 2\varphi$. Because PO_1, QO_1, RO_2, SO_2 are internal or external bisectors in the triangles PQO and RSO, we have

$$u + v = x + y \ (= 90^\circ - \varphi). \tag{1}$$



By the law of sines

$$\frac{OP}{OO_1} = \frac{\sin(u+\varphi)}{\sin u} \quad \text{and} \quad \frac{OO_2}{OR} = \frac{\sin(x+\varphi)}{\sin x}.$$

Therefore, since x, u and φ are acute,

$$OP \cdot OR \geq p \Leftrightarrow \frac{OP}{OO_1} \geq \frac{OO_2}{OR} \Leftrightarrow \sin x \sin(u+\varphi) \geq \sin u \sin(x+\varphi) \Leftrightarrow \sin(x-u) \geq 0 \Leftrightarrow x \geq u.$$

Thus $OP \cdot OR \geq p$ is equivalent to $x \geq u$, with $OP \cdot OR = p$ if and only if $x = u$.

Analogously, $p \geq OQ \cdot OS$ is equivalent to $v \geq y$, with $p = OQ \cdot OS$ if and only if $v = y$. On the other hand $x \geq u$ and $v \geq y$ are equivalent by (1), with $x = u$ if and only if $v = y$. The conclusion of the lemma follows from here. $\square$

Going back to the problem, apply the lemma to the quadruples $\{B, E, D, F_1\}$, $\{A, B, C, D\}$ and $\{A, E, C, F_2\}$. Assuming $OE \cdot OF_1 > p$, we obtain

$$OE \cdot OF_1 > p \;\Rightarrow\; OB \cdot OD < p \;\Rightarrow\; OA \cdot OC > p \;\Rightarrow\; OE \cdot OF_2 < p.$$

In other words, $OE \cdot OF_1 > p$ implies

$$OB \cdot OD < p < OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 > p > OE \cdot OF_2.$$

Similarly, $OE \cdot OF_1 < p$ implies

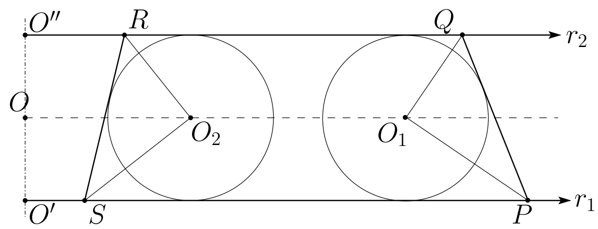
$$OB \cdot OD > p > OA \cdot OC \quad \text{and} \quad OE \cdot OF_1 < p < OE \cdot OF_2.$$

In these cases $F_1 \neq F_2$ and $OB \cdot OD \neq OA \cdot OC$, so the lines AB and CD are not parallel.

There remains the case $OE \cdot OF_1 = p$. Here the lemma leads to $OB \cdot OD = p = OA \cdot OC$ and $OE \cdot OF_1 = p = OE \cdot OF_2$. Therefore $F_1 = F_2$ and $AB \parallel CD$.

Comment. The conclusion is also true if BC and AD are parallel. One can prove a limit case of the lemma for the configuration shown in the figure below, where r_1 and r_2 are parallel rays starting at O' and O'', with $O'O'' \perp r_1, r_2$ and O the midpoint of $O'O''$. Two circles with centers O_1 and O_2 are inscribed in the strip between r_1 and r_2. The lines PQ and RS are tangent to the circles, with P, S on r_1, and Q, R on r_2, so that O, O_1 are on the same side of PQ and O, O_2 are on different sides of RS. Denote $s = OO_1 + OO_2$. Then exactly one of the following relations holds:

$$O'P + O''R < s < O''Q + O'S, \quad O'P + O''R > s > O''Q + O'S, \quad O'P + O''R = s = O''Q + O'S.$$

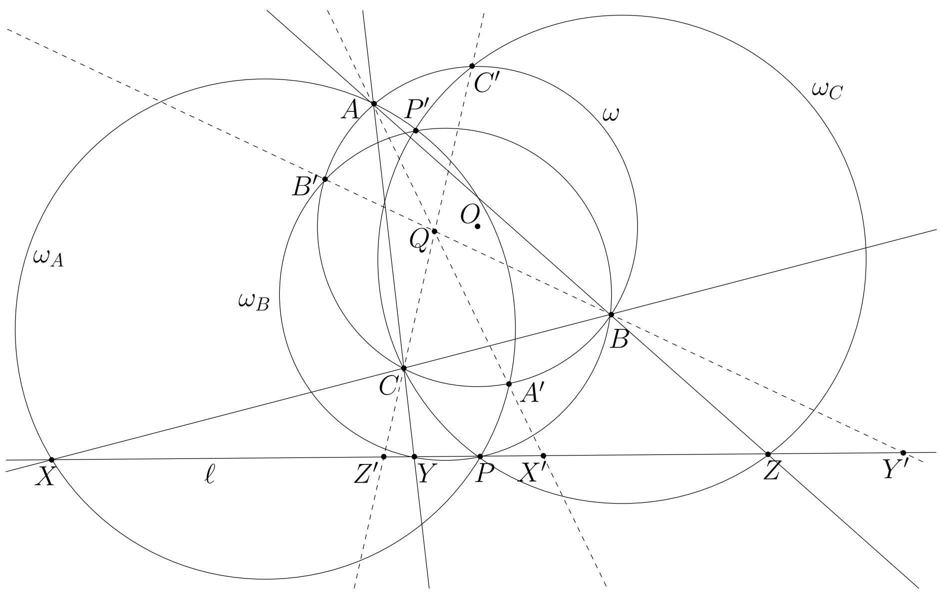


Once this is established, the proof of the original statement for $BC \parallel AD$ is analogous to the one in the intersecting case. One replaces products by sums of relevant segments.

G8. Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

Solution 1. Let $\omega_A, \omega_B, \omega_C$ and ω be the circumcircles of triangles AXP, BYP, CZP and ABC respectively. The strategy of the proof is to construct a point Q with the same power with respect to the four circles. Then each of P and Q has the same power with respect to $\omega_A, \omega_B, \omega_C$ and hence the three circles are coaxial. In other words they have another common point P' or the three of them are tangent at P.

We first give a description of the point Q. Let $A' \neq A$ be the second intersection of ω and ω_A; define B' and C' analogously. We claim that AA', BB' and CC' have a common point. Once this claim is established, the point just constructed will be on the radical axes of the three pairs of circles $\{\omega, \omega_A\}, \{\omega, \omega_B\}, \{\omega, \omega_C\}$. Hence it will have the same power with respect to $\omega, \omega_A, \omega_B, \omega_C$.



We proceed to prove that AA', BB' and CC' intersect at one point. Let r be the circumradius of triangle ABC. Define the points X', Y', Z' as the intersections of AA', BB', CC' with ℓ. Observe that X', Y', Z' do exist. If AA' is parallel to ℓ then ω_A is tangent to ℓ; hence $X = P$ which is a contradiction. Similarly, BB' and CC' are not parallel to ℓ.

From the powers of the point X' with respect to the circles ω_A and ω we get

$$X'P \cdot (X'P + PX) = X'P \cdot X'X = X'A' \cdot X'A = X'O^2 - r^2,$$

hence

$$X'P \cdot PX = X'O^2 - r^2 - X'P^2 = OP^2 - r^2.$$

We argue analogously for the points Y' and Z', obtaining

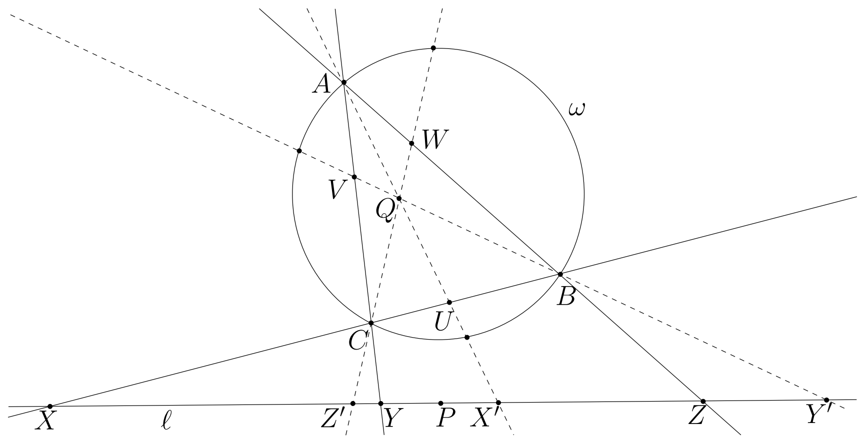
$$X'P \cdot PX = Y'P \cdot PY = Z'P \cdot PZ = OP^2 - r^2 = k^2. \tag{1}$$

In these computations all segments are regarded as directed segments. We keep the same convention for the sequel.

We prove that the lines AA', BB', CC' intersect at one point by Ceva's theorem. To avoid distracting remarks we interpret everything projectively, i. e. whenever two lines are parallel they meet at a point on the line at infinity.

Let U, V, W be the intersections of AA', BB', CC' with BC, CA, AB respectively. The idea is that although it is difficult to calculate the ratio $\frac{BU}{CU}$, it is easier to deal with the cross-ratio $\frac{BU}{CU} / \frac{BX}{CX}$ because we can send it to the line ℓ. With this in mind we apply MENELAUS' theorem to the triangle ABC and obtain $\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1$. Hence Ceva's ratio can be expressed as

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{BU}{CU} \Big/ \frac{BX}{CX} \cdot \frac{CV}{AV} \Big/ \frac{CY}{AY} \cdot \frac{AW}{BW} \Big/ \frac{AZ}{BZ}.$$



Project the line BC to ℓ from A. The cross-ratio between BC and UX equals the cross-ratio between ZY and $X'X$. Repeating the same argument with the lines CA and AB gives

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = \frac{ZX'}{YX'} \Big/ \frac{ZX}{YX} \cdot \frac{XY'}{ZY'} \Big/ \frac{XY}{ZY} \cdot \frac{YZ'}{XZ'} \Big/ \frac{YZ}{XZ}$$

and hence

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'} \cdot \frac{XY'}{ZY'} \cdot \frac{YZ'}{XZ'}.$$

The equations (1) reduce the problem to a straightforward computation on the line ℓ. For instance, the transformation $t \mapsto -k^2/t$ preserves cross-ratio and interchanges the points X, Y, Z with the points X', Y', Z'. Then

$$\frac{BU}{CU} \cdot \frac{CV}{AV} \cdot \frac{AW}{BW} = (-1) \cdot \frac{ZX'}{YX'} \Big/ \frac{ZZ'}{YZ'} \cdot \frac{XY'}{ZY'} \Big/ \frac{XZ'}{ZZ'} = -1.$$

We proved that CEVA's ratio equals -1, so AA', BB', CC' intersect at one point Q.

Comment 1. There is a nice projective argument to prove that AX', BY', CZ' intersect at one point. Suppose that ℓ and ω intersect at a pair of complex conjugate points D and E. Consider a projective transformation that takes D and E to $[i; 1, 0]$ and $[-i, 1, 0]$. Then ℓ is the line at infinity, and ω is a conic through the special points $[i; 1, 0]$ and $[-i, 1, 0]$, hence it is a circle. So one can assume that AX, BY, CZ are parallel to BC, CA, AB. The involution on ℓ taking X, Y, Z to X', Y', Z' and leaving D, E fixed is the involution changing each direction to its perpendicular one. Hence AX, BY, CZ are also perpendicular to AX', BY', CZ'.

It follows from the above that AX', BY', CZ' intersect at the orthocenter of triangle ABC.

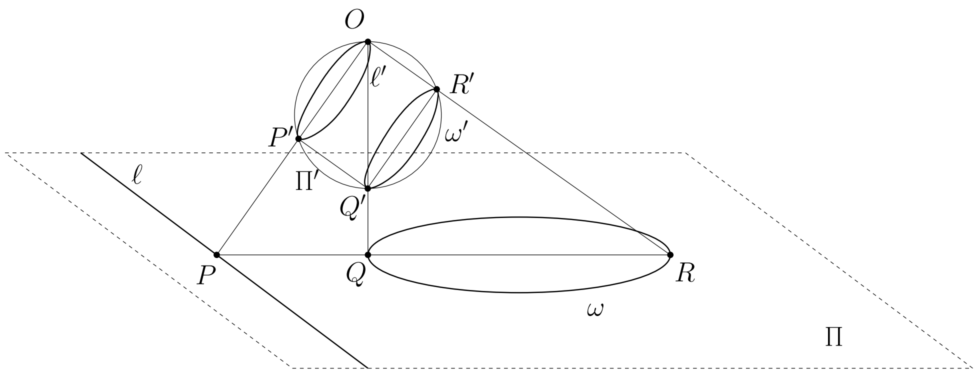
Comment 2. The restriction that the line ℓ does not intersect the circumcricle ω is unnecessary. The proof above works in general. In case ℓ intersects ω at D and E point P is the midpoint of DE, and some equations can be interpreted differently. For instance

$$X'P \cdot X'X = X'A' \cdot X'A = X'D \cdot X'E,$$

and hence the pairs $X'X$ and DE are harmonic conjugates. This means that X', Y', Z' are the harmonic conjugates of X, Y, Z with respect to the segment DE.

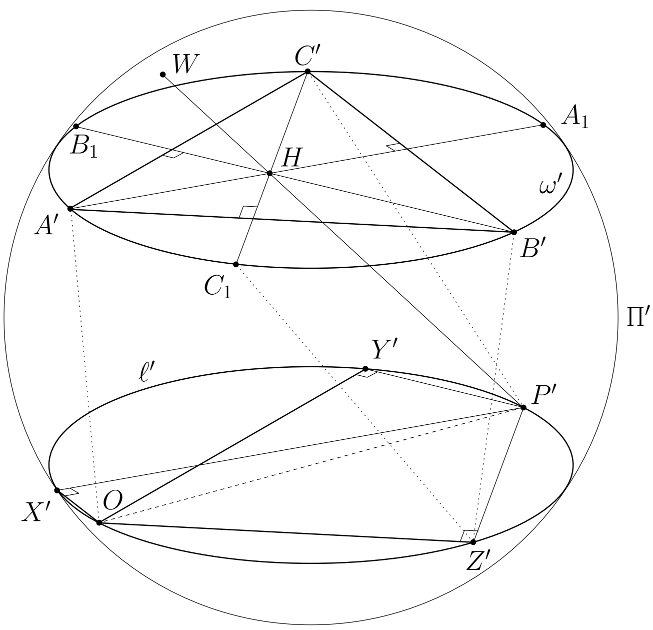
Solution 2. First we prove that there is an inversion in space that takes ℓ and ω to parallel circles on a sphere. Let QR be the diameter of ω whose extension beyond Q passes through P. Let Π be the plane carrying our objects. In space, choose a point O such that the line QO is perpendicular to Π and $\angle POR = 90°$, and apply an inversion with pole O (the radius of the inversion does not matter). For any object $\mathcal{T}$ denote by $\mathcal{T}'$ the image of $\mathcal{T}$ under this inversion.

The inversion takes the plane Π to a sphere Π'. The lines in Π are taken to circles through O, and the circles in Π also are taken to circles on Π'.



Since the line ℓ and the circle ω are perpendicular to the plane OPQ, the circles ℓ' and ω' also are perpendicular to this plane. Hence, the planes of the circles ℓ' and ω' are parallel.

Now consider the circles $A'X'P'$, $B'Y'P'$ and $C'Z'P'$. We want to prove that either they have a common point (on Π'), different from P', or they are tangent to each other.



The point X' is the second intersection of the circles $B'C'O$ and ℓ', other than O. Hence, the lines OX' and $B'C'$ are coplanar. Moreover, they lie in the parallel planes of ℓ' and ω'. Therefore, OX' and $B'C'$ are parallel. Analogously, OY' and OZ' are parallel to $A'C'$ and $A'B'$.

Let A_1 be the second intersection of the circles $A'X'P'$ and ω', other than A'. The segments $A'A_1$ and $P'X'$ are coplanar, and therefore parallel. Now we know that $B'C'$ and $A'A_1$ are parallel to OX' and $X'P'$ respectively, but these two segments are perpendicular because OP' is a diameter in ℓ'. We found that $A'A_1$ and $B'C'$ are perpendicular, hence $A'A_1$ is the altitude in the triangle $A'B'C'$, starting from A.

Analogously, let B_1 and C_1 be the second intersections of ω' with the circles $B'P'Y'$ and $C'P'Z'$, other than B' and C' respectively. Then $B'B_1$ and $C'C_1$ are the other two altitudes in the triangle $A'B'C'$.

Let H be the orthocenter of the triangle $A'B'C'$. Let W be the second intersection of the line $P'H$ with the sphere Π', other than P'. The point W lies on the sphere Π', in the plane of the circle $A'P'X'$, so W lies on the circle $A'P'X'$. Similarly, W lies on the circles $B'P'Y'$ and $C'P'Z'$ as well; indeed W is the second common point of the three circles.

If the line $P'H$ is tangent to the sphere then W coincides with P', and $P'H$ is the common tangent of the three circles.

Number Theory

N1. Call admissible a set A of integers that has the following property:

If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

Solution. A pair of integers m, n fulfills the condition if and only if $gcd(m, n) = 1$. Suppose that $gcd(m, n) = d > 1$. The set

$$A = \{\ldots, -2d, -d, 0, d, 2d, \ldots\}$$

is admissible, because if d divides x and y then it divides $x^2 + kxy + y^2$ for every integer k. Also $m, n \in A$ and $A \neq \mathbb{Z}$.

Now let $gcd(m, n) = 1$, and let A be an admissible set containing m and n. We use the following observations to prove that $A = \mathbb{Z}$:

(i) $kx^2 \in A$ for every $x \in A$ and every integer k.

(ii) $(x + y)^2 \in A$ for all $x, y \in A$.

To justify (i) let $y = x$ in the definition of an admissible set; to justify (ii) let $k = 2$.

Since $gcd(m, n) = 1$, we also have $gcd(m^2, n^2) = 1$. Hence one can find integers a, b such that $am^2 + bn^2 = 1$. It follows from (i) that $am^2 \in A$ and $bn^2 \in A$. Now we deduce from (ii) that $1 = (am^2 + bn^2)^2 \in A$. But if $1 \in A$ then (i) implies $k \in A$ for every integer k.

N2. Find all triples (x, y, z) of positive integers such that $x \leq y \leq z$ and

$$x^3(y^3 + z^3) = 2012(xyz + 2).$$

Solution. First note that x divides $2012 \cdot 2 = 2^3 \cdot 503$. If $503 \mid x$ then the right-hand side of the equation is divisible by 503^3, and it follows that $503^2 \mid xyz + 2$. This is false as $503 \mid x$. Hence $x = 2^m$ with $m \in \{0, 1, 2, 3\}$. If $m \geq 2$ then $2^6 \mid 2012(xyz + 2)$. However the highest powers of 2 dividing 2012 and $xyz + 2 = 2^m yz + 2$ are 2^2 and 2^1 respectively. So $x = 1$ or $x = 2$, yielding the two equations

$$y^3 + z^3 = 2012(yz + 2), \qquad \text{and} \qquad y^3 + z^3 = 503(yz + 1).$$

In both cases the prime $503 = 3 \cdot 167 + 2$ divides $y^3 + z^3$. We claim that $503 \mid y + z$. This is clear if $503 \mid y$, so let $503 \nmid y$ and $503 \nmid z$. Then $y^{502} \equiv z^{502} \pmod{503}$ by Fermat's little theorem. On the other hand $y^3 \equiv -z^3 \pmod{503}$ implies $y^{3 \cdot 167} \equiv -z^{3 \cdot 167} \pmod{503}$, i. e. $y^{501} \equiv -z^{501} \pmod{503}$. It follows that $y \equiv -z \pmod{503}$ as claimed.

Therefore $y + z = 503k$ with $k \geq 1$. In view of $y^3 + z^3 = (y + z)\big((y - z)^2 + yz\big)$ the two equations take the form

$$k(y - z)^2 + (k - 4)yz = 8, \tag{1}$$

$$k(y - z)^2 + (k - 1)yz = 1. \tag{2}$$

In (1) we have $(k - 4)yz \leq 8$, which implies $k \leq 4$. Indeed if $k > 4$ then $1 \leq (k - 4)yz \leq 8$, so that $y \leq 8$ and $z \leq 8$. This is impossible as $y + z = 503k \geq 503$. Note next that $y^3 + z^3$ is even in the first equation. Hence $y + z = 503k$ is even too, meaning that k is even. Thus $k = 2$ or $k = 4$. Clearly (1) has no integer solutions for $k = 4$. If $k = 2$ then (1) takes the form $(y + z)^2 - 5yz = 4$. Since $y + z = 503k = 503 \cdot 2$, this leads to $5yz = 503^2 \cdot 2^2 - 4$. However $503^2 \cdot 2^2 - 4$ is not a multiple of 5. Therefore (1) has no integer solutions.

Equation (2) implies $0 \leq (k - 1)yz \leq 1$, so that $k = 1$ or $k = 2$. Also $0 \leq k(y - z)^2 \leq 1$, hence $k = 2$ only if $y = z$. However then $y = z = 1$, which is false in view of $y + z \geq 503$. Therefore $k = 1$ and (2) takes the form $(y - z)^2 = 1$, yielding $z - y = |y - z| = 1$. Combined with $k = 1$ and $y + z = 503k$, this leads to $y = 251$, $z = 252$.

In summary the triple $(2, 251, 252)$ is the only solution.

N3. Determine all integers $m \geq 2$ such that every n with $\frac{m}{3} \leq n \leq \frac{m}{2}$ divides the binomial coefficient $\binom{n}{m-2n}$.

Solution. The integers in question are all prime numbers.

First we check that all primes satisfy the condition, and even a stronger one. Namely, if p is a prime then every n with $1 \leq n \leq \frac{p}{2}$ divides $\binom{n}{p-2n}$. This is true for $p = 2$ where $n = 1$ is the only possibility. For an odd prime p take $n \in [1, \frac{p}{2}]$ and consider the following identity of binomial coefficients:

$$(p-2n) \cdot \binom{n}{p-2n} = n \cdot \binom{n-1}{p-2n-1}.$$

Since $p \geq 2n$ and p is odd, all factors are non-zero. If $d = gcd(p-2n, n)$ then d divides p, but $d \leq n < p$ and hence $d = 1$. It follows that $p - 2n$ and n are relatively prime, and so the factor n in the right-hand side divides the binomial coefficient $\binom{n}{p-2n}$.

Next we show that no composite number m has the stated property. Consider two cases.

- If $m = 2k$ with $k > 1$, pick $n = k$. Then $\frac{m}{3} \leq n \leq \frac{m}{2}$ but $\binom{n}{m-2n} = \binom{k}{0} = 1$ is not divisible by $k > 1$.

- If m is odd then there exist an odd prime p and an integer $k \geq 1$ with $m = p(2k+1)$. Pick $n = pk$, then $\frac{m}{3} \leq n \leq \frac{m}{2}$ by $k \geq 1$. However

$$\frac{1}{n}\binom{n}{m-2n} = \frac{1}{pk}\binom{pk}{p} = \frac{(pk-1)(pk-2)\cdots(pk-(p-1))}{p!}$$

is not an integer, because p divides the denominator but not the numerator.

N4. An integer a is called friendly if the equation $(m^2+n)(n^2+m)=a(m-n)^3$ has a solution over the positive integers.

a) Prove that there are at least 500 friendly integers in the set $\{1,2,\ldots,2012\}$.

b) Decide whether $a=2$ is friendly.

Solution. a) Every a of the form $a=4k-3$ with $k\geq 2$ is friendly. Indeed the numbers $m=2k-1>0$ and $n=k-1>0$ satisfy the given equation with $a=4k-3$:

$$(m^2+n)(n^2+m)=\big((2k-1)^2+(k-1)\big)\big((k-1)^2+(2k-1)\big)=(4k-3)k^3=a(m-n)^3.$$

Hence $5,9,\ldots,2009$ are friendly and so $\{1,2,\ldots,2012\}$ contains at least 502 friendly numbers.

b) We show that $a=2$ is not friendly. Consider the equation with $a=2$ and rewrite its left-hand side as a difference of squares:

$$\frac{1}{4}\left((m^2+n+n^2+m)^2-(m^2+n-n^2-m)^2\right)=2(m-n)^3.$$

Since $m^2+n-n^2-m=(m-n)(m+n-1)$, we can further reformulate the equation as

$$(m^2+n+n^2+m)^2=(m-n)^2\left(8(m-n)+(m+n-1)^2\right).$$

It follows that $8(m-n)+(m+n-1)^2$ is a perfect square. Clearly $m>n$, hence there is an integer $s\geq 1$ such that

$$(m+n-1+2s)^2=8(m-n)+(m+n-1)^2.$$

Subtracting the squares gives $s(m+n-1+s)=2(m-n)$. Since $m+n-1+s>m-n$, we conclude that $s<2$. Therefore the only possibility is $s=1$ and $m=3n$. However then the left-hand side of the given equation (with $a=2$) is greater than $m^3=27n^3$, whereas its right-hand side equals $16n^3$. The contradiction proves that $a=2$ is not friendly.

Comment. A computer search shows that there are 561 friendly numbers in $\{1,2,\ldots,2012\}$.

N5. For a nonnegative integer n define $rad(n) = 1$ if $n = 0$ or $n = 1$, and $rad(n) = p_1 p_2 \cdots p_k$ where $p_1 < p_2 < \cdots < p_k$ are all prime factors of n. Find all polynomials $f(x)$ with nonnegative integer coefficients such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for every nonnegative integer n.

Solution 1. We are going to prove that $f(x) = ax^m$ for some nonnegative integers a and m. If $f(x)$ is the zero polynomial we are done, so assume that $f(x)$ has at least one positive coefficient. In particular $f(1) > 0$.

Let p be a prime number. The condition is that $f(n) \equiv 0 \pmod{p}$ implies

$$f(n^{rad(n)}) \equiv 0 \pmod{p}. \tag{1}$$

Since $rad(n^{rad(n)^k}) = rad(n)$ for all k, repeated applications of the preceding implication show that if p divides $f(n)$ then

$$f(n^{rad(n)^k}) \equiv 0 \pmod{p} \quad \text{for all } k.$$

The idea is to construct a prime p and a positive integer n such that $p - 1$ divides n and p divides $f(n)$. In this case, for k large enough $p - 1$ divides $rad(n)^k$. Hence if $(p, n) = 1$ then $n^{rad(n)^k} \equiv 1 \pmod{p}$ by FERMAT's little theorem, so that

$$f(1) \equiv f(n^{rad(n)^k}) \equiv 0 \pmod{p}. \tag{2}$$

Suppose that $f(x) = g(x)x^m$ with $g(0) \neq 0$. Let t be a positive integer, p any prime factor of $g(-t)$ and $n = (p-1)t$. So $p-1$ divides n and $f(n) = f((p-1)t) \equiv f(-t) \equiv 0 \pmod{p}$, hence either $(p, n) > 1$ or (2) holds. If $(p, (p-1)t) > 1$ then p divides t and $g(0) \equiv g(-t) \equiv 0 \pmod{p}$, meaning that p divides $g(0)$.

In conclusion we proved that each prime factor of $g(-t)$ divides $g(0)f(1) \neq 0$, and thus the set of prime factors of $g(-t)$ when t ranges through the positive integers is finite. This is known to imply that $g(x)$ is a constant polynomial, and so $f(x) = ax^m$.

Solution 2. Let $f(x)$ be a polynomial with integer coefficients (not necessarily nonnegative) such that $rad(f(n))$ divides $rad(f(n^{rad(n)}))$ for any nonnegative integer n. We give a complete description of all polynomials with this property. More precisely, we claim that if $f(x)$ is such a polynomial and ξ is a root of $f(x)$ then so is ξ^d for every positive integer d.

Therefore each root of $f(x)$ is zero or a root of unity. In particular, if a root of unity ξ is a root of $f(x)$ then $1 = \xi^d$ is a root too (for some positive integer d). In the original problem $f(x)$ has nonnegative coefficients. Then either $f(x)$ is the zero polynomial or $f(1) > 0$ and $\xi = 0$ is the only possible root. In either case $f(x) = ax^m$ with a and m nonnegative integers.

To prove the claim let ξ be a root of $f(x)$, and let $g(x)$ be an irreducible factor of $f(x)$ such that $g(\xi) = 0$. If 0 or 1 are roots of $g(x)$ then either $\xi = 0$ or $\xi = 1$ (because $g(x)$ is irreducible) and we are done. So assume that $g(0), g(1) \neq 0$. By decomposing d as a product of prime numbers, it is enough to consider the case $d = p$ prime. We argue for $p = 2$. Since $rad(2^k) = 2$ for every k, we have

$$rad(f(2^k)) \mid rad(f(2^{2k})).$$

Now we prove that $g(x)$ divides $f(x^2)$. Suppose that this is not the case. Then, since $g(x)$ is irreducible, there are integer-coefficient polynomials $a(x)$, $b(x)$ and an integer N such that

$$a(x)g(x) + b(x)f(x^2) = N. \tag{3}$$

Each prime factor p of $g(2^k)$ divides $f(2^k)$, so by $rad(f(2^k)) | rad(f(2^{2k}))$ it also divides $f(2^{2k})$. From the equation above with $x = 2^k$ it follows that p divides N.

In summary, each prime divisor of $g(2^k)$ divides N, for all $k \geq 0$. Let $p_1, \ldots, p_n$ be the odd primes dividing N, and suppose that

$$g(1) = 2^{\alpha} p_1^{\alpha_1} \cdots p_n^{\alpha_n}.$$

If k is divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ then

$$2^k \equiv 1 \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}},$$

yielding

$$g(2^k) \equiv g(1) \pmod{p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1}}.$$

It follows that for each i the maximal power of p_i dividing $g(2^k)$ and $g(1)$ is the same, namely $p_i^{\alpha_i}$. On the other hand, for large enough k, the maximal power of 2 dividing $g(2^k)$ and $g(0) \neq 0$ is the same. From the above, for k divisible by $\varphi(p_1^{\alpha_1+1} \cdots p_n^{\alpha_n+1})$ and large enough, we obtain that $g(2^k)$ divides $g(0) \cdot g(1)$. This is impossible because $g(0), g(1) \neq 0$ are fixed and $g(2^k)$ is arbitrarily large.

In conclusion, $g(x)$ divides $f(x^2)$. Recall that ξ is a root of $f(x)$ such that $g(\xi) = 0$; then $f(\xi^2) = 0$, i. e. ξ^2 is a root of $f(x)$.

Likewise if ξ is a root of $f(x)$ and p an arbitrary prime then ξ^p is a root too. The argument is completely analogous, in the proof above just replace 2 by p and "odd prime" by "prime different from p."

Comment. The claim in the second solution can be proved by varying $n \pmod{p}$ in (1). For instance, we obtain

$$f(n^{rad(n+pk)}) \equiv 0 \pmod{p}$$

for every positive integer k. One can prove that if $(n, p) = 1$ then $rad(n + pk)$ runs through all residue classes $r \pmod{p-1}$ with $(r, p-1)$ squarefree. Hence if $f(n) \equiv 0 \pmod{p}$ then $f(n^r) \equiv 0 \pmod{p}$ for all integers r. This implies the claim by an argument leading to the identity (3).

N6. Let x and y be positive integers. If $x^{2^n}-1$ is divisible by $2^n y+1$ for every positive integer n, prove that $x=1$.

Solution. First we prove the following fact: For every positive integer y there exist infinitely many primes $p \equiv 3 \pmod 4$ such that p divides some number of the form $2^n y+1$.

Clearly it is enough to consider the case y odd. Let

$$2y+1 = p_1^{e_1} \cdots p_r^{e_r}$$

be the prime factorization of $2y+1$. Suppose on the contrary that there are finitely many primes $p_{r+1}, \ldots, p_{r+s} \equiv 3 \pmod 4$ that divide some number of the form $2^n y+1$ but do not divide $2y+1$.

We want to find an n such that $p_i^{e_i} || 2^n y+1$ for $1 \le i \le r$ and $p_i \nmid 2^n y+1$ for $r+1 \le i \le r+s$. For this it suffices to take

$$n = 1+\varphi(p_1^{e_1+1} \cdots p_r^{e_r+1} p_{r+1}^1 \cdots p_{r+s}^1),$$

because then

$$2^n y+1 \equiv 2y+1 \pmod{p_1^{e_1+1} \cdots p_r^{e_r+1} p_{r+1}^1 \cdots p_{r+s}^1}.$$

The last congruence means that $p_1^{e_1}, \ldots, p_r^{e_r}$ divide exactly $2^n y+1$ and no prime $p_{r+1}, \ldots, p_{r+s}$ divides $2^n y+1$. It follows that the prime factorization of $2^n y+1$ consists of the prime powers $p_1^{e_1}, \ldots, p_r^{e_r}$ and powers of primes $\equiv 1 \pmod 4$. Because y is odd, we obtain

$$2^n y+1 \equiv p_1^{e_1} \cdots p_r^{e_r} \equiv 2y+1 \equiv 3 \pmod 4.$$

This is a contradiction since $n>1$, and so $2^n y+1 \equiv 1 \pmod 4$.

Now we proceed to the problem. If p is a prime divisor of $2^n y+1$ the problem statement implies that $x^d \equiv 1 \pmod p$ for $d=2^n$. By FERMAT's little theorem the same congruence holds for $d=p-1$, so it must also hold for $d=(2^n, p-1)$. For $p \equiv 3 \pmod 4$ we have $(2^n, p-1)=2$, therefore in this case $x^2 \equiv 1 \pmod p$.

In summary, we proved that every prime $p \equiv 3 \pmod 4$ that divides some number of the form $2^n y+1$ also divides x^2-1. This is possible only if $x=1$, otherwise by the above x^2-1 would be a positive integer with infinitely many prime factors.

Comment. For each x and each odd prime p the maximal power of p dividing $x^{2^n}-1$ for some n is bounded and hence the same must be true for the numbers $2^n y+1$. We infer that p^2 divides $2^{p-1}-1$ for each prime divisor p of $2^n y+1$. However trying to reach a contradiction with this conclusion alone seems hopeless, since it is not even known if there are infinitely many primes p *without* this property.

N7. Find all $n \in \mathbb{N}$ for which there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solution. Such numbers $a_1, a_2, \ldots, a_n$ exist if and only if $n \equiv 1 \pmod 4$ or $n \equiv 2 \pmod 4$.

Let $\sum_{k=1}^{n} \frac{k}{3^{a_k}} = 1$ with $a_1, a_2, \ldots, a_n$ nonnegative integers. Then $1 \cdot x_1 + 2 \cdot x_2 + \cdots + n \cdot x_n = 3^a$ with $x_1, \ldots, x_n$ powers of 3 and $a \geq 0$. The right-hand side is odd, and the left-hand side has the same parity as $1 + 2 + \cdots + n$. Hence the latter sum is odd, which implies $n \equiv 1, 2 \pmod 4$. Now we prove the converse.

Call *feasible* a sequence $b_1, b_2, \ldots, b_n$ if there are nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{b_1}{3^{a_1}} + \frac{b_2}{3^{a_2}} + \cdots + \frac{b_n}{3^{a_n}} = 1.$$

Let b_k be a term of a feasible sequence $b_1, b_2, \ldots, b_n$ with exponents $a_1, a_2, \ldots, a_n$ like above, and let u, v be nonnegative integers with sum $3b_k$. Observe that

$$\frac{1}{2^{a_k+1}} + \frac{1}{2^{a_k+1}} = \frac{1}{2^{a_k}} \qquad \text{and} \qquad \frac{u}{3^{a_k+1}} + \frac{v}{3^{a_k+1}} = \frac{b_k}{3^{a_k}}.$$

It follows that the sequence $b_1, \ldots, b_{k-1}, u, v, b_{k+1}, \ldots, b_n$ is feasible. The exponents a_i are the same for the unchanged terms b_i, $i \neq k$; the new terms u, v have exponents $a_k + 1$.

We state the conclusion in reverse. If two terms u, v of a sequence are replaced by one term $\frac{u+v}{3}$ and the obtained sequence is feasible, then the original sequence is feasible too. Denote by α_n the sequence $1, 2, \ldots, n$. To show that α_n is feasible for $n \equiv 1, 2 \pmod 4$, we transform it by $n - 1$ replacements $\{u, v\} \mapsto \frac{u+v}{3}$ to the one-term sequence α_1. The latter is feasible, with $a_1 = 0$. Note that if m and $2m$ are terms of a sequence then $\{m, 2m\} \mapsto m$, so $2m$ can be ignored if necessary.

Let $n \geq 16$. We prove that α_n can be reduced to α_{n-12} by 12 operations. Write $n = 12k + r$ where $k \geq 1$ and $0 \leq r \leq 11$. If $0 \leq r \leq 5$ then the last 12 terms of α_n can be partitioned into 2 singletons $\{12k - 6\}$, $\{12k\}$ and the following 5 pairs:

$$\{12k - 6 - i, 12k - 6 + i\}, i = 1, \ldots, 5 - r; \qquad \{12k - j, 12k + j\}, j = 1, \ldots, r.$$

(There is only one kind of pairs if $r \in \{0, 5\}$.) One can ignore $12k - 6$ and $12k$ since α_n contains $6k - 3$ and $6k$. Furthermore the 5 operations $\{12k - 6 - i, 12k - 6 + i\} \mapsto 8k - 4$ and $\{12k - j, 12k + j\} \mapsto 8k$ remove the 10 terms in the pairs and bring in 5 new terms equal to $8k - 4$ or $8k$. All of these can be ignored too as $4k - 2$ and $4k$ are still present in the sequence. Indeed $4k \leq n - 12$ is equivalent to $8k \geq 12 - r$, which is true for $r \in \{4, 5\}$. And if $r \in \{0, 1, 2, 3\}$ then $n \geq 16$ implies $k \geq 2$, so $8k \geq 12 - r$ also holds. Thus α_n reduces to α_{n-12}.

The case $6 \leq r \leq 11$ is analogous. Consider the singletons $\{12k\}$, $\{12k + 6\}$ and the 5 pairs

$$\{12k - i, 12k + i\}, i = 1, \ldots, 11 - r; \qquad \{12k + 6 - j, 12k + 6 + j\}, j = 1, \ldots, r - 6.$$

Ignore the singletons like before, then remove the pairs via operations $\{12k - i, 12k + i\} \mapsto 8k$ and $\{12k + 6 - j, 12k + 6 + j\} \mapsto 8k + 4$. The 5 newly-appeared terms $8k$ and $8k + 4$ can be ignored too since $4k + 2 \leq n - 12$ (this follows from $k \geq 1$ and $r \geq 6$). We obtain α_{n-12} again.

The problem reduces to $2 \leq n \leq 15$. In fact $n \in \{2, 5, 6, 9, 10, 13, 14\}$ by $n \equiv 1, 2 \pmod 4$. The cases $n = 2, 6, 10, 14$ reduce to $n = 1, 5, 9, 13$ respectively because the last even term of α_n can be ignored. For $n = 5$ apply $\{4, 5\} \mapsto 3$, then $\{3, 3\} \mapsto 2$, then ignore the 2 occurrences of 2. For $n = 9$ ignore 6 first, then apply $\{5, 7\} \mapsto 4$, $\{4, 8\} \mapsto 4$, $\{3, 9\} \mapsto 4$. Now ignore the 3 occurrences of 4, then ignore 2. Finally $n = 13$ reduces to $n = 10$ by $\{11, 13\} \mapsto 8$ and ignoring 8 and 12. The proof is complete.

N8. Prove that for every prime $p > 100$ and every integer r there exist two integers a and b such that p divides $a^2 + b^5 - r$.

Solution 1. Throughout the solution, all congruence relations are meant modulo p.

Fix p, and let $\mathcal{P} = \{0, 1, \ldots, p-1\}$ be the set of residue classes modulo p. For every $r \in \mathcal{P}$, let $S_r = \{(a, b) \in \mathcal{P} \times \mathcal{P} : a^2 + b^5 \equiv r\}$, and let $s_r = |S_r|$. Our aim is to prove $s_r > 0$ for all $r \in \mathcal{P}$.

We will use the well-known fact that for every residue class $r \in \mathcal{P}$ and every positive integer k, there are at most k values $x \in \mathcal{P}$ such that $x^k \equiv r$.

Lemma. Let N be the number of quadruples $(a, b, c, d) \in \mathcal{P}^4$ for which $a^2 + b^5 \equiv c^2 + d^5$. Then

$$N = \sum_{r \in \mathcal{P}} s_r^2 \tag{a}$$

and

$$N \leq p(p^2 + 4p - 4). \tag{b}$$

Proof. (a) For each residue class r there exist exactly s_r pairs (a, b) with $a^2 + b^5 \equiv r$ and s_r pairs (c, d) with $c^2 + d^5 \equiv r$. So there are s_r^2 quadruples with $a^2 + b^5 \equiv c^2 + d^5 \equiv r$. Taking the sum over all $r \in \mathcal{P}$, the statement follows.

(b) Choose an arbitrary pair $(b, d) \in \mathcal{P}$ and look for the possible values of a, c.

1. Suppose that $b^5 \equiv d^5$, and let k be the number of such pairs (b, d). The value b can be chosen in p different ways. For $b \equiv 0$ only $d = 0$ has this property; for the nonzero values of b there are at most 5 possible values for d. So we have $k \leq 1 + 5(p-1) = 5p - 4$.

The values a and c must satisfy $a^2 \equiv c^2$, so $a \equiv \pm c$, and there are exactly $2p - 1$ such pairs (a, c).

2. Now suppose $b^5 \not\equiv d^5$. In this case a and c must be distinct. By $(a - c)(a + c) = d^5 - b^5$, the value of $a - c$ uniquely determines $a + c$ and thus a and c as well. Hence, there are $p - 1$ suitable pairs (a, c).

Thus, for each of the k pairs (b, d) with $b^5 \equiv d^5$ there are $2p - 1$ pairs (a, c), and for each of the other $p^2 - k$ pairs (b, d) there are $p - 1$ pairs (a, c). Hence,

$$N = k(2p-1) + (p^2 - k)(p-1) = p^2(p-1) + kp \leq p^2(p-1) + (5p-4)p = p(p^2 + 4p - 4). \quad \square$$

To prove the statement of the problem, suppose that $S_r = \emptyset$ for some $r \in \mathcal{P}$; obviously $r \not\equiv 0$. Let $T = \{x^{10} : x \in \mathcal{P} \setminus \{0\}\}$ be the set of nonzero 10th powers modulo p. Since each residue class is the 10th power of at most 10 elements in $\mathcal{P}$, we have $|T| \geq \frac{p-1}{10} \geq 4$ by $p > 100$.

For every $t \in T$, we have $S_{tr} = \emptyset$. Indeed, if $(x, y) \in S_{tr}$ and $t \equiv z^{10}$ then

$$(z^{-5}x)^2 + (z^{-2}y)^5 \equiv t^{-1}(x^2 + y^5) \equiv r,$$

so $(z^{-5}x, z^{-2}y) \in S_r$. So, there are at least $\frac{p-1}{10} \geq 4$ empty sets among $S_1, \ldots, S_{p-1}$, and there are at most $p - 4$ nonzero values among $s_0, s_2, \ldots, s_{p-1}$. Then by the AM-QM inequality we obtain

$$N = \sum_{r \in \mathcal{P} \setminus rT} s_r^2 \geq \frac{1}{p-4} \left(\sum_{r \in \mathcal{P} \setminus rT} s_r \right)^2 = \frac{|\mathcal{P} \times \mathcal{P}|^2}{p-4} = \frac{p^4}{p-4} > p(p^2 + 4p - 4),$$

which is impossible by the lemma.

Solution 2. If $5 \nmid p-1$, then all modulo p residue classes are complete fifth powers and the statement is trivial. So assume that $p = 10k+1$ where $k \geq 10$. Let g be a primitive root modulo p.

We will use the following facts:

(F1) If some residue class x is not quadratic then $x^{(p-1)/2} \equiv -1 \pmod{p}$.

(F2) For every integer d, as a simple corollary of the summation formula for geometric progressions,
$$\sum_{i=0}^{2k-1} g^{5di} \equiv \begin{cases} 2k & \text{if } 2k \mid d \\ 0 & \text{if } 2k \nmid d \end{cases} \pmod{p}.$$

Suppose that, contrary to the statement, some modulo p residue class r cannot be expressed as a^2+b^5. Of course $r \not\equiv 0 \pmod{p}$. By (F1) we have $(r-b^5)^{(p-1)/2} = (r-b^5)^{5k} \equiv -1 \pmod{p}$ for all residue classes b.

For $t = 1, 2 \ldots, k-1$ consider the sums
$$S(t) = \sum_{i=0}^{2k-1} \left(r - g^{5i}\right)^{5k} g^{5ti}.$$

By the indirect assumption and (F2),
$$S(t) = \sum_{i=0}^{2k-1} \left(r - (g^i)^5\right)^{5k} g^{5ti} \equiv \sum_{i=0}^{2k-1} (-1) g^{5ti} \equiv -\sum_{i=0}^{2k-1} g^{5ti} \equiv 0 \pmod{p}$$

because $2k$ cannot divide t.

On the other hand, by the binomial theorem,
$$\begin{aligned} S(t) &= \sum_{i=0}^{2k-1} \left(\sum_{j=0}^{5k} \binom{5k}{j} r^{5k-j} \big(-g^{5i}\big)^j \right) g^{5ti} = \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \left(\sum_{i=0}^{2k-1} g^{5(j+t)i} \right) \equiv \\ &\equiv \sum_{j=0}^{5k} (-1)^j \binom{5k}{j} r^{5k-j} \begin{cases} 2k & \text{if } 2k \mid j+t \\ 0 & \text{if } 2k \nmid j+t \end{cases} \pmod{p}. \end{aligned}$$

Since $1 \leq j+t < 6k$, the number $2k$ divides $j+t$ only for $j = 2k-t$ and $j = 4k-t$. Hence,
$$0 \equiv S(t) \equiv (-1)^t \left(\binom{5k}{2k-t} r^{3k+t} + \binom{5k}{4k-t} r^{k+t} \right) \cdot 2k \pmod{p},$$
$$\binom{5k}{2k-t} r^{2k} + \binom{5k}{4k-t} \equiv 0 \pmod{p}.$$

Taking this for $t = 1, 2$ and eliminating r, we get
$$\begin{aligned} 0 &\equiv \binom{5k}{2k-2} \left(\binom{5k}{2k-1} r^{2k} + \binom{5k}{4k-1} \right) - \binom{5k}{2k-1} \left(\binom{5k}{2k-2} r^{2k} + \binom{5k}{4k-2} \right) \\ &= \binom{5k}{2k-2}\binom{5k}{4k-1} - \binom{5k}{2k-1}\binom{5k}{4k-2} \\ &= \frac{(5k)!^2}{(2k-1)!(3k+2)!(4k-1)!(k+2)!} \Big((2k-1)(k+2) - (3k+2)(4k-1) \Big) \\ &= \frac{-(5k)!^2 \cdot 2k(5k+1)}{(2k-1)!(3k+2)!(4k-1)!(k+2)!} \pmod{p}. \end{aligned}$$

But in the last expression none of the numbers is divisible by $p = 10k+1$, a contradiction.

Comment 1. The argument in the second solution is valid whenever $k \geq 3$, that is for all primes $p = 10k + 1$ except $p = 11$. This is an exceptional case when the statement is not true; $r = 7$ cannot be expressed as desired.

Comment 2. The statement is true in a more general setting: for every positive integer n, for all sufficiently large p, each residue class modulo p can be expressed as $a^2 + b^n$. Choosing $t = 3$ would allow using the Cauchy-Davenport theorem (together with some analysis on the case of equality).

In the literature more general results are known. For instance, the statement easily follows from the *Hasse-Weil bound*.

Shortlisted Problems with Solutions

54^{th} International Mathematical Olympiad

Santa Marta, Colombia 2013

Note of Confidentiality

The Shortlisted Problems should be kept strictly confidential until IMO 2014.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2013 thank the following 50 countries for contributing 149 problem proposals.

Argentina, Armenia, Australia, Austria, Belgium, Belarus, Brazil, Bulgaria, Croatia, Cyprus, Czech Republic, Denmark, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hungary, India, Indonesia, Iran, Ireland, Israel, Italy, Japan, Latvia, Lithuania, Luxembourg, Malaysia, Mexico, Netherlands, Nicaragua, Pakistan, Panama, Poland, Romania, Russia, Saudi Arabia, Serbia, Slovenia, Sweden, Switzerland, Tajikistan, Thailand, Turkey, U.S.A., Ukraine, United Kingdom

Problem Selection Committee

Federico Ardila (chairman)
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Problems

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

(Lithuania)

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy) \quad \text{and} \quad f(x+y) \geqslant f(x) + f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n,$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f\colon \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1) + 1$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x)$$

for all real numbers x.

(Serbia)

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \dots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \dots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

C4. Let n be a positive integer, and let A be a subset of $\{1, \dots, n\}$. An A-*partition* of n *into* k *parts* is a representation of n as a sum $n = a_1 + \cdots + a_k$, where the parts $a_1, \dots, a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1, a_2, \dots, a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r < k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

C5. Let r be a positive integer, and let $a_0, a_1, \dots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i.e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n + 1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a + c = b + d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x + y \leqslant n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

(Russia)

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k + 1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Geometry

G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

(Ukraine)

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

(Japan)

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x) + a}{b}\right) = f\left(\frac{x + a}{b}\right)$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu \rceil - b\lfloor a\nu \rfloor = m.$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x \rfloor$ and $\lceil x \rceil$ we denote the integer numbers such that $x - 1 < \lfloor x \rfloor \leqslant x$ and $x \leqslant \lceil x \rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solutions

Algebra

A1. Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = u_1 = v_0 = v_1 = 1$, and

$$u_{k+1} = u_k + a_k u_{k-1}, \qquad v_{k+1} = v_k + a_{n-k} v_{k-1} \qquad \text{for } k = 1, \ldots, n-1.$$

Prove that $u_n = v_n$.

(France)

Solution 1. We prove by induction on k that

$$u_k = \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{1}$$

Note that we have one trivial summand equal to 1 (which corresponds to $t = 0$ and the empty sequence, whose product is 1).

For $k = 0, 1$ the sum on the right-hand side only contains the empty product, so (1) holds due to $u_0 = u_1 = 1$. For $k \geqslant 1$, assuming the result is true for $0, 1, \ldots, k$, we have

$$\begin{aligned} u_{k+1} &= \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k-1,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t} \cdot a_k \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2,\\ k\notin\{i_1,\ldots,i_t\}}} a_{i_1} \ldots a_{i_t} + \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2,\\ k\in\{i_1,\ldots,i_t\}}} a_{i_1} \ldots a_{i_t} \\ &= \sum_{\substack{0<i_1<\ldots<i_t<k+1,\\ i_{j+1}-i_j \geqslant 2}} a_{i_1} \ldots a_{i_t}, \end{aligned}$$

as required.

Applying (1) to the sequence $b_1, \ldots, b_n$ given by $b_k = a_{n-k}$ for $1 \leqslant k \leqslant n$, we get

$$v_k = \sum_{\substack{0<i_1<\ldots<i_t<k,\\ i_{j+1}-i_j \geqslant 2}} b_{i_1} \ldots b_{i_t} = \sum_{\substack{n>i_1>\ldots>i_t>n-k,\\ i_j-i_{j+1} \geqslant 2}} a_{i_1} \ldots a_{i_t}. \tag{2}$$

For $k = n$ the expressions (1) and (2) coincide, so indeed $u_n = v_n$.

Solution 2. Define recursively a sequence of multivariate polynomials by

$$P_0 = P_1 = 1, \qquad P_{k+1}(x_1, \ldots, x_k) = P_k(x_1, \ldots, x_{k-1}) + x_k P_{k-1}(x_1, \ldots, x_{k-2}),$$

so P_n is a polynomial in $n - 1$ variables for each $n \geqslant 1$. Two easy inductive arguments show that

$$u_n = P_n(a_1, \ldots, a_{n-1}), \qquad v_n = P_n(a_{n-1}, \ldots, a_1),$$

so we need to prove $P_n(x_1,\dots,x_{n-1}) = P_n(x_{n-1},\dots,x_1)$ for every positive integer n. The cases $n = 1, 2$ are trivial, and the cases $n = 3, 4$ follow from $P_3(x, y) = 1 + x + y$ and $P_4(x, y, z) = 1 + x + y + z + xz$.

Now we proceed by induction, assuming that $n \geqslant 5$ and the claim hold for all smaller cases. Using $F(a, b)$ as an abbreviation for $P_{|a-b|+1}(x_a,\dots,x_b)$ (where the indices $a,\dots,b$ can be either in increasing or decreasing order),

$$\begin{aligned} F(n,1) &= F(n,2) + x_1F(n,3) = F(2,n) + x_1F(3,n) \\ &= \big(F(2,n-1) + x_nF(2,n-2)\big) + x_1\big(F(3,n-1) + x_nF(3,n-2)\big) \\ &= \big(F(n-1,2) + x_1F(n-1,3)\big) + x_n\big(F(n-2,2) + x_1F(n-2,3)\big) \\ &= F(n-1,1) + x_nF(n-2,1) = F(1,n-1) + x_nF(1,n-2) \\ &= F(1,n), \end{aligned}$$

as we wished to show.

Solution 3. Using matrix notation, we can rewrite the recurrence relation as

$$\begin{pmatrix} u_{k+1} \\ u_{k+1} - u_k \end{pmatrix} = \begin{pmatrix} u_k + a_ku_{k-1} \\ a_ku_{k-1} \end{pmatrix} = \begin{pmatrix} 1 + a_k & -a_k \\ a_k & -a_k \end{pmatrix} \begin{pmatrix} u_k \\ u_k - u_{k-1} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$, and similarly

$$(v_{k+1}; v_k - v_{k+1}) = \big(v_k + a_{n-k}v_{k-1}; -a_{n-k}v_{k-1}\big) = (v_k; v_{k-1} - v_k) \begin{pmatrix} 1 + a_{n-k} & -a_{n-k} \\ a_{n-k} & -a_{n-k} \end{pmatrix}$$

for $1 \leqslant k \leqslant n-1$. Hence, introducing the 2×2 matrices $A_k = \begin{pmatrix} 1 + a_k & -a_k \\ a_k & -a_k \end{pmatrix}$ we have

$$\begin{pmatrix} u_{k+1} \\ u_{k+1} - u_k \end{pmatrix} = A_k \begin{pmatrix} u_k \\ u_k - u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k - v_{k+1}) = (v_k; v_{k-1} - v_k)A_{n-k}.$$

for $1 \leqslant k \leqslant n-1$. Since $\binom{u_1}{u_1-u_0} = \binom{1}{0}$ and $(v_1; v_0 - v_1) = (1; 0)$, we get

$$\begin{pmatrix} u_n \\ u_n - u_{n-1} \end{pmatrix} = A_{n-1}A_{n-2}\cdots A_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad (v_n; v_{n-1} - v_n) = (1; 0) \cdot A_{n-1}A_{n-2}\cdots A_1.$$

It follows that

$$(u_n) = (1; 0)\begin{pmatrix} u_n \\ u_n - u_{n-1} \end{pmatrix} = (1; 0) \cdot A_{n-1}A_{n-2}\cdots A_1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n; v_{n-1} - v_n)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = (v_n).$$

Comment 1. These sequences are related to the Fibonacci sequence; when $a_1 = \cdots = a_{n-1} = 1$, we have $u_k = v_k = F_{k+1}$, the $(k+1)$st Fibonacci number. Also, for every positive integer k, the polynomial $P_k(x_1,\dots,x_{k-1})$ from Solution 2 is the sum of F_{k+1} monomials.

Comment 2. One may notice that the condition is equivalent to

$$\frac{u_{k+1}}{u_k} = 1 + \cfrac{a_k}{1 + \cfrac{a_{k-1}}{1 + \ldots + \cfrac{a_2}{1 + a_1}}} \qquad \text{and} \qquad \frac{v_{k+1}}{v_k} = 1 + \cfrac{a_{n-k}}{1 + \cfrac{a_{n-k+1}}{1 + \ldots + \cfrac{a_{n-2}}{1 + a_{n-1}}}}$$

so the problem claims that the corresponding continued fractions for u_n/u_{n-1} and v_n/v_{n-1} have the same numerator.

Comment 3. An alternative variant of the problem is the following.

> *Let n be a positive integer and let $a_1, \ldots, a_{n-1}$ be arbitrary real numbers. Define the sequences $u_0, \ldots, u_n$ and $v_0, \ldots, v_n$ inductively by $u_0 = v_0 = 0$, $u_1 = v_1 = 1$, and*
>
> $$u_{k+1} = a_k u_k + u_{k-1}, \qquad v_{k+1} = a_{n-k} v_k + v_{k-1} \qquad \textit{for } k = 1, \ldots, n-1.$$
>
> *Prove that $u_n = v_n$.*

All three solutions above can be reformulated to prove this statement; one may prove

$$u_n = v_n = \sum_{\substack{0=i_0<i_1<\ldots<i_t=n, \\ i_{j+1}-i_j \text{ is odd}}} a_{i_1} \ldots a_{i_{t-1}} \qquad \text{for } n > 0$$

or observe that

$$\begin{pmatrix} u_{k+1} \\ u_k \end{pmatrix} = \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_k \\ u_{k-1} \end{pmatrix} \quad \text{and} \quad (v_{k+1}; v_k) = (v_k; v_{k-1}) \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}.$$

Here we have

$$\frac{u_{k+1}}{u_k} = a_k + \cfrac{1}{a_{k-1} + \cfrac{1}{a_{k-2} + \ldots + \cfrac{1}{a_1}}} = [a_k; a_{k-1}, \ldots, a_1]$$

and

$$\frac{v_{k+1}}{v_k} = a_{n-k} + \cfrac{1}{a_{n-k+1} + \cfrac{1}{a_{n-k+2} + \ldots + \cfrac{1}{a_{n-1}}}} = [a_{n-k}; a_{n-k+1}, \ldots, a_{n-1}],$$

so this alternative statement is equivalent to the known fact that the continued fractions $[a_{n-1}; a_{n-2}, \ldots, a_1]$ and $[a_1; a_2, \ldots, a_{n-1}]$ have the same numerator.

A2. Prove that in any set of 2000 distinct real numbers there exist two pairs $a > b$ and $c > d$ with $a \neq c$ or $b \neq d$, such that

$$\left|\frac{a-b}{c-d} - 1\right| < \frac{1}{100000}.$$

(Lithuania)

Solution. For any set S of $n = 2000$ distinct real numbers, let $D_1 \leqslant D_2 \leqslant \cdots \leqslant D_m$ be the distances between them, displayed with their multiplicities. Here $m = n(n-1)/2$. By rescaling the numbers, we may assume that the smallest distance D_1 between two elements of S is $D_1 = 1$. Let $D_1 = 1 = y - x$ for $x, y \in S$. Evidently $D_m = v - u$ is the difference between the largest element v and the smallest element u of S.

If $D_{i+1}/D_i < 1 + 10^{-5}$ for some $i = 1, 2, \ldots, m-1$ then the required inequality holds, because $0 \leqslant D_{i+1}/D_i - 1 < 10^{-5}$. Otherwise, the reverse inequality

$$\frac{D_{i+1}}{D_i} \geqslant 1 + \frac{1}{10^5}$$

holds for each $i = 1, 2, \ldots, m-1$, and therefore

$$v - u = D_m = \frac{D_m}{D_1} = \frac{D_m}{D_{m-1}} \cdots \frac{D_3}{D_2} \cdot \frac{D_2}{D_1} \geqslant \left(1 + \frac{1}{10^5}\right)^{m-1}.$$

From $m - 1 = n(n-1)/2 - 1 = 1000 \cdot 1999 - 1 > 19 \cdot 10^5$, together with the fact that for all $n \geqslant 1$, $\left(1 + \frac{1}{n}\right)^n \geqslant 1 + \binom{n}{1} \cdot \frac{1}{n} = 2$, we get

$$\left(1 + \frac{1}{10^5}\right)^{19 \cdot 10^5} = \left(\left(1 + \frac{1}{10^5}\right)^{10^5}\right)^{19} \geqslant 2^{19} = 2^9 \cdot 2^{10} > 500 \cdot 1000 > 2 \cdot 10^5,$$

and so $v - u = D_m > 2 \cdot 10^5$.

Since the distance of x to at least one of the numbers u, v is at least $(u - v)/2 > 10^5$, we have

$$|x - z| > 10^5.$$

for some $z \in \{u, v\}$. Since $y - x = 1$, we have either $z > y > x$ (if $z = v$) or $y > x > z$ (if $z = u$). If $z > y > x$, selecting $a = z, b = y, c = z$ and $d = x$ (so that $b \neq d$), we obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{z-y}{z-x} - 1\right| = \left|\frac{x-y}{z-x}\right| = \frac{1}{z-x} < 10^{-5}.$$

Otherwise, if $y > x > z$, we may choose $a = y, b = z, c = x$ and $d = z$ (so that $a \neq c$), and obtain

$$\left|\frac{a-b}{c-d} - 1\right| = \left|\frac{y-z}{x-z} - 1\right| = \left|\frac{y-x}{x-z}\right| = \frac{1}{x-z} < 10^{-5}.$$

The desired result follows.

Comment. As the solution shows, the numbers 2000 and $\frac{1}{100000}$ appearing in the statement of the problem may be replaced by any $n \in \mathbb{Z}_{>0}$ and $\delta > 0$ satisfying

$$\delta(1 + \delta)^{n(n-1)/2 - 1} > 2.$$

A3. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f\colon \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the conditions

$$f(x)f(y) \geqslant f(xy), \tag{1}$$
$$f(x+y) \geqslant f(x) + f(y) \tag{2}$$

for all $x, y \in \mathbb{Q}_{>0}$. Given that $f(a) = a$ for some rational $a > 1$, prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

(Bulgaria)

Solution. Denote by $\mathbb{Z}_{>0}$ the set of positive integers.

Plugging $x = 1$, $y = a$ into (1) we get $f(1) \geqslant 1$. Next, by an easy induction on n we get from (2) that
$$f(nx) \geqslant nf(x) \quad \text{for all } n \in \mathbb{Z}_{>0} \text{ and } x \in \mathbb{Q}_{>0}. \tag{3}$$
In particular, we have
$$f(n) \geqslant nf(1) \geqslant n \quad \text{for all } n \in \mathbb{Z}_{>0}. \tag{4}$$
From (1) again we have $f(m/n)f(n) \geqslant f(m)$, so $f(q) > 0$ for all $q \in \mathbb{Q}_{>0}$.

Now, (2) implies that f is strictly increasing; this fact together with (4) yields

$$f(x) \geqslant f(\lfloor x \rfloor) \geqslant \lfloor x \rfloor > x - 1 \quad \text{for all } x \geqslant 1.$$

By an easy induction we get from (1) that $f(x)^n \geqslant f(x^n)$, so

$$f(x)^n \geqslant f(x^n) > x^n - 1 \quad \Longrightarrow \quad f(x) \geqslant \sqrt[n]{x^n - 1} \quad \text{for all } x > 1 \text{ and } n \in \mathbb{Z}_{>0}.$$

This yields
$$f(x) \geqslant x \quad \text{for every } x > 1. \tag{5}$$
(Indeed, if $x > y > 1$ then $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^n) > n(x-y)$, so for a large n we have $x^n - 1 > y^n$ and thus $f(x) > y$.)

Now, (1) and (5) give $a^n = f(a)^n \geqslant f(a^n) \geqslant a^n$, so $f(a^n) = a^n$. Now, for $x > 1$ let us choose $n \in \mathbb{Z}_{>0}$ such that $a^n - x > 1$. Then by (2) and (5) we get

$$a^n = f(a^n) \geqslant f(x) + f(a^n - x) \geqslant x + (a^n - x) = a^n$$

and therefore $f(x) = x$ for $x > 1$. Finally, for every $x \in \mathbb{Q}_{>0}$ and every $n \in \mathbb{Z}_{>0}$, from (1) and (3) we get
$$nf(x) = f(n)f(x) \geqslant f(nx) \geqslant nf(x),$$
which gives $f(nx) = nf(x)$. Therefore $f(m/n) = f(m)/n = m/n$ for all $m, n \in \mathbb{Z}_{>0}$.

Comment. The condition $f(a) = a > 1$ is essential. Indeed, for $b \geqslant 1$ the function $f(x) = bx^2$ satisfies (1) and (2) for all $x, y \in \mathbb{Q}_{>0}$, and it has a unique fixed point $1/b \leqslant 1$.

A4. Let n be a positive integer, and consider a sequence $a_1, a_2, \ldots, a_n$ of positive integers. Extend it periodically to an infinite sequence $a_1, a_2, \ldots$ by defining $a_{n+i} = a_i$ for all $i \geqslant 1$. If

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n \leqslant a_1 + n \tag{1}$$

and

$$a_{a_i} \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n, \tag{2}$$

prove that

$$a_1 + \cdots + a_n \leqslant n^2.$$

(Germany)

Solution 1. First, we claim that

$$a_i \leqslant n + i - 1 \qquad \text{for } i = 1, 2, \ldots, n. \tag{3}$$

Assume contrariwise that i is the smallest counterexample. From $a_n \geqslant a_{n-1} \geqslant \cdots \geqslant a_i \geqslant n + i$ and $a_{a_i} \leqslant n + i - 1$, taking into account the periodicity of our sequence, it follows that

$$a_i \text{ cannot be congruent to } i,\ i+1, \ldots, n-1, \text{or } n \pmod{n}. \tag{4}$$

Thus our assumption that $a_i \geqslant n + i$ implies the stronger statement that $a_i \geqslant 2n + 1$, which by $a_1 + n \geqslant a_n \geqslant a_i$ gives $a_1 \geqslant n + 1$. The minimality of i then yields $i = 1$, and (4) becomes contradictory. This establishes our first claim.

In particular we now know that $a_1 \leqslant n$. If $a_n \leqslant n$, then $a_1 \leqslant \cdots \leqslant \cdots a_n \leqslant n$ and the desired inequality holds trivially. Otherwise, consider the number t with $1 \leqslant t \leqslant n - 1$ such that

$$a_1 \leqslant a_2 \leqslant \ldots \leqslant a_t \leqslant n < a_{t+1} \leqslant \ldots \leqslant a_n. \tag{5}$$

Since $1 \leqslant a_1 \leqslant n$ and $a_{a_1} \leqslant n$ by (2), we have $a_1 \leqslant t$ and hence $a_n \leqslant n + t$. Therefore if for each positive integer i we let b_i be the number of indices $j \in \{t+1, \ldots, n\}$ satisfying $a_j \geqslant n + i$, we have

$$b_1 \geqslant b_2 \geqslant \ldots \geqslant b_t \geqslant b_{t+1} = 0.$$

Next we claim that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$. Indeed, by $n + i - 1 \geqslant a_{a_i}$ and $a_i \leqslant n$, each j with $a_j \geqslant n + i$ (thus $a_j > a_{a_i}$) belongs to $\{a_i + 1, \ldots, n\}$, and for this reason $b_i \leqslant n - a_i$.

It follows from the definition of the b_is and (5) that

$$a_{t+1} + \ldots + a_n \leqslant n(n - t) + b_1 + \ldots + b_t.$$

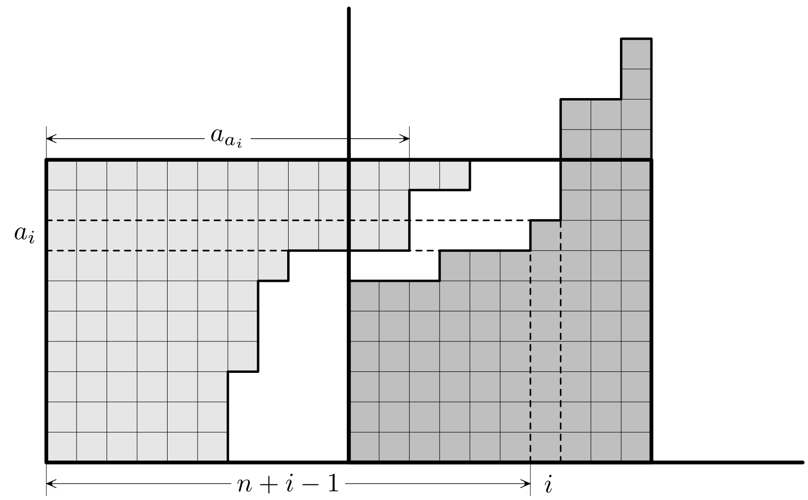
Adding $a_1 + \ldots + a_t$ to both sides and using that $a_i + b_i \leqslant n$ for $1 \leqslant i \leqslant t$, we get

$$a_1 + a_2 + \cdots + a_n \leqslant n(n - t) + nt = n^2$$

as we wished to prove.

Solution 2. In the first quadrant of an infinite grid, consider the increasing "staircase" obtained by shading in dark the bottom a_i cells of the ith column for $1 \leqslant i \leqslant n$. We will prove that there are at most n^2 dark cells.

To do it, consider the $n \times n$ square S in the first quadrant with a vertex at the origin. Also consider the $n \times n$ square directly to the left of S. Starting from its lower left corner, shade in light the leftmost a_j cells of the jth row for $1 \leqslant j \leqslant n$. Equivalently, the light shading is obtained by reflecting the dark shading across the line $x = y$ and translating it n units to the left. The figure below illustrates this construction for the sequence $6, 6, 6, 7, 7, 7, 8, 12, 12, 14$.



We claim that there is no cell in S which is both dark and light. Assume, contrariwise, that there is such a cell in column i. Consider the highest dark cell in column i which is inside S. Since it is above a light cell and inside S, it must be light as well. There are two cases:

Case 1. $a_i \leqslant n$

If $a_i \leqslant n$ then this dark and light cell is (i, a_i), as highlighted in the figure. However, this is the $(n+i)$-th cell in row a_i, and we only shaded $a_{a_i} < n + i$ light cells in that row, a contradiction.

Case 2. $a_i \geqslant n + 1$

If $a_i \geqslant n + 1$, this dark and light cell is (i, n). This is the $(n + i)$-th cell in row n and we shaded $a_n \leqslant a_1 + n$ light cells in this row, so we must have $i \leqslant a_1$. But $a_1 \leqslant a_{a_1} \leqslant n$ by (1) and (2), so $i \leqslant a_1$ implies $a_i \leqslant a_{a_1} \leqslant n$, contradicting our assumption.

We conclude that there are no cells in S which are both dark and light. It follows that the number of shaded cells in S is at most n^2.

Finally, observe that if we had a light cell to the right of S, then by symmetry we would have a dark cell above S, and then the cell (n, n) would be dark and light. It follows that the number of light cells in S equals the number of dark cells outside of S, and therefore the number of shaded cells in S equals $a_1 + \cdots + a_n$. The desired result follows.

Solution 3. As in Solution 1, we first establish that $a_i \leqslant n + i - 1$ for $1 \leqslant i \leqslant n$. Now define $c_i = \max(a_i, i)$ for $1 \leqslant i \leqslant n$ and extend the sequence $c_1, c_2, \ldots$ periodically modulo n. We claim that this sequence also satisfies the conditions of the problem.

For $1 \leqslant i < j \leqslant n$ we have $a_i \leqslant a_j$ and $i < j$, so $c_i \leqslant c_j$. Also $a_n \leqslant a_1 + n$ and $n < 1 + n$ imply $c_n \leqslant c_1 + n$. Finally, the definitions imply that $c_{c_i} \in \{a_{a_i}, a_i, a_i - n, i\}$ so $c_{c_i} \leqslant n + i - 1$ by (2) and (3). This establishes (1) and (2) for $c_1, c_2, \ldots$.

Our new sequence has the additional property that

$$c_i \geqslant i \qquad \text{for } i = 1, 2, \ldots, n, \tag{6}$$

which allows us to construct the following visualization: Consider n equally spaced points on a circle, sequentially labelled 1, 2, $\ldots$, n $\pmod n$, so point k is also labelled $n+k$. We draw arrows from vertex i to vertices $i+1, \ldots, c_i$ for $1 \leqslant i \leqslant n$, keeping in mind that $c_i \geqslant i$ by (6). Since $c_i \leqslant n+i-1$ by (3), no arrow will be drawn twice, and there is no arrow from a vertex to itself. The total number of arrows is

$$\text{number of arrows } = \sum_{i=1}^{n}(c_i - i) = \sum_{i=1}^{n} c_i - \binom{n+1}{2}$$

Now we show that we never draw both arrows $i \to j$ and $j \to i$ for $1 \leqslant i < j \leqslant n$. Assume contrariwise. This means, respectively, that

$$i < j \leqslant c_i \qquad \text{and} \qquad j < n+i \leqslant c_j.$$

We have $n+i \leqslant c_j \leqslant c_1 + n$ by (1), so $i \leqslant c_1$. Since $c_1 \leqslant n$ by (3), this implies that $c_i \leqslant c_{c_1} \leqslant n$ using (1) and (3). But then, using (1) again, $j \leqslant c_i \leqslant n$ implies $c_j \leqslant c_{c_i}$, which combined with $n+i \leqslant c_j$ gives us that $n+i \leqslant c_{c_i}$. This contradicts (2).

This means that the number of arrows is at most $\binom{n}{2}$, which implies that

$$\sum_{i=1}^{n} c_i \leqslant \binom{n}{2} + \binom{n+1}{2} = n^2.$$

Recalling that $a_i \leqslant c_i$ for $1 \leqslant i \leqslant n$, the desired inequality follows.

Comment 1. We sketch an alternative proof by induction. Begin by verifying the initial case $n = 1$ and the simple cases when $a_1 = 1$, $a_1 = n$, or $a_n \leqslant n$. Then, as in Solution 1, consider the index t such that $a_1 \leqslant \cdots \leqslant a_t \leqslant n < a_{t+1} \leqslant \cdots \leqslant a_n$. Observe again that $a_1 \leqslant t$. Define the sequence $d_1, \ldots, d_{n-1}$ by

$$d_i = \begin{cases} a_{i+1} - 1 & \text{if } i \leqslant t-1 \\ a_{i+1} - 2 & \text{if } i \geqslant t \end{cases}$$

and extend it periodically modulo $n-1$. One may verify that this sequence also satisfies the hypotheses of the problem. The induction hypothesis then gives $d_1 + \cdots + d_{n-1} \leqslant (n-1)^2$, which implies that

$$\sum_{i=1}^{n} a_i = a_1 + \sum_{i=2}^{t}(d_{i-1}+1) + \sum_{i=t+1}^{n}(d_{i-1}+2) \leqslant t + (t-1) + 2(n-t) + (n-1)^2 = n^2.$$

Comment 2. One unusual feature of this problem is that there are many different sequences for which equality holds. The discovery of such *optimal sequences* is not difficult, and it is useful in guiding the steps of a proof.

In fact, Solution 2 gives a complete description of the optimal sequences. Start with any lattice path P from the lower left to the upper right corner of the $n \times n$ square S using only steps up and right, such that the total number of steps along the left and top edges of S is at least n. Shade the cells of S below P dark, and the cells of S above P light. Now reflect the light shape across the line $x = y$ and shift it up n units, and shade it dark. As Solution 2 shows, the dark region will then correspond to an optimal sequence, and every optimal sequence arises in this way.

A5. Let $\mathbb{Z}_{\geqslant 0}$ be the set of all nonnegative integers. Find all the functions $f\colon \mathbb{Z}_{\geqslant 0} \rightarrow \mathbb{Z}_{\geqslant 0}$ satisfying the relation

$$f(f(f(n))) = f(n+1)+1 \tag{$*$}$$

for all $n \in \mathbb{Z}_{\geqslant 0}$.

(Serbia)

Answer. There are two such functions: $f(n) = n+1$ for all $n \in \mathbb{Z}_{\geqslant 0}$, and

$$f(n) = \begin{cases} n+1, & n \equiv 0 \pmod 4 \text{ or } n \equiv 2 \pmod 4, \\ n+5, & n \equiv 1 \pmod 4, \\ n-3, & n \equiv 3 \pmod 4 \end{cases} \qquad \text{for all } n \in \mathbb{Z}_{\geqslant 0}. \tag{1}$$

Throughout all the solutions, we write $h^k(x)$ to abbreviate the kth iteration of function h, so h^0 is the identity function, and $h^k(x) = \underbrace{h(\ldots h}_{k \text{ times}}(x)\ldots))$ for $k \geqslant 1$.

Solution 1. To start, we get from $(*)$ that

$$f^4(n) = f(f^3(n)) = f\big(f(n+1)+1\big) \quad \text{and} \quad f^4(n+1) = f^3(f(n+1)) = f\big(f(n+1)+1\big)+1,$$

thus

$$f^4(n)+1 = f^4(n+1). \tag{2}$$

I. Let us denote by R_i the range of f^i; note that $R_0 = \mathbb{Z}_{\geqslant 0}$ since f^0 is the identity function. Obviously, $R_0 \supseteq R_1 \supseteq \ldots$. Next, from (2) we get that if $a \in R_4$ then also $a+1 \in R_4$. This implies that $\mathbb{Z}_{\geqslant 0} \backslash R_4$ — and hence $\mathbb{Z}_{\geqslant 0} \backslash R_1$ — is finite. In particular, R_1 is unbounded.

Assume that $f(m) = f(n)$ for some distinct m and n. Then from $(*)$ we obtain $f(m+1) = f(n+1)$; by an easy induction we then get that $f(m+c) = f(n+c)$ for every $c \geqslant 0$. So the function $f(k)$ is periodic with period $|m-n|$ for $k \geqslant m$, and thus R_1 should be bounded, which is false. So, f is injective.

II. Denote now $S_i = R_{i-1} \backslash R_i$; all these sets are finite for $i \leqslant 4$. On the other hand, by the injectivity we have $n \in S_i \iff f(n) \in S_{i+1}$. By the injectivity again, f implements a bijection between S_i and S_{i+1}, thus $|S_1| = |S_2| = \ldots$; denote this common cardinality by k. If $0 \in R_3$ then $0 = f(f(f(n)))$ for some n, thus from $(*)$ we get $f(n+1) = -1$ which is impossible. Therefore $0 \in R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$, thus $k \geqslant 1$.

Next, let us describe the elements b of $R_0 \backslash R_3 = S_1 \cup S_2 \cup S_3$. We claim that each such element satisfies at least one of three conditions (i) $b = 0$, (ii) $b = f(0)+1$, and (iii) $b-1 \in S_1$. Otherwise $b-1 \in \mathbb{Z}_{\geqslant 0}$, and there exists some $n > 0$ such that $f(n) = b-1$; but then $f^3(n-1) = f(n)+1 = b$, so $b \in R_3$.

This yields

$$3k = |S_1 \cup S_2 \cup S_3| \leqslant 1+1+|S_1| = k+2,$$

or $k \leqslant 1$. Therefore $k = 1$, and the inequality above comes to equality. So we have $S_1 = \{a\}$, $S_2 = \{f(a)\}$, and $S_3 = \{f^2(a)\}$ for some $a \in \mathbb{Z}_{\geqslant 0}$, and each one of the three options (i), (ii), and (iii) should be realized exactly once, which means that

$$\{a, f(a), f^2(a)\} = \{0, a+1, f(0)+1\}. \tag{3}$$

III. From (3), we get $a+1 \in \{f(a), f^2(a)\}$ (the case $a+1=a$ is impossible). If $a+1=f^2(a)$ then we have $f(a+1)=f^3(a)=f(a+1)+1$ which is absurd. Therefore

$$f(a)=a+1. \tag{4}$$

Next, again from (3) we have $0 \in \{a, f^2(a)\}$. Let us consider these two cases separately.

Case 1. Assume that $a=0$, then $f(0)=f(a)=a+1=1$. Also from (3) we get $f(1)=f^2(a)=f(0)+1=2$. Now, let us show that $f(n)=n+1$ by induction on n; the base cases $n \leqslant 1$ are established. Next, if $n \geqslant 2$ then the induction hypothesis implies

$$n+1=f(n-1)+1=f^3(n-2)=f^2(n-1)=f(n),$$

establishing the step. In this case we have obtained the first of two answers; checking that is satisfies $(*)$ is straightforward.

Case 2. Assume now that $f^2(a)=0$; then by (3) we get $a=f(0)+1$. By (4) we get $f(a+1)=f^2(a)=0$, then $f(0)=f^3(a)=f(a+1)+1=1$, hence $a=f(0)+1=2$ and $f(2)=3$ by (4). To summarize,

$$f(0)=1, \quad f(2)=3, \quad f(3)=0.$$

Now let us prove by induction on m that (1) holds for all $n=4k, 4k+2, 4k+3$ with $k \leqslant m$ and for all $n=4k+1$ with $k<m$. The base case $m=0$ is established above. For the step, assume that $m \geqslant 1$. From $(*)$ we get $f^3(4m-3)=f(4m-2)+1=4m$. Next, by (2) we have

$$f(4m)=f^4(4m-3)=f^4(4m-4)+1=f^3(4m-3)+1=4m+1.$$

Then by the induction hypothesis together with $(*)$ we successively obtain

$$\begin{aligned}
f(4m-3)&=f^3(4m-1)=f(4m)+1=4m+2,\\
f(4m+2)&=f^3(4m-4)=f(4m-3)+1=4m+3,\\
f(4m+3)&=f^3(4m-3)=f(4m-2)+1=4m,
\end{aligned}$$

thus finishing the induction step.

Finally, it is straightforward to check that the constructed function works:

$$\begin{aligned}
f^3(4k)&=4k+7=f(4k+1)+1, &\qquad f^3(4k+1)&=4k+4=f(4k+2)+1,\\
f^3(4k+2)&=4k+1=f(4k+3)+1, &\qquad f^3(4k+3)&=4k+6=f(4k+4)+1.
\end{aligned}$$

Solution 2. I. For convenience, let us introduce the function $g(n)=f(n)+1$. Substituting $f(n)$ instead of n into $(*)$ we obtain

$$f^4(n)=f\big(f(n)+1\big)+1, \quad \text{or} \quad f^4(n)=g^2(n). \tag{5}$$

Applying f to both parts of $(*)$ and using (5) we get

$$f^4(n)+1=f\big(f(n+1)+1\big)+1=f^4(n+1). \tag{6}$$

Thus, if $g^2(0)=f^4(0)=c$ then an easy induction on n shows that

$$g^2(n)=f^4(n)=n+c, \qquad n \in \mathbb{Z}_{\geqslant 0}. \tag{7}$$

This relation implies that both f and g are injective: if, say, $f(m) = f(n)$ then $m + c = f^4(m) = f^4(n) = n + c$. Next, since $g(n) \geqslant 1$ for every n, we have $c = g^2(0) \geqslant 1$. Thus from (7) again we obtain $f(n) \neq n$ and $g(n) \neq n$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

II. Next, application of f and g to (7) yields

$$f(n+c) = f^5(n) = f^4(f(n)) = f(n) + c \quad \text{and} \quad g(n+c) = g^3(n) = g(n) + c. \tag{8}$$

In particular, this means that if $m \equiv n \pmod c$ then $f(m) \equiv f(n) \pmod c$. Conversely, if $f(m) \equiv f(n) \pmod c$ then we get $m + c = f^4(m) \equiv f^4(n) = n + c \pmod c$. Thus,

$$m \equiv n \pmod c \iff f(m) \equiv f(n) \pmod c \iff g(m) \equiv g(n) \pmod c. \tag{9}$$

Now, let us introduce the function $\delta(n) = f(n) - n = g(n) - n - 1$. Set

$$S = \sum_{n=0}^{c-1} \delta(n).$$

Using (8), we get that for every complete residue system $n_1, \ldots, n_c$ modulo c we also have

$$S = \sum_{i=1}^{c} \delta(n_i).$$

By (9), we get that $\{f^k(n) \colon n = 0, \ldots, c-1\}$ and $\{g^k(n) \colon n = 0, \ldots, c-1\}$ are complete residue systems modulo c for all k. Thus we have

$$c^2 = \sum_{n=0}^{c-1} \big(f^4(n) - n\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \big(f^{k+1}(n) - f^k(n)\big) = \sum_{k=0}^{3} \sum_{n=0}^{c-1} \delta(f^k(n)) = 4S$$

and similarly

$$c^2 = \sum_{n=0}^{c-1} \big(g^2(n) - n\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(g^{k+1}(n) - g^k(n)\big) = \sum_{k=0}^{1} \sum_{n=0}^{c-1} \big(\delta(g^k(n)) + 1\big) = 2S + 2c.$$

Therefore $c^2 = 4S = 2 \cdot 2S = 2(c^2 - 2c)$, or $c^2 = 4c$. Since $c \neq 0$, we get $c = 4$. Thus, in view of (8) it is sufficient to determine the values of f on the numbers $0, 1, 2, 3$.

III. Let $d = g(0) \geqslant 1$. Then $g(d) = g^2(0) = 0 + c = 4$. Now, if $d \geqslant 4$, then we would have $g(d-4) = g(d) - 4 = 0$ which is impossible. Thus $d \in \{1, 2, 3\}$. If $d = 1$ then we have $f(0) = g(0) - 1 = 0$ which is impossible since $f(n) \neq n$ for all n. If $d = 3$ then $g(3) = g^2(0) = 4$ and hence $f(3) = 3$ which is also impossible. Thus $g(0) = 2$ and hence $g(2) = g^2(0) = 4$.

Next, if $g(1) = 1 + 4k$ for some integer k, then $5 = g^2(1) = g(1 + 4k) = g(1) + 4k = 1 + 8k$ which is impossible. Thus, since $\{g(n) \colon n = 0, 1, 2, 3\}$ is a complete residue system modulo 4, we get $g(1) = 3 + 4k$ and hence $g(3) = g^2(1) - 4k = 5 - 4k$, leading to $k = 0$ or $k = 1$. So, we obtain iether

$$f(0) = 1,\ f(1) = 2,\ f(2) = 3,\ f(3) = 4, \quad \text{or} \quad f(0) = 1,\ f(1) = 6,\ f(2) = 3,\ f(3) = 0,$$

thus arriving to the two functions listed in the answer.

Finally, one can check that these two function work as in Solution 1. One may simplify the checking by noticing that (8) allows us to reduce it to $n = 0, 1, 2, 3$.

A6. Let $m \neq 0$ be an integer. Find all polynomials $P(x)$ with real coefficients such that

$$(x^3 - mx^2 + 1)P(x+1) + (x^3 + mx^2 + 1)P(x-1) = 2(x^3 - mx + 1)P(x) \qquad (1)$$

for all real numbers x.

(Serbia)

Answer. $P(x) = tx$ for any real number t.

Solution. Let $P(x) = a_n x^n + \cdots + a_0 x^0$ with $a_n \neq 0$. Comparing the coefficients of x^{n+1} on both sides gives $a_n(n-2m)(n-1) = 0$, so $n = 1$ or $n = 2m$.

If $n = 1$, one easily verifies that $P(x) = x$ is a solution, while $P(x) = 1$ is not. Since the given condition is linear in P, this means that the linear solutions are precisely $P(x) = tx$ for $t \in \mathbb{R}$.

Now assume that $n = 2m$. The polynomial $xP(x+1) - (x+1)P(x) = (n-1)a_n x^n + \cdots$ has degree n, and therefore it has at least one (possibly complex) root r. If $r \notin \{0, -1\}$, define $k = P(r)/r = P(r+1)/(r+1)$. If $r = 0$, let $k = P(1)$. If $r = -1$, let $k = -P(-1)$. We now consider the polynomial $S(x) = P(x) - kx$. It also satisfies (1) because $P(x)$ and kx satisfy it. Additionally, it has the useful property that r and $r+1$ are roots.

Let $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$. Plugging in $x = s$ into (1) implies that:

If $s-1$ and s are roots of S and s is not a root of A, then $s+1$ is a root of S.

If s and $s+1$ are roots of S and s is not a root of B, then $s-1$ is a root of S.

Let $a \geqslant 0$ and $b \geqslant 1$ be such that $r-a, r-a+1, \ldots, r, r+1, \ldots, r+b-1, r+b$ are roots of S, while $r-a-1$ and $r+b+1$ are not. The two statements above imply that $r-a$ is a root of B and $r+b$ is a root of A.

Since $r-a$ is a root of $B(x)$ and of $A(x+a+b)$, it is also a root of their greatest common divisor $C(x)$ as integer polynomials. If $C(x)$ was a non-trivial divisor of $B(x)$, then B would have a rational root α. Since the first and last coefficients of B are 1, α can only be 1 or -1; but $B(-1) = m > 0$ and $B(1) = m+2 > 0$ since $n = 2m$.

Therefore $B(x) = A(x+a+b)$. Writing $c = a+b \geqslant 1$ we compute

$$0 = A(x+c) - B(x) = (3c-2m)x^2 + c(3c-2m)x + c^2(c-m).$$

Then we must have $3c - 2m = c - m = 0$, which gives $m = 0$, a contradiction. We conclude that $f(x) = tx$ is the only solution.

Solution 2. Multiplying (1) by x, we rewrite it as

$$x(x^3 - mx^2 + 1)P(x+1) + x(x^3 + mx^2 + 1)P(x-1) = \left[(x+1) + (x-1)\right](x^3 - mx + 1)P(x).$$

After regrouping, it becomes

$$(x^3 - mx^2 + 1)Q(x) = (x^3 + mx^2 + 1)Q(x-1), \qquad (2)$$

where $Q(x) = xP(x+1) - (x+1)P(x)$. If $\deg P \geqslant 2$ then $\deg Q = \deg P$, so $Q(x)$ has a finite multiset of complex roots, which we denote R_Q. Each root is taken with its multiplicity. Then the multiset of complex roots of $Q(x-1)$ is $R_Q + 1 = \{z + 1 : z \in R_Q\}$.

Let $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ be the multisets of roots of the polynomials $A(x) = x^3 - mx^2 + 1$ and $B(x) = x^3 + mx^2 + 1$, respectively. From (2) we get the equality of multisets

$$\{x_1, x_2, x_3\} \cup R_Q = \{y_1, y_2, y_3\} \cup (R_Q + 1).$$

For every $r \in R_Q$, since $r + 1$ is in the set of the right hand side, we must have $r + 1 \in R_Q$ or $r + 1 = x_i$ for some i. Similarly, since r is in the set of the left hand side, either $r - 1 \in R_Q$ or $r = y_i$ for some i. This implies that, possibly after relabelling y_1, y_2, y_3, all the roots of (2) may be partitioned into three chains of the form $\{y_i,\ y_i + 1, \ldots, y_i + k_i = x_i\}$ for $i = 1, 2, 3$ and some integers $k_1, k_2, k_3 \geqslant 0$.

Now we analyze the roots of the polynomial $A_a(x) = x^3 + ax^2 + 1$. Using calculus or elementary methods, we find that the local extrema of $A_a(x)$ occur at $x = 0$ and $x = -2a/3$; their values are $A_a(0) = 1 > 0$ and $A_a(-2a/3) = 1 + 4a^3/27$, which is positive for integers $a \geqslant -1$ and negative for integers $a \leqslant -2$. So when $a \in \mathbb{Z}$, A_a has three real roots if $a \leqslant -2$ and one if $a \geqslant -1$.

Now, since $y_i - x_i \in \mathbb{Z}$ for $i = 1, 2, 3$, the cubics A_m and A_{-m} must have the same number of real roots. The previous analysis then implies that $m = 1$ or $m = -1$. Therefore the real root α of $A_1(x) = x^3 + x^2 + 1$ and the real root β of $A_{-1}(x) = x^3 - x^2 + 1$ must differ by an integer. But this is impossible, because $A_1\left(-\frac{3}{2}\right) = -\frac{1}{8}$ and $A_1(-1) = 1$ so $-1.5 < \alpha < -1$, while $A_{-1}(-1) = -1$ and $A_{-1}\left(-\frac{1}{2}\right) = \frac{5}{8}$, so $-1 < \beta < -0.5$.

It follows that $\deg P \leqslant 1$. Then, as shown in Solution 1, we conclude that the solutions are $P(x) = tx$ for all real numbers t.

Combinatorics

C1. Let n be a positive integer. Find the smallest integer k with the following property: Given any real numbers $a_1, \ldots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leqslant a_i \leqslant 1$ for $i = 1, 2, \ldots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.

(Poland)

Answer. $k = 2n - 1$.

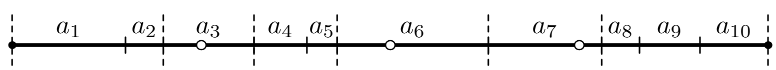
Solution 1. If $d = 2n-1$ and $a_1 = \cdots = a_{2n-1} = n/(2n-1)$, then each group in such a partition can contain at most one number, since $2n/(2n-1) > 1$. Therefore $k \geqslant 2n-1$. It remains to show that a suitable partition into $2n-1$ groups always exists.

We proceed by induction on d. For $d \leqslant 2n-1$ the result is trivial. If $d \geqslant 2n$, then since

$$(a_1 + a_2) + \ldots + (a_{2n-1} + a_{2n}) \leqslant n$$

we may find two numbers a_i, a_{i+1} such that $a_i + a_{i+1} \leqslant 1$. We "merge" these two numbers into one new number $a_i + a_{i+1}$. By the induction hypothesis, a suitable partition exists for the $d-1$ numbers $a_1, \ldots, a_{i-1}, a_i + a_{i+1}, a_{i+2}, \ldots, a_d$. This induces a suitable partition for $a_1, \ldots, a_d$.

Solution 2. We will show that it is even possible to split the sequence $a_1, \ldots, a_d$ into $2n-1$ *contiguous* groups so that the sum of the numbers in each groups does not exceed 1. Consider a segment S of length n, and partition it into segments $S_1, \ldots, S_d$ of lengths $a_1, \ldots, a_d$, respectively, as shown below. Consider a second partition of S into n equal parts by $n-1$ "empty dots".



Assume that the $n-1$ empty dots are in segments $S_{i_1}, \ldots, S_{i_{n-1}}$. (If a dot is on the boundary of two segments, we choose the right segment). These $n-1$ segments are distinct because they have length at most 1. Consider the partition:

$$\{a_1, \ldots, a_{i_1-1}\}, \{a_{i_1}\}, \{a_{i_1+1}, \ldots, a_{i_2-1}\}, \{a_{i_2}\}, \ldots \{a_{i_{n-1}}\}, \{a_{i_{n-1}+1}, \ldots, a_d\}.$$

In the example above, this partition is $\{a_1, a_2\}, \{a_3\}, \{a_4, a_5\}, \{a_6\}, \varnothing, \{a_7\}, \{a_8, a_9, a_{10}\}$. We claim that in this partition, the sum of the numbers in this group is at most 1.

For the sets $\{a_{i_t}\}$ this is obvious since $a_{i_t} \leqslant 1$. For the sets $\{a_{i_t} + 1, \ldots, a_{i_{t+1}-1}\}$ this follows from the fact that the corresponding segments lie between two neighboring empty dots, or between an endpoint of S and its nearest empty dot. Therefore the sum of their lengths cannot exceed 1.

Solution 3. First put all numbers greater than $\frac{1}{2}$ in their own groups. Then, form the remaining groups as follows: For each group, add new a_is one at a time until their sum exceeds $\frac{1}{2}$. Since the last summand is at most $\frac{1}{2}$, this group has sum at most 1. Continue this procedure until we have used all the a_is. Notice that the last group may have sum less than $\frac{1}{2}$. If the sum of the numbers in the last two groups is less than or equal to 1, we merge them into one group. In the end we are left with m groups. If $m = 1$ we are done. Otherwise the first $m-2$ have sums greater than $\frac{1}{2}$ and the last two have total sum greater than 1. Therefore $n > (m-2)/2 + 1$ so $m \leqslant 2n-1$ as desired.

Comment 1. The original proposal asked for the minimal value of k when $n = 2$.

Comment 2. More generally, one may ask the same question for real numbers between 0 and 1 whose sum is a real number r. In this case the smallest value of k is $k = \lceil 2r \rceil - 1$, as Solution 3 shows.

Solutions 1 and 2 lead to the slightly weaker bound $k \leqslant 2\lceil r \rceil - 1$. This is actually the optimal bound for partitions into *consecutive* groups, which are the ones contemplated in these two solutions. To see this, assume that r is not an integer and let $c = (r + 1 - \lceil r \rceil)/(1 + \lceil r \rceil)$. One easily checks that $0 < c < \frac{1}{2}$ and $\lceil r \rceil(2c) + (\lceil r \rceil - 1)(1 - c) = r$, so the sequence

$$2c,\ 1 - c,\ 2c,\ 1 - c,\ \ldots,\ 1 - c,\ 2c$$

of $2\lceil r \rceil - 1$ numbers satisfies the given conditions. For this sequence, the only suitable partition into consecutive groups is the trivial partition, which requires $2\lceil r \rceil - 1$ groups.

C2. In the plane, 2013 red points and 2014 blue points are marked so that no three of the marked points are collinear. One needs to draw k lines not passing through the marked points and dividing the plane into several regions. The goal is to do it in such a way that no region contains points of both colors.

Find the minimal value of k such that the goal is attainable for every possible configuration of 4027 points.

(Australia)

Answer. $k = 2013$.

Solution 1. Firstly, let us present an example showing that $k \geqslant 2013$. Mark 2013 red and 2013 blue points on some circle alternately, and mark one more blue point somewhere in the plane. The circle is thus split into 4026 arcs, each arc having endpoints of different colors. Thus, if the goal is reached, then each arc should intersect some of the drawn lines. Since any line contains at most two points of the circle, one needs at least $4026/2 = 2013$ lines.

It remains to prove that one can reach the goal using 2013 lines. First of all, let us mention that for every two points A and B having the same color, one can draw two lines separating these points from all other ones. Namely, it suffices to take two lines parallel to AB and lying on different sides of AB sufficiently close to it: the only two points between these lines will be A and B.

Now, let P be the convex hull of all marked points. Two cases are possible.

Case 1. Assume that P has a red vertex A. Then one may draw a line separating A from all the other points, pair up the other 2012 red points into 1006 pairs, and separate each pair from the other points by two lines. Thus, 2013 lines will be used.

Case 2. Assume now that all the vertices of P are blue. Consider any two consecutive vertices of P, say A and B. One may separate these two points from the others by a line parallel to AB. Then, as in the previous case, one pairs up all the other 2012 blue points into 1006 pairs, and separates each pair from the other points by two lines. Again, 2013 lines will be used.

Comment 1. Instead of considering the convex hull, one may simply take a line containing two marked points A and B such that all the other marked points are on one side of this line. If one of A and B is red, then one may act as in Case 1; otherwise both are blue, and one may act as in Case 2.

Solution 2. Let us present a different proof of the fact that $k = 2013$ suffices. In fact, we will prove a more general statement:

If n points in the plane, no three of which are collinear, are colored in red and blue arbitrarily, then it suffices to draw $\lfloor n/2 \rfloor$ lines to reach the goal.

We proceed by induction on n. If $n \leqslant 2$ then the statement is obvious. Now assume that $n \geqslant 3$, and consider a line ℓ containing two marked points A and B such that all the other marked points are on one side of ℓ; for instance, any line containing a side of the convex hull works.

Remove for a moment the points A and B. By the induction hypothesis, for the remaining configuration it suffices to draw $\lfloor n/2 \rfloor - 1$ lines to reach the goal. Now return the points A and B back. Three cases are possible.

Case 1. If A and B have the same color, then one may draw a line parallel to ℓ and separating A and B from the other points. Obviously, the obtained configuration of $\lfloor n/2 \rfloor$ lines works.

Case 2. If A and B have different colors, but they are separated by some drawn line, then again the same line parallel to ℓ works.

Case 3. Finally, assume that A and B have different colors and lie in one of the regions defined by the drawn lines. By the induction assumption, this region contains no other points of one of the colors — without loss of generality, the only blue point it contains is A. Then it suffices to draw a line separating A from all other points.

Thus the step of the induction is proved.

Comment 2. One may ask a more general question, replacing the numbers 2013 and 2014 by any positive integers m and n, say with $m \leqslant n$. Denote the answer for this problem by $f(m, n)$.

One may show along the lines of Solution 1 that $m \leqslant f(m, n) \leqslant m + 1$; moreover, if m is even then $f(m, n) = m$. On the other hand, for every odd m there exists an N such that $f(m, n) = m$ for all $m \leqslant n \leqslant N$, and $f(m, n) = m + 1$ for all $n > N$.

C3. A crazy physicist discovered a new kind of particle which he called an *imon*, after some of them mysteriously appeared in his lab. Some pairs of imons in the lab can be *entangled*, and each imon can participate in many entanglement relations. The physicist has found a way to perform the following two kinds of operations with these particles, one operation at a time.

(i) If some imon is entangled with an odd number of other imons in the lab, then the physicist can destroy it.

(ii) At any moment, he may double the whole family of imons in his lab by creating a copy I' of each imon I. During this procedure, the two copies I' and J' become entangled if and only if the original imons I and J are entangled, and each copy I' becomes entangled with its original imon I; no other entanglements occur or disappear at this moment.

Prove that the physicist may apply a sequence of such operations resulting in a family of imons, no two of which are entangled.

(Japan)

Solution 1. Let us consider a graph with the imons as vertices, and two imons being connected if and only if they are entangled. Recall that a *proper coloring* of a graph G is a coloring of its vertices in several colors so that every two connected vertices have different colors.

Lemma. Assume that a graph G admits a proper coloring in n colors ($n > 1$). Then one may perform a sequence of operations resulting in a graph which admits a proper coloring in $n-1$ colors.

Proof. Let us apply repeatedly operation (i) to any appropriate vertices while it is possible. Since the number of vertices decreases, this process finally results in a graph where all the degrees are even. Surely this graph also admits a proper coloring in n colors $1, \ldots, n$; let us fix this coloring.

Now apply the operation (ii) to this graph. A proper coloring of the resulting graph in n colors still exists: one may preserve the colors of the original vertices and color the vertex I' in a color $k+1 \pmod{n}$ if the vertex I has color k. Then two connected original vertices still have different colors, and so do their two connected copies. On the other hand, the vertices I and I' have different colors since $n > 1$.

All the degrees of the vertices in the resulting graph are odd, so one may apply operation (i) to delete consecutively all the vertices of color n one by one; no two of them are connected by an edge, so their degrees do not change during the process. Thus, we obtain a graph admitting a proper coloring in $n-1$ colors, as required. The lemma is proved. □

Now, assume that a graph G has n vertices; then it admits a proper coloring in n colors. Applying repeatedly the lemma we finally obtain a graph admitting a proper coloring in one color, that is — a graph with no edges, as required.

Solution 2. Again, we will use the graph language.

I. We start with the following observation.

Lemma. Assume that a graph G contains an isolated vertex A, and a graph G° is obtained from G by deleting this vertex. Then, if one can apply a sequence of operations which makes a graph with no edges from G°, then such a sequence also exists for G.

Proof. Consider any operation applicable to G° resulting in a graph G_1°; then there exists a sequence of operations applicable to G and resulting in a graph G_1 differing from G_1° by an addition of an isolated vertex A. Indeed, if this operation is of type (i), then one may simply repeat it in G.

Otherwise, the operation is of type (ii), and one may apply it to G and then delete the vertex A' (it will have degree 1).

Thus one may change the process for G° into a corresponding process for G step by step. $\square$

In view of this lemma, if at some moment a graph contains some isolated vertex, then we may simply delete it; let us call this operation (iii).

II. Let $V = \{A_1^0, \ldots, A_n^0\}$ be the vertices of the initial graph. Let us describe which graphs can appear during our operations. Assume that operation (ii) was applied m times. If these were the only operations applied, then the resulting graph G_n^m has the set of vertices which can be enumerated as

$$V_n^m = \{A_i^j \colon 1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant 2^m - 1\},$$

where A_i^0 is the common "ancestor" of all the vertices A_i^j, and the binary expansion of j (adjoined with some zeroes at the left to have m digits) "keeps the history" of this vertex: the dth digit from the right is 0 if at the dth doubling the ancestor of A_i^j was in the original part, and this digit is 1 if it was in the copy.

Next, the two vertices A_i^j and A_k^ℓ in G_n^m are connected with an edge exactly if either (1) $j = \ell$ and there was an edge between A_i^0 and A_k^0 (so these vertices appeared at the same application of operation (ii)); or (2) $i = k$ and the binary expansions of j and ℓ differ in exactly one digit (so their ancestors became connected as a copy and the original vertex at some application of (ii)).

Now, if some operations (i) were applied during the process, then simply some vertices in G_n^m disappeared. So, in any case the resulting graph is some induced subgraph of G_n^m.

III. Finally, we will show that from each (not necessarily induced) subgraph of G_n^m one can obtain a graph with no vertices by applying operations (i), (ii) and (iii). We proceed by induction on n; the base case $n = 0$ is trivial.

For the induction step, let us show how to apply several operations so as to obtain a graph containing no vertices of the form A_n^j for $j \in \mathbb{Z}$. We will do this in three steps.

Step 1. We apply repeatedly operation (i) to any appropriate vertices while it is possible. In the resulting graph, all vertices have even degrees.

Step 2. Apply operation (ii) obtaining a subgraph of G_n^{m+1} with all degrees being odd. In this graph, we delete one by one all the vertices A_n^j where the sum of the binary digits of j is even; it is possible since there are no edges between such vertices, so all their degrees remain odd. After that, we delete all isolated vertices.

Step 3. Finally, consider any remaining vertex A_n^j (then the sum of digits of j is odd). If its degree is odd, then we simply delete it. Otherwise, since A_n^j is not isolated, we consider any vertex adjacent to it. It has the form A_k^j for some $k < n$ (otherwise it would have the form A_n^ℓ, where ℓ has an even digit sum; but any such vertex has already been deleted at Step 2). No neighbor of A_k^j was deleted at Steps 2 and 3, so it has an odd degree. Then we successively delete A_k^j and A_n^j.

Notice that this deletion does not affect the applicability of this step to other vertices, since no two vertices A_i^j and A_k^ℓ for different j, ℓ with odd digit sum are connected with an edge. Thus we will delete all the remaining vertices of the form A_n^j, obtaining a subgraph of G_{n-1}^{m+1}. The application of the induction hypothesis finishes the proof.

Comment. In fact, the graph G_n^m is a Cartesian product of G and the graph of an m-dimensional hypercube.

C4. Let n be a positive integer, and let A be a subset of $\{1,\ldots,n\}$. An *A-partition* of n *into* k *parts* is a representation of n as a sum $n=a_1+\cdots+a_k$, where the parts $a_1,\ldots,a_k$ belong to A and are not necessarily distinct. The *number of different parts* in such a partition is the number of (distinct) elements in the set $\{a_1,a_2,\ldots,a_k\}$.

We say that an A-partition of n into k parts is *optimal* if there is no A-partition of n into r parts with $r<k$. Prove that any optimal A-partition of n contains at most $\sqrt[3]{6n}$ different parts.

(Germany)

Solution 1. If there are no A-partitions of n, the result is vacuously true. Otherwise, let $k_{\min}$ be the minimum number of parts in an A-partition of n, and let $n=a_1+\cdots+a_{k_{\min}}$ be an optimal partition. Denote by s the number of different parts in this partition, so we can write $S=\{a_1,\ldots,a_{k_{\min}}\}=\{b_1,\ldots,b_s\}$ for some pairwise different numbers $b_1<\cdots<b_s$ in A.

If $s>\sqrt[3]{6n}$, we will prove that there exist subsets X and Y of S such that $|X|<|Y|$ and $\sum_{x\in X}x=\sum_{y\in Y}y$. Then, deleting the elements of Y from our partition and adding the elements of X to it, we obtain an A-partition of n into less than $k_{\min}$ parts, which is the desired contradiction.

For each positive integer $k\leqslant s$, we consider the k-element subset

$$S_{1,0}^k:=\{b_1,\ldots,b_k\}$$

as well as the following k-element subsets $S_{i,j}^k$ of S:

$$S_{i,j}^k:=\bigl\{b_1,\ldots,b_{k-i},b_{k-i+j+1},b_{s-i+2},\ldots,b_s\bigr\},\quad i=1,\ldots,k,\quad j=1,\ldots,s-k.$$

Pictorially, if we represent the elements of S by a sequence of dots in increasing order, and represent a subset of S by shading in the appropriate dots, we have:

$$S_{i,j}^k=\underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{k-i}\underbrace{\circ\circ\circ\circ\circ}_{j}\bullet\underbrace{\circ\circ\circ\circ\circ\circ\circ}_{s-k-j}\underbrace{\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet\bullet}_{i-1}$$

Denote by $\Sigma_{i,j}^k$ the sum of elements in $S_{i,j}^k$. Clearly, $\Sigma_{1,0}^k$ is the minimum sum of a k-element subset of S. Next, for all appropriate indices i and j we have

$$\Sigma_{i,j}^k=\Sigma_{i,j+1}^k+b_{k-i+j+1}-b_{k-i+j+2}<\Sigma_{i,j+1}^k\quad\text{and}\quad\Sigma_{i,s-k}^k=\Sigma_{i+1,1}^k+b_{k-i}-b_{k-i+1}<\Sigma_{i+1,1}^k.$$

Therefore

$$1\leqslant\Sigma_{1,0}^k<\Sigma_{1,1}^k<\Sigma_{1,2}^k<\cdots<\Sigma_{1,s-k}^k<\Sigma_{2,1}^k<\cdots<\Sigma_{2,s-k}^k<\Sigma_{3,1}^k<\cdots<\Sigma_{k,s-k}^k\leqslant n.$$

To see this in the picture, we start with the k leftmost points marked. At each step, we look for the rightmost point which can move to the right, and move it one unit to the right. We continue until the k rightmost points are marked. As we do this, the corresponding sums clearly increase.

For each k we have found $k(s-k)+1$ different integers of the form $\Sigma_{i,j}^k$ between 1 and n. As we vary k, the total number of integers we are considering is

$$\sum_{k=1}^{s}\bigl(k(s-k)+1\bigr)=s\cdot\frac{s(s+1)}{2}-\frac{s(s+1)(2s+1)}{6}+s=\frac{s(s^2+5)}{6}>\frac{s^3}{6}>n.$$

Since they are between 1 and n, at least two of these integers are equal. Consequently, there exist $1\leqslant k<k'\leqslant s$ and $X=S_{i,j}^k$ as well as $Y=S_{i',j'}^{k'}$ such that

$$\sum_{x\in X}x=\sum_{y\in Y}y,\quad\text{but}\quad|X|=k<k'=|Y|,$$

as required. The result follows.

Solution 2. Assume, to the contrary, that the statement is false, and choose the minimum number n for which it fails. So there exists a set $A \subseteq \{1, \ldots, n\}$ together with an optimal A-partition $n = a_1 + \cdots + a_{k_{\min}}$ of n refuting our statement, where, of course, $k_{\min}$ is the minimum number of parts in an A-partition of n. Again, we define $S = \{a_1, \ldots, a_{k_{\min}}\} = \{b_1, \ldots, b_s\}$ with $b_1 < \cdots < b_s$; by our assumption we have $s > \sqrt[3]{6n} > 1$. Without loss of generality we assume that $a_{k_{\min}} = b_s$. Let us distinguish two cases.

Case 1. $b_s \geqslant \frac{s(s-1)}{2} + 1$.

Consider the partition $n - b_s = a_1 + \cdots + a_{k_{\min}-1}$, which is clearly a minimum A-partition of $n - b_s$ with at least $s - 1 \geqslant 1$ different parts. Now, from $n < \frac{s^3}{6}$ we obtain

$$n - b_s \leqslant n - \frac{s(s-1)}{2} - 1 < \frac{s^3}{6} - \frac{s(s-1)}{2} - 1 < \frac{(s-1)^3}{6},$$

so $s - 1 > \sqrt[3]{6(n - b_s)}$, which contradicts the choice of n.

Case 2. $b_s \leqslant \frac{s(s-1)}{2}$.

Set $b_0 = 0$, $\Sigma_{0,0} = 0$, and $\Sigma_{i,j} = b_1 + \cdots + b_{i-1} + b_j$ for $1 \leqslant i \leqslant j < s$. There are $\frac{s(s-1)}{2} + 1 > b_s$ such sums; so at least two of them, say $\Sigma_{i,j}$ and $\Sigma_{i',j'}$, are congruent modulo b_s (where $(i, j) \neq (i', j')$). This means that $\Sigma_{i,j} - \Sigma_{i',j'} = rb_s$ for some integer r. Notice that for $i \leqslant j < k < s$ we have

$$0 < \Sigma_{i,k} - \Sigma_{i,j} = b_k - b_j < b_s,$$

so the indices i and i' are distinct, and we may assume that $i > i'$. Next, we observe that $\Sigma_{i,j} - \Sigma_{i',j'} = (b_{i'} - b_{j'}) + b_j + b_{i'+1} + \cdots + b_{i-1}$ and $b_{i'} \leqslant b_{j'}$ imply

$$-b_s < -b_{j'} < \Sigma_{i,j} - \Sigma_{i',j'} < (i - i')b_s,$$

so $0 \leqslant r \leqslant i - i' - 1$.

Thus, we may remove the i terms of $\Sigma_{i,j}$ in our A-partition, and replace them by the i' terms of $\Sigma_{i',j'}$ and r terms equal to b_s, for a total of $r + i' < i$ terms. The result is an A-partition of n into a smaller number of parts, a contradiction.

Comment. The original proposal also contained a second part, showing that the estimate appearing in the problem has the correct order of magnitude:

For every positive integer n, there exist a set A and an optimal A-partition of n that contains $\lfloor \sqrt[3]{2n} \rfloor$ different parts.

The Problem Selection Committee removed this statement from the problem, since it seems to be less suitable for the competiton; but for completeness we provide an outline of its proof here.

Let $k = \lfloor \sqrt[3]{2n} \rfloor - 1$. The statement is trivial for $n < 4$, so we assume $n \geqslant 4$ and hence $k \geqslant 1$. Let $h = \lfloor \frac{n-1}{k} \rfloor$. Notice that $h \geqslant \frac{n}{k} - 1$.

Now let $A = \{1, \ldots, h\}$, and set $a_1 = h$, $a_2 = h - 1$, $\ldots$, $a_k = h - k + 1$, and $a_{k+1} = n - (a_1 + \cdots + a_k)$. It is not difficult to prove that $a_k > a_{k+1} \geqslant 1$, which shows that

$$n = a_1 + \ldots + a_{k+1}$$

is an A-partition of n into $k + 1$ different parts. Since $kh < n$, any A-partition of n has at least $k + 1$ parts. Therefore our A-partition is optimal, and it has $\lfloor \sqrt[3]{2n} \rfloor$ distinct parts, as desired.

C5. Let r be a positive integer, and let $a_0, a_1, \dots$ be an infinite sequence of real numbers. Assume that for all nonnegative integers m and s there exists a positive integer $n \in [m+1, m+r]$ such that

$$a_m + a_{m+1} + \cdots + a_{m+s} = a_n + a_{n+1} + \cdots + a_{n+s}.$$

Prove that the sequence is periodic, i. e. there exists some $p \geqslant 1$ such that $a_{n+p} = a_n$ for all $n \geqslant 0$.

(India)

Solution. For every indices $m \leqslant n$ we will denote $S(m,n) = a_m + a_{m+1} + \cdots + a_{n-1}$; thus $S(n,n) = 0$. Let us start with the following lemma.

Lemma. Let $b_0, b_1, \dots$ be an infinite sequence. Assume that for every nonnegative integer m there exists a nonnegative integer $n \in [m+1, m+r]$ such that $b_m = b_n$. Then for every indices $k \leqslant \ell$ there exists an index $t \in [\ell, \ell+r-1]$ such that $b_t = b_k$. Moreover, there are at most r distinct numbers among the terms of (b_i).

Proof. To prove the first claim, let us notice that there exists an infinite sequence of indices $k_1 = k, k_2, k_3, \dots$ such that $b_{k_1} = b_{k_2} = \cdots = b_k$ and $k_i < k_{i+1} \leqslant k_i + r$ for all $i \geqslant 1$. This sequence is unbounded from above, thus it hits each segment of the form $[\ell, \ell+r-1]$ with $\ell \geqslant k$, as required.

To prove the second claim, assume, to the contrary, that there exist $r+1$ distinct numbers $b_{i_1}, \dots, b_{i_{r+1}}$. Let us apply the first claim to $k = i_1, \dots, i_{r+1}$ and $\ell = \max\{i_1, \dots, i_{r+1}\}$; we obtain that for every $j \in \{1, \dots, r+1\}$ there exists $t_j \in [s, s+r-1]$ such that $b_{t_j} = b_{i_j}$. Thus the segment $[s, s+r-1]$ should contain $r+1$ distinct integers, which is absurd. □

Setting $s = 0$ in the problem condition, we see that the sequence (a_i) satisfies the condition of the lemma, thus it attains at most r distinct values. Denote by A_i the ordered r-tuple $(a_i, \dots, a_{i+r-1})$; then among A_i's there are at most r^r distinct tuples, so for every $k \geqslant 0$ two of the tuples $A_k, A_{k+1}, \dots, A_{k+r^r}$ are identical. This means that there exists a positive integer $p \leqslant r^r$ such that the equality $A_d = A_{d+p}$ holds infinitely many times. Let D be the set of indices d satisfying this relation.

Now we claim that D coincides with the set of all nonnegative integers. Since D is unbounded, it suffices to show that $d \in D$ whenever $d+1 \in D$. For that, denote $b_k = S(k, p+k)$. The sequence $b_0, b_1, \dots$ satisfies the lemma conditions, so there exists an index $t \in [d+1, d+r]$ such that $S(t, t+p) = S(d, d+p)$. This last relation rewrites as $S(d,t) = S(d+p, t+p)$. Since $A_{d+1} = A_{d+p+1}$, we have $S(d+1, t) = S(d+p+1, t+p)$, therefore we obtain

$$a_d = S(d,t) - S(d+1,t) = S(d+p, t+p) - S(d+p+1, t+p) = a_{d+p}$$

and thus $A_d = A_{d+p}$, as required.

Finally, we get $A_d = A_{d+p}$ for all d, so in particular $a_d = a_{d+p}$ for all d, QED.

Comment 1. In the present proof, the upper bound for the minimal period length is r^r. This bound is not sharp; for instance, one may improve it to $(r-1)^r$ for $r \geqslant 3$..

On the other hand, this minimal length may happen to be greater than r. For instance, it is easy to check that the sequence with period $(3, -3, 3, -3, 3, -1, -1, -1)$ satisfies the problem condition for $r = 7$.

Comment 2. The conclusion remains true even if the problem condition only holds for every $s \geqslant N$ for some positive integer N. To show that, one can act as follows. Firstly, the sums of the form $S(i, i+N)$ attain at most r values, as well as the sums of the form $S(i, i+N+1)$. Thus the terms $a_i = S(i, i+N+1) - S(i+1, i+N+1)$ attain at most r^2 distinct values. Then, among the tuples $A_k, A_{k+N}, \dots, A_{k+r^{2r}N}$ two are identical, so for some $p \leqslant r^{2r}$ the set $D = \{d\colon A_d = A_{d+Np}\}$ is infinite. The further arguments apply almost literally, with p being replaced by Np.

After having proved that such a sequence is also necessarily periodic, one may reduce the bound for the minimal period length to r^r — essentially by verifying that the sequence satisfies the original version of the condition.

C6. In some country several pairs of cities are connected by direct two-way flights. It is possible to go from any city to any other by a sequence of flights. The *distance* between two cities is defined to be the least possible number of flights required to go from one of them to the other. It is known that for any city there are at most 100 cities at distance exactly three from it. Prove that there is no city such that more than 2550 other cities have distance exactly four from it.

(Russia)

Solution. Let us denote by $d(a, b)$ the distance between the cities a and b, and by

$$S_i(a) = \{c\colon d(a, c) = i\}$$

the set of cities at distance exactly i from city a.

Assume that for some city x the set $D = S_4(x)$ has size at least 2551. Let $A = S_1(x)$. A subset A' of A is said to be *substantial*, if every city in D can be reached from x with four flights while passing through some member of A'; in other terms, every city in D has distance 3 from some member of A', or $D \subseteq \bigcup_{a\in A'} S_3(a)$. For instance, A itself is substantial. Now let us fix some substantial subset A^* of A having the minimal cardinality $m = |A^*|$.

Since

$$m(101 - m) \leqslant 50 \cdot 51 = 2550,$$

there has to be a city $a \in A^*$ such that $|S_3(a) \cap D| \geqslant 102 - m$. As $|S_3(a)| \leqslant 100$, we obtain that $S_3(a)$ may contain at most $100 - (102 - m) = m - 2$ cities c with $d(c, x) \leqslant 3$. Let us denote by $T = \{c \in S_3(a)\colon d(x, c) \leqslant 3\}$ the set of all such cities, so $|T| \leqslant m - 2$. Now, to get a contradiction, we will construct $m - 1$ distinct elements in T, corresponding to $m - 1$ elements of the set $A_a = A^*\backslash\{a\}$.

Firstly, due to the minimality of A^*, for each $y \in A_a$ there exists some city $d_y \in D$ which can only be reached with four flights from x by passing through y. So, there is a way to get from x to d_y along x–y–b_y–c_y–d_y for some cities b_y and c_y; notice that $d(x, b_y) = 2$ and $d(x, c_y) = 3$ since this path has the minimal possible length.

Now we claim that all $2(m - 1)$ cities of the form b_y, c_y with $y \in A_a$ are distinct. Indeed, no b_y may coincide with any c_z since their distances from x are different. On the other hand, if one had $b_y = b_z$ for $y \neq z$, then there would exist a path of length 4 from x to d_z via y, namely x–y–b_z–c_z–d_z; this is impossible by the choice of d_z. Similarly, $c_y \neq c_z$ for $y \neq z$.

So, it suffices to prove that for every $y \in A_a$, one of the cities b_y and c_y has distance 3 from a (and thus belongs to T). For that, notice that $d(a, y) \leqslant 2$ due to the path a–x–y, while $d(a, d_y) \geqslant d(x, d_y) - d(x, a) = 3$. Moreover, $d(a, d_y) \neq 3$ by the choice of d_y; thus $d(a, d_y) > 3$. Finally, in the sequence $d(a, y)$, $d(a, b_y)$, $d(a, c_y)$, $d(a, d_y)$ the neighboring terms differ by at most 1, the first term is less than 3, and the last one is greater than 3; thus there exists one which is equal to 3, as required.

Comment 1. The upper bound 2550 is sharp. This can be seen by means of various examples; one of them is the "Roman Empire": it has one capital, called "Rome", that is connected to 51 semicapitals by internally disjoint paths of length 3. Moreover, each of these semicapitals is connected to 50 rural cities by direct flights.

Comment 2. Observe that, under the conditions of the problem, there exists no bound for the size of $S_1(x)$ or $S_2(x)$.

Comment 3. The numbers 100 and 2550 appearing in the statement of the problem may be replaced by n and $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ for any positive integer n. Still more generally, one can also replace the pair $(3,4)$ of distances under consideration by any pair (r,s) of positive integers satisfying $r < s \leqslant \frac{3}{2}r$.

To adapt the above proof to this situation, one takes $A = S_{s-r}(x)$ and defines the concept of substantiality as before. Then one takes A^* to be a minimal substantial subset of A, and for each $y \in A^*$ one fixes an element $d_y \in S_s(x)$ which is only reachable from x by a path of length s by passing through y. As before, it suffices to show that for distinct $a, y \in A^*$ and a path $y = y_0 - y_1 - \ldots - y_r = d_y$, at least one of the cities $y_0, \ldots, y_{r-1}$ has distance r from a. This can be done as above; the relation $s \leqslant \frac{3}{2}r$ is used here to show that $d(a, y_0) \leqslant r$.

Moreover, the estimate $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ is also sharp for every positive integer n and every positive integers r, s with $r < s \leqslant \frac{3}{2}r$. This may be shown by an example similar to that in the previous comment.

C7. Let $n \geqslant 2$ be an integer. Consider all circular arrangements of the numbers $0, 1, \ldots, n$; the $n+1$ rotations of an arrangement are considered to be equal. A circular arrangement is called *beautiful* if, for any four distinct numbers $0 \leqslant a, b, c, d \leqslant n$ with $a+c=b+d$, the chord joining numbers a and c does not intersect the chord joining numbers b and d.

Let M be the number of beautiful arrangements of $0, 1, \ldots, n$. Let N be the number of pairs (x, y) of positive integers such that $x+y \leqslant n$ and $\gcd(x, y)=1$. Prove that

$$M=N+1.$$

(Russia)

Solution 1. Given a circular arrangement of $[0, n]=\{0, 1, \ldots, n\}$, we define a *$k$-chord* to be a (possibly degenerate) chord whose (possibly equal) endpoints add up to k. We say that three chords of a circle are *aligned* if one of them separates the other two. Say that $m \geqslant 3$ chords are aligned if any three of them are aligned. For instance, in Figure 1, A, B, and C are aligned, while B, C, and D are not.

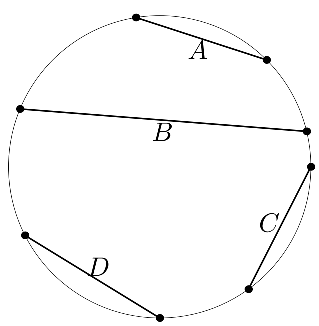


Figure 1

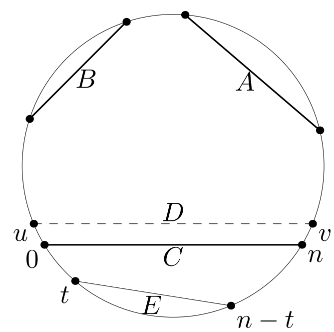


Figure 2

Claim. In a beautiful arrangement, the k–chords are aligned for any integer k.

Proof. We proceed by induction. For $n \leqslant 3$ the statement is trivial. Now let $n \geqslant 4$, and proceed by contradiction. Consider a beautiful arrangement S where the three k–chords A, B, C are not aligned. If n is not among the endpoints of A, B, and C, then by deleting n from S we obtain a beautiful arrangement $S \backslash\{n\}$ of $[0, n-1]$, where A, B, and C are aligned by the induction hypothesis. Similarly, if 0 is not among these endpoints, then deleting 0 and decreasing all the numbers by 1 gives a beautiful arrangement $S \backslash\{0\}$ where A, B, and C are aligned. Therefore both 0 and n are among the endpoints of these segments. If x and y are their respective partners, we have $n \geqslant 0+x=k=n+y \geqslant n$. Thus 0 and n are the endpoints of one of the chords; say it is C.

Let D be the chord formed by the numbers u and v which are adjacent to 0 and n and on the same side of C as A and B, as shown in Figure 2. Set $t=u+v$. If we had $t=n$, the n–chords A, B, and D would not be aligned in the beautiful arrangement $S \backslash\{0, n\}$, contradicting the induction hypothesis. If $t<n$, then the t-chord from 0 to t cannot intersect D, so the chord C separates t and D. The chord E from t to $n-t$ does not intersect C, so t and $n-t$ are on the same side of C. But then the chords A, B, and E are not aligned in $S \backslash\{0, n\}$, a contradiction. Finally, the case $t>n$ is equivalent to the case $t<n$ via the beauty-preserving relabelling $x \mapsto n-x$ for $0 \leqslant x \leqslant n$, which sends t-chords to $(2n-t)$–chords. This proves the Claim.

Having established the Claim, we prove the desired result by induction. The case $n=2$ is trivial. Now assume that $n \geqslant 3$. Let S be a beautiful arrangement of $[0, n]$ and delete n to obtain the beautiful arrangement T of $[0, n-1]$. The n–chords of T are aligned, and they contain every point except 0. Say T is of *Type 1* if 0 lies between two of these n–chords, and it is of *Type 2* otherwise; *i.e.*, if 0 is aligned with these n–chords. We will show that each Type 1 arrangement of $[0, n-1]$ arises from a unique arrangement of $[0, n]$, and each Type 2 arrangement of $[0, n-1]$ arises from exactly two beautiful arrangements of $[0, n]$.

If T is of Type 1, let 0 lie between chords A and B. Since the chord from 0 to n must be aligned with A and B in S, n must be on the other arc between A and B. Therefore S can be recovered uniquely from T. In the other direction, if T is of Type 1 and we insert n as above, then we claim the resulting arrangement S is beautiful. For $0 < k < n$, the k–chords of S are also k–chords of T, so they are aligned. Finally, for $n < k < 2n$, notice that the n–chords of S are parallel by construction, so there is an antisymmetry axis ℓ such that x is symmetric to $n - x$ with respect to ℓ for all x. If we had two k–chords which intersect, then their reflections across ℓ would be two $(2n - k)$-chords which intersect, where $0 < 2n - k < n$, a contradiction.

If T is of Type 2, there are two possible positions for n in S, on either side of 0. As above, we check that both positions lead to beautiful arrangements of $[0, n]$.

Hence if we let M_n be the number of beautiful arrangements of $[0, n]$, and let L_n be the number of beautiful arrangements of $[0, n-1]$ of Type 2, we have

$$M_n = (M_{n-1} - L_{n-1}) + 2L_{n-1} = M_{n-1} + L_{n-1}.$$

It then remains to show that L_{n-1} is the number of pairs (x, y) of positive integers with $x + y = n$ and $\gcd(x, y) = 1$. Since $n \geqslant 3$, this number equals $\varphi(n) = \#\{x : 1 \leqslant x \leqslant n, \gcd(x, n) = 1\}$.

To prove this, consider a Type 2 beautiful arrangement of $[0, n-1]$. Label the positions $0, \ldots, n-1 \pmod n$ clockwise around the circle, so that number 0 is in position 0. Let $f(i)$ be the number in position i; note that f is a permutation of $[0, n-1]$. Let a be the position such that $f(a) = n - 1$.

Since the n–chords are aligned with 0, and every point is in an n–chord, these chords are all parallel and

$$f(i) + f(-i) = n \qquad \text{for all } i.$$

Similarly, since the $(n-1)$–chords are aligned and every point is in an $(n-1)$–chord, these chords are also parallel and

$$f(i) + f(a - i) = n - 1 \qquad \text{for all } i.$$

Therefore $f(a - i) = f(-i) - 1$ for all i; and since $f(0) = 0$, we get

$$f(-ak) = k \qquad \text{for all } k. \tag{1}$$

Recall that this is an equality modulo n. Since f is a permutation, we must have $(a, n) = 1$. Hence $L_{n-1} \leqslant \varphi(n)$.

To prove equality, it remains to observe that the labeling (1) is beautiful. To see this, consider four numbers w, x, y, z on the circle with $w + y = x + z$. Their positions around the circle satisfy $(-aw) + (-ay) = (-ax) + (-az)$, which means that the chord from w to y and the chord from x to z are parallel. Thus (1) is beautiful, and by construction it has Type 2. The desired result follows.

Solution 2. Notice that there are exactly N irreducible fractions $f_1 < \cdots < f_N$ in $(0,1)$ whose denominator is at most n, since the pair (x,y) with $x+y \leqslant n$ and $(x,y)=1$ corresponds to the fraction $x/(x+y)$. Write $f_i = \frac{a_i}{b_i}$ for $1 \leqslant i \leqslant N$.

We begin by constructing $N+1$ beautiful arrangements. Take any $\alpha \in (0,1)$ which is not one of the above N fractions. Consider a circle of perimeter 1. Successively mark points $0,1,2,\ldots,n$ where 0 is arbitrary, and the clockwise distance from i to $i+1$ is α. The point k will be at clockwise distance $\{k\alpha\}$ from 0, where $\{r\}$ denotes the fractional part of r. Call such a circular arrangement *cyclic* and denote it by $A(\alpha)$. If the clockwise order of the points is the same in $A(\alpha_1)$ and $A(\alpha_2)$, we regard them as the same circular arrangement. Figure 3 shows the cyclic arrangement $A(3/5+\epsilon)$ of $[0,13]$ where $\epsilon > 0$ is very small.

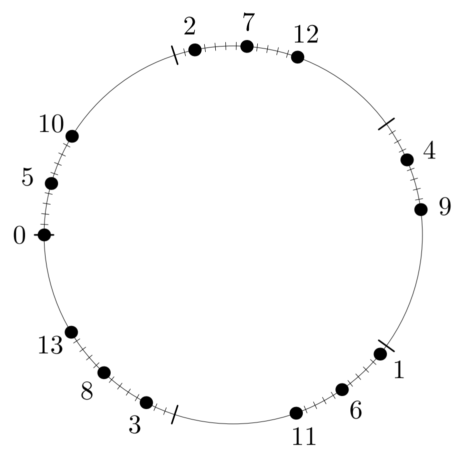


Figure 3

If $0 \leqslant a,\, b,\, c,\, d \leqslant n$ satisfy $a+c=b+d$, then $a\alpha + c\alpha = b\alpha + d\alpha$, so the chord from a to c is parallel to the chord from b to d in $A(\alpha)$. Hence in a cyclic arrangement all k—chords are parallel. In particular every cyclic arrangement is beautiful.

Next we show that there are exactly $N+1$ distinct cyclic arrangements. To see this, let us see how $A(\alpha)$ changes as we increase α from 0 to 1. The order of points p and q changes precisely when we cross a value $\alpha = f$ such that $\{pf\} = \{qf\}$; this can only happen if f is one of the N fractions $f_1,\ldots,f_N$. Therefore there are at most $N+1$ different cyclic arrangements.

To show they are all distinct, recall that $f_i = a_i/b_i$ and let $\epsilon > 0$ be a very small number. In the arrangement $A(f_i+\epsilon)$, point k lands at $\frac{ka_i \pmod{b_i}}{b_i} + k\epsilon$. Therefore the points are grouped into b_i clusters next to the points $0, \frac{1}{b_i}, \ldots, \frac{b_i-1}{b_i}$ of the circle. The cluster following $\frac{k}{b_i}$ contains the numbers congruent to ka_i^{-1} modulo b_i, listed clockwise in increasing order. It follows that the first number after 0 in $A(f_i+\epsilon)$ is b_i, and the first number after 0 which is less than b_i is $a_i^{-1} \pmod{b_i}$, which uniquely determines a_i. In this way we can recover f_i from the cyclic arrangement. Note also that $A(f_i+\epsilon)$ is not the trivial arrangement where we list $0,1,\ldots,n$ in order clockwise. It follows that the $N+1$ cyclic arrangements $A(\epsilon), A(f_1+\epsilon),\ldots,A(f_N+\epsilon)$ are distinct.

Let us record an observation which will be useful later:

$$\text{if } f_i < \alpha < f_{i+1} \text{ then 0 is immediately after } b_{i+1} \text{ and before } b_i \text{ in } A(\alpha). \tag{2}$$

Indeed, we already observed that b_i is the first number after 0 in $A(f_i+\epsilon) = A(\alpha)$. Similarly we see that b_{i+1} is the last number before 0 in $A(f_{i+1}-\epsilon) = A(\alpha)$.

Finally, we show that any beautiful arrangement of $[0,n]$ is cyclic by induction on n. For $n \leqslant 2$ the result is clear. Now assume that all beautiful arrangements of $[0,n-1]$ are cyclic, and consider a beautiful arrangement A of $[0,n]$. The subarrangement $A_{n-1} = A\backslash\{n\}$ of $[0,n-1]$ obtained by deleting n is cyclic; say $A_{n-1} = A_{n-1}(\alpha)$.

Let α be between the consecutive fractions $\frac{p_1}{q_1} < \frac{p_2}{q_2}$ among the irreducible fractions of denominator at most $n-1$. There is at most one fraction $\frac{i}{n}$ in $(\frac{p_1}{q_1}, \frac{p_2}{q_2})$, since $\frac{i}{n} < \frac{i}{n-1} \leqslant \frac{i+1}{n}$ for $0 < i \leqslant n-1$.

Case 1. There is no fraction with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$.

In this case the only cyclic arrangement extending $A_{n-1}(\alpha)$ is $A_n(\alpha)$. We know that A and $A_n(\alpha)$ can only differ in the position of n. Assume n is immediately after x and before y in $A_n(\alpha)$. Since the neighbors of 0 are q_1 and q_2 by (2), we have $x, y \geqslant 1$.

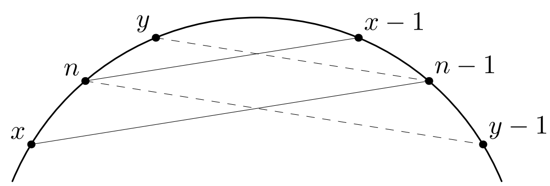


Figure 4

In $A_n(\alpha)$ the chord from $n-1$ to x is parallel and adjacent to the chord from n to $x-1$, so $n-1$ is between $x-1$ and x in clockwise order, as shown in Figure 4. Similarly, $n-1$ is between y and $y-1$. Therefore x, y, $x-1$, $n-1$, and $y-1$ occur in this order in $A_n(\alpha)$ and hence in A (possibly with $y = x-1$ or $x = y-1$).

Now, A may only differ from $A_n(\alpha)$ in the location of n. In A, since the chord from $n-1$ to x and the chord from n to $x-1$ do not intersect, n is between x and $n-1$. Similarly, n is between $n-1$ and y. Then n must be between x and y and $A = A_n(\alpha)$. Therefore A is cyclic as desired.

Case 2. There is exactly one i with $\frac{p_1}{q_1} < \frac{i}{n} < \frac{p_2}{q_2}$.

In this case there exist two cyclic arrangements $A_n(\alpha_1)$ and $A_n(\alpha_2)$ of the numbers $0, \ldots, n$ extending $A_{n-1}(\alpha)$, where $\frac{p_1}{q_1} < \alpha_1 < \frac{i}{n}$ and $\frac{i}{n} < \alpha_2 < \frac{p_2}{q_2}$. In $A_{n-1}(\alpha)$, 0 is the only number between q_2 and q_1 by (2). For the same reason, n is between q_2 and 0 in $A_n(\alpha_1)$, and between 0 and q_1 in $A_n(\alpha_2)$.

Letting $x = q_2$ and $y = q_1$, the argument of Case 1 tells us that n must be between x and y in A. Therefore A must equal $A_n(\alpha_1)$ or $A_n(\alpha_2)$, and therefore it is cyclic.

This concludes the proof that every beautiful arrangement is cyclic. It follows that there are exactly $N+1$ beautiful arrangements of $[0,n]$ as we wished to show.

C8. Players A and B play a paintful game on the real line. Player A has a pot of paint with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In every round, player A picks some positive integer m and provides $1/2^m$ units of ink from the pot. Player B then picks an integer k and blackens the interval from $k/2^m$ to $(k+1)/2^m$ (some parts of this interval may have been blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0,1]$ is not completely blackened.

Decide whether there exists a strategy for player A to win in a finite number of moves.

(Austria)

Answer. No. Such a strategy for player A does not exist.

Solution. We will present a strategy for player B that guarantees that the interval $[0,1]$ is completely blackened, once the paint pot has become empty.

At the beginning of round r, let x_r denote the largest real number for which the interval between 0 and x_r has already been blackened; for completeness we define $x_1 = 0$. Let m be the integer picked by player A in this round; we define an integer y_r by

$$\frac{y_r}{2^m} \leqslant x_r < \frac{y_r + 1}{2^m}.$$

Note that $I_0^r = [y_r/2^m, (y_r+1)/2^m]$ is the leftmost interval that may be painted in round r and that still contains some uncolored point.

Player B now looks at the *next* interval $I_1^r = [(y_r+1)/2^m, (y_r+2)/2^m]$. If I_1^r still contains an uncolored point, then player B blackens the interval I_1^r; otherwise he blackens the interval I_0^r. We make the convention that, at the beginning of the game, the interval $[1,2]$ is already blackened; thus, if $y_r + 1 = 2^m$, then B blackens I_0^r.

Our aim is to estimate the amount of ink used after each round. Firstly, we will prove by induction that, if before rth round the segment $[0,1]$ is not completely colored, then, before this move,

(i) the amount of ink used for the segment $[0, x_r]$ is at most $3x_r$; and

(ii) for every m, B has blackened at most one interval of length $1/2^m$ to the right of x_r.

Obviously, these conditions are satisfied for $r = 0$. Now assume that they were satisfied before the rth move, and consider the situation after this move; let m be the number A has picked at this move.

If B has blackened the interval I_1^r at this move, then $x_{r+1} = x_r$, and (i) holds by the induction hypothesis. Next, had B blackened before the rth move any interval of length $1/2^m$ to the right of x_r, this interval would necessarily coincide with I_1^r. By our strategy, this cannot happen. So, condition (ii) also remains valid.

Assume now that B has blackened the interval I_0^r at the rth move, but the interval $[0,1]$ still contains uncolored parts (which means that I_1^r is contained in $[0,1]$). Then condition (ii) clearly remains true, and we need to check (i) only. In our case, the intervals I_0^r and I_1^r are completely colored after the rth move, so x_{r+1} either reaches the right endpoint of I_1 or moves even further to the right. So, $x_{r+1} = x_r + \alpha$ for some $\alpha > 1/2^m$.

Next, any interval blackened by B before the rth move which intersects (x_r, x_{r+1}) should be contained in $[x_r, x_{r+1}]$; by (ii), all such intervals have different lengths not exceeding $1/2^m$, so the total amount of ink used for them is less than $2/2^m$. Thus, the amount of ink used for the segment $[0, x_{r+1}]$ does not exceed the sum of $2/2^m$, $3x_r$ (used for $[0, x_r]$), and $1/2^m$ used for the segment I_0^r. In total it gives at most $3(x_r + 1/2^m) < 3(x_r + \alpha) = 3x_{r+1}$. Thus condition (i) is also verified in this case. The claim is proved.

Finally, we can perform the desired estimation. Consider any situation in the game, say after the $(r-1)$st move; assume that the segment $[0, 1]$ is not completely black. By (ii), in the segment $[x_r, 1]$ player B has colored several segments of different lengths; all these lengths are negative powers of 2 not exceeding $1 - x_r$; thus the total amount of ink used for this interval is at most $2(1 - x_r)$. Using (i), we obtain that the total amount of ink used is at most $3x_r + 2(1 - x_r) < 3$. Thus the pot is not empty, and therefore A never wins.

Comment 1. Notice that this strategy works even if the pot contains initially only 3 units of ink.

Comment 2. There exist other strategies for B allowing him to prevent emptying the pot before the whole interval is colored. On the other hand, let us mention some idea which *does not* work.

Player B could try a strategy in which the set of blackened points in each round is an interval of the type $[0, x]$. Such a strategy cannot work (even if there is more ink available). Indeed, under the assumption that B uses such a strategy, let us prove by induction on s the following statement:

For any positive integer s, player A has a strategy picking only positive integers $m \leqslant s$ in which, if player B ever paints a point $x \geqslant 1 - 1/2^s$ then after some move, exactly the interval $[0, 1 - 1/2^s]$ is blackened, and the amount of ink used up to this moment is at least $s/2$.

For the base case $s = 1$, player A just picks $m = 1$ in the first round. If for some positive integer k player A has such a strategy, for $s + 1$ he can first rescale his strategy to the interval $[0, 1/2]$ (sending in each round half of the amount of ink he would give by the original strategy). Thus, after some round, the interval $[0, 1/2 - 1/2^{s+1}]$ becomes blackened, and the amount of ink used is at least $s/4$. Now player A picks $m = 1/2$, and player B spends $1/2$ unit of ink to blacken the interval $[0, 1/2]$. After that, player A again rescales his strategy to the interval $[1/2, 1]$, and player B spends at least $s/4$ units of ink to blacken the interval $[1/2, 1 - 1/2^{s+1}]$, so he spends in total at least $s/4 + 1/2 + s/4 = (s + 1)/2$ units of ink.

Comment 3. In order to avoid finiteness issues, the statement could be replaced by the following one:

> *Players A and B play a paintful game on the real numbers. Player A has a paint pot with four units of black ink. A quantity p of this ink suffices to blacken a (closed) real interval of length p. In the beginning of the game, player A chooses (and announces) a positive integer N. In every round, player A picks some positive integer $m \leqslant N$ and provides $1/2^m$ units of ink from the pot. The player B picks an integer k and blackens the interval from $k/2^m$ to $(k + 1)/2^m$ (some parts of this interval may happen to be blackened before). The goal of player A is to reach a situation where the pot is empty and the interval $[0, 1]$ is not completely blackened.*
>
> *Decide whether there exists a strategy for player A to win.*

However, the Problem Selection Committee believes that this version may turn out to be harder than the original one.

Geometry

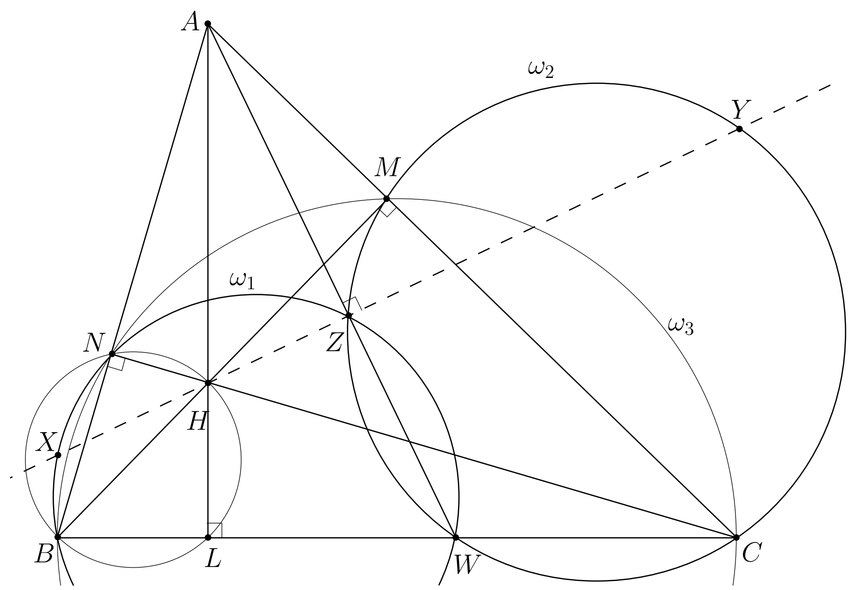
G1. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on side BC. Denote by M and N the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 which is diametrically opposite to W. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 which is diametrically opposite to W. Prove that X, Y and H are collinear.

(Thaliand)

Solution. Let L be the foot of the altitude from A, and let Z be the second intersection point of circles ω_1 and ω_2, other than W. We show that X, Y, Z and H lie on the same line.

Due to $\angle BNC = \angle BMC = 90^\circ$, the points B, C, N and M are concyclic; denote their circle by ω_3. Observe that the line WZ is the radical axis of ω_1 and ω_2; similarly, BN is the radical axis of ω_1 and ω_3, and CM is the radical axis of ω_2 and ω_3. Hence $A = BN \cap CM$ is the radical center of the three circles, and therefore WZ passes through A.

Since WX and WY are diameters in ω_1 and ω_2, respectively, we have $\angle WZX = \angle WZY = 90^\circ$, so the points X and Y lie on the line through Z, perpendicular to WZ.



The quadrilateral $BLHN$ is cyclic, because it has two opposite right angles. From the power of A with respect to the circles ω_1 and $BLHN$ we find $AL \cdot AH = AB \cdot AN = AW \cdot AZ$. If H lies on the line AW then this implies $H = Z$ immediately. Otherwise, by $\frac{AZ}{AH} = \frac{AL}{AW}$ the triangles AHZ and AWL are similar. Then $\angle HZA = \angle WLA = 90^\circ$, so the point H also lies on the line XYZ.

Comment. The original proposal also included a second statement:

> *Let P be the point on ω_1 such that WP is parallel to CN, and let Q be the point on ω_2 such that WQ is parallel to BM. Prove that P, Q and H are collinear if and only if $BW = CW$ or $AW \perp BC$.*

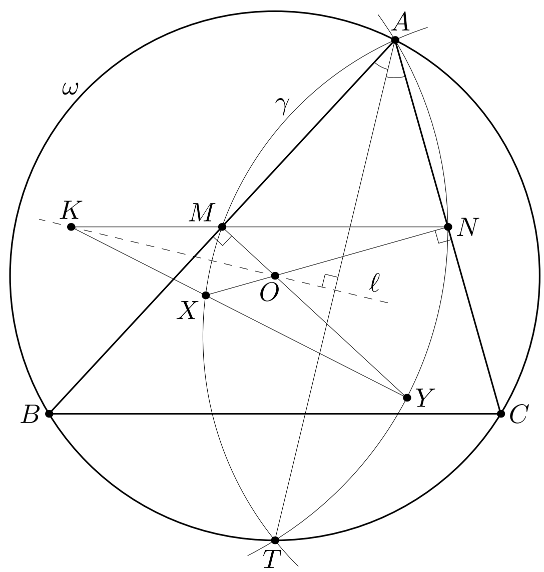
The Problem Selection Committee considered the first part more suitable for the competition.

G2. Let ω be the circumcircle of a triangle ABC. Denote by M and N the midpoints of the sides AB and AC, respectively, and denote by T the midpoint of the arc BC of ω not containing A. The circumcircles of the triangles AMT and ANT intersect the perpendicular bisectors of AC and AB at points X and Y, respectively; assume that X and Y lie inside the triangle ABC. The lines MN and XY intersect at K. Prove that $KA = KT$.

(Iran)

Solution 1. Let O be the center of ω, thus $O = MY \cap NX$. Let ℓ be the perpendicular bisector of AT (it also passes through O). Denote by r the operation of reflection about ℓ. Since AT is the angle bisector of $\angle BAC$, the line $r(AB)$ is parallel to AC. Since $OM \perp AB$ and $ON \perp AC$, this means that the line $r(OM)$ is parallel to the line ON and passes through O, so $r(OM) = ON$. Finally, the circumcircle γ of the triangle AMT is symmetric about ℓ, so $r(\gamma) = \gamma$. Thus the point M maps to the common point of ON with the arc AMT of γ — that is, $r(M) = X$.

Similarly, $r(N) = Y$. Thus, we get $r(MN) = XY$, and the common point K of MN nd XY lies on ℓ. This means exactly that $KA = KT$.



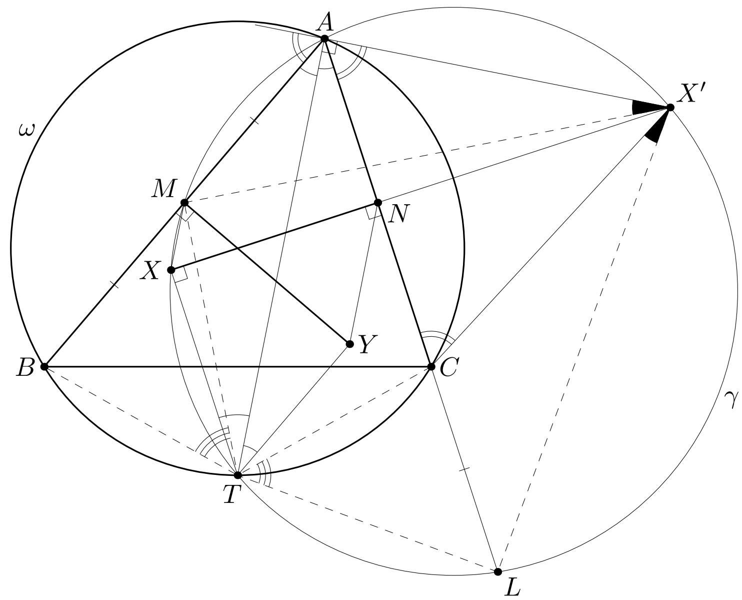
Solution 2. Let L be the second common point of the line AC with the circumcircle γ of the triangle AMT. From the cyclic quadrilaterals $ABTC$ and $AMTL$ we get $\angle BTC = 180^\circ - \angle BAC = \angle MTL$, which implies $\angle BTM = \angle CTL$. Since AT is an angle bisector in these quadrilaterals, we have $BT = TC$ and $MT = TL$. Thus the triangles BTM and CTL are congruent, so $CL = BM = AM$.

Let X' be the common point of the line NX with the external bisector of $\angle BAC$; notice that it lies outside the triangle ABC. Then we have $\angle TAX' = 90^\circ$ and $X'A = X'C$, so we get $\angle X'AM = 90^\circ + \angle BAC/2 = 180^\circ - \angle X'AC = 180^\circ - \angle X'CA = \angle X'CL$. Thus the triangles $X'AM$ and $X'CL$ are congruent, and therefore

$$\angle MX'L = \angle AX'C + (\angle CX'L - \angle AX'M) = \angle AX'C = 180^\circ - 2\angle X'AC = \angle BAC = \angle MAL.$$

This means that X' lies on γ.

Thus we have $\angle TXN = \angle TXX' = \angle TAX' = 90^\circ$, so $TX \parallel AC$. Then $\angle XTA = \angle TAC = \angle TAM$, so the cyclic quadrilateral $MATX$ is an isosceles trapezoid. Similarly, $NATY$ is an isosceles trapezoid, so again the lines MN and XY are the reflections of each other about the perpendicular bisector of AT. Thus K belongs to this perpendicular bisector.



Comment. There are several different ways of showing that the points X and M are symmetrical with respect to ℓ. For instance, one can show that the quadrilaterals $AMON$ and $TXOY$ are congruent. We chose Solution 1 as a simple way of doing it. On the other hand, Solution 2 shows some other interesting properties of the configuration.

Let us define Y', analogously to X', as the common point of MY and the external bisector of $\angle BAC$. One may easily see that in general the lines MN and $X'Y'$ (which is the external bisector of $\angle BAC$) do not intersect on the perpendicular bisector of AT. Thus, any solution should involve some argument using the choice of the intersection points X and Y.

G3. In a triangle ABC, let D and E be the feet of the angle bisectors of angles A and B, respectively. A rhombus is inscribed into the quadrilateral $AEDB$ (all vertices of the rhombus lie on different sides of $AEDB$). Let φ be the non-obtuse angle of the rhombus. Prove that $\varphi \leqslant \max\{\angle BAC, \angle ABC\}$.

(Serbia)

Solution 1. Let K, L, M, and N be the vertices of the rhombus lying on the sides AE, ED, DB, and BA, respectively. Denote by $d(X, YZ)$ the distance from a point X to a line YZ. Since D and E are the feet of the bisectors, we have $d(D, AB) = d(D, AC)$, $d(E, AB) = d(E, BC)$, and $d(D, BC) = d(E, AC) = 0$, which implies

$$d(D, AC) + d(D, BC) = d(D, AB) \quad \text{and} \quad d(E, AC) + d(E, BC) = d(E, AB).$$

Since L lies on the segment DE and the relation $d(X, AC) + d(X, BC) = d(X, AB)$ is linear in X inside the triangle, these two relations imply

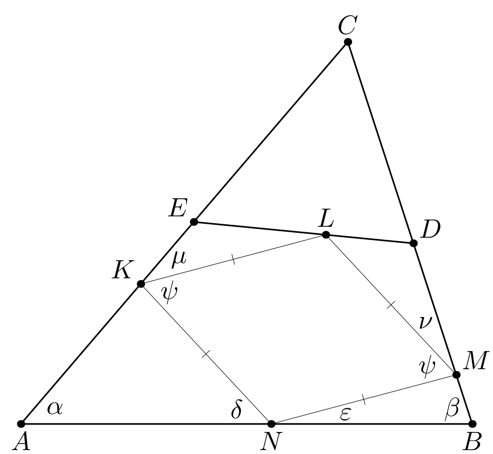
$$d(L, AC) + d(L, BC) = d(L, AB). \tag{1}$$

Denote the angles as in the figure below, and denote $a = KL$. Then we have $d(L, AC) = a \sin \mu$ and $d(L, BC) = a \sin \nu$. Since $KLMN$ is a parallelogram lying on one side of AB, we get

$$d(L, AB) = d(L, AB) + d(N, AB) = d(K, AB) + d(M, AB) = a(\sin \delta + \sin \varepsilon).$$

Thus the condition (1) reads

$$\sin \mu + \sin \nu = \sin \delta + \sin \varepsilon. \tag{2}$$



If one of the angles α and β is non-acute, then the desired inequality is trivial. So we assume that $\alpha, \beta < \pi/2$. It suffices to show then that $\psi = \angle NKL \leqslant \max\{\alpha, \beta\}$.

Assume, to the contrary, that $\psi > \max\{\alpha, \beta\}$. Since $\mu + \psi = \angle CKN = \alpha + \delta$, by our assumption we obtain $\mu = (\alpha - \psi) + \delta < \delta$. Similarly, $\nu < \varepsilon$. Next, since $KN \parallel ML$, we have $\beta = \delta + \nu$, so $\delta < \beta < \pi/2$. Similarly, $\varepsilon < \pi/2$. Finally, by $\mu < \delta < \pi/2$ and $\nu < \varepsilon < \pi/2$, we obtain

$$\sin \mu < \sin \delta \quad \text{and} \quad \sin \nu < \sin \varepsilon.$$

This contradicts (2).

Comment. One can see that the equality is achieved if $\alpha = \beta$ for every rhombus inscribed into the quadrilateral $AEDB$.

G4. Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

(Georgia)

Solution 1. Denote by ω the circumcircle of the triangle ABC, and let $\angle ACB = \gamma$. Note that the condition $\gamma < \angle CBA$ implies $\gamma < 90^\circ$. Since $\angle PBA = \gamma$, the line PB is tangent to ω, so $PA \cdot PC = PB^2 = PD^2$. By $\frac{PA}{PD} = \frac{PD}{PC}$ the triangles PAD and PDC are similar, and $\angle ADP = \angle DCP$.

Next, since $\angle ABQ = \angle ACB$, the triangles ABC and AQB are also similar. Then $\angle AQB = \angle ABC = \angle ARC$, which means that the points D, R, C, and Q are concyclic. Therefore $\angle DRQ = \angle DCQ = \angle ADP$.

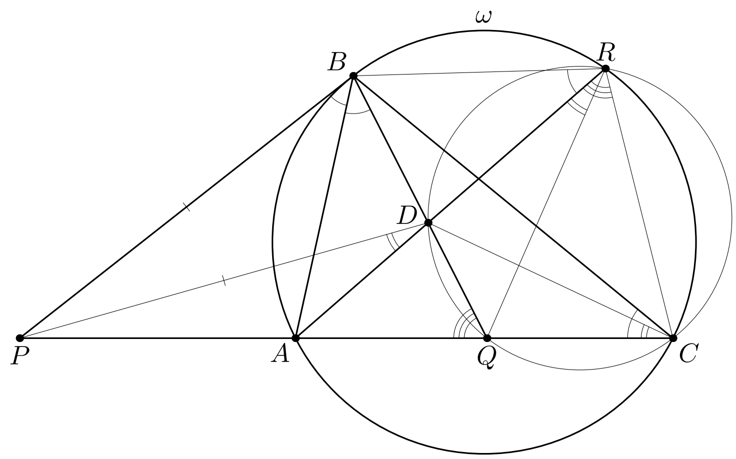


Figure 1

Now from $\angle ARB = \angle ACB = \gamma$ and $\angle PDB = \angle PBD = 2\gamma$ we get

$$\angle QBR = \angle ADB - \angle ARB = \angle ADP + \angle PDB - \angle ARB = \angle DRQ + \gamma = \angle QRB,$$

so the triangle QRB is isosceles, which yields $QB = QR$.

Solution 2. Again, denote by ω the circumcircle of the triangle ABC. Denote $\angle ACB = \gamma$. Since $\angle PBA = \gamma$, the line PB is tangent to ω.

Let E be the second intersection point of BQ with ω. If V' is any point on the ray CE beyond E, then $\angle BEV' = 180^\circ - \angle BEC = 180^\circ - \angle BAC = \angle PAB$; together with $\angle ABQ = \angle PBA$ this shows firstly, that the rays BA and CE intersect at some point V, and secondly that the triangle VEB is similar to the triangle PAB. Thus we have $\angle BVE = \angle BPA$. Next, $\angle AEV = \angle BEV - \gamma = \angle PAB - \angle ABQ = \angle AQB$; so the triangles PBQ and VAE are also similar.

Let PH be an altitude in the isosceles triangle PBD; then $BH = HD$. Let G be the intersection point of PH and AB. By the symmetry with respect to PH, we have $\angle BDG = \angle DBG = \gamma = \angle BEA$; thus $DG \parallel AE$ and hence $\frac{BG}{GA} = \frac{BD}{DE}$. Thus the points G and D correspond to each other in the similar triangles PAB and VEB, so $\angle DVB = \angle GPB = 90^\circ - \angle PBQ = 90^\circ - \angle VAE$. Thus $VD \perp AE$.

Let T be the common point of VD and AE, and let DS be an altitude in the triangle BDR. The points S and T are the feet of corresponding altitudes in the similar triangles ADE and BDR, so $\frac{BS}{SR} = \frac{AT}{TE}$. On the other hand, the points T and H are feet of corresponding altitudes in the similar triangles VAE and PBQ, so $\frac{AT}{TE} = \frac{BH}{HQ}$. Thus $\frac{BS}{SR} = \frac{AT}{TE} = \frac{BH}{HQ}$, and the triangles BHS and BQR are similar.

Finally, SH is a median in the right-angled triangle SBD; so $BH = HS$, and hence $BQ = QR$.

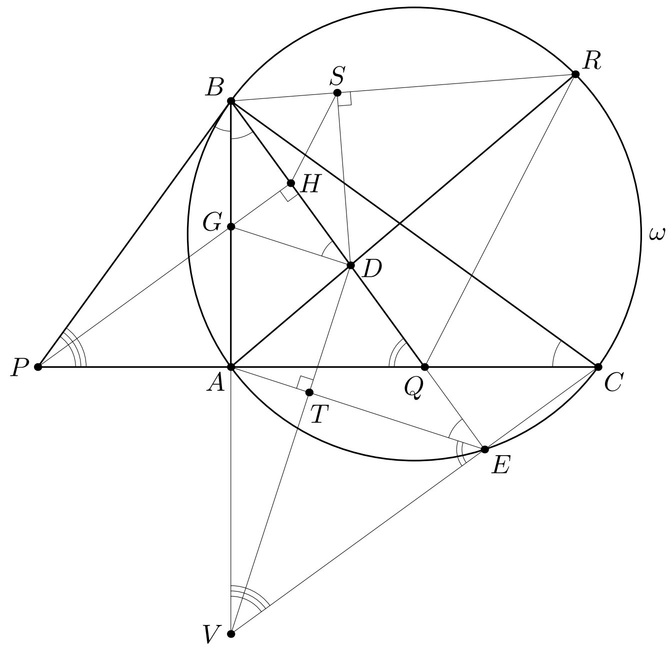


Figure 2

Solution 3. Denote by ω and O the circumcircle of the triangle ABC and its center, respectively. From the condition $\angle PBA = \angle BCA$ we know that BP is tangent to ω.

Let E be the second point of intersection of ω and BD. Due to the isosceles triangle BDP, the tangent of ω at E is parallel to DP and consequently it intersects BP at some point L. Of course, $PD \parallel LE$. Let M be the midpoint of BE, and let H be the midpoint of BR. Notice that $\angle AEB = \angle ACB = \angle ABQ = \angle ABE$, so A lies on the perpendicular bisector of BE; thus the points L, A, M, and O are collinear. Let ω_1 be the circle with diameter BO. Let $Q' = HO \cap BE$; since HO is the perpendicular bisector of BR, the statement of the problem is equivalent to $Q' = Q$.

Consider the following sequence of projections (see Fig. 3).

1. Project the line BE to the line LB through the center A. (This maps Q to P.)
2. Project the line LB to BE in parallel direction with LE. ($P \mapsto D$.)
3. Project the line BE to the circle ω through its point A. ($D \mapsto R$.)
4. Scale ω by the ratio $\frac{1}{2}$ from the point B to the circle ω_1. ($R \mapsto H$.)
5. Project ω_1 to the line BE through its point O. ($H \mapsto Q'$.)

We prove that the composition of these transforms, which maps the line BE to itself, is the identity. To achieve this, it suffices to show three fixed points. An obvious fixed point is B which is fixed by all the transformations above. Another fixed point is M, its path being $M \mapsto L \mapsto E \mapsto E \mapsto M \mapsto M$.

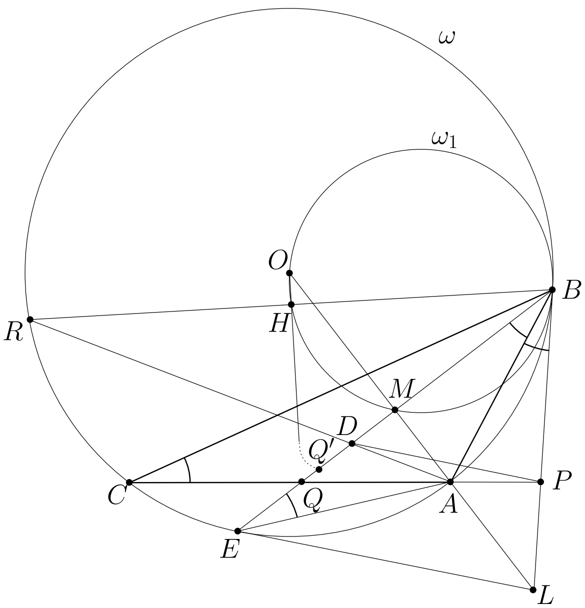


Figure 3

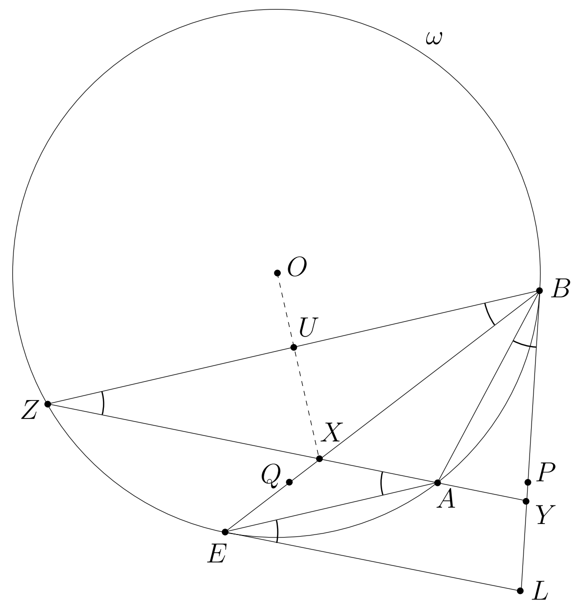


Figure 4

In order to show a third fixed point, draw a line parallel with LE through A; let that line intersect BE, LB and ω at X, Y and $Z \neq A$, respectively (see Fig. 4). We show that X is a fixed point. The images of X at the first three transformations are $X \mapsto Y \mapsto X \mapsto Z$. From $\angle XBZ = \angle EAZ = \angle AEL = \angle LBA = \angle BZX$ we can see that the triangle XBZ is isosceles. Let U be the midpoint of BZ; then the last two transformations do $Z \mapsto U \mapsto X$, and the point X is fixed.

Comment. Verifying that the point E is fixed seems more natural at first, but it appears to be less straightforward. Here we outline a possible proof.

Let the images of E at the first three transforms above be F, G and I. After comparing the angles depicted in Fig. 5 (noticing that the quadrilateral $AFBG$ is cyclic) we can observe that the tangent LE of ω is parallel to BI. Then, similarly to the above reasons, the point E is also fixed.

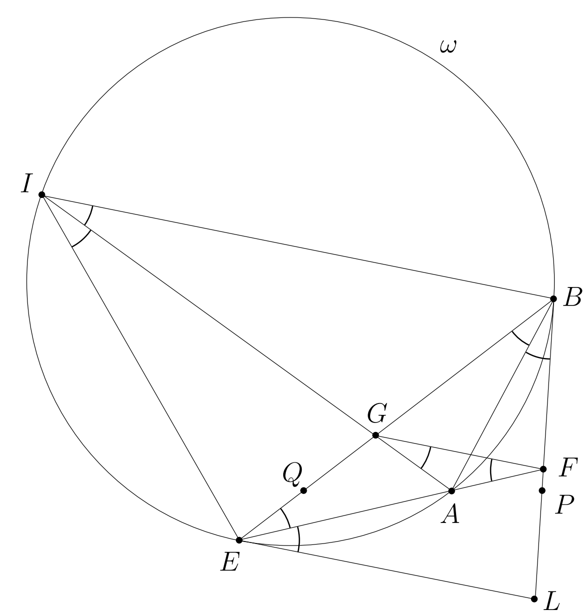


Figure 5

G5. Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

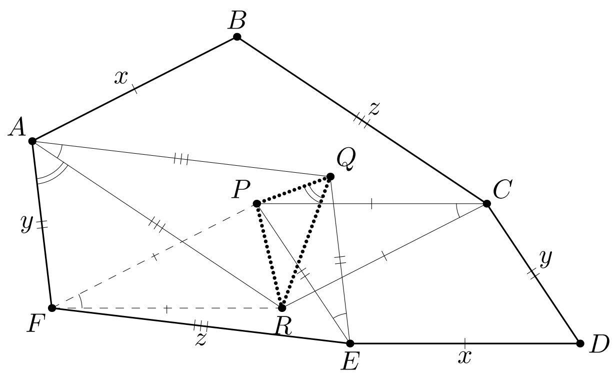
(Ukraine)

In all three solutions, we denote $\theta = \angle A - \angle D = \angle C - \angle F = \angle E - \angle B$ and assume without loss of generality that $\theta \geqslant 0$.

Solution 1. Let $x = AB = DE$, $y = CD = FA$, $z = EF = BC$. Consider the points P, Q, and R such that the quadrilaterals $CDEP$, $EFAQ$, and $ABCR$ are parallelograms. We compute

$$\begin{aligned}\angle PEQ &= \angle FEQ + \angle DEP - \angle E = (180^\circ - \angle F) + (180^\circ - \angle D) - \angle E \\ &= 360^\circ - \angle D - \angle E - \angle F = \tfrac{1}{2}\big(\angle A + \angle B + \angle C - \angle D - \angle E - \angle F\big) = \theta/2.\end{aligned}$$

Similarly, $\angle QAR = \angle RCP = \theta/2$.



If $\theta = 0$, since $\triangle RCP$ is isosceles, $R = P$. Therefore $AB \parallel RC = PC \parallel ED$, so $ABDE$ is a parallelogram. Similarly, $BCEF$ and $CDFA$ are parallelograms. It follows that AD, BE and CF meet at their common midpoint.

Now assume $\theta > 0$. Since $\triangle PEQ$, $\triangle QAR$, and $\triangle RCP$ are isosceles and have the same angle at the apex, we have $\triangle PEQ \sim \triangle QAR \sim \triangle RCP$ with ratios of similarity $y : z : x$. Thus

$$\triangle PQR \text{ is similar to the triangle with sidelengths } y,\ z, \text{ and } x. \tag{1}$$

Next, notice that

$$\frac{RQ}{QP} = \frac{z}{y} = \frac{RA}{AF}$$

and, using directed angles between rays,

$$\begin{aligned}\measuredangle(RQ, QP) &= \measuredangle(RQ, QE) + \measuredangle(QE, QP) \\ &= \measuredangle(RQ, QE) + \measuredangle(RA, RQ) = \measuredangle(RA, QE) = \measuredangle(RA, AF).\end{aligned}$$

Thus $\triangle PQR \sim \triangle FAR$. Since $FA = y$ and $AR = z$, (1) then implies that $FR = x$. Similarly $FP = x$. Therefore $CRFP$ is a rhombus.

We conclude that CF is the perpendicular bisector of PR. Similarly, BE is the perpendicular bisector of PQ and AD is the perpendicular bisector of QR. It follows that AD, BE, and CF are concurrent at the circumcenter of PQR.

Solution 2. Let $X = CD \cap EF$, $Y = EF \cap AB$, $Z = AB \cap CD$, $X' = FA \cap BC$, $Y' = BC \cap DE$, and $Z' = DE \cap FA$. From $\angle A + \angle B + \angle C = 360^\circ + \theta/2$ we get $\angle A + \angle B > 180^\circ$ and $\angle B + \angle C > 180^\circ$, so Z and X' are respectively on the opposite sides of BC and AB from the hexagon. Similar conclusions hold for X, Y, Y', and Z'. Then

$$\angle YZX = \angle B + \angle C - 180^\circ = \angle E + \angle F - 180^\circ = \angle Y'Z'X',$$

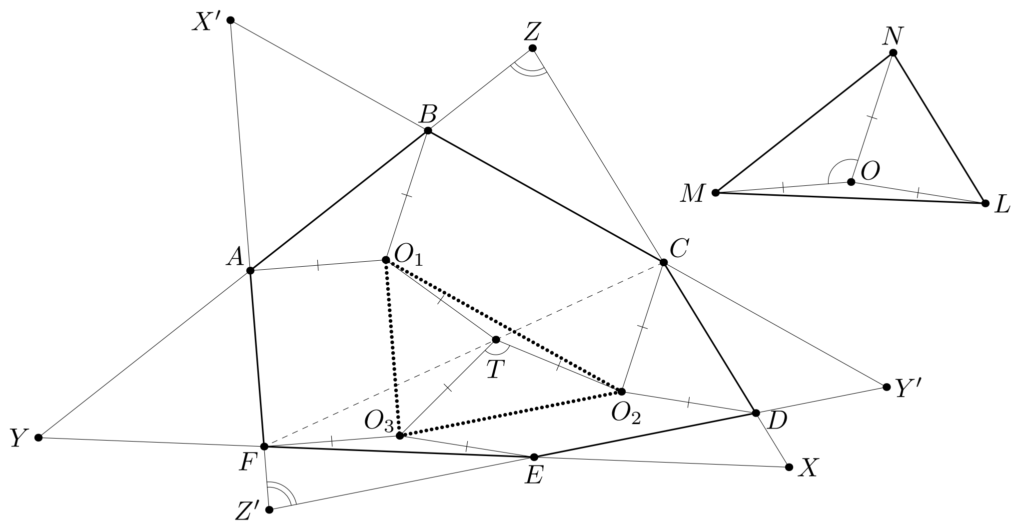
and similarly $\angle ZXY = \angle Z'X'Y'$ and $\angle XYZ = \angle X'Y'Z'$, so $\triangle XYZ \sim \triangle X'Y'Z'$. Thus there is a rotation R which sends $\triangle XYZ$ to a triangle with sides parallel to $\triangle X'Y'Z'$. Since $AB = DE$ we have $R(\overrightarrow{AB}) = \overrightarrow{DE}$. Similarly, $R(\overrightarrow{CD}) = \overrightarrow{FA}$ and $R(\overrightarrow{EF}) = \overrightarrow{BC}$. Therefore

$$\overrightarrow{0} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right) + R\left(\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF}\right).$$

If R is a rotation by 180°, then any two opposite sides of our hexagon are equal and parallel, so the three diagonals meet at their common midpoint. Otherwise, we must have

$$\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{EF} = \overrightarrow{0},$$

or else we would have two vectors with different directions whose sum is $\overrightarrow{0}$.



This allows us to consider a triangle LMN with $\overrightarrow{LM} = \overrightarrow{EF}$, $\overrightarrow{MN} = \overrightarrow{AB}$, and $\overrightarrow{NL} = \overrightarrow{CD}$. Let O be the circumcenter of $\triangle LMN$ and consider the points O_1, O_2, O_3 such that $\triangle AO_1B$, $\triangle CO_2D$, and $\triangle EO_3F$ are translations of $\triangle MON$, $\triangle NOL$, and $\triangle LOM$, respectively. Since FO_3 and AO_1 are translations of MO, quadrilateral AFO_3O_1 is a parallelogram and $O_3O_1 = FA = CD = NL$. Similarly, $O_1O_2 = LM$ and $O_2O_3 = MN$. Therefore $\triangle O_1O_2O_3 \cong \triangle LMN$. Moreover, by means of the rotation R one may check that these triangles have the same orientation.

Let T be the circumcenter of $\triangle O_1O_2O_3$. We claim that AD, BE, and CF meet at T. Let us show that C, T, and F are collinear. Notice that $CO_2 = O_2T = TO_3 = O_3F$ since they are all equal to the circumradius of $\triangle LMN$. Therefore $\triangle TO_3F$ and $\triangle CO_2T$ are isosceles. Using directed angles between rays again, we get

$$\measuredangle(TF, TO_3) = \measuredangle(FO_3, FT) \qquad \text{and} \qquad \measuredangle(TO_2, TC) = \measuredangle(CT, CO_2). \tag{2}$$

Also, T and O are the circumcenters of the congruent triangles $\triangle O_1O_2O_3$ and $\triangle LMN$ so we have $\measuredangle(TO_3, TO_2) = \measuredangle(ON, OM)$. Since CO_2 and FO_3 are translations of NO and MO respectively, this implies

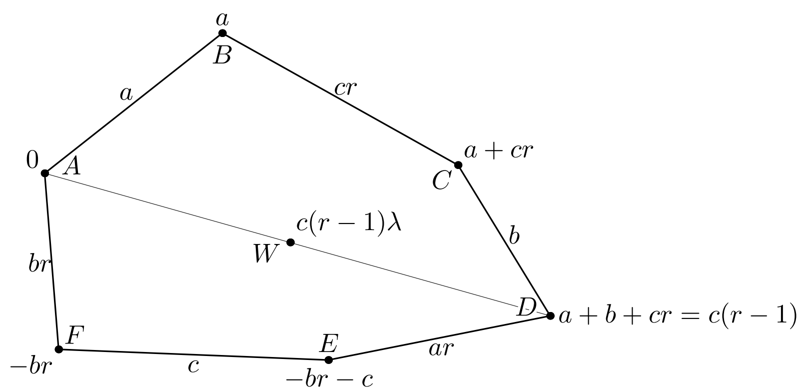
$$\measuredangle(TO_3, TO_2) = \measuredangle(CO_2, FO_3). \tag{3}$$

Adding the three equations in (2) and (3) gives

$$\measuredangle(TF, TC) = \measuredangle(CT, FT) = -\measuredangle(TF, TC)$$

which implies that T is on CF. Analogous arguments show that it is on AD and BE also. The desired result follows.

Solution 3. Place the hexagon on the complex plane, with A at the origin and vertices labelled clockwise. Now A, B, C, D, E, F represent the corresponding complex numbers. Also consider the complex numbers a, b, c, a', b', c' given by $B - A = a$, $D - C = b$, $F - E = c$, $E - D = a'$, $A - F = b'$, and $C - B = c'$. Let $k = |a|/|b|$. From $a/b' = -ke^{i\angle A}$ and $a'/b = -ke^{i\angle D}$ we get that $(a'/a)(b'/b) = e^{-i\theta}$ and similarly $(b'/b)(c'/c) = e^{-i\theta}$ and $(c'/c)(a'/a) = e^{-i\theta}$. It follows that $a' = ar$, $b' = br$, and $c' = cr$ for a complex number r with $|r| = 1$, as shown below.



We have

$$0 = a + cr + b + ar + c + br = (a + b + c)(1 + r).$$

If $r = -1$, then the hexagon is centrally symmetric and its diagonals intersect at its center of symmetry. Otherwise

$$a + b + c = 0.$$

Therefore

$$A = 0, \quad B = a, \quad C = a + cr, \quad D = c(r-1), \quad E = -br - c, \quad F = -br.$$

Now consider a point W on AD given by the complex number $c(r-1)\lambda$, where λ is a real number with $0 < \lambda < 1$. Since $D \neq A$, we have $r \neq 1$, so we can define $s = 1/(r-1)$. From $r\overline{r} = |r|^2 = 1$ we get

$$1 + s = \frac{r}{r-1} = \frac{r}{r - r\overline{r}} = \frac{1}{1-\overline{r}} = -\overline{s}.$$

Now,

$$\begin{aligned} W \text{ is on } BE &\iff c(r-1)\lambda - a \parallel a - (-br - c) = b(r-1) \iff c\lambda - as \parallel b \\ &\iff -a\lambda - b\lambda - as \parallel b \iff a(\lambda + s) \parallel b. \end{aligned}$$

One easily checks that $r \neq \pm 1$ implies that $\lambda + s \neq 0$ since s is not real. On the other hand,

$$\begin{aligned} W \text{ on } CF &\iff c(r-1)\lambda + br \parallel -br - (a + cr) = a(r-1) \iff c\lambda + b(1+s) \parallel a \\ &\iff -a\lambda - b\lambda - b\overline{s} \parallel a \iff b(\lambda + \overline{s}) \parallel a \iff b \parallel a(\lambda + s), \end{aligned}$$

where in the last step we use that $(\lambda + s)(\lambda + \overline{s}) = |\lambda + s|^2 \in \mathbb{R}_{>0}$. We conclude that $AD \cap BE = CF \cap BE$, and the desired result follows.

G6. Let the excircle of the triangle ABC lying opposite to A touch its side BC at the point A_1. Define the points B_1 and C_1 analogously. Suppose that the circumcentre of the triangle $A_1B_1C_1$ lies on the circumcircle of the triangle ABC. Prove that the triangle ABC is right-angled.

(Russia)

Solution 1. Denote the circumcircles of the triangles ABC and $A_1B_1C_1$ by Ω and Γ, respectively. Denote the midpoint of the arc CB of Ω containing A by A_0, and define B_0 as well as C_0 analogously. By our hypothesis the centre Q of Γ lies on Ω.

Lemma. One has $A_0B_1 = A_0C_1$. Moreover, the points A, A_0, B_1, and C_1 are concyclic. Finally, the points A and A_0 lie on the same side of B_1C_1. Similar statements hold for B and C.

Proof. Let us consider the case $A = A_0$ first. Then the triangle ABC is isosceles at A, which implies $AB_1 = AC_1$ while the remaining assertions of the Lemma are obvious. So let us suppose $A \neq A_0$ from now on.

By the definition of A_0, we have $A_0B = A_0C$. It is also well known and easy to show that $BC_1 = CB_1$. Next, we have $\angle C_1BA_0 = \angle ABA_0 = \angle ACA_0 = \angle B_1CA_0$. Hence the triangles A_0BC_1 and A_0CB_1 are congruent. This implies $A_0C_1 = A_0B_1$, establishing the first part of the Lemma. It also follows that $\angle A_0C_1A = \angle A_0B_1A$, as these are exterior angles at the corresponding vertices C_1 and B_1 of the congruent triangles A_0BC_1 and A_0CB_1. For that reason the points A, A_0, B_1, and C_1 are indeed the vertices of some cyclic quadrilateral two opposite sides of which are AA_0 and B_1C_1. $\square$

Now we turn to the solution. Evidently the points A_1, B_1, and C_1 lie interior to some semicircle arc of Γ, so the triangle $A_1B_1C_1$ is obtuse-angled. Without loss of generality, we will assume that its angle at B_1 is obtuse. Thus Q and B_1 lie on different sides of A_1C_1; obviously, the same holds for the points B and B_1. So, the points Q and B are on the same side of A_1C_1.

Notice that the perpendicular bisector of A_1C_1 intersects Ω at two points lying on different sides of A_1C_1. By the first statement from the Lemma, both points B_0 and Q are among these points of intersection; since they share the same side of A_1C_1, they coincide (see Figure 1).

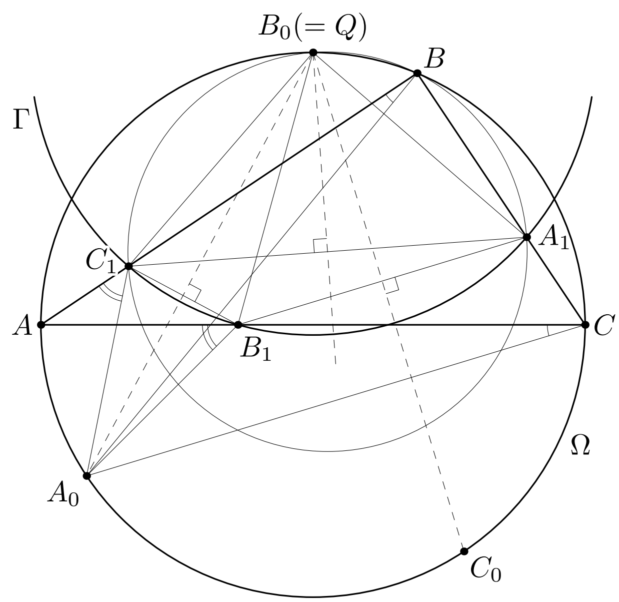


Figure 1

Now, by the first part of the Lemma again, the lines QA_0 and QC_0 are the perpendicular bisectors of B_1C_1 and A_1B_1, respectively. Thus

$$\angle C_1B_0A_1 = \angle C_1B_0B_1 + \angle B_1B_0A_1 = 2\angle A_0B_0B_1 + 2\angle B_1B_0C_0 = 2\angle A_0B_0C_0 = 180^\circ - \angle ABC,$$

recalling that A_0 and C_0 are the midpoints of the arcs CB and BA, respectively.

On the other hand, by the second part of the Lemma we have

$$\angle C_1B_0A_1 = \angle C_1BA_1 = \angle ABC.$$

From the last two equalities, we get $\angle ABC = 90^\circ$, whereby the problem is solved.

Solution 2. Let Q again denote the centre of the circumcircle of the triangle $A_1B_1C_1$, that lies on the circumcircle Ω of the triangle ABC. We first consider the case where Q coincides with one of the vertices of ABC, say $Q = B$. Then $BC_1 = BA_1$ and consequently the triangle ABC is isosceles at B. Moreover we have $BC_1 = B_1C$ in any triangle, and hence $BB_1 = BC_1 = B_1C$; similarly, $BB_1 = B_1A$. It follows that B_1 is the centre of Ω and that the triangle ABC has a right angle at B.

So from now on we may suppose $Q \notin \{A, B, C\}$. We start with the following well known fact.

Lemma. Let XYZ and $X'Y'Z'$ be two triangles with $XY = X'Y'$ and $YZ = Y'Z'$.

(i) If $XZ \neq X'Z'$ and $\angle YZX = \angle Y'Z'X'$, then $\angle ZXY + \angle Z'X'Y' = 180^\circ$.

(ii) If $\angle YZX + \angle X'Z'Y' = 180^\circ$, then $\angle ZXY = \angle Y'X'Z'$.

Proof. For both parts, we may move the triangle XYZ through the plane until $Y = Y'$ and $Z = Z'$. Possibly after reflecting one of the two triangles about YZ, we may also suppose that X and X' lie on the same side of YZ if we are in case (i) and on different sides if we are in case (ii). In both cases, the points X, Z, and X' are collinear due to the angle condition (see Fig. 2). Moreover we have $X \neq X'$, because in case (i) we assumed $XZ \neq X'Z'$ and in case (ii) these points even lie on different sides of YZ. Thus the triangle $XX'Y$ is isosceles at Y. The claim now follows by considering the equal angles at its base. □

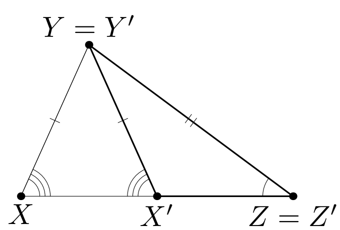


Figure 2(i)

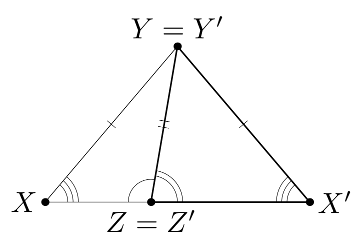


Figure 2(ii)

Relabeling the vertices of the triangle ABC if necessary we may suppose that Q lies in the interior of the arc AB of Ω not containing C. We will sometimes use tacitly that the six triangles QBA_1, QA_1C, QCB_1, QB_1A, QC_1A, and QBC_1 have the same orientation.

As Q cannot be the circumcentre of the triangle ABC, it is impossible that $QA = QB = QC$ and thus we may also suppose that $QC \neq QB$. Now the above Lemma (i) is applicable to the triangles QB_1C and QC_1B, since $QB_1 = QC_1$ and $B_1C = C_1B$, while $\angle B_1CQ = \angle C_1BQ$ holds as both angles appear over the same side of the chord QA in Ω (see Fig. 3). So we get

$$\angle CQB_1 + \angle BQC_1 = 180^\circ. \tag{1}$$

We claim that $QC = QA$. To see this, let us assume for the sake of a contradiction that $QC \neq QA$. Then arguing similarly as before but now with the triangles QA_1C and QC_1A we get

$$\angle A_1QC + \angle C_1QA = 180^\circ.$$

Adding this equation to (1), we get $\angle A_1QB_1 + \angle BQA = 360^\circ$, which is absurd as both summands lie in the interval $(0^\circ, 180^\circ)$.

This proves $QC = QA$; so the triangles QA_1C and QC_1A are congruent their sides being equal, which in turn yields

$$\angle A_1QC = \angle C_1QA. \tag{2}$$

Finally our Lemma (ii) is applicable to the triangles QA_1B and QB_1A. Indeed we have $QA_1 = QB_1$ and $A_1B = B_1A$ as usual, and the angle condition $\angle A_1BQ + \angle QAB_1 = 180^\circ$ holds as A and B lie on different sides of the chord QC in Ω. Consequently we have

$$\angle BQA_1 = \angle B_1QA. \tag{3}$$

From (1) and (3) we get

$$(\angle B_1QC + \angle B_1QA) + (\angle C_1QB - \angle BQA_1) = 180^\circ,$$

i.e. $\angle CQA + \angle A_1QC_1 = 180^\circ$. In light of (2) this may be rewritten as $2\angle CQA = 180^\circ$ and as Q lies on Ω this implies that the triangle ABC has a right angle at B.

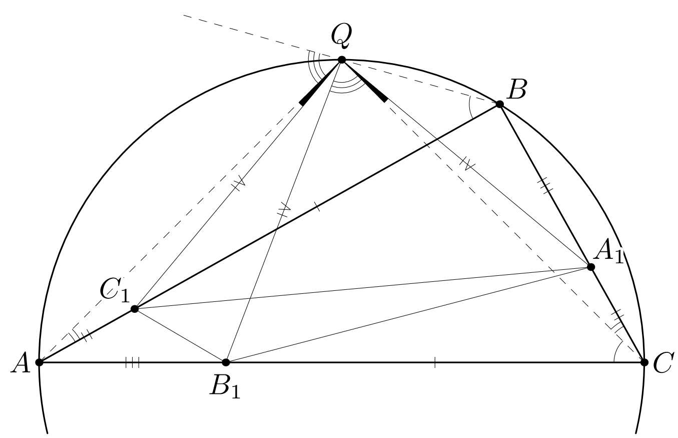


Figure 3

Comment 1. One may also check that Q is in the interior of Ω if and only if the triangle ABC is acute-angled.

Comment 2. The original proposal asked to prove the converse statement as well: if the triangle ABC is right-angled, then the point Q lies on its circumcircle. The Problem Selection Committee thinks that the above simplified version is more suitable for the competition.

Number Theory

N1. Let $\mathbb{Z}_{>0}$ be the set of positive integers. Find all functions $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that

$$m^2 + f(n) \mid mf(m) + n$$

for all positive integers m and n.

(Malaysia)

Answer. $f(n) = n$.

Solution 1. Setting $m = n = 2$ tells us that $4 + f(2) \mid 2f(2) + 2$. Since $2f(2) + 2 < 2(4 + f(2))$, we must have $2f(2) + 2 = 4 + f(2)$, so $f(2) = 2$. Plugging in $m = 2$ then tells us that $4 + f(n) \mid 4 + n$, which implies that $f(n) \leqslant n$ for all n.

Setting $m = n$ gives $n^2 + f(n) \mid nf(n) + n$, so $nf(n) + n \geqslant n^2 + f(n)$ which we rewrite as $(n-1)(f(n) - n) \geqslant 0$. Therefore $f(n) \geqslant n$ for all $n \geqslant 2$. This is trivially true for $n = 1$ also.

It follows that $f(n) = n$ for all n. This function obviously satisfies the desired property.

Solution 2. Setting $m = f(n)$ we get $f(n)(f(n) + 1) \mid f(n)f(f(n)) + n$. This implies that $f(n) \mid n$ for all n.

Now let m be any positive integer, and let $p > 2m^2$ be a prime number. Note that $p > mf(m)$ also. Plugging in $n = p - mf(m)$ we learn that $m^2 + f(n)$ divides p. Since $m^2 + f(n)$ cannot equal 1, it must equal p. Therefore $p - m^2 = f(n) \mid n = p - mf(m)$. But $p - mf(m) < p < 2(p - m^2)$, so we must have $p - mf(m) = p - m^2$, i.e., $f(m) = m$.

Solution 3. Plugging $m = 1$ we obtain $1 + f(n) \leqslant f(1) + n$, so $f(n) \leqslant n + c$ for the constant $c = f(1) - 1$. Assume that $f(n) \neq n$ for some fixed n. When m is large enough (e.g. $m \geqslant \max(n, c+1)$) we have

$$mf(m) + n \leqslant m(m + c) + n \leqslant 2m^2 < 2(m^2 + f(n)),$$

so we must have $mf(m) + n = m^2 + f(n)$. This implies that

$$0 \neq f(n) - n = m(f(m) - m),$$

which is impossible for $m > |f(n) - n|$. It follows that f is the identity function.

N2. Prove that for any pair of positive integers k and n there exist k positive integers $m_1, m_2, \ldots, m_k$ such that

$$1+\frac{2^k-1}{n}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_k}\right).$$

(Japan)

Solution 1. We proceed by induction on k. For $k=1$ the statement is trivial. Assuming we have proved it for $k=j-1$, we now prove it for $k=j$.

Case 1. $n=2t-1$ for some positive integer t.

Observe that

$$1+\frac{2^j-1}{2t-1}=\frac{2(t+2^{j-1}-1)}{2t}\cdot\frac{2t}{2t-1}=\left(1+\frac{2^{j-1}-1}{t}\right)\left(1+\frac{1}{2t-1}\right).$$

By the induction hypothesis we can find $m_1, \ldots, m_{j-1}$ such that

$$1+\frac{2^{j-1}-1}{t}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_{j-1}}\right),$$

so setting $m_j=2t-1$ gives the desired expression.

Case 2. $n=2t$ for some positive integer t.

Now we have

$$1+\frac{2^j-1}{2t}=\frac{2t+2^j-1}{2t+2^j-2}\cdot\frac{2t+2^j-2}{2t}=\left(1+\frac{1}{2t+2^j-2}\right)\left(1+\frac{2^{j-1}-1}{t}\right),$$

noting that $2t+2^j-2>0$. Again, we use that

$$1+\frac{2^{j-1}-1}{t}=\left(1+\frac{1}{m_1}\right)\left(1+\frac{1}{m_2}\right)\cdots\left(1+\frac{1}{m_{j-1}}\right).$$

Setting $m_j=2t+2^j-2$ then gives the desired expression.

Solution 2. Consider the base 2 expansions of the residues of $n-1$ and $-n$ modulo 2^k:

$$\begin{aligned} n-1 &\equiv 2^{a_1}+2^{a_2}+\cdots+2^{a_r} \pmod{2^k} \qquad &&\text{where } \ 0\leqslant a_1<a_2<\ldots<a_r\leqslant k-1,\\ -n &\equiv 2^{b_1}+2^{b_2}+\cdots+2^{b_s} \pmod{2^k} \qquad &&\text{where } \ 0\leqslant b_1<b_2<\ldots<b_s\leqslant k-1. \end{aligned}$$

Since $-1\equiv 2^0+2^1+\cdots+2^{k-1} \pmod{2^k}$, we have $\{a_1,\ldots,a_r\}\cup\{b_1\ldots,b_s\}=\{0,1,\ldots,k-1\}$ and $r+s=k$. Write

$$\begin{aligned} S_p &= 2^{a_p}+2^{a_{p+1}}+\cdots+2^{a_r} \qquad &&\text{for } \ 1\leqslant p\leqslant r,\\ T_q &= 2^{b_1}+2^{b_2}+\cdots+2^{b_q} \qquad &&\text{for } \ 1\leqslant q\leqslant s. \end{aligned}$$

Also set $S_{r+1} = T_0 = 0$. Notice that $S_1 + T_s = 2^k - 1$ and $n + T_s \equiv 0 \pmod{2^k}$. We have

$$\begin{aligned} 1 + \frac{2^k - 1}{n} = \frac{n + S_1 + T_s}{n} &= \frac{n + S_1 + T_s}{n + T_s} \cdot \frac{n + T_s}{n} \\ &= \prod_{p=1}^{r} \frac{n + S_p + T_s}{n + S_{p+1} + T_s} \cdot \prod_{q=1}^{s} \frac{n + T_q}{n + T_{q-1}} \\ &= \prod_{p=1}^{r} \left(1 + \frac{2^{a_p}}{n + S_{p+1} + T_s}\right) \cdot \prod_{q=1}^{s} \left(1 + \frac{2^{b_q}}{n + T_{q-1}}\right), \end{aligned}$$

so if we define

$$m_p = \frac{n + S_{p+1} + T_s}{2^{a_p}} \quad \text{for } 1 \leqslant p \leqslant r \quad \text{and} \quad m_{r+q} = \frac{n + T_{q-1}}{2^{b_q}} \quad \text{for } 1 \leqslant q \leqslant s,$$

the desired equality holds. It remains to check that every m_i is an integer. For $1 \leqslant p \leqslant r$ we have

$$n + S_{p+1} + T_s \equiv n + T_s \equiv 0 \pmod{2^{a_p}}$$

and for $1 \leqslant q \leqslant r$ we have

$$n + T_{q-1} \equiv n + T_s \equiv 0 \pmod{2^{b_q}}.$$

The desired result follows.

N3. Prove that there exist infinitely many positive integers n such that the largest prime divisor of $n^4 + n^2 + 1$ is equal to the largest prime divisor of $(n+1)^4 + (n+1)^2 + 1$.

(Belgium)

Solution. Let p_n be the largest prime divisor of $n^4 + n^2 + 1$ and let q_n be the largest prime divisor of $n^2 + n + 1$. Then $p_n = q_{n^2}$, and from

$$n^4 + n^2 + 1 = (n^2 + 1)^2 - n^2 = (n^2 - n + 1)(n^2 + n + 1) = ((n-1)^2 + (n-1) + 1)(n^2 + n + 1)$$

it follows that $p_n = \max\{q_n, q_{n-1}\}$ for $n \geqslant 2$. Keeping in mind that $n^2 - n + 1$ is odd, we have

$$\gcd(n^2 + n + 1, n^2 - n + 1) = \gcd(2n, n^2 - n + 1) = \gcd(n, n^2 - n + 1) = 1.$$

Therefore $q_n \neq q_{n-1}$.

To prove the result, it suffices to show that the set

$$S = \{n \in \mathbb{Z}_{\geqslant 2} \mid q_n > q_{n-1} \text{ and } q_n > q_{n+1}\}$$

is infinite, since for each $n \in S$ one has

$$p_n = \max\{q_n, q_{n-1}\} = q_n = \max\{q_n, q_{n+1}\} = p_{n+1}.$$

Suppose on the contrary that S is finite. Since $q_2 = 7 < 13 = q_3$ and $q_3 = 13 > 7 = q_4$, the set S is non-empty. Since it is finite, we can consider its largest element, say m.

Note that it is impossible that $q_m > q_{m+1} > q_{m+2} > \ldots$ because all these numbers are positive integers, so there exists a $k \geqslant m$ such that $q_k < q_{k+1}$ (recall that $q_k \neq q_{k+1}$). Next observe that it is impossible to have $q_k < q_{k+1} < q_{k+2} < \ldots$, because $q_{(k+1)^2} = p_{k+1} = \max\{q_k, q_{k+1}\} = q_{k+1}$, so let us take the smallest $\ell \geqslant k+1$ such that $q_\ell > q_{\ell+1}$. By the minimality of ℓ we have $q_{\ell-1} < q_\ell$, so $\ell \in S$. Since $\ell \geqslant k+1 > k \geqslant m$, this contradicts the maximality of m, and hence S is indeed infinite.

Comment. Once the factorization of $n^4 + n^2 + 1$ is found and the set S is introduced, the problem is mainly about ruling out the case that

$$q_k < q_{k+1} < q_{k+2} < \ldots \tag{1}$$

might hold for some $k \in \mathbb{Z}_{>0}$. In the above solution, this is done by observing $q_{(k+1)^2} = \max(q_k, q_{k+1})$. Alternatively one may notice that (1) implies that $q_{j+2} - q_j \geqslant 6$ for $j \geqslant k+1$, since every prime greater than 3 is congruent to -1 or 1 modulo 6. Then there is some integer $C \geqslant 0$ such that $q_n \geqslant 3n - C$ for all $n \geqslant k$.

Now let the integer t be sufficiently large (e.g. $t = \max\{k+1, C+3\}$) and set $p = q_{t-1} \geqslant 2t$. Then $p \mid (t-1)^2 + (t-1) + 1$ implies that $p \mid (p-t)^2 + (p-t) + 1$, so p and q_{p-t} are prime divisors of $(p-t)^2 + (p-t) + 1$. But $p - t > t - 1 \geqslant k$, so $q_{p-t} > q_{t-1} = p$ and $p \cdot q_{p-t} > p^2 > (p-t)^2 + (p-t) + 1$, a contradiction.

N4. Determine whether there exists an infinite sequence of nonzero digits $a_1, a_2, a_3, \ldots$ and a positive integer N such that for every integer $k > N$, the number $\overline{a_k a_{k-1} \ldots a_1}$ is a perfect square.

(Iran)

Answer. No.

Solution. Assume that $a_1, a_2, a_3, \ldots$ is such a sequence. For each positive integer k, let $y_k = \overline{a_k a_{k-1} \ldots a_1}$. By the assumption, for each $k > N$ there exists a positive integer x_k such that $y_k = x_k^2$.

I. For every n, let 5^{γ_n} be the greatest power of 5 dividing x_n. Let us show first that $2\gamma_n \geqslant n$ for every positive integer $n > N$.

Assume, to the contrary, that there exists a positive integer $n > N$ such that $2\gamma_n < n$, which yields

$$y_{n+1} = \overline{a_{n+1} a_n \ldots a_1} = 10^n a_{n+1} + \overline{a_n a_{n-1} \ldots a_1} = 10^n a_{n+1} + y_n = 5^{2\gamma_n}\left(2^n 5^{n-2\gamma_n} a_{n+1} + \frac{y_n}{5^{2\gamma_n}}\right).$$

Since $5 \nmid y_n / 5^{2\gamma_n}$, we obtain $\gamma_{n+1} = \gamma_n < n < n+1$. By the same arguments we obtain that $\gamma_n = \gamma_{n+1} = \gamma_{n+2} = \ldots$. Denote this common value by γ.

Now, for each $k \geqslant n$ we have

$$(x_{k+1} - x_k)(x_{k+1} + x_k) = x_{k+1}^2 - x_k^2 = y_{k+1} - y_k = a_{k+1} \cdot 10^k.$$

One of the numbers $x_{k+1} - x_k$ and $x_{k+1} + x_k$ is not divisible by $5^{\gamma+1}$ since otherwise one would have $5^{\gamma+1} \mid \big((x_{k+1} - x_k) + (x_{k+1} + x_k)\big) = 2x_{k+1}$. On the other hand, we have $5^k \mid (x_{k+1} - x_k)(x_{k+1} + x_k)$, so $5^{k-\gamma}$ divides one of these two factors. Thus we get

$$5^{k-\gamma} \leqslant \max\{x_{k+1} - x_k, x_{k+1} + x_k\} < 2x_{k+1} = 2\sqrt{y_{k+1}} < 2 \cdot 10^{(k+1)/2},$$

which implies $5^{2k} < 4 \cdot 5^{2\gamma} \cdot 10^{k+1}$, or $(5/2)^k < 40 \cdot 5^{2\gamma}$. The last inequality is clearly false for sufficiently large values of k. This contradiction shows that $2\gamma_n \geqslant n$ for all $n > N$.

II. Consider now any integer $k > \max\{N/2, 2\}$. Since $2\gamma_{2k+1} \geqslant 2k+1$ and $2\gamma_{2k+2} \geqslant 2k+2$, we have $\gamma_{2k+1} \geqslant k+1$ and $\gamma_{2k+2} \geqslant k+1$. So, from $y_{2k+2} = a_{2k+2} \cdot 10^{2k+1} + y_{2k+1}$ we obtain $5^{2k+2} \mid y_{2k+2} - y_{2k+1} = a_{2k+2} \cdot 10^{2k+1}$ and thus $5 \mid a_{2k+2}$, which implies $a_{2k+2} = 5$. Therefore,

$$(x_{2k+2} - x_{2k+1})(x_{2k+2} + x_{2k+1}) = x_{2k+2}^2 - x_{2k+1}^2 = y_{2k+2} - y_{2k+1} = 5 \cdot 10^{2k+1} = 2^{2k+1} \cdot 5^{2k+2}.$$

Setting $A_k = x_{2k+2}/5^{k+1}$ and $B_k = x_{2k+1}/5^{k+1}$, which are integers, we obtain

$$(A_k - B_k)(A_k + B_k) = 2^{2k+1}. \tag{1}$$

Both A_k and B_k are odd, since otherwise y_{2k+2} or y_{2k+1} would be a multiple of 10 which is false by $a_1 \neq 0$; so one of the numbers $A_k - B_k$ and $A_k + B_k$ is not divisible by 4. Therefore (1) yields $A_k - B_k = 2$ and $A_k + B_k = 2^{2k}$, hence $A_k = 2^{2k-1} + 1$ and thus

$$x_{2k+2} = 5^{k+1} A_k = 10^{k+1} \cdot 2^{k-2} + 5^{k+1} > 10^{k+1},$$

since $k \geqslant 2$. This implies that $y_{2k+2} > 10^{2k+2}$ which contradicts the fact that y_{2k+2} contains $2k+2$ digits. The desired result follows.

Solution 2. Again, we assume that a sequence $a_1, a_2, a_3, \ldots$ satisfies the problem conditions, introduce the numbers x_k and y_k as in the previous solution, and notice that

$$y_{k+1} - y_k = (x_{k+1} - x_k)(x_{k+1} + x_k) = 10^k a_{k+1} \tag{2}$$

for all $k > N$. Consider any such k. Since $a_1 \neq 0$, the numbers x_k and x_{k+1} are not multiples of 10, and therefore the numbers $p_k = x_{k+1} - x_k$ and $q_k = x_{k+1} + x_k$ cannot be simultaneously multiples of 20, and hence one of them is not divisible either by 4 or by 5. In view of (2), this means that the other one is divisible by either 5^k or by 2^{k-1}. Notice also that p_k and q_k have the same parity, so both are even.

On the other hand, we have $x_{k+1}^2 = x_k^2 + 10^k a_{k+1} \geqslant x_k^2 + 10^k > 2x_k^2$, so $x_{k+1}/x_k > \sqrt{2}$, which implies that

$$1 < \frac{q_k}{p_k} = 1 + \frac{2}{x_{k+1}/x_k - 1} < 1 + \frac{2}{\sqrt{2}-1} < 6. \tag{3}$$

Thus, if one of the numbers p_k and q_k is divisible by 5^k, then we have

$$10^{k+1} > 10^k a_{k+1} = p_k q_k \geqslant \frac{(5^k)^2}{6}$$

and hence $(5/2)^k < 60$ which is false for sufficiently large k. So, assuming that k is large, we get that 2^{k-1} divides one of the numbers p_k and q_k. Hence

$$\{p_k,\, q_k\} = \{2^{k-1} \cdot 5^{r_k} b_k,\, 2 \cdot 5^{k-r_k} c_k\} \quad \text{with nonnegative integers } b_k,\, c_k,\, r_k \text{ such that } b_k c_k = a_{k+1}.$$

Moreover, from (3) we get

$$6 > \frac{2^{k-1} \cdot 5^{r_k} b_k}{2 \cdot 5^{k-r_k} c_k} \geqslant \frac{1}{36} \cdot \left(\frac{2}{5}\right)^k \cdot 5^{2r_k} \quad \text{and} \quad 6 > \frac{2 \cdot 5^{k-r_k} c_k}{2^{k-1} \cdot 5^{r_k} b_k} \geqslant \frac{4}{9} \cdot \left(\frac{5}{2}\right)^k \cdot 5^{-2r_k},$$

so

$$\alpha k + c_1 < r_k < \alpha k + c_2 \qquad \text{for } \alpha = \tfrac{1}{2}\log_5\left(\tfrac{5}{2}\right) < 1 \text{ and some constants } c_2 > c_1. \tag{4}$$

Consequently, for $C = c_2 - c_1 + 1 - \alpha > 0$ we have

$$(k+1) - r_{k+1} \leqslant k - r_k + C. \tag{5}$$

Next, we will use the following easy lemma.

Lemma. Let s be a positive integer. Then $5^{s+2^s} \equiv 5^s \pmod{10^s}$.

Proof. Euler's theorem gives $5^{2^s} \equiv 1 \pmod{2^s}$, so $5^{s+2^s} - 5^s = 5^s(5^{2^s} - 1)$ is divisible by 2^s and 5^s.

Now, for every large k we have

$$x_{k+1} = \frac{p_k + q_k}{2} = 5^{r_k} \cdot 2^{k-2} b_k + 5^{k-r_k} c_k \equiv 5^{k-r_k} c_k \pmod{10^{r_k}} \tag{6}$$

since $r_k \leqslant k-2$ by (4); hence $y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 \pmod{10^{r_k}}$. Let us consider some large integer s, and choose the minimal k such that $2(k - r_k) \geqslant s + 2^s$; it exists by (4). Set $d = 2(k-r_k) - (s+2^s)$. By (4) we have $2^s < 2(k - r_k) < \left(\frac{2}{\alpha} - 2\right) r_k - \frac{2c_1}{\alpha}$; if s is large this implies $r_k > s$, so (6) also holds modulo 10^s. Then (6) and the lemma give

$$y_{k+1} \equiv 5^{2(k-r_k)} c_k^2 = 5^{s+2^s} \cdot 5^d c_k^2 \equiv 5^s \cdot 5^d c_k^2 \pmod{10^s}. \tag{7}$$

By (5) and the minimality of k we have $d \leqslant 2C$, so $5^d c_k^2 \leqslant 5^{2C} \cdot 81 = D$. Using $5^4 < 10^3$ we obtain

$$5^s \cdot 5^d c_k^2 < 10^{3s/4} D < 10^{s-1}$$

for sufficiently large s. This, together with (7), shows that the sth digit from the right in y_{k+1}, which is a_s, is zero. This contradicts the problem condition.

N5. Fix an integer $k \geqslant 2$. Two players, called Ana and Banana, play the following *game of numbers*: Initially, some integer $n \geqslant k$ gets written on the blackboard. Then they take moves in turn, with Ana beginning. A player making a move erases the number m just written on the blackboard and replaces it by some number m' with $k \leqslant m' < m$ that is coprime to m. The first player who cannot move anymore loses.

An integer $n \geqslant k$ is called *good* if Banana has a winning strategy when the initial number is n, and *bad* otherwise.

Consider two integers $n, n' \geqslant k$ with the property that each prime number $p \leqslant k$ divides n if and only if it divides n'. Prove that either both n and n' are good or both are bad.

(Italy)

Solution 1. Let us first observe that the number appearing on the blackboard decreases after every move; so the game necessarily ends after at most n steps, and consequently there always has to be some player possessing a winning strategy. So if some $n \geqslant k$ is bad, then Ana has a winning strategy in the game with starting number n.

More precisely, if $n \geqslant k$ is such that there is a good integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$, then n itself is bad, for Ana has the following winning strategy in the game with initial number n: She proceeds by first playing m and then using Banana's strategy for the game with starting number m.

Otherwise, if some integer $n \geqslant k$ has the property that every integer m with $n > m \geqslant k$ and $\gcd(m, n) = 1$ is bad, then n is good. Indeed, if Ana can make a first move at all in the game with initial number n, then she leaves it in a position where the first player has a winning strategy, so that Banana can defeat her.

In particular, this implies that any two good numbers have a non–trivial common divisor. Also, k itself is good.

For brevity, we say that $n \longrightarrow x$ is a *move* if n and x are two coprime integers with $n > x \geqslant k$.

Claim 1. If n is good and n' is a multiple of n, then n' is also good.

Proof. If n' were bad, there would have to be some move $n' \longrightarrow x$, where x is good. As n' is a multiple of n this implies that the two good numbers n and x are coprime, which is absurd. □

Claim 2. If r and s denote two positive integers for which $rs \geqslant k$ is bad, then r^2s is also bad.

Proof. Since rs is bad, there is a move $rs \longrightarrow x$ for some good x. Evidently x is coprime to r^2s as well, and hence the move $r^2s \longrightarrow x$ shows that r^2s is indeed bad. □

Claim 3. If $p > k$ is prime and $n \geqslant k$ is bad, then np is also bad.

Proof. Otherwise we choose a counterexample with n being as small as possible. In particular, np is good. Since n is bad, there is a move $n \longrightarrow x$ for some good x. Now $np \longrightarrow x$ cannot be a valid move, which tells us that x has to be divisible by p. So we can write $x = p^r y$, where r and y denote some positive integers, the latter of which is not divisible by p.

Note that $y = 1$ is impossible, for then we would have $x = p^r$ and the move $x \longrightarrow k$ would establish that x is bad. In view of this, there is a least power y^α of y that is at least as large as k. Since the numbers np and y^α are coprime and the former is good, the latter has to be bad. Moreover, the minimality of α implies $y^\alpha < ky < py = \frac{x}{p^{r-1}} < \frac{n}{p^{r-1}}$. So $p^{r-1} \cdot y^\alpha < n$ and consequently all the numbers $y^\alpha, py^\alpha, \ldots, p^r \cdot y^\alpha = p(p^{r-1} \cdot y^\alpha)$ are bad due to the minimal choice of n. But now by Claim 1 the divisor x of $p^r \cdot y^\alpha$ cannot be good, whereby we have reached a contradiction that proves Claim 3. □

We now deduce the statement of the problem from these three claims. To this end, we call two integers $a, b \geqslant k$ *similar* if they are divisible by the same prime numbers not exceeding k. We are to prove that if a and b are similar, then either both of them are good or both are bad. As in this case the product ab is similar to both a and b, it suffices to show the following: if $c \geqslant k$ is similar to some of its multiples d, then either both c and d are good or both are bad.

Assuming that this is not true in general, we choose a counterexample (c_0, d_0) with d_0 being as small as possible. By Claim 1, c_0 is bad whilst d_0 is good. Plainly d_0 is strictly greater than c_0 and hence the quotient $\frac{d_0}{c_0}$ has some prime factor p. Clearly p divides d_0. If $p \leqslant k$, then p divides c_0 as well due to the similarity, and hence d_0 is actually divisible by p^2. So $\frac{d_0}{p}$ is good by the contrapositive of Claim 2. Since $c_0 \mid \frac{d_0}{p}$, the pair $(c_0, \frac{d_0}{p})$ contradicts the supposed minimality of d_0. This proves $p > k$, but now we get the same contradiction using Claim 3 instead of Claim 2. Thereby the problem is solved.

Solution 2. We use the same analysis of the game of numbers as in the first five paragraphs of the first solution. Let us call a prime number p *small* in case $p \leqslant k$ and *big* otherwise. We again call two integers *similar* if their sets of small prime factors coincide.

Claim 4. For each integer $b \geqslant k$ having some small prime factor, there exists an integer x similar to it with $b \geqslant x \geqslant k$ and having no big prime factors.

Proof. Unless b has a big prime factor we may simply choose $x = b$. Now let p and q denote a small and a big prime factor of b, respectively. Let a be the product of all small prime factors of b. Further define n to be the least non–negative integer for which the number $x = p^n a$ is at least as large as k. It suffices to show that $b > x$. This is clear in case $n = 0$, so let us assume $n > 0$ from now on. Then we have $x < pk$ due to the minimality of n, $p \leqslant a$ because p divides a by construction, and $k < q$. Therefore $x < aq$ and, as the right hand side is a product of distinct prime factors of b, this implies indeed $x < b$. □

Let us now assume that there is a pair (a, b) of similar numbers such that a is bad and b is good. Take such a pair with $\max(a, b)$ being as small as possible. Since a is bad, there exists a move $a \longrightarrow r$ for some good r. Since the numbers k and r are both good, they have a common prime factor, which necessarily has to be small. Thus Claim 4 is applicable to r, which yields an integer r' similar to r containing small prime factors only and satisfying $r \geqslant r' \geqslant k$. Since $\max(r, r') = r < a \leqslant \max(a, b)$ the number r' is also good. Now let p denote a common prime factor of the good numbers r' and b. By our construction of r', this prime is small and due to the similarities it consequently divides a and r, contrary to $a \longrightarrow r$ being a move. Thereby the problem is solved.

Comment 1. Having reached Claim 4 of Solution 2, there are various other ways to proceed. For instance, one may directly obtain the following fact, which seems to be interesting in its own right:

Claim 5. Any two good numbers have a common small prime factor.

Proof. Otherwise there exists a pair (b, b') of good numbers with $b' \geqslant b \geqslant k$ all of whose common prime factors are big. Choose such a pair with b' being as small as possible. Since b and k are both good, there has to be a common prime factor p of b and k. Evidently p is small and thus it cannot divide b', which in turn tells us $b' > b$. Applying Claim 4 to b we get an integer x with $b \geqslant x \geqslant k$ that is similar to b and has no big prime divisors at all. By our assumption, b' and x are coprime, and as b' is good this implies that x is bad. Consequently there has to be some move $x \longrightarrow b^*$ such that b^* is good. But now all the small prime factors of b also appear in x and thus they cannot divide b^*. Therefore the pair (b^*, b) contradicts the supposed minimality of b'. □

From that point, it is easy to complete the solution: assume that there are two similar integers a and b such that a is bad and b is good. Since a is bad, there is a move $a \longrightarrow b'$ for some good b'. By Claim 5, there is a small prime p dividing b and b'. Due to the similarity of a and b, the prime p has to divide a as well, but this contradicts the fact that $a \longrightarrow b'$ is a valid move. Thereby the problem is solved.

Comment 2. There are infinitely many good numbers, e.g. all multiples of k. The increasing sequence $b_0, b_1, \ldots,$ of all good numbers may be constructed recursively as follows:

- Start with $b_0 = k$.
- If b_n has just been defined for some $n \geqslant 0$, then b_{n+1} is the smallest number $b > b_n$ that is coprime to none of $b_0, \ldots, b_n$.

This construction can be used to determine the set of good numbers for any specific k as explained in the next comment. It is already clear that if $k = p^\alpha$ is a prime power, then a number $b \geqslant k$ is good if and only if it is divisible by p.

Comment 3. Let $P > 1$ denote the product of all small prime numbers. Then any two integers $a, b \geqslant k$ that are congruent modulo P are similar. Thus the infinite word $W_k = (X_k, X_{k+1}, \ldots)$ defined by

$$X_i = \begin{cases} A & \text{if } i \text{ is bad} \\ B & \text{if } i \text{ is good} \end{cases}$$

for all $i \geqslant k$ is periodic and the length of its period divides P. As the prime power example shows, the true period can sometimes be much smaller than P. On the other hand, there are cases where the period is rather large; e.g., if $k = 15$, the sequence of good numbers begins with $15, 18, 20, 24, 30, 36, 40, 42, 45$ and the period of W_{15} is 30.

Comment 4. The original proposal contained two questions about the game of numbers, namely (a) to show that if two numbers have the same prime factors then either both are good or both are bad, and (b) to show that the word W_k introduced in the previous comment is indeed periodic. The Problem Selection Committee thinks that the above version of the problem is somewhat easier, even though it demands to prove a stronger result.

N6. Determine all functions $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfying

$$f\left(\frac{f(x)+a}{b}\right) = f\left(\frac{x+a}{b}\right) \tag{1}$$

for all $x \in \mathbb{Q}$, $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{>0}$. (Here, $\mathbb{Z}_{>0}$ denotes the set of positive integers.)

(Israel)

Answer. There are three kinds of such functions, which are: all constant functions, the floor function, and the ceiling function.

Solution 1. I. We start by verifying that these functions do indeed satisfy (1). This is clear for all constant functions. Now consider any triple $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$ and set

$$q = \left\lfloor \frac{x+a}{b} \right\rfloor.$$

This means that q is an integer and $bq \leqslant x + a < b(q+1)$. It follows that $bq \leqslant \lfloor x \rfloor + a < b(q+1)$ holds as well, and thus we have

$$\left\lfloor \frac{\lfloor x \rfloor + a}{b} \right\rfloor = \left\lfloor \frac{x+a}{b} \right\rfloor,$$

meaning that the floor function does indeed satisfy (1). One can check similarly that the ceiling function has the same property.

II. Let us now suppose conversely that the function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1) for all $(x, a, b) \in \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z}_{>0}$. According to the behaviour of the restriction of f to the integers we distinguish two cases.

Case 1: There is some $m \in \mathbb{Z}$ such that $f(m) \neq m$.

Write $f(m) = C$ and let $\eta \in \{-1, +1\}$ and b denote the sign and absolute value of $f(m) - m$, respectively. Given any integer r, we may plug the triple $(m, rb - C, b)$ into (1), thus getting $f(r) = f(r - \eta)$. Starting with m and using induction in both directions, we deduce from this that the equation $f(r) = C$ holds for all integers r. Now any rational number y can be written in the form $y = \frac{p}{q}$ with $(p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0}$, and substituting $(C - p, p - C, q)$ into (1) we get $f(y) = f(0) = C$. Thus f is the constant function whose value is always C.

Case 2: One has $f(m) = m$ for all integers m.

Note that now the special case $b = 1$ of (1) takes a particularly simple form, namely

$$f(x) + a = f(x + a) \qquad \text{for all } (x, a) \in \mathbb{Q} \times \mathbb{Z}. \tag{2}$$

Defining $f\left(\frac{1}{2}\right) = \omega$ we proceed in three steps.

Step A. We show that $\omega \in \{0, 1\}$.

If $\omega \leqslant 0$, we may plug $\left(\frac{1}{2}, -\omega, 1 - 2\omega\right)$ into (1), obtaining $0 = f(0) = f\left(\frac{1}{2}\right) = \omega$. In the contrary case $\omega \geqslant 1$ we argue similarly using the triple $\left(\frac{1}{2}, \omega - 1, 2\omega - 1\right)$.

Step B. We show that $f(x) = \omega$ for all rational numbers x with $0 < x < 1$.

Assume that this fails and pick some rational number $\frac{a}{b} \in (0, 1)$ with minimal b such that $f\left(\frac{a}{b}\right) \neq \omega$. Obviously, $\gcd(a, b) = 1$ and $b \geqslant 2$. If b is even, then a has to be odd and we can substitute $\left(\frac{1}{2}, \frac{a-1}{2}, \frac{b}{2}\right)$ into (1), which yields

$$f\left(\frac{\omega + (a-1)/2}{b/2}\right) = f\left(\frac{a}{b}\right) \neq \omega. \tag{3}$$

Recall that $0 \leqslant (a-1)/2 < b/2$. Thus, in both cases $\omega = 0$ and $\omega = 1$, the left-hand part of (3) equals ω either by the minimality of b, or by $f(\omega) = \omega$. A contradiction.

Thus b has to be odd, so $b = 2k+1$ for some $k \geqslant 1$. Applying (1) to $\left(\frac{1}{2}, k, b\right)$ we get

$$f\left(\frac{\omega + k}{b}\right) = f\left(\frac{1}{2}\right) = \omega. \tag{4}$$

Since a and b are coprime, there exist integers $r \in \{1, 2, \ldots, b\}$ and m such that $ra - mb = k + \omega$. Note that we actually have $1 \leqslant r < b$, since the right hand side is not a multiple of b. If m is negative, then we have $ra - mb > b \geqslant k + \omega$, which is absurd. Similarly, $m \geqslant r$ leads to $ra - mb < br - br = 0$, which is likewise impossible; so we must have $0 \leqslant m \leqslant r - 1$.

We finally substitute $\left(\frac{k+\omega}{b}, m, r\right)$ into (1) and use (4) to learn

$$f\left(\frac{\omega + m}{r}\right) = f\left(\frac{a}{b}\right) \neq \omega.$$

But as above one may see that the left hand side has to equal ω due to the minimality of b. This contradiction concludes our step B.

Step C. Now notice that if $\omega = 0$, then $f(x) = \lfloor x \rfloor$ holds for all rational x with $0 \leqslant x < 1$ and hence by (2) this even holds for all rational numbers x. Similarly, if $\omega = 1$, then $f(x) = \lceil x \rceil$ holds for all $x \in \mathbb{Q}$. Thereby the problem is solved.

Comment 1. An alternative treatment of Steps B and C from the second case, due to the proposer, proceeds as follows. Let square brackets indicate the floor function in case $\omega = 0$ and the ceiling function if $\omega = 1$. We are to prove that $f(x) = [x]$ holds for all $x \in \mathbb{Q}$, and because of Step A and (2) we already know this in case $2x \in \mathbb{Z}$. Applying (1) to $(2x, 0, 2)$ we get

$$f(x) = f\left(\frac{f(2x)}{2}\right),$$

and by the previous observation this yields

$$f(x) = \left[\frac{f(2x)}{2}\right] \qquad \text{for all } x \in \mathbb{Q}. \tag{5}$$

An easy induction now shows

$$f(x) = \left[\frac{f(2^n x)}{2^n}\right] \qquad \text{for all } (x, n) \in \mathbb{Q} \times \mathbb{Z}_{>0}. \tag{6}$$

Now suppose first that x is not an integer but can be written in the form $\frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$ both being odd. Let d denote the multiplicative order of 2 modulo q and let m be any large integer. Plugging $n = dm$ into (6) and using (2) we get

$$f(x) = \left[\frac{f(2^{dm}x)}{2^{dm}}\right] = \left[\frac{f(x) + (2^{dm} - 1)x}{2^{dm}}\right] = \left[x + \frac{f(x) - x}{2^{dm}}\right].$$

Since x is not an integer, the square bracket function is continuous at x; hence as m tends to infinity the above fomula gives $f(x) = [x]$. To complete the argument we just need to observe that if some $y \in \mathbb{Q}$ satisfies $f(y) = [y]$, then (5) yields $f\left(\frac{y}{2}\right) = f\left(\frac{[y]}{2}\right) = \left[\frac{[y]}{2}\right] = \left[\frac{y}{2}\right]$.

Solution 2. Here we just give another argument for the second case of the above solution. Again we use equation (2). It follows that the set S of all zeros of f contains for each $x \in \mathbb{Q}$ exactly one term from the infinite sequence $\ldots, x-2, x-1, x, x+1, x+2, \ldots$.

Next we claim that

$$\text{if } (p, q) \in \mathbb{Z} \times \mathbb{Z}_{>0} \text{ and } \tfrac{p}{q} \in S, \text{ then } \tfrac{p}{q+1} \in S \text{ holds as well.} \tag{7}$$

To see this we just plug $\left(\frac{p}{q}, p, q+1\right)$ into (1), thus getting $f\left(\frac{p}{q+1}\right) = f\left(\frac{p}{q}\right) = 0$.

From this we get that

$$\text{if } x, y \in \mathbb{Q},\ x > y > 0, \text{ and } x \in S, \text{ then } y \in S. \tag{8}$$

Indeed, if we write $x = \frac{p}{q}$ and $y = \frac{r}{s}$ with $p, q, r, s \in \mathbb{Z}_{>0}$, then $ps > qr$ and (7) tells us

$$0 = f\left(\frac{p}{q}\right) = f\left(\frac{pr}{qr}\right) = f\left(\frac{pr}{qr+1}\right) = \ldots = f\left(\frac{pr}{ps}\right) = f\left(\frac{r}{s}\right).$$

Essentially the same argument also establishes that

$$\text{if } x, y \in \mathbb{Q},\ x < y < 0, \text{ and } x \in S, \text{ then } y \in S. \tag{9}$$

From (8) and (9) we get $0 \in S \subseteq (-1, +1)$ and hence the real number $\alpha = \sup(S)$ exists and satisfies $0 \leqslant \alpha \leqslant 1$.

Let us assume that we actually had $0 < \alpha < 1$. Note that $f(x) = 0$ if $x \in (0, \alpha) \cap \mathbb{Q}$ by (8), and $f(x) = 1$ if $x \in (\alpha, 1) \cap \mathbb{Q}$ by (9) and (2). Let K denote the unique positive integer satisfying $K\alpha < 1 \leqslant (K+1)\alpha$. The first of these two inequalities entails $\alpha < \frac{1+\alpha}{K+1}$, and thus there is a rational number $x \in \left(\alpha, \frac{1+\alpha}{K+1}\right)$. Setting $y = (K+1)x - 1$ and substituting $(y, 1, K+1)$ into (1) we learn

$$f\left(\frac{f(y)+1}{K+1}\right) = f\left(\frac{y+1}{K+1}\right) = f(x).$$

Since $\alpha < x < 1$ and $0 < y < \alpha$, this simplifies to

$$f\left(\frac{1}{K+1}\right) = 1.$$

But, as $0 < \frac{1}{K+1} \leqslant \alpha$, this is only possible if $\alpha = \frac{1}{K+1}$ and $f(\alpha) = 1$. From this, however, we get the contradiction

$$0 = f\left(\frac{1}{(K+1)^2}\right) = f\left(\frac{\alpha+0}{K+1}\right) = f\left(\frac{f(\alpha)+0}{K+1}\right) = f(\alpha) = 1.$$

Thus our assumption $0 < \alpha < 1$ has turned out to be wrong and it follows that $\alpha \in \{0, 1\}$. If $\alpha = 0$, then we have $S \subseteq (-1, 0]$, whence $S = (-1, 0] \cap \mathbb{Q}$, which in turn yields $f(x) = \lceil x \rceil$ for all $x \in \mathbb{Q}$ due to (2). Similarly, $\alpha = 1$ entails $S = [0, 1) \cap \mathbb{Q}$ and $f(x) = \lfloor x \rfloor$ for all $x \in \mathbb{Q}$. Thereby the solution is complete.

Comment 2. It seems that all solutions to this problems involve some case distinction separating the constant solutions from the unbounded ones, though the "descriptions" of the cases may be different depending on the work that has been done at the beginning of the solution. For instance, these two cases can also be "f is periodic on the integers" and "f is not periodic on the integers". The case leading to the unbounded solutions appears to be the harder one.

In most approaches, the cases leading to the two functions $x \longmapsto \lfloor x \rfloor$ and $x \longmapsto \lceil x \rceil$ can easily be treated parallelly, but sometimes it may be useful to know that there is some symmetry in the problem interchanging these two functions. Namely, if a function $f\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ satisfies (1), then so does the function $g\colon \mathbb{Q} \longrightarrow \mathbb{Z}$ defined by $g(x) = -f(-x)$ for all $x \in \mathbb{Q}$. For that reason, we could have restricted our attention to the case $\omega = 0$ in the first solution and, once $\alpha \in \{0, 1\}$ had been obtained, to the case $\alpha = 0$ in the second solution.

N7. Let ν be an irrational positive number, and let m be a positive integer. A pair (a, b) of positive integers is called *good* if

$$a\lceil b\nu\rceil - b\lfloor a\nu\rfloor = m. \tag{$*$}$$

A good pair (a, b) is called *excellent* if neither of the pairs $(a-b, b)$ and $(a, b-a)$ is good. (As usual, by $\lfloor x\rfloor$ and $\lceil x\rceil$ we denote the integer numbers such that $x - 1 < \lfloor x\rfloor \leqslant x$ and $x \leqslant \lceil x\rceil < x + 1$.)

Prove that the number of excellent pairs is equal to the sum of the positive divisors of m.

(U.S.A.)

Solution. For positive integers a and b, let us denote

$$f(a, b) = a\lceil b\nu\rceil - b\lfloor a\nu\rfloor.$$

We will deal with various values of m; thus it is convenient to say that a pair (a, b) is m-*good* or m-*excellent* if the corresponding conditions are satisfied.

To start, let us investigate how the values $f(a + b, b)$ and $f(a, b + a)$ are related to $f(a, b)$. If $\{a\nu\} + \{b\nu\} < 1$, then we have $\lfloor(a + b)\nu\rfloor = \lfloor a\nu\rfloor + \lfloor b\nu\rfloor$ and $\lceil(a + b)\nu\rceil = \lceil a\nu\rceil + \lceil b\nu\rceil - 1$, so

$$f(a + b, b) = (a + b)\lceil b\nu\rceil - b(\lfloor a\nu\rfloor + \lfloor b\nu\rfloor) = f(a, b) + b(\lceil b\nu\rceil - \lfloor b\nu\rfloor) = f(a, b) + b$$

and

$$f(a, b + a) = a(\lceil b\nu\rceil + \lceil a\nu\rceil - 1) - (b + a)\lfloor a\nu\rfloor = f(a, b) + a(\lceil a\nu\rceil - 1 - \lfloor a\nu\rfloor) = f(a, b).$$

Similarly, if $\{a\nu\} + \{b\nu\} \geqslant 1$ then one obtains

$$f(a + b, b) = f(a, b) \quad \text{and} \quad f(a, b + a) = f(a, b) + a.$$

So, in both cases one of the numbers $f(a + b, a)$ and $f(a, b + a)$ is equal to $f(a, b)$ while the other is greater than $f(a, b)$ by one of a and b. Thus, exactly one of the pairs $(a + b, b)$ and $(a, b + a)$ is excellent (for an appropriate value of m).

Now let us say that the pairs $(a + b, b)$ and $(a, b + a)$ are the *children* of the pair (a, b), while this pair is their *parent*. Next, if a pair (c, d) can be obtained from (a, b) by several passings from a parent to a child, we will say that (c, d) is a *descendant* of (a, b), while (a, b) is an *ancestor* of (c, d) (a pair is neither an ancestor nor a descendant of itself). Thus each pair (a, b) has two children, it has a unique parent if $a \neq b$, and no parents otherwise. Therefore, each pair of distinct positive integers has a unique ancestor of the form (a, a); our aim is now to find how many m-excellent descendants each such pair has.

Notice now that if a pair (a, b) is m-excellent then $\min\{a, b\} \leqslant m$. Indeed, if $a = b$ then $f(a, a) = a = m$, so the statement is valid. Otherwise, the pair (a, b) is a child of some pair (a', b'). If $b = b'$ and $a = a' + b'$, then we should have $m = f(a, b) = f(a', b') + b'$, so $b = b' = m - f(a', b') < m$. Similarly, if $a = a'$ and $b = b' + a'$ then $a < m$.

Let us consider the set S_m of all pairs (a, b) such that $f(a, b) \leqslant m$ and $\min\{a, b\} \leqslant m$. Then all the ancestors of the elements in S_m are again in S_m, and each element in S_m either is of the form (a, a) with $a \leqslant m$, or has a unique ancestor of this form. From the arguments above we see that all m-excellent pairs lie in S_m.

We claim now that the set S_m is finite. Indeed, assume, for instance, that it contains infinitely many pairs (c, d) with $d > 2m$. Such a pair is necessarily a child of $(c, d - c)$, and thus a descendant of some pair (c, d') with $m < d' \leqslant 2m$. Therefore, one of the pairs $(a, b) \in S_m$ with $m < b \leqslant 2m$

has infinitely many descendants in S_m, and all these descendants have the form $(a, b+ka)$ with k a positive integer. Since $f(a, b+ka)$ does not decrease as k grows, it becomes constant for $k \geqslant k_0$, where k_0 is some positive integer. This means that $\{a\nu\} + \{(b+ka)\nu\} < 1$ for all $k \geqslant k_0$. But this yields $1 > \{(b+ka)\nu\} = \{(b+k_0a)\nu\} + (k-k_0)\{a\nu\}$ for all $k > k_0$, which is absurd.

Similarly, one can prove that S_m contains finitely many pairs (c, d) with $c > 2m$, thus finitely many elements at all.

We are now prepared for proving the following crucial lemma.

Lemma. Consider any pair (a, b) with $f(a, b) \neq m$. Then the number $g(a, b)$ of its m-excellent descendants is equal to the number $h(a, b)$ of ways to represent the number $t = m - f(a, b)$ as $t = ka + \ell b$ with k and ℓ being some nonnegative integers.

Proof. We proceed by induction on the number N of descendants of (a, b) in S_m. If $N = 0$ then clearly $g(a, b) = 0$. Assume that $h(a, b) > 0$; without loss of generality, we have $a \leqslant b$. Then, clearly, $m - f(a, b) \geqslant a$, so $f(a, b+a) \leqslant f(a, b) + a \leqslant m$ and $a \leqslant m$, hence $(a, b+a) \in S_m$ which is impossible. Thus in the base case we have $g(a, b) = h(a, b) = 0$, as desired.

Now let $N > 0$. Assume that $f(a+b, b) = f(a, b) + b$ and $f(a, b+a) = f(a, b)$ (the other case is similar). If $f(a, b) + b \neq m$, then by the induction hypothesis we have

$$g(a, b) = g(a+b, b) + g(a, b+a) = h(a+b, b) + h(a, b+a).$$

Notice that both pairs $(a+b, b)$ and $(a, b+a)$ are descendants of (a, b) and thus each of them has strictly less descendants in S_m than (a, b) does.

Next, each one of the $h(a+b, b)$ representations of $m - f(a+b, b) = m - b - f(a, b)$ as the sum $k'(a+b) + \ell' b$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' < k' + \ell' + 1 = \ell$. Similarly, each one of the $h(a, b+a)$ representations of $m - f(a, b+a) = m - f(a, b)$ as the sum $k'a + \ell'(b+a)$ provides the representation $m - f(a, b) = ka + \ell b$ with $k = k' + \ell' \geqslant \ell' = \ell$. This correspondence is obviously bijective, so

$$h(a, b) = h(a+b, b) + h(a, b+a) = g(a, b),$$

as required.

Finally, if $f(a, b) + b = m$ then $(a+b, b)$ is m-excellent, so $g(a, b) = 1 + g(a, b+a) = 1 + h(a, b+a)$ by the induction hypothesis. On the other hand, the number $m - f(a, b) = b$ has a representation $0 \cdot a + 1 \cdot b$ and sometimes one more representation as $ka + 0 \cdot b$; this last representation exists simultaneously with the representation $m - f(a, b+a) = ka + 0 \cdot (b+a)$, so $h(a, b) = 1 + h(a, b+a)$ as well. Thus in this case the step is also proved. $\square$

Now it is easy to finish the solution. There exists a unique m-excellent pair of the form (a, a), and each other m-excellent pair (a, b) has a unique ancestor of the form (x, x) with $x < m$. By the lemma, for every $x < m$ the number of its m-excellent descendants is $h(x, x)$, which is the number of ways to represent $m - f(x, x) = m - x$ as $kx + \ell x$ (with nonnegative integer k and ℓ). This number is 0 if $x \nmid m$, and m/x otherwise. So the total number of excellent pairs is

$$1 + \sum_{x \mid m,\ x < m} \frac{m}{x} = 1 + \sum_{d \mid m,\ d > 1} d = \sum_{d \mid m} d,$$

as required.

Comment. Let us present a sketch of an outline of a different solution. The plan is to check that the number of excellent pairs does not depend on the (irrational) number ν, and to find this number for some appropriate value of ν. For that, we first introduce some geometrical language. We deal only with the excellent pairs (a,b) with $a \neq b$.

Part I. Given an irrational positive ν, for every positive integer n we introduce two integral points $F_\nu(n) = (n, \lfloor n\nu \rfloor)$ and $C_\nu(n) = (n, \lceil n\nu \rceil)$ on the coordinate plane Oxy. Then $(*)$ reads as $[OF_\nu(a)C_\nu(b)] = m/2$; here $[\cdot]$ stands for the signed area. Next, we rewrite in these terms the condition on a pair (a,b) to be excellent. Let ℓ_ν, ℓ_ν^+, and ℓ_ν^- be the lines determined by the equations $y = \nu x$, $y = \nu x + 1$, and $y = \nu x - 1$, respectively.

a). Firstly, we deal with all excellent pairs (a,b) with $a < b$. Given some value of a, all the points C such that $[OF_\nu(a)C] = m/2$ lie on some line $f_\nu(a)$; if there exist any good pairs (a,b) at all, this line has to contain at least one integral point, which happens exactly when $\gcd(a, \lfloor a\nu \rfloor) \mid m$.

Let $P_\nu(a)$ be the point of intersection of ℓ_ν^+ and $f_\nu(a)$, and let $p_\nu(a)$ be its abscissa; notice that $p_\nu(a)$ is irrational if it is nonzero. Now, if (a,b) is good, then the point $C_\nu(b)$ lies on $f_\nu(a)$, which means that the point of $f_\nu(a)$ with abscissa b lies between ℓ_ν and ℓ_ν^+ and is integral. If in addition the pair $(a, b-a)$ is not good, then the point of $f_\nu(a)$ with abscissa $b-a$ lies above ℓ_ν^+ (see Fig. 1). Thus, the pair (a,b) with $b > a$ is excellent exactly when $p_\nu(a)$ lies between $b-a$ and b, and the point of $f_\nu(a)$ with abscissa b is integral (which means that this point is $C_\nu(b)$).

Notice now that, if $p_\nu(a) > a$, then the number of excellent pairs of the form (a,b) (with $b > a$) is $\gcd(a, \lfloor a\nu \rfloor)$.

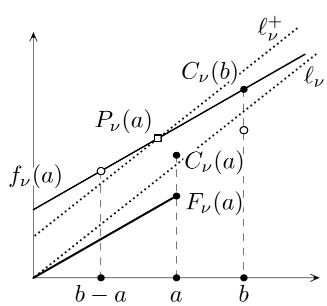


Figure 1

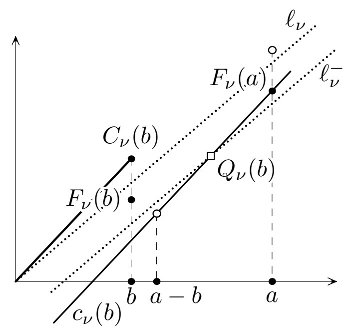


Figure 2

b). Analogously, considering the pairs (a,b) with $a > b$, we fix the value of b, introduce the line $c_\nu(b)$ containing all the points F with $[OFC_\nu(b)] = m/2$, assume that this line contains an integral point (which means $\gcd(b, \lceil b\nu \rceil) \mid m$), and denote the common point of $c_\nu(b)$ and ℓ_ν^- by $Q_\nu(b)$, its abscissa being $q_\nu(b)$. Similarly to the previous case, we obtain that the pair (a,b) is excellent exactly when $q_\nu(a)$ lies between $a-b$ and a, and the point of $c_\nu(b)$ with abscissa a is integral (see Fig. 2). Again, if $q_\nu(b) > b$, then the number of excellent pairs of the form (a,b) (with $a > b$) is $\gcd(b, \lceil b\nu \rceil)$.

Part II, sketchy. Having obtained such a description, one may check how the number of excellent pairs changes as ν grows. (Having done that, one may find this number for one appropriate value of ν; for instance, it is relatively easy to make this calculation for $\nu \in \left(1, 1 + \frac{1}{m}\right)$.)

Consider, for the initial value of ν, some excellent pair (a,t) with $a > t$. As ν grows, this pair eventually stops being excellent; this happens when the point $Q_\nu(t)$ passes through $F_\nu(a)$. At the same moment, the pair $(a+t, t)$ becomes excellent instead.

This process halts when the point $Q_\nu(t)$ eventually disappears, i.e. when ν passes through the ratio of the coordinates of the point $T = C_\nu(t)$. Hence, the point T afterwards is regarded as $F_\nu(t)$. Thus, all the old excellent pairs of the form (a,t) with $a > t$ disappear; on the other hand, the same number of excellent pairs with the first element being t just appear.

Similarly, if some pair (t, b) with $t < b$ is initially ν-excellent, then at some moment it stops being excellent when $P_\nu(t)$ passes through $C_\nu(b)$; at the same moment, the pair $(t, b-t)$ becomes excellent. This process eventually stops when $b - t < t$. At this moment, again the second element of the pair becomes fixed, and the first one starts to increase.

These ideas can be made precise enough to show that the number of excellent pairs remains unchanged, as required.

We should warn the reader that the rigorous elaboration of Part II is technically quite involved, mostly by the reason that the set of moments when the collection of excellent pairs changes is infinite. Especially much care should be applied to the limit points of this set, which are exactly the points when the line ℓ_ν passes through some point of the form $C_\nu(b)$.

The same ideas may be explained in an algebraic language instead of a geometrical one; the same technicalities remain in this way as well.

55th International Mathematical Olympiad

PROBLEMS SHORT LIST

WITH SOLUTIONS

IMO 2014

Cape Town - South Africa

Shortlisted Problems with Solutions

55$^{\text{th}}$ International Mathematical Olympiad

Cape Town, South Africa, 2014

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2015.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2014 thank the following 43 countries for contributing 141 problem proposals.

Australia, Austria, Belgium, Benin, Bulgaria, Colombia, Croatia, Cyprus, Czech Republic, Denmark, Ecuador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, Iceland, India, Indonesia, Iran, Ireland, Japan, Lithuania, Luxembourg, Malaysia, Mongolia, Netherlands, Nigeria, Pakistan, Russia, Saudi Arabia, Serbia, Slovakia, Slovenia, South Korea, Thailand, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Johan Meyer
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Problems

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}.$$

(Austria)

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

A3. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leqslant i \leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m) + n\big) + f(m) = f(n) + f(3m) + 2014$$

for all integers m and n.

(Netherlands)

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2|y|.$$

Determine all possible values of $P(0)$.

(Belgium)

A6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2$$

for all $n \in \mathbb{Z}$.

(United Kingdom)

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

(Hungary)

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\lceil \sqrt{n/2} \rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.

2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.

3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ .$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC} .$$

(Iran)

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1\,.$$

(U.S.A.)

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k \geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n + 2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\| \frac{a}{p^k} \right\| + \left\| \frac{b}{p^k} \right\| + \left\| \frac{a+b}{p^k} \right\| = 1.$$

(Hungary)

Solutions

Algebra

A1. Let $z_0 < z_1 < z_2 < \cdots$ be an infinite sequence of positive integers. Prove that there exists a unique integer $n \geqslant 1$ such that

$$z_n < \frac{z_0 + z_1 + \cdots + z_n}{n} \leqslant z_{n+1}. \tag{1}$$

(Austria)

Solution. For $n = 1, 2, \ldots$ define

$$d_n = (z_0 + z_1 + \cdots + z_n) - n z_n.$$

The sign of d_n indicates whether the first inequality in (1) holds; i.e., it is satisfied if and only if $d_n > 0$.

Notice that

$$n z_{n+1} - (z_0 + z_1 + \cdots + z_n) = (n+1) z_{n+1} - (z_0 + z_1 + \cdots + z_n + z_{n+1}) = -d_{n+1},$$

so the second inequality in (1) is equivalent to $d_{n+1} \leqslant 0$. Therefore, we have to prove that there is a unique index $n \geqslant 1$ that satisfies $d_n > 0 \geqslant d_{n+1}$.

By its definition the sequence $d_1, d_2, \ldots$ consists of integers and we have

$$d_1 = (z_0 + z_1) - 1 \cdot z_1 = z_0 > 0.$$

From

$$d_{n+1} - d_n = \big((z_0 + \cdots + z_n + z_{n+1}) - (n+1) z_{n+1}\big) - \big((z_0 + \cdots + z_n) - n z_n\big) = n(z_n - z_{n+1}) < 0$$

we can see that $d_{n+1} < d_n$ and thus the sequence strictly decreases.

Hence, we have a decreasing sequence $d_1 > d_2 > \ldots$ of integers such that its first element d_1 is positive. The sequence must drop below 0 at some point, and thus there is a unique index n, that is the index of the last positive term, satisfying $d_n > 0 \geqslant d_{n+1}$.

Comment. Omitting the assumption that $z_1, z_2, \ldots$ are integers allows the numbers d_n to be all positive. In such cases the desired n does not exist. This happens for example if $z_n = 2 - \frac{1}{2^n}$ for all integers $n \geqslant 0$.

A2. Define the function $f\colon (0,1) \to (0,1)$ by

$$f(x) = \begin{cases} x + \frac{1}{2} & \text{if } x < \frac{1}{2}, \\ x^2 & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Let a and b be two real numbers such that $0 < a < b < 1$. We define the sequences a_n and b_n by $a_0 = a$, $b_0 = b$, and $a_n = f(a_{n-1})$, $b_n = f(b_{n-1})$ for $n > 0$. Show that there exists a positive integer n such that

$$(a_n - a_{n-1})(b_n - b_{n-1}) < 0.$$

(Denmark)

Solution. Note that

$$f(x) - x = \tfrac{1}{2} > 0$$

if $x < \frac{1}{2}$ and

$$f(x) - x = x^2 - x < 0$$

if $x \geqslant \frac{1}{2}$. So if we consider $(0,1)$ as being divided into the two subintervals $I_1 = (0, \frac{1}{2})$ and $I_2 = [\frac{1}{2}, 1)$, the inequality

$$(a_n - a_{n-1})(b_n - b_{n-1}) = \big(f(a_{n-1}) - a_{n-1}\big)\big(f(b_{n-1}) - b_{n-1}\big) < 0$$

holds if and only if a_{n-1} and b_{n-1} lie in distinct subintervals.

Let us now assume, to the contrary, that a_k and b_k always lie in the same subinterval. Consider the distance $d_k = |a_k - b_k|$. If both a_k and b_k lie in I_1, then

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k + \tfrac{1}{2} - b_k - \tfrac{1}{2}\right| = d_k.$$

If, on the other hand, a_k and b_k both lie in I_2, then $\min(a_k, b_k) \geqslant \frac{1}{2}$ and $\max(a_k, b_k) = \min(a_k, b_k) + d_k \geqslant \frac{1}{2} + d_k$, which implies

$$d_{k+1} = |a_{k+1} - b_{k+1}| = \left|a_k^2 - b_k^2\right| = \left|(a_k - b_k)(a_k + b_k)\right| \geqslant |a_k - b_k|\left(\tfrac{1}{2} + \tfrac{1}{2} + d_k\right) = d_k(1 + d_k) \geqslant d_k.$$

This means that the difference d_k is non-decreasing, and in particular $d_k \geqslant d_0 > 0$ for all k.

We can even say more. If a_k and b_k lie in I_2, then

$$d_{k+2} \geqslant d_{k+1} \geqslant d_k(1 + d_k) \geqslant d_k(1 + d_0).$$

If a_k and b_k both lie in I_1, then a_{k+1} and b_{k+1} both lie in I_2, and so we have

$$d_{k+2} \geqslant d_{k+1}(1 + d_{k+1}) \geqslant d_{k+1}(1 + d_0) = d_k(1 + d_0).$$

In either case, $d_{k+2} \geqslant d_k(1 + d_0)$, and inductively we get

$$d_{2m} \geqslant d_0(1 + d_0)^m.$$

For sufficiently large m, the right-hand side is greater than 1, but since a_{2m}, b_{2m} both lie in $(0,1)$, we must have $d_{2m} < 1$, a contradiction.

Thus there must be a positive integer n such that a_{n-1} and b_{n-1} do not lie in the same subinterval, which proves the desired statement.

A3. For a sequence $x_1, x_2, \ldots, x_n$ of real numbers, we define its *price* as

$$\max_{1 \leqslant i \leqslant n} |x_1 + \cdots + x_i|.$$

Given n real numbers, Dave and George want to arrange them into a sequence with a low price. Diligent Dave checks all possible ways and finds the minimum possible price D. Greedy George, on the other hand, chooses x_1 such that $|x_1|$ is as small as possible; among the remaining numbers, he chooses x_2 such that $|x_1 + x_2|$ is as small as possible, and so on. Thus, in the i^{th} step he chooses x_i among the remaining numbers so as to minimise the value of $|x_1 + x_2 + \cdots + x_i|$. In each step, if several numbers provide the same value, George chooses one at random. Finally he gets a sequence with price G.

Find the least possible constant c such that for every positive integer n, for every collection of n real numbers, and for every possible sequence that George might obtain, the resulting values satisfy the inequality $G \leqslant cD$.

(Georgia)

Answer. $c = 2$.

Solution. If the initial numbers are 1, -1, 2, and -2, then Dave may arrange them as $1, -2, 2, -1$, while George may get the sequence $1, -1, 2, -2$, resulting in $D = 1$ and $G = 2$. So we obtain $c \geqslant 2$.

Therefore, it remains to prove that $G \leqslant 2D$. Let $x_1, x_2, \ldots, x_n$ be the numbers Dave and George have at their disposal. Assume that Dave and George arrange them into sequences $d_1, d_2, \ldots, d_n$ and $g_1, g_2, \ldots, g_n$, respectively. Put

$$M = \max_{1 \leqslant i \leqslant n} |x_i|, \quad S = |x_1 + \cdots + x_n|, \quad \text{and} \quad N = \max\{M, S\}.$$

We claim that

$$D \geqslant S, \tag{1}$$

$$D \geqslant \frac{M}{2}, \qquad \text{and} \tag{2}$$

$$G \leqslant N = \max\{M, S\}. \tag{3}$$

These inequalities yield the desired estimate, as $G \leqslant \max\{M, S\} \leqslant \max\{M, 2S\} \leqslant 2D$.

The inequality (1) is a direct consequence of the definition of the price.

To prove (2), consider an index i with $|d_i| = M$. Then we have

$$M = |d_i| = \big|(d_1 + \cdots + d_i) - (d_1 + \cdots + d_{i-1})\big| \leqslant |d_1 + \cdots + d_i| + |d_1 + \cdots + d_{i-1}| \leqslant 2D,$$

as required.

It remains to establish (3). Put $h_i = g_1 + g_2 + \cdots + g_i$. We will prove by induction on i that $|h_i| \leqslant N$. The base case $i = 1$ holds, since $|h_1| = |g_1| \leqslant M \leqslant N$. Notice also that $|h_n| = S \leqslant N$.

For the induction step, assume that $|h_{i-1}| \leqslant N$. We distinguish two cases.

Case 1. Assume that no two of the numbers g_i, g_{i+1}, $\ldots$, g_n have opposite signs.

Without loss of generality, we may assume that they are all nonnegative. Then one has $h_{i-1} \leqslant h_i \leqslant \cdots \leqslant h_n$, thus

$$|h_i| \leqslant \max\big\{|h_{i-1}|, |h_n|\big\} \leqslant N.$$

Case 2. Among the numbers $g_i, g_{i+1}, \ldots, g_n$ there are positive and negative ones.

Then there exists some index $j \geqslant i$ such that $h_{i-1}g_j \leqslant 0$. By the definition of George's sequence we have

$$|h_i| = |h_{i-1} + g_i| \leqslant |h_{i-1} + g_j| \leqslant \max\bigl\{|h_{i-1}|, |g_j|\bigr\} \leqslant N.$$

Thus, the induction step is established.

Comment 1. One can establish the weaker inequalities $D \geqslant \frac{M}{2}$ and $G \leqslant D + \frac{M}{2}$ from which the result also follows.

Comment 2. One may ask a more specific question to find the maximal suitable c if the number n is fixed. For $n = 1$ or 2, the answer is $c = 1$. For $n = 3$, the answer is $c = \frac{3}{2}$, and it is reached e.g., for the collection $1, 2, -4$. Finally, for $n \geqslant 4$ the answer is $c = 2$. In this case the arguments from the solution above apply, and the answer is reached e.g., for the same collection $1, -1, 2, -2$, augmented by several zeroes.

A4. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(m)+n\big)+f(m)=f(n)+f(3m)+2014 \tag{1}$$

for all integers m and n.

(Netherlands)

Answer. There is only one such function, namely $n \longmapsto 2n+1007$.

Solution. Let f be a function satisfying (1). Set $C=1007$ and define the function $g\colon \mathbb{Z} \to \mathbb{Z}$ by $g(m)=f(3m)-f(m)+2C$ for all $m \in \mathbb{Z}$; in particular, $g(0)=2C$. Now (1) rewrites as

$$f\big(f(m)+n\big)=g(m)+f(n)$$

for all $m, n \in \mathbb{Z}$. By induction in both directions it follows that

$$f\big(tf(m)+n\big)=tg(m)+f(n) \tag{2}$$

holds for all $m, n, t \in \mathbb{Z}$. Applying this, for any $r \in \mathbb{Z}$, to the triples $\big(r, 0, f(0)\big)$ and $\big(0, 0, f(r)\big)$ in place of (m, n, t) we obtain

$$f(0)g(r)=f\big(f(r)f(0)\big)-f(0)=f(r)g(0)\,.$$

Now if $f(0)$ vanished, then $g(0)=2C>0$ would entail that f vanishes identically, contrary to (1). Thus $f(0) \neq 0$ and the previous equation yields $g(r)=\alpha\, f(r)$, where $\alpha=\frac{g(0)}{f(0)}$ is some nonzero constant.

So the definition of g reveals $f(3m)=(1+\alpha)f(m)-2C$, i.e.,

$$f(3m)-\beta=(1+\alpha)\big(f(m)-\beta\big) \tag{3}$$

for all $m \in \mathbb{Z}$, where $\beta=\frac{2C}{\alpha}$. By induction on k this implies

$$f(3^k m)-\beta=(1+\alpha)^k\big(f(m)-\beta\big) \tag{4}$$

for all integers $k \geqslant 0$ and m.

Since $3 \nmid 2014$, there exists by (1) some value $d=f(a)$ attained by f that is not divisible by 3. Now by (2) we have $f(n+td)=f(n)+tg(a)=f(n)+\alpha \cdot tf(a)$, i.e.,

$$f(n+td)=f(n)+\alpha \cdot td \tag{5}$$

for all $n, t \in \mathbb{Z}$.

Let us fix any positive integer k with $d \mid (3^k-1)$, which is possible, since $\gcd(3, d)=1$. E.g., by the Euler–Fermat theorem, we may take $k=\varphi(|d|)$. Now for each $m \in \mathbb{Z}$ we get

$$f(3^k m)=f(m)+\alpha(3^k-1)m$$

from (5), which in view of (4) yields $\big((1+\alpha)^k-1\big)\big(f(m)-\beta\big)=\alpha(3^k-1)m$. Since $\alpha \neq 0$, the right hand side does not vanish for $m \neq 0$, wherefore the first factor on the left hand side cannot vanish either. It follows that

$$f(m)=\frac{\alpha(3^k-1)}{(1+\alpha)^k-1} \cdot m+\beta\,.$$

So f is a linear function, say $f(m) = Am + \beta$ for all $m \in \mathbb{Z}$ with some constant $A \in \mathbb{Q}$. Plugging this into (1) one obtains $(A^2 - 2A)m + (A\beta - 2C) = 0$ for all m, which is equivalent to the conjunction of

$$A^2 = 2A \qquad \text{and} \qquad A\beta = 2C\,. \tag{6}$$

The first equation is equivalent to $A \in \{0, 2\}$, and as $C \neq 0$ the second one gives

$$A = 2 \qquad \text{and} \qquad \beta = C\,. \tag{7}$$

This shows that f is indeed the function mentioned in the answer and as the numbers found in (7) do indeed satisfy the equations (6) this function is indeed as desired.

Comment 1. One may see that $\alpha = 2$. A more pedestrian version of the above solution starts with a direct proof of this fact, that can be obtained by substituting some special values into (1), e.g., as follows.

Set $D = f(0)$. Plugging $m = 0$ into (1) and simplifying, we get

$$f(n + D) = f(n) + 2C \tag{8}$$

for all $n \in \mathbb{Z}$. In particular, for $n = 0, D, 2D$ we obtain $f(D) = 2C + D$, $f(2D) = f(D) + 2C = 4C + D$, and $f(3D) = f(2D) + 2C = 6C + D$. So substituting $m = D$ and $n = r - D$ into (1) and applying (8) with $n = r - D$ afterwards we learn

$$f(r + 2C) + 2C + D = \big(f(r) - 2C\big) + (6C + D) + 2C\,,$$

i.e., $f(r + 2C) = f(r) + 4C$. By induction in both directions it follows that

$$f(n + 2Ct) = f(n) + 4Ct \tag{9}$$

holds for all $n, t \in \mathbb{Z}$.

Claim. If a and b denote two integers with the property that $f(n + a) = f(n) + b$ holds for all $n \in \mathbb{Z}$, then $b = 2a$.

Proof. Applying induction in both directions to the assumption we get $f(n + ta) = f(n) + tb$ for all $n, t \in \mathbb{Z}$. Plugging $(n, t) = (0, 2C)$ into this equation and $(n, t) = (0, a)$ into (9) we get $f(2aC) - f(0) = 2bC = 4aC$, and, as $C \neq 0$, the claim follows. □

Now by (1), for any $m \in \mathbb{Z}$, the numbers $a = f(m)$ and $b = f(3m) - f(m) + 2C$ have the property mentioned in the claim, whence we have

$$f(3m) - C = 3\big(f(m) - C\big)\,.$$

In view of (3) this tells us indeed that $\alpha = 2$.

Now the solution may be completed as above, but due to our knowledge of $\alpha = 2$ we get the desired formula $f(m) = 2m + C$ directly without having the need to go through all linear functions. Now it just remains to check that this function does indeed satisfy (1).

Comment 2. It is natural to wonder what happens if one replaces the number 2014 appearing in the statement of the problem by some arbitrary integer B.

If B is odd, there is no such function, as can be seen by using the same ideas as in the above solution.

If $B \neq 0$ is even, however, then the only such function is given by $n \longmapsto 2n + B/2$. In case $3 \nmid B$ this was essentially proved above, but for the general case one more idea seems to be necessary. Writing $B = 3^\nu \cdot k$ with some integers ν and k such that $3 \nmid k$ one can obtain $f(n) = 2n + B/2$ for all n that are divisible by 3^ν in the same manner as usual; then one may use the formula $f(3n) = 3f(n) - B$ to establish the remaining cases.

Finally, in case $B = 0$ there are more solutions than just the function $n \longmapsto 2n$. It can be shown that all these other functions are periodic; to mention just one kind of example, for any even integers r and s the function

$$f(n) = \begin{cases} r & \text{if } n \text{ is even,} \\ s & \text{if } n \text{ is odd,} \end{cases}$$

also has the property under discussion.

A5. Consider all polynomials $P(x)$ with real coefficients that have the following property: for any two real numbers x and y one has

$$|y^2 - P(x)| \leqslant 2|x| \quad \text{if and only if} \quad |x^2 - P(y)| \leqslant 2|y|. \tag{1}$$

Determine all possible values of $P(0)$.

(Belgium)

Answer. The set of possible values of $P(0)$ is $(-\infty, 0) \cup \{1\}$.

Solution.

Part I. We begin by verifying that these numbers are indeed possible values of $P(0)$. To see that each negative real number $-C$ can be $P(0)$, it suffices to check that for every $C > 0$ the polynomial $P(x) = -\left(\frac{2x^2}{C} + C\right)$ has the property described in the statement of the problem. Due to symmetry it is enough for this purpose to prove $|y^2 - P(x)| > 2|x|$ for any two real numbers x and y. In fact we have

$$|y^2 - P(x)| = y^2 + \frac{x^2}{C} + \frac{(|x| - C)^2}{C} + 2|x| \geqslant \frac{x^2}{C} + 2|x| \geqslant 2|x|,$$

where in the first estimate equality can only hold if $|x| = C$, whilst in the second one it can only hold if $x = 0$. As these two conditions cannot be met at the same time, we have indeed $|y^2 - P(x)| > 2|x|$.

To show that $P(0) = 1$ is possible as well, we verify that the polynomial $P(x) = x^2 + 1$ satisfies (1). Notice that for all real numbers x and y we have

$$\begin{aligned} |y^2 - P(x)| \leqslant 2|x| &\iff (y^2 - x^2 - 1)^2 \leqslant 4x^2 \\ &\iff 0 \leqslant \left((y^2 - (x-1)^2\right)\left((x+1)^2 - y^2\right) \\ &\iff 0 \leqslant (y - x + 1)(y + x - 1)(x + 1 - y)(x + 1 + y) \\ &\iff 0 \leqslant \left((x+y)^2 - 1\right)\left(1 - (x-y)^2\right). \end{aligned}$$

Since this inequality is symmetric in x and y, we are done.

Part II. Now we show that no values other than those mentioned in the answer are possible for $P(0)$. To reach this we let P denote any polynomial satisfying (1) and $P(0) \geqslant 0$; as we shall see, this implies $P(x) = x^2 + 1$ for all real x, which is actually more than what we want.

First step: We prove that P is even.

By (1) we have

$$|y^2 - P(x)| \leqslant 2|x| \iff |x^2 - P(y)| \leqslant 2|y| \iff |y^2 - P(-x)| \leqslant 2|x|$$

for all real numbers x and y. Considering just the equivalence of the first and third statement and taking into account that y^2 may vary through $\mathbb{R}_{\geqslant 0}$ we infer that

$$\big[P(x) - 2|x|, P(x) + 2|x|\big] \cap \mathbb{R}_{\geqslant 0} = \big[P(-x) - 2|x|, P(-x) + 2|x|\big] \cap \mathbb{R}_{\geqslant 0}$$

holds for all $x \in \mathbb{R}$. We claim that there are infinitely many real numbers x such that $P(x) + 2|x| \geqslant 0$. This holds in fact for any real polynomial with $P(0) \geqslant 0$; in order to see this, we may assume that the coefficient of P appearing in front of x is nonnegative. In this case the desired inequality holds for all sufficiently small positive real numbers.

For such numbers x satisfying $P(x) + 2|x| \geqslant 0$ we have $P(x) + 2|x| = P(-x) + 2|x|$ by the previous displayed formula, and hence also $P(x) = P(-x)$. Consequently the polynomial $P(x) - P(-x)$ has infinitely many zeros, wherefore it has to vanish identically. Thus P is indeed even.

Second step: We prove that $P(t) > 0$ for all $t \in \mathbb{R}$.

Let us assume for a moment that there exists a real number $t \neq 0$ with $P(t) = 0$. Then there is some open interval I around t such that $|P(y)| \leqslant 2|y|$ holds for all $y \in I$. Plugging $x = 0$ into (1) we learn that $y^2 = P(0)$ holds for all $y \in I$, which is clearly absurd. We have thus shown $P(t) \neq 0$ for all $t \neq 0$.

In combination with $P(0) \geqslant 0$ this informs us that our claim could only fail if $P(0) = 0$. In this case there is by our first step a polynomial $Q(x)$ such that $P(x) = x^2 Q(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|y\,Q(y)| > 2$, which is surely false when y is sufficiently small.

Third step: We prove that P is a quadratic polynomial.

Notice that P cannot be constant, for otherwise if $x = \sqrt{P(0)}$ and y is sufficiently large, the first part of (1) is false whilst the second part is true. So the degree n of P has to be at least 1. By our first step n has to be even as well, whence in particular $n \geqslant 2$.

Now assume that $n \geqslant 4$. Plugging $y = \sqrt{P(x)}$ into (1) we get $\left|x^2 - P\left(\sqrt{P(x)}\right)\right| \leqslant 2\sqrt{P(x)}$ and hence

$$P\left(\sqrt{P(x)}\right) \leqslant x^2 + 2\sqrt{P(x)}$$

for all real x. Choose positive real numbers x_0, a, and b such that if $x \in (x_0, \infty)$, then $ax^n < P(x) < bx^n$; this is indeed possible, for if $d > 0$ denotes the leading coefficient of P, then $\lim\limits_{x\to\infty} \frac{P(x)}{x^n} = d$, whence for instance the numbers $a = \frac{d}{2}$ and $b = 2d$ work provided that x_0 is chosen large enough.

Now for all sufficiently large real numbers x we have

$$a^{n/2+1} x^{n^2/2} < aP(x)^{n/2} < P\left(\sqrt{P(x)}\right) \leqslant x^2 + 2\sqrt{P(x)} < x^{n/2} + 2b^{1/2} x^{n/2}\,,$$

i.e.

$$x^{(n^2-n)/2} < \frac{1 + 2b^{1/2}}{a^{n/2+1}}\,,$$

which is surely absurd. Thus P is indeed a quadratic polynomial.

Fourth step: We prove that $P(x) = x^2 + 1$.

In the light of our first three steps there are two real numbers $a > 0$ and b such that $P(x) = ax^2 + b$. Now if x is large enough and $y = \sqrt{a}\,x$, the left part of (1) holds and the right part reads $|(1-a^2)x^2 - b| \leqslant 2\sqrt{a}\,x$. In view of the fact that $a > 0$ this is only possible if $a = 1$. Finally, substituting $y = x + 1$ with $x > 0$ into (1) we get

$$|2x + 1 - b| \leqslant 2x \iff |2x + 1 + b| \leqslant 2x + 2\,,$$

i.e.,

$$b \in [1, 4x + 1] \iff b \in [-4x - 3, 1]$$

for all $x > 0$. Choosing x large enough, we can achieve that at least one of these two statements holds; then both hold, which is only possible if $b = 1$, as desired.

Comment 1. There are some issues with this problem in that its most natural solutions seem to use some basic facts from analysis, such as the continuity of polynomials or the intermediate value theorem. Yet these facts are intuitively obvious and implicitly clear to the students competing at this level of difficulty, so that the Problem Selection Committee still thinks that the problem is suitable for the IMO.

Comment 2. It seems that most solutions will in the main case, where $P(0)$ is nonnegative, contain an argument that is somewhat asymptotic in nature showing that P is quadratic, and some part narrowing that case down to $P(x) = x^2 + 1$.

Comment 3. It is also possible to skip the first step and start with the second step directly, but then one has to work a bit harder to rule out the case $P(0) = 0$. Let us sketch one possibility of doing this: Take the auxiliary polynomial $Q(x)$ such that $P(x) = xQ(x)$. Applying (1) to $x = 0$ and an arbitrary $y \neq 0$ we get $|Q(y)| > 2$. Hence we either have $Q(z) \geqslant 2$ for all real z or $Q(z) \leqslant -2$ for all real z. In particular there is some $\eta \in \{-1, +1\}$ such that $P(\eta) \geqslant 2$ and $P(-\eta) \leqslant -2$. Substituting $x = \pm\eta$ into (1) we learn

$$|y^2 - P(\eta)| \leqslant 2 \iff |1 - P(y)| \leqslant 2\,|y| \iff |y^2 - P(-\eta)| \leqslant 2\,.$$

But for $y = \sqrt{P(\eta)}$ the first statement is true, whilst the third one is false.

Also, if one has not obtained the evenness of P before embarking on the fourth step, one needs to work a bit harder there, but not in a way that is likely to cause major difficulties.

Comment 4. Truly curious people may wonder about the set of all polynomials having property (1). As explained in the solution above, $P(x) = x^2 + 1$ is the only one with $P(0) = 1$. On the other hand, it is not hard to notice that for negative $P(0)$ there are more possibilities than those mentioned above. E.g., as remarked by the proposer, if a and b denote two positive real numbers with $ab > 1$ and Q denotes a polynomial attaining nonnegative values only, then $P(x) = -\big(ax^2 + b + Q(x)\big)$ works.

More generally, it may be proved that if $P(x)$ satisfies (1) and $P(0) < 0$, then $-P(x) > 2\,|x|$ holds for all $x \in \mathbb{R}$ so that one just considers the equivalence of two false statements. One may generate all such polynomials P by going through all combinations of a solution of the polynomial equation

$$x = A(x)B(x) + C(x)D(x)$$

and a real $E > 0$, and setting

$$P(x) = -\big(A(x)^2 + B(x)^2 + C(x)^2 + D(x)^2 + E\big)$$

for each of them.

A6. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$n^2 + 4f(n) = f(f(n))^2 \tag{1}$$

for all $n \in \mathbb{Z}$.

(United Kingdom)

Answer. The possibilities are:

- $f(n) = n + 1$ for all n;
- or, for some $a \geqslant 1$, $f(n) = \begin{cases} n+1, & n > -a, \\ -n+1, & n \leqslant -a; \end{cases}$
- or $f(n) = \begin{cases} n+1, & n > 0, \\ 0, & n = 0, \\ -n+1, & n < 0. \end{cases}$

Solution 1.

Part I. Let us first check that each of the functions above really satisfies the given functional equation. If $f(n) = n + 1$ for all n, then we have

$$n^2 + 4f(n) = n^2 + 4n + 4 = (n+2)^2 = f(n+1)^2 = f(f(n))^2.$$

If $f(n) = n + 1$ for $n > -a$ and $f(n) = -n + 1$ otherwise, then we have the same identity for $n > -a$ and

$$n^2 + 4f(n) = n^2 - 4n + 4 = (2-n)^2 = f(1-n)^2 = f(f(n))^2$$

otherwise. The same applies to the third solution (with $a = 0$), where in addition one has

$$0^2 + 4f(0) = 0 = f(f(0))^2.$$

Part II. It remains to prove that these are really the only functions that satisfy our functional equation. We do so in three steps:

Step 1: We prove that $f(n) = n + 1$ for $n > 0$.

Consider the sequence (a_k) given by $a_k = f^k(1)$ for $k \geqslant 0$. Setting $n = a_k$ in (1), we get

$$a_k^2 + 4a_{k+1} = a_{k+2}^2.$$

Of course, $a_0 = 1$ by definition. Since $a_2^2 = 1 + 4a_1$ is odd, a_2 has to be odd as well, so we set $a_2 = 2r + 1$ for some $r \in \mathbb{Z}$. Then $a_1 = r^2 + r$ and consequently

$$a_3^2 = a_1^2 + 4a_2 = (r^2 + r)^2 + 8r + 4.$$

Since $8r + 4 \neq 0$, $a_3^2 \neq (r^2 + r)^2$, so the difference between a_3^2 and $(r^2 + r)^2$ is at least the distance from $(r^2 + r)^2$ to the nearest even square (since $8r + 4$ and $r^2 + r$ are both even). This implies that

$$|8r + 4| = \left|a_3^2 - (r^2 + r)^2\right| \geqslant (r^2 + r)^2 - (r^2 + r - 2)^2 = 4(r^2 + r - 1),$$

(for $r = 0$ and $r = -1$, the estimate is trivial, but this does not matter). Therefore, we ave

$$4r^2 \leqslant |8r + 4| - 4r + 4.$$

If $|r| \geqslant 4$, then

$$4r^2 \geqslant 16|r| \geqslant 12|r| + 16 > 8|r| + 4 + 4|r| + 4 \geqslant |8r+4| - 4r + 4,$$

a contradiction. Thus $|r| < 4$. Checking all possible remaining values of r, we find that $(r^2+r)^2 + 8r + 4$ is only a square in three cases: $r = -3$, $r = 0$ and $r = 1$. Let us now distinguish these three cases:

- $r = -3$, thus $a_1 = 6$ and $a_2 = -5$. For each $k \geqslant 1$, we have

$$a_{k+2} = \pm\sqrt{a_k^2 + 4a_{k+1}},$$

  and the sign needs to be chosen in such a way that $a_{k+1}^2 + 4a_{k+2}$ is again a square. This yields $a_3 = -4$, $a_4 = -3$, $a_5 = -2$, $a_6 = -1$, $a_7 = 0$, $a_8 = 1$, $a_9 = 2$. At this point we have reached a contradiction, since $f(1) = f(a_0) = a_1 = 6$ and at the same time $f(1) = f(a_8) = a_9 = 2$.

- $r = 0$, thus $a_1 = 0$ and $a_2 = 1$. Then $a_3^2 = a_1^2 + 4a_2 = 4$, so $a_3 = \pm 2$. This, however, is a contradiction again, since it gives us $f(1) = f(a_0) = a_1 = 0$ and at the same time $f(1) = f(a_2) = a_3 = \pm 2$.

- $r = 1$, thus $a_1 = 2$ and $a_2 = 3$. We prove by induction that $a_k = k + 1$ for all $k \geqslant 0$ in this case, which we already know for $k \leqslant 2$ now. For the induction step, assume that $a_{k-1} = k$ and $a_k = k+1$. Then

$$a_{k+1}^2 = a_{k-1}^2 + 4a_k = k^2 + 4k + 4 = (k+2)^2,$$

  so $a_{k+1} = \pm(k+2)$. If $a_{k+1} = -(k+2)$, then

$$a_{k+2}^2 = a_k^2 + 4a_{k+1} = (k+1)^2 - 4k - 8 = k^2 - 2k - 7 = (k-1)^2 - 8.$$

  The latter can only be a square if $k = 4$ (since 1 and 9 are the only two squares whose difference is 8). Then, however, $a_4 = 5$, $a_5 = -6$ and $a_6 = \pm 1$, so

$$a_7^2 = a_5^2 + 4a_6 = 36 \pm 4,$$

  but neither 32 nor 40 is a perfect square. Thus $a_{k+1} = k + 2$, which completes our induction. This also means that $f(n) = f(a_{n-1}) = a_n = n + 1$ for all $n \geqslant 1$.

Step 2: We prove that either $f(0) = 1$, or $f(0) = 0$ and $f(n) \neq 0$ for $n \neq 0$.

Set $n = 0$ in (1) to get

$$4f(0) = f(f(0))^2.$$

This means that $f(0) \geqslant 0$. If $f(0) = 0$, then $f(n) \neq 0$ for all $n \neq 0$, since we would otherwise have

$$n^2 = n^2 + 4f(n) = f(f(n))^2 = f(0)^2 = 0.$$

If $f(0) > 0$, then we know that $f(f(0)) = f(0) + 1$ from the first step, so

$$4f(0) = \big(f(0) + 1\big)^2,$$

which yields $f(0) = 1$.

Step 3: We discuss the values of $f(n)$ for $n < 0$.

Lemma. For every $n \geqslant 1$, we have $f(-n) = -n+1$ or $f(-n) = n+1$. Moreover, if $f(-n) = -n+1$ for some $n \geqslant 1$, then also $f(-n+1) = -n+2$.

Proof. We prove this statement by strong induction on n. For $n = 1$, we get

$$1 + 4f(-1) = f(f(-1))^2.$$

Thus $f(-1)$ needs to be nonnegative. If $f(-1) = 0$, then $f(f(-1)) = f(0) = \pm 1$, so $f(0) = 1$ (by our second step). Otherwise, we know that $f(f(-1)) = f(-1) + 1$, so

$$1 + 4f(-1) = \big(f(-1) + 1\big)^2,$$

which yields $f(-1) = 2$ and thus establishes the base case. For the induction step, we consider two cases:

- If $f(-n) \leqslant -n$, then

  $$f(f(-n))^2 = (-n)^2 + 4f(-n) \leqslant n^2 - 4n < (n-2)^2,$$

  so $|f(f(-n))| \leqslant n - 3$ (for $n = 2$, this case cannot even occur). If $f(f(-n)) \geqslant 0$, then we already know from the first two steps that $f(f(f(-n))) = f(f(-n)) + 1$, unless perhaps if $f(0) = 0$ and $f(f(-n)) = 0$. However, the latter would imply $f(-n) = 0$ (as shown in Step 2) and thus $n = 0$, which is impossible. If $f(f(-n)) < 0$, we can apply the induction hypothesis to $f(f(-n))$. In either case, $f(f(f(-n))) = \pm f(f(-n)) + 1$. Therefore,

  $$f(-n)^2 + 4f(f(-n)) = f(f(f(-n)))^2 = \big(\pm f(f(-n)) + 1\big)^2,$$

  which gives us

  $$\begin{aligned} n^2 &\leqslant f(-n)^2 = \big(\pm f(f(-n)) + 1\big)^2 - 4f(f(-n)) \leqslant f(f(-n))^2 + 6|f(f(-n))| + 1 \\ &\leqslant (n-3)^2 + 6(n-3) + 1 = n^2 - 8, \end{aligned}$$

  a contradiction.

- Thus, we are left with the case that $f(-n) > -n$. Now we argue as in the previous case: if $f(-n) \geqslant 0$, then $f(f(-n)) = f(-n) + 1$ by the first two steps, since $f(0) = 0$ and $f(-n) = 0$ would imply $n = 0$ (as seen in Step 2) and is thus impossible. If $f(-n) < 0$, we can apply the induction hypothesis, so in any case we can infer that $f(f(-n)) = \pm f(-n) + 1$. We obtain

  $$(-n)^2 + 4f(-n) = \big(\pm f(-n) + 1\big)^2,$$

  so either

  $$n^2 = f(-n)^2 - 2f(-n) + 1 = \big(f(-n) - 1\big)^2,$$

  which gives us $f(-n) = \pm n + 1$, or

  $$n^2 = f(-n)^2 - 6f(-n) + 1 = \big(f(-n) - 3\big)^2 - 8.$$

  Since 1 and 9 are the only perfect squares whose difference is 8, we must have $n = 1$, which we have already considered.

Finally, suppose that $f(-n) = -n+1$ for some $n \geqslant 2$. Then

$$f(-n+1)^2 = f(f(-n))^2 = (-n)^2 + 4f(-n) = (n-2)^2,$$

so $f(-n+1) = \pm(n-2)$. However, we already know that $f(-n+1) = -n+2$ or $f(-n+1) = n$, so $f(-n+1) = -n+2$. $\square$

Combining everything we know, we find the solutions as stated in the answer:

- One solution is given by $f(n) = n + 1$ for all n.
- If $f(n)$ is not always equal to $n + 1$, then there is a largest integer m (which cannot be positive) for which this is not the case. In view of the lemma that we proved, we must then have $f(n) = -n+1$ for any integer $n < m$. If $m = -a < 0$, we obtain $f(n) = -n+1$ for $n \leqslant -a$ (and $f(n) = n + 1$ otherwise). If $m = 0$, we have the additional possibility that $f(0) = 0$, $f(n) = -n + 1$ for negative n and $f(n) = n + 1$ for positive n.

Solution 2. Let us provide an alternative proof for Part II, which also proceeds in several steps.

Step 1. Let a be an arbitrary integer and $b = f(a)$. We first concentrate on the case where $|a|$ is sufficiently large.

1. If $b = 0$, then (1) applied to a yields $a^2 = f(f(a))^2$, thus
$$f(a) = 0 \quad \Rightarrow \quad a = \pm f(0). \tag{2}$$
From now on, we set $D = |f(0)|$. Throughout Step 1, we will assume that $a \notin \{-D, 0, D\}$, thus $b \neq 0$.

2. From (1), noticing that $f(f(a))$ and a have the same parity, we get
$$0 \neq 4|b| = \left|f(f(a))^2 - a^2\right| \geqslant a^2 - \big(|a| - 2\big)^2 = 4|a| - 4.$$
Hence we have
$$|b| = |f(a)| \geqslant |a| - 1 \qquad \text{for } a \notin \{-D, 0, D\}. \tag{3}$$
For the rest of Step 1, we also assume that $|a| \geqslant E = \max\{D + 2, 10\}$. Then by (3) we have $|b| \geqslant D + 1$ and thus $|f(b)| \geqslant D$.

3. Set $c = f(b)$; by (3), we have $|c| \geqslant |b| - 1$. Thus (1) yields
$$a^2 + 4b = c^2 \geqslant \big(|b| - 1\big)^2,$$
which implies
$$a^2 \geqslant \big(|b| - 1\big)^2 - 4|b| = \big(|b| - 3\big)^2 - 8 > \big(|b| - 4\big)^2$$
because $|b| \geqslant |a| - 1 \geqslant 9$. Thus (3) can be refined to
$$|a| + 3 \geqslant |f(a)| \geqslant |a| - 1 \qquad \text{for } |a| \geqslant E.$$
Now, from $c^2 = a^2 + 4b$ with $|b| \in \big[|a| - 1, |a| + 3\big]$ we get $c^2 = (a \pm 2)^2 + d$, where $d \in \{-16, -12, -8, -4, 0, 4, 8\}$. Since $|a \pm 2| \geqslant 8$, this can happen only if $c^2 = (a \pm 2)^2$, which in turn yields $b = \pm a + 1$. To summarise,
$$f(a) = 1 \pm a \qquad \text{for } |a| \geqslant E. \tag{4}$$

We have shown that, with at most finitely many exceptions, $f(a) = 1 \pm a$. Thus it will be convenient for our second step to introduce the sets

$$Z_+ = \big\{a \in \mathbb{Z} \colon f(a) = a + 1\big\}, \quad Z_- = \big\{a \in \mathbb{Z} \colon f(a) = 1 - a\big\}, \quad \text{and} \quad Z_0 = \mathbb{Z} \backslash \big(Z_+ \cup Z_-\big).$$

Step 2. Now we investigate the structure of the sets Z_+, Z_-, and Z_0.

4. Note that $f(E+1) = 1 \pm (E+1)$. If $f(E+1) = E+2$, then $E+1 \in Z_+$. Otherwise we have $f(1+E) = -E$; then the original equation (1) with $n = E+1$ gives us $(E-1)^2 = f(-E)^2$, so $f(-E) = \pm(E-1)$. By (4) this may happen only if $f(-E) = 1 - E$, so in this case $-E \in Z_+$. In any case we find that $Z_+ \neq \varnothing$.

5. Now take any $a \in Z_+$. We claim that every integer $x \geqslant a$ also lies in Z_+. We proceed by induction on x, the base case $x = a$ being covered by our assumption. For the induction step, assume that $f(x-1) = x$ and plug $n = x - 1$ into (1). We get $f(x)^2 = (x+1)^2$, so either $f(x) = x + 1$ or $f(x) = -(x+1)$.

   Assume that $f(x) = -(x+1)$ and $x \neq -1$, since otherwise we already have $f(x) = x + 1$. Plugging $n = x$ into (1), we obtain $f(-x-1)^2 = (x-2)^2 - 8$, which may happen only if $x-2 = \pm 3$ and $f(-x-1) = \pm 1$. Plugging $n = -x-1$ into (1), we get $f(\pm 1)^2 = (x+1)^2 \pm 4$, which in turn may happen only if $x + 1 \in \{-2, 0, 2\}$.

   Thus $x \in \{-1, 5\}$ and at the same time $x \in \{-3, -1, 1\}$, which gives us $x = -1$. Since this has already been excluded, we must have $f(x) = x + 1$, which completes our induction.

6. Now we know that either $Z_+ = \mathbb{Z}$ (if Z_+ is not bounded below), or $Z_+ = \{a \in \mathbb{Z} \colon a \geqslant a_0\}$, where a_0 is the smallest element of Z_+. In the former case, $f(n) = n + 1$ for all $n \in \mathbb{Z}$, which is our first solution. So we assume in the following that Z_+ is bounded below and has a smallest element a_0.

   If $Z_0 = \varnothing$, then we have $f(x) = x + 1$ for $x \geqslant a_0$ and $f(x) = 1 - x$ for $x < a_0$. In particular, $f(0) = 1$ in any case, so $0 \in Z_+$ and thus $a_0 \leqslant 0$. Thus we end up with the second solution listed in the answer. It remains to consider the case where $Z_0 \neq \varnothing$.

7. Assume that there exists some $a \in Z_0$ with $b = f(a) \notin Z_0$, so that $f(b) = 1 \pm b$. Then we have $a^2 + 4b = (1 \pm b)^2$, so either $a^2 = (b-1)^2$ or $a^2 = (b-3)^2 - 8$. In the former case we have $b = 1 \pm a$, which is impossible by our choice of a. So we get $a^2 = (b-3)^2 - 8$, which implies $f(b) = 1 - b$ and $|a| = 1$, $|b - 3| = 3$.

   If $b = 0$, then we have $f(b) = 1$, so $b \in Z_+$ and therefore $a_0 \leqslant 0$; hence $a = -1$. But then $f(a) = 0 = a + 1$, so $a \in Z_+$, which is impossible.

   If $b = 6$, then we have $f(6) = -5$, so $f(-5)^2 = 16$ and $f(-5) \in \{-4, 4\}$. Then $f(f(-5))^2 = 25 + 4f(-5) \in \{9, 41\}$, so $f(-5) = -4$ and $-5 \in Z_+$. This implies $a_0 \leqslant -5$, which contradicts our assumption that $\pm 1 = a \notin Z_+$.

8. Thus we have shown that $f(Z_0) \subseteq Z_0$, and Z_0 is finite. Take any element $c \in Z_0$, and consider the sequence defined by $c_i = f^i(c)$. All elements of the sequence (c_i) lie in Z_0, hence it is bounded. Choose an index k for which $|c_k|$ is maximal, so that in particular $|c_{k+1}| \leqslant |c_k|$ and $|c_{k+2}| \leqslant |c_k|$. Our functional equation (1) yields

   $$(|c_k| - 2)^2 - 4 = |c_k|^2 - 4|c_k| \leqslant c_k^2 + 4c_{k+1} = c_{k+2}^2.$$

   Since c_k and c_{k+2} have the same parity and $|c_{k+2}| \leqslant |c_k|$, this leaves us with three possibilities: $|c_{k+2}| = |c_k|$, $|c_{k+2}| = |c_k| - 2$, and $|c_k| - 2 = \pm 2$, $c_{k+2} = 0$.

   If $|c_{k+2}| = |c_k| - 2$, then $f(c_k) = c_{k+1} = 1 - |c_k|$, which means that $c_k \in Z_-$ or $c_k \in Z_+$, and we reach a contradiction.

   If $|c_{k+2}| = |c_k|$, then $c_{k+1} = 0$, thus $c_{k+3}^2 = 4c_{k+2}$. So either $c_{k+3} \neq 0$ or (by maximality of $|c_{k+2}| = |c_k|$) $c_i = 0$ for all i. In the former case, we can repeat the entire argument with c_{k+2} in the place of c_k. Now $|c_{k+4}| = |c_{k+2}|$ is not possible any more since $c_{k+3} \neq 0$, leaving us with the only possibility $|c_k| - 2 = |c_{k+2}| - 2 = \pm 2$.

Thus we know now that either all c_i are equal to 0, or $|c_k| = 4$. If $c_k = \pm 4$, then either $c_{k+1} = 0$ and $|c_{k+2}| = |c_k| = 4$, or $c_{k+2} = 0$ and $c_{k+1} = -4$. From this point onwards, all elements of the sequence are either 0 or ± 4.

Let c_r be the last element of the sequence that is not equal to 0 or ± 4 (if such an element exists). Then $c_{r+1}, c_{r+2} \in \{-4, 0, 4\}$, so

$$c_r^2 = c_{r+2}^2 - 4c_{r+1} \in \{-16, 0, 16, 32\},$$

which gives us a contradiction. Thus all elements of the sequence are equal to 0 or ± 4, and since the choice of $c_0 = c$ was arbitrary, $Z_0 \subseteq \{-4, 0, 4\}$.

9. Finally, we show that $4 \notin Z_0$ and $-4 \notin Z_0$. Suppose that $4 \in Z_0$. Then in particular a_0 (the smallest element of Z_+) cannot be less than 4, since this would imply $4 \in Z_+$. So $-3 \in Z_-$, which means that $f(-3) = 4$. Then $25 = (-3)^2 + 4f(-3) = f(f(-3))^2 = f(4)^2$, so $f(4) = \pm 5 \notin Z_0$, and we reach a contradiction.

   Suppose that $-4 \in Z_0$. The only possible values for $f(-4)$ that are left are 0 and -4. Note that $4f(0) = f(f(0))^2$, so $f(0) \geqslant 0$. If $f(-4) = 0$, then we get $16 = (-4)^2 + 0 = f(0)^2$, thus $f(0) = 4$. But then $f(f(-4)) \notin Z_0$, which is impossible. Thus $f(-4) = -4$, which gives us $0 = (-4)^2 + 4f(-4) = f(f(-4))^2 = 16$, and this is clearly absurd.

   Now we are left with $Z_0 = \{0\}$ and $f(0) = 0$ as the only possibility. If $1 \in Z_-$, then $f(1) = 0$, so $1 = 1^2 + 4f(1) = f(f(1))^2 = f(0)^2 = 0$, which is another contradiction. Thus $1 \in Z_+$, meaning that $a_0 \leqslant 1$. On the other hand, $a_0 \leqslant 0$ would imply $0 \in Z_+$, so we can only have $a_0 = 1$. Thus Z_+ comprises all positive integers, and Z_- comprises all negative integers. This gives us the third solution.

Comment. All solutions known to the Problem Selection Committee are quite lengthy and technical, as the two solutions presented here show. It is possible to make the problem easier by imposing additional assumptions, such as $f(0) \neq 0$ or $f(n) \geqslant 1$ for all $n \geqslant 0$, to remove some of the technicalities.

Combinatorics

C1. Let n points be given inside a rectangle R such that no two of them lie on a line parallel to one of the sides of R. The rectangle R is to be dissected into smaller rectangles with sides parallel to the sides of R in such a way that none of these rectangles contains any of the given points in its interior. Prove that we have to dissect R into at least $n+1$ smaller rectangles.

(Serbia)

Solution 1. Let k be the number of rectangles in the dissection. The set of all points that are corners of one of the rectangles can be divided into three disjoint subsets:

- A, which consists of the four corners of the original rectangle R, each of which is the corner of exactly one of the smaller rectangles,
- B, which contains points where exactly two of the rectangles have a common corner (T-junctions, see the figure below),
- C, which contains points where four of the rectangles have a common corner (crossings, see the figure below).

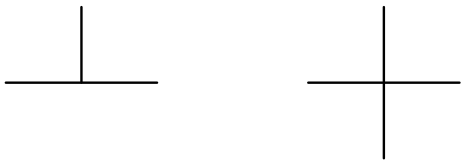


Figure 1: A T-junction and a crossing

We denote the number of points in B by b and the number of points in C by c. Since each of the k rectangles has exactly four corners, we get

$$4k = 4 + 2b + 4c.$$

It follows that $2b \leqslant 4k - 4$, so $b \leqslant 2k - 2$.

Each of the n given points has to lie on a side of one of the smaller rectangles (but not of the original rectangle R). If we extend this side as far as possible along borders between rectangles, we obtain a line segment whose ends are T-junctions. Note that every point in B can only be an endpoint of at most one such segment containing one of the given points, since it is stated that no two of them lie on a common line parallel to the sides of R. This means that

$$b \geqslant 2n.$$

Combining our two inequalities for b, we get

$$2k - 2 \geqslant b \geqslant 2n,$$

thus $k \geqslant n + 1$, which is what we wanted to prove.

Solution 2. Let k denote the number of rectangles. In the following, we refer to the directions of the sides of R as 'horizontal' and 'vertical' respectively. Our goal is to prove the inequality $k \geqslant n+1$ for fixed n. Equivalently, we can prove the inequality $n \leqslant k-1$ for each k, which will be done by induction on k. For $k=1$, the statement is trivial.

Now assume that $k>1$. If none of the line segments that form the borders between the rectangles is horizontal, then we have $k-1$ vertical segments dividing R into k rectangles. On each of them, there can only be one of the n points, so $n \leqslant k-1$, which is exactly what we want to prove.

Otherwise, consider the lowest horizontal line h that contains one or more of these line segments. Let R' be the rectangle that results when everything that lies below h is removed from R (see the example in the figure below).

The rectangles that lie entirely below h form blocks of rectangles separated by vertical line segments. Suppose there are r blocks and k_i rectangles in the i^{th} block. The left and right border of each block has to extend further upwards beyond h. Thus we can move any points that lie on these borders upwards, so that they now lie inside R'. This can be done without violating the conditions, one only needs to make sure that they do not get to lie on a common horizontal line with one of the other given points.

All other borders between rectangles in the i^{th} block have to lie entirely below h. There are k_i-1 such line segments, each of which can contain at most one of the given points. Finally, there can be one point that lies on h. All other points have to lie in R' (after moving some of them as explained in the previous paragraph).

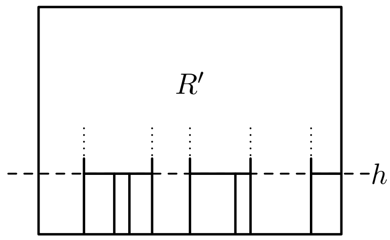


Figure 2: Illustration of the inductive argument

We see that R' is divided into $k-\sum_{i=1}^{r} k_i$ rectangles. Applying the induction hypothesis to R', we find that there are at most

$$\left(k-\sum_{i=1}^{r} k_i\right)-1+\sum_{i=1}^{r}(k_i-1)+1=k-r$$

points. Since $r \geqslant 1$, this means that $n \leqslant k-1$, which completes our induction.

C2. We have 2^m sheets of paper, with the number 1 written on each of them. We perform the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

(Iran)

Solution. Let P_k be the product of the numbers on the sheets after k steps.

Suppose that in the $(k+1)^{\text{th}}$ step the numbers a and b are replaced by $a+b$. In the product, the number ab is replaced by $(a+b)^2$, and the other factors do not change. Since $(a+b)^2 \geqslant 4ab$, we see that $P_{k+1} \geqslant 4P_k$. Starting with $P_0 = 1$, a straightforward induction yields

$$P_k \geqslant 4^k$$

for all integers $k \geqslant 0$; in particular

$$P_{m\cdot 2^{m-1}} \geqslant 4^{m\cdot 2^{m-1}} = \left(2^m\right)^{2^m},$$

so by the AM–GM inequality, the sum of the numbers written on the sheets after $m2^{m-1}$ steps is at least

$$2^m \cdot \sqrt[2^m]{P_{m\cdot 2^{m-1}}} \geqslant 2^m \cdot 2^m = 4^m.$$

Comment 1. It is possible to achieve the sum 4^m in $m2^{m-1}$ steps. For example, starting from 2^m equal numbers on the sheets, in 2^{m-1} consecutive steps we can double all numbers. After m such doubling rounds we have the number 2^m on every sheet.

Comment 2. There are several versions of the solution above. E.g., one may try to assign to each positive integer n a *weight* w_n in such a way that the sum of the weights of the numbers written on the sheets increases, say, by at least 2 in each step. For this purpose, one needs the inequality

$$2w_{a+b} \geqslant w_a + w_b + 2 \tag{1}$$

to be satisfied for all positive integers a and b.

Starting from $w_1 = 1$ and trying to choose the weights as small as possible, one may find that these weights can be defined as follows: For every positive integer n, one chooses k to be the maximal integer such that $n \geqslant 2^k$, and puts

$$w_n = k + \frac{n}{2^k} = \min_{d\in\mathbb{Z}_{\geqslant 0}} \left(d + \frac{n}{2^d}\right). \tag{2}$$

Now, in order to prove that these weights satisfy (1), one may take arbitrary positive integers a and b, and choose an integer $d \geqslant 0$ such that $w_{a+b} = d + \frac{a+b}{2^d}$. Then one has

$$2w_{a+b} = 2d + 2\cdot\frac{a+b}{2^d} = \left((d-1) + \frac{a}{2^{d-1}}\right) + \left((d-1) + \frac{b}{2^{d-1}}\right) + 2 \geqslant w_a + w_b + 2.$$

Since the initial sum of the weights was 2^m, after $m2^{m-1}$ steps the sum is at least $(m+1)2^m$. To finish the solution, one may notice that by (2) for every positive integer a one has

$$w_a \leqslant m + \frac{a}{2^m}, \qquad \text{i.e.,} \quad a \geqslant 2^m(-m + w_a). \tag{3}$$

So the sum of the numbers $a_1, a_2, \ldots, a_{2^m}$ on the sheets can be estimated as

$$\sum_{i=1}^{2^m} a_i \geqslant \sum_{i=1}^{2^m} 2^m(-m + w_{a_i}) = -m2^m \cdot 2^m + 2^m \sum_{i=1}^{2^m} w_{a_i} \geqslant -m4^m + (m+1)4^m = 4^m,$$

as required.

For establishing the inequalities (1) and (3), one may also use the convexity argument, instead of the second definition of w_n in (2).

One may check that $\log_2 n \leqslant w_n \leqslant \log_2 n + 1$; thus, in some rough sense, this approach is obtained by "taking the logarithm" of the solution above.

Comment 3. An intuitive strategy to minimise the sum of numbers is that in every step we choose the two smallest numbers. We may call this the *greedy strategy.* In the following paragraphs we prove that the greedy strategy indeed provides the least possible sum of numbers.

Claim. Starting from any sequence $x_1, \dots, x_N$ of positive real numbers on N sheets, for any number k of steps, the greedy strategy achieves the lowest possible sum of numbers.

Proof. We apply induction on k; for $k = 1$ the statement is obvious. Let $k \geqslant 2$, and assume that the claim is true for smaller values.

Every sequence of k steps can be encoded as $S = \big((i_1, j_1), \dots, (i_k, j_k)\big)$, where, for $r = 1, 2, \dots, k$, the numbers i_r and j_r are the indices of the two sheets that are chosen in the r^{th} step. The resulting final sum will be some linear combination of $x_1, \dots, x_N$, say, $c_1x_1 + \cdots + c_Nx_N$ with positive integers $c_1, \dots, c_N$ that depend on S only. Call the numbers $(c_1, \dots, c_N)$ the *characteristic vector* of S.

Choose a sequence $S_0 = \big((i_1, j_1), \dots, (i_k, j_k)\big)$ of steps that produces the minimal sum, starting from $x_1, \dots, x_N$, and let $(c_1, \dots, c_N)$ be the characteristic vector of S. We may assume that the sheets are indexed in such an order that $c_1 \geqslant c_2 \geqslant \cdots \geqslant c_N$. If the sheets (and the numbers) are permuted by a permutation π of the indices $(1, 2, \dots, N)$ and then the same steps are performed, we can obtain the sum $\sum\limits_{t=1}^{N} c_t x_{\pi(t)}$. By the rearrangement inequality, the smallest possible sum can be achieved when the numbers $(x_1, \dots, x_N)$ are in non-decreasing order. So we can assume that also $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_N$.

Let ℓ be the largest index with $c_1 = \cdots = c_\ell$, and let the r^{th} step be the first step for which $c_{i_r} = c_1$ or $c_{j_r} = c_1$. The role of i_r and j_r is symmetrical, so we can assume $c_{i_r} = c_1$ and thus $i_r \leqslant \ell$. We show that $c_{j_r} = c_1$ and $j_r \leqslant \ell$ hold, too.

Before the r^{th} step, on the ${i_r}^{\text{th}}$ sheet we had the number x_{i_r}. On the ${j_r}^{\text{th}}$ sheet there was a linear combination that contains the number x_{j_r} with a positive integer coefficient, and possibly some other terms. In the r^{th} step, the number x_{i_r} joins that linear combination. From this point, each sheet contains a linear combination of $x_1, \dots, x_N$, with the coefficient of x_{j_r} being not smaller than the coefficient of x_{i_r}. This is preserved to the end of the procedure, so we have $c_{j_r} \geqslant c_{i_r}$. But $c_{i_r} = c_1$ is maximal among the coefficients, so we have $c_{j_r} = c_{i_r} = c_1$ and thus $j_r \leqslant \ell$.

Either from $c_{j_r} = c_{i_r} = c_1$ or from the arguments in the previous paragraph we can see that none of the ${i_r}^{\text{th}}$ and the ${j_r}^{\text{th}}$ sheets were used before step r. Therefore, the final linear combination of the numbers does not change if the step (i_r, j_r) is performed first: the sequence of steps

$$S_1 = \big((i_r, j_r), (i_1, j_1), \dots, (i_{r-1}, j_{r-1}), (i_{r+1}, j_{r+1}), \dots, (i_N, j_N)\big)$$

also produces the same minimal sum at the end. Therefore, we can replace S_0 by S_1 and we may assume that $r = 1$ and $c_{i_1} = c_{j_1} = c_1$.

As $i_1 \neq j_1$, we can see that $\ell \geqslant 2$ and $c_1 = c_2 = c_{i_1} = c_{j_1}$. Let π be such a permutation of the indices $(1, 2, \dots, N)$ that exchanges $1, 2$ with i_r, j_r and does not change the remaining indices. Let

$$S_2 = \big((\pi(i_1), \pi(j_1)), \dots, (\pi(i_N), \pi(j_N))\big).$$

Since $c_{\pi(i)} = c_i$ for all indices i, this sequence of steps produces the same, minimal sum. Moreover, in the first step we chose $x_{\pi(i_1)} = x_1$ and $x_{\pi(j_1)} = x_2$, the two smallest numbers.

Hence, it is possible to achieve the optimal sum if we follow the greedy strategy in the first step. By the induction hypothesis, following the greedy strategy in the remaining steps we achieve the optimal sum.

C3. Let $n \geqslant 2$ be an integer. Consider an $n \times n$ chessboard divided into n^2 unit squares. We call a configuration of n rooks on this board *happy* if every row and every column contains exactly one rook. Find the greatest positive integer k such that for every happy configuration of rooks, we can find a $k \times k$ square without a rook on any of its k^2 unit squares.

(Croatia)

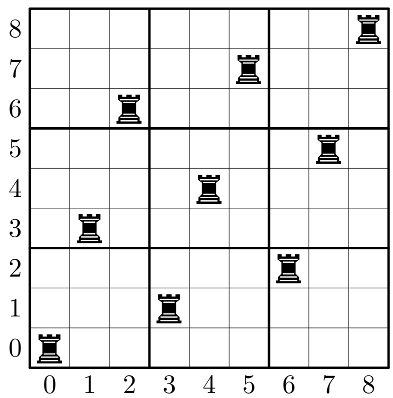
Answer. $\lfloor \sqrt{n-1} \rfloor$.

Solution. Let ℓ be a positive integer. We will show that **(i)** if $n > \ell^2$ then each happy configuration contains an empty $\ell \times \ell$ square, but **(ii)** if $n \leqslant \ell^2$ then there exists a happy configuration not containing such a square. These two statements together yield the answer.

(i). Assume that $n > \ell^2$. Consider any happy configuration. There exists a row R containing a rook in its leftmost square. Take ℓ consecutive rows with R being one of them. Their union U contains exactly ℓ rooks. Now remove the $n - \ell^2 \geqslant 1$ leftmost columns from U (thus at least one rook is also removed). The remaining part is an $\ell^2 \times \ell$ rectangle, so it can be split into ℓ squares of size $\ell \times \ell$, and this part contains at most $\ell - 1$ rooks. Thus one of these squares is empty.

(ii). Now we assume that $n \leqslant \ell^2$. Firstly, we will construct a happy configuration with no empty $\ell \times \ell$ square for the case $n = \ell^2$. After that we will modify it to work for smaller values of n.

Let us enumerate the rows from bottom to top as well as the columns from left to right by the numbers $0, 1, \ldots, \ell^2 - 1$. Every square will be denoted, as usual, by the pair (r, c) of its row and column numbers. Now we put the rooks on all squares of the form $(i\ell + j, j\ell + i)$ with $i, j = 0, 1, \ldots, \ell - 1$ (the picture below represents this arrangement for $\ell = 3$). Since each number from 0 to $\ell^2 - 1$ has a unique representation of the form $i\ell + j$ $(0 \leqslant i, j \leqslant \ell - 1)$, each row and each column contains exactly one rook.



Next, we show that each $\ell \times \ell$ square A on the board contains a rook. Consider such a square A, and consider ℓ consecutive rows the union of which contains A. Let the lowest of these rows have number $p\ell + q$ with $0 \leqslant p, q \leqslant \ell - 1$ (notice that $p\ell + q \leqslant \ell^2 - \ell$). Then the rooks in this union are placed in the columns with numbers $q\ell + p$, $(q+1)\ell + p$, $\ldots$, $(\ell-1)\ell + p$, $p + 1$, $\ell + (p+1)$, $\ldots$, $(q-1)\ell + p + 1$, or, putting these numbers in increasing order,

$$p + 1,\ \ell + (p+1),\ \ldots,\ (q-1)\ell + (p+1),\ q\ell + p,\ (q+1)\ell + p,\ \ldots,\ (\ell - 1)\ell + p.$$

One readily checks that the first number in this list is at most $\ell - 1$ (if $p = \ell - 1$, then $q = 0$, and the first listed number is $q\ell + p = \ell - 1$), the last one is at least $(\ell - 1)\ell$, and the difference between any two consecutive numbers is at most ℓ. Thus, one of the ℓ consecutive columns intersecting A contains a number listed above, and the rook in this column is inside A, as required. The construction for $n = \ell^2$ is established.

It remains to construct a happy configuration of rooks not containing an empty $\ell \times \ell$ square for $n < \ell^2$. In order to achieve this, take the construction for an $\ell^2 \times \ell^2$ square described above and remove the $\ell^2 - n$ bottom rows together with the $\ell^2 - n$ rightmost columns. We will have a rook arrangement with no empty $\ell \times \ell$ square, but several rows and columns may happen to be empty. Clearly, the number of empty rows is equal to the number of empty columns, so one can find a bijection between them, and put a rook on any crossing of an empty row and an empty column corresponding to each other.

Comment. Part **(i)** allows several different proofs. E.g., in the last paragraph of the solution, it suffices to deal only with the case $n = \ell^2 + 1$. Notice now that among the four corner squares, at least one is empty. So the rooks in its row and in its column are distinct. Now, deleting this row and column we obtain an $\ell^2 \times \ell^2$ square with $\ell^2 - 1$ rooks in it. This square can be partitioned into ℓ^2 squares of size $\ell \times \ell$, so one of them is empty.

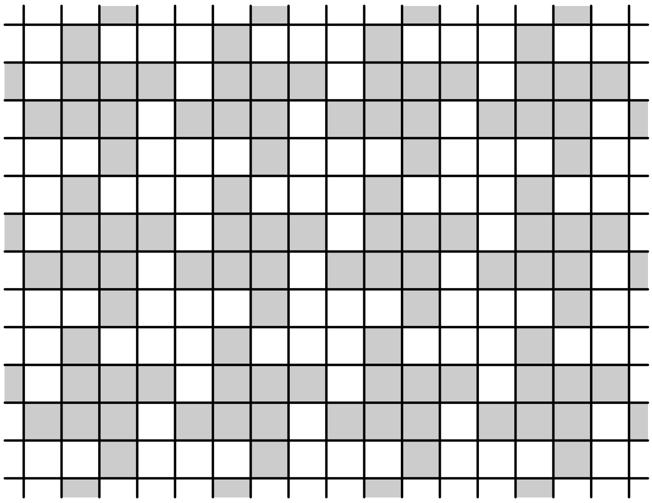
C4. Construct a tetromino by attaching two 2×1 dominoes along their longer sides such that the midpoint of the longer side of one domino is a corner of the other domino. This construction yields two kinds of tetrominoes with opposite orientations. Let us call them S- and Z-tetrominoes, respectively.

S-tetrominoes Z-tetrominoes

Assume that a lattice polygon P can be tiled with S-tetrominoes. Prove than no matter how we tile P using only S- and Z-tetrominoes, we always use an even number of Z-tetrominoes.

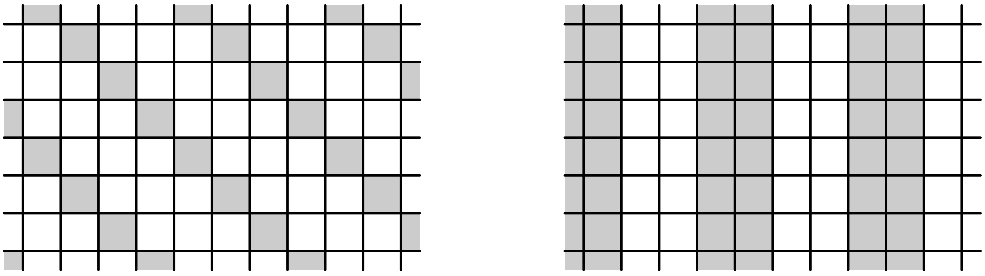
(Hungary)

Solution 1. We may assume that polygon P is the union of some squares of an infinite chessboard. Colour the squares of the chessboard with two colours as the figure below illustrates.



Observe that no matter how we tile P, any S-tetromino covers an even number of black squares, whereas any Z-tetromino covers an odd number of them. As P can be tiled exclusively by S-tetrominoes, it contains an even number of black squares. But if some S-tetrominoes and some Z-tetrominoes cover an even number of black squares, then the number of Z-tetrominoes must be even.

Comment. An alternative approach makes use of the following two colourings, which are perhaps somewhat more natural:



Let s_1 and s_2 be the number of S-tetrominoes of the first and second type (as shown in the figure above) respectively that are used in a tiling of P. Likewise, let z_1 and z_2 be the number of Z-tetrominoes of the first and second type respectively. The first colouring shows that $s_1 + z_2$ is invariant modulo 2, the second colouring shows that $s_1 + z_1$ is invariant modulo 2. Adding these two conditions, we find that $z_1 + z_2$ is invariant modulo 2, which is what we have to prove. Indeed, the sum of the two colourings (regarding white as 0 and black as 1 and adding modulo 2) is the colouring shown in the solution.

Solution 2. Let us assign coordinates to the squares of the infinite chessboard in such a way that the squares of P have nonnegative coordinates only, and that the first coordinate increases as one moves to the right, while the second coordinate increases as one moves upwards. Write the integer $3^i \cdot (-3)^j$ into the square with coordinates (i, j), as in the following figure:

$\vdots$					
81	$\vdots$				
-27	-81	$\vdots$			
9	27	81	$\cdots$		
-3	-9	-27	-81	$\cdots$	
1	3	9	27	81	$\cdots$

The sum of the numbers written into four squares that can be covered by an S-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(1 + 3 + 3 \cdot (-3) + 3^2 \cdot (-3)\big) = -32 \cdot 3^i \cdot (-3)^j$$

(for the first type of S-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3 \cdot (-3) + (-3) + (-3)^2\big) = 0,$$

and thus divisible by 32. For this reason, the sum of the numbers written into the squares of P, and thus also the sum of the numbers covered by Z-tetrominoes in the second covering, is likewise divisible by 32. Now the sum of the entries of a Z-tetromino is either of the form

$$3^i \cdot (-3)^j \cdot \big(3 + 3^2 + (-3) + 3 \cdot (-3)\big) = 0$$

(for the first type of Z-tetrominoes), or of the form

$$3^i \cdot (-3)^j \cdot \big(1 + (-3) + 3 \cdot (-3) + 3 \cdot (-3)^2\big) = 16 \cdot 3^i \cdot (-3)^j,$$

i.e., 16 times an odd number. Thus in order to obtain a total that is divisible by 32, an even number of the latter kind of Z-tetrominoes needs to be used. Rotating everything by 90°, we find that the number of Z-tetrominoes of the first kind is even as well. So we have even proven slightly more than necessary.

Comment 1. In the second solution, 3 and -3 can be replaced by other combinations as well. For example, for any positive integer $a \equiv 3 \pmod 4$, we can write $a^i \cdot (-a)^j$ into the square with coordinates (i, j) and apply the same argument.

Comment 2. As the second solution shows, we even have the stronger result that the parity of the number of each of the four types of tetrominoes in a tiling of P by S- and Z-tetrominoes is an invariant of P. This also remains true if there is no tiling of P that uses only S-tetrominoes.

C5. Consider $n \geqslant 3$ lines in the plane such that no two lines are parallel and no three have a common point. These lines divide the plane into polygonal regions; let $\mathcal{F}$ be the set of regions having finite area. Prove that it is possible to colour $\lceil \sqrt{n/2} \rceil$ of the lines blue in such a way that no region in $\mathcal{F}$ has a completely blue boundary. (For a real number x, $\lceil x \rceil$ denotes the least integer which is not smaller than x.)

(Austria)

Solution. Let L be the given set of lines. Choose a maximal (by inclusion) subset $B \subseteq L$ such that when we colour the lines of B blue, no region in $\mathcal{F}$ has a completely blue boundary. Let $|B| = k$. We claim that $k \geqslant \lceil \sqrt{n/2} \rceil$.

Let us colour all the lines of $L \backslash B$ red. Call a point *blue* if it is the intersection of two blue lines. Then there are $\binom{k}{2}$ blue points.

Now consider any red line ℓ. By the maximality of B, there exists at least one region $A \in \mathcal{F}$ whose only red side lies on ℓ. Since A has at least three sides, it must have at least one blue vertex. Let us take one such vertex and associate it to ℓ.

Since each blue point belongs to four regions (some of which may be unbounded), it is associated to at most four red lines. Thus the total number of red lines is at most $4\binom{k}{2}$. On the other hand, this number is $n - k$, so

$$n - k \leqslant 2k(k-1), \quad \text{thus} \quad n \leqslant 2k^2 - k \leqslant 2k^2,$$

and finally $k \geqslant \lceil \sqrt{n/2} \rceil$, which gives the desired result.

Comment 1. The constant factor in the estimate can be improved in different ways; we sketch two of them below. On the other hand, the Problem Selection Committee is not aware of any results showing that it is sometimes impossible to colour k lines satisfying the desired condition for $k \gg \sqrt{n}$. In this situation we find it more suitable to keep the original formulation of the problem.

1. Firstly, we show that in the proof above one has in fact $k = |B| \geqslant \lceil \sqrt{2n/3} \rceil$.

Let us make weighted associations as follows. Let a region A whose only red side lies on ℓ have k vertices, so that $k-2$ of them are blue. We associate each of these blue vertices to ℓ, and put the weight $\frac{1}{k-2}$ on each such association. So the sum of the weights of all the associations is exactly $n - k$.

Now, one may check that among the four regions adjacent to a blue vertex v, at most two are triangles. This means that the sum of the weights of all associations involving v is at most $1 + 1 + \frac{1}{2} + \frac{1}{2} = 3$. This leads to the estimate

$$n - k \leqslant 3\binom{k}{2},$$

or

$$2n \leqslant 3k^2 - k < 3k^2,$$

which yields $k \geqslant \lceil \sqrt{2n/3} \rceil$.

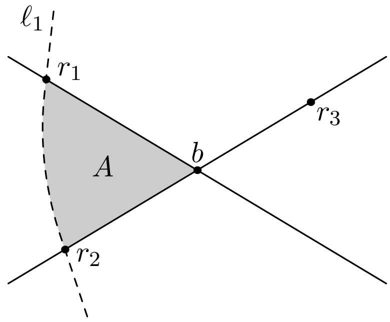
2. Next, we even show that $k = |B| \geqslant \lceil \sqrt{n} \rceil$. For this, we specify the process of associating points to red lines in one more different way.

Call a point *red* if it lies on a red line as well as on a blue line. Consider any red line ℓ, and take an arbitrary region $A \in \mathcal{F}$ whose only red side lies on ℓ. Let r', r, b_1, …, b_k be its vertices in clockwise order with $r', r \in \ell$; then the points r', r are red, while all the points b_1, …, b_k are blue. Let us associate to ℓ the red point r and the blue point b_1. One may notice that to each pair of a red point r and a blue point b, at most one red line can be associated, since there is at most one region A having r and b as two clockwise consecutive vertices.

We claim now that at most two red lines are associated to each blue point b; this leads to the desired bound

$$n - k \leqslant 2\binom{k}{2} \qquad \Longleftrightarrow \qquad n \leqslant k^2.$$

Assume, to the contrary, that three red lines ℓ_1, ℓ_2, and ℓ_3 are associated to the same blue point b. Let r_1, r_2, and r_3 respectively be the red points associated to these lines; all these points are distinct. The point b defines four blue rays, and each point r_i is the red point closest to b on one of these rays. So we may assume that the points r_2 and r_3 lie on one blue line passing through b, while r_1 lies on the other one.



Now consider the region A used to associate r_1 and b with ℓ_1. Three of its clockwise consecutive vertices are r_1, b, and either r_2 or r_3 (say, r_2). Since A has only one red side, it can only be the triangle r_1br_2; but then both ℓ_1 and ℓ_2 pass through r_2, as well as some blue line. This is impossible by the problem assumptions.

Comment 2. The condition that the lines be non-parallel is essentially not used in the solution, nor in the previous comment; thus it may be omitted.

C6. We are given an infinite deck of cards, each with a real number on it. For every real number x, there is exactly one card in the deck that has x written on it. Now two players draw disjoint sets A and B of 100 cards each from this deck. We would like to define a rule that declares one of them a winner. This rule should satisfy the following conditions:

1. The winner only depends on the relative order of the 200 cards: if the cards are laid down in increasing order face down and we are told which card belongs to which player, but not what numbers are written on them, we can still decide the winner.

2. If we write the elements of both sets in increasing order as $A = \{a_1, a_2, \ldots, a_{100}\}$ and $B = \{b_1, b_2, \ldots, b_{100}\}$, and $a_i > b_i$ for all i, then A beats B.

3. If three players draw three disjoint sets A, B, C from the deck, A beats B and B beats C, then A also beats C.

How many ways are there to define such a rule? Here, we consider two rules as different if there exist two sets A and B such that A beats B according to one rule, but B beats A according to the other.

(Russia)

Answer. 100.

Solution 1. We prove a more general statement for sets of cardinality n (the problem being the special case $n = 100$, then the answer is n). In the following, we write $A > B$ or $B < A$ for "A beats B".

Part I. Let us first define n different rules that satisfy the conditions. To this end, fix an index $k \in \{1, 2, \ldots, n\}$. We write both A and B in increasing order as $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ and say that A beats B if and only if $a_k > b_k$. This rule clearly satisfies all three conditions, and the rules corresponding to different k are all different. Thus there are at least n different rules.

Part II. Now we have to prove that there is no other way to define such a rule. Suppose that our rule satisfies the conditions, and let $k \in \{1, 2, \ldots, n\}$ be minimal with the property that

$$A_k = \{1, 2, \ldots, k, n+k+1, n+k+2, \ldots, 2n\} \prec B_k = \{k+1, k+2, \ldots, n+k\}.$$

Clearly, such a k exists, since this holds for $k = n$ by assumption. Now consider two disjoint sets $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, both in increasing order (i.e., $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_n$). We claim that $X \prec Y$ if (and only if – this follows automatically) $x_k < y_k$.

To prove this statement, pick arbitrary real numbers $u_i, v_i, w_i \notin X \cup Y$ such that

$$u_1 < u_2 < \cdots < u_{k-1} < \min(x_1, y_1), \quad \max(x_n, y_n) < v_{k+1} < v_{k+2} < \cdots < v_n,$$

and

$$x_k \;<\; v_1 < v_2 < \cdots < v_k \;<\; w_1 < w_2 < \cdots < w_n \;<\; u_k < u_{k+1} < \cdots < u_n \;<\; y_k,$$

and set

$$U = \{u_1, u_2, \ldots, u_n\}, \; V = \{v_1, v_2, \ldots, v_n\}, \; W = \{w_1, w_2, \ldots, w_n\}.$$

Then

- $u_i < y_i$ and $x_i < v_i$ for all i, so $U \prec Y$ and $X \prec V$ by the second condition.

- The elements of $U \cup W$ are ordered in the same way as those of $A_{k-1} \cup B_{k-1}$, and since $A_{k-1} \succ B_{k-1}$ by our choice of k, we also have $U \succ W$ (if $k = 1$, this is trivial).
- The elements of $V \cup W$ are ordered in the same way as those of $A_k \cup B_k$, and since $A_k \prec B_k$ by our choice of k, we also have $V \prec W$.

It follows that
$$X \prec V \prec W \prec U \prec Y,$$
so $X \prec Y$ by the third condition, which is what we wanted to prove.

Solution 2. Another possible approach to Part II of this problem is induction on n. For $n = 1$, there is trivially only one rule in view of the second condition.

In the following, we assume that our claim (namely, that there are no possible rules other than those given in Part I) holds for $n-1$ in place of n. We start with the following observation:

Claim. At least one of the two relations
$$\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$$
and
$$\big(\{2i-1 \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \,|\, 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$$
holds.

Proof. Suppose that the first relation does not hold. Since our rule may only depend on the relative order, we must also have
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big).$$
Likewise, if the second relation does not hold, then we must also have
$$\big(\{1\} \cup \{3i-1 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big).$$
Now condition 3 implies that
$$\big(\{2\} \cup \{3i-2 \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-2\}\big) \succ \big(\{3\} \cup \{3i \,|\, 2 \leqslant i \leqslant n-1\} \cup \{3n-1\}\big),$$
which contradicts the second condition. □

Now we distinguish two cases, depending on which of the two relations actually holds:

First case: $\big(\{2\} \cup \{2i-1 \,|\, 2 \leqslant i \leqslant n\}\big) \prec \big(\{1\} \cup \{2i \,|\, 2 \leqslant i \leqslant n\}\big)$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two disjoint sets, both in increasing order. We claim that the winner can be decided only from the values of $a_2, \ldots, a_n$ and $b_2, \ldots, b_n$, while a_1 and b_1 are actually irrelevant. Suppose that this was not the case, and assume without loss of generality that $a_2 < b_2$. Then the relative order of $a_1, a_2, \ldots, a_n, b_2, \ldots, b_n$ is fixed, and the position of b_1 has to decide the winner. Suppose that for some value $b_1 = x$, B wins, while for some other value $b_1 = y$, A wins.

Write $B_x = \{x, b_2, \ldots, b_n\}$ and $B_y = \{y, b_2, \ldots, b_n\}$, and let $\varepsilon > 0$ be smaller than half the distance between any two of the numbers in $B_x \cup B_y \cup A$. For any set M, let $M \pm \varepsilon$ be the set obtained by adding/subtracting ε to all elements of M. By our choice of ε, the relative order of the elements of $(B_y + \varepsilon) \cup A$ is still the same as for $B_y \cup A$, while the relative order of the elements of $(B_x - \varepsilon) \cup A$ is still the same as for $B_x \cup A$. Thus $A \prec B_x - \varepsilon$, but $A \succ B_y + \varepsilon$. Moreover, if $y > x$, then $B_x - \varepsilon \prec B_y + \varepsilon$ by condition 2, while otherwise the relative order of the elements in $(B_x - \varepsilon) \cup (B_y + \varepsilon)$ is the same as for the two sets $\{2\} \cup \{2i-1 \mid 2 \leqslant i \leqslant n\}$ and $\{1\} \cup \{2i \mid 2 \leqslant i \leqslant n\}$, so that $B_x - \varepsilon \prec B_y + \varepsilon$. In either case, we obtain

$$A \prec B_x - \varepsilon \prec B_y + \varepsilon \prec A,$$

which contradicts condition 3.

So we know now that the winner does not depend on a_1, b_1. Therefore, we can define a new rule $\prec^*$ on sets of cardinality $n-1$ by saying that $A \prec^* B$ if and only if $A \cup \{a\} \prec B \cup \{b\}$ for some a, b (or equivalently, all a, b) such that $a < \min A$, $b < \min B$ and $A \cup \{a\}$ and $B \cup \{b\}$ are disjoint. The rule $\prec^*$ satisfies all conditions again, so by the induction hypothesis, there exists an index i such that $A \prec^* B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B. This implies that $C \prec D$ if and only if the $(i+1)^{\text{th}}$ smallest element of C is less than the $(i+1)^{\text{th}}$ smallest element of D, which completes our induction.

Second case: $\big(\{2i-1 \mid 1 \leqslant i \leqslant n-1\} \cup \{2n\}\big) \prec \big(\{2i \mid 1 \leqslant i \leqslant n-1\} \cup \{2n-1\}\big)$.

Set $-A = \{-a \mid a \in A\}$ for any $A \subseteq \mathbb{R}$. For any two disjoint sets $A, B \subseteq \mathbb{R}$ of cardinality n, we write $A \prec^\circ B$ to mean $(-B) \prec (-A)$. It is easy to see that $\prec^\circ$ defines a rule to determine a winner that satisfies the three conditions of our problem as well as the relation of the first case. So it follows in the same way as in the first case that for some i, $A \prec^\circ B$ if and only if the i^{th} smallest element of A is less than the i^{th} smallest element of B, which is equivalent to the condition that the i^{th} largest element of $-A$ is greater than the i^{th} largest element of $-B$. This proves that the original rule $\prec$ also has the desired form.

Comment. The problem asks for all possible partial orders on the set of n-element subsets of $\mathbb{R}$ such that any two disjoint sets are comparable, the order relation only depends on the relative order of the elements, and $\{a_1, a_2, \ldots, a_n\} \prec \{b_1, b_2, \ldots, b_n\}$ whenever $a_i < b_i$ for all i.

As the proposer points out, one may also ask for all **total** orders on all n-element subsets of $\mathbb{R}$ (dropping the condition of disjointness and requiring that $\{a_1, a_2, \ldots, a_n\} \preceq \{b_1, b_2, \ldots, b_n\}$ whenever $a_i \leqslant b_i$ for all i). It turns out that the number of possibilities in this case is $n!$, and all possible total orders are obtained in the following way. Fix a permutation $\sigma \in S_n$. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two subsets of $\mathbb{R}$ with $a_1 < a_2 < \cdots < a_n$ and $b_1 < b_2 < \cdots < b_n$. Then we say that $A \succ_\sigma B$ if and only if $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ is lexicographically greater than $(b_{\sigma(1)}, \ldots, b_{\sigma(n)})$.

It seems, however, that this formulation adds rather more technicalities to the problem than additional ideas.

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C7. Let M be a set of $n \geqslant 4$ points in the plane, no three of which are collinear. Initially these points are connected with n segments so that each point in M is the endpoint of exactly two segments. Then, at each step, one may choose two segments AB and CD sharing a common interior point and replace them by the segments AC and BD if none of them is present at this moment. Prove that it is impossible to perform $n^3/4$ or more such moves.

(Russia)

Solution. A line is said to be *red* if it contains two points of M. As no three points of M are collinear, each red line determines a unique pair of points of M. Moreover, there are precisely $\binom{n}{2} < \frac{n^2}{2}$ red lines. By the *value of a segment* we mean the number of red lines intersecting it in its interior, and the *value of a set of segments* is defined to be the sum of the values of its elements. We will prove that (i) the value of the initial set of segments is smaller than $n^3/2$ and that (ii) each step decreases the value of the set of segments present by at least 2. Since such a value can never be negative, these two assertions imply the statement of the problem.

To show (i) we just need to observe that each segment has a value that is smaller than $n^2/2$. Thus the combined value of the n initial segments is indeed below $n \cdot n^2/2 = n^3/2$.

It remains to establish (ii). Suppose that at some moment we have two segments AB and CD sharing an interior point S, and that at the next moment we have the two segments AC and BD instead. Let X_{AB} denote the set of red lines intersecting the segment AB in its interior and let the sets X_{AC}, X_{BD}, and X_{CD} be defined similarly. We are to prove that $|X_{AC}| + |X_{BD}| + 2 \leqslant |X_{AB}| + |X_{CD}|$.

As a first step in this direction, we claim that

$$|X_{AC} \cup X_{BD}| + 2 \leqslant |X_{AB} \cup X_{CD}|\,. \tag{1}$$

Indeed, if g is a red line intersecting, e.g. the segment AC in its interior, then it has to intersect the triangle ACS once again, either in the interior of its side AS, or in the interior of its side CS, or at S, meaning that it belongs to X_{AB} or to X_{CD} (see Figure 1). Moreover, the red lines AB and CD contribute to $X_{AB} \cup X_{CD}$ but not to $X_{AC} \cup X_{BD}$. Thereby (1) is proved.

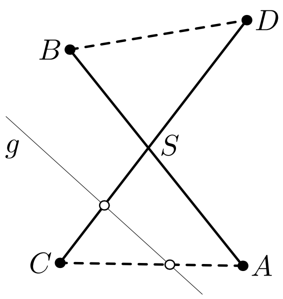


Figure 1

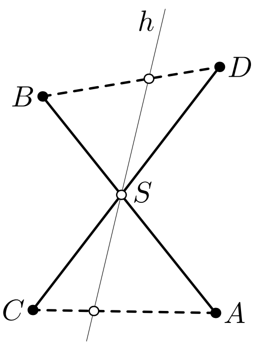


Figure 2

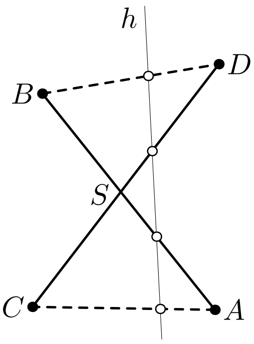


Figure 3

Similarly but more easily one obtains

$$|X_{AC} \cap X_{BD}| \leqslant |X_{AB} \cap X_{CD}|\,. \tag{2}$$

Indeed, a red line h appearing in $X_{AC} \cap X_{BD}$ belongs, for similar reasons as above, also to $X_{AB} \cap X_{CD}$. To make the argument precise, one may just distinguish the cases $S \in h$ (see Figure 2) and $S \notin h$ (see Figure 3). Thereby (2) is proved.

Adding (1) and (2) we obtain the desired conclusion, thus completing the solution of this problem.

Comment 1. There is a problem belonging to the folklore, in the solution of which one may use the same kind of operation:

Given n red and n green points in the plane, prove that one may draw n nonintersecting segments each of which connects a red point with a green point.

A standard approach to this problem consists in taking n arbitrary segments connecting the red points with the green points, and to perform the same operation as in the above proposal whenever an intersection occurs. Now each time one performs such a step, the total length of the segments that are present decreases due to the triangle inequality. So, as there are only finitely many possibilities for the set of segments present, the process must end at some stage.

In the above proposal, however, considering the sum of the Euclidean lengths of the segment that are present does not seem to help much, for even though it shows that the process must necessarily terminate after some finite number of steps, it does not seem to easily yield any upper bound on the number of these steps that grows polynomially with n.

One may regard the concept of the value of a segment introduced in the above solution as an appropriately discretised version of Euclidean length suitable for obtaining such a bound.

The Problem Selection Committee still believes the problem to be sufficiently original for the competition.

Comment 2. There are some other essentially equivalent ways of presenting the same solution. E.g., put $M = \{A_1, A_2, \ldots, A_n\}$, denote the set of segments present at any moment by $\{e_1, e_2, \ldots, e_n\}$, and called a triple (i, j, k) of indices with $i \neq j$ *intersecting*, if the line A_iA_j intersects the segment e_k. It may then be shown that the number S of intersecting triples satisfies $0 \leqslant S < n^3$ at the beginning and decreases by at least 4 in each step.

Comment 3. It is not difficult to construct an example where cn^2 moves are possible (for some absolute constant $c > 0$). It would be interesting to say more about the gap between cn^2 and cn^3.

C8. A card deck consists of 1024 cards. On each card, a set of distinct decimal digits is written in such a way that no two of these sets coincide (thus, one of the cards is empty). Two players alternately take cards from the deck, one card per turn. After the deck is empty, each player checks if he can throw out one of his cards so that each of the ten digits occurs on an even number of his remaining cards. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine all possible first moves of the first player after which he has a winning strategy.

(Russia)

Answer. All the moves except for taking the empty card.

Solution. Let us identify each card with the set of digits written on it. For any collection of cards $C_1, C_2, \ldots, C_k$ denote by their *sum* the set $C_1 \triangle C_2 \triangle \cdots \triangle C_k$ consisting of all elements belonging to an odd number of the C_i's. Denote the first and the second player by $\mathcal{F}$ and $\mathcal{S}$, respectively.

Since each digit is written on exactly 512 cards, the sum of all the cards is $\varnothing$. Therefore, at the end of the game the sum of all the cards of $\mathcal{F}$ will be the same as that of $\mathcal{S}$; denote this sum by C. Then the player who took C can throw it out and get the desired situation, while the other one cannot. Thus, the player getting card C wins, and no draw is possible.

Now, given a nonempty card B, one can easily see that all the cards can be split into 512 pairs of the form $(X, X \triangle B)$ because $(X \triangle B) \triangle B = X$. The following lemma shows a property of such a partition that is important for the solution.

Lemma. Let $B \neq \varnothing$ be some card. Let us choose 512 cards so that exactly one card is chosen from every pair $(X, X \triangle B)$. Then the sum of all chosen cards is either $\varnothing$ or B.

Proof. Let b be some element of B. Enumerate the pairs; let X_i be the card not containing b in the i^{th} pair, and let Y_i be the other card in this pair. Then the sets X_i are exactly all the sets not containing b, therefore each digit $a \neq b$ is written on exactly 256 of these cards, so $X_1 \triangle X_2 \triangle \cdots \triangle X_{512} = \varnothing$. Now, if we replace some summands in this sum by the other elements from their pairs, we will simply add B several times to this sum, thus the sum will either remain unchanged or change by B, as required. $\square$

Now we consider two cases.

Case 1. Assume that $\mathcal{F}$ takes the card $\varnothing$ on his first move. In this case, we present a winning strategy for $\mathcal{S}$.

Let $\mathcal{S}$ take an arbitrary card A. Assume that $\mathcal{F}$ takes card B after that; then $\mathcal{S}$ takes $A \triangle B$. Split all 1024 cards into 512 pairs of the form $(X, X \triangle B)$; we call two cards in one pair *partners*. Then the four cards taken so far form two pairs $(\varnothing, B)$ and $(A, A \triangle B)$ belonging to $\mathcal{F}$ and $\mathcal{S}$, respectively. On each of the subsequent moves, when $\mathcal{F}$ takes some card, $\mathcal{S}$ should take the partner of this card in response.

Consider the situation at the end of the game. Let us for a moment replace card A belonging to $\mathcal{S}$ by $\varnothing$. Then he would have one card from each pair; by our lemma, the sum of all these cards would be either $\varnothing$ or B. Now, replacing $\varnothing$ back by A we get that the actual sum of the cards of $\mathcal{S}$ is either A or $A \triangle B$, and he has both these cards. Thus $\mathcal{S}$ wins.

Case 2. Now assume that $\mathcal{F}$ takes some card $A \neq \varnothing$ on his first move. Let us present a winning strategy for $\mathcal{F}$ in this case.

Assume that $\mathcal{S}$ takes some card $B \neq \varnothing$ on his first move; then $\mathcal{F}$ takes $A \triangle B$. Again, let us split all the cards into pairs of the form $(X, X \triangle B)$; then the cards which have not been taken yet form several complete pairs and one extra element (card $\varnothing$ has not been taken while its partner B has). Now, on each of the subsequent moves, if $\mathcal{S}$ takes some element from a complete pair, then $\mathcal{F}$ takes its partner. If $\mathcal{S}$ takes the extra element, then $\mathcal{F}$ takes an arbitrary card Y, and the partner of Y becomes the new extra element.

Thus, on his last move $\mathcal{S}$ is forced to take the extra element. After that player $\mathcal{F}$ has cards A and $A \triangle B$, player $\mathcal{S}$ has cards B and $\varnothing$, and $\mathcal{F}$ has exactly one element from every other pair. Thus the situation is the same as in the previous case with roles reversed, and $\mathcal{F}$ wins.

Finally, if $\mathcal{S}$ takes $\varnothing$ on his first move then $\mathcal{F}$ denotes any card which has not been taken yet by B and takes $A \triangle B$. After that, the same strategy as above is applicable.

Comment 1. If one wants to avoid the unusual question about the first move, one may change the formulation as follows. (The difficulty of the problem would decrease somewhat.)

A card deck consists of 1023 cards; on each card, a nonempty set of distinct decimal digits is written in such a way that no two of these sets coincide. Two players alternately take cards from the deck, one card per turn. When the deck is empty, each player checks if he can throw out one of his cards so that for each of the ten digits, he still holds an even number of cards with this digit. If one player can do this but the other one cannot, the one who can is the winner; otherwise a draw is declared.

Determine which of the players (if any) has a winning strategy.

The winner in this version is the first player. The analysis of the game from the first two paragraphs of the previous solution applies to this version as well, except for the case $C = \varnothing$ in which the result is a draw. Then the strategy for $\mathcal{S}$ in Case 1 works for $\mathcal{F}$ in this version: the sum of all his cards at the end is either A or $A \triangle B$, thus nonempty in both cases.

Comment 2. Notice that all the cards form a vector space over $\mathbb{F}_2$, with $\triangle$ the operation of addition. Due to the automorphisms of this space, all possibilities for $\mathcal{F}$'s first move except $\varnothing$ are equivalent. The same holds for the response by $\mathcal{S}$ if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

Comment 3. It is not that hard to show that in the initial game, $\mathcal{F}$ has a winning move, by the idea of "strategy stealing".

Namely, assume that $\mathcal{S}$ has a winning strategy. Let us take two card decks and start two games, in which $\mathcal{S}$ will act by his strategy. In the first game, $\mathcal{F}$ takes an arbitrary card A_1; assume that $\mathcal{S}$ takes some B_1 in response. Then $\mathcal{F}$ takes the card B_1 at the second game; let the response by $\mathcal{S}$ be A_2. Then $\mathcal{F}$ takes A_2 in the first game and gets a response B_2, and so on.

This process stops at some moment when in the second game $\mathcal{S}$ takes $A_i = A_1$. At this moment the players hold the same sets of cards in both games, but with roles reversed. Now, if some cards remain in the decks, $\mathcal{F}$ takes an arbitrary card from the first deck starting a similar cycle.

At the end of the game, player $\mathcal{F}$'s cards in the first game are exactly player $\mathcal{S}$'s cards in the second game, and vice versa. Thus in one of the games $\mathcal{F}$ will win, which is impossible by our assumption.

One may notice that the strategy in Case 2 is constructed exactly in this way from the strategy in Case 1. This is possible since every response by $\mathcal{S}$ wins if $\mathcal{F}$ takes the card $\varnothing$ on his first move.

C9. There are n circles drawn on a piece of paper in such a way that any two circles intersect in two points, and no three circles pass through the same point. Turbo the snail slides along the circles in the following fashion. Initially he moves on one of the circles in clockwise direction. Turbo always keeps sliding along the current circle until he reaches an intersection with another circle. Then he continues his journey on this new circle and also changes the direction of moving, i.e. from clockwise to anticlockwise or *vice versa*.

Suppose that Turbo's path entirely covers all circles. Prove that n must be odd.

(India)

Solution 1. Replace every cross (i.e. intersection of two circles) by two small circle arcs that indicate the direction in which the snail should leave the cross (see Figure 1.1). Notice that the placement of the small arcs does not depend on the direction of moving on the curves; no matter which direction the snail is moving on the circle arcs, he will follow the same curves (see Figure 1.2). In this way we have a set of curves, that are the possible paths of the snail. Call these curves *snail orbits* or just *orbits*. Every snail orbit is a simple closed curve that has no intersection with any other orbit.

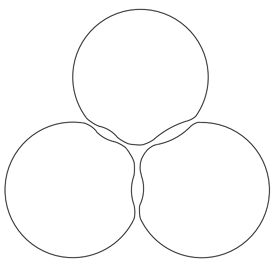


Figure 1.1

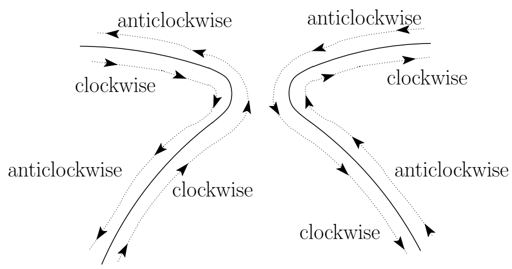


Figure 1.2

We prove the following, more general statement.

> $(*)$ *In any configuration of n circles such that no two of them are tangent, the number of snail orbits has the same parity as the number n.* (Note that it is not assumed that all circle pairs intersect.)

This immediately solves the problem.

Let us introduce the following operation that will be called *flipping a cross*. At a cross, remove the two small arcs of the orbits, and replace them by the other two arcs. Hence, when the snail arrives at a flipped cross, he will continue on the other circle as before, but he will preserve the orientation in which he goes along the circle arcs (see Figure 2).

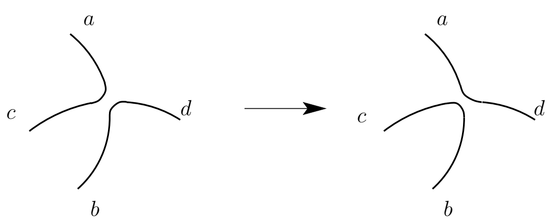


Figure 2

Consider what happens to the number of orbits when a cross is flipped. Denote by a, b, c, and d the four arcs that meet at the cross such that a and b belong to the same circle. Before the flipping a and b were connected to c and d, respectively, and after the flipping a and b are connected to d and c, respectively.

The orbits passing through the cross are closed curves, so each of the arcs a, b, c, and d is connected to another one by orbits outside the cross. We distinguish three cases.

Case 1: a is connected to b and c is connected to d by the orbits outside the cross (see Figure 3.1).

We show that this case is impossible. Remove the two small arcs at the cross, connect a to b, and connect c to d at the cross. Let γ be the new closed curve containing a and b, and let δ be the new curve that connects c and d. These two curves intersect at the cross. So one of c and d is inside γ and the other one is outside γ. Then the two closed curves have to meet at least one more time, but this is a contradiction, since no orbit can intersect itself.

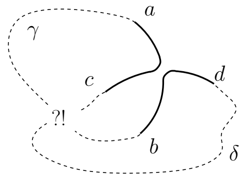


Figure 3.1

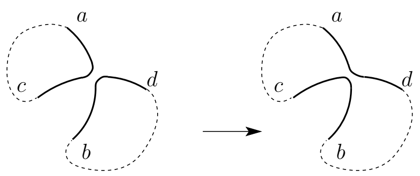


Figure 3.2

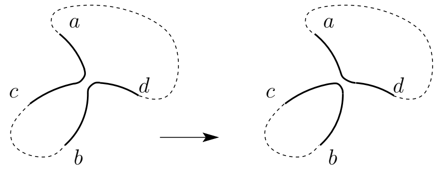


Figure 3.3

Case 2: a is connected to c and b is connected to d (see Figure 3.2).

Before the flipping a and c belong to one orbit and b and d belong to another orbit. Flipping the cross merges the two orbits into a single orbit. Hence, the number of orbits decreases by 1.

Case 3: a is connected to d and b is connected to c (see Figure 3.3).

Before the flipping the arcs a, b, c, and d belong to a single orbit. Flipping the cross splits that orbit in two. The number of orbits increases by 1.

As can be seen, every flipping decreases or increases the number of orbits by one, thus changes its parity.

Now flip every cross, one by one. Since every pair of circles has 0 or 2 intersections, the number of crosses is even. Therefore, when all crosses have been flipped, the original parity of the number of orbits is restored. So it is sufficient to prove $(*)$ for the new configuration, where all crosses are flipped. Of course also in this new configuration the (modified) orbits are simple closed curves not intersecting each other.

Orient the orbits in such a way that the snail always moves anticlockwise along the circle arcs. Figure 4 shows the same circles as in Figure 1 after flipping all crosses and adding orientation. (Note that this orientation may be different from the orientation of the orbit as a planar curve; the orientation of every orbit may be negative as well as positive, like the middle orbit in Figure 4.) If the snail moves around an orbit, the total angle change in his moving direction, the *total curvature*, is either $+2\pi$ or -2π, depending on the orientation of the orbit. Let P and N be the number of orbits with positive and negative orientation, respectively. Then the total curvature of all orbits is $(P-N)\cdot 2\pi$.

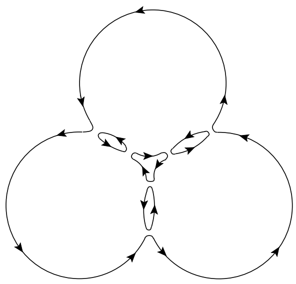


Figure 4

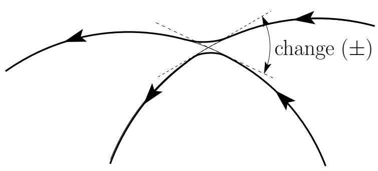


Figure 5

Double-count the total curvature of all orbits. Along every circle the total curvature is 2π. At every cross, the two turnings make two changes with some angles having the same absolute value but opposite signs, as depicted in Figure 5. So the changes in the direction at the crosses cancel out. Hence, the total curvature is $n\cdot 2\pi$.

Now we have $(P-N)\cdot 2\pi = n\cdot 2\pi$, so $P-N=n$. The number of (modified) orbits is $P+N$, that has a same parity as $P-N=n$.

Solution 2. We present a different proof of $(*)$.

We perform a sequence of small modification steps on the configuration of the circles in such a way that at the end they have no intersection at all (see Figure 6.1). We use two kinds of local changes to the structure of the orbits (see Figure 6.2):

- *Type-1 step:* An arc of a circle is moved over an arc of another circle; such a step creates or removes two intersections.

- *Type-2 step:* An arc of a circle is moved through the intersection of two other circles.

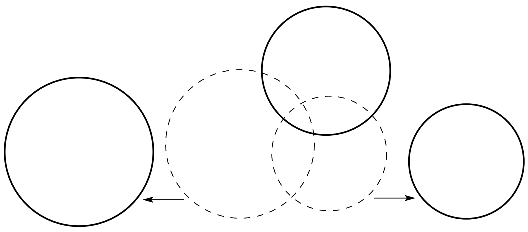


Figure 6.1

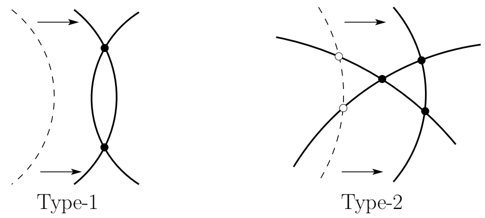


Figure 6.2

We assume that in every step only one circle is moved, and that this circle is moved over at most one arc or intersection point of other circles.

We will show that the parity of the number of orbits does not change in any step. As every circle becomes a separate orbit at the end of the procedure, this fact proves $(*)$.

Consider what happens to the number of orbits when a Type-1 step is performed. The two intersection points are created or removed in a small neighbourhood. Denote some points of the two circles where they enter or leave this neighbourhood by a, b, c, and d in this order around the neighbourhood; let a and b belong to one circle and let c and d belong to the other circle. The two circle arcs may have the same or opposite orientations. Moreover, the four end-points of the two arcs are connected by the other parts of the orbits. This can happen in two ways without intersection: either a is connected to d and b is connected to c, or a is connected to b and c is connected to d. Altogether we have four cases, as shown in Figure 7.

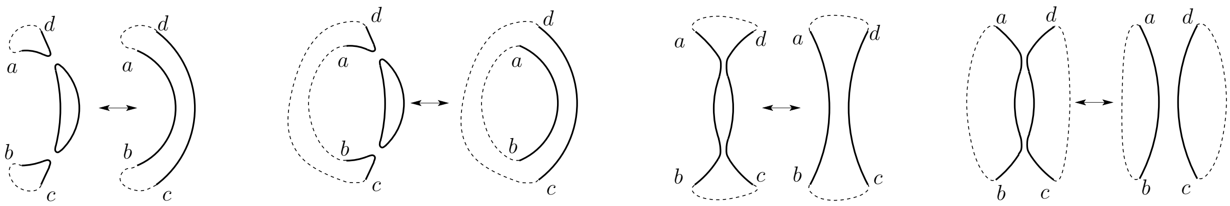


Figure 7

We can see that the number of orbits is changed by -2 or $+2$ in the leftmost case when the arcs have the same orientation, a is connected to d, and b is connected to c. In the other three cases the number of orbits is not changed. Hence, Type-1 steps do not change the parity of the number of orbits.

Now consider a Type-2 step. The three circles enclose a small, triangular region; by the step, this triangle is replaced by another triangle. Again, the modification of the orbits is done in some small neighbourhood; the structure does not change outside. Each side of the triangle shaped region can be convex or concave; the number of concave sides can be 0, 1, 2 or 3, so there are 4 possible arrangements of the orbits inside the neighbourhood, as shown in Figure 8.

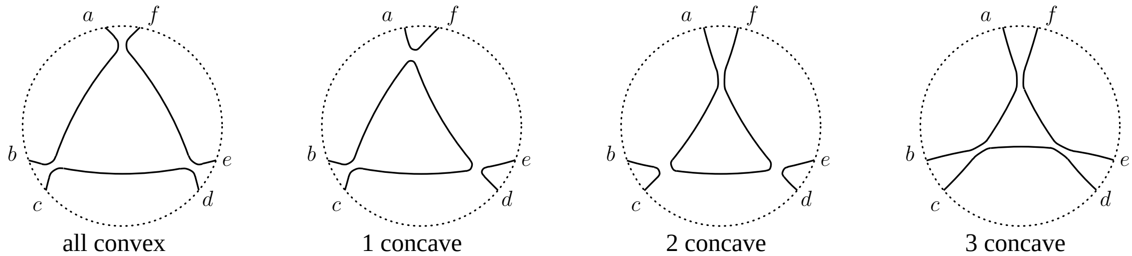


Figure 8

Denote the points where the three circles enter or leave the neighbourhood by a, b, c, d, e, and f in this order around the neighbourhood. As can be seen in Figure 8, there are only two essentially different cases; either a, c, e are connected to b, d, f, respectively, or a, c, e are connected to f, b, d, respectively. The step either preserves the set of connections or switches to the other arrangement. Obviously, in the earlier case the number of orbits is not changed; therefore we have to consider only the latter case.

The points a, b, c, d, e, and f are connected by the orbits outside, without intersection. If a was connected to c, say, then this orbit would isolate b, so this is impossible. Hence, each of a, b, c, d, e and f must be connected either to one of its neighbours or to the opposite point. If say a is connected to d, then this orbit separates b and c from e and f, therefore b must be connected to c and e must be connected to f. Altogether there are only two cases and their reverses: either each point is connected to one of its neighbours or two opposite points are connected and the the remaining neigh boring pairs are connected to each other. See Figure 9.

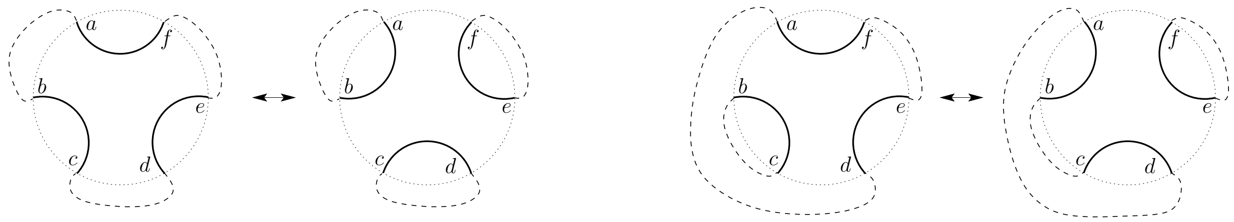


Figure 9

We can see that if only neighbouring points are connected, then the number of orbits is changed by $+2$ or -2. If two opposite points are connected (a and d in the figure), then the orbits are re-arranged, but their number is unchanged. Hence, Type-2 steps also preserve the parity. This completes the proof of $(*)$.

Solution 3. Like in the previous solutions, we do not need all circle pairs to intersect but we assume that the circles form a connected set. Denote by $\mathcal{C}$ and $\mathcal{P}$ the sets of circles and their intersection points, respectively.

The circles divide the plane into several simply connected, bounded regions and one unbounded region. Denote the set of these regions by $\mathcal{R}$. We say that an intersection point or a region is *odd* or *even* if it is contained inside an odd or even number of circles, respectively. Let $\mathcal{P}_{\text{odd}}$ and $\mathcal{R}_{\text{odd}}$ be the sets of odd intersection points and odd regions, respectively.

Claim.

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv n \pmod 2. \tag{1}$$

Proof. For each circle $c \in \mathcal{C}$, denote by R_c, P_c, and X_c the number of regions inside c, the number of intersection points inside c, and the number of circles intersecting c, respectively. The circles divide each other into several arcs; denote by A_c the number of such arcs inside c. By double counting the regions and intersection points inside the circles we get

$$|\mathcal{R}_{\text{odd}}| \equiv \sum_{c \in \mathcal{C}} R_c \pmod 2 \quad \text{and} \quad |\mathcal{P}_{\text{odd}}| \equiv \sum_{c \in \mathcal{C}} P_c \pmod 2.$$

For each circle c, apply Euler's polyhedron theorem to the (simply connected) regions in c. There are $2X_c$ intersection points on c; they divide the circle into $2X_c$ arcs. The polyhedron theorem yields $(R_c + 1) + (P_c + 2X_c) = (A_c + 2X_c) + 2$, considering the exterior of c as a single region. Therefore,

$$R_c + P_c = A_c + 1. \tag{2}$$

Moreover, we have four arcs starting from every interior points inside c and a single arc starting into the interior from each intersection point on the circle. By double-counting the end-points of the interior arcs we get $2A_c = 4P_c + 2X_c$, so

$$A_c = 2P_c + X_c. \tag{3}$$

The relations (2) and (3) together yield

$$R_c - P_c = X_c + 1. \tag{4}$$

By summing up (4) for all circles we obtain

$$\sum_{c\in\mathcal{C}} R_c - \sum_{c\in\mathcal{C}} P_c = \sum_{c\in\mathcal{C}} X_c + |\mathcal{C}|,$$

which yields

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| \equiv \sum_{c\in\mathcal{C}} X_c + n \pmod 2. \tag{5}$$

Notice that in $\sum\limits_{c\in\mathcal{C}} X_c$ each intersecting circle pair is counted twice, i.e., for both circles in the pair, so

$$\sum_{c\in\mathcal{C}} X_c \equiv 0 \pmod 2,$$

which finishes the proof of the Claim. □

Now insert the same small arcs at the intersections as in the first solution, and suppose that there is a single snail orbit b.

First we show that the odd regions are inside the curve b, while the even regions are outside. Take a region $r \in \mathcal{R}$ and a point x in its interior, and draw a ray y, starting from x, that does not pass through any intersection point of the circles and is neither tangent to any of the circles. As is well-known, x is inside the curve b if and only if y intersects b an odd number of times (see Figure 10). Notice that if an arbitrary circle c contains x in its interior, then c intersects y at a single point; otherwise, if x is outside c, then c has 2 or 0 intersections with y. Therefore, y intersects b an odd number of times if and only if x is contained in an odd number of circles, so if and only if r is odd.

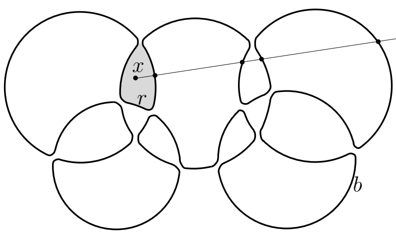


Figure 10

Now consider an intersection point p of two circles c_1 and c_2 and a small neighbourhood around p. Suppose that p is contained inside k circles.

We have four regions that meet at p. Let r_1 be the region that lies outside both c_1 and c_2, let r_2 be the region that lies inside both c_1 and c_2, and let r_3 and r_4 be the two remaining regions, each lying inside exactly one of c_1 and c_2. The region r_1 is contained inside the same k circles as p; the region r_2 is contained also by c_1 and c_2, so by $k+2$ circles in total; each of the regions r_3 and r_4 is contained inside $k+1$ circles. After the small arcs have been inserted at p, the regions r_1 and r_2 get connected, and the regions r_3 and r_4 remain separated at p (see Figure 11). If p is an odd point, then r_1 and r_2 are odd, so two odd regions are connected at p. Otherwise, if p is even, then we have two even regions connected at p.

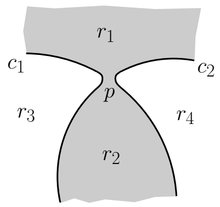


Figure 11

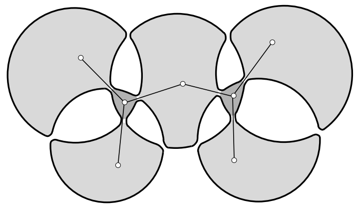


Figure 12

Consider the system of odd regions and their connections at the odd points as a graph. In this graph the odd regions are the vertices, and each odd point establishes an edge that connects two vertices (see Figure 12). As b is a single closed curve, this graph is connected and contains no cycle, so the graph is a tree. Then the number of vertices must be by one greater than the number of edges, so

$$|\mathcal{R}_{\text{odd}}| - |\mathcal{P}_{\text{odd}}| = 1. \tag{9}$$

The relations (1) and (9) together prove that n must be odd.

Comment. For every odd n there exists at least one configuration of n circles with a single snail orbit. Figure 13 shows a possible configuration with 5 circles. In general, if a circle is rotated by $k \cdot \frac{360^\circ}{n}$ $(k = 1, 2, \ldots, n-1)$ around an interior point other than the centre, the circle and its rotated copies together provide a single snail orbit.

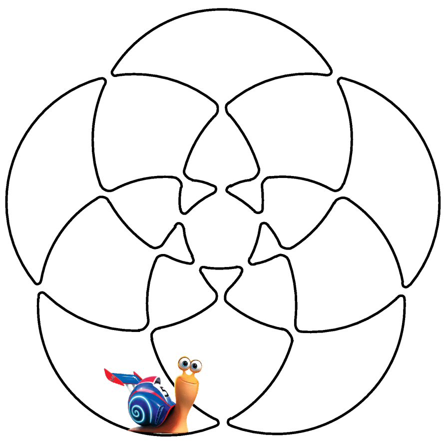


Figure 13

Geometry

G1. The points P and Q are chosen on the side BC of an acute-angled triangle ABC so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points M and N are taken on the rays AP and AQ, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines BM and CN intersect on the circumcircle of the triangle ABC.

(Georgia)

Solution 1. Denote by S the intersection point of the lines BM and CN. Let moreover $\beta = \angle QAC = \angle CBA$ and $\gamma = \angle PAB = \angle ACB$. From these equalities it follows that the triangles ABP and CAQ are similar (see Figure 1). Therefore we obtain

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{NQ}{QC}\,.$$

Moreover,

$$\angle BPM = \beta + \gamma = \angle CQN\,.$$

Hence the triangles BPM and NQC are similar. This gives $\angle BMP = \angle NCQ$, so the triangles BPM and BSC are also similar. Thus we get

$$\angle CSB = \angle BPM = \beta + \gamma = 180^\circ - \angle BAC\,,$$

which completes the solution.

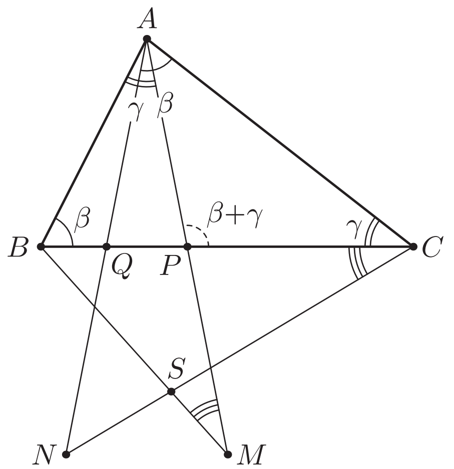


Figure 1

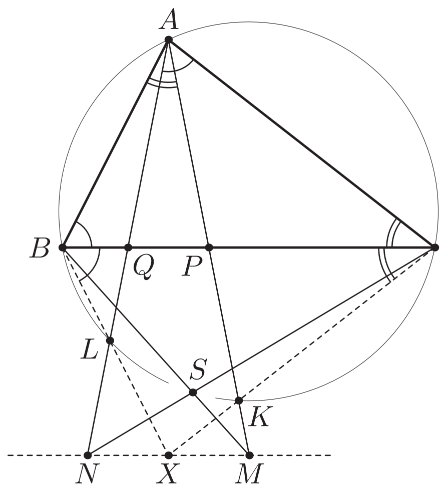


Figure 2

Solution 2. As in the previous solution, denote by S the intersection point of the lines BM and NC. Let moreover the circumcircle of the triangle ABC intersect the lines AP and AQ again at K and L, respectively (see Figure 2).

Note that $\angle LBC = \angle LAC = \angle CBA$ and similarly $\angle KCB = \angle KAB = \angle BCA$. It implies that the lines BL and CK meet at a point X, being symmetric to the point A with respect to the line BC. Since $AP = PM$ and $AQ = QN$, it follows that X lies on the line MN. Therefore, using Pascal's theorem for the hexagon $ALBSCK$, we infer that S lies on the circumcircle of the triangle ABC, which finishes the proof.

Comment. Both solutions can be modified to obtain a more general result, with the equalities

$$AP = PM \quad \text{and} \quad AQ = QN$$

replaced by

$$\frac{AP}{PM} = \frac{QN}{AQ}\,.$$

G2. Let ABC be a triangle. The points K, L, and M lie on the segments BC, CA, and AB, respectively, such that the lines AK, BL, and CM intersect in a common point. Prove that it is possible to choose two of the triangles ALM, BMK, and CKL whose inradii sum up to at least the inradius of the triangle ABC.

(Estonia)

Solution. Denote

$$a = \frac{BK}{KC}, \qquad b = \frac{CL}{LA}, \qquad c = \frac{AM}{MB}.$$

By CEVA's theorem, $abc = 1$, so we may, without loss of generality, assume that $a \geqslant 1$. Then at least one of the numbers b or c is not greater than 1. Therefore at least one of the pairs (a, b), (b, c) has its first component not less than 1 and the second one not greater than 1. Without loss of generality, assume that $1 \leqslant a$ and $b \leqslant 1$.

Therefore, we obtain $bc \leqslant 1$ and $1 \leqslant ca$, or equivalently

$$\frac{AM}{MB} \leqslant \frac{LA}{CL} \qquad \text{and} \qquad \frac{MB}{AM} \leqslant \frac{BK}{KC}.$$

The first inequality implies that the line passing through M and parallel to BC intersects the *segment* AL at a point X (see Figure 1). Therefore the inradius of the triangle ALM is not less than the inradius r_1 of triangle AMX.

Similarly, the line passing through M and parallel to AC intersects the *segment* BK at a point Y, so the inradius of the triangle BMK is not less than the inradius r_2 of the triangle BMY. Thus, to complete our solution, it is enough to show that $r_1 + r_2 \geqslant r$, where r is the inradius of the triangle ABC. We prove that in fact $r_1 + r_2 = r$.

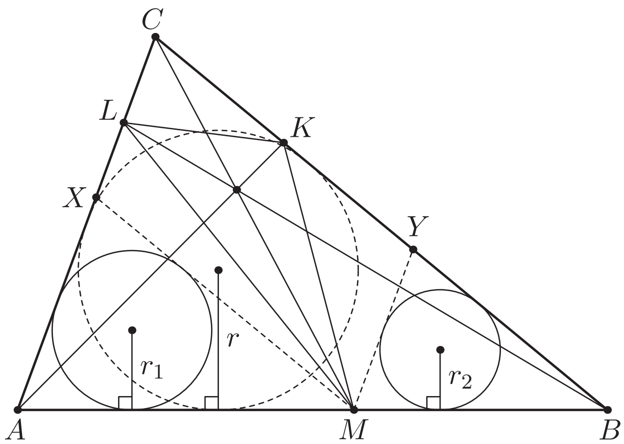


Figure 1

Since $MX \parallel BC$, the dilation with centre A that takes M to B takes the incircle of the triangle AMX to the incircle of the triangle ABC. Therefore

$$\frac{r_1}{r} = \frac{AM}{AB}, \qquad \text{and similarly} \qquad \frac{r_2}{r} = \frac{MB}{AB}.$$

Adding these equalities gives $r_1 + r_2 = r$, as required.

Comment. Alternatively, one can use DESARGUES' theorem instead of CEVA's theorem, as follows: The lines AB, BC, CA dissect the plane into seven regions. One of them is bounded, and amongst the other six, three are two-sided and three are three-sided. Now define the points $P = BC \cap LM$, $Q = CA \cap MK$, and $R = AB \cap KL$ (in the projective plane). By DESARGUES' theorem, the points P, Q, R lie on a common line ℓ. This line intersects only unbounded regions. If we now assume (without loss of generality) that P, Q and R lie on ℓ in that order, then one of the segments PQ or QR lies inside a two-sided region. If, for example, this segment is PQ, then the triangles ALM and BMK will satisfy the statement of the problem for the same reason.

G3. Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$. (Here we always assume that an angle bisector is a ray.)

(Russia)

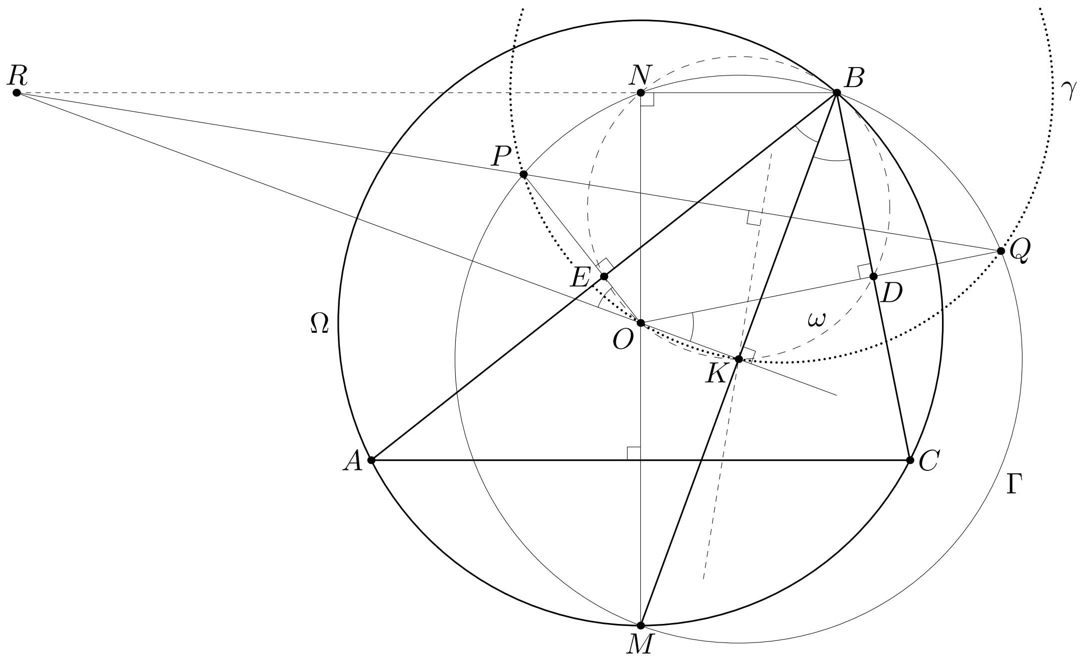
Solution. Let K be the midpoint of BM, i.e., the centre of Γ. Notice that $AB \neq BC$ implies $K \neq O$. Clearly, the lines OM and OK are the perpendicular bisectors of AC and BM, respectively. Therefore, R is the intersection point of PQ and OK.

Let N be the second point of intersection of Γ with the line OM. Since BM is a diameter of Γ, the lines BN and AC are both perpendicular to OM. Hence $BN \parallel AC$, and it suffices to prove that BN passes through R. Our plan for doing this is to interpret the lines BN, OK, and PQ as the radical axes of three appropriate circles.

Let ω be the circle with diameter BO. Since $\angle BNO = \angle BKO = 90^\circ$, the points N and K lie on ω.

Next we show that the points O, K, P, and Q are concyclic. To this end, let D and E be the midpoints of BC and AB, respectively. Clearly, D and E lie on the rays OQ and OP, respectively. By our assumptions about the triangle ABC, the points B, E, O, K, and D lie in this order on ω. It follows that $\angle EOR = \angle EBK = \angle KBD = \angle KOD$, so the line KO externally bisects the angle POQ. Since the point K is the centre of Γ, it also lies on the perpendicular bisector of PQ. So K coincides with the midpoint of the arc POQ of the circumcircle γ of triangle POQ.

Thus the lines OK, BN, and PQ are pairwise radical axes of the circles ω, γ, and Γ. Hence they are concurrent at R, as required.



G4. Consider a fixed circle Γ with three fixed points A, B, and C on it. Also, let us fix a real number $\lambda \in (0,1)$. For a variable point $P \notin \{A, B, C\}$ on Γ, let M be the point on the segment CP such that $CM = \lambda \cdot CP$. Let Q be the second point of intersection of the circumcircles of the triangles AMP and BMC. Prove that as P varies, the point Q lies on a fixed circle.

(United Kingdom)

Solution 1. Throughout the solution, we denote by $\measuredangle(a, b)$ the directed angle between the lines a and b.

Let D be the point on the segment AB such that $BD = \lambda \cdot BA$. We will show that either $Q = D$, or $\measuredangle(DQ, QB) = \measuredangle(AB, BC)$; this would mean that the point Q varies over the constant circle through D tangent to BC at B, as required.

Denote the circumcircles of the triangles AMP and BMC by ω_A and ω_B, respectively. The lines AP, BC, and MQ are pairwise radical axes of the circles Γ, ω_A, and ω_B, thus either they are parallel, or they share a common point X.

Assume that these lines are parallel (see Figure 1). Then the segments AP, QM, and BC have a common perpendicular bisector; the reflection in this bisector maps the segment CP to BA, and maps M to Q. Therefore, in this case Q lies on AB, and $BQ/AB = CM/CP = BD/AB$; so we have $Q = D$.

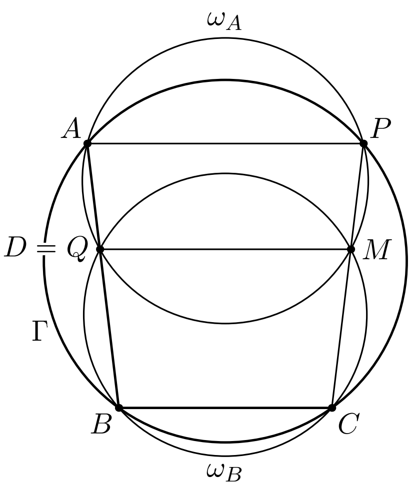


Figure 1

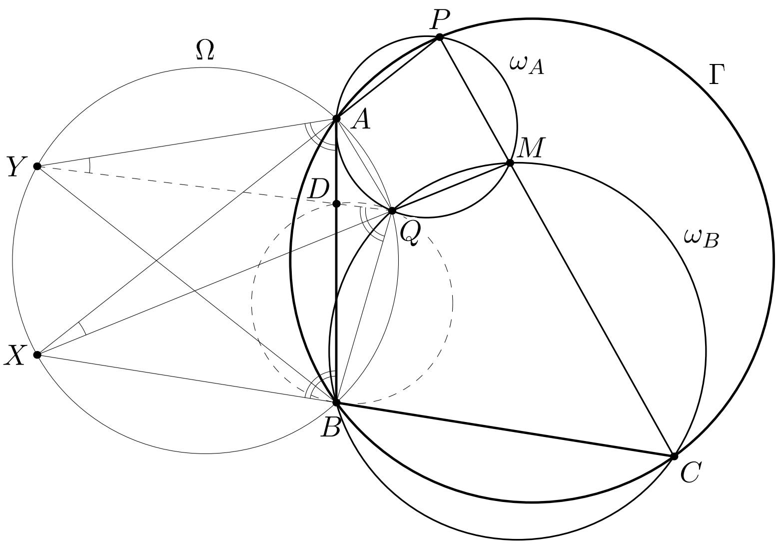


Figure 2

Now assume that the lines AP, QM, and BC are concurrent at some point X (see Figure 2). Notice that the points A, B, Q, and X lie on a common circle Ω by Miquel's theorem applied to the triangle XPC. Let us denote by Y the symmetric image of X about the perpendicular bisector of AB. Clearly, Y lies on Ω, and the triangles YAB and $\triangle XBA$ are congruent. Moreover, the triangle XPC is similar to the triangle XBA, so it is also similar to the triangle YAB.

Next, the points D and M correspond to each other in similar triangles YAB and XPC, since $BD/BA = CM/CP = \lambda$. Moreover, the triangles YAB and XPC are equi-oriented, so $\measuredangle(MX, XP) = \measuredangle(DY, YA)$. On the other hand, since the points A, Q, X, and Y lie on Ω, we have $\measuredangle(QY, YA) = \measuredangle(MX, XP)$. Therefore, $\measuredangle(QY, YA) = \measuredangle(DY, YA)$, so the points Y, D, and Q are collinear.

Finally, we have $\measuredangle(DQ, QB) = \measuredangle(YQ, QB) = \measuredangle(YA, AB) = \measuredangle(AB, BX) = \measuredangle(AB, BC)$, as desired.

Comment. In the original proposal, λ was supposed to be an arbitrary real number distinct from 0 and 1, and the point M was defined by $\overrightarrow{CM} = \lambda \cdot \overrightarrow{CP}$. The Problem Selection Committee decided to add the restriction $\lambda \in (0,1)$ in order to avoid a large case distinction.

Solution 2. As in the previous solution, we introduce the radical centre $X = AP \cap BC \cap MQ$ of the circles ω_A, ω_B, and Γ. Next, we also notice that the points A, Q, B, and X lie on a common circle Ω.

If the point P lies on the arc BAC of Γ, then the point X is outside Γ, thus the point Q belongs to the ray XM, and therefore the points P, A, and Q lie on the same side of BC. Otherwise, if P lies on the arc BC not containing A, then X lies inside Γ, so M and Q lie on different sides of BC; thus again Q and A lie on the same side of BC. So, in each case the points Q and A lie on the same side of BC.

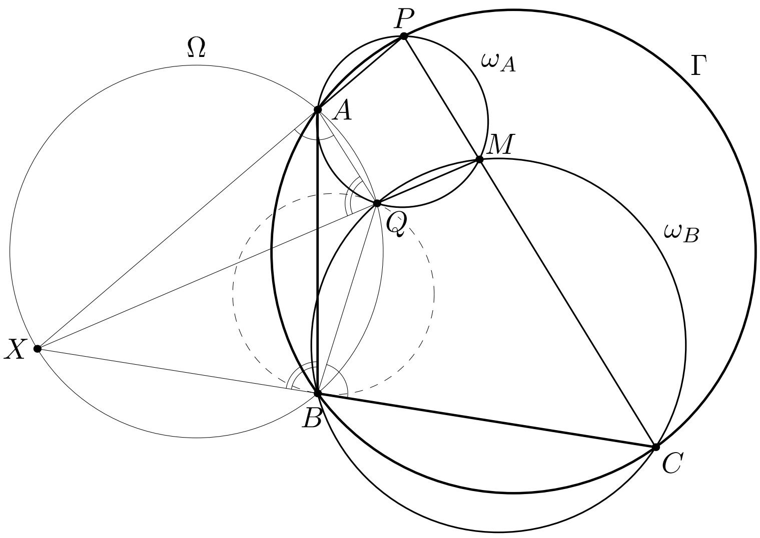


Figure 3

Now we prove that the ratio

$$\frac{QB}{\sin \angle QBC} = \frac{QB}{QX} \cdot \frac{QX}{\sin \angle QBX}$$

is constant. Since the points A, Q, B, and X are concyclic, we have

$$\frac{QX}{\sin \angle QBX} = \frac{AX}{\sin \angle ABC}.$$

Next, since the points B, Q, M, and C are concyclic, the triangles XBQ and XMC are similar, so

$$\frac{QB}{QX} = \frac{CM}{CX} = \lambda \cdot \frac{CP}{CX}.$$

Analogously, the triangles XCP and XAB are also similar, so

$$\frac{CP}{CX} = \frac{AB}{AX}.$$

Therefore, we obtain

$$\frac{QB}{\sin \angle QBC} = \lambda \cdot \frac{AB}{AX} \cdot \frac{AX}{\sin \angle ABC} = \lambda \cdot \frac{AB}{\sin \angle ABC},$$

so this ratio is indeed constant. Thus the circle passing through Q and tangent to BC at B is also constant, and Q varies over this fixed circle.

Comment. It is not hard to guess that the desired circle should be tangent to BC at B. Indeed, the second paragraph of this solution shows that this circle lies on one side of BC; on the other hand, in the limit case $P = B$, the point Q also coincides with B.

Solution 3. Let us perform an inversion centred at C. Denote by X' the image of a point X under this inversion.

The circle Γ maps to the line Γ' passing through the constant points A' and B', and containing the variable point P'. By the problem condition, the point M varies over the circle γ which is the homothetic image of Γ with centre C and coefficient λ. Thus M' varies over the constant line $\gamma' \parallel A'B'$ which is the homothetic image of $A'B'$ with centre C and coefficient $1/\lambda$, and $M = \gamma' \cap CP'$. Next, the circumcircles ω_A and ω_B of the triangles AMP and BMC map to the circumcircle ω'_A of the triangle $A'M'P'$ and to the line $B'M'$, respectively; the point Q thus maps to the second point of intersection of $B'M'$ with ω'_A (see Figure 4).

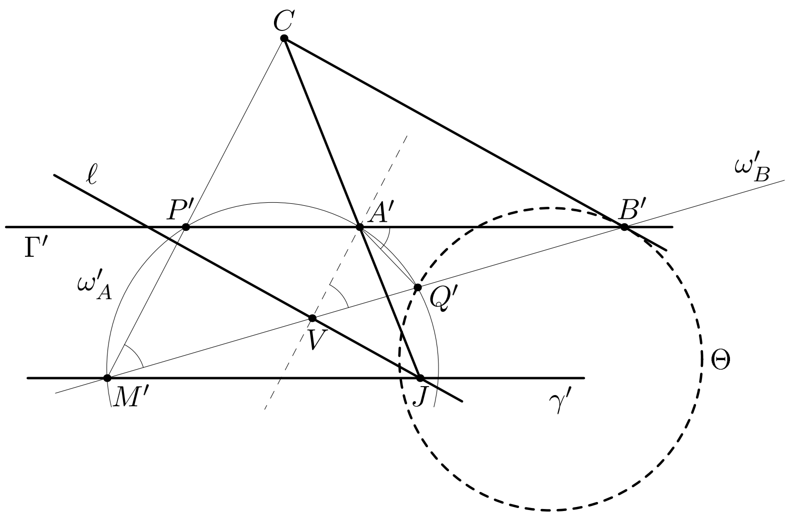


Figure 4

Let J be the (constant) common point of the lines γ' and CA', and let ℓ be the (constant) line through J parallel to CB'. Let V be the common point of the lines ℓ and $B'M'$. Applying Pappus' theorem to the triples (C, J, A') and (V, B', M') we get that the points $CB' \cap JV$, $JM' \cap A'B'$, and $CM' \cap A'V$ are collinear. The first two of these points are ideal, hence so is the third, which means that $CM' \parallel A'V$.

Now we have $\measuredangle(Q'A', A'P') = \measuredangle(Q'M', M'P') = \angle(VM', A'V)$, which means that the triangles $B'Q'A'$ and $B'A'V$ are similar, and $(B'A')^2 = B'Q' \cdot B'V$. Thus Q' is the image of V under the second (fixed) inversion with centre B' and radius $B'A'$. Since V varies over the constant line ℓ, Q' varies over some constant circle Θ. Thus, applying the first inversion back we get that Q also varies over some fixed circle.

One should notice that this last circle is not a line; otherwise Θ would contain C, and thus ℓ would contain the image of C under the second inversion. This is impossible, since $CB' \parallel \ell$.

G5. Let $ABCD$ be a convex quadrilateral with $\angle B = \angle D = 90^\circ$. Point H is the foot of the perpendicular from A to BD. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ\,.$$

Prove that the circumcircle of triangle SHT is tangent to the line BD.

(Iran)

Solution. Let the line passing through C and perpendicular to the line SC intersect the line AB at Q (see Figure 1). Then

$$\angle SQC = 90^\circ - \angle BSC = 180^\circ - \angle SHC\,,$$

which implies that the points C, H, S, and Q lie on a common circle. Moreover, since SQ is a diameter of this circle, we infer that the circumcentre K of triangle SHC lies on the line AB. Similarly, we prove that the circumcentre L of triangle CHT lies on the line AD.

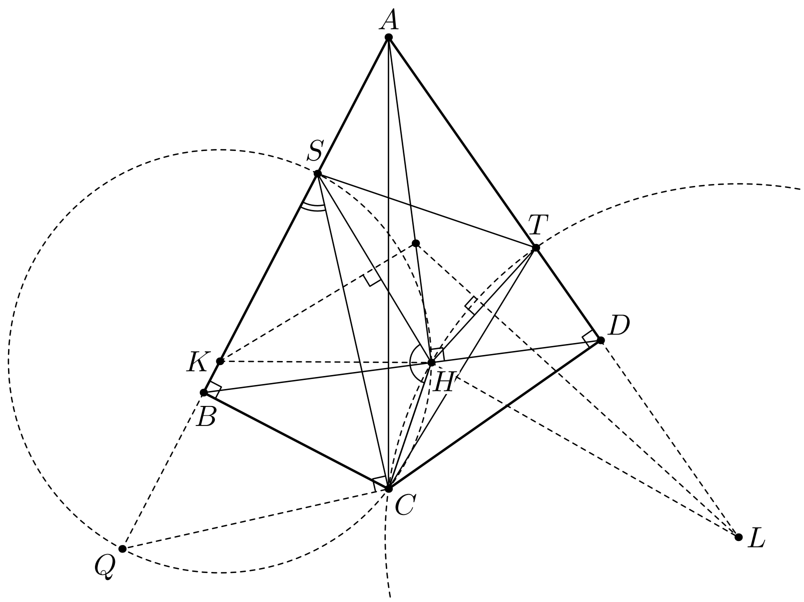


Figure 1

In order to prove that the circumcircle of triangle SHT is tangent to BD, it suffices to show that the perpendicular bisectors of HS and HT intersect on the line AH. However, these two perpendicular bisectors coincide with the angle bisectors of angles AKH and ALH. Therefore, in order to complete the solution, it is enough (by the bisector theorem) to show that

$$\frac{AK}{KH} = \frac{AL}{LH}\,. \tag{1}$$

We present two proofs of this equality.

First proof. Let the lines KL and HC intersect at M (see Figure 2). Since $KH = KC$ and $LH = LC$, the points H and C are symmetric to each other with respect to the line KL. Therefore M is the midpoint of HC. Denote by O the circumcentre of quadrilateral $ABCD$. Then O is the midpoint of AC. Therefore we have $OM \parallel AH$ and hence $OM \perp BD$. This together with the equality $OB = OD$ implies that OM is the perpendicular bisector of BD and therefore $BM = DM$.

Since $CM \perp KL$, the points B, C, M, and K lie on a common circle with diameter KC. Similarly, the points L, C, M, and D lie on a circle with diameter LC. Thus, using the sine law, we obtain

$$\frac{AK}{AL} = \frac{\sin\angle ALK}{\sin\angle AKL} = \frac{DM}{CL}\cdot\frac{CK}{BM} = \frac{CK}{CL} = \frac{KH}{LH}\,,$$

which finishes the proof of (1).

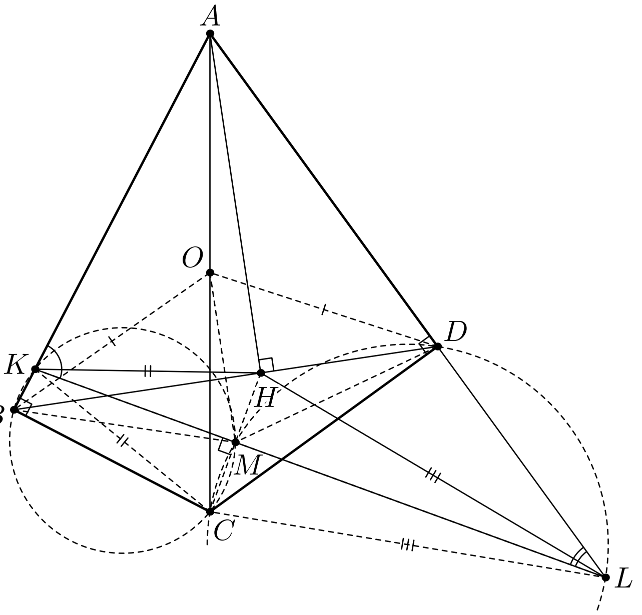


Figure 2

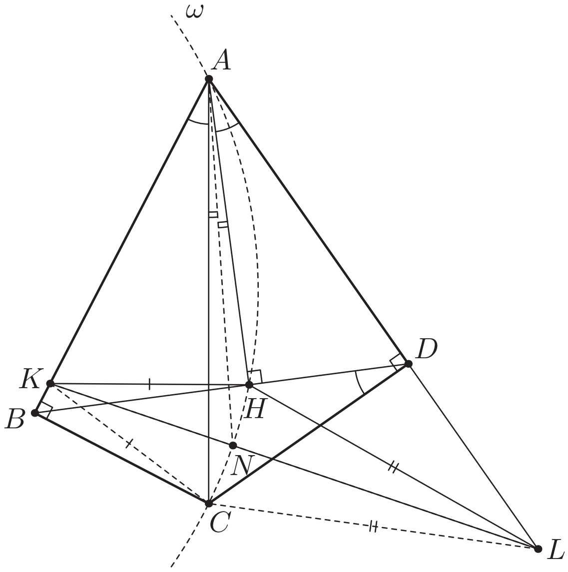


Figure 3

Second proof. If the points A, H, and C are collinear, then $AK = AL$ and $KH = LH$, so the equality (1) follows. Assume therefore that the points A, H, and C do not lie in a line and consider the circle ω passing through them (see Figure 3). Since the quadrilateral $ABCD$ is cyclic,

$$\angle BAC = \angle BDC = 90^\circ - \angle ADH = \angle HAD\,.$$

Let $N \neq A$ be the intersection point of the circle ω and the angle bisector of $\angle CAH$. Then AN is also the angle bisector of $\angle BAD$. Since H and C are symmetric to each other with respect to the line KL and $HN = NC$, it follows that both N and the centre of ω lie on the line KL. This means that the circle ω is an Apollonius circle of the points K and L. This immediately yields (1).

Comment. Either proof can be used to obtain the following generalised result:

Let $ABCD$ be a convex quadrilateral and let H be a point in its interior with $\angle BAC = \angle DAH$. The points S and T are chosen on the sides AB and AD, respectively, in such a way that H lies inside triangle SCT and

$$\angle SHC - \angle BSC = 90^\circ, \quad \angle THC - \angle DTC = 90^\circ\,.$$

Then the circumcentre of triangle SHT lies on the line AH (and moreover the circumcentre of triangle SCT lies on AC).

G6. Let ABC be a fixed acute-angled triangle. Consider some points E and F lying on the sides AC and AB, respectively, and let M be the midpoint of EF. Let the perpendicular bisector of EF intersect the line BC at K, and let the perpendicular bisector of MK intersect the lines AC and AB at S and T, respectively. We call the pair (E, F) *interesting*, if the quadrilateral $KSAT$ is cyclic.

Suppose that the pairs (E_1, F_1) and (E_2, F_2) are interesting. Prove that

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC}.$$

(Iran)

Solution 1. For any interesting pair (E, F), we will say that the corresponding triangle EFK is also *interesting*.

Let EFK be an interesting triangle. Firstly, we prove that $\angle KEF = \angle KFE = \angle A$, which also means that the circumcircle ω_1 of the triangle AEF is tangent to the lines KE and KF.

Denote by ω the circle passing through the points K, S, A, and T. Let the line AM intersect the line ST and the circle ω (for the second time) at N and L, respectively (see Figure 1).

Since $EF \parallel TS$ and M is the midpoint of EF, N is the midpoint of ST. Moreover, since K and M are symmetric to each other with respect to the line ST, we have $\angle KNS = \angle MNS = \angle LNT$. Thus the points K and L are symmetric to each other with respect to the perpendicular bisector of ST. Therefore $KL \parallel ST$.

Let G be the point symmetric to K with respect to N. Then G lies on the line EF, and we may assume that it lies on the ray MF. One has

$$\angle KGE = \angle KNS = \angle SNM = \angle KLA = 180^\circ - \angle KSA$$

(if $K = L$, then the angle KLA is understood to be the angle between AL and the tangent to ω at L). This means that the points K, G, E, and S are concyclic. Now, since $KSGT$ is a parallelogram, we obtain $\angle KEF = \angle KSG = 180^\circ - \angle TKS = \angle A$. Since $KE = KF$, we also have $\angle KFE = \angle KEF = \angle A$.

After having proved this fact, one may finish the solution by different methods.

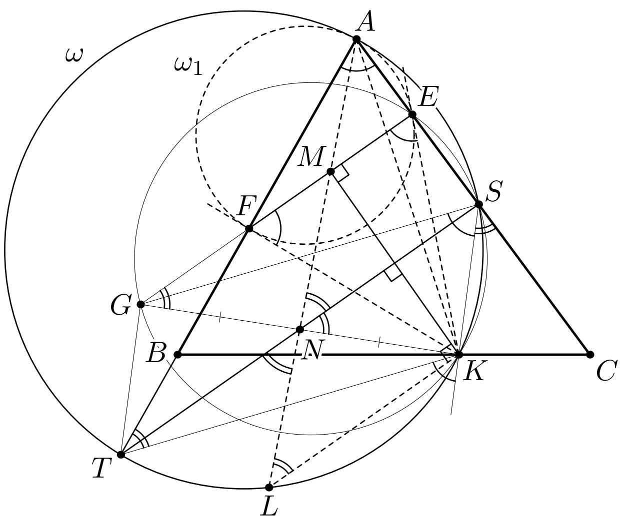


Figure 1

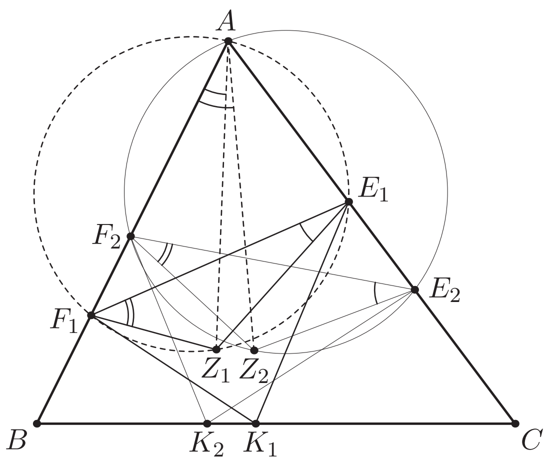


Figure 2

First method. We have just proved that all interesting triangles are similar to each other. This allows us to use the following lemma.

Lemma. Let ABC be an arbitrary triangle. Choose two points E_1 and E_2 on the side AC, two points F_1 and F_2 on the side AB, and two points K_1 and K_2 on the side BC, in a way that the triangles $E_1F_1K_1$ and $E_2F_2K_2$ are similar. Then the six circumcircles of the triangles AE_iF_i, BF_iK_i, and CE_iK_i $(i = 1, 2)$ meet at a common point Z. Moreover, Z is the centre of the spiral similarity that takes the triangle $E_1F_1K_1$ to the triangle $E_2F_2K_2$.

Proof. Firstly, notice that for each $i = 1, 2$, the circumcircles of the triangles AE_iF_i, BF_iK_i, and CK_iE_i have a common point Z_i by Miquel's theorem. Moreover, we have

$$\measuredangle(Z_iF_i, Z_iE_i) = \measuredangle(AB, CA), \quad \measuredangle(Z_iK_i, Z_iF_i) = \measuredangle(BC, AB), \quad \measuredangle(Z_iE_i, Z_iK_i) = \measuredangle(CA, BC).$$

This yields that the points Z_1 and Z_2 correspond to each other in similar triangles $E_1F_1K_1$ and $E_2F_2K_2$. Thus, if they coincide, then this common point is indeed the desired centre of a spiral similarity.

Finally, in order to show that $Z_1 = Z_2$, one may notice that $\measuredangle(AB, AZ_1) = \measuredangle(E_1F_1, E_1Z_1) = \measuredangle(E_2F_2, E_2Z_2) = \measuredangle(AB, AZ_2)$ (see Figure 2). Similarly, one has $\measuredangle(BC, BZ_1) = \measuredangle(BC, BZ_2)$ and $\measuredangle(CA, CZ_1) = \measuredangle(CA, CZ_2)$. This yields $Z_1 = Z_2$. $\square$

Now, let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively, and let R be the midpoint of BC (see Figure 3). Then R is the circumcentre of the cyclic quadrilateral $BCPQ$. Thus we obtain $\angle APQ = \angle B$ and $\angle RPC = \angle C$, which yields $\angle QPR = \angle A$. Similarly, we show that $\angle PQR = \angle A$. Thus, all interesting triangles are similar to the triangle PQR.

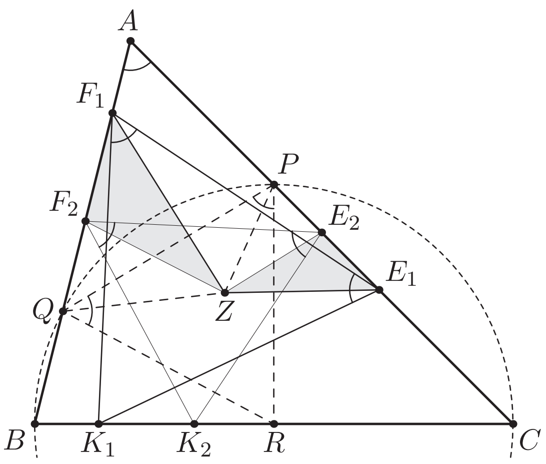


Figure 3

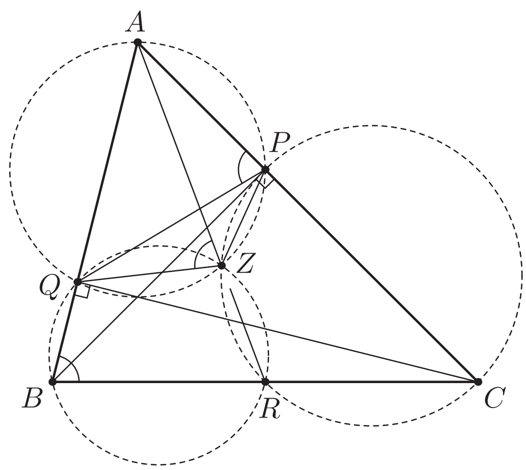


Figure 4

Denote now by Z the common point of the circumcircles of APQ, BQR, and CPR. Let $E_1F_1K_1$ and $E_2F_2K_2$ be two interesting triangles. By the lemma, Z is the centre of any spiral similarity taking one of the triangles $E_1F_1K_1$, $E_2F_2K_2$, and PQR to some other of them. Therefore the triangles ZE_1E_2 and ZF_1F_2 are similar, as well as the triangles ZE_1F_1 and ZPQ. Hence

$$\frac{E_1E_2}{F_1F_2} = \frac{ZE_1}{ZF_1} = \frac{ZP}{ZQ}.$$

Moreover, the equalities $\angle AZQ = \angle APQ = \angle ABC = 180^\circ - \angle QZR$ show that the point Z lies on the line AR (see Figure 4). Therefore the triangles AZP and ACR are similar, as well as the triangles AZQ and ABR. This yields

$$\frac{ZP}{ZQ} = \frac{ZP}{RC} \cdot \frac{RB}{ZQ} = \frac{AZ}{AC} \cdot \frac{AB}{AZ} = \frac{AB}{AC},$$

which completes the solution.

Second method. Now we will start from the fact that ω_1 is tangent to the lines KE and KF (see Figure 5). We prove that if (E, F) is an interesting pair, then

$$\frac{AE}{AB} + \frac{AF}{AC} = 2\cos\angle A. \tag{1}$$

Let Y be the intersection point of the segments BE and CF. The points B, K, and C are collinear, hence applying Pascal's theorem to the degenerated hexagon $AFFYEE$, we infer that Y lies on the circle ω_1.

Denote by Z the second intersection point of the circumcircle of the triangle BFY with the line BC (see Figure 6). By Miquel's theorem, the points C, Z, Y, and E are concyclic. Therefore we obtain

$$BF \cdot AB + CE \cdot AC = BY \cdot BE + CY \cdot CF = BZ \cdot BC + CZ \cdot BC = BC^2 .$$

On the other hand, $BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos\angle A$, by the cosine law. Hence

$$(AB - AF) \cdot AB + (AC - AE) \cdot AC = AB^2 + AC^2 - 2AB \cdot AC \cos\angle A ,$$

which simplifies to the desired equality (1).

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs of points. Then we get

$$\frac{AE_1}{AB} + \frac{AF_1}{AC} = \frac{AE_2}{AB} + \frac{AF_2}{AC} ,$$

which gives the desired result.

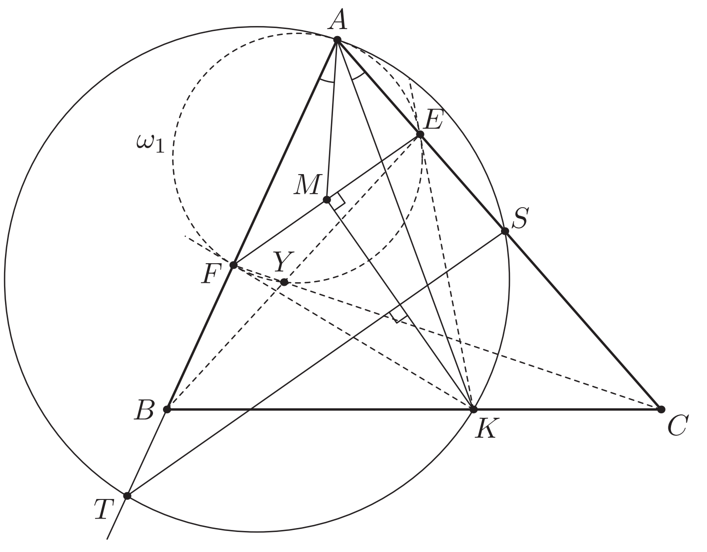


Figure 5

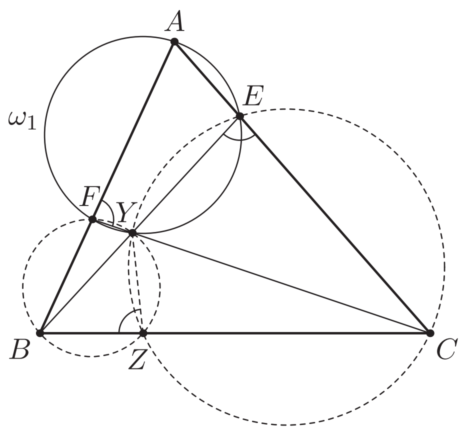


Figure 6

Third method. Again, we make use of the fact that all interesting triangles are similar (and equi-oriented). Let us put the picture onto a complex plane such that A is at the origin, and identify each point with the corresponding complex number.

Let EFK be any interesting triangle. The equalities $\angle KEF = \angle KFE = \angle A$ yield that the ratio $\nu = \frac{K-E}{F-E}$ is the same for all interesting triangles. This in turn means that the numbers E, F, and K satisfy the linear equation

$$K = \mu E + \nu F, \qquad \text{where} \quad \mu = 1 - \nu. \tag{2}$$

Now let us choose the points X and Y on the rays AB and AC, respectively, so that $\angle CXA = \angle AYB = \angle A = \angle KEF$ (see Figure 7). Then each of the triangles AXC and YAB is similar to any interesting triangle, which also means that

$$C = \mu A + \nu X = \nu X \quad \text{and} \quad B = \mu Y + \nu A = \mu Y. \tag{3}$$

Moreover, one has $X/Y = \overline{C/B}$.

Since the points E, F, and K lie on AC, AB, and BC, respectively, one gets

$$E = \rho Y, \quad F = \sigma X, \quad \text{and} \quad K = \lambda B + (1-\lambda)C$$

for some real ρ, σ, and λ. In view of (3), the equation (2) now reads $\lambda B + (1-\lambda)C = K = \mu E + \nu F = \rho B + \sigma C$, or

$$(\lambda - \rho)B = (\sigma + \lambda - 1)C.$$

Since the nonzero complex numbers B and C have different arguments, the coefficients in the brackets vanish, so $\rho = \lambda$ and $\sigma = 1 - \lambda$. Therefore,

$$\frac{E}{Y} + \frac{F}{X} = \rho + \sigma = 1. \tag{4}$$

Now, if (E_1, F_1) and (E_2, F_2) are two distinct interesting pairs, one may apply (4) to both pairs. Subtracting, we get

$$\frac{E_1 - E_2}{Y} = \frac{F_2 - F_1}{X}, \qquad \text{so} \quad \frac{E_1 - E_2}{F_2 - F_1} = \frac{Y}{X} = \frac{\overline{B}}{\overline{C}}.$$

Taking absolute values provides the required result.

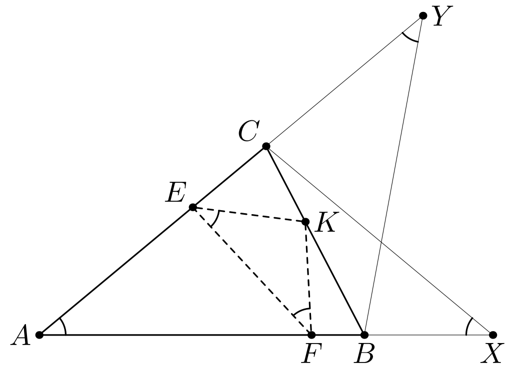


Figure 7

Comment 1. One may notice that the triangle PQR is also interesting.

Comment 2. In order to prove that $\angle KEF = \angle KFE = \angle A$, one may also use the following well-known fact:

Let AEF be a triangle with $AE \neq AF$, and let K be the common point of the symmedian taken from A and the perpendicular bisector of EF. Then the lines KE and KF are tangent to the circumcircle ω_1 of the triangle AEF.

In this case, however, one needs to deal with the case $AE = AF$ separately.

Solution 2. Let (E, F) be an interesting pair. This time we prove that

$$\frac{AM}{AK} = \cos\angle A\,. \tag{5}$$

As in Solution 1, we introduce the circle ω passing through the points K, S, A, and T, together with the points N and L at which the line AM intersect the line ST and the circle ω for the second time, respectively. Let moreover O be the centre of ω (see Figures 8 and 9). As in Solution 1, we note that N is the midpoint of ST and show that $KL \parallel ST$, which implies $\angle FAM = \angle EAK$.

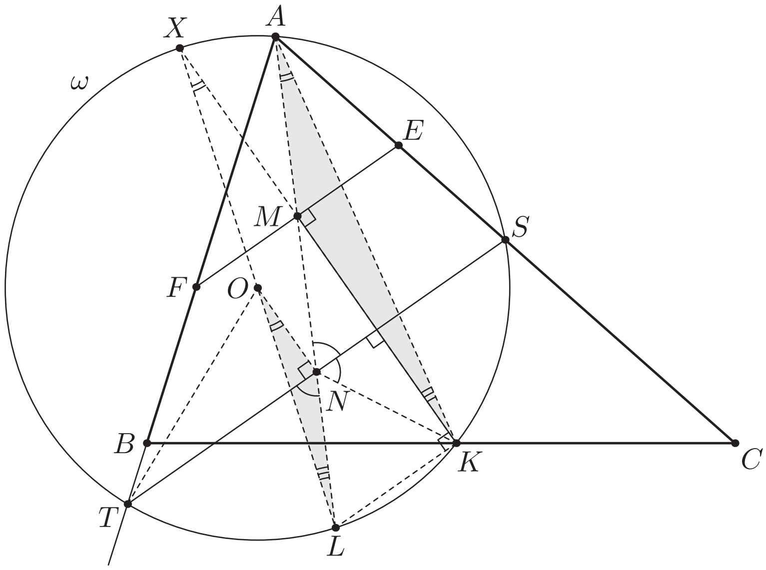


Figure 8

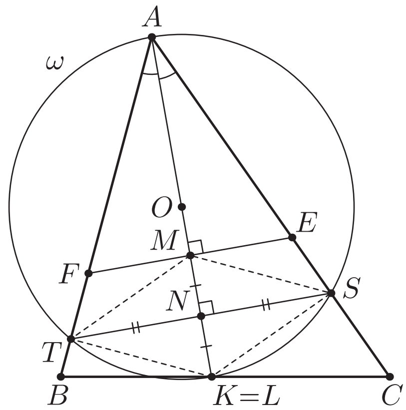


Figure 9

Suppose now that $K \neq L$ (see Figure 8). Then $KL \parallel ST$, and consequently the lines KM and KL are perpendicular. It implies that the lines LO and KM meet at a point X lying on the circle ω. Since the lines ON and XM are both perpendicular to the line ST, they are parallel to each other, and hence $\angle LON = \angle LXK = \angle MAK$. On the other hand, $\angle OLN = \angle MKA$, so we infer that triangles NOL and MAK are similar. This yields

$$\frac{AM}{AK} = \frac{ON}{OL} = \frac{ON}{OT} = \cos\angle TON = \cos\angle A\,.$$

If, on the other hand, $K = L$, then the points A, M, N, and K lie on a common line, and this line is the perpendicular bisector of ST (see Figure 9). This implies that AK is a diameter of ω, which yields $AM = 2OK - 2NK = 2ON$. So also in this case we obtain

$$\frac{AM}{AK} = \frac{2ON}{2OT} = \cos\angle TON = \cos\angle A\,.$$

Thus (5) is proved.

Let P and Q be the feet of the perpendiculars from B and C onto AC and AB, respectively (see Figure 10). We claim that the point M lies on the line PQ. Consider now the composition of the dilatation with factor $\cos\angle A$ and centre A, and the reflection with respect to the angle bisector of $\angle BAC$. This transformation is a similarity that takes B, C, and K to P, Q, and M, respectively. Since K lies on the line BC, the point M lies on the line PQ.

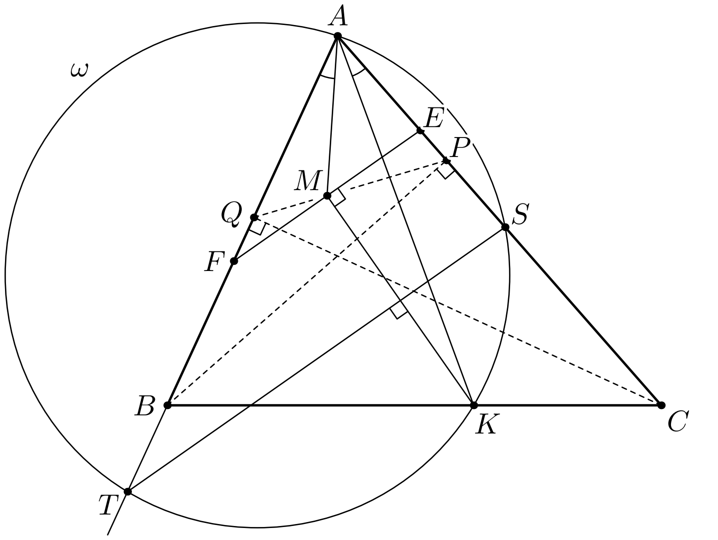


Figure 10

Suppose that $E \neq P$. Then also $F \neq Q$, and by MENELAUS' theorem, we obtain

$$\frac{AQ}{FQ} \cdot \frac{FM}{EM} \cdot \frac{EP}{AP} = 1 \,.$$

Using the similarity of the triangles APQ and ABC, we infer that

$$\frac{EP}{FQ} = \frac{AP}{AQ} = \frac{AB}{AC}, \quad \text{and hence} \quad \frac{EP}{AB} = \frac{FQ}{AC} \,.$$

The last equality holds obviously also in case $E = P$, because then $F = Q$. Moreover, since the line PQ intersects the *segment* EF, we infer that the point E lies on the segment AP if and only if the point F lies outside of the segment AQ.

Let now (E_1, F_1) and (E_2, F_2) be two interesting pairs. Then we obtain

$$\frac{E_1P}{AB} = \frac{F_1Q}{AC} \qquad \text{and} \qquad \frac{E_2P}{AB} = \frac{F_2Q}{AC} \,.$$

If P lies between the points E_1 and E_2, we add the equalities above, otherwise we subtract them. In any case we obtain

$$\frac{E_1E_2}{AB} = \frac{F_1F_2}{AC} \,,$$

which completes the solution.

G7. Let ABC be a triangle with circumcircle Ω and incentre I. Let the line passing through I and perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at points U and V, respectively. Let the line passing through U and parallel to AI intersect AV at X, and let the line passing through V and parallel to AI intersect AB at Y. Let W and Z be the midpoints of AX and BC, respectively. Prove that if the points I, X, and Y are collinear, then the points I, W, and Z are also collinear.

(U.S.A.)

Solution 1. We start with some general observations. Set $\alpha = \angle A/2$, $\beta = \angle B/2$, $\gamma = \angle C/2$. Then obviously $\alpha + \beta + \gamma = 90^\circ$. Since $\angle UIC = 90^\circ$, we obtain $\angle IUC = \alpha + \beta$. Therefore $\angle BIV = \angle IUC - \angle IBC = \alpha = \angle BAI = \angle BYV$, which implies that the points B, Y, I, and V lie on a common circle (see Figure 1).

Assume now that the points I, X and Y are collinear. We prove that $\angle YIA = 90^\circ$.

Let the line XU intersect AB at N. Since the lines AI, UX, and VY are parallel, we get

$$\frac{NX}{AI} = \frac{YN}{YA} = \frac{VU}{VI} = \frac{XU}{AI},$$

implying $NX = XU$. Moreover, $\angle BIU = \alpha = \angle BNU$. This implies that the quadrilateral $BUIN$ is cyclic, and since BI is the angle bisector of $\angle UBN$, we infer that $NI = UI$. Thus in the isosceles triangle NIU, the point X is the midpoint of the base NU. This gives $\angle IXN = 90^\circ$, i.e., $\angle YIA = 90^\circ$.

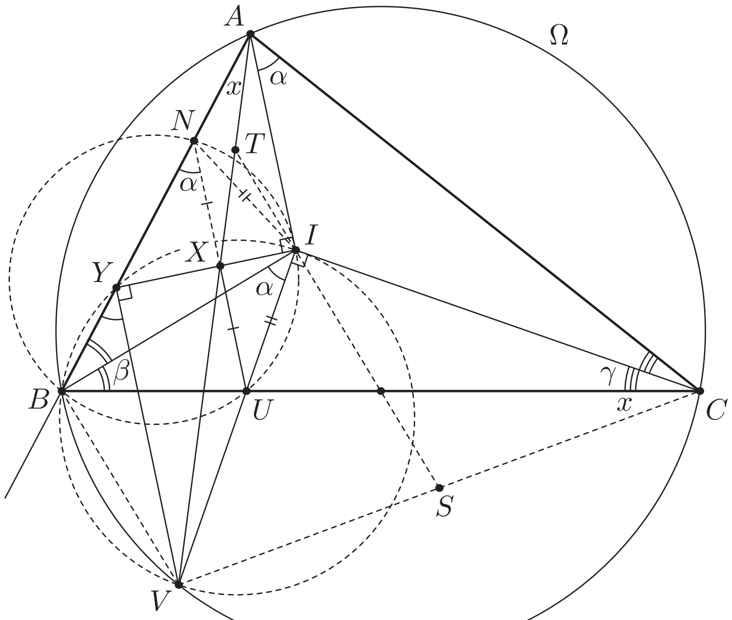


Figure 1

Let S be the midpoint of the segment VC. Let moreover T be the intersection point of the lines AX and SI, and set $x = \angle BAV = \angle BCV$. Since $\angle CIA = 90^\circ + \beta$ and $SI = SC$, we obtain

$$\angle TIA = 180^\circ - \angle AIS = 90^\circ - \beta - \angle CIS = 90^\circ - \beta - \gamma - x = \alpha - x = \angle TAI,$$

which implies that $TI = TA$. Therefore, since $\angle XIA = 90^\circ$, the point T is the midpoint of AX, i.e., $T = W$.

To complete our solution, it remains to show that the intersection point of the lines IS and BC coincide with the midpoint of the segment BC. But since S is the midpoint of the segment VC, it suffices to show that the lines BV and IS are parallel.

Since the quadrilateral $BYIV$ is cyclic, $\angle VBI = \angle VYI = \angle YIA = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which yields $\angle VAC = \angle VCA$. Therefore $2\alpha - x = 2\gamma + x$, which gives $\alpha = \gamma + x$. Hence $\angle SCI = \alpha$, so $\angle VSI = 2\alpha$.

On the other hand, $\angle BVC = 180^\circ - \angle BAC = 180^\circ - 2\alpha$, which implies that the lines BV and IS are parallel. This completes the solution.

Solution 2. As in Solution 1, we first prove that the points B, Y, I, V lie on a common circle and $\angle YIA = 90^\circ$. The remaining part of the solution is based on the following lemma, which holds true for any triangle ABC, not necessarily with the property that I, X, Y are collinear.

Lemma. Let ABC be the triangle inscribed in a circle Γ and let I be its incentre. Assume that the line passing through I and perpendicular to the line AI intersects the side AB at the point Y. Let the circumcircle of the triangle BYI intersect the circle Γ for the second time at V, and let the excircle of the triangle ABC opposite to the vertex A be tangent to the side BC at E. Then

$$\angle BAV = \angle CAE\,.$$

Proof. Let ρ be the composition of the inversion with centre A and radius $\sqrt{AB \cdot AC}$, and the symmetry with respect to AI. Clearly, ρ interchanges B and C.

Let J be the excentre of the triangle ABC opposite to A (see Figure 2). Then we have $\angle JAC = \angle BAI$ and $\angle JCA = 90^\circ + \gamma = \angle BIA$, so the triangles ACJ and AIB are similar, and therefore $AB \cdot AC = AI \cdot AJ$. This means that ρ interchanges I and J. Moreover, since Y lies on AB and $\angle AIY = 90^\circ$, the point $Y' = \rho(Y)$ lies on AC, and $\angle JY'A = 90^\circ$. Thus ρ maps the circumcircle γ of the triangle BYI to a circle γ' with diameter JC.

Finally, since V lies on both Γ and γ, the point $V' = \rho(V)$ lies on the line $\rho(\Gamma) = AB$ as well as on γ', which in turn means that $V' = E$. This implies the desired result. $\square$

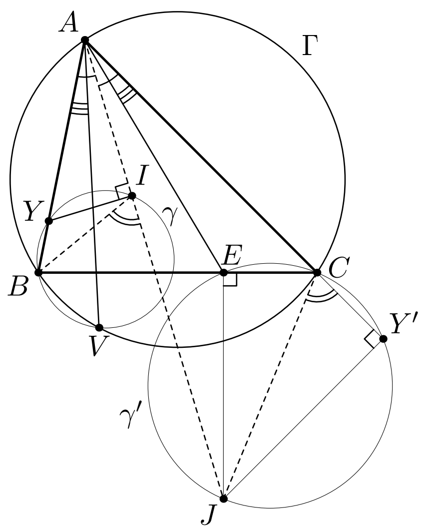


Figure 2

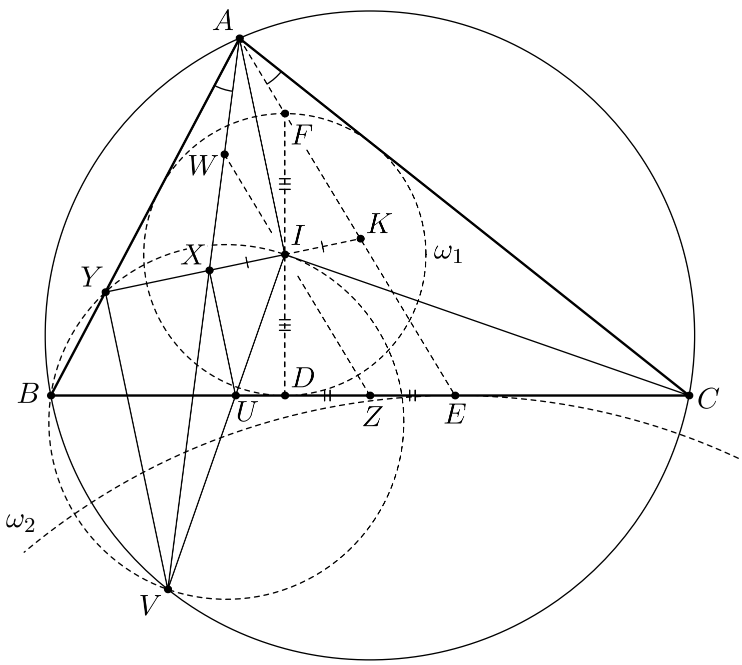


Figure 3

Now we turn to the solution of the problem.

Assume that the incircle ω_1 of the triangle ABC is tangent to BC at D, and let the excircle ω_2 of the triangle ABC opposite to the vertex A touch the side BC at E (see Figure 3). The homothety with centre A that takes ω_2 to ω_1 takes the point E to some point F, and the tangent to ω_1 at F is parallel to BC. Therefore DF is a diameter of ω_1. Moreover, Z is the midpoint of DE. This implies that the lines IZ and FE are parallel.

Let $K = YI \cap AE$. Since $\angle YIA = 90^\circ$, the lemma yields that I is the midpoint of XK. This implies that the segments IW and AK are parallel. Therefore, the points W, I and Z are collinear.

Comment 1. The properties $\angle YIA = 90^\circ$ and $VA = VC$ can be established in various ways. The main difficulty of the problem seems to find out how to use these properties in connection to the points W and Z.

In Solution 2 this principal part is more or less covered by the lemma, for which we have presented a direct proof. On the other hand, this lemma appears to be a combination of two well-known facts; let us formulate them in terms of the lemma statement.

Let the line IY intersect AC at P (see Figure 4). The first fact states that the circumcircle ω of the triangle VYP is tangent to the segments AB and AC, as well as to the circle Γ. The second fact states that for such a circle, the angles BAV and CAE are equal.

The awareness of this lemma may help a lot in solving this problem; so the Jury might also consider a variation of the proposed problem, for which the lemma does not seem to be useful; see Comment 3.

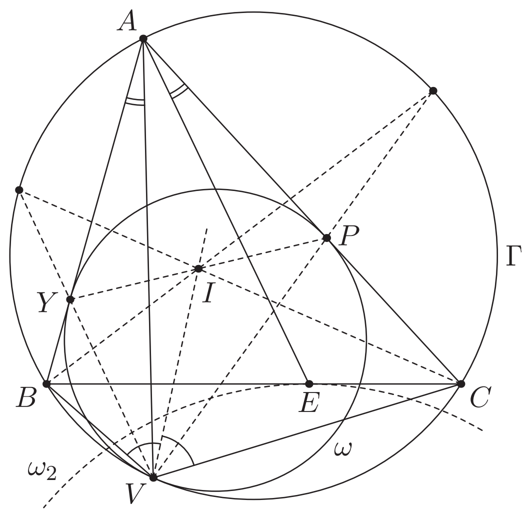


Figure 4

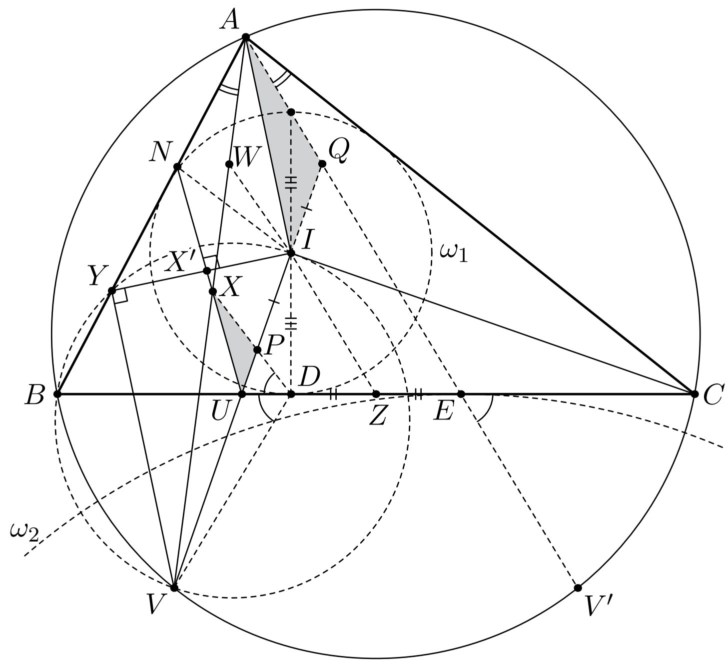


Figure 5

Comment 2. The proposed problem stated the equivalence: the point I lies on the line XY if and only if I lies on the line WZ. Here we sketch the proof of the "if" part (see Figure 5).

As in Solution 2, let BC touch the circles ω_1 and ω_2 at D and E, respectively. Since $IZ \parallel AE$ and W lies on IZ, the line DX is also parallel to AE. Therefore, the triangles XUP and AIQ are similar. Moreover, the line DX is symmetric to AE with respect to I, so $IP = IQ$, where $P = UV \cap XD$ and $Q = UV \cap AE$. Thus we obtain

$$\frac{UV}{VI} = \frac{UX}{IA} = \frac{UP}{IQ} = \frac{UP}{IP}.$$

So the pairs IU and PV are harmonic conjugates, and since $\angle UDI = 90^\circ$, we get $\angle VDB = \angle BDX = \angle BEA$. Therefore the point V' symmetric to V with respect to the perpendicular bisector of BC lies on the line AE. So we obtain $\angle BAV = \angle CAE$.

The rest can be obtained by simply reversing the arguments in Solution 2. The points B, V, I, and Y are concyclic. The lemma implies that $\angle YIA = 90^\circ$. Moreover, the points B, U, I, and N, where $N = UX \cap AB$, lie on a common circle, so $IN = IU$. Since $IY \perp UN$, the point $X' = IY \cap UN$ is the midpoint of UN. But in the trapezoid $AYVI$, the line XU is parallel to the sides AI and YV, so $NX = UX'$. This yields $X = X'$.

The reasoning presented in Solution 1 can also be reversed, but it requires a lot of technicalities. Therefore the Problem Selection Committee proposes to consider only the "only if" part of the original proposal, which is still challenging enough.

Comment 3. The Jury might also consider the following variation of the proposed problem.

Let ABC be a triangle with circumcircle Ω and incentre I. Let the line through I perpendicular to CI intersect the segment BC and the arc BC (not containing A) of Ω at U and V, respectively. Let the line through U parallel to AI intersect AV at X. Prove that if the lines XI and AI are perpendicular, then the midpoint of the segment AC lies on the line XI (see Figure 6).

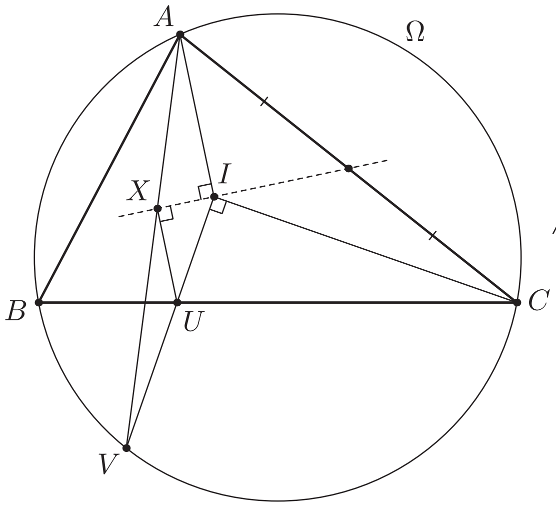


Figure 6

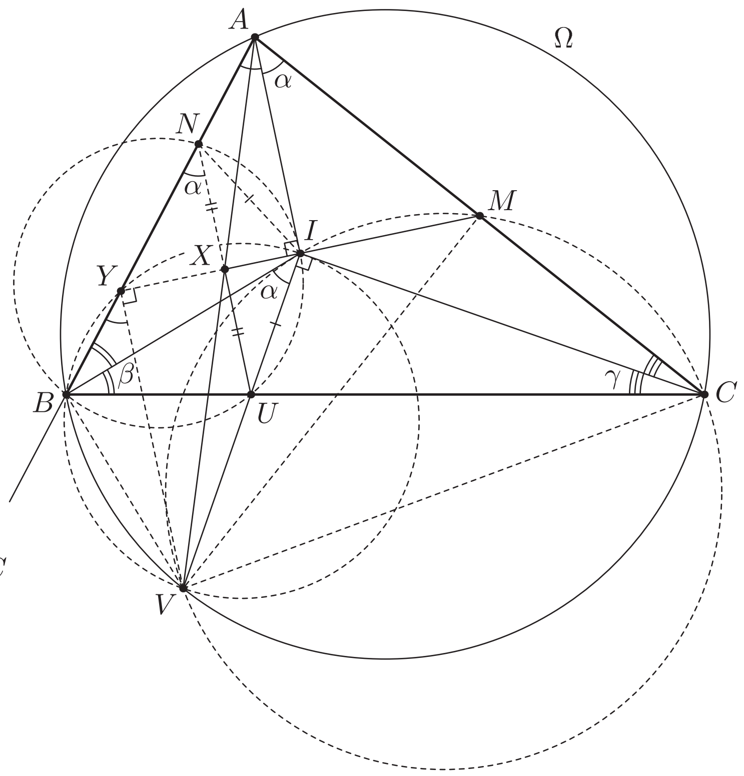


Figure 7

Since the solution contains the arguments used above, we only sketch it.

Let $N = XU \cap AB$ (see Figure 7). Then $\angle BNU = \angle BAI = \angle BIU$, so the points B, U, I, and N lie on a common circle. Therefore $IU = IN$, and since $IX \perp NU$, it follows that $NX = XU$.

Now set $Y = XI \cap AB$. The equality $NX = XU$ implies that

$$\frac{VX}{VA} = \frac{XU}{AI} = \frac{NX}{AI} = \frac{YX}{YI},$$

and therefore $YV \parallel AI$. Hence $\angle BYV = \angle BAI = \angle BIV$, so the points B, V, I, Y are concyclic. Next we have $IY \perp YV$, so $\angle IBV = 90^\circ$. This implies that BV is the external angle bisector of the angle ABC, which gives $\angle VAC = \angle VCA$.

So in order to show that $M = XI \cap AC$ is the midpoint of AC, it suffices to prove that $\angle VMC = 90^\circ$. But this follows immediately from the observation that the points V, C, M, and I are concyclic, as $\angle MIV = \angle YBV = 180^\circ - \angle ACV$.

The converse statement is also true, but its proof requires some technicalities as well.

Number Theory

N1. Let $n \geqslant 2$ be an integer, and let A_n be the set

$$A_n = \{2^n - 2^k \mid k \in \mathbb{Z},\ 0 \leqslant k < n\}.$$

Determine the largest positive integer that cannot be written as the sum of one or more (not necessarily distinct) elements of A_n.

(Serbia)

Answer. $(n-2)2^n + 1$.

Solution 1.

Part I. First we show that every integer greater than $(n-2)2^n + 1$ can be represented as such a sum. This is achieved by induction on n.

For $n = 2$, the set A_n consists of the two elements 2 and 3. Every positive integer m except for 1 can be represented as the sum of elements of A_n in this case: as $m = 2 + 2 + \cdots + 2$ if m is even, and as $m = 3 + 2 + 2 + \cdots + 2$ if m is odd.

Now consider some $n > 2$, and take an integer $m > (n-2)2^n + 1$. If m is even, then consider

$$\frac{m}{2} \geqslant \frac{(n-2)2^n + 2}{2} = (n-2)2^{n-1} + 1 > (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}),$$

giving us the desired representation as a sum of elements of A_n. If m is odd, we consider

$$\frac{m - (2^n - 1)}{2} > \frac{(n-2)2^n + 1 - (2^n - 1)}{2} = (n-3)2^{n-1} + 1.$$

By the induction hypothesis, there is a representation of the form

$$\frac{m - (2^n - 1)}{2} = (2^{n-1} - 2^{k_1}) + (2^{n-1} - 2^{k_2}) + \cdots + (2^{n-1} - 2^{k_r})$$

for some k_i with $0 \leqslant k_i < n-1$. It follows that

$$m = (2^n - 2^{k_1+1}) + (2^n - 2^{k_2+1}) + \cdots + (2^n - 2^{k_r+1}) + (2^n - 1),$$

giving us the desired representation of m once again.

Part II. It remains to show that there is no representation for $(n-2)2^n + 1$. Let N be the smallest positive integer that satisfies $N \equiv 1 \pmod{2^n}$, and which can be represented as a sum of elements of A_n. Consider a representation of N, i.e.,

$$N = (2^n - 2^{k_1}) + (2^n - 2^{k_2}) + \cdots + (2^n - 2^{k_r}), \tag{1}$$

where $0 \leqslant k_1, k_2, \ldots, k_r < n$. Suppose first that two of the terms in the sum are the same, i.e., $k_i = k_j$ for some $i \neq j$. If $k_i = k_j = n-1$, then we can simply remove these two terms to get a representation for

$$N - 2(2^n - 2^{n-1}) = N - 2^n$$

as a sum of elements of A_n, which contradicts our choice of N. If $k_i = k_j = k < n-1$, replace the two terms by $2^n - 2^{k+1}$, which is also an element of A_n, to get a representation for

$$N - 2(2^n - 2^k) + 2^n - 2^{k+1} = N - 2^n.$$

This is a contradiction once again. Therefore, all k_i have to be distinct, which means that

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \leqslant 2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

On the other hand, taking (1) modulo 2^n, we find

$$2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} \equiv -N \equiv -1 \pmod{2^n}.$$

Thus we must have $2^{k_1} + 2^{k_2} + \cdots + 2^{k_r} = 2^n - 1$, which is only possible if each element of $\{0, 1, \ldots, n-1\}$ occurs as one of the k_i. This gives us

$$N = n2^n - (2^0 + 2^1 + \cdots + 2^{n-1}) = (n-1)2^n + 1.$$

In particular, this means that $(n-2)2^n + 1$ cannot be represented as a sum of elements of A_n.

Solution 2. The fact that $m = (n-2)2^n + 1$ cannot be represented as a sum of elements of A_n can also be shown in other ways. We prove the following statement by induction on n:

Claim. If a, b are integers with $a \geqslant 0$, $b \geqslant 1$, and $a + b < n$, then $a2^n + b$ cannot be written as a sum of elements of A_n.

Proof. The claim is clearly true for $n = 2$ (since $a = 0$, $b = 1$ is the only possibility). For $n > 2$, assume that there exist integers a, b with $a \geqslant 0$, $b \geqslant 1$ and $a + b < n$ as well as elements $m_1, m_2, \ldots, m_r$ of A_n such that

$$a2^n + b = m_1 + m_2 + \cdots + m_r.$$

We can suppose, without loss of generality, that $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_r$. Let ℓ be the largest index for which $m_\ell = 2^n - 1$ ($\ell = 0$ if $m_1 \neq 2^n - 1$). Clearly, ℓ and b must have the same parity. Now

$$(a - \ell)2^n + (b + \ell) = m_{\ell+1} + m_{\ell+2} + \cdots + m_r$$

and thus

$$(a - \ell)2^{n-1} + \frac{b + \ell}{2} = \frac{m_{\ell+1}}{2} + \frac{m_{\ell+2}}{2} + \cdots + \frac{m_r}{2}.$$

Note that $m_{\ell+1}/2, m_{\ell+2}/2, \ldots, m_r/2$ are elements of A_{n-1}. Moreover, $a - \ell$ and $(b + \ell)/2$ are integers, and $(b + \ell)/2 \geqslant 1$. If $a - \ell$ was negative, then we would have

$$a2^n + b \geqslant \ell(2^n - 1) \geqslant (a + 1)(2^n - 1) = a2^n + 2^n - a - 1,$$

thus $n \geqslant a + b + 1 \geqslant 2^n$, which is impossible. So $a - \ell \geqslant 0$. By the induction hypothesis, we must have $a - \ell + \frac{b+\ell}{2} \geqslant n - 1$, which gives us a contradiction, since

$$a - \ell + \frac{b + \ell}{2} \leqslant a - \ell + b + \ell - 1 = a + b - 1 < n - 1. \qquad \square$$

Considering the special case $a = n - 2$, $b = 1$ now completes the proof.

Solution 3. Denote by B_n the set of all positive integers that can be written as a sum of elements of A_n. In this solution, we explicitly describe all the numbers in B_n by an argument similar to the first solution.

For a positive integer n, we denote by $\sigma_2(n)$ the sum of its digits in the binary representation. Notice that every positive integer m has a unique representation of the form $m = s2^n - t$ with some positive integer s and $0 \leqslant t \leqslant 2^n - 1$.

Lemma. For any two integers $s \geqslant 1$ and $0 \leqslant t \leqslant 2^n - 1$, the number $m = s2^n - t$ belongs to B_n if and only if $s \geqslant \sigma_2(t)$.

Proof. For $t = 0$, the statement of the Lemma is obvious, since $m = 2s \cdot (2^n - 2^{n-1})$.

Now suppose that $t \geqslant 1$, and let

$$t = 2^{k_1} + \cdots + 2^{k_\sigma} \qquad (0 \leqslant k_1 < \cdots < k_\sigma \leqslant n-1, \quad \sigma = \sigma_2(t))$$

be its binary expansion. If $s \geqslant \sigma$, then $m \in B_n$ since

$$m = (s-\sigma)2^n + (\sigma 2^n - t) = 2(s-\sigma)\cdot(2^n - 2^{n-1}) + \sum_{i=1}^{\sigma}(2^n - 2^{k_i}).$$

Assume now that there exist integers s and t with $1 \leqslant s < \sigma_2(t)$ and $0 \leqslant t \leqslant 2^n - 1$ such that the number $m = s2^n - t$ belongs to B_n. Among all such instances, choose the one for which m is smallest, and let

$$m = \sum_{i=1}^{d}(2^n - 2^{\ell_i}) \qquad (0 \leqslant \ell_i \leqslant n-1)$$

be the corresponding representation. If all the ℓ_i's are distinct, then $\sum_{i=1}^{d} 2^{\ell_i} \leqslant \sum_{j=0}^{n-1} 2^j = 2^n - 1$, so one has $s = d$ and $t = \sum_{i=1}^{d} 2^{\ell_i}$, whence $s = d = \sigma_2(t)$; this is impossible. Therefore, two of the ℓ_i's must be equal, say $\ell_{d-1} = \ell_d$. Then $m \geqslant 2(2^n - 2^{\ell_d}) \geqslant 2^n$, so $s \geqslant 2$.

Now we claim that the number $m' = m - 2^n = (s-1)2^n - t$ also belongs to B_n, which contradicts the minimality assumption. Indeed, one has

$$(2^n - 2^{\ell_{d-1}}) + (2^n - 2^{\ell_d}) = 2(2^n - 2^{\ell_d}) = 2^n + (2^n - 2^{\ell_d+1}),$$

so

$$m' = \sum_{i=1}^{d-2}(2^n - 2^{\ell_i}) + (2^n - 2^{\ell_d+1})$$

is the desired representation of m' (if $\ell_d = n-1$, then the last summand is simply omitted). This contradiction finishes the proof. □

By our lemma, the largest number M which does not belong to B_n must have the form

$$m_t = (\sigma_2(t) - 1)2^n - t$$

for some t with $1 \leqslant t \leqslant 2^n - 1$, so M is just the largest of these numbers. For $t_0 = 2^n - 1$ we have $m_{t_0} = (n-1)2^n - (2^n - 1) = (n-2)2^n + 1$; for every other value of t one has $\sigma_2(t) \leqslant n-1$, thus $m_t \leqslant (\sigma(t) - 1)2^n \leqslant (n-2)2^n < m_{t_0}$. This means that $M = m_{t_0} = (n-2)2^n + 1$.

N2. Determine all pairs (x, y) of positive integers such that

$$\sqrt[3]{7x^2 - 13xy + 7y^2} = |x - y| + 1\,. \tag{1}$$

(U.S.A.)

Answer. Either $(x, y) = (1, 1)$ or $\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\}$ for some positive integer $m \geqslant 2$.

Solution. Let (x, y) be any pair of positive integers solving (1). We shall prove that it appears in the list displayed above. The converse assertion that all these pairs do actually satisfy (1) either may be checked directly by means of a somewhat laborious calculation, or it can be seen by going in reverse order through the displayed equations that follow.

In case $x = y$ the given equation reduces to $x^{2/3} = 1$, which is equivalent to $x = 1$, whereby he have found the first solution.

To find the solutions with $x \neq y$ we may assume $x > y$ due to symmetry. Then the integer $n = x - y$ is positive and (1) may be rewritten as

$$\sqrt[3]{7(y + n)^2 - 13(y + n)y + 7y^2} = n + 1\,.$$

Raising this to the third power and simplifying the result one obtains

$$y^2 + yn = n^3 - 4n^2 + 3n + 1\,.$$

To complete the square on the left hand side, we multiply by 4 and add n^2, thus getting

$$(2y + n)^2 = 4n^3 - 15n^2 + 12n + 4 = (n - 2)^2(4n + 1)\,.$$

This shows that the cases $n = 1$ and $n = 2$ are impossible, whence $n > 2$, and $4n + 1$ is the square of the rational number $\frac{2y+n}{n-2}$. Consequently, it has to be a perfect square, and, since it is odd as well, there has to exist some nonnegative integer m such that $4n + 1 = (2m + 1)^2$, i.e.

$$n = m^2 + m\,.$$

Notice that $n > 2$ entails $m \geqslant 2$. Substituting the value of n just found into the previous displayed equation we arrive at

$$(2y + m^2 + m)^2 = (m^2 + m - 2)^2(2m + 1)^2 = (2m^3 + 3m^2 - 3m - 2)^2\,.$$

Extracting square roots and taking $2m^3 + 3m^2 - 3m - 2 = (m - 1)(2m^2 + 5m + 2) > 0$ into account we derive $2y + m^2 + m = 2m^3 + 3m^2 - 3m - 2$, which in turn yields

$$y = m^3 + m^2 - 2m - 1\,.$$

Notice that $m \geqslant 2$ implies that $y = (m^3 - 1) + (m - 2)m$ is indeed positive, as it should be. In view of $x = y + n = y + m^2 + m$ it also follows that

$$x = m^3 + 2m^2 - m - 1\,,$$

and that this integer is positive as well.

Comment. Alternatively one could ask to find all pairs (x, y) of – not necessarily positive – integers solving (1). The answer to that question is a bit nicer than the answer above: the set of solutions are now described by

$$\{x, y\} = \{m^3 + m^2 - 2m - 1, m^3 + 2m^2 - m - 1\}\,,$$

where m varies through $\mathbb{Z}$. This may be shown using essentially the same arguments as above. We finally observe that the pair $(x, y) = (1, 1)$, that appears to be sporadic above, corresponds to $m = -1$.

N3. A coin is called a *Cape Town coin* if its value is $1/n$ for some positive integer n. Given a collection of Cape Town coins of total value at most $99 + \frac{1}{2}$, prove that it is possible to split this collection into at most 100 groups each of total value at most 1.

(Luxembourg)

Solution. We will show that for every positive integer N any collection of Cape Town coins of total value at most $N - \frac{1}{2}$ can be split into N groups each of total value at most 1. The problem statement is a particular case for $N = 100$.

We start with some preparations. If several given coins together have a total value also of the form $\frac{1}{k}$ for a positive integer k, then we may merge them into one new coin. Clearly, if the resulting collection can be split in the required way then the initial collection can also be split.

After each such merging, the total number of coins decreases, thus at some moment we come to a situation when no more merging is possible. At this moment, for every even k there is at most one coin of value $\frac{1}{k}$ (otherwise two such coins may be merged), and for every odd $k > 1$ there are at most $k - 1$ coins of value $\frac{1}{k}$ (otherwise k such coins may also be merged).

Now, clearly, each coin of value 1 should form a single group; if there are d such coins then we may remove them from the collection and replace N by $N - d$. So from now on we may assume that there are no coins of value 1.

Finally, we may split all the coins in the following way. For each $k = 1, 2, \ldots, N$ we put all the coins of values $\frac{1}{2k-1}$ and $\frac{1}{2k}$ into a group G_k; the total value of G_k does not exceed

$$(2k-2) \cdot \frac{1}{2k-1} + \frac{1}{2k} < 1.$$

It remains to distribute the "small" coins of values which are less than $\frac{1}{2N}$; we will add them one by one. In each step, take any remaining small coin. The total value of coins in the groups at this moment is at most $N - \frac{1}{2}$, so there exists a group of total value at most $\frac{1}{N}\left(N - \frac{1}{2}\right) = 1 - \frac{1}{2N}$; thus it is possible to put our small coin into this group. Acting so, we will finally distribute all the coins.

Comment 1. The algorithm may be modified, at least the step where one distributes the coins of values $\geqslant \frac{1}{2N}$. One different way is to put into G_k all the coins of values $\frac{1}{(2k-1)2^s}$ for all integer $s \geqslant 0$. One may easily see that their total value also does not exceed 1.

Comment 2. The original proposal also contained another part, suggesting to show that a required splitting may be impossible if the total value of coins is at most 100. There are many examples of such a collection, e.g. one may take 98 coins of value 1, one coin of value $\frac{1}{2}$, two coins of value $\frac{1}{3}$, and four coins of value $\frac{1}{5}$.

The Problem Selection Committee thinks that this part is less suitable for the competition.

N4. Let $n > 1$ be a given integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geqslant 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd. (For a real number x, $\lfloor x \rfloor$ denotes the largest integer not exceeding x.)

(Hong Kong)

Solution 1. If n is odd, let $k = n^m$ for $m = 1, 2, \ldots$. Then $a_k = n^{n^m - m}$, which is odd for each m.

Henceforth, assume that n is even, say $n = 2t$ for some integer $t \geqslant 1$. Then, for any $m \geqslant 2$, the integer $n^{2^m} - 2^m = 2^m(2^{2^m - m} \cdot t^{2^m} - 1)$ has an odd prime divisor p, since $2^m - m > 1$. Then, for $k = p \cdot 2^m$, we have

$$n^k = (n^{2^m})^p \equiv (2^m)^p = (2^p)^m \equiv 2^m,$$

where the congruences are taken modulo p (recall that $2^p \equiv 2 \pmod{p}$, by Fermat's little theorem). Also, from $n^k - 2^m < n^k < n^k + 2^m(p-1)$, we see that the fraction $\dfrac{n^k}{k}$ lies strictly between the consecutive integers $\dfrac{n^k - 2^m}{p \cdot 2^m}$ and $\dfrac{n^k + 2^m(p-1)}{p \cdot 2^m}$, which gives

$$\left\lfloor \frac{n^k}{k} \right\rfloor = \frac{n^k - 2^m}{p \cdot 2^m}.$$

We finally observe that $\dfrac{n^k - 2^m}{p \cdot 2^m} = \dfrac{\frac{n^k}{2^m} - 1}{p}$ is an odd integer, since the integer $\dfrac{n^k}{2^m} - 1$ is odd (recall that $k > m$). Note that for different values of m, we get different values of k, due to the different powers of 2 in the prime factorisation of k.

Solution 2. Treat the (trivial) case when n is odd as in Solution 1.

Now assume that n is even and $n > 2$. Let p be a prime divisor of $n - 1$.

Proceed by induction on i to prove that p^{i+1} is a divisor of $n^{p^i} - 1$ for every $i \geqslant 0$. The case $i = 0$ is true by the way in which p is chosen. Suppose the result is true for some $i \geqslant 0$. The factorisation

$$n^{p^{i+1}} - 1 = (n^{p^i} - 1)[n^{p^i(p-1)} + n^{p^i(p-2)} + \cdots + n^{p^i} + 1],$$

together with the fact that each of the p terms between the square brackets is congruent to 1 modulo p, implies that the result is also true for $i + 1$.

Hence $\left\lfloor \dfrac{n^{p^i}}{p^i} \right\rfloor = \dfrac{n^{p^i} - 1}{p^i}$, an odd integer for each $i \geqslant 1$.

Finally, we consider the case $n = 2$. We observe that $3 \cdot 4^i$ is a divisor of $2^{3 \cdot 4^i} - 4^i$ for every $i \geqslant 1$: Trivially, 4^i is a divisor of $2^{3 \cdot 4^i} - 4^i$, since $3 \cdot 4^i > 2i$. Furthermore, since $2^{3 \cdot 4^i}$ and 4^i are both congruent to 1 modulo 3, we have $3 \mid 2^{3 \cdot 4^i} - 4^i$. Hence, $\left\lfloor \dfrac{2^{3 \cdot 4^i}}{3 \cdot 4^i} \right\rfloor = \dfrac{2^{3 \cdot 4^i} - 4^i}{3 \cdot 4^i} = \dfrac{2^{3 \cdot 4^i - 2i} - 1}{3}$, which is odd for every $i \geqslant 1$.

Comment. The case n even and $n > 2$ can also be solved by recursively defining the sequence $(k_i)_{i\geqslant 1}$ by $k_1 = 1$ and $k_{i+1} = n^{k_i} - 1$ for $i \geqslant 1$. Then (k_i) is strictly increasing and it follows (by induction on i) that $k_i \mid n^{k_i} - 1$ for all $i \geqslant 1$, so the k_i are as desired.

The case $n = 2$ can also be solved as follows: Let $i \geqslant 2$. By Bertrand's postulate, there exists a prime number p such that $2^{2^i - 1} < p \cdot 2^i < 2^{2^i}$. This gives

$$p \cdot 2^i < 2^{2^i} < 2p \cdot 2^i. \tag{1}$$

Also, we have that $p \cdot 2^i$ is a divisor of $2^{p\cdot 2^i} - 2^{2^i}$, hence, using (1), we get that

$$\left\lfloor \frac{2^{p\cdot 2^i}}{p \cdot 2^i} \right\rfloor = \frac{2^{p\cdot 2^i} - 2^{2^i} + p \cdot 2^i}{p \cdot 2^i} = \frac{2^{p\cdot 2^i - i} - 2^{2^i - i} + p}{p},$$

which is an odd integer.

Solution 3. Treat the (trivial) case when n is odd as in Solution 1.

Let n be even, and let p be a prime divisor of $n+1$. Define the sequence $(a_i)_{i\geqslant 1}$ by

$$a_i = \min\left\{a \in \mathbb{Z}_{>0} \colon 2^i \text{ divides } ap + 1\right\}.$$

Recall that there exists a with $1 \leqslant a < 2^i$ such that $ap \equiv -1 \pmod{2^i}$, so each a_i satisfies $1 \leqslant a_i < 2^i$. This implies that $a_i p + 1 < p \cdot 2^i$. Also, $a_i \to \infty$ as $i \to \infty$, whence there are infinitely many i such that $a_i < a_{i+1}$. From now on, we restrict ourselves only to these i.

Notice that p is a divisor of $n^p + 1$, which, in turn, divides $n^{p\cdot 2^i} - 1$. It follows that $p \cdot 2^i$ is a divisor of $n^{p\cdot 2^i} - (a_i p + 1)$, and we consequently see that the integer $\left\lfloor \dfrac{n^{p\cdot 2^i}}{p \cdot 2^i} \right\rfloor = \dfrac{n^{p\cdot 2^i} - (a_i p + 1)}{p \cdot 2^i}$ is odd, since 2^{i+1} divides $n^{p\cdot 2^i}$, but not $a_i p + 1$.

N5. Find all triples (p, x, y) consisting of a prime number p and two positive integers x and y such that $x^{p-1} + y$ and $x + y^{p-1}$ are both powers of p.

(Belgium)

Answer. $(p, x, y) \in \{(3, 2, 5), (3, 5, 2)\} \cup \{(2, n, 2^k - n) \mid 0 < n < 2^k\}$.

Solution 1. For $p = 2$, clearly all pairs of two positive integers x and y whose sum is a power of 2 satisfy the condition. Thus we assume in the following that $p > 2$, and we let a and b be positive integers such that $x^{p-1} + y = p^a$ and $x + y^{p-1} = p^b$. Assume further, without loss of generality, that $x \leqslant y$, so that $p^a = x^{p-1} + y \leqslant x + y^{p-1} = p^b$, which means that $a \leqslant b$ (and thus $p^a \mid p^b$).

Now we have

$$p^b = y^{p-1} + x = (p^a - x^{p-1})^{p-1} + x.$$

We take this equation modulo p^a and take into account that $p - 1$ is even, which gives us

$$0 \equiv x^{(p-1)^2} + x \pmod{p^a}.$$

If $p \mid x$, then $p^a \mid x$, since $x^{(p-1)^2-1} + 1$ is not divisible by p in this case. However, this is impossible, since $x \leqslant x^{p-1} < p^a$. Thus we know that $p \nmid x$, which means that

$$p^a \mid x^{(p-1)^2-1} + 1 = x^{p(p-2)} + 1.$$

By FERMAT's little theorem, $x^{(p-1)^2} \equiv 1 \pmod{p}$, thus p divides $x+1$. Let p^r be the highest power of p that divides $x + 1$. By the binomial theorem, we have

$$x^{p(p-2)} = \sum_{k=0}^{p(p-2)} \binom{p(p-2)}{k} (-1)^{p(p-2)-k}(x + 1)^k.$$

Except for the terms corresponding to $k = 0$, $k = 1$ and $k = 2$, all terms in the sum are clearly divisible by p^{3r} and thus by p^{r+2}. The remaining terms are

$$-\frac{p(p-2)(p^2 - 2p - 1)}{2}(x + 1)^2,$$

which is divisible by p^{2r+1} and thus also by p^{r+2},

$$p(p-2)(x + 1),$$

which is divisible by p^{r+1}, but not p^{r+2} by our choice of r, and the final term -1 corresponding to $k = 0$. It follows that the highest power of p that divides $x^{p(p-2)} + 1$ is p^{r+1}.

On the other hand, we already know that p^a divides $x^{p(p-2)}+1$, which means that $a \leqslant r+1$. Moreover,

$$p^r \leqslant x + 1 \leqslant x^{p-1} + y = p^a.$$

Hence we either have $a = r$ or $a = r + 1$.

If $a = r$, then $x = y = 1$ needs to hold in the inequality above, which is impossible for $p > 2$. Thus $a = r + 1$. Now since $p^r \leqslant x + 1$, we get

$$x = \frac{x^2 + x}{x + 1} \leqslant \frac{x^{p-1} + y}{x + 1} = \frac{p^a}{x + 1} \leqslant \frac{p^a}{p^r} = p,$$

so we must have $x = p - 1$ for p to divide $x + 1$.

It follows that $r = 1$ and $a = 2$. If $p \geqslant 5$, we obtain

$$p^a = x^{p-1} + y > (p - 1)^4 = (p^2 - 2p + 1)^2 > (3p)^2 > p^2 = p^a,$$

a contradiction. So the only case that remains is $p = 3$, and indeed $x = 2$ and $y = p^a - x^{p-1} = 5$ satisfy the conditions.

Comment 1. In this solution, we are implicitly using a special case of the following lemma known as "lifting the exponent":

Lemma. Let n be a positive integer, let p be an odd prime, and let $v_p(m)$ denote the exponent of the highest power of p that divides m.

If x and y are integers not divisible by p such that $p \mid x - y$, then we have

$$v_p(x^n - y^n) = v_p(x - y) + v_p(n).$$

Likewise, if x and y are integers not divisible by p such that $p \mid x + y$, then we have

$$v_p(x^n + y^n) = v_p(x + y) + v_p(n).$$

Comment 2. There exist various ways of solving the problem involving the "lifting the exponent" lemma. Let us sketch another one.

The cases $x = y$ and $p \mid x$ are ruled out easily, so we assume that $p > 2$, $x < y$, and $p \nmid x$. In this case we also have $p^a < p^b$ and $p \mid x + 1$.

Now one has

$$y^p - x^p \equiv y(y^{p-1} + x) - x(x^{p-1} + y) \equiv 0 \pmod{p^a},$$

so by the lemma mentioned above one has $p^{a-1} \mid y - x$ and hence $y = x + tp^{a-1}$ for some positive integer t. Thus one gets

$$x(x^{p-2} + 1) = x^{p-1} + x = (x^{p-1} + y) - (y - x) = p^{a-1}(p - t).$$

The factors on the left-hand side are coprime. So if $p \mid x$, then $x^{p-2} + 1 \mid p - t$, which is impossible since $x < x^{p-2} + 1$. Therefore, $p \nmid x$, and thus $x \mid p - t$. Since $p \mid x + 1$, the only remaining case is $x = p - 1$, $t = 1$, and $y = p^{a-1} + p - 1$. Now the solution can be completed in the same way as before.

Solution 2. Again, we can focus on the case that $p > 2$. If $p \mid x$, then also $p \mid y$. In this case, let p^k and p^ℓ be the highest powers of p that divide x and y respectively, and assume without loss of generality that $k \leqslant \ell$. Then p^k divides $x + y^{p-1}$ while p^{k+1} does not, but $p^k < x + y^{p-1}$, which yields a contradiction. So x and y are not divisible by p. Fermat's little theorem yields $0 \equiv x^{p-1} + y \equiv 1 + y \pmod{p}$, so $y \equiv -1 \pmod{p}$ and for the same reason $x \equiv -1 \pmod{p}$.

In particular, $x, y \geqslant p - 1$ and thus $x^{p-1} + y \geqslant 2(p-1) > p$, so $x^{p-1} + y$ and $y^{p-1} + x$ are both at least equal to p^2. Now we have

$$x^{p-1} \equiv -y \pmod{p^2} \qquad \text{and} \qquad y^{p-1} \equiv -x \pmod{p^2}.$$

These two congruences, together with the Euler–Fermat theorem, give us

$$1 \equiv x^{p(p-1)} \equiv (-y)^p \equiv -y^p \equiv xy \pmod{p^2}.$$

Since $x \equiv y \equiv -1 \pmod{p}$, $x - y$ is divisible by p, so $(x - y)^2$ is divisible by p^2. This means that

$$(x + y)^2 = (x - y)^2 + 4xy \equiv 4 \pmod{p^2},$$

so p^2 divides $(x + y - 2)(x + y + 2)$. We already know that $x + y \equiv -2 \pmod{p}$, so $x + y - 2 \equiv -4 \not\equiv 0 \pmod{p}$. This means that p^2 divides $x + y + 2$.

Using the same notation as in the first solution, we subtract the two original equations to obtain

$$p^b - p^a = y^{p-1} - x^{p-1} + x - y = (y - x)(y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1). \tag{1}$$

The second factor is symmetric in x and y, so it can be written as a polynomial of the elementary symmetric polynomials $x + y$ and xy with integer coefficients. In particular, its value modulo

p^2 is characterised by the two congruences $xy \equiv 1 \pmod{p^2}$ and $x + y \equiv -2 \pmod{p^2}$. Since both congruences are satisfied when $x = y = -1$, we must have

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv (-1)^{p-2} + (-1)^{p-3}(-1) + \cdots + (-1)^{p-2} - 1 \pmod{p^2},$$

which simplifies to $y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv -p \pmod{p^2}$. Thus the second factor in (1) is divisible by p, but not p^2.

This means that p^{a-1} has to divide the other factor $y - x$. It follows that

$$0 \equiv x^{p-1} + y \equiv x^{p-1} + x \equiv x(x+1)(x^{p-3} - x^{p-4} + \cdots + 1) \pmod{p^{a-1}}.$$

Since $x \equiv -1 \pmod{p}$, the last factor is $x^{p-3} - x^{p-4} + \cdots + 1 \equiv p - 2 \pmod{p}$ and in particular not divisible by p. We infer that $p^{a-1} \mid x + 1$ and continue as in the first solution.

Comment. Instead of reasoning by means of elementary symmetric polynomials, it is possible to provide a more direct argument as well. For odd r, $(x+1)^2$ divides $(x^r + 1)^2$, and since p divides $x + 1$, we deduce that p^2 divides $(x^r + 1)^2$. Together with the fact that $xy \equiv 1 \pmod{p^2}$, we obtain

$$0 \equiv y^r(x^r + 1)^2 \equiv x^{2r}y^r + 2x^r y^r + y^r \equiv x^r + 2 + y^r \pmod{p^2}.$$

We apply this congruence with $r = p - 2 - 2k$ (where $0 \leqslant k < (p-2)/2$) to find that

$$x^k y^{p-2-k} + x^{p-2-k}y^k \equiv (xy)^k(x^{p-2-2k} + y^{p-2-2k}) \equiv 1^k \cdot (-2) \equiv -2 \pmod{p^2}.$$

Summing over all k yields

$$y^{p-2} + y^{p-3}x + \cdots + x^{p-2} - 1 \equiv \tfrac{p-1}{2} \cdot (-2) - 1 \equiv -p \pmod{p^2}$$

once again.

N6. Let $a_1 < a_2 < \cdots < a_n$ be pairwise coprime positive integers with a_1 being prime and $a_1 \geqslant n+2$. On the segment $I = [0, a_1 a_2 \cdots a_n]$ of the real line, mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$. These points split I into a number of smaller segments. Prove that the sum of the squares of the lengths of these segments is divisible by a_1.

(Serbia)

Solution 1. Let $A = a_1 \cdots a_n$. Throughout the solution, all intervals will be nonempty and have integer end-points. For any interval X, the length of X will be denoted by $|X|$.

Define the following two families of intervals:

$$\mathcal{S} = \big\{[x,y]\colon x<y \text{ are consecutive marked points}\big\}$$
$$\mathcal{T} = \big\{[x,y]\colon x<y \text{ are integers, } 0 \leqslant x \leqslant A-1, \text{ and no point is marked in } (x,y)\big\}$$

We are interested in computing $\sum\limits_{X\in\mathcal{S}} |X|^2$ modulo a_1.

Note that the number A is marked, so in the definition of $\mathcal{T}$ the condition $y \leqslant A$ is enforced without explicitly prescribing it.

Assign weights to the intervals in $\mathcal{T}$, depending only on their lengths. The weight of an arbitrary interval $Y \in \mathcal{T}$ will be $w\big(|Y|\big)$, where

$$w(k) = \begin{cases} 1 & \text{if } k=1, \\ 2 & \text{if } k \geqslant 2\,. \end{cases}$$

Consider an arbitrary interval $X \in \mathcal{S}$ and its sub-intervals $Y \in \mathcal{T}$. Clearly, X has one sub-interval of length $|X|$, two sub-intervals of length $|X|-1$ and so on; in general X has $|X|-d+1$ sub-intervals of length d for every $d = 1, 2, \ldots, |X|$. The sum of the weights of the sub-intervals of X is

$$\sum_{Y\in\mathcal{T},\, Y\subseteq X} w\big(|Y|\big) = \sum_{d=1}^{|X|} (|X|-d+1)\cdot w(d) = |X|\cdot 1 + \big((|X|-1)+(|X|-2)+\cdots+1\big)\cdot 2 = |X|^2.$$

Since the intervals in $\mathcal{S}$ are non-overlapping, every interval $Y \in \mathcal{T}$ is a sub-interval of a single interval $X \in \mathcal{S}$. Therefore,

$$\sum_{X\in\mathcal{S}} |X|^2 = \sum_{X\in\mathcal{S}} \left(\sum_{Y\in\mathcal{T},\, Y\subseteq X} w\big(|Y|\big) \right) = \sum_{Y\in\mathcal{T}} w\big(|Y|\big). \tag{1}$$

For every $d = 1, 2, \ldots, a_1$, we count how many intervals in $\mathcal{T}$ are of length d. Notice that the multiples of a_1 are all marked, so the lengths of the intervals in $\mathcal{S}$ and $\mathcal{T}$ cannot exceed a_1. Let x be an arbitrary integer with $0 \leqslant x \leqslant A-1$ and consider the interval $[x, x+d]$. Let r_1, $\ldots$, r_n be the remainders of x modulo $a_1, \ldots, a_n$, respectively. Since $a_1, \ldots, a_n$ are pairwise coprime, the number x is uniquely identified by the sequence $(r_1, \ldots, r_n)$, due to the Chinese remainder theorem.

For every $i = 1, \ldots, n$, the property that the interval $(x, x+d)$ does not contain any multiple of a_i is equivalent with $r_i + d \leqslant a_i$, i.e. $r_i \in \{0, 1, \ldots, a_i - d\}$, so there are $a_i - d + 1$ choices for the number r_i for each i. Therefore, the number of the remainder sequences $(r_1, \ldots, r_n)$ that satisfy $[x, x+d] \in \mathcal{T}$ is precisely $(a_1+1-d)\cdots(a_n+1-d)$. Denote this product by $f(d)$.

Now we can group the last sum in (1) by length of the intervals. As we have seen, for every $d = 1, \ldots, a_1$ there are $f(d)$ intervals $Y \in \mathcal{T}$ with $|Y| = d$. Therefore, (1) can be continued as

$$\sum_{X \in \mathcal{S}} |X|^2 = \sum_{Y \in \mathcal{T}} w\big(|Y|\big) = \sum_{d=1}^{a_1} f(d) \cdot w(d) = 2 \sum_{d=1}^{a_1} f(d) - f(1). \tag{2}$$

Having the formula (2), the solution can be finished using the following well-known fact:

Lemma. If p is a prime, $F(x)$ is a polynomial with integer coefficients, and $\deg F \leqslant p-2$, then $\sum\limits_{x=1}^{p} F(x)$ is divisible by p.

Proof. Obviously, it is sufficient to prove the lemma for monomials of the form x^k with $k \leqslant p-2$. Apply induction on k. If $k = 0$ then $F = 1$, and the statement is trivial.

Let $1 \leqslant k \leqslant p-2$, and assume that the lemma is proved for all lower degrees. Then

$$\begin{aligned} 0 \equiv p^{k+1} &= \sum_{x=1}^{p} \big(x^{k+1} - (x-1)^{k+1}\big) = \sum_{x=1}^{p} \left(\sum_{\ell=0}^{k} (-1)^{k-\ell} \binom{k+1}{\ell} x^{\ell} \right) \\ &= (k+1) \sum_{x=1}^{p} x^k + \sum_{\ell=0}^{k-1} (-1)^{k-\ell} \binom{k+1}{\ell} \sum_{x=1}^{p} x^{\ell} \equiv (k+1) \sum_{x=1}^{p} x^k \pmod{p}. \end{aligned}$$

Since $0 < k+1 < p$, this proves $\sum\limits_{x=1}^{p} x^k \equiv 0 \pmod{p}$. $\square$

In (2), by applying the lemma to the polynomial f and the prime a_1, we obtain that $\sum\limits_{d=1}^{a_1} f(d)$ is divisible by a_1. The term $f(1) = a_1 \cdots a_n$ is also divisible by a_1; these two facts together prove that $\sum\limits_{X \in \mathcal{S}} |X|^2$ is divisible by a_1.

Comment 1. With suitable sets of weights, the same method can be used to sum up other expressions on the lengths of the segments. For example, $w(1) = 1$ and $w(k) = 6(k-1)$ for $k \geqslant 2$ can be used to compute $\sum\limits_{X \in \mathcal{S}} |X|^3$ and to prove that this sum is divisible by a_1 if a_1 is a prime with $a_1 \geqslant n+3$. See also Comment 2 after the second solution.

Solution 2. The conventions from the first paragraph of the first solution are still in force. We shall prove the following more general statement:

> ($\boxplus$) *Let p denote a prime number, let $p = a_1 < a_2 < \cdots < a_n$ be n pairwise coprime positive integers, and let d be an integer with $1 \leqslant d \leqslant p-n$. Mark all integers that are divisible by at least one of the numbers $a_1, \ldots, a_n$ on the interval $I = [0, a_1 a_2 \cdots a_n]$ of the real line. These points split I into a number of smaller segments, say of lengths $b_1, \ldots, b_k$. Then the sum $\sum\limits_{i=1}^{k} \binom{b_i}{d}$ is divisible by p.*

Applying ($\boxplus$) to $d = 1$ and $d = 2$ and using the equation $x^2 = 2\binom{x}{2} + \binom{x}{1}$, one easily gets the statement of the problem.

To prove ($\boxplus$) itself, we argue by induction on n. The base case $n = 1$ follows from the known fact that the binomial coefficient $\binom{p}{d}$ is divisible by p whenever $1 \leqslant d \leqslant p-1$.

Let us now assume that $n \geqslant 2$, and that the statement is known whenever $n-1$ rather than n coprime integers are given together with some integer $d \in [1, p-n+1]$. Suppose that the numbers $p = a_1 < a_2 < \cdots < a_n$ and d are as above. Write $A' = \prod_{i=1}^{n-1} a_i$ and $A = A'\, a_n$. Mark the points on the real axis divisible by one of the numbers $a_1, \ldots, a_{n-1}$ *green* and those divisible by a_n *red*. The green points divide $[0, A']$ into certain sub-intervals, say $J_1,\ J_2,\ \ldots,$ and J_ℓ.

To translate intervals we use the notation $[a, b] + m = [a + m, b + m]$ whenever $a, b, m \in \mathbb{Z}$.

For each $i \in \{1, 2, \ldots, \ell\}$ let $\mathcal{F}_i$ be the family of intervals into which the red points partition the intervals $J_i,\ J_i + A',\ \ldots,$ and $J_i + (a_n - 1)A'$. We are to prove that

$$\sum_{i=1}^{\ell} \sum_{X \in \mathcal{F}_i} \binom{|X|}{d}$$

is divisible by p.

Let us fix any index i with $1 \leqslant i \leqslant \ell$ for a while. Since the numbers A' and a_n are coprime by hypothesis, the numbers $0, A', \ldots, (a_n - 1)A'$ form a complete system of residues modulo a_n. Moreover, we have $|J_i| \leqslant p < a_n$, as in particular all multiples of p are green. So each of the intervals $J_i,\ J_i + A',\ \ldots,$ and $J_i + (a_n - 1)A'$ contains at most one red point. More precisely, for each $j \in \{1, \ldots, |J_i| - 1\}$ there is exactly one amongst those intervals containing a red point splitting it into an interval of length j followed by an interval of length $|J_i| - j$, while the remaining $a_n - |J_i| + 1$ such intervals have no red points in their interiors. For these reasons

$$\begin{aligned}
\sum_{X \in \mathcal{F}_i} \binom{|X|}{d} &= 2\left(\binom{1}{d} + \cdots + \binom{|J_i| - 1}{d}\right) + (a_n - |J_i| + 1)\binom{|J_i|}{d} \\
&= 2\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d} - (d + 1)\binom{|J_i|}{d+1} \\
&= (1 - d)\binom{|J_i|}{d+1} + (a_n - d + 1)\binom{|J_i|}{d}.
\end{aligned}$$

So it remains to prove that

$$(1 - d)\sum_{i=1}^{\ell} \binom{|J_i|}{d+1} + (a_n - d + 1)\sum_{i=1}^{\ell} \binom{|J_i|}{d}$$

is divisible by p. By the induction hypothesis, however, it is even true that both summands are divisible by p, for $1 \leqslant d < d + 1 \leqslant p - (n - 1)$. This completes the proof of ($\boxplus$) and hence the solution of the problem.

Comment 2. The statement ($\boxplus$) can also be proved by the method of the first solution, using the weights $w(x) = \binom{x-2}{d-2}$.

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N7. Let $c \geqslant 1$ be an integer. Define a sequence of positive integers by $a_1 = c$ and

$$a_{n+1} = a_n^3 - 4c \cdot a_n^2 + 5c^2 \cdot a_n + c$$

for all $n \geqslant 1$. Prove that for each integer $n \geqslant 2$ there exists a prime number p dividing a_n but none of the numbers $a_1, \ldots, a_{n-1}$.

(Austria)

Solution. Let us define $x_0 = 0$ and $x_n = a_n/c$ for all integers $n \geqslant 1$. It is easy to see that the sequence (x_n) thus obtained obeys the recursive law

$$x_{n+1} = c^2(x_n^3 - 4x_n^2 + 5x_n) + 1 \tag{1}$$

for all integers $n \geqslant 0$. In particular, all of its terms are positive integers; notice that $x_1 = 1$ and $x_2 = 2c^2 + 1$. Since

$$x_{n+1} = c^2 x_n (x_n - 2)^2 + c^2 x_n + 1 > x_n \tag{2}$$

holds for all integers $n \geqslant 0$, it is also strictly increasing. Since x_{n+1} is by (1) coprime to c for any $n \geqslant 0$, it suffices to prove that for each $n \geqslant 2$ there exists a prime number p dividing x_n but none of the numbers $x_1, \ldots, x_{n-1}$. Let us begin by establishing three preliminary claims.

Claim 1. If $i \equiv j \pmod{m}$ holds for some integers $i, j \geqslant 0$ and $m \geqslant 1$, then $x_i \equiv x_j \pmod{x_m}$ holds as well.

Proof. Evidently, it suffices to show $x_{i+m} \equiv x_i \pmod{x_m}$ for all integers $i \geqslant 0$ and $m \geqslant 1$. For this purpose we may argue for fixed m by induction on i using $x_0 = 0$ in the base case $i = 0$. Now, if we have $x_{i+m} \equiv x_i \pmod{x_m}$ for some integer i, then the recursive equation (1) yields

$$x_{i+m+1} \equiv c^2(x_{i+m}^3 - 4x_{i+m}^2 + 5x_{i+m}) + 1 \equiv c^2(x_i^3 - 4x_i^2 + 5x_i) + 1 \equiv x_{i+1} \pmod{x_m},$$

which completes the induction. □

Claim 2. If the integers $i, j \geqslant 2$ and $m \geqslant 1$ satisfy $i \equiv j \pmod{m}$, then $x_i \equiv x_j \pmod{x_m^2}$ holds as well.

Proof. Again it suffices to prove $x_{i+m} \equiv x_i \pmod{x_m^2}$ for all integers $i \geqslant 2$ and $m \geqslant 1$. As above, we proceed for fixed m by induction on i. The induction step is again easy using (1), but this time the base case $i = 2$ requires some calculation. Set $L = 5c^2$. By (1) we have $x_{m+1} \equiv Lx_m + 1 \pmod{x_m^2}$, and hence

$$\begin{aligned} x_{m+1}^3 - 4x_{m+1}^2 + 5x_{m+1} &\equiv (Lx_m + 1)^3 - 4(Lx_m + 1)^2 + 5(Lx_m + 1) \\ &\equiv (3Lx_m + 1) - 4(2Lx_m + 1) + 5(Lx_m + 1) \equiv 2 \pmod{x_m^2}, \end{aligned}$$

which in turn gives indeed $x_{m+2} \equiv 2c^2 + 1 \equiv x_2 \pmod{x_m^2}$. □

Claim 3. For each integer $n \geqslant 2$, we have $x_n > x_1 \cdot x_2 \cdots x_{n-2}$.

Proof. The cases $n = 2$ and $n = 3$ are clear. Arguing inductively, we assume now that the claim holds for some $n \geqslant 3$. Recall that $x_2 \geqslant 3$, so by monotonicity and (2) we get $x_n \geqslant x_3 \geqslant x_2(x_2 - 2)^2 + x_2 + 1 \geqslant 7$. It follows that

$$x_{n+1} > x_n^3 - 4x_n^2 + 5x_n > 7x_n^2 - 4x_n^2 > x_n^2 > x_n x_{n-1},$$

which by the induction hypothesis yields $x_{n+1} > x_1 \cdot x_2 \cdots x_{n-1}$, as desired. □

Now we direct our attention to the problem itself: let any integer $n \geqslant 2$ be given. By Claim 3 there exists a prime number p appearing with a higher exponent in the prime factorisation of x_n than in the prime factorisation of $x_1 \cdots x_{n-2}$. In particular, $p \mid x_n$, and it suffices to prove that p divides none of $x_1, \ldots, x_{n-1}$.

Otherwise let $k \in \{1, \ldots, n-1\}$ be minimal such that p divides x_k. Since x_{n-1} and x_n are coprime by (1) and $x_1 = 1$, we actually have $2 \leqslant k \leqslant n-2$. Write $n = qk + r$ with some integers $q \geqslant 0$ and $0 \leqslant r < k$. By Claim 1 we have $x_n \equiv x_r \pmod{x_k}$, whence $p \mid x_r$. Due to the minimality of k this entails $r = 0$, i.e. $k \mid n$.

Thus from Claim 2 we infer

$$x_n \equiv x_k \pmod{x_k^2}.$$

Now let $\alpha \geqslant 1$ be maximal with the property $p^\alpha \mid x_k$. Then x_k^2 is divisible by $p^{\alpha+1}$ and by our choice of p so is x_n. So by the previous congruence x_k is a multiple of $p^{\alpha+1}$ as well, contrary to our choice of α. This is the final contradiction concluding the solution.

N8. For every real number x, let $\|x\|$ denote the distance between x and the nearest integer. Prove that for every pair (a, b) of positive integers there exist an odd prime p and a positive integer k satisfying

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = 1. \tag{1}$$

(Hungary)

Solution. Notice first that $\left\lfloor x + \frac{1}{2} \right\rfloor$ is an integer nearest to x, so $\|x\| = \left|\left\lfloor x + \frac{1}{2} \right\rfloor - x\right|$. Thus we have

$$\left\lfloor x + \frac{1}{2} \right\rfloor = x \pm \|x\|. \tag{2}$$

For every rational number r and every prime number p, denote by $v_p(r)$ the exponent of p in the prime factorisation of r. Recall the notation $(2n-1)!!$ for the product of all odd positive integers not exceeding $2n-1$, i.e., $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$.

Lemma. For every positive integer n and every odd prime p, we have

$$v_p\big((2n-1)!!\big) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Proof. For every positive integer k, let us count the multiples of p^k among the factors $1, 3, \ldots, 2n-1$. If ℓ is an arbitrary integer, the number $(2\ell - 1)p^k$ is listed above if and only if

$$0 < (2\ell - 1)p^k \leqslant 2n \quad \Longleftrightarrow \quad \frac{1}{2} < \ell \leqslant \frac{n}{p^k} + \frac{1}{2} \quad \Longleftrightarrow \quad 1 \leqslant \ell \leqslant \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor.$$

Hence, the number of multiples of p^k among the factors is precisely $m_k = \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor$. Thus we obtain

$$v_p\big((2n-1)!!\big) = \sum_{i=1}^{n} v_p(2i-1) = \sum_{i=1}^{n} \sum_{k=1}^{v_p(2i-1)} 1 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} 1 = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} + \frac{1}{2} \right\rfloor. \qquad \square$$

In order to prove the problem statement, consider the rational number

$$N = \frac{(2a+2b-1)!!}{(2a-1)!! \cdot (2b-1)!!} = \frac{(2a+1)(2a+3)\cdots(2a+2b-1)}{1 \cdot 3 \cdots (2b-1)}.$$

Obviously, $N > 1$, so there exists a prime p with $v_p(N) > 0$. Since N is a fraction of two odd numbers, p is odd.

By our lemma,

$$0 < v_p(N) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor \right).$$

Therefore, there exists some positive integer k such that the integer number

$$d_k = \left\lfloor \frac{a+b}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{a}{p^k} + \frac{1}{2} \right\rfloor - \left\lfloor \frac{b}{p^k} + \frac{1}{2} \right\rfloor$$

is positive, so $d_k \geqslant 1$. By (2) we have

$$1 \leqslant d_k = \frac{a+b}{p^k} - \frac{a}{p^k} - \frac{b}{p^k} \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\| = \pm \left\|\frac{a+b}{p^k}\right\| \pm \left\|\frac{a}{p^k}\right\| \pm \left\|\frac{b}{p^k}\right\|. \tag{3}$$

Since $\|x\| < \frac{1}{2}$ for every rational x with odd denominator, the relation (3) can only be satisfied if all three signs on the right-hand side are positive and $d_k = 1$. Thus we get

$$\left\|\frac{a}{p^k}\right\| + \left\|\frac{b}{p^k}\right\| + \left\|\frac{a+b}{p^k}\right\| = d_k = 1,$$

as required.

Comment 1. There are various choices for the number N in the solution. Here we sketch such a version.

Let x and y be two rational numbers with odd denominators. It is easy to see that the condition $\|x\| + \|y\| + \|x+y\| = 1$ is satisfied if and only if

$$\text{either} \quad \{x\} < \tfrac{1}{2}, \quad \{y\} < \tfrac{1}{2}, \quad \{x+y\} > \tfrac{1}{2}, \qquad \text{or} \quad \{x\} > \tfrac{1}{2}, \quad \{y\} > \tfrac{1}{2}, \quad \{x+y\} < \tfrac{1}{2},$$

where $\{x\}$ denotes the fractional part of x.

In the context of our problem, the first condition seems easier to deal with. Also, one may notice that

$$\{x\} < \tfrac{1}{2} \iff \varkappa(x) = 0 \qquad \text{and} \qquad \{x\} \geqslant \tfrac{1}{2} \iff \varkappa(x) = 1, \tag{4}$$

where

$$\varkappa(x) = \lfloor 2x \rfloor - 2\lfloor x \rfloor.$$

Now it is natural to consider the number

$$M = \frac{\binom{2a+2b}{a+b}}{\binom{2a}{a}\binom{2b}{b}},$$

since

$$v_p(M) = \sum_{k=1}^{\infty} \left(\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) \right).$$

One may see that $M > 1$, and that $v_2(M) \leqslant 0$. Thus, there exist an odd prime p and a positive integer k with

$$\varkappa\left(\frac{2(a+b)}{p^k}\right) - \varkappa\left(\frac{2a}{p^k}\right) - \varkappa\left(\frac{2b}{p^k}\right) > 0.$$

In view of (4), the last inequality yields

$$\left\{\frac{a}{p^k}\right\} < \frac{1}{2}, \quad \left\{\frac{b}{p^k}\right\} < \frac{1}{2}, \quad \text{and} \quad \left\{\frac{a+b}{p^k}\right\} > \frac{1}{2}, \tag{5}$$

which is what we wanted to obtain.

Comment 2. Once one tries to prove the existence of suitable p and k satisfying (5), it seems somehow natural to suppose that $a \leqslant b$ and to add the restriction $p^k > a$. In this case the inequalities (5) can be rewritten as

$$2a < p^k, \quad 2mp^k < 2b < (2m+1)p^k, \quad \text{and} \quad (2m+1)p^k < 2(a+b) < (2m+2)p^k$$

for some positive integer m. This means exactly that one of the numbers $2a+1, 2a+3, \ldots, 2a+2b-1$ is divisible by some number of the form p^k which is greater than $2a$.

Using more advanced techniques, one can show that such a number p^k exists even with $k = 1$. This was shown in 2004 by Laishram and Shorey; the methods used for this proof are elementary but still quite involved. In fact, their result generalises a theorem by Sylvester which states that for every pair of integers (n, k) with $n \geqslant k \geqslant 1$, the product $(n+1)(n+2)\cdots(n+k)$ is divisible by some prime $p > k$. We would like to mention here that Sylvester's theorem itself does not seem to suffice for solving the problem.

Shortlisted Problems
with Solutions

56th International Mathematical Olympiad

IMO
THAILAND 2015

Shortlisted Problems with Solutions

56^{th} International Mathematical Olympiad

Chiang Mai, Thailand, 4–16

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2016.

Contributing Countries

The Organizing Committee and the Problem Selection Committee of IMO 2015 thank the following 53 countries for contributing 155 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Brazil, Bulgaria, Canada, Costa Rica, Croatia, Cyprus, Denmark, El Salvador, Estonia, Finland, France, Georgia, Germany, Greece, Hong Kong, Hungary, India, Iran, Ireland, Israel, Italy, Japan, Kazakhstan, Lithuania, Luxembourg, Montenegro, Morocco, Netherlands, Pakistan, Poland, Romania, Russia, Saudi Arabia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Sweden, Turkey, Turkmenistan, Taiwan, Tanzania, Ukraine, United Kingdom, U.S.A., Uzbekistan

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Problems

Algebra

A1. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geqslant \frac{ka_k}{a_k^2 + (k-1)}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geqslant n$ for every $n \geqslant 2$.

(Serbia)

A2. Determine all functions $f\colon \mathbb{Z} \to \mathbb{Z}$ with the property that

$$f\big(x - f(y)\big) = f\big(f(x)\big) - f(y) - 1$$

holds for all $x, y \in \mathbb{Z}$.

(Croatia)

A3. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1 \leqslant r < s \leqslant 2n} (s - r - n) x_r x_s\,,$$

where $-1 \leqslant x_i \leqslant 1$ for all $i = 1, 2, \ldots, 2n$.

(Austria)

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) = x + f(x+y) + yf(x)$$

for all real numbers x and y.

(Albania)

A5. Let $2\mathbb{Z} + 1$ denote the set of odd integers. Find all functions $f\colon \mathbb{Z} \to 2\mathbb{Z} + 1$ satisfying

$$f\big(x + f(x) + y\big) + f\big(x - f(x) - y\big) = f(x+y) + f(x-y)$$

for every $x, y \in \mathbb{Z}$.

(U.S.A.)

A6. Let n be a fixed integer with $n \geqslant 2$. We say that two polynomials P and Q with real coefficients are *block-similar* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$\begin{aligned} &P(2015i), P(2015i - 1), \ldots, P(2015i - 2014) \quad \text{and} \\ &Q(2015i), Q(2015i - 1), \ldots, Q(2015i - 2014) \end{aligned}$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n+1$.

(b) Prove that there do not exist distinct block-similar polynomials of degree n.

(Canada)

Combinatorics

C1. In Lineland there are $n \geqslant 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep town B away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

(Estonia)

C2. Let $\mathcal{V}$ be a finite set of points in the plane. We say that $\mathcal{V}$ is *balanced* if for any two distinct points $A, B \in \mathcal{V}$, there exists a point $C \in \mathcal{V}$ such that $AC = BC$. We say that $\mathcal{V}$ is *center-free* if for any distinct points $A, B, C \in \mathcal{V}$, there does not exist a point $P \in \mathcal{V}$ such that $PA = PB = PC$.

(a) Show that for all $n \geqslant 3$, there exists a balanced set consisting of n points.

(b) For which $n \geqslant 3$ does there exist a balanced, center-free set consisting of n points?

(Netherlands)

C3. For a finite set A of positive integers, we call a partition of A into two disjoint nonempty subsets A_1 and A_2 *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

(Ukraine)

C4. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leqslant n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.

(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

(Finland)

C5. Consider an infinite sequence $a_1, a_2, \dots$ of positive integers with $a_i \leqslant 2015$ for all $i \geqslant 1$. Suppose that for any two distinct indices i and j we have $i + a_i \neq j + a_j$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{i=m+1}^{n} (a_i - b) \right| \leqslant 1007^2$$

whenever $n > m \geqslant N$.

(Australia)

C6. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

(U.S.A.)

C7. In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

(Russia)

Geometry

G1. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

(Australia)

G2. Let ABC be a triangle inscribed into a circle Ω with center O. A circle Γ with center A meets the side BC at points D and E such that D lies between B and E. Moreover, let F and G be the common points of Γ and Ω. We assume that F lies on the arc AB of Ω not containing C, and G lies on the arc AC of Ω not containing B. The circumcircles of the triangles BDF and CEG meet the sides AB and AC again at K and L, respectively. Suppose that the lines FK and GL are distinct and intersect at X. Prove that the points A, X, and O are collinear.

(Greece)

G3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

(Georgia)

G4. Let ABC be an acute triangle, and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC again at P and Q, respectively. Let T be the point such that the quadrilateral $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of the triangle ABC. Determine all possible values of BT/BM.

(Russia)

G5. Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC, respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

(El Salvador)

G6. Let ABC be an acute triangle with $AB > AC$, and let Γ be its circumcircle. Let H, M, and F be the orthocenter of the triangle, the midpoint of BC, and the foot of the altitude from A, respectively. Let Q and K be the two points on Γ that satisfy $\angle AQH = 90^\circ$ and $\angle QKH = 90^\circ$. Prove that the circumcircles of the triangles KQH and KFM are tangent to each other.

(Ukraine)

G7. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

(Bulgaria)

G8. A *triangulation* of a convex polygon Π is a partitioning of Π into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a *Thaiangulation* if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon Π differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

(Bulgaria)

Number Theory

N1. Determine all positive integers M for which the sequence $a_0, a_1, a_2, \ldots$, defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k \lfloor a_k \rfloor$ for $k = 0, 1, 2, \ldots$, contains at least one integer term.

(Luxembourg)

N2. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geqslant 2b + 2$.

(United Kingdom)

N3. Let m and n be positive integers such that $m > n$. Define $x_k = (m+k)/(n+k)$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

(Austria)

N4. Suppose that $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are two sequences of positive integers satisfying $a_0, b_0 \geqslant 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1$$

for all $n \geqslant 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geqslant 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geqslant N$.

(France)

N5. Determine all triples (a, b, c) of positive integers for which $ab - c$, $bc - a$, and $ca - b$ are powers of 2.

Explanation: A power of 2 is an integer of the form 2^n, where n denotes some nonnegative integer.

(Serbia)

N6. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(i) If $m, n \in \mathbb{Z}_{>0}$, then $\dfrac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;

(ii) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

(Singapore)

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called *k-good* if $\gcd\big(f(m) + n, f(n) + m\big) \leqslant k$ for all $m \neq n$. Find all k such that there exists a k-good function.

(Canada)

N8. For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i\colon p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

Find all strictly increasing functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho\big(f(a) - f(b)\big) \leqslant \mho(a - b) \qquad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

(Brazil)

Solutions

Algebra

A1. Suppose that a sequence $a_1, a_2, \ldots$ of positive real numbers satisfies

$$a_{k+1} \geqslant \frac{ka_k}{a_k^2 + (k-1)} \tag{1}$$

for every positive integer k. Prove that $a_1 + a_2 + \cdots + a_n \geqslant n$ for every $n \geqslant 2$.

(Serbia)

Solution. From the constraint (1), it can be seen that

$$\frac{k}{a_{k+1}} \leqslant \frac{a_k^2 + (k-1)}{a_k} = a_k + \frac{k-1}{a_k},$$

and so

$$a_k \geqslant \frac{k}{a_{k+1}} - \frac{k-1}{a_k}.$$

Summing up the above inequality for $k = 1, \ldots, m$, we obtain

$$a_1 + a_2 + \cdots + a_m \geqslant \left(\frac{1}{a_2} - \frac{0}{a_1}\right) + \left(\frac{2}{a_3} - \frac{1}{a_2}\right) + \cdots + \left(\frac{m}{a_{m+1}} - \frac{m-1}{a_m}\right) = \frac{m}{a_{m+1}}. \tag{2}$$

Now we prove the problem statement by induction on n. The case $n = 2$ can be done by applying (1) to $k = 1$:

$$a_1 + a_2 \geqslant a_1 + \frac{1}{a_1} \geqslant 2.$$

For the induction step, assume that the statement is true for some $n \geqslant 2$. If $a_{n+1} \geqslant 1$, then the induction hypothesis yields

$$\big(a_1 + \cdots + a_n\big) + a_{n+1} \geqslant n + 1. \tag{3}$$

Otherwise, if $a_{n+1} < 1$ then apply (2) as

$$\big(a_1 + \cdots + a_n\big) + a_{n+1} \geqslant \frac{n}{a_{n+1}} + a_{n+1} = \frac{n-1}{a_{n+1}} + \left(\frac{1}{a_{n+1}} + a_{n+1}\right) > (n-1) + 2.$$

That completes the solution.

Comment 1. It can be seen easily that having equality in the statement requires $a_1 = a_2 = 1$ in the base case $n = 2$, and $a_{n+1} = 1$ in (3). So the equality $a_1 + \cdots + a_n = n$ is possible only in the trivial case $a_1 = \cdots = a_n = 1$.

Comment 2. After obtaining (2), there are many ways to complete the solution. We outline three such possibilities.

- With defining $s_n = a_1 + \cdots + a_n$, the induction step can be replaced by

  $$s_{n+1} = s_n + a_{n+1} \geqslant s_n + \frac{n}{s_n} \geqslant n + 1,$$

  because the function $x \mapsto x + \dfrac{n}{x}$ increases on $[n, \infty)$.

- By applying the AM–GM inequality to the numbers $a_1 + \cdots + a_k$ and ka_{k+1}, we can conclude

$$a_1 + \cdots + a_k + ka_{k+1} \geqslant 2k$$

and sum it up for $k = 1, \ldots, n-1$.

- We can derive the symmetric estimate

$$\sum_{1 \leqslant i < j \leqslant n} a_i a_j = \sum_{j=2}^{n} (a_1 + \cdots + a_{j-1}) a_j \geqslant \sum_{j=2}^{n} (j-1) = \frac{n(n-1)}{2}$$

and combine it with the AM–QM inequality.

A2. Determine all functions $f\colon \mathbb{Z} \rightarrow \mathbb{Z}$ with the property that

$$f\big(x - f(y)\big) = f\big(f(x)\big) - f(y) - 1 \tag{1}$$

holds for all $x, y \in \mathbb{Z}$.

(Croatia)

Answer. There are two such functions, namely the constant function $x \mapsto -1$ and the successor function $x \mapsto x + 1$.

Solution 1. It is immediately checked that both functions mentioned in the answer are as desired.

Now let f denote any function satisfying (1) for all $x, y \in \mathbb{Z}$. Substituting $x = 0$ and $y = f(0)$ into (1) we learn that the number $z = -f\big(f(0)\big)$ satisfies $f(z) = -1$. So by plugging $y = z$ into (1) we deduce that

$$f(x + 1) = f\big(f(x)\big) \tag{2}$$

holds for all $x \in \mathbb{Z}$. Thereby (1) simplifies to

$$f\big(x - f(y)\big) = f(x + 1) - f(y) - 1\,. \tag{3}$$

We now work towards showing that f is linear by contemplating the difference $f(x+1)-f(x)$ for any $x \in \mathbb{Z}$. By applying (3) with $y = x$ and (2) in this order, we obtain

$$f(x + 1) - f(x) = f\big(x - f(x)\big) + 1 = f\big(f(x - 1 - f(x))\big) + 1\,.$$

Since (3) shows $f\big(x - 1 - f(x)\big)= f(x) - f(x) - 1 = -1$, this simplifies to

$$f(x + 1) = f(x) + A\,,$$

where $A = f(-1) + 1$ is some absolute constant.

Now a standard induction in both directions reveals that f is indeed linear and that in fact we have $f(x) = Ax + B$ for all $x \in \mathbb{Z}$, where $B = f(0)$. Substituting this into (2) we obtain that

$$Ax + (A + B) = A^2x + (AB + B)$$

holds for all $x \in \mathbb{Z}$; applying this to $x = 0$ and $x = 1$ we infer $A + B = AB + B$ and $A^2 = A$. The second equation leads to $A = 0$ or $A = 1$. In case $A = 1$, the first equation gives $B = 1$, meaning that f has to be the successor function. If $A = 0$, then f is constant and (1) shows that its constant value has to be -1. Thereby the solution is complete.

Comment. After (2) and (3) have been obtained, there are several other ways to combine them so as to obtain linearity properties of f. For instance, using (2) thrice in a row and then (3) with $x = f(y)$ one may deduce that

$$f(y + 2) = f\big(f(y + 1)\big) = f\big(f\big(f(y)\big)\big) = f\big(f(y) + 1\big) = f(y) + f(0) + 1$$

holds for all $y \in \mathbb{Z}$. It follows that f behaves linearly on the even numbers and on the odd numbers separately, and moreover that the slopes of these two linear functions coincide. From this point, one may complete the solution with some straightforward case analysis.

A different approach using the equations (2) and (3) will be presented in Solution 2. To show that it is also possible to start in a completely different way, we will also present a third solution that avoids these equations entirely.

Solution 2. We commence by deriving (2) and (3) as in the first solution. Now provided that f is injective, (2) tells us that f is the successor function. Thus we may assume from now on that f is not injective, i.e., that there are two integers $a > b$ with $f(a) = f(b)$. A straightforward induction using (2) in the induction step reveals that we have $f(a + n) = f(b + n)$ for all nonnegative integers n. Consequently, the sequence $\gamma_n = f(b + n)$ is periodic and thus in particular bounded, which means that the numbers

$$\varphi = \min_{n \geqslant 0} \gamma_n \quad \text{and} \quad \psi = \max_{n \geqslant 0} \gamma_n$$

exist.

Let us pick any integer y with $f(y) = \varphi$ and then an integer $x \geqslant a$ with $f\big(x - f(y)\big) = \varphi$. Due to the definition of φ and (3) we have

$$\varphi \leqslant f(x + 1) = f\big(x - f(y)\big) + f(y) + 1 = 2\varphi + 1\,,$$

whence $\varphi \geqslant -1$. The same reasoning applied to ψ yields $\psi \leqslant -1$. Since $\varphi \leqslant \psi$ holds trivially, it follows that $\varphi = \psi = -1$, or in other words that we have $f(t) = -1$ for all integers $t \geqslant a$.

Finally, if any integer y is given, we may find an integer x which is so large that $x + 1 \geqslant a$ and $x - f(y) \geqslant a$ hold. Due to (3) and the result from the previous paragraph we get

$$f(y) = f(x + 1) - f\big(x - f(y)\big) - 1 = (-1) - (-1) - 1 = -1\,.$$

Thereby the problem is solved.

Solution 3. Set $d = f(0)$. By plugging $x = f(y)$ into (1) we obtain

$$f^3(y) = f(y) + d + 1 \tag{4}$$

for all $y \in \mathbb{Z}$, where the left-hand side abbreviates $f\big(f(f(y))\big)$. When we replace x in (1) by $f(x)$ we obtain $f\big(f(x) - f(y)\big) = f^3(x) - f(y) - 1$ and as a consequence of (4) this simplifies to

$$f\big(f(x) - f(y)\big) = f(x) - f(y) + d\,. \tag{5}$$

Now we consider the set

$$E = \{f(x) - d \mid x \in \mathbb{Z}\}\,.$$

Given two integers a and b from E, we may pick some integers x and y with $f(x) = a + d$ and $f(y) = b + d$; now (5) tells us that $f(a - b) = (a - b) + d$, which means that $a - b$ itself exemplifies $a - b \in E$. Thus,

$$E \text{ is closed under taking differences.} \tag{6}$$

Also, the definitions of d and E yield $0 \in E$. If $E = \{0\}$, then f is a constant function and (1) implies that the only value attained by f is indeed -1.

So let us henceforth suppose that E contains some number besides zero. It is known that in this case (6) entails E to be the set of all integer multiples of some positive integer k. Indeed, this holds for

$$k = \min\big\{|x| \;\big|\; x \in E \text{ and } x \neq 0\big\}\,,$$

as one may verify by an argument based on division with remainder.

Thus we have

$$\{f(x) \mid x \in \mathbb{Z}\} = \{k \cdot t + d \mid t \in \mathbb{Z}\}\,. \tag{7}$$

Due to (5) and (7) we get

$$f(k \cdot t) = k \cdot t + d$$

for all $t \in \mathbb{Z}$, whence in particular $f(k) = k + d$. So by comparing the results of substituting $y = 0$ and $y = k$ into (1) we learn that

$$f(z + k) = f(z) + k \tag{8}$$

holds for all integers z. In plain English, this means that on any residue class modulo k the function f is linear with slope 1.

Now by (7) the set of all values attained by f is such a residue class. Hence, there exists an absolute constant c such that $f\big(f(x)\big) = f(x) + c$ holds for all $x \in \mathbb{Z}$. Thereby (1) simplifies to

$$f\big(x - f(y)\big) = f(x) - f(y) + c - 1\,. \tag{9}$$

On the other hand, considering (1) modulo k we obtain $d \equiv -1 \pmod{k}$ because of (7). So by (7) again, f attains the value -1.

Thus we may apply (9) to some integer y with $f(y) = -1$, which gives $f(x+1) = f(x) + c$. So f is a linear function with slope c. Hence, (8) leads to $c = 1$, wherefore there is an absolute constant d' with $f(x) = x + d'$ for all $x \in \mathbb{Z}$. Using this for $x = 0$ we obtain $d' = d$ and finally (4) discloses $d = 1$, meaning that f is indeed the successor function.

A3. Let n be a fixed positive integer. Find the maximum possible value of

$$\sum_{1\leqslant r<s\leqslant 2n}(s-r-n)x_rx_s\,,$$

where $-1\leqslant x_i\leqslant 1$ for all $i=1,2,\ldots,2n$.

(Austria)

Answer. $n(n-1)$.

Solution 1. Let Z be the expression to be maximized. Since this expression is linear in every variable x_i and $-1\leqslant x_i\leqslant 1$, the maximum of Z will be achieved when $x_i=-1$ or 1. Therefore, it suffices to consider only the case when $x_i\in\{-1,1\}$ for all $i=1,2,\ldots,2n$.

For $i=1,2,\ldots,2n$, we introduce auxiliary variables

$$y_i=\sum_{r=1}^{i}x_r-\sum_{r=i+1}^{2n}x_r\,.$$

Taking squares of both sides, we have

$$\begin{aligned}y_i^2&=\sum_{r=1}^{2n}x_r^2+\sum_{r<s\leqslant i}2x_rx_s+\sum_{i<r<s}2x_rx_s-\sum_{r\leqslant i<s}2x_rx_s\\&=2n+\sum_{r<s\leqslant i}2x_rx_s+\sum_{i<r<s}2x_rx_s-\sum_{r\leqslant i<s}2x_rx_s\,,\end{aligned}\tag{1}$$

where the last equality follows from the fact that $x_r\in\{-1,1\}$. Notice that for every $r<s$, the coefficient of x_rx_s in (1) is 2 for each $i=1,\ldots,r-1,s,\ldots,2n$, and this coefficient is -2 for each $i=r,\ldots,s-1$. This implies that the coefficient of x_rx_s in $\sum_{i=1}^{2n}y_i^2$ is $2(2n-s+r)-2(s-r)=4(n-s+r)$. Therefore, summing (1) for $i=1,2,\ldots,2n$ yields

$$\sum_{i=1}^{2n}y_i^2=4n^2+\sum_{1\leqslant r<s\leqslant 2n}4(n-s+r)x_rx_s=4n^2-4Z.\tag{2}$$

Hence, it suffices to find the minimum of the left-hand side.

Since $x_r\in\{-1,1\}$, we see that y_i is an even integer. In addition, $y_i-y_{i-1}=2x_i=\pm 2$, and so y_{i-1} and y_i are consecutive even integers for every $i=2,3,\ldots,2n$. It follows that $y_{i-1}^2+y_i^2\geqslant 4$, which implies

$$\sum_{i=1}^{2n}y_i^2=\sum_{j=1}^{n}\left(y_{2j-1}^2+y_{2j}^2\right)\geqslant 4n.\tag{3}$$

Combining (2) and (3), we get

$$4n\leqslant\sum_{i=1}^{2n}y_i^2=4n^2-4Z.\tag{4}$$

Hence, $Z\leqslant n(n-1)$.

If we set $x_i=1$ for odd indices i and $x_i=-1$ for even indices i, then we obtain equality in (3) (and thus in (4)). Therefore, the maximum possible value of Z is $n(n-1)$, as desired.

Comment 1. $Z=n(n-1)$ can be achieved by several other examples. In particular, x_i needs not be ± 1. For instance, setting $x_i=(-1)^i$ for all $2\leqslant i\leqslant 2n$, we find that the coefficient of x_1 in Z is 0. Therefore, x_1 can be chosen arbitrarily in the interval $[-1,1]$.

Nevertheless, if $x_i\in\{-1,1\}$ for all $i=1,2,\ldots,2n$, then the equality $Z=n(n-1)$ holds only when $(y_1,y_2,\ldots,y_{2n})=(0,\pm 2,0,\pm 2,\ldots,0,\pm 2)$ or $(\pm 2,0,\pm 2,0,\ldots,\pm 2,0)$. In each case, we can reconstruct x_i accordingly. The sum $\sum_{i=1}^{2n}x_i$ in the optimal cases needs not be 0, but it must equal 0 or ± 2.

Comment 2. Several variations in setting up the auxiliary variables are possible. For instance, one may let $x_{2n+i} = -x_i$ and $y'_i = x_i + x_{i+1} + \cdots + x_{i+n-1}$ for any $1 \leqslant i \leqslant 2n$. Similarly to Solution 1, we obtain $Y := y_1'^2 + y_2'^2 + \cdots + y_{2n}'^2 = 2n^2 - 2Z$. Then, it suffices to show that $Y \geqslant 2n$. If n is odd, then each y'_i is odd, and so $y_i'^2 \geqslant 1$. If n is even, then each y'_i is even. We can check that at least one of y'_i, y'_{i+1}, y'_{n+i}, and y'_{n+i+1} is nonzero, so that $y_i'^2 + y_{i+1}'^2 + y_{n+i}'^2 + y_{n+i+1}'^2 \geqslant 4$; summing these up for $i = 1, 3, \dots, n-1$ yields $Y \geqslant 2n$.

Solution 2. We present a different method of obtaining the bound $Z \leqslant n(n-1)$. As in the previous solution, we reduce the problem to the case $x_i \in \{-1, 1\}$. For brevity, we use the notation $[2n] = \{1, 2, \dots, 2n\}$.

Consider any $x_1, x_2, \dots, x_{2n} \in \{-1, 1\}$. Let

$$A = \{i \in [2n] : x_i = 1\} \quad \text{and} \quad B = \{i \in [2n] : x_i = -1\}.$$

For any subsets X and Y of $[2n]$ we define

$$e(X, Y) = \sum_{r<s,\ r \in X,\ s \in Y} (s - r - n).$$

One may observe that

$$e(A,A)+e(A,B)+e(B,A)+e(B,B) = e([2n],[2n]) = \sum_{1 \leqslant r<s \leqslant 2n} (s-r-n) = -\frac{(n-1)n(2n-1)}{3}.$$

Therefore, we have

$$Z = e(A,A) - e(A,B) - e(B,A) + e(B,B) = 2\big(e(A,A) + e(B,B)\big) + \frac{(n-1)n(2n-1)}{3}. \tag{5}$$

Thus, we need to maximize $e(A,A) + e(B,B)$, where A and B form a partition of $[2n]$.

Due to the symmetry, we may assume that $|A| = n - p$ and $|B| = n + p$, where $0 \leqslant p \leqslant n$. From now on, we fix the value of p and find an upper bound for Z in terms of n and p.

Let $a_1 < a_2 < \cdots < a_{n-p}$ and $b_1 < b_2 < \cdots < b_{n+p}$ list all elements of A and B, respectively. Then

$$e(A,A) = \sum_{1 \leqslant i<j \leqslant n-p} (a_j - a_i - n) = \sum_{i=1}^{n-p} (2i - 1 - n + p)a_i - \binom{n-p}{2} \cdot n \tag{6}$$

and similarly

$$e(B,B) = \sum_{i=1}^{n+p} (2i - 1 - n - p)b_i - \binom{n+p}{2} \cdot n. \tag{7}$$

Thus, now it suffices to maximize the value of

$$M = \sum_{i=1}^{n-p} (2i - 1 - n + p)a_i + \sum_{i=1}^{n+p} (2i - 1 - n - p)b_i. \tag{8}$$

In order to get an upper bound, we will apply the rearrangement inequality to the sequence $a_1, a_2, \dots, a_{n-p}, b_1, b_2, \dots, b_{n+p}$ (which is a permutation of $1, 2, \dots, 2n$), together with the sequence of coefficients of these numbers in (8). The coefficients of a_i form the sequence

$$n-p-1, n-p-3, \dots, 1-n+p,$$

and those of b_i form the sequence

$$n+p-1, n+p-3, \dots, 1-n-p.$$

Altogether, these coefficients are, in descending order:

- $n+p+1-2i$, for $i=1,2,\dots,p$;
- $n-p+1-2i$, counted twice, for $i=1,2,\dots,n-p$; and
- $-(n+p+1-2i)$, for $i=p,p-1,\dots,1$.

Thus, the rearrangement inequality yields

$$\begin{aligned} M \leqslant \sum_{i=1}^{p}(n+p+1-2i)(2n+1-i) \\ + \sum_{i=1}^{n-p}(n-p+1-2i)\big((2n+2-p-2i)+(2n+1-p-2i)\big) \\ - \sum_{i=1}^{p}(n+p+1-2i)i\,. \end{aligned} \tag{9}$$

Finally, combining the information from (5), (6), (7), and (9), we obtain

$$\begin{aligned} Z \leqslant \frac{(n-1)n(2n-1)}{3} - 2n\left(\binom{n-p}{2}+\binom{n+p}{2}\right) \\ + 2\sum_{i=1}^{p}(n+p+1-2i)(2n+1-2i) + 2\sum_{i=1}^{n-p}(n-p+1-2i)(4n-2p+3-4i)\,, \end{aligned}$$

which can be simplified to

$$Z \leqslant n(n-1) - \frac{2}{3}\,p(p-1)(p+1)\,.$$

Since p is a nonnegative integer, this yields $Z \leqslant n(n-1)$.

A4. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f\big(x + f(x+y)\big) + f(xy) = x + f(x+y) + yf(x) \tag{1}$$

for all real numbers x and y.

(Albania)

Answer. There are two such functions, namely the identity function and $x \mapsto 2 - x$.

Solution. Clearly, each of the functions $x \mapsto x$ and $x \mapsto 2 - x$ satisfies (1). It suffices now to show that they are the only solutions to the problem.

Suppose that f is any function satisfying (1). Then setting $y = 1$ in (1), we obtain

$$f\big(x + f(x+1)\big) = x + f(x+1); \tag{2}$$

in other words, $x + f(x+1)$ is a fixed point of f for every $x \in \mathbb{R}$.

We distinguish two cases regarding the value of $f(0)$.

Case 1. $f(0) \neq 0$.

By letting $x = 0$ in (1), we have

$$f\big(f(y)\big) + f(0) = f(y) + yf(0).$$

So, if y_0 is a fixed point of f, then substituting $y = y_0$ in the above equation we get $y_0 = 1$. Thus, it follows from (2) that $x + f(x+1) = 1$ for all $x \in \mathbb{R}$. That is, $f(x) = 2 - x$ for all $x \in \mathbb{R}$.

Case 2. $f(0) = 0$.

By letting $y = 0$ and replacing x by $x + 1$ in (1), we obtain

$$f\big(x + f(x+1) + 1\big) = x + f(x+1) + 1. \tag{3}$$

From (1), the substitution $x = 1$ yields

$$f\big(1 + f(y+1)\big) + f(y) = 1 + f(y+1) + yf(1). \tag{4}$$

By plugging $x = -1$ into (2), we see that $f(-1) = -1$. We then plug $y = -1$ into (4) and deduce that $f(1) = 1$. Hence, (4) reduces to

$$f\big(1 + f(y+1)\big) + f(y) = 1 + f(y+1) + y. \tag{5}$$

Accordingly, if both y_0 and $y_0 + 1$ are fixed points of f, then so is $y_0 + 2$. Thus, it follows from (2) and (3) that $x + f(x+1) + 2$ is a fixed point of f for every $x \in \mathbb{R}$; i.e.,

$$f\big(x + f(x+1) + 2\big) = x + f(x+1) + 2.$$

Replacing x by $x - 2$ simplifies the above equation to

$$f\big(x + f(x-1)\big) = x + f(x-1).$$

On the other hand, we set $y = -1$ in (1) and get

$$f\big(x + f(x-1)\big) = x + f(x-1) - f(x) - f(-x).$$

Therefore, $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Finally, we substitute (x, y) by $(-1, -y)$ in (1) and use the fact that $f(-1) = -1$ to get

$$f\big(-1 + f(-y-1)\big) + f(y) = -1 + f(-y-1) + y.$$

Since f is an odd function, the above equation becomes

$$-f\big(1 + f(y+1)\big) + f(y) = -1 - f(y+1) + y.$$

By adding this equation to (5), we conclude that $f(y) = y$ for all $y \in \mathbb{R}$.

A5. Let $2\mathbb{Z}+1$ denote the set of odd integers. Find all functions $f\colon \mathbb{Z} \to 2\mathbb{Z}+1$ satisfying

$$f\big(x+f(x)+y\big)+f\big(x-f(x)-y\big)=f(x+y)+f(x-y) \tag{1}$$

for every $x, y \in \mathbb{Z}$.

(U.S.A.)

Answer. Fix an odd positive integer d, an integer k, and odd integers $\ell_0, \ell_1, \ldots, \ell_{d-1}$. Then the function defined as

$$f(md+i)=2kmd+\ell_i d \qquad (m \in \mathbb{Z}, \quad i=0,1,\ldots,d-1)$$

satisfies the problem requirements, and these are all such functions.

Solution. Throughout the solution, all functions are assumed to map integers to integers.

For any function g and any nonzero integer t, define

$$\Delta_t g(x)=g(x+t)-g(x).$$

For any nonzero integers a and b, notice that $\Delta_a\Delta_b g=\Delta_b\Delta_a g$. Moreover, if $\Delta_a g=0$ and $\Delta_b g=0$, then $\Delta_{a+b} g=0$ and $\Delta_{at} g=0$ for all nonzero integers t. We say that g is *t-quasi-periodic* if $\Delta_t g$ is a constant function (in other words, if $\Delta_1\Delta_t g=0$, or $\Delta_1 g$ is t-periodic). In this case, we call t a *quasi-period* of g. We say that g is *quasi-periodic* if it is t-quasi-periodic for some nonzero integer t.

Notice that a quasi-period of g is a period of $\Delta_1 g$. So if g is quasi-periodic, then its minimal positive quasi-period t divides all its quasi-periods.

We now assume that f satisfies (1). First, by setting $a=x+y$, the problem condition can be rewritten as

$$\Delta_{f(x)} f(a)=\Delta_{f(x)} f\big(2x-a-f(x)\big) \qquad \text{for all } x, a \in \mathbb{Z}. \tag{2}$$

Let b be an arbitrary integer and let k be an arbitrary positive integer. Applying (2) when a is substituted by $b, b+f(x), \ldots, b+(k-1)f(x)$ and summing up all these equations, we get

$$\Delta_{kf(x)} f(b)=\Delta_{kf(x)} f\big(2x-b-kf(x)\big).$$

Notice that a similar argument works when k is negative, so that

$$\Delta_M f(b)=\Delta_M f(2x-b-M) \qquad \text{for any nonzero integer } M \text{ such that } f(x) \mid M. \tag{3}$$

We now prove two lemmas.

Lemma 1. For any distinct integers x and y, the function $\Delta_{\operatorname{lcm}(f(x),f(y))} f$ is $2(y-x)$-periodic.

Proof. Denote $L=\operatorname{lcm}\big(f(x),f(y)\big)$. Applying (3) twice, we obtain

$$\Delta_L f(b)=\Delta_L f(2x-b-L)=\Delta_L f\big(2y-(b+2(y-x))-L\big)=\Delta_L f\big(b+2(y-x)\big).$$

Thus, the function $\Delta_L f$ is $2(y-x)$-periodic, as required. □

Lemma 2. Let g be a function. If t and s are nonzero integers such that $\Delta_{ts} g=0$ and $\Delta_t\Delta_t g=0$, then $\Delta_t g=0$.

Proof. Assume, without loss of generality, that s is positive. Let a be an arbitrary integer. Since $\Delta_t\Delta_t g=0$, we have

$$\Delta_t g(a)=\Delta_t g(a+t)=\cdots=\Delta_t g\big(a+(s-1)t\big).$$

The sum of these s equal numbers is $\Delta_{ts} g(a)=0$, so each of them is zero, as required. □

We now return to the solution.

Step 1. We prove that f is quasi-periodic.

Let $Q = \operatorname{lcm}\big(f(0), f(1)\big)$. Applying Lemma 1, we get that the function $g = \Delta_Q f$ is 2-periodic. In other words, the values of g are constant on even numbers and on odd numbers separately. Moreover, setting $M = Q$ and $x = b = 0$ in (3), we get $g(0) = g(-Q)$. Since 0 and $-Q$ have different parities, the value of g at even numbers is the same as that at odd numbers. Thus, g is constant, which means that Q is a quasi-period of f.

Step 2. Denote the minimal positive quasi-period of f by T. We prove that $T \mid f(x)$ for all integers x.

Since an odd number Q is a quasi-period of f, the number T is also odd. Now suppose, to the contrary, that there exist an odd prime p, a positive integer α, and an integer u such that $p^\alpha \mid T$ but $p^\alpha \nmid f(u)$. Setting $x = u$ and $y = 0$ in (1), we have $2f(u) = f\big(u+f(u)\big)+f\big(u-f(u)\big)$, so p^α does not divide the value of f at one of the points $u + f(u)$ or $u - f(u)$. Denote this point by v.

Let $L = \operatorname{lcm}\big(f(u), f(v)\big)$. Since $|u - v| = f(u)$, from Lemma 1 we get $\Delta_{2f(u)}\Delta_L f = 0$. Hence the function $\Delta_L f$ is $2f(u)$-periodic as well as T-periodic, so it is $\gcd\big(T, 2f(u)\big)$-periodic, or $\Delta_{\gcd(T,2f(u))}\Delta_L f = 0$. Similarly, observe that the function $\Delta_{\gcd(T,2f(u))} f$ is L-periodic as well as T-periodic, so we may conclude that $\Delta_{\gcd(T,L)}\Delta_{\gcd(T,2f(u))} f = 0$. Since $p^\alpha \nmid L$, both $\gcd\big(T, 2f(u)\big)$ and $\gcd(T, L)$ divide T/p. We thus obtain $\Delta_{T/p}\Delta_{T/p} f = 0$, which yields

$$\Delta_{T/p}\Delta_{T/p}\Delta_1 f = 0.$$

Since $\Delta_T\Delta_1 f = 0$, we can apply Lemma 2 to the function $\Delta_1 f$, obtaining $\Delta_{T/p}\Delta_1 f = 0$. However, this means that f is (T/p)-quasi-periodic, contradicting the minimality of T. Our claim is proved.

Step 3. We describe all functions f.

Let d be the greatest common divisor of all values of f. Then d is odd. By Step 2, d is a quasi-period of f, so that $\Delta_d f$ is constant. Since the value of $\Delta_d f$ is even and divisible by d, we may denote this constant by $2dk$, where k is an integer. Next, for all $i = 0, 1, \ldots, d-1$, define $\ell_i = f(i)/d$; notice that ℓ_i is odd. Then

$$f(md + i) = \Delta_{md} f(i) + f(i) = 2kmd + \ell_i d \qquad \text{for all } m \in \mathbb{Z} \quad \text{and } i = 0, 1, \ldots, d-1.$$

This shows that all functions satisfying (1) are listed in the answer.

It remains to check that all such functions indeed satisfy (1). This is equivalent to checking (2), which is true because for every integer x, the value of $f(x)$ is divisible by d, so that $\Delta_{f(x)} f$ is constant.

Comment. After obtaining Lemmas 1 and 2, it is possible to complete the steps in a different order. Here we sketch an alternative approach.

For any function g and any nonzero integer t, we say that g is *t-pseudo-periodic* if $\Delta_t\Delta_t g = 0$. In this case, we call t a *pseudo-period* of g, and we say that g is *pseudo-periodic*.

Let us first prove a basic property: if a function g is pseudo-periodic, then its minimal positive pseudo-period divides all its pseudo-periods. To establish this, it suffices to show that if t and s are pseudo-periods of g with $t \neq s$, then so is $t - s$. Indeed, suppose that $\Delta_t\Delta_t g = \Delta_s\Delta_s g = 0$. Then $\Delta_t\Delta_t\Delta_s g = \Delta_{ts}\Delta_s g = 0$, so that $\Delta_t\Delta_s g = 0$ by Lemma 2. Taking differences, we obtain $\Delta_t\Delta_{t-s} g = \Delta_s\Delta_{t-s} g = 0$, and thus $\Delta_{t-s}\Delta_{t-s} g = 0$.

Now let f satisfy the problem condition. We will show that f is pseudo-periodic. When this is done, we will let T' be the minimal pseudo-period of f, and show that T' divides $2f(x)$ for every integer x, using arguments similar to Step 2 of the solution. Then we will come back to Step 1 by showing that T' is also a quasi-period of f.

First, Lemma 1 yields that $\Delta_{2(y-x)}\Delta_{\operatorname{lcm}(f(x),f(y))}f=0$ for every distinct integers x and y. Hence f is pseudo-periodic with pseudo-period $L_{x,y}=\operatorname{lcm}\bigl(2(y-x),f(x),f(y)\bigr)$.

We now show that $T'\mid 2f(x)$ for every integer x. Suppose, to the contrary, that there exists an integer u, a prime p, and a positive integer α such that $p^\alpha\mid T'$ and $p^\alpha\nmid 2f(u)$. Choose v as in Step 2 and employ Lemma 1 to obtain $\Delta_{2f(u)}\Delta_{\operatorname{lcm}(f(u),f(v))}f=0$. However, this implies that $\Delta_{T'/p}\Delta_{T'/p}f=0$, a contradiction with the minimality of T'.

We now claim that $\Delta_{T'}\Delta_2 f=0$. Indeed, Lemma 1 implies that there exists an integer s such that $\Delta_s\Delta_2 f=0$. Hence $\Delta_{T's}\Delta_2 f=\Delta_{T'}\Delta_{T'}\Delta_2 f=0$, which allows us to conclude that $\Delta_{T'}\Delta_2 f=0$ by Lemma 2. (The last two paragraphs are similar to Step 2 of the solution.)

Now, it is not difficult to finish the solution, though more work is needed to eliminate the factors of 2 from the subscripts of $\Delta_{T'}\Delta_2 f=0$. Once this is done, we will obtain an odd quasi-period of f that divides $f(x)$ for all integers x. Then we can complete the solution as in Step 3.

A6. Let n be a fixed integer with $n \geqslant 2$. We say that two polynomials P and Q with real coefficients are *block-similar* if for each $i \in \{1, 2, \ldots, n\}$ the sequences

$$P(2015i), P(2015i-1), \ldots, P(2015i-2014) \quad \text{and}$$
$$Q(2015i), Q(2015i-1), \ldots, Q(2015i-2014)$$

are permutations of each other.

(a) Prove that there exist distinct block-similar polynomials of degree $n+1$.

(b) Prove that there do not exist distinct block-similar polynomials of degree n.

(Canada)

Solution 1. For convenience, we set $k = 2015 = 2\ell + 1$.

Part (a). Consider the following polynomials of degree $n+1$:

$$P(x) = \prod_{i=0}^{n}(x - ik) \qquad \text{and} \qquad Q(x) = \prod_{i=0}^{n}(x - ik - 1).$$

Since $Q(x) = P(x-1)$ and $P(0) = P(k) = P(2k) = \cdots = P(nk)$, these polynomials are block-similar (and distinct).

Part (b). For every polynomial $F(x)$ and every nonnegative integer m, define $\Sigma_F(m) = \sum_{i=1}^{m} F(i)$; in particular, $\Sigma_F(0) = 0$. It is well-known that for every nonnegative integer d the sum $\sum_{i=1}^{m} i^d$ is a polynomial in m of degree $d+1$. Thus Σ_F may also be regarded as a real polynomial of degree $\deg F + 1$ (with the exception that if $F = 0$, then $\Sigma_F = 0$ as well). This allows us to consider the values of Σ_F at all real points (where the initial definition does not apply).

Assume for the sake of contradiction that there exist two distinct block-similar polynomials $P(x)$ and $Q(x)$ of degree n. Then both polynomials $\Sigma_{P-Q}(x)$ and $\Sigma_{P^2-Q^2}(x)$ have roots at the points $0, k, 2k, \ldots, nk$. This motivates the following lemma, where we use the special polynomial

$$T(x) = \prod_{i=0}^{n}(x - ik).$$

Lemma. Assume that $F(x)$ is a nonzero polynomial such that $0, k, 2k, \ldots, nk$ are among the roots of the polynomial $\Sigma_F(x)$. Then $\deg F \geqslant n$, and there exists a polynomial $G(x)$ such that $\deg G = \deg F - n$ and $F(x) = T(x)G(x) - T(x-1)G(x-1)$.

Proof. If $\deg F < n$, then $\Sigma_F(x)$ has at least $n+1$ roots, while its degree is less than $n+1$. Therefore, $\Sigma_F(x) = 0$ and hence $F(x) = 0$, which is impossible. Thus $\deg F \geqslant n$.

The lemma condition yields that $\Sigma_F(x) = T(x)G(x)$ for some polynomial $G(x)$ such that $\deg G = \deg \Sigma_F - (n+1) = \deg F - n$.

Now, let us define $F_1(x) = T(x)G(x) - T(x-1)G(x-1)$. Then for every positive integer n we have

$$\Sigma_{F_1}(n) = \sum_{i=1}^{n}\big(T(x)G(x) - T(x-1)G(x-1)\big) = T(n)G(n) - T(0)G(0) = T(n)G(n) = \Sigma_F(n),$$

so the polynomial $\Sigma_{F-F_1}(x) = \Sigma_F(x) - \Sigma_{F_1}(x)$ has infinitely many roots. This means that this polynomial is zero, which in turn yields $F(x) = F_1(x)$, as required. □

First, we apply the lemma to the nonzero polynomial $R_1(x) = P(x) - Q(x)$. Since the degree of $R_1(x)$ is at most n, we conclude that it is exactly n. Moreover, $R_1(x) = \alpha \cdot \big(T(x) - T(x-1)\big)$ for some nonzero constant α.

Our next aim is to prove that the polynomial $S(x) = P(x) + Q(x)$ is constant. Assume the contrary. Then, notice that the polynomial $R_2(x) = P(x)^2 - Q(x)^2 = R_1(x)S(x)$ is also nonzero and satisfies the lemma condition. Since $n < \deg R_1 + \deg S = \deg R_2 \leqslant 2n$, the lemma yields

$$R_2(x) = T(x)G(x) - T(x-1)G(x-1)$$

with some polynomial $G(x)$ with $0 < \deg G \leqslant n$.

Since the polynomial $R_1(x) = \alpha\big(T(x) - T(x-1)\big)$ divides the polynomial

$$R_2(x) = T(x)\big(G(x) - G(x-1)\big) + G(x-1)\big(T(x) - T(x-1)\big),$$

we get $R_1(x) \mid T(x)\big(G(x) - G(x-1)\big)$. On the other hand,

$$\gcd\big(T(x), R_1(x)\big) = \gcd\big(T(x), T(x) - T(x-1)\big) = \gcd\big(T(x), T(x-1)\big) = 1,$$

since both $T(x)$ and $T(x-1)$ are the products of linear polynomials, and their roots are distinct. Thus $R_1(x) \mid G(x) - G(x-1)$. However, this is impossible since $G(x) - G(x-1)$ is a nonzero polynomial of degree less than $n = \deg R_1$.

Thus, our assumption is wrong, and $S(x)$ is a constant polynomial, say $S(x) = \beta$. Notice that the polynomials $\big(2P(x) - \beta\big)/\alpha$ and $\big(2Q(x) - \beta\big)/\alpha$ are also block-similar and distinct. So we may replace the initial polynomials by these ones, thus obtaining two block-similar polynomials $P(x)$ and $Q(x)$ with $P(x) = -Q(x) = T(x) - T(x-1)$. It remains to show that this is impossible.

For every $i = 1, 2 \ldots, n$, the values $T(ik - k + 1)$ and $T(ik - 1)$ have the same sign. This means that the values $P(ik - k + 1) = T(ik - k + 1)$ and $P(ik) = -T(ik - 1)$ have opposite signs, so $P(x)$ has a root in each of the n segments $[ik - k + 1, ik]$. Since $\deg P = n$, it must have exactly one root in each of them.

Thus, the sequence $P(1), P(2), \ldots, P(k)$ should change sign exactly once. On the other hand, since $P(x)$ and $-P(x)$ are block-similar, this sequence must have as many positive terms as negative ones. Since $k = 2\ell + 1$ is odd, this shows that the middle term of the sequence above must be zero, so $P(\ell + 1) = 0$, or $T(\ell + 1) = T(\ell)$. However, this is not true since

$$|T(\ell+1)| = |\ell+1| \cdot |\ell| \cdot \prod_{i=2}^{n} |\ell + 1 - ik| < |\ell| \cdot |\ell+1| \cdot \prod_{i=2}^{n} |\ell - ik| = |T(\ell)|,$$

where the strict inequality holds because $n \geqslant 2$. We come to the final contradiction.

Comment 1. In the solution above, we used the fact that $k > 1$ is odd. One can modify the arguments of the last part in order to work for every (not necessarily odd) sufficiently large value of k; namely, when k is even, one may show that the sequence $P(1), P(2), \ldots, P(k)$ has different numbers of positive and negative terms.

On the other hand, the problem statement with k replaced by 2 is false, since the polynomials $P(x) = T(x) - T(x-1)$ and $Q(x) = T(x-1) - T(x)$ *are* block-similar in this case, due to the fact that $P(2i-1) = -P(2i) = Q(2i) = -Q(2i-1) = T(2i-1)$ for all $i = 1, 2, \ldots, n$. Thus, every complete solution should use the relation $k > 2$.

One may easily see that the condition $n \geqslant 2$ is also substantial, since the polynomials x and $k + 1 - x$ become block-similar if we set $n = 1$.

It is easily seen from the solution that the result still holds if we assume that the polynomials have degree *at most* n.

Solution 2. We provide an alternative argument for part (b).

Assume again that there exist two distinct block-similar polynomials $P(x)$ and $Q(x)$ of degree n. Let $R(x) = P(x) - Q(x)$ and $S(x) = P(x) + Q(x)$. For brevity, we also denote the segment $\big[(i-1)k+1, ik\big]$ by I_i, and the set $\big\{(i-1)k+1, (i-1)k+2, \ldots, ik\big\}$ of all integer points in I_i by Z_i.

Step 1. *We prove that $R(x)$ has exactly one root in each segment I_i, $i = 1, 2, \ldots, n$, and all these roots are simple.*

Indeed, take any $i \in \{1, 2, \ldots, n\}$ and choose some points $p^-, p^+ \in Z_i$ so that

$$P(p^-) = \min_{x \in Z_i} P(x) \quad \text{and} \qquad P(p^+) = \max_{x \in Z_i} P(x).$$

Since the sequences of values of P and Q in Z_i are permutations of each other, we have $R(p^-) = P(p^-) - Q(p^-) \leqslant 0$ and $R(p^+) = P(p^+) - Q(p^+) \geqslant 0$. Since $R(x)$ is continuous, there exists at least one root of $R(x)$ between p^- and p^+ — thus in I_i.

So, $R(x)$ has at least one root in each of the n disjoint segments I_i with $i = 1, 2, \ldots, n$. Since $R(x)$ is nonzero and its degree does not exceed n, it should have exactly one root in each of these segments, and all these roots are simple, as required.

Step 2. *We prove that $S(x)$ is constant.*

We start with the following claim.

Claim. For every $i = 1, 2, \ldots, n$, the sequence of values $S\big((i-1)k+1\big)$, $S\big((i-1)k+2\big)$, $\ldots$, $S(ik)$ cannot be strictly increasing.

Proof. Fix any $i \in \{1, 2, \ldots, n\}$. Due to the symmetry, we may assume that $P\big(ik\big) \leqslant Q(ik)$. Choose now p^- and p^+ as in Step 1. If we had $P(p^+) = P(p^-)$, then P would be constant on Z_i, so all the elements of Z_i would be the roots of $R(x)$, which is not the case. In particular, we have $p^+ \neq p^-$. If $p^- > p^+$, then $S(p^-) = P(p^-) + Q(p^-) \leqslant Q(p^+) + P(p^+) = S(p^+)$, so our claim holds.

We now show that the remaining case $p^- < p^+$ is impossible. Assume first that $P(p^+) > Q(p^+)$. Then, like in Step 1, we have $R(p^-) \leqslant 0$, $R(p^+) > 0$, and $R(ik) \leqslant 0$, so $R(x)$ has a root in each of the intervals $[p^-, p^+)$ and $(p^+, ik]$. This contradicts the result of Step 1.

We are left only with the case $p^- < p^+$ and $P(p^+) = Q(p^+)$ (thus p^+ is the unique root of $R(x)$ in I_i). If $p^+ = ik$, then the values of $R(x)$ on $Z_i \setminus \{ik\}$ are all of the same sign, which is absurd since their sum is zero. Finally, if $p^- < p^+ < ik$, then $R(p^-)$ and $R(ik)$ are both negative. This means that $R(x)$ should have an even number of roots in $[p^-, ik]$, counted with multiplicity. This also contradicts the result of Step 1. $\square$

In a similar way, one may prove that for every $i = 1, 2, \ldots, n$, the sequence $S\big((i-1)k+1\big)$, $S\big((i-1)k+2\big)$, $\ldots$, $S(ik)$ cannot be strictly decreasing. This means that the polynomial $\Delta S(x) = S(x) - S(x-1)$ attains at least one nonnegative value, as well as at least one nonpositive value, on the set Z_i (and even on $Z_i \setminus \big\{(i-1)k+1\big\}$); so ΔS has a root in I_i.

Thus ΔS has at least n roots; however, its degree is less than n, so ΔS should be identically zero. This shows that $S(x)$ is a constant, say $S(x) \equiv \beta$.

Step 3. Notice that the polynomials $P(x) - \beta/2$ and $Q(x) - \beta/2$ are also block-similar and distinct. So we may replace the initial polynomials by these ones, thus reaching $P(x) = -Q(x)$.

Then $R(x) = 2P(x)$, so $P(x)$ has exactly one root in each of the segments I_i, $i = 1, 2, \ldots, n$. On the other hand, $P(x)$ and $-P(x)$ should attain the same number of positive values on Z_i. Since k is odd, this means that Z_i contains exactly one root of $P(x)$; moreover, this root should be at the center of Z_i, because $P(x)$ has the same number of positive and negative values on Z_i.

Thus we have found all n roots of $P(x)$, so

$$P(x) = c\prod_{i=1}^{n}(x - ik + \ell) \qquad \text{for some } c \in \mathbb{R} \setminus \{0\},$$

where $\ell = (k-1)/2$. It remains to notice that for every $t \in Z_1 \setminus \{1\}$ we have

$$|P(t)| = |c| \cdot |t - \ell - 1| \cdot \prod_{i=2}^{n} |t - ik + \ell| < |c| \cdot \ell \cdot \prod_{i=2}^{n} |1 - ik + \ell| = |P(1)|,$$

so $P(1) \neq -P(t)$ for all $t \in Z_1$. This shows that $P(x)$ is not block-similar to $-P(x)$. The final contradiction.

Comment 2. One may merge Steps 1 and 2 in the following manner. As above, we set $R(x) = P(x) - Q(x)$ and $S(x) = P(x) + Q(x)$.

We aim to prove that the polynomial $S(x) = 2P(x) - R(x) = 2Q(x) + R(x)$ is constant. Since the degrees of $R(x)$ and $S(x)$ do not exceed n, it suffices to show that the total number of roots of $R(x)$ and $\Delta S(x) = S(x) - S(x-1)$ is at least $2n$. For this purpose, we prove the following claim.

Claim. For every $i = 1, 2, \ldots, n$, either each of R and ΔS has a root in I_i, or R has at least two roots in I_i.

Proof. Fix any $i \in \{1, 2, \ldots, n\}$. Let $r \in Z_i$ be a point such that $|R(r)| = \max_{x \in Z_i} |R(x)|$; we may assume that $R(r) > 0$. Next, let $p^-, q^+ \in I_i$ be some points such that $P(p^-) = \min_{x \in Z_i} P(x)$ and $Q(q^+) = \max_{x \in Z_i} Q(x)$. Notice that $P(p^-) \leqslant Q(r) < P(r)$ and $Q(q^+) \geqslant P(r) > Q(r)$, so r is different from p^- and q^+.

Without loss of generality, we may assume that $p^- < r$. Then we have $R(p^-) = P(p^-) - Q(p^-) \leqslant 0 < R(r)$, so $R(x)$ has a root in $[p^-, r)$. If $q^+ > r$, then, similarly, $R(q^+) \leqslant 0 < R(r)$, and $R(x)$ also has a root in $(r, q^+]$; so $R(x)$ has two roots in I_i, as required.

In the remaining case we have $q^+ < r$; it suffices now to show that in this case ΔS has a root in I_i. Since $P(p^-) \leqslant Q(r)$ and $|R(p^-)| \leqslant R(r)$, we have $S(p^-) = 2P(p^-) - R(p^-) \leqslant 2Q(r) + R(r) = S(r)$. Similarly, we get $S(q^+) = 2Q(q^+) + R(q^+) \geqslant 2P(r) - R(r) = S(r)$. Therefore, the sequence of values of S on Z_i is neither strictly increasing nor strictly decreasing, which shows that ΔS has a root in I_i. $\square$

Comment 3. After finding the relation $P(x) - Q(x) = \alpha\big(T(x) - T(x-1)\big)$ from Solution 1, one may also follow the approach presented in Solution 2. Knowledge of the difference of polynomials may simplify some steps; e.g., it is clear now that $P(x) - Q(x)$ has exactly one root in each of the segments I_i.

Combinatorics

C1. In Lineland there are $n \geqslant 1$ towns, arranged along a road running from left to right. Each town has a *left bulldozer* (put to the left of the town and facing left) and a *right bulldozer* (put to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a right and a left bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, the bulldozers are quite unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

Let A and B be two towns, with B being to the right of A. We say that town A can *sweep town B away* if the right bulldozer of A can move over to B pushing off all bulldozers it meets. Similarly, B can sweep A away if the left bulldozer of B can move to A pushing off all bulldozers of all towns on its way.

Prove that there is exactly one town which cannot be swept away by any other one.

(Estonia)

Solution 1. Let $T_1, T_2, \ldots, T_n$ be the towns enumerated from left to right. Observe first that, if town T_i can sweep away town T_j, then T_i also can sweep away every town located between T_i and T_j.

We prove the problem statement by strong induction on n. The base case $n = 1$ is trivial.

For the induction step, we first observe that the left bulldozer in T_1 and the right bulldozer in T_n are completely useless, so we may forget them forever. Among the other $2n-2$ bulldozers, we choose the largest one. Without loss of generality, it is the right bulldozer of some town T_k with $k < n$.

Surely, with this large bulldozer T_k can sweep away all the towns to the right of it. Moreover, none of these towns can sweep T_k away; so they also cannot sweep away any town to the left of T_k. Thus, if we remove the towns $T_{k+1}, T_{k+2}, \ldots, T_n$, none of the remaining towns would change its status of being (un)sweepable away by the others.

Applying the induction hypothesis to the remaining towns, we find a unique town among $T_1, T_2, \ldots, T_k$ which cannot be swept away. By the above reasons, it is also the unique such town in the initial situation. Thus the induction step is established.

Solution 2. We start with the same enumeration and the same observation as in Solution 1. We also denote by ℓ_i and r_i the sizes of the left and the right bulldozers belonging to T_i, respectively. One may easily see that no two towns T_i and T_j with $i < j$ can sweep each other away, for this would yield $r_i > \ell_j > r_i$.

Clearly, there is no town which can sweep T_n away from the right. Then we may choose the leftmost town T_k which cannot be swept away from the right. One can observe now that no town T_i with $i > k$ may sweep away some town T_j with $j < k$, for otherwise T_i would be able to sweep T_k away as well.

Now we prove two claims, showing together that T_k is the unique town which cannot be swept away, and thus establishing the problem statement.

Claim 1. T_k also cannot be swept away from the left.

Proof. Let T_m be some town to the left of T_k. By the choice of T_k, town T_m can be swept away from the right by some town T_p with $p > m$. As we have already observed, p cannot be greater than k. On the other hand, T_m cannot sweep T_p away, so *a fortiori* it cannot sweep T_k away. □

Claim 2. Any town T_m with $m \neq k$ can be swept away by some other town.

Proof. If $m < k$, then T_m can be swept away from the right due to the choice of T_k. In the remaining case we have $m > k$.

Let T_p be a town among $T_k, T_{k+1}, \ldots, T_{m-1}$ having the largest right bulldozer. We claim that T_p can sweep T_m away. If this is not the case, then $r_p < \ell_q$ for some q with $p < q \leqslant m$. But this means that ℓ_q is greater than all the numbers r_i with $k \leqslant i \leqslant m-1$, so T_q can sweep T_k away. This contradicts the choice of T_k. □

Comment 1. One may employ the same ideas within the inductive approach. Here we sketch such a solution.

Assume that the problem statement holds for the collection of towns $T_1, T_2, \ldots, T_{n-1}$, so that there is a unique town T_i among them which cannot be swept away by any other of them. Thus we need to prove that in the full collection $T_1, T_2, \ldots, T_n$, exactly one of the towns T_i and T_n cannot be swept away.

If T_n cannot sweep T_i away, then it remains to prove that T_n can be swept away by some other town. This can be established as in the second paragraph of the proof of Claim 2.

If T_n can sweep T_i away, then it remains to show that T_n cannot be swept away by any other town. Since T_n can sweep T_i away, it also can sweep all the towns $T_i, T_{i+1}, \ldots, T_{n-1}$ away, so T_n cannot be swept away by any of those. On the other hand, none of the remaining towns $T_1, T_2, \ldots, T_{i-1}$ can sweep T_i away, so that they cannot sweep T_n away as well.

Comment 2. Here we sketch yet another inductive approach. Assume that $n > 1$. Firstly, we find a town which can be swept away by each of its neighbors (each town has two neighbors, except for the bordering ones each of which has one); we call such town a *loser*. Such a town exists, because there are $n-1$ pairs of neighboring towns, and in each of them there is only one which can sweep the other away; so there exists a town which is a winner in none of these pairs.

Notice that a loser can be swept away, but it cannot sweep any other town away (due to its neighbors' protection). Now we remove a loser, and *suggest* its left bulldozer to its *right* neighbor (if it exists), and its right bulldozer to a *left* one (if it exists). Surely, a town accepts a suggestion if a suggested bulldozer is larger than the town's one of the same orientation.

Notice that suggested bulldozers are useless in attack (by the definition of a loser), but may serve for defensive purposes. Moreover, each suggested bulldozer's protection works for the same pairs of remaining towns as before the removal.

By the induction hypothesis, the new configuration contains exactly one town which cannot be swept away. The arguments above show that the initial one also satisfies this property.

Solution 3. We separately prove that (i) there exists a town which cannot be swept away, and that (ii) there is at most one such town. We also make use of the two observations from the previous solutions.

To prove (i), assume contrariwise that every town can be swept away. Let t_1 be the leftmost town; next, for every $k = 1, 2, \ldots$ we inductively choose t_{k+1} to be some town which can sweep t_k away. Now we claim that for every $k = 1, 2, \ldots$, the town t_{k+1} is to the right of t_k; this leads to the contradiction, since the number of towns is finite.

Induction on k. The base case $k = 1$ is clear due to the choice of t_1. Assume now that for all j with $1 \leqslant j < k$, the town t_{j+1} is to the right of t_j. Suppose that t_{k+1} is situated to the left of t_k; then it lies between t_j and t_{j+1} (possibly coinciding with t_j) for some $j < k$. Therefore, t_{k+1} can be swept away by t_{j+1}, which shows that it cannot sweep t_{j+1} away — so t_{k+1} also cannot sweep t_k away. This contradiction proves the induction step.

To prove (ii), we also argue indirectly and choose two towns A and B neither of which can be swept away, with A being to the left of B. Consider the largest bulldozer b between them (taking into consideration the right bulldozer of A and the left bulldozer of B). Without loss of generality, b is a left bulldozer; then it is situated in some town to the right of A, and this town may sweep A away since nothing prevents it from doing that. A contradiction.

Comment 3. The Problem Selection Committee decided to reformulate this problem. The original formulation was as follows.

Let n be a positive integer. There are n cards in a deck, enumerated from bottom to top with numbers $1, 2, \ldots, n$. For each $i = 1, 2, \ldots, n$, an even number a_i is printed on the lower side and an odd number b_i is printed on the upper side of the i^{th} card. We say that the i^{th} card opens *the j^{th} card, if $i < j$ and $b_i < a_k$ for every $k = i+1, i+2, \ldots, j$. Similarly, we say that the i^{th} card* closes *the j^{th} card, if $i > j$ and $a_i < b_k$ for every $k = i-1, i-2, \ldots, j$. Prove that the deck contains exactly one card which is neither opened nor closed by any other card.*

C2. Let $\mathcal{V}$ be a finite set of points in the plane. We say that $\mathcal{V}$ is *balanced* if for any two distinct points $A, B \in \mathcal{V}$, there exists a point $C \in \mathcal{V}$ such that $AC = BC$. We say that $\mathcal{V}$ is *center-free* if for any distinct points $A, B, C \in \mathcal{V}$, there does not exist a point $P \in \mathcal{V}$ such that $PA = PB = PC$.

(*a*) Show that for all $n \geqslant 3$, there exists a balanced set consisting of n points.

(*b*) For which $n \geqslant 3$ does there exist a balanced, center-free set consisting of n points?

(Netherlands)

Answer for part (*b*). All odd integers $n \geqslant 3$.

Solution.

Part (*a*). Assume that n is odd. Consider a regular n-gon. Label the vertices of the n-gon as $A_1, A_2, \ldots, A_n$ in counter-clockwise order, and set $\mathcal{V} = \{A_1, \ldots, A_n\}$. We check that $\mathcal{V}$ is balanced. For any two distinct vertices A_i and A_j, let $k \in \{1, 2, \ldots, n\}$ be the solution of $2k \equiv i + j \pmod{n}$. Then, since $k - i \equiv j - k \pmod{n}$, we have $A_iA_k = A_jA_k$, as required.

Now assume that n is even. Consider a regular $(3n-6)$-gon, and let O be its circumcenter. Again, label its vertices as $A_1, \ldots, A_{3n-6}$ in counter-clockwise order, and choose $\mathcal{V} = \{O, A_1, A_2, \ldots, A_{n-1}\}$. We check that $\mathcal{V}$ is balanced. For any two distinct vertices A_i and A_j, we always have $OA_i = OA_j$. We now consider the vertices O and A_i. First note that the triangle $OA_iA_{n/2-1+i}$ is equilateral for all $i \leqslant \frac{n}{2}$. Hence, if $i \leqslant \frac{n}{2}$, then we have $OA_{n/2-1+i} = A_iA_{n/2-1+i}$; otherwise, if $i > \frac{n}{2}$, then we have $OA_{i-n/2+1} = A_iA_{i-n/2+1}$. This completes the proof.

An example of such a construction when $n = 10$ is shown in Figure 1.

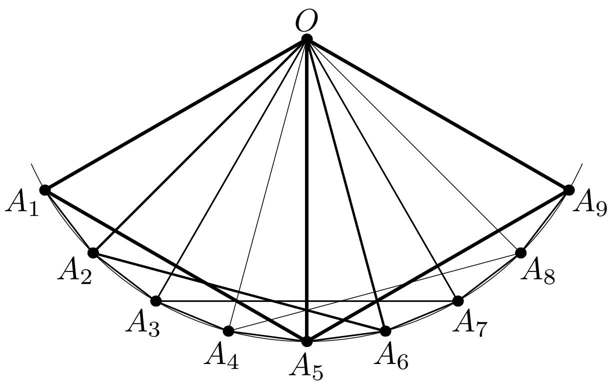

Figure 1

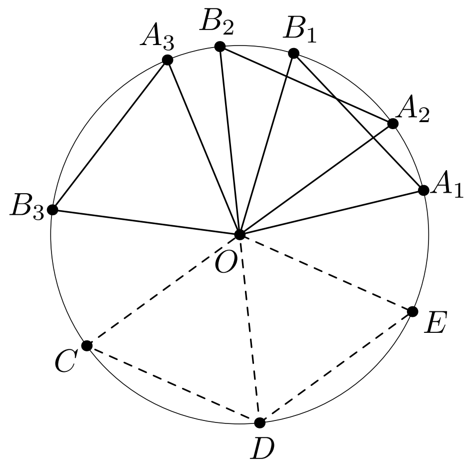

Figure 2

Comment (*a*). There are many ways to construct an example by placing equilateral triangles in a circle. Here we present one general method.

Let O be the center of a circle and let $A_1, B_1, \ldots, A_k, B_k$ be distinct points on the circle such that the triangle OA_iB_i is equilateral for each i. Then $\mathcal{V} = \{O, A_1, B_1, \ldots, A_k, B_k\}$ is balanced. To construct a set of even cardinality, put extra points C, D, E on the circle such that triangles OCD and ODE are equilateral (see Figure 2). Then $\mathcal{V} = \{O, A_1, B_1, \ldots, A_k, B_k, C, D, E\}$ is balanced.

Part (*b*). We now show that there exists a balanced, center-free set containing n points for all odd $n \geqslant 3$, and that one does not exist for any even $n \geqslant 3$.

If n is odd, then let $\mathcal{V}$ be the set of vertices of a regular n-gon. We have shown in part (a) that $\mathcal{V}$ is balanced. We claim that $\mathcal{V}$ is also center-free. Indeed, if P is a point such that

$PA = PB = PC$ for some three distinct vertices A, B and C, then P is the circumcenter of the n-gon, which is not contained in $\mathcal{V}$.

Now suppose that $\mathcal{V}$ is a balanced, center-free set of even cardinality n. We will derive a contradiction. For a pair of distinct points $A, B \in \mathcal{V}$, we say that a point $C \in \mathcal{V}$ is *associated* with the pair $\{A, B\}$ if $AC = BC$. Since there are $\frac{n(n-1)}{2}$ pairs of points, there exists a point $P \in \mathcal{V}$ which is associated with at least $\left\lceil \frac{n(n-1)}{2} / n \right\rceil = \frac{n}{2}$ pairs. Note that none of these $\frac{n}{2}$ pairs can contain P, so that the union of these $\frac{n}{2}$ pairs consists of at most $n - 1$ points. Hence there exist two such pairs that share a point. Let these two pairs be $\{A, B\}$ and $\{A, C\}$. Then $PA = PB = PC$, which is a contradiction.

Comment (*b*). We can rephrase the argument in graph theoretic terms as follows. Let $\mathcal{V}$ be a balanced, center-free set consisting of n points. For any pair of distinct vertices $A, B \in \mathcal{V}$ and for any $C \in \mathcal{V}$ such that $AC = BC$, draw directed edges $A \to C$ and $B \to C$. Then all pairs of vertices generate altogether at least $n(n-1)$ directed edges; since the set is center-free, these edges are distinct. So we must obtain a graph in which any two vertices are connected in both directions. Now, each vertex has exactly $n - 1$ incoming edges, which means that $n - 1$ is even. Hence n is odd.

C3. For a finite set A of positive integers, we call a partition of A into two disjoint nonempty subsets A_1 and A_2 *good* if the least common multiple of the elements in A_1 is equal to the greatest common divisor of the elements in A_2. Determine the minimum value of n such that there exists a set of n positive integers with exactly 2015 good partitions.

(Ukraine)

Answer. 3024.

Solution. Let $A = \{a_1, a_2, \ldots, a_n\}$, where $a_1 < a_2 < \cdots < a_n$. For a finite nonempty set B of positive integers, denote by $\operatorname{lcm} B$ and $\gcd B$ the least common multiple and the greatest common divisor of the elements in B, respectively.

Consider any good partition (A_1, A_2) of A. By definition, $\operatorname{lcm} A_1 = d = \gcd A_2$ for some positive integer d. For any $a_i \in A_1$ and $a_j \in A_2$, we have $a_i \leqslant d \leqslant a_j$. Therefore, we have $A_1 = \{a_1, a_2, \ldots, a_k\}$ and $A_2 = \{a_{k+1}, a_{k+2}, \ldots, a_n\}$ for some k with $1 \leqslant k < n$. Hence, each good partition is determined by an element a_k, where $1 \leqslant k < n$. We call such a_k *partitioning*.

It is convenient now to define $\ell_k = \operatorname{lcm}(a_1, a_2, \ldots, a_k)$ and $g_k = \gcd(a_{k+1}, a_{k+2}, \ldots, a_n)$ for $1 \leqslant k \leqslant n-1$. So a_k is partitioning exactly when $\ell_k = g_k$.

We proceed by proving some properties of partitioning elements, using the following claim.

Claim. If a_{k-1} and a_k are partitioning where $2 \leqslant k \leqslant n-1$, then $g_{k-1} = g_k = a_k$.

Proof. Assume that a_{k-1} and a_k are partitioning. Since $\ell_{k-1} = g_{k-1}$, we have $\ell_{k-1} \mid a_k$. Therefore, $g_k = \ell_k = \operatorname{lcm}(\ell_{k-1}, a_k) = a_k$, and $g_{k-1} = \gcd(a_k, g_k) = a_k$, as desired. □

Property 1. For every $k = 2, 3, \ldots, n-2$, at least one of a_{k-1}, a_k, and a_{k+1} is not partitioning.

Proof. Suppose, to the contrary, that all three numbers a_{k-1}, a_k, and a_{k+1} are partitioning. The claim yields that $a_{k+1} = g_k = a_k$, a contradiction. □

Property 2. The elements a_1 and a_2 cannot be simultaneously partitioning. Also, a_{n-2} and a_{n-1} cannot be simultaneously partitioning

Proof. Assume that a_1 and a_2 are partitioning. By the claim, it follows that $a_2 = g_1 = \ell_1 = \operatorname{lcm}(a_1) = a_1$, a contradiction.

Similarly, assume that a_{n-2} and a_{n-1} are partitioning. The claim yields that $a_{n-1} = g_{n-1} = \gcd(a_n) = a_n$, a contradiction. □

Now let A be an n-element set with exactly 2015 good partitions. Clearly, we have $n \geqslant 5$. Using Property 2, we find that there is at most one partitioning element in each of $\{a_1, a_2\}$ and $\{a_{n-2}, a_{n-1}\}$. By Property 1, there are at least $\left\lfloor \frac{n-5}{3} \right\rfloor$ non-partitioning elements in $\{a_3, a_4, \ldots, a_{n-3}\}$. Therefore, there are at most $(n-1) - 2 - \left\lfloor \frac{n-5}{3} \right\rfloor = \left\lceil \frac{2(n-2)}{3} \right\rceil$ partitioning elements in A. Thus, $\left\lceil \frac{2(n-2)}{3} \right\rceil \geqslant 2015$, which implies that $n \geqslant 3024$.

Finally, we show that there exists a set of 3024 positive integers with exactly 2015 partitioning elements. Indeed, in the set $A = \{2 \cdot 6^i, 3 \cdot 6^i, 6^{i+1} \mid 0 \leqslant i \leqslant 1007\}$, each element of the form $3 \cdot 6^i$ or 6^i, except 6^{1008}, is partitioning.

Therefore, the minimum possible value of n is 3024.

Comment. Here we will work out the general case when 2015 is replaced by an arbitrary positive integer m. Note that the bound $\left\lceil \frac{2(n-2)}{3} \right\rceil \geqslant m$ obtained in the solution is, in fact, true for any positive integers m and n. Using this bound, one can find that $n \geqslant \left\lceil \frac{3m}{2} \right\rceil + 1$.

To show that the bound is sharp, one constructs a set of $\left\lceil \frac{3m}{2} \right\rceil + 1$ elements with exactly m good partitions. Indeed, the minimum is attained on the set $\{6^i, 2 \cdot 6^i, 3 \cdot 6^i \mid 0 \leqslant i \leqslant t-1\} \cup \{6^t\}$ for every even $m = 2t$, and $\{2 \cdot 6^i, 3 \cdot 6^i, 6^{i+1} \mid 0 \leqslant i \leqslant t-1\}$ for every odd $m = 2t-1$.

C4. Let n be a positive integer. Two players A and B play a game in which they take turns choosing positive integers $k \leqslant n$. The rules of the game are:

(i) A player cannot choose a number that has been chosen by either player on any previous turn.

(ii) A player cannot choose a number consecutive to any of those the player has already chosen on any previous turn.

(iii) The game is a draw if all numbers have been chosen; otherwise the player who cannot choose a number anymore loses the game.

The player A takes the first turn. Determine the outcome of the game, assuming that both players use optimal strategies.

(Finland)

Answer. The game ends in a draw when $n = 1, 2, 4, 6$; otherwise B wins.

Solution. For brevity, we denote by $[n]$ the set $\{1, 2, \ldots, n\}$.

Firstly, we show that B wins whenever $n \neq 1, 2, 4, 6$. For this purpose, we provide a strategy which guarantees that B can always make a move after A's move, and also guarantees that the game does not end in a draw.

We begin with an important observation.

Lemma. Suppose that B's first pick is n and that A has made the k^{th} move where $k \geqslant 2$. Then B can also make the k^{th} move.

Proof. Let $\mathcal{S}$ be the set of the first k numbers chosen by A. Since $\mathcal{S}$ does not contain consecutive integers, we see that the set $[n] \setminus \mathcal{S}$ consists of k "contiguous components" if $1 \in \mathcal{S}$, and $k + 1$ components otherwise. Since B has chosen only $k - 1$ numbers, there is at least one component of $[n] \setminus \mathcal{S}$ consisting of numbers not yet picked by B. Hence, B can choose a number from this component. $\square$

We will now describe a winning strategy for B, when $n \neq 1, 2, 4, 6$. By symmetry, we may assume that A's first choice is a number not exceeding $\frac{n+1}{2}$. So B can pick the number n in B's first turn. We now consider two cases.

Case 1. n is odd and $n \geqslant 3$. The only way the game ends in a draw is that A eventually picks all the odd numbers from the set $[n]$. However, B has already chosen n, so this cannot happen. Thus B can continue to apply the lemma until A cannot make a move.

Case 2. n is even and $n \geqslant 8$. Since B has picked n, the game is a draw only if A can eventually choose all the odd numbers from the set $[n-1]$. So B picks a number from the set $\{1, 3, 5, \ldots, n-3\}$ not already chosen by A, on B's second move. This is possible since the set consists of $\frac{n-2}{2} \geqslant 3$ numbers and A has chosen only 2 numbers. Hereafter B can apply the lemma until A cannot make a move.

Hence, in both cases A loses.

We are left with the cases $n = 1, 2, 4, 6$. The game is trivially a draw when $n = 1, 2$. When $n = 4$, A has to first pick 1 to avoid losing. Similarly, B has to choose 4 as well. It then follows that the game ends in a draw.

When $n = 6$, B gets at least a draw by the lemma or by using a mirror strategy. On the other hand, A may also get at least a draw in the following way. In the first turn, A chooses 1. After B's response by a number b, A finds a neighbor c of b which differs from 1 and 2, and reserves c for A's third move. Now, clearly A can make the second move by choosing a number different from $1, 2, c-1, c, c+1$. Therefore A will not lose.

Comment 1. We present some explicit winning strategies for B.

We start with the case n is odd and $n \geqslant 3$. B starts by picking n in the first turn. On the k^{th} move for $k \geqslant 2$, B chooses the number exactly 1 less than A's k^{th} pick. The only special case is when A's k^{th} choice is 1. In this situation, A's first pick was a number $a > 1$ and B can respond by choosing $a - 1$ on the k^{th} move instead.

We now give an alternative winning strategy in the case n is even and $n \geqslant 8$. We first present a winning strategy for the case when A's first pick is 1. We consider two cases depending on A's second move.

Case 1. A's second pick is 3. Then B chooses $n-3$ on the second move. On the k^{th} move, B chooses the number exactly 1 less than A's k^{th} pick except that B chooses 2 if A's k^{th} pick is $n-2$ or $n-1$.

Case 2. A's second pick is $a > 3$. Then B chooses $a - 2$ on the second move. Afterwards on the k^{th} move, B picks the number exactly 1 less than A's k^{th} pick.

One may easily see that this strategy guarantees B's victory, when A's first pick is 1.

The following claim shows how to extend the strategy to the general case.

Claim. Assume that B has an explicit strategy leading to a victory after A picks 1 on the first move. Then B also has an explicit strategy leading to a victory after any first moves of A.

Proof. Let S be an optimal strategy of B after A picks 1 on the first move. Assume that A picks some number $a > 1$ on this move; we show how B can make use of S in order to win in this case.

In parallel to the real play, B starts an *imaginary* play. The positions in these plays differ by flipping the segment $[1, a]$; so, if a player chooses some number x in the real play, then the same player chooses a number x or $a + 1 - x$ in the imaginary play, depending on whether $x > a$ or $x \leqslant a$. Thus A's first pick in the imaginary play is 1.

Clearly, a number is chosen in the real play exactly if the corresponding number is chosen in the imaginary one. Next, if an unchosen number is neighboring to one chosen by A in the imaginary play, then the corresponding number also has this property in the real play, so A also cannot choose it. One can easily see that a similar statement with real and imaginary plays interchanged holds for B instead of A.

Thus, when A makes some move in the real play, B may imagine the corresponding legal move in the imaginary one. Then B chooses the response according to S in the imaginary game and makes the corresponding legal move in the real one. Acting so, B wins the imaginary game, thus B will also win the real one. $\square$

Hence, B has a winning strategy for all even n greater or equal to 8.

Notice that the claim can also be used to simplify the argument when n is odd.

Comment 2. One may also employ symmetry when n is odd. In particular, B could use a mirror strategy. However, additional ideas are required to modify the strategy after A picks $\frac{n+1}{2}$.

C5. Consider an infinite sequence $a_1, a_2, \dots$ of positive integers with $a_i \leqslant 2015$ for all $i \geqslant 1$. Suppose that for any two distinct indices i and j we have $i + a_i \neq j + a_j$.

Prove that there exist two positive integers b and N such that

$$\left| \sum_{i=m+1}^{n} (a_i - b) \right| \leqslant 1007^2$$

whenever $n > m \geqslant N$.

(Australia)

Solution 1. We visualize the set of positive integers as a sequence of points. For each n we draw an arrow emerging from n that points to $n + a_n$; so the *length* of this arrow is a_n. Due to the condition that $m + a_m \neq n + a_n$ for $m \neq n$, each positive integer receives at most one arrow. There are some positive integers, such as 1, that receive no arrows; these will be referred to as *starting points* in the sequel. When one starts at any of the starting points and keeps following the arrows, one is led to an infinite path, called its *ray*, that visits a strictly increasing sequence of positive integers. Since the length of any arrow is at most 2015, such a ray, say with starting point s, meets every interval of the form $[n, n + 2014]$ with $n \geqslant s$ at least once.

Suppose for the sake of contradiction that there would be at least 2016 starting points. Then we could take an integer n that is larger than the first 2016 starting points. But now the interval $[n, n + 2014]$ must be met by at least 2016 rays in distinct points, which is absurd. We have thereby shown that the number b of starting points satisfies $1 \leqslant b \leqslant 2015$. Let N denote any integer that is larger than all starting points. We contend that b and N are as required.

To see this, let any two integers m and n with $n > m \geqslant N$ be given. The sum $\sum_{i=m+1}^{n} a_i$ gives the total length of the arrows emerging from $m + 1, \dots, n$. Taken together, these arrows form b subpaths of our rays, some of which may be empty. Now on each ray we look at the first number that is larger than m; let $x_1, \dots, x_b$ denote these numbers, and let $y_1, \dots, y_b$ enumerate in corresponding order the numbers defined similarly with respect to n. Then the list of differences $y_1 - x_1, \dots, y_b - x_b$ consists of the lengths of these paths and possibly some zeros corresponding to empty paths. Consequently, we obtain

$$\sum_{i=m+1}^{n} a_i = \sum_{j=1}^{b} (y_j - x_j)\,,$$

whence

$$\sum_{i=m+1}^{n} (a_i - b) = \sum_{j=1}^{b} (y_j - n) - \sum_{j=1}^{b} (x_j - m)\,.$$

Now each of the b rays meets the interval $[m + 1, m + 2015]$ at some point and thus $x_1 - m, \dots, x_b - m$ are b distinct members of the set $\{1, 2, \dots, 2015\}$. Moreover, since $m + 1$ is not a starting point, it must belong to some ray; so 1 has to appear among these numbers, wherefore

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (x_j - m) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j)\,.$$

The same argument applied to n and $y_1, \dots, y_b$ yields

$$1 + \sum_{j=1}^{b-1} (j + 1) \leqslant \sum_{j=1}^{b} (y_j - n) \leqslant 1 + \sum_{j=1}^{b-1} (2016 - b + j)\,.$$

So altogether we get

$$\left|\sum_{i=m+1}^{n}(a_i-b)\right| \leqslant \sum_{j=1}^{b-1}\big((2016-b+j)-(j+1)\big) = (b-1)(2015-b)$$
$$\leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2\,,$$

as desired.

Solution 2. Set $s_n = n + a_n$ for all positive integers n. By our assumptions, we have

$$n+1 \leqslant s_n \leqslant n+2015$$

for all $n \in \mathbb{Z}_{>0}$. The members of the sequence $s_1, s_2, \ldots$ are distinct. We shall investigate the set

$$M = \mathbb{Z}_{>0} \setminus \{s_1, s_2, \ldots\}\,.$$

Claim. At most 2015 numbers belong to M.

Proof. Otherwise let $m_1 < m_2 < \cdots < m_{2016}$ be any 2016 distinct elements from M. For $n = m_{2016}$ we have

$$\{s_1, \ldots, s_n\} \cup \{m_1, \ldots, m_{2016}\} \subseteq \{1, 2, \ldots, n+2015\}\,,$$

where on the left-hand side we have a disjoint union containing altogether $n + 2016$ elements. But the set on the right-hand side has only $n + 2015$ elements. This contradiction proves our claim. □

Now we work towards proving that the positive integers $b = |M|$ and $N = \max(M)$ are as required. Recall that we have just shown $b \leqslant 2015$.

Let us consider any integer $r \geqslant N$. As in the proof of the above claim, we see that

$$B_r = M \cup \{s_1, \ldots, s_r\} \tag{1}$$

is a subset of $[1, r+2015] \cap \mathbb{Z}$ with precisely $b+r$ elements. Due to the definitions of M and N, we also know $[1, r+1] \cap \mathbb{Z} \subseteq B_r$. It follows that there is a set $C_r \subseteq \{1, 2, \ldots, 2014\}$ with $|C_r| = b - 1$ and

$$B_r = \big([1, r+1] \cap \mathbb{Z}\big) \cup \big\{r+1+x \,\big|\, x \in C_r\big\}\,. \tag{2}$$

For any finite set of integers J we denote the sum of its elements by $\sum J$. Now the equations (1) and (2) give rise to two ways of computing $\sum B_r$ and the comparison of both methods leads to

$$\sum M + \sum_{i=1}^{r} s_i = \sum_{i=1}^{r} i + b(r+1) + \sum C_r\,,$$

or in other words to

$$\sum M + \sum_{i=1}^{r}(a_i - b) = b + \sum C_r\,. \tag{3}$$

After this preparation, we consider any two integers m and n with $n > m \geqslant N$. Plugging $r = n$ and $r = m$ into (3) and subtracting the estimates that result, we deduce

$$\sum_{i=m+1}^{n}(a_i - b) = \sum C_n - \sum C_m\,.$$

Since C_n and C_m are subsets of $\{1, 2, \ldots, 2014\}$ with $|C_n| = |C_m| = b - 1$, it is clear that the absolute value of the right-hand side of the above inequality attains its largest possible value if either $C_m = \{1, 2, \ldots, b-1\}$ and $C_n = \{2016 - b, \ldots, 2014\}$, or the other way around. In these two cases we have

$$\left|\sum C_n - \sum C_m\right| = (b-1)(2015-b)\,,$$

so in the general case we find

$$\left|\sum_{i=m+1}^{n} (a_i - b)\right| \leqslant (b-1)(2015-b) \leqslant \left(\frac{(b-1)+(2015-b)}{2}\right)^2 = 1007^2\,,$$

as desired.

Comment. The sets C_n may be visualized by means of the following process: Start with an empty blackboard. For $n \geqslant 1$, the following happens during the n^{th} step. The number a_n gets written on the blackboard, then all numbers currently on the blackboard are decreased by 1, and finally all zeros that have arisen get swept away.

It is not hard to see that the numbers present on the blackboard after n steps are distinct and form the set C_n. Moreover, it is possible to complete a solution based on this idea.

C6. Let S be a nonempty set of positive integers. We say that a positive integer n is *clean* if it has a unique representation as a sum of an odd number of distinct elements from S. Prove that there exist infinitely many positive integers that are not clean.

(U.S.A.)

Solution 1. Define an *odd* (respectively, *even*) *representation* of n to be a representation of n as a sum of an odd (respectively, even) number of distinct elements of S. Let $\mathbb{Z}_{>0}$ denote the set of all positive integers.

Suppose, to the contrary, that there exist only finitely many positive integers that are not clean. Therefore, there exists a positive integer N such that every integer $n > N$ has exactly one odd representation.

Clearly, S is infinite. We now claim the following properties of odd and even representations.

Property 1. Any positive integer n has at most one odd and at most one even representation.

Proof. We first show that every integer n has at most one even representation. Since S is infinite, there exists $x \in S$ such that $x > \max\{n, N\}$. Then, the number $n + x$ must be clean, and x does not appear in any even representation of n. If n has more than one even representation, then we obtain two distinct odd representations of $n + x$ by adding x to the even representations of n, which is impossible. Therefore, n can have at most one even representation.

Similarly, there exist two distinct elements $y, z \in S$ such that $y, z > \max\{n, N\}$. If n has more than one odd representation, then we obtain two distinct odd representations of $n + y + z$ by adding y and z to the odd representations of n. This is again a contradiction. □

Property 2. Fix $s \in S$. Suppose that a number $n > N$ has no even representation. Then $n + 2as$ has an even representation containing s for all integers $a \geqslant 1$.

Proof. It is sufficient to prove the following statement: If n has no even representation without s, then $n+2s$ has an even representation containing s (and hence no even representation without s by Property 1).

Notice that the odd representation of $n + s$ does not contain s; otherwise, we have an even representation of n without s. Then, adding s to this odd representation of $n + s$, we get that $n + 2s$ has an even representation containing s, as desired. □

Property 3. Every sufficiently large integer has an even representation.

Proof. Fix any $s \in S$, and let r be an arbitrary element in $\{1, 2, \ldots, 2s\}$. Then, Property 2 implies that the set $Z_r = \{r + 2as : a \geqslant 0\}$ contains at most one number exceeding N with no even representation. Therefore, Z_r contains finitely many positive integers with no even representation, and so does $\mathbb{Z}_{>0} = \bigcup_{r=1}^{2s} Z_r$. □

In view of Properties 1 and 3, we may assume that N is chosen such that every $n > N$ has exactly one odd and exactly one even representation. In particular, each element $s > N$ of S has an even representation.

Property 4. For any $s, t \in S$ with $N < s < t$, the even representation of t contains s.

Proof. Suppose the contrary. Then, $s + t$ has at least two odd representations: one obtained by adding s to the even representation of t and one obtained by adding t to the even representation of s. Since the latter does not contain s, these two odd representations of $s + t$ are distinct, a contradiction. □

Let $s_1 < s_2 < \cdots$ be all the elements of S, and set $\sigma_n = \sum_{i=1}^{n} s_i$ for each nonnegative integer n. Fix an integer k such that $s_k > N$. Then, Property 4 implies that for every $i > k$ the even representation of s_i contains all the numbers $s_k, s_{k+1}, \ldots, s_{i-1}$. Therefore,

$$s_i = s_k + s_{k+1} + \cdots + s_{i-1} + R_i = \sigma_{i-1} - \sigma_{k-1} + R_i, \tag{1}$$

where R_i is a sum of some of $s_1, \ldots, s_{k-1}$. In particular, $0 \leqslant R_i \leqslant s_1 + \cdots + s_{k-1} = \sigma_{k-1}$.

Let j_0 be an integer satisfying $j_0 > k$ and $\sigma_{j_0} > 2\sigma_{k-1}$. Then (1) shows that, for every $j > j_0$,

$$s_{j+1} \geqslant \sigma_j - \sigma_{k-1} > \sigma_j/2. \tag{2}$$

Next, let $p > j_0$ be an index such that $R_p = \min_{i>j_0} R_i$. Then,

$$s_{p+1} = s_k + s_{k+1} + \cdots + s_p + R_{p+1} = (s_p - R_p) + s_p + R_{p+1} \geqslant 2s_p.$$

Therefore, there is no element of S larger than s_p but smaller than $2s_p$. It follows that the even representation τ of $2s_p$ does not contain any element larger than s_p. On the other hand, inequality (2) yields $2s_p > s_1 + \cdots + s_{p-1}$, so τ must contain a term larger than s_{p-1}. Thus, it must contain s_p. After removing s_p from τ, we have that s_p has an odd representation not containing s_p, which contradicts Property 1 since s_p itself also forms an odd representation of s_p.

Solution 2. We will also use Property 1 from Solution 1.

We first define some terminology and notations used in this solution. Let $\mathbb{Z}_{\geqslant 0}$ denote the set of all nonnegative integers. All sums mentioned are regarded as sums of distinct elements of S. Moreover, a sum is called *even* or *odd* depending on the parity of the number of terms in it. All closed or open intervals refer to sets of all integers inside them, e.g., $[a, b] = \{x \in \mathbb{Z} \colon a \leqslant x \leqslant b\}$.

Again, let $s_1 < s_2 < \cdots$ be all elements of S, and denote $\sigma_n = \sum_{i=1}^{n} s_i$ for each positive integer n. Let O_n (respectively, E_n) be the set of numbers representable as an odd (respectively, even) sum of elements of $\{s_1, \ldots, s_n\}$. Set $E = \bigcup_{n=1}^{\infty} E_n$ and $O = \bigcup_{n=1}^{\infty} O_n$. We assume that $0 \in E_n$ since 0 is representable as a sum of 0 terms.

We now proceed to our proof. Assume, to the contrary, that there exist only finitely many positive integers that are not clean and denote the number of non-clean positive integers by $m - 1$. Clearly, S is infinite. By Property 1 from Solution 1, every positive integer n has at most one odd and at most one even representation.

<u>*Step 1.*</u> We estimate s_{n+1} and σ_{n+1}.

Upper bounds: Property 1 yields $|O_n| = |E_n| = 2^{n-1}$, so $\big|[1, 2^{n-1} + m] \setminus O_n\big| \geqslant m$. Hence, there exists a clean integer $x_n \in [1, 2^{n-1} + m] \setminus O_n$. The definition of O_n then yields that the odd representation of x_n contains a term larger than s_n. Therefore, $s_{n+1} \leqslant x_n \leqslant 2^{n-1} + m$ for every positive integer n. Moreover, since s_1 is the smallest clean number, we get $\sigma_1 = s_1 \leqslant m$. Then,

$$\sigma_{n+1} = \sum_{i=2}^{n+1} s_i + s_1 \leqslant \sum_{i=2}^{n+1} (2^{i-2} + m) + m = 2^n - 1 + (n+1)m$$

for every positive integer n. Notice that this estimate also holds for $n = 0$.

Lower bounds: Since $O_{n+1} \subseteq [1, \sigma_{n+1}]$, we have $\sigma_{n+1} \geqslant |O_{n+1}| = 2^n$ for all positive integers n. Then,

$$s_{n+1} = \sigma_{n+1} - \sigma_n \geqslant 2^n - (2^{n-1} - 1 + nm) = 2^{n-1} + 1 - nm$$

for every positive integer n.

Combining the above inequalities, we have

$$2^{n-1} + 1 - nm \leqslant s_{n+1} \leqslant 2^{n-1} + m \qquad \text{and} \qquad 2^n \leqslant \sigma_{n+1} \leqslant 2^n - 1 + (n+1)m, \tag{3}$$

for every positive integer n.

<u>*Step 2.*</u> We prove Property 3 from Solution 1.

For every integer x and set of integers Y, define $x \pm Y = \{x \pm y \colon y \in Y\}$.

In view of Property 1, we get

$$E_{n+1} = E_n \sqcup (s_{n+1} + O_n) \qquad \text{and} \qquad O_{n+1} = O_n \sqcup (s_{n+1} + E_n),$$

where $\sqcup$ denotes the disjoint union operator. Notice also that $s_{n+2} \geqslant 2^n + 1 - (n+1)m > 2^{n-1} - 1 + nm \geqslant \sigma_n$ for every sufficiently large n. We now claim the following.

Claim 1. $(\sigma_n - s_{n+1}, s_{n+2} - s_{n+1}) \subseteq E_n$ for every sufficiently large n.

Proof. For sufficiently large n, all elements of (σ_n, s_{n+2}) are clean. Clearly, the elements of (σ_n, s_{n+2}) can be in neither O_n nor $O \setminus O_{n+1}$. So, $(\sigma_n, s_{n+2}) \subseteq O_{n+1} \setminus O_n = s_{n+1} + E_n$, which yields the claim. □

Now, Claim 1 together with inequalities (3) implies that, for all sufficiently large n,

$$E \supseteq E_n \supseteq (\sigma_n - s_{n+1}, s_{n+2} - s_{n+1}) \supseteq \big(2nm, 2^{n-1} - (n+2)m\big).$$

This easily yields that $\mathbb{Z}_{\geqslant 0} \setminus E$ is also finite. Since $\mathbb{Z}_{\geqslant 0} \setminus O$ is also finite, by Property 1, there exists a positive integer N such that every integer $n > N$ has exactly one even and one odd representation.

<u>*Step 3.*</u> We investigate the structures of E_n and O_n.

Suppose that $z \in E_{2n}$. Since z can be represented as an even sum using $\{s_1, s_2, \ldots, s_{2n}\}$, so can its complement $\sigma_{2n} - z$. Thus, we get $E_{2n} = \sigma_{2n} - E_{2n}$. Similarly, we have

$$E_{2n} = \sigma_{2n} - E_{2n}, \quad O_{2n} = \sigma_{2n} - O_{2n}, \qquad E_{2n+1} = \sigma_{2n+1} - O_{2n+1}, \quad O_{2n+1} = \sigma_{2n+1} - E_{2n+1}. \tag{4}$$

Claim 2. For every sufficiently large n, we have

$$[0, \sigma_n] \supseteq O_n \supseteq (N, \sigma_n - N) \qquad \text{and} \qquad [0, \sigma_n] \supseteq E_n \supseteq (N, \sigma_n - N).$$

Proof. Clearly $O_n, E_n \subseteq [0, \sigma_n]$ for every positive integer n. We now prove $O_n, E_n \supseteq (N, \sigma_n - N)$. Taking n sufficiently large, we may assume that $s_{n+1} \geqslant 2^{n-1} + 1 - nm > \frac{1}{2}(2^{n-1} - 1 + nm) \geqslant \sigma_n/2$. Therefore, the odd representation of every element of $(N, \sigma_n/2]$ cannot contain a term larger than s_n. Thus, $(N, \sigma_n/2] \subseteq O_n$. Similarly, since $s_{n+1} + s_1 > \sigma_n/2$, we also have $(N, \sigma_n/2] \subseteq E_n$. Equations (4) then yield that, for sufficiently large n, the interval $(N, \sigma_n - N)$ is a subset of both O_n and E_n, as desired. □

<u>*Step 4.*</u> We obtain a final contradiction.

Notice that $0 \in \mathbb{Z}_{\geqslant 0} \setminus O$ and $1 \in \mathbb{Z}_{\geqslant 0} \setminus E$. Therefore, the sets $\mathbb{Z}_{\geqslant 0} \setminus O$ and $\mathbb{Z}_{\geqslant 0} \setminus E$ are nonempty. Denote $o = \max(\mathbb{Z}_{\geqslant 0} \setminus O)$ and $e = \max(\mathbb{Z}_{\geqslant 0} \setminus E)$. Observe also that $e, o \leqslant N$.

Taking k sufficiently large, we may assume that $\sigma_{2k} > 2N$ and that Claim 2 holds for all $n \geqslant 2k$. Due to (4) and Claim 2, we have that $\sigma_{2k} - e$ is the minimal number greater than N which is not in E_{2k}, i.e., $\sigma_{2k} - e = s_{2k+1} + s_1$. Similarly,

$$\sigma_{2k} - o = s_{2k+1}, \qquad \sigma_{2k+1} - e = s_{2k+2}, \qquad \text{and} \qquad \sigma_{2k+1} - o = s_{2k+2} + s_1.$$

Therefore, we have

$$\begin{aligned} s_1 &= (s_{2k+1} + s_1) - s_{2k+1} = (\sigma_{2k} - e) - (\sigma_{2k} - o) = o - e \\ &= (\sigma_{2k+1} - e) - (\sigma_{2k+1} - o) = s_{2k+2} - (s_{2k+2} + s_1) = -s_1, \end{aligned}$$

which is impossible since $s_1 > 0$.

C7. In a company of people some pairs are enemies. A group of people is called *unsociable* if the number of members in the group is odd and at least 3, and it is possible to arrange all its members around a round table so that every two neighbors are enemies. Given that there are at most 2015 unsociable groups, prove that it is possible to partition the company into 11 parts so that no two enemies are in the same part.

(Russia)

Solution 1. Let $G = (V, E)$ be a graph where V is the set of people in the company and E is the set of the enemy pairs — the edges of the graph. In this language, partitioning into 11 disjoint enemy-free subsets means properly coloring the vertices of this graph with 11 colors.

We will prove the following more general statement.

Claim. Let G be a graph with chromatic number $k \geqslant 3$. Then G contains at least $2^{k-1} - k$ unsociable groups.

Recall that the chromatic number of G is the least k such that a proper coloring

$$V = V_1 \sqcup \cdots \sqcup V_k \tag{1}$$

exists. In view of $2^{11} - 12 > 2015$, the claim implies the problem statement.

Let G be a graph with chromatic number k. We say that a proper coloring (1) of G is *leximinimal*, if the k-tuple $(|V_1|, |V_2|, \ldots, |V_k|)$ is lexicographically minimal; in other words, the following conditions are satisfied: the number $n_1 = |V_1|$ is minimal; the number $n_2 = |V_2|$ is minimal, subject to the previously chosen value of n_1; $\ldots$; the number $n_{k-1} = |V_{k-1}|$ is minimal, subject to the previously chosen values of $n_1, \ldots, n_{k-2}$.

The following lemma is the core of the proof.

Lemma 1. Suppose that $G = (V, E)$ is a graph with odd chromatic number $k \geqslant 3$, and let (1) be one of its leximinimal colorings. Then G contains an odd cycle which visits all color classes $V_1, V_2, \ldots, V_k$.

Proof of Lemma 1. Let us call a cycle *colorful* if it visits all color classes.

Due to the definition of the chromatic number, V_1 is nonempty. Choose an arbitrary vertex $v \in V_1$. We construct a colorful odd cycle that has only one vertex in V_1, and this vertex is v.

We draw a subgraph of G as follows. Place v in the center, and arrange the sets $V_2, V_3, \ldots, V_k$ in counterclockwise circular order around it. For convenience, let $V_{k+1} = V_2$. We will draw arrows to add direction to some edges of G, and mark the vertices these arrows point to. First we draw arrows from v to all its neighbors in V_2, and mark all those neighbors. If some vertex $u \in V_i$ with $i \in \{2, 3, \ldots, k\}$ is already marked, we draw arrows from u to all its neighbors in V_{i+1} which are not marked yet, and we mark all of them. We proceed doing this as long as it is possible. The process of marking is exemplified in Figure 1.

Notice that by the rules of our process, in the final state, marked vertices in V_i cannot have unmarked neighbors in V_{i+1}. Moreover, v is connected to all marked vertices by directed paths.

Now move each marked vertex to the next color class in circular order (see an example in Figure 3). In view of the arguments above, the obtained coloring $V_1 \sqcup W_2 \sqcup \cdots \sqcup W_k$ is proper. Notice that v has a neighbor $w \in W_2$, because otherwise

$$\big(V_1 \setminus \{v\}\big) \sqcup \big(W_2 \cup \{v\}\big) \sqcup W_3 \sqcup \cdots \sqcup W_k$$

would be a proper coloring lexicographically smaller than (1). If w was unmarked, i.e., w was an element of V_2, then it would be marked at the beginning of the process and thus moved to V_3, which did not happen. Therefore, w is marked and $w \in V_k$.

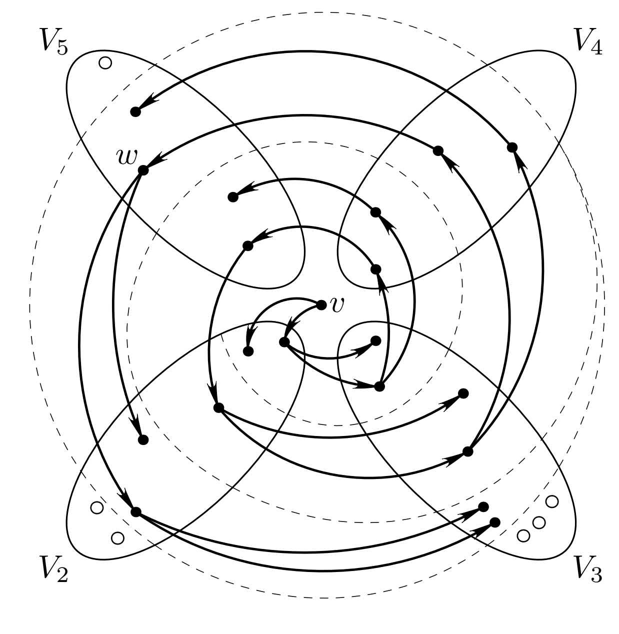

Figure 1

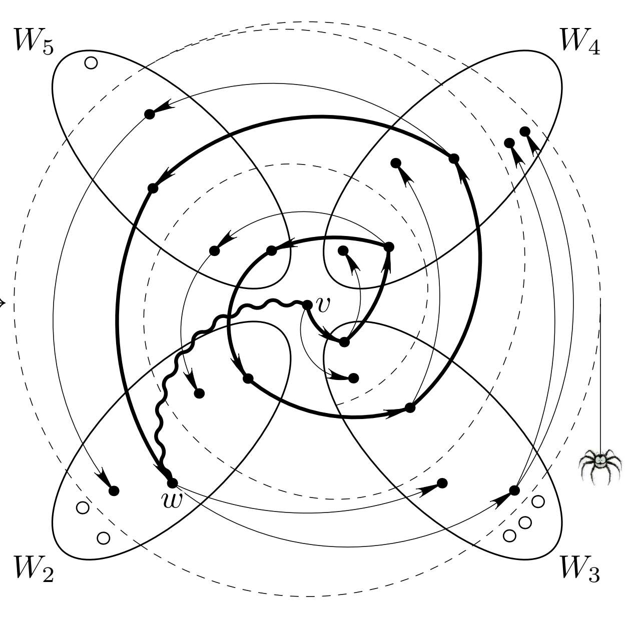

Figure 2

Since w is marked, there exists a directed path from v to w. This path moves through the sets $V_2, \ldots, V_k$ in circular order, so the number of edges in it is divisible by $k-1$ and thus even. Closing this path by the edge $w \to v$, we get a colorful odd cycle, as required. $\square$

Proof of the claim. Let us choose a leximinimal coloring (1) of G. For every set $C \subseteq \{1, 2, \ldots, k\}$ such that $|C|$ is odd and greater than 1, we will provide an odd cycle visiting exactly those color classes whose indices are listed in the set C. This property ensures that we have different cycles for different choices of C, and it proves the claim because there are $2^{k-1} - k$ choices for the set C.

Let $V_C = \bigcup_{c \in C} V_c$, and let G_C be the induced subgraph of G on the vertex set V_C. We also have the induced coloring of V_C with $|C|$ colors; this coloring is of course proper. Notice further that the induced coloring is leximinimal: if we had a lexicographically smaller coloring $(W_c)_{c \in C}$ of G_C, then these classes, together the original color classes V_i for $i \notin C$, would provide a proper coloring which is lexicographically smaller than (1). Hence Lemma 1, applied to the subgraph G_C and its leximinimal coloring $(V_c)_{c \in C}$, provides an odd cycle that visits exactly those color classes that are listed in the set C. $\square$

Solution 2. We provide a different proof of the claim from the previous solution.

We say that a graph is *critical* if deleting any vertex from the graph decreases the graph's chromatic number. Obviously every graph contains a critical induced subgraph with the same chromatic number.

Lemma 2. Suppose that $G = (V, E)$ is a critical graph with chromatic number $k \geqslant 3$. Then every vertex v of G is contained in at least $2^{k-2} - 1$ unsociable groups.

Proof. For every set $X \subseteq V$, denote by $n(X)$ the number of neighbors of v in the set X.

Since G is critical, there exists a proper coloring of $G \setminus \{v\}$ with $k-1$ colors, so there exists a proper coloring $V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ of G such that $V_1 = \{v\}$. Among such colorings, take one for which the sequence $\big(n(V_2), n(V_3), \ldots, n(V_k)\big)$ is lexicographically minimal. Clearly, $n(V_i) > 0$ for every $i = 2, 3, \ldots, k$; otherwise $V_2 \sqcup \ldots \sqcup V_{i-1} \sqcup (V_i \cup V_1) \sqcup V_{i+1} \sqcup \ldots V_k$ would be a proper coloring of G with $k-1$ colors.

We claim that for every $C \subseteq \{2, 3, \ldots, k\}$ with $|C| \geqslant 2$ being even, G contains an unsociable group so that the set of its members' colors is precisely $C \cup \{1\}$. Since the number of such sets C is $2^{k-2} - 1$, this proves the lemma. Denote the elements of C by $c_1, \ldots, c_{2\ell}$ in increasing order. For brevity, let $U_i = V_{c_i}$. Denote by N_i the set of neighbors of v in U_i.

We show that for every $i = 1, \ldots, 2\ell - 1$ and $x \in N_i$, the subgraph induced by $U_i \cup U_{i+1}$ contains a path that connects x with another point in N_{i+1}. For the sake of contradiction, suppose that no such path exists. Let S be the set of vertices that lie in the connected component of x in the subgraph induced by $U_i \cup U_{i+1}$, and let $P = U_i \cap S$, and $Q = U_{i+1} \cap S$ (see Figure 3). Since x is separated from N_{i+1}, the sets Q and N_{i+1} are disjoint. So, if we re-color G by replacing U_i and U_{i+1} by $(U_i \cup Q) \setminus P$ and $(U_{i+1} \cup P) \setminus Q$, respectively, we obtain a proper coloring such that $n(U_i) = n(V_{c_i})$ is decreased and only $n(U_{i+1}) = n(V_{c_{i+1}})$ is increased. That contradicts the lexicographical minimality of $\big(n(V_2), n(V_3), \ldots, n(V_k)\big)$.

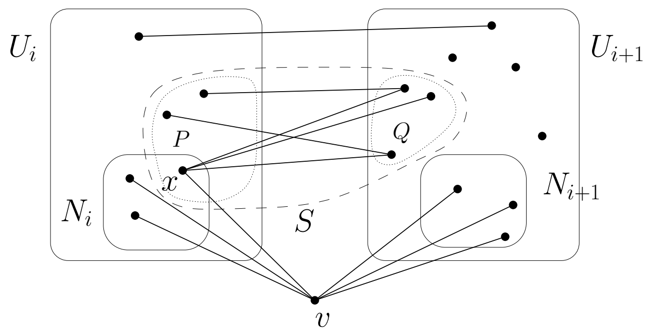

Figure 3

Next, we build a path through $U_1, U_2, \ldots, U_{2\ell}$ as follows. Let the starting point of the path be an arbitrary vertex v_1 in the set N_1. For $i \leqslant 2\ell - 1$, if the vertex $v_i \in N_i$ is already defined, connect v_i to some vertex in N_{i+1} in the subgraph induced by $U_i \cup U_{i+1}$, and add these edges to the path. Denote the new endpoint of the path by v_{i+1}; by the construction we have $v_{i+1} \in N_{i+1}$ again, so the process can be continued. At the end we have a path that starts at $v_1 \in N_1$ and ends at some $v_{2\ell} \in N_{2\ell}$. Moreover, all edges in this path connect vertices in neighboring classes: if a vertex of the path lies in U_i, then the next vertex lies in U_{i+1} or U_{i-1}. Notice that the path is not necessary simple, so take a minimal subpath of it. The minimal subpath is simple and connects the same endpoints v_1 and $v_{2\ell}$. The property that every edge steps to a neighboring color class (i.e., from U_i to U_{i+1} or U_{i-1}) is preserved. So the resulting path also visits all of $U_1, \ldots, U_{2\ell}$, and its length must be odd. Closing the path with the edges vv_1 and $v_{2\ell}v$ we obtain the desired odd cycle (see Figure 4). $\square$

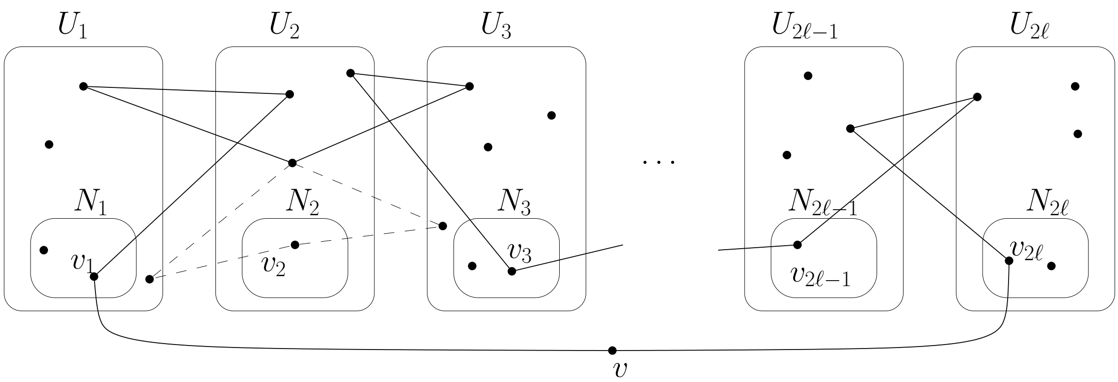

Figure 4

Now we prove the claim by induction on $k \geqslant 3$. The base case $k = 3$ holds by applying Lemma 2 to a critical subgraph. For the induction step, let G_0 be a critical k-chromatic subgraph of G, and let v be an arbitrary vertex of G_0. By Lemma 2, G_0 has at least $2^{k-2} - 1$ unsociable groups containing v. On the other hand, the graph $G_0 \setminus \{v\}$ has chromatic number $k - 1$, so it contains at least $2^{k-2} - (k-1)$ unsociable groups by the induction hypothesis. Altogether, this gives $2^{k-2} - 1 + 2^{k-2} - (k-1) = 2^{k-1} - k$ distinct unsociable groups in G_0 (and thus in G).

Comment 1. The claim we proved is sharp. The complete graph with k vertices has chromatic number k and contains exactly $2^{k-1} - k$ unsociable groups.

Comment 2. The proof of Lemma 2 works for odd values of $|C| \geqslant 3$ as well. Hence, the second solution shows the analogous statement that the number of even sized unsociable groups is at least $2^k - 1 - \binom{k}{2}$.

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Geometry

G1. Let ABC be an acute triangle with orthocenter H. Let G be the point such that the quadrilateral $ABGH$ is a parallelogram. Let I be the point on the line GH such that AC bisects HI. Suppose that the line AC intersects the circumcircle of the triangle GCI at C and J. Prove that $IJ = AH$.

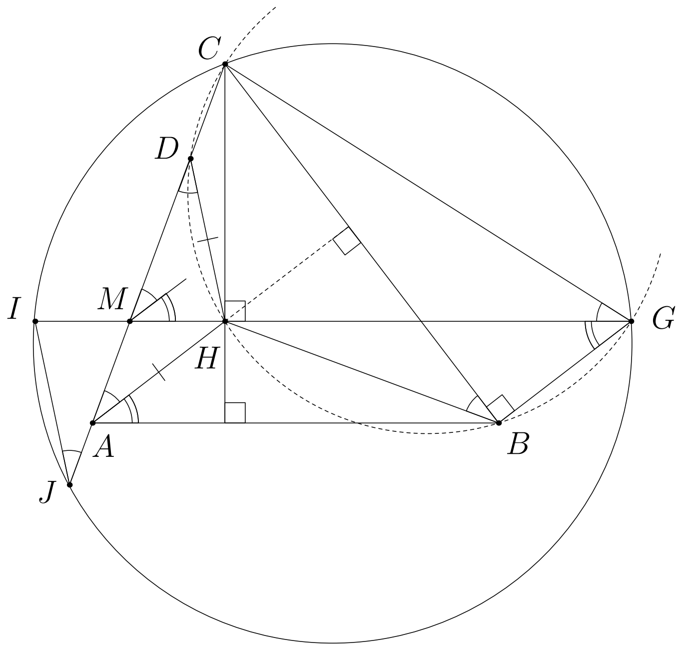
(Australia)

Solution 1. Since $HG \parallel AB$ and $BG \parallel AH$, we have $BG \perp BC$ and $CH \perp GH$. Therefore, the quadrilateral $BGCH$ is cyclic. Since H is the orthocenter of the triangle ABC, we have $\angle HAC = 90^\circ - \angle ACB = \angle CBH$. Using that $BGCH$ and $CGJI$ are cyclic quadrilaterals, we get

$$\angle CJI = \angle CGH = \angle CBH = \angle HAC.$$

Let M be the intersection of AC and GH, and let $D \neq A$ be the point on the line AC such that $AH = HD$. Then $\angle MJI = \angle HAC = \angle MDH$.

Since $\angle MJI = \angle MDH$, $\angle IMJ = \angle HMD$, and $IM = MH$, the triangles IMJ and HMD are congruent, and thus $IJ = HD = AH$.



Comment. Instead of introducing the point D, one can complete the solution by using the law of sines in the triangles IJM and AMH, yielding

$$\frac{IJ}{IM} = \frac{\sin\angle IMJ}{\sin\angle MJI} = \frac{\sin\angle AMH}{\sin\angle HAM} = \frac{AH}{MH} = \frac{AH}{IM}.$$

Solution 2. Obtain $\angle CGH = \angle HAC$ as in the previous solution. In the parallelogram $ABGH$ we have $\angle BAH = \angle HGB$. It follows that

$$\angle HMC = \angle BAC = \angle BAH + \angle HAC = \angle HGB + \angle CGH = \angle CGB.$$

So the right triangles CMH and CGB are similar. Also, in the circumcircle of triangle GCI we have similar triangles MIJ and MCG. Therefore,

$$\frac{IJ}{CG} = \frac{MI}{MC} = \frac{MH}{MC} = \frac{GB}{GC} = \frac{AH}{CG}.$$

Hence $IJ = AH$.

G2. Let ABC be a triangle inscribed into a circle Ω with center O. A circle Γ with center A meets the side BC at points D and E such that D lies between B and E. Moreover, let F and G be the common points of Γ and Ω. We assume that F lies on the arc AB of Ω not containing C, and G lies on the arc AC of Ω not containing B. The circumcircles of the triangles BDF and CEG meet the sides AB and AC again at K and L, respectively. Suppose that the lines FK and GL are distinct and intersect at X. Prove that the points A, X, and O are collinear.

(Greece)

Solution 1. It suffices to prove that the lines FK and GL are symmetric about AO. Now the segments AF and AG, being chords of Ω with the same length, are clearly symmetric with respect to AO. Hence it is enough to show

$$\angle KFA = \angle AGL\,. \tag{1}$$

Let us denote the circumcircles of BDF and CEG by ω_B and ω_C, respectively. To prove (1), we start from

$$\angle KFA = \angle DFG + \angle GFA - \angle DFK\,.$$

In view of the circles ω_B, Γ, and Ω, this may be rewritten as

$$\angle KFA = \angle CEG + \angle GBA - \angle DBK = \angle CEG - \angle CBG\,.$$

Due to the circles ω_C and Ω, we obtain $\angle KFA = \angle CLG - \angle CAG = \angle AGL$. Thereby the problem is solved.

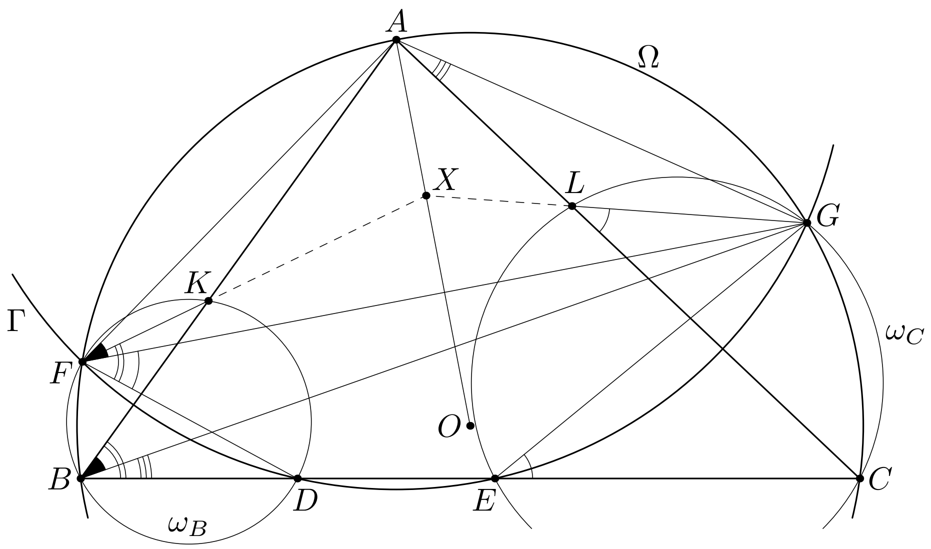


Figure 1

Solution 2. Again, we denote the circumcircle of $BDKF$ by ω_B. In addition, we set $\alpha = \angle BAC$, $\varphi = \angle ABF$, and $\psi = \angle EDA = \angle AED$ (see Figure 2). Notice that $AF = AG$ entails $\varphi = \angle GCA$, so all three of α, φ, and ψ respect the "symmetry" between B and C of our configuration. Again, we reduce our task to proving (1).

This time, we start from

$$2\angle KFA = 2(\angle DFA - \angle DFK)\,.$$

Since the triangle AFD is isosceles, we have

$$\angle DFA = \angle ADF = \angle EDF - \psi = \angle BFD + \angle EBF - \psi\,.$$

Moreover, because of the circle ω_B we have $\angle DFK = \angle CBA$. Altogether, this yields

$$2\angle KFA = \angle DFA + \big(\angle BFD + \angle EBF - \psi\big) - 2\angle CBA\,,$$

which simplifies to

$$2\angle KFA = \angle BFA + \varphi - \psi - \angle CBA\,.$$

Now the quadrilateral $AFBC$ is cyclic, so this entails $2\angle KFA = \alpha + \varphi - \psi$.

Due to the "symmetry" between B and C alluded to above, this argument also shows that $2\angle AGL = \alpha + \varphi - \psi$. This concludes the proof of (1).

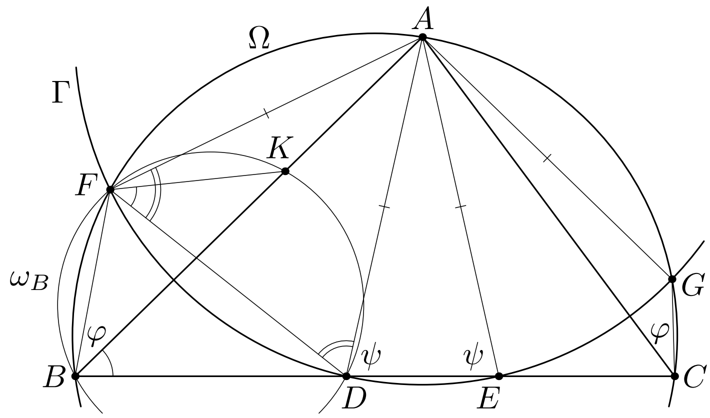


Figure 2

Comment 1. As the first solution shows, the assumption that A be the center of Γ may be weakened to the following one: The center of Γ lies on the line OA. The second solution may be modified to yield the same result.

Comment 2. It might be interesting to remark that $\angle GDK = 90^\circ$. To prove this, let G' denote the point on Γ diametrically opposite to G. Because of $\angle KDF = \angle KBF = \angle AGF = \angle G'DF$, the points D, K, and G' are collinear, which leads to the desired result. Notice that due to symmetry we also have $\angle LEF = 90^\circ$.

Moreover, a standard argument shows that the triangles AGL and BGE are similar. By symmetry again, also the triangles AFK and CDF are similar.

There are several ways to derive a solution from these facts. For instance, one may argue that

$$\begin{aligned}\angle KFA &= \angle BFA - \angle BFK = \angle BFA - \angle EDG' = (180^\circ - \angle AGB) - (180^\circ - \angle G'GE)\\ &= \angle AGE - \angle AGB = \angle BGE = \angle AGL\,.\end{aligned}$$

Comment 3. The original proposal did not contain the point X in the assumption and asked instead to prove that the lines FK, GL, and AO are concurrent. This differs from the version given above only insofar as it also requires to show that these lines cannot be parallel. The Problem Selection Committee removed this part from the problem intending to make it thus more suitable for the Olympiad.

For the sake of completeness, we would still like to sketch one possibility for proving $FK \nparallel AO$ here. As the points K and O lie in the angular region $\angle FAG$, it suffices to check $\angle KFA + \angle FAO < 180^\circ$. Multiplying by 2 and making use of the formulae from the second solution, we see that this is equivalent to $(\alpha + \varphi - \psi) + (180^\circ - 2\varphi) < 360^\circ$, which in turn is an easy consequence of $\alpha < 180^\circ$.

G3. Let ABC be a triangle with $\angle C = 90^\circ$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

(Georgia)

Solution 1. Let K be the projection of D onto AB; then $AH = HK$ (see Figure 1). Since $PH \parallel DK$, we have

$$\frac{PD}{PB} = \frac{HK}{HB} = \frac{AH}{HB}. \tag{1}$$

Let L be the projection of Q onto DB. Since PQ is tangent to ω and $\angle DQB = \angle BLQ = 90^\circ$, we have $\angle PQD = \angle QBP = \angle DQL$. Therefore, QD and QB are respectively the internal and the external bisectors of $\angle PQL$. By the angle bisector theorem, we obtain

$$\frac{PD}{DL} = \frac{PQ}{QL} = \frac{PB}{BL}. \tag{2}$$

The relations (1) and (2) yield $\dfrac{AH}{HB} = \dfrac{PD}{PB} = \dfrac{DL}{LB}$. So, the spiral similarity τ centered at B and sending A to D maps H to L. Moreover, τ sends the semicircle with diameter AB passing through C to ω. Due to $CH \perp AB$ and $QL \perp DB$, it follows that $\tau(C) = Q$.

Hence, the triangles ABD and CBQ are similar, so $\angle ADB = \angle CQB$. This means that the lines AD and CQ meet at some point T, and this point satisfies $\angle BDT = \angle BQT$. Therefore, T lies on ω, as needed.

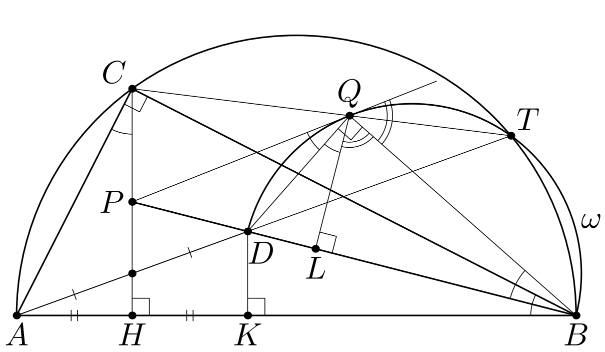


Figure 1

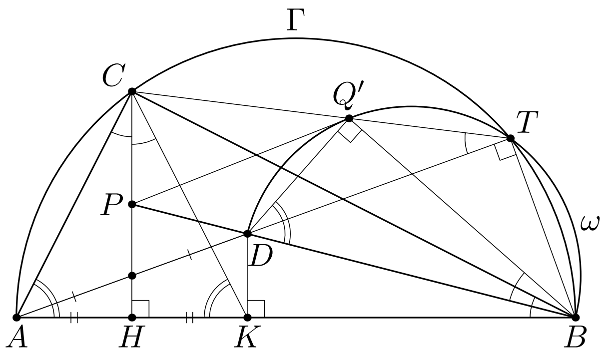


Figure 2

Comment 1. Since $\angle BAD = \angle BCQ$, the point T lies also on the circumcircle of the triangle ABC.

Solution 2. Let Γ be the circumcircle of ABC, and let AD meet ω at T. Then $\angle ATB = \angle ACB = 90^\circ$, so T lies on Γ as well. As in the previous solution, let K be the projection of D onto AB; then $AH = HK$ (see Figure 2).

Our goal now is to prove that the points C, Q, and T are collinear. Let CT meet ω again at Q'. Then, it suffices to show that PQ' is tangent to ω, or that $\angle PQ'D = \angle Q'BD$.

Since the quadrilateral $BDQ'T$ is cyclic and the triangles AHC and KHC are congruent, we have $\angle Q'BD = \angle Q'TD = \angle CTA = \angle CBA = \angle ACH = \angle HCK$. Hence, the right triangles CHK and $BQ'D$ are similar. This implies that $\dfrac{HK}{CK} = \dfrac{Q'D}{BD}$, and thus $HK \cdot BD = CK \cdot Q'D$.

Notice that $PH \parallel DK$; therefore, we have $\dfrac{PD}{BD} = \dfrac{HK}{BK}$, and so $PD \cdot BK = HK \cdot BD$. Consequently, $PD \cdot BK = HK \cdot BD = CK \cdot Q'D$, which yields $\dfrac{PD}{Q'D} = \dfrac{CK}{BK}$.

Since $\angle CKA = \angle KAC = \angle BDQ'$, the triangles CKB and PDQ' are similar, so $\angle PQ'D = \angle CBA = \angle Q'BD$, as required.

Comment 2. There exist several other ways to prove that PQ' is tangent to ω. For instance, one may compute $\dfrac{PD}{PB}$ and $\dfrac{PQ'}{PB}$ in terms of AH and HB to verify that $PQ'^2 = PD \cdot PB$, concluding that PQ' is tangent to ω.

Another possible approach is the following. As in Solution 2, we introduce the points T and Q' and mention that the triangles ABC and DBQ' are similar (see Figure 3).

Let M be the midpoint of AD, and let L be the projection of Q' onto AB. Construct E on the line AB so that EP is parallel to AD. Projecting from P, we get $(A, B; H, E) = (A, D; M, \infty) = -1$.

Since $\dfrac{EA}{AB} = \dfrac{PD}{DB}$, the point P is the image of E under the similarity transform mapping ABC to DBQ'. Therefore, we have $(D, B; L, P) = (A, B; H, E) = -1$, which means that $Q'D$ and $Q'B$ are respectively the internal and the external bisectors of $\angle PQ'L$. This implies that PQ' is tangent to ω, as required.

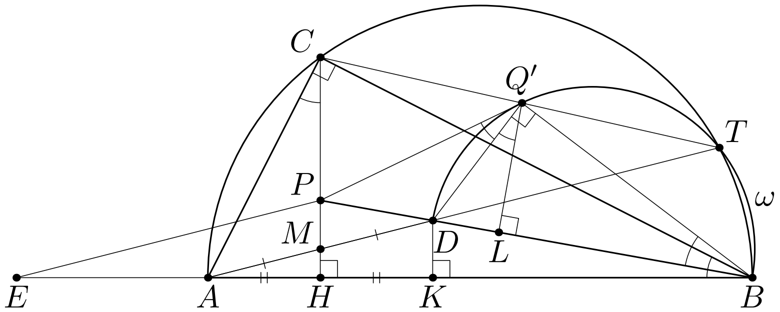


Figure 3

Solution 3. Introduce the points T and Q' as in the previous solution. Note that T lies on the circumcircle of ABC. Here we present yet another proof that PQ' is tangent to ω.

Let Ω be the circle completing the semicircle ω. Construct a point F symmetric to C with respect to AB. Let $S \neq T$ be the second intersection point of FT and Ω (see Figure 4).

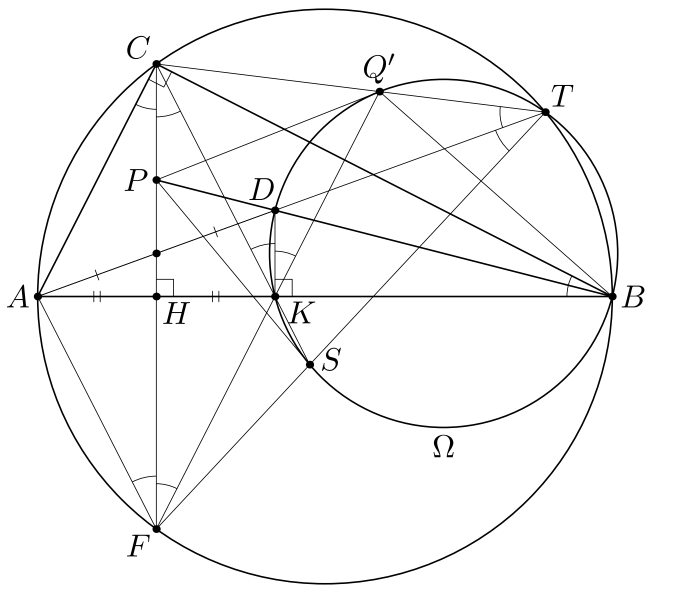


Figure 4

Since $AC = AF$, we have $\angle DKC = \angle HCK = \angle CBA = \angle CTA = \angle DTS = 180^\circ - \angle SKD$. Thus, the points C, K, and S are collinear. Notice also that $\angle Q'KD = \angle Q'TD = \angle HCK = \angle KFH = 180^\circ - \angle DKF$. This implies that the points F, K, and Q' are collinear.

Applying Pascal's theorem to the degenerate hexagon $KQ'Q'TSS$, we get that the tangents to Ω passing through Q' and S intersect on CF. The relation $\angle Q'TD = \angle DTS$ yields that Q' and S are symmetric with respect to BD. Therefore, the two tangents also intersect on BD. Thus, the two tangents pass through P. Hence, PQ' is tangent to ω, as needed.

G4. Let ABC be an acute triangle, and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC again at P and Q, respectively. Let T be the point such that the quadrilateral $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of the triangle ABC. Determine all possible values of BT/BM.

(Russia)

Answer. $\sqrt{2}$.

Solution 1. Let S be the center of the parallelogram $BPTQ$, and let $B' \neq B$ be the point on the ray BM such that $BM = MB'$ (see Figure 1). It follows that $ABCB'$ is a parallelogram. Then, $\angle ABB' = \angle PQM$ and $\angle BB'A = \angle B'BC = \angle MPQ$, and so the triangles ABB' and MQP are similar. It follows that AM and MS are corresponding medians in these triangles. Hence,

$$\angle SMP = \angle B'AM = \angle BCA = \angle BTA. \tag{1}$$

Since $\angle ACT = \angle PBT$ and $\angle TAC = \angle TBC = \angle BTP$, the triangles TCA and PBT are similar. Again, as TM and PS are corresponding medians in these triangles, we have

$$\angle MTA = \angle TPS = \angle BQP = \angle BMP. \tag{2}$$

Now we deal separately with two cases.

Case 1. S does not lie on BM. Since the configuration is symmetric between A and C, we may assume that S and A lie on the same side with respect to the line BM.

Applying (1) and (2), we get

$$\angle BMS = \angle BMP - \angle SMP = \angle MTA - \angle BTA = \angle MTB,$$

and so the triangles BSM and BMT are similar. We now have $BM^2 = BS \cdot BT = BT^2/2$, so $BT = \sqrt{2}BM$.

Case 2. S lies on BM. It follows from (2) that $\angle BCA = \angle MTA = \angle BQP = \angle BMP$ (see Figure 2). Thus, $PQ \parallel AC$ and $PM \parallel AT$. Hence, $BS/BM = BP/BA = BM/BT$, so $BT^2 = 2BM^2$ and $BT = \sqrt{2}BM$.

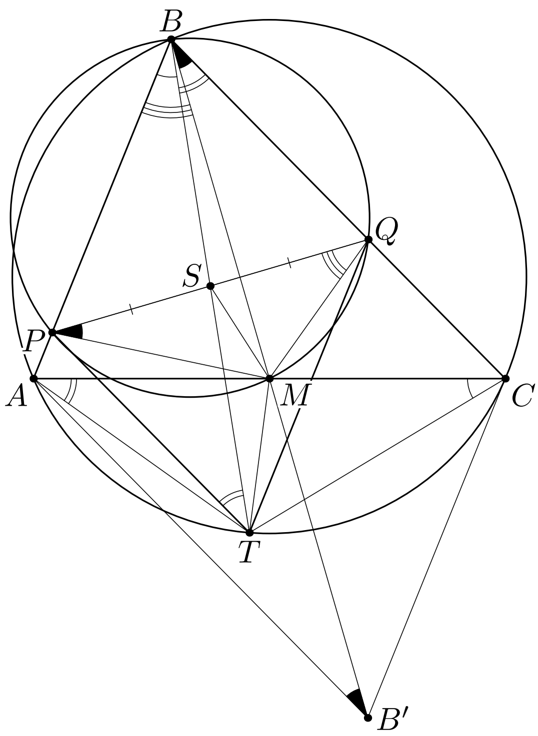


Figure 1

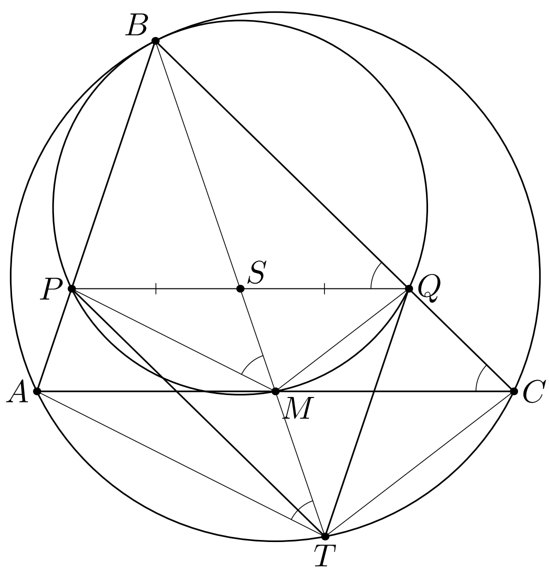


Figure 2

Comment 1. Here is another way to show that the triangles BSM and BMT are similar. Denote by Ω the circumcircle of the triangle ABC. Let R be the second point of intersection of ω and Ω, and let τ be the spiral similarity centered at R mapping ω to Ω. Then, one may show that τ maps each point X on ω to a point Y on Ω such that B, X, and Y are collinear (see Figure 3). If we let K and L be the second points of intersection of BM with Ω and of BT with ω, respectively, then it follows that the triangle MKT is the image of SML under τ. We now obtain $\angle BSM = \angle TMB$, which implies the desired result.

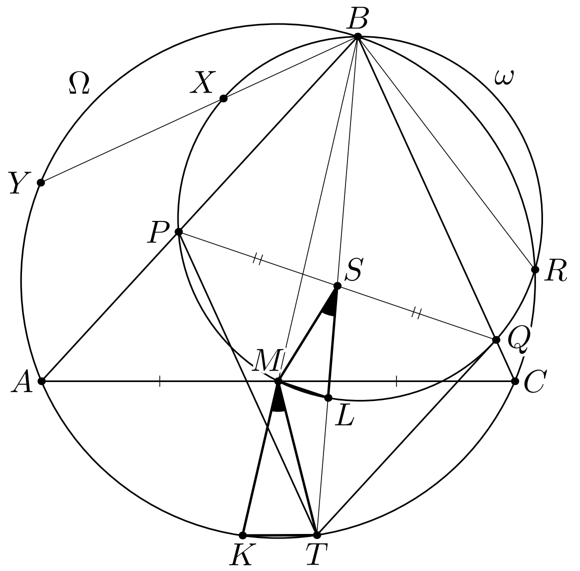


Figure 3

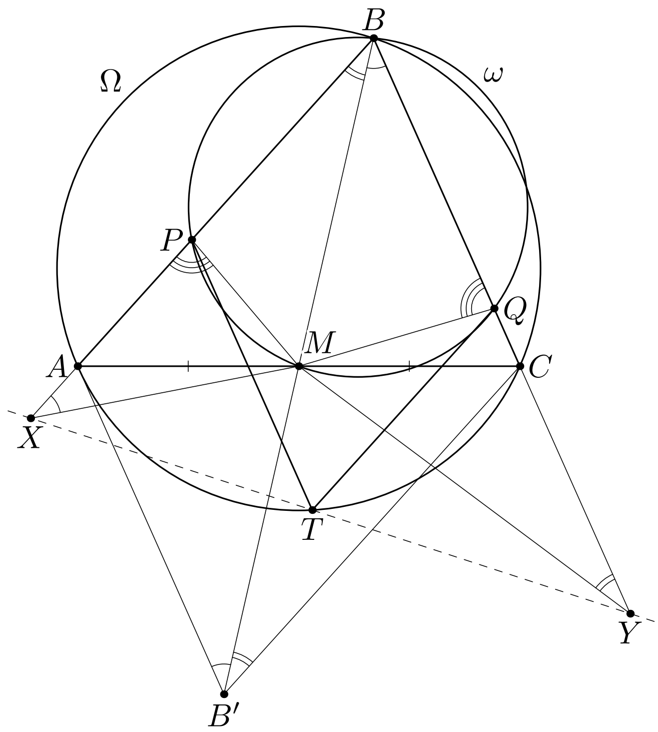


Figure 4

Solution 2. Again, we denote by Ω the circumcircle of the triangle ABC.

Choose the points X and Y on the rays BA and BC respectively, so that $\angle MXB = \angle MBC$ and $\angle BYM = \angle ABM$ (see Figure 4). Then the triangles BMX and YMB are similar. Since $\angle XPM = \angle BQM$, the points P and Q correspond to each other in these triangles. So, if $\overrightarrow{BP} = \mu \cdot \overrightarrow{BX}$, then $\overrightarrow{BQ} = (1-\mu) \cdot \overrightarrow{BY}$. Thus

$$\overrightarrow{BT} = \overrightarrow{BP} + \overrightarrow{BQ} = \overrightarrow{BY} + \mu \cdot (\overrightarrow{BX} - \overrightarrow{BY}) = \overrightarrow{BY} + \mu \cdot \overrightarrow{YX},$$

which means that T lies on the line XY.

Let $B' \neq B$ be the point on the ray BM such that $BM = MB'$. Then $\angle MB'A = \angle MBC = \angle MXB$ and $\angle CB'M = \angle ABM = \angle BYM$. This means that the triangles BMX, BAB', YMB, and $B'CB$ are all similar; hence $BA \cdot BX = BM \cdot BB' = BC \cdot BY$. Thus there exists an inversion centered at B which swaps A with X, M with B', and C with Y. This inversion then swaps Ω with the line XY, and hence it preserves T. Therefore, we have $BT^2 = BM \cdot BB' = 2BM^2$, and $BT = \sqrt{2}BM$.

Solution 3. We begin with the following lemma.

Lemma. Let $ABCT$ be a cyclic quadrilateral. Let P and Q be points on the sides BA and BC respectively, such that $BPTQ$ is a parallelogram. Then $BP \cdot BA + BQ \cdot BC = BT^2$.

Proof. Let the circumcircle of the triangle QTC meet the line BT again at J (see Figure 5). The power of B with respect to this circle yields

$$BQ \cdot BC = BJ \cdot BT. \tag{3}$$

We also have $\angle TJQ = 180^\circ - \angle QCT = \angle TAB$ and $\angle QTJ = \angle ABT$, and so the triangles TJQ and BAT are similar. We now have $TJ/TQ = BA/BT$. Therefore,

$$TJ \cdot BT = TQ \cdot BA = BP \cdot BA. \tag{4}$$

Combining (3) and (4) now yields the desired result. $\square$

Let X and Y be the midpoints of BA and BC respectively (see Figure 6). Applying the lemma to the cyclic quadrilaterals $PBQM$ and $ABCT$, we obtain

$$BX \cdot BP + BY \cdot BQ = BM^2$$

and

$$BP \cdot BA + BQ \cdot BC = BT^2.$$

Since $BA = 2BX$ and $BC = 2BY$, we have $BT^2 = 2BM^2$, and so $BT = \sqrt{2}BM$.

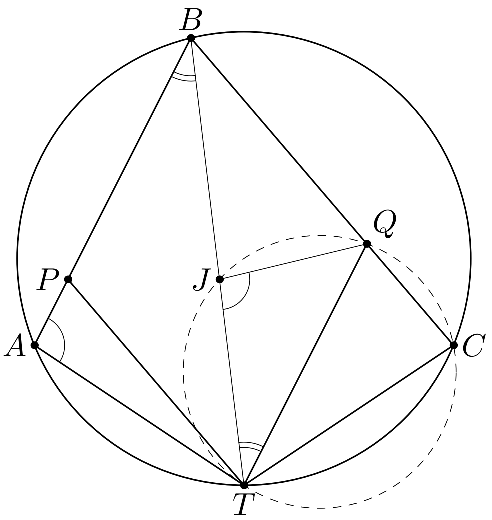


Figure 5

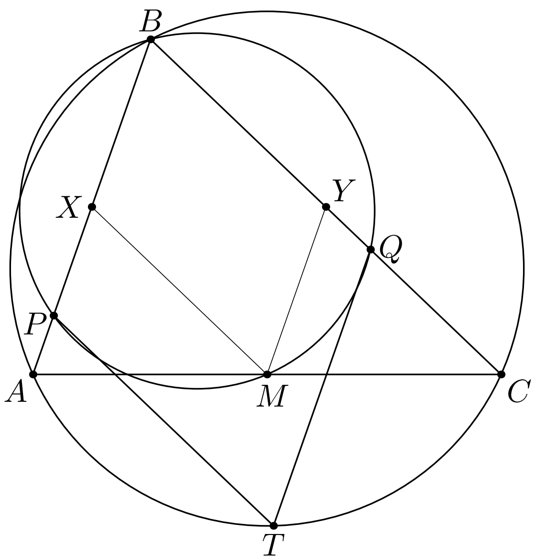


Figure 6

Comment 2. Here we give another proof of the lemma using Ptolemy's theorem. We readily have

$$TC \cdot BA + TA \cdot BC = AC \cdot BT.$$

The lemma now follows from

$$\frac{BP}{TC} = \frac{BQ}{TA} = \frac{BT}{AC} = \frac{\sin \angle BCT}{\sin \angle ABC}.$$

G5. Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC, respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

(El Salvador)

Solution 1. We may assume that $CA > CB$. Observe that H' and I' lie inside the segments CF and CG, respectively. Therefore, M lies outside $\triangle ABC$ (see Figure 1).

Due to the powers of points A and B with respect to the circle Γ, we have

$$CH' \cdot CA = AH \cdot AC = AD^2 = BD^2 = BI \cdot BC = CI' \cdot CB.$$

Therefore, $CH' \cdot CF = CI' \cdot CG$. Hence, the quadrilateral $H'I'GF$ is cyclic, and so $\angle I'H'C = \angle CGF$.

Let DF and DG meet Γ again at R and S, respectively. We claim that the points R and S lie on the line $H'I'$.

Observe that $FH' \cdot FA = FH \cdot FC = FR \cdot FD$. Thus, the quadrilateral $ADH'R$ is cyclic, and hence $\angle RH'F = \angle FDA = \angle CGF = \angle I'H'C$. Therefore, the points R, H', and I' are collinear. Similarly, the points S, H', and I' are also collinear, and so all the points R, H', Q, I', S, and M are all collinear.

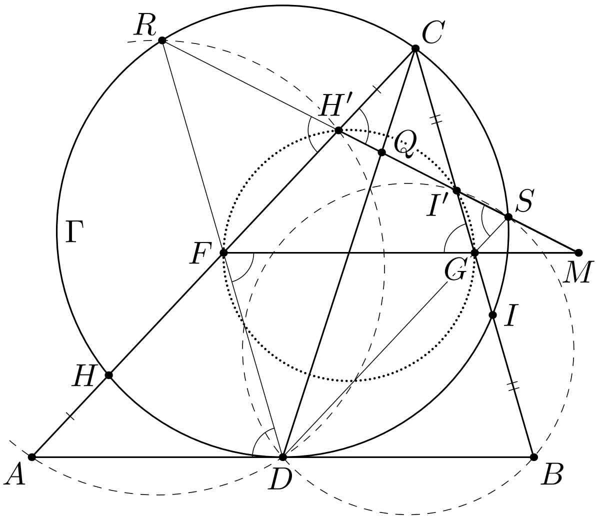


Figure 1

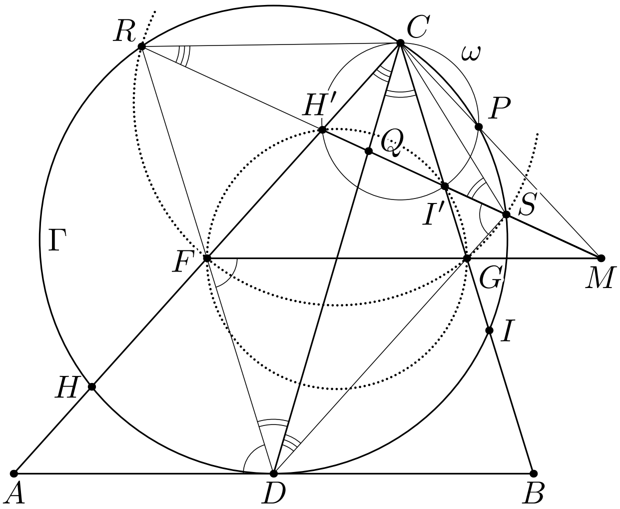


Figure 2

Then, $\angle RSD = \angle RDA = \angle DFG$. Hence, the quadrilateral $RSGF$ is cyclic (see Figure 2). Therefore, $MH' \cdot MI' = MF \cdot MG = MR \cdot MS = MP \cdot MC$. Thus, the quadrilateral $CPI'H'$ is also cyclic. Let ω be its circumcircle.

Notice that $\angle H'CQ = \angle SDC = \angle SRC$ and $\angle QCI' = \angle CDR = \angle CSR$. Hence, $\triangle CH'Q \sim \triangle RCQ$ and $\triangle CI'Q \sim \triangle SCQ$, and therefore $QH' \cdot QR = QC^2 = QI' \cdot QS$.

We apply the inversion with center Q and radius QC. Observe that the points R, C, and S are mapped to H', C, and I', respectively. Therefore, the circumcircle Γ of $\triangle RCS$ is mapped to the circumcircle ω of $\triangle H'CI'$. Since P and C belong to both circles and the point C is preserved by the inversion, we have that P is also mapped to itself. We then get $QP^2 = QC^2$. Hence, $QP = QC$.

Comment 1. The problem statement still holds when Γ intersects the sides CA and CB outside segments AF and BG, respectively.

Solution 2. Let $X = HI \cap AB$, and let the tangent to Γ at C meet AB at Y. Let XC meet Γ again at X' (see Figure 3). Projecting from C, X, and C again, we have $(X, A; D, B) = (X', H; D, I) = (C, I; D, H) = (Y, B; D, A)$. Since A and B are symmetric about D, it follows that X and Y are also symmetric about D.

Now, Menelaus' theorem applied to $\triangle ABC$ with the line HIX yields

$$1 = \frac{CH}{HA} \cdot \frac{BI}{IC} \cdot \frac{AX}{XB} = \frac{AH'}{H'C} \cdot \frac{CI'}{I'B} \cdot \frac{BY}{YA}.$$

By the converse of Menelaus' theorem applied to $\triangle ABC$ with points H', I', Y, we get that the points H', I', Y are collinear.

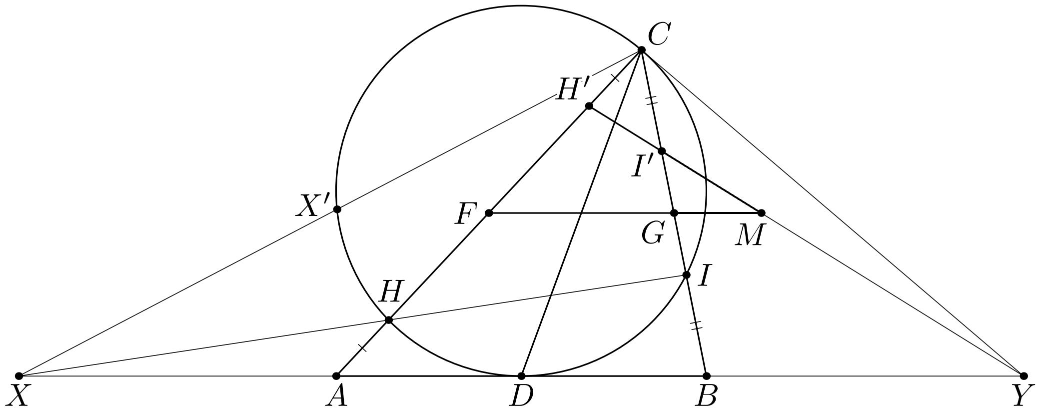


Figure 3

Let T be the midpoint of CD, and let O be the center of Γ. Let CM meet TY at N. To avoid confusion, we clean some superfluous details out of the picture (see Figure 4).

Let $V = MT \cap CY$. Since $MT \parallel YD$ and $DT = TC$, we get $CV = VY$. Then Ceva's theorem applied to $\triangle CTY$ with the point M yields

$$1 = \frac{TQ}{QC} \cdot \frac{CV}{VY} \cdot \frac{YN}{NT} = \frac{TQ}{QC} \cdot \frac{YN}{NT}.$$

Therefore, $\frac{TQ}{QC} = \frac{TN}{NY}$. So, $NQ \parallel CY$, and thus $NQ \perp OC$.

Note that the points O, N, T, and Y are collinear. Therefore, $CQ \perp ON$. So, Q is the orthocenter of $\triangle OCN$, and hence $OQ \perp CP$. Thus, Q lies on the perpendicular bisector of CP, and therefore $CQ = QP$, as required.

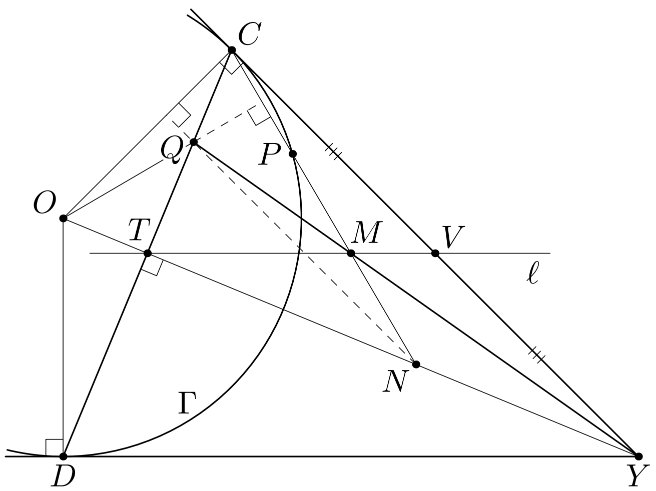


Figure 4

Comment 2. The second part of Solution 2 provides a proof of the following more general statement, which does not involve a specific choice of Q on CD.

Let YC and YD be two tangents to a circle Γ with center O (see Figure 4). Let ℓ be the midline of $\triangle YCD$ parallel to YD. Let Q and M be two points on CD and ℓ, respectively, such that the line QM passes through Y. Then $OQ \perp CM$.

G6. Let ABC be an acute triangle with $AB > AC$, and let Γ be its circumcircle. Let H, M, and F be the orthocenter of the triangle, the midpoint of BC, and the foot of the altitude from A, respectively. Let Q and K be the two points on Γ that satisfy $\angle AQH = 90^\circ$ and $\angle QKH = 90^\circ$. Prove that the circumcircles of the triangles KQH and KFM are tangent to each other.

(Ukraine)

Solution 1. Let A' be the point diametrically opposite to A on Γ. Since $\angle AQA' = 90^\circ$ and $\angle AQH = 90^\circ$, the points Q, H, and A' are collinear. Similarly, if Q' denotes the point on Γ diametrically opposite to Q, then K, H, and Q' are collinear. Let the line AHF intersect Γ again at E; it is known that M is the midpoint of the segment HA' and that F is the midpoint of HE. Let J be the midpoint of HQ'.

Consider any point T such that TK is tangent to the circle KQH at K with Q and T lying on different sides of KH (see Figure 1). Then $\angle HKT = \angle HQK$ and we are to prove that $\angle MKT = \angle CFK$. Thus it remains to show that $\angle HQK = \angle CFK + \angle HKM$. Due to $\angle HQK = 90^\circ - \angle Q'HA'$ and $\angle CFK = 90^\circ - \angle KFA$, this means the same as $\angle Q'HA' = \angle KFA - \angle HKM$. Now, since the triangles KHE and AHQ' are similar with F and J being the midpoints of corresponding sides, we have $\angle KFA = \angle HJA$, and analogously one may obtain $\angle HKM = \angle JQH$. Thereby our task is reduced to verifying

$$\angle Q'HA' = \angle HJA - \angle JQH\,.$$

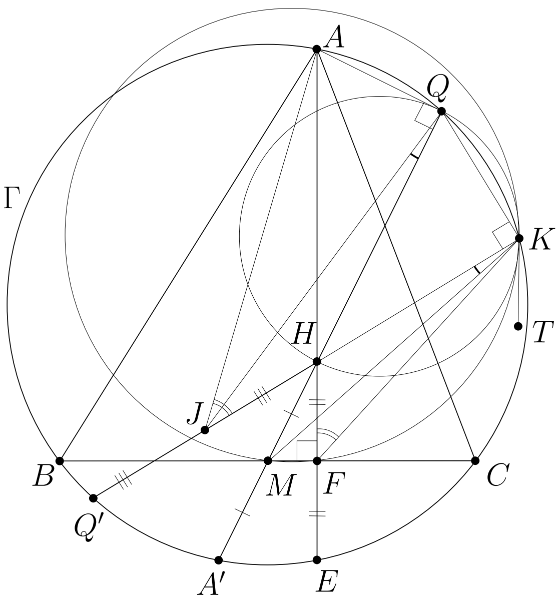


Figure 1

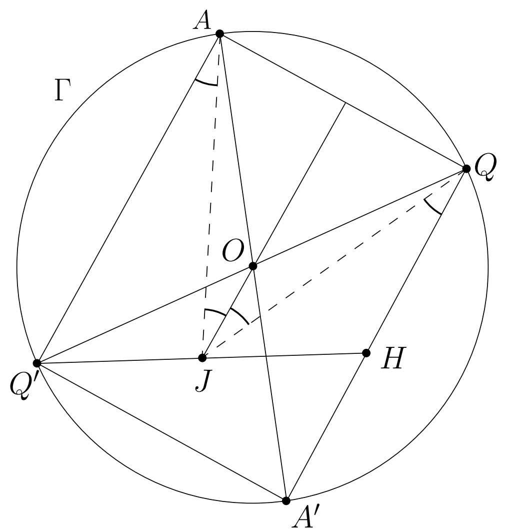


Figure 2

To avoid confusion, let us draw a new picture at this moment (see Figure 2). Owing to $\angle Q'HA' = \angle JQH + \angle HJQ$ and $\angle HJA = \angle QJA + \angle HJQ$, we just have to show that $2\angle JQH = \angle QJA$. To this end, it suffices to remark that $AQA'Q'$ is a rectangle and that J, being defined to be the midpoint of HQ', has to lie on the mid parallel of QA' and $Q'A$.

Solution 2. We define the points A' and E and prove that the ray MH passes through Q in the same way as in the first solution. Notice that the points A' and E can play analogous roles to the points Q and K, respectively: point A' is the second intersection of the line MH with Γ, and E is the point on Γ with the property $\angle HEA' = 90^\circ$ (see Figure 3).

In the circles KQH and $EA'H$, the line segments HQ and HA' are diameters, respectively; so, these circles have a common tangent t at H, perpendicular to MH. Let R be the radical center of the circles ABC, KQH and $EA'H$. Their pairwise radical axes are the lines QK, $A'E$ and the line t; they all pass through R. Let S be the midpoint of HR; by $\angle QKH =$

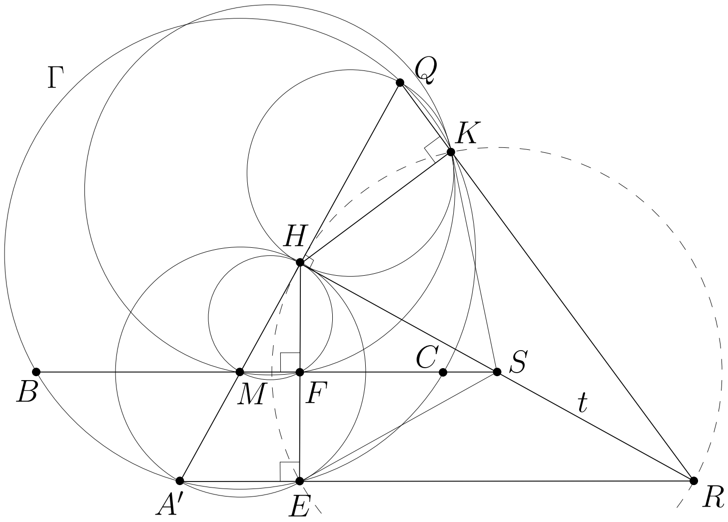


Figure 3

$\angle HEA' = 90^\circ$, the quadrilateral $HERK$ is cyclic and its circumcenter is S; hence we have $SK = SE = SH$. The line BC, being the perpendicular bisector of HE, passes through S.

The circle HMF also is tangent to t at H; from the power of S with respect to the circle HMF we have

$$SM \cdot SF = SH^2 = SK^2.$$

So, the power of S with respect to the circles KQH and KFM is SK^2. Therefore, the line segment SK is tangent to both circles at K.

G7. Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segments PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

(Bulgaria)

Solution 1. Denote by γ_A, γ_B, γ_C, and γ_D the incircles of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$, respectively.

We start with proving that the quadrilateral $ABCD$ also has an incircle which will be referred to as Ω. Denote the points of tangency as in Figure 1. It is well-known that $QQ_1 = OO_1$ (if $BC \parallel PR$, this is obvious; otherwise, one may regard the two circles involved as the incircle and an excircle of the triangle formed by the lines OQ, PR, and BC). Similarly, $OO_1 = PP_1$. Hence we have $QQ_1 = PP_1$. The other equalities of segment lengths marked in Figure 1 can be proved analogously. These equalities, together with $AP_1 = AS_1$ and similar ones, yield $AB + CD = AD + BC$, as required.

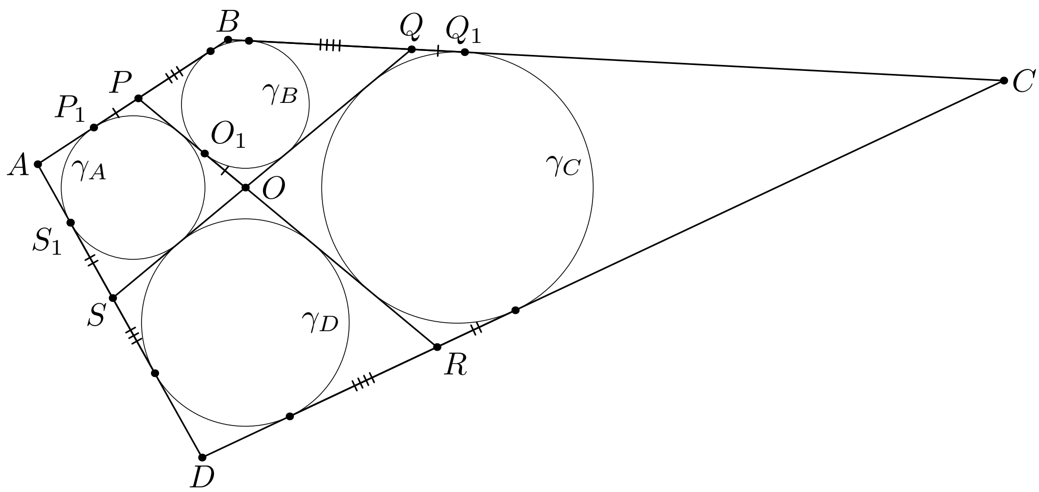


Figure 1

Next, let us draw the lines parallel to QS through P and R, and also draw the lines parallel to PR through Q and S. These lines form a parallelogram; denote its vertices by A', B', C', and D' as shown in Figure 2.

Since the quadrilateral $APOS$ has an incircle, we have $AP - AS = OP - OS = A'S - A'P$. It is well-known that in this case there also exists a circle ω_A tangent to the four rays AP, AS, $A'P$, and $A'S$. It is worth mentioning here that in case when, say, the lines AB and $A'B'$ coincide, the circle ω_A is just tangent to AB at P. We introduce the circles ω_B, ω_C, and ω_D in a similar manner.

Assume that the radii of the circles ω_A and ω_C are different. Let X be the center of the homothety having a positive scale factor and mapping ω_A to ω_C.

Now, MONGE's theorem applied to the circles ω_A, Ω, and ω_C shows that the points A, C, and X are collinear. Applying the same theorem to the circles ω_A, ω_B, and ω_C, we see that the points P, Q, and X are also collinear. Similarly, the points R, S, and X are collinear, as required.

If the radii of ω_A and ω_C are equal but these circles do not coincide, then the degenerate version of the same theorem yields that the three lines AC, PQ, and RS are parallel to the line of centers of ω_A and ω_C.

Finally, we need to say a few words about the case when ω_A and ω_C coincide (and thus they also coincide with Ω, ω_B, and ω_D). It may be regarded as the limit case in the following manner.

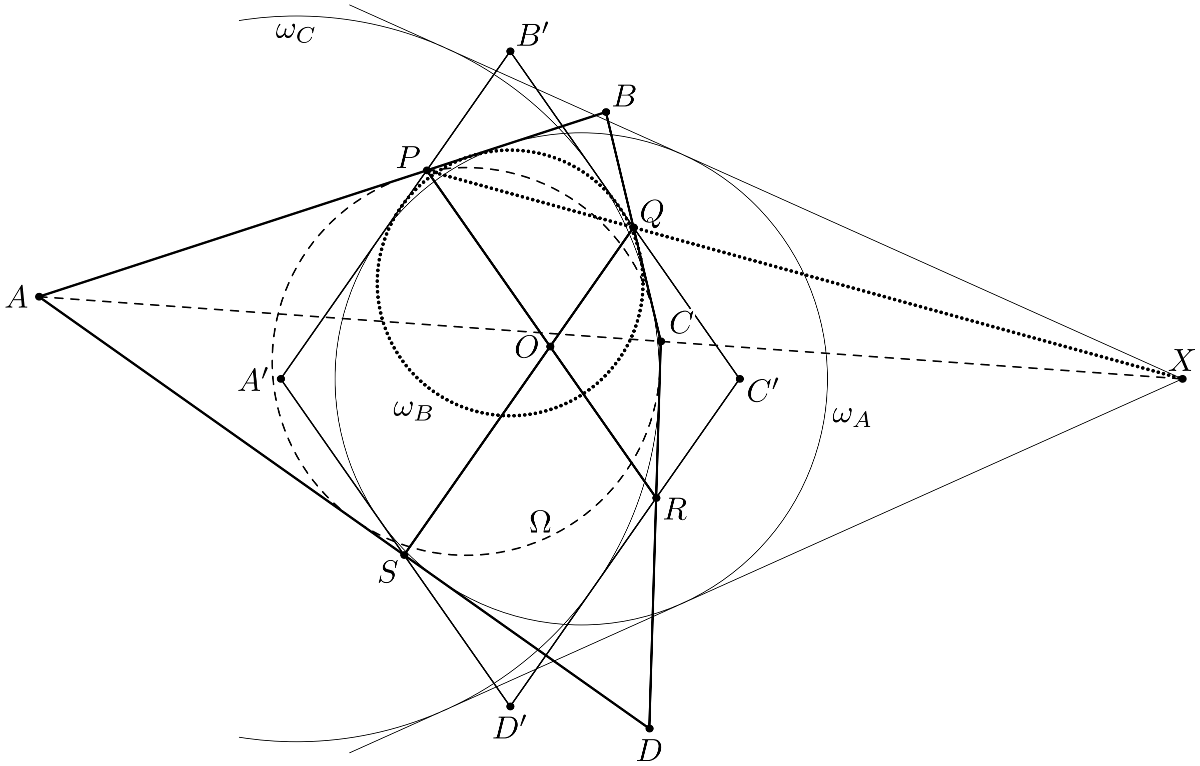


Figure 2

Let us fix the positions of A, P, O, and S (thus we also fix the circles ω_A, γ_A, γ_B, and γ_D). Now we vary the circle γ_C inscribed into $\angle QOR$; for each of its positions, one may reconstruct the lines BC and CD as the external common tangents to γ_B, γ_C and γ_C, γ_D different from PR and QS, respectively. After such variation, the circle Ω changes, so the result obtained above may be applied.

Solution 2. Applying Menelaus' theorem to $\triangle ABC$ with the line PQ and to $\triangle ACD$ with the line RS, we see that the line AC meets PQ and RS at the same point (possibly at infinity) if and only if

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = 1. \tag{1}$$

So, it suffices to prove (1).

We start with the following result.

Lemma 1. Let $EFGH$ be a circumscribed quadrilateral, and let M be its incenter. Then

$$\frac{EF \cdot FG}{GH \cdot HE} = \frac{FM^2}{HM^2}.$$

Proof. Notice that $\angle EMH + \angle GMF = \angle FME + \angle HMG = 180^\circ$, $\angle FGM = \angle MGH$, and $\angle HEM = \angle MEF$ (see Figure 3). By the law of sines, we get

$$\frac{EF}{FM} \cdot \frac{FG}{FM} = \frac{\sin \angle FME \cdot \sin \angle GMF}{\sin \angle MEF \cdot \sin \angle FGM} = \frac{\sin \angle HMG \cdot \sin \angle EMH}{\sin \angle MGH \cdot \sin \angle HEM} = \frac{GH}{HM} \cdot \frac{HE}{HM}. \qquad \square$$

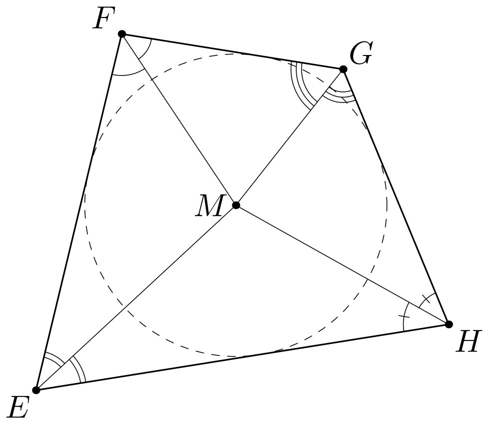


Figure 3

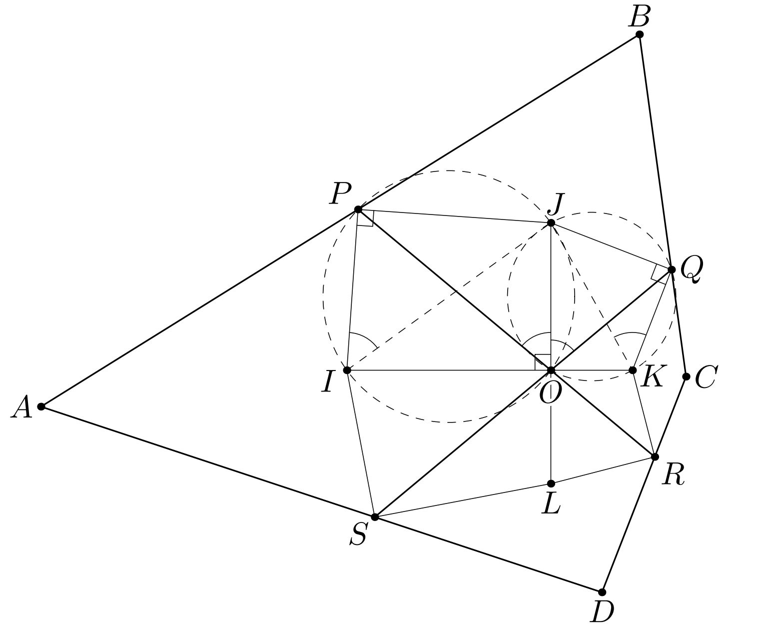


Figure 4

We denote by I, J, K, and L the incenters of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$, respectively. Applying Lemma 1 to these four quadrilaterals we get

$$\frac{AP\cdot PO}{OS\cdot SA}\cdot\frac{BQ\cdot QO}{OP\cdot PB}\cdot\frac{CR\cdot RO}{OQ\cdot QC}\cdot\frac{DS\cdot SO}{OR\cdot RD}=\frac{PI^2}{SI^2}\cdot\frac{QJ^2}{PJ^2}\cdot\frac{RK^2}{QK^2}\cdot\frac{SL^2}{RL^2},$$

which reduces to

$$\frac{AP}{PB}\cdot\frac{BQ}{QC}\cdot\frac{CR}{RD}\cdot\frac{DS}{SA}=\frac{PI^2}{PJ^2}\cdot\frac{QJ^2}{QK^2}\cdot\frac{RK^2}{RL^2}\cdot\frac{SL^2}{SI^2}. \tag{2}$$

Next, we have $\angle IPJ=\angle JOI=90^\circ$, and the line OP separates I and J (see Figure 4). This means that the quadrilateral $IPJO$ is cyclic. Similarly, we get that the quadrilateral $JQKO$ is cyclic with $\angle JQK=90^\circ$. Thus, $\angle QKJ=\angle QOJ=\angle JOP=\angle JIP$. Hence, the right triangles IPJ and KQJ are similar. Therefore, $\frac{PI}{PJ}=\frac{QK}{QJ}$. Likewise, we obtain $\frac{RK}{RL}=\frac{SI}{SL}$. These two equations together with (2) yield (1).

Comment. Instead of using the sine law, one may prove Lemma 1 by the following approach.

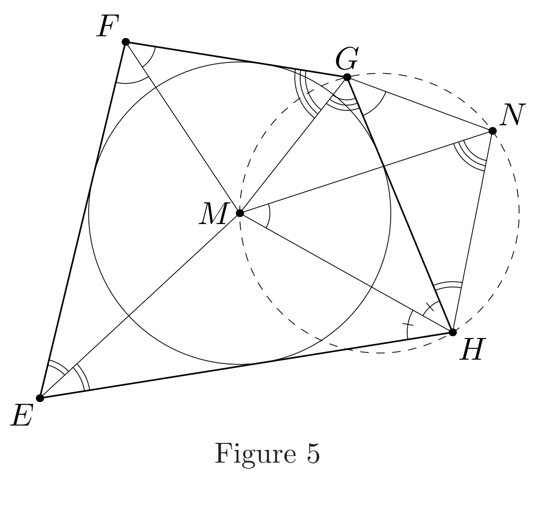


Figure 5

Let N be the point such that $\triangle NHG\sim\triangle MEF$ and such that N and M lie on different sides of the line GH, as shown in Figure 5. Then $\angle GNH+\angle HMG=\angle FME+\angle HMG=180^\circ$. So, the quadrilateral $GNHM$ is cyclic. Thus, $\angle MNH = \angle MGH = \angle FGM$ and $\angle HMN = \angle HGN = \angle EFM = \angle MFG$. Hence, $\triangle HMN \sim \triangle MFG$. Therefore, $\frac{HM}{HG} = \frac{HM}{HN} \cdot \frac{HN}{HG} = \frac{MF}{MG} \cdot \frac{EM}{EF}$. Similarly, we obtain $\frac{HM}{HE} = \frac{MF}{ME} \cdot \frac{GM}{GF}$. By multiplying these two equations, we complete the proof.

Solution 3. We present another approach for showing (1) from Solution 2.

Lemma 2. Let $EFGH$ and $E'F'G'H'$ be circumscribed quadrilaterals such that $\angle E + \angle E' = \angle F + \angle F' = \angle G + \angle G' = \angle H + \angle H' = 180^\circ$. Then

$$\frac{EF \cdot GH}{FG \cdot HE} = \frac{E'F' \cdot G'H'}{F'G' \cdot H'E'}.$$

Proof. Let M and M' be the incenters of $EFGH$ and $E'F'G'H'$, respectively. We use the notation $[XYZ]$ for the area of a triangle XYZ.

Taking into account the relation $\angle FME + \angle F'M'E' = 180^\circ$ together with the analogous ones, we get

$$\begin{aligned}\frac{EF \cdot GH}{FG \cdot HE} &= \frac{[MEF] \cdot [MGH]}{[MFG] \cdot [MHE]} = \frac{ME \cdot MF \cdot \sin\angle FME \cdot MG \cdot MH \cdot \sin\angle HMG}{MF \cdot MG \cdot \sin\angle GMF \cdot MH \cdot ME \cdot \sin\angle EMH} \\ &= \frac{M'E' \cdot M'F' \cdot \sin\angle F'M'E' \cdot M'G' \cdot M'H' \cdot \sin\angle H'M'G'}{M'F' \cdot M'G' \cdot \sin\angle G'M'F' \cdot M'H' \cdot M'E' \cdot \sin\angle E'M'H'} = \frac{E'F' \cdot G'H'}{F'G' \cdot H'E'}.\end{aligned}$$ □

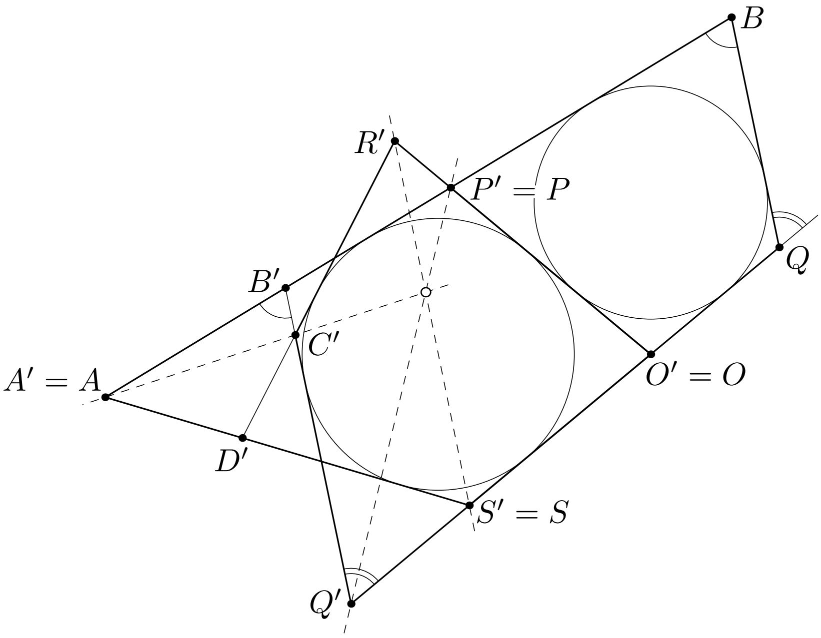


Figure 6

Denote by h the homothety centered at O that maps the incircle of $CROQ$ to the incircle of $APOS$. Let $Q' = h(Q)$, $C' = h(C)$, $R' = h(R)$, $O' = O$, $S' = S$, $A' = A$, and $P' = P$. Furthermore, define $B' = A'P' \cap C'Q'$ and $D' = A'S' \cap C'R'$ as shown in Figure 6. Then

$$\frac{AP \cdot OS}{PO \cdot SA} = \frac{A'P' \cdot O'S'}{P'O' \cdot S'A'}$$

holds trivially. We also have

$$\frac{CR \cdot OQ}{RO \cdot QC} = \frac{C'R' \cdot O'Q'}{R'O' \cdot Q'C'}$$

by the similarity of the quadrilaterals $CROQ$ and $C'R'O'Q'$.

Next, consider the circumscribed quadrilaterals $BQOP$ and $B'Q'O'P'$ whose incenters lie on different sides of the quadrilaterals' shared side line $OP = O'P'$. Observe that $BQ \parallel B'Q'$ and that B' and Q' lie on the lines BP and QO, respectively. It is now easy to see that the two quadrilaterals satisfy the hypotheses of Lemma 2. Thus, we deduce

$$\frac{BQ \cdot OP}{QO \cdot PB} = \frac{B'Q' \cdot O'P'}{Q'O' \cdot P'B'}.$$

Similarly, we get

$$\frac{DS \cdot OR}{SO \cdot RD} = \frac{D'S' \cdot O'R'}{S'O' \cdot R'D'}.$$

Multiplying these four equations, we obtain

$$\frac{AP}{PB} \cdot \frac{BQ}{QC} \cdot \frac{CR}{RD} \cdot \frac{DS}{SA} = \frac{A'P'}{P'B'} \cdot \frac{B'Q'}{Q'C'} \cdot \frac{C'R'}{R'D'} \cdot \frac{D'S'}{S'A'}. \tag{3}$$

Finally, we apply Brianchon's theorem to the circumscribed hexagon $A'P'R'C'Q'S'$ and deduce that the lines $A'C'$, $P'Q'$, and $R'S'$ are either concurrent or parallel to each other. So, by Menelaus' theorem, we obtain

$$\frac{A'P'}{P'B'} \cdot \frac{B'Q'}{Q'C'} \cdot \frac{C'R'}{R'D'} \cdot \frac{D'S'}{S'A'} = 1.$$

This equation together with (3) yield (1).

G8. A *triangulation* of a convex polygon Π is a partitioning of Π into triangles by diagonals having no common points other than the vertices of the polygon. We say that a triangulation is a *Thaiangulation* if all triangles in it have the same area.

Prove that any two different Thaiangulations of a convex polygon Π differ by exactly two triangles. (In other words, prove that it is possible to replace one pair of triangles in the first Thaiangulation with a different pair of triangles so as to obtain the second Thaiangulation.)

(Bulgaria)

Solution 1. We denote by $[S]$ the area of a polygon S.

Recall that each triangulation of a convex n-gon has exactly $n-2$ triangles. This means that all triangles in any two Thaiangulations of a convex polygon Π have the same area.

Let $\mathcal{T}$ be a triangulation of a convex polygon Π. If four vertices A, B, C, and D of Π form a parallelogram, and $\mathcal{T}$ contains two triangles whose union is this parallelogram, then we say that $\mathcal{T}$ *contains* parallelogram $ABCD$. Notice here that if two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π differ by two triangles, then the union of these triangles is a quadrilateral each of whose diagonals bisects its area, i.e., a parallelogram.

We start with proving two properties of triangulations.

Lemma 1. A triangulation of a convex polygon Π cannot contain two parallelograms.

Proof. Arguing indirectly, assume that P_1 and P_2 are two parallelograms contained in some triangulation $\mathcal{T}$. If they have a common triangle in $\mathcal{T}$, then we may assume that P_1 consists of triangles ABC and ADC of $\mathcal{T}$, while P_2 consists of triangles ADC and CDE (see Figure 1). But then $BC \parallel AD \parallel CE$, so the three vertices B, C, and E of Π are collinear, which is absurd.

Assume now that P_1 and P_2 contain no common triangle. Let $P_1 = ABCD$. The sides AB, BC, CD, and DA partition Π into several parts, and P_2 is contained in one of them; we may assume that this part is cut off from P_1 by AD. Then one may label the vertices of P_2 by X, Y, Z, and T so that the polygon $ABCDXYZT$ is convex (see Figure 2; it may happen that $D = X$ and/or $T = A$, but still this polygon has at least six vertices). But the sum of the external angles of this polygon at B, C, Y, and Z is already $360°$, which is impossible. A final contradiction. $\square$

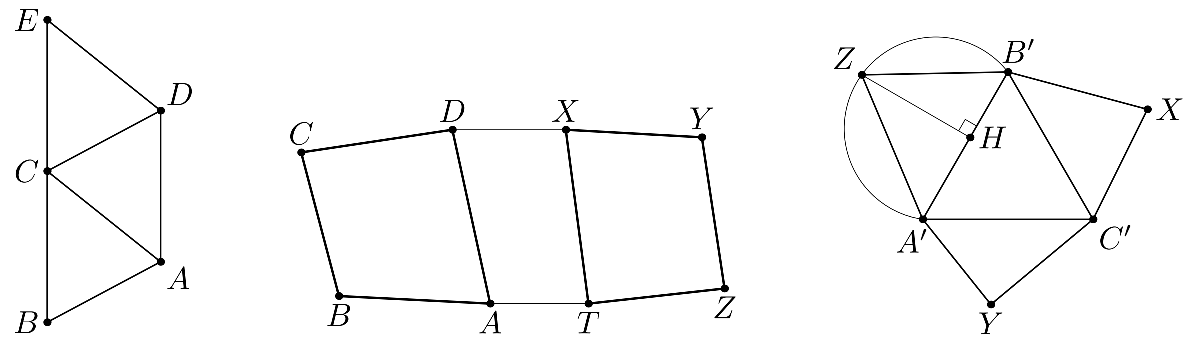


Figure 1 Figure 2 Figure 3

Lemma 2. Every triangle in a Thaiangulation $\mathcal{T}$ of Π contains a side of Π.

Proof. Let ABC be a triangle in $\mathcal{T}$. Apply an affine transform such that ABC maps to an equilateral triangle; let $A'B'C'$ be the image of this triangle, and Π' be the image of Π. Clearly, $\mathcal{T}$ maps into a Thaiangulation $\mathcal{T}'$ of Π'.

Assume that none of the sides of $\triangle A'B'C'$ is a side of Π'. Then $\mathcal{T}'$ contains some other triangles with these sides, say, $A'B'Z$, $C'A'Y$, and $B'C'X$; notice that $A'ZB'XC'Y$ is a convex hexagon (see Figure 3). The sum of its external angles at X, Y, and Z is less than $360°$. So one of these angles (say, at Z) is less than $120°$, hence $\angle A'ZB' > 60°$. Then Z lies on a circular arc subtended by $A'B'$ and having angular measure less than $240°$; consequently, the altitude ZH of $\triangle A'B'Z$ is less than $\sqrt{3}\,A'B'/2$. Thus $[A'B'Z] < [A'B'C']$, and $\mathcal{T}'$ is not a Thaiangulation. A contradiction. $\square$

Now we pass to the solution. We say that a triangle in a triangulation of Π is an *ear* if it contains two sides of Π. Note that each triangulation of a polygon contains some ear.

Arguing indirectly, we choose a convex polygon Π with the least possible number of sides such that some two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π violate the problem statement (thus Π has at least five sides). Consider now any ear ABC in $\mathcal{T}_1$, with AC being a diagonal of Π. If $\mathcal{T}_2$ also contains $\triangle ABC$, then one may cut $\triangle ABC$ off from Π, getting a polygon with a smaller number of sides which also violates the problem statement. This is impossible; thus $\mathcal{T}_2$ does not contain $\triangle ABC$.

Next, $\mathcal{T}_1$ contains also another triangle with side AC, say $\triangle ACD$. By Lemma 2, this triangle contains a side of Π, so D is adjacent to either A or C on the boundary of Π. We may assume that D is adjacent to C.

Assume that $\mathcal{T}_2$ does not contain the triangle BCD. Then it contains two different triangles BCX and CDY (possibly, with $X = Y$); since these triangles have no common interior points, the polygon $ABCDYX$ is convex (see Figure 4). But, since $[ABC] = [BCX] = [ACD] = [CDY]$, we get $AX \parallel BC$ and $AY \parallel CD$ which is impossible. Thus $\mathcal{T}_2$ contains $\triangle BCD$.

Therefore, $[ABD] = [ABC] + [ACD] - [BCD] = [ABC]$, and $ABCD$ is a parallelogram contained in $\mathcal{T}_1$. Let $\mathcal{T}'$ be the Thaiangulation of Π obtained from $\mathcal{T}_1$ by replacing the diagonal AC with BD; then $\mathcal{T}'$ is distinct from $\mathcal{T}_2$ (otherwise $\mathcal{T}_1$ and $\mathcal{T}_2$ would differ by two triangles). Moreover, $\mathcal{T}'$ shares a common ear BCD with $\mathcal{T}_2$. As above, cutting this ear away we obtain that $\mathcal{T}_2$ and $\mathcal{T}'$ differ by two triangles forming a parallelogram different from $ABCD$. Thus $\mathcal{T}'$ contains two parallelograms, which contradicts Lemma 1.

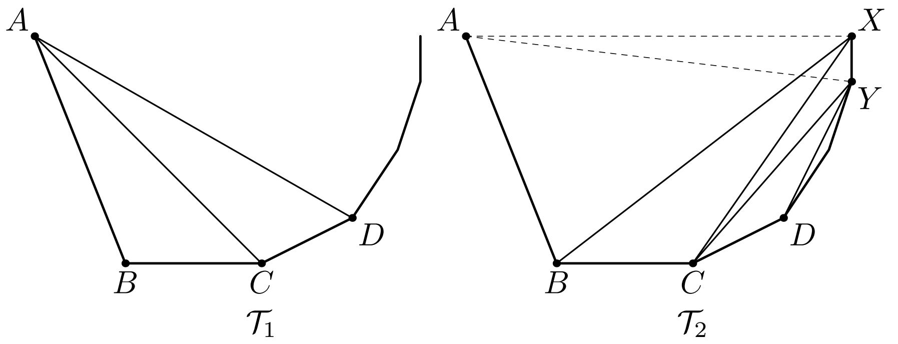


Figure 4

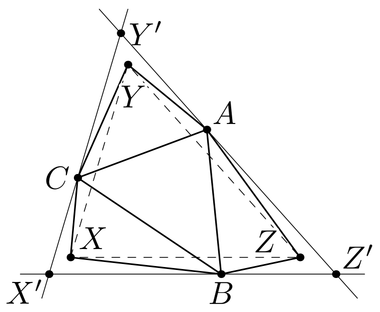


Figure 5

Comment 1. Lemma 2 is equivalent to the well-known Erdős–Debrunner inequality stating that for any triangle PQR and any points A, B, C lying on the sides QR, RP, and PQ, respectively, we have

$$[ABC] \geqslant \min\{[ABR], [BCP], [CAQ]\}. \tag{1}$$

To derive this inequality from Lemma 2, one may assume that (1) does not hold, and choose some points X, Y, and Z inside the triangles BCP, CAQ, and ABR, respectively, so that $[ABC] = [ABZ] = [BCX] = [CAY]$. Then a convex hexagon $AZBXCY$ has a Thaiangulation containing $\triangle ABC$, which contradicts Lemma 2.

Conversely, assume that a Thaiangulation $\mathcal{T}$ of Π contains a triangle ABC none of whose sides is a side of Π, and let ABZ, AYC, and XBC be other triangles in $\mathcal{T}$ containing the corresponding sides. Then $AZBXCY$ is a convex hexagon.

Consider the lines through A, B, and C parallel to YZ, ZX, and XY, respectively. They form a triangle $X'Y'Z'$ similar to $\triangle XYZ$ (see Figure 5). By (1) we have

$$[ABC] \geqslant \min\{[ABZ'], [BCX'], [CAY']\} > \min\{[ABZ], [BCX], [CAY]\},$$

so $\mathcal{T}$ is not a Thaiangulation.

Solution 2. We will make use of the preliminary observations from Solution 1, together with Lemma 1.

Arguing indirectly, we choose a convex polygon Π with the least possible number of sides such that some two Thaiangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of Π violate the statement (thus Π has at least five sides). Assume that $\mathcal{T}_1$ and $\mathcal{T}_2$ share a diagonal d splitting Π into two smaller polygons Π_1 and Π_2. Since the problem statement holds for any of them, the induced Thaiangulations of each of Π_i differ by two triangles forming a parallelogram (the Thaiangulations induced on Π_i by $\mathcal{T}_1$ and T_2 may not coincide, otherwise $\mathcal{T}_1$ and $\mathcal{T}_2$ would differ by at most two triangles). But both these parallelograms are contained in $\mathcal{T}_1$; this contradicts Lemma 1. Therefore, $\mathcal{T}_1$ and $\mathcal{T}_2$ share no diagonal. Hence they also share no triangle.

We consider two cases.

Case 1. Assume that some vertex B of Π is an endpoint of some diagonal in $\mathcal{T}_1$, as well as an endpoint of some diagonal in $\mathcal{T}_2$.

Let A and C be the vertices of Π adjacent to B. Then $\mathcal{T}_1$ contains some triangles ABX and BCY, while $\mathcal{T}_2$ contains some triangles ABX' and BCY'. Here, some of the points X, X', Y, and Y' may coincide; however, in view of our assumption together with the fact that $\mathcal{T}_1$ and $\mathcal{T}_2$ share no triangle, all four triangles ABX, BCY, ABX', and BCY' are distinct.

Since $[ABX] = [BCY] = [ABX'] = [BCY']$, we have $XX' \parallel AB$ and $YY' \parallel BC$. Now, if $X = Y$, then X' and Y' lie on different lines passing through X and are distinct from that point, so that $X' \neq Y'$. In this case, we may switch the two Thaiangulations. So, hereafter we assume that $X \neq Y$.

In the convex pentagon $ABCYX$ we have either $\angle BAX + \angle AXY > 180^\circ$ or $\angle XYC + \angle YCB > 180^\circ$ (or both); due to the symmetry, we may assume that the first inequality holds. Let r be the ray emerging from X and co-directed with $\overrightarrow{AB}$; our inequality shows that r points to the interior of the pentagon (and thus to the interior of Π). Therefore, the ray opposite to r points outside Π, so X' lies on r; moreover, X' lies on the "arc" CY of Π not containing X. So the segments XX' and YB intersect (see Figure 6).

Let O be the intersection point of the rays r and BC. Since the triangles ABX' and BCY' have no common interior points, Y' must lie on the "arc" CX' which is situated inside the triangle XBO. Therefore, the line YY' meets two sides of $\triangle XBO$, none of which may be XB (otherwise the diagonals XB and YY' would share a common point). Thus YY' intersects BO, which contradicts $YY' \parallel BC$.

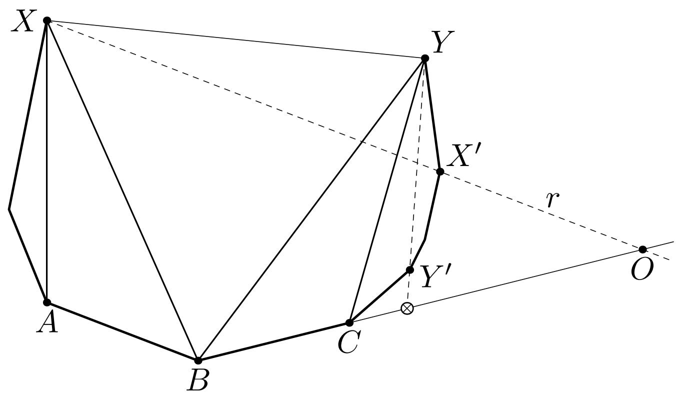


Figure 6

Case 2. In the remaining case, each vertex of Π is an endpoint of a diagonal in at most one of $\mathcal{T}_1$ and $\mathcal{T}_2$. On the other hand, a triangulation cannot contain two consecutive vertices with no diagonals from each. Therefore, the vertices of Π alternatingly emerge diagonals in $\mathcal{T}_1$ and in $\mathcal{T}_2$. In particular, Π has an even number of sides.

Next, we may choose five consecutive vertices A, B, C, D, and E of Π in such a way that

$$\angle ABC + \angle BCD > 180^\circ \quad \text{and} \quad \angle BCD + \angle CDE > 180^\circ. \tag{2}$$

In order to do this, it suffices to choose three consecutive vertices B, C, and D of Π such that the sum of their external angles is at most 180°. This is possible, since Π has at least six sides.

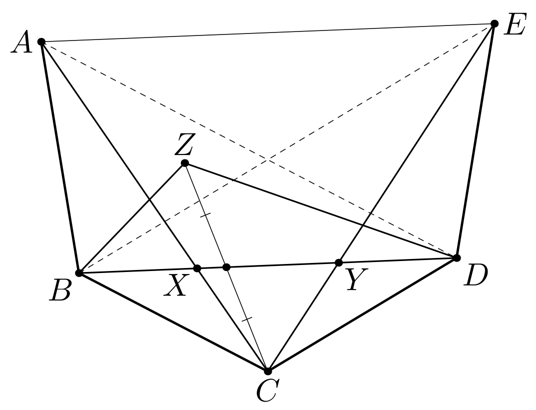


Figure 7

We may assume that $\mathcal{T}_1$ has no diagonals from B and D (and thus contains the triangles ABC and CDE), while $\mathcal{T}_2$ has no diagonals from A, C, and E (and thus contains the triangle BCD). Now, since $[ABC] = [BCD] = [CDE]$, we have $AD \parallel BC$ and $BE \parallel CD$ (see Figure 7). By (2) this yields that $AD > BC$ and $BE > CD$. Let $X = AC \cap BD$ and $Y = CE \cap BD$; then the inequalities above imply that $AX > CX$ and $EY > CY$.

Finally, $\mathcal{T}_2$ must also contain some triangle BDZ with $Z \neq C$; then the ray CZ lies in the angle ACE. Since $[BCD] = [BDZ]$, the diagonal BD bisects CZ. Together with the inequalities above, this yields that Z lies inside the triangle ACE (but Z is distinct from A and E), which is impossible. The final contradiction.

Comment 2. Case 2 may also be accomplished with the use of Lemma 2. Indeed, since each triangulation of an n-gon contains $n-2$ triangles neither of which may contain three sides of Π, Lemma 2 yields that each Thaiangulation contains exactly two ears. But each vertex of Π is a vertex of an ear either in $\mathcal{T}_1$ or in $\mathcal{T}_2$, so Π cannot have more than four vertices.

Number Theory

N1. Determine all positive integers M for which the sequence $a_0, a_1, a_2, \ldots$, defined by $a_0 = \frac{2M+1}{2}$ and $a_{k+1} = a_k\lfloor a_k \rfloor$ for $k = 0, 1, 2, \ldots$, contains at least one integer term.

(Luxembourg)

Answer. All integers $M \geqslant 2$.

Solution 1. Define $b_k = 2a_k$ for all $k \geqslant 0$. Then

$$b_{k+1} = 2a_{k+1} = 2a_k\lfloor a_k \rfloor = b_k \left\lfloor \frac{b_k}{2} \right\rfloor.$$

Since b_0 is an integer, it follows that b_k is an integer for all $k \geqslant 0$.

Suppose that the sequence $a_0, a_1, a_2, \ldots$ does not contain any integer term. Then b_k must be an odd integer for all $k \geqslant 0$, so that

$$b_{k+1} = b_k \left\lfloor \frac{b_k}{2} \right\rfloor = \frac{b_k(b_k - 1)}{2}. \tag{1}$$

Hence

$$b_{k+1} - 3 = \frac{b_k(b_k - 1)}{2} - 3 = \frac{(b_k - 3)(b_k + 2)}{2} \tag{2}$$

for all $k \geqslant 0$.

Suppose that $b_0 - 3 > 0$. Then equation (2) yields $b_k - 3 > 0$ for all $k \geqslant 0$. For each $k \geqslant 0$, define c_k to be the highest power of 2 that divides $b_k - 3$. Since $b_k - 3$ is even for all $k \geqslant 0$, the number c_k is positive for every $k \geqslant 0$.

Note that $b_k + 2$ is an odd integer. Therefore, from equation (2), we have that $c_{k+1} = c_k - 1$. Thus, the sequence $c_0, c_1, c_2, \ldots$ of positive integers is strictly decreasing, a contradiction. So, $b_0 - 3 \leqslant 0$, which implies $M = 1$.

For $M = 1$, we can check that the sequence is constant with $a_k = \frac{3}{2}$ for all $k \geqslant 0$. Therefore, the answer is $M \geqslant 2$.

Solution 2. We provide an alternative way to show $M = 1$ once equation (1) has been reached. We claim that $b_k \equiv 3 \pmod{2^m}$ for all $k \geqslant 0$ and $m \geqslant 1$. If this is true, then we would have $b_k = 3$ for all $k \geqslant 0$ and hence $M = 1$.

To establish our claim, we proceed by induction on m. The base case $b_k \equiv 3 \pmod 2$ is true for all $k \geqslant 0$ since b_k is odd. Now suppose that $b_k \equiv 3 \pmod{2^m}$ for all $k \geqslant 0$. Hence $b_k = 2^m d_k + 3$ for some integer d_k. We have

$$3 \equiv b_{k+1} \equiv (2^m d_k + 3)(2^{m-1} d_k + 1) \equiv 3 \cdot 2^{m-1} d_k + 3 \pmod{2^m},$$

so that d_k must be even. This implies that $b_k \equiv 3 \pmod{2^{m+1}}$, as required.

Comment. The reason the number 3 which appears in both solutions is important, is that it is a nontrivial fixed point of the recurrence relation for b_k.

N2. Let a and b be positive integers such that $a!b!$ is a multiple of $a! + b!$. Prove that $3a \geqslant 2b + 2$.

(United Kingdom)

Solution 1. If $a > b$, we immediately get $3a \geqslant 2b + 2$. In the case $a = b$, the required inequality is equivalent to $a \geqslant 2$, which can be checked easily since $(a, b) = (1, 1)$ does not satisfy $a! + b! \mid a!b!$. We now assume $a < b$ and denote $c = b - a$. The required inequality becomes $a \geqslant 2c + 2$.

Suppose, to the contrary, that $a \leqslant 2c + 1$. Define $M = \frac{b!}{a!} = (a + 1)(a + 2) \cdots (a + c)$. Since $a! + b! \mid a!b!$ implies $1 + M \mid a!M$, we obtain $1 + M \mid a!$. Note that we must have $c < a$; otherwise $1 + M > a!$, which is impossible. We observe that $c! \mid M$ since M is a product of c consecutive integers. Thus $\gcd(1 + M, c!) = 1$, which implies

$$1 + M \,\Bigg|\, \frac{a!}{c!} = (c + 1)(c + 2) \cdots a. \tag{1}$$

If $a \leqslant 2c$, then $\frac{a!}{c!}$ is a product of $a - c \leqslant c$ integers not exceeding a whereas M is a product of c integers exceeding a. Therefore, $1 + M > \frac{a!}{c!}$, which is a contradiction.

It remains to exclude the case $a = 2c + 1$. Since $a + 1 = 2(c + 1)$, we have $c + 1 \mid M$. Hence, we can deduce from (1) that $1 + M \mid (c + 2)(c + 3) \cdots a$. Now $(c + 2)(c + 3) \cdots a$ is a product of $a - c - 1 = c$ integers not exceeding a; thus it is smaller than $1 + M$. Again, we arrive at a contradiction.

Comment 1. One may derive a weaker version of (1) and finish the problem as follows. After assuming $a \leqslant 2c + 1$, we have $\left\lfloor \frac{a}{2} \right\rfloor \leqslant c$, so $\left\lfloor \frac{a}{2} \right\rfloor! \mid M$. Therefore,

$$1 + M \,\Bigg|\, \left(\left\lfloor \frac{a}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{a}{2} \right\rfloor + 2\right) \cdots a.$$

Observe that $\left(\left\lfloor \frac{a}{2} \right\rfloor + 1\right)\left(\left\lfloor \frac{a}{2} \right\rfloor + 2\right) \cdots a$ is a product of $\left\lceil \frac{a}{2} \right\rceil$ integers not exceeding a. This leads to a contradiction when a is even since $\left\lceil \frac{a}{2} \right\rceil = \frac{a}{2} \leqslant c$ and M is a product of c integers exceeding a.

When a is odd, we can further deduce that $1 + M \mid \left(\frac{a+3}{2}\right)\left(\frac{a+5}{2}\right) \cdots a$ since $\left\lfloor \frac{a}{2} \right\rfloor + 1 = \frac{a+1}{2} \,\big|\, a + 1$. Now $\left(\frac{a+3}{2}\right)\left(\frac{a+5}{2}\right) \cdots a$ is a product of $\frac{a-1}{2} \leqslant c$ numbers not exceeding a, and we get a contradiction.

Solution 2. As in Solution 1, we may assume that $a < b$ and let $c = b - a$. Suppose, to the contrary, that $a \leqslant 2c + 1$. From $a! + b! \mid a!b!$, we have

$$N = 1 + (a + 1)(a + 2) \cdots (a + c) \,\big|\, (a + c)!,$$

which implies that all prime factors of N are at most $a + c$.

Let p be a prime factor of N. If $p \leqslant c$ or $p \geqslant a + 1$, then p divides one of $a + 1, \ldots, a + c$ which is impossible. Hence $a \geqslant p \geqslant c + 1$. Furthermore, we must have $2p > a + c$; otherwise, $a + 1 \leqslant 2c + 2 \leqslant 2p \leqslant a + c$ so $p \mid N - 1$, again impossible. Thus, we have $p \in \left(\frac{a+c}{2}, a\right]$, and $p^2 \nmid (a + c)!$ since $2p > a + c$. Therefore, $p^2 \nmid N$ as well.

If $a \leqslant c + 2$, then the interval $\left(\frac{a+c}{2}, a\right]$ contains at most one integer and hence at most one prime number, which has to be a. Since $p^2 \nmid N$, we must have $N = p = a$ or $N = 1$, which is absurd since $N > a \geqslant 1$. Thus, we have $a \geqslant c + 3$, and so $\frac{a+c+1}{2} \geqslant c + 2$. It follows that p lies in the interval $[c + 2, a]$.

Thus, every prime appearing in the prime factorization of N lies in the interval $[c + 2, a]$, and its exponent is exactly 1. So we must have $N \mid (c + 2)(c + 3) \cdots a$. However, $(c + 2)(c + 3) \cdots a$ is a product of $a - c - 1 \leqslant c$ numbers not exceeding a, so it is less than N. This is a contradiction.

Comment 2. The original problem statement also asks to determine when the equality $3a = 2b + 2$ holds. It can be checked that the answer is $(a, b) = (2, 2), (4, 5)$.

N3. Let m and n be positive integers such that $m > n$. Define $x_k = (m+k)/(n+k)$ for $k = 1, 2, \ldots, n+1$. Prove that if all the numbers $x_1, x_2, \ldots, x_{n+1}$ are integers, then $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

(Austria)

Solution. Assume that $x_1, x_2, \ldots, x_{n+1}$ are integers. Define the integers

$$a_k = x_k - 1 = \frac{m+k}{n+k} - 1 = \frac{m-n}{n+k} > 0$$

for $k = 1, 2, \ldots, n+1$.

Let $P = x_1 x_2 \cdots x_{n+1} - 1$. We need to prove that P is divisible by an odd prime, or in other words, that P is not a power of 2. To this end, we investigate the powers of 2 dividing the numbers a_k.

Let 2^d be the largest power of 2 dividing $m-n$, and let 2^c be the largest power of 2 not exceeding $2n+1$. Then $2n+1 \leqslant 2^{c+1}-1$, and so $n+1 \leqslant 2^c$. We conclude that 2^c is one of the numbers $n+1, n+2, \ldots, 2n+1$, and that it is the only multiple of 2^c appearing among these numbers. Let ℓ be such that $n+\ell = 2^c$. Since $\frac{m-n}{n+\ell}$ is an integer, we have $d \geqslant c$. Therefore, $2^{d-c+1} \nmid a_\ell = \frac{m-n}{n+\ell}$, while $2^{d-c+1} \mid a_k$ for all $k \in \{1, \ldots, n+1\} \setminus \{\ell\}$.

Computing modulo 2^{d-c+1}, we get

$$P = (a_1+1)(a_2+1)\cdots(a_{n+1}+1) - 1 \equiv (a_\ell+1)\cdot 1^n - 1 \equiv a_\ell \not\equiv 0 \pmod{2^{d-c+1}}.$$

Therefore, $2^{d-c+1} \nmid P$.

On the other hand, for any $k \in \{1, \ldots, n+1\} \setminus \{\ell\}$, we have $2^{d-c+1} \mid a_k$. So $P \geqslant a_k \geqslant 2^{d-c+1}$, and it follows that P is not a power of 2.

Comment. Instead of attempting to show that P is not a power of 2, one may try to find an odd factor of P (greater than 1) as follows:

From $a_k = \frac{m-n}{n+k} \in \mathbb{Z}_{>0}$, we get that $m-n$ is divisible by $n+1, n+2, \ldots, 2n+1$, and thus it is also divisible by their least common multiple L. So $m - n = qL$ for some positive integer q; hence $x_k = q \cdot \frac{L}{n+k} + 1$.

Then, since $n+1 \leqslant 2^c = n+\ell \leqslant 2n+1 \leqslant 2^{c+1}-1$, we have $2^c \mid L$, but $2^{c+1} \nmid L$. So $\frac{L}{n+\ell}$ is odd, while $\frac{L}{n+k}$ is even for $k \neq \ell$. Computing modulo $2q$ yields

$$x_1 x_2 \cdots x_{n+1} - 1 \equiv (q+1)\cdot 1^n - 1 \equiv q \pmod{2q}.$$

Thus, $x_1 x_2 \cdots x_{n+1} - 1 = 2qr + q = q(2r+1)$ for some integer r.

Since $x_1 x_2 \cdots x_{n+1} - 1 \geqslant x_1 x_2 - 1 \geqslant (q+1)^2 - 1 > q$, we have $r \geqslant 1$. This implies that $x_1 x_2 \cdots x_{n+1} - 1$ is divisible by an odd prime.

N4. Suppose that $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ are two sequences of positive integers satisfying $a_0, b_0 \geqslant 2$ and

$$a_{n+1} = \gcd(a_n, b_n) + 1, \qquad b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1$$

for all $n \geqslant 0$. Prove that the sequence (a_n) is eventually periodic; in other words, there exist integers $N \geqslant 0$ and $t > 0$ such that $a_{n+t} = a_n$ for all $n \geqslant N$.

(France)

Solution 1. Let $s_n = a_n + b_n$. Notice that if $a_n \mid b_n$, then $a_{n+1} = a_n + 1$, $b_{n+1} = b_n - 1$ and $s_{n+1} = s_n$. So, a_n increases by 1 and s_n does not change until the first index is reached with $a_n \nmid s_n$. Define

$$W_n = \left\{m \in \mathbb{Z}_{>0} \colon m \geqslant a_n \text{ and } m \nmid s_n\right\} \quad \text{and} \quad w_n = \min W_n.$$

Claim 1. The sequence (w_n) is non-increasing.

Proof. If $a_n \mid b_n$ then $a_{n+1} = a_n + 1$. Due to $a_n \mid s_n$, we have $a_n \notin W_n$. Moreover $s_{n+1} = s_n$; therefore, $W_{n+1} = W_n$ and $w_{n+1} = w_n$.

Otherwise, if $a_n \nmid b_n$, then $a_n \nmid s_n$, so $a_n \in W_n$ and thus $w_n = a_n$. We show that $a_n \in W_{n+1}$; this implies $w_{n+1} \leqslant a_n = w_n$. By the definition of W_{n+1}, we need that $a_n \geqslant a_{n+1}$ and $a_n \nmid s_{n+1}$. The first relation holds because of $\gcd(a_n, b_n) < a_n$. For the second relation, observe that in $s_{n+1} = \gcd(a_n, b_n) + \operatorname{lcm}(a_n, b_n)$, the second term is divisible by a_n, but the first term is not. So $a_n \nmid s_{n+1}$; that completes the proof of the claim. $\square$

Let $w = \min\limits_n w_n$ and let N be an index with $w = w_N$. Due to Claim 1, we have $w_n = w$ for all $n \geqslant N$.

Let $g_n = \gcd(w, s_n)$. As we have seen, starting from an arbitrary index $n \geqslant N$, the sequence $a_n, a_{n+1}, \ldots$ increases by 1 until it reaches w, which is the first value not dividing s_n; then it drops to $\gcd(w, s_n) + 1 = g_n + 1$.

Claim 2. The sequence (g_n) is constant for $n \geqslant N$.

Proof. If $a_n \mid b_n$, then $s_{n+1} = s_n$ and hence $g_{n+1} = g_n$. Otherwise we have $a_n = w$,

$$\begin{aligned}
\gcd(a_n, b_n) &= \gcd(a_n, s_n) = \gcd(w, s_n) = g_n, \\
s_{n+1} &= \gcd(a_n, b_n) + \operatorname{lcm}(a_n, b_n) = g_n + \frac{a_n b_n}{g_n} = g_n + \frac{w(s_n - w)}{g_n}, \qquad (1) \\
\text{and} \quad g_{n+1} &= \gcd(w, s_{n+1}) = \gcd\left(w,\ g_n + \frac{s_n - w}{g_n} w\right) = \gcd(w, g_n) = g_n. \qquad \square
\end{aligned}$$

Let $g = g_N$. We have proved that the sequence (a_n) eventually repeats the following cycle:

$$g + 1 \mapsto g + 2 \mapsto \ldots \mapsto w \mapsto g + 1.$$

Solution 2. By Claim 1 in the first solution, we have $a_n \leqslant w_n \leqslant w_0$, so the sequence (a_n) is bounded, and hence it has only finitely many values.

Let $M = \operatorname{lcm}(a_1, a_2, \ldots)$, and consider the sequence b_n modulo M. Let r_n be the remainder of b_n, divided by M. For every index n, since $a_n \mid M \mid b_n - r_n$, we have $\gcd(a_n, b_n) = \gcd(a_n, r_n)$, and therefore

$$a_{n+1} = \gcd(a_n, r_n) + 1.$$

Moreover,

$$\begin{aligned}
r_{n+1} \equiv b_{n+1} = \operatorname{lcm}(a_n, b_n) - 1 &= \frac{a_n}{\gcd(a_n, b_n)} b_n - 1 \\
&= \frac{a_n}{\gcd(a_n, r_n)} b_n - 1 \equiv \frac{a_n}{\gcd(a_n, r_n)} r_n - 1 \pmod{M}.
\end{aligned}$$

Hence, the pair (a_n, r_n) uniquely determines the pair (a_{n+1}, r_{n+1}). Since there are finitely many possible pairs, the sequence of pairs (a_n, r_n) is eventually periodic; in particular, the sequence (a_n) is eventually periodic.

Comment. We show that there are only four possibilities for g and w (as defined in Solution 1), namely

$$(w, g) \in \{(2,1), (3,1), (4,2), (5,1)\}. \tag{2}$$

This means that the sequence (a_n) eventually repeats one of the following cycles:

$$(2), \quad (2,3), \quad (3,4), \quad \text{or} \quad (2,3,4,5). \tag{3}$$

Using the notation of Solution 1, for $n \geqslant N$ the sequence (a_n) has a cycle $(g+1, g+2, \ldots, w)$ such that $g = \gcd(w, s_n)$. By the observations in the proof of Claim 2, the numbers $g+1, \ldots, w-1$ all divide s_n; so the number $L = \operatorname{lcm}(g+1, g+2, \ldots, w-1)$ also divides s_n. Moreover, g also divides w.

Now choose any $n \geqslant N$ such that $a_n = w$. By (1), we have

$$s_{n+1} = g + \frac{w(s_n - w)}{g} = s_n \cdot \frac{w}{g} - \frac{w^2 - g^2}{g}.$$

Since L divides both s_n and s_{n+1}, it also divides the number $T = \frac{w^2 - g^2}{g}$.

Suppose first that $w \geqslant 6$, which yields $g + 1 \leqslant \frac{w}{2} + 1 \leqslant w - 2$. Then $(w-2)(w-1) \mid L \mid T$, so we have either $w^2 - g^2 \geqslant 2(w-1)(w-2)$, or $g = 1$ and $w^2 - g^2 = (w-1)(w-2)$. In the former case we get $(w-1)(w-5) + (g^2 - 1) \leqslant 0$ which is false by our assumption. The latter equation rewrites as $3w = 3$, so $w = 1$, which is also impossible.

Now we are left with the cases when $w \leqslant 5$ and $g \mid w$. The case $(w, g) = (4, 1)$ violates the condition $L \mid \frac{w^2 - g^2}{g}$; all other such pairs are listed in (2).

In the table below, for each pair (w, g), we provide possible sequences (a_n) and (b_n). That shows that the cycles shown in (3) are indeed possible.

$w = 2$	$g = 1$	$a_n = 2$	$b_n = 2 \cdot 2^n + 1$
$w = 3$	$g = 1$	$(a_{2k}, a_{2k+1}) = (2, 3)$	$(b_{2k}, b_{2k+1}) = (6 \cdot 3^k + 2, 6 \cdot 3^k + 1)$
$w = 4$	$g = 2$	$(a_{2k}, a_{2k+1}) = (3, 4)$	$(b_{2k}, b_{2k+1}) = (12 \cdot 2^k + 3, 12 \cdot 2^k + 2)$
$w = 5$	$g = 1$	$(a_{4k}, \ldots, a_{4k+3}) = (2, 3, 4, 5)$	$(b_{4k}, \ldots, b_{4k+3}) = (6 \cdot 5^k + 4, \ldots, 6 \cdot 5^k + 1)$

N5. Determine all triples (a, b, c) of positive integers for which $ab - c$, $bc - a$, and $ca - b$ are powers of 2.

Explanation: A power of 2 is an integer of the form 2^n, where n denotes some nonnegative integer.

(Serbia)

Answer. There are sixteen such triples, namely $(2, 2, 2)$, the three permutations of $(2, 2, 3)$, and the six permutations of each of $(2, 6, 11)$ and $(3, 5, 7)$.

Solution 1. It can easily be verified that these sixteen triples are as required. Now let (a, b, c) be any triple with the desired property. If we would have $a = 1$, then both $b - c$ and $c - b$ were powers of 2, which is impossible since their sum is zero; because of symmetry, this argument shows $a, b, c \geqslant 2$.

Case 1. Among a, b, and c there are at least two equal numbers.

Without loss of generality we may suppose that $a = b$. Then $a^2 - c$ and $a(c - 1)$ are powers of 2. The latter tells us that actually a and $c - 1$ are powers of 2. So there are nonnegative integers α and γ with $a = 2^\alpha$ and $c = 2^\gamma + 1$. Since $a^2 - c = 2^{2\alpha} - 2^\gamma - 1$ is a power of 2 and thus incongruent to -1 modulo 4, we must have $\gamma \leqslant 1$. Moreover, each of the terms $2^{2\alpha} - 2$ and $2^{2\alpha} - 3$ can only be a power of 2 if $\alpha = 1$. It follows that the triple (a, b, c) is either $(2, 2, 2)$ or $(2, 2, 3)$.

Case 2. The numbers a, b, and c are distinct.

Due to symmetry we may suppose that

$$2 \leqslant a < b < c\,. \tag{1}$$

We are to prove that the triple (a, b, c) is either $(2, 6, 11)$ or $(3, 5, 7)$. By our hypothesis, there exist three nonnegative integers α, β, and γ such that

$$bc - a = 2^\alpha\,, \tag{2}$$

$$ac - b = 2^\beta\,, \tag{3}$$

$$\text{and} \quad ab - c = 2^\gamma\,. \tag{4}$$

Evidently we have

$$\alpha > \beta > \gamma\,. \tag{5}$$

Depending on how large a is, we divide the argument into two further cases.

Case 2.1. $a = 2$.

We first prove that $\gamma = 0$. Assume for the sake of contradiction that $\gamma > 0$. Then c is even by (4) and, similarly, b is even by (5) and (3). So the left-hand side of (2) is congruent to 2 modulo 4, which is only possible if $bc = 4$. As this contradicts (1), we have thereby shown that $\gamma = 0$, i.e., that $c = 2b - 1$.

Now (3) yields $3b - 2 = 2^\beta$. Due to $b > 2$ this is only possible if $\beta \geqslant 4$. If $\beta = 4$, then we get $b = 6$ and $c = 2 \cdot 6 - 1 = 11$, which is a solution. It remains to deal with the case $\beta \geqslant 5$. Now (2) implies

$$9 \cdot 2^\alpha = 9b(2b - 1) - 18 = (3b - 2)(6b + 1) - 16 = 2^\beta(2^{\beta+1} + 5) - 16\,,$$

and by $\beta \geqslant 5$ the right-hand side is not divisible by 32. Thus $\alpha \leqslant 4$ and we get a contradiction to (5).

Case 2.2. $a \geqslant 3$.
Pick an integer $\vartheta \in \{-1, +1\}$ such that $c - \vartheta$ is not divisible by 4. Now

$$2^\alpha + \vartheta \cdot 2^\beta = (bc - a\vartheta^2) + \vartheta(ca - b) = (b + a\vartheta)(c - \vartheta)$$

is divisible by 2^β and, consequently, $b + a\vartheta$ is divisible by $2^{\beta-1}$. On the other hand, $2^\beta = ac - b > (a-1)c \geqslant 2c$ implies in view of (1) that a and b are smaller than $2^{\beta-1}$. All this is only possible if $\vartheta = 1$ and $a + b = 2^{\beta-1}$. Now (3) yields

$$ac - b = 2(a + b)\,, \tag{6}$$

whence $4b > a + 3b = a(c-1) \geqslant ab$, which in turn yields $a = 3$.

So (6) simplifies to $c = b + 2$ and (2) tells us that $b(b+2) - 3 = (b-1)(b+3)$ is a power of 2. Consequently, the factors $b-1$ and $b+3$ are powers of 2 themselves. Since their difference is 4, this is only possible if $b = 5$ and thus $c = 7$. Thereby the solution is complete.

Solution 2. As in the beginning of the first solution, we observe that $a, b, c \geqslant 2$. Depending on the parities of a, b, and c we distinguish three cases.

Case 1. The numbers a, b, and c are even.
Let 2^A, 2^B, and 2^C be the largest powers of 2 dividing a, b, and c respectively. We may assume without loss of generality that $1 \leqslant A \leqslant B \leqslant C$. Now 2^B is the highest power of 2 dividing $ac - b$, whence $ac - b = 2^B \leqslant b$. Similarly, we deduce $bc - a = 2^A \leqslant a$. Adding both estimates we get $(a + b)c \leqslant 2(a + b)$, whence $c \leqslant 2$. So $c = 2$ and thus $A = B = C = 1$; moreover, we must have had equality throughout, i.e., $a = 2^A = 2$ and $b = 2^B = 2$. We have thereby found the solution $(a, b, c) = (2, 2, 2)$.

Case 2. The numbers a, b, and c are odd.
If any two of these numbers are equal, say $a = b$, then $ac - b = a(c-1)$ has a nontrivial odd divisor and cannot be a power of 2. Hence a, b, and c are distinct. So we may assume without loss of generality that $a < b < c$.

Let α and β denote the nonnegative integers for which $bc - a = 2^\alpha$ and $ac - b = 2^\beta$ hold. Clearly, we have $\alpha > \beta$, and thus 2^β divides

$$a \cdot 2^\alpha - b \cdot 2^\beta = a(bc - a) - b(ac - b) = b^2 - a^2 = (b + a)(b - a)\,.$$

Since a is odd, it is not possible that both factors $b + a$ and $b - a$ are divisible by 4. Consequently, one of them has to be a multiple of $2^{\beta-1}$. Hence one of the numbers $2(b + a)$ and $2(b - a)$ is divisible by 2^β and in either case we have

$$ac - b = 2^\beta \leqslant 2(a + b)\,. \tag{7}$$

This in turn yields $(a-1)b < ac - b < 4b$ and thus $a = 3$ (recall that a is odd and larger than 1). Substituting this back into (7) we learn $c \leqslant b + 2$. But due to the parity $b < c$ entails that $b + 2 \leqslant c$ holds as well. So we get $c = b + 2$ and from $bc - a = (b-1)(b+3)$ being a power of 2 it follows that $b = 5$ and $c = 7$.

Case 3. Among a, b, and c both parities occur.
Without loss of generality, we suppose that c is odd and that $a \leqslant b$. We are to show that (a, b, c) is either $(2, 2, 3)$ or $(2, 6, 11)$. As at least one of a and b is even, the expression $ab - c$ is odd; since it is also a power of 2, we obtain

$$ab - c = 1\,. \tag{8}$$

If $a = b$, then $c = a^2 - 1$, and from $ac - b = a(a^2 - 2)$ being a power of 2 it follows that both a and $a^2 - 2$ are powers of 2, whence $a = 2$. This gives rise to the solution $(2, 2, 3)$.

We may suppose $a < b$ from now on. As usual, we let $\alpha > \beta$ denote the integers satisfying

$$2^\alpha = bc - a \quad \text{and} \quad 2^\beta = ac - b\,. \tag{9}$$

If $\beta = 0$ it would follow that $ac - b = ab - c = 1$ and hence that $b = c = 1$, which is absurd. So β and α are positive and consequently a and b are even. Substituting $c = ab - 1$ into (9) we obtain

$$2^\alpha = ab^2 - (a+b)\,, \tag{10}$$

$$\text{and} \quad 2^\beta = a^2b - (a+b)\,. \tag{11}$$

The addition of both equation yields $2^\alpha + 2^\beta = (ab-2)(a+b)$. Now $ab - 2$ is even but not divisible by 4, so the highest power of 2 dividing $a + b$ is $2^{\beta-1}$. For this reason, the equations (10) and (11) show that the highest powers of 2 dividing either of the numbers ab^2 and a^2b is likewise $2^{\beta-1}$. Thus there is an integer $\tau \geqslant 1$ together with odd integers A, B, and C such that $a = 2^\tau A$, $b = 2^\tau B$, $a + b = 2^{3\tau}C$, and $\beta = 1 + 3\tau$.

Notice that $A + B = 2^{2\tau}C \geqslant 4C$. Moreover, (11) entails $A^2B - C = 2$. Thus $8 = 4A^2B - 4C \geqslant 4A^2B - A - B \geqslant A^2(3B - 1)$. Since A and B are odd with $A < B$, this is only possible if $A = 1$ and $B = 3$. Finally, one may conclude $C = 1$, $\tau = 1$, $a = 2$, $b = 6$, and $c = 11$. We have thereby found the triple $(2, 6, 11)$. This completes the discussion of the third case, and hence the solution.

Comment. In both solutions, there are many alternative ways to proceed in each of its cases. Here we present a different treatment of the part "$a < b$" of Case 3 in Solution 2, assuming that (8) and (9) have already been written down:

Put $d = \gcd(a, b)$ and define the integers p and q by $a = dp$ and $b = dq$; notice that $p < q$ and $\gcd(p, q) = 1$. Now (8) implies $c = d^2pq - 1$ and thus we have

$$2^\alpha = d(d^2pq^2 - p - q)$$

$$\text{and} \quad 2^\beta = d(d^2p^2q - p - q)\,. \tag{12}$$

Now 2^β divides $2^\alpha - 2^\beta = d^3pq(q - p)$ and, as p and q are easily seen to be coprime to $d^2p^2q - p - q$, it follows that

$$(d^2p^2q - p - q) \mid d^2(q - p)\,. \tag{13}$$

In particular, we have $d^2p^2q - p - q \leqslant d^2(q - p)$, i.e., $d^2(p^2q + p - q) \leqslant p + q$. As $p^2q + p - q > 0$, this may be weakened to $p^2q + p - q \leqslant p + q$. Hence $p^2q \leqslant 2q$, which is only possible if $p = 1$.

Going back to (13), we get

$$(d^2q - q - 1) \mid d^2(q - 1)\,. \tag{14}$$

Now $2(d^2q - q - 1) \leqslant d^2(q - 1)$ would entail $d^2(q + 1) \leqslant 2(q + 1)$ and thus $d = 1$. But this would tell us that $a = dp = 1$, which is absurd. This argument proves $2(d^2q - q - 1) > d^2(q - 1)$ and in the light of (14) it follows that $d^2q - q - 1 = d^2(q - 1)$, i.e., $q = d^2 - 1$. Plugging this together with $p = 1$ into (12) we infer $2^\beta = d^3(d^2 - 2)$. Hence d and $d^2 - 2$ are powers of 2. Consequently, $d = 2$, $q = 3$, $a = 2$, $b = 6$, and $c = 11$, as desired.

N6. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\ldots f}_{n}(m)\ldots))$. Suppose that f has the following two properties:

(*i*) If $m, n \in \mathbb{Z}_{>0}$, then $\dfrac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;

(*ii*) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \ldots$ is periodic.

(Singapore)

Solution. We split the solution into three steps. In the first of them, we show that the function f is injective and explain how this leads to a useful visualization of f. Then comes the second step, in which most of the work happens: its goal is to show that for any $n \in \mathbb{Z}_{>0}$ the sequence $n, f(n), f^2(n), \ldots$ is an arithmetic progression. Finally, in the third step we put everything together, thus solving the problem.

Step 1. We commence by checking that f is injective. For this purpose, we consider any $m, k \in \mathbb{Z}_{>0}$ with $f(m) = f(k)$. By (*i*), every positive integer n has the property that

$$\frac{k-m}{n} = \frac{f^n(m) - m}{n} - \frac{f^n(k) - k}{n}$$

is a difference of two integers and thus integral as well. But for $n = |k - m| + 1$ this is only possible if $k = m$. Thereby, the injectivity of f is established.

Now recall that due to condition (*ii*) there are finitely many positive integers $a_1, \ldots, a_k$ such that $\mathbb{Z}_{>0}$ is the disjoint union of $\{a_1, \ldots, a_k\}$ and $\{f(n) \mid n \in \mathbb{Z}_{>0}\}$. Notice that by plugging $n = 1$ into condition (*i*) we get $f(m) > m$ for all $m \in \mathbb{Z}_{>0}$.

We contend that every positive integer n may be expressed uniquely in the form $n = f^j(a_i)$ for some $j \geqslant 0$ and $i \in \{1, \ldots, k\}$. The uniqueness follows from the injectivity of f. The existence can be proved by induction on n in the following way. If $n \in \{a_1, \ldots, a_k\}$, then we may take $j = 0$; otherwise there is some $n' < n$ with $f(n') = n$ to which the induction hypothesis may be applied.

The result of the previous paragraph means that every positive integer appears exactly once in the following infinite picture, henceforth referred to as "the Table":

a_1	$f(a_1)$	$f^2(a_1)$	$f^3(a_1)$	$\ldots$
a_2	$f(a_2)$	$f^2(a_2)$	$f^3(a_2)$	$\ldots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
a_k	$f(a_k)$	$f^2(a_k)$	$f^3(a_k)$	$\ldots$

The Table

Step 2. Our next goal is to prove that each row of the Table is an arithmetic progression. Assume contrariwise that the number t of rows which are arithmetic progressions would satisfy $0 \leqslant t < k$. By permuting the rows if necessary we may suppose that precisely the first t rows are arithmetic progressions, say with steps $T_1, \ldots, T_t$. Our plan is to find a further row that is "not too sparse" in an asymptotic sense, and then to prove that such a row has to be an arithmetic progression as well.

Let us write $T = \operatorname{lcm}(T_1, T_2, \ldots, T_t)$ and $A = \max\{a_1, a_2, \ldots, a_t\}$ if $t > 0$; and $T = 1$ and $A = 0$ if $t = 0$. For every integer $n \geqslant A$, the interval $\Delta_n = [n+1, n+T]$ contains exactly T/T_i

elements of the i^{th} row ($1 \leqslant i \leqslant t$). Therefore, the number of elements from the last $(k-t)$ rows of the Table contained in Δ_n does not depend on $n \geqslant A$. It is not possible that none of these intervals Δ_n contains an element from the $k-t$ last rows, because infinitely many numbers appear in these rows. It follows that for each $n \geqslant A$ the interval Δ_n contains at least one member from these rows.

This yields that for every positive integer d, the interval $\big[A+1, A+(d+1)(k-t)T\big]$ contains at least $(d+1)(k-t)$ elements from the last $k-t$ rows; therefore, there exists an index x with $t+1 \leqslant x \leqslant k$, possibly depending on d, such that our interval contains at least $d+1$ elements from the x^{th} row. In this situation we have

$$f^d(a_x) \leqslant A + (d+1)(k-t)T\,.$$

Finally, since there are finitely many possibilities for x, there exists an index $x \geqslant t+1$ such that the set

$$X = \big\{d \in \mathbb{Z}_{>0} \,\big|\, f^d(a_x) \leqslant A + (d+1)(k-t)T\big\}$$

is infinite. Thereby we have found the "dense row" promised above.

By assumption (i), for every $d \in X$ the number

$$\beta_d = \frac{f^d(a_x) - a_x}{d}$$

is a positive integer not exceeding

$$\frac{A + (d+1)(k-t)T}{d} \leqslant \frac{Ad + 2d(k-t)T}{d} = A + 2(k-t)T\,.$$

This leaves us with finitely many choices for β_d, which means that there exists a number T_x such that the set

$$Y = \big\{d \in X \,\big|\, \beta_d = T_x\big\}$$

is infinite. Notice that we have $f^d(a_x) = a_x + d \cdot T_x$ for all $d \in Y$.

Now we are prepared to prove that the numbers in the x^{th} row form an arithmetic progression, thus coming to a contradiction with our assumption. Let us fix any positive integer j. Since the set Y is infinite, we can choose a number $y \in Y$ such that $y - j > \big|f^j(a_x) - (a_x + jT_x)\big|$. Notice that both numbers

$$f^y(a_x) - f^j(a_x) = f^{y-j}\big(f^j(a_x)\big) - f^j(a_x) \quad \text{and} \quad f^y(a_x) - (a_x + jT_x) = (y-j)T_x$$

are divisible by $y - j$. Thus, the difference between these numbers is also divisible by $y - j$. Since the absolute value of this difference is less than $y - j$, it has to vanish, so we get $f^j(a_x) = a_x + j \cdot T_x$.

Hence, it is indeed true that all rows of the Table are arithmetic progressions.

Step 3. Keeping the above notation in force, we denote the step of the i^{th} row of the table by T_i. Now we claim that we have $f(n) - n = f(n+T) - (n+T)$ for all $n \in \mathbb{Z}_{>0}$, where

$$T = \operatorname{lcm}(T_1, \ldots, T_k)\,.$$

To see this, let any $n \in \mathbb{Z}_{>0}$ be given and denote the index of the row in which it appears in the Table by i. Then we have $f^j(n) = n + j \cdot T_i$ for all $j \in \mathbb{Z}_{>0}$, and thus indeed

$$f(n+T) - f(n) = f^{1+T/T_i}(n) - f(n) = (n + T + T_i) - (n + T_i) = T\,.$$

This concludes the solution.

Comment 1. There are some alternative ways to complete the second part once the index x corresponding to a "dense row" is found. For instance, one may show that for some integer T_x^* the set

$$Y^* = \{ j \in \mathbb{Z}_{>0} \mid f^{j+1}(a_x) - f^j(a_x) = T_x^* \}$$

is infinite, and then one may conclude with a similar divisibility argument.

Comment 2. It may be checked that, conversely, any way to fill out the Table with finitely many arithmetic progressions so that each positive integer appears exactly once, gives rise to a function f satisfying the two conditions mentioned in the problem. For example, we may arrange the positive integers as follows:

2	4	6	8	10	...
1	5	9	13	17	...
3	7	11	15	19	...

This corresponds to the function

$$f(n) = \begin{cases} n+2 & \text{if } n \text{ is even;} \\ n+4 & \text{if } n \text{ is odd.} \end{cases}$$

As this example shows, it is not true that the function $n \mapsto f(n) - n$ has to be constant.

N7. Let $\mathbb{Z}_{>0}$ denote the set of positive integers. For any positive integer k, a function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ is called *k-good* if $\gcd\big(f(m)+n, f(n)+m\big) \leqslant k$ for all $m \neq n$. Find all k such that there exists a k-good function.

(Canada)

Answer. $k \geqslant 2$.

Solution 1. For any function $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$, let $G_f(m,n) = \gcd\big(f(m)+n, f(n)+m\big)$. Note that a k-good function is also $(k+1)$-good for any positive integer k. Hence, it suffices to show that there does not exist a 1-good function and that there exists a 2-good function.

We first show that there is no 1-good function. Suppose that there exists a function f such that $G_f(m,n) = 1$ for all $m \neq n$. Now, if there are two distinct even numbers m and n such that $f(m)$ and $f(n)$ are both even, then $2 \mid G_f(m,n)$, a contradiction. A similar argument holds if there are two distinct odd numbers m and n such that $f(m)$ and $f(n)$ are both odd. Hence we can choose an even m and an odd n such that $f(m)$ is odd and $f(n)$ is even. This also implies that $2 \mid G_f(m,n)$, a contradiction.

We now construct a 2-good function. Define $f(n) = 2^{g(n)+1} - n - 1$, where g is defined recursively by $g(1) = 1$ and $g(n+1) = (2^{g(n)+1})!$.

For any positive integers $m > n$, set

$$A = f(m) + n = 2^{g(m)+1} - m + n - 1, \qquad B = f(n) + m = 2^{g(n)+1} - n + m - 1.$$

We need to show that $\gcd(A,B) \leqslant 2$. First, note that $A + B = 2^{g(m)+1} + 2^{g(n)+1} - 2$ is not divisible by 4, so that $4 \nmid \gcd(A,B)$. Now we suppose that there is an odd prime p for which $p \mid \gcd(A,B)$ and derive a contradiction.

We first claim that $2^{g(m-1)+1} \geqslant B$. This is a rather weak bound; one way to prove it is as follows. Observe that $g(k+1) > g(k)$ and hence $2^{g(k+1)+1} \geqslant 2^{g(k)+1} + 1$ for every positive integer k. By repeatedly applying this inequality, we obtain $2^{g(m-1)+1} \geqslant 2^{g(n)+1} + (m-1) - n = B$.

Now, since $p \mid B$, we have $p - 1 < B \leqslant 2^{g(m-1)+1}$, so that $p - 1 \mid (2^{g(m-1)+1})! = g(m)$. Hence $2^{g(m)} \equiv 1 \pmod{p}$, which yields $A + B \equiv 2^{g(n)+1} \pmod{p}$. However, since $p \mid A + B$, this implies that $p = 2$, a contradiction.

Solution 2. We provide an alternative construction of a 2-good function f.

Let $\mathcal{P}$ be the set consisting of 4 and all odd primes. For every $p \in \mathcal{P}$, we say that a number $a \in \{0, 1, \ldots, p-1\}$ is *p-useful* if $a \not\equiv -a \pmod{p}$. Note that a residue modulo p which is neither 0 nor 2 is p-useful (the latter is needed only when $p = 4$).

We will construct f recursively; in some steps, we will also define a p-useful number a_p. After the m^{th} step, the construction will satisfy the following conditions:

(i) The values of $f(n)$ have already been defined for all $n \leqslant m$, and p-useful numbers a_p have already been defined for all $p \leqslant m+2$;

(ii) If $n \leqslant m$ and $p \leqslant m+2$, then $f(n) + n \not\equiv a_p \pmod{p}$;

(iii) $\gcd\big(f(n_1) + n_2, f(n_2) + n_1\big) \leqslant 2$ for all $n_1 < n_2 \leqslant m$.

If these conditions are satisfied, then f will be a 2-good function.

Step 1. Set $f(1) = 1$ and $a_3 = 1$. Clearly, all the conditions are satisfied.

Step m, for $m \geqslant 2$. We need to determine $f(m)$ and, if $m+2 \in \mathcal{P}$, the number a_{m+2}.

Defining $f(m)$. Let $X_m = \{p \in \mathcal{P} \colon p \mid f(n) + m \text{ for some } n < m\}$. We will determine $f(m) \bmod p$ for all $p \in X_m$ and then choose $f(m)$ using the Chinese Remainder Theorem.

Take any $p \in X_m$. If $p \leqslant m+1$, then we define $f(m) \equiv -a_p - m \pmod p$. Otherwise, if $p \geqslant m+2$, then we define $f(m) \equiv 0 \pmod p$.

Defining a_{m+2}. Now let $p = m+2$ and suppose that $p \in \mathcal{P}$. We choose a_p to be a residue modulo p that is not congruent to 0, 2, or $f(n)+n$ for any $n \leqslant m$. Since $f(1)+1=2$, there are at most $m+1<p$ residues to avoid, so we can always choose a remaining residue.

We first check that (*ii*) is satisfied. We only need to check it if $p=m+2$ or $n=m$. In the former case, we have $f(n)+n \not\equiv a_p \pmod p$ by construction. In the latter case, if $n=m$ and $p \leqslant m+1$, then we have $f(m)+m \equiv -a_p \not\equiv a_p \pmod p$, where we make use of the fact that a_p is p-useful.

Now we check that (*iii*) holds. Suppose, to the contrary, that $p \mid \gcd\bigl(f(n)+m, f(m)+n\bigr)$ for some $n<m$. Then $p \in X_m$ and $p \mid f(m)+n$. If $p \geqslant m+2$, then $0 \equiv f(m)+n \equiv n \pmod p$, which is impossible since $n<m<p$.

Otherwise, if $p \leqslant m+1$, then

$$0 \equiv \bigl(f(m)+n\bigr)+\bigl(f(n)+m\bigr) \equiv \bigl(f(n)+n\bigr)+\bigl(f(m)+m\bigr) \equiv \bigl(f(n)+n\bigr)-a_p \pmod p.$$

This implies that $f(n)+n \equiv a_p \pmod p$, a contradiction with (*ii*).

Comment 1. For any $p \in \mathcal{P}$, we may also define a_p at step m for an arbitrary $m \leqslant p-2$. The construction will work as long as we define a finite number of a_p at each step.

Comment 2. When attempting to construct a 2-good function f recursively, the following way seems natural. Start with setting $f(1)=1$. Next, for each integer $m>1$, introduce the set X_m like in Solution 2 and define $f(m)$ so as to satisfy

$$\begin{aligned} f(m) &\equiv f(m-p) \pmod p \qquad \text{for all } p \in X_m \text{ with } p<m, \quad \text{and} \\ f(m) &\equiv 0 \pmod p \qquad \text{for all } p \in X_m \text{ with } p \geqslant m. \end{aligned}$$

This construction might seem to work. Indeed, consider a fixed $p \in \mathcal{P}$, and suppose that p divides $\gcd\bigl(f(n)+m, f(m)+n\bigr)$ for some $n<m$. Choose such m and n so that $\max(m,n)$ is minimal. Then $p \in X_m$. We can check that $p<m$, so that the construction implies that p divides $\gcd\bigl(f(n)+(m-p), f(m-p)+n\bigr)$. Since $\max(n, m-p) < \max(m,n)$, this *almost* leads to a contradiction—the only trouble is the possibility that $n = m-p$. However, this flaw may happen to be not so easy to fix.

We will present one possible way to repair this argument in the next comment.

Comment 3. There are many recursive constructions for a 2-good function f. Here we sketch one general approach which may be specified in different ways. For convenience, we denote by $\mathbb{Z}_p$ the set of residues modulo p; all operations on elements of $\mathbb{Z}_p$ are also performed modulo p.

The general structure is the same as in Solution 2, i.e. using the Chinese Remainder Theorem to successively determine $f(m)$. But instead of designating a common "safe" residue a_p for future steps, we act as follows.

For every $p \in \mathcal{P}$, in some step of the process we define p subsets $B_p^{(1)}, B_p^{(2)}, \ldots, B_p^{(p)} \subset \mathbb{Z}_p$. The meaning of these sets is that

$$f(m)+m \text{ should be congruent to some element in } B_p^{(i)} \text{ whenever } m \equiv i \pmod p \text{ for } i \in \mathbb{Z}_p. \tag{1}$$

Moreover, in every such subset we specify a *safe* element $b_p^{(i)} \in B_p^{(i)}$. The meaning now is that in future steps, it is safe to set $f(m)+m \equiv b_p^{(i)} \pmod p$ whenever $m \equiv i \pmod p$. In view of (1), this safety will follow from the condition that $p \nmid \gcd\bigl(b_p^{(i)}+(j-i), c^{(j)}-(j-i)\bigr)$ for all $j \in \mathbb{Z}_p$ and all $c^{(j)} \in B_p^{(j)}$. In turn, this condition can be rewritten as

$$-b_p^{(i)} \notin B_p^{(j)}, \qquad \text{where} \quad j \equiv i - b_p^{(i)} \pmod p. \tag{2}$$

The construction in Solution 2 is equivalent to setting $b_p^{(i)} = -a_p$ and $B_p^{(i)} = \mathbb{Z}_p \setminus \{a_p\}$ for all i. However, there are different, more technical specifications of our approach.

One may view the (incomplete) construction in Comment 2 as defining $B_p^{(i)}$ and $b_p^{(i)}$ at step $p-1$ by setting $B_p^{(0)} = \{b_p^{(0)}\} = \{0\}$ and $B_p^{(i)} = \{b_p^{(i)}\} = \{f(i) + i \mod p\}$ for every $i = 1, 2, \ldots, p-1$. However, this construction violates (2) as soon as some number of the form $f(i) + i$ is divisible by some p with $i + 2 \leqslant p \in \mathcal{P}$, since then $-b_p^{(i)} = b_p^{(i)} \in B_p^{(i)}$.

Here is one possible way to repair this construction. For all $p \in \mathcal{P}$, we define the sets $B_p^{(i)}$ and the elements $b_p^{(i)}$ at step $(p-2)$ as follows. Set $B_p^{(1)} = \{b_p^{(1)}\} = \{2\}$ and $B_p^{(-1)} = B_p^{(0)} = \{b_p^{(-1)}\} = \{b_p^{(0)}\} = \{-1\}$. Next, for all $i = 2, \ldots, p-2$, define $B_p^{(i)} = \{i, f(i) + i \mod p\}$ and $b_p^{(i)} = i$. One may see that these definitions agree with both (1) and (2).

N8. For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i\colon p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

Find all strictly increasing functions $f\colon \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho\big(f(a) - f(b)\big) \leqslant \mho(a-b) \qquad \text{for all integers } a \text{ and } b \text{ with } a > b. \tag{1}$$

(Brazil)

Answer. $f(x) = ax + b$, where b is an arbitrary integer, and a is an arbitrary positive integer with $\mho(a) = 0$.

Solution. A straightforward check shows that all the functions listed in the answer satisfy the problem condition. It remains to show the converse.

Assume that f is a function satisfying the problem condition. Notice that the function $g(x) = f(x) - f(0)$ also satisfies this condition. Replacing f by g, we assume from now on that $f(0) = 0$; then $f(n) > 0$ for any positive integer n. Thus, we aim to prove that there exists a positive integer a with $\mho(a) = 0$ such that $f(n) = an$ for all $n \in \mathbb{Z}$.

We start by introducing some notation. Set $N = 10^{100}$. We say that a prime p is *large* if $p > N$, and p is *small* otherwise; let $\mathcal{S}$ be the set of all small primes. Next, we say that a positive integer is *large* or *small* if all its prime factors are such (thus, the number 1 is the unique number which is both large and small). For a positive integer k, we denote the greatest large divisor of k and the greatest small divisor of k by $L(k)$ and $S(k)$, respectively; thus, $k = L(k)S(k)$.

We split the proof into three steps.

Step 1. *We prove that for every large k, we have $k \mid f(a) - f(b) \iff k \mid a - b$. In other words, $L\big(f(a) - f(b)\big) = L(a-b)$ for all integers a and b with $a > b$.*

We use induction on k. The base case $k = 1$ is trivial. For the induction step, assume that k_0 is a large number, and that the statement holds for all large numbers k with $k < k_0$.

Claim 1. For any integers x and y with $0 < x - y < k_0$, the number k_0 does not divide $f(x) - f(y)$.

Proof. Assume, to the contrary, that $k_0 \mid f(x) - f(y)$. Let $\ell = L(x-y)$; then $\ell \leqslant x - y < k_0$. By the induction hypothesis, $\ell \mid f(x) - f(y)$, and thus $\operatorname{lcm}(k_0, \ell) \mid f(x) - f(y)$. Notice that $\operatorname{lcm}(k_0, \ell)$ is large, and $\operatorname{lcm}(k_0, \ell) \geqslant k_0 > \ell$. But then

$$\mho\big(f(x) - f(y)\big) \geqslant \mho\big(\operatorname{lcm}(k_0, \ell)\big) > \mho(\ell) = \mho(x - y),$$

which is impossible. □

Now we complete the induction step. By Claim 1, for every integer a each of the sequences

$$f(a), f(a+1), \ldots, f(a + k_0 - 1) \quad \text{and} \quad f(a+1), f(a+2), \ldots, f(a + k_0)$$

forms a complete residue system modulo k_0. This yields $f(a) \equiv f(a + k_0) \pmod{k_0}$. Thus, $f(a) \equiv f(b) \pmod{k_0}$ whenever $a \equiv b \pmod{k_0}$.

Finally, if $a \not\equiv b \pmod{k_0}$ then there exists an integer b' such that $b' \equiv b \pmod{k_0}$ and $|a - b'| < k_0$. Then $f(b) \equiv f(b') \not\equiv f(a) \pmod{k_0}$. The induction step is proved.

Step 2. *We prove that for some small integer a there exist infinitely many integers n such that $f(n) = an$. In other words, f is linear on some infinite set.*

We start with the following general statement.

Claim 2. There exists a constant c such that $f(t) < ct$ for every positive integer $t > N$.

Proof. Let d be the product of all small primes, and let α be a positive integer such that $2^\alpha > f(N)$. Then, for every $p \in \mathcal{S}$ the numbers $f(0), f(1), \ldots, f(N)$ are distinct modulo p^α. Set $P = d^\alpha$ and $c = P + f(N)$.

Choose any integer $t > N$. Due to the choice of α, for every $p \in \mathcal{S}$ there exists at most one nonnegative integer $i \leqslant N$ with $p^\alpha \mid f(t) - f(i)$. Since $|\mathcal{S}| < N$, we can choose a nonnegative integer $j \leqslant N$ such that $p^\alpha \nmid f(t) - f(j)$ for all $p \in \mathcal{S}$. Therefore, $S\big(f(t) - f(j)\big) < P$.

On the other hand, Step 1 shows that $L\big(f(t) - f(j)\big) = L(t-j) \leqslant t - j$. Since $0 \leqslant j \leqslant N$, this yields

$$f(t) = f(j) + L\big(f(t) - f(j)\big) \cdot S\big(f(t) - f(j)\big) < f(N) + (t-j)P \leqslant \big(P + f(N)\big)t = ct. \quad \square$$

Now let $\mathcal{T}$ be the set of large primes. For every $t \in \mathcal{T}$, Step 1 implies $L\big(f(t)\big) = t$, so the ratio $f(t)/t$ is an integer. Now Claim 2 leaves us with only finitely many choices for this ratio, which means that there exists an infinite subset $\mathcal{T}' \subseteq \mathcal{T}$ and a positive integer a such that $f(t) = at$ for all $t \in \mathcal{T}'$, as required.

Since $L(t) = L\big(f(t)\big) = L(a)L(t)$ for all $t \in \mathcal{T}'$, we get $L(a) = 1$, so the number a is small.

Step 3. *We show that $f(x) = ax$ for all $x \in \mathbb{Z}$.*

Let $R_i = \big\{x \in \mathbb{Z} \colon x \equiv i \pmod{N!}\big\}$ denote the residue class of i modulo $N!$.

Claim 3. Assume that for some r, there are infinitely many $n \in R_r$ such that $f(n) = an$. Then $f(x) = ax$ for all $x \in R_{r+1}$.

Proof. Choose any $x \in R_{r+1}$. By our assumption, we can select $n \in R_r$ such that $f(n) = an$ and $|n - x| > \big|f(x) - ax\big|$. Since $n - x \equiv r - (r+1) = -1 \pmod{N!}$, the number $|n - x|$ is large. Therefore, by Step 1 we have $f(x) \equiv f(n) = an \equiv ax \pmod{n - x}$, so $n - x \mid f(x) - ax$. Due to the choice of n, this yields $f(x) = ax$. $\square$

To complete Step 3, notice that the set $\mathcal{T}'$ found in Step 2 contains infinitely many elements of some residue class R_i. Applying Claim 3, we successively obtain that $f(x) = ax$ for all $x \in R_{i+1}, R_{i+2}, \ldots, R_{i+N!} = R_i$. This finishes the solution.

Comment 1. As the proposer also mentions, one may also consider the version of the problem where the condition (1) is replaced by the condition that $L\big(f(a) - f(b)\big) = L(a - b)$ for all integers a and b with $a > b$. This allows to remove of Step 1 from the solution.

Comment 2. Step 2 is the main step of the solution. We sketch several different approaches allowing to perform this step using statements which are weaker than Claim 2.

Approach 1. Let us again denote the product of all small primes by d. We focus on the values $f(d^i)$, $i \geqslant 0$. In view of Step 1, we have $L\big(f(d^i) - f(d^k)\big) = L(d^i - d^k) = d^{i-k} - 1$ for all $i > k \geqslant 0$.

Acting similarly to the beginning of the proof of Claim 2, one may choose a number $\alpha \geqslant 0$ such that the residues of the numbers $f(d^i)$, $i = 0, 1, \ldots, N$, are distinct modulo p^α for each $p \in \mathcal{S}$. Then, for every $i > N$, there exists an exponent $k = k(i) \leqslant N$ such that $S\big(f(d^i) - f(d^k)\big) < P = d^\alpha$.

Since there are only finitely many options for $k(i)$, as well as for the corresponding numbers $S\big(f(d^i) - f(d^k)\big)$, there exists an infinite set I of exponents $i > N$ such that $k(i)$ attains the same value k_0 for all $i \in I$, and such that, moreover, $S\big(f(d^i) - f(d^{k_0})\big)$ attains the same value s_0 for all $i \in I$. Therefore, for all such i we have

$$f(d^i) = f(d^{k_0}) + L\big(f(d^i) - f(d^{k_0})\big) \cdot S\big(f(d^i) - f(d^{k_0})\big) = f(d^{k_0}) + \big(d^{i-k_0} - 1\big)s_0,$$

which means that f is linear on the infinite set $\{d^i : i \in I\}$ (although with rational coefficients).

Finally, one may implement the relation $f(d^i) \equiv f(1) \pmod{d^i - 1}$ in order to establish that in fact $f(d^i)/d^i$ is a (small and fixed) integer for all $i \in I$.

Approach 2. Alternatively, one may start with the following lemma.

Lemma. There exists a positive constant c such that

$$L\left(\prod_{i=1}^{3N}\big(f(k)-f(i)\big)\right)=\prod_{i=1}^{3N}L\big(f(k)-f(i)\big)\geqslant c\big(f(k)\big)^{2N}$$

for all $k>3N$.

Proof. Let k be an integer with $k>3N$. Set $\Pi=\prod_{i=1}^{3N}\big(f(k)-f(i)\big)$.

Notice that for every prime $p\in\mathcal{S}$, at most one of the numbers in the set

$$\mathcal{H}=\big\{f(k)-f(i)\colon 1\leqslant i\leqslant 3N\big\}$$

is divisible by a power of p which is greater than $f(3N)$; we say that such elements of $\mathcal{H}$ are *bad*. Now, for each element $h\in\mathcal{H}$ which is not bad we have $S(h)\leqslant f(3N)^N$, while the bad elements do not exceed $f(k)$. Moreover, there are less than N bad elements in $\mathcal{H}$. Therefore,

$$S(\Pi)=\prod_{h\in\mathcal{H}}S(h)\leqslant\big(f(3N)\big)^{3N^2}\cdot\big(f(k)\big)^N.$$

This easily yields the lemma statement in view of the fact that $L(\Pi)S(\Pi)=\Pi\geqslant\mu\big(f(k)\big)^{3N}$ for some absolute constant μ. □

As a corollary of the lemma, one may get a weaker version of Claim 2 stating that there exists a positive constant C such that $f(k)\leqslant Ck^{3/2}$ for all $k>3N$. Indeed, from Step 1 we have

$$k^{3N}\geqslant\prod_{i=1}^{3N}L(k-i)=\prod_{i=1}^{3N}L\big(f(k)-f(i)\big)\geqslant c\big(f(k)\big)^{2N},$$

so $f(k)\leqslant c^{-1/(2N)}k^{3/2}$.

To complete Step 2 now, set $a=f(1)$. Due to the estimates above, we may choose a positive integer n_0 such that $\big|f(n)-an\big|<\frac{n(n-1)}{2}$ for all $n\geqslant n_0$.

Take any $n\geqslant n_0$ with $n\equiv 2 \pmod{N!}$. Then $L\big(f(n)-f(0)\big)=L(n)=n/2$ and $L\big(f(n)-f(1)\big)=L(n-1)=n-1$; these relations yield $f(n)\equiv f(0)=0\equiv an \pmod{n/2}$ and $f(n)\equiv f(1)=a\equiv an \pmod{n-1}$, respectively. Thus, $\frac{n(n-1)}{2}\mid f(n)-an$, which shows that $f(n)=an$ in view of the estimate above.

Comment 3. In order to perform Step 3, it suffices to establish the equality $f(n)=an$ for any infinite set of values of n. However, if this set has some good structure, then one may find easier ways to complete this step.

For instance, after showing, as in Approach 2, that $f(n)=an$ for all $n\geqslant n_0$ with $n\equiv 2\pmod{N!}$, one may proceed as follows. Pick an arbitrary integer x and take any large prime p which is greater than $|f(x)-ax|$. By the Chinese Remainder Theorem, there exists a positive integer $n>\max(x,n_0)$ such that $n\equiv 2\pmod{N!}$ and $n\equiv x\pmod{p}$. By Step 1, we have $f(x)\equiv f(n)=an\equiv ax\pmod{p}$. Due to the choice of p, this is possible only if $f(x)=ax$.

CHIANG MAI, THAILAND

4 - 16 JULY 2015

Shortlisted Problems with Solutions

57$^{\text{th}}$ International Mathematical Olympiad
Hong Kong, 2016

Note of Confidentiality

The shortlisted problems should be kept strictly confidential until IMO 2017.

Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2016 thank the following 40 countries for contributing 121 problem proposals:

Albania, Algeria, Armenia, Australia, Austria, Belarus, Belgium, Bulgaria, Colombia, Cyprus, Czech Republic, Denmark, Estonia, France, Georgia, Greece, Iceland, India, Iran, Ireland, Israel, Japan, Latvia, Luxembourg, Malaysia, Mexico, Mongolia, Netherlands, Philippines, Russia, Serbia, Slovakia, Slovenia, South Africa, Taiwan, Tanzania, Thailand, Trinidad and Tobago, Turkey, Ukraine.

Problem Selection Committee

Front row from left: Yong-Gao Chen, Andy Liu, Tat Wing Leung (Chairman).
Back row from left: Yi-Jun Yao, Yun-Hao Fu, Yi-Jie He, Zhongtao Wu, Heung Wing Joseph Lee, Chi Hong Chow, Ka Ho Law, Tak Wing Ching.

Problems

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1.$$

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C.$$

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1.$$

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$x f(x^2) f(f(y)) + f(y f(x)) = f(xy)\left(f(f(x^2)) + f(f(y^2))\right)$$

for all positive real numbers x and y.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n} + 1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that $f(0)\neq 0$ and

$$f(x+y)^2=2f(x)f(y)+\max\left\{f(x^2)+f(y^2),f(x^2+y^2)\right\}$$

for all real numbers x and y.

A8. Determine the largest real number a such that for all $n\geqslant 1$ and for all real numbers $x_0,x_1,\ldots,x_n$ satisfying $0=x_0<x_1<x_2<\cdots<x_n$, we have

$$\frac{1}{x_1-x_0}+\frac{1}{x_2-x_1}+\cdots+\frac{1}{x_n-x_{n-1}}\geqslant a\left(\frac{2}{x_1}+\frac{3}{x_2}+\cdots+\frac{n+1}{x_n}\right).$$

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90°$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA.$$

Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120^\circ$.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

N3. Define $P(n) = n^2 + n + 1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l | k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \tag{1}$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x < \sqrt{a}$. Prove that $A = B$.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and all squares of its side lengths are divisible by n. Prove that $2S$ is an integer divisible by n.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Solutions

Algebra

A1. Let a, b and c be positive real numbers such that $\min\{ab, bc, ca\} \geqslant 1$. Prove that

$$\sqrt[3]{(a^2+1)(b^2+1)(c^2+1)} \leqslant \left(\frac{a+b+c}{3}\right)^2 + 1. \tag{1}$$

Solution 1. We first show the following.

- **Claim.** For any positive real numbers x, y with $xy \geqslant 1$, we have

$$(x^2+1)(y^2+1) \leqslant \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2. \tag{2}$$

Proof. Note that $xy \geqslant 1$ implies $(\frac{x+y}{2})^2 - 1 \geqslant xy - 1 \geqslant 0$. We find that

$$(x^2+1)(y^2+1) = (xy-1)^2 + (x+y)^2 \leqslant \left(\left(\frac{x+y}{2}\right)^2 - 1\right)^2 + (x+y)^2 = \left(\left(\frac{x+y}{2}\right)^2 + 1\right)^2.$$

$\square$

Without loss of generality, assume $a \geqslant b \geqslant c$. This implies $a \geqslant 1$. Let $d = \frac{a+b+c}{3}$. Note that

$$ad = \frac{a(a+b+c)}{3} \geqslant \frac{1+1+1}{3} = 1.$$

Then we can apply (2) to the pair (a, d) and the pair (b, c). We get

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+d}{2}\right)^2 + 1\right)^2 \left(\left(\frac{b+c}{2}\right)^2 + 1\right)^2. \tag{3}$$

Next, from

$$\frac{a+d}{2} \cdot \frac{b+c}{2} \geqslant \sqrt{ad} \cdot \sqrt{bc} \geqslant 1,$$

we can apply (2) again to the pair $(\frac{a+d}{2}, \frac{b+c}{2})$. Together with (3), we have

$$(a^2+1)(d^2+1)(b^2+1)(c^2+1) \leqslant \left(\left(\frac{a+b+c+d}{4}\right)^2 + 1\right)^4 = (d^2+1)^4.$$

Therefore, $(a^2+1)(b^2+1)(c^2+1) \leqslant (d^2+1)^3$, and (1) follows by taking cube root of both sides.

Comment. After justifying the Claim, one may also obtain (1) by mixing variables. Indeed, the function involved is clearly continuous, and hence it suffices to check that the condition $xy \geqslant 1$ is preserved under each mixing step. This is true since whenever $ab, bc, ca \geqslant 1$, we have

$$\frac{a+b}{2} \cdot \frac{a+b}{2} \geqslant ab \geqslant 1 \quad \text{and} \quad \frac{a+b}{2} \cdot c \geqslant \frac{1+1}{2} = 1.$$

Solution 2. Let $f(x) = \ln(1+x^2)$. Then the inequality (1) to be shown is equivalent to

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant f\left(\frac{a+b+c}{3}\right),$$

while (2) becomes

$$\frac{f(x)+f(y)}{2} \leqslant f\left(\frac{x+y}{2}\right)$$

for $xy \geqslant 1$.

Without loss of generality, assume $a \geqslant b \geqslant c$. From the Claim in Solution 1, we have

$$\frac{f(a)+f(b)+f(c)}{3} \leqslant \frac{f(a)+2f\left(\frac{b+c}{2}\right)}{3}.$$

Note that $a \geqslant 1$ and $\frac{b+c}{2} \geqslant \sqrt{bc} \geqslant 1$. Since

$$f''(x) = \frac{2(1-x^2)}{(1+x^2)^2},$$

we know that f is concave on $[1, \infty)$. Then we can apply Jensen's Theorem to get

$$\frac{f(a)+2f\left(\frac{b+c}{2}\right)}{3} \leqslant f\left(\frac{a+2\cdot\frac{b+c}{2}}{3}\right) = f\left(\frac{a+b+c}{3}\right).$$

This completes the proof.

A2. Find the smallest real constant C such that for any positive real numbers a_1, a_2, a_3, a_4 and a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k and l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \leqslant C. \tag{1}$$

Answer. The smallest C is $\frac{1}{2}$.

Solution. We first show that $C \leqslant \frac{1}{2}$. For any positive real numbers $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4 \leqslant a_5$, consider the five fractions

$$\frac{a_1}{a_2}, \frac{a_3}{a_4}, \frac{a_1}{a_5}, \frac{a_2}{a_3}, \frac{a_4}{a_5}. \tag{2}$$

Each of them lies in the interval $(0, 1]$. Therefore, by the Pigeonhole Principle, at least three of them must lie in $(0, \frac{1}{2}]$ or lie in $(\frac{1}{2}, 1]$ simultaneously. In particular, there must be two consecutive terms in (2) which belong to an interval of length $\frac{1}{2}$ (here, we regard $\frac{a_1}{a_2}$ and $\frac{a_4}{a_5}$ as consecutive). In other words, the difference of these two fractions is less than $\frac{1}{2}$. As the indices involved in these two fractions are distinct, we can choose them to be i, j, k, l and conclude that $C \leqslant \frac{1}{2}$.

Next, we show that $C = \frac{1}{2}$ is best possible. Consider the numbers $1, 2, 2, 2, n$ where n is a large real number. The fractions formed by two of these numbers in ascending order are $\frac{1}{n}, \frac{2}{n}, \frac{1}{2}, \frac{2}{2}, \frac{2}{1}, \frac{n}{2}, \frac{n}{1}$. Since the indices i, j, k, l are distinct, $\frac{1}{n}$ and $\frac{2}{n}$ cannot be chosen simultaneously. Therefore the minimum value of the left-hand side of (1) is $\frac{1}{2} - \frac{2}{n}$. When n tends to infinity, this value approaches $\frac{1}{2}$, and so C cannot be less than $\frac{1}{2}$.

These conclude that $C = \frac{1}{2}$ is the smallest possible choice.

Comment. The conclusion still holds if $a_1, a_2, \ldots, a_5$ are pairwise distinct, since in the construction, we may replace the 2's by real numbers sufficiently close to 2.

There are two possible simplifications for this problem:

(i) the answer $C = \frac{1}{2}$ is given to the contestants; or

(ii) simply ask the contestants to prove the inequality (1) for $C = \frac{1}{2}$.

A3. Find all integers $n \geqslant 3$ with the following property: for all real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ satisfying $|a_k| + |b_k| = 1$ for $1 \leqslant k \leqslant n$, there exist $x_1, x_2, \ldots, x_n$, each of which is either -1 or 1, such that

$$\left|\sum_{k=1}^{n} x_k a_k\right| + \left|\sum_{k=1}^{n} x_k b_k\right| \leqslant 1. \tag{1}$$

Answer. n can be any odd integer greater than or equal to 3.

Solution 1. For any even integer $n \geqslant 4$, we consider the case

$$a_1 = a_2 = \cdots = a_{n-1} = b_n = 0 \quad \text{and} \quad b_1 = b_2 = \cdots = b_{n-1} = a_n = 1.$$

The condition $|a_k| + |b_k| = 1$ is satisfied for each $1 \leqslant k \leqslant n$. No matter how we choose each x_k, both sums $\sum_{k=1}^{n} x_k a_k$ and $\sum_{k=1}^{n} x_k b_k$ are odd integers. This implies $\left|\sum_{k=1}^{n} x_k a_k\right| \geqslant 1$ and $\left|\sum_{k=1}^{n} x_k b_k\right| \geqslant 1$, which shows (1) cannot hold.

For any odd integer $n \geqslant 3$, we may assume without loss of generality $b_k \geqslant 0$ for $1 \leqslant k \leqslant n$ (this can be done by flipping the pair (a_k, b_k) to $(-a_k, -b_k)$ and x_k to $-x_k$ if necessary) and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m \geqslant 0 > a_{m+1} \geqslant \cdots \geqslant a_n$. We claim that the choice $x_k = (-1)^{k+1}$ for $1 \leqslant k \leqslant n$ will work. Define

$$s = \sum_{k=1}^{m} x_k a_k \quad \text{and} \quad t = -\sum_{k=m+1}^{n} x_k a_k.$$

Note that

$$s = (a_1 - a_2) + (a_3 - a_4) + \cdots \geqslant 0$$

by the assumption $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_m$ (when m is odd, there is a single term a_m at the end, which is also positive). Next, we have

$$s = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots \leqslant a_1 \leqslant 1.$$

Similarly,

$$t = (-a_n + a_{n-1}) + (-a_{n-2} + a_{n-3}) + \cdots \geqslant 0$$

and

$$t = -a_n + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-4}) + \cdots \leqslant -a_n \leqslant 1.$$

From the condition, we have $a_k + b_k = 1$ for $1 \leqslant k \leqslant m$ and $-a_k + b_k = 1$ for $m+1 \leqslant k \leqslant n$. It follows that $\sum_{k=1}^{n} x_k a_k = s - t$ and $\sum_{k=1}^{n} x_k b_k = 1 - s - t$. Hence it remains to prove

$$|s - t| + |1 - s - t| \leqslant 1$$

under the constraint $0 \leqslant s, t \leqslant 1$. By symmetry, we may assume $s \geqslant t$. If $1 - s - t \geqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t + 1 - s - t = 1 - 2t \leqslant 1.$$

If $1 - s - t \leqslant 0$, then we have

$$|s - t| + |1 - s - t| = s - t - 1 + s + t = 2s - 1 \leqslant 1.$$

Hence, the inequality is true in both cases.

These show n can be any odd integer greater than or equal to 3.

Solution 2. The even case can be handled in the same way as Solution 1. For the odd case, we prove by induction on n.

Firstly, for $n = 3$, we may assume without loss of generality $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ and $b_1 = a_1 - 1$ (if $b_1 = 1 - a_1$, we may replace each b_k by $-b_k$).

- **Case 1.** $b_2 = a_2 - 1$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

  Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Then $|b_1 - b_2 + b_3| = |a_1 - a_2 + a_3 - 1| = 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| = 1$.

- **Case 2.** $b_2 = 1 - a_2$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (1, -1, 1)$.

  Let $c = a_1 - a_2 + a_3$ so that $0 \leqslant c \leqslant 1$. Since $a_3 \leqslant a_2$ and $a_1 \leqslant 1$, we have

$$c - 1 \leqslant b_1 - b_2 + b_3 = a_1 + a_2 - a_3 - 1 \leqslant 1 - c.$$

This gives $|b_1 - b_2 + b_3| \leqslant 1 - c$ and hence $|c| + |b_1 - b_2 + b_3| \leqslant 1$.

- **Case 3.** $b_2 = a_2 - 1$ and $b_3 = 1 - a_3$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

  Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_3 \leqslant 1$ and $a_2 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 - c.$$

  If $c < 0$, then $a_1 \leqslant a_2 + 1$ and $a_3 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 + a_2 - a_3 + 1 \leqslant 1 + c.$$

  In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

- **Case 4.** $b_2 = 1 - a_2$ and $b_3 = a_3 - 1$, in which case we take $(x_1, x_2, x_3) = (-1, 1, 1)$.

  Let $c = -a_1 + a_2 + a_3$. If $c \geqslant 0$, then $a_2 \leqslant 1$ and $a_3 \leqslant a_1$ imply

$$c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 - c.$$

  If $c < 0$, then $a_1 \leqslant a_3 + 1$ and $a_2 \geqslant 0$ imply

$$-c - 1 \leqslant -b_1 + b_2 + b_3 = -a_1 - a_2 + a_3 + 1 \leqslant 1 + c.$$

  In both cases, we get $|-b_1 + b_2 + b_3| \leqslant 1 - |c|$ and hence $|c| + |-b_1 + b_2 + b_3| \leqslant 1$.

We have found x_1, x_2, x_3 satisfying (1) in each case for $n = 3$.

Now, let $n \geqslant 5$ be odd and suppose the result holds for any smaller odd cases. Again we may assume $a_k \geqslant 0$ for each $1 \leqslant k \leqslant n$. By the Pigeonhole Principle, there are at least three indices k for which $b_k = a_k - 1$ or $b_k = 1 - a_k$. Without loss of generality, suppose $b_k = a_k - 1$ for $k = 1, 2, 3$. Again by the Pigeonhole Principle, as a_1, a_2, a_3 lies between 0 and 1, the difference of two of them is at most $\frac{1}{2}$. By changing indices if necessary, we may assume $0 \leqslant d = a_1 - a_2 \leqslant \frac{1}{2}$.

By the inductive hypothesis, we can choose $x_3, x_4, \ldots, x_n$ such that $a' = \sum_{k=3}^{n} x_k a_k$ and $b' = \sum_{k=3}^{n} x_k b_k$ satisfy $|a'| + |b'| \leqslant 1$. We may further assume $a' \geqslant 0$.

- **Case 1.** $b' \geqslant 0$, in which case we take $(x_1, x_2) = (-1, 1)$.
  We have $|-a_1 + a_2 + a'| + |-(a_1 - 1) + (a_2 - 1) + b'| = |-d + a'| + |-d + b'| \leqslant \max\{a' + b' - 2d, a' - b', b' - a', 2d - a' - b'\} \leqslant 1$ since $0 \leqslant a', b', a' + b' \leqslant 1$ and $0 \leqslant d \leqslant \frac{1}{2}$.

- **Case 2.** $0 > b' \geqslant -a'$, in which case we take $(x_1, x_2) = (-1, 1)$.
  We have $|-a_1+a_2+a'|+|-(a_1-1)+(a_2-1)+b'| = |-d+a'|+|-d+b'|$. If $-d+a' \geqslant 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$. If $-d + a' < 0$, this equals $2d - a' - b' \leqslant 2d \leqslant 1$.

- **Case 3.** $b' < -a'$, in which case we take $(x_1, x_2) = (1, -1)$.
  We have $|a_1 - a_2 + a'| + |(a_1 - 1) - (a_2 - 1) + b'| = |d + a'| + |d + b'|$. If $d + b' \geqslant 0$, this equals $2d + a' + b' < 2d \leqslant 1$. If $d + b' < 0$, this equals $a' - b' = |a'| + |b'| \leqslant 1$.

Therefore, we have found $x_1, x_2, \ldots, x_n$ satisfying (1) in each case. By induction, the property holds for all odd integers $n \geqslant 3$.

A4. Denote by $\mathbb{R}^+$ the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x^2)f(f(y)) + f(yf(x)) = f(xy)\big(f(f(x^2)) + f(f(y^2))\big) \tag{1}$$

for all positive real numbers x and y.

Answer. $f(x) = \frac{1}{x}$ for any $x \in \mathbb{R}^+$.

Solution 1. Taking $x = y = 1$ in (1), we get $f(1)f(f(1)) + f(f(1)) = 2f(1)f(f(1))$ and hence $f(1) = 1$. Swapping x and y in (1) and comparing with (1) again, we find

$$xf(x^2)f(f(y)) + f(yf(x)) = yf(y^2)f(f(x)) + f(xf(y)). \tag{2}$$

Taking $y = 1$ in (2), we have $xf(x^2) + f(f(x)) = f(f(x)) + f(x)$, that is,

$$f(x^2) = \frac{f(x)}{x}. \tag{3}$$

Take $y = 1$ in (1) and apply (3) to $xf(x^2)$. We get $f(x) + f(f(x)) = f(x)(f(f(x^2)) + 1)$, which implies

$$f(f(x^2)) = \frac{f(f(x))}{f(x)}. \tag{4}$$

For any $x \in \mathbb{R}^+$, we find that

$$f(f(x)^2) \overset{(3)}{=} \frac{f(f(x))}{f(x)} \overset{(4)}{=} f(f(x^2)) \overset{(3)}{=} f\left(\frac{f(x)}{x}\right). \tag{5}$$

It remains to show the following key step.

• **Claim.** The function f is injective.

Proof. Using (3) and (4), we rewrite (1) as

$$f(x)f(f(y)) + f(yf(x)) = f(xy)\left(\frac{f(f(x))}{f(x)} + \frac{f(f(y))}{f(y)}\right). \tag{6}$$

Take $x = y$ in (6) and apply (3). This gives $f(x)f(f(x)) + f(xf(x)) = 2\frac{f(f(x))}{x}$, which means

$$f(xf(x)) = f(f(x))\left(\frac{2}{x} - f(x)\right). \tag{7}$$

Using (3), equation (2) can be rewritten as

$$f(x)f(f(y)) + f(yf(x)) = f(y)f(f(x)) + f(xf(y)). \tag{8}$$

Suppose $f(x) = f(y)$ for some $x, y \in \mathbb{R}^+$. Then (8) implies

$$f(yf(y)) = f(yf(x)) = f(xf(y)) = f(xf(x)).$$

Using (7), this gives

$$f(f(y))\left(\frac{2}{y} - f(y)\right) = f(f(x))\left(\frac{2}{x} - f(x)\right).$$

Noting $f(x) = f(y)$, we find $x = y$. This establishes the injectivity. $\square$

By the Claim and (5), we get the only possible solution $f(x)=\frac{1}{x}$. It suffices to check that this is a solution. Indeed, the left-hand side of (1) becomes

$$x\cdot\frac{1}{x^2}\cdot y+\frac{x}{y}=\frac{y}{x}+\frac{x}{y},$$

while the right-hand side becomes

$$\frac{1}{xy}(x^2+y^2)=\frac{x}{y}+\frac{y}{x}.$$

The two sides agree with each other.

<u>Solution 2</u>. Taking $x=y=1$ in (1), we get $f(1)f(f(1))+f(f(1))=2f(1)f(f(1))$ and hence $f(1)=1$. Putting $x=1$ in (1), we have $f(f(y))+f(y)=f(y)(1+f(f(y^2)))$ so that

$$f(f(y))=f(y)f(f(y^2)). \tag{9}$$

Putting $y=1$ in (1), we get $xf(x^2)+f(f(x))=f(x)(f(f(x^2))+1)$. Using (9), this gives

$$xf(x^2)=f(x). \tag{10}$$

Replace y by $\frac{1}{x}$ in (1). Then we have

$$xf(x^2)f\left(f\left(\frac{1}{x}\right)\right)+f\left(\frac{f(x)}{x}\right)=f(f(x^2))+f\left(f\left(\frac{1}{x^2}\right)\right).$$

The relation (10) shows $f(\frac{f(x)}{x})=f(f(x^2))$. Also, using (9) with $y=\frac{1}{x}$ and using (10) again, the last equation reduces to

$$f(x)f\left(\frac{1}{x}\right)=1. \tag{11}$$

Replace x by $\frac{1}{x}$ and y by $\frac{1}{y}$ in (1) and apply (11). We get

$$\frac{1}{xf(x^2)f(f(y))}+\frac{1}{f(yf(x))}=\frac{1}{f(xy)}\left(\frac{1}{f(f(x^2))}+\frac{1}{f(f(y^2))}\right).$$

Clearing denominators, we can use (1) to simplify the numerators and obtain

$$f(xy)^2f(f(x^2))f(f(y^2))=xf(x^2)f(f(y))f(yf(x)).$$

Using (9) and (10), this is the same as

$$f(xy)^2f(f(x))=f(x)^2f(y)f(yf(x)). \tag{12}$$

Substitute $y=f(x)$ in (12) and apply (10) (with x replaced by $f(x)$). We have

$$f(xf(x))^2=f(x)f(f(x)). \tag{13}$$

Taking $y=x$ in (12), squaring both sides, and using (10) and (13), we find that

$$f(f(x))=x^4f(x)^3. \tag{14}$$

Finally, we combine (9), (10) and (14) to get

$$y^4f(y)^3\overset{(14)}{=}f(f(y))\overset{(9)}{=}f(y)f(f(y^2))\overset{(14)}{=}f(y)y^8f(y^2)^3\overset{(10)}{=}y^5f(y)^4,$$

which implies $f(y)=\frac{1}{y}$. This is a solution by the checking in Solution 1.

A5.

(a) Prove that for every positive integer n, there exists a fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}+1$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

(b) Prove that there are infinitely many positive integers n such that there is no fraction $\frac{a}{b}$ where a and b are integers satisfying $0 < b \leqslant \sqrt{n}$ and $\sqrt{n} \leqslant \frac{a}{b} \leqslant \sqrt{n+1}$.

Solution.

(a) Let r be the unique positive integer for which $r^2 \leqslant n < (r+1)^2$. Write $n = r^2 + s$. Then we have $0 \leqslant s \leqslant 2r$. We discuss in two cases according to the parity of s.

- **Case 1.** s is even.

  Consider the number $(r + \frac{s}{2r})^2 = r^2 + s + (\frac{s}{2r})^2$. We find that

  $$n = r^2 + s \leqslant r^2 + s + \left(\frac{s}{2r}\right)^2 \leqslant r^2 + s + 1 = n + 1.$$

  It follows that

  $$\sqrt{n} \leqslant r + \frac{s}{2r} \leqslant \sqrt{n+1}.$$

  Since s is even, we can choose the fraction $r + \frac{s}{2r} = \frac{r^2+(s/2)}{r}$ since $r \leqslant \sqrt{n}$.

- **Case 2.** s is odd.

  Consider the number $(r+1-\frac{2r+1-s}{2(r+1)})^2 = (r+1)^2 - (2r+1-s) + (\frac{2r+1-s}{2(r+1)})^2$. We find that

  $$\begin{aligned} n = r^2 + s = (r+1)^2 - (2r+1-s) &\leqslant (r+1)^2 - (2r+1-s) + \left(\frac{2r+1-s}{2(r+1)}\right)^2 \\ &\leqslant (r+1)^2 - (2r+1-s) + 1 = n+1. \end{aligned}$$

  It follows that

  $$\sqrt{n} \leqslant r + 1 - \frac{2r+1-s}{2(r+1)} \leqslant \sqrt{n+1}.$$

  Since s is odd, we can choose the fraction $(r+1) - \frac{2r+1-s}{2(r+1)} = \frac{(r+1)^2 - r + ((s-1)/2)}{r+1}$ since $r + 1 \leqslant \sqrt{n} + 1$.

(b) We show that for every positive integer r, there is no fraction $\frac{a}{b}$ with $b \leqslant \sqrt{r^2+1}$ such that $\sqrt{r^2+1} \leqslant \frac{a}{b} \leqslant \sqrt{r^2+2}$. Suppose on the contrary that such a fraction exists. Since $b \leqslant \sqrt{r^2+1} < r+1$ and b is an integer, we have $b \leqslant r$. Hence,

$$(br)^2 < b^2(r^2+1) \leqslant a^2 \leqslant b^2(r^2+2) \leqslant b^2r^2 + 2br < (br+1)^2.$$

This shows the square number a^2 is strictly bounded between the two consecutive squares $(br)^2$ and $(br+1)^2$, which is impossible. Hence, we have found infinitely many $n = r^2+1$ for which there is no fraction of the desired form.

A6. The equation

$$(x-1)(x-2)\cdots(x-2016)=(x-1)(x-2)\cdots(x-2016)$$

is written on the board. One tries to erase some linear factors from both sides so that each side still has at least one factor, and the resulting equation has no real roots. Find the least number of linear factors one needs to erase to achieve this.

Answer. 2016.

Solution. Since there are 2016 common linear factors on both sides, we need to erase at least 2016 factors. We claim that the equation has no real roots if we erase all factors $(x-k)$ on the left-hand side with $k \equiv 2,3 \pmod 4$, and all factors $(x-m)$ on the right-hand side with $m \equiv 0,1 \pmod 4$. Therefore, it suffices to show that no real number x satisfies

$$\prod_{j=0}^{503}(x-4j-1)(x-4j-4)=\prod_{j=0}^{503}(x-4j-2)(x-4j-3). \tag{1}$$

- **Case 1.** $x=1,2,\ldots,2016$.
  In this case, one side of (1) is zero while the other side is not. This shows x cannot satisfy (1).

- **Case 2.** $4k+1<x<4k+2$ or $4k+3<x<4k+4$ for some $k=0,1,\ldots,503$.
  For $j=0,1,\ldots,503$ with $j\neq k$, the product $(x-4j-1)(x-4j-4)$ is positive. For $j=k$, the product $(x-4k-1)(x-4k-4)$ is negative. This shows the left-hand side of (1) is negative. On the other hand, each product $(x-4j-2)(x-4j-3)$ on the right-hand side of (1) is positive. This yields a contradiction.

- **Case 3.** $x<1$ or $x>2016$ or $4k<x<4k+1$ for some $k=1,2,\ldots,503$.
  The equation (1) can be rewritten as

$$1=\prod_{j=0}^{503}\frac{(x-4j-1)(x-4j-4)}{(x-4j-2)(x-4j-3)}=\prod_{j=0}^{503}\left(1-\frac{2}{(x-4j-2)(x-4j-3)}\right).$$

Note that $(x-4j-2)(x-4j-3)>2$ for $0\leqslant j\leqslant 503$ in this case. So each term in the product lies strictly between 0 and 1, and the whole product must be less than 1, which is impossible.

- **Case 4.** $4k+2<x<4k+3$ for some $k=0,1,\ldots,503$.
  This time we rewrite (1) as

$$\begin{aligned}1&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\frac{(x-4j)(x-4j-1)}{(x-4j+1)(x-4j-2)}\\&=\frac{x-1}{x-2}\cdot\frac{x-2016}{x-2015}\prod_{j=1}^{503}\left(1+\frac{2}{(x-4j+1)(x-4j-2)}\right).\end{aligned}$$

Clearly, $\frac{x-1}{x-2}$ and $\frac{x-2016}{x-2015}$ are both greater than 1. For the range of x in this case, each term in the product is also greater than 1. Then the right-hand side must be greater than 1 and hence a contradiction arises.

From the four cases, we conclude that (1) has no real roots. Hence, the minimum number of linear factors to be erased is 2016.

Comment. We discuss the general case when 2016 is replaced by a positive integer n. The above solution works equally well when n is divisible by 4.

If $n \equiv 2 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n}{2})$ on the left-hand side and $r(x) = (x-\frac{n}{2}-1)(x-\frac{n}{2}-2)\cdots(x-n)$ on the right-hand side. One checks that for $x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$, while for $x > \frac{n+1}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 3 \pmod 4$, one may leave $l(x) = (x-1)(x-2)\cdots(x-\frac{n+1}{2})$ on the left-hand side and $r(x) = (x-\frac{n+3}{2})(x-\frac{x+5}{2})\cdots(x-n)$ on the right-hand side. For $x < 1$ or $\frac{n+1}{2} < x < \frac{n+3}{2}$, we have $l(x) > 0 > r(x)$. For $1 < x < \frac{n+1}{2}$, we have $|l(x)| < |r(x)|$. For $x > \frac{n+3}{2}$, we have $|l(x)| > |r(x)|$.

If $n \equiv 1 \pmod 4$, as the proposer mentioned, the situation is a bit more out of control. Since the construction for $n-1 \equiv 0 \pmod 4$ works, the answer can be either n or $n-1$. For $n = 5$, we can leave the products $(x-1)(x-2)(x-3)(x-4)$ and $(x-5)$. For $n = 9$, the only example that works is $l(x) = (x-1)(x-2)(x-9)$ and $r(x) = (x-3)(x-4)\cdots(x-8)$, while there seems to be no such partition for $n = 13$.

A7. Denote by $\mathbb{R}$ the set of all real numbers. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) \neq 0$ and

$$f(x+y)^2 = 2f(x)f(y) + \max\left\{f(x^2)+f(y^2), f(x^2+y^2)\right\} \tag{1}$$

for all real numbers x and y.

Answer.

- $f(x) = -1$ for any $x \in \mathbb{R}$; or
- $f(x) = x - 1$ for any $x \in \mathbb{R}$.

Solution 1. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have $f(x)^2 = -2f(x) + f(x^2)$, which is the same as $(f(x)+1)^2 = f(x^2)+1$. Let $g(x) = f(x)+1$. Then for any $x \in \mathbb{R}$, we have

$$g(x^2) = g(x)^2 \geqslant 0. \tag{2}$$

From (1), we find that $f(x+y)^2 \geqslant 2f(x)f(y) + f(x^2) + f(y^2)$. In terms of g, this becomes $(g(x+y)-1)^2 \geqslant 2(g(x)-1)(g(y)-1) + g(x^2) + g(y^2) - 2$. Using (2), this means

$$(g(x+y)-1)^2 \geqslant (g(x)+g(y)-1)^2 - 1. \tag{3}$$

Putting $x = 1$ in (2), we get $g(1) = 0$ or 1. The two cases are handled separately.

- **Case 1.** $g(1) = 0$, which is the same as $f(1) = -1$.

We put $x = -1$ and $y = 0$ in (1). This gives $f(-1)^2 = -2f(-1) - 1$, which forces $f(-1) = -1$. Next, we take $x = -1$ and $y = 1$ in (1) to get $1 = 2 + \max\{-2, f(2)\}$. This clearly implies $1 = 2 + f(2)$ and hence $f(2) = -1$, that is, $g(2) = 0$. From (2), we can prove inductively that $g(2^{2^n}) = g(2)^{2^n} = 0$ for any $n \in \mathbb{N}$. Substitute $y = 2^{2^n} - x$ in (3). We obtain

$$(g(x) + g(2^{2^n} - x) - 1)^2 \leqslant (g(2^{2^n}) - 1)^2 + 1 = 2.$$

For any fixed $x \geqslant 0$, we consider n to be sufficiently large so that $2^{2^n} - x > 0$. From (2), this implies $g(2^{2^n} - x) \geqslant 0$ so that $g(x) \leqslant 1 + \sqrt{2}$. Using (2) again, we get

$$g(x)^{2^n} = g(x^{2^n}) \leqslant 1 + \sqrt{2}$$

for any $n \in \mathbb{N}$. Therefore, $|g(x)| \leqslant 1$ for any $x \geqslant 0$.

If there exists $a \in \mathbb{R}$ for which $g(a) \neq 0$, then for sufficiently large n we must have $g((a^2)^{\frac{1}{2^n}}) = g(a^2)^{\frac{1}{2^n}} > \frac{1}{2}$. By taking $x = -y = -(a^2)^{\frac{1}{2^n}}$ in (1), we obtain

$$\begin{aligned} 1 &= 2f(x)f(-x) + \max\left\{2f(x^2), f(2x^2)\right\} \\ &= 2(g(x)-1)(g(-x)-1) + \max\left\{2(g(x^2)-1), g(2x^2)-1\right\} \\ &\leqslant 2\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + 0 = \frac{1}{2} \end{aligned}$$

since $|g(-x)| = |g(x)| \in (\frac{1}{2}, 1]$ by (2) and the choice of x, and since $g(z) \leqslant 1$ for $z \geqslant 0$. This yields a contradiction and hence $g(x) = 0$ must hold for any x. This means $f(x) = -1$ for any $x \in \mathbb{R}$, which clearly satisfies (1).

- **Case 2.** $g(1)=1$, which is the same as $f(1)=0$.

We put $x=-1$ and $y=1$ in (1) to get $1=\max\{0,f(2)\}$. This clearly implies $f(2)=1$ and hence $g(2)=2$. Setting $x=2n$ and $y=2$ in (3), we have

$$(g(2n+2)-1)^2 \geqslant (g(2n)+1)^2-1.$$

By induction on n, it is easy to prove that $g(2n)\geqslant n+1$ for all $n\in\mathbb{N}$. For any real number $a>1$, we choose a large $n\in\mathbb{N}$ and take k to be the positive integer such that $2k\leqslant a^{2^n}<2k+2$. From (2) and (3), we have

$$(g(a)^{2^n}-1)^2+1=(g(a^{2^n})-1)^2+1\geqslant(g(2k)+g(a^{2^n}-2k)-1)^2\geqslant k^2>\frac{1}{4}(a^{2^n}-2)^2$$

since $g(a^{2^n}-2k)\geqslant 0$. For large n, this clearly implies $g(a)^{2^n}>1$. Thus,

$$(g(a)^{2^n})^2>(g(a)^{2^n}-1)^2+1>\frac{1}{4}(a^{2^n}-2)^2.$$

This yields

$$g(a)^{2^n}>\frac{1}{2}(a^{2^n}-2). \tag{4}$$

Note that

$$\frac{a^{2^n}}{a^{2^n}-2}=1+\frac{2}{a^{2^n}-2}\leqslant\left(1+\frac{2}{2^n(a^{2^n}-2)}\right)^{2^n}$$

by binomial expansion. This can be rewritten as

$$(a^{2^n}-2)^{\frac{1}{2^n}}\geqslant\frac{a}{1+\frac{2}{2^n(a^{2^n}-2)}}.$$

Together with (4), we conclude $g(a)\geqslant a$ by taking n sufficiently large.

Consider $x=na$ and $y=a>1$ in (3). This gives $(g((n+1)a)-1)^2\geqslant(g(na)+g(a)-1)^2-1$. By induction on n, it is easy to show $g(na)\geqslant(n-1)(g(a)-1)+a$ for any $n\in\mathbb{N}$. We choose a large $n\in\mathbb{N}$ and take k to be the positive integer such that $ka\leqslant 2^{2^n}<(k+1)a$. Using (2) and (3), we have

$$2^{2^{n+1}}>(2^{2^n}-1)^2+1=(g(2^{2^n})-1)^2+1\geqslant(g(2^{2^n}-ka)+g(ka)-1)^2\geqslant((k-1)(g(a)-1)+a-1)^2,$$

from which it follows that

$$2^{2^n}\geqslant(k-1)(g(a)-1)+a-1>\frac{2^{2^n}}{a}(g(a)-1)-2(g(a)-1)+a-1$$

holds for sufficiently large n. Hence, we must have $\frac{g(a)-1}{a}\leqslant 1$, which implies $g(a)\leqslant a+1$ for any $a>1$. Then for large $n\in\mathbb{N}$, from (3) and (2) we have

$$4a^{2^{n+1}}=(2a^{2^n})^2\geqslant(g(2a^{2^n})-1)^2\geqslant(2g(a^{2^n})-1)^2-1=(2g(a)^{2^n}-1)^2-1.$$

This implies

$$2a^{2^n} > \frac{1}{2}(1+\sqrt{4a^{2^{n+1}}+1}) \geqslant g(a)^{2^n}.$$

When n tends to infinity, this forces $g(a) \leqslant a$. Together with $g(a) \geqslant a$, we get $g(a) = a$ for all real numbers $a > 1$, that is, $f(a) = a - 1$ for all $a > 1$.

Finally, for any $x \in \mathbb{R}$, we choose y sufficiently large in (1) so that $y, x+y > 1$. This gives $(x+y-1)^2 = 2f(x)(y-1) + \max\{f(x^2)+y^2-1, x^2+y^2-1\}$, which can be rewritten as

$$2(x-1-f(x))y = -x^2+2x-2-2f(x)+\max\{f(x^2), x^2\}.$$

As the right-hand side is fixed, this can only hold for all large y when $f(x) = x - 1$. We now check that this function satisfies (1). Indeed, we have

$$\begin{aligned} f(x+y)^2 = (x+y-1)^2 &= 2(x-1)(y-1) + (x^2+y^2-1) \\ &= 2f(x)f(y) + \max\{f(x^2)+f(y^2), f(x^2+y^2)\}. \end{aligned}$$

Solution 2. Taking $x = y = 0$ in (1), we get $f(0)^2 = 2f(0)^2 + \max\{2f(0), f(0)\}$. If $f(0) > 0$, then $f(0)^2 + 2f(0) = 0$ gives no positive solution. If $f(0) < 0$, then $f(0)^2 + f(0) = 0$ gives $f(0) = -1$. Putting $y = 0$ in (1), we have

$$f(x)^2 = -2f(x) + f(x^2). \tag{5}$$

Replace x by $-x$ in (5) and compare with (5) again. We get $f(x)^2 + 2f(x) = f(-x)^2 + 2f(-x)$, which implies

$$f(x) = f(-x) \quad \text{or} \quad f(x) + f(-x) = -2. \tag{6}$$

Taking $x = y$ and $x = -y$ respectively in (1) and comparing the two equations obtained, we have

$$f(2x)^2 - 2f(x)^2 = 1 - 2f(x)f(-x). \tag{7}$$

Combining (6) and (7) to eliminate $f(-x)$, we find that $f(2x)$ can be ± 1 (when $f(x) = f(-x)$) or $\pm(2f(x)+1)$ (when $f(x) + f(-x) = -2$).

We prove the following.

- **Claim.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

Proof. Suppose there exists $a \in \mathbb{R}$ such that $f(a) + f(-a) \neq -2$. Then $f(a) = f(-a) \neq -1$ and we may assume $a > 0$. We first show that $f(a) \neq 1$. Suppose $f(a) = 1$. Consider $y = a$ in (7). We get $f(2a)^2 = 1$. Taking $x = y = a$ in (1), we have $1 = 2 + \max\{2f(a^2), f(2a^2)\}$. From (5), $f(a^2) = 3$ so that $1 \geqslant 2 + 6$. This is impossible, and thus $f(a) \neq 1$.

As $f(a) \neq \pm 1$, we have $f(a) = \pm(2f(\frac{a}{2})+1)$. Similarly, $f(-a) = \pm(2f(-\frac{a}{2})+1)$. These two expressions are equal since $f(a) = f(-a)$. If $f(\frac{a}{2}) = f(-\frac{a}{2})$, then the above argument works when we replace a by $\frac{a}{2}$. In particular, we have $f(a)^2 = f(2 \cdot \frac{a}{2})^2 = 1$, which is a contradiction. Therefore, (6) forces $f(\frac{a}{2}) + f(-\frac{a}{2}) = -2$. Then we get

$$\pm\left(2f\left(\frac{a}{2}\right)+1\right) = \pm\left(-2f\left(\frac{a}{2}\right)-3\right).$$

For any choices of the two signs, we either get a contradiction or $f(\frac{a}{2}) = -1$, in which case $f(\frac{a}{2}) = f(-\frac{a}{2})$ and hence $f(a) = \pm 1$ again. Therefore, there is no such real number a and the Claim follows. $\square$

Replace x and y by $-x$ and $-y$ in (1) respectively and compare with (1). We get

$$f(x+y)^2 - 2f(x)f(y) = f(-x-y)^2 - 2f(-x)f(-y).$$

Using the Claim, this simplifies to $f(x+y) = f(x)+f(y)+1$. In addition, (5) can be rewritten as $(f(x)+1)^2 = f(x^2)+1$. Therefore, the function g defined by $g(x) = f(x)+1$ satisfies $g(x+y) = g(x)+g(y)$ and $g(x)^2 = g(x^2)$. The latter relation shows $g(y)$ is nonnegative for $y \geqslant 0$. For such a function satisfying the Cauchy Equation $g(x+y) = g(x)+g(y)$, it must be monotonic increasing and hence $g(x) = cx$ for some constant c.

From $(cx)^2 = g(x)^2 = g(x^2) = cx^2$, we get $c = 0$ or 1, which corresponds to the two functions $f(x) = -1$ and $f(x) = x-1$ respectively, both of which are solutions to (1) as checked in Solution 1.

Solution 3. As in Solution 2, we find that $f(0) = -1$,

$$(f(x)+1)^2 = f(x^2)+1 \tag{8}$$

and

$$f(x) = f(-x) \quad \text{or} \quad f(x)+f(-x) = -2 \tag{9}$$

for any $x \in \mathbb{R}$. We shall show that one of the statements in (9) holds for all $x \in \mathbb{R}$. Suppose $f(a) = f(-a)$ but $f(a)+f(-a) \neq -2$, while $f(b) \neq f(-b)$ but $f(b)+f(-b) = -2$. Clearly, $a, b \neq 0$ and $f(a), f(b) \neq -1$.

Taking $y = a$ and $y = -a$ in (1) respectively and comparing the two equations obtained, we have $f(x+a)^2 = f(x-a)^2$, that is, $f(x+a) = \pm f(x-a)$. This implies $f(x+2a) = \pm f(x)$ for all $x \in \mathbb{R}$. Putting $x = b$ and $x = -2a-b$ respectively, we find $f(2a+b) = \pm f(b)$ and $f(-2a-b) = \pm f(-b) = \pm(-2-f(b))$. Since $f(b) \neq -1$, the term $\pm(-2-f(b))$ is distinct from $\pm f(b)$ in any case. So $f(2a+b) \neq f(-2a-b)$. From (9), we must have $f(2a+b)+f(-2a-b) = -2$. Note that we also have $f(b)+f(-b) = -2$ where $|f(b)|, |f(-b)|$ are equal to $|f(2a+b)|, |f(-2a-b)|$ respectively. The only possible case is $f(2a+b) = f(b)$ and $f(-2a-b) = f(-b)$.

Applying the argument to $-a$ instead of a and using induction, we have $f(2ka+b) = f(b)$ and $f(2ka-b) = f(-b)$ for any integer k. Note that $f(b)+f(-b) = -2$ and $f(b) \neq -1$ imply one of $f(b), f(-b)$ is less than -1. Without loss of generality, assume $f(b) < -1$. We consider $x = \sqrt{2ka+b}$ in (8) for sufficiently large k so that

$$(f(x)+1)^2 = f(2ka+b)+1 = f(b)+1 < 0$$

yields a contradiction. Therefore, one of the statements in (9) must hold for all $x \in \mathbb{R}$.

- **Case 1.** $f(x) = f(-x)$ for any $x \in \mathbb{R}$.

  For any $a \in \mathbb{R}$, setting $x = y = \frac{a}{2}$ and $x = -y = \frac{a}{2}$ in (1) respectively and comparing these, we obtain $f(a)^2 = f(0)^2 = 1$, which means $f(a) = \pm 1$ for all $a \in \mathbb{R}$. If $f(a) = 1$ for some a, we may assume $a > 0$ since $f(a) = f(-a)$. Taking $x = y = \sqrt{a}$ in (1), we get

$$f(2\sqrt{a})^2 = 2f(\sqrt{a})^2 + \max\{2, f(2a)\} = 2f(\sqrt{a})^2 + 2.$$

Note that the left-hand side is ± 1 while the right-hand side is an even integer. This is a contradiction. Therefore, $f(x) = -1$ for all $x \in \mathbb{R}$, which is clearly a solution.

- **Case 2.** $f(x) + f(-x) = -2$ for any $x \in \mathbb{R}$.

  This case can be handled in the same way as in Solution 2, which yields another solution $f(x) = x - 1$.

A8. Determine the largest real number a such that for all $n \geqslant 1$ and for all real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 = x_0 < x_1 < x_2 < \cdots < x_n$, we have

$$\frac{1}{x_1 - x_0} + \frac{1}{x_2 - x_1} + \cdots + \frac{1}{x_n - x_{n-1}} \geqslant a\left(\frac{2}{x_1} + \frac{3}{x_2} + \cdots + \frac{n+1}{x_n}\right). \tag{1}$$

Answer. The largest a is $\frac{4}{9}$.

Solution 1. We first show that $a = \frac{4}{9}$ is admissible. For each $2 \leqslant k \leqslant n$, by the Cauchy-Schwarz Inequality, we have

$$(x_{k-1} + (x_k - x_{k-1}))\left(\frac{(k-1)^2}{x_{k-1}} + \frac{3^2}{x_k - x_{k-1}}\right) \geqslant (k-1+3)^2,$$

which can be rewritten as

$$\frac{9}{x_k - x_{k-1}} \geqslant \frac{(k+2)^2}{x_k} - \frac{(k-1)^2}{x_{k-1}}. \tag{2}$$

Summing (2) over $k = 2, 3, \ldots, n$ and adding $\frac{9}{x_1}$ to both sides, we have

$$9\sum_{k=1}^{n} \frac{1}{x_k - x_{k-1}} \geqslant 4\sum_{k=1}^{n} \frac{k+1}{x_k} + \frac{n^2}{x_n} > 4\sum_{k=1}^{n} \frac{k+1}{x_k}.$$

This shows (1) holds for $a = \frac{4}{9}$.

Next, we show that $a = \frac{4}{9}$ is the optimal choice. Consider the sequence defined by $x_0 = 0$ and $x_k = x_{k-1} + k(k+1)$ for $k \geqslant 1$, that is, $x_k = \frac{1}{3}k(k+1)(k+2)$. Then the left-hand side of (1) equals

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 - \frac{1}{n+1},$$

while the right-hand side equals

$$a\sum_{k=1}^{n} \frac{k+1}{x_k} = 3a\sum_{k=1}^{n} \frac{1}{k(k+2)} = \frac{3}{2}a\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+2}\right) = \frac{3}{2}\left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right)a.$$

When n tends to infinity, the left-hand side tends to 1 while the right-hand side tends to $\frac{9}{4}a$. Therefore a has to be at most $\frac{4}{9}$.

Hence the largest value of a is $\frac{4}{9}$.

Solution 2. We shall give an alternative method to establish (1) with $a = \frac{4}{9}$. We define $y_k = x_k - x_{k-1} > 0$ for $1 \leqslant k \leqslant n$. By the Cauchy-Schwarz Inequality, for $1 \leqslant k \leqslant n$, we have

$$(y_1 + y_2 + \cdots + y_k)\left(\sum_{j=1}^{k} \frac{1}{y_j}\binom{j+1}{2}^2\right) \geqslant \left(\binom{2}{2} + \binom{3}{2} + \cdots + \binom{k+1}{2}\right)^2 = \binom{k+2}{3}^2.$$

This can be rewritten as

$$\frac{k+1}{y_1+y_2+\cdots+y_k} \leqslant \frac{36}{k^2(k+1)(k+2)^2}\left(\sum_{j=1}^{k} \frac{1}{y_j}\binom{j+1}{2}^2\right). \tag{3}$$

Summing (3) over $k=1,2,\ldots,n$, we get

$$\frac{2}{y_1}+\frac{3}{y_1+y_2}+\cdots+\frac{n+1}{y_1+y_2+\cdots+y_n} \leqslant \frac{c_1}{y_1}+\frac{c_2}{y_2}+\cdots+\frac{c_n}{y_n} \tag{4}$$

where for $1 \leqslant m \leqslant n$,

$$\begin{aligned} c_m &= 36\binom{m+1}{2}^2 \sum_{k=m}^{n} \frac{1}{k^2(k+1)(k+2)^2} \\ &= \frac{9m^2(m+1)^2}{4} \sum_{k=m}^{n}\left(\frac{1}{k^2(k+1)^2}-\frac{1}{(k+1)^2(k+2)^2}\right) \\ &= \frac{9m^2(m+1)^2}{4}\left(\frac{1}{m^2(m+1)^2}-\frac{1}{(n+1)^2(n+2)^2}\right) < \frac{9}{4}. \end{aligned}$$

From (4), the inequality (1) holds for $a=\frac{4}{9}$. This is also the upper bound as can be verified in the same way as Solution 1.

Combinatorics

C1. The leader of an IMO team chooses positive integers n and k with $n > k$, and announces them to the deputy leader and a contestant. The leader then secretly tells the deputy leader an n-digit binary string, and the deputy leader writes down all n-digit binary strings which differ from the leader's in exactly k positions. (For example, if $n = 3$ and $k = 1$, and if the leader chooses 101, the deputy leader would write down 001, 111 and 100.) The contestant is allowed to look at the strings written by the deputy leader and guess the leader's string. What is the minimum number of guesses (in terms of n and k) needed to guarantee the correct answer?

Answer. The minimum number of guesses is 2 if $n = 2k$ and 1 if $n \neq 2k$.

Solution 1. Let X be the binary string chosen by the leader and let X' be the binary string of length n every digit of which is different from that of X. The strings written by the deputy leader are the same as those in the case when the leader's string is X' and k is changed to $n - k$. In view of this, we may assume $k \geqslant \frac{n}{2}$. Also, for the particular case $k = \frac{n}{2}$, this argument shows that the strings X and X' cannot be distinguished, and hence in that case the contestant has to guess at least twice.

It remains to show that the number of guesses claimed suffices. Consider any string Y which differs from X in m digits where $0 < m < 2k$. Without loss of generality, assume the first m digits of X and Y are distinct. Let Z be the binary string obtained from X by changing its first k digits. Then Z is written by the deputy leader. Note that Z differs from Y by $|m - k|$ digits where $|m - k| < k$ since $0 < m < 2k$. From this observation, the contestant must know that Y is not the desired string.

As we have assumed $k \geqslant \frac{n}{2}$, when $n < 2k$, every string $Y \neq X$ differs from X in fewer than $2k$ digits. When $n = 2k$, every string except X and X' differs from X in fewer than $2k$ digits. Hence, the answer is as claimed.

Solution 2. Firstly, assume $n \neq 2k$. Without loss of generality suppose the first digit of the leader's string is 1. Then among the $\binom{n}{k}$ strings written by the deputy leader, $\binom{n-1}{k}$ will begin with 1 and $\binom{n-1}{k-1}$ will begin with 0. Since $n \neq 2k$, we have $k + (k-1) \neq n - 1$ and so $\binom{n-1}{k} \neq \binom{n-1}{k-1}$. Thus, by counting the number of strings written by the deputy leader that start with 0 and 1, the contestant can tell the first digit of the leader's string. The same can be done on the other digits, so 1 guess suffices when $n \neq 2k$.

Secondly, for the case $n = 2$ and $k = 1$, the answer is clearly 2. For the remaining cases where $n = 2k > 2$, the deputy leader would write down the same strings if the leader's string X is replaced by X' obtained by changing each digit of X. This shows at least 2 guesses are needed. We shall show that 2 guesses suffice in this case. Suppose the first two digits of the leader's string are the same. Then among the strings written by the deputy leader, the prefices 01 and 10 will occur $\binom{2k-2}{k-1}$ times each, while the prefices 00 and 11 will occur $\binom{2k-2}{k}$ times each. The two numbers are interchanged if the first two digits of the leader's string are different. Since $\binom{2k-2}{k-1} \neq \binom{2k-2}{k}$, the contestant can tell whether the first two digits of the leader's string are the same or not. He can work out the relation of the first digit and the other digits in the same way and reduce the leader's string to only 2 possibilities. The proof is complete.

C2. Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

- each cell contains a distinct divisor;
- the sums of all rows are equal; and
- the sums of all columns are equal.

Answer. 1.

Solution 1. Suppose all positive divisors of n can be arranged into a rectangular table of size $k \times l$ where the number of rows k does not exceed the number of columns l. Let the sum of numbers in each column be s. Since n belongs to one of the columns, we have $s \geqslant n$, where equality holds only when $n = 1$.

For $j = 1, 2, \ldots, l$, let d_j be the largest number in the j-th column. Without loss of generality, assume $d_1 > d_2 > \cdots > d_l$. Since these are divisors of n, we have

$$d_l \leqslant \frac{n}{l}. \tag{1}$$

As d_l is the maximum entry of the l-th column, we must have

$$d_l \geqslant \frac{s}{k} \geqslant \frac{n}{k}. \tag{2}$$

The relations (1) and (2) combine to give $\frac{n}{l} \geqslant \frac{n}{k}$, that is, $k \geqslant l$. Together with $k \leqslant l$, we conclude that $k = l$. Then all inequalities in (1) and (2) are equalities. In particular, $s = n$ and so $n = 1$, in which case the conditions are clearly satisfied.

Solution 2. Clearly $n = 1$ works. Then we assume $n > 1$ and let its prime factorization be $n = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}$. Suppose the table has k rows and l columns with $1 < k \leqslant l$. Note that kl is the number of positive divisors of n and the sum of all entries is the sum of positive divisors of n, which we denote by $\sigma(n)$. Consider the column containing n. Since the column sum is $\frac{\sigma(n)}{l}$, we must have $\frac{\sigma(n)}{l} > n$. Therefore, we have

$$\begin{aligned}(r_1+1)(r_2+1)\cdots(r_t+1) = kl \leqslant l^2 &< \left(\frac{\sigma(n)}{n}\right)^2 \\ &= \left(1 + \frac{1}{p_1} + \cdots + \frac{1}{p_1^{r_1}}\right)^2 \cdots \left(1 + \frac{1}{p_t} + \cdots + \frac{1}{p_t^{r_t}}\right)^2.\end{aligned}$$

This can be rewritten as

$$f(p_1, r_1) f(p_2, r_2) \cdots f(p_t, r_t) < 1 \tag{3}$$

where

$$f(p, r) = \frac{r+1}{\left(1 + \frac{1}{p} + \cdots + \frac{1}{p^r}\right)^2} = \frac{(r+1)\left(1 - \frac{1}{p}\right)^2}{\left(1 - \frac{1}{p^{r+1}}\right)^2}.$$

Direct computation yields

$$f(2,1)=\frac{8}{9},\quad f(2,2)=\frac{48}{49},\quad f(3,1)=\frac{9}{8}.$$

Also, we find that

$$\begin{aligned}
f(2,r) &\geqslant \left(1-\frac{1}{2^{r+1}}\right)^{-2}>1 \quad \text{for } r\geqslant 3,\\
f(3,r) &\geqslant \frac{4}{3}\left(1-\frac{1}{3^{r+1}}\right)^{-2}>\frac{4}{3}>\frac{9}{8} \quad \text{for } r\geqslant 2, \text{ and}\\
f(p,r) &\geqslant \frac{32}{25}\left(1-\frac{1}{p^{r+1}}\right)^{-2}>\frac{32}{25}>\frac{9}{8} \quad \text{for } p\geqslant 5.
\end{aligned}$$

From these values and bounds, it is clear that (3) holds only when $n=2$ or 4. In both cases, it is easy to see that the conditions are not satisfied. Hence, the only possible n is 1.

C3. Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

Solution. For $k = 1, 2, 3$, let a_k be the number of isosceles triangles whose vertices contain exactly k colours. Suppose on the contrary that $a_3 = 0$. Let b, c, d be the number of vertices of the three different colours respectively. We now count the number of pairs $(\triangle, E)$ where $\triangle$ is an isosceles triangle and E is a side of $\triangle$ whose endpoints are of different colours.

On the one hand, since we have assumed $a_3 = 0$, each triangle in the pair must contain exactly two colours, and hence each triangle contributes twice. Thus the number of pairs is $2a_2$.

On the other hand, if we pick any two vertices A, B of distinct colours, then there are three isosceles triangles having these as vertices, two when AB is not the base and one when AB is the base since n is odd. Note that the three triangles are all distinct as $(n, 3) = 1$. In this way, we count the number of pairs to be $3(bc + cd + db)$. However, note that $2a_2$ is even while $3(bc+cd+db)$ is odd, as each of b, c, d is. This yields a contradiction and hence $a_3 \geqslant 1$.

Comment. A slightly stronger version of this problem is to replace the condition $(n, 6) = 1$ by n being odd (where equilateral triangles are regarded as isosceles triangles). In that case, the only difference in the proof is that by fixing any two vertices A, B, one can find exactly one or three isosceles triangles having these as vertices. But since only parity is concerned in the solution, the proof goes the same way.

The condition that there is an odd number of vertices of each colour is necessary, as can be seen from the following example. Consider $n = 25$ and we label the vertices $A_0, A_1, \ldots, A_{24}$. Suppose colour 1 is used for A_0, colour 2 is used for $A_5, A_{10}, A_{15}, A_{20}$, while colour 3 is used for the remaining vertices. Then any isosceles triangle having colours 1 and 2 must contain A_0 and one of $A_5, A_{10}, A_{15}, A_{20}$. Clearly, the third vertex must have index which is a multiple of 5 so it is not of colour 3.

C4. Find all positive integers n for which we can fill in the entries of an $n \times n$ table with the following properties:

- each entry can be one of I, M and O;
- in each row and each column, the letters I, M and O occur the same number of times; and
- in any diagonal whose number of entries is a multiple of three, the letters I, M and O occur the same number of times.

Answer. n can be any multiple of 9.

Solution. We first show that such a table exists when n is a multiple of 9. Consider the following 9×9 table.

$$\begin{pmatrix} I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \\ I & I & I & M & M & M & O & O & O \\ M & M & M & O & O & O & I & I & I \\ O & O & O & I & I & I & M & M & M \end{pmatrix} \tag{1}$$

It is a direct checking that the table (1) satisfies the requirements. For $n = 9k$ where k is a positive integer, we form an $n \times n$ table using $k \times k$ copies of (1). For each row and each column of the table of size n, since there are three I's, three M's and three O's for any nine consecutive entries, the numbers of I, M and O are equal. In addition, every diagonal of the large table whose number of entries is divisible by 3 intersects each copy of (1) at a diagonal with number of entries divisible by 3 (possibly zero). Therefore, every such diagonal also contains the same number of I, M and O.

Next, consider any $n \times n$ table for which the requirements can be met. As the number of entries of each row should be a multiple of 3, we let $n = 3k$ where k is a positive integer. We divide the whole table into $k \times k$ copies of 3×3 blocks. We call the entry at the centre of such a 3×3 square a *vital entry*. We also call any row, column or diagonal that contains at least one vital entry a *vital line*. We compute the number of pairs (l, c) where l is a vital line and c is an entry belonging to l that contains the letter M. We let this number be N.

On the one hand, since each vital line contains the same number of I, M and O, it is obvious that each vital row and each vital column contain k occurrences of M. For vital diagonals in either direction, we count there are exactly

$$1 + 2 + \cdots + (k-1) + k + (k-1) + \cdots + 2 + 1 = k^2$$

occurrences of M. Therefore, we have $N = 4k^2$.

On the other hand, there are $3k^2$ occurrences of M in the whole table. Note that each entry belongs to exactly 1 or 4 vital lines. Therefore, N must be congruent to $3k^2$ mod 3.

From the double counting, we get $4k^2 \equiv 3k^2 \pmod 3$, which forces k to be a multiple of 3. Therefore, n has to be a multiple of 9 and the proof is complete.

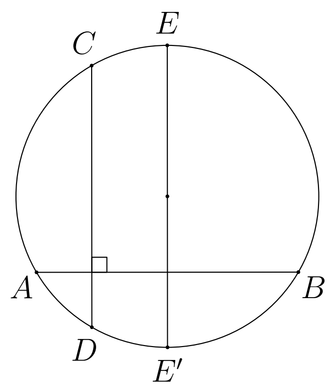
C5. Let $n \geqslant 3$ be a positive integer. Find the maximum number of diagonals of a regular n-gon one can select, so that any two of them do not intersect in the interior or they are perpendicular to each other.

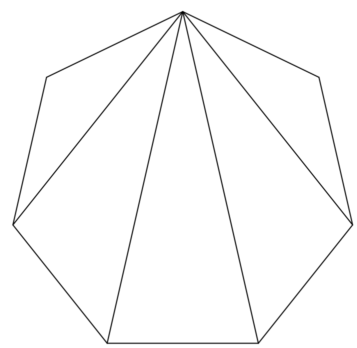
Answer. $n-2$ if n is even and $n-3$ if n is odd.

Solution 1. We consider two cases according to the parity of n.

- **Case 1.** n is odd.

We first claim that no pair of diagonals is perpendicular. Suppose A, B, C, D are vertices where AB and CD are perpendicular, and let E be the vertex lying on the perpendicular bisector of AB. Let E' be the opposite point of E on the circumcircle of the regular polygon. Since $EC = E'D$ and C, D, E are vertices of the regular polygon, E' should also belong to the polygon. This contradicts the fact that a regular polygon with an odd number of vertices does not contain opposite points on the circumcircle.



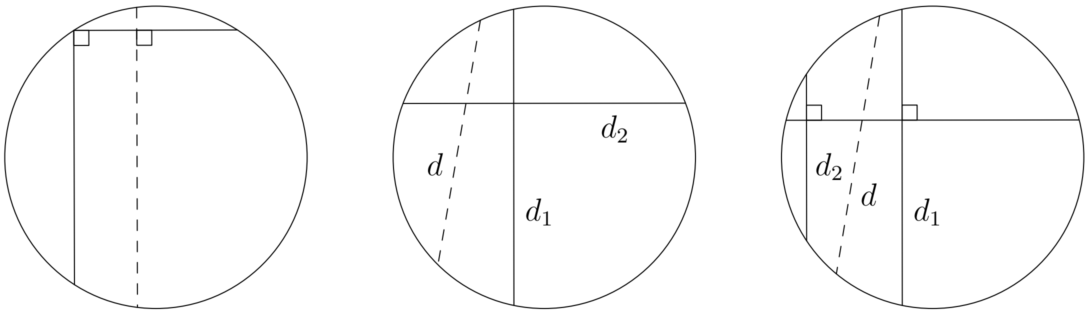


Therefore in the odd case we can only select diagonals which do not intersect. In the maximal case these diagonals should divide the regular n-gon into $n-2$ triangles, so we can select at most $n-3$ diagonals. This can be done, for example, by selecting all diagonals emanated from a particular vertex.

- **Case 2.** n is even.

If there is no intersection, then the proof in the odd case works. Suppose there are two perpendicular diagonals selected. We consider the set S of all selected diagonals parallel to one of them which intersect with some selected diagonals. Suppose S contains k diagonals and the number of distinct endpoints of the k diagonals is l.

Firstly, consider the longest diagonal in one of the two directions in S. No other diagonal in S can start from either endpoint of that diagonal, since otherwise it has to meet another longer diagonal in S. The same holds true for the other direction. Ignoring these two longest diagonals and their four endpoints, the remaining $k-2$ diagonals share $l-4$ endpoints where each endpoint can belong to at most two diagonals. This gives $2(l-4) \geqslant 2(k-2)$, so that $k \leqslant l-2$.

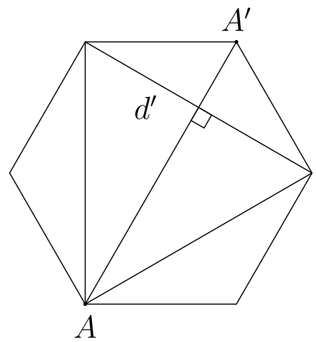


Consider a group of consecutive vertices of the regular n-gon so that each of the two outermost vertices is an endpoint of a diagonal in S, while the interior points are not. There are l such groups. We label these groups $P_1, P_2, \ldots, P_l$ in this order. We claim that each selected diagonal outside S must connect vertices of the same group P_i. Consider any diagonal d joining vertices from distinct groups P_i and P_j. Let d_1 and d_2 be two diagonals in S each having one of the outermost points of P_i as endpoint. Then d must meet either d_1, d_2 or a diagonal in S which is perpendicular to both d_1 and d_2. In any case d should belong to S by definition, which is a contradiction.

Within the same group P_i, there are no perpendicular diagonals since the vertices belong to the same side of a diameter of the circumcircle. Hence there can be at most $|P_i|-2$ selected diagonals within P_i, including the one joining the two outermost points of P_i when $|P_i| > 2$. Therefore, the maximum number of diagonals selected is

$$\sum_{i=1}^{l}(|P_i|-2)+k=\sum_{i=1}^{l}|P_i|-2l+k=(n+l)-2l+k=n-l+k\leqslant n-2.$$

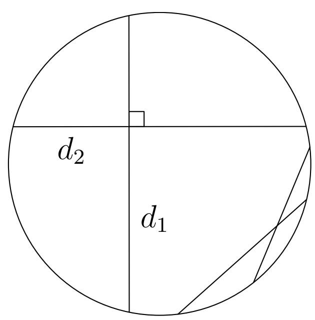
This upper bound can be attained as follows. We take any vertex A and let A' be the vertex for which AA' is a diameter of the circumcircle. If we select all diagonals emanated from A together with the diagonal d' joining the two neighbouring vertices of A', then the only pair of diagonals that meet each other is AA' and d', which are perpendicular to each other. In total we can take $n-2$ diagonals.

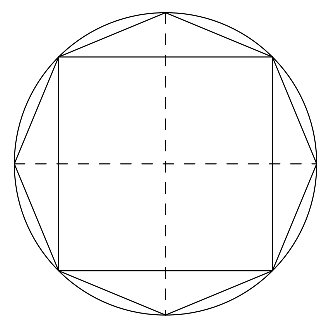


Solution 2. The constructions and the odd case are the same as Solution 1. Instead of dealing separately with the case where n is even, we shall prove by induction more generally that we can select at most $n-2$ diagonals for any cyclic n-gon with circumcircle Γ.

The base case $n = 3$ is trivial since there is no diagonal at all. Suppose the upper bound holds for any cyclic polygon with fewer than n sides. For a cyclic n-gon, if there is a selected diagonal which does not intersect any other selected diagonal, then this diagonal divides the n-gon into an m-gon and an l-gon (with $m+l = n+2$) so that each selected diagonal belongs to one of them. Without loss of generality, we may assume the m-gon lies on the same side of a diameter of Γ. Then no two selected diagonals of the m-gon can intersect, and hence we can select at most $m-3$ diagonals. Also, we can apply the inductive hypothesis to the l-gon. This shows the maximum number of selected diagonals is $(m-3)+(l-2)+1 = n-2$.

It remains to consider the case when all selected diagonals meet at least one other selected diagonal. Consider a pair of selected perpendicular diagonals d_1, d_2. They divide the circumference of Γ into four arcs, each of which lies on the same side of a diameter of Γ. If there are two selected diagonals intersecting each other and neither is parallel to d_1 or d_2, then their endpoints must belong to the same arc determined by d_1, d_2, and hence they cannot be perpendicular. This violates the condition, and hence all selected diagonals must have the same direction as one of d_1, d_2.





Take the longest selected diagonal in one of the two directions. We argue as in Solution 1 that its endpoints do not belong to any other selected diagonal. The same holds true for the longest diagonal in the other direction. Apart from these four endpoints, each of the remaining $n-4$ vertices can belong to at most two selected diagonals. Thus we can select at most $\frac{1}{2}(2(n-4)+4) = n-2$ diagonals. Then the proof follows by induction.

C6. There are $n \geqslant 3$ islands in a city. Initially, the ferry company offers some routes between some pairs of islands so that it is impossible to divide the islands into two groups such that no two islands in different groups are connected by a ferry route.

After each year, the ferry company will close a ferry route between some two islands X and Y. At the same time, in order to maintain its service, the company will open new routes according to the following rule: for any island which is connected by a ferry route to exactly one of X and Y, a new route between this island and the other of X and Y is added.

Suppose at any moment, if we partition all islands into two nonempty groups in any way, then it is known that the ferry company will close a certain route connecting two islands from the two groups after some years. Prove that after some years there will be an island which is connected to all other islands by ferry routes.

Solution. Initially, we pick any pair of islands A and B which are connected by a ferry route and put A in set $\mathcal{A}$ and B in set $\mathcal{B}$. From the condition, without loss of generality there must be another island which is connected to A. We put such an island C in set $\mathcal{B}$. We say that two sets of islands form a *network* if each island in one set is connected to each island in the other set.

Next, we shall included all islands to $\mathcal{A} \cup \mathcal{B}$ one by one. Suppose we have two sets $\mathcal{A}$ and $\mathcal{B}$ which form a network where $3 \leqslant |\mathcal{A} \cup \mathcal{B}| < n$. This relation no longer holds only when a ferry route between islands $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed. In that case, we define $\mathcal{A}' = \{A, B\}$, and $\mathcal{B}' = (\mathcal{A} \cup \mathcal{B}) - \{A, B\}$. Note that $\mathcal{B}'$ is nonempty. Consider any island $C \in \mathcal{A} - \{A\}$. From the relation of $\mathcal{A}$ and $\mathcal{B}$, we know that C is connected to B. If C was not connected to A before the route between A and B closes, then there will be a route added between C and A afterwards. Hence, C must now be connected to both A and B. The same holds true for any island in $\mathcal{B} - \{B\}$. Therefore, $\mathcal{A}'$ and $\mathcal{B}'$ form a network, and $\mathcal{A}' \cup \mathcal{B}' = \mathcal{A} \cup \mathcal{B}$. Hence these islands can always be partitioned into sets $\mathcal{A}$ and $\mathcal{B}$ which form a network.

As $|\mathcal{A} \cup \mathcal{B}| < n$, there are some islands which are not included in $\mathcal{A} \cup \mathcal{B}$. From the condition, after some years there must be a ferry route between an island A in $\mathcal{A} \cup \mathcal{B}$ and an island D outside $\mathcal{A} \cup \mathcal{B}$ which closes. Without loss of generality assume $A \in \mathcal{A}$. Then each island in $\mathcal{B}$ must then be connected to D, no matter it was or not before. Hence, we can put D in set $\mathcal{A}$ so that the new sets $\mathcal{A}$ and $\mathcal{B}$ still form a network and the size of $\mathcal{A} \cup \mathcal{B}$ is increased by 1. The same process can be done to increase the size of $\mathcal{A} \cup \mathcal{B}$. Eventually, all islands are included in this way so we may now assume $|\mathcal{A} \cup \mathcal{B}| = n$.

Suppose a ferry route between $A \in \mathcal{A}$ and $B \in \mathcal{B}$ is closed after some years. We put A and B in set $\mathcal{A}'$ and all remaining islands in set $\mathcal{B}'$. Then $\mathcal{A}'$ and $\mathcal{B}'$ form a network. This relation no longer holds only when a route between A, without loss of generality, and $C \in \mathcal{B}'$ is closed. Since this must eventually occur, at that time island B will be connected to all other islands and the result follows.

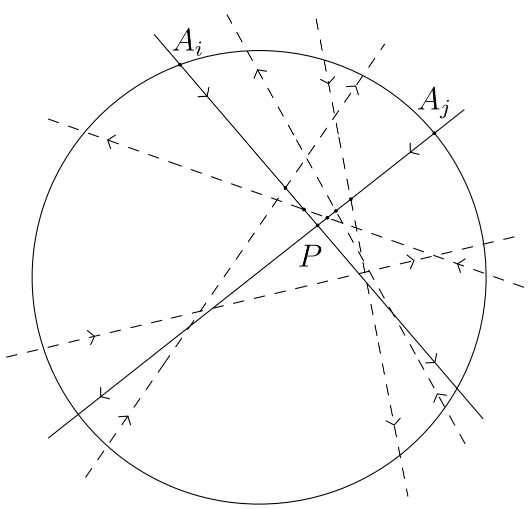
C7. Let $n \geqslant 2$ be an integer. In the plane, there are n segments given in such a way that any two segments have an intersection point in the interior, and no three segments intersect at a single point. Jeff places a snail at one of the endpoints of each of the segments and claps his hands $n-1$ times. Each time when he claps his hands, all the snails move along their own segments and stay at the next intersection points until the next clap. Since there are $n-1$ intersection points on each segment, all snails will reach the furthest intersection points from their starting points after $n-1$ claps.

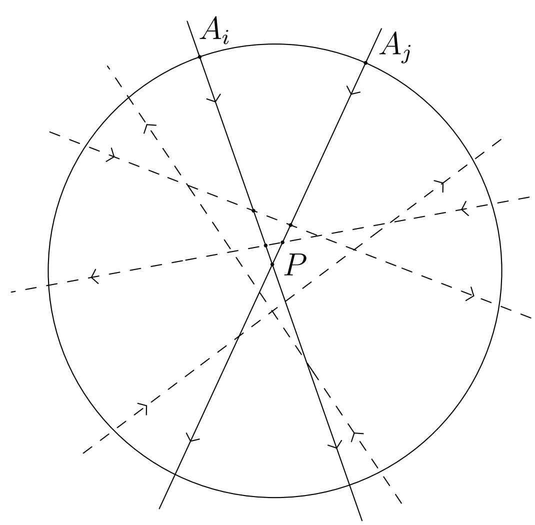
(a) Prove that if n is odd then Jeff can always place the snails so that no two of them ever occupy the same intersection point.

(b) Prove that if n is even then there must be a moment when some two snails occupy the same intersection point no matter how Jeff places the snails.

Solution. We consider a big disk which contains all the segments. We extend each segment to a line l_i so that each of them cuts the disk at two distinct points A_i, B_i.

(a) For odd n, we travel along the circumference of the disk and mark each of the points A_i or B_i 'in' and 'out' alternately. Since every pair of lines intersect in the disk, there are exactly $n-1$ points between A_i and B_i for any fixed $1 \leqslant i \leqslant n$. As n is odd, this means one of A_i and B_i is marked 'in' and the other is marked 'out'. Then Jeff can put a snail on the endpoint of each segment which is closer to the 'in' side of the corresponding line. We claim that the snails on l_i and l_j do not meet for any pairs i, j, hence proving part (a).

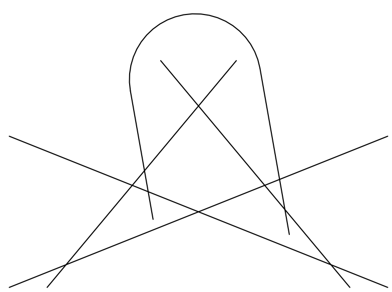


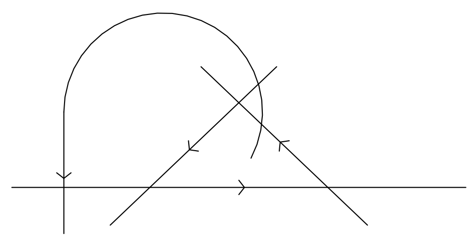


Without loss of generality, we may assume the snails start at A_i and A_j respectively. Let l_i intersect l_j at P. Note that there is an odd number of points between arc A_iA_j. Each of these points belongs to a line l_k. Such a line l_k must intersect exactly one of the segments A_iP and A_jP, making an odd number of intersections. For the other lines, they may intersect both segments A_iP and A_jP, or meet none of them. Therefore, the total number of intersection points on segments A_iP and A_jP (not counting P) is odd. However, if the snails arrive at P at the same time, then there should be the same number of intersections on A_iP and A_jP, which gives an even number of intersections. This is a contradiction so the snails do not meet each other.

(b) For even n, we consider any way that Jeff places the snails and mark each of the points A_i or B_i 'in' and 'out' according to the directions travelled by the snails. In this case there must be two neighbouring points A_i and A_j both of which are marked 'in'. Let P be the intersection of the segments A_iB_i and A_jB_j. Then any other segment meeting one of the segments A_iP and A_jP must also meet the other one, and so the number of intersections on A_iP and A_jP are the same. This shows the snails starting from A_i and A_j will meet at P.

Comment. The conclusions do not hold for pseudosegments, as can be seen from the following examples.



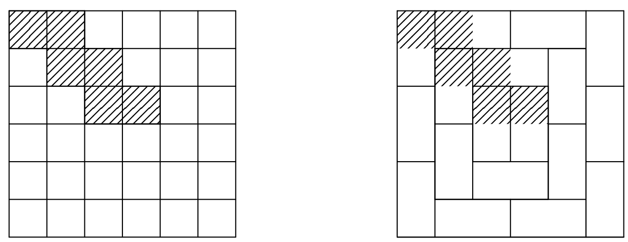


C8. Let n be a positive integer. Determine the smallest positive integer k with the following property: it is possible to mark k cells on a $2n \times 2n$ board so that there exists a unique partition of the board into 1×2 and 2×1 dominoes, none of which contains two marked cells.

Answer. $2n$.

Solution. We first construct an example of marking $2n$ cells satisfying the requirement. Label the rows and columns $1, 2, \ldots, 2n$ and label the cell in the i-th row and the j-th column (i, j).

For $i = 1, 2, \ldots, n$, we mark the cells (i, i) and $(i, i+1)$. We claim that the required partition exists and is unique. The two diagonals of the board divide the board into four regions. Note that the domino covering cell $(1, 1)$ must be vertical. This in turn shows that each domino covering $(2, 2), (3, 3), \ldots, (n, n)$ is vertical. By induction, the dominoes in the left region are all vertical. By rotational symmetry, the dominoes in the bottom region are horizontal, and so on. This shows that the partition exists and is unique.



It remains to show that this value of k is the smallest possible. Assume that only $k < 2n$ cells are marked, and there exists a partition P satisfying the requirement. It suffices to show there exists another desirable partition distinct from P. Let d be the main diagonal of the board.

Construct the following graph with edges of two colours. Its vertices are the cells of the board. Connect two vertices with a red edge if they belong to the same domino of P. Connect two vertices with a blue edge if their reflections in d are connected by a red edge. It is possible that two vertices are connected by edges of both colours. Clearly, each vertex has both red and blue degrees equal to 1. Thus the graph splits into cycles where the colours of edges in each cycle alternate (a cycle may have length 2).

Consider any cell c lying on the diagonal d. Its two edges are symmetrical with respect to d. Thus they connect c to different cells. This shows c belongs to a cycle $C(c)$ of length at least 4. Consider a part of this cycle $c_0, c_1, \ldots, c_m$ where $c_0 = c$ and m is the least positive integer such that c_m lies on d. Clearly, c_m is distinct from c. From the construction, the path symmetrical to this with respect to d also lies in the graph, and so these paths together form $C(c)$. Hence, $C(c)$ contains exactly two cells from d. Then all $2n$ cells in d belong to n cycles $C_1, C_2, \ldots, C_n$, each has length at least 4.

By the Pigeonhole Principle, there exists a cycle C_i containing at most one of the k marked cells. We modify P as follows. We remove all dominoes containing the vertices of C_i, which correspond to the red edges of C_i. Then we put the dominoes corresponding to the blue edges of C_i. Since C_i has at least 4 vertices, the resultant partition P' is different from P. Clearly, no domino in P' contains two marked cells as C_i contains at most one marked cell. This shows the partition is not unique and hence k cannot be less than $2n$.

Geometry

G1. In a convex pentagon $ABCDE$, let F be a point on AC such that $\angle FBC = 90^\circ$. Suppose triangles ABF, ACD and ADE are similar isosceles triangles with

$$\angle FAB = \angle FBA = \angle DAC = \angle DCA = \angle EAD = \angle EDA. \tag{1}$$

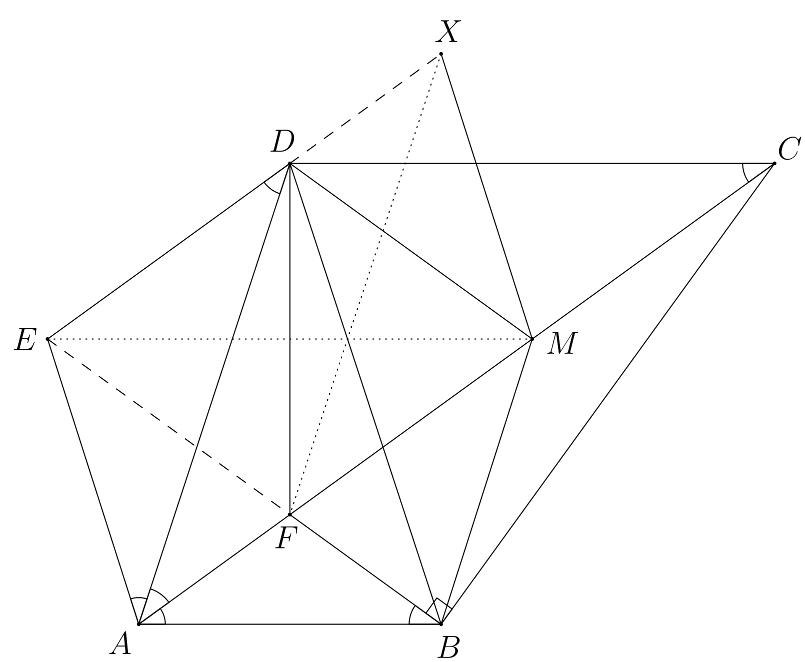
Let M be the midpoint of CF. Point X is chosen such that $AMXE$ is a parallelogram. Show that BD, EM and FX are concurrent.

Solution 1. Denote the common angle in (1) by θ. As $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. From $EA = ED$, we get

$$\angle AFD = \angle ABC = 90^\circ + \theta = 180^\circ - \frac{1}{2}\angle AED.$$

Hence, F lies on the circle with centre E and radius EA. In particular, $EF = EA = ED$. As $\angle EFA = \angle EAF = 2\theta = \angle BFC$, points B, F, E are collinear.

As $\angle EDA = \angle MAD$, we have $ED//AM$ and hence E, D, X are collinear. As M is the midpoint of CF and $\angle CBF = 90^\circ$, we get $MF = MB$. In the isosceles triangles EFA and MFB, we have $\angle EFA = \angle MFB$ and $AF = BF$. Therefore, they are congruent to each other. Then we have $BM = AE = XM$ and $BE = BF + FE = AF + FM = AM = EX$. This shows $\triangle EMB \cong \triangle EMX$. As F and D lie on EB and EX respectively and $EF = ED$, we know that lines BD and XF are symmetric with respect to EM. It follows that the three lines are concurrent.



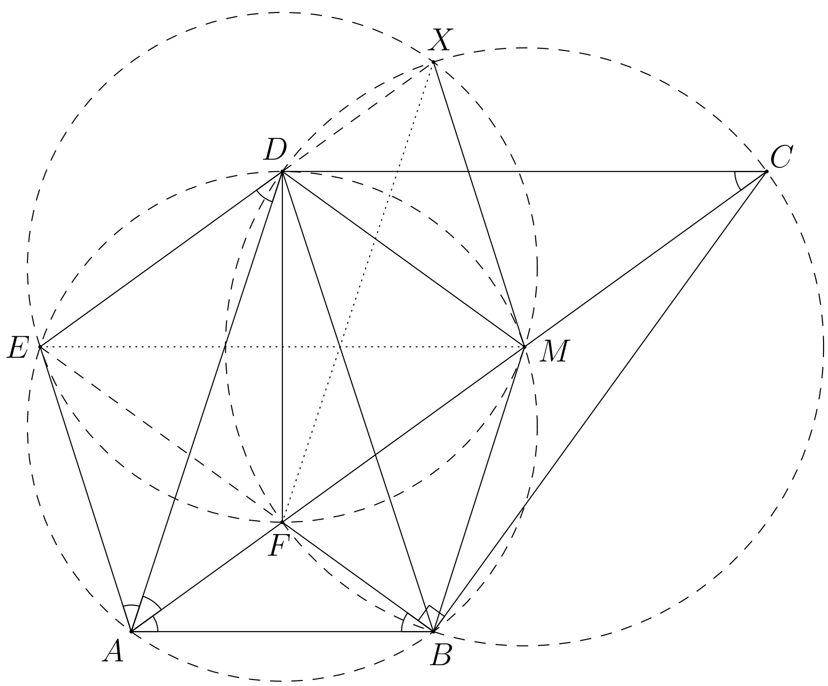
Solution 2. From $\angle CAD = \angle EDA$, we have $AC // ED$. Together with $AC // EX$, we know that E, D, X are collinear. Denote the common angle in (1) by θ. From $\triangle ABF \sim \triangle ACD$, we get $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. This yields $\angle AFD = \angle ABC = 90^\circ + \theta$ and hence $\angle FDC = 90^\circ$, implying that $BCDF$ is cyclic. Let Γ_1 be its circumcircle.

Next, from $\triangle ABF \sim \triangle ADE$, we have $\frac{AB}{AD} = \frac{AF}{AE}$ so that $\triangle ABD \sim \triangle AFE$. Therefore,

$$\angle AFE = \angle ABD = \theta + \angle FBD = \theta + \angle FCD = 2\theta = 180^\circ - \angle BFA.$$

This implies B, F, E are collinear. Note that F is the incentre of triangle DAB. Point E lies on the internal angle bisector of $\angle DBA$ and lies on the perpendicular bisector of AD. It follows that E lies on the circumcircle Γ_2 of triangle ABD, and $EA = EF = ED$.

Also, since CF is a diameter of Γ_1 and M is the midpoint of CF, M is the centre of Γ_1 and hence $\angle AMD = 2\theta = \angle ABD$. This shows M lies on Γ_2. Next, $\angle MDX = \angle MAE = \angle DXM$ since $AMXE$ is a parallelogram. Hence $MD = MX$ and X lies on Γ_1.



We now have two ways to complete the solution.

- **Method 1.** From $EF = EA = XM$ and $EX // FM$, $EFMX$ is an isosceles trapezoid and is cyclic. Denote its circumcircle by Γ_3. Since BD, EM, FX are the three radical axes of $\Gamma_1, \Gamma_2, \Gamma_3$, they must be concurrent.

- **Method 2.** As $\angle DMF = 2\theta = \angle BFM$, we have $DM // EB$. Also,

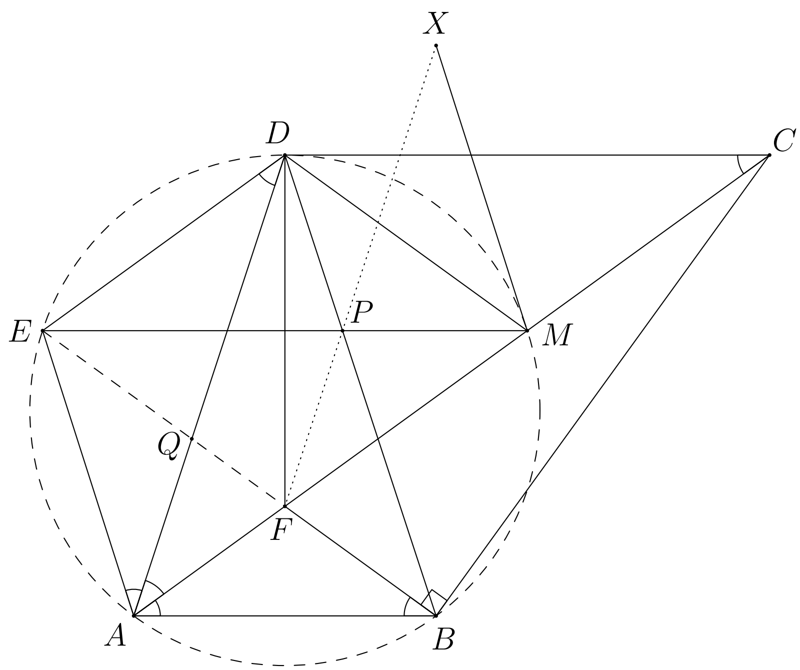
$$\angle BFD + \angle XBF = \angle BFC + \angle CFD + 90^\circ - \angle CBX = 2\theta + (90^\circ - \theta) + 90^\circ - \theta = 180^\circ$$

implies $DF // XB$. These show the corresponding sides of triangles DMF and BEX are parallel. By Desargues' Theorem, the two triangles are perspective and hence DB, ME, FX meet at a point.

Comment. In Solution 2, both the Radical Axis Theorem and Desargues' Theorem could imply DB, ME, FX are parallel. However, this is impossible as can be seen from the configuration. For example, it is obvious that DB and ME meet each other.

Solution 3. Let the common angle in (1) be θ. From $\triangle ABF \sim \triangle ACD$, we have $\frac{AB}{AC} = \frac{AF}{AD}$ so that $\triangle ABC \sim \triangle AFD$. Then $\angle ADF = \angle ACB = 90^\circ - 2\theta = 90^\circ - \angle BAD$ and hence $DF \perp AB$. As $FA = FB$, this implies $\triangle DAB$ is isosceles with $DA = DB$. Then F is the incentre of $\triangle DAB$.

Next, from $\angle AED = 180^\circ - 2\theta = 180^\circ - \angle DBA$, points A, B, D, E are concyclic. Since we also have $EA = ED$, this shows E, F, B are collinear and $EA = EF = ED$.



Note that C lies on the internal angle bisector of $\angle BAD$ and lies on the external angle bisector of $\angle DBA$. It follows that it is the A-excentre of triangle DAB. As M is the midpoint of CF, M lies on the circumcircle of triangle DAB and it is the centre of the circle passing through D, F, B, C. By symmetry, $DEFM$ is a rhombus. Then the midpoints of AX, EM and DF coincide, and it follows that $DAFX$ is a parallelogram.

Let P be the intersection of BD and EM, and Q be the intersection of AD and BE. From $\angle BAC = \angle DCA$, we know that DC, AB, EM are parallel. Thus we have $\frac{DP}{PB} = \frac{CM}{MA}$. This is further equal to $\frac{AE}{BE}$ since $CM = DM = DE = AE$ and $MA = BE$. From $\triangle AEQ \sim \triangle BEA$, we find that

$$\frac{DP}{PB} = \frac{AE}{BE} = \frac{AQ}{BA} = \frac{QF}{FB}$$

by the Angle Bisector Theorem. This implies $QD // FP$ and hence F, P, X are collinear, as desired.

G2. Let ABC be a triangle with circumcircle Γ and incentre I. Let M be the midpoint of side BC. Denote by D the foot of perpendicular from I to side BC. The line through I perpendicular to AI meets sides AB and AC at F and E respectively. Suppose the circumcircle of triangle AEF intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Solution 1. Let AM meet Γ again at Y and XY meet BC at D'. It suffices to show $D' = D$. We shall apply the following fact.

- **Claim.** For any cyclic quadrilateral $PQRS$ whose diagonals meet at T, we have

$$\frac{QT}{TS} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

Proof. We use $[W_1W_2W_3]$ to denote the area of $W_1W_2W_3$. Then

$$\frac{QT}{TS} = \frac{[PQR]}{[PSR]} = \frac{\frac{1}{2}PQ \cdot QR \sin\angle PQR}{\frac{1}{2}PS \cdot SR \sin\angle PSR} = \frac{PQ \cdot QR}{PS \cdot SR}.$$

□

Applying the Claim to $ABYC$ and $XBYC$ respectively, we have $1 = \frac{BM}{MC} = \frac{AB \cdot BY}{AC \cdot CY}$ and $\frac{BD'}{D'C} = \frac{XB \cdot BY}{XC \cdot CY}$. These combine to give

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{BY}{CY} = \frac{XB}{XC} \cdot \frac{AC}{AB}. \tag{1}$$

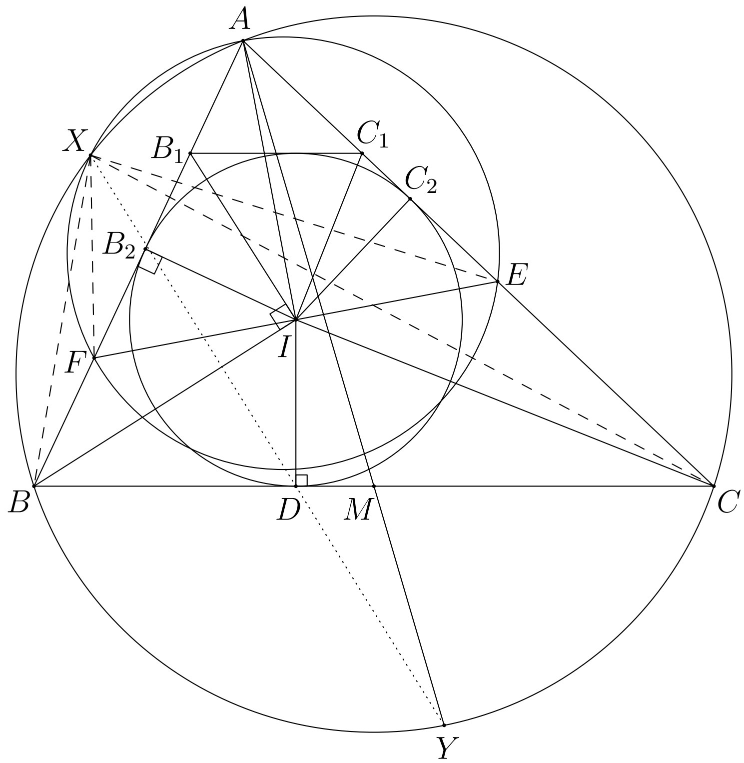
Next, we use directed angles to find that $\measuredangle XBF = \measuredangle XBA = \measuredangle XCA = \measuredangle XCE$ and $\measuredangle XFB = \measuredangle XFA = \measuredangle XEA = \measuredangle XEC$. This shows triangles XBF and XCE are directly similar. In particular, we have

$$\frac{XB}{XC} = \frac{BF}{CE}. \tag{2}$$

In the following, we give two ways to continue the proof.

- **Method 1.** Here is a geometrical method. As $\angle FIB = \angle AIB - 90° = \frac{1}{2}\angle ACB = \angle ICB$ and $\angle FBI = \angle IBC$, the triangles FBI and IBC are similar. Analogously, triangles EIC and IBC are also similar. Hence, we get

$$\frac{FB}{IB} = \frac{BI}{BC} \quad \text{and} \quad \frac{EC}{IC} = \frac{IC}{BC}. \tag{3}$$



Next, construct a line parallel to BC and tangent to the incircle. Suppose it meets sides AB and AC at B_1 and C_1 respectively. Let the incircle touch AB and AC at B_2 and C_2 respectively. By homothety, the line B_1I is parallel to the external angle bisector of $\angle ABC$, and hence $\angle B_1IB = 90^\circ$. Since $\angle BB_2I = 90^\circ$, we get $BB_2 \cdot BB_1 = BI^2$, and similarly $CC_2 \cdot CC_1 = CI^2$. Hence,

$$\frac{BI^2}{CI^2} = \frac{BB_2 \cdot BB_1}{CC_2 \cdot CC_1} = \frac{BB_1}{CC_1} \cdot \frac{BD}{CD} = \frac{AB}{AC} \cdot \frac{BD}{CD}. \tag{4}$$

Combining (1), (2), (3) and (4), we conclude

$$\frac{BD'}{CD'} = \frac{XB}{XC} \cdot \frac{AC}{AB} = \frac{BF}{CE} \cdot \frac{AC}{AB} = \frac{BI^2}{CI^2} \cdot \frac{AC}{AB} = \frac{BD}{CD}$$

so that $D' = D$. The result then follows.

• **Method 2.** We continue the proof of Solution 1 using trigonometry. Let $\beta = \frac{1}{2}\angle ABC$ and $\gamma = \frac{1}{2}\angle ACB$. Observe that $\angle FIB = \angle AIB - 90^\circ = \gamma$. Hence, $\frac{BF}{FI} = \frac{\sin \angle FIB}{\sin \angle IBF} = \frac{\sin\gamma}{\sin\beta}$. Similarly, $\frac{CE}{EI} = \frac{\sin\beta}{\sin\gamma}$. As $FI = EI$, we get

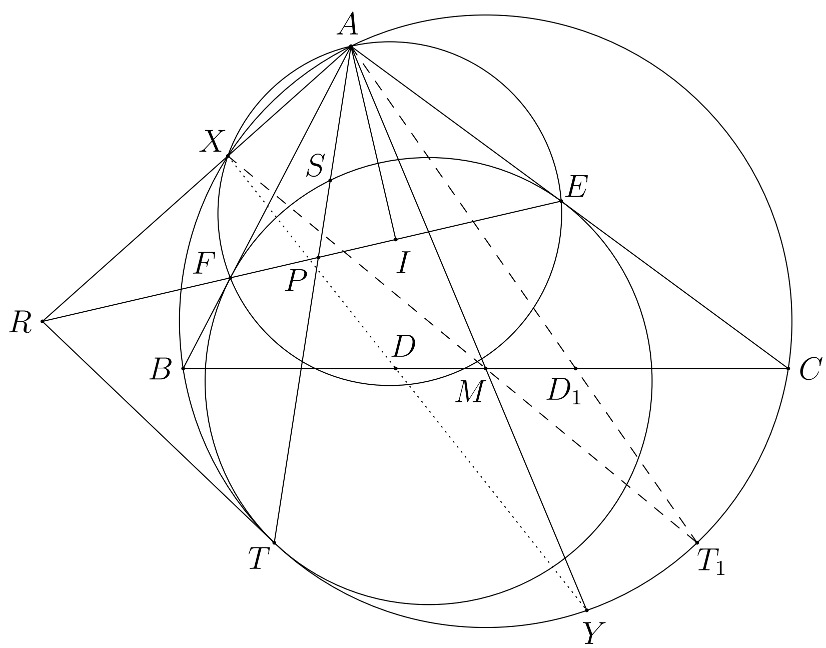
$$\frac{BF}{CE} = \frac{BF}{FI} \cdot \frac{EI}{CE} = \left(\frac{\sin\gamma}{\sin\beta}\right)^2. \tag{5}$$

Together with (1) and (2), we find that

$$\frac{BD'}{CD'} = \frac{AC}{AB} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\sin 2\beta}{\sin 2\gamma} \cdot \left(\frac{\sin\gamma}{\sin\beta}\right)^2 = \frac{\tan\gamma}{\tan\beta} = \frac{ID/CD}{ID/BD} = \frac{BD}{CD}.$$

This shows $D' = D$ and the result follows.

Solution 2. Let ω_A be the A-mixtilinear incircle of triangle ABC. From the properties of mixtilinear incircles, ω_A touches sides AB and AC at F and E respectively. Suppose ω_A is tangent to Γ at T. Let AM meet Γ again at Y, and let D_1, T_1 be the reflections of D and T with respect to the perpendicular bisector of BC respectively. It is well-known that $\angle BAT = \angle D_1AC$ so that A, D_1, T_1 are collinear.



We then show that X, M, T_1 are collinear. Let R be the radical centre of ω_A, Γ and the circumcircle of triangle AEF. Then R lies on AX, EF and the tangent at T to Γ. Let AT meet ω_A again at S and meet EF at P. Obviously, $SFTE$ is a harmonic quadrilateral. Projecting from T, the pencil $(R, P; F, E)$ is harmonic. We further project the pencil onto Γ from A, so that $XBTC$ is a harmonic quadrilateral. As $TT_1 // BC$, the projection from T_1 onto BC maps T to a point at infinity, and hence maps X to the midpoint of BC, which is M. This shows X, M, T_1 are collinear.

We have two ways to finish the proof.

• **Method 1.** Note that both AY and XT_1 are chords of Γ passing through the midpoint M of the chord BC. By the Butterfly Theorem, XY and AT_1 cut BC at a pair of symmetric points with respect to M, and hence X, D, Y are collinear. The proof is thus complete.

• **Method 2.** Here, we finish the proof without using the Butterfly Theorem. As DTT_1D_1 is an isosceles trapezoid, we have

$$\measuredangle YTD = \measuredangle YTT_1 + \measuredangle T_1TD = \measuredangle YAT_1 + \measuredangle AD_1D = \measuredangle YMD$$

so that D, T, Y, M are concyclic. As X, M, T_1 are collinear, we have

$$\measuredangle AYD = \measuredangle MTD = \measuredangle D_1T_1M = \measuredangle AT_1X = \measuredangle AYX.$$

This shows X, D, Y are collinear.

G3. Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

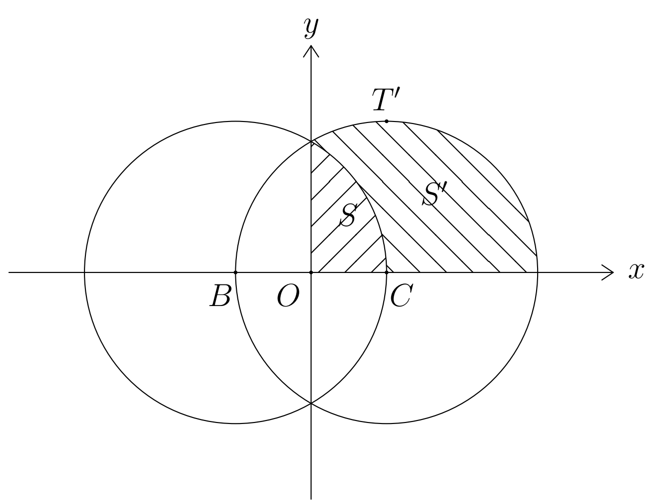
(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1, 2, 3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x, y) : x \geqslant 0, y \geqslant 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

Solution. If in the similarity of $\triangle ABC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the longest side of $\triangle P_1P_2P_3$, then we have $BC \geqslant AB \geqslant AC$. The condition $BC \geqslant AB$ is equivalent to $(x + 1)^2 + y^2 \leqslant 4$, while $AB \geqslant AC$ is trivially satisfied for any point in the first quadrant. Then we first define

$$S = \{(x, y) : (x + 1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

Note that S is the intersection of a disk and the first quadrant, so it is bounded and convex, and we can choose any $T \in S$ to meet condition (i). For any point A in S, the relation $BC \geqslant AB \geqslant AC$ always holds. Therefore, the point A in (ii) is uniquely determined, while its existence is guaranteed by the above construction.



Next, if in the similarity of $\triangle A'BC$ and $\triangle P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$, BC corresponds to the second longest side of $\triangle P_1P_2P_3$, then we have $A'B \geqslant BC \geqslant A'C$. The two inequalities are equivalent to $(x + 1)^2 + y^2 \geqslant 4$ and $(x - 1)^2 + y^2 \leqslant 4$ respectively. Then we define

$$S' = \{(x, y) : (x + 1)^2 + y^2 \geqslant 4, (x - 1)^2 + y^2 \leqslant 4, x \geqslant 0, y \geqslant 0\}.$$

The boundedness condition is satisfied while (ii) can be argued as in the previous case. For (i), note that S' contains points inside the disk $(x-1)^2+y^2 \leqslant 4$ and outside the disk $(x+1)^2+y^2 \geqslant 4$. This shows we can take $T'=(1,2)$ in (i), which is the topmost point of the circle $(x-1)^2+y^2=4$.

It remains to check that the product $BA \cdot BA'$ is a constant. Suppose we are given a triangle $P_1P_2P_3$ with $P_1P_2 \geqslant P_2P_3 \geqslant P_3P_1$. By the similarity, we have

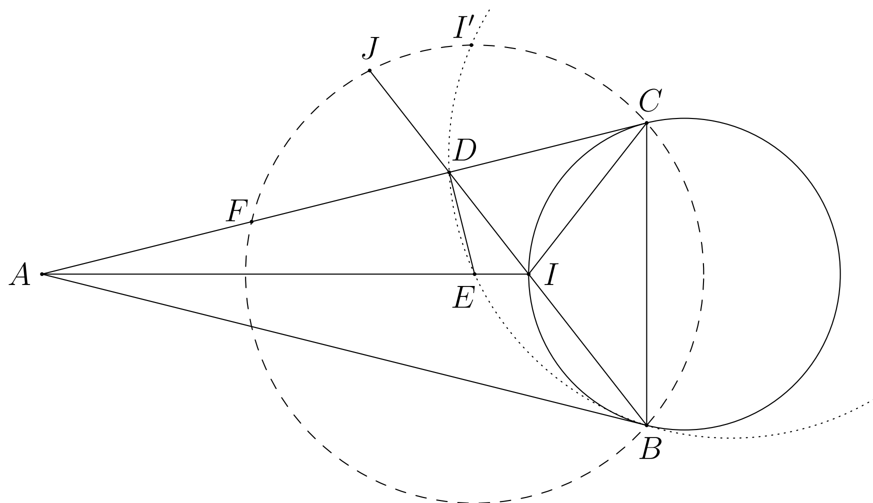
$$BA = BC \cdot \frac{P_2P_3}{P_1P_2} \quad \text{and} \quad BA' = BC \cdot \frac{P_1P_2}{P_2P_3}.$$

Thus $BA \cdot BA' = BC^2 = 4$, which is certainly independent of the triangle $P_1P_2P_3$.

Comment. The original version of this problem includes the condition that the interiors of S and S' are disjoint. We remove this condition since it is hard to define the interior of a point set rigorously.

G4. Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

Solution 1.



Let Γ be the circle with centre E passing through B and C. Since $ED \perp AC$, the point F symmetric to C with respect to D lies on Γ. From $\angle DCI = \angle ICB = \angle CBI$, the line DC is a tangent to the circumcircle of triangle IBC. Let J be the symmetric point of I with respect to D. Using directed lengths, from

$$DC \cdot DF = -DC^2 = -DI \cdot DB = DJ \cdot DB,$$

the point J also lies on Γ. Let I' be the reflection of I in AC. Since IJ and CF bisect each other, $CJFI$ is a parallelogram. From $\angle FI'C = \angle CIF = \angle FJC$, we find that I' lies on Γ. This gives $EI' = EB$.

Note that AC is the internal angle bisector of $\angle BDI'$. This shows DE is the external angle bisector of $\angle BDI'$ as $DE \perp AC$. Together with $EI' = EB$, it is well-known that E lies on the circumcircle of triangle BDI'.

Solution 2. Let I' be the reflection of I in AC and let S be the intersection of $I'C$ and AI. Using directed angles, we let $\theta = \measuredangle ACI = \measuredangle ICB = \measuredangle CBI$. We have

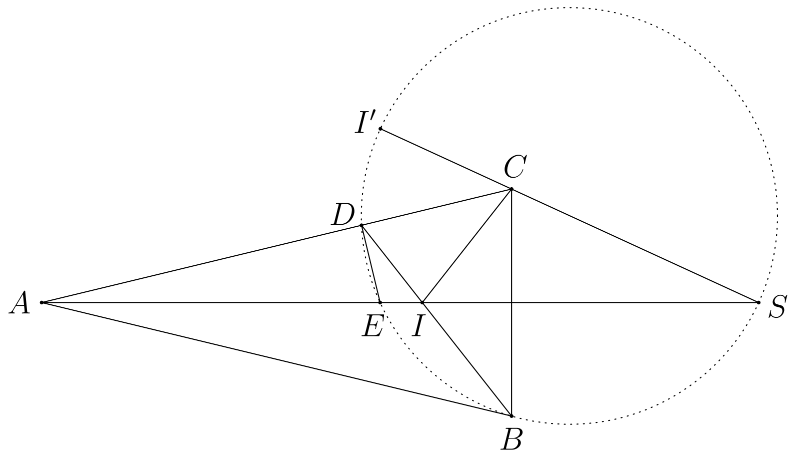
$$\measuredangle I'SE = \measuredangle I'CA + \measuredangle CAI = \theta + \left(\frac{\pi}{2} + 2\theta\right) = 3\theta + \frac{\pi}{2}$$

and

$$\measuredangle I'DE = \measuredangle I'DC + \frac{\pi}{2} = \measuredangle CDI + \frac{\pi}{2} = \measuredangle DCB + \measuredangle CBD + \frac{\pi}{2} = 3\theta + \frac{\pi}{2}.$$

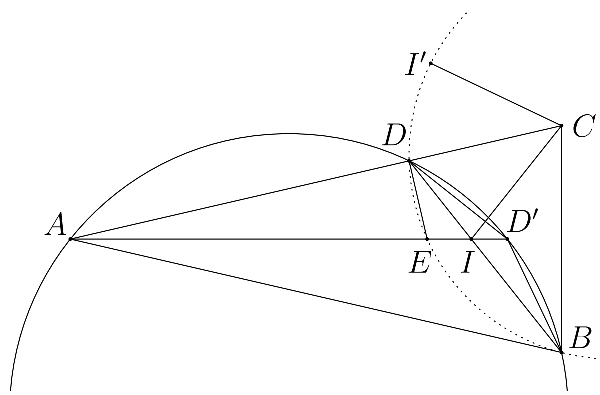
This shows I', D, E, S are concyclic.

Next, we find $\measuredangle I'SB = 2\measuredangle I'SE = 6\theta$ and $\measuredangle I'DB = 2\measuredangle CDI = 6\theta$. Therefore, I', D, B, S are concyclic so that I', D, E, B, S lie on the same circle. The result then follows.



Comment. The point S constructed in Solution 2 may lie on the same side as A of BC. Also, since S lies on the circumcircle of the non-degenerate triangle BDE, we implicitly know that S is not an ideal point. Indeed, one can verify that $I'C$ and AI are parallel if and only if triangle ABC is equilateral.

Solution 3. Let I' be the reflection of I in AC, and let D' be the second intersection of AI and the circumcircle of triangle ABD. Since AD' bisects $\angle BAD$, point D' is the midpoint of the arc BD and $DD' = BD' = CD'$. Obviously, A, E, D' lie on AI in this order.



We find that $\angle ED'D = \angle AD'D = \angle ABD = \angle IBC = \angle ICB$. Next, since D' is the circumcentre of triangle BCD, we have $\angle EDD' = 90^\circ - \angle D'DC = \angle CBD = \angle IBC$. The two relations show that triangles $ED'D$ and ICB are similar. Therefore, we have

$$\frac{BC}{CI'} = \frac{BC}{CI} = \frac{DD'}{D'E} = \frac{BD'}{D'E}.$$

Also, we get

$$\angle BCI' = \angle BCA + \angle ACI' = \angle BCA + \angle ICA = \angle BCA + \angle DBC = \angle BDA = \angle BD'E.$$

These show triangles BCI' and $BD'E$ are similar, and hence triangles BCD' and $BI'E$ are similar. As BCD' is isosceles, we obtain $BE = I'E$.

As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.

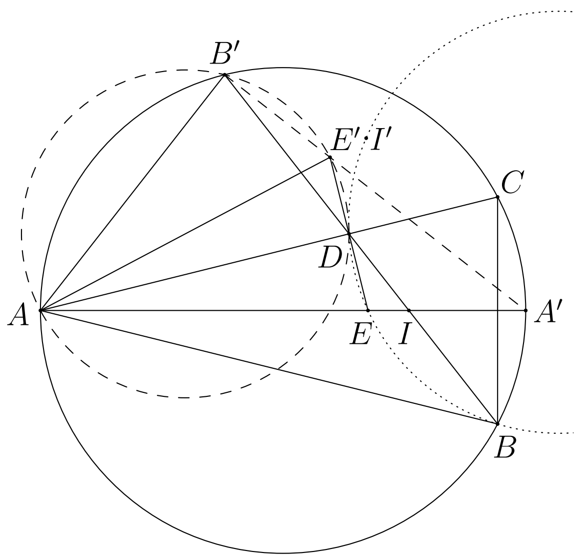
<u>Solution 4</u>. Let AI and BI meet the circumcircle of triangle ABC again at A' and B' respectively, and let E' be the reflection of E in AC. From

$$\begin{aligned}\angle B'AE' = \angle B'AD - \angle E'AD &= \frac{\angle ABC}{2} - \frac{\angle BAC}{2} = 90^\circ - \angle BAC - \frac{\angle ABC}{2}\\ &= 90^\circ - \angle B'DA = \angle B'DE',\end{aligned}$$

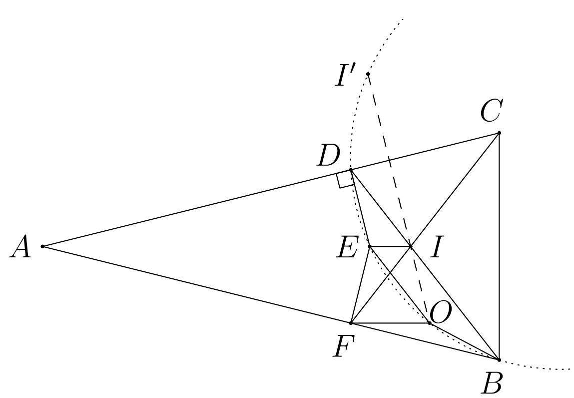
points B', A, D, E' are concyclic. Then

$$\angle DB'E' = \angle DAE' = \frac{\angle BAC}{2} = \angle BAA' = \angle DB'A'$$

and hence B', E', A' are collinear. It is well-known that $A'B'$ is the perpendicular bisector of CI, so that $CE' = IE'$. Let I' be the reflection of I in AC. This implies $BE = CE = I'E$. As DE is the external angle bisector of $\angle BDI'$ and $EI' = EB$, we know that E lies on the circumcircle of triangle BDI'.



<u>Solution 5</u>. Let F be the intersection of CI and AB. Clearly, F and D are symmetric with respect to AI. Let O be the circumcentre of triangle BIF, and let I' be the reflection of I in AC.



From $\angle BFO = 90^\circ - \angle FIB = \frac{1}{2}\angle BAC = \angle BAI$, we get $EI//FO$. Also, from the relation $\angle OIB = 90^\circ - \angle BFI = 90^\circ - \angle CDI = \angle I'ID$, we know that O, I, I' are collinear.

Note that $ED//OI$ since both are perpendicular to AC. Then $\angle FEI = \angle DEI = \angle OIE$. Together with $EI//FO$, the quadrilateral $EFOI$ is an isosceles trapezoid. Therefore, we find that $\angle DIE = \angle FIE = \angle OEI$ so $OE//ID$. Then $DEOI$ is a parallelogram. Hence, we have $DI' = DI = EO$, which shows $DEOI'$ is an isosceles trapezoid. In addition, $ED = OI = OB$ and $OE//BD$ imply $EOBD$ is another isosceles trapezoid. In particular, both $DEOI'$ and $EOBD$ are cyclic. This shows B, D, E, I' are concyclic.

Solution 6. Let I' be the reflection of I in AC. Denote by T and M the projections from I to sides AB and BC respectively. Since BI is the perpendicular bisector of TM, we have

$$DT = DM. \tag{1}$$

Since $\angle ADE = \angle ATI = 90^\circ$ and $\angle DAE = \angle TAI$, we have $\triangle ADE \sim \triangle ATI$. This shows $\frac{AD}{AE} = \frac{AT}{AI} = \frac{AT}{AI'}$. Together with $\angle DAT = 2\angle DAE = \angle EAI'$, this yields $\triangle DAT \sim \triangle EAI'$. In particular, we have

$$\frac{DT}{EI'} = \frac{AT}{AI'} = \frac{AT}{AI}. \tag{2}$$

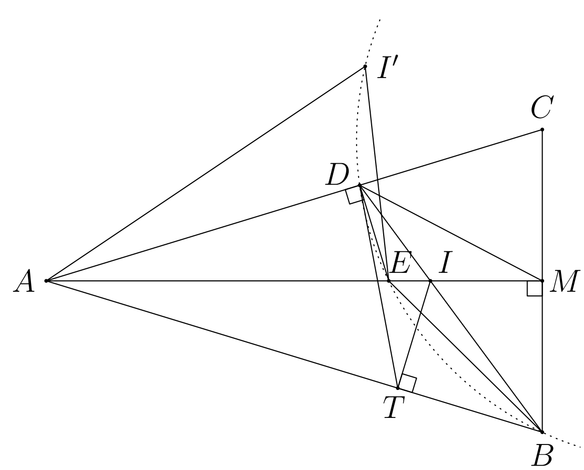
Obviously, the right-angled triangles AMB and ATI are similar. Then we get

$$\frac{AM}{AB} = \frac{AT}{AI}. \tag{3}$$

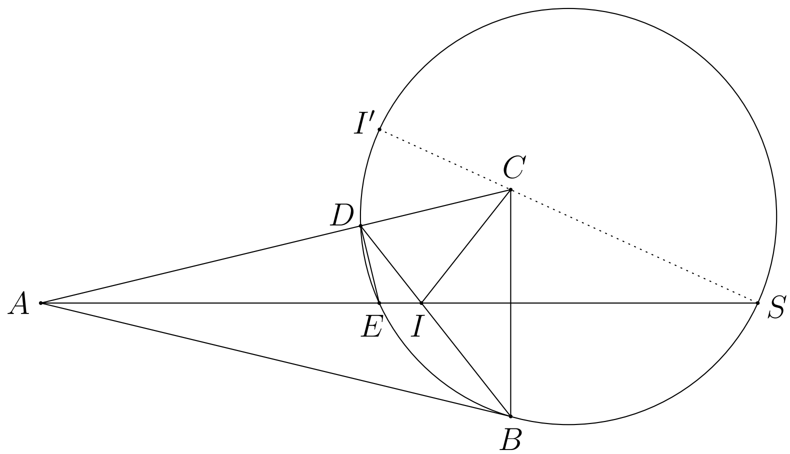
Next, from $\triangle AMB \sim \triangle ATI \sim \triangle ADE$, we have $\frac{AM}{AB} = \frac{AD}{AE}$ so that $\triangle ADM \sim \triangle AEB$. It follows that

$$\frac{DM}{EB} = \frac{AM}{AB}. \tag{4}$$

Combining (1), (2), (3) and (4), we get $EB = EI'$. As DE is the external angle bisector of $\angle BDI'$, we know that E lies on the circumcircle of triangle BDI'.



Comment. A stronger version of this problem is to ask the contestants to prove the reflection of I in AC lies on the circumcircle of triangle BDE if and only if $AB = AC$. Some of the above solutions can be modified to prove the converse statement to the original problem. For example, we borrow some ideas from Solution 2 to establish the converse as follows.



Let I' be the reflection of I in AC and suppose B, E, D, I' lie on a circle Γ. Let AI meet Γ again at S. As DE is the external angle bisector of $\angle BDI'$, we have $EB = EI'$. Using directed angles, we get

$$\measuredangle CI'S = \measuredangle CI'D + \measuredangle DI'S = \measuredangle DIC + \measuredangle DES = \measuredangle DIC + \measuredangle DEA = \measuredangle DIC + \measuredangle DCB = 0.$$

This means I', C, S are collinear. From this we get $\measuredangle BSE = \measuredangle ESI' = \measuredangle ESC$ and hence AS bisects both $\angle BAC$ and $\angle BSC$. Clearly, S and A are distinct points. It follows that $\triangle BAS \cong \triangle CAS$ and thus $AB = AC$.

As in some of the above solutions, an obvious way to prove the stronger version is to establish the following equivalence: $BE = EI'$ if and only if $AB = AC$. In addition to the ideas used in those solutions, one can use trigonometry to express the lengths of BE and EI' in terms of the side lengths of triangle ABC and to establish the equivalence.

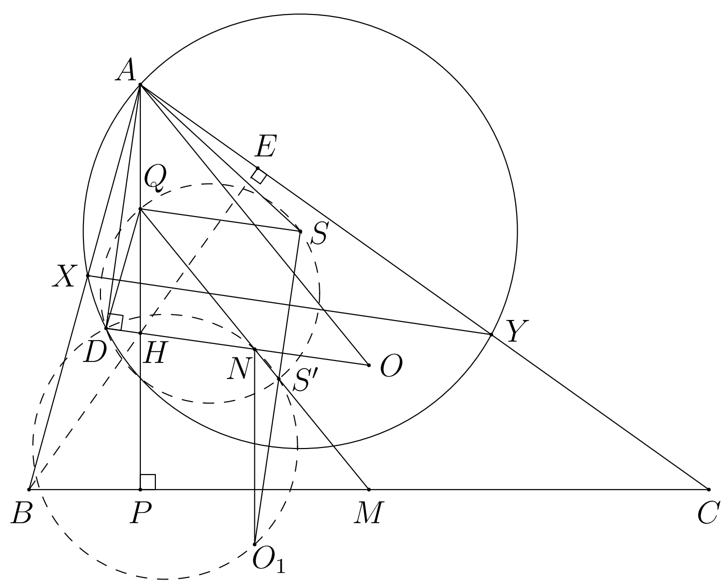
G5. Let D be the foot of perpendicular from A to the Euler line (the line passing through the circumcentre and the orthocentre) of an acute scalene triangle ABC. A circle ω with centre S passes through A and D, and it intersects sides AB and AC at X and Y respectively. Let P be the foot of altitude from A to BC, and let M be the midpoint of BC. Prove that the circumcentre of triangle XSY is equidistant from P and M.

__**Solution 1.**__ Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let Q be the midpoint of AH and N be the nine-point centre of triangle ABC. It is known that Q lies on the nine-point circle of triangle ABC, N is the midpoint of QM and that QM is parallel to AO.

Let the perpendicular from S to XY meet line QM at S'. Let E be the foot of altitude from B to side AC. Since Q and S lie on the perpendicular bisector of AD, using directed angles, we have

$$\begin{aligned}\measuredangle SDQ = \measuredangle QAS = \measuredangle XAS - \measuredangle XAQ &= \left(\frac{\pi}{2} - \measuredangle AYX\right) - \measuredangle BAP = \measuredangle CBA - \measuredangle AYX \\ &= (\measuredangle CBA - \measuredangle ACB) - \measuredangle BCA - \measuredangle AYX = \measuredangle PEM - (\measuredangle BCA + \measuredangle AYX) \\ &= \measuredangle PQM - \angle(BC, XY) = \frac{\pi}{2} - \angle(S'Q, BC) - \angle(BC, XY) = \measuredangle SS'Q.\end{aligned}$$

This shows D, S', S, Q are concyclic.



Let the perpendicular from N to BC intersect line SS' at O_1. (Note that the two lines coincide when S is the midpoint of AO, in which case the result is true since the circumcentre of triangle XSY must lie on this line.) It suffices to show that O_1 is the circumcentre of triangle XSY since N lies on the perpendicular bisector of PM. From

$$\measuredangle DS'O_1 = \measuredangle DQS = \measuredangle SQA = \angle(SQ, QA) = \angle(OD, O_1N) = \measuredangle DNO_1$$

since $SQ//OD$ and $QA//O_1N$, we know that D, O_1, S', N are concyclic. Therefore, we get

$$\measuredangle SDS' = \measuredangle SQS' = \angle(SQ, QS') = \angle(ND, NS') = \measuredangle DNS',$$

so that SD is a tangent to the circle through D, O_1, S', N. Then we have

$$SS' \cdot SO_1 = SD^2 = SX^2. \tag{1}$$

Next, we show that S and S' are symmetric with respect to XY. By the Sine Law, we have

$$\frac{SS'}{\sin\angle SQS'} = \frac{SQ}{\sin\angle SS'Q} = \frac{SQ}{\sin\angle SDQ} = \frac{SQ}{\sin\angle SAQ} = \frac{SA}{\sin\angle SQA}.$$

It follows that

$$SS' = SA \cdot \frac{\sin\angle SQS'}{\sin\angle SQA} = SA \cdot \frac{\sin\angle HOA}{\sin\angle OHA} = SA \cdot \frac{AH}{AO} = SA \cdot 2\cos A,$$

which is twice the distance from S to XY. Note that S and C lie on the same side of the perpendicular bisector of PM if and only if $\angle SAC < \angle OAC$ if and only if $\angle YXA > \angle CBA$. This shows S and O_1 lie on different sides of XY. As S' lies on ray SO_1, it follows that S and S' cannot lie on the same side of XY. Therefore, S and S' are symmetric with respect to XY.

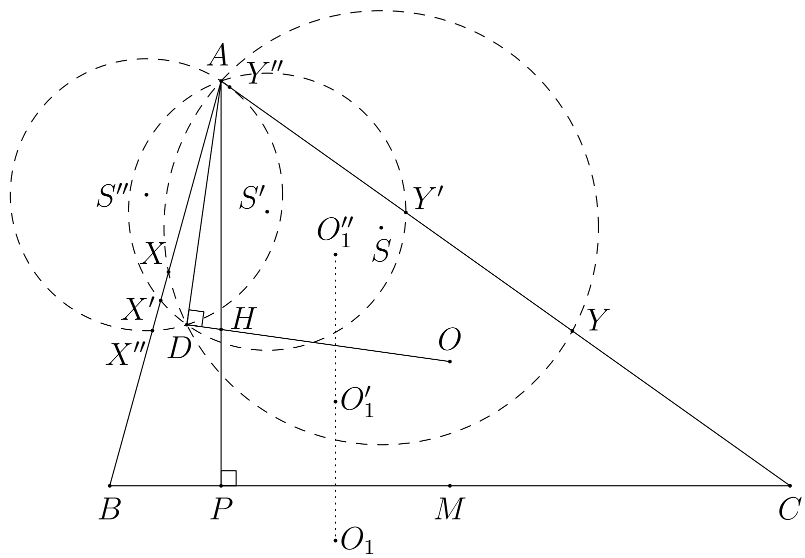
Let d be the diameter of the circumcircle of triangle XSY. As SS' is twice the distance from S to XY and $SX = SY$, we have $SS' = 2\frac{SX^2}{d}$. It follows from (1) that $d = 2SO_1$. As SO_1 is the perpendicular bisector of XY, point O_1 is the circumcentre of triangle XSY.

Solution 2. Denote the orthocentre and circumcentre of triangle ABC by H and O respectively. Let O_1 be the circumcentre of triangle XSY. Consider two other possible positions of S. We name them S' and S'' and define the analogous points $X', Y', O_1', X'', Y''O_1''$ accordingly. Note that S, S', S'' lie on the perpendicular bisector of AD.

As XX' and YY' meet at A and the circumcircles of triangles AXY and $AX'Y'$ meet at D, there is a spiral similarity with centre D mapping XY to $X'Y'$. We find that

$$\measuredangle SXY = \frac{\pi}{2} - \measuredangle YAX = \frac{\pi}{2} - \measuredangle Y'AX' = \measuredangle S'X'Y'$$

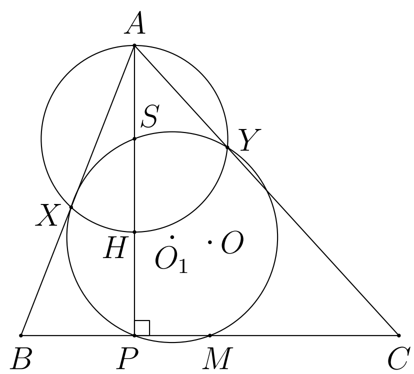
and similarly $\measuredangle SYX = \measuredangle S'Y'X'$. This shows triangles SXY and $S'X'Y'$ are directly similar. Then the spiral similarity with centre D takes points S, X, Y, O_1 to S', X', Y', O_1'. Similarly, there is a spiral similarity with centre D mapping S, X, Y, O_1 to S'', X'', Y'', O_1''. From these, we see that there is a spiral similarity taking the corresponding points S, S', S'' to points O_1, O_1', O_1''. In particular, O_1, O_1', O_1'' are collinear.

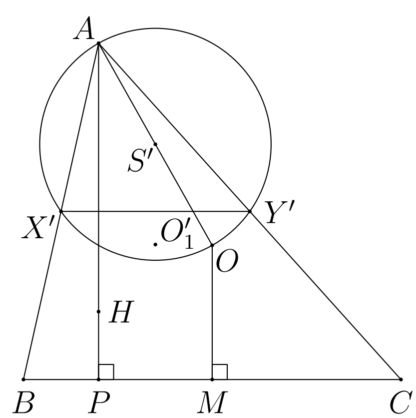


It now suffices to show that O_1 lies on the perpendicular bisector of PM for two special cases.

Firstly, we take S to be the midpoint of AH. Then X and Y are the feet of altitudes from C and B respectively. It is well-known that the circumcircle of triangle XSY is the nine-point circle of triangle ABC. Then O_1 is the nine-point centre and $O_1P = O_1M$. Indeed, P and M also lies on the nine-point circle.

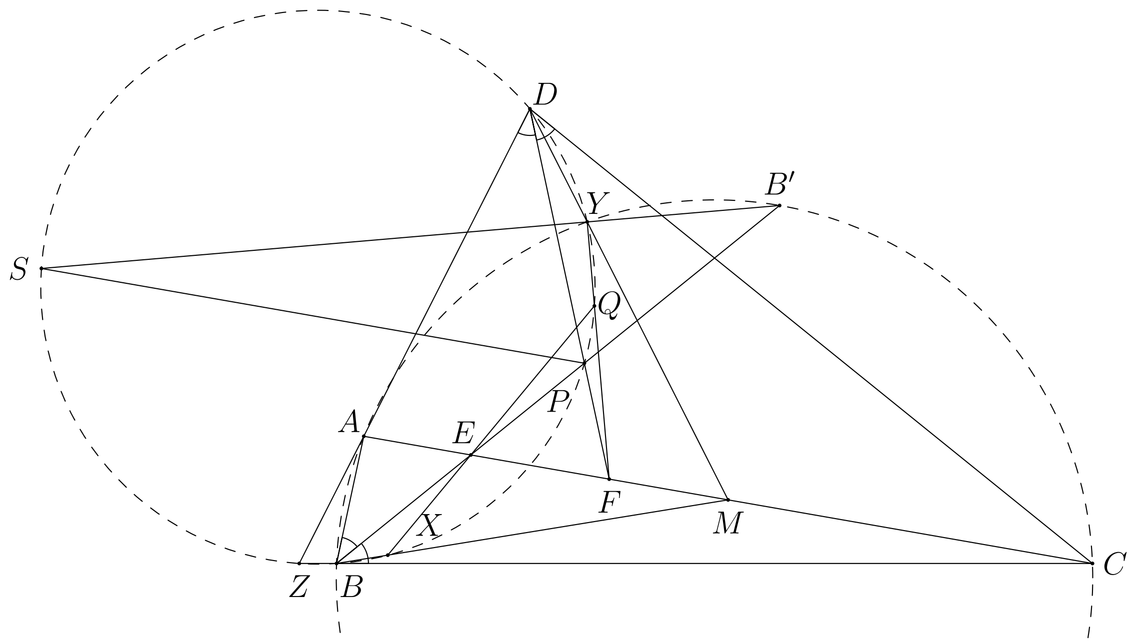
Secondly, we take S' to be the midpoint of AO. Then X' and Y' are the midpoints of AB and AC respectively. Then $X'Y'//BC$. Clearly, S' lies on the perpendicular bisector of PM. This shows the perpendicular bisectors of $X'Y'$ and PM coincide. Hence, we must have $O_1'P = O_1'M$.





G6. Let $ABCD$ be a convex quadrilateral with $\angle ABC = \angle ADC < 90^\circ$. The internal angle bisectors of $\angle ABC$ and $\angle ADC$ meet AC at E and F respectively, and meet each other at point P. Let M be the midpoint of AC and let ω be the circumcircle of triangle BPD. Segments BM and DM intersect ω again at X and Y respectively. Denote by Q the intersection point of lines XE and YF. Prove that $PQ \perp AC$.

Solution 1.



Let ω_1 be the circumcircle of triangle ABC. We first prove that Y lies on ω_1. Let Y' be the point on ray MD such that $MY' \cdot MD = MA^2$. Then triangles MAY' and MDA are oppositely similar. Since $MC^2 = MA^2 = MY' \cdot MD$, triangles MCY' and MDC are also oppositely similar. Therefore, using directed angles, we have

$$\measuredangle AY'C = \measuredangle AY'M + \measuredangle MY'C = \measuredangle MAD + \measuredangle DCM = \measuredangle CDA = \measuredangle ABC$$

so that Y' lies on ω_1.

Let Z be the intersection point of lines BC and AD. Since $\measuredangle PDZ = \measuredangle PBC = \measuredangle PBZ$, point Z lies on ω. In addition, from $\measuredangle Y'BZ = \measuredangle Y'BC = \measuredangle Y'AC = \measuredangle Y'AM = \measuredangle Y'DZ$, we also know that Y' lies on ω. Note that $\angle ADC$ is acute implies $MA \neq MD$ so $MY' \neq MD$. Therefore, Y' is the second intersection of DM and ω. Then $Y' = Y$ and hence Y lies on ω_1.

Next, by the Angle Bisector Theorem and the similar triangles, we have

$$\frac{FA}{FC} = \frac{AD}{CD} = \frac{AD}{AM} \cdot \frac{CM}{CD} = \frac{YA}{YM} \cdot \frac{YM}{YC} = \frac{YA}{YC}.$$

Hence, FY is the internal angle bisector of $\angle AYC$.

Let B' be the second intersection of the internal angle bisector of $\angle CBA$ and ω_1. Then B' is the midpoint of arc AC not containing B. Therefore, YB' is the external angle bisector of $\angle AYC$, so that $B'Y \perp FY$.

Denote by l the line through P parallel to AC. Suppose l meets line $B'Y$ at S. From

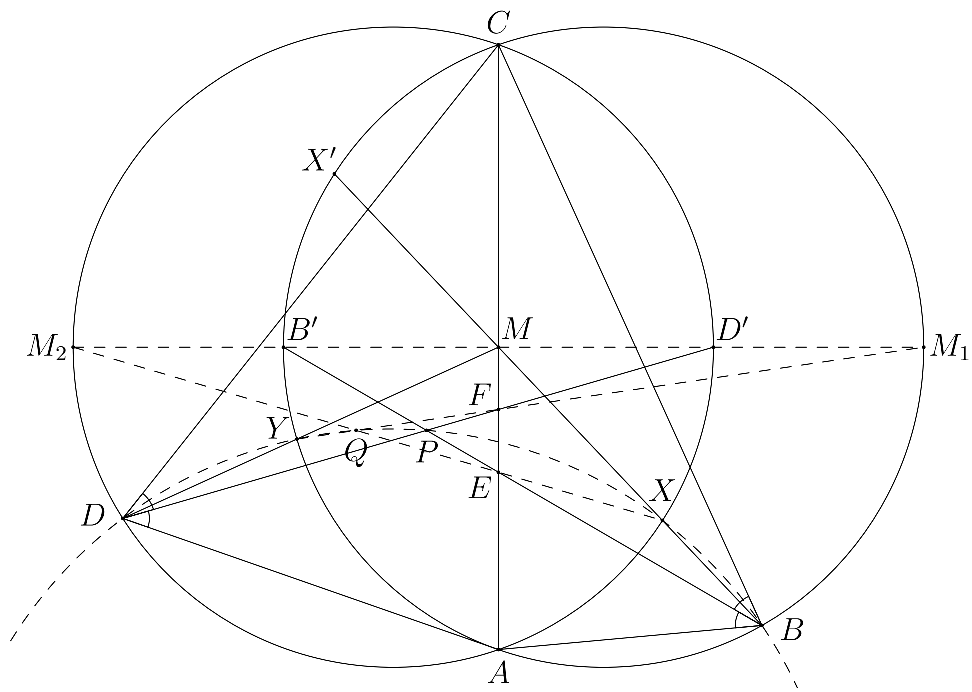
$$\begin{aligned}\measuredangle PSY &= \angle(AC, B'Y) = \measuredangle ACY + \measuredangle CYB' = \measuredangle ACY + \measuredangle CAB' = \measuredangle ACY + \measuredangle B'CA \\ &= \measuredangle B'CY = \measuredangle B'BY = \measuredangle PBY,\end{aligned}$$

the point S lies on ω. Similarly, the line through X perpendicular to XE also passes through the second intersection of l and ω, which is the point S. From $QY \perp YS$ and $QX \perp XS$, point Q lies on ω and QS is a diameter of ω. Therefore, $PQ \perp PS$ so that $PQ \perp AC$.

<u>Solution 2</u>. Denote by ω_1 and ω_2 the circumcircles of triangles ABC and ADC respectively. Since $\angle ABC = \angle ADC$, we know that ω_1 and ω_2 are symmetric with respect to the midpoint M of AC.

Firstly, we show that X lies on ω_2. Let X_1 be the second intersection of ray MB and ω_2 and X' be its symmetric point with respect to M. Then X' lies on ω_1 and $X'AX_1C$ is a parallelogram. Hence, we have

$$\begin{aligned}\measuredangle DX_1B &= \measuredangle DX_1A + \measuredangle AX_1B = \measuredangle DCA + \measuredangle AX_1X' = \measuredangle DCA + \measuredangle CX'X_1 \\ &= \measuredangle DCA + \measuredangle CAB = \angle(CD, AB).\end{aligned}$$



Also, we have

$$\measuredangle DPB = \measuredangle PDC + \angle(CD, AB) + \measuredangle ABP = \angle(CD, AB).$$

These yield $\measuredangle DX_1B = \measuredangle DPB$ and hence X_1 lies on ω. It follows that $X_1 = X$ and X lies on ω_2. Similarly, Y lies on ω_1.

Next, we prove that Q lies on ω. Suppose the perpendicular bisector of AC meet ω_1 at B' and M_1 and meet ω_2 at D' and M_2, so that B, M_1 and D' lie on the same side of AC. Note that B' lies on the angle bisector of $\angle ABC$ and similarly D' lies on DP.

If we denote the area of $W_1W_2W_3$ by $[W_1W_2W_3]$, then

$$\frac{BA \cdot X'A}{BC \cdot X'C} = \frac{\frac{1}{2}BA \cdot X'A \sin \angle BAX'}{\frac{1}{2}BC \cdot X'C \sin \angle BCX'} = \frac{[BAX']}{[BCX']} = \frac{MA}{MC} = 1.$$

As BE is the angle bisector of $\angle ABC$, we have

$$\frac{EA}{EC} = \frac{BA}{BC} = \frac{X'C}{X'A} = \frac{XA}{XC}.$$

Therefore, XE is the angle bisector of $\angle AXC$, so that M_2 lies on the line joining X, E, Q. Analogously, M_1, F, Q, Y are collinear. Thus,

$$\begin{aligned} \measuredangle XQY &= \measuredangle M_2QM_1 = \measuredangle QM_2M_1 + \measuredangle M_2M_1Q = \measuredangle XM_2D' + \measuredangle B'M_1Y \\ &= \measuredangle XDD' + \measuredangle B'BY = \measuredangle XDP + \measuredangle PBY = \measuredangle XBP + \measuredangle PBY = \measuredangle XBY, \end{aligned}$$

which implies Q lies on ω.

Finally, as M_1 and M_2 are symmetric with respect to M, the quadrilateral $X'M_2XM_1$ is a parallelogram. Consequently,

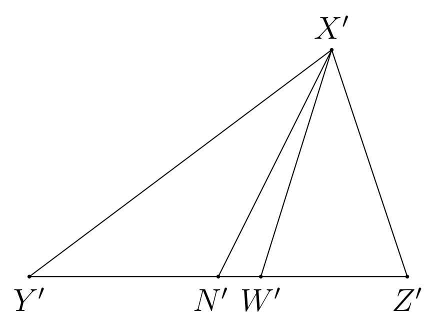
$$\measuredangle XQP = \measuredangle XBP = \measuredangle X'BB' = \measuredangle X'M_1B' = \measuredangle XM_2M_1.$$

This shows $QP // M_2M_1$. As $M_2M_1 \perp AC$, we get $QP \perp AC$.

<u>Solution 3</u>. We first state two results which will be needed in our proof.

• **Claim 1.** In $\triangle X'Y'Z'$ with $X'Y' \neq X'Z'$, let N' be the midpoint of $Y'Z'$ and W' be the foot of internal angle bisector from X'. Then $\tan^2 \measuredangle W'X'Z' = \tan \measuredangle N'X'W' \tan \measuredangle Z'W'X'$.

Proof.



Without loss of generality, assume $X'Y' > X'Z'$. Then W' lies between N' and Z'. The signs of both sides agree so it suffices to establish the relation for ordinary angles. Let $\angle W'X'Z' = \alpha$, $\angle N'X'W' = \beta$ and $\angle Z'W'X' = \gamma$. We have

$$\frac{\sin(\gamma - \alpha)}{\sin(\alpha - \beta)} = \frac{N'X'}{N'Y'} = \frac{N'X'}{N'Z'} = \frac{\sin(\gamma + \alpha)}{\sin(\alpha + \beta)}.$$

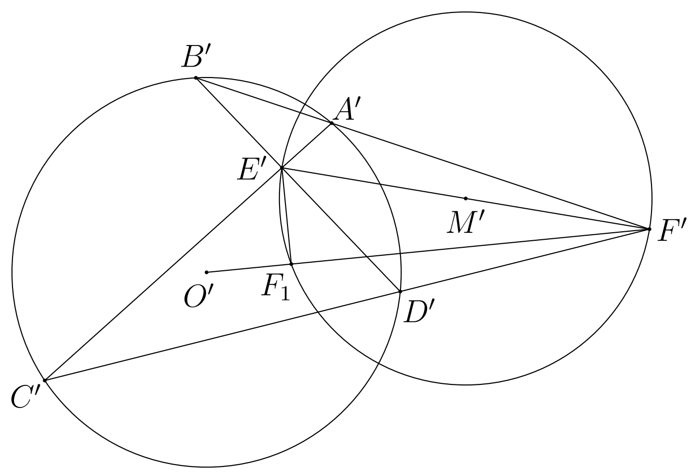
This implies

$$\frac{\tan\gamma-\tan\alpha}{\tan\gamma+\tan\alpha}=\frac{\sin\gamma\cos\alpha-\cos\gamma\sin\alpha}{\sin\gamma\cos\alpha+\cos\gamma\sin\alpha}=\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\cos\beta+\cos\alpha\sin\beta}=\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}.$$

Expanding and simplifying, we get the desired result $\tan^2\alpha=\tan\beta\tan\gamma$. □

• **Claim 2.** Let $A'B'C'D'$ be a quadrilateral inscribed in circle Γ. Let diagonals $A'C'$ and $B'D'$ meet at E', and F' be the intersection of lines $A'B'$ and $C'D'$. Let M' be the midpoint of $E'F'$. Then the power of M' with respect to Γ is equal to $(M'E')^2$.

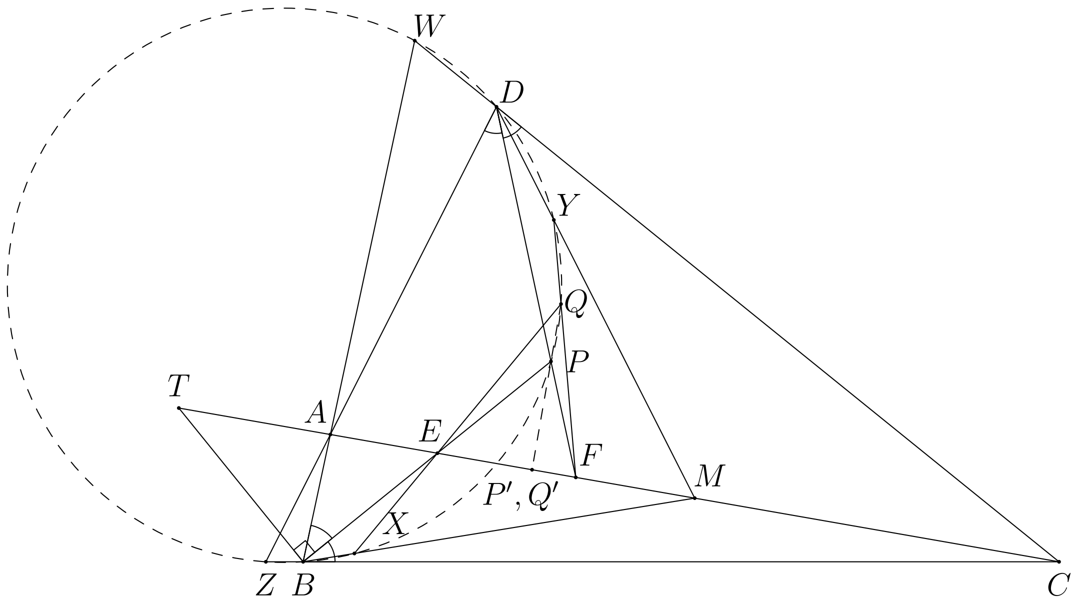
Proof.



Let O' be the centre of Γ and let Γ' be the circle with centre M' passing through E'. Let F_1 be the inversion image of F' with respect to Γ. It is well-known that E' lies on the polar of F' with respect to Γ. This shows $E'F_1 \perp O'F'$ and hence F_1 lies on Γ'. It follows that the inversion image of Γ' with respect to Γ is Γ' itself. This shows Γ' is orthogonal to Γ, and thus the power of M' with respect to Γ is the square of radius of Γ', which is $(M'E')^2$. □

We return to the main problem. Let Z be the intersection of lines AD and BC, and W be the intersection of lines AB and CD. Since $\measuredangle PDZ=\measuredangle PBC=\measuredangle PBZ$, point Z lies on ω. Similarly, W lies on ω. Applying Claim 2 to the cyclic quadrilateral $ZBDW$, we know that the power of M with respect to ω is MA^2. Hence, $MX\cdot MB=MA^2$.

Suppose the line through B perpendicular to BE meets line AC at T. Then BE and BT are the angle bisectors of $\angle CBA$. This shows $(T,E;A,C)$ is harmonic. Thus, we have $ME\cdot MT=MA^2=MX\cdot MB$. It follows that E,T,B,X are concyclic.



The result is trivial for the special case $AD = CD$ since P, Q lie on the perpendicular bisector of AC in that case. Similarly, the case $AB = CB$ is trivial. It remains to consider the general cases where we can apply Claim 1 in the latter part of the proof.

Let the projections from P and Q to AC be P' and Q' respectively. Then $PQ \perp AC$ if and only if $P' = Q'$ if and only if $\frac{EP'}{FP'} = \frac{EQ'}{FQ'}$ in terms of directed lengths. Note that

$$\frac{EP'}{FP'} = \frac{\tan \measuredangle EFP}{\tan \measuredangle FEP} = \frac{\tan \measuredangle AFD}{\tan \measuredangle AEB}.$$

Next, we have $\frac{EQ'}{FQ'} = \frac{\tan \measuredangle EFQ}{\tan \measuredangle FEQ}$ where $\measuredangle FEQ = \measuredangle TEX = \measuredangle TBX = \frac{\pi}{2} + \measuredangle EBM$ and by symmetry $\measuredangle EFQ = \frac{\pi}{2} + \measuredangle FDM$. Combining all these, it suffices to show

$$\frac{\tan \measuredangle AFD}{\tan \measuredangle AEB} = \frac{\tan \measuredangle MBE}{\tan \measuredangle MDF}.$$

We now apply Claim 1 twice to get

$$\tan \measuredangle AFD \tan \measuredangle MDF = \tan^2 \measuredangle FDC = \tan^2 \measuredangle EBA = \tan \measuredangle MBE \tan \measuredangle AEB.$$

The result then follows.

G7. Let I be the incentre of a non-equilateral triangle ABC, I_A be the A-excentre, I'_A be the reflection of I_A in BC, and l_A be the reflection of line AI'_A in AI. Define points I_B, I'_B and line l_B analogously. Let P be the intersection point of l_A and l_B.

(a) Prove that P lies on line OI where O is the circumcentre of triangle ABC.

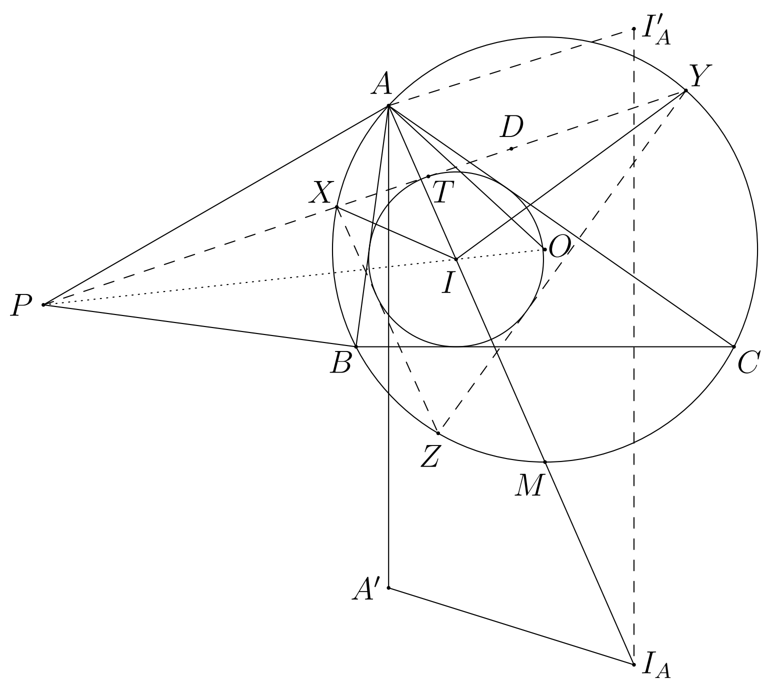
(b) Let one of the tangents from P to the incircle of triangle ABC meet the circumcircle at points X and Y. Show that $\angle XIY = 120°$.

Solution 1.

(a) Let A' be the reflection of A in BC and let M be the second intersection of line AI and the circumcircle Γ of triangle ABC. As triangles ABA' and AOC are isosceles with $\angle ABA' = 2\angle ABC = \angle AOC$, they are similar to each other. Also, triangles ABI_A and AIC are similar. Therefore we have

$$\frac{AA'}{AI_A} = \frac{AA'}{AB} \cdot \frac{AB}{AI_A} = \frac{AC}{AO} \cdot \frac{AI}{AC} = \frac{AI}{AO}.$$

Together with $\angle A'AI_A = \angle IAO$, we find that triangles $AA'I_A$ and AIO are similar.



Denote by P' the intersection of line AP and line OI. Using directed angles, we have

$$\begin{aligned}\measuredangle MAP' &= \measuredangle I'_A AI_A = \measuredangle I'_A AA' - \measuredangle I_A AA' = \measuredangle AA'I_A - \angle(AM, OM) \\ &= \measuredangle AIO - \measuredangle AMO = \measuredangle MOP'.\end{aligned}$$

This shows M, O, A, P' are concyclic.

Denote by R and r the circumradius and inradius of triangle ABC. Then

$$IP' = \frac{IA \cdot IM}{IO} = \frac{IO^2 - R^2}{IO}$$

is independent of A. Hence, BP also meets line OI at the same point P' so that $P' = P$, and P lies on OI.

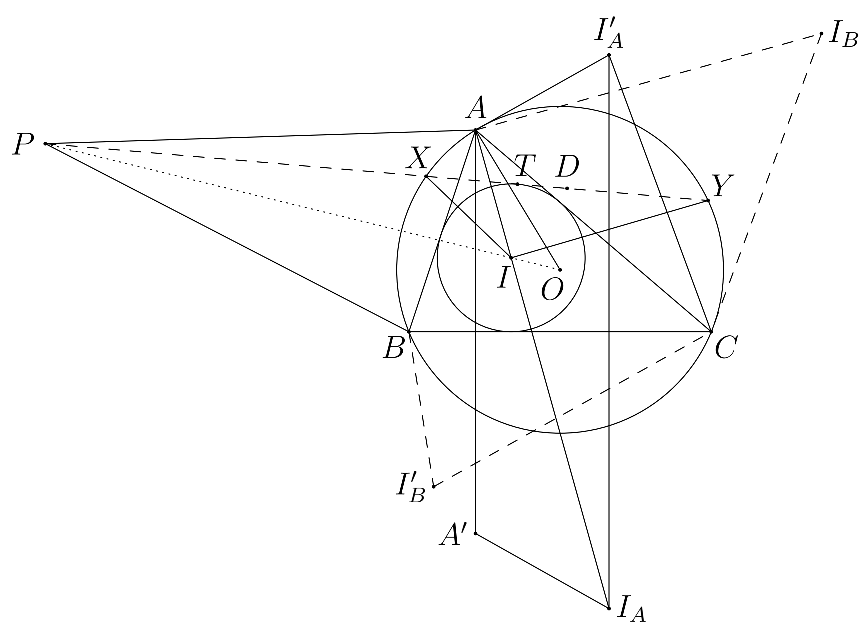
(b) By Poncelet's Porism, the other tangents to the incircle of triangle ABC from X and Y meet at a point Z on Γ. Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. We have

$$\begin{aligned} OD = IT \cdot \frac{OP}{IP} &= r\left(1 + \frac{OI}{IP}\right) = r\left(1 + \frac{OI^2}{OI \cdot IP}\right) = r\left(1 + \frac{R^2 - 2Rr}{R^2 - IO^2}\right) \\ &= r\left(1 + \frac{R^2 - 2Rr}{2Rr}\right) = \frac{R}{2} = \frac{OX}{2}. \end{aligned}$$

This shows $\angle XZY = 60^\circ$ and hence $\angle XIY = 120^\circ$.

<u>Solution 2</u>.

(a) Note that triangles AI_BC and I_ABC are similar since their corresponding interior angles are equal. Therefore, the four triangles AI'_BC, AI_BC, I_ABC and I'_ABC are all similar. From $\triangle AI'_BC \sim \triangle I'_ABC$, we get $\triangle AI'_AC \sim \triangle I'_BBC$. From $\measuredangle ABP = \measuredangle I'_BBC = \measuredangle AI'_AC$ and $\measuredangle BAP = \measuredangle I'_AAC$, the triangles ABP and AI'_AC are directly similar.



Consider the inversion with centre A and radius $\sqrt{AB \cdot AC}$ followed by the reflection in AI. Then B and C are mapped to each other, and I and I_A are mapped to each other.

From the similar triangles obtained, we have $AP \cdot AI'_A = AB \cdot AC$ so that P is mapped to I'_A under the transformation. In addition, line AO is mapped to the altitude from A, and hence O is mapped to the reflection of A in BC, which we call point A'. Note that $AA'I_AI'_A$ is an isosceles trapezoid, which shows it is inscribed in a circle. The preimage of this circle is a straight line, meaning that O, I, P are collinear.

(b) Denote by R and r the circumradius and inradius of triangle ABC. Note that by the above transformation, we have $\triangle APO \sim \triangle AA'I'_A$ and $\triangle AA'I_A \sim \triangle AIO$. Therefore, we find that

$$PO = A'I'_A \cdot \frac{AO}{AI'_A} = AI_A \cdot \frac{AO}{A'I_A} = \frac{AI_A}{A'I_A} \cdot AO = \frac{AO}{IO} \cdot AO.$$

This shows $PO \cdot IO = R^2$, and it follows that P and I are mapped to each other under the inversion with respect to the circumcircle Γ of triangle ABC. Then $PX \cdot PY$, which is the power of P with respect to Γ, equals $PI \cdot PO$. This yields X, I, O, Y are concyclic.

Let T be the touching point of the incircle to XY, and let D be the midpoint of XY. Then

$$OD = IT \cdot \frac{PO}{PI} = r \cdot \frac{PO}{PO - IO} = r \cdot \frac{R^2}{R^2 - IO^2} = r \cdot \frac{R^2}{2Rr} = \frac{R}{2}.$$

This shows $\angle DOX = 60°$ and hence $\angle XIY = \angle XOY = 120°$.

Comment. A simplification of this problem is to ask part (a) only. Note that the question in part (b) implicitly requires P to lie on OI, or otherwise the angle is not uniquely determined as we can find another tangent from P to the incircle.

G8. Let A_1, B_1 and C_1 be points on sides BC, CA and AB of an acute triangle ABC respectively, such that AA_1, BB_1 and CC_1 are the internal angle bisectors of triangle ABC. Let I be the incentre of triangle ABC, and H be the orthocentre of triangle $A_1B_1C_1$. Show that

$$AH + BH + CH \geqslant AI + BI + CI.$$

Solution. Without loss of generality, assume $\alpha = \angle BAC \leqslant \beta = \angle CBA \leqslant \gamma = \angle ACB$. Denote by a, b, c the lengths of BC, CA, AB respectively. We first show that triangle $A_1B_1C_1$ is acute.

Choose points D and E on side BC such that $B_1D // AB$ and B_1E is the internal angle bisector of $\angle BB_1C$. As $\angle B_1DB = 180^\circ - \beta$ is obtuse, we have $BB_1 > B_1D$. Thus,

$$\frac{BE}{EC} = \frac{BB_1}{B_1C} > \frac{DB_1}{B_1C} = \frac{BA}{AC} = \frac{BA_1}{A_1C}.$$

Therefore, $BE > BA_1$ and $\frac{1}{2}\angle BB_1C = \angle BB_1E > \angle BB_1A_1$. Similarly, $\frac{1}{2}\angle BB_1A > \angle BB_1C_1$. It follows that

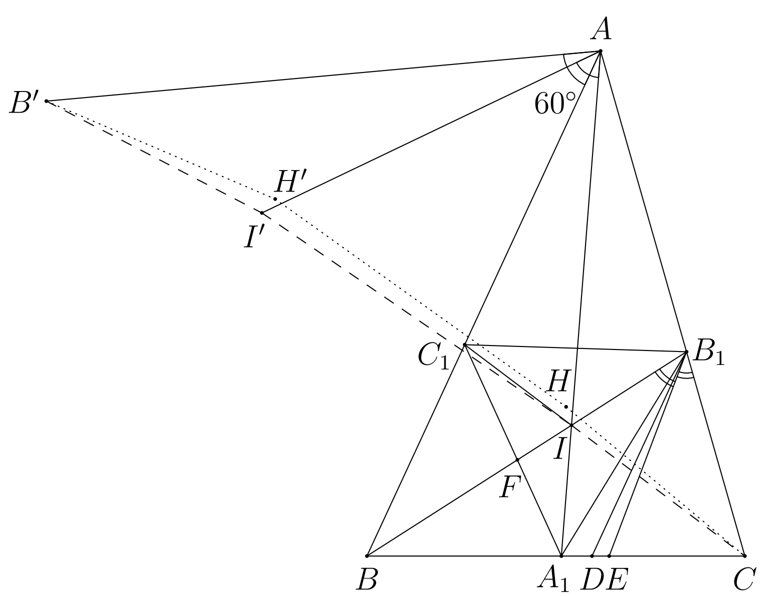
$$\angle A_1B_1C_1 = \angle BB_1A_1 + \angle BB_1C_1 < \frac{1}{2}(\angle BB_1C + \angle BB_1A) = 90^\circ$$

is acute. By symmetry, triangle $A_1B_1C_1$ is acute.

Let BB_1 meet A_1C_1 at F. From $\alpha \leqslant \gamma$, we get $a \leqslant c$, which implies

$$BA_1 = \frac{ca}{b+c} \leqslant \frac{ac}{a+b} = BC_1$$

and hence $\angle BC_1A_1 \leqslant \angle BA_1C_1$. As BF is the internal angle bisector of $\angle A_1BC_1$, this shows $\angle B_1FC_1 = \angle BFA_1 \leqslant 90^\circ$. Hence, H lies on the same side of BB_1 as C_1. This shows H lies inside triangle BB_1C_1. Similarly, from $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$, we know that H lies inside triangles CC_1B_1 and AA_1C_1.



As $\alpha \leqslant \beta \leqslant \gamma$, we have $\alpha \leqslant 60^\circ \leqslant \gamma$. Then $\angle BIC \leqslant 120^\circ \leqslant \angle AIB$. Firstly, suppose $\angle AIC \geqslant 120^\circ$.

Rotate points B, I, H through 60° about A to B', I', H' so that B' and C lie on different sides of AB. Since triangle $AI'I$ is equilateral, we have

$$AI + BI + CI = I'I + B'I' + IC = B'I' + I'I + IC. \tag{1}$$

Similarly,

$$AH + BH + CH = H'H + B'H' + HC = B'H' + H'H + HC. \tag{2}$$

As $\angle AII' = \angle AI'I = 60^\circ$, $\angle AI'B' = \angle AIB \geqslant 120^\circ$ and $\angle AIC \geqslant 120^\circ$, the quadrilateral $B'I'IC$ is convex and lies on the same side of $B'C$ as A.

Next, since H lies inside triangle ACC_1, H lies outside $B'I'IC$. Also, H lying inside triangle ABI implies H' lies inside triangle $AB'I'$. This shows H' lies outside $B'I'IC$ and hence the convex quadrilateral $B'I'IC$ is contained inside the quadrilateral $B'H'HC$. It follows that the perimeter of $B'I'IC$ cannot exceed the perimeter of $B'H'HC$. From (1) and (2), we conclude that

$$AH + BH + CH \geqslant AI + BI + CI.$$

For the case $\angle AIC < 120^\circ$, we can rotate B, I, H through 60° about C to B', I', H' so that B' and A lie on different sides of BC. The proof is analogous to the previous case and we still get the desired inequality.

Number Theory

N1. For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geqslant 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)). \tag{1}$$

Answer.

- $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer; or
- $P(x) = x$.

Solution 1. We consider three cases according to the degree of P.

- **Case 1.** $P(x)$ is a constant polynomial.

  Let $P(x) = c$ where c is an integer constant. Then (1) becomes $S(c) = c$. This holds if and only if $1 \leqslant c \leqslant 9$.

- **Case 2.** $\deg P = 1$.

  We have the following observation. For any positive integers m, n, we have

$$S(m+n) \leqslant S(m) + S(n), \tag{2}$$

and equality holds if and only if there is no carry in the addition $m + n$.

Let $P(x) = ax + b$ for some integers a, b where $a \neq 0$. As $P(n)$ is positive for large n, we must have $a \geqslant 1$. The condition (1) becomes $S(an + b) = aS(n) + b$ for all $n \geqslant 2016$. Setting $n = 2025$ and $n = 2020$ respectively, we get

$$S(2025a + b) - S(2020a + b) = (aS(2025) + b) - (aS(2020) + b) = 5a.$$

On the other hand, (2) implies

$$S(2025a + b) = S((2020a + b) + 5a) \leqslant S(2020a + b) + S(5a).$$

These give $5a \leqslant S(5a)$. As $a \geqslant 1$, this holds only when $a = 1$, in which case (1) reduces to $S(n + b) = S(n) + b$ for all $n \geqslant 2016$. Then we find that

$$S(n + 1 + b) - S(n + b) = (S(n + 1) + b) - (S(n) + b) = S(n + 1) - S(n). \tag{3}$$

If $b > 0$, we choose n such that $n + 1 + b = 10^k$ for some sufficiently large k. Note that all the digits of $n + b$ are 9's, so that the left-hand side of (3) equals $1 - 9k$. As n is a positive integer less than $10^k - 1$, we have $S(n) < 9k$. Therefore, the right-hand side of (3) is at least $1 - (9k - 1) = 2 - 9k$, which is a contradiction.

The case $b < 0$ can be handled similarly by considering $n + 1$ to be a large power of 10. Therefore, we conclude that $P(x) = x$, in which case (1) is trivially satisfied.

- **Case 3.** $\deg P \geqslant 2$.

Suppose the leading term of P is $a_d n^d$ where $a_d \neq 0$. Clearly, we have $a_d > 0$. Consider $n = 10^k - 1$ in (1). We get $S(P(n)) = P(9k)$. Note that $P(n)$ grows asymptotically as fast as n^d, so $S(P(n))$ grows asymptotically as no faster than a constant multiple of k. On the other hand, $P(9k)$ grows asymptotically as fast as k^d. This shows the two sides of the last equation cannot be equal for sufficiently large k since $d \geqslant 2$.

Therefore, we conclude that $P(x) = c$ where $1 \leqslant c \leqslant 9$ is an integer, or $P(x) = x$.

Solution 2. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Clearly $a_d > 0$. There exists an integer $m \geqslant 1$ such that $|a_i| < 10^m$ for all $0 \leqslant i \leqslant d$. Consider $n = 9 \times 10^k$ for a sufficiently large integer k in (1). If there exists an index $0 \leqslant i \leqslant d-1$ such that $a_i < 0$, then all digits of $P(n)$ in positions from 10^{ik+m+1} to $10^{(i+1)k-1}$ are all 9's. Hence, we have $S(P(n)) \geqslant 9(k-m-1)$. On the other hand, $P(S(n)) = P(9)$ is a fixed constant. Therefore, (1) cannot hold for large k. This shows $a_i \geqslant 0$ for all $0 \leqslant i \leqslant d-1$.

Hence, $P(n)$ is an integer formed by the nonnegative integers $a_d \times 9^d, a_{d-1} \times 9^{d-1}, \ldots, a_0$ by inserting some zeros in between. This yields

$$S(P(n)) = S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0).$$

Combining with (1), we have

$$S(a_d \times 9^d) + S(a_{d-1} \times 9^{d-1}) + \cdots + S(a_0) = P(9) = a_d \times 9^d + a_{d-1} \times 9^{d-1} + \cdots + a_0.$$

As $S(m) \leqslant m$ for any positive integer m, with equality when $1 \leqslant m \leqslant 9$, this forces each $a_i \times 9^i$ to be a positive integer between 1 and 9. In particular, this shows $a_i = 0$ for $i \geqslant 2$ and hence $d \leqslant 1$. Also, we have $a_1 \leqslant 1$ and $a_0 \leqslant 9$. If $a_1 = 1$ and $1 \leqslant a_0 \leqslant 9$, we take $n = 10^k + (10 - a_0)$ for sufficiently large k in (1). This yields a contradiction since

$$S(P(n)) = S(10^k + 10) = 2 \neq 11 = P(11 - a_0) = P(S(n)).$$

The zero polynomial is also rejected since $P(n)$ is positive for large n. The remaining candidates are $P(x) = x$ or $P(x) = a_0$ where $1 \leqslant a_0 \leqslant 9$, all of which satisfy (1), and hence are the only solutions.

N2. Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all possible integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

Answer. All composite numbers together with 2.

Solution. In this solution, we always use p_i to denote primes congruent to 1 mod 3, and use q_j to denote primes congruent to 2 mod 3. When we express a positive integer m using its prime factorization, we also include the special case $m = 1$ by allowing the exponents to be zeros. We first compute $\tau_1(m)$ for a positive integer m.

- **Claim.** Let $m = 3^x p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$ be the prime factorization of m. Then

$$\tau_1(m) = \prod_{i=1}^{s} (a_i + 1) \left\lceil \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \right\rceil . \tag{1}$$

Proof. To choose a divisor of m congruent to 1 mod 3, it cannot have the prime divisor 3, while there is no restriction on choosing prime factors congruent to 1 mod 3. Also, we have to choose an even number of prime factors (counted with multiplicity) congruent to 2 mod 3.

If $\prod_{j=1}^{t} (b_j + 1)$ is even, then we may assume without loss of generality $b_1 + 1$ is even. We can choose the prime factors $q_2, q_3, \ldots, q_t$ freely in $\prod_{j=2}^{t} (b_j + 1)$ ways. Then the parity of the number of q_1 is uniquely determined, and hence there are $\frac{1}{2}(b_1 + 1)$ ways to choose the exponent of q_1. Hence (1) is verified in this case.

If $\prod_{j=1}^{t} (b_j + 1)$ is odd, we use induction on t to count the number of choices. When $t = 1$, there are $\lceil \frac{b_1+1}{2} \rceil$ choices for which the exponent is even and $\lfloor \frac{b_1+1}{2} \rfloor$ choices for which the exponent is odd. For the inductive step, we find that there are

$$\left\lceil \frac{1}{2} \prod_{j=1}^{t-1} (b_j + 1) \right\rceil \cdot \left\lceil \frac{b_t + 1}{2} \right\rceil + \left\lfloor \frac{1}{2} \prod_{j=1}^{t-1} (b_j + 1) \right\rfloor \cdot \left\lfloor \frac{b_t + 1}{2} \right\rfloor = \left\lceil \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \right\rceil$$

choices with an even number of prime factors and hence $\lfloor \frac{1}{2} \prod_{j=1}^{t} (b_j + 1) \rfloor$ choices with an odd number of prime factors. Hence (1) is also true in this case. $\square$

Let $n = 3^x 2^y 5^z p_1^{a_1} p_2^{a_2} \cdots p_s^{a_s} q_1^{b_1} q_2^{b_2} \cdots q_t^{b_t}$. Using the well-known formula for computing the divisor function, we get

$$\tau(10n) = (x + 1)(y + 2)(z + 2) \prod_{i=1}^{s} (a_i + 1) \prod_{j=1}^{t} (b_j + 1). \tag{2}$$

By the Claim, we have

$$\tau_1(10n) = \prod_{i=1}^{s} (a_i + 1) \left\lceil \frac{1}{2}(y + 2)(z + 2) \prod_{j=1}^{t} (b_j + 1) \right\rceil . \tag{3}$$

If $c = (y+2)(z+2)\prod_{j=1}^{t}(b_j+1)$ is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = 2(x+1).$$

In this case $\frac{\tau(10n)}{\tau_1(10n)}$ can be any even positive integer as x runs through all nonnegative integers.

If c is odd, which means y, z are odd and each b_j is even, then (2) and (3) imply

$$\frac{\tau(10n)}{\tau_1(10n)} = \frac{2(x+1)c}{c+1}. \tag{4}$$

For this to be an integer, we need $c+1$ divides $2(x+1)$ since c and $c+1$ are relatively prime. Let $2(x+1) = k(c+1)$. Then (4) reduces to

$$\frac{\tau(10n)}{\tau_1(10n)} = kc = k(y+2)(z+2)\prod_{j=1}^{t}(b_j+1). \tag{5}$$

Noting that y, z are odd, the integers $y+2$ and $z+2$ are at least 3. This shows the integer in this case must be composite. On the other hand, for any odd composite number ab with $a, b \geqslant 3$, we may simply take $n = 3^{\frac{ab-1}{2}} \cdot 2^{a-2} \cdot 5^{b-2}$ so that $\frac{\tau(10n)}{\tau_1(10n)} = ab$ from (5).

We conclude that the fraction can be any even integer or any odd composite number. Equivalently, it can be 2 or any composite number.

N3. Define $P(n)=n^2+n+1$. For any positive integers a and b, the set

$$\{P(a), P(a+1), P(a+2), \ldots, P(a+b)\}$$

is said to be *fragrant* if none of its elements is relatively prime to the product of the other elements. Determine the smallest size of a fragrant set.

Answer. 6.

Solution. We have the following observations.

(i) $(P(n), P(n+1))=1$ for any n.

We have $(P(n), P(n+1))=(n^2+n+1, n^2+3n+3)=(n^2+n+1, 2n+2)$. Noting that n^2+n+1 is odd and $(n^2+n+1, n+1)=(1, n+1)=1$, the claim follows.

(ii) $(P(n), P(n+2))=1$ for $n \not\equiv 2 \pmod 7$ and $(P(n), P(n+2))=7$ for $n \equiv 2 \pmod 7$.

From $(2n+7)P(n)-(2n-1)P(n+2)=14$ and the fact that $P(n)$ is odd, $(P(n), P(n+2))$ must be a divisor of 7. The claim follows by checking $n \equiv 0,1,\ldots,6 \pmod 7$ directly.

(iii) $(P(n), P(n+3))=1$ for $n \not\equiv 1 \pmod 3$ and $3 \mid (P(n), P(n+3))$ for $n \equiv 1 \pmod 3$.

From $(n+5)P(n)-(n-1)P(n+3)=18$ and the fact that $P(n)$ is odd, $(P(n), P(n+3))$ must be a divisor of 9. The claim follows by checking $n \equiv 0,1,2 \pmod 3$ directly.

Suppose there exists a fragrant set with at most 5 elements. We may assume it contains exactly 5 elements $P(a), P(a+1), \ldots, P(a+4)$ since the following argument also works with fewer elements. Consider $P(a+2)$. From (i), it is relatively prime to $P(a+1)$ and $P(a+3)$. Without loss of generality, assume $(P(a), P(a+2))>1$. From (ii), we have $a \equiv 2 \pmod 7$. The same observation implies $(P(a+1), P(a+3))=1$. In order that the set is fragrant, $(P(a), P(a+3))$ and $(P(a+1), P(a+4))$ must both be greater than 1. From (iii), this holds only when both a and $a+1$ are congruent to 1 mod 3, which is a contradiction.

It now suffices to construct a fragrant set of size 6. By the Chinese Remainder Theorem, we can take a positive integer a such that

$$a \equiv 7 \pmod{19}, \quad a+1 \equiv 2 \pmod 7, \quad a+2 \equiv 1 \pmod 3.$$

For example, we may take $a=197$. From (ii), both $P(a+1)$ and $P(a+3)$ are divisible by 7. From (iii), both $P(a+2)$ and $P(a+5)$ are divisible by 3. One also checks from $19 \mid P(7)=57$ and $19 \mid P(11)=133$ that $P(a)$ and $P(a+4)$ are divisible by 19. Therefore, the set $\{P(a), P(a+1), \ldots, P(a+5)\}$ is fragrant.

Therefore, the smallest size of a fragrant set is 6.

Comment. "Fragrant Harbour" is the English translation of "Hong Kong".

A stronger version of this problem is to show that there exists a fragrant set of size k for any $k \geqslant 6$. We present a proof here.

For each even positive integer m which is not divisible by 3, since $m^2+3 \equiv 3 \pmod 4$, we can find a prime $p_m \equiv 3 \pmod 4$ such that $p_m \mid m^2+3$. Clearly, $p_m>3$.

If $b = 2t \geqslant 6$, we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for each $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$. For $0 \leqslant r \leqslant t$ and $3|r$, we have $a+t \pm r \equiv 1 \pmod 3$ so that $3|P(a+t \pm r)$. For $0 \leqslant r \leqslant t$ and $(r, 3) = 1$, we have

$$4P(a+t \pm r) \equiv (-1 \pm 2r)^2 + 2(-1 \pm 2r) + 4 = 4r^2 + 3 \equiv 0 \pmod{p_{2r}}.$$

Hence, $\{P(a), P(a+1), \ldots, P(a+b)\}$ is fragrant.

If $b = 2t+1 \geqslant 7$ (the case $b = 5$ has been done in the original problem), we choose a such that $3|2(a+t)+1$ and $p_m|2(a+t)+1$ for $1 \leqslant m \leqslant b$ with $m \equiv 2, 4 \pmod 6$, and that $a + b \equiv 9 \pmod{13}$. Note that a exists by the Chinese Remainder Theorem since $p_m \neq 13$ for all m. The even case shows that $\{P(a), P(a+1), \ldots, P(a+b-1)\}$ is fragrant. Also, one checks from $13|P(9) = 91$ and $13|P(3) = 13$ that $P(a+b)$ and $P(a+b-6)$ are divisible by 13. The proof is thus complete.

N4. Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

- $m = 1$ and $l = 2k$; or
- $l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

<u>**Solution 1**</u>. It is given that

$$n^k + mn^l + 1 | n^{k+l} - 1. \tag{1}$$

This implies

$$n^k + mn^l + 1 | (n^{k+l} - 1) + (n^k + mn^l + 1) = n^{k+l} + n^k + mn^l. \tag{2}$$

We have two cases to discuss.

- **Case 1.** $l \geqslant k$.
  Since $(n^k + mn^l + 1, n) = 1$, (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^l + mn^{l-k} + 1$. As $n \geqslant 2$ and $k \geqslant 1$, $(m-1)n^l$ is at least $2(m-1)n^{l-k}$. It follows that the inequality cannot hold when $m \geqslant 2$. For $m = 1$, the above divisibility becomes

$$n^k + n^l + 1 | n^l + n^{l-k} + 1.$$

Note that $n^l + n^{l-k} + 1 < n^l + n^l + 1 < 2(n^k + n^l + 1)$. Thus we must have $n^l + n^{l-k} + 1 = n^k + n^l + 1$ so that $l = 2k$, which gives the first result.

- **Case 2.** $l < k$.
  This time (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

In particular, we get $n^k + mn^l + 1 \leqslant n^k + n^{k-l} + m$, which implies

$$m \leqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{3}$$

On the other hand, from (1) we may let $n^{k+l} - 1 = (n^k + mn^l + 1)t$ for some positive integer t. Obviously, t is less than n^l, which means $t \leqslant n^l - 1$ as it is an integer. Then we have $n^{k+l} - 1 \leqslant (n^k + mn^l + 1)(n^l - 1)$, which is the same as

$$m \geqslant \frac{n^{k-l} - 1}{n^l - 1}. \tag{4}$$

Equations (3) and (4) combine to give $m = \frac{n^{k-l}-1}{n^l-1}$. As this is an integer, we have $l | k - l$. This means $l | k$ and it corresponds to the second result.

Solution 2. As in Solution 1, we begin with equation (2).

- **Case 1.** $l \geqslant k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^l + mn^{l-k} + 1.$$

Since $2(n^k+mn^l+1) > 2mn^l+1 > n^l+mn^{l-k}+1$, it follows that $n^k+mn^l+1 = n^l+mn^{l-k}+1$, that is,

$$m(n^l - n^{l-k}) = n^l - n^k.$$

If $m \geqslant 2$, then $m(n^l - n^{l-k}) \geqslant 2n^l - 2n^{l-k} \geqslant 2n^l - n^l > n^l - n^k$ gives a contradiction. Hence $m = 1$ and $l - k = k$, which means $m = 1$ and $l = 2k$.

- **Case 2.** $l < k$.
  Then (2) yields

$$n^k + mn^l + 1 | n^k + n^{k-l} + m.$$

Since $2(n^k+mn^l+1) > 2n^k+m > n^k+n^{k-l}+m$, it follows that $n^k+mn^l+1 = n^k+n^{k-l}+m$. This gives $m = \frac{n^{k-l}-1}{n^l-1}$. Note that $n^l - 1 | n^{k-l} - 1$ implies $l | k - l$ and hence $l | k$. The proof is thus complete.

Comment. Another version of this problem is as follows: let n, m, k and l be positive integers with $n \neq 1$ such that k and l do not divide each other. Show that $n^k + mn^l + 1$ does not divide $n^{k+l} - 1$.

N5. Let a be a positive integer which is not a square number. Denote by A the set of all positive integers k such that

$$k = \frac{x^2 - a}{x^2 - y^2} \tag{1}$$

for some integers x and y with $x > \sqrt{a}$. Denote by B the set of all positive integers k such that (1) is satisfied for some integers x and y with $0 \leqslant x < \sqrt{a}$. Prove that $A = B$.

<u>Solution 1</u>. We first prove the following preliminary result.

- **Claim.** For fixed k, let x, y be integers satisfying (1). Then the numbers x_1, y_1 defined by

$$x_1 = \frac{1}{2}\left(x - y + \frac{(x-y)^2 - 4a}{x+y}\right), \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right)$$

are integers and satisfy (1) (with x, y replaced by x_1, y_1 respectively).

Proof. Since $x_1 + y_1 = x - y$ and

$$x_1 = \frac{x^2 - xy - 2a}{x+y} = -x + \frac{2(x^2 - a)}{x+y} = -x + 2k(x-y),$$

both x_1 and y_1 are integers. Let $u = x + y$ and $v = x - y$. The relation (1) can be rewritten as

$$u^2 - (4k-2)uv + (v^2 - 4a) = 0.$$

By Vieta's Theorem, the number $z = \frac{v^2 - 4a}{u}$ satisfies

$$v^2 - (4k-2)vz + (z^2 - 4a) = 0.$$

Since x_1 and y_1 are defined so that $v = x_1 + y_1$ and $z = x_1 - y_1$, we can reverse the process and verify (1) for x_1, y_1. $\square$

We first show that $B \subset A$. Take any $k \in B$ so that (1) is satisfied for some integers x, y with $0 \leqslant x < \sqrt{a}$. Clearly, $y \neq 0$ and we may assume y is positive. Since a is not a square, we have $k > 1$. Hence, we get $0 \leqslant x < y < \sqrt{a}$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Also, we have

$$x_1 \geqslant -\frac{1}{2}\left(x - y + \frac{(x-y)^2 - 4a}{x+y}\right) = \frac{2a + x(y-x)}{x+y} \geqslant \frac{2a}{x+y} > \sqrt{a}.$$

This implies $k \in A$ and hence $B \subset A$.

Next, we shall show that $A \subset B$. Take any $k \in A$ so that (1) is satisfied for some integers x, y with $x > \sqrt{a}$. Again, we may assume y is positive. Among all such representations of k, we choose the one with smallest $x + y$. Define

$$x_1 = \frac{1}{2}\left|x - y + \frac{(x-y)^2 - 4a}{x+y}\right|, \quad y_1 = \frac{1}{2}\left(x - y - \frac{(x-y)^2 - 4a}{x+y}\right).$$

By the Claim, x_1, y_1 are integers satisfying (1). Since $k > 1$, we get $x > y > \sqrt{a}$. Therefore, we have $y_1 > \frac{4a}{x+y} > 0$ and $\frac{4a}{x+y} < x + y$. It follows that

$$x_1 + y_1 \leqslant \max\left\{x - y, \frac{4a - (x-y)^2}{x+y}\right\} < x + y.$$

If $x_1 > \sqrt{a}$, we get a contradiction due to the minimality of $x + y$. Therefore, we must have $0 \leqslant x_1 < \sqrt{a}$, which means $k \in B$ so that $A \subset B$.

The two subset relations combine to give $A = B$.

Solution 2. The relation (1) is equivalent to

$$ky^2 - (k-1)x^2 = a. \tag{2}$$

Motivated by Pell's Equation, we prove the following, which is essentially the same as the Claim in Solution 1.

• **Claim.** If (x_0, y_0) is a solution to (2), then $((2k-1)x_0 \pm 2ky_0, (2k-1)y_0 \pm 2(k-1)x_0)$ is also a solution to (2).

Proof. We check directly that

$$\begin{aligned} &k((2k-1)y_0 \pm 2(k-1)x_0)^2 - (k-1)((2k-1)x_0 \pm 2ky_0)^2 \\ &= (k(2k-1)^2 - (k-1)(2k)^2)y_0^2 + (k(2(k-1))^2 - (k-1)(2k-1)^2)x_0^2 \\ &= ky_0^2 - (k-1)x_0^2 = a. \end{aligned}$$

$\square$

If (2) is satisfied for some $0 \leqslant x < \sqrt{a}$ and nonnegative integer y, then clearly (1) implies $y > x$. Also, we have $k > 1$ since a is not a square number. By the Claim, consider another solution to (2) defined by

$$x_1 = (2k-1)x + 2ky, \quad y_1 = (2k-1)y + 2(k-1)x.$$

It satisfies $x_1 \geqslant (2k-1)x + 2k(x+1) = (4k-1)x + 2k > x$. Then we can replace the old solution by a new one which has a larger value in x. After a finite number of replacements, we must get a solution with $x > \sqrt{a}$. This shows $B \subset A$.

If (2) is satisfied for some $x > \sqrt{a}$ and nonnegative integer y, by the Claim we consider another solution to (2) defined by

$$x_1 = |(2k-1)x - 2ky|, \quad y_1 = (2k-1)y - 2(k-1)x.$$

From (2), we get $\sqrt{k}y > \sqrt{k-1}x$. This implies $ky > \sqrt{k(k-1)}x > (k-1)x$ and hence $(2k-1)x - 2ky < x$. On the other hand, the relation (1) implies $x > y$. Then it is clear that $(2k-1)x - 2ky > -x$. These combine to give $x_1 < x$, which means we have found a solution to (2) with x having a smaller absolute value. After a finite number of steps, we shall obtain a solution with $0 \leqslant x < \sqrt{a}$. This shows $A \subset B$.

The desired result follows from $B \subset A$ and $A \subset B$.

Solution 3. It suffices to show $A \cup B$ is a subset of $A \cap B$. We take any $k \in A \cup B$, which means there exist integers x, y satisfying (1). Since a is not a square, it follows that $k \neq 1$. As in Solution 2, the result follows readily once we have proved the existence of a solution (x_1, y_1) to (1) with $|x_1| > |x|$, and, in case of $x > \sqrt{a}$, another solution (x_2, y_2) with $|x_2| < |x|$.

Without loss of generality, assume $x, y \geqslant 0$. Let $u = x + y$ and $v = x - y$. Then $u \geqslant v$ and (1) becomes

$$k = \frac{(u+v)^2 - 4a}{4uv}. \tag{3}$$

This is the same as

$$v^2 + (2u - 4ku)v + u^2 - 4a = 0.$$

Let $v_1 = 4ku - 2u - v$. Then $u + v_1 = 4ku - u - v \geqslant 8u - u - v > u + v$. By Vieta's Theorem, v_1 satisfies

$$v_1^2 + (2u - 4ku)v_1 + u^2 - 4a = 0.$$

This gives $k = \frac{(u+v_1)^2 - 4a}{4uv_1}$. As k is an integer, $u + v_1$ must be even. Therefore, $x_1 = \frac{u+v_1}{2}$ and $y_1 = \frac{v_1 - u}{2}$ are integers. By reversing the process, we can see that (x_1, y_1) is a solution to (1), with $x_1 = \frac{u+v_1}{2} > \frac{u+v}{2} = x \geqslant 0$. This completes the first half of the proof.

Suppose $x > \sqrt{a}$. Then $u + v > 2\sqrt{a}$ and (3) can be rewritten as

$$u^2 + (2v - 4kv)u + v^2 - 4a = 0.$$

Let $u_2 = 4kv - 2v - u$. By Vieta's Theorem, we have $uu_2 = v^2 - 4a$ and

$$u_2^2 + (2v - 4kv)u_2 + v^2 - 4a = 0. \tag{4}$$

By $u > 0$, $u + v > 2\sqrt{a}$ and (3), we have $v > 0$. If $u_2 \geqslant 0$, then $vu_2 \leqslant uu_2 = v^2 - 4a < v^2$. This shows $u_2 < v \leqslant u$ and $0 < u_2 + v < u + v$. If $u_2 < 0$, then $(u_2 + v) + (u + v) = 4kv > 0$ and $u_2 + v < u + v$ imply $|u_2 + v| < u + v$. In any case, since $u_2 + v$ is even from (4), we can define $x_2 = \frac{u_2+v}{2}$ and $y_2 = \frac{u_2-v}{2}$ so that (1) is satisfied with $|x_2| < x$, as desired. The proof is thus complete.

N6. Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Answer. $f(n) = n^2$ for any $n \in \mathbb{N}$.

Solution. It is given that

$$f(m) + f(n) - mn | mf(m) + nf(n). \tag{1}$$

Taking $m = n = 1$ in (1), we have $2f(1) - 1 | 2f(1)$. Then $2f(1) - 1 | 2f(1) - (2f(1) - 1) = 1$ and hence $f(1) = 1$.

Let $p \geqslant 7$ be a prime. Taking $m = p$ and $n = 1$ in (1), we have $f(p) - p + 1 | pf(p) + 1$ and hence

$$f(p) - p + 1 | pf(p) + 1 - p(f(p) - p + 1) = p^2 - p + 1.$$

If $f(p) - p + 1 = p^2 - p + 1$, then $f(p) = p^2$. If $f(p) - p + 1 \neq p^2 - p + 1$, as $p^2 - p + 1$ is an odd positive integer, we have $p^2 - p + 1 \geqslant 3(f(p) - p + 1)$, that is,

$$f(p) \leqslant \frac{1}{3}(p^2 + 2p - 2). \tag{2}$$

Taking $m = n = p$ in (1), we have $2f(p) - p^2 | 2pf(p)$. This implies

$$2f(p) - p^2 | 2pf(p) - p(2f(p) - p^2) = p^3.$$

By (2) and $f(p) \geqslant 1$, we get

$$-p^2 < 2f(p) - p^2 \leqslant \frac{2}{3}(p^2 + 2p - 2) - p^2 < -p$$

since $p \geqslant 7$. This contradicts the fact that $2f(p) - p^2$ is a factor of p^3. Thus we have proved that $f(p) = p^2$ for all primes $p \geqslant 7$.

Let n be a fixed positive integer. Choose a sufficiently large prime p. Consider $m = p$ in (1). We obtain

$$f(p) + f(n) - pn | pf(p) + nf(n) - n(f(p) + f(n) - pn) = pf(p) - nf(p) + pn^2.$$

As $f(p) = p^2$, this implies $p^2 - pn + f(n) | p(p^2 - pn + n^2)$. As p is sufficiently large and n is fixed, p cannot divide $f(n)$, and so $(p, p^2 - pn + f(n)) = 1$. It follows that $p^2 - pn + f(n) | p^2 - pn + n^2$ and hence

$$p^2 - pn + f(n) | (p^2 - pn + n^2) - (p^2 - pn + f(n)) = n^2 - f(n).$$

Note that $n^2 - f(n)$ is fixed while $p^2 - pn + f(n)$ is chosen to be sufficiently large. Therefore, we must have $n^2 - f(n) = 0$ so that $f(n) = n^2$ for any positive integer n.

Finally, we check that when $f(n) = n^2$ for any positive integer n, then

$$f(m) + f(n) - mn = m^2 + n^2 - mn$$

and

$$mf(m) + nf(n) = m^3 + n^3 = (m + n)(m^2 + n^2 - mn).$$

The latter expression is divisible by the former for any positive integers m, n. This shows $f(n) = n^2$ is the only solution.

N7. Let n be an odd positive integer. In the Cartesian plane, a cyclic polygon P with area S is chosen. All its vertices have integral coordinates, and the squares of its side lengths are all divisible by n. Prove that $2S$ is an integer divisible by n.

Solution. Let $P = A_1A_2 \ldots A_k$ and let $A_{k+i} = A_i$ for $i \geqslant 1$. By the Shoelace Formula, the area of any convex polygon with integral coordinates is half an integer. Therefore, $2S$ is an integer. We shall prove by induction on $k \geqslant 3$ that $2S$ is divisible by n. Clearly, it suffices to consider $n = p^t$ where p is an odd prime and $t \geqslant 1$.

For the base case $k = 3$, let the side lengths of P be $\sqrt{na}, \sqrt{nb}, \sqrt{nc}$ where a, b, c are positive integers. By Heron's Formula,

$$16S^2 = n^2(2ab + 2bc + 2ca - a^2 - b^2 - c^2).$$

This shows $16S^2$ is divisible by n^2. Since n is odd, $2S$ is divisible by n.

Assume $k \geqslant 4$. If the square of length of one of the diagonals is divisible by n, then that diagonal divides P into two smaller polygons, to which the induction hypothesis applies. Hence we may assume that none of the squares of diagonal lengths is divisible by n. As usual, we denote by $\nu_p(r)$ the exponent of p in the prime decomposition of r. We claim the following.

- **Claim.** $\nu_p(A_1A_m^2) > \nu_p(A_1A_{m+1}^2)$ for $2 \leqslant m \leqslant k-1$.

Proof. The case $m = 2$ is obvious since $\nu_p(A_1A_2^2) \geqslant p^t > \nu_p(A_1A_3^2)$ by the condition and the above assumption.

Suppose $\nu_p(A_1A_2^2) > \nu_p(A_1A_3^2) > \cdots > \nu_p(A_1A_m^2)$ where $3 \leqslant m \leqslant k-1$. For the induction step, we apply Ptolemy's Theorem to the cyclic quadrilateral $A_1A_{m-1}A_mA_{m+1}$ to get

$$A_1A_{m+1} \times A_{m-1}A_m + A_1A_{m-1} \times A_mA_{m+1} = A_1A_m \times A_{m-1}A_{m+1},$$

which can be rewritten as

$$\begin{aligned} A_1A_{m+1}^2 \times A_{m-1}A_m^2 = {} & A_1A_{m-1}^2 \times A_mA_{m+1}^2 + A_1A_m^2 \times A_{m-1}A_{m+1}^2 \\ & - 2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}. \end{aligned} \tag{1}$$

From this, $2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}$ is an integer. We consider the component of p of each term in (1). By the inductive hypothesis, we have $\nu_p(A_1A_{m-1}^2) > \nu_p(A_1A_m^2)$. Also, we have $\nu_p(A_mA_{m+1}^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$. These give

$$\nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{2}$$

Next, we have $\nu_p(4A_1A_{m-1}^2 \times A_mA_{m+1}^2 \times A_1A_m^2 \times A_{m-1}A_{m+1}^2) = \nu_p(A_1A_{m-1}^2 \times A_mA_{m+1}^2) + \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2) > 2\nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2)$ from (2). This implies

$$\nu_p(2A_1A_{m-1} \times A_mA_{m+1} \times A_1A_m \times A_{m-1}A_{m+1}) > \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2). \tag{3}$$

Combining (1), (2) and (3), we conclude that

$$\nu_p(A_1A_{m+1}^2 \times A_{m-1}A_m^2) = \nu_p(A_1A_m^2 \times A_{m-1}A_{m+1}^2).$$

By $\nu_p(A_{m-1}A_m^2) \geqslant p^t > \nu_p(A_{m-1}A_{m+1}^2)$, we get $\nu_p(A_1A_{m+1}^2) < \nu_p(A_1A_m^2)$. The Claim follows by induction. $\square$

From the Claim, we get a chain of inequalities

$$p^t > \nu_p(A_1A_3^2) > \nu_p(A_1A_4^2) > \cdots > \nu_p(A_1A_k^2) \geqslant p^t,$$

which yields a contradiction. Therefore, we can show by induction that $2S$ is divisible by n.

Comment. The condition that P is cyclic is crucial. As a counterexample, consider the rhombus with vertices $(0,3),(4,0),(0,-3),(-4,0)$. Each of its squares of side lengths is divisible by 5, while $2S=48$ is not.

The proposer also gives a proof for the case n is even. One just needs an extra technical step for the case $p=2$.

N8. Find all polynomials $P(x)$ of odd degree d and with integer coefficients satisfying the following property: for each positive integer n, there exist n positive integers $x_1, x_2, \ldots, x_n$ such that $\frac{1}{2} < \frac{P(x_i)}{P(x_j)} < 2$ and $\frac{P(x_i)}{P(x_j)}$ is the d-th power of a rational number for every pair of indices i and j with $1 \leqslant i, j \leqslant n$.

Answer. $P(x) = a(rx+s)^d$ where a, r, s are integers with $a \neq 0$, $r \geqslant 1$ and $(r, s) = 1$.

Solution. Let $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$. Consider the substitution $y = da_d x + a_{d-1}$. By defining $Q(y) = P(x)$, we find that Q is a polynomial with rational coefficients without the term y^{d-1}. Let $Q(y) = b_d y^d + b_{d-2}y^{d-2} + b_{d-3}y^{d-3} + \cdots + b_0$ and $B = \max_{0 \leqslant i \leqslant d} \{|b_i|\}$ (where $b_{d-1} = 0$).

The condition shows that for each $n \geqslant 1$, there exist integers $y_1, y_2, \ldots, y_n$ such that $\frac{1}{2} < \frac{Q(y_i)}{Q(y_j)} < 2$ and $\frac{Q(y_i)}{Q(y_j)}$ is the d-th power of a rational number for $1 \leqslant i, j \leqslant n$. Since n can be arbitrarily large, we may assume all x_i's and hence y_i's are integers larger than some absolute constant in the following.

By Dirichlet's Theorem, since d is odd, we can find a sufficiently large prime p such that $p \equiv 2 \pmod{d}$. In particular, we have $(p-1, d) = 1$. For this fixed p, we choose n to be sufficiently large. Then by the Pigeonhole Principle, there must be $d+1$ of $y_1, y_2, \ldots, y_n$ which are congruent mod p. Without loss of generality, assume $y_i \equiv y_j \pmod{p}$ for $1 \leqslant i, j \leqslant d+1$. We shall establish the following.

- **Claim.** $\frac{Q(y_i)}{Q(y_1)} = \frac{y_i^d}{y_1^d}$ for $2 \leqslant i \leqslant d+1$.

Proof. Let $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d}$ where $(l, m) = 1$ and $l, m > 0$. This can be rewritten in the expanded form

$$b_d(m^d y_i^d - l^d y_1^d) = -\sum_{j=0}^{d-2} b_j (m^d y_i^j - l^d y_1^j). \tag{1}$$

Let c be the common denominator of Q, so that $cQ(k)$ is an integer for any integer k. Note that c depends only on P and so we may assume $(p, c) = 1$. Then $y_1 \equiv y_i \pmod{p}$ implies $cQ(y_1) \equiv cQ(y_i) \pmod{p}$.

- **Case 1.** $p | cQ(y_1)$.

In this case, there is a cancellation of p in the numerator and denominator of $\frac{cQ(y_i)}{cQ(y_1)}$, so that $m^d \leqslant p^{-1}|cQ(y_1)|$. Noting $|Q(y_1)| < 2By_1^d$ as y_1 is large, we get

$$m \leqslant p^{-\frac{1}{d}} (2cB)^{\frac{1}{d}} y_1. \tag{2}$$

For large y_1 and y_i, the relation $\frac{1}{2} < \frac{Q(y_i)}{Q(y_1)} < 2$ implies

$$\frac{1}{3} < \frac{y_i^d}{y_1^d} < 3. \tag{3}$$

We also have

$$\frac{1}{2} < \frac{l^d}{m^d} < 2. \tag{4}$$

Now, the left-hand side of (1) is

$$b_d(my_i - ly_1)(m^{d-1}y_i^{d-1} + m^{d-2}y_i^{d-2}ly_1 + \cdots + l^{d-1}y_1^{d-1}).$$

Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the absolute value of the above expression is at least $|b_d|m^{d-1}y_i^{d-1}$. On the other hand, the absolute value of the right-hand side of (1) is at most

$$\begin{aligned}\sum_{j=0}^{d-2} B(m^d y_i^j + l^d y_1^j) &\leqslant (d-1)B(m^d y_i^{d-2} + l^d y_1^{d-2})\\ &\leqslant (d-1)B(7m^d y_i^{d-2})\\ &\leqslant 7(d-1)B(p^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}y_1)m^{d-1}y_i^{d-2}\\ &\leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1}\end{aligned}$$

by using successively (3), (4), (2) and again (3). This shows

$$|b_d|m^{d-1}y_i^{d-1} \leqslant 21(d-1)Bp^{-\frac{1}{d}}(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which is a contradiction for large p as b_d, B, c, d depend only on the polynomial P. Therefore, we have $my_i - ly_1 = 0$ in this case.

- **Case 2.** $(p, cQ(y_1)) = 1$.

From $cQ(y_1) \equiv cQ(y_i) \pmod p$, we have $l^d \equiv m^d \pmod p$. Since $(p-1, d) = 1$, we use Fermat Little Theorem to conclude $l \equiv m \pmod p$. Then $p | my_i - ly_1$. Suppose on the contrary that $my_i - ly_1 \neq 0$. Then the left-hand side of (1) has absolute value at least $|b_d|pm^{d-1}y_i^{d-1}$. Similar to Case 1, the right-hand side of (1) has absolute value at most

$$21(d-1)B(2cB)^{\frac{1}{d}}m^{d-1}y_i^{d-1},$$

which must be smaller than $|b_d|pm^{d-1}y_i^{d-1}$ for large p. Again this yields a contradiction and hence $my_i - ly_1 = 0$.

In both cases, we find that $\frac{Q(y_i)}{Q(y_1)} = \frac{l^d}{m^d} = \frac{y_i^d}{y_1^d}$. □

From the Claim, the polynomial $Q(y_1)y^d - y_1^d Q(y)$ has roots $y = y_1, y_2, \ldots, y_{d+1}$. Since its degree is at most d, this must be the zero polynomial. Hence, $Q(y) = b_d y^d$. This implies $P(x) = a_d(x + \frac{a_{d-1}}{da_d})^d$. Let $\frac{a_{d-1}}{da_d} = \frac{s}{r}$ with integers r, s where $r \geqslant 1$ and $(r, s) = 1$. Since P has integer coefficients, we need $r^d | a_d$. Let $a_d = r^d a$. Then $P(x) = a(rx+s)^d$. It is obvious that such a polynomial satisfies the conditions.

Comment. In the proof, the use of prime and Dirichlet's Theorem can be avoided. One can easily show that each $P(x_i)$ can be expressed in the form uv_i^d where u, v_i are integers and u cannot be divisible by the d-th power of a prime (note that u depends only on P). By fixing a large integer q and by choosing a large n, we can apply the Pigeonhole Principle and assume

$x_1 \equiv x_2 \equiv \cdots \equiv x_{d+1} \pmod{q}$ and $v_1 \equiv v_2 \equiv \cdots \equiv v_{d+1} \pmod{q}$. Then the remaining proof is similar to Case 2 of the Solution.

Alternatively, we give another modification of the proof as follows.

We take a sufficiently large n and consider the corresponding positive integers $y_1, y_2, \ldots, y_n$. For each $2 \leqslant i \leqslant n$, let $\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m_i^d}$.

As in Case 1, if there are d indices i such that the integers $\frac{c|Q(y_1)|}{m_i^d}$ are bounded below by a constant depending only on P, we can establish the Claim using those y_i's and complete the proof. Similarly, as in Case 2, if there are d indices i such that the integers $|m_i y_i - l_i y_1|$ are bounded below, then the proof goes the same. So it suffices to consider the case where $\frac{c|Q(y_1)|}{m_i^d} \leqslant M$ and $|m_i y_i - l_i y_1| \leqslant N$ for all $2 \leqslant i \leqslant n'$ where M, N are fixed constants and n' is large. Since there are only finitely many choices for m_i and $m_i y_i - l_i y_1$, by the Pigeonhole Principle, we can assume without loss of generality $m_i = m$ and $m_i y_i - l_i y_1 = t$ for $2 \leqslant i \leqslant d+2$. Then

$$\frac{Q(y_i)}{Q(y_1)} = \frac{l_i^d}{m^d} = \frac{(my_i - t)^d}{m^d y_1^d}$$

so that $Q(y_1)(my-t)^d - m^d y_1^d Q(y)$ has roots $y = y_2, y_3, \ldots, y_{d+2}$. Its degree is at most d and hence it is the zero polynomial. Therefore, $Q(y) = \frac{b_d}{m^d}(my-t)^d$. Indeed, Q does not have the term y^{d-1}, which means t should be 0. This gives the corresponding $P(x)$ of the desired form.

The two modifications of the Solution work equally well when the degree d is even.

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Zhuo Qun (Alex) Song, Ralph Costa Teixeira, Eduardo Tengan

Problems

Algebra

A1. Let $a_1, a_2, \ldots, a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \ldots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

(Trinidad and Tobago)

A2. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
- In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

(Austria)

A3. Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

(India)

A4. A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all } n > 2017.$$

Prove that this sequence is bounded, i.e., there is a constant M such that $|a_n| \leqslant M$ for all positive integers n.

(Russia)

A5. An integer $n \geqslant 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1 y_2 + y_2 y_3 + y_3 y_4 + \cdots + y_{n-1} y_n \geqslant -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j \geqslant K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

(Serbia)

A6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(y)) + f(x+y) = f(xy)$$

for all $x, y \in \mathbb{R}$.

(Albania)

A7. Let $a_0, a_1, a_2, \dots$ be a sequence of integers and $b_0, b_1, b_2, \dots$ be a sequence of *positive* integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1}, & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1}, & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \dots.$$

Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

(Australia)

A8. Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

For every $x, y \in \mathbb{R}$ such that $\big(f(x)+y\big)\big(f(y)+x\big) > 0$, we have $f(x)+y = f(y)+x$.

Prove that $f(x) + y \leqslant f(y) + x$ whenever $x > y$.

(Netherlands)

Combinatorics

C1. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

(Singapore)

C2. Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

(Australia)

C3. Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

(1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

(2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

(Thailand)

C4. Let $N \geqslant 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row in some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between the two tallest ones, no one stands between the third and the fourth tallest ones, ..., and finally no one stands between the two shortest ones. Show that this is always possible.

(Russia)

C5. A hunter and an invisible rabbit play a game in the Euclidean plane. The hunter's starting point H_0 coincides with the rabbit's starting point R_0. In the n^{th} round of the game $(n \geqslant 1)$, the following happens.

(1) First the invisible rabbit moves secretly and unobserved from its current point R_{n-1} to some new point R_n with $R_{n-1}R_n = 1$.

(2) The hunter has a tracking device (e.g. dog) that returns an approximate position R_n' of the rabbit, so that $R_nR_n' \leqslant 1$.

(3) The hunter then visibly moves from point H_{n-1} to a new point H_n with $H_{n-1}H_n = 1$.

Is there a strategy for the hunter that guarantees that after 10^9 such rounds the distance between the hunter and the rabbit is below 100?

(Austria)

C6. Let $n > 1$ be an integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one color. For each $n \times n \times 1$ box consisting of n^2 unit cubes (of any of the three possible orientations), we consider the set of the colors present in that box (each color is listed only once). This way, we get $3n$ sets of colors, split into three groups according to the orientation. It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colors that are present.

(Russia)

C7. For any finite sets X and Y of positive integers, denote by $f_X(k)$ the k^{th} smallest positive integer not in X, and let

$$X * Y = X \cup \{f_X(y) \colon y \in Y\}.$$

Let A be a set of $a > 0$ positive integers, and let B be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then

$$\underbrace{A * (A * \cdots * (A * (A * A)) \ldots)}_{A \text{ appears } b \text{ times}} = \underbrace{B * (B * \cdots * (B * (B * B)) \ldots)}_{B \text{ appears } a \text{ times}}.$$

(U.S.A.)

C8. Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The *neighborhood* of a lattice point c consists of all lattice points within the axis-aligned $(2n+1) \times (2n+1)$ square centered at c, apart from c itself. We call a butterfly *lonely*, *crowded*, or *comfortable*, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N.

Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

(Bulgaria)

Geometry

G1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

(Italy)

G2. Let R and S be distinct points on circle Ω, and let t denote the tangent line to Ω at R. Point R' is the reflection of R with respect to S. A point I is chosen on the smaller arc RS of Ω so that the circumcircle Γ of triangle ISR' intersects t at two different points. Denote by A the common point of Γ and t that is closest to R. Line AI meets Ω again at J. Show that JR' is tangent to Γ.

(Luxembourg)

G3. Let O be the circumcenter of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

(Ukraine)

G4. In triangle ABC, let ω be the excircle opposite A. Let D, E, and F be the points where ω is tangent to lines BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

(Denmark)

G5. Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

(Ukraine)

G6. Let $n \geqslant 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

(Czech Republic)

G7. A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c, and I_d be the incenters of the triangles DAB, ABC, BCD, and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

(Kazakhstan)

G8. There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A *tangent segment* is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when he stops drawing.

(Australia)

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Number Theory

N1. The sequence $a_0, a_1, a_2, \ldots$ of positive integers satisfies

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer} \\ a_n + 3, & \text{otherwise} \end{cases} \qquad \text{for every } n \geqslant 0.$$

Determine all values of $a_0 > 1$ for which there is at least one number a such that $a_n = a$ for infinitely many values of n.

(South Africa)

N2. Let $p \geqslant 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

(Morocco)

N3. Determine all integers $n \geqslant 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leqslant i \leqslant n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

(Thailand)

N4. Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is m-*tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leqslant k < t$. Let $S(m)$ be the set of m-tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

(Turkey)

N5. Find all pairs (p, q) of prime numbers with $p > q$ for which the number

$$\frac{(p+q)^{p+q}(p-q)^{p-q} - 1}{(p+q)^{p-q}(p-q)^{p+q} - 1}$$

is an integer.

(Japan)

N6. Find the smallest positive integer n, or show that no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

(Singapore)

N7. Say that an ordered pair (x, y) of integers is an *irreducible lattice point* if x and y are relatively prime. For any finite set S of irreducible lattice points, show that there is a homogenous polynomial in two variables, $f(x, y)$, with integer coefficients, of degree at least 1, such that $f(x, y) = 1$ for each (x, y) in the set S.

Note: A homogenous polynomial of degree n is any nonzero polynomial of the form

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n.$$

(U.S.A.)

N8. Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f\colon \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0, 1\}$ satisfies the following properties:

- $f(1, 1) = 0$;
- $f(a, b) + f(b, a) = 1$ for any pair of relatively prime positive integers (a, b) not both equal to 1;
- $f(a + b, b) = f(a, b)$ for any pair of relatively prime positive integers (a, b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \sqrt{2p} - 2.$$

(Italy)

Solutions

Algebra

A1. Let $a_1, a_2, \ldots, a_n, k$, and M be positive integers such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = k \quad \text{and} \quad a_1 a_2 \ldots a_n = M.$$

If $M > 1$, prove that the polynomial

$$P(x) = M(x+1)^k - (x+a_1)(x+a_2)\cdots(x+a_n)$$

has no positive roots.

(Trinidad and Tobago)

Solution 1. We first prove that, for $x > 0$,

$$a_i(x+1)^{1/a_i} \leqslant x + a_i, \tag{1}$$

with equality if and only if $a_i = 1$. It is clear that equality occurs if $a_i = 1$.

If $a_i > 1$, the AM–GM inequality applied to a single copy of $x+1$ and $a_i - 1$ copies of 1 yields

$$\frac{(x+1) + \overbrace{1 + 1 + \cdots + 1}^{a_i - 1 \text{ ones}}}{a_i} \geqslant \sqrt[a_i]{(x+1) \cdot 1^{a_i - 1}} \implies a_i(x+1)^{1/a_i} \leqslant x + a_i.$$

Since $x + 1 > 1$, the inequality is strict for $a_i > 1$.

Multiplying the inequalities (1) for $i = 1, 2, \ldots, n$ yields

$$\prod_{i=1}^{n} a_i(x+1)^{1/a_i} \leqslant \prod_{i=1}^{n}(x+a_i) \iff M(x+1)^{\sum_{i=1}^{n} 1/a_i} - \prod_{i=1}^{n}(x+a_i) \leqslant 0 \iff P(x) \leqslant 0$$

with equality iff $a_i = 1$ for all $i \in \{1, 2, \ldots, n\}$. But this implies $M = 1$, which is not possible. Hence $P(x) < 0$ for all $x \in \mathbb{R}^+$, and P has no positive roots.

Comment 1. Inequality (1) can be obtained in several ways. For instance, we may also use the binomial theorem: since $a_i \geqslant 1$,

$$\left(1 + \frac{x}{a_i}\right)^{a_i} = \sum_{j=0}^{a_i} \binom{a_i}{j} \left(\frac{x}{a_i}\right)^j \geqslant \binom{a_i}{0} + \binom{a_i}{1} \cdot \frac{x}{a_i} = 1 + x.$$

Both proofs of (1) mimic proofs to Bernoulli's inequality for a positive integer exponent a_i; we can use this inequality directly:

$$\left(1 + \frac{x}{a_i}\right)^{a_i} \geqslant 1 + a_i \cdot \frac{x}{a_i} = 1 + x,$$

and so

$$x + a_i = a_i\left(1 + \frac{x}{a_i}\right) \geqslant a_i(1+x)^{1/a_i},$$

or its (reversed) formulation, with exponent $1/a_i \leqslant 1$:

$$(1+x)^{1/a_i} \leqslant 1 + \frac{1}{a_i} \cdot x = \frac{x + a_i}{a_i} \implies a_i(1+x)^{1/a_i} \leqslant x + a_i.$$

Solution 2. We will prove that, in fact, all coefficients of the polynomial $P(x)$ are non-positive, and at least one of them is negative, which implies that $P(x) < 0$ for $x > 0$.

Indeed, since $a_j \geqslant 1$ for all j and $a_j > 1$ for some j (since $a_1 a_2 \ldots a_n = M > 1$), we have $k = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} < n$, so the coefficient of x^n in $P(x)$ is $-1 < 0$. Moreover, the coefficient of x^r in $P(x)$ is negative for $k < r \leqslant n = \deg(P)$.

For $0 \leqslant r \leqslant k$, the coefficient of x^r in $P(x)$ is

$$M \cdot \binom{k}{r} - \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-r} \leqslant n} a_{i_1} a_{i_2} \cdots a_{i_{n-r}} = a_1 a_2 \cdots a_n \cdot \binom{k}{r} - \sum_{1 \leqslant i_1 < i_2 < \cdots < i_{n-r} \leqslant n} a_{i_1} a_{i_2} \cdots a_{i_{n-r}},$$

which is non-positive iff

$$\binom{k}{r} \leqslant \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}}. \tag{2}$$

We will prove (2) by induction on r. For $r = 0$ it is an equality because the constant term of $P(x)$ is $P(0) = 0$, and if $r = 1$, (2) becomes $k = \sum_{i=1}^{n} \frac{1}{a_i}$. For $r > 1$, if (2) is true for a given $r < k$, we have

$$\binom{k}{r+1} = \frac{k-r}{r+1} \cdot \binom{k}{r} \leqslant \frac{k-r}{r+1} \cdot \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}},$$

and it suffices to prove that

$$\frac{k-r}{r+1} \cdot \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \leqslant \sum_{1 \leqslant j_1 < \cdots < j_r < j_{r+1} \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}},$$

which is equivalent to

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - r\right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \leqslant (r+1) \sum_{1 \leqslant j_1 < \cdots < j_r < j_{r+1} \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}.$$

Since there are $r+1$ ways to choose a fraction $\frac{1}{a_{j_i}}$ from $\frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}$ to factor out, every term $\frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r} a_{j_{r+1}}}$ in the right hand side appears exactly $r+1$ times in the product

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \sum_{1 \leqslant j_1 < j_2 < \cdots < j_r \leqslant n} \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}}.$$

Hence all terms in the right hand side cancel out.

The remaining terms in the left hand side can be grouped in sums of the type

$$\begin{aligned} &\frac{1}{a_{j_1}^2 a_{j_2} \cdots a_{j_r}} + \frac{1}{a_{j_1} a_{j_2}^2 \cdots a_{j_r}} + \cdots + \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}^2} - \frac{r}{a_{j_1} a_{j_2} \cdots a_{j_r}} \\ &\qquad = \frac{1}{a_{j_1} a_{j_2} \cdots a_{j_r}} \left(\frac{1}{a_{j_1}} + \frac{1}{a_{j_2}} + \cdots + \frac{1}{a_{j_r}} - r\right), \end{aligned}$$

which are all non-positive because $a_i \geqslant 1 \implies \frac{1}{a_i} \leqslant 1$, $i = 1, 2, \ldots, n$.

Comment 2. The result is valid for any real numbers a_i, $i = 1, 2, \ldots, n$ with $a_i \geqslant 1$ and product M greater than 1. A variation of Solution 1, namely using *weighted* AM–GM (or the Bernoulli inequality for real exponents), actually proves that $P(x) < 0$ for $x > -1$ and $x \neq 0$.

A2. Let q be a real number. Gugu has a napkin with ten distinct real numbers written on it, and he writes the following three lines of real numbers on the blackboard:

- In the first line, Gugu writes down every number of the form $a-b$, where a and b are two (not necessarily distinct) numbers on his napkin.
- In the second line, Gugu writes down every number of the form qab, where a and b are two (not necessarily distinct) numbers from the *first line*.
- In the third line, Gugu writes down every number of the form $a^2+b^2-c^2-d^2$, where a, b, c, d are four (not necessarily distinct) numbers from the *first line*.

Determine all values of q such that, regardless of the numbers on Gugu's napkin, every number in the second line is also a number in the third line.

(Austria)

Answer: $-2, 0, 2$.

Solution 1. Call a number q *good* if every number in the second line appears in the third line unconditionally. We first show that the numbers 0 and ± 2 are good. The third line necessarily contains 0, so 0 is good. For any two numbers a, b in the first line, write $a=x-y$ and $b=u-v$, where x, y, u, v are (not necessarily distinct) numbers on the napkin. We may now write

$$2ab = 2(x-y)(u-v) = (x-v)^2+(y-u)^2-(x-u)^2-(y-v)^2,$$

which shows that 2 is good. By negating both sides of the above equation, we also see that -2 is good.

We now show that $-2, 0$, and 2 are the only good numbers. Assume for sake of contradiction that q is a good number, where $q \notin \{-2, 0, 2\}$. We now consider some particular choices of numbers on Gugu's napkin to arrive at a contradiction.

Assume that the napkin contains the integers $1, 2, \ldots, 10$. Then, the first line contains the integers $-9, -8, \ldots, 9$. The second line then contains q and $81q$, so the third line must also contain both of them. But the third line only contains integers, so q must be an integer. Furthermore, the third line contains no number greater than $162 = 9^2+9^2-0^2-0^2$ or less than -162, so we must have $-162 \leqslant 81q \leqslant 162$. This shows that the only possibilities for q are ± 1.

Now assume that $q = \pm 1$. Let the napkin contain $0, 1, 4, 8, 12, 16, 20, 24, 28, 32$. The first line contains ± 1 and ± 4, so the second line contains ± 4. However, for every number a in the first line, $a \not\equiv 2 \pmod 4$, so we may conclude that $a^2 \equiv 0, 1 \pmod 8$. Consequently, every number in the third line must be congruent to $-2, -1, 0, 1, 2 \pmod 8$; in particular, ± 4 cannot be in the third line, which is a contradiction.

Solution 2. Let q be a good number, as defined in the first solution, and define the polynomial $P(x_1, \ldots, x_{10})$ as

$$\prod_{i<j}(x_i-x_j)\prod_{a_i \in S}\left(q(x_1-x_2)(x_3-x_4)-(a_1-a_2)^2-(a_3-a_4)^2+(a_5-a_6)^2+(a_7-a_8)^2\right),$$

where $S = \{x_1, \ldots, x_{10}\}$.

We claim that $P(x_1, \ldots, x_{10}) = 0$ for every choice of real numbers $(x_1, \ldots, x_{10})$. If any two of the x_i are equal, then $P(x_1, \ldots, x_{10}) = 0$ trivially. If no two are equal, assume that Gugu has those ten numbers $x_1, \ldots, x_{10}$ on his napkin. Then, the number $q(x_1-x_2)(x_3-x_4)$ is in the second line, so we must have some $a_1, \ldots, a_8$ so that

$$q(x_1-x_2)(x_3-x_4)-(a_1-a_2)^2-(a_3-a_4)^2+(a_5-a_6)^2+(a_7-a_8)^2 = 0,$$

and hence $P(x_1, \dots, x_{10}) = 0$.

Since every polynomial that evaluates to zero everywhere is the zero polynomial, and the product of two nonzero polynomials is necessarily nonzero, we may define F such that

$$F(x_1, \dots, x_{10}) \equiv q(x_1 - x_2)(x_3 - x_4) - (a_1 - a_2)^2 - (a_3 - a_4)^2 + (a_5 - a_6)^2 + (a_7 - a_8)^2 \equiv 0 \quad (1)$$

for some particular choice $a_i \in S$.

Each of the sets $\{a_1, a_2\}$, $\{a_3, a_4\}$, $\{a_5, a_6\}$, and $\{a_7, a_8\}$ is equal to at most one of the four sets $\{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_4\}$, and $\{x_2, x_4\}$. Thus, without loss of generality, we may assume that at most one of the sets $\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}$, and $\{a_7, a_8\}$ is equal to $\{x_1, x_3\}$. Let u_1, u_3, u_5, u_7 be the indicator functions for this equality of sets: that is, $u_i = 1$ if and only if $\{a_i, a_{i+1}\} = \{x_1, x_3\}$. By assumption, at least three of the u_i are equal to 0.

We now compute the coefficient of x_1x_3 in F. It is equal to $q + 2(u_1 + u_3 - u_5 - u_7) = 0$, and since at least three of the u_i are zero, we must have that $q \in \{-2, 0, 2\}$, as desired.

A3. Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

(India)

Solution. For $n \geqslant 1$, denote the n-th composition of f with itself by

$$f^n \stackrel{\text{def}}{=} \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}.$$

By hypothesis, if $g \in \mathcal{A}$ satisfies $f \circ g \circ f = g \circ f \circ g$, then $g = f$. A natural idea is to try to plug in $g = f^n$ for some n in the expression $f \circ g \circ f = g \circ f \circ g$ in order to get $f^n = f$, which solves the problem:

Claim. If there exists $n \geqslant 3$ such that $f^{n+2} = f^{2n+1}$, then the restriction $f \colon T \to T$ of f to T is a bijection.

Proof. Indeed, by hypothesis, $f^{n+2} = f^{2n+1} \iff f \circ f^n \circ f = f^n \circ f \circ f^n \implies f^n = f$. Since $n - 2 \geqslant 1$, the image of f^{n-2} is contained in $T = f(S)$, hence f^{n-2} restricts to a function $f^{n-2} \colon T \to T$. This is the inverse of $f \colon T \to T$. In fact, given $t \in T$, say $t = f(s)$ with $s \in S$, we have

$$t = f(s) = f^n(s) = f^{n-2}(f(t)) = f(f^{n-2}(t)), \qquad \text{i.e.,} \qquad f^{n-2} \circ f = f \circ f^{n-2} = \text{id on } T$$

(here id stands for the identity function). Hence, the restriction $f \colon T \to T$ of f to T is bijective with inverse given by $f^{n-2} \colon T \to T$. $\square$

It remains to show that n as in the claim exists. For that, define

$$S_m \stackrel{\text{def}}{=} f^m(S) \qquad\qquad (S_m \text{ is image of } f^m)$$

Clearly the image of f^{m+1} is contained in the image of f^m, i.e., there is a descending chain of subsets of S

$$S \supseteq S_1 \supseteq S_2 \supseteq S_3 \supseteq S_4 \supseteq \cdots,$$

which must eventually stabilise since S is finite, i.e., there is a $k \geqslant 1$ such that

$$S_k = S_{k+1} = S_{k+2} = S_{k+3} = \cdots \stackrel{\text{def}}{=} S_\infty.$$

Hence f restricts to a *surjective* function $f \colon S_\infty \to S_\infty$, which is also *bijective* since $S_\infty \subseteq S$ is finite. To sum up, $f \colon S_\infty \to S_\infty$ is a permutation of the elements of the finite set S_∞, hence there exists an integer $r \geqslant 1$ such that $f^r = \text{id}$ on S_∞ (for example, we may choose $r = |S_\infty|!$). In other words,

$$f^{m+r} = f^m \text{ on } S \text{ for all } m \geqslant k. \tag{$*$}$$

Clearly, $(*)$ also implies that $f^{m+tr} = f^m$ for all integers $t \geqslant 1$ and $m \geqslant k$. So, to find n as in the claim and finish the problem, it is enough to choose m and t in order to ensure that there exists $n \geqslant 3$ satisfying

$$\begin{cases} 2n + 1 = m + tr \\ n + 2 = m \end{cases} \iff \begin{cases} m = 3 + tr \\ n = m - 2. \end{cases}$$

This can be clearly done by choosing m large enough with $m \equiv 3 \pmod{r}$. For instance, we may take $n = 2kr + 1$, so that

$$f^{n+2} = f^{2kr+3} = f^{4kr+3} = f^{2n+1}$$

where the middle equality follows by $(*)$ since $2kr + 3 \geqslant k$.

A4. A sequence of real numbers $a_1, a_2, \ldots$ satisfies the relation

$$a_n = -\max_{i+j=n}(a_i + a_j) \qquad \text{for all } n > 2017.$$

Prove that this sequence is bounded, i.e., there is a constant M such that $|a_n| \leqslant M$ for all positive integers n.

(Russia)

Solution 1. Set $D = 2017$. Denote

$$M_n = \max_{k<n} a_k \qquad \text{and} \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Clearly, the sequences (m_n) and (M_n) are nondecreasing. We need to prove that both are bounded.

Consider an arbitrary $n > D$; our first aim is to bound a_n in terms of m_n and M_n.

(i) There exist indices p and q such that $a_n = -(a_p + a_q)$ and $p + q = n$. Since $a_p, a_q \leqslant M_n$, we have $a_n \geqslant -2M_n$.

(ii) On the other hand, choose an index $k < n$ such that $a_k = M_n$. Then, we have

$$a_n = -\max_{\ell<n}(a_{n-\ell} + a_\ell) \leqslant -(a_{n-k} + a_k) = -a_{n-k} - M_n \leqslant m_n - M_n.$$

Summarizing (i) and (ii), we get

$$-2M_n \leqslant a_n \leqslant m_n - M_n,$$

whence

$$m_n \leqslant m_{n+1} \leqslant \max\{m_n, 2M_n\} \quad \text{and} \quad M_n \leqslant M_{n+1} \leqslant \max\{M_n, m_n - M_n\}. \tag{1}$$

Now, say that an index $n > D$ is *lucky* if $m_n \leqslant 2M_n$. Two cases are possible.

Case 1. Assume that there exists a lucky index n. In this case, (1) yields $m_{n+1} \leqslant 2M_n$ and $M_n \leqslant M_{n+1} \leqslant M_n$. Therefore, $M_{n+1} = M_n$ and $m_{n+1} \leqslant 2M_n = 2M_{n+1}$. So, the index $n+1$ is also lucky, and $M_{n+1} = M_n$. Applying the same arguments repeatedly, we obtain that all indices $k > n$ are lucky (i.e., $m_k \leqslant 2M_k$ for all these indices), and $M_k = M_n$ for all such indices. Thus, all of the m_k and M_k are bounded by $2M_n$.

Case 2. Assume now that there is no lucky index, i.e., $2M_n < m_n$ for all $n > D$. Then (1) shows that for all $n > D$ we have $m_n \leqslant m_{n+1} \leqslant m_n$, so $m_n = m_{D+1}$ for all $n > D$. Since $M_n < m_n/2$ for all such indices, all of the m_n and M_n are bounded by m_{D+1}.

Thus, in both cases the sequences (m_n) and (M_n) are bounded, as desired.

Solution 2. As in the previous solution, let $D = 2017$. If the sequence is bounded above, say, by Q, then we have that $a_n \geqslant \min\{a_1, \ldots, a_D, -2Q\}$ for all n, so the sequence is bounded. Assume for sake of contradiction that the sequence is not bounded above. Let $\ell = \min\{a_1, \ldots, a_D\}$, and $L = \max\{a_1, \ldots, a_D\}$. Call an index n *good* if the following criteria hold:

$$a_n > a_i \text{ for each } i < n, \quad a_n > -2\ell, \quad \text{and} \quad n > D \tag{2}$$

We first show that there must be some good index n. By assumption, we may take an index N such that $a_N > \max\{L, -2\ell\}$. Choose n minimally such that $a_n = \max\{a_1, a_2, \ldots, a_N\}$. Now, the first condition in (2) is satisfied because of the minimality of n, and the second and third conditions are satisfied because $a_n \geqslant a_N > L, -2\ell$, and $L \geqslant a_i$ for every i such that $1 \leqslant i \leqslant D$.

Let n be a good index. We derive a contradiction. We have that

$$a_n + a_u + a_v \leqslant 0, \tag{3}$$

whenever $u + v = n$.

We define the index u to maximize a_u over $1 \leqslant u \leqslant n-1$, and let $v = n-u$. Then, we note that $a_u \geqslant a_v$ by the maximality of a_u.

Assume first that $v \leqslant D$. Then, we have that

$$a_N + 2\ell \leqslant 0,$$

because $a_u \geqslant a_v \geqslant \ell$. But this contradicts our assumption that $a_n > -2\ell$ in the second criteria of (2).

Now assume that $v > D$. Then, there exist some indices w_1, w_2 summing up to v such that

$$a_v + a_{w_1} + a_{w_2} = 0.$$

But combining this with (3), we have

$$a_n + a_u \leqslant a_{w_1} + a_{w_2}.$$

Because $a_n > a_u$, we have that $\max\{a_{w_1}, a_{w_2}\} > a_u$. But since each of the w_i is less than v, this contradicts the maximality of a_u.

Comment 1. We present two harder versions of this problem below.

Version 1. Let $a_1, a_2, \ldots$ be a sequence of numbers that satisfies the relation

$$a_n = -\max_{i+j+k=n}(a_i + a_j + a_k) \qquad \text{for all } n > 2017.$$

Then, this sequence is bounded.

Proof. Set $D = 2017$. Denote

$$M_n = \max_{k<n} a_k \qquad \text{and} \qquad m_n = -\min_{k<n} a_k = \max_{k<n}(-a_k).$$

Clearly, the sequences (m_n) and (M_n) are nondecreasing. We need to prove that both are bounded.

Consider an arbitrary $n > 2D$; our first aim is to bound a_n in terms of m_i and M_i. Set $k = \lfloor n/2 \rfloor$.

(i) Choose indices p, q, and r such that $a_n = -(a_p + a_q + a_r)$ and $p + q + r = n$. Without loss of generality, $p \geqslant q \geqslant r$.

Assume that $p \geqslant k + 1 (> D)$; then $p > q + r$. Hence

$$-a_p = \max_{i_1+i_2+i_3=p}(a_{i_1} + a_{i_2} + a_{i_3}) \geqslant a_q + a_r + a_{p-q-r},$$

and therefore $a_n = -(a_p + a_q + a_r) \geqslant (a_q + a_r + a_{p-q-r}) - a_q - a_r = a_{p-q-r} \geqslant -m_n$.

Otherwise, we have $k \geqslant p \geqslant q \geqslant r$. Since $n < 3k$, we have $r < k$. Then $a_p, a_q \leqslant M_{k+1}$ and $a_r \leqslant M_k$, whence $a_n \geqslant -2M_{k+1} - M_k$.

Thus, in any case $a_n \geqslant -\max\{m_n, 2M_{k+1} + M_k\}$.

(ii) On the other hand, choose $p \leqslant k$ and $q \leqslant k-1$ such that $a_p = M_{k+1}$ and $a_q = M_k$. Then $p+q < n$, so $a_n \leqslant -(a_p + a_q + a_{n-p-q}) = -a_{n-p-q} - M_{k+1} - M_k \leqslant m_n - M_{k+1} - M_k$.

To summarize,

$$-\max\{m_n, 2M_{k+1} + M_k\} \leqslant a_n \leqslant m_n - M_{k+1} - M_k,$$

whence

$$m_n \leqslant m_{n+1} \leqslant \max\{m_n, 2M_{k+1} + M_k\} \quad \text{and} \quad M_n \leqslant M_{n+1} \leqslant \max\{M_n, m_n - M_{k+1} - M_k\}. \tag{4}$$

Now, say that an index $n > 2D$ is *lucky* if $m_n \leqslant 2M_{\lfloor n/2\rfloor+1} + M_{\lfloor n/2\rfloor}$. Two cases are possible.

Case 1. Assume that there exists a lucky index n; set $k = \lfloor n/2\rfloor$. In this case, (4) yields $m_{n+1} \leqslant 2M_{k+1} + M_k$ and $M_n \leqslant M_{n+1} \leqslant M_n$ (the last relation holds, since $m_n - M_{k+1} - M_k \leqslant (2M_{k+1} + M_k) - M_{k+1} - M_k = M_{k+1} \leqslant M_n$). Therefore, $M_{n+1} = M_n$ and $m_{n+1} \leqslant 2M_{k+1} + M_k$; the last relation shows that the index $n+1$ is also lucky.

Thus, all indices $N > n$ are lucky, and $M_N = M_n \geqslant m_N/3$, whence all the m_N and M_N are bounded by $3M_n$.

Case 2. Conversely, assume that there is no lucky index, i.e., $2M_{\lfloor n/2\rfloor+1} + M_{\lfloor n/2\rfloor} < m_n$ for all $n > 2D$. Then (4) shows that for all $n > 2D$ we have $m_n \leqslant m_{n+1} \leqslant m_n$, i.e., $m_N = m_{2D+1}$ for all $N > 2D$. Since $M_N < m_{2N+1}/3$ for all such indices, all the m_N and M_N are bounded by m_{2D+1}.

Thus, in both cases the sequences (m_n) and (M_n) are bounded, as desired. □

Version 2. Let $a_1, a_2, \ldots$ be a sequence of numbers that satisfies the relation

$$a_n = -\max_{i_1+\cdots+i_k=n}(a_{i_1} + \cdots + a_{i_k}) \qquad \text{for all } n > 2017.$$

Then, this sequence is bounded.

Proof. As in the solutions above, let $D = 2017$. If the sequence is bounded above, say, by Q, then we have that $a_n \geqslant \min\{a_1, \ldots, a_D, -kQ\}$ for all n, so the sequence is bounded. Assume for sake of contradiction that the sequence is not bounded above. Let $\ell = \min\{a_1, \ldots, a_D\}$, and $L = \max\{a_1, \ldots, a_D\}$. Call an index n *good* if the following criteria hold:

$$a_n > a_i \text{ for each } i < n, \quad a_n > -k\ell, \quad \text{and} \quad n > D \tag{5}$$

We first show that there must be some good index n. By assumption, we may take an index N such that $a_N > \max\{L, -k\ell\}$. Choose n minimally such that $a_n = \max\{a_1, a_2, \ldots, a_N\}$. Now, the first condition is satisfied because of the minimality of n, and the second and third conditions are satisfied because $a_n \geqslant a_N > L, -k\ell$, and $L \geqslant a_i$ for every i such that $1 \leqslant i \leqslant D$.

Let n be a good index. We derive a contradiction. We have that

$$a_n + a_{v_1} + \cdots + a_{v_k} \leqslant 0, \tag{6}$$

whenever $v_1 + \cdots + v_k = n$.

We define the sequence of indices $v_1, \ldots, v_{k-1}$ to greedily maximize a_{v_1}, then a_{v_2}, and so forth, selecting only from indices such that the equation $v_1 + \cdots + v_k = n$ can be satisfied by positive integers $v_1, \ldots, v_k$. More formally, we define them inductively so that the following criteria are satisfied by the v_i:

1. $1 \leqslant v_i \leqslant n - (k-i) - (v_1 + \cdots + v_{i-1})$.
2. a_{v_i} is maximal among all choices of v_i from the first criteria.

First of all, we note that for each i, the first criteria is always satisfiable by some v_i, because we are guaranteed that

$$v_{i-1} \leqslant n - (k-(i-1)) - (v_1 + \cdots + v_{i-2}),$$

which implies

$$1 \leqslant n - (k-i) - (v_1 + \cdots + v_{i-1}).$$

Secondly, the sum $v_1 + \cdots + v_{k-1}$ is at most $n-1$. Define $v_k = n - (v_1 + \cdots + v_{k-1})$. Then, (6) is satisfied by the v_i. We also note that $a_{v_i} \geqslant a_{v_j}$ for all $i < j$; otherwise, in the definition of v_i, we could have selected v_j instead.

Assume first that $v_k \leqslant D$. Then, from (6), we have that

$$a_n + k\ell \leqslant 0,$$

by using that $a_{v_1} \geqslant \cdots \geqslant a_{v_k} \geqslant \ell$. But this contradicts our assumption that $a_n > -k\ell$ in the second criteria of (5).

Now assume that $v_k > D$, and then we must have some indices $w_1, \dots, w_k$ summing up to v_k such that

$$a_{v_k} + a_{w_1} + \cdots + a_{w_k} = 0.$$

But combining this with (6), we have

$$a_n + a_{v_1} + \cdots + a_{v_{k-1}} \leqslant a_{w_1} + \cdots + a_{w_k}.$$

Because $a_n > a_{v_1} \geqslant \cdots \geqslant a_{v_{k-1}}$, we have that $\max\{a_{w_1}, \dots, a_{w_k}\} > a_{v_{k-1}}$. But since each of the w_i is less than v_k, in the definition of the v_{k-1} we could have chosen one of the w_i instead, which is a contradiction. $\square$

Comment 2. It seems that each sequence satisfying the condition in Version 2 is eventually periodic, at least when its terms are integers.

However, up to this moment, the Problem Selection Committee is not aware of a proof for this fact (even in the case $k = 2$).

A5. An integer $n \geqslant 3$ is given. We call an n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ *Shiny* if for each permutation $y_1, y_2, \dots, y_n$ of these numbers we have

$$\sum_{i=1}^{n-1} y_i y_{i+1} = y_1y_2 + y_2y_3 + y_3y_4 + \cdots + y_{n-1}y_n \geqslant -1.$$

Find the largest constant $K = K(n)$ such that

$$\sum_{1 \leqslant i < j \leqslant n} x_i x_j \geqslant K$$

holds for every Shiny n-tuple $(x_1, x_2, \dots, x_n)$.

(Serbia)

Answer: $K = -(n-1)/2$.

Solution 1. First of all, we show that we may not take a larger constant K. Let t be a positive number, and take $x_2 = x_3 = \cdots = t$ and $x_1 = -1/(2t)$. Then, every product x_ix_j $(i \neq j)$ is equal to either t^2 or $-1/2$. Hence, for every permutation y_i of the x_i, we have

$$y_1y_2 + \cdots + y_{n-1}y_n \geqslant (n-3)t^2 - 1 \geqslant -1.$$

This justifies that the n-tuple $(x_1, \dots, x_n)$ is Shiny. Now, we have

$$\sum_{i<j} x_ix_j = -\frac{n-1}{2} + \frac{(n-1)(n-2)}{2}t^2.$$

Thus, as t approaches 0 from above, $\sum_{i<j} x_ix_j$ gets arbitrarily close to $-(n-1)/2$. This shows that we may not take K any larger than $-(n-1)/2$. It remains to show that $\sum_{i<j} x_ix_j \geqslant -(n-1)/2$ for any Shiny choice of the x_i.

From now onward, assume that $(x_1, \dots, x_n)$ is a Shiny n-tuple. Let the z_i $(1 \leqslant i \leqslant n)$ be some permutation of the x_i to be chosen later. The indices for z_i will always be taken modulo n. We will first split up the sum $\sum_{i<j} x_ix_j = \sum_{i<j} z_iz_j$ into $\lfloor (n-1)/2 \rfloor$ expressions, each of the form $y_1y_2 + \cdots + y_{n-1}y_n$ for some permutation y_i of the z_i, and some leftover terms. More specifically, write

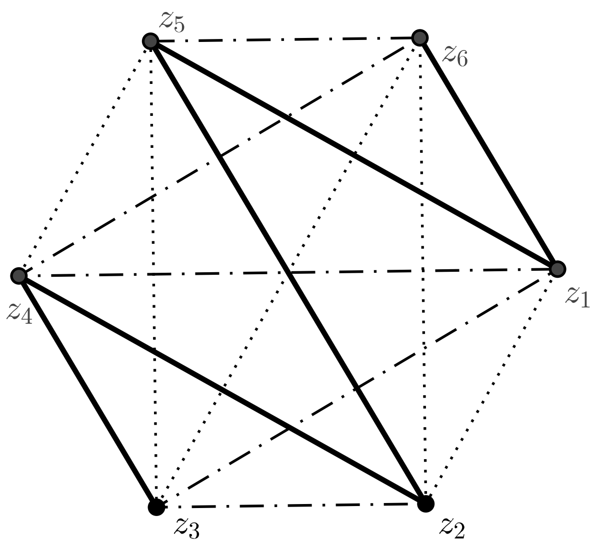
$$\sum_{i<j} z_iz_j = \sum_{q=0}^{n-1} \sum_{\substack{i+j \equiv q \pmod n \\ i \not\equiv j \pmod n}} z_iz_j = \sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{\substack{i+j \equiv 2p-1, 2p \pmod n \\ i \not\equiv j \pmod n}} z_iz_j + L, \tag{1}$$

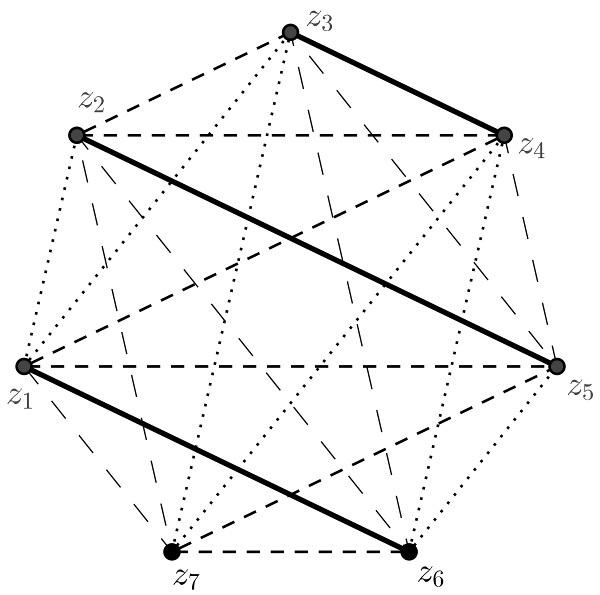
where $L = z_1z_{-1} + z_2z_{-2} + \cdots + z_{(n-1)/2}z_{-(n-1)/2}$ if n is odd, and $L = z_1z_{-1} + z_1z_{-2} + z_2z_{-2} + \cdots + z_{(n-2)/2}z_{-n/2}$ if n is even. We note that for each $p = 1, 2, \dots, \lfloor (n-1)/2 \rfloor$, there is some permutation y_i of the z_i such that

$$\sum_{\substack{i+j \equiv 2p-1, 2p \pmod n \\ i \not\equiv j \pmod n}} z_iz_j = \sum_{k=1}^{n-1} y_ky_{k+1},$$

because we may choose $y_{2i-1} = z_{i+p-1}$ for $1 \leqslant i \leqslant (n+1)/2$ and $y_{2i} = z_{p-i}$ for $1 \leqslant i \leqslant n/2$.

We show (1) graphically for $n = 6, 7$ in the diagrams below. The edges of the graphs each represent a product z_iz_j, and the dashed and dotted series of lines represents the sum of the edges, which is of the form $y_1y_2 + \cdots + y_{n-1}y_n$ for some permutation y_i of the z_i precisely when the series of lines is a Hamiltonian path. The filled edges represent the summands of L.





Now, because the z_i are Shiny, we have that (1) yields the following bound:

$$\sum_{i<j} z_i z_j \geqslant -\left\lfloor \frac{n-1}{2} \right\rfloor + L.$$

It remains to show that, for each n, there exists some permutation z_i of the x_i such that $L \geqslant 0$ when n is odd, and $L \geqslant -1/2$ when n is even. We now split into cases based on the parity of n and provide constructions of the permutations z_i.

Since we have not made any assumptions yet about the x_i, we may now assume without loss of generality that

$$x_1 \leqslant x_2 \leqslant \cdots \leqslant x_k \leqslant 0 \leqslant x_{k+1} \leqslant \cdots \leqslant x_n. \tag{2}$$

Case 1: n is odd.

Without loss of generality, assume that k (from (2)) is even, because we may negate all the x_i if k is odd. We then have $x_1x_2, x_3x_4, \ldots, x_{n-2}x_{n-1} \geqslant 0$ because the factors are of the same sign. Let $L = x_1x_2 + x_3x_4 + \cdots + x_{n-2}x_{n-1} \geqslant 0$. We choose our z_i so that this definition of L agrees with the sum of the leftover terms in (1). Relabel the x_i as z_i such that

$$\{z_1, z_{n-1}\}, \{z_2, z_{n-2}\}, \ldots, \{z_{(n-1)/2}, z_{(n+1)/2}\}$$

are some permutation of

$$\{x_1, x_2\}, \{x_3, x_4\}, \ldots, \{x_{n-2}, x_{n-1}\},$$

and $z_n = x_n$. Then, we have $L = z_1z_{n-1} + \cdots + z_{(n-1)/2}z_{(n+1)/2}$, as desired.

Case 2: n is even.

Let $L = x_1x_2 + x_2x_3 + \cdots + x_{n-1}x_n$. Assume without loss of generality $k \neq 1$. Now, we have

$$2L = (x_1x_2 + \cdots + x_{n-1}x_n) + (x_1x_2 + \cdots + x_{n-1}x_n) \geqslant (x_2x_3 + \cdots + x_{n-1}x_n) + x_kx_{k+1}$$

$$\geqslant x_2x_3 + \cdots + x_{n-1}x_n + x_nx_1 \geqslant -1,$$

where the first inequality holds because the only negative term in L is x_kx_{k+1}, the second inequality holds because $x_1 \leqslant x_k \leqslant 0 \leqslant x_{k+1} \leqslant x_n$, and the third inequality holds because the x_i are assumed to be Shiny. We thus have that $L \geqslant -1/2$. We now choose a suitable z_i such that the definition of L matches the leftover terms in (1).

Relabel the x_i with z_i in the following manner: $x_{2i-1} = z_{-i}, x_{2i} = z_i$ (again taking indices modulo n). We have that

$$L = \sum_{\substack{i+j \equiv 0,-1 \pmod n \\ i \not\equiv j \pmod n}} z_i z_j,$$

as desired.

Solution 2. We present another proof that $\sum_{i<j} x_i x_j \geqslant -(n-1)/2$ for any Shiny n-tuple $(x_1, \ldots, x_n)$. Assume an ordering of the x_i as in (2), and let $\ell = n-k$. Assume without loss of generality that $k \geqslant \ell$. Also assume $k \neq n$, (as otherwise, all of the x_i are nonpositive, and so the inequality is trivial). Define the sets of indices $S = \{1, 2, \ldots, k\}$ and $T = \{k+1, \ldots, n\}$. Define the following sums:

$$K = \sum_{\substack{i<j \\ i,j \in S}} x_i x_j, \quad M = \sum_{\substack{i \in S \\ j \in T}} x_i x_j, \quad \text{and} \quad L = \sum_{\substack{i<j \\ i,j \in T}} x_i x_j$$

By definition, $K, L \geqslant 0$ and $M \leqslant 0$. We aim to show that $K + L + M \geqslant -(n-1)/2$.

We split into cases based on whether $k = \ell$ or $k > \ell$.

Case 1: $k > \ell$.

Consider all permutations $\phi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\phi^{-1}(T) = \{2, 4, \ldots, 2\ell\}$. Note that there are $k!\ell!$ such permutations ϕ. Define

$$f(\phi) = \sum_{i=1}^{n-1} x_{\phi(i)} x_{\phi(i+1)}.$$

We know that $f(\phi) \geqslant -1$ for every permutation ϕ with the above property. Averaging $f(\phi)$ over all ϕ gives

$$-1 \leqslant \frac{1}{k!\ell!} \sum_{\phi} f(\phi) = \frac{2\ell}{k\ell} M + \frac{2(k-\ell-1)}{k(k-1)} K,$$

where the equality holds because there are $k\ell$ products in M, of which 2ℓ are selected for each ϕ, and there are $k(k-1)/2$ products in K, of which $k - \ell - 1$ are selected for each ϕ. We now have

$$K + L + M \geqslant K + L + \left(-\frac{k}{2} - \frac{k-\ell-1}{k-1} K\right) = -\frac{k}{2} + \frac{\ell}{k-1} K + L.$$

Since $k \leqslant n-1$ and $K, L \geqslant 0$, we get the desired inequality.

Case 2: $k = \ell = n/2$.

We do a similar approach, considering all $\phi\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that $\phi^{-1}(T) = \{2, 4, \ldots, 2\ell\}$, and defining f the same way. Analogously to Case 1, we have

$$-1 \leqslant \frac{1}{k!\ell!} \sum_{\phi} f(\phi) = \frac{2\ell - 1}{k\ell} M,$$

because there are $k\ell$ products in M, of which $2\ell - 1$ are selected for each ϕ. Now, we have that

$$K + L + M \geqslant M \geqslant -\frac{n^2}{4(n-1)} \geqslant -\frac{n-1}{2},$$

where the last inequality holds because $n \geqslant 4$.

A6. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)f(y)) + f(x+y) = f(xy) \qquad (*)$$

for all $x, y \in \mathbb{R}$.

(Albania)

Answer: There are 3 solutions:

$$x \mapsto 0 \qquad \text{or} \qquad x \mapsto x-1 \qquad \text{or} \qquad x \mapsto 1-x \qquad (x \in \mathbb{R}).$$

Solution. An easy check shows that all the 3 above mentioned functions indeed satisfy the original equation $(*)$.

In order to show that these are the only solutions, first observe that if $f(x)$ is a solution then $-f(x)$ is also a solution. Hence, without loss of generality we may (and will) assume that $f(0) \leqslant 0$ from now on. We have to show that either f is identically zero or $f(x) = x - 1$ $(\forall x \in \mathbb{R})$.

Observe that, for a fixed $x \neq 1$, we may choose $y \in \mathbb{R}$ so that $x + y = xy \iff y = \frac{x}{x-1}$, and therefore from the original equation $(*)$ we have

$$f\Big(f(x) \cdot f\Big(\frac{x}{x-1}\Big)\Big) = 0 \qquad (x \neq 1). \qquad (1)$$

In particular, plugging in $x = 0$ in (1), we conclude that f has at least one zero, namely $(f(0))^2$:

$$f\big((f(0))^2\big) = 0. \qquad (2)$$

We analyze two cases (recall that $f(0) \leqslant 0$):

Case 1: $f(0) = 0$.

Setting $y = 0$ in the original equation we get the identically zero solution:

$$f(f(x)f(0)) + f(x) = f(0) \implies f(x) = 0 \text{ for all } x \in \mathbb{R}.$$

From now on, we work on the main

Case 2: $f(0) < 0$.

We begin with the following

Claim 1.

$$f(1) = 0, \qquad f(a) = 0 \implies a = 1, \qquad \text{and} \qquad f(0) = -1. \qquad (3)$$

Proof. We need to show that 1 is the *unique* zero of f. First, observe that f has at least one zero a by (2); if $a \neq 1$ then setting $x = a$ in (1) we get $f(0) = 0$, a contradiction. Hence from (2) we get $(f(0))^2 = 1$. Since we are assuming $f(0) < 0$, we conclude that $f(0) = -1$. $\square$

Setting $y = 1$ in the original equation $(*)$ we get

$$f(f(x)f(1)) + f(x+1) = f(x) \iff f(0) + f(x+1) = f(x) \iff f(x+1) = f(x) + 1 \qquad (x \in \mathbb{R}).$$

An easy induction shows that

$$f(x+n) = f(x) + n \qquad (x \in \mathbb{R},\ n \in \mathbb{Z}). \qquad (4)$$

Now we make the following

Claim 2. f is injective.

Proof. Suppose that $f(a) = f(b)$ with $a \neq b$. Then by (4), for all $N \in \mathbb{Z}$,

$$f(a + N + 1) = f(b + N) + 1.$$

Choose any integer $N < -b$; then there exist $x_0, y_0 \in \mathbb{R}$ with $x_0 + y_0 = a + N + 1$, $x_0 y_0 = b + N$. Since $a \neq b$, we have $x_0 \neq 1$ and $y_0 \neq 1$. Plugging in x_0 and y_0 in the original equation $(*)$ we get

$$\begin{aligned} f(f(x_0)f(y_0)) + f(a + N + 1) = f(b + N) &\iff f(f(x_0)f(y_0)) + 1 = 0 \\ &\iff f(f(x_0)f(y_0) + 1) = 0 \qquad \text{by (4)} \\ &\iff f(x_0)f(y_0) = 0 \qquad \text{by (3)}. \end{aligned}$$

However, by Claim 1 we have $f(x_0) \neq 0$ and $f(y_0) \neq 0$ since $x_0 \neq 1$ and $y_0 \neq 1$, a contradiction. $\square$

Now the end is near. For any $t \in \mathbb{R}$, plug in $(x, y) = (t, -t)$ in the original equation $(*)$ to get

$$\begin{aligned} f(f(t)f(-t)) + f(0) = f(-t^2) &\iff f(f(t)f(-t)) = f(-t^2) + 1 \qquad \text{by (3)} \\ &\iff f(f(t)f(-t)) = f(-t^2 + 1) \qquad \text{by (4)} \\ &\iff f(t)f(-t) = -t^2 + 1 \qquad \text{by injectivity of } f. \end{aligned}$$

Similarly, plugging in $(x, y) = (t, 1 - t)$ in $(*)$ we get

$$\begin{aligned} f(f(t)f(1-t)) + f(1) = f(t(1-t)) &\iff f(f(t)f(1-t)) = f(t(1-t)) \quad \text{by (3)} \\ &\iff f(t)f(1-t) = t(1-t) \qquad \text{by injectivity of } f. \end{aligned}$$

But since $f(1 - t) = 1 + f(-t)$ by (4), we get

$$\begin{aligned} f(t)f(1-t) = t(1-t) &\iff f(t)(1 + f(-t)) = t(1-t) \iff f(t) + (-t^2 + 1) = t(1-t) \\ &\iff f(t) = t - 1, \end{aligned}$$

as desired.

Comment. Other approaches are possible. For instance, after Claim 1, we may define

$$g(x) \stackrel{\text{def}}{=} f(x) + 1.$$

Replacing $x + 1$ and $y + 1$ in place of x and y in the original equation $(*)$, we get

$$f(f(x+1)f(y+1)) + f(x + y + 2) = f(xy + x + y + 1) \qquad (x, y \in \mathbb{R}),$$

and therefore, using (4) (so that in particular $g(x) = f(x + 1)$), we may rewrite $(*)$ as

$$g(g(x)g(y)) + g(x + y) = g(xy + x + y) \qquad (x, y \in \mathbb{R}). \qquad (**)$$

We are now to show that $g(x) = x$ for all $x \in \mathbb{R}$ under the assumption (Claim 1) that 0 is the unique zero of g.

Claim 3. Let $n \in \mathbb{Z}$ and $x \in \mathbb{R}$. Then

(a) $g(x + n) = x + n$, and the conditions $g(x) = n$ and $x = n$ are equivalent.

(b) $g(nx) = ng(x)$.

Proof. For part (a), just note that $g(x+n) = x+n$ is just a reformulation of (4). Then $g(x) = n \iff g(x-n) = 0 \iff x-n = 0$ since 0 is the unique zero of g. For part (b), we may assume that $x \neq 0$ since the result is obvious when $x = 0$. Plug in $y = n/x$ in (**) and use part (a) to get

$$g\left(g(x)g\Big(\frac{n}{x}\Big)\right) + g\Big(x + \frac{n}{x}\Big) = g\Big(n + x + \frac{n}{x}\Big) \iff g\left(g(x)g\Big(\frac{n}{x}\Big)\right) = n \iff g(x)g\Big(\frac{n}{x}\Big) = n.$$

In other words, for $x \neq 0$ we have

$$g(x) = \frac{n}{g\big(n/x\big)}.$$

In particular, for $n = 1$, we get $g(1/x) = 1/g(x)$, and therefore replacing $x \leftarrow nx$ in the last equation we finally get

$$g(nx) = \frac{n}{g\big(1/x\big)} = ng(x),$$

as required.

Claim 4. The function g is *additive*, i.e., $g(a+b) = g(a) + g(b)$ for all $a, b \in \mathbb{R}$.

Proof. Set $x \leftarrow -x$ and $y \leftarrow -y$ in (**); since g is an odd function (by Claim 3(b) with $n = -1$), we get

$$g(g(x)g(y)) - g(x+y) = -g(-xy + x + y).$$

Subtracting the last relation from (**) we have

$$2g(x+y) = g(xy + x + y) + g(-xy + x + y)$$

and since by Claim 3(b) we have $2g(x+y) = g(2(x+y))$, we may rewrite the last equation as

$$g(\alpha + \beta) = g(\alpha) + g(\beta) \qquad \text{where} \qquad \begin{cases} \alpha = xy + x + y \\ \beta = -xy + x + y. \end{cases}$$

In other words, we have additivity for all $\alpha, \beta \in \mathbb{R}$ for which there are real numbers x and y satisfying

$$x + y = \frac{\alpha + \beta}{2} \qquad \text{and} \qquad xy = \frac{\alpha - \beta}{2},$$

i.e., for all $\alpha, \beta \in \mathbb{R}$ such that $(\frac{\alpha+\beta}{2})^2 - 4 \cdot \frac{\alpha-\beta}{2} \geqslant 0$. Therefore, given any $a, b \in \mathbb{R}$, we may choose $n \in \mathbb{Z}$ large enough so that we have additivity for $\alpha = na$ and $\beta = nb$, i.e.,

$$g(na) + g(nb) = g(na + nb) \iff ng(a) + ng(b) = ng(a+b)$$

by Claim 3(b). Cancelling n, we get the desired result. (Alternatively, setting either $(\alpha, \beta) = (a, b)$ or $(\alpha, \beta) = (-a, -b)$ will ensure that $(\frac{\alpha+\beta}{2})^2 - 4 \cdot \frac{\alpha-\beta}{2} \geqslant 0$). □

Now we may finish the solution. Set $y = 1$ in (**), and use Claim 3 to get

$$g(g(x)g(1)) + g(x+1) = g(2x+1) \iff g(g(x)) + g(x) + 1 = 2g(x) + 1 \iff g(g(x)) = g(x).$$

By additivity, this is equivalent to $g(g(x) - x) = 0$. Since 0 is the unique zero of g by assumption, we finally get $g(x) - x = 0 \iff g(x) = x$ for all $x \in \mathbb{R}$.

A7. Let $a_0, a_1, a_2, \ldots$ be a sequence of integers and $b_0, b_1, b_2, \ldots$ be a sequence of *positive* integers such that $a_0 = 0, a_1 = 1$, and

$$a_{n+1} = \begin{cases} a_n b_n + a_{n-1}, & \text{if } b_{n-1} = 1 \\ a_n b_n - a_{n-1}, & \text{if } b_{n-1} > 1 \end{cases} \qquad \text{for } n = 1, 2, \ldots.$$

Prove that at least one of the two numbers a_{2017} and a_{2018} must be greater than or equal to 2017.

(Australia)

Solution 1. The value of b_0 is irrelevant since $a_0 = 0$, so we may assume that $b_0 = 1$.

Lemma. We have $a_n \geqslant 1$ for all $n \geqslant 1$.

Proof. Let us suppose otherwise in order to obtain a contradiction. Let

$$n \geqslant 1 \text{ be the smallest integer with } a_n \leqslant 0. \tag{1}$$

Note that $n \geqslant 2$. It follows that $a_{n-1} \geqslant 1$ and $a_{n-2} \geqslant 0$. Thus we cannot have $a_n = a_{n-1}b_{n-1} + a_{n-2}$, so we must have $a_n = a_{n-1}b_{n-1} - a_{n-2}$. Since $a_n \leqslant 0$, we have $a_{n-1} \leqslant a_{n-2}$. Thus we have $a_{n-2} \geqslant a_{n-1} \geqslant a_n$.

Let

$$r \text{ be the smallest index with } a_r \geqslant a_{r+1} \geqslant a_{r+2}. \tag{2}$$

Then $r \leqslant n-2$ by the above, but also $r \geqslant 2$: if $b_1 = 1$, then $a_2 = a_1 = 1$ and $a_3 = a_2 b_2 + a_1 > a_2$; if $b_1 > 1$, then $a_2 = b_1 > 1 = a_1$.

By the minimal choice (2) of r, it follows that $a_{r-1} < a_r$. And since $2 \leqslant r \leqslant n-2$, by the minimal choice (1) of n we have $a_{r-1}, a_r, a_{r+1} > 0$. In order to have $a_{r+1} \geqslant a_{r+2}$, we must have $a_{r+2} = a_{r+1}b_{r+1} - a_r$ so that $b_r \geqslant 2$. Putting everything together, we conclude that

$$a_{r+1} = a_r b_r \pm a_{r-1} \geqslant 2a_r - a_{r-1} = a_r + (a_r - a_{r-1}) > a_r,$$

which contradicts (2). $\square$

To complete the problem, we prove that $\max\{a_n, a_{n+1}\} \geqslant n$ by induction. The cases $n = 0, 1$ are given. Assume it is true for all non-negative integers strictly less than n, where $n \geqslant 2$. There are two cases:

Case 1: $b_{n-1} = 1$.

Then $a_{n+1} = a_n b_n + a_{n-1}$. By the inductive assumption one of a_{n-1}, a_n is at least $n-1$ and the other, by the lemma, is at least 1. Hence

$$a_{n+1} = a_n b_n + a_{n-1} \geqslant a_n + a_{n-1} \geqslant (n-1) + 1 = n.$$

Thus $\max\{a_n, a_{n+1}\} \geqslant n$, as desired.

Case 2: $b_{n-1} > 1$.

Since we defined $b_0 = 1$ there is an index r with $1 \leqslant r \leqslant n-1$ such that

$$b_{n-1}, b_{n-2}, \ldots, b_r \geqslant 2 \qquad \text{and} \qquad b_{r-1} = 1.$$

We have $a_{r+1} = a_r b_r + a_{r-1} \geqslant 2a_r + a_{r-1}$. Thus $a_{r+1} - a_r \geqslant a_r + a_{r-1}$.

Now we claim that $a_r + a_{r-1} \geqslant r$. Indeed, this holds by inspection for $r = 1$; for $r \geqslant 2$, one of a_r, a_{r-1} is at least $r-1$ by the inductive assumption, while the other, by the lemma, is at least 1. Hence $a_r + a_{r-1} \geqslant r$, as claimed, and therefore $a_{r+1} - a_r \geqslant r$ by the last inequality in the previous paragraph.

Since $r \geqslant 1$ and, by the lemma, $a_r \geqslant 1$, from $a_{r+1} - a_r \geqslant r$ we get the following two inequalities:

$$a_{r+1} \geqslant r + 1 \qquad \text{and} \qquad a_{r+1} > a_r.$$

Now observe that

$$a_m > a_{m-1} \implies a_{m+1} > a_m \text{ for } m = r+1, r+2, \ldots, n-1,$$

since $a_{m+1} = a_m b_m - a_{m-1} \geqslant 2a_m - a_{m-1} = a_m + (a_m - a_{m-1}) > a_m$. Thus

$$a_n > a_{n-1} > \cdots > a_{r+1} \geqslant r+1 \implies a_n \geqslant n.$$

So $\max\{a_n, a_{n+1}\} \geqslant n$, as desired.

Solution 2. We say that an index $n > 1$ is *bad* if $b_{n-1} = 1$ and $b_{n-2} > 1$; otherwise n is *good.* The value of b_0 is irrelevant to the definition of (a_n) since $a_0 = 0$; so we assume that $b_0 > 1$.

Lemma 1. (a) $a_n \geqslant 1$ for all $n > 0$.

(b) If $n > 1$ is good, then $a_n > a_{n-1}$.

Proof. Induction on n. In the base cases $n = 1, 2$ we have $a_1 = 1 \geqslant 1$, $a_2 = b_1 a_1 \geqslant 1$, and finally $a_2 > a_1$ if 2 is good, since in this case $b_1 > 1$.

Now we assume that the lemma statement is proved for $n = 1, 2, \ldots, k$ with $k \geqslant 2$, and prove it for $n = k+1$. Recall that a_k and a_{k-1} are positive by the induction hypothesis.

Case 1: k is bad.

We have $b_{k-1} = 1$, so $a_{k+1} = b_k a_k + a_{k-1} \geqslant a_k + a_{k-1} > a_k \geqslant 1$, as required.

Case 2: k is good.

We already have $a_k > a_{k-1} \geqslant 1$ by the induction hypothesis. We consider three easy subcases.

Subcase 2.1: $b_k > 1$.

Then $a_{k+1} \geqslant b_k a_k - a_{k-1} \geqslant a_k + (a_k - a_{k-1}) > a_k \geqslant 1$.

Subcase 2.2: $b_k = b_{k-1} = 1$.

Then $a_{k+1} = a_k + a_{k-1} > a_k \geqslant 1$.

Subcase 2.3: $b_k = 1$ but $b_{k-1} > 1$.

Then $k+1$ is bad, and we need to prove only (a), which is trivial: $a_{k+1} = a_k - a_{k-1} \geqslant 1$.

So, in all three subcases we have verified the required relations. □

Lemma 2. Assume that $n > 1$ is bad. Then there exists a $j \in \{1, 2, 3\}$ such that $a_{n+j} \geqslant a_{n-1} + j + 1$, and $a_{n+i} \geqslant a_{n-1} + i$ for all $1 \leqslant i < j$.

Proof. Recall that $b_{n-1} = 1$. Set

$$m = \inf\{i > 0 \colon b_{n+i-1} > 1\}$$

(possibly $m = +\infty$). We claim that $j = \min\{m, 3\}$ works. Again, we distinguish several cases, according to the value of m; in each of them we use Lemma 1 without reference.

Case 1: $m = 1$, so $b_n > 1$.

Then $a_{n+1} \geqslant 2a_n + a_{n-1} \geqslant a_{n-1} + 2$, as required.

Case 2: $m = 2$, so $b_n = 1$ and $b_{n+1} > 1$.

Then we successively get

$$a_{n+1} = a_n + a_{n-1} \geqslant a_{n-1} + 1,$$
$$a_{n+2} \geqslant 2a_{n+1} + a_n \geqslant 2(a_{n-1} + 1) + a_n = a_{n-1} + (a_{n-1} + a_n + 2) \geqslant a_{n-1} + 4,$$

which is even better than we need.

Case 3: $m > 2$, *so* $b_n = b_{n+1} = 1$.

Then we successively get

$$a_{n+1} = a_n + a_{n-1} \geqslant a_{n-1} + 1, \quad a_{n+2} = a_{n+1} + a_n \geqslant a_{n-1} + 1 + a_n \geqslant a_{n-1} + 2,$$
$$a_{n+3} \geqslant a_{n+2} + a_{n+1} \geqslant (a_{n-1} + 1) + (a_{n-1} + 2) \geqslant a_{n-1} + 4,$$

as required. □

Lemmas 1(b) and 2 provide enough information to prove that $\max\{a_n, a_{n+1}\} \geqslant n$ for all n and, moreover, that $a_n \geqslant n$ often enough. Indeed, assume that we have found some n with $a_{n-1} \geqslant n-1$. If n is good, then by Lemma 1(b) we have $a_n \geqslant n$ as well. If n is bad, then Lemma 2 yields $\max\{a_{n+i}, a_{n+i+1}\} \geqslant a_{n-1}+i+1 \geqslant n+i$ for all $0 \leqslant i < j$ and $a_{n+j} \geqslant a_{n-1}+j+1 \geqslant n+j$; so $n + j$ is the next index to start with.

A8. Assume that a function $f\colon \mathbb{R} \to \mathbb{R}$ satisfies the following condition:

For every $x, y \in \mathbb{R}$ such that $\big(f(x)+y\big)\big(f(y)+x\big) > 0$, we have $f(x)+y = f(y)+x$.

Prove that $f(x) + y \leqslant f(y) + x$ whenever $x > y$.

(Netherlands)

Solution 1. Define $g(x) = x - f(x)$. The condition on f then rewrites as follows:

For every $x, y \in \mathbb{R}$ such that $\big((x+y) - g(x)\big)\big((x+y) - g(y)\big) > 0$, we have $g(x) = g(y)$.

This condition may in turn be rewritten in the following form:

If $g(x) \neq g(y)$, then the number $x + y$ lies (non-strictly) between $g(x)$ and $g(y)$. $(*)$

Notice here that the function $g_1(x) = -g(-x)$ also satisfies $(*)$, since

$$\begin{aligned} g_1(x) \neq g_1(y) \quad &\Longrightarrow \quad g(-x) \neq g(-y) \quad \Longrightarrow \quad -(x+y) \text{ lies between } g(-x) \text{ and } g(-y) \\ &\Longrightarrow \quad x + y \text{ lies between } g_1(x) \text{ and } g_1(y). \end{aligned}$$

On the other hand, the relation we need to prove reads now as

$$g(x) \leqslant g(y) \qquad \text{whenever } x < y. \tag{1}$$

Again, this condition is equivalent to the same one with g replaced by g_1.

If $g(x) = 2x$ for all $x \in \mathbb{R}$, then $(*)$ is obvious; so in what follows we consider the other case. We split the solution into a sequence of lemmas, strengthening one another. We always consider some value of x with $g(x) \neq 2x$ and denote $X = g(x)$.

Lemma 1. Assume that $X < 2x$. Then on the interval $(X - x; x]$ the function g attains at most two values — namely, X and, possibly, some $Y > X$. Similarly, if $X > 2x$, then g attains at most two values on $[x; X - x)$ — namely, X and, possibly, some $Y < X$.

Proof. We start with the first claim of the lemma. Notice that $X - x < x$, so the considered interval is nonempty.

Take any $a \in (X - x; x)$ with $g(a) \neq X$ (if it exists). If $g(a) < X$, then $(*)$ yields $g(a) \leqslant a + x \leqslant g(x) = X$, so $a \leqslant X - x$ which is impossible. Thus, $g(a) > X$ and hence by $(*)$ we get $X \leqslant a + x \leqslant g(a)$.

Now, for any $b \in (X - x; x)$ with $g(b) \neq X$ we similarly get $b + x \leqslant g(b)$. Therefore, the number $a + b$ (which is smaller than each of $a + x$ and $b + x$) cannot lie between $g(a)$ and $g(b)$, which by $(*)$ implies that $g(a) = g(b)$. Hence g may attain only two values on $(X - x; x]$, namely X and $g(a) > X$.

To prove the second claim, notice that $g_1(-x) = -X < 2 \cdot (-x)$, so g_1 attains at most two values on $(-X + x, -x]$, i.e., $-X$ and, possibly, some $-Y > -X$. Passing back to g, we get what we need. $\square$

Lemma 2. If $X < 2x$, then g is constant on $(X - x; x)$. Similarly, if $X > 2x$, then g is constant on $(x; X - x)$.

Proof. Again, it suffices to prove the first claim only. Assume, for the sake of contradiction, that there exist $a, b \in (X - x; x)$ with $g(a) \neq g(b)$; by Lemma 1, we may assume that $g(a) = X$ and $Y = g(b) > X$.

Notice that $\min\{X - a, X - b\} > X - x$, so there exists a $u \in (X - x; x)$ such that $u < \min\{X - a, X - b\}$. By Lemma 1, we have either $g(u) = X$ or $g(u) = Y$. In the former case, by $(*)$ we have $X \leqslant u + b \leqslant Y$ which contradicts $u < X - b$. In the second case, by $(*)$ we have $X \leqslant u + a \leqslant Y$ which contradicts $u < X - a$. Thus the lemma is proved. $\square$

Lemma 3. If $X < 2x$, then $g(a) = X$ for all $a \in (X - x; x)$. Similarly, if $X > 2x$, then $g(a) = X$ for all $a \in (x; X - x)$.

Proof. Again, we only prove the first claim.

By Lemmas 1 and 2, this claim may be violated only if g takes on a constant value $Y > X$ on $(X - x, x)$. Choose any $a, b \in (X - x; x)$ with $a < b$. By $(*)$, we have

$$Y \geqslant b + x \geqslant X. \tag{2}$$

In particular, we have $Y \geqslant b + x > 2a$. Applying Lemma 2 to a in place of x, we obtain that g is constant on $(a, Y - a)$. By (2) again, we have $x \leqslant Y - b < Y - a$; so $x, b \in (a; Y - a)$. But $X = g(x) \neq g(b) = Y$, which is a contradiction. □

Now we are able to finish the solution. Assume that $g(x) > g(y)$ for some $x < y$. Denote $X = g(x)$ and $Y = g(y)$; by $(*)$, we have $X \geqslant x + y \geqslant Y$, so $Y - y \leqslant x < y \leqslant X - x$, and hence $(Y - y; y) \cap (x; X - x) = (x, y) \neq \varnothing$. On the other hand, since $Y - y < y$ and $x < X - x$, Lemma 3 shows that g should attain a constant value X on $(x; X - x)$ and a constant value $Y \neq X$ on $(Y - y; y)$. Since these intervals overlap, we get the final contradiction.

Solution 2. As in the previous solution, we pass to the function g satisfying $(*)$ and notice that we need to prove the condition (1). We will also make use of the function g_1.

If g is constant, then (1) is clearly satisfied. So, in the sequel we assume that g takes on at least two different values. Now we collect some information about the function g.

Claim 1. For any $c \in \mathbb{R}$, all the solutions of $g(x) = c$ are bounded.

Proof. Fix any $y \in \mathbb{R}$ with $g(y) \neq c$. Assume first that $g(y) > c$. Now, for any x with $g(x) = c$, by $(*)$ we have $c \leqslant x + y \leqslant g(y)$, or $c - y \leqslant x \leqslant g(y) - y$. Since c and y are constant, we get what we need.

If $g(y) < c$, we may switch to the function g_1 for which we have $g_1(-y) > -c$. By the above arguments, we obtain that all the solutions of $g_1(-x) = -c$ are bounded, which is equivalent to what we need. □

As an immediate consequence, the function g takes on infinitely many values, which shows that the next claim is indeed widely applicable.

Claim 2. If $g(x) < g(y) < g(z)$, then $x < z$.

Proof. By $(*)$, we have $g(x) \leqslant x + y \leqslant g(y) \leqslant z + y \leqslant g(z)$, so $x + y \leqslant z + y$, as required. □

Claim 3. Assume that $g(x) > g(y)$ for some $x < y$. Then $g(a) \in \{g(x), g(y)\}$ for all $a \in [x; y]$.

Proof. If $g(y) < g(a) < g(x)$, then the triple (y, a, x) violates Claim 2. If $g(a) < g(y) < g(x)$, then the triple (a, y, x) violates Claim 2. If $g(y) < g(x) < g(a)$, then the triple (y, x, a) violates Claim 2. The only possible cases left are $g(a) \in \{g(x), g(y)\}$. □

In view of Claim 3, we say that an interval I (which may be open, closed, or semi-open) is a *Dirichlet interval** if the function g takes on just two values on I.

Assume now, for the sake of contradiction, that (1) is violated by some $x < y$. By Claim 3, $[x; y]$ is a Dirichlet interval. Set

$$r = \inf\{a\colon (a; y] \text{ is a Dirichlet interval}\} \quad \text{and} \quad s = \sup\{b\colon [x; b) \text{ is a Dirichlet interval}\}.$$

Clearly, $r \leqslant x < y \leqslant s$. By Claim 1, r and s are finite. Denote $X = g(x)$, $Y = g(y)$, and $\Delta = (y - x)/2$.

Suppose first that there exists a $t \in (r; r + \Delta)$ with $f(t) = Y$. By the definition of r, the interval $(r - \Delta; y]$ is not Dirichlet, so there exists an $r' \in (r - \Delta; r]$ such that $g(r') \notin \{X, Y\}$.

*The name *Dirichlet interval* is chosen for the reason that g theoretically might act similarly to the Dirichlet function on this interval.

The function g attains at least three distinct values on $[r';y]$, namely $g(r')$, $g(x)$, and $g(y)$. Claim 3 now yields $g(r') \leqslant g(y)$; the equality is impossible by the choice of r', so in fact $g(r') < Y$. Applying $(*)$ to the pairs (r', y) and (t, x) we obtain $r' + y \leqslant Y \leqslant t + x$, whence $r - \Delta + y < r' + y \leqslant t + x < r + \Delta + x$, or $y - x < 2\Delta$. This is a contradiction.

Thus, $g(t) = X$ for all $t \in (r; r + \Delta)$. Applying the same argument to g_1, we get $g(t) = Y$ for all $t \in (s - \Delta; s)$.

Finally, choose some $s_1, s_2 \in (s - \Delta; s)$ with $s_1 < s_2$ and denote $\delta = (s_2 - s_1)/2$. As before, we choose $r' \in (r - \delta; r)$ with $g(r') \notin \{X, Y\}$ and obtain $g(r') < Y$. Choose any $t \in (r; r + \delta)$; by the above arguments, we have $g(t) = X$ and $g(s_1) = g(s_2) = Y$. As before, we apply $(*)$ to the pairs (r', s_2) and (t, s_1) obtaining $r - \delta + s_2 < r' + s_2 \leqslant Y \leqslant t + s_1 < r + \delta + s_1$, or $s_2 - s_1 < 2\delta$. This is a final contradiction.

Comment 1. The original submission discussed the same functions f, but the question was different — namely, the following one:

Prove that the equation $f(x) = 2017x$ has at most one solution, and the equation $f(x) = -2017x$ has at least one solution.

The Problem Selection Committee decided that the question we are proposing is more natural, since it provides more natural information about the function g (which is indeed the main character in this story). On the other hand, the new problem statement is strong enough in order to imply the original one easily.

Namely, we will deduce from the new problem statement (along with the facts used in the solutions) that (i) for every $N > 0$ the equation $g(x) = -Nx$ has at most one solution, and (ii) for every $N > 1$ the equation $g(x) = Nx$ has at least one solution.

Claim (i) is now trivial. Indeed, g is proven to be non-decreasing, so $g(x) + Nx$ is strictly increasing and thus has at most one zero.

We proceed on claim (ii). If $g(0) = 0$, then the required root has been already found. Otherwise, we may assume that $g(0) > 0$ and denote $c = g(0)$. We intend to prove that $x = c/N$ is the required root. Indeed, by monotonicity we have $g(c/N) \geqslant g(0) = c$; if we had $g(c/N) > c$, then $(*)$ would yield $c \leqslant 0 + c/N \leqslant g(c/N)$ which is false. Thus, $g(x) = c = Nx$.

Comment 2. There are plenty of functions g satisfying $(*)$ (and hence of functions f satisfying the problem conditions). One simple example is $g_0(x) = 2x$. Next, for any increasing sequence $A = (\dots, a_{-1}, a_0, a_1, \dots)$ which is unbounded in both directions (i.e., for every N this sequence contains terms greater than N, as well as terms smaller than $-N$), the function g_A defined by

$$g_A(x) = a_i + a_{i+1} \qquad \text{whenever } x \in [a_i; a_{i+1})$$

satisfies $(*)$. Indeed, pick any $x < y$ with $g(x) \neq g(y)$; this means that $x \in [a_i; a_{i+1})$ and $y \in [a_j; a_{j+1})$ for some $i < j$. Then we have $g(x) = a_i + a_{i+1} \leqslant x + y < a_j + a_{j+1} = g(y)$, as required.

There also exist examples of the mixed behavior; e.g., for an arbitrary sequence A as above and an arbitrary subset $I \subseteq \mathbb{Z}$ the function

$$g_{A,I}(x) = \begin{cases} g_0(x), & x \in [a_i; a_{i+1}) \quad \text{with } i \in I; \\ g_A(x), & x \in [a_i; a_{i+1}) \quad \text{with } i \notin I \end{cases}$$

also satisfies $(*)$.

Finally, it is even possible to provide a complete description of all functions g satisfying $(*)$ (and hence of all functions f satisfying the problem conditions); however, it seems to be far out of scope for the IMO. This description looks as follows.

Let A be any closed subset of $\mathbb{R}$ which is unbounded in both directions. Define the functions i_A, s_A, and g_A as follows:

$$i_A(x) = \inf\{a \in A \colon a \geqslant x\}, \quad s_A(x) = \sup\{a \in A \colon a \leqslant x\}, \quad g_A(x) = i_A(x) + s_A(x).$$

It is easy to see that for different sets A and B the functions g_A and g_B are also different (since, e.g., for any $a \in A \setminus B$ the function g_B is constant in a small neighborhood of a, but the function g_A is not). One may check, similarly to the arguments above, that each such function satisfies $(*)$.

Finally, one more modification is possible. Namely, for any $x \in A$ one may redefine $g_A(x)$ (which is $2x$) to be any of the numbers

$$g_{A+}(x) = i_{A+}(x) + x \quad \text{or} \quad g_{A-}(x) = x + s_{A-}(x),$$
$$\text{where} \qquad i_{A+}(x) = \inf\{a \in A\colon a > x\} \quad \text{and} \quad s_{A-}(x) = \sup\{a \in A\colon a < x\}.$$

This really changes the value if x has some right (respectively, left) semi-neighborhood disjoint from A, so there are at most countably many possible changes; all of them can be performed independently.

With some effort, one may show that the construction above provides all functions g satisfying $(*)$.

Combinatorics

C1. A rectangle $\mathcal{R}$ with odd integer side lengths is divided into small rectangles with integer side lengths. Prove that there is at least one among the small rectangles whose distances from the four sides of $\mathcal{R}$ are either all odd or all even.

(Singapore)

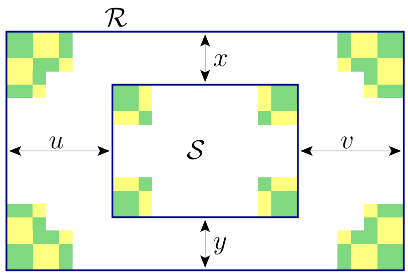
Solution. Let the width and height of $\mathcal{R}$ be odd numbers a and b. Divide $\mathcal{R}$ into ab unit squares and color them green and yellow in a checkered pattern. Since the side lengths of a and b are odd, the corner squares of $\mathcal{R}$ will all have the same color, say green.

Call a rectangle (either $\mathcal{R}$ or a small rectangle) *green* if its corners are all green; call it *yellow* if the corners are all yellow, and call it *mixed* if it has both green and yellow corners. In particular, $\mathcal{R}$ is a green rectangle.

We will use the following trivial observations.

- Every mixed rectangle contains the same number of green and yellow squares;
- Every green rectangle contains one more green square than yellow square;
- Every yellow rectangle contains one more yellow square than green square.

The rectangle $\mathcal{R}$ is green, so it contains more green unit squares than yellow unit squares. Therefore, among the small rectangles, at least one is green. Let $\mathcal{S}$ be such a small green rectangle, and let its distances from the sides of $\mathcal{R}$ be x, y, u and v, as shown in the picture. The top-left corner of $\mathcal{R}$ and the top-left corner of $\mathcal{S}$ have the same color, which happen if and only if x and u have the same parity. Similarly, the other three green corners of $\mathcal{S}$ indicate that x and v have the same parity, y and u have the same parity, i.e. x, y, u and v are all odd or all even.



C2. Let n be a positive integer. Define a *chameleon* to be any sequence of $3n$ letters, with exactly n occurrences of each of the letters a, b, and c. Define a *swap* to be the transposition of two adjacent letters in a chameleon. Prove that for any chameleon X, there exists a chameleon Y such that X cannot be changed to Y using fewer than $3n^2/2$ swaps.

(Australia)

Solution 1. To start, notice that the swap of two identical letters does not change a chameleon, so we may assume there are no such swaps.

For any two chameleons X and Y, define their *distance* $d(X,Y)$ to be the minimal number of swaps needed to transform X into Y (or vice versa). Clearly, $d(X,Y)+d(Y,Z)\geqslant d(X,Z)$ for any three chameleons X, Y, and Z.

Lemma. Consider two chameleons

$$P=\underbrace{aa\ldots a}_{n}\underbrace{bb\ldots b}_{n}\underbrace{cc\ldots c}_{n}\quad\text{and}\quad Q=\underbrace{cc\ldots c}_{n}\underbrace{bb\ldots b}_{n}\underbrace{aa\ldots a}_{n}.$$

Then $d(P,Q)\geqslant 3n^2$.

Proof. For any chameleon X and any pair of distinct letters $u,v\in\{a,b,c\}$, we define $f_{u,v}(X)$ to be the number of pairs of positions in X such that the left one is occupied by u, and the right one is occupied by v. Define $f(X)=f_{a,b}(X)+f_{a,c}(X)+f_{b,c}(X)$. Notice that $f_{a,b}(P)=f_{a,c}(P)=f_{b,c}(P)=n^2$ and $f_{a,b}(Q)=f_{a,c}(Q)=f_{b,c}(Q)=0$, so $f(P)=3n^2$ and $f(Q)=0$.

Now consider some swap changing a chameleon X to X'; say, the letters a and b are swapped. Then $f_{a,b}(X)$ and $f_{a,b}(X')$ differ by exactly 1, while $f_{a,c}(X)=f_{a,c}(X')$ and $f_{b,c}(X)=f_{b,c}(X')$. This yields $|f(X)-f(X')|=1$, i.e., on any swap the value of f changes by 1. Hence $d(X,Y)\geqslant |f(X)-f(Y)|$ for any two chameleons X and Y. In particular, $d(P,Q)\geqslant |f(P)-f(Q)|=3n^2$, as desired. □

Back to the problem, take any chameleon X and notice that $d(X,P)+d(X,Q)\geqslant d(P,Q)\geqslant 3n^2$ by the lemma. Consequently, $\max\{d(X,P),d(X,Q)\}\geqslant \frac{3n^2}{2}$, which establishes the problem statement.

Comment 1. The problem may be reformulated in a graph language. Construct a graph G with the chameleons as vertices, two vertices being connected with an edge if and only if these chameleons differ by a single swap. Then $d(X,Y)$ is the usual distance between the vertices X and Y in this graph. Recall that the *radius* of a connected graph G is defined as

$$r(G)=\min_{v\in V}\max_{u\in V} d(u,v).$$

So we need to prove that the radius of the constructed graph is at least $3n^2/2$.

It is well-known that the radius of any connected graph is at least the half of its diameter (which is simply $\max_{u,v\in V} d(u,v)$). Exactly this fact has been used above in order to finish the solution.

Solution 2. We use the notion of distance from Solution 1, but provide a different lower bound for it.

In any chameleon X, we enumerate the positions in it from left to right by $1,2,\ldots,3n$. Define $s_c(X)$ as the sum of positions occupied by c. The value of s_c changes by at most 1 on each swap, but this fact alone does not suffice to solve the problem; so we need an improvement.

For every chameleon X, denote by $X_{\overline{c}}$ the sequence obtained from X by removing all n letters c. Enumerate the positions in $X_{\overline{c}}$ from left to right by $1,2,\ldots,2n$, and define $s_{\overline{c},b}(X)$ as the sum of positions in $X_{\overline{c}}$ occupied by b. (In other words, here we consider the positions of the b's relatively to the a's only.) Finally, denote

$$d'(X,Y):=|s_c(X)-s_c(Y)|+|s_{\overline{c},b}(X)-s_{\overline{c},b}(Y)|.$$

Now consider any swap changing a chameleon X to X'. If no letter c is involved into this swap, then $s_c(X) = s_c(X')$; on the other hand, exactly one letter b changes its position in $X_{\overline{c}}$, so $|s_{\overline{c},b}(X) - s_{\overline{c},b}(X')| = 1$. If a letter c is involved into a swap, then $X_{\overline{c}} = X'_{\overline{c}}$, so $s_{\overline{c},b}(X) = s_{\overline{c},b}(X')$ and $|s_c(X) - s_c(X')| = 1$. Thus, in all cases we have $d'(X, X') = 1$.

As in the previous solution, this means that $d(X, Y) \geqslant d'(X, Y)$ for any two chameleons X and Y. Now, for any chameleon X we will indicate a chameleon Y with $d'(X, Y) \geqslant 3n^2/2$, thus finishing the solution.

The function s_c attains all integer values from $1 + \cdots + n = \frac{n(n+1)}{2}$ to $(2n+1) + \cdots + 3n = 2n^2 + \frac{n(n+1)}{2}$. If $s_c(X) \leqslant n^2 + \frac{n(n+1)}{2}$, then we put the letter c into the last n positions in Y; otherwise we put the letter c into the first n positions in Y. In either case we already have $|s_c(X) - s_c(Y)| \geqslant n^2$.

Similarly, $s_{\overline{c},b}$ ranges from $\frac{n(n+1)}{2}$ to $n^2 + \frac{n(n+1)}{2}$. So, if $s_{\overline{c},b}(X) \leqslant \frac{n^2}{2} + \frac{n(n+1)}{2}$, then we put the letter b into the last n positions in Y which are still free; otherwise, we put the letter b into the first n such positions. The remaining positions are occupied by a. In any case, we have $|s_{\overline{c},b}(X) - s_{\overline{c},b}(Y)| \geqslant \frac{n^2}{2}$, thus $d'(X, Y) \geqslant n^2 + \frac{n^2}{2} = \frac{3n^2}{2}$, as desired.

Comment 2. The two solutions above used two lower bounds $|f(X) - f(Y)|$ and $d'(X, Y)$ for the number $d(X, Y)$. One may see that these bounds are closely related to each other, as

$$f_{a,c}(X) + f_{b,c}(X) = s_c(X) - \frac{n(n+1)}{2} \quad \text{and} \quad f_{a,b}(X) = s_{\overline{c},b}(X) - \frac{n(n+1)}{2}.$$

One can see that, e.g., the bound $d'(X, Y)$ could as well be used in the proof of the lemma in Solution 1.

Let us describe here an even sharper bound which also can be used in different versions of the solutions above.

In each chameleon X, enumerate the occurrences of a from the left to the right as $a_1, a_2, \ldots, a_n$. Since we got rid of swaps of identical letters, the relative order of these letters remains the same during the swaps. Perform the same operation with the other letters, obtaining new letters $b_1, \ldots, b_n$ and $c_1, \ldots, c_n$. Denote by A the set of the $3n$ obtained letters.

Since all $3n$ letters became different, for any chameleon X and any $s \in A$ we may define the position $N_s(X)$ of s in X (thus $1 \leqslant N_s(X) \leqslant 3n$). Now, for any two chameleons X and Y we say that a pair of letters $(s, t) \in A \times A$ is an (X, Y)-*inversion* if $N_s(X) < N_t(X)$ but $N_s(Y) > N_t(Y)$, and define $d^*(X, Y)$ to be the number of (X, Y)-inversions. Then for any two chameleons Y and Y' differing by a single swap, we have $|d^*(X, Y) - d^*(X, Y')| = 1$. Since $d^*(X, X) = 0$, this yields $d(X, Y) \geqslant d^*(X, Y)$ for any pair of chameleons X and Y. The bound d^* may also be used in both Solution 1 and Solution 2.

Comment 3. In fact, one may prove that the distance d^* defined in the previous comment *coincides* with d. Indeed, if $X \neq Y$, then there exist an (X, Y)-inversion (s, t). One can show that such s and t may be chosen to occupy consecutive positions in Y. Clearly, s and t correspond to different letters among $\{a, b, c\}$. So, swapping them in Y we get another chameleon Y' with $d^*(X, Y') = d^*(X, Y) - 1$. Proceeding in this manner, we may change Y to X in $d^*(X, Y)$ steps.

Using this fact, one can show that the estimate in the problem statement is sharp for all $n \geqslant 2$. (For $n = 1$ it is not sharp, since any permutation of three letters can be changed to an opposite one in no less than three swaps.) We outline the proof below.

For any $k \geqslant 0$, define

$$X_{2k} = \underbrace{abc\,abc \ldots abc}_{3k \text{ letters}}\underbrace{cba\,cba \ldots cba}_{3k \text{ letters}} \quad \text{and} \quad X_{2k+3} = \underbrace{abc\,abc \ldots abc}_{3k \text{ letters}}\,abc\,bca\,cab\,\underbrace{cba\,cba \ldots cba}_{3k \text{ letters}}.$$

We claim that for every $n \geqslant 2$ and every chameleon Y, we have $d^*(X_n, Y) \leqslant \lceil 3n^2/2 \rceil$. This will mean that for every $n \geqslant 2$ the number $3n^2/2$ in the problem statement cannot be changed by any number larger than $\lceil 3n^2/2 \rceil$.

For any distinct letters $u, v \in \{a, b, c\}$ and any two chameleons X and Y, we define $d^*_{u,v}(X, Y)$ as the number of (X, Y)-inversions (s, t) such that s and t are instances of u and v (in any of the two possible orders). Then $d^*(X, Y) = d^*_{a,b}(X, Y) + d^*_{b,c}(X, Y) + d^*_{c,a}(X, Y)$.

We start with the case when $n = 2k$ is even; denote $X = X_{2k}$. We show that $d^*_{a,b}(X,Y) \leqslant 2k^2$ for any chameleon Y; this yields the required estimate. Proceed by the induction on k with the trivial base case $k = 0$. To perform the induction step, notice that $d^*_{a,b}(X,Y)$ is indeed the minimal number of swaps needed to change $Y_{\overline{c}}$ into $X_{\overline{c}}$. One may show that moving a_1 and a_{2k} in Y onto the first and the last positions in Y, respectively, takes at most $2k$ swaps, and that subsequent moving b_1 and b_{2k} onto the second and the second last positions takes at most $2k-2$ swaps. After performing that, one may delete these letters from both $X_{\overline{c}}$ and $Y_{\overline{c}}$ and apply the induction hypothesis; so $X_{\overline{c}}$ can be obtained from $Y_{\overline{c}}$ using at most $2(k-1)^2 + 2k + (2k-2) = 2k^2$ swaps, as required.

If $n = 2k+3$ is odd, the proof is similar but more technically involved. Namely, we claim that $d^*_{a,b}(X_{2k+3}, Y) \leqslant 2k^2 + 6k + 5$ for any chameleon Y, and that the equality is achieved only if $Y_{\overline{c}} = bb\ldots b\,aa\ldots a$. The proof proceeds by a similar induction, with some care taken of the base case, as well as of extracting the equality case. Similar estimates hold for $d^*_{b,c}$ and $d^*_{c,a}$. Summing three such estimates, we obtain

$$d^*(X_{2k+3}, Y) \leqslant 3(2k^2 + 6k + 5) = \left\lceil \frac{3n^2}{2} \right\rceil + 1,$$

which is by 1 more than we need. But the equality could be achieved only if $Y_{\overline{c}} = bb\ldots b\,aa\ldots a$ and, similarly, $Y_{\overline{b}} = aa\ldots a\,cc\ldots c$ and $Y_{\overline{a}} = cc\ldots c\,bb\ldots b$. Since these three equalities cannot hold simultaneously, the proof is finished.

C3. Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

(1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

(2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

(Thailand)

Answer: $2\sum_{j=0}^{8}\binom{n}{j}-1$.

Solution 1. We will solve a more general problem, replacing the row of 9 cells with a row of k cells, where k is a positive integer. Denote by $m(n,k)$ the maximum possible number of moves Sir Alex can make starting with a row of k empty cells, and ending with one cell containing the number 2^n and all the other $k-1$ cells empty. Call an operation of type (1) an *insertion*, and an operation of type (2) a *merge*.

Only one move is possible when $k=1$, so we have $m(n,1)=1$. From now on we consider $k\geqslant 2$, and we may assume Sir Alex's last move was a merge. Then, just before the last move, there were exactly two cells with the number 2^{n-1}, and the other $k-2$ cells were empty.

Paint one of those numbers 2^{n-1} blue, and the other one red. Now trace back Sir Alex's moves, always painting the numbers blue or red following this rule: *if a and b merge into c, paint a and b with the same color as c.* Notice that in this backward process new numbers are produced only by reversing merges, since reversing an insertion simply means deleting one of the numbers. Therefore, all numbers appearing in the whole process will receive one of the two colors.

Sir Alex's first move is an insertion. Without loss of generality, assume this first number inserted is blue. Then, from this point on, until the last move, there is always at least one cell with a blue number.

Besides the last move, there is no move involving a blue and a red number, since all merges involves numbers with the same color, and insertions involve only one number. Call an insertion of a blue number or merge of two blue numbers a *blue move*, and define a *red move* analogously.

The whole sequence of blue moves could be repeated on another row of k cells to produce one cell with the number 2^{n-1} and all the others empty, so there are at most $m(n-1,k)$ blue moves.

Now we look at the red moves. Since every time we perform a red move there is at least one cell occupied with a blue number, the whole sequence of red moves could be repeated on a row of $k-1$ cells to produce one cell with the number 2^{n-1} and all the others empty, so there are at most $m(n-1,k-1)$ red moves. This proves that

$$m(n,k)\leqslant m(n-1,k)+m(n-1,k-1)+1.$$

On the other hand, we can start with an empty row of k cells and perform $m(n-1,k)$ moves to produce one cell with the number 2^{n-1} and all the others empty, and after that perform $m(n-1,k-1)$ moves on those $k-1$ empty cells to produce the number 2^{n-1} in one of them, leaving $k-2$ empty. With one more merge we get one cell with 2^n and the others empty, proving that

$$m(n,k)\geqslant m(n-1,k)+m(n-1,k-1)+1.$$

It follows that

$$m(n,k)=m(n-1,k)+m(n-1,k-1)+1, \tag{1}$$

for $n\geqslant 1$ and $k\geqslant 2$.

If $k=1$ or $n=0$, we must insert 2^n on our first move and immediately get the final configuration, so $m(0,k)=1$ and $m(n,1)=1$, for $n\geqslant 0$ and $k\geqslant 1$. These initial values, together with the recurrence relation (1), determine $m(n,k)$ uniquely.

Finally, we show that

$$m(n,k)=2\sum_{j=0}^{k-1}\binom{n}{j}-1, \tag{2}$$

for all integers $n\geqslant 0$ and $k\geqslant 1$.

We use induction on n. Since $m(0,k)=1$ for $k\geqslant 1$, (2) is true for the base case. We make the induction hypothesis that (2) is true for some fixed positive integer n and all $k\geqslant 1$. We have $m(n+1,1)=1=2\binom{n+1}{0}-1$, and for $k\geqslant 2$ the recurrence relation (1) and the induction hypothesis give us

$$m(n+1,k)=m(n,k)+m(n,k-1)+1=2\sum_{j=0}^{k-1}\binom{n}{j}-1+2\sum_{j=0}^{k-2}\binom{n}{j}-1+1$$
$$=2\sum_{j=0}^{k-1}\binom{n}{j}+2\sum_{j=0}^{k-1}\binom{n}{j-1}-1=2\sum_{j=0}^{k-1}\left(\binom{n}{j}+\binom{n}{j-1}\right)-1=2\sum_{j=0}^{k-1}\binom{n+1}{j}-1,$$

which completes the proof.

Comment 1. After deducing the recurrence relation (1), it may be convenient to homogenize the recurrence relation by defining $h(n,k)=m(n,k)+1$. We get the new relation

$$h(n,k)=h(n-1,k)+h(n-1,k), \tag{3}$$

for $n\geqslant 1$ and $k\geqslant 2$, with initial values $h(0,k)=h(n,1)=2$, for $n\geqslant 0$ and $k\geqslant 1$.

This may help one to guess the answer, and also with other approaches like the one we develop next.

Comment 2. We can use a generating function to find the answer without guessing. We work with the homogenized recurrence relation (3). Define $h(n,0)=0$ so that (3) is valid for $k=1$ as well. Now we set up the generating function $f(x,y)=\sum_{n,k\geqslant 0}h(n,k)x^ny^k$. Multiplying the recurrence relation (3) by x^ny^k and summing over $n,k\geqslant 1$, we get

$$\sum_{n,k\geqslant 1}h(n,k)x^ny^k=x\sum_{n,k\geqslant 1}h(n-1,k)x^{n-1}y^k+xy\sum_{n,k\geqslant 1}h(n-1,k-1)x^{n-1}y^{k-1}.$$

Completing the missing terms leads to the following equation on $f(x,y)$:

$$f(x,y)-\sum_{n\geqslant 0}h(n,0)x^n-\sum_{k\geqslant 1}h(0,k)y^k=xf(x,y)-x\sum_{n\geqslant 0}h(n,0)x^n+xyf(x,y).$$

Substituting the initial values, we obtain

$$f(x,y)=\frac{2y}{1-y}\cdot\frac{1}{1-x(1+y)}.$$

Developing as a power series, we get

$$f(x,y)=2\sum_{j\geqslant 1}y^j\cdot\sum_{n\geqslant 0}(1+y)^nx^n.$$

The coefficient of x^n in this power series is

$$2\sum_{j\geqslant 1} y^j \cdot (1+y)^n = 2\sum_{j\geqslant 1} y^j \cdot \sum_{i\geqslant 0}\binom{n}{i} y^i,$$

and extracting the coefficient of y^k in this last expression we finally obtain the value for $h(n,k)$,

$$h(n,k) = 2\sum_{j=0}^{k-1}\binom{n}{j}.$$

This proves that

$$m(n,k) = 2\sum_{j=0}^{k-1}\binom{n}{j} - 1.$$

The generating function approach also works if applied to the non-homogeneous recurrence relation (1), but the computations are less straightforward.

Solution 2. Define merges and insertions as in Solution 1. After each move made by Sir Alex we compute the number N of empty cells, and the sum S of all the numbers written in the cells. Insertions always increase S by some power of 2, and increase N exactly by 1. Merges do not change S and decrease N exactly by 1. Since the initial value of N is 0 and its final value is 1, the total number of insertions exceeds that of merges by exactly one. So, to maximize the number of moves, we need to maximize the number of insertions.

We will need the following lemma.

Lemma. If the binary representation of a positive integer A has d nonzero digits, then A cannot be represented as a sum of fewer than d powers of 2. Moreover, any representation of A as a sum of d powers of 2 must coincide with its binary representation.

Proof. Let s be the minimum number of summands in all possible representations of A as sum of powers of 2. Suppose there is such a representation with s summands, where two of the summands are equal to each other. Then, replacing those two summands with the result of their sum, we obtain a representation with fewer than s summands, which is a contradiction. We deduce that in any representation with s summands, the summands are all distinct, so any such representation must coincide with the unique binary representation of A, and $s = d$. $\square$

Now we split the solution into a sequence of claims.

Claim 1. After every move, the number S is the sum of at most $k-1$ distinct powers of 2.

Proof. If S is the sum of k (or more) distinct powers of 2, the Lemma implies that the k cells are filled with these numbers. This is a contradiction since no more merges or insertions can be made. $\square$

Let $A(n,k-1)$ denote the set of all positive integers not exceeding 2^n with at most $k-1$ nonzero digits in its base 2 representation. Since every insertion increases the value of S, by Claim 1, the total number of insertions is at most $|A(n,k-1)|$. We proceed to prove that it is possible to achieve this number of insertions.

Claim 2. Let $A(n,k-1) = \{a_1, a_2, \ldots, a_m\}$, with $a_1 < a_2 < \cdots < a_m$. If after some of Sir Alex's moves the value of S is a_j, with $j \in \{1, 2, \ldots, m-1\}$, then there is a sequence of moves after which the value of S is exactly a_{j+1}.

Proof. Suppose $S = a_j$. Performing all possible merges, we eventually get different powers of 2 in all nonempty cells. After that, by Claim 1 there will be at least one empty cell, in which we want to insert $a_{j+1} - a_j$. It remains to show that $a_{j+1} - a_j$ is a power of 2.

For this purpose, we notice that if a_j has less than $k-1$ nonzero digits in base 2 then $a_{j+1} = a_j + 1$. Otherwise, we have $a_j = 2^{b_{k-1}} + \cdots + 2^{b_2} + 2^{b_1}$ with $b_1 < b_2 < \cdots < b_{k-1}$. Then, adding any number less than 2^{b_1} to a_j will result in a number with more than $k-1$ nonzero binary digits. On the other hand, $a_j + 2^{b_1}$ is a sum of k powers of 2, not all distinct, so by the Lemma it will be a sum of less then k distinct powers of 2. This means that $a_{j+1} - a_j = 2^{b_1}$, completing the proof. □

Claims 1 and 2 prove that the maximum number of insertions is $|A(n, k-1)|$. We now compute this number.

Claim 3. $|A(n, k-1)| = \sum_{j=0}^{k-1} \binom{n}{j}$.

Proof. The number 2^n is the only element of $A(n, k-1)$ with $n+1$ binary digits. Any other element has at most n binary digits, at least one and at most $k-1$ of them are nonzero (so they are ones). For each $j \in \{1, 2, \ldots, k-1\}$, there are $\binom{n}{j}$ such elements with exactly j binary digits equal to one. We conclude that $|A(n, k-1)| = 1 + \sum_{j=1}^{k-1} \binom{n}{j} = \sum_{j=0}^{k-1} \binom{n}{j}$. □

Recalling that the number of insertions exceeds that of merges by exactly 1, we deduce that the maximum number of moves is $2\sum_{j=0}^{k-1} \binom{n}{j} - 1$.

C4. Let $N \geqslant 2$ be an integer. $N(N+1)$ soccer players, no two of the same height, stand in a row in some order. Coach Ralph wants to remove $N(N-1)$ people from this row so that in the remaining row of $2N$ players, no one stands between the two tallest ones, no one stands between the third and the fourth tallest ones, ..., and finally no one stands between the two shortest ones. Show that this is always possible.

(Russia)

Solution 1. Split the row into N blocks with $N+1$ consecutive people each. We will show how to remove $N-1$ people from each block in order to satisfy the coach's wish.

First, construct a $(N+1) \times N$ matrix where $x_{i,j}$ is the height of the i^{th} tallest person of the j^{th} block—in other words, each column lists the heights within a single block, sorted in decreasing order from top to bottom.

We will reorder this matrix by repeatedly swapping whole columns. First, by column permutation, make sure that $x_{2,1} = \max\{x_{2,i} \colon i = 1, 2, \ldots, N\}$ (the first column contains the largest height of the *second* row). With the first column fixed, permute the other ones so that $x_{3,2} = \max\{x_{3,i} \colon i = 2, \ldots, N\}$ (the second column contains the tallest person of the *third* row, first column excluded). In short, at step k ($k = 1, 2, \ldots, N-1$), we permute the columns from k to N so that $x_{k+1,k} = \max\{x_{i,k} \colon i = k, k+1, \ldots, N\}$, and end up with an array like this:

$$
\begin{array}{cccccccc}
\boldsymbol{x_{1,1}} & & x_{1,2} & & x_{1,3} & \cdots & x_{1,N-1} & & x_{1,N} \\
\boldsymbol{\vee} & & \vee & & \vee & & \vee & & \vee \\
\boldsymbol{x_{2,1}} & \boldsymbol{>} & \boldsymbol{x_{2,2}} & & x_{2,3} & \cdots & x_{2,N-1} & & x_{2,N} \\
\vee & & \boldsymbol{\vee} & & \vee & & \vee & & \vee \\
x_{3,1} & & \boldsymbol{x_{3,2}} & \boldsymbol{>} & \boldsymbol{x_{3,3}} & \cdots & x_{3,N-1} & & x_{3,N} \\
\vee & & \vee & & \boldsymbol{\vee} & & \vee & & \vee \\
\vdots & & \vdots & & \vdots & \ddots & \vdots & & \vdots \\
\vee & & \vee & & \vee & & \boldsymbol{\vee} & & \vee \\
x_{N,1} & & x_{N,2} & & x_{N,3} & \cdots & \boldsymbol{x_{N,N-1}} & \boldsymbol{>} & \boldsymbol{x_{N,N}} \\
\vee & & \vee & & \vee & & \vee & & \boldsymbol{\vee} \\
x_{N+1,1} & & x_{N+1,2} & & x_{N+1,3} & \cdots & x_{N+1,N-1} & & \boldsymbol{x_{N+1,N}}
\end{array}
$$

Now we make the bold choice: from the original row of people, remove everyone but those with heights

$$x_{1,1} > x_{2,1} > x_{2,2} > x_{3,2} > \cdots > x_{N,N-1} > x_{N,N} > x_{N+1,N} \qquad (*)$$

Of course this *height* order $(*)$ is not necessarily their *spatial* order in the new row. We now need to convince ourselves that each pair $(x_{k,k}; x_{k+1,k})$ remains spatially together in this new row. But $x_{k,k}$ and $x_{k+1,k}$ belong to the same column/block of consecutive $N+1$ people; the only people that could possibly stand between them were also in this block, and they are all gone.

Solution 2. Split the people into N groups *by height*: group G_1 has the $N+1$ tallest ones, group G_2 has the next $N+1$ tallest, and so on, up to group G_N with the $N+1$ shortest people.

Now scan the original row from left to right, stopping as soon as you have scanned two people (consecutively or not) from the same group, say, G_i. Since we have N groups, this must happen before or at the $(N+1)^{\text{th}}$ person of the row. Choose this pair of people, removing all the other people from the same group G_i and also all people that have been scanned so far. The only people that could separate this pair's heights were in group G_i (and they are gone); the only people that could separate this pair's positions were already scanned (and they are gone too).

We are now left with $N-1$ groups (all except G_i). Since each of them lost at most one person, each one has at least N unscanned people left in the row. Repeat the scanning process from left to right, choosing the next two people from the same group, removing this group and everyone scanned up to that point. Once again we end up with two people who are next to each other in the remaining row and whose heights cannot be separated by anyone else who remains (since the rest of their group is gone). After picking these 2 pairs, we still have $N-2$ groups with at least $N-1$ people each.

If we repeat the scanning process a total of N times, it is easy to check that we will end up with 2 people from each group, for a total of $2N$ people remaining. The height order is guaranteed by the grouping, and the scanning construction from left to right guarantees that each pair from a group stand next to each other in the final row. We are done.

Solution 3. This is essentially the same as solution 1, but presented inductively. The essence of the argument is the following lemma.

Lemma. Assume that we have N disjoint groups of at least $N+1$ people in each, all people have distinct heights. Then one can choose two people from each group so that among the chosen people, the two tallest ones are in one group, the third and the fourth tallest ones are in one group, ..., and the two shortest ones are in one group.

Proof. Induction on $N \geqslant 1$; for $N=1$, the statement is trivial.

Consider now N groups $G_1, \ldots, G_N$ with at least $N+1$ people in each for $N \geqslant 2$. Enumerate the people by $1, 2, \ldots, N(N+1)$ according to their height, say, from tallest to shortest. Find the least s such that two people among $1, 2, \ldots, s$ are in one group (without loss of generality, say this group is G_N). By the minimality of s, the two mentioned people in G_N are s and some $i<s$.

Now we choose people i and s in G_N, forget about this group, and remove the people $1, 2, \ldots, s$ from $G_1, \ldots, G_{N-1}$. Due to minimality of s again, each of the obtained groups $G'_1, \ldots, G'_{N-1}$ contains at least N people. By the induction hypothesis, one can choose a pair of people from each of $G'_1, \ldots, G'_{N-1}$ so as to satisfy the required conditions. Since all these people have numbers greater than s, addition of the pair (s, i) from G_N does not violate these requirements. $\square$

To solve the problem, it suffices now to split the row into N contiguous groups with $N+1$ people in each and apply the Lemma to those groups.

Comment 1. One can identify each person with a pair of indices (p, h) $(p, h \in \{1, 2, \ldots, N(N+1)\})$ so that the p^{th} person in the row (say, from left to right) is the h^{th} tallest person in the group. Say that (a, b) *separates* (x_1, y_1) and (x_2, y_2) whenever a is strictly between x_1 and y_1, *or* b is strictly between x_2 and y_2. So the coach wants to pick $2N$ people $(p_i, h_i)(i=1, 2, \ldots, 2N)$ such that no chosen person separates (p_1, h_1) from (p_2, h_2), no chosen person separates (p_3, h_3) and (p_4, h_4), and so on. This formulation reveals a duality between positions and heights. In that sense, solutions 1 and 2 are dual of each other.

Comment 2. The number $N(N+1)$ is sharp for $N=2$ and $N=3$, due to arrangements $1, 5, 3, 4, 2$ and $1, 10, 6, 4, 3, 9, 5, 8, 7, 2, 11$.

C5. A hunter and an invisible rabbit play a game in the Euclidean plane. The hunter's starting point H_0 coincides with the rabbit's starting point R_0. In the n^{th} round of the game ($n \geqslant 1$), the following happens.

(1) First the invisible rabbit moves secretly and unobserved from its current point R_{n-1} to some new point R_n with $R_{n-1}R_n = 1$.

(2) The hunter has a tracking device (e.g. dog) that returns an approximate position R'_n of the rabbit, so that $R_nR'_n \leqslant 1$.

(3) The hunter then visibly moves from point H_{n-1} to a new point H_n with $H_{n-1}H_n = 1$.

Is there a strategy for the hunter that guarantees that after 10^9 such rounds the distance between the hunter and the rabbit is below 100?

(Austria)

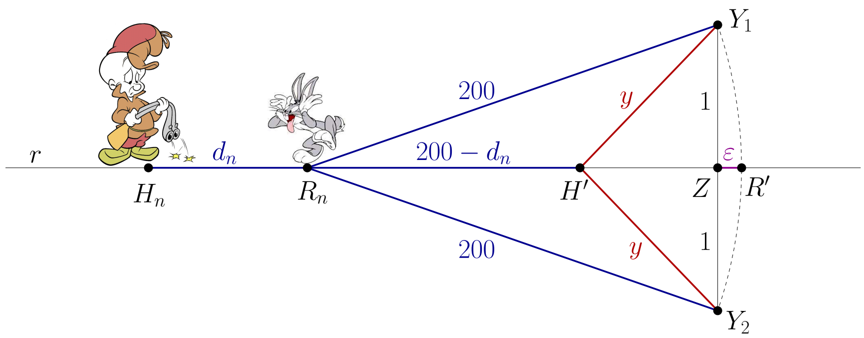
Answer: There is no such strategy for the hunter. The rabbit "wins".

Solution. If the answer were "yes", the hunter would have a strategy that would "work", no matter how the rabbit moved or where the radar pings R'_n appeared. We will show the opposite: with bad luck from the radar pings, there is no strategy for the hunter that guarantees that the distance stays below 100 in 10^9 rounds.

So, let d_n be the distance between the hunter and the rabbit after n rounds. Of course, if $d_n \geqslant 100$ for any $n < 10^9$, the rabbit has won — it just needs to move straight away from the hunter, and the distance will be kept at or above 100 thereon.

We will now show that, while $d_n < 100$, whatever given strategy the hunter follows, the rabbit has a way of increasing d_n^2 by at least $\frac{1}{2}$ every 200 rounds (as long as the radar pings are lucky enough for the rabbit). This way, d_n^2 will reach 10^4 in less than $2 \cdot 10^4 \cdot 200 = 4 \cdot 10^6 < 10^9$ rounds, and the rabbit wins.

Suppose the hunter is at H_n and the rabbit is at R_n. Suppose even that the rabbit *reveals* its position at this moment to the hunter (this allows us to ignore all information from previous radar pings). Let r be the line H_nR_n, and Y_1 and Y_2 be points which are 1 unit away from r and 200 units away from R_n, as in the figure below.



The rabbit's plan is simply to choose one of the points Y_1 or Y_2 and hop 200 rounds straight towards it. Since all hops stay within 1 distance unit from r, it is possible that all radar pings stay on r. In particular, in this case, the hunter has no way of knowing whether the rabbit chose Y_1 or Y_2.

Looking at such pings, what is the hunter going to do? If the hunter's strategy tells him to go 200 rounds straight to the right, he ends up at point H' in the figure. Note that the hunter does not have a better alternative! Indeed, after these 200 rounds he will always end up at a point to the left of H'. If his strategy took him to a point above r, he would end up even further from Y_2; and if his strategy took him below r, he would end up even further from Y_1. In other words, no matter what strategy the hunter follows, he can never be sure his distance to the rabbit will be less than $y \stackrel{\text{def}}{=} H'Y_1 = H'Y_2$ after these 200 rounds.

To estimate y^2, we take Z as the midpoint of segment Y_1Y_2, we take R' as a point 200 units to the right of R_n and we define $\varepsilon = ZR'$ (note that $H'R' = d_n$). Then

$$y^2 = 1 + (H'Z)^2 = 1 + (d_n - \varepsilon)^2$$

where

$$\varepsilon = 200 - R_n Z = 200 - \sqrt{200^2 - 1} = \frac{1}{200 + \sqrt{200^2 - 1}} > \frac{1}{400}.$$

In particular, $\varepsilon^2 + 1 = 400\varepsilon$, so

$$y^2 = d_n^2 - 2\varepsilon d_n + \varepsilon^2 + 1 = d_n^2 + \varepsilon(400 - 2d_n).$$

Since $\varepsilon > \frac{1}{400}$ and we assumed $d_n < 100$, this shows that $y^2 > d_n^2 + \frac{1}{2}$. So, as we claimed, with this list of radar pings, no matter what the hunter does, the rabbit might achieve $d_{n+200}^2 > d_n^2 + \frac{1}{2}$. The wabbit wins.

Comment 1. Many different versions of the solution above can be found by replacing 200 with some other number N for the number of hops the rabbit takes between reveals. If this is done, we have:

$$\varepsilon = N - \sqrt{N^2 - 1} > \frac{1}{N + \sqrt{N^2 - 1}} > \frac{1}{2N}$$

and

$$\varepsilon^2 + 1 = 2N\varepsilon,$$

so, as long as $N > d_n$, we would find

$$y^2 = d_n^2 + \varepsilon(2N - 2d_n) > d_n^2 + \frac{N - d_n}{N}.$$

For example, taking $N = 101$ is already enough—the squared distance increases by at least $\frac{1}{101}$ every 101 rounds, and $101^2 \cdot 10^4 = 1.0201 \cdot 10^8 < 10^9$ rounds are enough for the rabbit. If the statement is made sharper, some such versions might not work any longer.

Comment 2. The original statement asked whether the distance could be kept under 10^{10} in 10^{100} rounds.

C6. Let $n > 1$ be an integer. An $n \times n \times n$ cube is composed of n^3 unit cubes. Each unit cube is painted with one color. For each $n \times n \times 1$ box consisting of n^2 unit cubes (of any of the three possible orientations), we consider the set of the colors present in that box (each color is listed only once). This way, we get $3n$ sets of colors, split into three groups according to the orientation. It happens that for every set in any group, the same set appears in both of the other groups. Determine, in terms of n, the maximal possible number of colors that are present.

(Russia)

Answer: The maximal number is $\frac{n(n+1)(2n+1)}{6}$.

Solution 1. Call a $n \times n \times 1$ box an *x-box*, a *y-box*, or a *z-box*, according to the direction of its short side. Let C be the number of colors in a valid configuration. We start with the upper bound for C.

Let $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ be the sets of colors which appear in the big cube exactly once, exactly twice, and at least thrice, respectively. Let M_i be the set of unit cubes whose colors are in $\mathcal{C}_i$, and denote $n_i = |M_i|$.

Consider any x-box X, and let Y and Z be a y- and a z-box containing the same set of colors as X does.

Claim. $4|X \cap M_1| + |X \cap M_2| \leqslant 3n + 1$.

Proof. We distinguish two cases.

Case 1: $X \cap M_1 \neq \varnothing$.

A cube from $X \cap M_1$ should appear in all three boxes X, Y, and Z, so it should lie in $X \cap Y \cap Z$. Thus $X \cap M_1 = X \cap Y \cap Z$ and $|X \cap M_1| = 1$.

Consider now the cubes in $X \cap M_2$. There are at most $2(n-1)$ of them lying in $X \cap Y$ or $X \cap Z$ (because the cube from $X \cap Y \cap Z$ is in M_1). Let a be some *other* cube from $X \cap M_2$. Recall that there is just one other cube a' sharing a color with a. But both Y and Z should contain such cube, so $a' \in Y \cap Z$ (but $a' \notin X \cap Y \cap Z$). The map $a \mapsto a'$ is clearly injective, so the number of cubes a we are interested in does not exceed $|(Y \cap Z) \setminus X| = n - 1$. Thus $|X \cap M_2| \leqslant 2(n-1) + (n-1) = 3(n-1)$, and hence $4|X \cap M_1| + |X \cap M_2| \leqslant 4 + 3(n-1) = 3n+1$.

Case 2: $X \cap M_1 = \varnothing$.

In this case, the same argument applies with several changes. Indeed, $X \cap M_2$ contains at most $2n - 1$ cubes from $X \cap Y$ or $X \cap Z$. Any other cube a in $X \cap M_2$ corresponds to some $a' \in Y \cap Z$ (possibly with $a' \in X$), so there are at most n of them. All this results in $|X \cap M_2| \leqslant (2n-1) + n = 3n - 1$, which is even better than we need (by the assumptions of our case). □

Summing up the inequalities from the Claim over all x-boxes X, we obtain

$$4n_1 + n_2 \leqslant n(3n+1).$$

Obviously, we also have $n_1 + n_2 + n_3 = n^3$.

Now we are prepared to estimate C. Due to the definition of the M_i, we have $n_i \geqslant i|\mathcal{C}_i|$, so

$$C \leqslant n_1 + \frac{n_2}{2} + \frac{n_3}{3} = \frac{n_1 + n_2 + n_3}{3} + \frac{4n_1 + n_2}{6} \leqslant \frac{n^3}{3} + \frac{3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

It remains to present an example of an appropriate coloring in the above-mentioned number of colors. For each color, we present the set of all cubes of this color. These sets are:

1. n singletons of the form $S_i = \{(i,i,i)\}$ (with $1 \leqslant i \leqslant n$);
2. $3\binom{n}{2}$ doubletons of the forms $D^1_{i,j} = \{(i,j,j),(j,i,i)\}$, $D^2_{i,j} = \{(j,i,j),(i,j,i)\}$, and $D^3_{i,j} = \{(j,j,i),(i,i,j)\}$ (with $1 \leqslant i < j \leqslant n$);

3. $2\binom{n}{3}$ triplets of the form $T_{i,j,k} = \{(i,j,k),(j,k,i),(k,i,j)\}$ (with $1 \leqslant i < j < k \leqslant n$ or $1 \leqslant i < k < j \leqslant n$).

One may easily see that the i^{th} boxes of each orientation contain the same set of colors, and that

$$n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3} = \frac{n(n+1)(2n+1)}{6}$$

colors are used, as required.

Solution 2. We will approach a new version of the original problem. In this new version, each cube may have a color, or be *invisible* (not both). Now we make sets of colors for each $n \times n \times 1$ box as before (where "invisible" is not considered a color) and group them by orientation, also as before. Finally, we require that, for every *non-empty* set in any group, the same set must appear in the other 2 groups. What is the maximum number of colors present with these new requirements?

Let us call *strange* a big $n \times n \times n$ cube whose painting scheme satisfies the new requirements, and let D be the number of colors in a strange cube. Note that any cube that satisfies the original requirements is also strange, so $\max(D)$ is an upper bound for the original answer.

Claim. $D \leqslant \frac{n(n+1)(2n+1)}{6}$.

Proof. The proof is by induction on n. If $n = 1$, we must paint the cube with at most 1 color.

Now, pick a $n \times n \times n$ strange cube A, where $n \geqslant 2$. If A is completely invisible, $D = 0$ and we are done. Otherwise, pick a non-empty set of colors $\mathcal{S}$ which corresponds to, say, the boxes X, Y and Z of different orientations.

Now find all cubes in A whose colors are in $\mathcal{S}$ and make them invisible. Since X, Y and Z are now completely invisible, we can throw them away and focus on the remaining $(n-1) \times (n-1) \times (n-1)$ cube B. The sets of colors in all the groups for B are the same as the sets for A, removing exactly the colors in $\mathcal{S}$, and no others! Therefore, every nonempty set that appears in one group for B still shows up in all possible orientations (it is possible that an empty set of colors in B only matched X, Y or Z before these were thrown away, but remember we do not require empty sets to match anyway). In summary, B is also strange.

By the induction hypothesis, we may assume that B has at most $\frac{(n-1)n(2n-1)}{6}$ colors. Since there were at most n^2 different colors in $\mathcal{S}$, we have that A has at most $\frac{(n-1)n(2n-1)}{6} + n^2 = \frac{n(n+1)(2n+1)}{6}$ colors. $\square$

Finally, the construction in the previous solution shows a painting scheme (with no invisible cubes) that reaches this maximum, so we are done.

C7. For any finite sets X and Y of positive integers, denote by $f_X(k)$ the k^{th} smallest positive integer not in X, and let

$$X * Y = X \cup \{f_X(y) : y \in Y\}.$$

Let A be a set of $a > 0$ positive integers, and let B be a set of $b > 0$ positive integers. Prove that if $A * B = B * A$, then

$$\underbrace{A * (A * \cdots * (A * (A * A)) \ldots)}_{A \text{ appears } b \text{ times}} = \underbrace{B * (B * \cdots * (B * (B * B)) \ldots)}_{B \text{ appears } a \text{ times}}.$$

(U.S.A.)

Solution 1. For any function $g : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and any subset $X \subset \mathbb{Z}_{>0}$, we define $g(X) = \{g(x) : x \in X\}$. We have that the image of f_X is $f_X(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus X$. We now show a general lemma about the operation $*$, with the goal of showing that $*$ is associative.

Lemma 1. Let X and Y be finite sets of positive integers. The functions f_{X*Y} and $f_X \circ f_Y$ are equal.

Proof. We have

$$f_{X*Y}(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \backslash (X*Y) = (\mathbb{Z}_{>0} \backslash X) \backslash f_X(Y) = f_X(\mathbb{Z}_{>0}) \backslash f_X(Y) = f_X(\mathbb{Z}_{>0} \backslash Y) = f_X(f_Y(\mathbb{Z}_{>0})).$$

Thus, the functions f_{X*Y} and $f_X \circ f_Y$ are strictly increasing functions with the same range. Because a strictly function is uniquely defined by its range, we have $f_{X*Y} = f_X \circ f_Y$. $\square$

Lemma 1 implies that $*$ is associative, in the sense that $(A * B) * C = A * (B * C)$ for any finite sets A, B, and C of positive integers. We prove the associativity by noting

$$\mathbb{Z}_{>0} \setminus ((A * B) * C) = f_{(A*B)*C}(\mathbb{Z}_{>0}) = f_{A*B}(f_C(\mathbb{Z}_{>0})) = f_A(f_B(f_C(\mathbb{Z}_{>0})))$$

$$= f_A(f_{B*C}(\mathbb{Z}_{>0}) = f_{A*(B*C)}(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus (A * (B * C)).$$

In light of the associativity of $*$, we may drop the parentheses when we write expressions like $A * (B * C)$. We also introduce the notation

$$X^{*k} = \underbrace{X * (X * \cdots * (X * (X * X)) \ldots)}_{X \text{ appears } k \text{ times}}.$$

Our goal is then to show that $A * B = B * A$ implies $A^{*b} = B^{*a}$. We will do so via the following general lemma.

Lemma 2. Suppose that X and Y are finite sets of positive integers satisfying $X * Y = Y * X$ and $|X| = |Y|$. Then, we must have $X = Y$.

Proof. Assume that X and Y are not equal. Let s be the largest number in exactly one of X and Y. Without loss of generality, say that $s \in X \setminus Y$. The number $f_X(s)$ counts the s^{th} number not in X, which implies that

$$f_X(s) = s + \big|X \cap \{1, 2, \ldots, f_X(s)\}\big|. \tag{1}$$

Since $f_X(s) \geqslant s$, we have that

$$\big\{f_X(s) + 1, f_X(s) + 2, \ldots\big\} \cap X = \big\{f_X(s) + 1, f_X(s) + 2, \ldots\big\} \cap Y,$$

which, together with the assumption that $|X| = |Y|$, gives

$$\big|X \cap \{1, 2, \ldots, f_X(s)\}\big| = \big|Y \cap \{1, 2, \ldots, f_X(s)\}\big|. \tag{2}$$

Now consider the equation

$$t - \left|Y \cap \{1, 2, \ldots, t\}\right| = s.$$

This equation is satisfied only when $t \in \left[f_Y(s), f_Y(s+1)\right)$, because the left hand side counts the number of elements up to t that are not in Y. We have that the value $t = f_X(s)$ satisfies the above equation because of (1) and (2). Furthermore, since $f_X(s) \notin X$ and $f_X(s) \geqslant s$, we have that $f_X(s) \notin Y$ due to the maximality of s. Thus, by the above discussion, we must have $f_X(s) = f_Y(s)$.

Finally, we arrive at a contradiction. The value $f_X(s)$ is neither in X nor in $f_X(Y)$, because s is not in Y by assumption. Thus, $f_X(s) \notin X * Y$. However, since $s \in X$, we have $f_Y(s) \in Y * X$, a contradiction. $\square$

We are now ready to finish the proof. Note first of all that $|A^{*b}| = ab = |B^{*a}|$. Moreover, since $A * B = B * A$, and $*$ is associative, it follows that $A^{*b} * B^{*a} = B^{*a} * A^{*b}$. Thus, by Lemma 2, we have $A^{*b} = B^{*a}$, as desired.

Comment 1. Taking $A = X^{*k}$ and $B = X^{*l}$ generates many non-trivial examples where $A*B = B*A$. There are also other examples not of this form. For example, if $A = \{1, 2, 4\}$ and $B = \{1, 3\}$, then $A * B = \{1, 2, 3, 4, 6\} = B * A$.

Solution 2. We will use Lemma 1 from Solution 1. Additionally, let X^{*k} be defined as in Solution 1. If X and Y are finite sets, then

$$f_X = f_Y \iff f_X(\mathbb{Z}_{>0}) = f_Y(\mathbb{Z}_{>0}) \iff (\mathbb{Z}_{>0} \setminus X) = (\mathbb{Z}_{>0} \setminus Y) \iff X = Y, \tag{3}$$

where the first equivalence is because f_X and f_Y are strictly increasing functions, and the second equivalence is because $f_X(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus X$ and $f_Y(\mathbb{Z}_{>0}) = \mathbb{Z}_{>0} \setminus Y$.

Denote $g = f_A$ and $h = f_B$. The given relation $A * B = B * A$ is equivalent to $f_{A*B} = f_{B*A}$ because of (3), and by Lemma 1 of the first solution, this is equivalent to $g \circ h = h \circ g$. Similarly, the required relation $A^{*b} = B^{*a}$ is equivalent to $g^b = h^a$. We will show that

$$g^b(n) = h^a(n) \tag{4}$$

for all $n \in \mathbb{Z}_{>0}$, which suffices to solve the problem.

To start, we claim that (4) holds for all sufficiently large n. Indeed, let p and q be the maximal elements of A and B, respectively; we may assume that $p \geqslant q$. Then, for every $n \geqslant p$ we have $g(n) = n + a$ and $h(n) = n + b$, whence $g^b(n) = n + ab = h^a(n)$, as was claimed.

In view of this claim, if (4) is not identically true, then there exists a maximal s with $g^b(s) \neq h^a(s)$. Without loss of generality, we may assume that $g(s) \neq s$, for if we had $g(s) = h(s) = s$, then s would satisfy (4). As g is increasing, we then have $g(s) > s$, so (4) holds for $n = g(s)$. But then we have

$$g(g^b(s)) = g^{b+1}(s) = g^b(n) = h^a(n) = h^a(g(s)) = g(h^a(s)),$$

where the last equality holds in view of $g \circ h = h \circ g$. By the injectivity of g, the above equality yields $g^b(s) = h^a(s)$, which contradicts the choice of s. Thus, we have proved that (4) is identically true on $\mathbb{Z}_{>0}$, as desired.

Comment 2. We present another proof of Lemma 2 of the first solution.

Let $x = |X| = |Y|$. Say that u is the smallest number in X and v is the smallest number in Y; assume without loss of generality that $u \leqslant v$.

Let T be any finite set of positive integers, and define $t = |T|$. Enumerate the elements of X as $x_1 < x_2 < \cdots < x_n$. Define $S_m = f_{(T*X^{*(m-1)})}(X)$, and enumerate its elements $s_{m,1} < s_{m,2} < \cdots < s_{m,n}$. Note that the S_m are pairwise disjoint; indeed, if we have $m < m'$, then

$$S_m \subset T * X^{*m} \subset T * X^{*(m'-1)} \quad \text{and} \quad S_{m'} = (T * X^{*m'}) \setminus (T * X^{*(m'-1)})$$

We claim the following statement, which essentially says that the S_m are eventually linear translates of each other:

Claim. For every i, there exists some m_i and c_i such that for all $m > m_i$, we have that $s_{m,i} = t+mn-c_i$. Furthermore, the c_i do not depend on the choice of T.

First, we show that this claim implies Lemma 2. We may choose $T = X$ and $T = Y$. Then, there is some m' such that for all $m \geqslant m'$, we have

$$f_{X^{*m}}(X) = f_{(Y*X^{*(m-1)})}(X). \tag{5}$$

Because u is the minimum element of X, v is the minimum element of Y, and $u \leqslant v$, we have that

$$\left(\bigcup_{m=m'}^{\infty} f_{X^{*m}}(X)\right) \cup X^{*m'} = \left(\bigcup_{m=m'}^{\infty} f_{(Y*X^{*(m-1)})}(X)\right) \cup \left(Y * X^{*(m'-1)}\right) = \{u, u+1, \dots\},$$

and in both the first and second expressions, the unions are of pairwise distinct sets. By (5), we obtain $X^{*m'} = Y * X^{*(m'-1)}$. Now, because X and Y commute, we get $X^{*m'} = X^{*(m'-1)} * Y$, and so $X = Y$.

We now prove the claim.

Proof of the claim. We induct downwards on i, first proving the statement for $i = n$, and so on.

Assume that m is chosen so that all elements of S_m are greater than all elements of T (which is possible because T is finite). For $i = n$, we have that $s_{m,n} > s_{k,n}$ for every $k < m$. Thus, all $(m-1)n$ numbers of the form $s_{k,u}$ for $k < m$ and $1 \leqslant u \leqslant n$ are less than $s_{m,n}$. We then have that $s_{m,n}$ is the $((m-1)n + x_n)^{th}$ number not in T, which is equal to $t + (m-1)n + x_n$. So we may choose $c_n = x_n - n$, which does not depend on T, which proves the base case for the induction.

For $i < n$, we have again that all elements $s_{m,j}$ for $j < i$ and $s_{p,i}$ for $p < m$ are less than $s_{m,i}$, so $s_{m,i}$ is the $((m-1)i + x_i)^{th}$ element not in T or of the form $s_{p,j}$ for $j > i$ and $p < m$. But by the inductive hypothesis, each of the sequences $s_{p,j}$ is eventually periodic with period n, and thus the sequence $s_{m,i}$ such must be as well. Since each of the sequences $s_{p,j} - t$ with $j > i$ eventually do not depend on T, the sequence $s_{m,i} - t$ eventually does not depend on T either, so the inductive step is complete. This proves the claim and thus Lemma 2. $\square$

C8. Let n be a given positive integer. In the Cartesian plane, each lattice point with nonnegative coordinates initially contains a butterfly, and there are no other butterflies. The *neighborhood* of a lattice point c consists of all lattice points within the axis-aligned $(2n+1)\times(2n+1)$ square centered at c, apart from c itself. We call a butterfly *lonely*, *crowded*, or *comfortable*, depending on whether the number of butterflies in its neighborhood N is respectively less than, greater than, or equal to half of the number of lattice points in N.

Every minute, all lonely butterflies fly away simultaneously. This process goes on for as long as there are any lonely butterflies. Assuming that the process eventually stops, determine the number of comfortable butterflies at the final state.

(Bulgaria)

Answer: n^2+1.

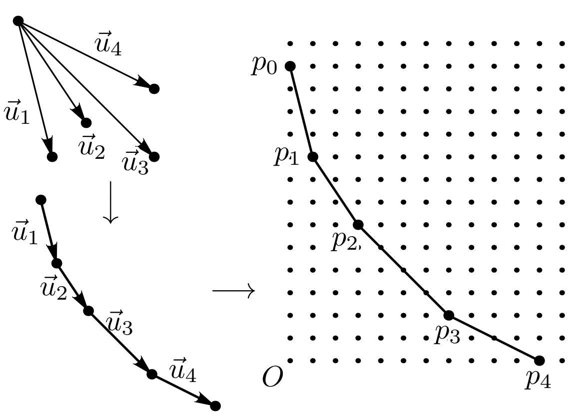
Solution. We always identify a butterfly with the lattice point it is situated at. For two points p and q, we write $p \geqslant q$ if each coordinate of p is at least the corresponding coordinate of q. Let O be the origin, and let $\mathcal{Q}$ be the set of initially occupied points, i.e., of all lattice points with nonnegative coordinates. Let $\mathcal{R}_{\mathrm{H}} = \{(x,0)\colon x \geqslant 0\}$ and $\mathcal{R}_{\mathrm{V}} = \{(0,y)\colon y \geqslant 0\}$ be the sets of the lattice points lying on the horizontal and vertical boundary rays of $\mathcal{Q}$. Denote by $N(a)$ the neighborhood of a lattice point a.

1. Initial observations. We call a set of lattice points *up-right closed* if its points stay in the set after being shifted by any lattice vector (i,j) with $i,j \geqslant 0$. Whenever the butterflies form a up-right closed set $\mathcal{S}$, we have $|N(p)\cap\mathcal{S}| \geqslant |N(q)\cap\mathcal{S}|$ for any two points $p,q \in \mathcal{S}$ with $p \geqslant q$. So, since $\mathcal{Q}$ is up-right closed, the set of butterflies at any moment also preserves this property. We assume all forthcoming sets of lattice points to be up-right closed.

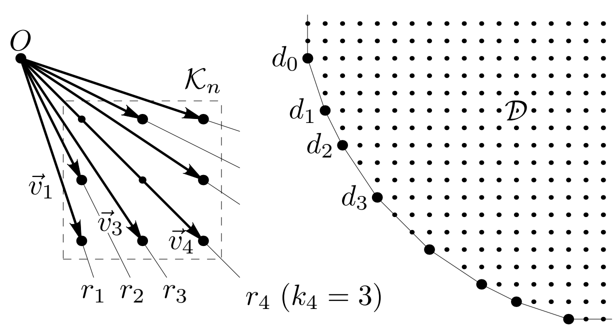
When speaking of some set $\mathcal{S}$ of lattice points, we call its points *lonely*, *comfortable*, or *crowded* with respect to this set (i.e., as if the butterflies were exactly at all points of $\mathcal{S}$). We call a set $\mathcal{S} \subset \mathcal{Q}$ *stable* if it contains no lonely points. In what follows, we are interested only in those stable sets whose complements in $\mathcal{Q}$ are finite, because one can easily see that only a finite number of butterflies can fly away on each minute.

If the initial set $\mathcal{Q}$ of butterflies contains some stable set $\mathcal{S}$, then, clearly no butterfly of this set will fly away. On the other hand, the set $\mathcal{F}$ of all butterflies in the end of the process is stable. This means that $\mathcal{F}$ is the largest (with respect to inclusion) stable set within $\mathcal{Q}$, and we are about to describe this set.

2. A description of a final set. The following notion will be useful. Let $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d\}$ be a set of d pairwise non-parallel lattice vectors, each having a positive x- and a negative y-coordinate. Assume that they are numbered in increasing order according to slope. We now define a *$\mathcal{U}$-curve* to be the broken line $p_0p_1\ldots p_d$ such that $p_0 \in \mathcal{R}_{\mathrm{V}}$, $p_d \in \mathcal{R}_{\mathrm{H}}$, and $\overrightarrow{p_{i-1}p_i} = \vec{u}_i$ for all $i = 1,2,\ldots,m$ (see the Figure below to the left).



Construction of $\mathcal{U}$-curve



Construction of $\mathcal{D}$

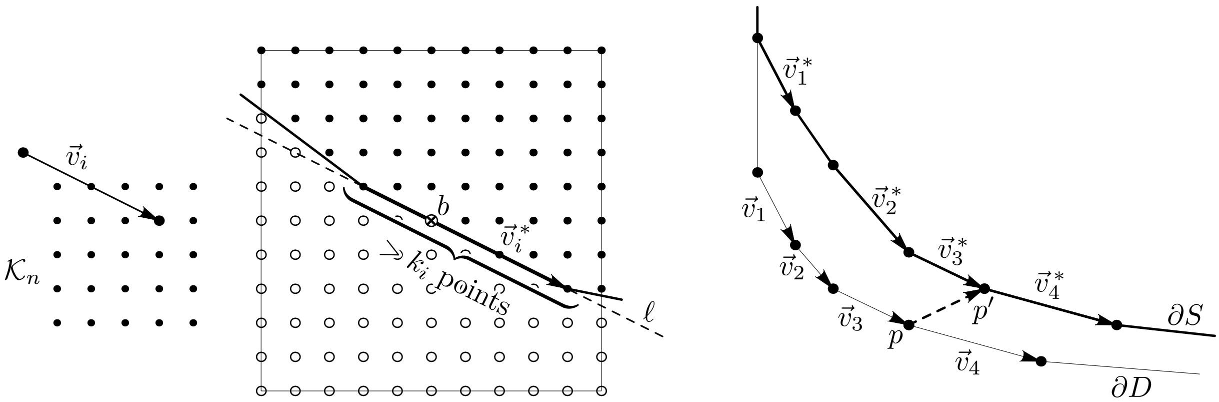
Now, let $\mathcal{K}_n = \{(i,j)\colon 1 \leqslant i \leqslant n,\ -n \leqslant j \leqslant -1\}$. Consider all the rays emerging at O and passing through a point from $\mathcal{K}_n$; number them as $r_1, \ldots, r_m$ in increasing order according to slope. Let A_i be the farthest from O lattice point in $r_i \cap \mathcal{K}_n$, set $k_i = |r_i \cap \mathcal{K}_n|$, let $\vec{v}_i = \overrightarrow{OA_i}$, and finally denote $\mathcal{V} = \{\vec{v}_i\colon 1 \leqslant i \leqslant m\}$; see the Figure above to the right. We will concentrate on the $\mathcal{V}$-curve $d_0d_1\ldots d_m$; let $\mathcal{D}$ be the set of all lattice points p such that $p \geqslant p'$ for some (not necessarily lattice) point p' on the $\mathcal{V}$-curve. In fact, we will show that $\mathcal{D} = \mathcal{F}$.

Clearly, the $\mathcal{V}$-curve is symmetric in the line $y = x$. Denote by D the convex hull of $\mathcal{D}$.

3. We prove that the set $\mathcal{D}$ contains all stable sets. Let $\mathcal{S} \subset \mathcal{Q}$ be a stable set (recall that it is assumed to be up-right closed and to have a finite complement in $\mathcal{Q}$). Denote by S its convex hull; clearly, the vertices of S are lattice points. The boundary of S consists of two rays (horizontal and vertical ones) along with some $\mathcal{V}_*$-curve for some set of lattice vectors $\mathcal{V}_*$.

Claim 1. For every $\vec{v}_i \in \mathcal{V}$, there is a $\vec{v}_i^{\,*} \in \mathcal{V}_*$ co-directed with $\vec{v}$ with $|\vec{v}_i^{\,*}| \geqslant |\vec{v}|$.

Proof. Let ℓ be the supporting line of S parallel to $\vec{v}_i$ (i.e., ℓ contains some point of S, and the set S lies on one side of ℓ). Take any point $b \in \ell \cap \mathcal{S}$ and consider $N(b)$. The line ℓ splits the set $N(b) \setminus \ell$ into two congruent parts, one having an empty intersection with $\mathcal{S}$. Hence, in order for b not to be lonely, at least half of the set $\ell \cap N(b)$ (which contains $2k_i$ points) should lie in S. Thus, the boundary of S contains a segment $\ell \cap S$ with at least $k_i + 1$ lattice points (including b) on it; this segment corresponds to the required vector $\vec{v}_i^{\,*} \in \mathcal{V}_*$. $\square$



Proof of Claim 1 Proof of Claim 2

Claim 2. Each stable set $\mathcal{S} \subseteq \mathcal{Q}$ lies in $\mathcal{D}$.

Proof. To show this, it suffices to prove that the $\mathcal{V}_*$-curve lies in D, i.e., that all its vertices do so. Let p' be an arbitrary vertex of the $\mathcal{V}_*$-curve; p' partitions this curve into two parts, $\mathcal{X}$ (being down-right of p) and $\mathcal{Y}$ (being up-left of p). The set $\mathcal{V}$ is split now into two parts: $\mathcal{V}_\mathcal{X}$ consisting of those $\vec{v}_i \in \mathcal{V}$ for which $\vec{v}_i^{\,*}$ corresponds to a segment in $\mathcal{X}$, and a similar part $\mathcal{V}_\mathcal{Y}$. Notice that the $\mathcal{V}$-curve consists of several segments corresponding to $\mathcal{V}_\mathcal{X}$, followed by those corresponding to $\mathcal{V}_\mathcal{Y}$. Hence there is a vertex p of the $\mathcal{V}$-curve separating $\mathcal{V}_\mathcal{X}$ from $\mathcal{V}_\mathcal{Y}$. Claim 1 now yields that $p' \geqslant p$, so $p' \in \mathcal{D}$, as required. $\square$

Claim 2 implies that the final set $\mathcal{F}$ is contained in $\mathcal{D}$.

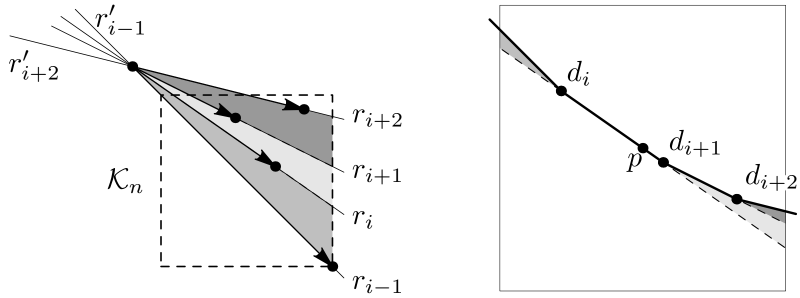
4. $\mathcal{D}$ is stable, and its comfortable points are known. Recall the definitions of r_i; let r_i' be the ray complementary to r_i. By our definitions, the set $N(O)$ contains no points between the rays r_i and r_{i+1}, as well as between r_i' and r_{i+1}'.

Claim 3. In the set $\mathcal{D}$, all lattice points of the $\mathcal{V}$-curve are comfortable.

Proof. Let p be any lattice point of the $\mathcal{V}$-curve, belonging to some segment d_id_{i+1}. Draw the line ℓ containing this segment. Then $\ell \cap \mathcal{D}$ contains exactly $k_i + 1$ lattice points, all of which lie in $N(p)$ except for p. Thus, exactly half of the points in $N(p) \cap \ell$ lie in $\mathcal{D}$. It remains to show that all points of $N(p)$ above ℓ lie in $\mathcal{D}$ (recall that all the points below ℓ lack this property).

Notice that each vector in $\mathcal{V}$ has one coordinate greater than $n/2$; thus the neighborhood of p contains parts of at most two segments of the $\mathcal{V}$-curve succeeding $d_i d_{i+1}$, as well as at most two of those preceding it.

The angles formed by these consecutive segments are obtained from those formed by r_j and r'_{j-1} (with $i-1 \leqslant j \leqslant i+2$) by shifts; see the Figure below. All the points in $N(p)$ above ℓ which could lie outside $\mathcal{D}$ lie in shifted angles between r_j, r_{j+1} or r'_j, r'_{j-1}. But those angles, restricted to $N(p)$, have no lattice points due to the above remark. The claim is proved. $\square$

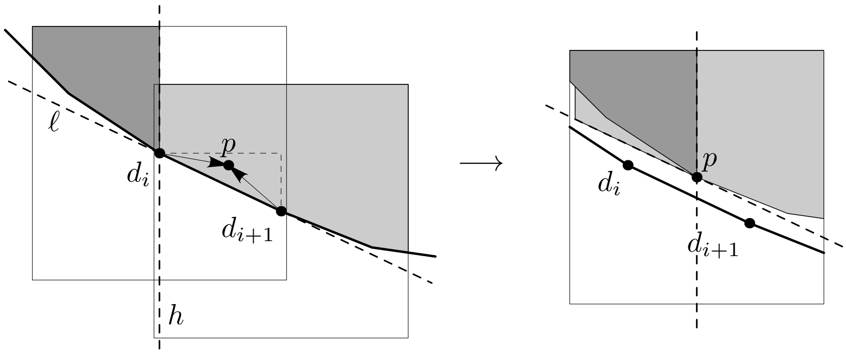


Proof of Claim 3

Claim 4. All the points of $\mathcal{D}$ which are not on the boundary of D are crowded.

Proof. Let $p \in \mathcal{D}$ be such a point. If it is to the up-right of some point p' on the curve, then the claim is easy: the shift of $N(p') \cap \mathcal{D}$ by $\overrightarrow{p'p}$ is still in $\mathcal{D}$, and $N(p)$ contains at least one more point of $\mathcal{D}$ — either below or to the left of p. So, we may assume that p lies in a right triangle constructed on some hypothenuse $d_i d_{i+1}$. Notice here that $d_i, d_{i+1} \in N(p)$.

Draw a line $\ell \parallel d_i d_{i+1}$ through p, and draw a vertical line h through d_i; see Figure below. Let $\mathcal{D}_{\mathrm{L}}$ and $\mathcal{D}_{\mathrm{R}}$ be the parts of $\mathcal{D}$ lying to the left and to the right of h, respectively (points of $\mathcal{D} \cap h$ lie in both parts).



Proof of Claim 4

Notice that the vectors $\overrightarrow{d_i p}$, $\overrightarrow{d_{i+1}d_{i+2}}$, $\overrightarrow{d_i d_{i+1}}$, $\overrightarrow{d_{i-1}d_i}$, and $\overrightarrow{p d_{i+1}}$ are arranged in non-increasing order by slope. This means that $\mathcal{D}_{\mathrm{L}}$ shifted by $\overrightarrow{d_i p}$ still lies in $\mathcal{D}$, as well as $\mathcal{D}_{\mathrm{R}}$ shifted by $\overrightarrow{d_{i+1}p}$. As we have seen in the proof of Claim 3, these two shifts cover all points of $N(p)$ above ℓ, along with those on ℓ to the left of p. Since $N(p)$ contains also d_i and d_{i+1}, the point p is crowded. $\square$

Thus, we have proved that $\mathcal{D} = \mathcal{F}$, and have shown that the lattice points on the $\mathcal{V}$-curve are exactly the comfortable points of $\mathcal{D}$. It remains to find their number.

Recall the definition of $\mathcal{K}_n$ (see Figure on the first page of the solution). Each segment $d_i d_{i+1}$ contains k_i lattice points different from d_i. Taken over all i, these points exhaust all the lattice points in the $\mathcal{V}$-curve, except for d_1, and thus the number of lattice points on the $\mathcal{V}$-curve is $1 + \sum_{i=1}^{m} k_i$. On the other hand, $\sum_{i=1}^{m} k_i$ is just the number of points in $\mathcal{K}_n$, so it equals n^2. Hence the answer to the problem is $n^2 + 1$.

Comment 1. The assumption that the process eventually stops is unnecessary for the problem, as one can see that, in fact, the process stops for every $n \geqslant 1$. Indeed, the proof of Claims 3 and 4 do not rely essentially on this assumption, and they together yield that the set $\mathcal{D}$ is stable. So, only butterflies that are not in $\mathcal{D}$ may fly away, and this takes only a finite time.

This assumption has been inserted into the problem statement in order to avoid several technical details regarding finiteness issues. It may also simplify several other arguments.

Comment 2. The description of the final set $\mathcal{F}(= \mathcal{D})$ seems to be crucial for the solution; the Problem Selection Committee is not aware of any solution that completely avoids such a description.

On the other hand, after the set $\mathcal{D}$ has been defined, the further steps may be performed in several ways. For example, in order to prove that all butterflies outside $\mathcal{D}$ will fly away, one may argue as follows. (Here we will also make use of the assumption that the process eventually stops.)

First of all, notice that the process can be modified in the following manner: *Each minute, exactly one of the lonely butterflies flies away, until there are no more lonely butterflies.* The modified process necessarily stops at the same state as the initial one. Indeed, one may observe, as in solution above, that the (unique) largest stable set is still the final set for the modified process.

Thus, in order to prove our claim, it suffices to indicate an order in which the butterflies should fly away in the new process; if we are able to exhaust the whole set $\mathcal{Q} \setminus \mathcal{D}$, we are done.

Let $\mathcal{C}_0 = d_0 d_1 \ldots d_m$ be the $\mathcal{V}$-curve. Take its copy $\mathcal{C}$ and shift it downwards so that d_0 comes to some point below the origin O. Now we start moving $\mathcal{C}$ upwards continuously, until it comes back to its initial position $\mathcal{C}_0$. At each moment when $\mathcal{C}$ meets some lattice points, we convince all the butterflies at those points to fly away in a certain order. We will now show that we always have enough arguments for butterflies to do so, which will finish our argument for the claim..

Let $\mathcal{C}' = d'_0 d'_1 \ldots d'_m$ be a position of $\mathcal{C}$ when it meets some butterflies. We assume that all butterflies under this current position of $\mathcal{C}$ were already convinced enough and flied away. Consider the lowest butterfly b on $\mathcal{C}'$. Let $d'_i d'_{i+1}$ be the segment it lies on; we choose i so that $b \neq d'_{i+1}$ (this is possible because $\mathcal{C}$ as not yet reached $\mathcal{C}_0$).

Draw a line ℓ containing the segment $d'_i d'_{i+1}$. Then all the butterflies in $N(b)$ are situated on or above ℓ; moreover, those on ℓ all lie on the segment $d_i d_{i+1}$. But this segment now contains at most k_i butterflies (including b), since otherwise some butterfly had to occupy d'_{i+1} which is impossible by the choice of b. Thus, b is lonely and hence may be convinced to fly away.

After b has flied away, we switch to the lowest of the remaining butterflies on $\mathcal{C}'$, and so on.

Claims 3 and 4 also allow some different proofs which are not presented here.

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Geometry

G1. Let $ABCDE$ be a convex pentagon such that $AB = BC = CD$, $\angle EAB = \angle BCD$, and $\angle EDC = \angle CBA$. Prove that the perpendicular line from E to BC and the line segments AC and BD are concurrent.

(Italy)

Solution 1. Throughout the solution, we refer to $\angle A$, $\angle B$, $\angle C$, $\angle D$, and $\angle E$ as internal angles of the pentagon $ABCDE$. Let the perpendicular bisectors of AC and BD, which pass respectively through B and C, meet at point I. Then $BD \perp CI$ and, similarly, $AC \perp BI$. Hence AC and BD meet at the orthocenter H of the triangle BIC, and $IH \perp BC$. It remains to prove that E lies on the line IH or, equivalently, $EI \perp BC$.

Lines IB and IC bisect $\angle B$ and $\angle C$, respectively. Since $IA = IC$, $IB = ID$, and $AB = BC = CD$, the triangles IAB, ICB and ICD are congruent. Hence $\angle IAB = \angle ICB = \angle C/2 = \angle A/2$, so the line IA bisects $\angle A$. Similarly, the line ID bisects $\angle D$. Finally, the line IE bisects $\angle E$ because I lies on all the other four internal bisectors of the angles of the pentagon.

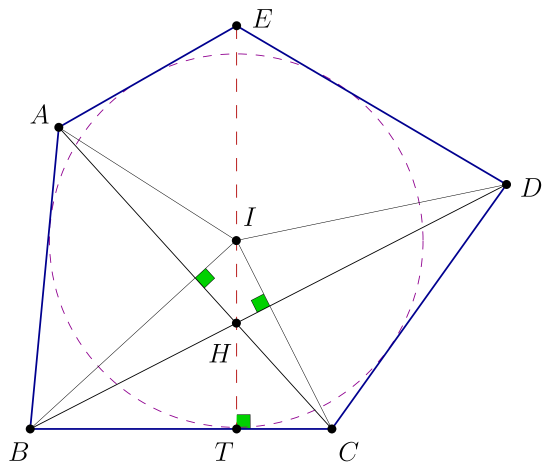
The sum of the internal angles in a pentagon is 540°, so

$$\angle E = 540^\circ - 2\angle A + 2\angle B.$$

In quadrilateral $ABIE$,

$$\begin{aligned}\angle BIE &= 360^\circ - \angle EAB - \angle ABI - \angle AEI = 360^\circ - \angle A - \frac{1}{2}\angle B - \frac{1}{2}\angle E\\ &= 360^\circ - \angle A - \frac{1}{2}\angle B - (270^\circ - \angle A - \angle B)\\ &= 90^\circ + \frac{1}{2}\angle B = 90^\circ + \angle IBC,\end{aligned}$$

which means that $EI \perp BC$, completing the proof.



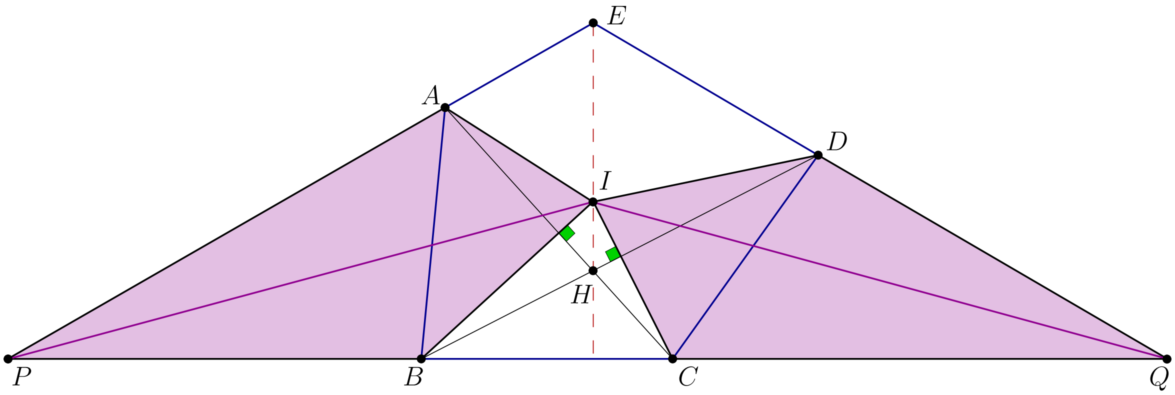
Solution 2. We present another proof of the fact that E lies on line IH. Since all five internal bisectors of $ABCDE$ meet at I, this pentagon has an inscribed circle with center I. Let this circle touch side BC at T.

Applying Brianchon's theorem to the (degenerate) hexagon $ABTCDE$ we conclude that AC, BD and ET are concurrent, so point E also lies on line IHT, completing the proof.

Solution 3. We present yet another proof that $EI \perp BC$. In pentagon $ABCDE$, $\angle E < 180^\circ \iff \angle A + \angle B + \angle C + \angle D > 360^\circ$. Then $\angle A + \angle B = \angle C + \angle D > 180^\circ$, so rays EA and CB meet at a point P, and rays BC and ED meet at a point Q. Now,

$$\angle PBA = 180^\circ - \angle B = 180^\circ - \angle D = \angle QDC$$

and, similarly, $\angle PAB = \angle QCD$. Since $AB = CD$, the triangles PAB and QCD are congruent with the same orientation. Moreover, PQE is isosceles with $EP = EQ$.



In Solution 1 we have proved that triangles IAB and ICD are also congruent with the same orientation. Then we conclude that quadrilaterals $PBIA$ and $QDIC$ are congruent, which implies $IP = IQ$. Then EI is the perpendicular bisector of PQ and, therefore, $EI \perp PQ \iff EI \perp BC$.

Comment. Even though all three solutions used the point I, there are solutions that do not need it. We present an outline of such a solution: if J is the incenter of $\triangle QCD$ (with P and Q as defined in Solution 3), then a simple angle chasing shows that triangles CJD and BHC are congruent. Then if S is the projection of J onto side CD and T is the orthogonal projection of H onto side BC, one can verify that

$$QT = QC + CT = QC + DS = QC + \frac{CD + DQ - QC}{2} = \frac{PB + BC + QC}{2} = \frac{PQ}{2},$$

so T is the midpoint of PQ, and E, H and T all lie on the perpendicular bisector of PQ.

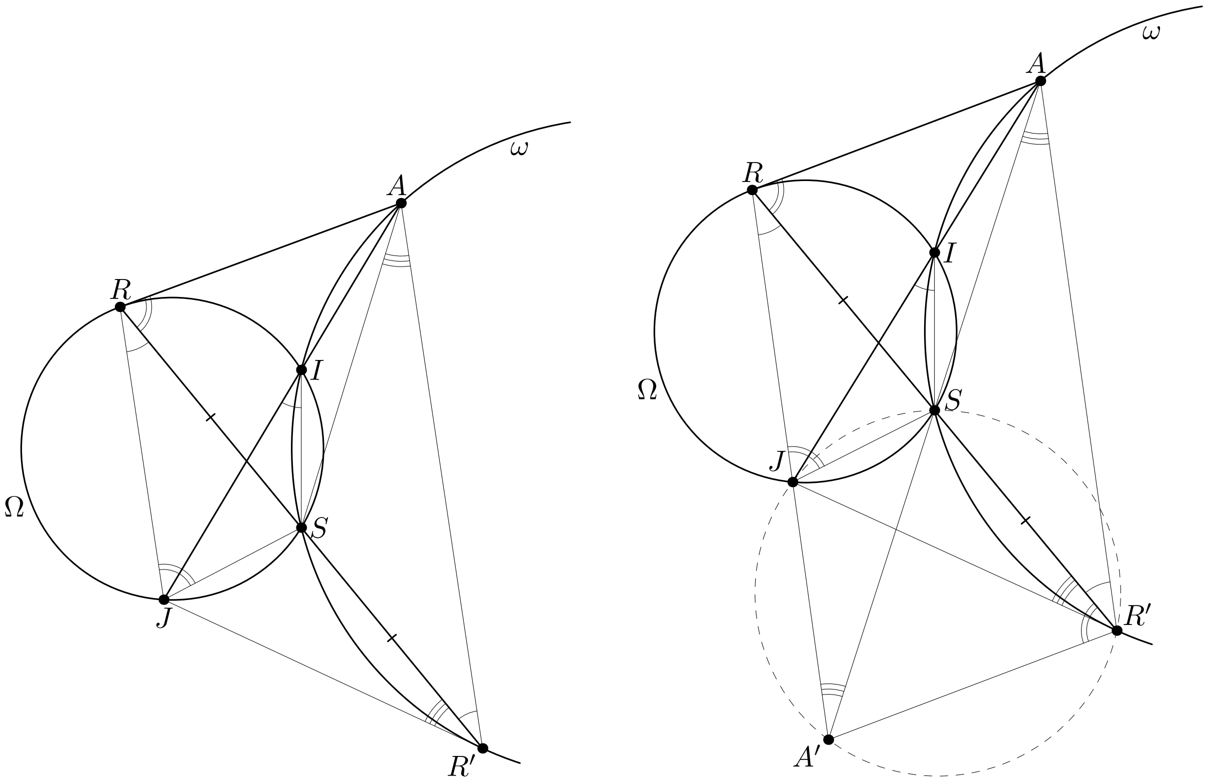
G2. Let R and S be distinct points on circle Ω, and let t denote the tangent line to Ω at R. Point R' is the reflection of R with respect to S. A point I is chosen on the smaller arc RS of Ω so that the circumcircle Γ of triangle ISR' intersects t at two different points. Denote by A the common point of Γ and t that is closest to R. Line AI meets Ω again at J. Show that JR' is tangent to Γ.

(Luxembourg)

Solution 1. In the circles Ω and Γ we have $\angle JRS = \angle JIS = \angle AR'S$. On the other hand, since RA is tangent to Ω, we get $\angle SJR = \angle SRA$. So the triangles ARR' and SJR are similar, and

$$\frac{R'R}{RJ} = \frac{AR'}{SR} = \frac{AR'}{SR'}.$$

The last relation, together with $\angle AR'S = \angle JRR'$, yields $\triangle ASR' \sim \triangle R'JR$, hence $\angle SAR' = \angle RR'J$. It follows that JR' is tangent to Γ at R'.



Solution 1 Solution 2

Solution 2. As in Solution 1, we notice that $\angle JRS = \angle JIS = \angle AR'S$, so we have $RJ \parallel AR'$.

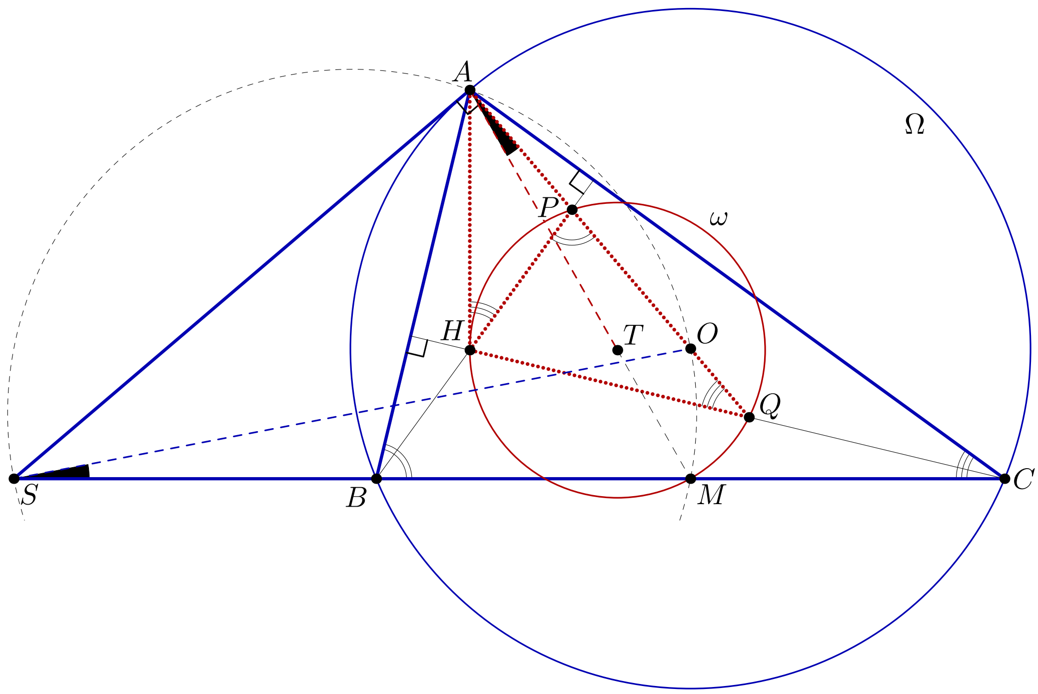
Let A' be the reflection of A about S; then $ARA'R'$ is a parallelogram with center S, and hence the point J lies on the line RA'.

From $\angle SR'A' = \angle SRA = \angle SJR$ we get that the points S, J, A', R' are concyclic. This proves that $\angle SR'J = \angle SA'J = \angle SA'R = \angle SAR'$, so JR' is tangent to Γ at R'.

G3. Let O be the circumcenter of an acute scalene triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

(Ukraine)

Solution. Suppose, without loss of generality, that $AB < AC$. We have $\angle PQH = 90^\circ - \angle QAB = 90^\circ - \angle OAB = \frac{1}{2}\angle AOB = \angle ACB$, and similarly $\angle QPH = \angle ABC$. Thus triangles ABC and HPQ are similar. Let Ω and ω be the circumcircles of ABC and HPQ, respectively. Since $\angle AHP = 90^\circ - \angle HAC = \angle ACB = \angle HQP$, line AH is tangent to ω.



Let T be the center of ω and let lines AT and BC meet at M. We will take advantage of the similarity between ABC and HPQ and the fact that AH is tangent to ω at H, with A on line PQ. Consider the corresponding tangent AS to Ω, with $S \in BC$. Then S and A correspond to each other in $\triangle ABC \sim \triangle HPQ$, and therefore $\angle OSM = \angle OAT = \angle OAM$. Hence quadrilateral $SAOM$ is cyclic, and since the tangent line AS is perpendicular to AO, $\angle OMS = 180^\circ - \angle OAS = 90^\circ$. This means that M is the orthogonal projection of O onto BC, which is its midpoint. So T lies on median AM of triangle ABC.

G4. In triangle ABC, let ω be the excircle opposite A. Let D, E, and F be the points where ω is tangent to lines BC, CA, and AB, respectively. The circle AEF intersects line BC at P and Q. Let M be the midpoint of AD. Prove that the circle MPQ is tangent to ω.

(Denmark)

Solution 1. Denote by Ω the circle $AEFPQ$, and denote by γ the circle PQM. Let the line AD meet ω again at $T \neq D$. We will show that γ is tangent to ω at T.

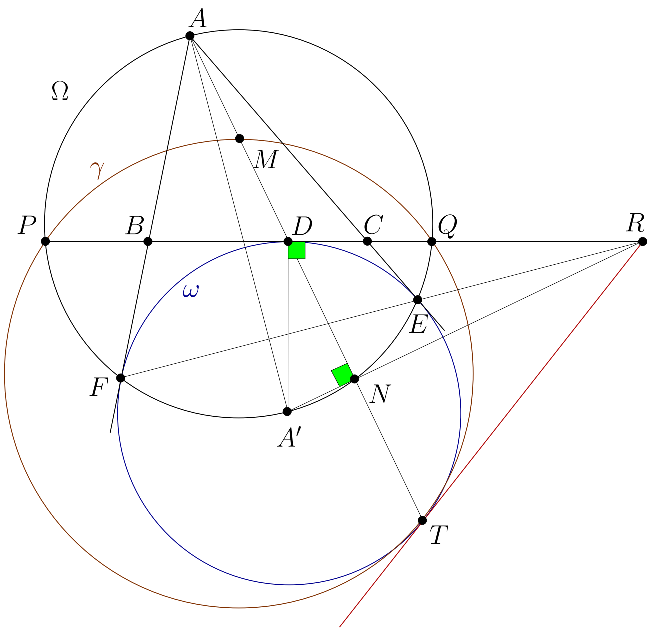
We first prove that points P, Q, M, T are concyclic. Let A' be the center of ω. Since $A'E \perp AE$ and $A'F \perp AF$, AA' is a diameter in Ω. Let N be the midpoint of DT; from $A'D = A'T$ we can see that $\angle A'NA = 90^\circ$ and therefore N also lies on the circle Ω. Now, from the power of D with respect to the circles γ and Ω we get

$$DP \cdot DQ = DA \cdot DN = 2DM \cdot \frac{DT}{2} = DM \cdot DT,$$

so P, Q, M, T are concyclic.

If $EF \parallel BC$, then ABC is isosceles and the problem is now immediate by symmetry. Otherwise, let the tangent line to ω at T meet line BC at point R. The tangent line segments RD and RT have the same length, so $A'R$ is the perpendicular bisector of DT; since $ND = NT$, N lies on this perpendicular bisector.

In right triangle $A'RD$, $RD^2 = RN \cdot RA' = RP \cdot RQ$, in which the last equality was obtained from the power of R with respect to Ω. Hence $RT^2 = RP \cdot RQ$, which implies that RT is also tangent to γ. Because RT is a common tangent to ω and γ, these two circles are tangent at T.



Solution 2. After proving that P, Q, M, T are concyclic, we finish the problem in a different fashion. We only consider the case in which EF and BC are not parallel. Let lines PQ and EF meet at point R. Since PQ and EF are radical axes of Ω, γ and ω, γ, respectively, R is the radical center of these three circles.

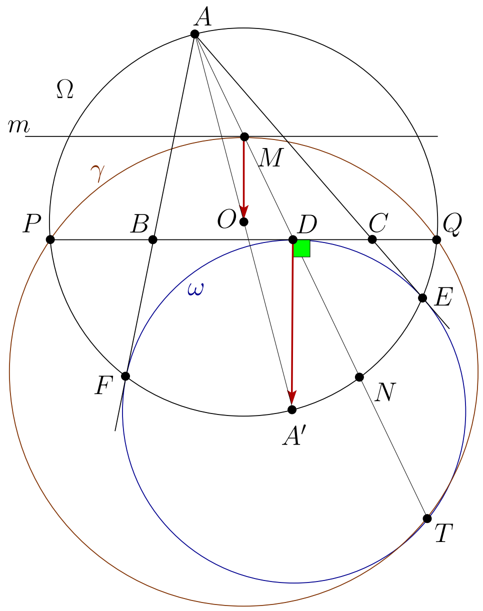
With respect to the circle ω, the line DR is the polar of D, and the line EF is the polar of A. So the pole of line ADT is $DR \cap EF = R$, and therefore RT is tangent to ω.

Finally, since T belongs to γ and ω and R is the radical center of γ, ω and Ω, line RT is the radical axis of γ and ω, and since it is tangent to ω, it is also tangent to γ. Because RT is a common tangent to ω and γ, these two circles are tangent at T.

Comment. In Solution 2 we defined the point R from Solution 1 in a different way.

Solution 3. We give an alternative proof that the circles are tangent at the common point T. Again, we start from the fact that P, Q, M, T are concyclic. Let point O be the midpoint of diameter AA'. Then MO is the midline of triangle ADA', so $MO \parallel A'D$. Since $A'D \perp PQ$, MO is perpendicular to PQ as well.

Looking at circle Ω, which has center O, $MO \perp PQ$ implies that MO is the perpendicular bisector of the chord PQ. Thus M is the midpoint of arc $\widehat{PQ}$ from γ, and the tangent line m to γ at M is parallel to PQ.



Consider the homothety with center T and ratio $\frac{TD}{TM}$. It takes D to M, and the line PQ to the line m. Since the circle that is tangent to a line at a given point and that goes through another given point is unique, this homothety also takes ω (tangent to PQ and going through T) to γ (tangent to m and going through T). We conclude that ω and γ are tangent at T.

G5. Let $ABCC_1B_1A_1$ be a convex hexagon such that $AB = BC$, and suppose that the line segments AA_1, BB_1, and CC_1 have the same perpendicular bisector. Let the diagonals AC_1 and A_1C meet at D, and denote by ω the circle ABC. Let ω intersect the circle A_1BC_1 again at $E \neq B$. Prove that the lines BB_1 and DE intersect on ω.

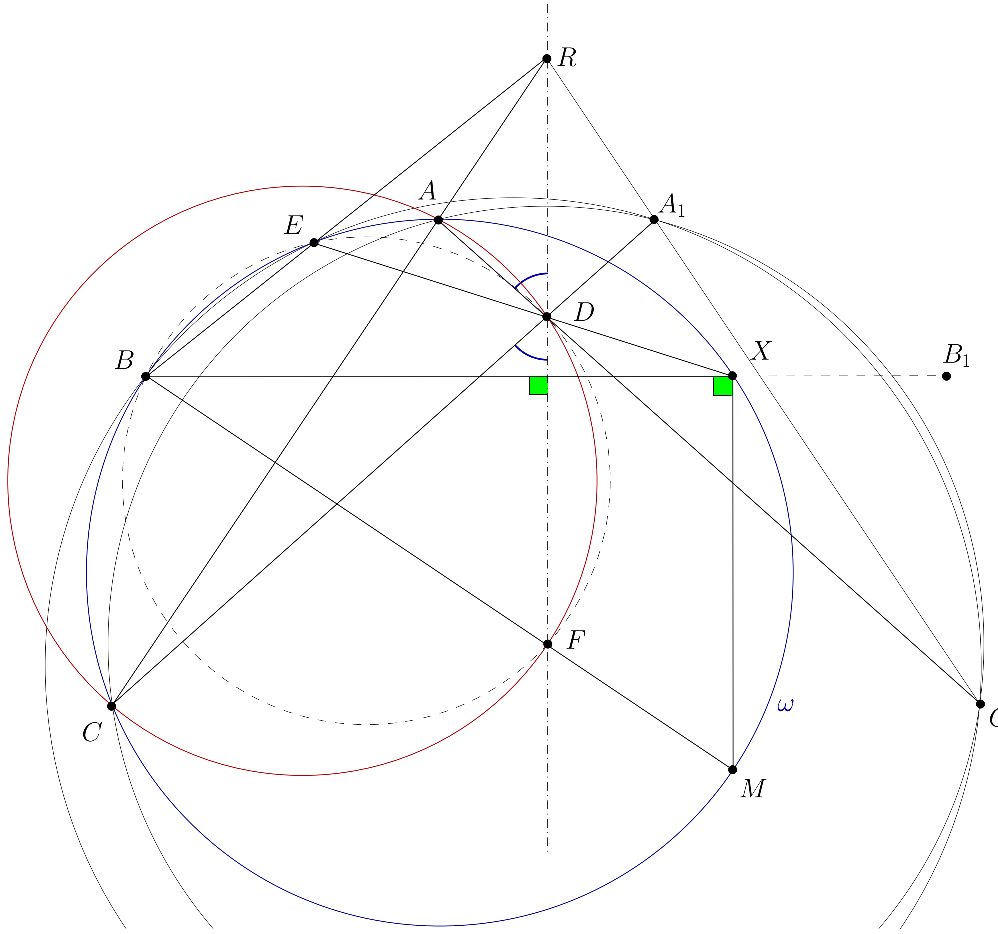
(Ukraine)

Solution 1. If $AA_1 = CC_1$, then the hexagon is symmetric about the line BB_1; in particular the circles ABC and A_1BC_1 are tangent to each other. So AA_1 and CC_1 must be different. Since the points A and A_1 can be interchanged with C and C_1, respectively, we may assume $AA_1 < CC_1$.

Let R be the radical center of the circles $AEBC$ and A_1EBC_1, and the circumcircle of the symmetric trapezoid ACC_1A_1; that is the common point of the pairwise radical axes AC, A_1C_1, and BE. By the symmetry of AC and A_1C_1, the point R lies on the common perpendicular bisector of AA_1 and CC_1, which is the external bisector of $\angle ADC$.

Let F be the second intersection of the line DR and the circle ACD. From the power of R with respect to the circles ω and $ACFD$ we have $RB \cdot RE = RA \cdot RC = RD \cdot DF$, so the points B, E, D and F are concyclic.

The line RDF is the external bisector of $\angle ADC$, so the point F bisects the arc $\widehat{CDA}$. By $AB = BC$, on circle ω, the point B is the midpoint of arc $\widehat{AEC}$; let M be the point diametrically opposite to B, that is the midpoint of the opposite arc $\widehat{CA}$ of ω. Notice that the points B, F and M lie on the perpendicular bisector of AC, so they are collinear.



Finally, let X be the second intersection point of ω and the line DE. Since BM is a diameter in ω, we have $\angle BXM = 90^\circ$. Moreover,

$$\angle EXM = 180^\circ - \angle MBE = 180^\circ - \angle FBE = \angle EDF,$$

so MX and FD are parallel. Since BX is perpendicular to MX and BB_1 is perpendicular to FD, this shows that X lies on line BB_1.

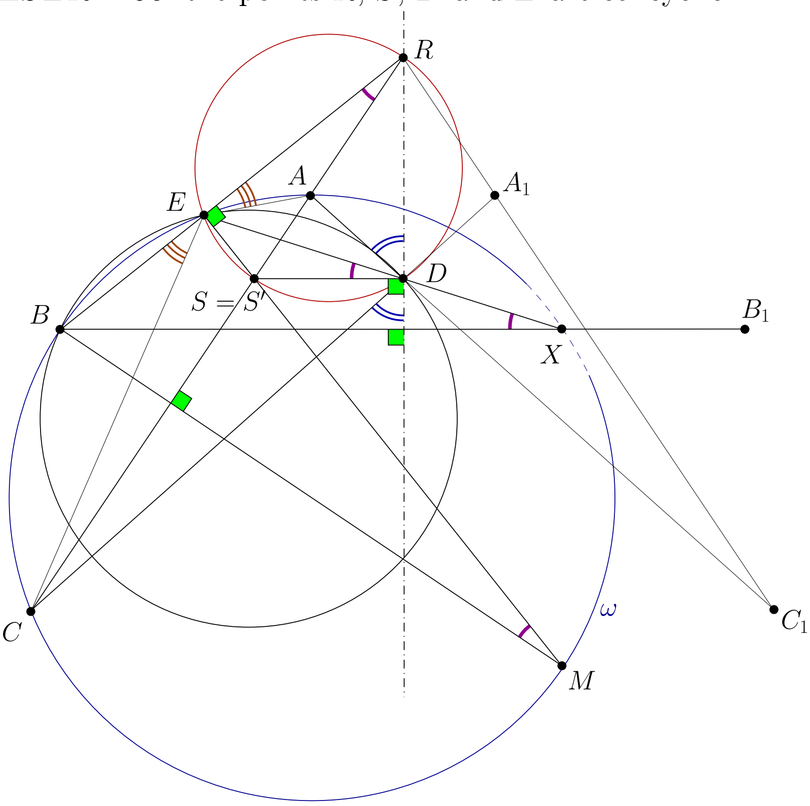
Solution 2. Define point M as the point opposite to B on circle ω, and point R as the intersection of lines AC, A_1C_1 and BE, and show that R lies on the external bisector of $\angle ADC$, like in the first solution.

Since B is the midpoint of the arc $\widehat{AEC}$, the line BER is the external bisector of $\angle CEA$. Now we show that the internal angle bisectors of $\angle ADC$ and $\angle CEA$ meet on the segment AC. Let the angle bisector of $\angle ADC$ meet AC at S, and let the angle bisector of $\angle CEA$, which is line EM, meet AC at S'. By applying the angle bisector theorem to both internal and external bisectors of $\angle ADC$ and $\angle CEA$,

$$AS : CS = AD : CD = AR : CR = AE : CE = AS' : CS',$$

so indeed $S = S'$.

By $\angle RDS = \angle SER = 90^\circ$ the points R, S, D and E are concyclic.



Now let the lines BB_1 and DE meet at point X. Notice that $\angle EXB = \angle EDS$ because both BB_1 and DS are perpendicular to the line DR, we have that $\angle EDS = \angle ERS$ in circle $SRDE$, and $\angle ERS = \angle EMB$ because $SR \perp BM$ and $ER \perp ME$. Therefore, $\angle EXB = \angle EMB$, so indeed, the point X lies on ω.

G6. Let $n \geqslant 3$ be an integer. Two regular n-gons $\mathcal{A}$ and $\mathcal{B}$ are given in the plane. Prove that the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

(That is, prove that there exists a line separating those vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary from the other vertices of $\mathcal{A}$.)

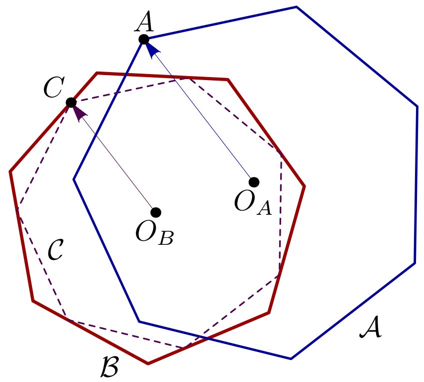
(Czech Republic)

Solution 1. In both solutions, by a *polygon* we always mean its interior together with its boundary.

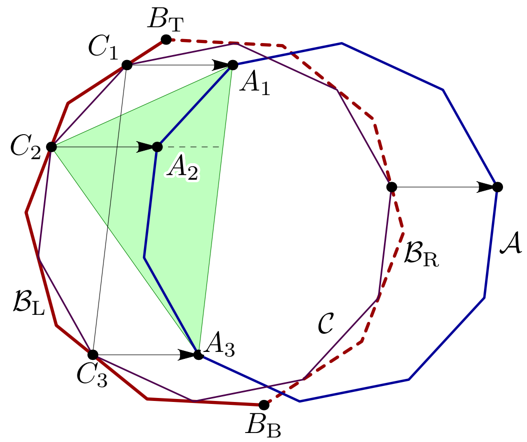
We start with finding a regular n-gon $\mathcal{C}$ which (i) is inscribed into $\mathcal{B}$ (that is, all vertices of $\mathcal{C}$ lie on the perimeter of $\mathcal{B}$); and (ii) is either a translation of $\mathcal{A}$, or a homothetic image of $\mathcal{A}$ with a positive factor.

Such a polygon may be constructed as follows. Let O_A and O_B be the centers of $\mathcal{A}$ and $\mathcal{B}$, respectively, and let A be an arbitrary vertex of $\mathcal{A}$. Let $\overrightarrow{O_B C}$ be the vector co-directional to $\overrightarrow{O_A A}$, with C lying on the perimeter of $\mathcal{B}$. The rotations of C around O_B by multiples of $2\pi/n$ form the required polygon. Indeed, it is regular, inscribed into $\mathcal{B}$ (due to the rotational symmetry of $\mathcal{B}$), and finally the translation/homothety mapping $\overrightarrow{O_A A}$ to $\overrightarrow{O_B C}$ maps $\mathcal{A}$ to $\mathcal{C}$.

Now we separate two cases.



Construction of $\mathcal{C}$



Case 1: Translation

Case 1: $\mathcal{C}$ is a translation of $\mathcal{A}$ by a vector $\vec{v}$.

Denote by t the translation transform by vector $\vec{v}$. We need to prove that the vertices of $\mathcal{C}$ which stay in $\mathcal{B}$ under t are consecutive. To visualize the argument, we refer the plane to Cartesian coordinates so that the x-axis is co-directional with $\vec{v}$. This way, the notions of right/left and top/bottom are also introduced, according to the x- and y-coordinates, respectively.

Let B_{T} and B_{B} be the top and the bottom vertices of $\mathcal{B}$ (if several vertices are extremal, we take the rightmost of them). They split the perimeter of $\mathcal{B}$ into the *right part* $\mathcal{B}_{\mathrm{R}}$ and the *left part* $\mathcal{B}_{\mathrm{L}}$ (the vertices B_{T} and B_{B} are assumed to lie in both parts); each part forms a connected subset of the perimeter of $\mathcal{B}$. So the vertices of $\mathcal{C}$ are also split into two parts $\mathcal{C}_{\mathrm{L}} \subset \mathcal{B}_{\mathrm{L}}$ and $\mathcal{C}_{\mathrm{R}} \subset \mathcal{B}_{\mathrm{R}}$, each of which consists of consecutive vertices.

Now, all the points in $\mathcal{B}_{\mathrm{R}}$ (and hence in $\mathcal{C}_{\mathrm{R}}$) move out from $\mathcal{B}$ under t, since they are the rightmost points of $\mathcal{B}$ on the corresponding horizontal lines. It remains to prove that the vertices of $\mathcal{C}_{\mathrm{L}}$ which stay in $\mathcal{B}$ under t are consecutive.

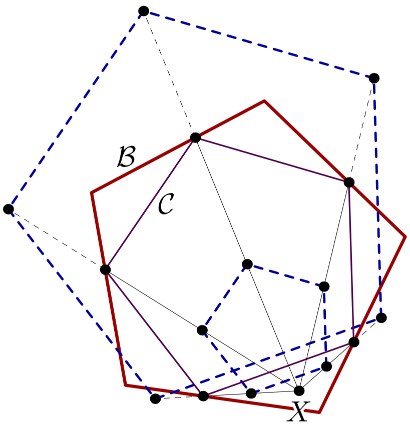
For this purpose, let C_1, C_2, and C_3 be three vertices in $\mathcal{C}_{\mathrm{L}}$ such that C_2 is between C_1 and C_3, and $t(C_1)$ and $t(C_3)$ lie in $\mathcal{B}$; we need to prove that $t(C_2) \in \mathcal{B}$ as well. Let $A_i = t(C_i)$. The line through C_2 parallel to $\vec{v}$ crosses the segment C_1C_3 to the right of C_2; this means that this line crosses A_1A_3 to the right of A_2, so A_2 lies inside the triangle $A_1C_2A_3$ which is contained in $\mathcal{B}$. This yields the desired result.

Case 2: $\mathcal{C}$ is a homothetic image of $\mathcal{A}$ centered at X with factor $k > 0$.

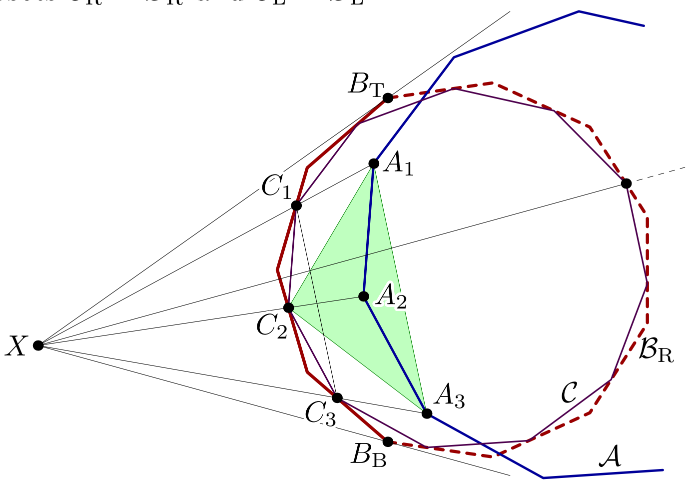
Denote by h the homothety mapping $\mathcal{C}$ to $\mathcal{A}$. We need now to prove that the vertices of $\mathcal{C}$ which stay in $\mathcal{B}$ after applying h are consecutive. If $X \in \mathcal{B}$, the claim is easy. Indeed, if $k < 1$, then the vertices of $\mathcal{A}$ lie on the segments of the form XC (C being a vertex of $\mathcal{C}$) which lie in $\mathcal{B}$. If $k > 1$, then the vertices of $\mathcal{A}$ lie on the extensions of such segments XC beyond C, and almost all these extensions lie outside $\mathcal{B}$. The exceptions may occur only in case when X lies on the boundary of $\mathcal{B}$, and they may cause one or two vertices of $\mathcal{A}$ stay on the boundary of $\mathcal{B}$. But even in this case those vertices are still consecutive.

So, from now on we assume that $X \notin \mathcal{B}$.

Now, there are two vertices B_{T} and $\mathcal{B}_{\mathrm{B}}$ of $\mathcal{B}$ such that $\mathcal{B}$ is contained in the angle $\angle B_{\mathrm{T}}XB_{\mathrm{B}}$; if there are several options, say, for B_{T}, then we choose the farthest one from X if $k > 1$, and the nearest one if $k < 1$. For the visualization purposes, we refer the plane to Cartesian coordinates so that the y-axis is co-directional with $\overrightarrow{B_{\mathrm{B}}B_{\mathrm{T}}}$, and X lies to the left of the line $B_{\mathrm{T}}B_{\mathrm{B}}$. Again, the perimeter of $\mathcal{B}$ is split by B_{T} and B_{B} into the *right part* $\mathcal{B}_{\mathrm{R}}$ and the *left part* $\mathcal{B}_{\mathrm{L}}$, and the set of vertices of $\mathcal{C}$ is split into two subsets $\mathcal{C}_{\mathrm{R}} \subset \mathcal{B}_{\mathrm{R}}$ and $\mathcal{C}_{\mathrm{L}} \subset \mathcal{B}_{\mathrm{L}}$.



Case 2, X inside $\mathcal{B}$

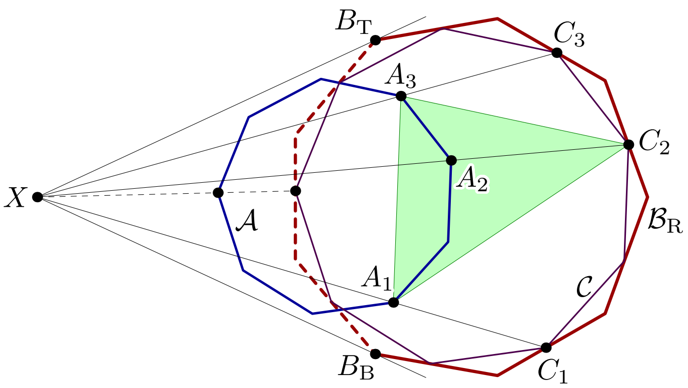


Subcase 2.1: $k > 1$

Subcase 2.1: $k > 1$.

In this subcase, all points from $\mathcal{B}_{\mathrm{R}}$ (and hence from $\mathcal{C}_{\mathrm{R}}$) move out from $\mathcal{B}$ under h, because they are the farthest points of $\mathcal{B}$ on the corresponding rays emanated from X. It remains to prove that the vertices of $\mathcal{C}_{\mathrm{L}}$ which stay in $\mathcal{B}$ under h are consecutive.

Again, let C_1, C_2, C_3 be three vertices in $\mathcal{C}_{\mathrm{L}}$ such that C_2 is between C_1 and C_3, and $h(C_1)$ and $h(C_3)$ lie in $\mathcal{B}$. Let $A_i = h(C_i)$. Then the ray XC_2 crosses the segment C_1C_3 beyond C_2, so this ray crosses A_1A_3 beyond A_2; this implies that A_2 lies in the triangle $A_1C_2A_3$, which is contained in $\mathcal{B}$.



Subcase 2.2: $k < 1$

Subcase 2.2: $k < 1$.

This case is completely similar to the previous one. All points from $\mathcal{B}_{\mathrm{L}}$ (and hence from $\mathcal{C}_{\mathrm{L}}$ move out from $\mathcal{B}$ under h, because they are the nearest points of $\mathcal{B}$ on the corresponding rays emanated from X. Assume that C_1, C_2, and C_3 are three vertices in $\mathcal{C}_{\mathrm{R}}$ such that C_2 lies between C_1 and C_3, and $h(C_1)$ and $h(C_3)$ lie in $\mathcal{B}$; let $A_i = h(C_i)$. Then A_2 lies on the segment XC_2, and the segments XA_2 and A_1A_3 cross each other. Thus A_2 lies in the triangle $A_1C_2A_3$, which is contained in $\mathcal{B}$.

Comment 1. In fact, Case 1 can be reduced to Case 2 via the following argument.

Assume that $\mathcal{A}$ and $\mathcal{C}$ are congruent. Apply to $\mathcal{A}$ a homothety centered at O_B with a factor slightly smaller than 1 to obtain a polygon $\mathcal{A}'$. With appropriately chosen factor, the vertices of $\mathcal{A}$ which were outside/inside $\mathcal{B}$ stay outside/inside it, so it suffices to prove our claim for $\mathcal{A}'$ instead of $\mathcal{A}$. And now, the polygon $\mathcal{A}'$ is a homothetic image of $\mathcal{C}$, so the arguments from Case 2 apply.

Comment 2. After the polygon $\mathcal{C}$ has been found, the rest of the solution uses only the convexity of the polygons, instead of regularity. Thus, it proves a more general statement:

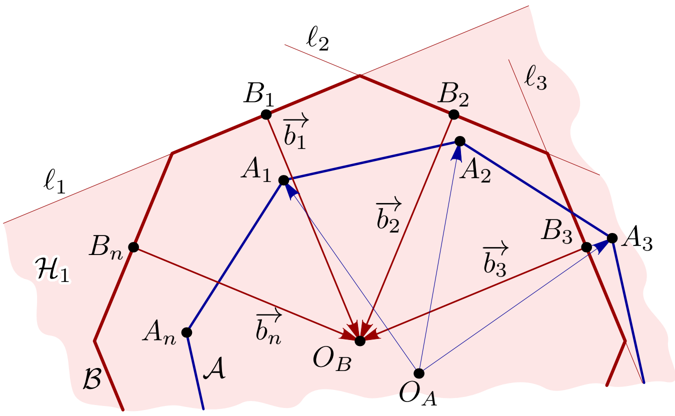
Assume that $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are three convex polygons in the plane such that $\mathcal{C}$ is inscribed into $\mathcal{B}$, and $\mathcal{A}$ can be obtained from it via either translation or positive homothety. Then the vertices of $\mathcal{A}$ that lie inside $\mathcal{B}$ or on its boundary are consecutive.

Solution 2. Let O_A and O_B be the centers of $\mathcal{A}$ and $\mathcal{B}$, respectively. Denote $[n] = \{1, 2, \dots, n\}$.

We start with introducing appropriate enumerations and notations. Enumerate the *sidelines* of $\mathcal{B}$ clockwise as $\ell_1, \ell_2, \dots, \ell_n$. Denote by $\mathcal{H}_i$ the half-plane of ℓ_i that contains $\mathcal{B}$ ($\mathcal{H}_i$ is assumed to contain ℓ_i); by B_i the *midpoint* of the side belonging to ℓ_i; and finally denote $\overrightarrow{b_i} = \overrightarrow{B_iO_B}$. (As usual, the numbering is cyclic modulo n, so $\ell_{n+i} = \ell_i$ etc.)

Now, choose a vertex A_1 of $\mathcal{A}$ such that the vector $\overrightarrow{O_AA_1}$ points "mostly outside $\mathcal{H}_1$"; strictly speaking, this means that the scalar product $\langle \overrightarrow{O_AA_1}, \overrightarrow{b_1} \rangle$ is minimal. Starting from A_1, enumerate the vertices of $\mathcal{A}$ clockwise as $A_1, A_2, \dots, A_n$; by the rotational symmetry, the choice of A_1 yields that the vector $\overrightarrow{O_AA_i}$ points "mostly outside $\mathcal{H}_i$", i.e.,

$$\langle \overrightarrow{O_AA_i}, \overrightarrow{b_i} \rangle = \min_{j\in[n]} \langle \overrightarrow{O_AA_j}, \overrightarrow{b_i} \rangle. \tag{1}$$



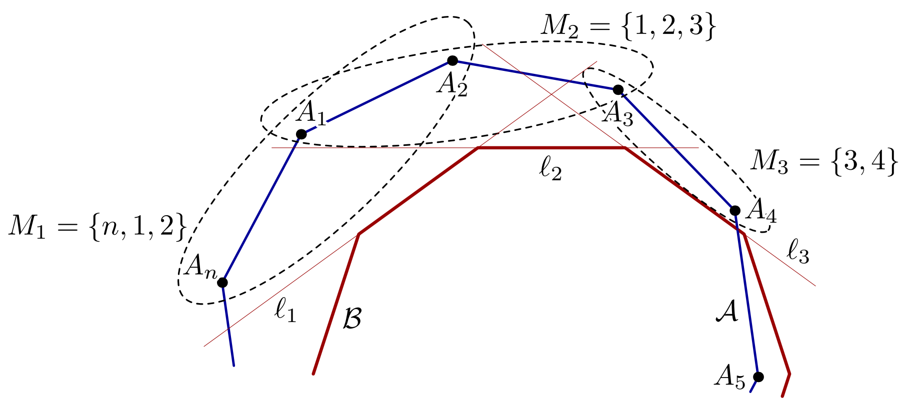
Enumerations and notations

We intend to reformulate the problem in more combinatorial terms, for which purpose we introduce the following notion. Say that a subset $I \subseteq [n]$ is *connected* if the elements of this set are consecutive in the cyclic order (in other words, if we join each i with $i+1 \mod n$ by an edge, this subset is connected in the usual graph sense). Clearly, the union of two connected subsets sharing at least one element is connected too. Next, for any half-plane $\mathcal{H}$ the indices of vertices of, say, $\mathcal{A}$ that lie in $\mathcal{H}$ form a connected set.

To access the problem, we denote

$$M = \{j \in [n] \colon A_j \notin \mathcal{B}\}, \qquad M_i = \{j \in [n] \colon A_j \notin \mathcal{H}_i\} \quad \text{for } i \in [n].$$

We need to prove that $[n] \setminus M$ is connected, which is equivalent to M being connected. On the other hand, since $\mathcal{B} = \bigcap_{i\in[n]} \mathcal{H}_i$, we have $M = \bigcup_{i\in[n]} M_i$, where the sets M_i are easier to investigate. We will utilize the following properties of these sets; the first one holds by the definition of M_i, along with the above remark.



The sets M_i

Property 1: Each set M_i is connected. □

Property 2: If M_i is nonempty, then $i \in M_i$.

Proof. Indeed, we have

$$j \in M_i \iff A_j \notin \mathcal{H}_i \iff \langle \overrightarrow{B_i A_j}, \overrightarrow{b_i} \rangle < 0 \iff \langle \overrightarrow{O_A A_j}, \overrightarrow{b_i} \rangle < \langle \overrightarrow{O_A B_i}, \overrightarrow{b_i} \rangle. \tag{2}$$

The right-hand part of the last inequality does not depend on j. Therefore, if some j lies in M_i, then by (1) so does i. □

In view of Property 2, it is useful to define the set

$$M' = \{i \in [n] : i \in M_i\} = \{i \in [n] : M_i \neq \varnothing\}.$$

Property 3: The set M' is connected.

Proof. To prove this property, we proceed on with the investigation started in (2) to write

$$i \in M' \iff A_i \in M_i \iff \langle \overrightarrow{B_i A_i}, \overrightarrow{b_i} \rangle < 0 \iff \langle \overrightarrow{O_B O_A}, \overrightarrow{b_i} \rangle < \langle \overrightarrow{O_B B_i}, \overrightarrow{b_i} \rangle + \langle \overrightarrow{A_i O_A}, \overrightarrow{b_i} \rangle.$$

The right-hand part of the obtained inequality does not depend on i, due to the rotational symmetry; denote its constant value by μ. Thus, $i \in M'$ if and only if $\langle \overrightarrow{O_B O_A}, \overrightarrow{b_i} \rangle < \mu$. This condition is in turn equivalent to the fact that B_i lies in a certain (open) half-plane whose boundary line is orthogonal to $O_B O_A$; thus, it defines a connected set. □

Now we can finish the solution. Since $M' \subseteq M$, we have

$$M = \bigcup_{i \in [n]} M_i = M' \cup \bigcup_{i \in [n]} M_i,$$

so M can be obtained from M' by adding all the sets M_i one by one. All these sets are connected, and each nonempty M_i contains an element of M' (namely, i). Thus their union is also connected.

Comment 3. Here we present a way in which one can come up with a solution like the one above.

Assume, for sake of simplicity, that O_A lies inside $\mathcal{B}$. Let us first put onto the plane a very small regular n-gon $\mathcal{A}'$ centered at O_A and aligned with $\mathcal{A}$; all its vertices lie inside $\mathcal{B}$. Now we start blowing it up, looking at the order in which the vertices leave $\mathcal{B}$. To go out of $\mathcal{B}$, a vertex should cross a certain side of $\mathcal{B}$ (which is hard to describe), or, equivalently, to cross at least one *sideline* of $\mathcal{B}$ — and this event is easier to describe. Indeed, the first vertex of $\mathcal{A}'$ to cross ℓ_i is the vertex A_i' (corresponding to A_i in $\mathcal{A}$); more generally, the vertices A_j' cross ℓ_i in such an order that the scalar product $\langle \overrightarrow{O_A A_j}, \overrightarrow{b_i} \rangle$ does not increase. For different indices i, these orders are just cyclic shifts of each other; and this provides some intuition for the notions and claims from Solution 2.

G7. A convex quadrilateral $ABCD$ has an inscribed circle with center I. Let I_a, I_b, I_c, and I_d be the incenters of the triangles DAB, ABC, BCD, and CDA, respectively. Suppose that the common external tangents of the circles AI_bI_d and CI_bI_d meet at X, and the common external tangents of the circles BI_aI_c and DI_aI_c meet at Y. Prove that $\angle XIY = 90^\circ$.

(Kazakhstan)

Solution. Denote by ω_a, ω_b, ω_c and ω_d the circles AI_bI_d, BI_aI_c, CI_bI_d, and DI_aI_c, let their centers be O_a, O_b, O_c and O_d, and let their radii be r_a, r_b, r_c and r_d, respectively.

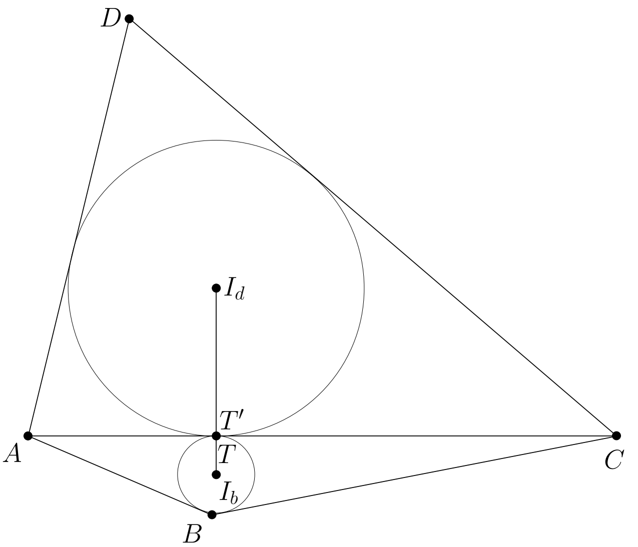
Claim 1. $I_bI_d \perp AC$ and $I_aI_c \perp BD$.

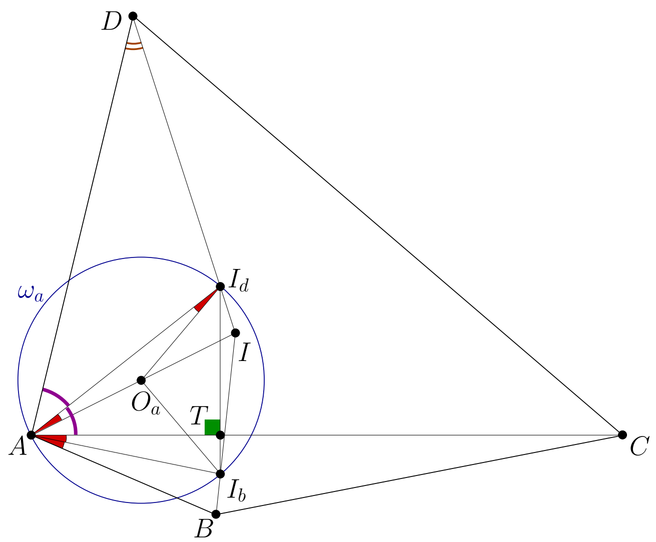
Proof. Let the incircles of triangles ABC and ACD be tangent to the line AC at T and T', respectively. (See the figure to the left.) We have $AT = \frac{AB+AC-BC}{2}$ in triangle ABC, $AT' = \frac{AD+AC-CD}{2}$ in triangle ACD, and $AB - BC = AD - CD$ in quadrilateral $ABCD$, so

$$AT = \frac{AC + AB - BC}{2} = \frac{AC + AD - CD}{2} = AT'.$$

This shows $T = T'$. As an immediate consequence, $I_bI_d \perp AC$.

The second statement can be shown analogously. $\square$





Claim 2. The points O_a, O_b, O_c and O_d lie on the lines AI, BI, CI and DI, respectively.

Proof. By symmetry it suffices to prove the claim for O_a. (See the figure to the right above.)

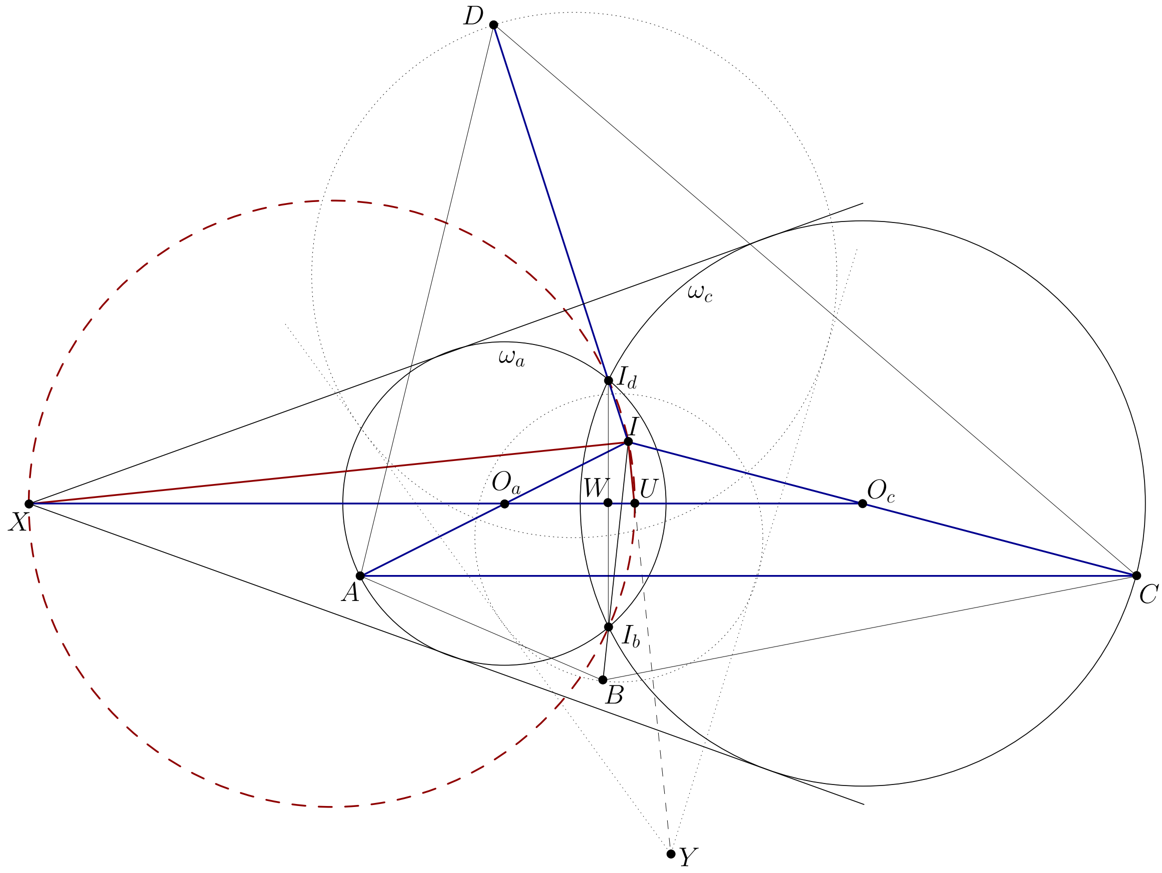
Notice first that the incircles of triangles ABC and ACD can be obtained from the incircle of the quadrilateral $ABCD$ with homothety centers B and D, respectively, and homothety factors less than 1, therefore the points I_b and I_d lie on the line segments BI and DI, respectively.

As is well-known, in every triangle the altitude and the diameter of the circumcircle starting from the same vertex are symmetric about the angle bisector. By Claim 1, in triangle AI_dI_b, the segment AT is the altitude starting from A. Since the foot T lies inside the segment I_bI_d, the circumcenter O_a of triangle AI_dI_b lies in the angle domain I_bAI_d in such a way that $\angle I_bAT = \angle O_aAI_d$. The points I_b and I_d are the incenters of triangles ABC and ACD, so the lines AI_b and AI_d bisect the angles $\angle BAC$ and $\angle CAD$, respectively. Then

$$\angle O_aAD = \angle O_aAI_d + \angle I_dAD = \angle I_bAT + \angle I_dAD = \tfrac{1}{2}\angle BAC + \tfrac{1}{2}\angle CAD = \tfrac{1}{2}\angle BAD,$$

so O_a lies on the angle bisector of $\angle BAD$, that is, on line AI. $\square$

The point X is the external similitude center of ω_a and ω_c; let U be their internal similitude center. The points O_a and O_c lie on the perpendicular bisector of the common chord I_bI_d of ω_a and ω_c, and the two similitude centers X and U lie on the same line; by Claim 2, that line is parallel to AC.



From the similarity of the circles ω_a and ω_c, from $O_aI_b = O_aI_d = O_aA = r_a$ and $O_cI_b = O_cI_d = O_cC = r_c$, and from $AC \parallel O_aO_c$ we can see that

$$\frac{O_aX}{O_cX} = \frac{O_aU}{O_cU} = \frac{r_a}{r_c} = \frac{O_aI_b}{O_cI_b} = \frac{O_aI_d}{O_cI_d} = \frac{O_aA}{O_cC} = \frac{O_aI}{O_cI}.$$

So the points X, U, I_b, I_d, I lie on the Apollonius circle of the points O_a, O_c with ratio $r_a : r_c$. In this Apollonius circle XU is a diameter, and the lines IU and IX are respectively the internal and external bisectors of $\angle O_aIO_c = \angle AIC$, according to the angle bisector theorem. Moreover, in the Apollonius circle the diameter UX is the perpendicular bisector of I_bI_d, so the lines IX and IU are the internal and external bisectors of $\angle I_bII_d = \angle BID$, respectively.

Repeating the same argument for the points B, D instead of A, C, we get that the line IY is the internal bisector of $\angle AIC$ and the external bisector of $\angle BID$. Therefore, the lines IX and IY respectively are the internal and external bisectors of $\angle BID$, so they are perpendicular.

Comment. In fact the points O_a, O_b, O_c and O_d lie on the *line segments* AI, BI, CI and DI, respectively. For the point O_a this can be shown for example by $\angle I_dO_aA + \angle AO_aI_b = (180^\circ - 2\angle O_aAI_d) + (180^\circ - 2\angle I_bAO_a) = 360^\circ - \angle BAD = \angle ADI + \angle DIA + \angle AIB + \angle IBA > \angle I_dIA + \angle AII_b$.

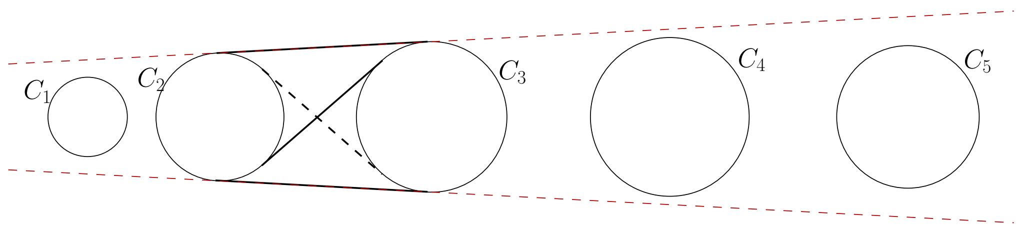
The solution also shows that the line IY passes through the point U, and analogously, IX passes through the internal similitude center of ω_b and ω_d.

G8. There are 2017 mutually external circles drawn on a blackboard, such that no two are tangent and no three share a common tangent. A *tangent segment* is a line segment that is a common tangent to two circles, starting at one tangent point and ending at the other one. Luciano is drawing tangent segments on the blackboard, one at a time, so that no tangent segment intersects any other circles or previously drawn tangent segments. Luciano keeps drawing tangent segments until no more can be drawn. Find all possible numbers of tangent segments when he stops drawing.

(Australia)

Answer: If there were n circles, there would always be exactly $3(n-1)$ segments; so the only possible answer is $3 \cdot 2017 - 3 = 6048$.

Solution 1. First, consider a particular arrangement of circles $C_1, C_2, \ldots, C_n$ where all the centers are aligned and each C_i is eclipsed from the other circles by its neighbors – for example, taking C_i with center $(i^2, 0)$ and radius $i/2$ works. Then the only tangent segments that can be drawn are between adjacent circles C_i and C_{i+1}, and exactly three segments can be drawn for each pair. So Luciano will draw exactly $3(n-1)$ segments in this case.



For the general case, start from a *final* configuration (that is, an arrangement of circles and segments in which no further segments can be drawn). The idea of the solution is to continuously resize and move the circles around the plane, one by one (in particular, making sure we never have 4 circles with a common tangent line), and show that the number of segments drawn remains constant as the picture changes. This way, we can reduce any circle/segment configuration to the particular one mentioned above, and the final number of segments must remain at $3n-3$.

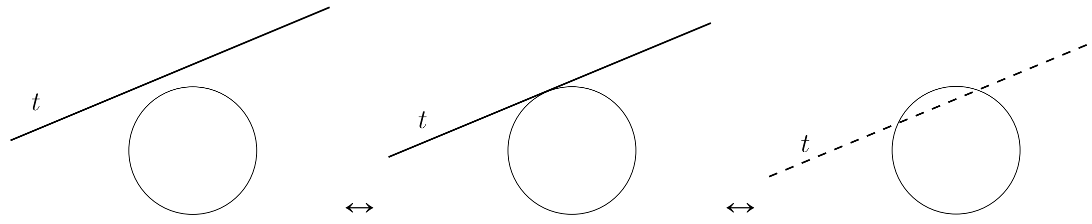
Some preliminary considerations: look at all possible tangent segments joining any two circles. A segment that is tangent to a circle A can do so in two possible orientations — it may come out of A in clockwise or counterclockwise orientation. Two segments touching the same circle with the same orientation will never intersect each other. Each pair (A, B) of circles has 4 choices of tangent segments, which can be identified by their orientations — for example, $(A+, B-)$ would be the segment which comes out of A in clockwise orientation and comes out of B in counterclockwise orientation. In total, we have $2n(n-1)$ possible segments, disregarding intersections.

Now we pick a circle C and start to continuously move and resize it, maintaining all existing tangent segments according to their identifications, including those involving C. We can keep our choice of tangent segments until the configuration reaches a *transition*. We lose nothing if we assume that C is kept at least ε units away from any other circle, where ε is a positive, fixed constant; therefore at a transition either: (1) a currently drawn tangent segment t suddenly becomes obstructed; or (2) a currently absent tangent segment t suddenly becomes unobstructed and available.

Claim. A transition can only occur when three circles C_1, C_2, C_3 are tangent to a common line ℓ containing t, in a way such that the three tangent segments lying on ℓ (joining the three circles pairwise) are not obstructed by any other circles or tangent segments (other than C_1, C_2, C_3).

Proof. Since (2) is effectively the reverse of (1), it suffices to prove the claim for (1). Suppose t has suddenly become obstructed, and let us consider two cases.

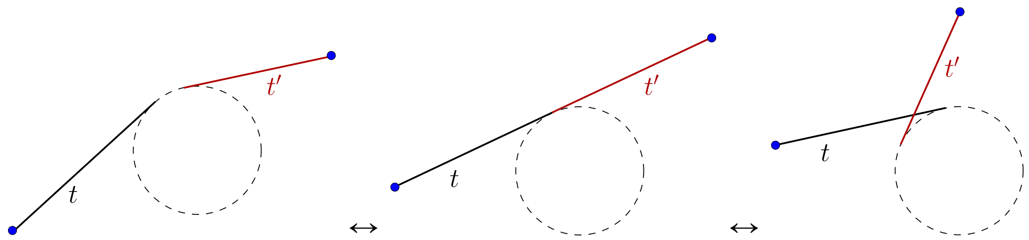
Case 1: t becomes obstructed by a circle



Then the new circle becomes the third circle tangent to ℓ, and no other circles or tangent segments are obstructing t.

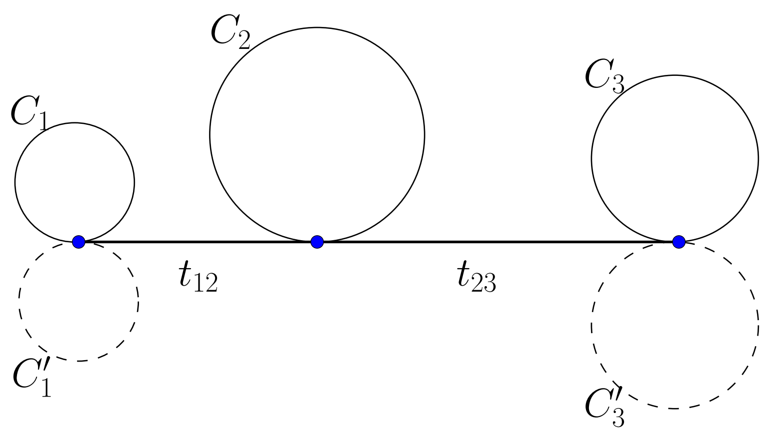
Case 2: t becomes obstructed by another tangent segment t'

When two segments t and t' first intersect each other, they must do so at a vertex of one of them. But if a vertex of t' first crossed an interior point of t, the circle associated to this vertex was already blocking t (absurd), or is about to (we already took care of this in case 1). So we only have to analyze the possibility of t and t' suddenly having a common vertex. However, if that happens, this vertex must belong to a single circle (remember we are keeping different circles at least ε units apart from each other throughout the moving/resizing process), and therefore they must have different orientations with respect to that circle.



Thus, at the transition moment, both t and t' are tangent to the same circle at a common point, that is, they must be on the same line ℓ and hence we again have three circles simultaneously tangent to ℓ. Also no other circles or tangent segments are obstructing t or t' (otherwise, they would have disappeared before this transition). $\square$

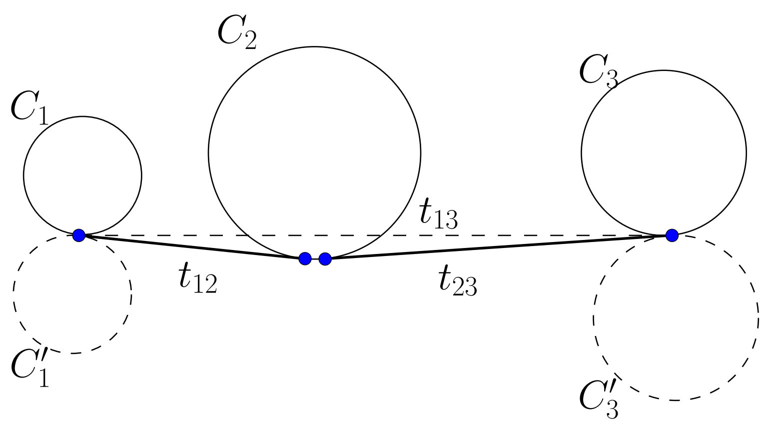
Next, we focus on the maximality of a configuration immediately before and after a transition, where three circles share a common tangent line ℓ. Let the three circles be C_1, C_2, C_3, ordered by their tangent points. The only possibly affected segments are the ones lying on ℓ, namely t_{12}, t_{23} and t_{13}. Since C_2 is in the middle, t_{12} and t_{23} must have different orientations with respect to C_2. For C_1, t_{12} and t_{13} must have the same orientation, while for C_3, t_{13} and t_{23} must have the same orientation. The figure below summarizes the situation, showing alternative positions for C_1 (namely, C_1 and C_1') and for C_3 (C_3 and C_3').



Now perturb the diagram slightly so the three circles no longer have a common tangent, while preserving the definition of t_{12}, t_{23} and t_{13} according to their identifications. First note that no other circles or tangent segments can obstruct any of these segments. Also recall that tangent segments joining the same circle at the same orientation will never obstruct each other.

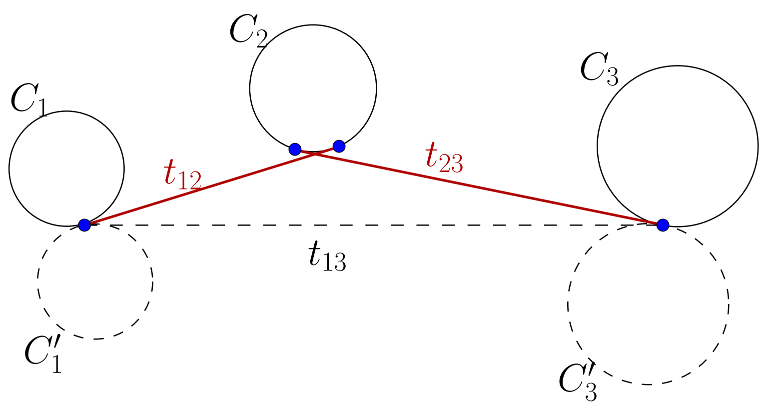
The availability of the tangent segments can now be checked using simple diagrams.

Case 1: t_{13} passes through C_2



In this case, t_{13} is not available, but both t_{12} and t_{23} are.

Case 2: t_{13} does not pass through C_2

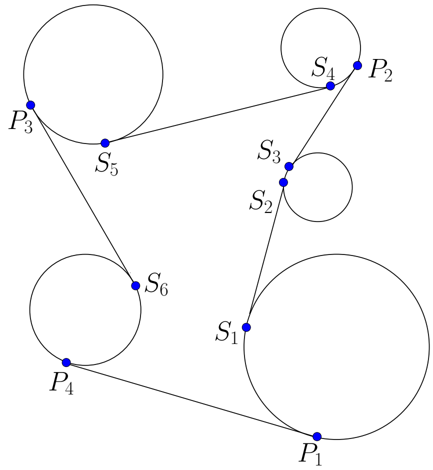


Now t_{13} is available, but t_{12} and t_{23} obstruct each other, so only one can be drawn.

In any case, exactly 2 out of these 3 segments can be drawn. Thus the maximal number of segments remains constant as we move or resize the circles, and we are done.

Solution 2. First note that all tangent segments lying on the boundary of the convex hull of the circles are always drawn since they do not intersect anything else. Now in the final picture, aside from the n circles, the blackboard is divided into regions. We can consider the picture as a plane (multi-)graph G in which the circles are the vertices and the tangent segments are the edges. The idea of this solution is to find a relation between the number of edges and the number of regions in G; then, once we prove that G is connected, we can use Euler's formula to finish the problem.

The boundary of each region consists of 1 or more (for now) simple closed curves, each made of arcs and tangent segments. The segment and the arc might meet smoothly (as in S_i, $i = 1, 2, \ldots, 6$ in the figure below) or not (as in P_1, P_2, P_3, P_4; call such points *sharp corners* of the boundary). In other words, if a person walks along the border, her direction would suddenly turn an angle of π at a sharp corner.



Claim 1. The outer boundary B_1 of any internal region has at least 3 sharp corners.

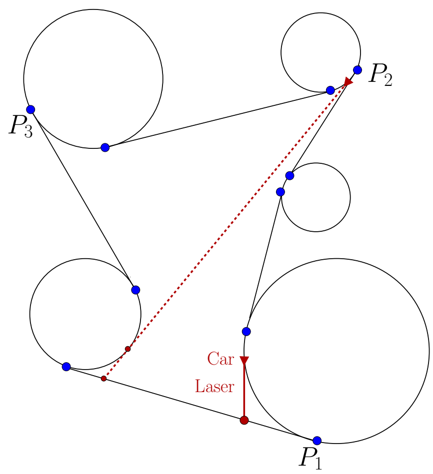
Proof. Let a person walk one lap along B_1 in the counterclockwise orientation. As she does so, she will turn *clockwise* as she moves along the circle arcs, and not turn at all when moving along the lines. On the other hand, her total rotation after one lap is 2π in the *counterclockwise* direction! Where could she be turning counterclockwise? She can only do so at sharp corners, and, even then, she turns only an angle of π there. But two sharp corners are not enough, since at least one arc must be present—so she must have gone through at least 3 sharp corners. □

Claim 2. Each internal region is simply connected, that is, has only one boundary curve.

Proof. Suppose, by contradiction, that some region has an outer boundary B_1 and inner boundaries $B_2, B_3, \ldots, B_m$ ($m \geqslant 2$). Let P_1 be one of the sharp corners of B_1.

Now consider a car starting at P_1 and traveling counterclockwise along B_1. It starts in reverse, i.e., it is initially facing the corner P_1. Due to the tangent conditions, the car may travel in a way so that its orientation only changes when it is moving along an arc. In particular, this means the car will sometimes travel forward. For example, if the car approaches a sharp corner when driving in reverse, it would continue travel forward after the corner, instead of making an immediate half-turn. This way, the orientation of the car only changes in a clockwise direction since the car always travels clockwise around each arc.

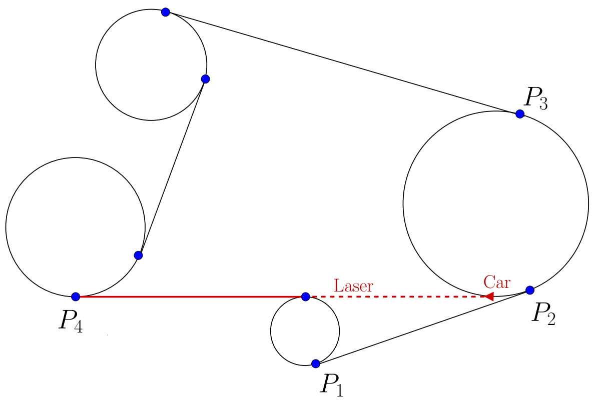
Now imagine there is a laser pointer at the front of the car, pointing directly ahead. Initially, the laser endpoint hits P_1, but, as soon as the car hits an arc, the endpoint moves clockwise around B_1. In fact, the laser endpoint must move continuously along B_1! Indeed, if the endpoint ever jumped (within B_1, or from B_1 to one of the inner boundaries), at the moment of the jump the interrupted laser would be a drawable tangent segment that Luciano missed (see figure below for an example).



Now, let P_2 and P_3 be the next two sharp corners the car goes through, after P_1 (the previous lemma assures their existence). At P_2 the car starts moving forward, and at P_3 it will start to move in reverse again. So, at P_3, the laser endpoint is at P_3 itself. So while the car moved counterclockwise between P_1 and P_3, the laser endpoint moved clockwise between P_1 and P_3. That means the laser beam itself scanned the whole region within B_1, and it should have crossed some of the inner boundaries. $\square$

Claim 3. Each region has exactly 3 sharp corners.

Proof. Consider again the car of the previous claim, with its laser still firmly attached to its front, traveling the same way as before and going through the same *consecutive* sharp corners P_1, P_2 and P_3. As we have seen, as the car goes counterclockwise from P_1 to P_3, the laser endpoint goes clockwise from P_1 to P_3, so together they cover the whole boundary. If there were a fourth sharp corner P_4, at some moment the laser endpoint would pass through it. But, since P_4 is a sharp corner, this means the car must be on the extension of a tangent segment going through P_4. Since the car is not on that segment itself (the car never goes through P_4), we would have 3 circles with a common tangent line, which is not allowed.



$\square$

We are now ready to finish the solution. Let r be the number of internal regions, and s be the number of tangent segments. Since each tangent segment contributes exactly 2 sharp corners to the diagram, and each region has exactly 3 sharp corners, we must have $2s = 3r$. Since the graph corresponding to the diagram is connected, we can use Euler's formula $n - s + r = 1$ and find $s = 3n - 3$ and $r = 2n - 2$.

Number Theory

N1. The sequence $a_0, a_1, a_2, \ldots$ of positive integers satisfies

$$a_{n+1} = \begin{cases} \sqrt{a_n}, & \text{if } \sqrt{a_n} \text{ is an integer} \\ a_n + 3, & \text{otherwise} \end{cases} \qquad \text{for every } n \geqslant 0.$$

Determine all values of $a_0 > 1$ for which there is at least one number a such that $a_n = a$ for infinitely many values of n.

(South Africa)

Answer: All positive multiples of 3.

Solution. Since the value of a_{n+1} only depends on the value of a_n, if $a_n = a_m$ for two different indices n and m, then the sequence is eventually periodic. So we look for the values of a_0 for which the sequence is eventually periodic.

Claim 1. If $a_n \equiv -1 \pmod 3$, then, for all $m > n$, a_m is not a perfect square. It follows that the sequence is eventually strictly increasing, so it is not eventually periodic.

Proof. A square cannot be congruent to -1 modulo 3, so $a_n \equiv -1 \pmod 3$ implies that a_n is not a square, therefore $a_{n+1} = a_n + 3 > a_n$. As a consequence, $a_{n+1} \equiv a_n \equiv -1 \pmod 3$, so a_{n+1} is not a square either. By repeating the argument, we prove that, from a_n on, all terms of the sequence are not perfect squares and are greater than their predecessors, which completes the proof. □

Claim 2. If $a_n \not\equiv -1 \pmod 3$ and $a_n > 9$ then there is an index $m > n$ such that $a_m < a_n$.

Proof. Let t^2 be the largest perfect square which is less than a_n. Since $a_n > 9$, t is at least 3. The first square in the sequence $a_n, a_n + 3, a_n + 6, \ldots$ will be $(t+1)^2$, $(t+2)^2$ or $(t+3)^2$, therefore there is an index $m > n$ such that $a_m \leqslant t + 3 < t^2 < a_n$, as claimed. □

Claim 3. If $a_n \equiv 0 \pmod 3$, then there is an index $m > n$ such that $a_m = 3$.

Proof. First we notice that, by the definition of the sequence, a multiple of 3 is always followed by another multiple of 3. If $a_n \in \{3, 6, 9\}$ the sequence will eventually follow the periodic pattern $3, 6, 9, 3, 6, 9, \ldots$. If $a_n > 9$, let j be an index such that a_j is equal to the minimum value of the set $\{a_{n+1}, a_{n+2}, \ldots\}$. We must have $a_j \leqslant 9$, otherwise we could apply Claim 2 to a_j and get a contradiction on the minimality hypothesis. It follows that $a_j \in \{3, 6, 9\}$, and the proof is complete. □

Claim 4. If $a_n \equiv 1 \pmod 3$, then there is an index $m > n$ such that $a_m \equiv -1 \pmod 3$.

Proof. In the sequence, 4 is always followed by $2 \equiv -1 \pmod 3$, so the claim is true for $a_n = 4$. If $a_n = 7$, the next terms will be $10, 13, 16, 4, 2, \ldots$ and the claim is also true. For $a_n \geqslant 10$, we again take an index $j > n$ such that a_j is equal to the minimum value of the set $\{a_{n+1}, a_{n+2}, \ldots\}$, which by the definition of the sequence consists of non-multiples of 3. Suppose $a_j \equiv 1 \pmod 3$. Then we must have $a_j \leqslant 9$ by Claim 2 and the minimality of a_j. It follows that $a_j \in \{4, 7\}$, so $a_m = 2 < a_j$ for some $m > j$, contradicting the minimality of a_j. Therefore, we must have $a_j \equiv -1 \pmod 3$. □

It follows from the previous claims that if a_0 is a multiple of 3 the sequence will eventually reach the periodic pattern $3, 6, 9, 3, 6, 9, \ldots$; if $a_0 \equiv -1 \pmod 3$ the sequence will be strictly increasing; and if $a_0 \equiv 1 \pmod 3$ the sequence will be eventually strictly increasing.

So the sequence will be eventually periodic if, and only if, a_0 is a multiple of 3.

N2. Let $p \geqslant 2$ be a prime number. Eduardo and Fernando play the following game making moves alternately: in each move, the current player chooses an index i in the set $\{0, 1, \ldots, p-1\}$ that was not chosen before by either of the two players and then chooses an element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Eduardo has the first move. The game ends after all the indices $i \in \{0, 1, \ldots, p-1\}$ have been chosen. Then the following number is computed:

$$M = a_0 + 10 \cdot a_1 + \cdots + 10^{p-1} \cdot a_{p-1} = \sum_{j=0}^{p-1} a_j \cdot 10^j.$$

The goal of Eduardo is to make the number M divisible by p, and the goal of Fernando is to prevent this.

Prove that Eduardo has a winning strategy.

(Morocco)

Solution. We say that a player makes the move (i, a_i) if he chooses the index i and then the element a_i of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ in this move.

If $p = 2$ or $p = 5$ then Eduardo chooses $i = 0$ and $a_0 = 0$ in the first move, and wins, since, independently of the next moves, M will be a multiple of 10.

Now assume that the prime number p does not belong to $\{2, 5\}$. Eduardo chooses $i = p-1$ and $a_{p-1} = 0$ in the first move. By Fermat's Little Theorem, $(10^{(p-1)/2})^2 = 10^{p-1} \equiv 1 \pmod{p}$, so $p \mid (10^{(p-1)/2})^2 - 1 = (10^{(p-1)/2} + 1)(10^{(p-1)/2} - 1)$. Since p is prime, either $p \mid 10^{(p-1)/2} + 1$ or $p \mid 10^{(p-1)/2} - 1$. Thus we have two cases:

Case a: $10^{(p-1)/2} \equiv -1 \pmod{p}$

In this case, for each move (i, a_i) of Fernando, Eduardo immediately makes the move $(j, a_j) = (i + \frac{p-1}{2}, a_i)$, if $0 \leqslant i \leqslant \frac{p-3}{2}$, or $(j, a_j) = (i - \frac{p-1}{2}, a_i)$, if $\frac{p-1}{2} \leqslant i \leqslant p-2$. We will have $10^j \equiv -10^i \pmod{p}$, and so $a_j \cdot 10^j = a_i \cdot 10^j \equiv -a_i \cdot 10^i \pmod{p}$. Notice that this move by Eduardo is always possible. Indeed, immediately before a move by Fernando, for any set of the type $\{r, r + (p-1)/2\}$ with $0 \leqslant r \leqslant (p-3)/2$, either no element of this set was chosen as an index by the players in the previous moves or else both elements of this set were chosen as indices by the players in the previous moves. Therefore, after each of his moves, Eduardo always makes the sum of the numbers $a_k \cdot 10^k$ corresponding to the already chosen pairs (k, a_k) divisible by p, and thus wins the game.

Case b: $10^{(p-1)/2} \equiv 1 \pmod{p}$

In this case, for each move (i, a_i) of Fernando, Eduardo immediately makes the move $(j, a_j) = (i + \frac{p-1}{2}, 9 - a_i)$, if $0 \leqslant i \leqslant \frac{p-3}{2}$, or $(j, a_j) = (i - \frac{p-1}{2}, 9 - a_i)$, if $\frac{p-1}{2} \leqslant i \leqslant p-2$. The same argument as above shows that Eduardo can always make such move. We will have $10^j \equiv 10^i \pmod{p}$, and so $a_j \cdot 10^j + a_i \cdot 10^i \equiv (a_i + a_j) \cdot 10^i = 9 \cdot 10^i \pmod{p}$. Therefore, at the end of the game, the sum of all terms $a_k \cdot 10^k$ will be congruent to

$$\sum_{i=0}^{\frac{p-3}{2}} 9 \cdot 10^i = 10^{(p-1)/2} - 1 \equiv 0 \pmod{p},$$

and Eduardo wins the game.

N3. Determine all integers $n \geqslant 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leqslant i \leqslant n$ such that none of the numbers

$$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

(Thailand)

Answer: These integers are exactly the prime numbers.

Solution. Let us first show that, if $n = ab$, with $a, b \geqslant 2$ integers, then the property in the statement of the problem does not hold. Indeed, in this case, let $a_k = a$ for $1 \leqslant k \leqslant n-1$ and $a_n = 0$. The sum $a_1 + a_2 + \cdots + a_n = a \cdot (n-1)$ is not divisible by n. Let i with $1 \leqslant i \leqslant n$ be an arbitrary index. Taking $j = b$ if $1 \leqslant i \leqslant n-b$, and $j = b+1$ if $n-b < i \leqslant n$, we have

$$a_i + a_{i+1} + \cdots + a_{i+j-1} = a \cdot b = n \equiv 0 \pmod{n}.$$

It follows that the given example is indeed a counterexample to the property of the statement.

Now let n be a prime number. Suppose by contradiction that the property in the statement of the problem does not hold. Then there are integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n such that for each i, $1 \leqslant i \leqslant n$, there is j, $1 \leqslant j \leqslant n$, for which the number $a_i + a_{i+1} + \cdots + a_{i+j-1}$ is divisible by n. Notice that, in any such case, we should have $1 \leqslant j \leqslant n-1$, since $a_1 + a_2 + \cdots + a_n$ is not divisible by n. So we may construct recursively a finite sequence of integers $0 = i_0 < i_1 < i_2 < \cdots < i_n$ with $i_{s+1} - i_s \leqslant n-1$ for $0 \leqslant s \leqslant n-1$ such that, for $0 \leqslant s \leqslant n-1$,

$$a_{i_s+1} + a_{i_s+2} + \cdots + a_{i_{s+1}} \equiv 0 \pmod{n}$$

(where we take indices modulo n). Indeed, for $0 \leqslant s < n$, we apply the previous observation to $i = i_s + 1$ in order to define $i_{s+1} = i_s + j$.

In the sequence of $n+1$ indices $i_0, i_1, i_2, \ldots, i_n$, by the pigeonhole principle, we have two distinct elements which are congruent modulo n. So there are indices r, s with $0 \leqslant r < s \leqslant n$ such that $i_s \equiv i_r \pmod{n}$ and

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = \sum_{j=r}^{s-1} (a_{i_j+1} + a_{i_j+2} + \cdots + a_{i_{j+1}}) \equiv 0 \pmod{n}.$$

Since $i_s \equiv i_r \pmod{n}$, we have $i_s - i_r = k \cdot n$ for some positive integer k, and, since $i_{j+1} - i_j \leqslant n-1$ for $0 \leqslant j \leqslant n-1$, we have $i_s - i_r \leqslant (n-1) \cdot n$, so $k \leqslant n-1$. But in this case

$$a_{i_r+1} + a_{i_r+2} + \cdots + a_{i_s} = k \cdot (a_1 + a_2 + \cdots + a_n)$$

cannot be a multiple of n, since n is prime and neither k nor $a_1 + a_2 + \cdots + a_n$ is a multiple of n. A contradiction.

N4. Call a rational number *short* if it has finitely many digits in its decimal expansion. For a positive integer m, we say that a positive integer t is m-*tastic* if there exists a number $c \in \{1, 2, 3, \ldots, 2017\}$ such that $\dfrac{10^t - 1}{c \cdot m}$ is short, and such that $\dfrac{10^k - 1}{c \cdot m}$ is not short for any $1 \leqslant k < t$. Let $S(m)$ be the set of m-tastic numbers. Consider $S(m)$ for $m = 1, 2, \ldots$. What is the maximum number of elements in $S(m)$?

(Turkey)

Answer: 807.

Solution. First notice that $x \in \mathbb{Q}$ is short if and only if there are exponents $a, b \geqslant 0$ such that $2^a \cdot 5^b \cdot x \in \mathbb{Z}$. In fact, if x is short, then $x = \frac{n}{10^k}$ for some k and we can take $a = b = k$; on the other hand, if $2^a \cdot 5^b \cdot x = q \in \mathbb{Z}$ then $x = \frac{2^b \cdot 5^a q}{10^{a+b}}$, so x is short.

If $m = 2^a \cdot 5^b \cdot s$, with $\gcd(s, 10) = 1$, then $\frac{10^t - 1}{m}$ is short if and only if s divides $10^t - 1$. So we may (and will) suppose without loss of generality that $\gcd(m, 10) = 1$. Define

$$C = \{1 \leqslant c \leqslant 2017\colon\ \gcd(c, 10) = 1\}.$$

The m-tastic numbers are then precisely the smallest exponents $t > 0$ such that $10^t \equiv 1 \pmod{cm}$ for some integer $c \in C$, that is, the set of *orders of* 10 *modulo* cm. In other words,

$$S(m) = \{\operatorname{ord}_{cm}(10)\colon c \in C\}.$$

Since there are $4 \cdot 201 + 3 = 807$ numbers c with $1 \leqslant c \leqslant 2017$ and $\gcd(c, 10) = 1$, namely those such that $c \equiv 1, 3, 7, 9 \pmod{10}$,

$$|S(m)| \leqslant |C| = 807.$$

Now we find m such that $|S(m)| = 807$. Let

$$P = \{1 < p \leqslant 2017\colon p \text{ is prime}, p \neq 2, 5\}$$

and choose a positive integer α such that every $p \in P$ divides $10^\alpha - 1$ (e.g. $\alpha = \varphi(T)$, T being the product of all primes in P), and let $m = 10^\alpha - 1$.

Claim. For every $c \in C$, we have

$$\operatorname{ord}_{cm}(10) = c\alpha.$$

As an immediate consequence, this implies $|S(m)| = |C| = 807$, finishing the problem.

Proof. Obviously $\operatorname{ord}_m(10) = \alpha$. Let $t = \operatorname{ord}_{cm}(10)$. Then

$$cm \mid 10^t - 1 \quad \Longrightarrow \quad m \mid 10^t - 1 \quad \Longrightarrow \quad \alpha \mid t.$$

Hence $t = k\alpha$ for some $k \in \mathbb{Z}_{>0}$. We will show that $k = c$.

Denote by $\nu_p(n)$ the number of prime factors p in n, that is, the maximum exponent β for which $p^\beta \mid n$. For every $\ell \geqslant 1$ and $p \in P$, the Lifting the Exponent Lemma provides

$$\nu_p(10^{\ell\alpha} - 1) = \nu_p((10^\alpha)^\ell - 1) = \nu_p(10^\alpha - 1) + \nu_p(\ell) = \nu_p(m) + \nu_p(\ell),$$

so

$$\begin{aligned} cm \mid 10^{k\alpha} - 1 &\iff \forall p \in P;\ \nu_p(cm) \leqslant \nu_p(10^{k\alpha} - 1) \\ &\iff \forall p \in P;\ \nu_p(m) + \nu_p(c) \leqslant \nu_p(m) + \nu_p(k) \\ &\iff \forall p \in P;\ \nu_p(c) \leqslant \nu_p(k) \\ &\iff c \mid k. \end{aligned}$$

The first such k is $k = c$, so $\operatorname{ord}_{cm}(10) = c\alpha$. $\square$

Comment. The Lifting the Exponent Lemma states that, for any odd prime p, any integers a, b coprime with p such that $p \mid a - b$, and any positive integer exponent n,

$$\nu_p(a^n - b^n) = \nu_p(a - b) + \nu_p(n),$$

and, for $p = 2$,

$$\nu_2(a^n - b^n) = \nu_2(a^2 - b^2) + \nu_p(n) - 1.$$

Both claims can be proved by induction on n.

N5. Find all pairs (p, q) of prime numbers with $p > q$ for which the number

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

is an integer.

(Japan)

Answer: The only such pair is $(3, 2)$.

Solution. Let $M = (p+q)^{p-q}(p-q)^{p+q} - 1$, which is relatively prime with both $p+q$ and $p-q$. Denote by $(p-q)^{-1}$ the multiplicative inverse of $(p-q)$ modulo M.

By eliminating the term -1 in the numerator,

$$(p+q)^{p+q}(p-q)^{p-q} - 1 \equiv (p+q)^{p-q}(p-q)^{p+q} - 1 \pmod{M}$$
$$(p+q)^{2q} \equiv (p-q)^{2q} \pmod{M} \tag{1}$$
$$\Big((p+q)\cdot(p-q)^{-1}\Big)^{2q} \equiv 1 \pmod{M}. \tag{2}$$

Case 1: $q \geqslant 5$.

Consider an arbitrary prime divisor r of M. Notice that M is odd, so $r \geqslant 3$. By (2), the multiplicative order of $\Big((p+q)\cdot(p-q)^{-1}\Big)$ modulo r is a divisor of the exponent $2q$ in (2), so it can be 1, 2, q or $2q$.

By Fermat's theorem, the order divides $r-1$. So, if the order is q or $2q$ then $r \equiv 1 \pmod{q}$. If the order is 1 or 2 then $r \mid (p+q)^2 - (p-q)^2 = 4pq$, so $r = p$ or $r = q$. The case $r = p$ is not possible, because, by applying Fermat's theorem,

$$M = (p+q)^{p-q}(p-q)^{p+q} - 1 \equiv q^{p-q}(-q)^{p+q} - 1 = \big(q^2\big)^p - 1 \equiv q^2 - 1 = (q+1)(q-1) \pmod{p}$$

and the last factors $q-1$ and $q+1$ are less than p and thus $p \nmid M$. Hence, all prime divisors of M are either q or of the form $kq+1$; it follows that all positive divisors of M are congruent to 0 or 1 modulo q.

Now notice that

$$M = \Big((p+q)^{\frac{p-q}{2}}(p-q)^{\frac{p+q}{2}} - 1\Big)\Big((p+q)^{\frac{p-q}{2}}(p-q)^{\frac{p+q}{2}} + 1\Big)$$

is the product of two consecutive positive odd numbers; both should be congruent to 0 or 1 modulo q. But this is impossible by the assumption $q \geqslant 5$. So, there is no solution in Case 1.

Case 2: $q = 2$.

By (1), we have $M \mid (p+q)^{2q} - (p-q)^{2q} = (p+2)^4 - (p-2)^4$, so

$$(p+2)^{p-2}(p-2)^{p+2} - 1 = M \leqslant (p+2)^4 - (p-2)^4 \leqslant (p+2)^4 - 1,$$
$$(p+2)^{p-6}(p-2)^{p+2} \leqslant 1.$$

If $p \geqslant 7$ then the left-hand side is obviously greater than 1. For $p = 5$ we have $(p+2)^{p-6}(p-2)^{p+2} = 7^{-1}\cdot 3^7$ which is also too large.

There remains only one candidate, $p = 3$, which provides a solution:

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1} = \frac{5^5\cdot 1^1 - 1}{5^1\cdot 1^5 - 1} = \frac{3124}{4} = 781.$$

So in Case 2 the only solution is $(p, q) = (3, 2)$.

Case 3: $q = 3$.

Similarly to Case 2, we have

$$M \mid (p+q)^{2q} - (p-q)^{2q} = 64 \cdot \left(\left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6 \right).$$

Since M is odd, we conclude that

$$M \mid \left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6$$

and

$$(p+3)^{p-3}(p-3)^{p+3} - 1 = M \leqslant \left(\frac{p+3}{2}\right)^6 - \left(\frac{p-3}{2}\right)^6 \leqslant \left(\frac{p+3}{2}\right)^6 - 1,$$
$$64(p+3)^{p-9}(p-3)^{p+3} \leqslant 1.$$

If $p \geqslant 11$ then the left-hand side is obviously greater than 1. If $p = 7$ then the left-hand side is $64 \cdot 10^{-2} \cdot 4^{10} > 1$. If $p = 5$ then the left-hand side is $64 \cdot 8^{-4} \cdot 2^8 = 2^2 > 1$. Therefore, there is no solution in Case 3.

N6. Find the smallest positive integer n, or show that no such n exists, with the following property: there are infinitely many distinct n-tuples of positive rational numbers $(a_1, a_2, \ldots, a_n)$ such that both

$$a_1 + a_2 + \cdots + a_n \qquad \text{and} \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

are integers.

(Singapore)

Answer: $n = 3$.

Solution 1. For $n = 1$, $a_1 \in \mathbb{Z}_{>0}$ and $\frac{1}{a_1} \in \mathbb{Z}_{>0}$ if and only if $a_1 = 1$. Next we show that

(i) *There are finitely many $(x, y) \in \mathbb{Q}^2_{>0}$ satisfying $x + y \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} \in \mathbb{Z}$*

Write $x = \frac{a}{b}$ and $y = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Z}_{>0}$ and $\gcd(a, b) = \gcd(c, d) = 1$. Then $x + y \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} \in \mathbb{Z}$ is equivalent to the two divisibility conditions

$$bd \mid ad + bc \quad (1) \qquad \text{and} \qquad ac \mid ad + bc \quad (2)$$

Condition (1) implies that $d \mid ad + bc \iff d \mid bc \iff d \mid b$ since $\gcd(c, d) = 1$. Still from (1) we get $b \mid ad + bc \iff b \mid ad \iff b \mid d$ since $\gcd(a, b) = 1$. From $b \mid d$ and $d \mid b$ we have $b = d$.

An analogous reasoning with condition (2) shows that $a = c$. Hence $x = \frac{a}{b} = \frac{c}{d} = y$, i.e., the problem amounts to finding all $x \in \mathbb{Q}_{>0}$ such that $2x \in \mathbb{Z}_{>0}$ and $\frac{2}{x} \in \mathbb{Z}_{>0}$. Letting $n = 2x \in \mathbb{Z}_{>0}$, we have that $\frac{2}{x} \in \mathbb{Z}_{>0} \iff \frac{4}{n} \in \mathbb{Z}_{>0} \iff n = 1$, 2 or 4, and there are finitely many solutions, namely $(x, y) = (\frac{1}{2}, \frac{1}{2})$, $(1, 1)$ or $(2, 2)$.

(ii) *There are infinitely many triples $(x, y, z) \in \mathbb{Q}^2_{>0}$ such that $x + y + z \in \mathbb{Z}$ and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \in \mathbb{Z}$.*
We will look for triples such that $x + y + z = 1$, so we may write them in the form

$$(x, y, z) = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}\right) \qquad \text{with } a, b, c \in \mathbb{Z}_{>0}$$

We want these to satisfy

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c} \in \mathbb{Z} \iff \frac{b+c}{a} + \frac{a+c}{b} + \frac{a+b}{c} \in \mathbb{Z}$$

Fixing $a = 1$, it suffices to find infinitely many pairs $(b, c) \in \mathbb{Z}^2_{>0}$ such that

$$\frac{1}{b} + \frac{1}{c} + \frac{c}{b} + \frac{b}{c} = 3 \iff b^2 + c^2 - 3bc + b + c = 0 \qquad (*)$$

To show that equation $(*)$ has infinitely many solutions, we use *Vieta jumping* (also known as *root flipping*): starting with $b = 2$, $c = 3$, the following algorithm generates infinitely many solutions. Let $c \geqslant b$, and view $(*)$ as a quadratic equation in b for c fixed:

$$b^2 - (3c - 1) \cdot b + (c^2 + c) = 0 \qquad (**)$$

Then there exists another root $b_0 \in \mathbb{Z}$ of $(**)$ which satisfies $b + b_0 = 3c - 1$ and $b \cdot b_0 = c^2 + c$. Since $c \geqslant b$ by assumption,

$$b_0 = \frac{c^2 + c}{b} \geqslant \frac{c^2 + c}{c} > c$$

Hence from the solution (b, c) we obtain another one (c, b_0) with $b_0 > c$, and we can then "jump" again, this time with c as the "variable" in the quadratic $(*)$. This algorithm will generate an infinite sequence of distinct solutions, whose first terms are

$(2, 3)$, $(3, 6)$, $(6, 14)$, $(14, 35)$, $(35, 90)$, $(90, 234)$, $(234, 611)$, $(611, 1598)$, $(1598, 4182)$, …

Comment. Although not needed for solving this problem, we may also explicitly solve the recursion given by the Vieta jumping. Define the sequence (x_n) as follows:

$$x_0 = 2, \quad x_1 = 3 \qquad \text{and} \qquad x_{n+2} = 3x_{n+1} - x_n - 1 \text{ for } n \geqslant 0$$

Then the triple

$$(x, y, z) = \left(\frac{1}{1 + x_n + x_{n+1}}, \frac{x_n}{1 + x_n + x_{n+1}}, \frac{x_{n+1}}{1 + x_n + x_{n+1}}\right)$$

satisfies the problem conditions for all $n \in \mathbb{N}$. It is easy to show that $x_n = F_{2n+1} + 1$, where F_n denotes the n-th term of the Fibonacci sequence ($F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geqslant 0$).

Solution 2. Call the n-tuples $(a_1, a_2, \ldots, a_n) \in \mathbb{Q}_{>0}^n$ satisfying the conditions of the problem statement *good*, and those for which

$$f(a_1, \ldots, a_n) \stackrel{\text{def}}{=} (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

is an integer *pretty*. Then good n-tuples are pretty, and if $(b_1, \ldots, b_n)$ is pretty then

$$\left(\frac{b_1}{b_1 + b_2 + \cdots + b_n}, \frac{b_2}{b_1 + b_2 + \cdots + b_n}, \ldots, \frac{b_n}{b_1 + b_2 + \cdots + b_n}\right)$$

is good since the sum of its components is 1, and the sum of the reciprocals of its components equals $f(b_1, \ldots, b_n)$. We declare pretty n-tuples proportional to each other *equivalent* since they are precisely those which give rise to the same good n-tuple. Clearly, each such equivalence class contains exactly one n-tuple of positive integers having no common prime divisors. Call such n-tuple a *primitive* pretty tuple. Our task is to find infinitely many primitive pretty n-tuples.

For $n = 1$, there is clearly a single primitive 1-tuple. For $n = 2$, we have $f(a, b) = \frac{(a+b)^2}{ab}$, which can be integral (for coprime $a, b \in \mathbb{Z}_{>0}$) only if $a = b = 1$ (see for instance (i) in the first solution).

Now we construct infinitely many primitive pretty triples for $n = 3$. Fix $b, c, k \in \mathbb{Z}_{>0}$; we will try to find sufficient conditions for the existence of an $a \in \mathbb{Q}_{>0}$ such that $f(a, b, c) = k$. Write $\sigma = b + c$, $\tau = bc$. From $f(a, b, c) = k$, we have that a should satisfy the quadratic equation

$$a^2 \cdot \sigma + a \cdot (\sigma^2 - (k-1)\tau) + \sigma\tau = 0 \tag{1}$$

whose discriminant is

$$\Delta = (\sigma^2 - (k-1)\tau)^2 - 4\sigma^2\tau = ((k+1)\tau - \sigma^2)^2 - 4k\tau^2.$$

We need it to be a square of an integer, say, $\Delta = M^2$ for some $M \in \mathbb{Z}$, i.e., we want

$$((k+1)\tau - \sigma^2)^2 - M^2 = 2k \cdot 2\tau^2$$

so that it suffices to set

$$(k+1)\tau - \sigma^2 = \tau^2 + k, \qquad M = \tau^2 - k.$$

The first relation reads $\sigma^2 = (\tau - 1)(k - \tau)$, so if b and c satisfy

$$\tau - 1 \mid \sigma^2 \qquad \text{i.e.} \qquad bc - 1 \mid (b+c)^2 \tag{2}$$

then $k = \frac{\sigma^2}{\tau - 1} + \tau$ will be integral, and we find rational solutions to (1), namely

$$a = \frac{\sigma}{\tau - 1} = \frac{b+c}{bc-1} \qquad \text{or} \qquad a = \frac{\tau^2 - \tau}{\sigma} = \frac{bc \cdot (bc-1)}{b+c}$$

We can now find infinitely many pairs (b, c) satisfying (2) by Vieta jumping. For example, if we impose

$$(b + c)^2 = 5 \cdot (bc - 1)$$

then all pairs $(b, c) = (v_i, v_{i+1})$ satisfy the above condition, where

$$v_1 = 2,\ v_2 = 3, \qquad v_{i+2} = 3v_{i+1} - v_i \quad \text{for } i \geqslant 0$$

For $(b, c) = (v_i, v_{i+1})$, one of the solutions to (1) will be $a = (b + c)/(bc - 1) = 5/(b + c) = 5/(v_i + v_{i+1})$. Then the pretty triple (a, b, c) will be equivalent to the integral pretty triple

$$(5, v_i(v_i + v_{i+1}), v_{i+1}(v_i + v_{i+1}))$$

After possibly dividing by 5, we obtain infinitely many primitive pretty triples, as required.

Comment. There are many other infinite series of $(b, c) = (v_i, v_{i+1})$ with $bc - 1 \mid (b + c)^2$. Some of them are:

$$\begin{array}{llll}
v_1 = 1, & v_2 = 3, & v_{i+1} = 6v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 8 \cdot (v_i v_{i+1} - 1); \\
v_1 = 1, & v_2 = 2, & v_{i+1} = 7v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 9 \cdot (v_i v_{i+1} - 1); \\
v_1 = 1, & v_2 = 5, & v_{i+1} = 7v_i - v_{i-1}, & (v_i + v_{i+1})^2 = 9 \cdot (v_i v_{i+1} - 1)
\end{array}$$

(the last two are in fact one sequence prolonged in two possible directions).

N7. Say that an ordered pair (x, y) of integers is an *irreducible lattice point* if x and y are relatively prime. For any finite set S of irreducible lattice points, show that there is a homogenous polynomial in two variables, $f(x, y)$, with integer coefficients, of degree at least 1, such that $f(x, y) = 1$ for each (x, y) in the set S.

Note: A homogenous polynomial of degree n is any nonzero polynomial of the form

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_{n-1}xy^{n-1} + a_ny^n.$$

(U.S.A.)

Solution 1. First of all, we note that finding a homogenous polynomial $f(x, y)$ such that $f(x, y) = \pm 1$ is enough, because we then have $f^2(x, y) = 1$. Label the irreducible lattice points (x_1, y_1) through (x_n, y_n). If any two of these lattice points (x_i, y_i) and (x_j, y_j) lie on the same line through the origin, then $(x_j, y_j) = (-x_i, -y_i)$ because both of the points are irreducible. We then have $f(x_j, y_j) = \pm f(x_i, y_i)$ whenever f is homogenous, so we can assume that no two of the lattice points are collinear with the origin by ignoring the extra lattice points.

Consider the homogenous polynomials $\ell_i(x, y) = y_ix - x_iy$ and define

$$g_i(x, y) = \prod_{j \neq i} \ell_j(x, y).$$

Then $\ell_i(x_j, y_j) = 0$ if and only if $j = i$, because there is only one lattice point on each line through the origin. Thus, $g_i(x_j, y_j) = 0$ for all $j \neq i$. Define $a_i = g_i(x_i, y_i)$, and note that $a_i \neq 0$.

Note that $g_i(x, y)$ is a degree $n - 1$ polynomial with the following two properties:

1. $g_i(x_j, y_j) = 0$ if $j \neq i$.
2. $g_i(x_i, y_i) = a_i$.

For any $N \geqslant n - 1$, there also exists a polynomial of degree N with the same two properties. Specifically, let $I_i(x, y)$ be a degree 1 homogenous polynomial such that $I_i(x_i, y_i) = 1$, which exists since (x_i, y_i) is irreducible. Then $I_i(x, y)^{N-(n-1)}g_i(x, y)$ satisfies both of the above properties and has degree N.

We may now reduce the problem to the following claim:

Claim: For each positive integer a, there is a homogenous polynomial $f_a(x, y)$, with integer coefficients, of degree at least 1, such that $f_a(x, y) \equiv 1 \pmod{a}$ for all relatively prime (x, y).

To see that this claim solves the problem, take a to be the least common multiple of the numbers a_i $(1 \leqslant i \leqslant n)$. Take f_a given by the claim, choose some power $f_a(x, y)^k$ that has degree at least $n - 1$, and subtract appropriate multiples of the g_i constructed above to obtain the desired polynomial.

We prove the claim by factoring a. First, if a is a power of a prime $(a = p^k)$, then we may choose either:

- $f_a(x, y) = (x^{p-1} + y^{p-1})^{\phi(a)}$ if p is odd;
- $f_a(x, y) = (x^2 + xy + y^2)^{\phi(a)}$ if $p = 2$.

Now suppose a is any positive integer, and let $a = q_1q_2 \cdots q_k$, where the q_i are prime powers, pairwise relatively prime. Let f_{q_i} be the polynomials just constructed, and let F_{q_i} be powers of these that all have the same degree. Note that

$$\frac{a}{q_i}F_{q_i}(x, y) \equiv \frac{a}{q_i} \pmod{a}$$

for any relatively prime x, y. By Bézout's lemma, there is an integer linear combination of the $\frac{a}{q_i}$ that equals 1. Thus, there is a linear combination of the F_{q_i} such that $F_{q_i}(x, y) \equiv 1 \pmod{a}$ for any relatively prime (x, y); and this polynomial is homogenous because all the F_{q_i} have the same degree.

Solution 2. As in the previous solution, label the irreducible lattice points $(x_1, y_1), \ldots, (x_n, y_n)$ and assume without loss of generality that no two of the points are collinear with the origin. We induct on n to construct a homogenous polynomial $f(x, y)$ such that $f(x_i, y_i) = 1$ for all $1 \leqslant i \leqslant n$.

If $n = 1$: Since x_1 and y_1 are relatively prime, there exist some integers c, d such that $cx_1 + dy_1 = 1$. Then $f(x, y) = cx + dy$ is suitable.

If $n \geqslant 2$: By the induction hypothesis we already have a homogeneous polynomial $g(x, y)$ with $g(x_1, y_1) = \ldots = g(x_{n-1}, y_{n-1}) = 1$. Let $j = \deg g$,

$$g_n(x, y) = \prod_{k=1}^{n-1} (y_k x - x_k y),$$

and $a_n = g_n(x_n, y_n)$. By assumption, $a_n \neq 0$. Take some integers c, d such that $cx_n + dy_n = 1$. We will construct $f(x, y)$ in the form

$$f(x, y) = g(x, y)^K - C \cdot g_n(x, y) \cdot (cx + dy)^L,$$

where K and L are some positive integers and C is some integer. We assume that $L = Kj-n+1$ so that f is homogenous.

Due to $g(x_1, y_1) = \ldots = g(x_{n-1}, y_{n-1}) = 1$ and $g_n(x_1, y_1) = \ldots = g_n(x_{n-1}, y_{n-1}) = 0$, the property $f(x_1, y_1) = \ldots = f(x_{n-1}, y_{n-1}) = 1$ is automatically satisfied with any choice of K, L, and C.

Furthermore,

$$f(x_n, y_n) = g(x_n, y_n)^K - C \cdot g_n(x_n, y_n) \cdot (cx_n + dy_n)^L = g(x_n, y_n)^K - Ca_n.$$

If we have an exponent K such that $g(x_n, y_n)^K \equiv 1 \pmod{a_n}$, then we may choose C such that $f(x_n, y_n) = 1$. We now choose such a K.

Consider an arbitrary prime divisor p of a_n. By

$$p \mid a_n = g_n(x_n, y_n) = \prod_{k=1}^{n-1} (y_k x_n - x_k y_n),$$

there is some $1 \leqslant k < n$ such that $x_k y_n \equiv x_n y_k \pmod{p}$. We first show that $x_k x_n$ or $y_k y_n$ is relatively prime with p. This is trivial in the case $x_k y_n \equiv x_n y_k \not\equiv 0 \pmod{p}$. In the other case, we have $x_k y_n \equiv x_n y_k \equiv 0 \pmod{p}$, If, say $p \mid x_k$, then $p \nmid y_k$ because (x_k, y_k) is irreducible, so $p \mid x_n$; then $p \nmid y_n$ because (x_k, y_k) is irreducible. In summary, $p \mid x_k$ implies $p \nmid y_k y_n$. Similarly, $p \mid y_n$ implies $p \nmid x_k x_n$.

By the homogeneity of g we have the congruences

$$x_k^d \cdot g(x_n, y_n) = g(x_k x_n, x_k y_n) \equiv g(x_k x_n, y_k x_n) = x_n^d \cdot g(x_k, y_k) = x_n^d \pmod{p} \tag{1.1}$$

and

$$y_k^d \cdot g(x_n, y_n) = g(y_k x_n, y_k y_n) \equiv g(x_k y_n, y_k y_n) = y_n^d \cdot g(x_k, y_k) = y_n^d \pmod{p}. \tag{1.2}$$

If $p \nmid x_k x_n$, then take the $(p-1)^{st}$ power of (1.1); otherwise take the $(p-1)^{st}$ power of (1.2); by Fermat's theorem, in both cases we get

$$g(x_n, y_n)^{p-1} \equiv 1 \pmod{p}.$$

If $p^\alpha \mid m$, then we have

$$g(x_n, y_n)^{p^{\alpha-1}(p-1)} \equiv 1 \pmod{p^\alpha},$$

which implies that the exponent $K = n \cdot \varphi(a_n)$, which is a multiple of all $p^{\alpha-1}(p-1)$, is a suitable choice. (The factor n is added only so that $K \geqslant n$ and so $L > 0$.)

Comment. It is possible to show that there is no constant C for which, given any two irreducible lattice points, there is some homogenous polynomial f of degree at most C with integer coefficients that takes the value 1 on the two points. Indeed, if one of the points is $(1,0)$ and the other is (a,b), the polynomial $f(x,y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$ should satisfy $a_0 = 1$, and so $a^n \equiv 1 \pmod{b}$. If $a = 3$ and $b = 2^k$ with $k \geqslant 3$, then $n \geqslant 2^{k-2}$. If we choose $2^{k-2} > C$, this gives a contradiction.

N8. Let p be an odd prime number and $\mathbb{Z}_{>0}$ be the set of positive integers. Suppose that a function $f\colon \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \to \{0,1\}$ satisfies the following properties:

- $f(1,1) = 0$;
- $f(a,b) + f(b,a) = 1$ for any pair of relatively prime positive integers (a,b) not both equal to 1;
- $f(a+b,b) = f(a,b)$ for any pair of relatively prime positive integers (a,b).

Prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \sqrt{2p} - 2.$$

(Italy)

Solution 1. Denote by $\mathbb{A}$ the set of all pairs of coprime positive integers. Notice that for every $(a,b) \in \mathbb{A}$ there exists a pair $(u,v) \in \mathbb{Z}^2$ with $ua + vb = 1$. Moreover, if (u_0, v_0) is one such pair, then all such pairs are of the form $(u,v) = (u_0 + kb, v_0 - ka)$, where $k \in \mathbb{Z}$. So there exists a unique such pair (u,v) with $-b/2 < u \leqslant b/2$; we denote this pair by $(u,v) = g(a,b)$.

Lemma. Let $(a,b) \in \mathbb{A}$ and $(u,v) = g(a,b)$. Then $f(a,b) = 1 \iff u > 0$.

Proof. We induct on $a+b$. The base case is $a+b=2$. In this case, we have that $a=b=1$, $g(a,b) = g(1,1) = (0,1)$ and $f(1,1) = 0$, so the claim holds.

Assume now that $a+b > 2$, and so $a \neq b$, since a and b are coprime. Two cases are possible.

Case 1: $a > b$.

Notice that $g(a-b,b) = (u, v+u)$, since $u(a-b) + (v+u)b = 1$ and $u \in (-b/2, b/2]$. Thus $f(a,b) = 1 \iff f(a-b,b) = 1 \iff u > 0$ by the induction hypothesis.

Case 2: $a < b$. (Then, clearly, $b \geqslant 2$.)

Now we estimate v. Since $vb = 1 - ua$, we have

$$1 + \frac{ab}{2} > vb \geqslant 1 - \frac{ab}{2}, \quad \text{so} \quad \frac{1+a}{2} \geqslant \frac{1}{b} + \frac{a}{2} > v \geqslant \frac{1}{b} - \frac{a}{2} > -\frac{a}{2}.$$

Thus $1 + a > 2v > -a$, so $a \geqslant 2v > -a$, hence $a/2 \geqslant v > -a/2$, and thus $g(b,a) = (v,u)$.

Observe that $f(a,b) = 1 \iff f(b,a) = 0 \iff f(b-a,a) = 0$. We know from Case 1 that $g(b-a,a) = (v, u+v)$. We have $f(b-a,a) = 0 \iff v \leqslant 0$ by the inductive hypothesis. Then, since $b > a \geqslant 1$ and $ua + vb = 1$, we have $v \leqslant 0 \iff u > 0$, and we are done. $\square$

The Lemma proves that, for all $(a,b) \in \mathbb{A}$, $f(a,b) = 1$ if and only if the inverse of a modulo b, taken in $\{1, 2, \ldots, b-1\}$, is at most $b/2$. Then, for any odd prime p and integer n such that $n \not\equiv 0 \pmod{p}$, $f(n^2,p) = 1$ iff the inverse of $n^2 \bmod p$ is less than $p/2$. Since $\{n^2 \bmod p \colon 1 \leqslant n \leqslant p-1\} = \{n^{-2} \bmod p \colon 1 \leqslant n \leqslant p-1\}$, including multiplicities (two for each quadratic residue in each set), we conclude that the desired sum is twice the number of quadratic residues that are less than $p/2$, i.e.,

$$\sum_{n=1}^{p-1} f(n^2,p) = 2\left|\left\{k \colon 1 \leqslant k \leqslant \frac{p-1}{2} \text{ and } k^2 \bmod p < \frac{p}{2}\right\}\right|. \tag{1}$$

Since the number of perfect squares in the interval $[1, p/2)$ is $\lfloor \sqrt{p/2} \rfloor > \sqrt{p/2} - 1$, we conclude that

$$\sum_{n=1}^{p-1} f(n^2,p) > 2\left(\sqrt{\frac{p}{2}} - 1\right) = \sqrt{2p} - 2.$$

Solution 2. We provide a different proof for the Lemma. For this purpose, we use continued fractions to find $g(a,b) = (u,v)$ explicitly.

The function f is completely determined on $\mathbb{A}$ by the following

Claim. Represent a/b as a continued fraction; that is, let a_0 be an integer and $a_1, \ldots, a_k$ be positive integers such that $a_k \geqslant 2$ and

$$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}} = [a_0; a_1, a_2, \ldots, a_k].$$

Then $f(a,b) = 0 \iff k$ is even.

Proof. We induct on b. If $b = 1$, then $a/b = [a]$ and $k = 0$. Then, for $a \geqslant 1$, an easy induction shows that $f(a,1) = f(1,1) = 0$.

Now consider the case $b > 1$. Perform the Euclidean division $a = qb + r$, with $0 \leqslant r < b$. We have $r \neq 0$ because $\gcd(a,b) = 1$. Hence

$$f(a,b) = f(r,b) = 1 - f(b,r), \quad \frac{a}{b} = [q; a_1, \ldots, a_k], \quad \text{and} \quad \frac{b}{r} = [a_1; a_2, \ldots, a_k].$$

Then the number of terms in the continued fraction representations of a/b and b/r differ by one. Since $r < b$, the inductive hypothesis yields

$$f(b,r) = 0 \iff k - 1 \text{ is even,}$$

and thus

$$f(a,b) = 0 \iff f(b,r) = 1 \iff k - 1 \text{ is odd} \iff k \text{ is even.} \qquad \square$$

Now we use the following well-known properties of continued fractions to prove the Lemma:

Let p_i and q_i be coprime positive integers with $[a_0; a_1, a_2, \ldots, a_i] = p_i/q_i$, with the notation borrowed from the Claim. In particular, $a/b = [a_0; a_1, a_2, \ldots, a_k] = p_k/q_k$. Assume that $k > 0$ and define $q_{-1} = 0$ if necessary. Then

- $q_k = a_k q_{k-1} + q_{k-2}$, and
- $aq_{k-1} - bp_{k-1} = p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$.

Assume that $k > 0$. Then $a_k \geqslant 2$, and

$$b = q_k = a_k q_{k-1} + q_{k-2} \geqslant a_k q_{k-1} \geqslant 2q_{k-1} \implies q_{k-1} \leqslant \frac{b}{2},$$

with strict inequality for $k > 1$, and

$$(-1)^{k-1} q_{k-1} a + (-1)^k p_{k-1} b = 1.$$

Now we finish the proof of the Lemma. It is immediate for $k = 0$. If $k = 1$, then $(-1)^{k-1} = 1$, so

$$-b/2 < 0 \leqslant (-1)^{k-1} q_{k-1} \leqslant b/2.$$

If $k > 1$, we have $q_{k-1} < b/2$, so

$$-b/2 < (-1)^{k-1} q_{k-1} < b/2.$$

Thus, for any $k > 0$, we find that $g(a,b) = ((-1)^{k-1} q_{k-1}, (-1)^k p_{k-1})$, and so

$$f(a,b) = 1 \iff k \text{ is odd} \iff u = (-1)^{k-1} q_{k-1} > 0.$$

Comment 1. The Lemma can also be established by observing that f is uniquely defined on $\mathbb{A}$, defining $f_1(a,b) = 1$ if $u > 0$ in $g(a,b) = (u,v)$ and $f_1(a,b) = 0$ otherwise, and verifying that f_1 satisfies all the conditions from the statement.

It seems that the main difficulty of the problem is in conjecturing the Lemma.

Comment 2. The case $p \equiv 1 \pmod 4$ is, in fact, easier than the original problem. We have, in general, for $1 \leqslant a \leqslant p-1$,

$$f(a,p) = 1-f(p,a) = 1-f(p-a,a) = f(a,p-a) = f(a+(p-a),p-a) = f(p,p-a) = 1-f(p-a,p).$$

If $p \equiv 1 \pmod 4$, then a is a quadratic residue modulo p if and only if $p-a$ is a quadratic residue modulo p. Therefore, denoting by r_k (with $1 \leqslant r_k \leqslant p-1$) the remainder of the division of k^2 by p, we get

$$\sum_{n=1}^{p-1} f(n^2,p) = \sum_{n=1}^{p-1} f(r_n,p) = \frac{1}{2}\sum_{n=1}^{p-1}(f(r_n,p) + f(p-r_n,p)) = \frac{p-1}{2}.$$

Comment 3. The estimate for the sum $\sum_{n=1}^{p} f(n^2,p)$ can be improved by refining the final argument in Solution 1. In fact, one can prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \frac{p-1}{16}.$$

By counting the number of perfect squares in the intervals $[kp,(k+1/2)p)$, we find that

$$\sum_{n=1}^{p-1} f(n^2,p) = \sum_{k=0}^{p-1}\left(\left\lfloor\sqrt{\left(k+\frac{1}{2}\right)p}\right\rfloor - \left\lfloor\sqrt{kp}\right\rfloor\right). \tag{2}$$

Each summand of (2) is non-negative. We now estimate the number of positive summands. Suppose that a summand is zero, i.e.,

$$\left\lfloor\sqrt{\left(k+\frac{1}{2}\right)p}\right\rfloor = \left\lfloor\sqrt{kp}\right\rfloor =: q.$$

Then both of the numbers kp and $kp+p/2$ lie within the interval $[q^2,(q+1)^2)$. Hence

$$\frac{p}{2} < (q+1)^2 - q^2,$$

which implies

$$q \geqslant \frac{p-1}{4}.$$

Since $q \leqslant \sqrt{kp}$, if the k^{th} summand of (2) is zero, then

$$k \geqslant \frac{q^2}{p} \geqslant \frac{(p-1)^2}{16p} > \frac{p-2}{16} \implies k \geqslant \frac{p-1}{16}.$$

So at least the first $\lceil \frac{p-1}{16} \rceil$ summands (from $k=0$ to $k = \lceil \frac{p-1}{16} \rceil - 1$) are positive, and the result follows.

Comment 4. The bound can be further improved by using different methods. In fact, we prove that

$$\sum_{n=1}^{p-1} f(n^2,p) \geqslant \frac{p-3}{4}.$$

To that end, we use the Legendre symbol

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a nonzero quadratic residue mod } p \\ -1 & \text{otherwise.} \end{cases}$$

We start with the following Claim, which tells us that there are not too many consecutive quadratic residues or consecutive quadratic non-residues.

Claim. $\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$.

Proof. We have $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = \left(\frac{n(n+1)}{p}\right)$. For $1 \leqslant n \leqslant p-1$, we get that $n(n+1) \equiv n^2(1+n^{-1}) \pmod p$, hence $\left(\frac{n(n+1)}{p}\right) = \left(\frac{1+n^{-1}}{p}\right)$. Since $\{1+n^{-1} \bmod p \colon 1 \leqslant n \leqslant p-1\} = \{0, 2, 3, \ldots, p-1 \bmod p\}$, we find

$$\sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = \sum_{n=1}^{p-1}\left(\frac{1+n^{-1}}{p}\right) = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right) - 1 = -1,$$

because $\sum_{n=1}^{p}\left(\frac{n}{p}\right) = 0$. $\square$

Observe that (1) becomes

$$\sum_{n=1}^{p-1} f(n^2, p) = 2\,|S|, \quad S = \left\{r \colon 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = 1\right\}.$$

We connect S with the sum from the claim by pairing quadratic residues and quadratic non-residues. To that end, define

$$S' = \left\{r \colon 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = -1\right\}$$
$$T = \left\{r \colon \frac{p+1}{2} \leqslant r \leqslant p-1 \text{ and } \left(\frac{r}{p}\right) = 1\right\}$$
$$T' = \left\{r \colon \frac{p+1}{2} \leqslant r \leqslant p-1 \text{ and } \left(\frac{r}{p}\right) = -1\right\}$$

Since there are exactly $(p-1)/2$ nonzero quadratic residues modulo p, $|S| + |T| = (p-1)/2$. Also we obviously have $|T| + |T'| = (p-1)/2$. Then $|S| = |T'|$.

For the sake of brevity, define $t = |S| = |T'|$. If $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$, then exactly of one the numbers $\left(\frac{n}{p}\right)$ and $\left(\frac{n+1}{p}\right)$ is equal to 1, so

$$\left|\left\{n \colon 1 \leqslant n \leqslant \frac{p-3}{2} \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant |S| + |S-1| = 2t.$$

On the other hand, if $\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1$, then exactly one of $\left(\frac{n}{p}\right)$ and $\left(\frac{n+1}{p}\right)$ is equal to -1, and

$$\left|\left\{n \colon \frac{p+1}{2} \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant |T'| + |T'-1| = 2t.$$

Thus, taking into account that the middle term $\left(\frac{(p-1)/2}{p}\right)\left(\frac{(p+1)/2}{p}\right)$ may happen to be -1,

$$\left|\left\{n \colon 1 \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = -1\right\}\right| \leqslant 4t+1.$$

This implies that

$$\left|\left\{n \colon 1 \leqslant n \leqslant p-2 \text{ and } \left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) = 1\right\}\right| \geqslant (p-2) - (4t+1) = p - 4t - 3,$$

and so

$$-1 = \sum_{n=1}^{p-1}\left(\frac{n}{p}\right)\left(\frac{n+1}{p}\right) \geqslant p - 4t - 3 - (4t+1) = p - 8t - 4,$$

which implies $8t \geqslant p-3$, and thus

$$\sum_{n=1}^{p-1} f(n^2, p) = 2t \geqslant \frac{p-3}{4}.$$

Comment 5. It is possible to prove that

$$\sum_{n=1}^{p-1} f(n^2, p) \geqslant \frac{p-1}{2}.$$

The case $p \equiv 1 \pmod 4$ was already mentioned, and it is the equality case. If $p \equiv 3 \pmod 4$, then, by a theorem of Dirichlet, we have

$$\left|\left\{r\colon 1 \leqslant r \leqslant \frac{p-1}{2} \text{ and } \left(\frac{r}{p}\right) = 1\right\}\right| > \frac{p-1}{4},$$

which implies the result.

See https://en.wikipedia.org/wiki/Quadratic_residue#Dirichlet.27s_formulas for the full statement of the theorem. It seems that no elementary proof of it is known; a proof using complex analysis is available, for instance, in Chapter 7 of the book *Quadratic Residues and Non-Residues: Selected Topics*, by Steve Wright, available in https://arxiv.org/abs/1408.0235.

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Shortlisted Problems

(with solutions)

59th International Mathematical Olympiad

Cluj-Napoca — Romania, 3–14 July 2018

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The Shortlist has to be kept strictly confidential until the conclusion of the following International Mathematical Olympiad.

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The Organising Committee and the Problem Selection Committee of IMO 2018 thank the following 49 countries for contributing 168 problem proposals:

Armenia, Australia, Austria, Azerbaijan, Belarus, Belgium, Bosnia and Herzegovina, Brazil, Bulgaria, Canada, China, Croatia, Cyprus, Czech Republic, Denmark, Estonia, Germany, Greece, Hong Kong, Iceland, India, Indonesia, Iran, Ireland, Israel, Japan, Kosovo, Luxembourg, Mexico, Moldova, Mongolia, Netherlands, Nicaragua, Poland, Russia, Serbia, Singapore, Slovakia, Slovenia, South Africa, South Korea, Switzerland, Taiwan, Tanzania, Thailand, Turkey, Ukraine, United Kingdom, U.S.A.

Problem Selection Committee

Calin Popescu, Radu Gologan, Marian Andronache, Mihail Baluna,
Nicolae Beli, Ilya Bogdanov, Pavel Kozhevnikov, Géza Kós, Sever Moldoveanu

Problems

Algebra

A1. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y)$$

for all $x, y \in \mathbb{Q}_{>0}$.

(Switzerland)

A2. Find all positive integers $n \geqslant 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$, $a_{n+1} = a_1$, $a_{n+2} = a_2$ such that

$$a_i a_{i+1} + 1 = a_{i+2}$$

for all $i = 1, 2, \ldots, n$.

(Slovakia)

A3. Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$;

(2) There exists a positive rational number $r < 1$ such that $\sum_{x \in F} 1/x \neq r$ for all finite subsets F of S.

(Luxembourg)

A4. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geqslant 2$ there exists $1 \leqslant k \leqslant n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal possible value of $a_{2018} - a_{2017}$.

(Belgium)

A5. Determine all functions $f : (0, \infty) \to \mathbb{R}$ satisfying

$$\left(x + \frac{1}{x}\right) f(y) = f(xy) + f\left(\frac{y}{x}\right)$$

for all $x, y > 0$.

(South Korea)

A6. Let $m, n \geqslant 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \ldots + x_n}{m} \right\rfloor \quad \text{for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

(Brazil)

A7. Find the maximal value of

$$S = \sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{d+7}} + \sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a + b + c + d = 100$.

(Taiwan)

Combinatorics

C1. Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

(Iceland)

C2. Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

(Armenia)

C3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n + 1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

(Netherlands)

C4. An *anti-Pascal pyramid* is a finite set of numbers, placed in a triangle-shaped array so that the first row of the array contains one number, the second row contains two numbers, the third row contains three numbers and so on; and, except for the numbers in the bottom row, each number equals the absolute value of the difference of the two numbers below it. For instance, the triangle below is an anti-Pascal pyramid with four rows, in which every integer from 1 to $1 + 2 + 3 + 4 = 10$ occurs exactly once:

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9\,. \end{array}$$

Is it possible to form an anti-Pascal pyramid with 2018 rows, using every integer from 1 to $1 + 2 + \cdots + 2018$ exactly once?

(Iran)

C5. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

(Russia)

C6. Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

> (i) If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.
>
> (ii) If no such pair exists, we write down two times the number 0.

Prove that, no matter how we make the choices in (i), operation (ii) will be performed only finitely many times.

(Serbia)

C7. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice — once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

(India)

Geometry

G1. Let ABC be an acute-angled triangle with circumcircle Γ. Let D and E be points on the segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of the segments BD and CE intersect the small arcs $\widehat{AB}$ and $\widehat{AC}$ at points F and G respectively. Prove that $DE \parallel FG$.

(Greece)

G2. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

(Australia)

G3. A circle ω of radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

(i) each triangle from T is inscribed in ω;

(ii) no two triangles from T have a common interior point.

Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

(South Africa)

G4. A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

(Mongolia)

G5. Let ABC be a triangle with circumcircle ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to ω.

(Denmark)

G6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. A point X is chosen inside the quadrilateral so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180^\circ$.

(Poland)

G7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, and ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

(Russia)

Number Theory

N1. Determine all pairs (n, k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(Ukraine)

N2. Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

(*i*) Each number in the table is congruent to 1 modulo n;

(*ii*) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the numbers in the j^{th} column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

(Indonesia)

N3. Define the sequence $a_0, a_1, a_2, \ldots$ by $a_n = 2^n + 2^{\lfloor n/2 \rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

(Serbia)

N4. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all $n \geqslant k$, where k is some positive integer. Prove that there exists a positive integer m such that $a_n = a_{n+1}$ for all $n \geqslant m$.

(Mongolia)

N5. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t.$$

Is it possible that both xy and zt are perfect squares?

(Russia)

N6. Let $f\colon \{1, 2, 3, \ldots\} \to \{2, 3, \ldots\}$ be a function such that $f(m+n) \mid f(m) + f(n)$ for all pairs m, n of positive integers. Prove that there exists a positive integer $c > 1$ which divides all values of f.

(Mexico)

N7. Let $n \geqslant 2018$ be an integer, and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

(Thailand)

Solutions

Algebra

A1. Let $\mathbb{Q}_{>0}$ denote the set of all positive rational numbers. Determine all functions $f\colon \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ satisfying

$$f\left(x^2 f(y)^2\right) = f(x)^2 f(y) \tag{$*$}$$

for all $x, y \in \mathbb{Q}_{>0}$.

(Switzerland)

Answer: $f(x) = 1$ for all $x \in \mathbb{Q}_{>0}$.

Solution. Take any $a, b \in \mathbb{Q}_{>0}$. By substituting $x = f(a)$, $y = b$ and $x = f(b)$, $y = a$ into $(*)$ we get

$$f\big(f(a)\big)^2 f(b) = f\big(f(a)^2 f(b)^2\big) = f\big(f(b)\big)^2 f(a),$$

which yields

$$\frac{f\big(f(a)\big)^2}{f(a)} = \frac{f\big(f(b)\big)^2}{f(b)} \qquad \text{for all } a, b \in \mathbb{Q}_{>0}.$$

In other words, this shows that there exists a constant $C \in \mathbb{Q}_{>0}$ such that $f\big(f(a)\big)^2 = Cf(a)$, or

$$\left(\frac{f\big(f(a)\big)}{C}\right)^2 = \frac{f(a)}{C} \qquad \text{for all } a \in \mathbb{Q}_{>0}. \tag{1}$$

Denote by $f^n(x) = \underbrace{f(f(\ldots(f}_{n}(x))\ldots))$ the n^{th} iteration of f. Equality (1) yields

$$\frac{f(a)}{C} = \left(\frac{f^2(a)}{C}\right)^2 = \left(\frac{f^3(a)}{C}\right)^4 = \cdots = \left(\frac{f^{n+1}(a)}{C}\right)^{2^n}$$

for all positive integer n. So, $f(a)/C$ is the 2^n-th power of a rational number for all positive integer n. This is impossible unless $f(a)/C = 1$, since otherwise the exponent of some prime in the prime decomposition of $f(a)/C$ is not divisible by sufficiently large powers of 2. Therefore, $f(a) = C$ for all $a \in \mathbb{Q}_{>0}$.

Finally, after substituting $f \equiv C$ into $(*)$ we get $C = C^3$, whence $C = 1$. So $f(x) \equiv 1$ is the unique function satisfying $(*)$.

Comment 1. There are several variations of the solution above. For instance, one may start with finding $f(1) = 1$. To do this, let $d = f(1)$. By substituting $x = y = 1$ and $x = d^2$, $y = 1$ into $(*)$ we get $f(d^2) = d^3$ and $f(d^6) = f(d^2)^2 \cdot d = d^7$. By substituting now $x = 1$, $y = d^2$ we obtain $f(d^6) = d^2 \cdot d^3 = d^5$. Therefore, $d^7 = f(d^6) = d^5$, whence $d = 1$.

After that, the rest of the solution simplifies a bit, since we already know that $C = \frac{f(f(1))^2}{f(1)} = 1$. Hence equation (1) becomes merely $f(f(a))^2 = f(a)$, which yields $f(a) = 1$ in a similar manner.

Comment 2. There exist nonconstant functions $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $(*)$ for all real $x, y > 0$ — e.g., $f(x) = \sqrt{x}$.

A2. Find all positive integers $n \geqslant 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n$, $a_{n+1} = a_1$, $a_{n+2} = a_2$ such that
$$a_i a_{i+1} + 1 = a_{i+2}$$
for all $i = 1, 2, \ldots, n$.

(Slovakia)

Answer: n can be any multiple of 3.

Solution 1. For the sake of convenience, extend the sequence $a_1, \ldots, a_{n+2}$ to an infinite periodic sequence with period n. (n is not necessarily the shortest period.)

If n is divisible by 3, then $(a_1, a_2, \ldots) = (-1, -1, 2, -1, -1, 2, \ldots)$ is an obvious solution.

We will show that in every periodic sequence satisfying the recurrence, each positive term is followed by two negative values, and after them the next number is positive again. From this, it follows that n is divisible by 3.

If the sequence contains two consecutive positive numbers a_i, a_{i+1}, then $a_{i+2} = a_i a_{i+1} + 1 > 1$, so the next value is positive as well; by induction, all numbers are positive and greater than 1. But then $a_{i+2} = a_i a_{i+1} + 1 \geqslant 1 \cdot a_{i+1} + 1 > a_{i+1}$ for every index i, which is impossible: our sequence is periodic, so it cannot increase everywhere.

If the number 0 occurs in the sequence, $a_i = 0$ for some index i, then it follows that $a_{i+1} = a_{i-1}a_i + 1$ and $a_{i+2} = a_i a_{i+1} + 1$ are two consecutive positive elements in the sequences and we get the same contradiction again.

Notice that after any two consecutive negative numbers the next one must be positive: if $a_i < 0$ and $a_{i+1} < 0$, then $a_{i+2} = a_1 a_{i+1} + 1 > 1 > 0$. Hence, the positive and negative numbers follow each other in such a way that each positive term is followed by one or two negative values and then comes the next positive term.

Consider the case when the positive and negative values alternate. So, if a_i is a negative value then a_{i+1} is positive, a_{i+2} is negative and a_{i+3} is positive again.

Notice that $a_i a_{i+1} + 1 = a_{i+2} < 0 < a_{i+3} = a_{i+1}a_{i+2} + 1$; by $a_{i+1} > 0$ we conclude $a_i < a_{i+2}$. Hence, the negative values form an infinite increasing subsequence, $a_i < a_{i+2} < a_{i+4} < \ldots$, which is not possible, because the sequence is periodic.

The only case left is when there are consecutive negative numbers in the sequence. Suppose that a_i and a_{i+1} are negative; then $a_{i+2} = a_i a_{i+1} + 1 > 1$. The number a_{i+3} must be negative. We show that a_{i+4} also must be negative.

Notice that a_{i+3} is negative and $a_{i+4} = a_{i+2}a_{i+3} + 1 < 1 < a_i a_{i+1} + 1 = a_{i+2}$, so
$$a_{i+5} - a_{i+4} = (a_{i+3}a_{i+4} + 1) - (a_{i+2}a_{i+3} + 1) = a_{i+3}(a_{i+4} - a_{i+2}) > 0,$$
therefore $a_{i+5} > a_{i+4}$. Since at most one of a_{i+4} and a_{i+5} can be positive, that means that a_{i+4} must be negative.

Now a_{i+3} and a_{i+4} are negative and a_{i+5} is positive; so after two negative and a positive terms, the next three terms repeat the same pattern. That completes the solution.

Solution 2. We prove that the shortest period of the sequence must be 3. Then it follows that n must be divisible by 3.

Notice that the equation $x^2 + 1 = x$ has no real root, so the numbers $a_1, \ldots, a_n$ cannot be all equal, hence the shortest period of the sequence cannot be 1.

By applying the recurrence relation for i and $i + 1$,
$$(a_{i+2} - 1)a_{i+2} = a_i a_{i+1} a_{i+2} = a_i(a_{i+3} - 1), \quad \text{so}$$
$$a_{i+2}^2 - a_i a_{i+3} = a_{i+2} - a_i.$$

By summing over $i = 1, 2, \ldots, n$, we get

$$\sum_{i=1}^{n} (a_i - a_{i+3})^2 = 0.$$

That proves that $a_i = a_{i+3}$ for every index i, so the sequence $a_1, a_2, \ldots$ is indeed periodic with period 3. The shortest period cannot be 1, so it must be 3; therefore, n is divisible by 3.

Comment. By solving the system of equations $ab + 1 = c, \quad bc + 1 = a, \quad ca + 1 = b$, it can be seen that the pattern $(-1, -1, 2)$ is repeated in all sequences satisfying the problem conditions.

A3. Given any set S of positive integers, show that at least one of the following two assertions holds:

(1) There exist distinct finite subsets F and G of S such that $\sum_{x\in F} 1/x = \sum_{x\in G} 1/x$;

(2) There exists a positive rational number $r<1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

(Luxembourg)

Solution 1. Argue indirectly. Agree, as usual, that the empty sum is 0 to consider rationals in $[0,1)$; adjoining 0 causes no harm, since $\sum_{x\in F} 1/x = 0$ for no nonempty finite subset F of S. For every rational r in $[0,1)$, let F_r be the unique finite subset of S such that $\sum_{x\in F_r} 1/x = r$. The argument hinges on the lemma below.

Lemma. If x is a member of S and q and r are rationals in $[0,1)$ such that $q-r=1/x$, then x is a member of F_q if and only if it is not one of F_r.

Proof. If x is a member of F_q, then

$$\sum_{y\in F_q\smallsetminus\{x\}} \frac{1}{y} = \sum_{y\in F_q} \frac{1}{y} - \frac{1}{x} = q - \frac{1}{x} = r = \sum_{y\in F_r} \frac{1}{y},$$

so $F_r = F_q \smallsetminus \{x\}$, and x is not a member of F_r. Conversely, if x is not a member of F_r, then

$$\sum_{y\in F_r\cup\{x\}} \frac{1}{y} = \sum_{y\in F_r} \frac{1}{y} + \frac{1}{x} = r + \frac{1}{x} = q = \sum_{y\in F_q} \frac{1}{y},$$

so $F_q = F_r \cup \{x\}$, and x is a member of F_q. □

Consider now an element x of S and a positive rational $r<1$. Let $n=\lfloor rx\rfloor$ and consider the sets $F_{r-k/x}$, $k=0,\ldots,n$. Since $0\leqslant r-n/x<1/x$, the set $F_{r-n/x}$ does not contain x, and a repeated application of the lemma shows that the $F_{r-(n-2k)/x}$ do not contain x, whereas the $F_{r-(n-2k-1)/x}$ do. Consequently, x is a member of F_r if and only if n is odd.

Finally, consider $F_{2/3}$. By the preceding, $\lfloor 2x/3\rfloor$ is odd for each x in $F_{2/3}$, so $2x/3$ is not integral. Since $F_{2/3}$ is finite, there exists a positive rational ε such that $\lfloor (2/3-\varepsilon)x\rfloor = \lfloor 2x/3\rfloor$ for all x in $F_{2/3}$. This implies that $F_{2/3}$ is a subset of $F_{2/3-\varepsilon}$ which is impossible.

Comment. The solution above can be adapted to show that the problem statement still holds, if the condition $r<1$ in **(2)** is replaced with $r<\delta$, for an arbitrary positive δ. This yields that, if S does not satisfy **(1)**, then there exist *infinitely many* positive rational numbers $r<1$ such that $\sum_{x\in F} 1/x \neq r$ for all finite subsets F of S.

Solution 2. A finite S clearly satisfies **(2)**, so let S be infinite. If S fails both conditions, so does $S\smallsetminus\{1\}$. We may and will therefore assume that S consists of integers greater than 1. Label the elements of S increasingly $x_1<x_2<\cdots$, where $x_1\geqslant 2$.

We first show that S satisfies **(2)** if $x_{n+1}\geqslant 2x_n$ for all n. In this case, $x_n\geqslant 2^{n-1}x_1$ for all n, so

$$s = \sum_{n\geqslant 1} \frac{1}{x_n} \leqslant \sum_{n\geqslant 1} \frac{1}{2^{n-1}x_1} = \frac{2}{x_1}.$$

If $x_1\geqslant 3$, or $x_1=2$ and $x_{n+1}>2x_n$ for some n, then $\sum_{x\in F} 1/x < s < 1$ for every finite subset F of S, so S satisfies **(2)**; and if $x_1=2$ and $x_{n+1}=2x_n$ for all n, that is, $x_n=2^n$ for all n, then every finite subset F of S consists of powers of 2, so $\sum_{x\in F} 1/x \neq 1/3$ and again S satisfies **(2)**.

Finally, we deal with the case where $x_{n+1}<2x_n$ for some n. Consider the positive rational $r=1/x_n-1/x_{n+1}<1/x_{n+1}$. If $r=\sum_{x\in F} 1/x$ for no finite subset F of S, then S satisfies **(2)**.

We now assume that $r = \sum_{x \in F_0} 1/x$ for some finite subset F_0 of S, and show that S satisfies **(1)**. Since $\sum_{x \in F_0} 1/x = r < 1/x_{n+1}$, it follows that x_{n+1} is not a member of F_0, so

$$\sum_{x \in F_0 \cup \{x_{n+1}\}} \frac{1}{x} = \sum_{x \in F_0} \frac{1}{x} + \frac{1}{x_{n+1}} = r + \frac{1}{x_{n+1}} = \frac{1}{x_n}.$$

Consequently, $F = F_0 \cup \{x_{n+1}\}$ and $G = \{x_n\}$ are distinct finite subsets of S such that $\sum_{x \in F} 1/x = \sum_{x \in G} 1/x$, and S satisfies **(1)**.

A4. Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers such that $a_0 = 0$, $a_1 = 1$, and for every $n \geqslant 2$ there exists $1 \leqslant k \leqslant n$ satisfying

$$a_n = \frac{a_{n-1} + \cdots + a_{n-k}}{k}.$$

Find the maximal possible value of $a_{2018} - a_{2017}$.

(Belgium)

Answer: The maximal value is $\frac{2016}{2017^2}$.

Solution 1. The claimed maximal value is achieved at

$$a_1 = a_2 = \cdots = a_{2016} = 1, \quad a_{2017} = \frac{a_{2016} + \cdots + a_0}{2017} = 1 - \frac{1}{2017},$$
$$a_{2018} = \frac{a_{2017} + \cdots + a_1}{2017} = 1 - \frac{1}{2017^2}.$$

Now we need to show that this value is optimal. For brevity, we use the notation

$$S(n,k) = a_{n-1} + a_{n-2} + \cdots + a_{n-k} \quad \text{for nonnegative integers } k \leqslant n.$$

In particular, $S(n,0) = 0$ and $S(n,1) = a_{n-1}$. In these terms, for every integer $n \geqslant 2$ there exists a positive integer $k \leqslant n$ such that $a_n = S(n,k)/k$.

For every integer $n \geqslant 1$ we define

$$M_n = \max_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \qquad m_n = \min_{1 \leqslant k \leqslant n} \frac{S(n,k)}{k}, \quad \text{and} \quad \Delta_n = M_n - m_n \geqslant 0.$$

By definition, $a_n \in [m_n, M_n]$ for all $n \geqslant 2$; on the other hand, $a_{n-1} = S(n,1)/1 \in [m_n, M_n]$. Therefore,

$$a_{2018} - a_{2017} \leqslant M_{2018} - m_{2018} = \Delta_{2018},$$

and we are interested in an upper bound for Δ_{2018}.

Also by definition, for any $0 < k \leqslant n$ we have $km_n \leqslant S(n,k) \leqslant kM_n$; notice that these inequalities are also valid for $k = 0$.

Claim 1. For every $n > 2$, we have $\Delta_n \leqslant \frac{n-1}{n}\Delta_{n-1}$.

Proof. Choose positive integers $k, \ell \leqslant n$ such that $M_n = S(n,k)/k$ and $m_n = S(n,\ell)/\ell$. We have $S(n,k) = a_{n-1} + S(n-1,k-1)$, so

$$k(M_n - a_{n-1}) = S(n,k) - ka_{n-1} = S(n-1,k-1) - (k-1)a_{n-1} \leqslant (k-1)(M_{n-1} - a_{n-1}),$$

since $S(n-1,k-1) \leqslant (k-1)M_{n-1}$. Similarly, we get

$$\ell(a_{n-1} - m_n) = (\ell-1)a_{n-1} - S(n-1,\ell-1) \leqslant (\ell-1)(a_{n-1} - m_{n-1}).$$

Since $m_{n-1} \leqslant a_{n-1} \leqslant M_{n-1}$ and $k, \ell \leqslant n$, the obtained inequalities yield

$$M_n - a_{n-1} \leqslant \frac{k-1}{k}(M_{n-1} - a_{n-1}) \leqslant \frac{n-1}{n}(M_{n-1} - a_{n-1}) \quad \text{and}$$
$$a_{n-1} - m_n \leqslant \frac{\ell-1}{\ell}(a_{n-1} - m_{n-1}) \leqslant \frac{n-1}{n}(a_{n-1} - m_{n-1}).$$

Therefore,

$$\Delta_n = (M_n - a_{n-1}) + (a_{n-1} - m_n) \leqslant \frac{n-1}{n}\big((M_{n-1} - a_{n-1}) + (a_{n-1} - m_{n-1})\big) = \frac{n-1}{n}\Delta_{n-1}. \qquad \square$$

Back to the problem, if $a_n = 1$ for all $n \leqslant 2017$, then $a_{2018} \leqslant 1$ and hence $a_{2018} - a_{2017} \leqslant 0$. Otherwise, let $2 \leqslant q \leqslant 2017$ be the minimal index with $a_q < 1$. We have $S(q, i) = i$ for all $i = 1, 2, \ldots, q-1$, while $S(q, q) = q - 1$. Therefore, $a_q < 1$ yields $a_q = S(q, q)/q = 1 - \frac{1}{q}$.

Now we have $S(q+1, i) = i - \frac{1}{q}$ for $i = 1, 2, \ldots, q$, and $S(q+1, q+1) = q - \frac{1}{q}$. This gives us

$$m_{q+1} = \frac{S(q+1,1)}{1} = \frac{S(q+1,q+1)}{q+1} = \frac{q-1}{q} \quad \text{and} \quad M_{q+1} = \frac{S(q+1,q)}{q} = \frac{q^2-1}{q^2},$$

so $\Delta_{q+1} = M_{q+1} - m_{q+1} = (q-1)/q^2$. Denoting $N = 2017 \geqslant q$ and using Claim 1 for $n = q+2, q+3, \ldots, N+1$ we finally obtain

$$\Delta_{N+1} \leqslant \frac{q-1}{q^2} \cdot \frac{q+1}{q+2} \cdot \frac{q+2}{q+3} \cdots \frac{N}{N+1} = \frac{1}{N+1}\left(1 - \frac{1}{q^2}\right) \leqslant \frac{1}{N+1}\left(1 - \frac{1}{N^2}\right) = \frac{N-1}{N^2},$$

as required.

Comment 1. One may check that the maximal value of $a_{2018} - a_{2017}$ is attained at the unique sequence, which is presented in the solution above.

Comment 2. An easier question would be to determine the maximal value of $|a_{2018} - a_{2017}|$. In this version, the answer $\frac{1}{2018}$ is achieved at

$$a_1 = a_2 = \cdots = a_{2017} = 1, \quad a_{2018} = \frac{a_{2017} + \cdots + a_0}{2018} = 1 - \frac{1}{2018}.$$

To prove that this value is optimal, it suffices to notice that $\Delta_2 = \frac{1}{2}$ and to apply Claim 1 obtaining

$$|a_{2018} - a_{2017}| \leqslant \Delta_{2018} \leqslant \frac{1}{2} \cdot \frac{2}{3} \cdots \frac{2017}{2018} = \frac{1}{2018}.$$

Solution 2. We present a different proof of the estimate $a_{2018} - a_{2017} \leqslant \frac{2016}{2017^2}$. We keep the same notations of $S(n, k)$, m_n and M_n from the previous solution.

Notice that $S(n, n) = S(n, n-1)$, as $a_0 = 0$. Also notice that for $0 \leqslant k \leqslant \ell \leqslant n$ we have $S(n, \ell) = S(n, k) + S(n-k, \ell-k)$.

Claim 2. For every positive integer n, we have $m_n \leqslant m_{n+1}$ and $M_{n+1} \leqslant M_n$, so the segment $[m_{n+1}, M_{n+1}]$ is contained in $[m_n, M_n]$.

Proof. Choose a positive integer $k \leqslant n+1$ such that $m_{n+1} = S(n+1, k)/k$. Then we have

$$km_{n+1} = S(n+1, k) = a_n + S(n, k-1) \geqslant m_n + (k-1)m_n = km_n,$$

which establishes the first inequality in the Claim. The proof of the second inequality is similar. $\square$

Claim 3. For every positive integers $k \geqslant n$, we have $m_n \leqslant a_k \leqslant M_n$.

Proof. By Claim 2, we have $[m_k, M_k] \subseteq [m_{k-1}, M_{k-1}] \subseteq \cdots \subseteq [m_n, M_n]$. Since $a_k \in [m_k, M_k]$, the claim follows. $\square$

Claim 4. For every integer $n \geqslant 2$, we have $M_n = S(n, n-1)/(n-1)$ and $m_n = S(n, n)/n$.

Proof. We use induction on n. The base case $n = 2$ is routine. To perform the induction step, we need to prove the inequalities

$$\frac{S(n,n)}{n} \leqslant \frac{S(n,k)}{k} \quad \text{and} \quad \frac{S(n,k)}{k} \leqslant \frac{S(n,n-1)}{n-1} \tag{1}$$

for every positive integer $k \leqslant n$. Clearly, these inequalities hold for $k = n$ and $k = n-1$, as $S(n,n) = S(n,n-1) > 0$. In the sequel, we assume that $k < n-1$.

Now the first inequality in (1) rewrites as $nS(n,k) \geqslant kS(n,n) = k\big(S(n,k)+S(n-k,n-k)\big)$, or, cancelling the terms occurring on both parts, as

$$(n-k)S(n,k) \geqslant kS(n-k,n-k) \iff S(n,k) \geqslant k \cdot \frac{S(n-k,n-k)}{n-k}.$$

By the induction hypothesis, we have $S(n-k,n-k)/(n-k) = m_{n-k}$. By Claim 3, we get $a_{n-i} \geqslant m_{n-k}$ for all $i = 1, 2, \ldots, k$. Summing these k inequalities we obtain

$$S(n,k) \geqslant km_{n-k} = k \cdot \frac{S(n-k,n-k)}{n-k},$$

as required.

The second inequality in (1) is proved similarly. Indeed, this inequality is equivalent to

$$\begin{aligned}(n-1)S(n,k) \leqslant kS(n,n-1) &\iff (n-k-1)S(n,k) \leqslant kS(n-k,n-k-1)\\ &\iff S(n,k) \leqslant k \cdot \frac{S(n-k,n-k-1)}{n-k-1} = kM_{n-k};\end{aligned}$$

the last inequality follows again from Claim 3, as each term in $S(n,k)$ is at most M_{n-k}. □

Now we can prove the required estimate for $a_{2018} - a_{2017}$. Set $N = 2017$. By Claim 4,

$$\begin{aligned}a_{N+1} - a_N \leqslant M_{N+1} - a_N = \frac{S(N+1,N)}{N} - a_N &= \frac{a_N + S(N,N-1)}{N} - a_N\\ &= \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot a_N.\end{aligned}$$

On the other hand, the same Claim yields

$$a_N \geqslant m_N = \frac{S(N,N)}{N} = \frac{S(N,N-1)}{N}.$$

Noticing that each term in $S(N,N-1)$ is at most 1, so $S(N,N-1) \leqslant N-1$, we finally obtain

$$a_{N+1} - a_N \leqslant \frac{S(N,N-1)}{N} - \frac{N-1}{N} \cdot \frac{S(N,N-1)}{N} = \frac{S(N,N-1)}{N^2} \leqslant \frac{N-1}{N^2}.$$

Comment 1. Claim 1 in Solution 1 can be deduced from Claims 2 and 4 in Solution 2.

By Claim 4 we have $M_n = \frac{S(n,n-1)}{n-1}$ and $m_n = \frac{S(n,n)}{n} = \frac{S(n,n-1)}{n}$. It follows that $\Delta_n = M_n - m_n = \frac{S(n,n-1)}{(n-1)n}$ and so $M_n = n\Delta_n$ and $m_n = (n-1)\Delta_n$

Similarly, $M_{n-1} = (n-1)\Delta_{n-1}$ and $m_{n-1} = (n-2)\Delta_{n-1}$. Then the inequalities $m_{n-1} \leqslant m_n$ and $M_n \leqslant M_{n-1}$ from Claim 2 write as $(n-2)\Delta_{n-1} \leqslant (n-1)\Delta_n$ and $n\Delta_n \leqslant (n-1)\Delta_{n-1}$. Hence we have the double inequality

$$\frac{n-2}{n-1}\Delta_{n-1} \leqslant \Delta_n \leqslant \frac{n-1}{n}\Delta_{n-1}.$$

Comment 2. Both solutions above discuss the properties of an arbitrary sequence satisfying the problem conditions. Instead, one may investigate only an *optimal* sequence which maximises the value of $a_{2018} - a_{2017}$. Here we present an observation which allows to simplify such investigation — for instance, the proofs of Claim 1 in Solution 1 and Claim 4 in Solution 2.

The sequence (a_n) is uniquely determined by choosing, for every $n \geqslant 2$, a positive integer $k(n) \leqslant n$ such that $a_n = S(n, k(n))/k(n)$. Take an arbitrary $2 \leqslant n_0 \leqslant 2018$, and assume that all such integers $k(n)$, for $n \neq n_0$, are fixed. Then, for every n, the value of a_n is a linear function in a_{n_0} (whose possible values constitute some discrete subset of $[m_{n_0}, M_{n_0}]$ containing both endpoints). Hence, $a_{2018} - a_{2017}$ is also a linear function in a_{n_0}, so it attains its maximal value at one of the endpoints of the segment $[m_{n_0}, M_{n_0}]$.

This shows that, while dealing with an optimal sequence, we may assume $a_n \in \{m_n, M_n\}$ for all $2 \leqslant n \leqslant 2018$. Now one can easily see that, if $a_n = m_n$, then $m_{n+1} = m_n$ and $M_{n+1} \leqslant \frac{m_n + nM_n}{n+1}$; similar estimates hold in the case $a_n = M_n$. This already establishes Claim 1, and simplifies the inductive proof of Claim 4, both applied to an optimal sequence.

A5. Determine all functions $f:(0,\infty)\to\mathbb{R}$ satisfying

$$\left(x+\frac{1}{x}\right)f(y)=f(xy)+f\left(\frac{y}{x}\right) \tag{1}$$

for all $x,y>0$.

(South Korea)

Answer: $f(x)=C_1x+\dfrac{C_2}{x}$ with arbitrary constants C_1 and C_2.

Solution 1. Fix a real number $a>1$, and take a new variable t. For the values $f(t)$, $f(t^2)$, $f(at)$ and $f(a^2t^2)$, the relation (1) provides a system of linear equations:

$$\begin{array}{lrcll}
x=y=t: & \left(t+\frac{1}{t}\right)f(t) & = & f(t^2)+f(1) & \text{(2a)}\\
x=\frac{t}{a},\ y=at: & \left(\frac{t}{a}+\frac{a}{t}\right)f(at) & = & f(t^2)+f(a^2) & \text{(2b)}\\
x=a^2t,\ y=t: & \left(a^2t+\frac{1}{a^2t}\right)f(t) & = & f(a^2t^2)+f\left(\frac{1}{a^2}\right) & \text{(2c)}\\
x=y=at: & \left(at+\frac{1}{at}\right)f(at) & = & f(a^2t^2)+f(1) & \text{(2d)}
\end{array}$$

In order to eliminate $f(t^2)$, take the difference of (2a) and (2b); from (2c) and (2d) eliminate $f(a^2t^2)$; then by taking a linear combination, eliminate $f(at)$ as well:

$$\left(t+\frac{1}{t}\right)f(t)-\left(\frac{t}{a}+\frac{a}{t}\right)f(at)=f(1)-f(a^2)\quad\text{and}$$

$$\left(a^2t+\frac{1}{a^2t}\right)f(t)-\left(at+\frac{1}{at}\right)f(at)=f(1/a^2)-f(1),\quad\text{so}$$

$$\left(\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)\right)f(t)$$
$$=\left(at+\frac{1}{at}\right)\left(f(1)-f(a^2)\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(f(1/a^2)-f(1)\right).$$

Notice that on the left-hand side, the coefficient of $f(t)$ is nonzero and does not depend on t:

$$\left(at+\frac{1}{at}\right)\left(t+\frac{1}{t}\right)-\left(\frac{t}{a}+\frac{a}{t}\right)\left(a^2t+\frac{1}{a^2t}\right)=a+\frac{1}{a}-\left(a^3+\frac{1}{a^3}\right)<0.$$

After dividing by this fixed number, we get

$$f(t)=C_1t+\frac{C_2}{t} \tag{3}$$

where the numbers C_1 and C_2 are expressed in terms of a, $f(1)$, $f(a^2)$ and $f(1/a^2)$, and they do not depend on t.

The functions of the form (3) satisfy the equation:

$$\left(x+\frac{1}{x}\right)f(y)=\left(x+\frac{1}{x}\right)\left(C_1y+\frac{C_2}{y}\right)=\left(C_1xy+\frac{C_2}{xy}\right)+\left(C_1\frac{y}{x}+C_2\frac{x}{y}\right)=f(xy)+f\left(\frac{y}{x}\right).$$

Solution 2. We start with an observation. If we substitute $x = a \neq 1$ and $y = a^n$ in (1), we obtain

$$f(a^{n+1}) - \left(a + \frac{1}{a}\right) f(a^n) + f(a^{n-1}) = 0.$$

For the sequence $z_n = a^n$, this is a homogeneous linear recurrence of the second order, and its characteristic polynomial is $t^2 - \left(a + \frac{1}{a}\right) t + 1 = (t-a)(t-\frac{1}{a})$ with two distinct nonzero roots, namely a and $1/a$. As is well-known, the general solution is $z_n = C_1 a^n + C_2 (1/a)^n$ where the index n can be as well positive as negative. Of course, the numbers C_1 and C_2 may depend of the choice of a, so in fact we have two functions, C_1 and C_2, such that

$$f(a^n) = C_1(a) \cdot a^n + \frac{C_2(a)}{a^n} \quad \text{for every } a \neq 1 \text{ and every integer } n. \tag{4}$$

The relation (4) can be easily extended to rational values of n, so we may conjecture that C_1 and C_2 are constants, and whence $f(t) = C_1 t + \frac{C_2}{t}$. As it was seen in the previous solution, such functions indeed satisfy (1).

The equation (1) is linear in f; so if some functions f_1 and f_2 satisfy (1) and c_1, c_2 are real numbers, then $c_1 f_1(x) + c_2 f_2(x)$ is also a solution of (1). In order to make our formulas simpler, define

$$f_0(x) = f(x) - f(1) \cdot x.$$

This function is another one satisfying (1) and the extra constraint $f_0(1) = 0$. Repeating the same argument on linear recurrences, we can write $f_0(a) = K(a)a^n + \frac{L(a)}{a^n}$ with some functions K and L. By substituting $n = 0$, we can see that $K(a) + L(a) = f_0(1) = 0$ for every a. Hence,

$$f_0(a^n) = K(a)\left(a^n - \frac{1}{a^n}\right).$$

Now take two numbers $a > b > 1$ arbitrarily and substitute $x = (a/b)^n$ and $y = (ab)^n$ in (1):

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) f_0\big((ab)^n\big) = f_0\big(a^{2n}\big) + f_0\big(b^{2n}\big), \quad \text{so}$$

$$\left(\frac{a^n}{b^n} + \frac{b^n}{a^n}\right) K(ab)\left((ab)^n - \frac{1}{(ab)^n}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right), \quad \text{or equivalently}$$

$$K(ab)\left(a^{2n} - \frac{1}{a^{2n}} + b^{2n} - \frac{1}{b^{2n}}\right) = K(a)\left(a^{2n} - \frac{1}{a^{2n}}\right) + K(b)\left(b^{2n} - \frac{1}{b^{2n}}\right). \tag{5}$$

By dividing (5) by a^{2n} and then taking limit with $n \to +\infty$ we get $K(ab) = K(a)$. Then (5) reduces to $K(a) = K(b)$. Hence, $K(a) = K(b)$ for all $a > b > 1$.

Fix $a > 1$. For every $x > 0$ there is some b and an integer n such that $1 < b < a$ and $x = b^n$. Then

$$f_0(x) = f_0(b^n) = K(b)\left(b^n - \frac{1}{b^n}\right) = K(a)\left(x - \frac{1}{x}\right).$$

Hence, we have $f(x) = f_0(x) + f(1)x = C_1 x + \frac{C_2}{x}$ with $C_1 = K(a) + f(1)$ and $C_2 = -K(a)$.

Comment. After establishing (5), there are several variants of finishing the solution. For example, instead of taking a limit, we can obtain a system of linear equations for $K(a)$, $K(b)$ and $K(ab)$ by substituting two positive integers n in (5), say $n = 1$ and $n = 2$. This approach leads to a similar ending as in the first solution.

Optionally, we define another function $f_1(x) = f_0(x) - C\left(x - \frac{1}{x}\right)$ and prescribe $K(c) = 0$ for another fixed c. Then we can choose $ab = c$ and decrease the number of terms in (5).

A6. Let $m, n \geqslant 2$ be integers. Let $f(x_1, \ldots, x_n)$ be a polynomial with real coefficients such that

$$f(x_1, \ldots, x_n) = \left\lfloor \frac{x_1 + \ldots + x_n}{m} \right\rfloor \quad \text{for every } x_1, \ldots, x_n \in \{0, 1, \ldots, m-1\}.$$

Prove that the total degree of f is at least n.

(Brazil)

Solution. We transform the problem to a single variable question by the following

Lemma. Let $a_1, \ldots, a_n$ be nonnegative integers and let $G(x)$ be a nonzero polynomial with $\deg G \leqslant a_1 + \ldots + a_n$. Suppose that some polynomial $F(x_1, \ldots, x_n)$ satisfies

$$F(x_1, \ldots, x_n) = G(x_1 + \ldots + x_n) \quad \text{for } (x_1, \ldots, x_n) \in \{0, 1, \ldots, a_1\} \times \ldots \times \{0, 1, \ldots, a_n\}.$$

Then F cannot be the zero polynomial, and $\deg F \geqslant \deg G$.

For proving the lemma, we will use *forward differences* of polynomials. If $p(x)$ is a polynomial with a single variable, then define $(\Delta p)(x) = p(x+1) - p(x)$. It is well-known that if p is a nonconstant polynomial then $\deg \Delta p = \deg p - 1$.

If $p(x_1, \ldots, x_n)$ is a polynomial with n variables and $1 \leqslant k \leqslant n$ then let

$$(\Delta_k p)(x_1, \ldots, x_n) = p(x_1, \ldots, x_{k-1}, x_k + 1, x_{k+1}, \ldots, x_n) - p(x_1, \ldots, x_n).$$

It is also well-known that either $\Delta_k p$ is the zero polynomial or $\deg(\Delta_k p) \leqslant \deg p - 1$.

Proof of the lemma. We apply induction on the degree of G. If G is a constant polynomial then we have $F(0, \ldots, 0) = G(0) \neq 0$, so F cannot be the zero polynomial.

Suppose that $\deg G \geqslant 1$ and the lemma holds true for lower degrees. Since $a_1 + \ldots + a_n \geqslant \deg G > 0$, at least one of $a_1, \ldots, a_n$ is positive; without loss of generality suppose $a_1 \geqslant 1$.

Consider the polynomials $F_1 = \Delta_1 F$ and $G_1 = \Delta G$. On the grid $\{0, \ldots, a_1 - 1\} \times \{0, \ldots, a_2\} \times \ldots \times \{0, \ldots, a_n\}$ we have

$$\begin{aligned} F_1(x_1, \ldots, x_n) &= F(x_1 + 1, x_2, \ldots, x_n) - F(x_1, x_2, \ldots, x_n) = \\ &= G(x_1 + \ldots + x_n + 1) - G(x_1 + \ldots + x_n) = G_1(x_1 + \ldots + x_n). \end{aligned}$$

Since G is nonconstant, we have $\deg G_1 = \deg G - 1 \leqslant (a_1 - 1) + a_2 + \ldots + a_n$. Therefore we can apply the induction hypothesis to F_1 and G_1 and conclude that F_1 is not the zero polynomial and $\deg F_1 \geqslant \deg G_1$. Hence, $\deg F \geqslant \deg F_1 + 1 \geqslant \deg G_1 + 1 = \deg G$. That finishes the proof. $\square$

To prove the problem statement, take the unique polynomial $g(x)$ so that $g(x) = \left\lfloor \frac{x}{m} \right\rfloor$ for $x \in \{0, 1, \ldots, n(m-1)\}$ and $\deg g \leqslant n(m-1)$. Notice that precisely $n(m-1)+1$ values of g are prescribed, so $g(x)$ indeed exists and is unique. Notice further that the constraints $g(0) = g(1) = 0$ and $g(m) = 1$ together enforce $\deg g \geqslant 2$.

By applying the lemma to $a_1 = \ldots = a_n = m - 1$ and the polynomials f and g, we achieve $\deg f \geqslant \deg g$. Hence we just need a suitable lower bound on $\deg g$.

Consider the polynomial $h(x) = g(x+m) - g(x) - 1$. The degree of $g(x+m) - g(x)$ is $\deg g - 1 \geqslant 1$, so $\deg h = \deg g - 1 \geqslant 1$, and therefore h cannot be the zero polynomial. On the other hand, h vanishes at the points $0, 1, \ldots, n(m-1) - m$, so h has at least $(n-1)(m-1)$ roots. Hence,

$$\deg f \geqslant \deg g = \deg h + 1 \geqslant (n-1)(m-1) + 1 \geqslant n.$$

Comment 1. In the lemma we have equality for the choice $F(x_1,\dots,x_n)=G(x_1+\dots+x_n)$, so it indeed transforms the problem to an equivalent single-variable question.

Comment 2. If $m\geqslant 3$, the polynomial $h(x)$ can be replaced by Δg. Notice that

$$(\Delta g)(x)=\begin{cases}1 & \text{if } x\equiv -1 \pmod m\\ 0 & \text{otherwise}\end{cases}\qquad \text{for } x=0,1,\dots,n(m-1)-1.$$

Hence, Δg vanishes at all integers x with $0\leqslant x<n(m-1)$ and $x\not\equiv -1 \pmod m$. This leads to $\deg g\geqslant \frac{(m-1)^2n}{m}+1$.

If m is even then this lower bound can be improved to $n(m-1)$. For $0\leqslant N<n(m-1)$, the $(N+1)^{\text{st}}$ forward difference at $x=0$ is

$$(\Delta^{N+1})g(0)=\sum_{k=0}^{N}(-1)^{N-k}\binom{N}{k}(\Delta g)(k)=\sum_{\substack{0\leqslant k\leqslant N\\ k\equiv -1 \pmod m}}(-1)^{N-k}\binom{N}{k}. \qquad (*)$$

Since m is even, all signs in the last sum are equal; with $N=n(m-1)-1$ this proves $\Delta^{n(m-1)}g(0)\neq 0$, indicating that $\deg g\geqslant n(m-1)$.

However, there are infinitely many cases when all terms in $(*)$ cancel out, for example if m is an odd divisor of $n+1$. In such cases, $\deg f$ can be less than $n(m-1)$.

Comment 3. The lemma is closely related to the so-called

> **Alon–Füredi bound.** *Let $S_1,\dots,S_n$ be nonempty finite sets in a field and suppose that the polynomial $P(x_1,\dots,x_n)$ vanishes at the points of the grid $S_1\times\dots\times S_n$, except for a single point. Then $\deg P\geqslant \sum_{i=1}^{n}\big(|S_i|-1\big)$.*
>
> (A well-known application of the Alon–Füredi bound was the former IMO problem 2007/6. Since then, this result became popular among the students and is part of the IMO training for many IMO teams.)

The proof of the lemma can be replaced by an application of the Alon–Füredi bound as follows. Let $d=\deg G$, and let G_0 be the unique polynomial such that $G_0(x)=G(x)$ for $x\in\{0,1,\dots,d-1\}$ but $\deg G_0<d$. The polynomials G_0 and G are different because they have different degrees, and they attain the same values at $0,1,\dots,d-1$; that enforces $G_0(d)\neq G(d)$.

Choose some nonnegative integers $b_1,\dots,b_n$ so that $b_1\leqslant a_1$, $\dots$, $b_n\leqslant a_n$, and $b_1+\dots+b_n=d$, and consider the polynomial

$$H(x_1,\dots,x_n)=F(x_1,\dots,x_n)-G_0(x_1+\dots+x_n)$$

on the grid $\{0,1,\dots,b_1\}\times\dots\times\{0,1,\dots,b_n\}$.

At the point $(b_1,\dots,b_n)$ we have $H(b_1,\dots,b_n)=G(d)-G_0(d)\neq 0$. At all other points of the grid we have $F=G$ and therefore $H=G-G_0=0$. So, by the Alon–Füredi bound, $\deg H\geqslant b_1+\dots+b_n=d$. Since $\deg G_0<d$, this implies $\deg F=\deg(H+G_0)=\deg H\geqslant d=\deg G$. $\square$

A7. Find the maximal value of

$$S=\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{d+7}}+\sqrt[3]{\frac{d}{a+7}},$$

where a, b, c, d are nonnegative real numbers which satisfy $a+b+c+d=100$.

(Taiwan)

Answer: $\dfrac{8}{\sqrt[3]{7}}$, reached when (a, b, c, d) is a cyclic permutation of $(1,49,1,49)$.

Solution 1. Since the value $8/\sqrt[3]{7}$ is reached, it suffices to prove that $S \leqslant 8/\sqrt[3]{7}$.

Assume that x, y, z, t is a permutation of the variables, with $x \leqslant y \leqslant z \leqslant t$. Then, by the rearrangement inequality,

$$S \leqslant\left(\sqrt[3]{\frac{x}{t+7}}+\sqrt[3]{\frac{t}{x+7}}\right)+\left(\sqrt[3]{\frac{y}{z+7}}+\sqrt[3]{\frac{z}{y+7}}\right).$$

Claim. The first bracket above does not exceed $\sqrt[3]{\dfrac{x+t+14}{7}}$.

Proof. Since

$$X^{3}+Y^{3}+3 X Y Z-Z^{3}=\frac{1}{2}(X+Y-Z)\left((X-Y)^{2}+(X+Z)^{2}+(Y+Z)^{2}\right),$$

the inequality $X+Y \leqslant Z$ is equivalent (when $X, Y, Z \geqslant 0$) to $X^{3}+Y^{3}+3 X Y Z \leqslant Z^{3}$. Therefore, the claim is equivalent to

$$\frac{x}{t+7}+\frac{t}{x+7}+3 \sqrt[3]{\frac{x t(x+t+14)}{7(x+7)(t+7)}} \leqslant \frac{x+t+14}{7}.$$

Notice that

$$\begin{aligned}
3 \sqrt[3]{\frac{x t(x+t+14)}{7(x+7)(t+7)}} & =3 \sqrt[3]{\frac{t(x+7)}{7(t+7)} \cdot \frac{x(t+7)}{7(x+7)} \cdot \frac{7(x+t+14)}{(t+7)(x+7)}} \\
& \leqslant \frac{t(x+7)}{7(t+7)}+\frac{x(t+7)}{7(x+7)}+\frac{7(x+t+14)}{(t+7)(x+7)}
\end{aligned}$$

by the AM–GM inequality, so it suffices to prove

$$\frac{x}{t+7}+\frac{t}{x+7}+\frac{t(x+7)}{7(t+7)}+\frac{x(t+7)}{7(x+7)}+\frac{7(x+t+14)}{(t+7)(x+7)} \leqslant \frac{x+t+14}{7}.$$

A straightforward check verifies that the last inequality is in fact an equality. □

The claim leads now to

$$S \leqslant \sqrt[3]{\frac{x+t+14}{7}}+\sqrt[3]{\frac{y+z+14}{7}} \leqslant 2 \sqrt[3]{\frac{x+y+z+t+28}{14}}=\frac{8}{\sqrt[3]{7}},$$

the last inequality being due to the AM–CM inequality (or to the fact that $\sqrt[3]{}$ is concave on $[0, \infty)$).

Solution 2. We present a different proof for the estimate $S \leqslant 8/\sqrt[3]{7}$.

Start by using Hölder's inequality:

$$S^3=\left(\sum_{\text{cyc}}\frac{\sqrt[6]{a}\cdot\sqrt[6]{a}}{\sqrt[3]{b+7}}\right)^3 \leqslant \sum_{\text{cyc}}\big(\sqrt[6]{a}\big)^3\cdot\sum_{\text{cyc}}\big(\sqrt[6]{a}\big)^3\cdot\sum_{\text{cyc}}\left(\frac{1}{\sqrt[3]{b+7}}\right)^3=\left(\sum_{\text{cyc}}\sqrt{a}\right)^2\sum_{\text{cyc}}\frac{1}{b+7}.$$

Notice that

$$\frac{(x-1)^2(x-7)^2}{x^2+7}\geqslant 0 \iff x^2-16x+71\geqslant\frac{448}{x^2+7}$$

yields

$$\sum\frac{1}{b+7}\leqslant\frac{1}{448}\sum\big(b-16\sqrt{b}+71\big)=\frac{1}{448}\left(384-16\sum\sqrt{b}\right)=\frac{48-2\sum\sqrt{b}}{56}.$$

Finally,

$$S^3\leqslant\frac{1}{56}\left(\sum\sqrt{a}\right)^2\left(48-2\sum\sqrt{a}\right)\leqslant\frac{1}{56}\left(\frac{\sum\sqrt{a}+\sum\sqrt{a}+\big(48-2\sum\sqrt{a}\big)}{3}\right)^3=\frac{512}{7}$$

by the AM–GM inequality. The conclusion follows.

Comment. All the above works if we replace 7 and 100 with $k>0$ and $2(k^2+1)$, respectively; in this case, the answer becomes

$$2\sqrt[3]{\frac{(k+1)^2}{k}}.$$

Even further, a linear substitution allows to extend the solutions to a version with 7 and 100 being replaced with arbitrary positive real numbers p and q satisfying $q\geqslant 4p$.

Combinatorics

C1. Let $n \geqslant 3$ be an integer. Prove that there exists a set S of $2n$ positive integers satisfying the following property: For every $m = 2, 3, \ldots, n$ the set S can be partitioned into two subsets with equal sums of elements, with one of subsets of cardinality m.

(Iceland)

Solution. We show that one of possible examples is the set

$$S = \{1 \cdot 3^k,\ 2 \cdot 3^k \colon k = 1, 2, \ldots, n-1\} \cup \left\{1,\ \frac{3^n+9}{2} - 1\right\}.$$

It is readily verified that all the numbers listed above are distinct (notice that the last two are not divisible by 3).

The sum of elements in S is

$$\Sigma = 1 + \left(\frac{3^n+9}{2} - 1\right) + \sum_{k=1}^{n-1}(1 \cdot 3^k + 2 \cdot 3^k) = \frac{3^n+9}{2} + \sum_{k=1}^{n-1} 3^{k+1} = \frac{3^n+9}{2} + \frac{3^{n+1}-9}{2} = 2 \cdot 3^n.$$

Hence, in order to show that this set satisfies the problem requirements, it suffices to present, for every $m = 2, 3, \ldots, n$, an m-element subset $A_m \subset S$ whose sum of elements equals 3^n.

Such a subset is

$$A_m = \{2 \cdot 3^k \colon k = n-m+1, n-m+2, \ldots, n-1\} \cup \{1 \cdot 3^{n-m+1}\}.$$

Clearly, $|A_m| = m$. The sum of elements in A_m is

$$3^{n-m+1} + \sum_{k=n-m+1}^{n-1} 2 \cdot 3^k = 3^{n-m+1} + \frac{2 \cdot 3^n - 2 \cdot 3^{n-m+1}}{2} = 3^n,$$

as required.

Comment. Let us present a more general construction. Let $s_1, s_2, \ldots, s_{2n-1}$ be a sequence of pairwise distinct positive integers satisfying $s_{2i+1} = s_{2i} + s_{2i-1}$ for all $i = 2, 3, \ldots, n-1$. Set $s_{2n} = s_1 + s_2 + \cdots + s_{2n-4}$.

Assume that s_{2n} is distinct from the other terms of the sequence. Then the set $S = \{s_1, s_2, \ldots, s_{2n}\}$ satisfies the problem requirements. Indeed, the sum of its elements is

$$\Sigma = \sum_{i=1}^{2n-4} s_i + (s_{2n-3} + s_{2n-2}) + s_{2n-1} + s_{2n} = s_{2n} + s_{2n-1} + s_{2n-1} + s_{2n} = 2s_{2n} + 2s_{2n-1}.$$

Therefore, we have

$$\frac{\Sigma}{2} = s_{2n} + s_{2n-1} = s_{2n} + s_{2n-2} + s_{2n-3} = s_{2n} + s_{2n-2} + s_{2n-4} + s_{2n-5} = \ldots,$$

which shows that the required sets A_m can be chosen as

$$A_m = \{s_{2n}, s_{2n-2}, \ldots, s_{2n-2m+4}, s_{2n-2m+3}\}.$$

So, the only condition to be satisfied is $s_{2n} \notin \{s_1, s_2, \ldots, s_{2n-1}\}$, which can be achieved in many different ways (e.g., by choosing properly the number s_1 after specifying $s_2, s_3, \ldots, s_{2n-1}$).

The solution above is an instance of this general construction. Another instance, for $n > 3$, is the set

$$\{F_1, F_2, \ldots, F_{2n-1}, F_1 + \cdots + F_{2n-4}\},$$

where $F_1 = 1$, $F_2 = 2$, $F_{n+1} = F_n + F_{n-1}$ is the usual Fibonacci sequence.

C2. Queenie and Horst play a game on a 20×20 chessboard. In the beginning the board is empty. In every turn, Horst places a black knight on an empty square in such a way that his new knight does not attack any previous knights. Then Queenie places a white queen on an empty square. The game gets finished when somebody cannot move.

Find the maximal positive K such that, regardless of the strategy of Queenie, Horst can put at least K knights on the board.

(Armenia)

Answer: $K = 20^2/4 = 100$. In case of a $4N \times 4M$ board, the answer is $K = 4NM$.

Solution. We show two strategies, one for Horst to place at least 100 knights, and another strategy for Queenie that prevents Horst from putting more than 100 knights on the board.

A strategy for Horst: *Put knights only on black squares, until all black squares get occupied.*

Colour the squares of the board black and white in the usual way, such that the white and black squares alternate, and let Horst put his knights on black squares as long as it is possible. Two knights on squares of the same colour never attack each other. The number of black squares is $20^2/2 = 200$. The two players occupy the squares in turn, so Horst will surely find empty black squares in his first 100 steps.

A strategy for Queenie: *Group the squares into cycles of length 4, and after each step of Horst, occupy the opposite square in the same cycle.*

Consider the squares of the board as vertices of a graph; let two squares be connected if two knights on those squares would attack each other. Notice that in a 4×4 board the squares can be grouped into 4 cycles of length 4, as shown in Figure 1. Divide the board into parts of size 4×4, and perform the same grouping in every part; this way we arrange the 400 squares of the board into 100 cycles (Figure 2).

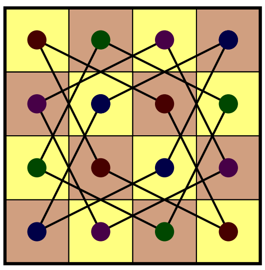

Figure 1

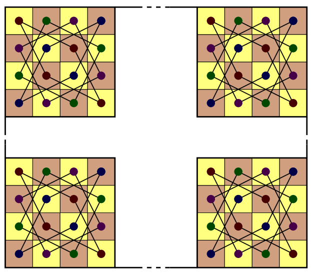

Figure 2

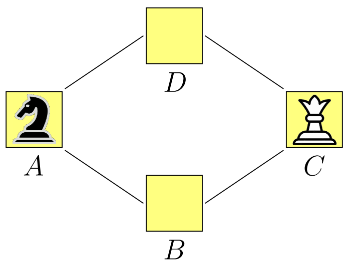

Figure 3

The strategy of Queenie can be as follows: Whenever Horst puts a new knight to a certain square A, which is part of some cycle $A - B - C - D - A$, let Queenie put her queen on the opposite square C in that cycle (Figure 3). From this point, Horst cannot put any knight on A or C because those squares are already occupied, neither on B or D because those squares are attacked by the knight standing on A. Hence, Horst can put at most one knight on each cycle, that is at most 100 knights in total.

Comment 1. Queenie's strategy can be prescribed by a simple rule: divide the board into 4×4 parts; whenever Horst puts a knight in a part P, Queenie reflects that square about the centre of P and puts her queen on the reflected square.

Comment 2. The result remains the same if Queenie moves first. In the first turn, she may put her first queen arbitrarily. Later, if she has to put her next queen on a square that already contains a queen, she may move arbitrarily again.

C3. Let n be a given positive integer. Sisyphus performs a sequence of turns on a board consisting of $n+1$ squares in a row, numbered 0 to n from left to right. Initially, n stones are put into square 0, and the other squares are empty. At every turn, Sisyphus chooses any nonempty square, say with k stones, takes one of those stones and moves it to the right by at most k squares (the stone should stay within the board). Sisyphus' aim is to move all n stones to square n.

Prove that Sisyphus cannot reach the aim in less than

$$\left\lceil \frac{n}{1} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{3} \right\rceil + \cdots + \left\lceil \frac{n}{n} \right\rceil$$

turns. (As usual, $\lceil x \rceil$ stands for the least integer not smaller than x.)

(Netherlands)

Solution. The stones are indistinguishable, and all have the same origin and the same final position. So, at any turn we can prescribe which stone from the chosen square to move. We do it in the following manner. Number the stones from 1 to n. At any turn, after choosing a square, Sisyphus moves the stone with the largest number from this square.

This way, when stone k is moved from some square, that square contains not more than k stones (since all their numbers are at most k). Therefore, stone k is moved by at most k squares at each turn. Since the total shift of the stone is exactly n, at least $\lceil n/k \rceil$ moves of stone k should have been made, for every $k = 1, 2, \ldots, n$.

By summing up over all $k = 1, 2, \ldots, n$, we get the required estimate.

Comment. The original submission contained the second part, asking for which values of n the equality can be achieved. The answer is $n = 1, 2, 3, 4, 5, 7$. The Problem Selection Committee considered this part to be less suitable for the competition, due to technicalities.

C4. An *anti-Pascal pyramid* is a finite set of numbers, placed in a triangle-shaped array so that the first row of the array contains one number, the second row contains two numbers, the third row contains three numbers and so on; and, except for the numbers in the bottom row, each number equals the absolute value of the difference of the two numbers below it. For instance, the triangle below is an anti-Pascal pyramid with four rows, in which every integer from 1 to $1+2+3+4=10$ occurs exactly once:

$$\begin{array}{ccccccc} & & & 4 & & & \\ & & 2 & & 6 & & \\ & 5 & & 7 & & 1 & \\ 8 & & 3 & & 10 & & 9\,. \end{array}$$

Is it possible to form an anti-Pascal pyramid with 2018 rows, using every integer from 1 to $1+2+\cdots+2018$ exactly once?

(Iran)

Answer: No, it is not possible.

Solution. Let T be an anti-Pascal pyramid with n rows, containing every integer from 1 to $1+2+\cdots+n$, and let a_1 be the topmost number in T (Figure 1). The two numbers below a_1 are some a_2 and $b_2=a_1+a_2$, the two numbers below b_2 are some a_3 and $b_3=a_1+a_2+a_3$, and so on and so forth all the way down to the bottom row, where some a_n and $b_n=a_1+a_2+\cdots+a_n$ are the two neighbours below $b_{n-1}=a_1+a_2+\cdots+a_{n-1}$. Since the a_k are n pairwise distinct positive integers whose sum does not exceed the largest number in T, which is $1+2+\cdots+n$, it follows that they form a permutation of $1,2,\ldots,n$.

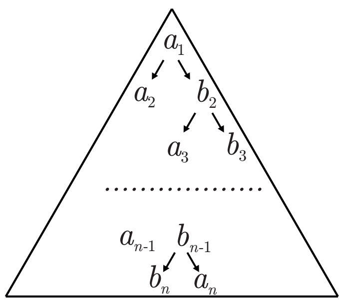

Figure 1

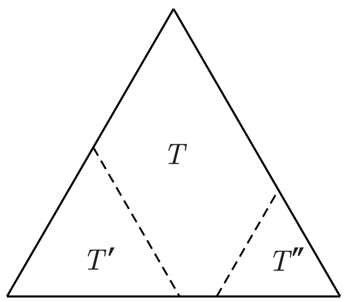

Figure 2

Consider now (Figure 2) the two 'equilateral' subtriangles of T whose bottom rows contain the numbers to the left, respectively right, of the pair a_n, b_n. (One of these subtriangles may very well be empty.) At least one of these subtriangles, say T', has side length $\ell \geqslant \lceil (n-2)/2 \rceil$. Since T' obeys the anti-Pascal rule, it contains ℓ pairwise distinct positive integers $a'_1, a'_2, \ldots, a'_\ell$, where a'_1 is at the apex, and a'_k and $b'_k=a'_1+a'_2+\cdots+a'_k$ are the two neighbours below b'_{k-1} for each $k=2,3\ldots,\ell$. Since the a_k all lie outside T', and they form a permutation of $1,2,\ldots,n$, the a'_k are all greater than n. Consequently,

$$\begin{aligned} b'_\ell &\geqslant (n+1)+(n+2)+\cdots+(n+\ell)=\frac{\ell(2n+\ell+1)}{2} \\ &\geqslant \frac{1}{2}\cdot\frac{n-2}{2}\left(2n+\frac{n-2}{2}+1\right)=\frac{5n(n-2)}{8}, \end{aligned}$$

which is greater than $1+2+\cdots+n=n(n+1)/2$ for $n=2018$. A contradiction.

Comment. The above estimate may be slightly improved by noticing that $b'_\ell \neq b_n$. This implies $n(n+1)/2=b_n>b'_\ell \geqslant \lceil (n-2)/2 \rceil \left(2n+\lceil (n-2)/2 \rceil+1\right)/2$, so $n \leqslant 7$ if n is odd, and $n \leqslant 12$ if n is even. It seems that the largest anti-Pascal pyramid whose entries are a permutation of the integers from 1 to $1+2+\cdots+n$ has 5 rows.

C5. Let k be a positive integer. The organising committee of a tennis tournament is to schedule the matches for $2k$ players so that every two players play once, each day exactly one match is played, and each player arrives to the tournament site the day of his first match, and departs the day of his last match. For every day a player is present on the tournament, the committee has to pay 1 coin to the hotel. The organisers want to design the schedule so as to minimise the total cost of all players' stays. Determine this minimum cost.

(Russia)

Answer: The required minimum is $k(4k^2+k-1)/2$.

Solution 1. Enumerate the days of the tournament $1, 2, \ldots, \binom{2k}{2}$. Let $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_{2k}$ be the days the players arrive to the tournament, arranged in *nondecreasing* order; similarly, let $e_1 \geqslant \cdots \geqslant e_{2k}$ be the days they depart arranged in *nonincreasing* order (it may happen that a player arrives on day b_i and departs on day e_j, where $i \neq j$). If a player arrives on day b and departs on day e, then his stay cost is $e-b+1$. Therefore, the total stay cost is

$$\Sigma = \sum_{i=1}^{2k} e_i - \sum_{i=1}^{2k} b_i + n = \sum_{i=1}^{2k} (e_i - b_i + 1).$$

Bounding the total cost from below. To this end, estimate $e_{i+1} - b_{i+1} + 1$. Before day b_{i+1}, only i players were present, so at most $\binom{i}{2}$ matches could be played. Therefore, $b_{i+1} \leqslant \binom{i}{2} + 1$. Similarly, at most $\binom{i}{2}$ matches could be played after day e_{i+1}, so $e_i \geqslant \binom{2k}{2} - \binom{i}{2}$. Thus,

$$e_{i+1} - b_{i+1} + 1 \geqslant \binom{2k}{2} - 2\binom{i}{2} = k(2k-1) - i(i-1).$$

This lower bound can be improved for $i > k$: List the i players who arrived first, and the i players who departed last; at least $2i - 2k$ players appear in both lists. The matches between these players were counted twice, though the players in each pair have played only once. Therefore, if $i > k$, then

$$e_{i+1} - b_{i+1} + 1 \geqslant \binom{2k}{2} - 2\binom{i}{2} + \binom{2i-2k}{2} = (2k-i)^2.$$

An optimal tournament, We now describe a schedule in which the lower bounds above are all achieved simultaneously. Split players into two groups X and Y, each of cardinality k. Next, partition the schedule into three parts. During the first part, the players from X arrive one by one, and each newly arrived player immediately plays with everyone already present. During the third part (after all players from X have already departed) the players from Y depart one by one, each playing with everyone still present just before departing.

In the middle part, everyone from X should play with everyone from Y. Let $S_1, S_2, \ldots, S_k$ be the players in X, and let $T_1, T_2, \ldots, T_k$ be the players in Y. Let $T_1, T_2, \ldots, T_k$ arrive in this order; after T_j arrives, he immediately plays with all the S_i, $i > j$. Afterwards, players S_k, $S_{k-1}, \ldots, S_1$ depart in this order; each S_i plays with all the T_j, $i \leqslant j$, just before his departure, and S_k departs the day T_k arrives. For $0 \leqslant s \leqslant k-1$, the number of matches played between T_{k-s}'s arrival and S_{k-s}'s departure is

$$\sum_{j=k-s}^{k-1} (k-j) + 1 + \sum_{j=k-s}^{k-1} (k-j+1) = \frac{1}{2}s(s+1) + 1 + \frac{1}{2}s(s+3) = (s+1)^2.$$

Thus, if $i > k$, then the number of matches that have been played between T_{i-k+1}'s arrival, which is b_{i+1}, and S_{i-k+1}'s departure, which is e_{i+1}, is $(2k-i)^2$; that is, $e_{i+1} - b_{i+1} + 1 = (2k-i)^2$, showing the second lower bound achieved for all $i > k$.

If $i \leqslant k$, then the matches between the i players present before b_{i+1} all fall in the first part of the schedule, so there are $\binom{i}{2}$ such, and $b_{i+1} = \binom{i}{2} + 1$. Similarly, after e_{i+1}, there are i players left, all $\binom{i}{2}$ matches now fall in the third part of the schedule, and $e_{i+1} = \binom{2k}{2} - \binom{i}{2}$. The first lower bound is therefore also achieved for all $i \leqslant k$.

Consequently, all lower bounds are achieved simultaneously, and the schedule is indeed optimal.

Evaluation. Finally, evaluate the total cost for the optimal schedule:

$$\Sigma = \sum_{i=0}^{k}\big(k(2k-1) - i(i-1)\big) + \sum_{i=k+1}^{2k-1}(2k-i)^2 = (k+1)k(2k-1) - \sum_{i=0}^{k} i(i-1) + \sum_{j=1}^{k-1} j^2$$
$$= k(k+1)(2k-1) - k^2 + \frac{1}{2}k(k+1) = \frac{1}{2}k(4k^2+k-1).$$

Solution 2. Consider any tournament schedule. Label players $P_1, P_2, \ldots, P_{2k}$ in order of their arrival, and label them again $Q_{2k}, Q_{2k-1}, \ldots, Q_1$ in order of their departure, to define a permutation $a_1, a_2, \ldots, a_{2k}$ of $1, 2, \ldots, 2k$ by $P_i = Q_{a_i}$.

We first describe an optimal tournament for any given permutation $a_1, a_2, \ldots, a_{2k}$ of the indices $1, 2, \ldots, 2k$. Next, we find an optimal permutation and an optimal tournament.

Optimisation for a fixed $a_1, \ldots, a_{2k}$. We say that the *cost* of the match between P_i and P_j is the number of players present at the tournament when this match is played. Clearly, the Committee pays for each day the cost of the match of that day. Hence, we are to minimise the total cost of all matches.

Notice that Q_{2k}'s departure does not precede P_{2k}'s arrival. Hence, the number of players at the tournament monotonically increases (non-strictly) until it reaches $2k$, and then monotonically decreases (non-strictly). So, the best time to schedule the match between P_i and P_j is either when $P_{\max(i,j)}$ arrives, or when $Q_{\max(a_i,a_j)}$ departs, in which case the cost is $\min\big(\max(i,j), \max(a_i,a_j)\big)$.

Conversely, assuming that $i > j$, if this match is scheduled between the arrivals of P_i and P_{i+1}, then its cost will be exactly $i = \max(i,j)$. Similarly, one can make it cost $\max(a_i,a_j)$. Obviously, these conditions can all be simultaneously satisfied, so the minimal cost for a fixed sequence $a_1, a_2, \ldots, a_{2k}$ is

$$\Sigma(a_1,\ldots,a_{2k}) = \sum_{1 \leqslant i<j \leqslant 2k} \min\big(\max(i,j), \max(a_i,a_j)\big). \tag{1}$$

Optimising the sequence (a_i). Optimisation hinges on the lemma below.

Lemma. If $a \leqslant b$ and $c \leqslant d$, then

$$\min\big(\max(a,x),\max(c,y)\big) + \min\big(\max(b,x),\max(d,y)\big)$$
$$\geqslant \min\big(\max(a,x),\max(d,y)\big) + \min\big(\max(b,x),\max(c,y)\big).$$

Proof. Write $a' = \max(a,x) \leqslant \max(b,x) = b'$ and $c' = \max(c,y) \leqslant \max(d,y) = d'$ and check that $\min(a',c') + \min(b',d') \geqslant \min(a',d') + \min(b',c')$. □

Consider a permutation $a_1, a_2, \ldots, a_{2k}$ such that $a_i < a_j$ for some $i < j$. Swapping a_i and a_j does not change the (i,j)th summand in (1), and for $\ell \notin \{i,j\}$ the sum of the (i,ℓ)th and the (j,ℓ)th summands does not increase by the Lemma. Hence the optimal value does not increase, but the number of disorders in the permutation increases. This process stops when $a_i = 2k+1-i$ for all i, so the required minimum is

$$S(2k, 2k-1, \ldots, 1) = \sum_{1 \leqslant i<j \leqslant 2k} \min\big(\max(i,j), \max(2k+1-i, 2k+1-j)\big)$$
$$= \sum_{1 \leqslant i<j \leqslant 2k} \min(j, 2k+1-i).$$

The latter sum is fairly tractable and yields the stated result; we omit the details.

Comment. If the number of players is odd, say, $2k-1$, the required minimum is $k(k-1)(4k-1)/2$. In this case, $|X| = k$, $|Y| = k-1$, the argument goes along the same lines, but some additional technicalities are to be taken care of.

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C6. Let a and b be distinct positive integers. The following infinite process takes place on an initially empty board.

(i) If there is at least a pair of equal numbers on the board, we choose such a pair and increase one of its components by a and the other by b.

(ii) If no such pair exists, we write down two times the number 0.

Prove that, no matter how we make the choices in (i), operation (ii) will be performed only finitely many times.

(Serbia)

Solution 1. We may assume $\gcd(a, b) = 1$; otherwise we work in the same way with multiples of $d = \gcd(a, b)$.

Suppose that after N moves of type (ii) and some moves of type (i) we have to add two new zeros. For each integer k, denote by $f(k)$ the number of times that the number k appeared on the board up to this moment. Then $f(0) = 2N$ and $f(k) = 0$ for $k < 0$. Since the board contains at most one $k - a$, every second occurrence of $k - a$ on the board produced, at some moment, an occurrence of k; the same stands for $k - b$. Therefore,

$$f(k) = \left\lfloor \frac{f(k-a)}{2} \right\rfloor + \left\lfloor \frac{f(k-b)}{2} \right\rfloor, \tag{1}$$

yielding

$$f(k) \geqslant \frac{f(k-a) + f(k-b)}{2} - 1. \tag{2}$$

Since $\gcd(a, b) = 1$, every integer $x > ab - a - b$ is expressible in the form $x = sa + tb$, with integer $s, t \geqslant 0$.

We will prove by induction on $s + t$ that if $x = sa + bt$, with s, t nonnegative integers, then

$$f(x) > \frac{f(0)}{2^{s+t}} - 2. \tag{3}$$

The base case $s+t = 0$ is trivial. Assume now that (3) is true for $s+t = v$. Then, if $s+t = v+1$ and $x = sa + tb$, at least one of the numbers s and t – say s – is positive, hence by (2),

$$f(x) = f(sa + tb) \geqslant \frac{f\big((s-1)a + tb\big)}{2} - 1 > \frac{1}{2}\left(\frac{f(0)}{2^{s+t-1}} - 2\right) - 1 = \frac{f(0)}{2^{s+t}} - 2.$$

Assume now that we must perform moves of type (ii) *ad infinitum*. Take $n = ab - a - b$ and suppose $b > a$. Since each of the numbers $n + 1, n + 2, \ldots, n + b$ can be expressed in the form $sa + tb$, with $0 \leqslant s \leqslant b$ and $0 \leqslant t \leqslant a$, after moves of type ($ii$) have been performed 2^{a+b+1} times and we have to add a new pair of zeros, each $f(n + k)$, $k = 1, 2, \ldots, b$, is at least 2. In this case (1) yields inductively $f(n + k) \geqslant 2$ for all $k \geqslant 1$. But this is absurd: after a finite number of moves, f cannot attain nonzero values at infinitely many points.

Solution 2. We start by showing that the result of the process in the problem does not depend on the way the operations are performed. For that purpose, it is convenient to modify the process a bit.

Claim 1. Suppose that the board initially contains a finite number of nonnegative integers, and one starts performing type (i) moves only. Assume that one had applied k moves which led to a *final* arrangement where no more type (i) moves are possible. Then, if one starts from the same initial arrangement, performing type (i) moves in an arbitrary fashion, then the process will necessarily stop at the same final arrangement

Proof. Throughout this proof, all moves are supposed to be of type (i).

Induct on k; the base case $k = 0$ is trivial, since no moves are possible. Assume now that $k \geqslant 1$. Fix some *canonical* process, consisting of k moves $M_1, M_2, \ldots, M_k$, and reaching the final arrangement A. Consider any *sample* process $m_1, m_2, \ldots$ starting with the same initial arrangement and proceeding as long as possible; clearly, it contains at least one move. We need to show that this process stops at A.

Let move m_1 consist in replacing two copies of x with $x + a$ and $x + b$. If move M_1 does the same, we may apply the induction hypothesis to the arrangement appearing after m_1. Otherwise, the canonical process should still contain at least one move consisting in replacing $(x, x) \mapsto (x + a, x + b)$, because the initial arrangement contains at least two copies of x, while the final one contains at most one such.

Let M_i be the first such move. Since the copies of x are indistinguishable and no other copy of x disappeared before M_i in the canonical process, the moves in this process can be permuted as $M_i, M_1, \ldots, M_{i-1}, M_{i+1}, \ldots, M_k$, without affecting the final arrangement. Now it suffices to perform the move $m_1 = M_i$ and apply the induction hypothesis as above. □

Claim 2. Consider any process starting from the empty board, which involved exactly n moves of type (ii) and led to a final arrangement where all the numbers are distinct. Assume that one starts with the board containing $2n$ zeroes (as if n moves of type (ii) were made in the beginning), applying type (i) moves in an arbitrary way. Then this process will reach the same final arrangement.

Proof. Starting with the board with $2n$ zeros, one may indeed model the first process mentioned in the statement of the claim, omitting the type (ii) moves. This way, one reaches the same final arrangement. Now, Claim 1 yields that this final arrangement will be obtained when type (i) moves are applied arbitrarily. □

Claim 2 allows now to reformulate the problem statement as follows: *There exists an integer n such that, starting from $2n$ zeroes, one may apply type (i) moves indefinitely.*

In order to prove this, we start with an obvious induction on $s + t = k \geqslant 1$ to show that if we start with 2^{s+t} zeros, then we can get simultaneously on the board, at some point, each of the numbers $sa + tb$, with $s + t = k$.

Suppose now that $a < b$. Then, an appropriate use of separate groups of zeros allows us to get two copies of each of the numbers $sa + tb$, with $1 \leqslant s, t \leqslant b$.

Define $N = ab - a - b$, and notice that after representing each of numbers $N + k$, $1 \leqslant k \leqslant b$, in the form $sa + tb$, $1 \leqslant s, t \leqslant b$ we can get, using enough zeros, the numbers $N+1, N+2, \ldots, N+a$ and the numbers $N + 1, N + 2, \ldots, N + b$.

From now on we can perform only moves of type (i). Indeed, if $n \geqslant N$, the occurrence of the numbers $n+1, n+2, \ldots, n+a$ and $n+1, n+2, \ldots, n+b$ and the replacement $(n+1, n+1) \mapsto (n + b + 1, n + a + 1)$ leads to the occurrence of the numbers $n + 2, n + 3, \ldots, n + a + 1$ and $n + 2, n + 3, \ldots, n + b + 1$.

Comment. The proofs of Claims 1 and 2 may be extended in order to show that in fact the number of moves in the canonical process is the same as in an arbitrary sample one.

C7. Consider 2018 pairwise crossing circles no three of which are concurrent. These circles subdivide the plane into regions bounded by circular *edges* that meet at *vertices*. Notice that there are an even number of vertices on each circle. Given the circle, alternately colour the vertices on that circle red and blue. In doing so for each circle, every vertex is coloured twice — once for each of the two circles that cross at that point. If the two colourings agree at a vertex, then it is assigned that colour; otherwise, it becomes yellow. Show that, if some circle contains at least 2061 yellow points, then the vertices of some region are all yellow.

(India)

Solution 1. Letting $n = 2018$, we will show that, if every region has at least one non-yellow vertex, then every circle contains at most $n + \lfloor\sqrt{n-2}\rfloor - 2$ yellow points. In the case at hand, the latter equals $2018 + 44 - 2 = 2060$, contradicting the hypothesis.

Consider the natural geometric graph G associated with the configuration of n circles. Fix any circle C in the configuration, let k be the number of yellow points on C, and find a suitable lower bound for the total number of yellow vertices of G in terms of k and n. It turns out that k is even, and G has at least

$$k + 2\binom{k/2}{2} + 2\binom{n - k/2 - 1}{2} = \frac{k^2}{2} - (n-2)k + (n-2)(n-1) \qquad (*)$$

yellow vertices. The proof hinges on the two lemmata below.

Lemma 1. Let two circles in the configuration cross at x and y. Then x and y are either both yellow or both non-yellow.

Proof. This is because the numbers of interior vertices on the four arcs x and y determine on the two circles have like parities. □

In particular, each circle in the configuration contains an even number of yellow vertices.

Lemma 2. If $\widehat{xy}$, $\widehat{yz}$, and $\widehat{zx}$ are circular arcs of three pairwise distinct circles in the configuration, then the number of yellow vertices in the set $\{x, y, z\}$ is odd.

Proof. Let C_1, C_2, C_3 be the three circles under consideration. Assume, without loss of generality, that C_2 and C_3 cross at x, C_3 and C_1 cross at y, and C_1 and C_2 cross at z. Let k_1, k_2, k_3 be the numbers of interior vertices on the three circular arcs under consideration. Since each circle in the configuration, different from the C_i, crosses the cycle $\widehat{xy} \cup \widehat{yz} \cup \widehat{zx}$ at an even number of points (recall that no three circles are concurrent), and self-crossings are counted twice, the sum $k_1 + k_2 + k_3$ is even.

Let Z_1 be the colour z gets from C_1 and define the other colours similarly. By the preceding, the number of bichromatic pairs in the list (Z_1, Y_1), (X_2, Z_2), (Y_3, X_3) is odd. Since the total number of colour changes in a cycle Z_1–Y_1–Y_3–X_3–X_2–Z_2–Z_1 is even, the number of bichromatic pairs in the list (X_2, X_3), (Y_1, Y_3), (Z_1, Z_2) is odd, and the lemma follows. □

We are now in a position to prove that $(*)$ bounds the total number of yellow vertices from below. Refer to Lemma 1 to infer that the k yellow vertices on C pair off to form the pairs of points where C is crossed by $k/2$ circles in the configuration. By Lemma 2, these circles cross pairwise to account for another $2\binom{k/2}{2}$ yellow vertices. Finally, the remaining $n - k/2 - 1$ circles in the configuration cross C at non-yellow vertices, by Lemma 1, and Lemma 2 applies again to show that these circles cross pairwise to account for yet another $2\binom{n-k/2-1}{2}$ yellow vertices. Consequently, there are at least $(*)$ yellow vertices.

Next, notice that G is a plane graph on $n(n-1)$ degree 4 vertices, having exactly $2n(n-1)$ edges and exactly $n(n-1) + 2$ faces (regions), the outer face inclusive (by Euler's formula for planar graphs).

Lemma 3. Each face of G has equally many red and blue vertices. In particular, each face has an even number of non-yellow vertices.

Proof. Trace the boundary of a face once in circular order, and consider the colours each vertex is assigned in the colouring of the two circles that cross at that vertex, to infer that colours of non-yellow vertices alternate. □

Consequently, if each region has at least one non-yellow vertex, then it has at least two such. Since each vertex of G has degree 4, consideration of vertex-face incidences shows that G has at least $n(n-1)/2+1$ non-yellow vertices, and hence at most $n(n-1)/2-1$ yellow vertices. (In fact, Lemma 3 shows that there are at least $n(n-1)/4+1/2$ red, respectively blue, vertices.)

Finally, recall the lower bound $(*)$ for the total number of yellow vertices in G, to write $n(n-1)/2-1 \geqslant k^2/2-(n-2)k+(n-2)(n-1)$, and conclude that $k \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$, as claimed in the first paragraph.

Solution 2. The first two lemmata in Solution 1 show that the circles in the configuration split into two classes: Consider any circle C along with all circles that cross C at yellow points to form one class; the remaining circles then form the other class. Lemma 2 shows that any pair of circles in the same class cross at yellow points; otherwise, they cross at non-yellow points.

Call the circles from the two classes *white* and *black*, respectively. Call a region *yellow* if its vertices are all yellow. Let w and b be the numbers of white and black circles, respectively; clearly, $w+b=n$. Assume that $w \geqslant b$, and that there is no yellow region. Clearly, $b \geqslant 1$, otherwise each region is yellow. The white circles subdivide the plane into $w(w-1)+2$ larger regions — call them *white*. The white regions (or rather their boundaries) subdivide each black circle into *black arcs*. Since there are no yellow regions, each white region contains at least one black arc.

Consider any white region; let it contain $t \geqslant 1$ black arcs. We claim that the number of points at which these t arcs cross does not exceed $t-1$. To prove this, consider a multigraph whose vertices are these black arcs, two vertices being joined by an edge for each point at which the corresponding arcs cross. If this graph had more than $t-1$ edges, it would contain a cycle, since it has t vertices; this cycle would correspond to a closed contour formed by black sub-arcs, lying inside the region under consideration. This contour would, in turn, define at least one yellow region, which is impossible.

Let t_i be the number of black arcs inside the i^{th} white region. The total number of black arcs is $\sum_i t_i = 2wb$, and they cross at $2\binom{b}{2}=b(b-1)$ points. By the preceding,

$$b(b-1) \leqslant \sum_{i=1}^{w^2-w+2}(t_i-1) = \sum_{i=1}^{w^2-w+2} t_i - (w^2-w+2) = 2wb-(w^2-w+2),$$

or, equivalently, $(w-b)^2 \leqslant w+b-2 = n-2$, which is the case if and only if $w-b \leqslant \lfloor\sqrt{n-2}\rfloor$. Consequently, $b \leqslant w \leqslant \left(n+\lfloor\sqrt{n-2}\rfloor\right)/2$, so there are at most $2(w-1) \leqslant n+\lfloor\sqrt{n-2}\rfloor-2$ yellow vertices on each circle — a contradiction.

Geometry

G1. Let ABC be an acute-angled triangle with circumcircle Γ. Let D and E be points on the segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of the segments BD and CE intersect the small arcs $\widehat{AB}$ and $\widehat{AC}$ at points F and G respectively. Prove that $DE \parallel FG$.

(Greece)

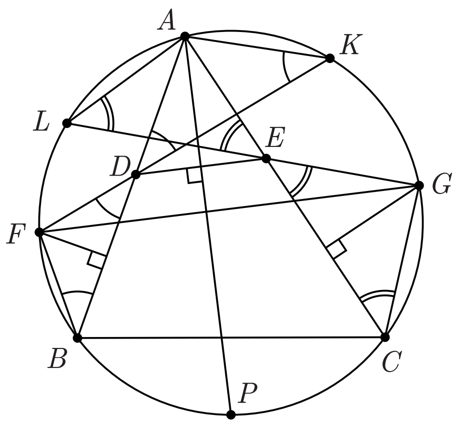
Solution 1. In the sequel, all the considered arcs are small arcs.

Let P be the midpoint of the arc $\widehat{BC}$. Then AP is the bisector of $\angle BAC$, hence, in the isosceles triangle ADE, $AP \perp DE$. So, the statement of the problem is equivalent to $AP \perp FG$.

In order to prove this, let K be the second intersection of Γ with FD. Then the triangle FBD is isosceles, therefore

$$\angle AKF = \angle ABF = \angle FDB = \angle ADK,$$

yielding $AK = AD$. In the same way, denoting by L the second intersection of Γ with GE, we get $AL = AE$. This shows that $AK = AL$.



Now $\angle FBD = \angle FDB$ gives $\widehat{AF} = \widehat{BF} + \widehat{AK} = \widehat{BF} + \widehat{AL}$, hence $\widehat{BF} = \widehat{LF}$. In a similar way, we get $\widehat{CG} = \widehat{GK}$. This yields

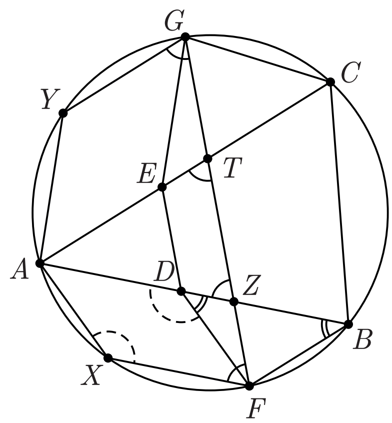
$$\angle(AP, FG) = \frac{\widehat{AF} + \widehat{PG}}{2} = \frac{\widehat{AL} + \widehat{LF} + \widehat{PC} + \widehat{CG}}{2} = \frac{\widehat{KL} + \widehat{LB} + \widehat{BC} + \widehat{CK}}{4} = 90^\circ.$$

Solution 2. Let $Z = AB \cap FG$, $T = AC \cap FG$. It suffices to prove that $\angle ATZ = \angle AZT$.

Let X be the point for which $FXAD$ is a parallelogram. Then

$$\angle FXA = \angle FDA = 180^\circ - \angle FDB = 180^\circ - \angle FBD,$$

where in the last equality we used that $FD = FB$. It follows that the quadrilateral $BFXA$ is cyclic, so X lies on Γ.

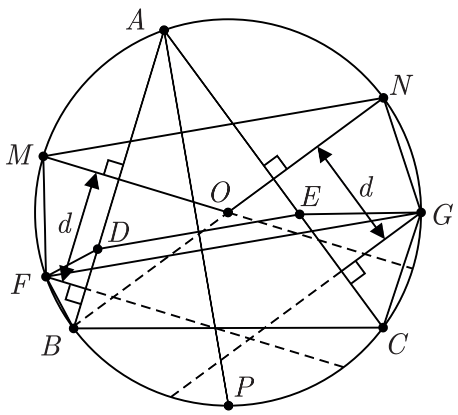


Analogously, if Y is the point for which $GYAE$ is a parallelogram, then Y lies on Γ. So the quadrilateral $XFGY$ is cyclic and $FX = AD = AE = GY$, hence $XFGY$ is an isosceles trapezoid.

Now, by $XF \parallel AZ$ and $YG \parallel AT$, it follows that $\angle ATZ = \angle YGF = \angle XFG = \angle AZT$.

Solution 3. As in the first solution, we prove that $FG \perp AP$, where P is the midpoint of the small arc $\widehat{BC}$.

Let O be the circumcentre of the triangle ABC, and let M and N be the midpoints of the small arcs $\widehat{AB}$ and $\widehat{AC}$, respectively. Then OM and ON are the perpendicular bisectors of AB and AC, respectively.



The distance d between OM and the perpendicular bisector of BD is $\frac{1}{2}AB - \frac{1}{2}BD = \frac{1}{2}AD$, hence it is equal to the distance between ON and the perpendicular bisector of CE.

This shows that the isosceles trapezoid determined by the diameter δ of Γ through M and the chord parallel to δ through F is congruent to the isosceles trapezoid determined by the diameter δ' of Γ through N and the chord parallel to δ' through G. Therefore $MF = NG$, yielding $MN \parallel FG$.

Now

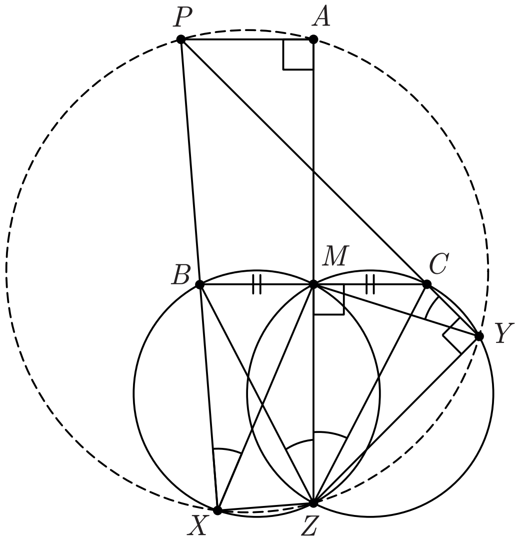
$$\angle(MN, AP) = \frac{1}{2}\left(\widehat{AM} + \widehat{PC} + \widehat{CN}\right) = \frac{1}{4}\left(\widehat{AB} + \widehat{BC} + \widehat{CA}\right) = 90^\circ,$$

hence $MN \perp AP$, and the conclusion follows.

G2. Let ABC be a triangle with $AB = AC$, and let M be the midpoint of BC. Let P be a point such that $PB < PC$ and PA is parallel to BC. Let X and Y be points on the lines PB and PC, respectively, so that B lies on the segment PX, C lies on the segment PY, and $\angle PXM = \angle PYM$. Prove that the quadrilateral $APXY$ is cyclic.

(Australia)

Solution. Since $AB = AC$, AM is the perpendicular bisector of BC, hence $\angle PAM = \angle AMC = 90^\circ$.



Now let Z be the common point of AM and the perpendicular through Y to PC (notice that Z lies on to the ray AM beyond M). We have $\angle PAZ = \angle PYZ = 90^\circ$. Thus the points P, A, Y, and Z are concyclic.

Since $\angle CMZ = \angle CYZ = 90^\circ$, the quadrilateral $CYZM$ is cyclic, hence $\angle CZM = \angle CYM$. By the condition in the statement, $\angle CYM = \angle BXM$, and, by symmetry in ZM, $\angle CZM = \angle BZM$. Therefore, $\angle BXM = \angle BZM$. It follows that the points B, X, Z, and M are concyclic, hence $\angle BXZ = 180^\circ - \angle BMZ = 90^\circ$.

Finally, we have $\angle PXZ = \angle PYZ = \angle PAZ = 90^\circ$, hence the five points P, A, X, Y, Z are concyclic. In particular, the quadrilateral $APXY$ is cyclic, as required.

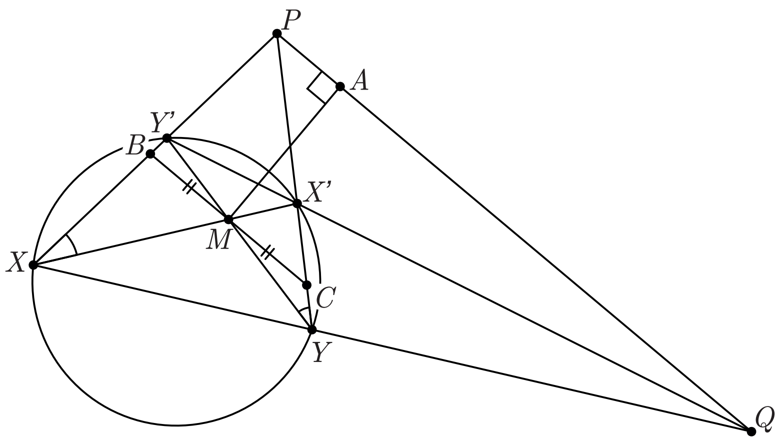
Comment 1. Clearly, the key point Z from the solution above can be introduced in several different ways, e.g., as the second meeting point of the circle CMY and the line AM, or as the second meeting point of the circles CMY and BMX, etc.

For some of definitions of Z its location is not obvious. For instance, if Z is defined as a common point of AM and the perpendicular through X to PX, it is not clear that Z lies on the ray AM beyond M. To avoid such slippery details some more restrictions on the construction may be required.

Comment 2. Let us discuss a connection to the Miquel point of a cyclic quadrilateral. Set $X' = MX \cap PC$, $Y' = MY \cap PB$, and $Q = XY \cap X'Y'$ (see the figure below).

We claim that $BC \parallel PQ$. (One way of proving this is the following. Notice that the quadruple of lines PX, PM, PY, PQ is harmonic, hence the quadruple B, M, C, $PQ \cap BC$ of their intersection points with BC is harmonic. Since M is the midpoint of BC, $PQ \cap BC$ is an ideal point, i.e., $PQ \parallel BC$.)

It follows from the given equality $\angle PXM = \angle PYM$ that the quadrilateral $XYX'Y'$ is cyclic. Note that A is the projection of M onto PQ. By a known description, A is the Miquel point for the sidelines $XY, XY', X'Y, X'Y'$. In particular, the circle PXY passes through A.



Comment 3. An alternative approach is the following. One can note that the (oriented) lengths of the segments CY and BX are both linear functions of a parameter $t = \cot \angle PXM$. As t varies, the intersection point S of the perpendicular bisectors of PX and PY traces a fixed line, thus the family of circles PXY has a fixed common point (other than P). By checking particular cases, one can show that this fixed point is A.

Comment 4. The problem states that $\angle PXM = \angle PYM$ implies that $APXY$ is cyclic. The original submission claims that these two conditions are in fact equivalent. The Problem Selection Committee omitted the converse part, since it follows easily from the direct one, by reversing arguments.

G3. A circle ω of radius 1 is given. A collection T of triangles is called *good*, if the following conditions hold:

(i) each triangle from T is inscribed in ω;

(ii) no two triangles from T have a common interior point.

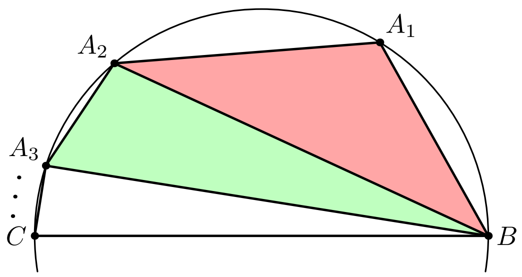
Determine all positive real numbers t such that, for each positive integer n, there exists a good collection of n triangles, each of perimeter greater than t.

(South Africa)

Answer: $t \in (0, 4]$.

Solution. First, we show how to construct a good collection of n triangles, each of perimeter greater than 4. This will show that all $t \leqslant 4$ satisfy the required conditions.

Construct inductively an $(n+2)$-gon $BA_1A_2 \ldots A_nC$ inscribed in ω such that BC is a diameter, and BA_1A_2, BA_2A_3, $\ldots$, $BA_{n-1}A_n$, BA_nC is a good collection of n triangles. For $n = 1$, take any triangle BA_1C inscribed in ω such that BC is a diameter; its perimeter is greater than $2BC = 4$. To perform the inductive step, assume that the $(n+2)$-gon $BA_1A_2 \ldots A_nC$ is already constructed. Since $A_nB + A_nC + BC > 4$, one can choose a point A_{n+1} on the small arc $\widehat{CA_n}$, close enough to C, so that $A_nB + A_nA_{n+1} + BA_{n+1}$ is still greater than 4. Thus each of these new triangles BA_nA_{n+1} and $BA_{n+1}C$ has perimeter greater than 4, which completes the induction step.



We proceed by showing that no $t > 4$ satisfies the conditions of the problem. To this end, we assume that there exists a good collection T of n triangles, each of perimeter greater than t, and then bound n from above.

Take $\varepsilon > 0$ such that $t = 4 + 2\varepsilon$.

Claim. There exists a positive constant $\sigma = \sigma(\varepsilon)$ such that any triangle Δ with perimeter $2s \geqslant 4 + 2\varepsilon$, inscribed in ω, has area $S(\Delta)$ at least σ.

Proof. Let a, b, c be the side lengths of Δ. Since Δ is inscribed in ω, each side has length at most 2. Therefore, $s - a \geqslant (2 + \varepsilon) - 2 = \varepsilon$. Similarly, $s - b \geqslant \varepsilon$ and $s - c \geqslant \varepsilon$. By Heron's formula, $S(\Delta) = \sqrt{s(s-a)(s-b)(s-c)} \geqslant \sqrt{(2+\varepsilon)\varepsilon^3}$. Thus we can set $\sigma(\varepsilon) = \sqrt{(2+\varepsilon)\varepsilon^3}$. □

Now we see that the total area S of all triangles from T is at least $n\sigma(\varepsilon)$. On the other hand, S does not exceed the area of the disk bounded by ω. Thus $n\sigma(\varepsilon) \leqslant \pi$, which means that n is bounded from above.

Comment 1. One may prove the Claim using the formula $S = \dfrac{abc}{4R}$ instead of Heron's formula.

Comment 2. In the statement of the problem condition (i) could be replaced by a weaker one: each triangle from T lies within ω. This does not affect the solution above, but reduces the number of ways to prove the Claim.

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G4. A point T is chosen inside a triangle ABC. Let A_1, B_1, and C_1 be the reflections of T in BC, CA, and AB, respectively. Let Ω be the circumcircle of the triangle $A_1B_1C_1$. The lines A_1T, B_1T, and C_1T meet Ω again at A_2, B_2, and C_2, respectively. Prove that the lines AA_2, BB_2, and CC_2 are concurrent on Ω.

(Mongolia)

Solution. By $\measuredangle(\ell, n)$ we always mean the directed angle of the lines ℓ and n, taken modulo 180°.

Let CC_2 meet Ω again at K (as usual, if CC_2 is tangent to Ω, we set $T = C_2$). We show that the line BB_2 contains K; similarly, AA_2 will also pass through K. For this purpose, it suffices to prove that

$$\measuredangle(C_2C, C_2A_1) = \measuredangle(B_2B, B_2A_1). \tag{1}$$

By the problem condition, CB and CA are the perpendicular bisectors of TA_1 and TB_1, respectively. Hence, C is the circumcentre of the triangle A_1TB_1. Therefore,

$$\measuredangle(CA_1, CB) = \measuredangle(CB, CT) = \measuredangle(B_1A_1, B_1T) = \measuredangle(B_1A_1, B_1B_2).$$

In circle Ω we have $\measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2)$. Thus,

$$\measuredangle(CA_1, CB) = \measuredangle(B_1A_1, B_1B_2) = \measuredangle(C_2A_1, C_2B_2). \tag{2}$$

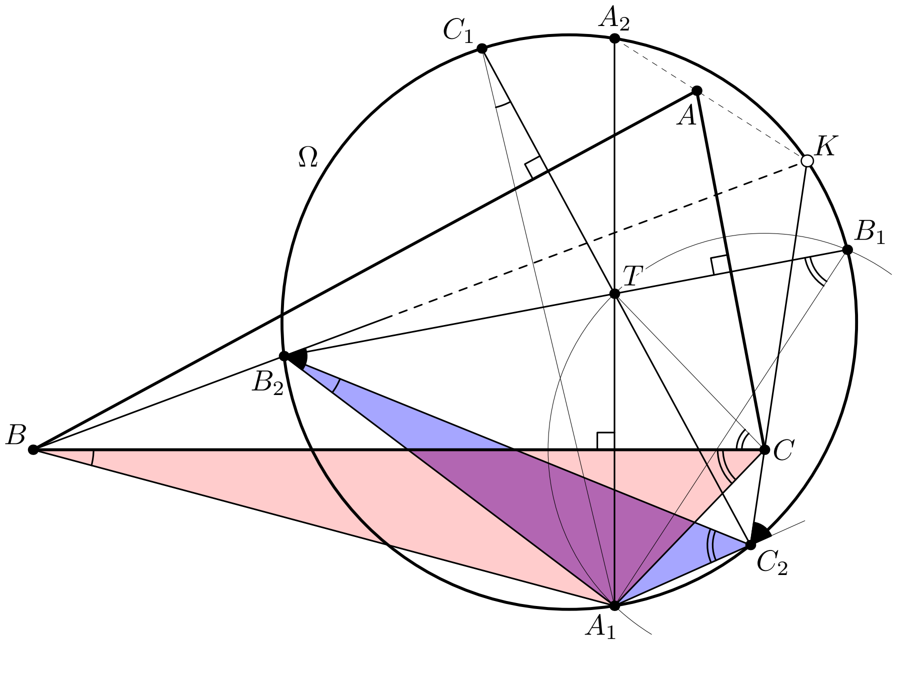
Similarly, we get

$$\measuredangle(BA_1, BC) = \measuredangle(C_1A_1, C_1C_2) = \measuredangle(B_2A_1, B_2C_2). \tag{3}$$

The two obtained relations yield that the triangles A_1BC and $A_1B_2C_2$ are similar and equioriented, hence

$$\frac{A_1B_2}{A_1B} = \frac{A_1C_2}{A_1C} \quad \text{and} \quad \measuredangle(A_1B, A_1C) = \measuredangle(A_1B_2, A_1C_2).$$

The second equality may be rewritten as $\measuredangle(A_1B, A_1B_2) = \measuredangle(A_1C, A_1C_2)$, so the triangles A_1BB_2 and A_1CC_2 are also similar and equioriented. This establishes (1).



Comment 1. In fact, the triangle A_1BC is an image of $A_1B_2C_2$ under a spiral similarity centred at A_1; in this case, the triangles ABB_2 and ACC_2 are also spirally similar with the same centre.

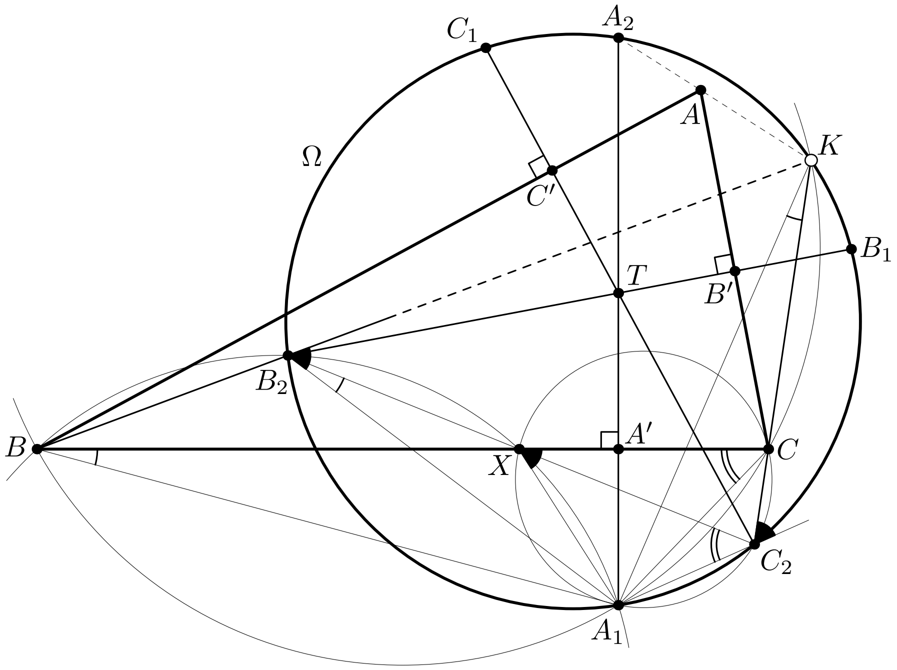
Comment 2. After obtaining (2) and (3), one can finish the solution in different ways.

For instance, introducing the point $X = BC \cap B_2C_2$, one gets from these relations that the 4-tuples (A_1, B, B_2, X) and (A_1, C, C_2, X) are both cyclic. Therefore, K is the Miquel point of the lines BB_2, CC_2, BC, and B_2C_2; this yields that the meeting point of BB_2 and CC_2 lies on Ω.

Yet another way is to show that the points A_1, B, C, and K are concyclic, as

$$\measuredangle(KC, KA_1) = \measuredangle(B_2C_2, B_2A_1) = \measuredangle(BC, BA_1).$$

By symmetry, the second point K' of intersection of BB_2 with Ω is also concyclic to A_1, B, and C, hence $K' = K$.



Comment 3. The requirement that the common point of the lines AA_2, BB_2, and CC_2 should lie on Ω may seem to make the problem easier, since it suggests some approaches. On the other hand, there are also different ways of showing that the lines AA_2, BB_2, and CC_2 are just concurrent.

In particular, the problem conditions yield that the lines A_2T, B_2T, and C_2T are perpendicular to the corresponding sides of the triangle ABC. One may show that the lines AT, BT, and CT are also perpendicular to the corresponding sides of the triangle $A_2B_2C_2$, i.e., the triangles ABC and $A_2B_2C_2$ are *orthologic*, and their orthology centres coincide. It is known that such triangles are also *perspective*, i.e. the lines AA_2, BB_2, and CC_2 are concurrent (in projective sense).

To show this mutual orthology, one may again apply angle chasing, but there are also other methods. Let A', B', and C' be the projections of T onto the sides of the triangle ABC. Then $A_2T \cdot TA' = B_2T \cdot TB' = C_2T \cdot TC'$, since all three products equal (minus) half the power of T with respect to Ω. This means that A_2, B_2, and C_2 are the poles of the sidelines of the triangle ABC with respect to some circle centred at T and having *pure imaginary* radius (in other words, the reflections of A_2, B_2, and C_2 in T are the poles of those sidelines with respect to some regular circle centred at T). Hence, dually, the vertices of the triangle ABC are also the poles of the sidelines of the triangle $A_2B_2C_2$.

G5. Let ABC be a triangle with circumcircle ω and incentre I. A line ℓ intersects the lines AI, BI, and CI at points D, E, and F, respectively, distinct from the points A, B, C, and I. The perpendicular bisectors x, y, and z of the segments AD, BE, and CF, respectively determine a triangle Θ. Show that the circumcircle of the triangle Θ is tangent to ω.

(Denmark)

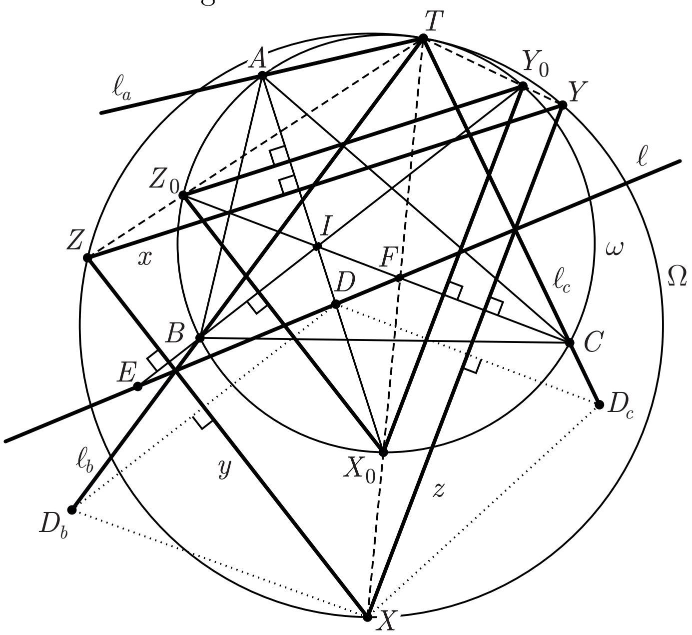
Preamble. Let $X = y \cap z$, $Y = x \cap z$, $Z = x \cap y$ and let Ω denote the circumcircle of the triangle XYZ. Denote by X_0, Y_0, and Z_0 the second intersection points of AI, BI and CI, respectively, with ω. It is known that Y_0Z_0 is the perpendicular bisector of AI, Z_0X_0 is the perpendicular bisector of BI, and X_0Y_0 is the perpendicular bisector of CI. In particular, the triangles XYZ and $X_0Y_0Z_0$ are homothetic, because their corresponding sides are parallel.

The solutions below mostly exploit the following approach. Consider the triangles XYZ and $X_0Y_0Z_0$, or some other pair of homothetic triangles Δ and δ inscribed into Ω and ω, respectively. In order to prove that Ω and ω are tangent, it suffices to show that the centre T of the homothety taking Δ to δ lies on ω (or Ω), or, in other words, to show that Δ and δ are perspective (i.e., the lines joining corresponding vertices are concurrent), with their perspector lying on ω (or Ω).

We use directed angles throughout all the solutions.

Solution 1.

Claim 1. The reflections ℓ_a, ℓ_b and ℓ_c of the line ℓ in the lines x, y, and z, respectively, are concurrent at a point T which belongs to ω.



Proof. Notice that $\sphericalangle(\ell_b, \ell_c) = \sphericalangle(\ell_b, \ell) + \sphericalangle(\ell, \ell_c) = 2\sphericalangle(y, \ell) + 2\sphericalangle(\ell, z) = 2\sphericalangle(y, z)$. But $y \perp BI$ and $z \perp CI$ implies $\sphericalangle(y, z) = \sphericalangle(BI, IC)$, so, since $2\sphericalangle(BI, IC) = \sphericalangle(BA, AC)$, we obtain

$$\sphericalangle(\ell_b, \ell_c) = \sphericalangle(BA, AC). \tag{1}$$

Since A is the reflection of D in x, A belongs to ℓ_a; similarly, B belongs to ℓ_b. Then (1) shows that the common point T' of ℓ_a and ℓ_b lies on ω; similarly, the common point T'' of ℓ_c and ℓ_b lies on ω.

If $B \notin \ell_a$ and $B \notin \ell_c$, then T' and T'' are the second point of intersection of ℓ_b and ω, hence they coincide. Otherwise, if, say, $B \in \ell_c$, then $\ell_c = BC$, so $\sphericalangle(BA, AC) = \sphericalangle(\ell_b, \ell_c) = \sphericalangle(\ell_b, BC)$, which shows that ℓ_b is tangent at B to ω and $T' = T'' = B$. So T' and T'' coincide in all the cases, and the conclusion of the claim follows. $\square$

Now we prove that X, X_0, T are collinear. Denote by D_b and D_c the reflections of the point D in the lines y and z, respectively. Then D_b lies on ℓ_b, D_c lies on ℓ_c, and

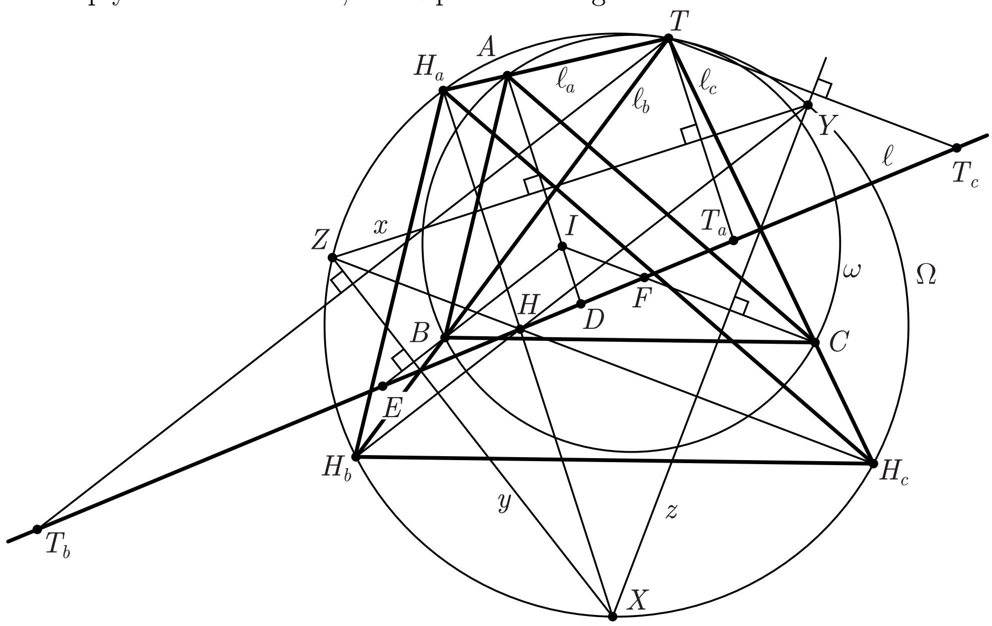
$$\begin{aligned}\measuredangle(D_bX, XD_c) &= \measuredangle(D_bX, DX) + \measuredangle(DX, XD_c) = 2\measuredangle(y, DX) + 2\measuredangle(DX, z) = 2\measuredangle(y, z)\\ &= \measuredangle(BA, AC) = \measuredangle(BT, TC),\end{aligned}$$

hence the quadrilateral XD_bTD_c is cyclic. Notice also that since $XD_b = XD = XD_c$, the points D, D_b, D_c lie on a circle with centre X. Using in this circle the diameter D_cD_c' yields $\measuredangle(D_bD_c, D_cX) = 90° + \measuredangle(D_bD_c', D_c'X) = 90° + \measuredangle(D_bD, DD_c)$. Therefore,

$$\begin{aligned}\measuredangle(\ell_b, XT) &= \measuredangle(D_bT, XT) = \measuredangle(D_bD_c, D_cX) = 90° + \measuredangle(D_bD, DD_c)\\ &= 90° + \measuredangle(BI, IC) = \measuredangle(BA, AI) = \measuredangle(BA, AX_0) = \measuredangle(BT, TX_0) = \measuredangle(\ell_b, X_0T),\end{aligned}$$

so the points X, X_0, T are collinear. By a similar argument, Y, Y_0, T and Z, Z_0, T are collinear. As mentioned in the preamble, the statement of the problem follows.

Comment 1. After proving Claim 1 one may proceed in another way. As it was shown, the reflections of ℓ in the sidelines of XYZ are concurrent at T. Thus ℓ is the Steiner line of T with respect to ΔXYZ (that is the line containing the reflections T_a, T_b, T_c of T in the sidelines of XYZ). The properties of the Steiner line imply that T lies on Ω, and ℓ passes through the orthocentre H of the triangle XYZ.



Let H_a, H_b, and H_c be the reflections of the point H in the lines x, y, and z, respectively. Then the triangle $H_aH_bH_c$ is inscribed in Ω and homothetic to ABC (by an easy angle chasing). Since $H_a \in \ell_a$, $H_b \in \ell_b$, and $H_c \in \ell_c$, the triangles $H_aH_bH_c$ and ABC form a required pair of triangles Δ and δ mentioned in the preamble.

Comment 2. The following observation shows how one may guess the description of the tangency point T from Solution 1.

Let us fix a direction and move the line ℓ parallel to this direction with constant speed.

Then the points D, E, and F are moving with constant speeds along the lines AI, BI, and CI, respectively. In this case x, y, and z are moving with constant speeds, defining a family of homothetic triangles XYZ with a common centre of homothety T. Notice that the triangle $X_0Y_0Z_0$ belongs to this family (for ℓ passing through I). We may specify the location of T considering the degenerate case when x, y, and z are concurrent. In this degenerate case all the lines x, y, z, ℓ, ℓ_a, ℓ_b, ℓ_c have a common point. Note that the lines ℓ_a, ℓ_b, ℓ_c remain constant as ℓ is moving (keeping its direction). Thus T should be the common point of ℓ_a, ℓ_b, and ℓ_c, lying on ω.

Solution 2. As mentioned in the preamble, it is sufficient to prove that the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to ω. Thus, it suffices to prove that $\measuredangle(TX_0, TY_0) = \measuredangle(Z_0X_0, Z_0Y_0)$, or, equivalently, $\measuredangle(XX_0, YY_0) = \measuredangle(Z_0X_0, Z_0Y_0)$.

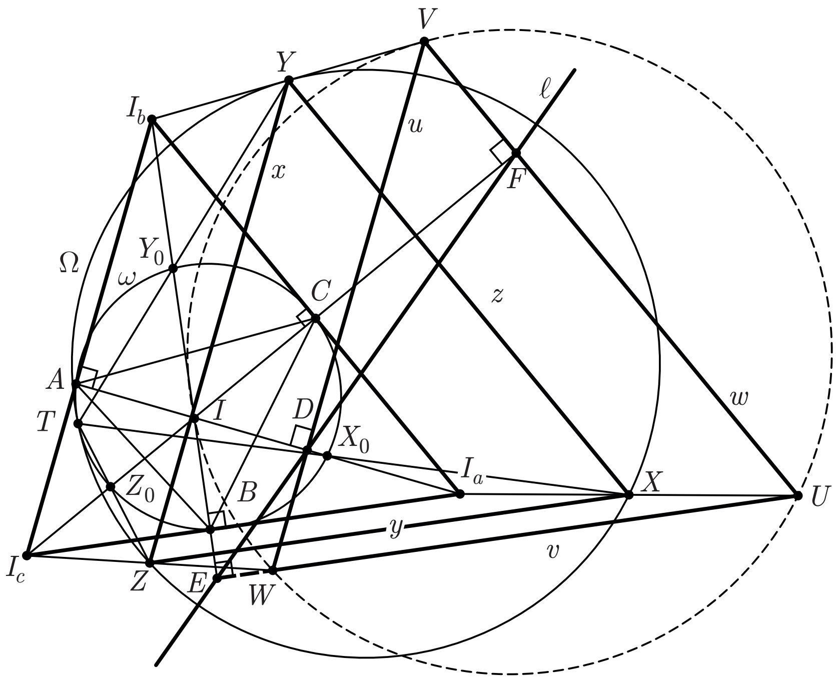
Recall that YZ and Y_0Z_0 are the perpendicular bisectors of AD and AI, respectively. Then, the vector $\overrightarrow{x}$ perpendicular to YZ and shifting the line Y_0Z_0 to YZ is equal to $\frac{1}{2}\overrightarrow{ID}$. Define the shifting vectors $\overrightarrow{y} = \frac{1}{2}\overrightarrow{IE}$, $\overrightarrow{z} = \frac{1}{2}\overrightarrow{IF}$ similarly. Consider now the triangle UVW formed by the perpendiculars to AI, BI, and CI through D, E, and F, respectively (see figure below). This is another triangle whose sides are parallel to the corresponding sides of XYZ.

Claim 2. $\overrightarrow{IU} = 2\overrightarrow{X_0X}$, $\overrightarrow{IV} = 2\overrightarrow{Y_0Y}$, $\overrightarrow{IW} = 2\overrightarrow{Z_0Z}$.

Proof. We prove one of the relations, the other proofs being similar. To prove the equality of two vectors it suffices to project them onto two non-parallel axes and check that their projections are equal.

The projection of $\overrightarrow{X_0X}$ onto IB equals $\vec{y}$, while the projection of $\overrightarrow{IU}$ onto IB is $\overrightarrow{IE} = 2\vec{y}$. The projections onto the other axis IC are $\vec{z}$ and $\overrightarrow{IF} = 2\vec{z}$. Then $\overrightarrow{IU} = 2\overrightarrow{X_0X}$ follows. □

Notice that the line ℓ is the Simson line of the point I with respect to the triangle UVW; thus U, V, W, and I are concyclic. It follows from Claim 2 that $\measuredangle(XX_0, YY_0) = \measuredangle(IU, IV) = \measuredangle(WU, WV) = \measuredangle(Z_0X_0, Z_0Y_0)$, and we are done.



Solution 3. Let I_a, I_b, and I_c be the excentres of triangle ABC corresponding to A, B, and C, respectively. Also, let u, v, and w be the lines through D, E, and F which are perpendicular to AI, BI, and CI, respectively, and let UVW be the triangle determined by these lines, where $u = VW$, $v = UW$ and $w = UV$ (see figure above).

Notice that the line u is the reflection of I_bI_c in the line x, because u, x, and I_bI_c are perpendicular to AD and x is the perpendicular bisector of AD. Likewise, v and I_aI_c are reflections of each other in y, while w and I_aI_b are reflections of each other in z. It follows that X, Y, and Z are the midpoints of UI_a, VI_b and WI_c, respectively, and that the triangles UVW, XYZ and $I_aI_bI_c$ are either translates of each other or homothetic with a common homothety centre.

Construct the points T and S such that the quadrilaterals $UVIW$, $XYTZ$ and $I_aI_bSI_c$ are homothetic. Then T is the midpoint of IS. Moreover, note that ℓ is the Simson line of the point I with respect to the triangle UVW, hence I belongs to the circumcircle of the triangle UVW, therefore T belongs to Ω.

Consider now the homothety or translation h_1 that maps $XYZT$ to $I_aI_bI_cS$ and the homothety h_2 with centre I and factor $\frac{1}{2}$. Furthermore, let $h = h_2 \circ h_1$. The transform h can be a homothety or a translation, and

$$h(T) = h_2(h_1(T)) = h_2(S) = T,$$

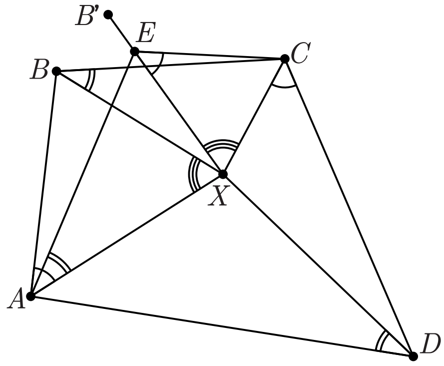
hence T is a fixed point of h. So, h is a homothety with centre T. Note that h_2 maps the excentres I_a, I_b, I_c to X_0, Y_0, Z_0 defined in the preamble. Thus the centre T of the homothety taking XYZ to $X_0Y_0Z_0$ belongs to Ω, and this completes the proof.

G6. A convex quadrilateral $ABCD$ satisfies $AB \cdot CD = BC \cdot DA$. A point X is chosen inside the quadrilateral so that $\angle XAB = \angle XCD$ and $\angle XBC = \angle XDA$. Prove that $\angle AXB + \angle CXD = 180^\circ$.

(Poland)

Solution 1. Let B' be the reflection of B in the internal angle bisector of $\angle AXC$, so that $\angle AXB' = \angle CXB$ and $\angle CXB' = \angle AXB$. If X, D, and B' are collinear, then we are done. Now assume the contrary.

On the ray XB' take a point E such that $XE \cdot XB = XA \cdot XC$, so that $\triangle AXE \sim \triangle BXC$ and $\triangle CXE \sim \triangle BXA$. We have $\angle XCE + \angle XCD = \angle XBA + \angle XAB < 180^\circ$ and $\angle XAE + \angle XAD = \angle XDA + \angle XAD < 180^\circ$, which proves that X lies inside the angles $\angle ECD$ and $\angle EAD$ of the quadrilateral $EADC$. Moreover, X lies in the interior of exactly one of the two triangles EAD, ECD (and in the exterior of the other).



The similarities mentioned above imply $XA \cdot BC = XB \cdot AE$ and $XB \cdot CE = XC \cdot AB$. Multiplying these equalities with the given equality $AB \cdot CD = BC \cdot DA$, we obtain $XA \cdot CD \cdot CE = XC \cdot AD \cdot AE$, or, equivalently,

$$\frac{XA \cdot DE}{AD \cdot AE} = \frac{XC \cdot DE}{CD \cdot CE}. \tag{$*$}$$

Lemma. Let PQR be a triangle, and let X be a point in the interior of the angle QPR such that $\angle QPX = \angle PRX$. Then $\dfrac{PX \cdot QR}{PQ \cdot PR} < 1$ if and only if X lies in the interior of the triangle PQR.

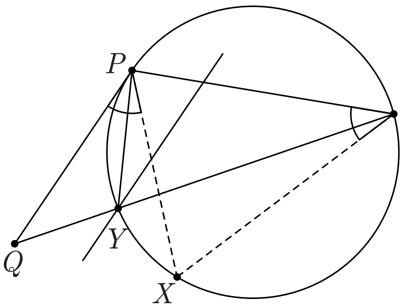
Proof. The locus of points X with $\angle QPX = \angle PRX$ lying inside the angle QPR is an arc α of the circle γ through R tangent to PQ at P. Let γ intersect the line QR again at Y (if γ is tangent to QR, then set $Y = R$). The similarity $\triangle QPY \sim \triangle QRP$ yields $PY = \dfrac{PQ \cdot PR}{QR}$. Now it suffices to show that $PX < PY$ if and only if X lies in the interior of the triangle PQR. Let m be a line through Y parallel to PQ. Notice that the points Z of γ satisfying $PZ < PY$ are exactly those between the lines m and PQ.

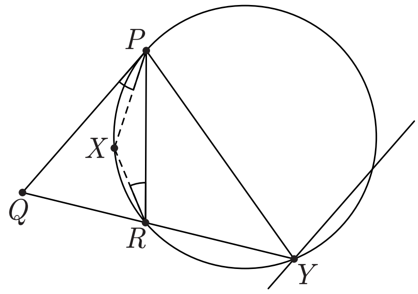
Case 1: Y lies in the segment QR (see the left figure below).

In this case Y splits α into two arcs $\overparen{PY}$ and $\overparen{YR}$. The arc $\overparen{PY}$ lies inside the triangle PQR, and $\overparen{PY}$ lies between m and PQ, hence $PX < PY$ for points $X \in \overparen{PY}$. The other arc $\overparen{YR}$ lies outside triangle PQR, and $\overparen{YR}$ is on the opposite side of m than P, hence $PX > PY$ for $X \in \overparen{YR}$.

Case 2: Y lies on the ray QR beyond R (see the right figure below).

In this case the whole arc α lies inside triangle PQR, and between m and PQ, thus $PX < PY$ for all $X \in \alpha$. $\square$





Applying the Lemma (to $\triangle EAD$ with the point X, and to $\triangle ECD$ with the point X), we obtain that exactly one of two expressions $\dfrac{XA \cdot DE}{AD \cdot AE}$ and $\dfrac{XC \cdot DE}{CD \cdot CE}$ is less than 1, which contradicts $(*)$.

Comment 1. One may show that $AB \cdot CD = XA \cdot XC + XB \cdot XD$. We know that D, X, E are collinear and $\angle DCE = \angle CXD = 180^\circ - \angle AXB$. Therefore,

$$AB \cdot CD = XB \cdot \frac{\sin \angle AXB}{\sin \angle BAX} \cdot DE \cdot \frac{\sin \angle CED}{\sin \angle DCE} = XB \cdot DE.$$

Furthermore, $XB \cdot DE = XB \cdot (XD + XE) = XB \cdot XD + XB \cdot XE = XB \cdot XD + XA \cdot XC$.

Comment 2. For a convex quadrilateral $ABCD$ with $AB \cdot CD = BC \cdot DA$, it is known that $\angle DAC + \angle ABD + \angle BCA + \angle CDB = 180^\circ$ (among other, it was used as a problem on the Regional round of All-Russian olympiad in 2012), but it seems that there is no essential connection between this fact and the original problem.

Solution 2. The solution consists of two parts. In Part 1 we show that it suffices to prove that

$$\frac{XB}{XD} = \frac{AB}{CD} \tag{1}$$

and

$$\frac{XA}{XC} = \frac{DA}{BC}. \tag{2}$$

In Part 2 we establish these equalities.

Part 1. Using the sine law and applying (1) we obtain

$$\frac{\sin \angle AXB}{\sin \angle XAB} = \frac{AB}{XB} = \frac{CD}{XD} = \frac{\sin \angle CXD}{\sin \angle XCD},$$

so $\sin \angle AXB = \sin \angle CXD$ by the problem conditions. Similarly, (2) yields $\sin \angle DXA = \sin \angle BXC$. If at least one of the pairs $(\angle AXB, \angle CXD)$ and $(\angle BXC, \angle DXA)$ consists of supplementary angles, then we are done. Otherwise, $\angle AXB = \angle CXD$ and $\angle DXA = \angle BXC$. In this case $X = AC \cap BD$, and the problem conditions yield that $ABCD$ is a parallelogram and hence a rhombus. In this last case the claim also holds.

Part 2. To prove the desired equality (1), invert $ABCD$ at centre X with unit radius; the images of points are denoted by primes.

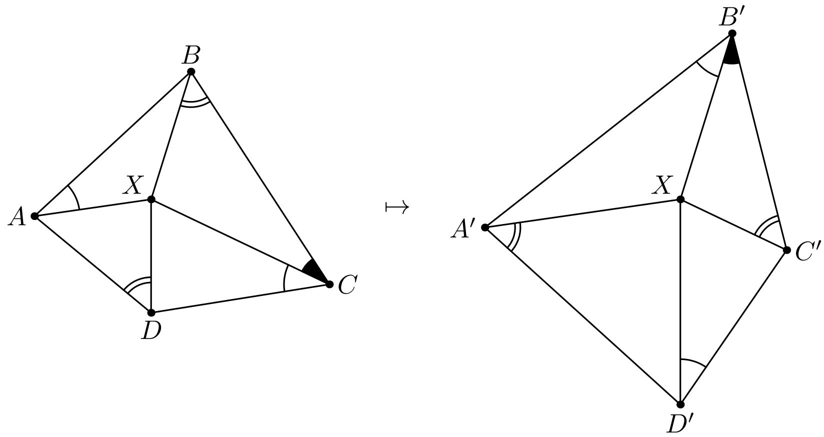
We have

$$\angle A'B'C' = \angle XB'A' + \angle XB'C' = \angle XAB + \angle XCB = \angle XCD + \angle XCB = \angle BCD.$$

Similarly, the corresponding angles of quadrilaterals $ABCD$ and $D'A'B'C'$ are equal.

Moreover, we have

$$A'B' \cdot C'D' = \frac{AB}{XA \cdot XB} \cdot \frac{CD}{XC \cdot XD} = \frac{BC}{XB \cdot XC} \cdot \frac{DA}{XD \cdot DA} = B'C' \cdot D'A'.$$



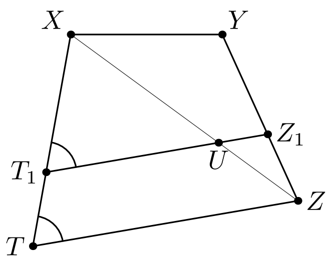
Now we need the following Lemma.

Lemma. Assume that the corresponding angles of convex quadrilaterals $XYZT$ and $X'Y'Z'T'$ are equal, and that $XY \cdot ZT = YZ \cdot TX$ and $X'Y' \cdot Z'T' = Y'Z' \cdot T'X'$. Then the two quadrilaterals are similar.

Proof. Take the quadrilateral XYZ_1T_1 similar to $X'Y'Z'T'$ and sharing the side XY with $XYZT$, such that Z_1 and T_1 lie on the rays YZ and XT, respectively, and $Z_1T_1 \parallel ZT$. We need to prove that $Z_1 = Z$ and $T_1 = T$. Assume the contrary. Without loss of generality, $TX > XT_1$. Let segments XZ and Z_1T_1 intersect at U. We have

$$\frac{T_1X}{T_1Z_1} < \frac{T_1X}{T_1U} = \frac{TX}{ZT} = \frac{XY}{YZ} < \frac{XY}{YZ_1},$$

thus $T_1X \cdot YZ_1 < T_1Z_1 \cdot XY$. A contradiction. □



It follows from the Lemma that the quadrilaterals $ABCD$ and $D'A'B'C'$ are similar, hence

$$\frac{BC}{AB} = \frac{A'B'}{D'A'} = \frac{AB}{XA \cdot XB} \cdot \frac{XD \cdot XA}{DA} = \frac{AB}{AD} \cdot \frac{XD}{XB},$$

and therefore

$$\frac{XB}{XD} = \frac{AB^2}{BC \cdot AD} = \frac{AB^2}{AB \cdot CD} = \frac{AB}{CD}.$$

We obtain (1), as desired; (2) is proved similarly.

Comment. Part 1 is an easy one, while part 2 seems to be crucial. On the other hand, after the proof of the similarity $D'A'B'C' \sim ABCD$ one may finish the solution in different ways, e.g., as follows. The similarity taking $D'A'B'C'$ to $ABCD$ maps X to the point X' *isogonally conjugate* of X with respect to $ABCD$ (i.e. to the point X' inside $ABCD$ such that $\angle BAX = \angle DAX'$, $\angle CBX = \angle ABX'$, $\angle DCX = \angle BCX'$, $\angle ADX = \angle CDX'$). It is known that the required equality $\angle AXB + \angle CXD = 180°$ is one of known conditions on a point X inside $ABCD$ equivalent to the existence of its isogonal conjugate.

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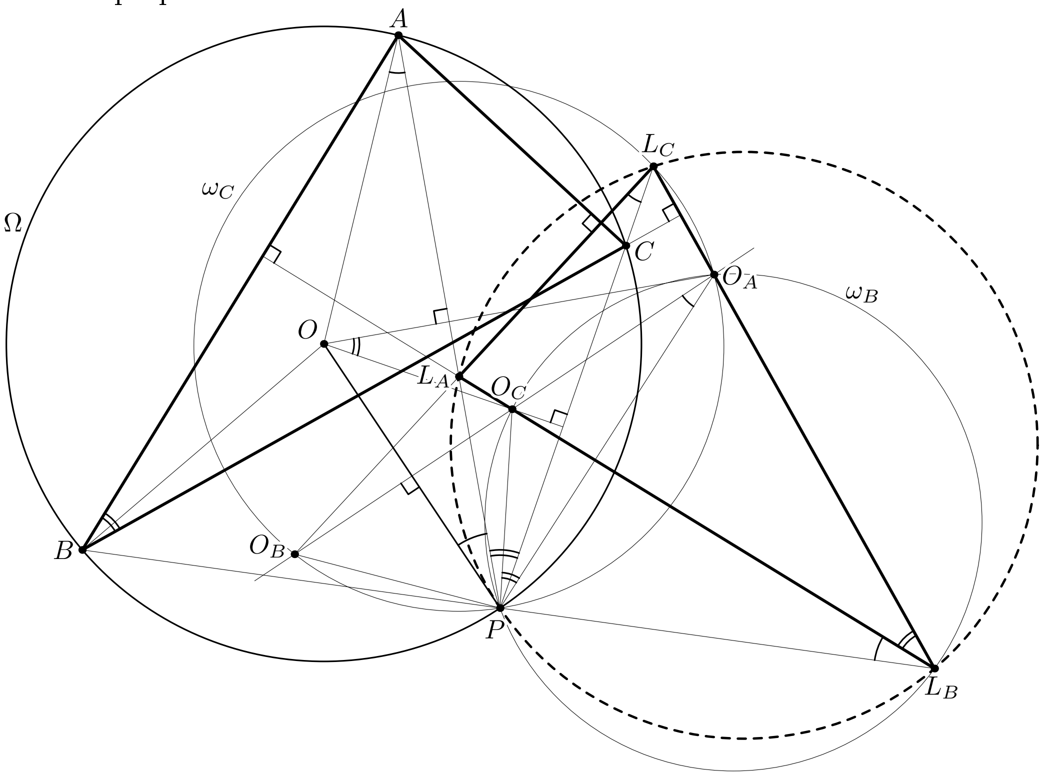
G7. Let O be the circumcentre, and Ω be the circumcircle of an acute-angled triangle ABC. Let P be an arbitrary point on Ω, distinct from A, B, C, and their antipodes in Ω. Denote the circumcentres of the triangles AOP, BOP, and COP by O_A, O_B, and O_C, respectively. The lines ℓ_A, ℓ_B, and ℓ_C perpendicular to BC, CA, and AB pass through O_A, O_B, and O_C, respectively. Prove that the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C is tangent to the line OP.

(Russia)

Solution. As usual, we denote the directed angle between the lines a and b by $\measuredangle(a,b)$. We frequently use the fact that $a_1 \perp a_2$ and $b_1 \perp b_2$ yield $\measuredangle(a_1,b_1) = \measuredangle(a_2,b_2)$.

Let the lines ℓ_B and ℓ_C meet at L_A; define the points L_B and L_C similarly. Note that the sidelines of the triangle $L_AL_BL_C$ are perpendicular to the corresponding sidelines of ABC. Points O_A, O_B, O_C are located on the corresponding sidelines of $L_AL_BL_C$; moreover, O_A, O_B, O_C all lie on the perpendicular bisector of OP.



Claim 1. The points L_B, P, O_A, and O_C are concyclic.

Proof. Since O is symmetric to P in O_AO_C, we have

$$\measuredangle(O_AP, O_CP) = \measuredangle(O_CO, O_AO) = \measuredangle(CP, AP) = \measuredangle(CB, AB) = \measuredangle(O_AL_B, O_CL_B). \qquad \square$$

Denote the circle through L_B, P, O_A, and O_C by ω_B. Define the circles ω_A and ω_C similarly.

Claim 2. The circumcircle of the triangle $L_AL_BL_C$ passes through P.

Proof. From cyclic quadruples of points in the circles ω_B and ω_C, we have

$$\begin{aligned}\measuredangle(L_CL_A, L_CP) &= \measuredangle(L_CO_B, L_CP) = \measuredangle(O_AO_B, O_AP)\\ &= \measuredangle(O_AO_C, O_AP) = \measuredangle(L_BO_C, L_BP) = \measuredangle(L_BL_A, L_BP).\end{aligned} \qquad \square$$

Claim 3. The points P, L_C, and C are collinear.

Proof. We have $\measuredangle(PL_C, L_CL_A) = \measuredangle(PL_C, L_CO_B) = \measuredangle(PO_A, O_AO_B)$. Further, since O_A is the centre of the circle AOP, $\measuredangle(PO_A, O_AO_B) = \measuredangle(PA, AO)$. As O is the circumcentre of the triangle PCA, $\measuredangle(PA, AO) = \pi/2 - \measuredangle(CA, CP) = \measuredangle(CP, L_CL_A)$. We obtain $\measuredangle(PL_C, L_CL_A) = \measuredangle(CP, L_CL_A)$, which shows that $P \in CL_C$. $\square$

Similarly, the points P, L_A, A are collinear, and the points P, L_B, B are also collinear.

Finally, the computation above also shows that

$$\measuredangle(OP, PL_A) = \measuredangle(PA, AO) = \measuredangle(PL_C, L_CL_A),$$

which means that OP is tangent to the circle $PL_AL_BL_C$.

Comment 1. The proof of Claim 2 may be replaced by the following remark: since P belongs to the circles ω_A and ω_C, P is the Miquel point of the four lines ℓ_A, ℓ_B, ℓ_C, and $O_AO_BO_C$.

Comment 2. Claims 2 and 3 can be proved in several different ways and, in particular, in the reverse order.

Claim 3 implies that the triangles ABC and $L_AL_BL_C$ are perspective with perspector P. Claim 2 can be derived from this observation using spiral similarity. Consider the centre Q of the spiral similarity that maps ABC to $L_AL_BL_C$. From known spiral similarity properties, the points L_A, L_B, P, Q are concyclic, and so are L_A, L_C, P, Q.

Comment 3. The final conclusion can also be proved it terms of spiral similarity: the spiral similarity with centre Q located on the circle ABC maps the circle ABC to the circle $PL_AL_BL_C$. Thus these circles are orthogonal.

Comment 4. Notice that the homothety with centre O and ratio 2 takes O_A to A' that is the common point of tangents to Ω at A and P. Similarly, let this homothety take O_B to B' and O_C to C'. Let the tangents to Ω at B and C meet at A'', and define the points B'' and C'' similarly. Now, replacing labels O with I, Ω with ω, and swapping labels $A \leftrightarrow A''$, $B \leftrightarrow B''$, $C \leftrightarrow C''$ we obtain the following

Reformulation. Let ω be the incircle, and let I be the incentre of a triangle ABC. Let P be a point of ω (other than the points of contact of ω with the sides of ABC). The tangent to ω at P meets the lines AB, BC, and CA at A', B', and C', respectively. Line ℓ_A parallel to the internal angle bisector of $\angle BAC$ passes through A'; define lines ℓ_B and ℓ_C similarly. Prove that the line IP is tangent to the circumcircle of the triangle formed by ℓ_A, ℓ_B, and ℓ_C.

Though this formulation is equivalent to the original one, it seems more challenging, since the point of contact is now “hidden”.

Number Theory

N1. Determine all pairs (n,k) of distinct positive integers such that there exists a positive integer s for which the numbers of divisors of sn and of sk are equal.

(Ukraine)

Answer: All pairs (n,k) such that $n \nmid k$ and $k \nmid n$.

Solution. As usual, the number of divisors of a positive integer n is denoted by $d(n)$. If $n = \prod_i p_i^{\alpha_i}$ is the prime factorisation of n, then $d(n) = \prod_i(\alpha_i + 1)$.

We start by showing that one cannot find any suitable number s if $k \mid n$ or $n \mid k$ (and $k \neq n$). Suppose that $n \mid k$, and choose any positive integer s. Then the set of divisors of sn is a proper subset of that of sk, hence $d(sn) < d(sk)$. Therefore, the pair (n,k) does not satisfy the problem requirements. The case $k \mid n$ is similar.

Now assume that $n \nmid k$ and $k \nmid n$. Let $p_1, \ldots, p_t$ be all primes dividing nk, and consider the prime factorisations

$$n = \prod_{i=1}^{t} p_i^{\alpha_i} \quad \text{and} \quad k = \prod_{i=1}^{t} p_i^{\beta_i}.$$

It is reasonable to search for the number s having the form

$$s = \prod_{i=1}^{t} p_i^{\gamma_i}.$$

The (nonnegative integer) exponents γ_i should be chosen so as to satisfy

$$\frac{d(sn)}{d(sk)} = \prod_{i=1}^{t} \frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = 1. \tag{1}$$

First of all, if $\alpha_i = \beta_i$ for some i, then, regardless of the value of γ_i, the corresponding factor in (1) equals 1 and does not affect the product. So we may assume that there is no such index i. For the other factors in (1), the following lemma is useful.

Lemma. Let $\alpha > \beta$ be nonnegative integers. Then, for every integer $M \geqslant \beta + 1$, there exists a nonnegative integer γ such that

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} = \frac{M+1}{M}.$$

Proof.

$$\frac{\alpha + \gamma + 1}{\beta + \gamma + 1} = 1 + \frac{1}{M} \iff \frac{\alpha - \beta}{\beta + \gamma + 1} = \frac{1}{M} \iff \gamma = M(\alpha - \beta) - (\beta + 1) \geqslant 0. \qquad \square$$

Now we can finish the solution. Without loss of generality, there exists an index u such that $\alpha_i > \beta_i$ for $i = 1, 2, \ldots, u$, and $\alpha_i < \beta_i$ for $i = u+1, \ldots, t$. The conditions $n \nmid k$ and $k \nmid n$ mean that $1 \leqslant u \leqslant t-1$.

Choose an integer X greater than all the α_i and β_i. By the lemma, we can define the numbers γ_i so as to satisfy

$$\frac{\alpha_i + \gamma_i + 1}{\beta_i + \gamma_i + 1} = \frac{uX + i}{uX + i - 1} \qquad \text{for } i = 1, 2, \ldots, u, \text{ and}$$

$$\frac{\beta_{u+i} + \gamma_{u+i} + 1}{\alpha_{u+i} + \gamma_{u+i} + 1} = \frac{(t-u)X + i}{(t-u)X + i - 1} \qquad \text{for } i = 1, 2, \ldots, t-u.$$

Then we will have

$$\frac{d(sn)}{d(sk)} = \prod_{i=1}^{u} \frac{uX+i}{uX+i-1} \cdot \prod_{i=1}^{t-u} \frac{(t-u)X+i-1}{(t-u)X+i} = \frac{u(X+1)}{uX} \cdot \frac{(t-u)X}{(t-u)(X+1)} = 1,$$

as required.

Comment. The lemma can be used in various ways, in order to provide a suitable value of s. In particular, one may apply induction on the number t of prime factors, using identities like

$$\frac{n}{n-1} = \frac{n^2}{n^2-1} \cdot \frac{n+1}{n}.$$

N2. Let $n > 1$ be a positive integer. Each cell of an $n \times n$ table contains an integer. Suppose that the following conditions are satisfied:

(i) Each number in the table is congruent to 1 modulo n;

(ii) The sum of numbers in any row, as well as the sum of numbers in any column, is congruent to n modulo n^2.

Let R_i be the product of the numbers in the i^{th} row, and C_j be the product of the numbers in the j^{th} column. Prove that the sums $R_1 + \cdots + R_n$ and $C_1 + \cdots + C_n$ are congruent modulo n^4.

(Indonesia)

Solution 1. Let $A_{i,j}$ be the entry in the i^{th} row and the j^{th} column; let P be the product of all n^2 entries. For convenience, denote $a_{i,j} = A_{i,j} - 1$ and $r_i = R_i - 1$. We show that

$$\sum_{i=1}^{n} R_i \equiv (n-1) + P \pmod{n^4}. \tag{1}$$

Due to symmetry of the problem conditions, the sum of all the C_j is also congruent to $(n-1)+P$ modulo n^4, whence the conclusion.

By condition (i), the number n divides $a_{i,j}$ for all i and j. So, every product of at least two of the $a_{i,j}$ is divisible by n^2, hence

$$R_i = \prod_{j=1}^{n}(1+a_{i,j}) = 1+\sum_{j=1}^{n} a_{i,j} + \sum_{1\leqslant j_1<j_2\leqslant n} a_{i,j_1}a_{i,j_2} + \cdots \equiv 1+\sum_{j=1}^{n} a_{i,j} \equiv 1-n+\sum_{j=1}^{n} A_{i,j} \pmod{n^2}$$

for every index i. Using condition (ii), we obtain $R_i \equiv 1 \pmod{n^2}$, and so $n^2 \mid r_i$.

Therefore, every product of at least two of the r_i is divisible by n^4. Repeating the same argument, we obtain

$$P = \prod_{i=1}^{n} R_i = \prod_{i=1}^{n}(1+r_i) \equiv 1 + \sum_{i=1}^{n} r_i \pmod{n^4},$$

whence

$$\sum_{i=1}^{n} R_i = n + \sum_{i=1}^{n} r_i \equiv n + (P-1) \pmod{n^4},$$

as desired.

Comment. The original version of the problem statement contained also the condition

(iii) The product of all the numbers in the table is congruent to 1 modulo n^4.

This condition appears to be superfluous, so it was omitted.

Solution 2. We present a more straightforward (though lengthier) way to establish (1). We also use the notation of $a_{i,j}$.

By condition (i), all the $a_{i,j}$ are divisible by n. Therefore, we have

$$\begin{aligned} P = \prod_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j}) \equiv 1 + \sum_{(i,j)} a_{i,j} + &\sum_{(i_1,j_1),\,(i_2,j_2)} a_{i_1,j_1}a_{i_2,j_2} \\ &+ \sum_{(i_1,j_1),\,(i_2,j_2),\,(i_3,j_3)} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \pmod{n^4}, \end{aligned}$$

where the last two sums are taken over all unordered pairs/triples of *pairwise different* pairs (i,j); such conventions are applied throughout the solution.

Similarly,

$$\sum_{i=1}^{n} R_i = \sum_{i=1}^{n}\prod_{j=1}^{n}(1+a_{i,j}) \equiv n + \sum_i\sum_j a_{i,j} + \sum_i \sum_{j_1,j_2} a_{i,j_1}a_{i,j_2} + \sum_i \sum_{j_1,j_2,j_3} a_{i,j_1}a_{i,j_2}a_{i,j_3} \pmod{n^4}.$$

Therefore,

$$\begin{aligned} P+(n-1)-\sum_i R_i \equiv \sum_{\substack{(i_1,j_1),(i_2,j_2)\\ i_1\neq i_2}} a_{i_1,j_1}a_{i_2,j_2} &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2\neq i_3\neq i_1}} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \\ &+ \sum_{\substack{(i_1,j_1),(i_2,j_2),(i_3,j_3)\\ i_1\neq i_2=i_3}} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} \pmod{n^4}. \end{aligned}$$

We show that in fact each of the three sums appearing in the right-hand part of this congruence is divisible by n^4; this yields (1). Denote those three sums by Σ_1, Σ_2, and Σ_3 in order of appearance. Recall that by condition (ii) we have

$$\sum_j a_{i,j} \equiv 0 \pmod{n^2} \qquad \text{for all indices } i.$$

For every two indices $i_1 < i_2$ we have

$$\sum_{j_1}\sum_{j_2} a_{i_1,j_1}a_{i_2,j_2} = \left(\sum_{j_1} a_{i_1,j_1}\right)\cdot\left(\sum_{j_2} a_{i_2,j_2}\right) \equiv 0 \pmod{n^4},$$

since each of the two factors is divisible by n^2. Summing over all pairs (i_1,i_2) we obtain $n^4 \mid \Sigma_1$.

Similarly, for every three indices $i_1 < i_2 < i_3$ we have

$$\sum_{j_1}\sum_{j_2}\sum_{j_3} a_{i_1,j_1}a_{i_2,j_2}a_{i_3,j_3} = \left(\sum_{j_1} a_{i_1,j_1}\right)\cdot\left(\sum_{j_2} a_{i_2,j_2}\right)\cdot\left(\sum_{j_3} a_{i_3,j_3}\right)$$

which is divisible even by n^6. Hence $n^4 \mid \Sigma_2$.

Finally, for every indices $i_1 \neq i_2 = i_3$ and $j_2 < j_3$ we have

$$a_{i_2,j_2}\cdot a_{i_2,j_3}\cdot \sum_{j_1} a_{i_1,j_1} \equiv 0 \pmod{n^4},$$

since the three factors are divisible by n, n, and n^2, respectively. Summing over all 4-tuples of indices (i_1,i_2,j_2,j_3) we get $n^4 \mid \Sigma_3$.

N3. Define the sequence $a_0, a_1, a_2, \dots$ by $a_n = 2^n + 2^{\lfloor n/2\rfloor}$. Prove that there are infinitely many terms of the sequence which can be expressed as a sum of (two or more) distinct terms of the sequence, as well as infinitely many of those which cannot be expressed in such a way.

(Serbia)

Solution 1. Call a nonnegative integer *representable* if it equals the sum of several (possibly 0 or 1) distinct terms of the sequence. We say that two nonnegative integers b and c are *equivalent* (written as $b \sim c$) if they are either both representable or both non-representable.

One can easily compute

$$S_{n-1} := a_0 + \cdots + a_{n-1} = 2^n + 2^{\lceil n/2\rceil} + 2^{\lfloor n/2\rfloor} - 3.$$

Indeed, we have $S_n - S_{n-1} = 2^n + 2^{\lfloor n/2\rfloor} = a_n$ so we can use the induction. In particular, $S_{2k-1} = 2^{2k} + 2^{k+1} - 3$.

Note that, if $n \geqslant 3$, then $2^{\lceil n/2\rceil} \geqslant 2^2 > 3$, so

$$S_{n-1} = 2^n + 2^{\lceil n/2\rceil} + 2^{\lfloor n/2\rfloor} - 3 > 2^n + 2^{\lfloor n/2\rfloor} = a_n.$$

Also notice that $S_{n-1} - a_n = 2^{\lceil n/2\rceil} - 3 < a_n$.

The main tool of the solution is the following claim.

Claim 1. Assume that b is a positive integer such that $S_{n-1} - a_n < b < a_n$ for some $n \geqslant 3$. Then $b \sim S_{n-1} - b$.

Proof. As seen above, we have $S_{n-1} > a_n$. Denote $c = S_{n-1} - b$; then $S_{n-1} - a_n < c < a_n$, so the roles of b and c are symmetrical.

Assume that b is representable. The representation cannot contain a_i with $i \geqslant n$, since $b < a_n$. So b is the sum of some subset of $\{a_0, a_1, \dots, a_{n-1}\}$; then c is the sum of the complement. The converse is obtained by swapping b and c. $\square$

We also need the following version of this claim.

Claim 2. For any $n \geqslant 3$, the number a_n can be represented as a sum of two or more distinct terms of the sequence if and only if $S_{n-1} - a_n = 2^{\lceil n/2\rceil} - 3$ is representable.

Proof. Denote $c = S_{n-1} - a_n < a_n$. If a_n satisfies the required condition, then it is the sum of some subset of $\{a_0, a_1, \dots, a_{n-1}\}$; then c is the sum of the complement. Conversely, if c is representable, then its representation consists only of the numbers from $\{a_0, \dots, a_{n-1}\}$, so a_n is the sum of the complement. $\square$

By Claim 2, in order to prove the problem statement, it suffices to find infinitely many representable numbers of the form $2^t - 3$, as well as infinitely many non-representable ones.

Claim 3. For every $t \geqslant 3$, we have $2^t - 3 \sim 2^{4t-6} - 3$, and $2^{4t-6} - 3 > 2^t - 3$.

Proof. The inequality follows from $t \geqslant 3$. In order to prove the equivalence, we apply Claim 1 twice in the following manner.

First, since $S_{2t-3} - a_{2t-2} = 2^{t-1} - 3 < 2^t - 3 < 2^{2t-2} + 2^{t-1} = a_{2t-2}$, by Claim 1 we have $2^t - 3 \sim S_{2t-3} - (2^t - 3) = 2^{2t-2}$.

Second, since $S_{4t-7} - a_{4t-6} = 2^{2t-3} - 3 < 2^{2t-2} < 2^{4t-6} + 2^{2t-3} = a_{4t-6}$, by Claim 1 we have $2^{2t-2} \sim S_{4t-7} - 2^{2t-2} = 2^{4t-6} - 3$.

Therefore, $2^t - 3 \sim 2^{2t-2} \sim 2^{4t-6} - 3$, as required. $\square$

Now it is easy to find the required numbers. Indeed, the number $2^3 - 3 = 5 = a_0 + a_1$ is representable, so Claim 3 provides an infinite sequence of representable numbers

$$2^3 - 3 \sim 2^6 - 3 \sim 2^{18} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots.$$

On the other hand, the number $2^7 - 3 = 125$ is non-representable (since by Claim 1 we have $125 \sim S_6 - 125 = 24 \sim S_4 - 24 = 17 \sim S_3 - 17 = 4$ which is clearly non-representable). So Claim 3 provides an infinite sequence of non-representable numbers

$$2^7 - 3 \sim 2^{22} - 3 \sim 2^{82} - 3 \sim \cdots \sim 2^t - 3 \sim 2^{4t-6} - 3 \sim \cdots.$$

Solution 2. We keep the notion of representability and the notation S_n from the previous solution. We say that an index n is *good* if a_n writes as a sum of smaller terms from the sequence $a_0, a_1, \ldots$. Otherwise we say it is *bad*. We must prove that there are infinitely many good indices, as well as infinitely many bad ones.

Lemma 1. If $m \geqslant 0$ is an integer, then 4^m is representable if and only if either of $2m+1$ and $2m+2$ is good.

Proof. The case $m = 0$ is obvious, so we may assume that $m \geqslant 1$. Let $n = 2m+1$ or $2m+2$. Then $n \geqslant 3$. We notice that

$$S_{n-1} < a_{n-2} + a_n.$$

The inequality writes as $2^n + 2^{\lceil n/2 \rceil} + 2^{\lfloor n/2 \rfloor} - 3 < 2^n + 2^{\lfloor n/2 \rfloor} + 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1}$, i.e. as $2^{\lceil n/2 \rceil} < 2^{n-2} + 2^{\lfloor n/2 \rfloor - 1} + 3$. If $n \geqslant 4$, then $n/2 \leqslant n-2$, so $\lceil n/2 \rceil \leqslant n-2$ and $2^{\lceil n/2 \rceil} \leqslant 2^{n-2}$. For $n = 3$ the inequality verifies separately.

If n is good, then a_n writes as $a_n = a_{i_1} + \cdots + a_{i_r}$, where $r \geqslant 2$ and $i_1 < \cdots < i_r < n$. Then $i_r = n-1$ and $i_{r-1} = n-2$, for if $n-1$ or $n-2$ is missing from the sequence $i_1, \ldots, i_r$, then $a_{i_1} + \cdots + a_{i_r} \leqslant a_0 + \cdots + a_{n-3} + a_{n-1} = S_{n-1} - a_{n-2} < a_n$. Thus, if n is good, then both $a_n - a_{n-1}$ and $a_n - a_{n-1} - a_{n-2}$ are representable.

We now consider the cases $n = 2m+1$ and $n = 2m+2$ separately.

If $n = 2m+1$, then $a_n - a_{n-1} = a_{2m+1} - a_{2m} = (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+1$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then $2^{2m} < a_{2m}$, so 2^{2m} is a sum of some distinct terms a_i with $i < 2m$. It follows that $a_{2m+1} = a_{2m} + 2^{2m}$ writes as a_{2m} plus a sum of some distinct terms a_i with $i < 2m$. Hence $2m+1$ is good.

If $n = 2m+2$, then $a_n - a_{n-1} - a_{n-2} = a_{2m+2} - a_{2m+1} - a_{2m} = (2^{2m+2} + 2^{m+1}) - (2^{2m+1} + 2^m) - (2^{2m} + 2^m) = 2^{2m}$. So we proved that, if $2m+2$ is good, then 2^{2m} is representable. Conversely, if 2^{2m} is representable, then, as seen in the previous case, it writes as a sum of some distinct terms a_i with $i < 2m$. Hence $a_{2m+2} = a_{2m+1} + a_{2m} + 2^{2m}$ writes as $a_{2m+1} + a_{2m}$ plus a sum of some distinct terms a_i with $i < 2m$. Thus $2m+2$ is good. $\square$

Lemma 2. If $k \geqslant 2$, then 2^{4k-2} is representable if and only if 2^{k+1} is representable.

In particular, if $s \geqslant 2$, then 4^s is representable if and only if 4^{4s-3} is representable. Also, $4^{4s-3} > 4^s$.

Proof. We have $2^{4k-2} < a_{4k-2}$, so in a representation of 2^{4k-2} we can have only terms a_i with $i \leqslant 4k-3$. Notice that

$$a_0 + \cdots + a_{4k-3} = 2^{4k-2} + 2^{2k} - 3 < 2^{4k-2} + 2^{2k} + 2^k = 2^{4k-2} + a_{2k}.$$

Hence, any representation of 2^{4k-2} must contain all terms from a_{2k} to a_{4k-3}. (If any of these terms is missing, then the sum of the remaining ones is $\leqslant (a_0 + \cdots + a_{4k-3}) - a_{2k} < 2^{4k-2}$.) Hence, if 2^{4k-2} is representable, then $2^{4k-2} - \sum_{i=2k}^{4k-3} a_i$ is representable. But

$$2^{4k-2} - \sum_{i=2k}^{4k-3} a_i = 2^{4k-2} - (S_{4k-3} - S_{2k-1}) = 2^{4k-2} - (2^{4k-2} + 2^{2k} - 3) + (2^{2k} + 2^{k+1} - 3) = 2^{k+1}.$$

So, if 2^{4k-2} is representable, then 2^{k+1} is representable. Conversely, if 2^{k+1} is representable, then $2^{k+1} < 2^{2k} + 2^k = a_{2k}$, so 2^{k+1} writes as a sum of some distinct terms a_i with $i < 2k$. It follows that $2^{4k-2} = \sum_{i=2k}^{4k-3} a_i + 2^{k+1}$ writes as $a_{4k-3} + a_{4k-4} + \cdots + a_{2k}$ plus the sum of some distinct terms a_i with $i < 2k$. Hence 2^{4k-2} is representable.

For the second statement, if $s \geqslant 2$, then we just take $k = 2s-1$ and we notice that $2^{k+1} = 4^s$ and $2^{4k-2} = 4^{4s-3}$. Also, $s \geqslant 2$ implies that $4s-3 > s$. $\square$

Now $4^2 = a_2 + a_3$ is representable, whereas $4^6 = 4096$ is not. Indeed, note that $4^6 = 2^{12} < a_{12}$, so the only available terms for a representation are $a_0, \ldots, a_{11}$, i.e., 2, 3, 6, 10, 20, 36, 72, 136, 272, 528, 1056, 2080. Their sum is $S_{11} = 4221$, which exceeds 4096 by 125. Then any representation of 4096 must contain all the terms from $a_0, \ldots, a_{11}$ that are greater that 125, i.e., 136, 272, 528, 1056, 2080. Their sum is 4072. Since $4096 - 4072 = 24$ and 24 is clearly not representable, 4096 is non-representable as well.

Starting with these values of m, by using Lemma 2, we can obtain infinitely many representable powers of 4, as well as infinitely many non-representable ones. By Lemma 1, this solves our problem.

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N4. Let $a_1, a_2, \ldots, a_n, \ldots$ be a sequence of positive integers such that

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer for all $n \geqslant k$, where k is some positive integer. Prove that there exists a positive integer m such that $a_n = a_{n+1}$ for all $n \geqslant m$.

(Mongolia)

Solution 1. The argument hinges on the following two facts: Let a, b, c be positive integers such that $N = b/c + (c - b)/a$ is an integer.

(1) If $\gcd(a, c) = 1$, then c divides b; and
(2) If $\gcd(a, b, c) = 1$, then $\gcd(a, b) = 1$.

To prove **(1)**, write $ab = c(aN + b - c)$. Since $\gcd(a, c) = 1$, it follows that c divides b. To prove **(2)**, write $c^2 - bc = a(cN - b)$ to infer that a divides $c^2 - bc$. Letting $d = \gcd(a, b)$, it follows that d divides c^2, and since the two are relatively prime by hypothesis, $d = 1$.

Now, let $s_n = a_1/a_2 + a_2/a_3 + \cdots + a_{n-1}/a_n + a_n/a_1$, let $\delta_n = \gcd(a_1, a_n, a_{n+1})$ and write

$$s_{n+1} - s_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1} - a_n}{a_1} = \frac{a_n/\delta_n}{a_{n+1}/\delta_n} + \frac{a_{n+1}/\delta_n - a_n/\delta_n}{a_1/\delta_n}.$$

Let $n \geqslant k$. Since $\gcd(a_1/\delta_n, a_n/\delta_n, a_{n+1}/\delta_n) = 1$, it follows by **(2)** that $\gcd(a_1/\delta_n, a_n/\delta_n) = 1$. Let $d_n = \gcd(a_1, a_n)$. Then $d_n = \delta_n \cdot \gcd(a_1/\delta_n, a_n/\delta_n) = \delta_n$, so d_n divides a_{n+1}, and therefore d_n divides d_{n+1}.

Consequently, from some rank on, the d_n form a nondecreasing sequence of integers not exceeding a_1, so $d_n = d$ for all $n \geqslant \ell$, where ℓ is some positive integer.

Finally, since $\gcd(a_1/d, a_{n+1}/d) = 1$, it follows by **(1)** that a_{n+1}/d divides a_n/d, so $a_n \geqslant a_{n+1}$ for all $n \geqslant \ell$. The conclusion follows.

Solution 2. We use the same notation s_n. This time, we explore the exponents of primes in the prime factorizations of the a_n for $n \geqslant k$.

To start, for every $n \geqslant k$, we know that the number

$$s_{n+1} - s_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} - \frac{a_n}{a_1} \qquad (*)$$

is integer. Multiplying it by a_1 we obtain that $a_1 a_n / a_{n+1}$ is integer as well, so that $a_{n+1} \mid a_1 a_n$. This means that $a_n \mid a_1^{n-k} a_k$, so all prime divisors of a_n are among those of $a_1 a_k$. There are finitely many such primes; therefore, it suffices to prove that the exponent of each of them in the prime factorization of a_n is eventually constant.

Choose any prime $p \mid a_1 a_k$. Recall that $v_p(q)$ is the standard notation for the exponent of p in the prime factorization of a nonzero rational number q. Say that an index $n \geqslant k$ is *large* if $v_p(a_n) \geqslant v_p(a_1)$. We separate two cases.

Case 1: There exists a large index n.

If $v_p(a_{n+1}) < v_p(a_1)$, then $v_p(a_n/a_{n+1})$ and $v_p(a_n/a_1)$ are nonnegative, while $v_p(a_{n+1}/a_1) < 0$; hence $(*)$ cannot be an integer. This contradiction shows that index $n+1$ is also large.

On the other hand, if $v_p(a_{n+1}) > v_p(a_n)$, then $v_p(a_n/a_{n+1}) < 0$, while $v_p\big((a_{n+1} - a_n)/a_1\big) \geqslant 0$, so $(*)$ is not integer again. Thus, $v_p(a_1) \leqslant v_p(a_{n+1}) \leqslant v_p(a_n)$.

The above arguments can now be applied successively to indices $n+1$, $n+2$, $\ldots$, showing that all the indices greater than n are large, and the sequence $v_p(a_n), v_p(a_{n+1}), v_p(a_{n+2}), \ldots$ is nonincreasing — hence eventually constant.

Case 2: There is no large index.

We have $v_p(a_1) > v_p(a_n)$ for all $n \geqslant k$. If we had $v_p(a_{n+1}) < v_p(a_n)$ for some $n \geqslant k$, then $v_p(a_{n+1}/a_1) < v_p(a_n/a_1) < 0 < v_p(a_n/a_{n+1})$ which would also yield that $(*)$ is not integer. Therefore, in this case the sequence $v_p(a_k), v_p(a_{k+1}), v_p(a_{k+2}), \ldots$ is nondecreasing and bounded by $v_p(a_1)$ from above; hence it is also eventually constant.

Comment. Given any positive odd integer m, consider the m-tuple $(2, 2^2, \ldots, 2^{m-1}, 2^m)$. Appending an infinite string of 1's to this m-tuple yields an eventually constant sequence of integers satisfying the condition in the statement, and shows that the rank from which the sequence stabilises may be arbitrarily large.

There are more sophisticated examples. The solution to part **(b)** of **10532**, *Amer. Math. Monthly*, Vol. 105 No. 8 (Oct. 1998), 775–777 (available at `https://www.jstor.org/stable/2589009`), shows that, for every integer $m \geqslant 5$, there exists an m-tuple $(a_1, a_2, \ldots, a_m)$ of pairwise distinct positive integers such that $\gcd(a_1, a_2) = \gcd(a_2, a_3) = \cdots = \gcd(a_{m-1}, a_m) = \gcd(a_m, a_1) = 1$, and the sum $a_1/a_2 + a_2/a_3 + \cdots + a_{m-1}/a_m + a_m/a_1$ is an integer. Letting $a_{m+k} = a_1$, $k = 1, 2, \ldots$, extends such an m-tuple to an eventually constant sequence of positive integers satisfying the condition in the statement of the problem at hand.

Here is the example given by the proposers of **10532**. Let $b_1 = 2$, let $b_{k+1} = 1 + b_1 \cdots b_k = 1 + b_k(b_k - 1)$, $k \geqslant 1$, and set $B_m = b_1 \cdots b_{m-4} = b_{m-3} - 1$. The m-tuple $(a_1, a_2, \ldots, a_m)$ defined below satisfies the required conditions:

$$a_1 = 1, \quad a_2 = (8B_m + 1)B_m + 8, \quad a_3 = 8B_m + 1, \quad a_k = b_{m-k} \quad \text{for } 4 \leqslant k \leqslant m-1,$$
$$a_m = \frac{a_2}{2} \cdot a_3 \cdot \frac{B_m}{2} = \left(\frac{1}{2}(8B_m + 1)B_m + 4\right) \cdot (8B_m + 1) \cdot \frac{B_m}{2}.$$

It is readily checked that $a_1 < a_{m-1} < a_{m-2} < \cdots < a_3 < a_2 < a_m$. For further details we refer to the solution mentioned above. Acquaintance with this example (or more elaborated examples derived from) offers no advantage in tackling the problem.

N5. Four positive integers x, y, z, and t satisfy the relations

$$xy - zt = x + y = z + t. \qquad (*)$$

Is it possible that both xy and zt are perfect squares?

(Russia)

Answer: No.

Solution 1. Arguing indirectly, assume that $xy = a^2$ and $zt = c^2$ with $a, c > 0$.

Suppose that the number $x + y = z + t$ is odd. Then x and y have opposite parity, as well as z and t. This means that both xy and zt are even, as well as $xy - zt = x + y$; a contradiction. Thus, $x + y$ is even, so the number $s = \frac{x+y}{2} = \frac{z+t}{2}$ is a positive integer.

Next, we set $b = \frac{|x-y|}{2}$, $d = \frac{|z-t|}{2}$. Now the problem conditions yield

$$s^2 = a^2 + b^2 = c^2 + d^2 \qquad (1)$$

and

$$2s = a^2 - c^2 = d^2 - b^2 \qquad (2)$$

(the last equality in (2) follows from (1)). We readily get from (2) that $a, d > 0$.

In the sequel we will use only the relations (1) and (2), along with the fact that a, d, s are positive integers, while b and c are nonnegative integers, at most one of which may be zero. Since both relations are symmetric with respect to the simultaneous swappings $a \leftrightarrow d$ and $b \leftrightarrow c$, we assume, without loss of generality, that $b \geqslant c$ (and hence $b > 0$). Therefore, $d^2 = 2s + b^2 > c^2$, whence

$$d^2 > \frac{c^2 + d^2}{2} = \frac{s^2}{2}. \qquad (3)$$

On the other hand, since $d^2 - b^2$ is even by (2), the numbers b and d have the same parity, so $0 < b \leqslant d - 2$. Therefore,

$$2s = d^2 - b^2 \geqslant d^2 - (d-2)^2 = 4(d-1), \qquad \text{i.e.,} \qquad d \leqslant \frac{s}{2} + 1. \qquad (4)$$

Combining (3) and (4) we obtain

$$2s^2 < 4d^2 \leqslant 4\left(\frac{s}{2} + 1\right)^2, \qquad \text{or} \qquad (s-2)^2 < 8,$$

which yields $s \leqslant 4$.

Finally, an easy check shows that each number of the form s^2 with $1 \leqslant s \leqslant 4$ has a unique representation as a sum of two squares, namely $s^2 = s^2 + 0^2$. Thus, (1) along with $a, d > 0$ imply $b = c = 0$, which is impossible.

Solution 2. We start with a complete description of all 4-tuples (x, y, z, t) of positive integers satisfying $(*)$. As in the solution above, we notice that the numbers

$$s = \frac{x+y}{2} = \frac{z+t}{2}, \quad p = \frac{x-y}{2}, \quad \text{and} \quad q = \frac{z-t}{2}$$

are integers (we may, and will, assume that $p, q \geqslant 0$). We have

$$2s = xy - zt = (s+p)(s-p) - (s+q)(s-q) = q^2 - p^2,$$

so p and q have the same parity, and $q > p$.

Set now $k=\frac{q-p}{2}$, $\ell=\frac{q+p}{2}$. Then we have $s=\frac{q^2-p^2}{2}=2k\ell$ and hence

$$\begin{aligned} x&=s+p=2k\ell-k+\ell, & y&=s-p=2k\ell+k-\ell,\\ z&=s+q=2k\ell+k+\ell, & t&=s-q=2k\ell-k-\ell. \end{aligned} \tag{5}$$

Recall here that $\ell \geqslant k>0$ and, moreover, $(k,\ell)\neq(1,1)$, since otherwise $t=0$.

Assume now that both xy and zt are squares. Then $xyzt$ is also a square. On the other hand, we have

$$\begin{aligned} xyzt&=(2k\ell-k+\ell)(2k\ell+k-\ell)(2k\ell+k+\ell)(2k\ell-k-\ell)\\ &=\bigl(4k^2\ell^2-(k-\ell)^2\bigr)\bigl(4k^2\ell^2-(k+\ell)^2\bigr)=(4k^2\ell^2-k^2-\ell^2)^2-4k^2\ell^2. \end{aligned} \tag{6}$$

Denote $D=4k^2\ell^2-k^2-\ell^2>0$. From (6) we get $D^2>xyzt$. On the other hand,

$$\begin{aligned} (D-1)^2=D^2-2(4k^2\ell^2-k^2-\ell^2)+1&=(D^2-4k^2\ell^2)-(2k^2-1)(2\ell^2-1)+2\\ &=xyzt-(2k^2-1)(2\ell^2-1)+2<xyzt, \end{aligned}$$

since $\ell \geqslant 2$ and $k \geqslant 1$. Thus $(D-1)^2<xyzt<D^2$, and $xyzt$ cannot be a perfect square; a contradiction.

Comment. The first part of Solution 2 shows that all 4-tuples of positive integers $x \geqslant y$, $z \geqslant t$ satisfying $(*)$ have the form (5), where $\ell \geqslant k>0$ and $\ell \geqslant 2$. The converse is also true: every pair of positive integers $\ell \geqslant k>0$, except for the pair $k=\ell=1$, generates via (5) a 4-tuple of positive integers satisfying $(*)$.

N6. Let $f\colon \{1,2,3,\ldots\} \to \{2,3,\ldots\}$ be a function such that $f(m+n) \mid f(m)+f(n)$ for all pairs m,n of positive integers. Prove that there exists a positive integer $c>1$ which divides all values of f.

(Mexico)

Solution 1. For every positive integer m, define $S_m = \{n\colon m \mid f(n)\}$.

Lemma. If the set S_m is infinite, then $S_m = \{d, 2d, 3d, \ldots\} = d \cdot \mathbb{Z}_{>0}$ for some positive integer d.

Proof. Let $d = \min S_m$; the definition of S_m yields $m \mid f(d)$.

Whenever $n \in S_m$ and $n > d$, we have $m \mid f(n) \mid f(n-d) + f(d)$, so $m \mid f(n-d)$ and therefore $n - d \in S_m$. Let $r \leqslant d$ be the least positive integer with $n \equiv r \pmod{d}$; repeating the same step, we can see that $n-d, n-2d, \ldots, r \in S_m$. By the minimality of d, this shows $r = d$ and therefore $d \mid n$.

Starting from an arbitrarily large element of S_m, the process above reaches all multiples of d; so they all are elements of S_m. $\square$

The solution for the problem will be split into two cases.

Case 1: The function f is bounded.

Call a prime p *frequent* if the set S_p is infinite, i.e., if p divides $f(n)$ for infinitely many positive integers n; otherwise call p *sporadic*. Since the function f is bounded, there are only a finite number of primes that divide at least one $f(n)$; so altogether there are finitely many numbers n such that $f(n)$ has a sporadic prime divisor. Let N be a positive integer, greater than all those numbers n.

Let $p_1, \ldots, p_k$ be the frequent primes. By the lemma we have $S_{p_i} = d_i \cdot \mathbb{Z}_{>0}$ for some d_i. Consider the number

$$n = Nd_1d_2\cdots d_k + 1.$$

Due to $n > N$, all prime divisors of $f(n)$ are frequent primes. Let p_i be any frequent prime divisor of $f(n)$. Then $n \in S_{p_i}$, and therefore $d_i \mid n$. But $n \equiv 1 \pmod{d_i}$, which means $d_i = 1$. Hence $S_{p_i} = 1 \cdot \mathbb{Z}_{>0} = \mathbb{Z}_{>0}$ and therefore p_i is a common divisor of all values $f(n)$.

Case 2: f is unbounded.

We prove that $f(1)$ divides all $f(n)$.

Let $a = f(1)$. Since $1 \in S_a$, by the lemma it suffices to prove that S_a is an infinite set.

Call a positive integer p a *peak* if $f(p) > \max\big(f(1), \ldots, f(p-1)\big)$. Since f is not bounded, there are infinitely many peaks. Let $1 = p_1 < p_2 < \ldots$ be the sequence of all peaks, and let $h_k = f(p_k)$. Notice that for any peak p_i and for any $k < p_i$, we have $f(p_i) \mid f(k) + f(p_i - k) < 2f(p_i)$, hence

$$f(k) + f(p_i - k) = f(p_i) = h_i. \tag{1}$$

By the pigeonhole principle, among the numbers $h_1, h_2, \ldots$ there are infinitely many that are congruent modulo a. Let $k_0 < k_1 < k_2 < \ldots$ be an infinite sequence of positive integers such that $h_{k_0} \equiv h_{k_1} \equiv \ldots \pmod{a}$. Notice that

$$f(p_{k_i} - p_{k_0}) = f(p_{k_i}) - f(p_{k_0}) = h_{k_i} - h_{k_0} \equiv 0 \pmod{a},$$

so $p_{k_i} - p_{k_0} \in S_a$ for all $i = 1, 2, \ldots$. This provides infinitely many elements in S_a.

Hence, S_a is an infinite set, and therefore $f(1) = a$ divides $f(n)$ for every n.

Comment. As an extension of the solution above, it can be proven that if f is not bounded then $f(n) = an$ with $a = f(1)$.

Take an arbitrary positive integer n; we will show that $f(n+1) = f(n) + a$. Then it follows by induction that $f(n) = an$.

Take a peak p such that $p > n + 2$ and $h = f(p) > f(n) + 2a$. By (1) we have $f(p-1) = f(p) - f(1) = h - a$ and $f(n+1) = f(p) - f(p-n-1) = h - f(p-n-1)$. From $h - a = f(p-1) \mid f(n) + f(p-n-1) < f(n) + h < 2(h-a)$ we get $f(n) + f(p-n-1) = h - a$. Then

$$f(n+1) - f(n) = \big(h - f(p-n-1)\big) - \big(h - a - f(p-n-1)\big) = a.$$

On the other hand, there exists a wide family of bounded functions satisfying the required properties. Here we present a few examples:

$$f(n) = c; \quad f(n) = \begin{cases} 2c & \text{if } n \text{ is even} \\ c & \text{if } n \text{ is odd;} \end{cases} \quad f(n) = \begin{cases} 2018c & \text{if } n \leqslant 2018 \\ c & \text{if } n > 2018. \end{cases}$$

Solution 2. Let $d_n = \gcd\big(f(n), f(1)\big)$. From $d_{n+1} \mid f(1)$ and $d_{n+1} \mid f(n+1) \mid f(n) + f(1)$, we can see that $d_{n+1} \mid f(n)$; then $d_{n+1} \mid \gcd\big(f(n), f(1)\big) = d_n$. So the sequence $d_1, d_2, \ldots$ is nonincreasing in the sense that every element is a divisor of the previous elements. Let $d = \min(d_1, d_2, \ldots) = \gcd(d_1.d_2, \ldots) = \gcd\big(f(1), f(2), \ldots\big)$; we have to prove $d \geqslant 2$.

For the sake of contradiction, suppose that the statement is wrong, so $d = 1$; that means there is some index n_0 such that $d_n = 1$ for every $n \geqslant n_0$, i.e., $f(n)$ is coprime with $f(1)$.

Claim 1. If $2^k \geqslant n_0$ then $f(2^k) \leqslant 2^k$.

Proof. By the condition, $f(2n) \mid 2f(n)$; a trivial induction yields $f(2^k) \mid 2^k f(1)$. If $2^k \geqslant n_0$ then $f(2^k)$ is coprime with $f(1)$, so $f(2^k)$ is a divisor of 2^k. □

Claim 2. There is a constant C such that $f(n) < n + C$ for every n.

Proof. Take the first power of 2 which is greater than or equal to n_0: let $K = 2^k \geqslant n_0$. By Claim 1, we have $f(K) \leqslant K$. Notice that $f(n+K) \mid f(n) + f(K)$ implies $f(n+K) \leqslant f(n) + f(K) \leqslant f(n) + K$. If $n = tK + r$ for some $t \geqslant 0$ and $1 \leqslant r \leqslant K$, then we conclude

$$f(n) \leqslant K + f(n-K) \leqslant 2K + f(n-2K) \leqslant \ldots \leqslant tK + f(r) < n + \max\big(f(1), f(2), \ldots, f(K)\big),$$

so the claim is true with $C = \max\big(f(1), \ldots, f(K)\big)$. □

Claim 3. If $a, b \in \mathbb{Z}_{>0}$ are coprime then $\gcd\big(f(a), f(b)\big) \mid f(1)$. In particular, if $a, b \geqslant n_0$ are coprime then $f(a)$ and $f(b)$ are coprime.

Proof. Let $d = \gcd\big(f(a), f(b)\big)$. We can replicate Euclid's algorithm. Formally, apply induction on $a + b$. If $a = 1$ or $b = 1$ then we already have $d \mid f(1)$.

Without loss of generality, suppose $1 < a < b$. Then $d \mid f(a)$ and $d \mid f(b) \mid f(a) + f(b-a)$, so $d \mid f(b-a)$. Therefore d divides $\gcd\big(f(a), f(b-a)\big)$ which is a divisor of $f(1)$ by the induction hypothesis. □

Let $p_1 < p_2 < \ldots$ be the sequence of all prime numbers; for every k, let q_k be the lowest power of p_k with $q_k \geqslant n_0$. (Notice that there are only finitely many positive integers with $q_k \neq p_k$.)

Take a positive integer N, and consider the numbers

$$f(1), f(q_1), f(q_2), \ldots, f(q_N).$$

Here we have $N + 1$ numbers, each being greater than 1, and they are pairwise coprime by Claim 3. Therefore, they have at least $N+1$ different prime divisors in total, and their greatest prime divisor is at least p_{N+1}. Hence, $\max(f(1), f(q_1), \ldots, f(q_N)) \geqslant p_{N+1}$.

Choose N such that $\max(q_1, \ldots, q_N) = p_N$ (this is achieved if N is sufficiently large), and $p_{N+1} - p_N > C$ (that is possible, because there are arbitrarily long gaps between the primes). Then we establish a contradiction

$$p_{N+1} \leqslant \max(f(1), f(q_1), \ldots, f(q_N)) < \max(1 + C, q_1 + C, \ldots, q_N + C) = p_N + C < p_{N+1}$$

which proves the statement.

N7. Let $n \geqslant 2018$ be an integer, and let $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ be pairwise distinct positive integers not exceeding $5n$. Suppose that the sequence

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_n}{b_n} \tag{1}$$

forms an arithmetic progression. Prove that the terms of the sequence are equal.

(Thailand)

Solution. Suppose that (1) is an arithmetic progression with nonzero difference. Let the difference be $\Delta = \frac{c}{d}$, where $d > 0$ and c, d are coprime.

We will show that too many denominators b_i should be divisible by d. To this end, for any $1 \leqslant i \leqslant n$ and any prime divisor p of d, say that *the index i is p-wrong*, if $v_p(b_i) < v_p(d)$. ($v_p(x)$ stands for the exponent of p in the prime factorisation of x.)

Claim 1. For any prime p, all p-wrong indices are congruent modulo p. In other words, the p-wrong indices (if they exist) are included in an arithmetic progression with difference p.

Proof. Let $\alpha = v_p(d)$. For the sake of contradiction, suppose that i and j are p-wrong indices (i.e., none of b_i and b_j is divisible by p^α) such that $i \not\equiv j \pmod p$. Then the least common denominator of $\frac{a_i}{b_i}$ and $\frac{a_j}{b_j}$ is not divisible by p^α. But this is impossible because in their difference, $(i-j)\Delta = \frac{(i-j)c}{d}$, the numerator is coprime to p, but p^α divides the denominator d. $\square$

Claim 2. d has no prime divisors greater than 5.

Proof. Suppose that $p \geqslant 7$ is a prime divisor of d. Among the indices $1, 2, \ldots, n$, at most $\left\lceil \frac{n}{p} \right\rceil < \frac{n}{p} + 1$ are p-wrong, so p divides at least $\frac{p-1}{p}n - 1$ of $b_1, \ldots, b_n$. Since these denominators are distinct,

$$5n \geqslant \max\{b_i : \ p \mid b_i\} \geqslant \left(\frac{p-1}{p}n - 1\right)p = (p-1)(n-1) - 1 \geqslant 6(n-1) - 1 > 5n,$$

a contradiction. $\square$

Claim 3. For every $0 \leqslant k \leqslant n - 30$, among the denominators $b_{k+1}, b_{k+2}, \ldots, b_{k+30}$, at least $\varphi(30) = 8$ are divisible by d.

Proof. By Claim 1, the 2-wrong, 3-wrong and 5-wrong indices can be covered by three arithmetic progressions with differences 2, 3 and 5. By a simple inclusion-exclusion, $(2-1)\cdot(3-1)\cdot(5-1) = 8$ indices are not covered; by Claim 2, we have $d \mid b_i$ for every uncovered index i. $\square$

Claim 4. $|\Delta| < \frac{20}{n-2}$ and $d > \frac{n-2}{20}$.

Proof. From the sequence (1), remove all fractions with $b_n < \frac{n}{2}$, There remain at least $\frac{n}{2}$ fractions, and they cannot exceed $\frac{5n}{n/2} = 10$. So we have at least $\frac{n}{2}$ elements of the arithmetic progression (1) in the interval $(0, 10]$, hence the difference must be below $\frac{10}{n/2-1} = \frac{20}{n-2}$.

The second inequality follows from $\frac{1}{d} \leqslant \frac{|c|}{d} = |\Delta|$. $\square$

Now we have everything to get the final contradiction. By Claim 3, we have $d \mid b_i$ for at least $\left\lfloor \frac{n}{30} \right\rfloor \cdot 8$ indices i. By Claim 4, we have $d \geqslant \frac{n-2}{20}$. Therefore,

$$5n \geqslant \max\{b_i : \ d \mid b_i\} \geqslant \left(\left\lfloor \frac{n}{30} \right\rfloor \cdot 8\right) \cdot d > \left(\frac{n}{30} - 1\right) \cdot 8 \cdot \frac{n-2}{20} > 5n.$$

Comment 1. It is possible that all terms in (1) are equal, for example with $a_i = 2i - 1$ and $b_i = 4i - 2$ we have $\frac{a_i}{b_i} = \frac{1}{2}$.

Comment 2. The bound $5n$ in the statement is far from sharp; the solution above can be modified to work for $9n$. For large n, the bound $5n$ can be replaced by $n^{\frac{3}{2}-\varepsilon}$.

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IMO 2019

60TH INTERNATIONAL MATHEMATICAL OLYMPIAD

July 11th - July 22nd,
Bath, United Kingdom

SHORTLISTED PROBLEMS WITH SOLUTIONS

Shortlisted Problems

(with solutions)

60th International Mathematical Olympiad

Bath — UK, 11th–22nd July 2019

Note of Confidentiality

The Shortlist has to be kept strictly confidential
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Contributing Countries

The Organising Committee and the Problem Selection Committee of IMO 2019 thank the following 58 countries for contributing 204 problem proposals:

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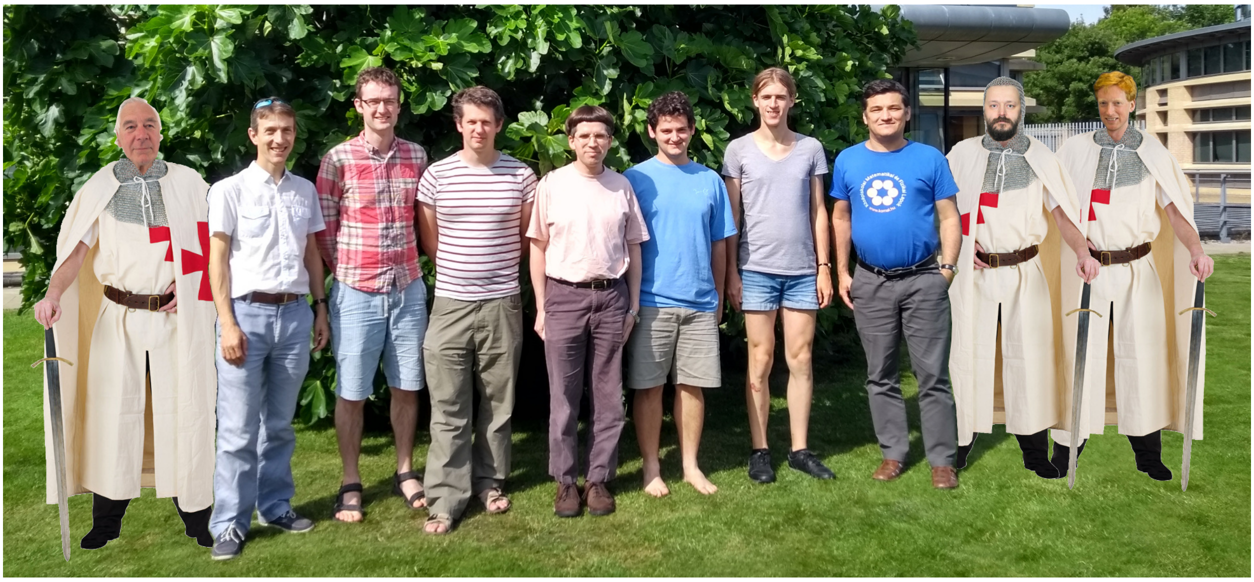
Problem Selection Committee



Tony Gardiner, Edward Crane, Alexander Betts, James Cranch, Joseph Myers (chair), James Aaronson, Andrew Carlotti, Géza Kós, Ilya I. Bogdanov, Jack Shotton

Problems

Algebra

A1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)).$$

(South Africa)

A2. Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

(Germany)

A3. Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

(New Zealand)

A4. Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \left\{(i, j) \,\middle|\, 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\right\}.$$

Prove that, if A is not empty, then

$$\sum_{(i,j) \in A} a_i a_j < 0.$$

(China)

A5. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

(Kazakhstan)

A6. A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z).$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

(Russia)

A7. Let $\mathbb{Z}$ be the set of integers. We consider functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(x + y) + y\big) = f\big(f(x) + y\big)$$

for all integers x and y. For such a function, we say that an integer v is f-*rare* if the set

$$X_v = \{x \in \mathbb{Z} : f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an f-rare integer.

(b) Prove that no such function f can have more than one f-rare integer.

(Netherlands)

Combinatorics

C1. The infinite sequence $a_0, a_1, a_2, \ldots$ of (not necessarily different) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

(Netherlands)

C2. You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

(Thailand)

C3. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k^{th} coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Letting C denote the initial configuration (a sequence of n H's and T's), write $\ell(C)$ for the number of steps needed before all coins show T. Show that this number $\ell(C)$ is finite, and determine its average value over all 2^n possible initial configurations C.

(USA)

C4. On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

(Canada)

C5. On a certain social network, there are 2019 users, some pairs of which are friends, where friendship is a symmetric relation. Initially, there are 1010 people with 1009 friends each and 1009 people with 1010 friends each. However, the friendships are rather unstable, so events of the following kind may happen repeatedly, one at a time:

> Let A, B, and C be people such that A is friends with both B and C, but B and C are not friends; then B and C become friends, but A is no longer friends with them.

Prove that, regardless of the initial friendships, there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

C6. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the $2n$ angles $\angle A_1A_2A_3, \angle A_2A_3A_4, \ldots, \angle A_{2n-2}A_{2n-1}A_{2n}, \angle A_{2n-1}A_{2n}A_1, \angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

(USA)

C7. There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game.

In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leqslant k \leqslant 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.

(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

(Czech Republic)

C8. Alice has a map of Wonderland, a country consisting of $n \geqslant 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

Comment. This problem could be posed with an explicit statement about points being awarded for weaker bounds cn for some $c > 4$, in the style of IMO 2014 Problem 6.

(Thailand)

C9. For any two different real numbers x and y, we define $D(x,y)$ to be the unique integer d satisfying $2^d \leqslant |x-y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the *scales* of x in $\mathcal{F}$ are the values of $D(x,y)$ for $y \in \mathcal{F}$ with $x \neq y$.

Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?

(Italy)

Geometry

G1. Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

(Nigeria)

G2. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

G3. In triangle ABC, let A_1 and B_1 be two points on sides BC and AC, and let P and Q be two points on segments AA_1 and BB_1, respectively, so that line PQ is parallel to AB. On ray PB_1, beyond B_1, let P_1 be a point so that $\angle PP_1C = \angle BAC$. Similarly, on ray QA_1, beyond A_1, let Q_1 be a point so that $\angle CQ_1Q = \angle CBA$. Show that points P, Q, P_1, and Q_1 are concyclic.

(Ukraine)

G4. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

(Australia)

G5. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\operatorname{area}(BCD) + \operatorname{area}(ADE) = \operatorname{area}(ABD) + \operatorname{area}(ABP)$.

(Hungary)

G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

G7. The incircle ω of acute-angled scalene triangle ABC has centre I and meets sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at $Q \neq P$. Prove that lines DI and PQ meet on the external bisector of angle BAC.

(India)

G8. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1)$, $f(\ell_2)$, $f(\ell_3)$ and X lie on a circle.

Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

(Australia)

Number Theory

N1. Find all pairs (m, n) of positive integers satisfying the equation

$$(2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) = m!$$

(El Salvador)

N2. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

(Nigeria)

N3. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

(Czech Republic)

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + b\, f(a).$$

(Croatia)

N5. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b} - 1$ is divisible by $an + 1$ for all positive integers n with $an \geqslant b$. Suppose b is a positive integer such that b is a-good, but $b + 2$ is not a-good. Prove that $b + 1$ is prime.

(Netherlands)

N6. Let $H = \left\{ \lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0} \right\} = \{1, 2, 4, 5, 7, \ldots\}$, and let n be a positive integer. Prove that there exists a constant C such that, if $A \subset \{1, 2, \ldots, n\}$ satisfies $|A| \geqslant C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Brazil)

N7. Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn \log(n)$ pairwise coprime n^{th} powers.

(Canada)

N8. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

(Russia)

Solutions

Algebra

A1. Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a and b,

$$f(2a) + 2f(b) = f(f(a+b)). \tag{1}$$

(South Africa)

Answer: The solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$.

Common remarks. Most solutions to this problem first prove that f must be linear, before determining all linear functions satisfying (1).

Solution 1. Substituting $a = 0, b = n+1$ gives $f(f(n+1)) = f(0) + 2f(n+1)$. Substituting $a = 1, b = n$ gives $f(f(n+1)) = f(2) + 2f(n)$.

In particular, $f(0) + 2f(n+1) = f(2) + 2f(n)$, and so $f(n+1) - f(n) = \frac{1}{2}(f(2) - f(0))$. Thus $f(n+1) - f(n)$ must be constant. Since f is defined only on $\mathbb{Z}$, this tells us that f must be a linear function; write $f(n) = Mn + K$ for arbitrary constants M and K, and we need only determine which choices of M and K work.

Now, (1) becomes

$$2Ma + K + 2(Mb + K) = M(M(a+b) + K) + K$$

which we may rearrange to form

$$(M-2)\big(M(a+b) + K\big) = 0.$$

Thus, either $M = 2$, or $M(a+b) + K = 0$ for all values of $a+b$. In particular, the only possible solutions are $f(n) = 0$ and $f(n) = 2n + K$ for any constant $K \in \mathbb{Z}$, and these are easily seen to work.

Solution 2. Let $K = f(0)$.

First, put $a = 0$ in (1); this gives

$$f(f(b)) = 2f(b) + K \tag{2}$$

for all $b \in \mathbb{Z}$.

Now put $b = 0$ in (1); this gives

$$f(2a) + 2K = f(f(a)) = 2f(a) + K,$$

where the second equality follows from (2). Consequently,

$$f(2a) = 2f(a) - K \tag{3}$$

for all $a \in \mathbb{Z}$.

Substituting (2) and (3) into (1), we obtain

$$\begin{aligned} f(2a) + 2f(b) &= f(f(a+b)) \\ 2f(a) - K + 2f(b) &= 2f(a+b) + K \\ f(a) + f(b) &= f(a+b) + K. \end{aligned}$$

Thus, if we set $g(n) = f(n) - K$ we see that g satisfies the Cauchy equation $g(a+b) = g(a) + g(b)$. The solution to the Cauchy equation over $\mathbb{Z}$ is well-known; indeed, it may be proven by an easy induction that $g(n) = Mn$ for each $n \in \mathbb{Z}$, where $M = g(1)$ is a constant.

Therefore, $f(n) = Mn + K$, and we may proceed as in Solution 1.

Comment 1. Instead of deriving (3) by substituting $b = 0$ into (1), we could instead have observed that the right hand side of (1) is symmetric in a and b, and thus

$$f(2a) + 2f(b) = f(2b) + 2f(a).$$

Thus, $f(2a) - 2f(a) = f(2b) - 2f(b)$ for any $a, b \in \mathbb{Z}$, and in particular $f(2a) - 2f(a)$ is constant. Setting $a = 0$ shows that this constant is equal to $-K$, and so we obtain (3).

Comment 2. Some solutions initially prove that $f(f(n))$ is linear (sometimes via proving that $f(f(n)) - 3K$ satisfies the Cauchy equation). However, one can immediately prove that f is linear by substituting something of the form $f(f(n)) = M'n + K'$ into (2).

A2. Let $u_1, u_2, \ldots, u_{2019}$ be real numbers satisfying

$$u_1 + u_2 + \cdots + u_{2019} = 0 \quad \text{and} \quad u_1^2 + u_2^2 + \cdots + u_{2019}^2 = 1.$$

Let $a = \min(u_1, u_2, \ldots, u_{2019})$ and $b = \max(u_1, u_2, \ldots, u_{2019})$. Prove that

$$ab \leqslant -\frac{1}{2019}.$$

(Germany)

Solution 1. Notice first that $b > 0$ and $a < 0$. Indeed, since $\sum\limits_{i=1}^{2019} u_i^2 = 1$, the variables u_i cannot be all zero, and, since $\sum\limits_{i=1}^{2019} u_i = 0$, the nonzero elements cannot be all positive or all negative.

Let $P = \{i : u_i > 0\}$ and $N = \{i : u_i \leqslant 0\}$ be the indices of positive and nonpositive elements in the sequence, and let $p = |P|$ and $n = |N|$ be the sizes of these sets; then $p + n = 2019$. By the condition $\sum\limits_{i=1}^{2019} u_i = 0$ we have $0 = \sum\limits_{i=1}^{2019} u_i = \sum\limits_{i \in P} u_i - \sum\limits_{i \in N} |u_i|$, so

$$\sum_{i \in P} u_i = \sum_{i \in N} |u_i|. \tag{1}$$

After this preparation, estimate the sum of squares of the positive and nonpositive elements as follows:

$$\sum_{i \in P} u_i^2 \leqslant \sum_{i \in P} b u_i = b \sum_{i \in P} u_i = b \sum_{i \in N} |u_i| \leqslant b \sum_{i \in N} |a| = -nab; \tag{2}$$

$$\sum_{i \in N} u_i^2 \leqslant \sum_{i \in N} |a| \cdot |u_i| = |a| \sum_{i \in N} |u_i| = |a| \sum_{i \in P} u_i \leqslant |a| \sum_{i \in P} b = -pab. \tag{3}$$

The sum of these estimates is

$$1 = \sum_{i=1}^{2019} u_i^2 = \sum_{i \in P} u_i^2 + \sum_{i \in N} u_i^2 \leqslant -(p+n)ab = -2019ab;$$

that proves $ab \leqslant \frac{-1}{2019}$.

Comment 1. After observing $\sum\limits_{i \in P} u_i^2 \leqslant b \sum\limits_{i \in P} u_i$ and $\sum\limits_{i \in N} u_i^2 \leqslant |a| \sum\limits_{i \in P} |u_i|$, instead of $(2, 3)$ an alternative continuation is

$$|ab| \geqslant \frac{\sum\limits_{i \in P} u_i^2}{\sum\limits_{i \in P} u_i} \cdot \frac{\sum\limits_{i \in N} u_i^2}{\sum\limits_{i \in N} |u_i|} = \frac{\sum\limits_{i \in P} u_i^2}{\left(\sum\limits_{i \in P} u_i\right)^2} \sum_{i \in N} u_i^2 \geqslant \frac{1}{p} \sum_{i \in N} u_i^2$$

(by the AM-QM or the Cauchy–Schwarz inequality) and similarly $|ab| \geqslant \frac{1}{n} \sum\limits_{i \in P} u_i^2$.

Solution 2. As in the previous solution we conclude that $a < 0$ and $b > 0$.

For every index i, the number u_i is a convex combination of a and b, so

$$u_i = x_i a + y_i b \quad \text{with some weights } 0 \leqslant x_i, y_i \leqslant 1, \text{ with } x_i + y_i = 1.$$

Let $X = \sum\limits_{i=1}^{2019} x_i$ and $Y = \sum\limits_{i=1}^{2019} y_i$. From $0 = \sum\limits_{i=1}^{2019} u_i = \sum\limits_{i=1}^{2019} (x_i a + y_i b) = -|a|X + bY$, we get

$$|a|X = bY. \tag{4}$$

From $\sum_{i=1}^{2019}(x_i+y_i)=2019$ we have

$$X+Y=2019. \tag{5}$$

The system of linear equations $(4,5)$ has a unique solution:

$$X=\frac{2019b}{|a|+b}, \quad Y=\frac{2019|a|}{|a|+b}.$$

Now apply the following estimate to every u_i^2 in their sum:

$$u_i^2=x_i^2a^2+2x_iy_iab+y_i^2b^2 \leqslant x_ia^2+y_ib^2;$$

we obtain that

$$1=\sum_{i=1}^{2019}u_i^2 \leqslant \sum_{i=1}^{2019}(x_ia^2+y_ib^2)=Xa^2+Yb^2=\frac{2019b}{|a|+b}|a|^2+\frac{2019|a|}{|a|+b}b^2=2019|a|b=-2019ab.$$

Hence, $ab \leqslant \dfrac{-1}{2019}$.

Comment 2. The idea behind Solution 2 is the following thought. Suppose we fix $a<0$ and $b>0$, fix $\sum u_i=0$ and vary the u_i to achieve the maximum value of $\sum u_i^2$. Considering varying any two of the u_i while preserving their sum: the maximum value of $\sum u_i^2$ is achieved when those two are as far apart as possible, so all but at most one of the u_i are equal to a or b. Considering a weighted version of the problem, we see the maximum (with fractional numbers of u_i having each value) is achieved when $\dfrac{2019b}{|a|+b}$ of them are a and $\dfrac{2019|a|}{|a|+b}$ are b.

In fact, this happens in the solution: the number u_i is replaced by x_i copies of a and y_i copies of b.

A3. Let $n \geqslant 3$ be a positive integer and let $(a_1, a_2, \ldots, a_n)$ be a strictly increasing sequence of n positive real numbers with sum equal to 2. Let X be a subset of $\{1, 2, \ldots, n\}$ such that the value of

$$\left|1 - \sum_{i \in X} a_i\right|$$

is minimised. Prove that there exists a strictly increasing sequence of n positive real numbers $(b_1, b_2, \ldots, b_n)$ with sum equal to 2 such that

$$\sum_{i \in X} b_i = 1.$$

(New Zealand)

Common remarks. In all solutions, we say an index set X is (a_i)-*minimising* if it has the property in the problem for the given sequence (a_i). Write X^c for the complement of X, and $[a, b]$ for the interval of integers k such that $a \leqslant k \leqslant b$. Note that

$$\left|1 - \sum_{i \in X} a_i\right| = \left|1 - \sum_{i \in X^c} a_i\right|,$$

so we may exchange X and X^c where convenient. Let

$$\Delta = \sum_{i \in X^c} a_i - \sum_{i \in X} a_i$$

and note that X is (a_i)-minimising if and only if it minimises $|\Delta|$, and that $\sum_{i \in X} a_i = 1$ if and only if $\Delta = 0$.

In some solutions, a scaling process is used. If we have a strictly increasing sequence of positive real numbers c_i (typically obtained by perturbing the a_i in some way) such that

$$\sum_{i \in X} c_i = \sum_{i \in X^c} c_i,$$

then we may put $b_i = 2c_i / \sum_{j=1}^{n} c_j$. So it suffices to construct such a sequence without needing its sum to be 2.

The solutions below show various possible approaches to the problem. Solutions 1 and 2 perturb a few of the a_i to form the b_i (with scaling in the case of Solution 1, without scaling in the case of Solution 2). Solutions 3 and 4 look at properties of the index set X. Solution 3 then perturbs many of the a_i to form the b_i, together with scaling. Rather than using such perturbations, Solution 4 constructs a sequence (b_i) directly from the set X with the required properties. Solution 4 can be used to give a complete description of sets X that are (a_i)-minimising for some (a_i).

Solution 1. Without loss of generality, assume $\sum_{i \in X} a_i \leqslant 1$, and we may assume strict inequality as otherwise $b_i = a_i$ works. Also, X clearly cannot be empty.

If $n \in X$, add Δ to a_n, producing a sequence of c_i with $\sum_{i \in X} c_i = \sum_{i \in X^c} c_i$, and then scale as described above to make the sum equal to 2. Otherwise, there is some k with $k \in X$ and $k + 1 \in X^c$. Let $\delta = a_{k+1} - a_k$.

- If $\delta > \Delta$, add Δ to a_k and then scale.
- If $\delta < \Delta$, then considering $X \cup \{k + 1\} \setminus \{k\}$ contradicts X being (a_i)-minimising.
- If $\delta = \Delta$, choose any $j \neq k, k + 1$ (possible since $n \geqslant 3$), and any ϵ less than the least of a_1 and all the differences $a_{i+1} - a_i$. If $j \in X$ then add $\Delta - \epsilon$ to a_k and ϵ to a_j, then scale; otherwise, add Δ to a_k and $\epsilon/2$ to a_{k+1}, and subtract $\epsilon/2$ from a_j, then scale.

Solution 2. This is similar to Solution 1, but without scaling. As in that solution, without loss of generality, assume $\sum_{i \in X} a_i < 1$.

Suppose there exists $1 \leqslant j \leqslant n-1$ such that $j \in X$ but $j+1 \in X^c$. Then $a_{j+1} - a_j \geqslant \Delta$, because otherwise considering $X \cup \{j+1\} \setminus \{j\}$ contradicts X being (a_i)-minimising.

If $a_{j+1} - a_j > \Delta$, put

$$b_i = \begin{cases} a_j + \Delta/2, & \text{if } i = j; \\ a_{j+1} - \Delta/2, & \text{if } i = j+1; \\ a_i, & \text{otherwise.} \end{cases}$$

If $a_{j+1} - a_j = \Delta$, choose any ϵ less than the least of $\Delta/2$, a_1 and all the differences $a_{i+1} - a_i$. If $|X| \geqslant 2$, choose $k \in X$ with $k \neq j$, and put

$$b_i = \begin{cases} a_j + \Delta/2 - \epsilon, & \text{if } i = j; \\ a_{j+1} - \Delta/2, & \text{if } i = j+1; \\ a_k + \epsilon, & \text{if } i = k; \\ a_i, & \text{otherwise.} \end{cases}$$

Otherwise, $|X^c| \geqslant 2$, so choose $k \in X^c$ with $k \neq j+1$, and put

$$b_i = \begin{cases} a_j + \Delta/2, & \text{if } i = j; \\ a_{j+1} - \Delta/2 + \epsilon, & \text{if } i = j+1; \\ a_k - \epsilon, & \text{if } i = k; \\ a_i, & \text{otherwise.} \end{cases}$$

If there is no $1 \leqslant j \leqslant n$ such that $j \in X$ but $j+1 \in X^c$, there must be some $1 < k \leqslant n$ such that $X = [k, n]$ (certainly X cannot be empty). We must have $a_1 > \Delta$, as otherwise considering $X \cup \{1\}$ contradicts X being (a_i)-minimising. Now put

$$b_i = \begin{cases} a_1 - \Delta/2, & \text{if } i = 1; \\ a_n + \Delta/2, & \text{if } i = n; \\ a_i, & \text{otherwise.} \end{cases}$$

Solution 3. Without loss of generality, assume $\sum_{i \in X} a_i \leqslant 1$, so $\Delta \geqslant 0$. If $\Delta = 0$ we can take $b_i = a_i$, so now assume that $\Delta > 0$.

Suppose that there is some $k \leqslant n$ such that $|X \cap [k, n]| > |X^c \cap [k, n]|$. If we choose the largest such k then $|X \cap [k, n]| - |X^c \cap [k, n]| = 1$. We can now find the required sequence (b_i) by starting with $c_i = a_i$ for $i < k$ and $c_i = a_i + \Delta$ for $i \geqslant k$, and then scaling as described above.

If no such k exists, we will derive a contradiction. For each $i \in X$ we can choose $i < j_i \leqslant n$ in such a way that $j_i \in X^c$ and all the j_i are different. (For instance, note that necessarily $n \in X^c$ and now just work downwards; each time an $i \in X$ is considered, let j_i be the least element of X^c greater than i and not yet used.) Let Y be the (possibly empty) subset of $[1, n]$ consisting of those elements in X^c that are also not one of the j_i. In any case

$$\Delta = \sum_{i \in X} (a_{j_i} - a_i) + \sum_{j \in Y} a_j$$

where each term in the sums is positive. Since $n \geqslant 3$ the total number of terms above is at least two. Take a least such term and its corresponding index i and consider the set Z which we form from X by removing i and adding j_i (if it is a term of the first type) or just by adding j if it is a term of the second type. The corresponding expression of Δ for Z has the sign of its least term changed, meaning that the sum is still nonnegative but strictly less than Δ, which contradicts X being (a_i)-minimising.

Solution 4. This uses some similar ideas to Solution 3, but describes properties of the index sets X that are sufficient to describe a corresponding sequence (b_i) that is not derived from (a_i).

Note that, for two subsets X, Y of $[1, n]$, the following are equivalent:

- $|X \cap [i, n]| \leqslant |Y \cap [i, n]|$ for all $1 \leqslant i \leqslant n$;
- Y is at least as large as X, and for all $1 \leqslant j \leqslant |Y|$, the j^{th} largest element of Y is at least as big as the j^{th} largest element of X;
- there is an injective function $f : X \to Y$ such that $f(i) \geqslant i$ for all $i \in X$.

If these equivalent conditions are satisfied, we write $X \preceq Y$. We write $X \prec Y$ if $X \preceq Y$ and $X \neq Y$.

Note that if $X \prec Y$, then $\sum_{i\in X} a_i < \sum_{i\in Y} a_i$ (the second description above makes this clear).

We claim first that, if $n \geqslant 3$ and $X \prec X^c$, then there exists Y with $X \prec Y \prec X^c$. Indeed, as $|X| \leqslant |X^c|$, we have $|X^c| \geqslant 2$. Define Y to consist of the largest element of X^c, together with all but the largest element of X; it is clear both that Y is distinct from X and X^c, and that $X \preceq Y \preceq X^c$, which is what we need.

But, in this situation, we have

$$\sum_{i\in X} a_i < \sum_{i\in Y} a_i < \sum_{i\in X^c} a_i \qquad \text{and} \qquad 1 - \sum_{i\in X} a_i = -\left(1 - \sum_{i\in X^c} a_i\right),$$

so $|1 - \sum_{i\in Y} a_i| < |1 - \sum_{i\in X} a_i|$.

Hence if X is (a_i)-minimising, we do not have $X \prec X^c$, and similarly we do not have $X^c \prec X$.

Considering the first description above, this immediately implies the following Claim.

Claim. There exist $1 \leqslant k, \ell \leqslant n$ such that $|X \cap [k, n]| > \frac{n-k+1}{2}$ and $|X \cap [\ell, n]| < \frac{n-\ell+1}{2}$.

We now construct our sequence (b_i) using this claim. Let k and ℓ be the greatest values satisfying the claim, and without loss of generality suppose $k = n$ and $\ell < n$ (otherwise replace X by its complement). As ℓ is maximal, $n - \ell$ is even and $|X \cap [\ell, n]| = \frac{n-\ell}{2}$. For sufficiently small positive ϵ, we take

$$b_i = i\epsilon + \begin{cases} 0, & \text{if } i < \ell; \\ \delta, & \text{if } \ell \leqslant i \leqslant n-1; \\ \gamma, & \text{if } i = n. \end{cases}$$

Let $M = \sum_{i\in X} i$. So we require

$$M\epsilon + \left(\frac{n-\ell}{2} - 1\right)\delta + \gamma = 1$$

and

$$\frac{n(n+1)}{2}\epsilon + (n-\ell)\delta + \gamma = 2.$$

These give

$$\gamma = 2\delta + \left(\frac{n(n+1)}{2} - 2M\right)\epsilon$$

and for sufficiently small positive ϵ, solving for γ and δ gives $0 < \delta < \gamma$ (since $\epsilon = 0$ gives $\delta = 1/(\frac{n-\ell}{2} + 1)$ and $\gamma = 2\delta$), so the sequence is strictly increasing and has positive values.

Comment. This solution also shows that the claim gives a complete description of sets X that are (a_i)-minimising for some (a_i).

Another approach to proving the claim is as follows. We prove the existence of ℓ with the claimed property; the existence of k follows by considering the complement of X.

Suppose, for a contradiction, that for all $1 \leqslant \ell \leqslant n$ we have $|X \cap [\ell, n]| \geqslant \left\lceil \frac{n-\ell+1}{2} \right\rceil$. If we ever have strict inequality, consider the set $Y = \{n, n-2, n-4, \ldots\}$. This set may be obtained from X by possibly removing some elements and reducing the values of others. (To see this, consider the largest $k \in X \setminus Y$, if any; remove it, and replace it by the greatest $j \in X^c$ with $j < k$, if any. Such steps preserve the given inequality, and are possible until we reach the set Y.) So if we had strict inequality, and so $X \neq Y$, we have

$$\sum_{i \in X} a_i > \sum_{i \in Y} a_i > 1,$$

contradicting X being (a_i)-minimising. Otherwise, we always have equality, meaning that $X = Y$. But now consider $Z = Y \cup \{n-1\} \setminus \{n\}$. Since $n \geqslant 3$, we have

$$\sum_{i \in Y} a_i > \sum_{i \in Z} a_i > \sum_{i \in Y^c} a_i = 2 - \sum_{i \in Y} a_i,$$

and so Z contradicts X being (a_i)-minimising.

A4. Let $n \geqslant 2$ be a positive integer and $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Define the set A by

$$A = \{(i,j) \mid 1 \leqslant i < j \leqslant n, |a_i - a_j| \geqslant 1\}.$$

Prove that, if A is not empty, then

$$\sum_{(i,j)\in A} a_i a_j < 0.$$

(China)

Solution 1. Define sets B and C by

$$\begin{aligned} B &= \{(i,j) \mid 1 \leqslant i, j \leqslant n, |a_i - a_j| \geqslant 1\}, \\ C &= \{(i,j) \mid 1 \leqslant i, j \leqslant n, |a_i - a_j| < 1\}. \end{aligned}$$

We have

$$\begin{aligned} \sum_{(i,j)\in A} a_i a_j &= \frac{1}{2} \sum_{(i,j)\in B} a_i a_j \\ \sum_{(i,j)\in B} a_i a_j &= \sum_{1\leqslant i,j\leqslant n} a_i a_j - \sum_{(i,j)\notin B} a_i a_j = 0 - \sum_{(i,j)\in C} a_i a_j. \end{aligned}$$

So it suffices to show that if A (and hence B) are nonempty, then

$$\sum_{(i,j)\in C} a_i a_j > 0.$$

Partition the indices into sets P, Q, R, and S such that

$$\begin{aligned} P &= \{i \mid a_i \leqslant -1\} & R &= \{i \mid 0 < a_i < 1\} \\ Q &= \{i \mid -1 < a_i \leqslant 0\} & S &= \{i \mid 1 \leqslant a_i\}. \end{aligned}$$

Then

$$\sum_{(i,j)\in C} a_i a_j \geqslant \sum_{i\in P\cup S} a_i^2 + \sum_{i,j\in Q\cup R} a_i a_j = \sum_{i\in P\cup S} a_i^2 + \left(\sum_{i\in Q\cup R} a_i\right)^2 \geqslant 0.$$

The first inequality holds because all of the positive terms in the RHS are also in the LHS, and all of the negative terms in the LHS are also in the RHS. The first inequality attains equality only if both sides have the same negative terms, which implies $|a_i - a_j| < 1$ whenever $i, j \in Q \cup R$; the second inequality attains equality only if $P = S = \varnothing$. But then we would have $A = \varnothing$. So A nonempty implies that the inequality holds strictly, as required.

Solution 2. Consider P, Q, R, S as in Solution 1, set

$$p = \sum_{i\in P} a_i, \quad q = \sum_{i\in Q} a_i, \quad r = \sum_{i\in R} a_i, \quad s = \sum_{i\in S} a_i,$$

and let

$$t_+ = \sum_{(i,j)\in A,\, a_i a_j \geqslant 0} a_i a_j, \quad t_- = \sum_{(i,j)\in A,\, a_i a_j \leqslant 0} a_i a_j.$$

We know that $p + q + r + s = 0$, and we need to prove that $t_+ + t_- < 0$.

Notice that $t_+ \leqslant p^2/2 + pq + rs + s^2/2$ (with equality only if $p = s = 0$), and $t_- \leqslant pr + ps + qs$ (with equality only if there do not exist $i \in Q$ and $j \in R$ with $a_j - a_i > 1$). Therefore,

$$t_+ + t_- \leqslant \frac{p^2 + s^2}{2} + pq + rs + pr + ps + qs = \frac{(p+q+r+s)^2}{2} - \frac{(q+r)^2}{2} = -\frac{(q+r)^2}{2} \leqslant 0.$$

If A is not empty and $p = s = 0$, then there must exist $i \in Q, j \in R$ with $|a_i - a_j| > 1$, and hence the earlier equality conditions cannot both occur.

Comment. The RHS of the original inequality cannot be replaced with any constant $c < 0$ (independent of n). Indeed, take

$$a_1 = -\frac{n}{n+2}, a_2 = \cdots = a_{n-1} = \frac{1}{n+2}, a_n = \frac{2}{n+2}.$$

Then $\sum_{(i,j)\in A} a_i a_j = -\frac{2n}{(n+2)^2}$, which converges to zero as $n \to \infty$.

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A5. Let $x_1, x_2, \ldots, x_n$ be different real numbers. Prove that

$$\sum_{1 \leqslant i \leqslant n} \prod_{j \neq i} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

(Kazakhstan)

Common remarks. Let $G(x_1, x_2, \ldots, x_n)$ be the function of the n variables $x_1, x_2, \ldots, x_n$ on the LHS of the required identity.

Solution 1 (Lagrange interpolation). Since both sides of the identity are rational functions, it suffices to prove it when all $x_i \notin \{\pm 1\}$. Define

$$f(t) = \prod_{i=1}^{n} (1 - x_i t),$$

and note that

$$f(x_i) = (1 - x_i^2) \prod_{j \neq i} 1 - x_i x_j .$$

Using the nodes $+1, -1, x_1, \ldots, x_n$, the Lagrange interpolation formula gives us the following expression for f:

$$\sum_{i=1}^{n} f(x_i) \frac{(x-1)(x+1)}{(x_i - 1)(x_i + 1)} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} + f(1) \frac{x+1}{1+1} \prod_{1 \leqslant i \leqslant n} \frac{x - x_i}{1 - x_i} + f(-1) \frac{x-1}{-1-1} \prod_{1 \leqslant i \leqslant n} \frac{x - x_i}{1 - x_i} .$$

The coefficient of t^{n+1} in $f(t)$ is 0, since f has degree n. The coefficient of t^{n+1} in the above expression of f is

$$\begin{aligned} 0 &= \sum_{1 \leqslant i \leqslant n} \frac{f(x_i)}{\prod_{j \neq i} (x_i - x_j) \cdot (x_i - 1)(x_i + 1)} + \frac{f(1)}{\prod_{1 \leqslant j \leqslant n} (1 - x_j) \cdot (1 + 1)} + \frac{f(-1)}{\prod_{1 \leqslant j \leqslant n} (-1 - x_j) \cdot (-1 - 1)} \\ &= -G(x_1, \ldots, x_n) + \frac{1}{2} + \frac{(-1)^{n+1}}{2}. \end{aligned}$$

Comment. The main difficulty is to think of including the two extra nodes ± 1 and evaluating the coefficient t^{n+1} in f when $n+1$ is higher than the degree of f.

It is possible to solve the problem using Lagrange interpolation on the nodes $x_1, \ldots, x_n$, but the definition of the polynomial being interpolated should depend on the parity of n. For n even, consider the polynomial

$$P(x) = \prod_i (1 - x x_i) - \prod_i (x - x_i).$$

Lagrange interpolation shows that G is the coefficient of x^{n-1} in the polynomial $P(x)/(1 - x^2)$, i.e. 0. For n odd, consider the polynomial

$$P(x) = \prod_i (1 - x x_i) - x \prod_i (x - x_i).$$

Now G is the coefficient of x^{n-1} in $P(x)/(1 - x^2)$, which is 1.

Solution 2 (using symmetries). Observe that G is symmetric in the variables $x_1, \ldots, x_n$. Define $V = \prod_{i<j}(x_j - x_i)$ and let $F = G \cdot V$, which is a polynomial in $x_1, \ldots, x_n$. Since V is alternating, F is also alternating (meaning that, if we exchange any two variables, then F changes sign). Every alternating polynomial in n variables $x_1, \ldots, x_n$ vanishes when any two variables x_i, x_j ($i \neq j$) are equal, and is therefore divisible by $x_i - x_j$ for each pair $i \neq j$. Since these linear factors are pairwise coprime, V divides F exactly as a polynomial. Thus G is in fact a symmetric polynomial in $x_1, \ldots, x_n$.

Now observe that if all x_i are nonzero and we set $y_i = 1/x_i$ for $i = 1, \ldots, n$, then we have

$$\frac{1 - y_i y_j}{y_i - y_j} = \frac{1 - x_i x_j}{x_i - x_j}\,,$$

so that

$$G\left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right) = G(x_1, \ldots, x_n)\,.$$

By continuity this is an identity of rational functions. Since G is a polynomial, it implies that G is constant. (If G were not constant, we could choose a point $(c_1, \ldots, c_n)$ with all $c_i \neq 0$, such that $G(c_1, \ldots, c_n) \neq G(0, \ldots, 0)$; then $g(x) := G(c_1 x, \ldots, c_n x)$ would be a nonconstant polynomial in the variable x, so $|g(x)| \to \infty$ as $x \to \infty$, hence $\left|G\left(\frac{y}{c_1}, \ldots, \frac{y}{c_n}\right)\right| \to \infty$ as $y \to 0$, which is impossible since G is a polynomial.)

We may identify the constant by substituting $x_i = \zeta^i$, where ζ is a primitive n^{th} root of unity in $\mathbb{C}$. In the i^{th} term in the sum in the original expression we have a factor $1 - \zeta^i \zeta^{n-i} = 0$, unless $i = n$ or $2i = n$. In the case where n is odd, the only exceptional term is $i = n$, which gives the value $\prod_{j \neq n} \frac{1-\zeta^j}{1-\zeta^j} = 1$. When n is even, we also have the term $\prod_{j \neq \frac{n}{2}} \frac{1+\zeta^j}{-1-\zeta^j} = (-1)^{n-1} = -1$, so the sum is 0.

Comment. If we write out an explicit expression for F,

$$F = \sum_{1 \leqslant i \leqslant n} (-1)^{n-i} \prod_{\substack{j<k \\ j,k \neq i}} (x_k - x_j) \prod_{j \neq i} (1 - x_i x_j)$$

then to prove directly that F vanishes when $x_i = x_j$ for some $i \neq j$, but no other pair of variables coincide, we have to check carefully that the two nonzero terms in this sum cancel.

A different and slightly less convenient way to identify the constant is to substitute $x_i = 1 + \epsilon \zeta^i$, and throw away terms that are $O(\epsilon)$ as $\epsilon \to 0$.

Solution 3 (breaking symmetry). Consider G as a rational function in x_n with coefficients that are rational functions in the other variables. We can write

$$G(x_1, \ldots, x_n) = \frac{P(x_n)}{\prod_{j \neq n} (x_n - x_j)}$$

where $P(x_n)$ is a polynomial in x_n whose coefficients are rational functions in the other variables. We then have

$$P(x_n) = \left(\prod_{j \neq n} (1 - x_n x_j)\right) + \sum_{1 \leqslant i \leqslant n-1} (x_i x_n - 1) \left(\prod_{j \neq i,n} (x_n - x_j)\right) \left(\prod_{j \neq i,n} \frac{1 - x_i x_j}{x_i - x_j}\right).$$

For any $k \neq n$, substituting $x_n = x_k$ (which is valid when manipulating the numerator $P(x_n)$

on its own), we have (noting that $x_n - x_j$ vanishes when $j = k$)

$$\begin{aligned}
P(x_k) &= \left(\prod_{j\neq n}(1 - x_k x_j)\right) + \sum_{1\leqslant i\leqslant n-1}(x_i x_k - 1)\left(\prod_{j\neq i,n}(x_k - x_j)\right)\left(\prod_{j\neq i,n}\frac{1 - x_i x_j}{x_i - x_j}\right)\\
&= \left(\prod_{j\neq n}(1 - x_k x_j)\right) + \left(x_k^2 - 1\right)\left(\prod_{j\neq k,n}(x_k - x_j)\right)\left(\prod_{j\neq k,n}\frac{1 - x_k x_j}{x_k - x_j}\right)\\
&= \left(\prod_{j\neq n}(1 - x_k x_j)\right) + \left(x_k^2 - 1\right)\left(\prod_{j\neq k,n}(1 - x_k x_j)\right)\\
&= 0.
\end{aligned}$$

Note that P is a polynomial in x_n of degree $n-1$. For any choice of distinct real numbers $x_1, \ldots, x_{n-1}$, P has those real numbers as its roots, and the denominator has the same degree and the same roots. This shows that G is constant in x_n, for any fixed choice of distinct $x_1, \ldots, x_{n-1}$. Now, G is symmetric in all n variables, so it must be also be constant in each of the other variables. G is therefore a constant that depends only on n. The constant may be identified as in the previous solution.

Comment. There is also a solution in which we recognise the expression for F in the comment after Solution 2 as the final column expansion of a certain matrix obtained by modifying the final column of the Vandermonde matrix. The task is then to show that the matrix can be modified by column operations either to make the final column identically zero (in the case where n even) or to recover the Vandermonde matrix (in the case where n odd). The polynomial $P/(1-x^2)$ is helpful for this task, where P is the parity-dependent polynomial defined in the comment after Solution 1.

A6. A polynomial $P(x, y, z)$ in three variables with real coefficients satisfies the identities

$$P(x, y, z) = P(x, y, xy - z) = P(x, zx - y, z) = P(yz - x, y, z). \qquad (*)$$

Prove that there exists a polynomial $F(t)$ in one variable such that

$$P(x, y, z) = F(x^2 + y^2 + z^2 - xyz).$$

(Russia)

Common remarks. The polynomial $x^2 + y^2 + z^2 - xyz$ satisfies the condition $(*)$, so every polynomial of the form $F(x^2+y^2+z^2-xyz)$ does satisfy $(*)$. We will use without comment the fact that two polynomials have the same coefficients if and only if they are equal as functions.

Solution 1. In the first two steps, we deal with any polynomial $P(x, y, z)$ satisfying $P(x, y, z) = P(x, y, xy - z)$. Call such a polynomial *weakly symmetric*, and call a polynomial satisfying the full conditions in the problem *symmetric*.

Step 1. We start with the description of weakly symmetric polynomials. We claim that they are exactly the polynomials in x, y, and $z(xy - z)$. Clearly, all such polynomials are weakly symmetric. For the converse statement, consider $P_1(x, y, z) := P(x, y, z + \frac{1}{2}xy)$, which satisfies $P_1(x, y, z) = P_1(x, y, -z)$ and is therefore a polynomial in x, y, and z^2. This means that P is a polynomial in x, y, and $(z - \frac{1}{2}xy)^2 = -z(xy - z) + \frac{1}{4}x^2y^2$, and therefore a polynomial in x, y, and $z(xy - z)$.

Step 2. Suppose that P is weakly symmetric. Consider the monomials in $P(x, y, z)$ of highest total degree. Our aim is to show that in each such monomial $\mu x^a y^b z^c$ we have $a, b \geqslant c$. Consider the expansion

$$P(x, y, z) = \sum_{i,j,k} \mu_{ijk} x^i y^j \big(z(xy - z)\big)^k. \qquad (1.1)$$

The maximal total degree of a summand in (1.1) is $m = \max_{i,j,k\colon \mu_{ijk} \neq 0}(i + j + 3k)$. Now, for any i, j, k satisfying $i + j + 3k = m$ the summand $\mu_{i,j,k} x^i y^j \big(z(xy - z)\big)^k$ has leading term of the form $\mu x^{i+k} y^{j+k} z^k$. No other nonzero summand in (1.1) may have a term of this form in its expansion, hence this term does not cancel in the whole sum. Therefore, $\deg P = m$, and the leading component of P is exactly

$$\sum_{i+j+3k=m} \mu_{i,j,k} x^{i+k} y^{j+k} z^k,$$

and each summand in this sum satisfies the condition claimed above.

Step 3. We now prove the problem statement by induction on $m = \deg P$. For $m = 0$ the claim is trivial. Consider now a symmetric polynomial P with $\deg P > 0$. By Step 2, each of its monomials $\mu x^a y^b z^c$ of the highest total degree satisfies $a, b \geqslant c$. Applying other weak symmetries, we obtain $a, c \geqslant b$ and $b, c \geqslant a$; therefore, P has a unique leading monomial of the form $\mu(xyz)^c$. The polynomial $P_0(x, y, z) = P(x, y, z) - \mu\big(xyz - x^2 - y^2 - z^2\big)^c$ has smaller total degree. Since P_0 is symmetric, it is representable as a polynomial function of $xyz - x^2 - y^2 - z^2$. Then P is also of this form, completing the inductive step.

Comment. We could alternatively carry out Step 1 by an induction on $n = \deg_z P$, in a manner similar to Step 3. If $n = 0$, the statement holds. Assume that $n > 0$ and check the leading component of P with respect to z:

$$P(x, y, z) = Q_n(x, y) z^n + R(x, y, z),$$

where $\deg_z R < n$. After the change $z \mapsto xy - z$, the leading component becomes $Q_n(x,y)(-z)^n$; on the other hand, it should remain the same. Hence n is even. Now consider the polynomial

$$P_0(x,y,z) = P(x,y,z) - Q_n(x,y)\cdot\big(z(z-xy)\big)^{n/2}.$$

It is also weakly symmetric, and $\deg_z P_0 < n$. By the inductive hypothesis, it has the form $P_0(x,y,z) = S\big(x,y,z(z-xy)\big)$. Hence the polynomial

$$P(x,y,z) = S\big(x,y,z(xy-z)\big) + Q_n(x,y)\big(z(z-xy)\big)^{n/2}$$

also has this form. This completes the inductive step.

Solution 2. We will rely on the well-known identity

$$\cos^2 u + \cos^2 v + \cos^2 w - 2\cos u\cos v\cos w - 1 = 0 \quad \text{whenever } u+v+w=0. \tag{2.1}$$

Claim 1. The polynomial $P(x,y,z)$ is constant on the surface

$$\mathfrak{S} = \big\{(2\cos u, 2\cos v, 2\cos w) : u+v+w=0\big\}.$$

Proof. Notice that for $x = 2\cos u$, $y = 2\cos v$, $z = 2\cos w$, the Vieta jumps $x \mapsto yz - x$, $y \mapsto zx - y$, $z \mapsto xy - z$ in $(*)$ replace (u,v,w) by $(v-w,-v,w)$, $(u,w-u,-w)$ and $(-u,v,u-v)$, respectively. For example, for the first type of jump we have

$$yz - x = 4\cos v\cos w - 2\cos u = 2\cos(v+w) + 2\cos(v-w) - 2\cos u = 2\cos(v-w).$$

Define $G(u,v,w) = P(2\cos u, 2\cos v, 2\cos w)$. For $u+v+w=0$, the jumps give

$$\begin{aligned} G(u,v,w) &= G(v-w,-v,w) = G(w-v,-v,(v-w)-(-v)) = G(-u-2v,-v,2v-w) \\ &= G(u+2v,v,w-2v). \end{aligned}$$

By induction,

$$G(u,v,w) = G\big(u+2kv, v, w-2kv\big) \quad (k\in\mathbb{Z}). \tag{2.2}$$

Similarly,

$$G(u,v,w) = G\big(u, v-2\ell u, w+2\ell u\big) \quad (\ell\in\mathbb{Z}). \tag{2.3}$$

And, of course, we have

$$G(u,v,w) = G\big(u+2p\pi, v+2q\pi, w-2(p+q)\pi\big) \quad (p,q\in\mathbb{Z}). \tag{2.4}$$

Take two nonzero real numbers u, v such that u, v and π are linearly independent over $\mathbb{Q}$. By combining (2.2–2.4), we can see that G is constant on a dense subset of the plane $u+v+w=0$. By continuity, G is constant on the entire plane and therefore P is constant on $\mathfrak{S}$. $\square$

Claim 2. The polynomial $T(x,y,z) = x^2+y^2+z^2-xyz-4$ divides $P(x,y,z) - P(2,2,2)$.

Proof. By dividing P by T with remainders, there exist some polynomials $R(x,y,z)$, $A(y,z)$ and $B(y,z)$ such that

$$P(x,y,z) - P(2,2,2) = T(x,y,z)\cdot R(x,y,z) + A(y,z)x + B(y,z). \tag{2.5}$$

On the surface $\mathfrak{S}$ the LHS of (2.5) is zero by Claim 1 (since $(2,2,2)\in\mathfrak{S}$) and $T=0$ by (2.1). Hence, $A(y,z)x + B(y,z)$ vanishes on $\mathfrak{S}$.

Notice that for every $y = 2\cos v$ and $z = 2\cos w$ with $\frac{\pi}{3} < v, w < \frac{2\pi}{3}$, there are two distinct values of x such that $(x,y,z)\in\mathfrak{S}$, namely $x_1 = 2\cos(v+w)$ (which is negative), and $x_2 = 2\cos(v-w)$ (which is positive). This can happen only if $A(y,z) = B(y,z) = 0$. Hence, $A(y,z) = B(y,z) = 0$ for $|y|<1$, $|z|<1$. The polynomials A and B vanish on an open set, so A and B are both the zero polynomial. $\square$

The quotient $(P(x,y,z) - P(2,2,2))/T(x,y,z)$ is a polynomial of lower degree than P and it also satisfies $(*)$. The problem statement can now be proven by induction on the degree of P.

Comment. In the proof of (2.2) and (2.3) we used two consecutive Vieta jumps; in fact from $(*)$ we used only $P(x,y,xy-z) = P(x,zx-y,z) = P(yz-x,y,z)$.

Solution 3 (using algebraic geometry, just for interest). Let $Q = x^2+y^2+z^2-xyz$ and let $t \in \mathbb{C}$. Checking where $Q-t$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$ and $\frac{\partial Q}{\partial z}$ vanish simultaneously, we find that the surface $Q = t$ is smooth except for the cases $t = 0$, when the only singular point is $(0,0,0)$, and $t = 4$, when the four points $(\pm 2, \pm 2, \pm 2)$ that satisfy $xyz = 8$ are the only singular points. The singular points are the fixed points of the group Γ of polynomial automorphisms of $\mathbb{C}^3$ generated by the three *Vieta involutions*

$$\iota_1 : (x,y,z) \mapsto (x,y,xy-z), \quad \iota_2 : (x,y,z) \mapsto (x,xz-y,z), \quad \iota_3 : (x,y,z) \mapsto (yz-x,y,z)\,.$$

Γ acts on each surface $\mathcal{V}_t : Q - t = 0$. If $Q-t$ were reducible then the surface $Q = t$ would contain a curve of singular points. Therefore $Q-t$ is irreducible in $\mathbb{C}[x,y,z]$. (One can also prove algebraically that $Q-t$ is irreducible, for example by checking that its discriminant as a quadratic polynomial in x is not a square in $\mathbb{C}[y,z]$, and likewise for the other two variables.) In the following solution we will only use the algebraic surface $\mathcal{V}_0$.

Let U be the Γ-orbit of $(3,3,3)$. Consider $\iota_3 \circ \iota_2$, which leaves z invariant. For each fixed value of z, $\iota_3 \circ \iota_2$ acts linearly on (x,y) by the matrix

$$M_z := \begin{pmatrix} z^2-1 & -z \\ z & -1 \end{pmatrix}.$$

The reverse composition $\iota_2 \circ \iota_3$ acts by $M_z^{-1} = M_z^{adj}$. Note $\det M_z = 1$ and $\mathrm{tr} M_z = z^2 - 2$. When z does not lie in the real interval $[-2,2]$, the eigenvalues of M_z do not have absolute value 1, so every orbit of the group generated by M_z on $\mathbb{C}^2 \setminus \{(0,0)\}$ is unbounded. For example, fixing $z = 3$ we find $(3F_{2k+1}, 3F_{2k-1}, 3) \in U$ for every $k \in \mathbb{Z}$, where $(F_n)_{n\in\mathbb{Z}}$ is the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$.

Now we may start at any point $(3F_{2k+1}, 3F_{2k-1}, 3)$ and iteratively apply $\iota_1 \circ \iota_2$ to generate another infinite sequence of distinct points of U, Zariski dense in the hyperbola cut out of $\mathcal{V}_0$ by the plane $x - 3F_{2k+1} = 0$. (The plane $x = a$ cuts out an irreducible conic when $a \notin \{-2, 0, 2\}$.) Thus the Zariski closure $\overline{U}$ of U contains infinitely many distinct algebraic curves in $\mathcal{V}_0$. Since $\mathcal{V}_0$ is an irreducible surface this implies that $\overline{U} = \mathcal{V}_0$.

For any polynomial P satisfying $(*)$, we have $P - P(3,3,3) = 0$ at each point of U. Since $\overline{U} = \mathcal{V}_0$, $P - P(3,3,3)$ vanishes on $\mathcal{V}_0$. Then Hilbert's Nullstellensatz and the irreducibility of Q imply that $P - P(3,3,3)$ is divisible by Q. Now $(P - P(3,3,3))/Q$ is a polynomial also satisfying $(*)$, so we may complete the proof by an induction on the total degree, as in the other solutions.

Comment. We remark that Solution 2 used a trigonometric parametrisation of a real component of $\mathcal{V}_4$; in contrast $\mathcal{V}_0$ is birationally equivalent to the projective space $\mathbb{P}^2$ under the maps

$$(x,y,z) \to (x:y:z), \quad (a:b:c) \to \left(\frac{a^2+b^2+c^2}{bc}, \frac{a^2+b^2+c^2}{ac}, \frac{a^2+b^2+c^2}{ab}\right).$$

The set U in Solution 3 is contained in $\mathbb{Z}^3$ so it is nowhere dense in $\mathcal{V}_0$ in the classical topology.

Comment (background to the problem). A triple $(a,b,c) \in \mathbb{Z}^3$ is called a *Markov triple* if $a^2 + b^2 + c^2 = 3abc$, and an integer that occurs as a coordinate of some Markov triple is called a *Markov number*. (The spelling *Markoff* is also frequent.) Markov triples arose in A. Markov's work in the 1870s on the reduction theory of indefinite binary quadratic forms. For every Markov triple,

$(3a, 3b, 3c)$ lies on $Q = 0$. It is well known that all nonzero Markov triples can be generated from $(1, 1, 1)$ by sequences of Vieta involutions, which are the substitutions described in equation $(*)$ in the problem statement. There has been recent work by number theorists about the properties of Markov numbers (see for example Jean Bourgain, Alex Gamburd and Peter Sarnak, *Markoff triples and strong approximation*, Comptes Rendus Math. **345**, no. 2, 131–135 (2016), arXiv:1505.06411). Each Markov number occurs in infinitely many triples, but a famous old open problem is the unicity conjecture, which asserts that each Markov number occurs in only one Markov triple (up to permutations and sign changes) as the largest coordinate in absolute value in that triple. It is a standard fact in the modern literature on Markov numbers that the Markov triples are Zariski dense in the Markov surface. Proving this is the main work of Solution 3. Algebraic geometry is definitely off-syllabus for the IMO, and one still has to work a bit to prove the Zariski density. On the other hand the approaches of Solutions 1 and 2 are elementary and only use tools expected to be known by IMO contestants. Therefore we do not think that the existence of a solution using algebraic geometry necessarily makes this problem unsuitable for the IMO.

A7. Let $\mathbb{Z}$ be the set of integers. We consider functions $f\colon \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f\big(f(x+y)+y\big) = f\big(f(x)+y\big)$$

for all integers x and y. For such a function, we say that an integer v is *f-rare* if the set

$$X_v = \{x \in \mathbb{Z}\colon f(x) = v\}$$

is finite and nonempty.

(a) Prove that there exists such a function f for which there is an f-rare integer.

(b) Prove that no such function f can have more than one f-rare integer.

(Netherlands)

Solution 1. a) Let f be the function where $f(0) = 0$ and $f(x)$ is the largest power of 2 dividing $2x$ for $x \neq 0$. The integer 0 is evidently f-rare, so it remains to verify the functional equation.

Since $f(2x) = 2f(x)$ for all x, it suffices to verify the functional equation when at least one of x and y is odd (the case $x = y = 0$ being trivial). If y is odd, then we have

$$f(f(x+y)+y) = 2 = f(f(x)+y)$$

since all the values attained by f are even. If, on the other hand, x is odd and y is even, then we already have

$$f(x+y) = 2 = f(x)$$

from which the functional equation follows immediately.

b) An easy inductive argument (substituting $x + ky$ for x) shows that

$$f(f(x+ky)+y) = f(f(x)+y) \tag{$*$}$$

for all integers x, y and k. If v is an f-rare integer and a is the least element of X_v, then by substituting $y = a - f(x)$ in the above, we see that

$$f(x + k \cdot (a - f(x))) - f(x) + a \in X_v$$

for all integers x and k, so that in particular

$$f(x + k \cdot (a - f(x))) \geqslant f(x)$$

for all integers x and k, by assumption on a. This says that on the (possibly degenerate) arithmetic progression through x with common difference $a - f(x)$, the function f attains its minimal value at x.

Repeating the same argument with a replaced by the greatest element b of X_v shows that

$$f(x + k \cdot (b - f(x)) \leqslant f(x)$$

for all integers x and k. Combined with the above inequality, we therefore have

$$f(x + k \cdot (a - f(x)) \cdot (b - f(x))) = f(x) \tag{$\dagger$}$$

for all integers x and k.

Thus if $f(x) \neq a, b$, then the set $X_{f(x)}$ contains a nondegenerate arithmetic progression, so is infinite. So the only possible f-rare integers are a and b.

In particular, the f-rare integer v we started with must be one of a or b, so that $f(v) = f(a) = f(b) = v$. This means that there cannot be any other f-rare integers v', as they would on the one hand have to be either a or b, and on the other would have to satisfy $f(v') = v'$. Thus v is the unique f-rare integer.

Comment 1. If f is a solution to the functional equation, then so too is any conjugate of f by a translation, i.e. any function $x \mapsto f(x+n) - n$ for an integer n. Thus in proving part (b), one is free to consider only functions f for which 0 is f-rare, as in the following solution.

Solution 2, part (b) only. Suppose v is f-rare, and let a and b be the least and greatest elements of X_v, respectively. Substituting $x = v$ and $y = a - v$ into the equation shows that

$$f(v) - v + a \in X_v$$

and in particular $f(v) \geqslant v$. Repeating the same argument with $x = v$ and $y = b - v$ shows that $f(v) \leqslant v$, and hence $f(v) = v$.

Suppose now that v' is a second f-rare integer. We may assume that $v = 0$ (see Comment 1). We've seen that $f(v') = v'$; we claim that in fact $f(kv') = v'$ for all positive integers k. This gives a contradiction unless $v' = v = 0$.

This claim is proved by induction on k. Supposing it to be true for k, we substitute $y = kv'$ and $x = 0$ into the functional equation to yield

$$f((k+1)v') = f(f(0) + kv') = f(kv') = v'$$

using that $f(0) = 0$. This completes the induction, and hence the proof.

Comment 2. There are many functions f satisfying the functional equation for which there is an f-rare integer. For instance, one may generalise the construction in part (a) of Solution 1 by taking a sequence $1 = a_0, a_1, a_2, \ldots$ of positive integers with each a_i a proper divisor of a_{i+1} and choosing arbitrary functions $f_i \colon (\mathbb{Z}/a_i\mathbb{Z}) \setminus \{0\} \to a_i\mathbb{Z} \setminus \{0\}$ from the nonzero residue classes modulo a_i to the nonzero multiples of a_i. One then defines a function $f \colon \mathbb{Z} \to \mathbb{Z}$ by

$$f(x) := \begin{cases} f_{i+1}(x \bmod a_{i+1}), & \text{if } a_i \mid x \text{ but } a_{i+1} \nmid x; \\ 0, & \text{if } x = 0. \end{cases}$$

If one writes $v(x)$ for the largest i such that $a_i \mid x$ (with $v(0) = \infty$), then it is easy to verify the functional equation for f separately in the two cases $v(y) > v(x)$ and $v(x) \geqslant v(y)$. Hence this f satisfies the functional equation and 0 is an f-rare integer.

Comment 3. In fact, if v is an f-rare integer for an f satisfying the functional equation, then its fibre $X_v = \{v\}$ must be a singleton. We may assume without loss of generality that $v = 0$. We've already seen in Solution 1 that 0 is either the greatest or least element of X_0; replacing f with the function $x \mapsto -f(-x)$ if necessary, we may assume that 0 is the least element of X_0. We write b for the largest element of X_0, supposing for contradiction that $b > 0$, and write $N = (2b)!$.

It now follows from $(*)$ that we have

$$f(f(Nb) + b) = f(f(0) + b) = f(b) = 0,$$

from which we see that $f(Nb) + b \in X_0 \subseteq [0, b]$. It follows that $f(Nb) \in [-b, 0)$, since by construction $Nb \notin X_v$. Now it follows that $(f(Nb) - 0) \cdot (f(Nb) - b)$ is a divisor of N, so from $(\dagger)$ we see that $f(Nb) = f(0) = 0$. This yields the desired contradiction.

Combinatorics

C1. The infinite sequence a_0, a_1, a_2, $\ldots$ of (not necessarily different) integers has the following properties: $0 \leqslant a_i \leqslant i$ for all integers $i \geqslant 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \geqslant 0$.

Prove that all integers $N \geqslant 0$ occur in the sequence (that is, for all $N \geqslant 0$, there exists $i \geqslant 0$ with $a_i = N$).

(Netherlands)

Solution. We prove by induction on k that every initial segment of the sequence, $a_0, a_1, \ldots, a_k$, consists of the following elements (counted with multiplicity, and not necessarily in order), for some $\ell \geqslant 0$ with $2\ell \leqslant k+1$:

$$0, 1, \ldots, \ell - 1, \quad 0, 1, \ldots, k - \ell.$$

For $k = 0$ we have $a_0 = 0$, which is of this form. Now suppose that for $k = m$ the elements $a_0, a_1, \ldots, a_m$ are $0, 0, 1, 1, 2, 2, \ldots, \ell - 1, \ell - 1, \ell, \ell + 1, \ldots, m - \ell - 1, m - \ell$ for some ℓ with $0 \leqslant 2\ell \leqslant m + 1$. It is given that

$$\binom{m+1}{a_0} + \binom{m+1}{a_1} + \cdots + \binom{m+1}{a_m} + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

which becomes

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{\ell-1}\right) + \left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m-\ell}\right) + \binom{m+1}{a_{m+1}} = 2^{m+1},$$

or, using $\binom{m+1}{i} = \binom{m+1}{m+1-i}$, that

$$\left(\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{\ell-1}\right) + \left(\binom{m+1}{m+1} + \binom{m+1}{m} + \cdots + \binom{m+1}{\ell+1}\right) + \binom{m+1}{a_{m+1}} = 2^{m+1}.$$

On the other hand, it is well known that

$$\binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m+1} = 2^{m+1},$$

and so, by subtracting, we get

$$\binom{m+1}{a_{m+1}} = \binom{m+1}{\ell}.$$

From this, using the fact that the binomial coefficients $\binom{m+1}{i}$ are increasing for $i \leqslant \frac{m+1}{2}$ and decreasing for $i \geqslant \frac{m+1}{2}$, we conclude that either $a_{m+1} = \ell$ or $a_{m+1} = m + 1 - \ell$. In either case, $a_0, a_1, \ldots, a_{m+1}$ is again of the claimed form, which concludes the induction.

As a result of this description, any integer $N \geqslant 0$ appears as a term of the sequence a_i for some $0 \leqslant i \leqslant 2N$.

C2. You are given a set of n blocks, each weighing at least 1; their total weight is $2n$. Prove that for every real number r with $0 \leqslant r \leqslant 2n-2$ you can choose a subset of the blocks whose total weight is at least r but at most $r+2$.

(Thailand)

Solution 1. We prove the following more general statement by induction on n.

Claim. Suppose that you have n blocks, each of weight at least 1, and of total weight $s \leqslant 2n$. Then for every r with $-2 \leqslant r \leqslant s$, you can choose some of the blocks whose total weight is at least r but at most $r+2$.

Proof. The base case $n = 1$ is trivial. To prove the inductive step, let x be the largest block weight. Clearly, $x \geqslant s/n$, so $s - x \leqslant \frac{n-1}{n}s \leqslant 2(n-1)$. Hence, if we exclude a block of weight x, we can apply the inductive hypothesis to show the claim holds (for this smaller set) for any $-2 \leqslant r \leqslant s - x$. Adding the excluded block to each of those combinations, we see that the claim also holds when $x - 2 \leqslant r \leqslant s$. So if $x - 2 \leqslant s - x$, then we have covered the whole interval $[-2, s]$. But each block weight is at least 1, so we have $x - 2 \leqslant (s - (n-1)) - 2 = s - (2n - (n-1)) \leqslant s - (s - (n-1)) \leqslant s - x$, as desired. $\square$

Comment. Instead of inducting on sets of blocks with total weight $s \leqslant 2n$, we could instead prove the result only for $s = 2n$. We would then need to modify the inductive step to scale up the block weights before applying the induction hypothesis.

Solution 2. Let $x_1, \ldots, x_n$ be the weights of the blocks in weakly increasing order. Consider the set S of sums of the form $\sum_{j \in J} x_j$ for a subset $J \subseteq \{1, 2, \ldots, n\}$. We want to prove that the *mesh* of S – i.e. the largest distance between two adjacent elements – is at most 2.

For $0 \leqslant k \leqslant n$, let S_k denote the set of sums of the form $\sum_{i \in J} x_i$ for a subset $J \subseteq \{1, 2, \ldots, k\}$. We will show by induction on k that the mesh of S_k is at most 2.

The base case $k = 0$ is trivial (as $S_0 = \{0\}$). For $k > 0$ we have

$$S_k = S_{k-1} \cup (x_k + S_{k-1})$$

(where $(x_k + S_{k-1})$ denotes $\{x_k + s : s \in S_{k-1}\}$), so it suffices to prove that $x_k \leqslant \sum_{j<k} x_j + 2$. But if this were not the case, we would have $x_l > \sum_{j<k} x_j + 2 \geqslant k + 1$ for all $l \geqslant k$, and hence

$$2n = \sum_{j=1}^{n} x_j > (n+1-k)(k+1) + k - 1.$$

This rearranges to $n > k(n+1-k)$, which is false for $1 \leqslant k \leqslant n$, giving the desired contradiction.

C3. Let n be a positive integer. Harry has n coins lined up on his desk, each showing heads or tails. He repeatedly does the following operation: if there are k coins showing heads and $k > 0$, then he flips the k^{th} coin over; otherwise he stops the process. (For example, the process starting with THT would be $THT \to HHT \to HTT \to TTT$, which takes three steps.)

Letting C denote the initial configuration (a sequence of n H's and T's), write $\ell(C)$ for the number of steps needed before all coins show T. Show that this number $\ell(C)$ is finite, and determine its average value over all 2^n possible initial configurations C.

(USA)

Answer: The average is $\frac{1}{4}n(n+1)$.

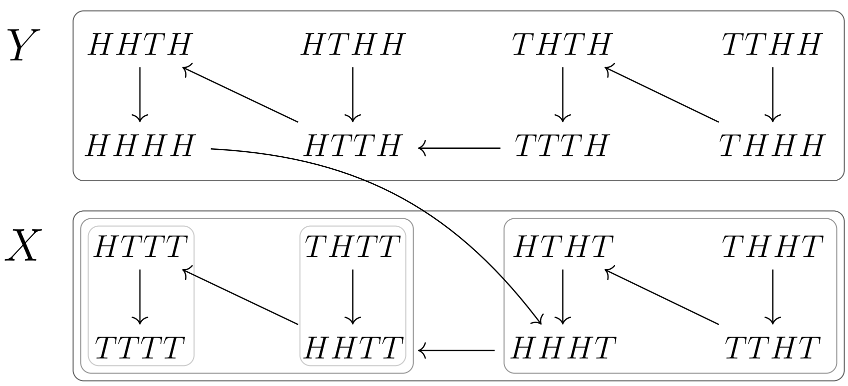
Common remarks. Throughout all these solutions, we let $E(n)$ denote the desired average value.

Solution 1. We represent the problem using a directed graph G_n whose vertices are the length-n strings of H's and T's. The graph features an edge from each string to its successor (except for $TT\cdots TT$, which has no successor). We will also write $\bar{H} = T$ and $\bar{T} = H$.

The graph G_0 consists of a single vertex: the empty string. The main claim is that G_n can be described explicitly in terms of G_{n-1}:

- We take two copies, X and Y, of G_{n-1}.
- In X, we take each string of $n-1$ coins and just append a T to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $s_1 \cdots s_{n-1}T$.
- In Y, we take each string of $n-1$ coins, flip every coin, reverse the order, and append an H to it. In symbols, we replace $s_1 \cdots s_{n-1}$ with $\bar{s}_{n-1}\bar{s}_{n-2}\cdots\bar{s}_1 H$.
- Finally, we add one new edge from Y to X, namely $HH\cdots HHH \to HH\cdots HHT$.

We depict G_4 below, in a way which indicates this recursive construction:



We prove the claim inductively. Firstly, X is correct as a subgraph of G_n, as the operation on coins is unchanged by an extra T at the end: if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $s_1 \cdots s_{n-1}T$ is sent to $t_1 \cdots t_{n-1}T$.

Next, Y is also correct as a subgraph of G_n, as if $s_1 \cdots s_{n-1}$ has k occurrences of H, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ has $(n-1-k)+1 = n-k$ occurrences of H, and thus (provided that $k > 0$), if $s_1 \cdots s_{n-1}$ is sent to $t_1 \cdots t_{n-1}$, then $\bar{s}_{n-1} \cdots \bar{s}_1 H$ is sent to $\bar{t}_{n-1} \cdots \bar{t}_1 H$.

Finally, the one edge from Y to X is correct, as the operation does send $HH\cdots HHH$ to $HH\cdots HHT$.

To finish, note that the sequences in X take an average of $E(n-1)$ steps to terminate, whereas the sequences in Y take an average of $E(n-1)$ steps to reach $HH\cdots H$ and then an additional n steps to terminate. Therefore, we have

$$E(n) = \frac{1}{2}\left(E(n-1) + (E(n-1)+n)\right) = E(n-1) + \frac{n}{2}.$$

We have $E(0)=0$ from our description of G_0. Thus, by induction, we have $E(n) = \frac{1}{2}(1+\cdots+n) = \frac{1}{4}n(n+1)$, which in particular is finite.

Solution 2. We consider what happens with configurations depending on the coins they start and end with.

- If a configuration starts with H, the last $n-1$ coins follow the given rules, as if they were all the coins, until they are all T, then the first coin is turned over.
- If a configuration ends with T, the last coin will never be turned over, and the first $n-1$ coins follow the given rules, as if they were all the coins.
- If a configuration starts with T and ends with H, the middle $n-2$ coins follow the given rules, as if they were all the coins, until they are all T. After that, there are $2n-1$ more steps: first coins 1, 2, …, $n-1$ are turned over in that order, then coins n, $n-1$, …, 1 are turned over in that order.

As this covers all configurations, and the number of steps is clearly finite for 0 or 1 coins, it follows by induction on n that the number of steps is always finite.

We define $E_{AB}(n)$, where A and B are each one of H, T or $*$, to be the average number of steps over configurations of length n restricted to those that start with A, if A is not $*$, and that end with B, if B is not $*$ (so $*$ represents "either H or T"). The above observations tell us that, for $n \geqslant 2$:

- $E_{H*}(n) = E(n-1)+1$.
- $E_{*T}(n) = E(n-1)$.
- $E_{HT}(n) = E(n-2)+1$ (by using both the observations for $H*$ and for $*T$).
- $E_{TH}(n) = E(n-2)+2n-1$.

Now $E_{H*}(n) = \frac{1}{2}(E_{HH}(n)+E_{HT}(n))$, so $E_{HH}(n) = 2E(n-1)-E(n-2)+1$. Similarly, $E_{TT}(n) = 2E(n-1)-E(n-2)-1$. So

$$E(n) = \frac{1}{4}(E_{HT}(n)+E_{HH}(n)+E_{TT}(n)+E_{TH}(n)) = E(n-1)+\frac{n}{2}.$$

We have $E(0)=0$ and $E(1)=\frac{1}{2}$, so by induction on n we have $E(n)=\frac{1}{4}n(n+1)$.

Solution 3. Let H_i be the number of heads in positions 1 to i inclusive (so H_n is the total number of heads), and let I_i be 1 if the i^{th} coin is a head, 0 otherwise. Consider the function

$$t(i) = I_i + 2(\min\{i, H_n\} - H_i).$$

We claim that $t(i)$ is the total number of times coin i is turned over (which implies that the process terminates). Certainly $t(i)=0$ when all coins are tails, and $t(i)$ is always a nonnegative integer, so it suffices to show that when the k^{th} coin is turned over (where $k=H_n$), $t(k)$ goes down by 1 and all the other $t(i)$ are unchanged. We show this by splitting into cases:

- If $i < k$, I_i and H_i are unchanged, and $\min\{i, H_n\} = i$ both before and after the coin flip, so $t(i)$ is unchanged.
- If $i > k$, $\min\{i, H_n\} = H_n$ both before and after the coin flip, and both H_n and H_i change by the same amount, so $t(i)$ is unchanged.
- If $i = k$ and the coin is heads, I_i goes down by 1, as do both $\min\{i, H_n\} = H_n$ and H_i; so $t(i)$ goes down by 1.
- If $i = k$ and the coin is tails, I_i goes up by 1, $\min\{i, H_n\} = i$ is unchanged and H_i goes up by 1; so $t(i)$ goes down by 1.

We now need to compute the average value of

$$\sum_{i=1}^{n} t(i) = \sum_{i=1}^{n} I_i + 2\sum_{i=1}^{n} \min\{i, H_n\} - 2\sum_{i=1}^{n} H_i.$$

The average value of the first term is $\frac{1}{2}n$, and that of the third term is $-\frac{1}{2}n(n+1)$. To compute the second term, we sum over choices for the total number of heads, and then over the possible values of i, getting

$$2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\sum_{i=1}^{n}\min\{i,j\} = 2^{1-n}\sum_{j=0}^{n}\binom{n}{j}\left(nj - \binom{j}{2}\right).$$

Now, in terms of trinomial coefficients,

$$\sum_{j=0}^{n} j\binom{n}{j} = \sum_{j=1}^{n}\binom{n}{n-j, j-1, 1} = n\sum_{j=0}^{n-1}\binom{n-1}{j} = 2^{n-1}n$$

and

$$\sum_{j=0}^{n}\binom{j}{2}\binom{n}{j} = \sum_{j=2}^{n}\binom{n}{n-j, j-2, 2} = \binom{n}{2}\sum_{j=0}^{n-2}\binom{n-2}{j} = 2^{n-2}\binom{n}{2}.$$

So the second term above is

$$2^{1-n}\left(2^{n-1}n^2 - 2^{n-2}\binom{n}{2}\right) = n^2 - \frac{n(n-1)}{4},$$

and the required average is

$$E(n) = \frac{1}{2}n + n^2 - \frac{n(n-1)}{4} - \frac{1}{2}n(n+1) = \frac{n(n+1)}{4}.$$

Solution 4. Harry has built a Turing machine to flip the coins for him. The machine is initially positioned at the k^{th} coin, where there are k heads (and the position before the first coin is considered to be the 0^{th} coin). The machine then moves according to the following rules, stopping when it reaches the position before the first coin: if the coin at its current position is H, it flips the coin and moves to the previous coin, while if the coin at its current position is T, it flips the coin and moves to the next position.

Consider the maximal sequences of consecutive moves in the same direction. Suppose the machine has a consecutive moves to the next coin, before a move to the previous coin. After those a moves, the a coins flipped in those moves are all heads, as is the coin the machine is now at, so at least the next $a+1$ moves will all be moves to the previous coin. Similarly, a consecutive moves to the previous coin are followed by at least $a+1$ consecutive moves to the next coin. There cannot be more than n consecutive moves in the same direction, so this proves that the process terminates (with a move from the first coin to the position before the first coin).

Thus we have a (possibly empty) sequence $a_1 < \cdots < a_t \leqslant n$ giving the lengths of maximal sequences of consecutive moves in the same direction, where the final a_t moves must be moves to the previous coin, ending before the first coin. We claim there is a bijection between initial configurations of the coins and such sequences. This gives

$$E(n) = \frac{1}{2}(1 + 2 + \cdots + n) = \frac{n(n+1)}{4}$$

as required, since each i with $1 \leqslant i \leqslant n$ will appear in half of the sequences, and will contribute i to the number of moves when it does.

To see the bijection, consider following the sequence of moves backwards, starting with the machine just before the first coin and all coins showing tails. This certainly determines a unique configuration of coins that could possibly correspond to the given sequence. Furthermore, every coin flipped as part of the a_j consecutive moves is also flipped as part of all subsequent sequences of a_k consecutive moves, for all $k > j$, meaning that, as we follow the moves backwards, each coin is always in the correct state when flipped to result in a move in the required direction. (Alternatively, since there are 2^n possible configurations of coins and 2^n possible such ascending sequences, the fact that the sequence of moves determines at most one configuration of coins, and thus that there is an injection from configurations of coins to such ascending sequences, is sufficient for it to be a bijection, without needing to show that coins are in the right state as we move backwards.)

Solution 5. We explicitly describe what happens with an arbitrary sequence C of n coins. Suppose that C contain k heads at positions $1 \leqslant c_1 < c_2 < \cdots < c_k \leqslant n$.

Let i be the minimal index such that $c_i \geqslant k$. Then the first few steps will consist of turning over the k^{th}, $(k+1)^{\text{th}}$, $\ldots$, $c_i{}^{\text{th}}$, $(c_i - 1)^{\text{th}}$, $(c_i - 2)^{\text{th}}$, $\ldots$, k^{th} coins in this order. After that we get a configuration with $k-1$ heads at the same positions as in the initial one, except for c_i. This part of the process takes $2(c_i - k) + 1$ steps.

After that, the process acts similarly; by induction on the number of heads we deduce that the process ends. Moreover, if the c_i disappear in order $c_{i_1}, \ldots, c_{i_k}$, the whole process takes

$$\ell(C) = \sum_{j=1}^{k} \bigl(2(c_{i_j} - (k+1-j)) + 1\bigr) = 2\sum_{j=1}^{k} c_j - 2\sum_{j=1}^{k}(k+1-j) + k = 2\sum_{j=1}^{k} c_j - k^2$$

steps.

Now let us find the total value S_k of $\ell(C)$ over all $\binom{n}{k}$ configurations with exactly k heads. To sum up the above expression over those, notice that each number $1 \leqslant i \leqslant n$ appears as c_j exactly $\binom{n-1}{k-1}$ times. Thus

$$\begin{aligned} S_k &= 2\binom{n-1}{k-1}\sum_{i=1}^{n} i - \binom{n}{k}k^2 = 2\frac{(n-1)\cdots(n-k+1)}{(k-1)!} \cdot \frac{n(n+1)}{2} - \frac{n\cdots(n-k+1)}{k!}k^2 \\ &= \frac{n(n-1)\cdots(n-k+1)}{(k-1)!}\bigl((n+1)-k\bigr) = n(n-1)\binom{n-2}{k-1} + n\binom{n-1}{k-1}. \end{aligned}$$

Therefore, the total value of $\ell(C)$ over all configurations is

$$\sum_{k=1}^{n} S_k = n(n-1)\sum_{k=1}^{n}\binom{n-2}{k-1} + n\sum_{k=1}^{n}\binom{n-1}{k-1} = n(n-1)2^{n-2} + n2^{n-1} = 2^n\frac{n(n+1)}{4}.$$

Hence the required average is $E(n) = \frac{n(n+1)}{4}$.

C4. On a flat plane in Camelot, King Arthur builds a labyrinth $\mathfrak{L}$ consisting of n walls, each of which is an infinite straight line. No two walls are parallel, and no three walls have a common point. Merlin then paints one side of each wall entirely red and the other side entirely blue.

At the intersection of two walls there are four corners: two diagonally opposite corners where a red side and a blue side meet, one corner where two red sides meet, and one corner where two blue sides meet. At each such intersection, there is a two-way door connecting the two diagonally opposite corners at which sides of different colours meet.

After Merlin paints the walls, Morgana then places some knights in the labyrinth. The knights can walk through doors, but cannot walk through walls.

Let $k(\mathfrak{L})$ be the largest number k such that, no matter how Merlin paints the labyrinth $\mathfrak{L}$, Morgana can always place at least k knights such that no two of them can ever meet. For each n, what are all possible values for $k(\mathfrak{L})$, where $\mathfrak{L}$ is a labyrinth with n walls?

(Canada)

Answer: The only possible value of k is $k = n + 1$, no matter what shape the labyrinth is.

Solution 1. First we show by induction that the n walls divide the plane into $\binom{n+1}{2}+1$ regions. The claim is true for $n = 0$ as, when there are no walls, the plane forms a single region. When placing the n^{th} wall, it intersects each of the $n-1$ other walls exactly once and hence splits each of n of the regions formed by those other walls into two regions. By the induction hypothesis, this yields $\left(\binom{n}{2}+1\right)+n=\binom{n+1}{2}+1$ regions, proving the claim.

Now let G be the graph with vertices given by the $\binom{n+1}{2}+1$ regions, and with two regions connected by an edge if there is a door between them.

We now show that no matter how Merlin paints the n walls, Morgana can place at least $n + 1$ knights. No matter how the walls are painted, there are exactly $\binom{n}{2}$ intersection points, each of which corresponds to a single edge in G. Consider adding the edges of G sequentially and note that each edge reduces the number of connected components by at most one. Therefore the number of connected components of G is at least $\binom{n+1}{2}+1-\binom{n}{2}=n+1$. If Morgana places a knight in regions corresponding to different connected components of G, then no two knights can ever meet.

Now we give a construction showing that, no matter what shape the labyrinth is, Merlin can colour it such that there are exactly $n + 1$ connected components, allowing Morgana to place at most $n + 1$ knights.

First, we choose a coordinate system on the labyrinth so that none of the walls run due north-south, or due east-west. We then have Merlin paint the west face of each wall red, and the east face of each wall blue. We label the regions according to how many walls the region is on the east side of: the labels are integers between 0 and n.

We claim that, for each i, the regions labelled i are connected by doors. First, we note that for each i with $0 \leqslant i \leqslant n$ there is a unique region labelled i which is unbounded to the north.

Now, consider a knight placed in some region with label i, and ask them to walk north (moving east or west by following the walls on the northern sides of regions, as needed). This knight will never get stuck: each region is convex, and so, if it is bounded to the north, it has a single northernmost vertex with a door northwards to another region with label i.

Eventually it will reach a region which is unbounded to the north, which will be the unique such region with label i. Hence every region with label i is connected to this particular region, and so all regions with label i are connected to each other.

As a result, there are exactly $n + 1$ connected components, and Morgana can place at most $n + 1$ knights.

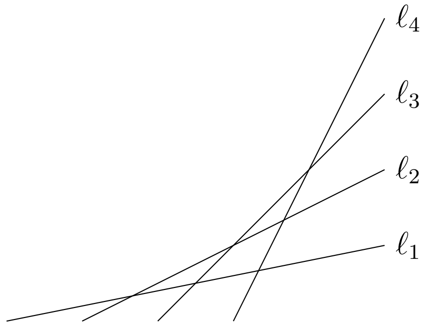
Comment. Variations on this argument exist: some of them capture more information, and some of them capture less information, about the connected components according to this system of numbering.

For example, it can be shown that the unbounded regions are numbered $0, 1, \ldots, n-1, n, n-1, \ldots, 1$ as one cycles around them, that the regions labelled 0 and n are the only regions in their connected components, and that each other connected component forms a single chain running between the two unbounded ones. It is also possible to argue that the regions are acyclic without revealing much about their structure.

Solution 2. We give another description of a strategy for Merlin to paint the walls so that Morgana can place no more than $n+1$ knights.

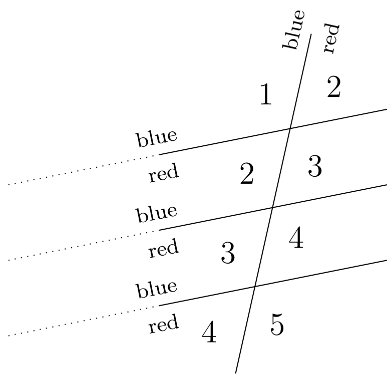
Merlin starts by building a labyrinth of n walls of his own design. He places walls in turn with increasing positive gradients, placing each so far to the right that all intersection points of previously-placed lines lie to the left of it. He paints each in such a way that blue is on the left and red is on the right.

For example, here is a possible sequence of four such lines $\ell_1, \ell_2, \ell_3, \ell_4$:



We say that a region is "on the right" if it has x-coordinate unbounded above (note that if we only have one wall, then both regions are on the right). We claim inductively that, after placing n lines, there are $n+1$ connected components in the resulting labyrinth, each of which contains exactly one region on the right. This is certainly true after placing 0 lines, as then there is only one region (and hence one connected component) and it is on the right.

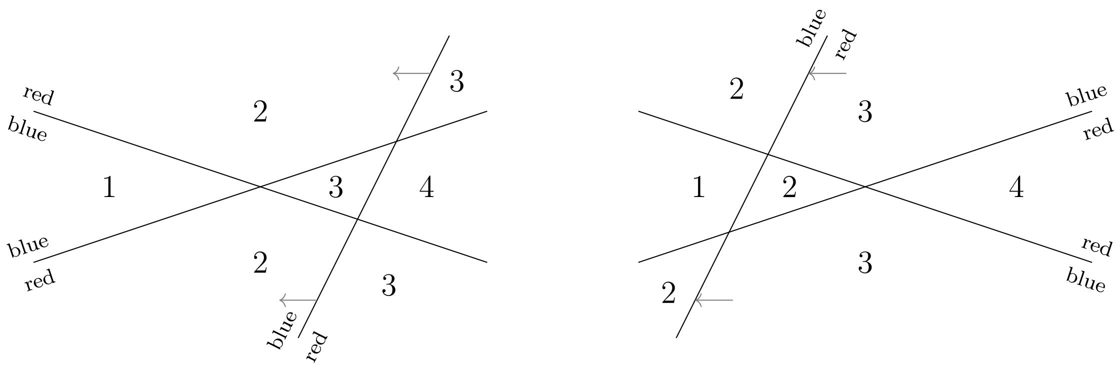
When placing the n^{th} line, it then cuts every one of the $n-1$ previously placed lines, and since it is to the right of all intersection points, the regions it cuts are exactly the n regions on the right.



The addition of this line leaves all previous connected components with exactly one region on the right, and creates a new connected component containing exactly one region, and that region is also on the right. As a result, by induction, this particular labyrinth will have $n+1$ connected components.

Having built this labyrinth, Merlin then moves the walls one-by-one (by a sequence of continuous translations and rotations of lines) into the proper position of the given labyrinth, in such a way that no two lines ever become parallel.

The only time the configuration is changed is when one wall is moved through an intersection point of two others:



Note that all moves really do switch between two configurations like this: all sets of three lines have this colour configuration initially, and the rules on rotations mean they are preserved (in particular, we cannot create three lines creating a triangle with three red edges inwards, or three blue edges inwards).

However, as can be seen, such a move preserves the number of connected components, so in the painting this provides for Arthur's actual labyrinth, Morgana can still only place at most $n + 1$ knights.

Comment. While these constructions are superficially distinct, they in fact result in the same colourings for any particular labyrinth. In fact, using the methods of Solution 2, it is possible to show that these are the only colourings that result in exactly $n + 1$ connected components.

C5. On a certain social network, there are 2019 users, some pairs of which are friends, where friendship is a symmetric relation. Initially, there are 1010 people with 1009 friends each and 1009 people with 1010 friends each. However, the friendships are rather unstable, so events of the following kind may happen repeatedly, one at a time:

> Let A, B, and C be people such that A is friends with both B and C, but B and C are not friends; then B and C become friends, but A is no longer friends with them.

Prove that, regardless of the initial friendships, there exists a sequence of such events after which each user is friends with at most one other user.

(Croatia)

Common remarks. The problem has an obvious rephrasing in terms of graph theory. One is given a graph G with 2019 vertices, 1010 of which have degree 1009 and 1009 of which have degree 1010. One is allowed to perform operations on G of the following kind:

> Suppose that vertex A is adjacent to two distinct vertices B and C which are not adjacent to each other. Then one may remove the edges AB and AC from G and add the edge BC into G.

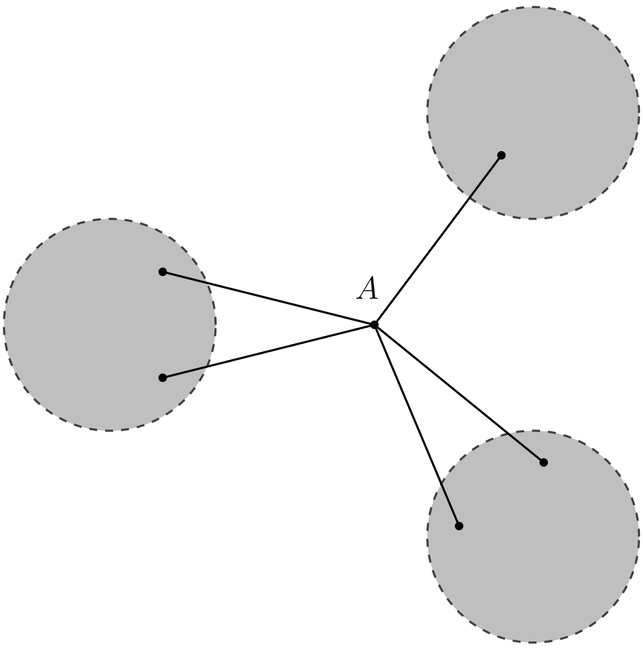
Call such an operation a *refriending*. One wants to prove that, via a sequence of such refriendings, one can reach a graph which is a disjoint union of single edges and vertices.

All of the solutions presented below will use this reformulation.

Solution 1. Note that the given graph is connected, since the total degree of any two vertices is at least 2018 and hence they are either adjacent or have at least one neighbour in common. Hence the given graph satisfies the following condition:

$$\text{Every connected component of } G \text{ with at least three vertices is not complete and has a vertex of odd degree.} \tag{1}$$

We will show that if a graph G satisfies condition (1) and has a vertex of degree at least 2, then there is a refriending on G that preserves condition (1). Since refriendings decrease the total number of edges of G, by using a sequence of such refriendings, we must reach a graph G with maximal degree at most 1, so we are done.



Pick a vertex A of degree at least 2 in a connected component G' of G. Since no component of G with at least three vertices is complete we may assume that not all of the neighbours of A are adjacent to one another. (For example, pick a maximal complete subgraph K of G'. Some vertex A of K has a neighbour outside K, and this neighbour is not adjacent to every vertex of K by maximality.) Removing A from G splits G' into smaller connected components $G_1, \ldots, G_k$ (possibly with $k = 1$), to each of which A is connected by at least one edge. We divide into several cases.

Case 1: $k \geqslant 2$ and A is connected to some G_i by at least two edges.

Choose a vertex B of G_i adjacent to A, and a vertex C in another component G_j adjacent to A. The vertices B and C are not adjacent, and hence removing edges AB and AC and adding in edge BC does not disconnect G'. It is easy to see that this preserves the condition, since the refriending does not change the parity of the degrees of vertices.

Case 2: $k \geqslant 2$ and A is connected to each G_i by exactly one edge.

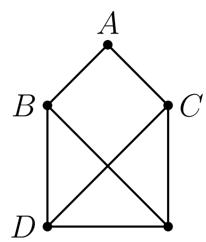
Consider the induced subgraph on any G_i and the vertex A. The vertex A has degree 1 in this subgraph; since the number of odd-degree vertices of a graph is always even, we see that G_i has a vertex of odd degree (in G). Thus if we let B and C be any distinct neighbours of A, then removing edges AB and AC and adding in edge BC preserves the above condition: the refriending creates two new components, and if either of these components has at least three vertices, then it cannot be complete and must contain a vertex of odd degree (since each G_i does).

Case 3: $k = 1$ and A is connected to G_1 by at least three edges.

By assumption, A has two neighbours B and C which are not adjacent to one another. Removing edges AB and AC and adding in edge BC does not disconnect G'. We are then done as in Case 1.

Case 4: $k = 1$ and A is connected to G_1 by exactly two edges.

Let B and C be the two neighbours of A, which are not adjacent. Removing edges AB and AC and adding in edge BC results in two new components: one consisting of a single vertex; and the other containing a vertex of odd degree. We are done unless this second component would be a complete graph on at least 3 vertices. But in this case, G_1 would be a complete graph minus the single edge BC, and hence has at least 4 vertices since G' is not a 4-cycle. If we let D be a third vertex of G_1, then removing edges BA and BD and adding in edge AD does not disconnect G'. We are then done as in Case 1.

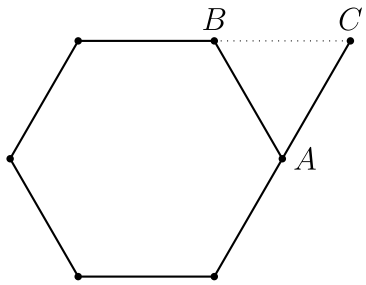


Comment. In fact, condition 1 above *precisely* characterises those graphs which can be reduced to a graph of maximal degree $\leqslant 1$ by a sequence of refriendings.

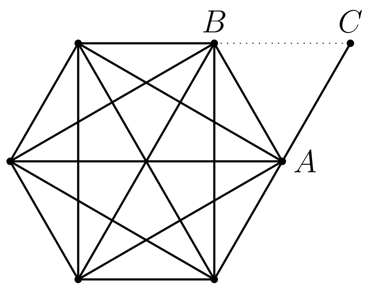
Solution 2. As in the previous solution, note that a refriending preserves the property that a graph has a vertex of odd degree and (trivially) the property that it is not complete; note also that our initial graph is connected. We describe an algorithm to reduce our initial graph to a graph of maximal degree at most 1, proceeding in two steps.

Step 1: There exists a sequence of refriendings reducing the graph to a tree.

Proof. Since the number of edges decreases with each refriending, it suffices to prove the following: as long as the graph contains a cycle, there exists a refriending such that the resulting graph is still connected. We will show that the graph in fact contains a cycle Z and vertices A, B, C such that A and B are adjacent in the cycle Z, C is not in Z, and is adjacent to A but not B. Removing edges AB and AC and adding in edge BC keeps the graph connected, so we are done.



To find this cycle Z and vertices A, B, C, we pursue one of two strategies. If the graph contains a triangle, we consider a largest complete subgraph K, which thus contains at least three vertices. Since the graph itself is not complete, there is a vertex C not in K connected to a vertex A of K. By maximality of K, there is a vertex B of K not connected to C, and hence we are done by choosing a cycle Z in K through the edge AB.



If the graph is triangle-free, we consider instead a smallest cycle Z. This cycle cannot be Hamiltonian (i.e. it cannot pass through every vertex of the graph), since otherwise by minimality the graph would then have no other edges, and hence would have even degree at every vertex. We may thus choose a vertex C not in Z adjacent to a vertex A of Z. Since the graph is triangle-free, it is not adjacent to any neighbour B of A in Z, and we are done. □

Step 2: Any tree may be reduced to a disjoint union of single edges and vertices by a sequence of refriendings.

Proof. The refriending preserves the property of being acyclic. Hence, after applying a sequence of refriendings, we arrive at an acyclic graph in which it is impossible to perform any further refriendings. The maximal degree of any such graph is 1: if it had a vertex A with two neighbours B, C, then B and C would necessarily be nonadjacent since the graph is cycle-free, and so a refriending would be possible. Thus we reach a graph with maximal degree at most 1 as desired. □

C6. Let $n > 1$ be an integer. Suppose we are given $2n$ points in a plane such that no three of them are collinear. The points are to be labelled $A_1, A_2, \ldots, A_{2n}$ in some order. We then consider the $2n$ angles $\angle A_1A_2A_3$, $\angle A_2A_3A_4$, $\ldots$, $\angle A_{2n-2}A_{2n-1}A_{2n}$, $\angle A_{2n-1}A_{2n}A_1$, $\angle A_{2n}A_1A_2$. We measure each angle in the way that gives the smallest positive value (i.e. between 0° and 180°). Prove that there exists an ordering of the given points such that the resulting $2n$ angles can be separated into two groups with the sum of one group of angles equal to the sum of the other group.

(USA)

Comment. The first three solutions all use the same construction involving a line separating the points into groups of n points each, but give different proofs that this construction works. Although Solution 1 is very short, the Problem Selection Committee does not believe any of the solutions is easy to find and thus rates this as a problem of medium difficulty.

Solution 1. Let ℓ be a line separating the points into two groups (L and R) with n points in each. Label the points $A_1, A_2, \ldots, A_{2n}$ so that $L = \{A_1, A_3, \ldots, A_{2n-1}\}$. We claim that this labelling works.

Take a line $s = A_{2n}A_1$.

(a) Rotate s around A_1 until it passes through A_2; the rotation is performed in a direction such that s is never parallel to ℓ.

(b) Then rotate the new s around A_2 until it passes through A_3 in a similar manner.

(c) Perform $2n-2$ more such steps, after which s returns to its initial position.

The total (directed) rotation angle Θ of s is clearly a multiple of 180°. On the other hand, s was never parallel to ℓ, which is possible only if $\Theta = 0$. Now it remains to partition all the $2n$ angles into those where s is rotated anticlockwise, and the others.

Solution 2. When tracing a cyclic path through the A_i in order, with straight line segments between consecutive points, let θ_i be the exterior angle at A_i, with a sign convention that it is positive if the path turns left and negative if the path turns right. Then $\sum_{i=1}^{2n} \theta_i = 360k^\circ$ for some integer k. Let $\phi_i = \angle A_{i-1}A_iA_{i+1}$ (indices mod $2n$), defined as in the problem; thus $\phi_i = 180^\circ - |\theta_i|$.

Let L be the set of i for which the path turns left at A_i and let R be the set for which it turns right. Then $S = \sum_{i\in L} \phi_i - \sum_{i\in R} \phi_i = (180(|L| - |R|) - 360k)^\circ$, which is a multiple of 360° since the number of points is even. We will show that the points can be labelled such that $S = 0$, in which case L and R satisfy the required condition of the problem.

Note that the value of S is defined for a slightly larger class of configurations: it is OK for two points to coincide, as long as they are not consecutive, and OK for three points to be collinear, as long as A_i, A_{i+1} and A_{i+2} do not appear on a line *in that order*. In what follows it will be convenient, although not strictly necessary, to consider such configurations.

Consider how S changes if a single one of the A_i is moved along some straight-line path (not passing through any A_j and not lying on any line A_jA_k, but possibly crossing such lines). Because S is a multiple of 360°, and the angles change continuously, S can only change when a point moves between R and L. Furthermore, if $\phi_j = 0$ when A_j moves between R and L, S is unchanged; it only changes if $\phi_j = 180^\circ$ when A_j moves between those sets.

For any starting choice of points, we will now construct a new configuration, with labels such that $S = 0$, that can be perturbed into the original one without any ϕ_i passing through 180°, so that $S = 0$ for the original configuration with those labels as well.

Take some line such that there are n points on each side of that line. The new configuration has n copies of a single point on each side of the line, and a path that alternates between sides of the line; all angles are 0, so this configuration has $S = 0$. Perturbing the points into their original positions, while keeping each point on its side of the line, no angle ϕ_i can pass through 180°, because no straight line can go from one side of the line to the other and back. So the perturbation process leaves $S = 0$.

Comment. More complicated variants of this solution are also possible; for example, a path defined using four quadrants of the plane rather than just two half-planes.

Solution 3. First, let ℓ be a line in the plane such that there are n points on one side and the other n points on the other side. For convenience, assume ℓ is horizontal (otherwise, we can rotate the plane). Then we can use the terms "above", "below", "left" and "right" in the usual way. We denote the n points above the line in an arbitrary order as $P_1, P_2, \ldots, P_n$, and the n points below the line as $Q_1, Q_2, \ldots, Q_n$.

If we connect P_i and Q_j with a line segment, the line segment will intersect with the line ℓ. Denote the intersection as I_{ij}. If P_i is connected to Q_j and Q_k, where $j < k$, then I_{ij} and I_{ik} are two different points, because P_i, Q_j and Q_k are not collinear.

Now we define a "sign" for each angle $\angle Q_j P_i Q_k$. Assume $j < k$. We specify that the sign is positive for the following two cases:

- if i is odd and I_{ij} is to the left of I_{ik},
- if i is even and I_{ij} is to the right of I_{ik}.

Otherwise the sign of the angle is negative. If $j > k$, then the sign of $\angle Q_j P_i Q_k$ is taken to be the same as for $\angle Q_k P_i Q_j$.

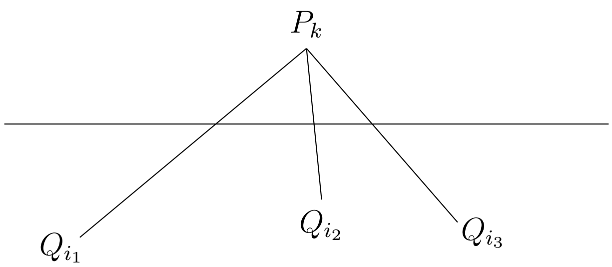
Similarly, we can define the sign of $\angle P_j Q_i P_k$ with $j < k$ (or equivalently $\angle P_k Q_i P_j$). For example, it is positive when i is odd and I_{ji} is to the left of I_{ki}.

Henceforth, whenever we use the notation $\angle Q_j P_i Q_k$ or $\angle P_j Q_i P_k$ for a numerical quantity, it is understood to denote either the (geometric) measure of the angle or the negative of this measure, depending on the sign as specified above.

We now have the following important fact for signed angle measures:

$$\angle Q_{i_1} P_k Q_{i_3} = \angle Q_{i_1} P_k Q_{i_2} + \angle Q_{i_2} P_k Q_{i_3} \tag{1}$$

for all points P_k, Q_{i_1}, Q_{i_2} and Q_{i_3} with $i_1 < i_2 < i_3$. The following figure shows a "natural" arrangement of the points. Equation (1) still holds for any other arrangement, as can be easily verified.



Similarly, we have

$$\angle P_{i_1} Q_k P_{i_3} = \angle P_{i_1} Q_k P_{i_2} + \angle P_{i_2} Q_k P_{i_3}, \tag{2}$$

for all points Q_k, P_{i_1}, P_{i_2} and P_{i_3}, with $i_1 < i_2 < i_3$.

We are now ready to specify the desired ordering $A_1, \ldots, A_{2n}$ of the points:

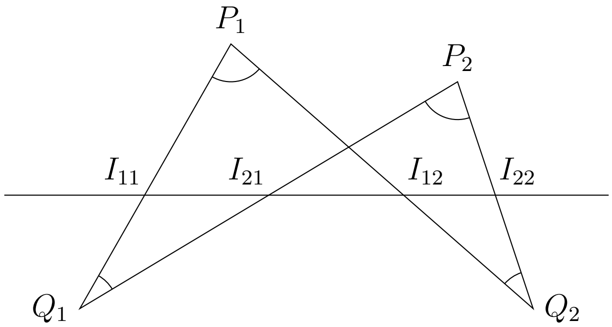
- if $i \leqslant n$ is odd, put $A_i = P_i$ and $A_{2n+1-i} = Q_i$;
- if $i \leqslant n$ is even, put $A_i = Q_i$ and $A_{2n+1-i} = P_i$.

For example, for $n = 3$ this ordering is $P_1, Q_2, P_3, Q_3, P_2, Q_1$. This sequence alternates between P's and Q's, so the above conventions specify a sign for each of the angles $A_{i-1}A_iA_{i+1}$. We claim that the sum of these $2n$ signed angles equals 0. If we can show this, it would complete the proof.

We prove the claim by induction. For brevity, we use the notation $\angle P_i$ to denote whichever of the $2n$ angles has its vertex at P_i, and $\angle Q_i$ similarly.

First let $n = 2$. If the four points can be arranged to form a convex quadrilateral, then the four line segments P_1Q_1, P_1Q_2, P_2Q_1 and P_2Q_2 constitute a self-intersecting quadrilateral. We use several figures to illustrate the possible cases.

The following figure is one possible arrangement of the points.



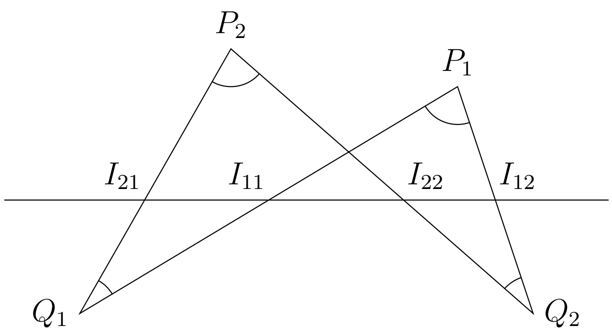
Then $\angle P_1$ and $\angle Q_1$ are positive, $\angle P_2$ and $\angle Q_2$ are negative, and we have

$$|\angle P_1| + |\angle Q_1| = |\angle P_2| + |\angle Q_2|.$$

With signed measures, we have

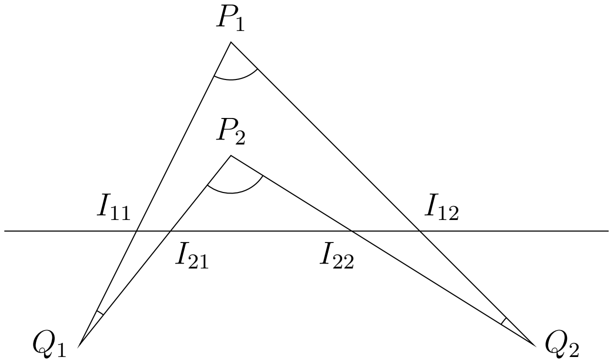
$$\angle P_1 + \angle Q_1 + \angle P_2 + \angle Q_2 = 0. \tag{3}$$

If we switch the labels of P_1 and P_2, we have the following picture:



Switching labels P_1 and P_2 has the effect of flipping the sign of all four angles (as well as swapping the magnitudes on the relabelled points); that is, the new values of $(\angle P_1, \angle P_2, \angle Q_1, \angle Q_2)$ equal the old values of $(-\angle P_2, -\angle P_1, -\angle Q_1, -\angle Q_2)$. Consequently, equation (3) still holds. Similarly, when switching the labels of Q_1 and Q_2, or both the P's and the Q's, equation (3) still holds.

The remaining subcase of $n = 2$ is that one point lies inside the triangle formed by the other three. We have the following picture.



We have

$$|\angle P_1| + |\angle Q_1| + |\angle Q_2| = |\angle P_2|.$$

and equation (3) holds.

Again, switching the labels for P's or the Q's will not affect the validity of equation (3). Also, if the point lying inside the triangle of the other three is one of the Q's rather than the P's, the result still holds, since our sign convention is preserved when we relabel Q's as P's and vice-versa and reflect across ℓ.

We have completed the proof of the claim for $n = 2$.

Assume the claim holds for $n = k$, and we wish to prove it for $n = k + 1$. Suppose we are given our $2(k + 1)$ points. First ignore P_{k+1} and Q_{k+1}, and form $2k$ angles from $P_1, \ldots, P_k$, $Q_1, \ldots, Q_k$ as in the $n = k$ case. By the induction hypothesis we have

$$\sum_{i=1}^{k} (\angle P_i + \angle Q_i) = 0.$$

When we add in the two points P_{k+1} and Q_{k+1}, this changes our angles as follows:

- the angle at P_k changes from $\angle Q_{k-1}P_kQ_k$ to $\angle Q_{k-1}P_kQ_{k+1}$;
- the angle at Q_k changes from $\angle P_{k-1}Q_kP_k$ to $\angle P_{k-1}Q_kP_{k+1}$;
- two new angles $\angle Q_kP_{k+1}Q_{k+1}$ and $\angle P_kQ_{k+1}P_{k+1}$ are added.

We need to prove the changes have no impact on the total sum. In other words, we need to prove

$$(\angle Q_{k-1}P_kQ_{k+1} - \angle Q_{k-1}P_kQ_k) + (\angle P_{k-1}Q_kP_{k+1} - \angle P_{k-1}Q_kP_k) + (\angle P_{k+1} + \angle Q_{k+1}) = 0. \quad (4)$$

In fact, from equations (1) and (2), we have

$$\angle Q_{k-1}P_kQ_{k+1} - \angle Q_{k-1}P_kQ_k = \angle Q_kP_kQ_{k+1},$$

and

$$\angle P_{k-1}Q_kP_{k+1} - \angle P_{k-1}Q_kP_k = \angle P_kQ_kP_{k+1}.$$

Therefore, the left hand side of equation (4) becomes $\angle Q_kP_kQ_{k+1} + \angle P_kQ_kP_{k+1} + \angle Q_kP_{k+1}Q_{k+1} + \angle P_kQ_{k+1}P_{k+1}$, which equals 0, simply by applying the $n = 2$ case of the claim. This completes the induction.

Solution 4. We shall think instead of the problem as asking us to assign a weight ± 1 to each angle, such that the weighted sum of all the angles is zero.

Given an ordering $A_1, \ldots, A_{2n}$ of the points, we shall assign weights according to the following recipe: walk in order from point to point, and assign the left turns $+1$ and the right turns -1. This is the same weighting as in Solution 3, and as in that solution, the weighted sum is a multiple of 360°.

We now aim to show the following:

Lemma. Transposing any two consecutive points in the ordering changes the weighted sum by $\pm 360^\circ$ or 0.

Knowing that, we can conclude quickly: if the ordering $A_1, \ldots, A_{2n}$ has weighted angle sum $360k^\circ$, then the ordering $A_{2n}, \ldots, A_1$ has weighted angle sum $-360k^\circ$ (since the angles are the same, but left turns and right turns are exchanged). We can reverse the ordering of A_1, $\ldots$, A_{2n} by a sequence of transpositions of consecutive points, and in doing so the weighted angle sum must become zero somewhere along the way.

We now prove that lemma:

Proof. Transposing two points amounts to taking a section $A_kA_{k+1}A_{k+2}A_{k+3}$ as depicted, reversing the central line segment $A_{k+1}A_{k+2}$, and replacing its two neighbours with the dotted lines.

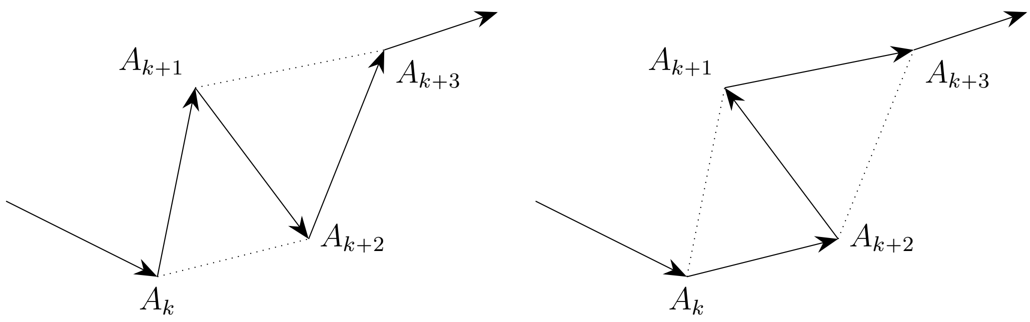


Figure 1: Transposing two consecutive vertices: before (left) and afterwards (right)

In each triangle, we alter the sum by $\pm 180^\circ$. Indeed, using (anticlockwise) directed angles modulo 360°, we either add or subtract all three angles of each triangle.

Hence both triangles together alter the sum by $\pm 180 \pm 180^\circ$, which is $\pm 360^\circ$ or 0. $\square$

C7. There are 60 empty boxes $B_1, \ldots, B_{60}$ in a row on a table and an unlimited supply of pebbles. Given a positive integer n, Alice and Bob play the following game.

In the first round, Alice takes n pebbles and distributes them into the 60 boxes as she wishes. Each subsequent round consists of two steps:

(a) Bob chooses an integer k with $1 \leqslant k \leqslant 59$ and splits the boxes into the two groups $B_1, \ldots, B_k$ and $B_{k+1}, \ldots, B_{60}$.

(b) Alice picks one of these two groups, adds one pebble to each box in that group, and removes one pebble from each box in the other group.

Bob wins if, at the end of any round, some box contains no pebbles. Find the smallest n such that Alice can prevent Bob from winning.

(Czech Republic)

Answer: $n = 960$. In general, if there are $N > 1$ boxes, the answer is $n = \left\lfloor \frac{N}{2} + 1 \right\rfloor \left\lceil \frac{N}{2} + 1 \right\rceil - 1$.

Common remarks. We present solutions for the general case of $N > 1$ boxes, and write $M = \left\lfloor \frac{N}{2} + 1 \right\rfloor \left\lceil \frac{N}{2} + 1 \right\rceil - 1$ for the claimed answer. For $1 \leqslant k < N$, say that Bob makes a *k-move* if he splits the boxes into a left group $\{B_1, \ldots, B_k\}$ and a right group $\{B_{k+1}, \ldots, B_N\}$. Say that one configuration *dominates* another if it has at least as many pebbles in each box, and say that it *strictly dominates* the other configuration if it also has more pebbles in at least one box. (Thus, if Bob wins in some configuration, he also wins in every configuration that it dominates.)

It is often convenient to consider 'V-shaped' configurations; for $1 \leqslant i \leqslant N$, let V_i be the configuration where B_j contains $1 + |j - i|$ pebbles (i.e. where the i^{th} box has a single pebble and the numbers increase by one in both directions, so the first box has i pebbles and the last box has $N + 1 - i$ pebbles). Note that V_i contains $\frac{1}{2}i(i+1) + \frac{1}{2}(N+1-i)(N+2-i) - 1$ pebbles. If $i = \left\lceil \frac{N}{2} \right\rceil$, this number equals M.

Solutions split naturally into a strategy for Alice (starting with M pebbles and showing she can prevent Bob from winning) and a strategy for Bob (showing he can win for any starting configuration with at most $M - 1$ pebbles). The following observation is also useful to simplify the analysis of strategies for Bob.

Observation A. Consider two consecutive rounds. Suppose that in the first round Bob made a k-move and Alice picked the left group, and then in the second round Bob makes an ℓ-move, with $\ell > k$. We may then assume, without loss of generality, that Alice again picks the left group.

Proof. Suppose Alice picks the right group in the second round. Then the combined effect of the two rounds is that each of the boxes $B_{k+1}, \ldots, B_\ell$ lost two pebbles (and the other boxes are unchanged). Hence this configuration is strictly dominated by that before the first round, and it suffices to consider only Alice's other response. $\square$

Solution 1 (Alice). Alice initially distributes pebbles according to $V_{\lceil \frac{N}{2} \rceil}$. Suppose the current configuration of pebbles dominates V_i. If Bob makes a k-move with $k \geqslant i$ then Alice picks the left group, which results in a configuration that dominates V_{i+1}. Likewise, if Bob makes a k-move with $k < i$ then Alice picks the right group, which results in a configuration that dominates V_{i-1}. Since none of $V_1, \ldots, V_N$ contains an empty box, Alice can prevent Bob from ever winning.

Solution 1 (Bob). The key idea in this solution is the following claim.

Claim. If there exist a positive integer k such that there are at least $2k$ boxes that have at most k pebbles each then Bob can force a win.

Proof. We ignore the other boxes. First, Bob makes a k-move (splits the $2k$ boxes into two groups of k boxes each). Without loss of generality, Alice picks the left group. Then Bob makes a $(k+1)$-move, ..., a $(2k-1)$-move. By Observation A, we may suppose Alice always picks the left group. After Bob's $(2k-1)$-move, the rightmost box becomes empty and Bob wins. □

Now, we claim that if $n < M$ then either there already exists an empty box, or there exist a positive integer k and $2k$ boxes with at most k pebbles each (and thus Bob can force a win). Otherwise, assume each box contains at least 1 pebble, and for each $1 \leqslant k \leqslant \lfloor \frac{N}{2} \rfloor$, at least $N-(2k-1) = N+1-2k$ boxes contain at least $k+1$ pebbles. Summing, there are at least as many pebbles in total as in $V_{\lceil \frac{N}{2} \rceil}$; that is, at least M pebbles, as desired.

Solution 2 (Alice). Let $K = \lfloor \frac{N}{2} + 1 \rfloor$. Alice starts with the boxes in the configuration V_K. For each of Bob's $N-1$ possible choices, consider the subset of rounds in which he makes that choice. In that subset of rounds, Alice alternates between picking the left group and picking the right group; the first time Bob makes that choice, Alice picks the group containing the K^{th} box. Thus, at any time during the game, the number of pebbles in each box depends only on which choices Bob has made an odd number of times. This means that the number of pebbles in a box could decrease by at most the number of choices for which Alice would have started by removing a pebble from the group containing that box. These numbers are, for each box,

$$\left\lfloor \tfrac{N}{2} \right\rfloor, \left\lfloor \tfrac{N}{2} - 1 \right\rfloor, \ldots, 1, 0, 1, \ldots, \left\lceil \tfrac{N}{2} - 1 \right\rceil.$$

These are pointwise less than the numbers of pebbles the boxes started with, meaning that no box ever becomes empty with this strategy.

Solution 2 (Bob). Let $K = \lfloor \frac{N}{2} + 1 \rfloor$. For Bob's strategy, we consider a configuration X with at most $M-1$ pebbles, and we make use of Observation A. Consider two configurations with M pebbles: V_K and V_{N+1-K} (if n is odd, they are the same configuration; if n is even, one is the reverse of the other). The configuration X has fewer pebbles than V_K in at least one box, and fewer pebbles than V_{N+1-K} in at least one box.

Suppose first that, with respect to one of those configurations (without loss of generality V_K), X has fewer pebbles in one of the boxes in the half where they have 1, 2, ..., $\lceil \frac{N}{2} \rceil$ pebbles (the right half in V_K if N is even; if N is odd, we can take it to be the right half, without loss of generality, as the configuration is symmetric). Note that the number cannot be fewer in the box with 1 pebble in V_K, because then it would have 0 pebbles. Bob then does a K-move. If Alice picks the right group, the total number of pebbles goes down and we restart Bob's strategy with a smaller number of pebbles. If Alice picks the left group, Bob follows with a $(K+1)$-move, a $(K+2)$-move, and so on; by Observation A we may assume Alice always picks the left group. But whichever box in the right half had fewer pebbles in X than in V_K ends up with 0 pebbles at some point in this sequence of moves.

Otherwise, N is even, and for both of those configurations, there are fewer pebbles in X only on the 2, 3, ..., $\frac{N}{2}+1$ side. That is, the numbers of pebbles in X are at least

$$\frac{N}{2}, \frac{N}{2} - 1, \ldots, 1, 1, \ldots, \frac{N}{2} \tag{C}$$

with equality occurring at least once on each side. Bob does an $\frac{N}{2}$-move. Whichever group Alice chooses, the total number of pebbles is unchanged, and the side from which pebbles are removed now has a box with fewer pebbles than in (C), so the previous case of Bob's strategy can now be applied.

Solution 3 (Bob). For any configuration C, define $L(C)$ to be the greatest integer such that, for all $0 \leqslant i \leqslant N-1$, the box B_{i+1} contains at least $L(C)-i$ pebbles. Similarly, define $R(C)$ to be greatest integer such that, for all $0 \leqslant i \leqslant N-1$, the box B_{N-i} contains at least $R(C)-i$ pebbles. (Thus, C dominates the 'left half' of $V_{L(C)}$ and the 'right half' of $V_{N+1-R(C)}$.) Then C dominates a 'V-shaped' configuration if and only if $L(C)+R(C) \geqslant N+1$. Note that if C dominates a V-shaped configuration, it has at least M pebbles.

Now suppose that there are fewer than M pebbles, so we have $L(C)+R(C) \leqslant N$. Then Bob makes an $L(C)$-move (or more generally any move with at least $L(C)$ boxes on the left and $R(C)$ boxes on the right). Let C' be the new configuration, and suppose that no box becomes empty (otherwise Bob has won). If Alice picks the left group, we have $L(C')=L(C)+1$ and $R(C')=R(C)-1$. Otherwise, we have $L(C')=L(C)-1$ and $R(C')=R(C)+1$. In either case, we have $L(C')+R(C') \leqslant N$.

Bob then repeats this strategy, until one of the boxes becomes empty. Since the condition in Observation A holds, we may assume that Alice picks a group on the same side each time. Then one of L and R is strictly decreasing; without loss of generality assume that L strictly decreases. At some point we reach $L=1$. If B_2 is still nonempty, then B_1 must contain a single pebble. Bob makes a 1-move, and by Observation A, Alice must (eventually) pick the right group, making this box empty.

C8. Alice has a map of Wonderland, a country consisting of $n \geqslant 2$ towns. For every pair of towns, there is a narrow road going from one town to the other. One day, all the roads are declared to be "one way" only. Alice has no information on the direction of the roads, but the King of Hearts has offered to help her. She is allowed to ask him a number of questions. For each question in turn, Alice chooses a pair of towns and the King of Hearts tells her the direction of the road connecting those two towns.

Alice wants to know whether there is at least one town in Wonderland with at most one outgoing road. Prove that she can always find out by asking at most $4n$ questions.

Comment. This problem could be posed with an explicit statement about points being awarded for weaker bounds cn for some $c > 4$, in the style of IMO 2014 Problem 6.

(Thailand)

Solution. We will show Alice needs to ask at most $4n - 7$ questions. Her strategy has the following phases. In what follows, S is the set of towns that Alice, so far, does not know to have more than one outgoing road (so initially $|S| = n$).

Phase 1. Alice chooses any two towns, say A and B. Without loss of generality, suppose that the King of Hearts' answer is that the road goes from A to B.

At the end of this phase, Alice has asked 1 question.

Phase 2. During this phase there is a single (variable) town T that is known to have at least one incoming road but not yet known to have any outgoing roads. Initially, T is B. Alice does the following $n - 2$ times: she picks a town X she has not asked about before, and asks the direction of the road between T and X. If it is from X to T, T is unchanged; if it is from T to X, X becomes the new choice of town T, as the previous T is now known to have an outgoing road.

At the end of this phase, Alice has asked a total of $n - 1$ questions. The final town T is not yet known to have any outgoing roads, while every other town has exactly one outgoing road known. The undirected graph of roads whose directions are known is a tree.

Phase 3. During this phase, Alice asks about the directions of all roads between T and another town she has not previously asked about, stopping if she finds two outgoing roads from T. This phase involves at most $n - 2$ questions. If she does not find two outgoing roads from T, she has answered her original question with at most $2n - 3 \leqslant 4n - 7$ questions, so in what follows we suppose that she does find two outgoing roads, asking a total of k questions in this phase, where $2 \leqslant k \leqslant n - 2$ (and thus $n \geqslant 4$ for what follows).

For every question where the road goes towards T, the town at the other end is removed from S (as it already had one outgoing road known), while the last question resulted in T being removed from S. So at the end of this phase, $|S| = n - k + 1$, while a total of $n + k - 1$ questions have been asked. Furthermore, the undirected graph of roads within S whose directions are known contains no cycles (as T is no longer a member of S, all questions asked in this phase involved T and the graph was a tree before this phase started). Every town in S has exactly one outgoing road known (not necessarily to another town in S).

Phase 4. During this phase, Alice repeatedly picks any pair of towns in S for which she does not know the direction of the road between them. Because every town in S has exactly one outgoing road known, this always results in the removal of one of those two towns from S. Because there are no cycles in the graph of roads of known direction within S, this can continue until there are at most 2 towns left in S.

If it ends with t towns left, $n - k + 1 - t$ questions were asked in this phase, so a total of $2n - t$ questions have been asked.

Phase 5. During this phase, Alice asks about all the roads from the remaining towns in S that she has not previously asked about. She has definitely already asked about any road between those towns (if $t = 2$). She must also have asked in one of the first two phases about at least one other road involving one of those towns (as those phases resulted in a tree with $n > 2$ vertices). So she asks at most $t(n-t)-1$ questions in this phase.

At the end of this phase, Alice knows whether any town has at most one outgoing road. If $t = 1$, at most $3n-3 \leqslant 4n-7$ questions were needed in total, while if $t = 2$, at most $4n-7$ questions were needed in total.

Comment 1. The version of this problem originally submitted asked only for an upper bound of $5n$, which is much simpler to prove. The Problem Selection Committee preferred a version with an asymptotically optimal constant. In the following comment, we will show that the constant is optimal.

Comment 2. We will show that Alice *cannot* always find out by asking at most $4n - 3(\log_2 n) - 15$ questions, if $n \geqslant 8$.

To show this, we suppose the King of Hearts is choosing the directions as he goes along, only picking the direction of a road when Alice asks about it for the first time. We provide a strategy for the King of Hearts that ensures that, after the given number of questions, the map is still consistent both with the existence of a town with at most one outgoing road, and with the nonexistence of such a town. His strategy has the following phases. When describing how the King of Hearts' answer to a question is determined below, we always assume he is being asked about a road for the first time (otherwise, he just repeats his previous answer for that road). This strategy is described throughout in graph-theoretic terms (vertices and edges rather than towns and roads).

Phase 1. In this phase, we consider the undirected graph formed by edges whose directions are known. The phase terminates when there are exactly 8 connected components whose undirected graphs are trees. The following invariant is maintained: in a component with k vertices whose undirected graph is a tree, every vertex has at most $\lfloor \log_2 k \rfloor$ edges into it.

- If the King of Hearts is asked about an edge between two vertices in the same component, or about an edge between two components at least one of which is not a tree, he chooses any direction for that edge arbitrarily.

- If he is asked about an edge between a vertex in component A that has a vertices and is a tree and a vertex in component B that has b vertices and is a tree, suppose without loss of generality that $a \geqslant b$. He then chooses the edge to go from A to B. In this case, the new number of edges into any vertex is at most $\max\{\lfloor \log_2 a \rfloor, \lfloor \log_2 b \rfloor + 1\} \leqslant \lfloor \log_2(a+b) \rfloor$.

In all cases, the invariant is preserved, and the number of tree components either remains unchanged or goes down by 1. Assuming Alice does not repeat questions, the process must eventually terminate with 8 tree components, and at least $n-8$ questions having been asked.

Note that each tree component contains at least one vertex with no outgoing edges. Colour one such vertex in each tree component red.

Phase 2. Let V_1, V_2 and V_3 be the three of the red vertices whose components are smallest (so their components together have at most $\left\lfloor \frac{3}{8}n \right\rfloor$ vertices, with each component having at most $\left\lfloor \frac{3}{8}n - 2 \right\rfloor$ vertices). Let sets C_1, C_2, … be the connected components after removing the V_j. By construction, there are no edges with known direction between C_i and C_j for $i \neq j$, and there are at least five such components.

If at any point during this phase, the King of Hearts is asked about an edge within one of the C_i, he chooses an arbitrary direction. If he is asked about an edge between C_i and C_j for $i \neq j$, he answers so that all edges go from C_i to C_{i+1} and C_{i+2}, with indices taken modulo the number of components, and chooses arbitrarily for other pairs. This ensures that all vertices other than the V_j will have more than one outgoing edge.

For edges involving one of the V_j he answers as follows, so as to remain consistent for as long as possible with both possibilities for whether one of those vertices has at most one outgoing edge. Note that as they were red vertices, they have no outgoing edges at the start of this phase. For edges between two of the V_j, he answers that the edges go from V_1 to V_2, from V_2 to V_3 and from V_3 to V_1. For edges between V_j and some other vertex, he always answers that the edge goes into V_j, except for the last such edge for which he is asked the question for any given V_j, for which he answers that the edge goes out of V_j. Thus, as long as at least one of the V_j has not had the question answered for all the vertices that are not among the V_j, his answers are still compatible both with all vertices having more than one outgoing edge, and with that V_j having only one outgoing edge.

At the start of this phase, each of the V_j has at most $\lfloor \log_2 \lfloor \frac{3}{8}n - 2 \rfloor \rfloor < (\log_2 n) - 1$ incoming edges. Thus, Alice cannot determine whether some vertex has only one outgoing edge within $3(n - 3 - ((\log_2 n) - 1)) - 1$ questions in this phase; that is, $4n - 3(\log_2 n) - 15$ questions total.

Comment 3. We can also improve the upper bound slightly, to $4n - 2(\log_2 n) + 1$. (We do not know where the precise minimum number of questions lies between $4n - 3(\log_2 n) + O(1)$ and $4n - 2(\log_2 n) + O(1)$.) Suppose $n \geqslant 5$ (otherwise no questions are required at all).

To do this, we replace Phases 1 and 2 of the given solution with a different strategy that also results in a spanning tree where one vertex V is not known to have any outgoing edges, and all other vertices have exactly one outgoing edge known, but where there is more control over the numbers of incoming edges. In Phases 3 and 4 we then take more care about the order in which pairs of towns are chosen, to ensure that each of the remaining towns has already had a question asked about at least $\log_2 n + O(1)$ edges.

Define trees T_m with 2^m vertices, exactly one of which (the *root*) has no outgoing edges and the rest of which have exactly one outgoing edge, as follows: T_0 is a single vertex, while T_m is constructed by joining the roots of two copies of T_{m-1} with an edge in either direction. If $n = 2^m$ we can readily ask $n - 1$ questions, resulting in a tree T_m for the edges with known direction: first ask about 2^{m-1} disjoint pairs of vertices, then about 2^{m-2} disjoint pairs of the roots of the resulting T_1 trees, and so on. For the general case, where n is not a power of 2, after k stages of this process we have $\lfloor n/2^k \rfloor$ trees, each of which is like T_k but may have some extra vertices (but, however, a unique root). If there are an even number of trees, then ask about pairs of their roots. If there are an odd number (greater than 1) of trees, when a single T_k is left over, ask about its root together with that of one of the T_{k+1} trees.

Say $m = \lfloor \log_2 n \rfloor$. The result of that process is a single T_m tree, possibly with some extra vertices but still a unique root V. That root has at least m incoming edges, and we may list vertices V_0, …, V_{m-1} with edges to V, such that, for all $0 \leqslant i < m$, vertex V_i itself has at least i incoming edges.

Now divide the vertices other than V into two parts: A has all vertices at an odd distance from V and B has all the vertices at an even distance from B. Both A and B are nonempty; A contains the V_i, while B contains a sequence of vertices with at least 0, 1, …, $m - 2$ incoming edges respectively, similar to the V_i. There are no edges with known direction within A or within B.

In Phase 3, then ask about edges between V and other vertices: first those in B, in order of increasing number of incoming edges to the other vertex, then those in A, again in order of increasing number of incoming edges, which involves asking at most $n - 1 - m$ questions in this phase. If two outgoing edges are not found from V, at most $2n - 2 - m \leqslant 4n - 2(\log_2 n) + 1$ questions needed to be asked in total, so we suppose that two outgoing edges were found, with k questions asked in this phase, where $2 \leqslant k \leqslant n - 1 - m$. The state of S is as described in the solution above, with the additional property that, since S must still contain all vertices with edges to V, it contains the vertices V_i described above.

In Phase 4, consider the vertices left in B, in increasing order of number of edges incoming to a vertex. If s is the least number of incoming edges to such a vertex, then, for any $s \leqslant t \leqslant m - 2$, there are at least $m - t - 2$ vertices with more than t incoming edges. Repeatedly asking about the pair of vertices left in B with the least numbers of incoming edges results in a single vertex left over (if any were in B at all at the start of this phase) with at least $m - 2$ incoming edges. Doing the same with A (which must be nonempty) leaves a vertex with at least $m - 1$ incoming edges.

Thus if only A is nonempty we ask at most $n - m$ questions in Phase 5, so in total at most $3n - m - 1$ questions, while if both are nonempty we ask at most $2n - 2m + 1$ questions in Phase 5, so in total at most $4n - 2m - 1 < 4n - 2(\log_2 n) + 1$ questions.

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C9. For any two different real numbers x and y, we define $D(x,y)$ to be the unique integer d satisfying $2^d \leqslant |x-y| < 2^{d+1}$. Given a set of reals $\mathcal{F}$, and an element $x \in \mathcal{F}$, we say that the *scales* of x in $\mathcal{F}$ are the values of $D(x,y)$ for $y \in \mathcal{F}$ with $x \neq y$.

Let k be a given positive integer. Suppose that each member x of $\mathcal{F}$ has at most k different scales in $\mathcal{F}$ (note that these scales may depend on x). What is the maximum possible size of $\mathcal{F}$?

(Italy)

Answer: The maximum possible size of $\mathcal{F}$ is 2^k.

Common remarks. For convenience, we extend the use of the word *scale*: we say that the scale between two reals x and y is $D(x,y)$.

Solution. We first construct a set $\mathcal{F}$ with 2^k members, each member having at most k different scales in $\mathcal{F}$. Take $\mathcal{F} = \{0, 1, 2, \ldots, 2^k - 1\}$. The scale between any two members of $\mathcal{F}$ is in the set $\{0, 1, \ldots, k-1\}$.

We now show that 2^k is an upper bound on the size of $\mathcal{F}$. For every finite set $\mathcal{S}$ of real numbers, and every real x, let $r_{\mathcal{S}}(x)$ denote the number of different scales of x in $\mathcal{S}$. That is, $r_{\mathcal{S}}(x) = |\{D(x,y) \colon x \neq y \in \mathcal{S}\}|$. Thus, for every element x of the set $\mathcal{F}$ in the problem statement, we have $r_{\mathcal{F}}(x) \leqslant k$. The condition $|\mathcal{F}| \leqslant 2^k$ is an immediate consequence of the following lemma.

Lemma. Let $\mathcal{S}$ be a finite set of real numbers, and define

$$w(\mathcal{S}) = \sum_{x \in \mathcal{S}} 2^{-r_{\mathcal{S}}(x)}.$$

Then $w(\mathcal{S}) \leqslant 1$.

Proof. Induction on $n = |\mathcal{S}|$. If $\mathcal{S} = \{x\}$, then $r_{\mathcal{S}}(x) = 0$, so $w(\mathcal{S}) = 1$.

Assume now $n \geqslant 2$, and let $x_1 < \cdots < x_n$ list the members of $\mathcal{S}$. Let d be the minimal scale between two distinct elements of $\mathcal{S}$; then there exist neighbours x_t and x_{t+1} with $D(x_t, x_{t+1}) = d$. Notice that for any two indices i and j with $j - i > 1$ we have $D(x_i, x_j) > d$, since

$$|x_i - x_j| = |x_{i+1} - x_i| + |x_j - x_{i+1}| \geqslant 2^d + 2^d = 2^{d+1}.$$

Now choose the minimal $i \leqslant t$ and the maximal $j \geqslant t+1$ such that $D(x_i, x_{i+1}) = D(x_{i+1}, x_{i+2}) = \cdots = D(x_{j-1}, x_j) = d$.

Let E be the set of all the x_s with even indices $i \leqslant s \leqslant j$, O be the set of those with odd indices $i \leqslant s \leqslant j$, and R be the rest of the elements (so that $\mathcal{S}$ is the disjoint union of E, O and R). Set $\mathcal{S}_O = R \cup O$ and $\mathcal{S}_E = R \cup E$; we have $|\mathcal{S}_O| < |\mathcal{S}|$ and $|\mathcal{S}_E| < |\mathcal{S}|$, so $w(\mathcal{S}_O), w(\mathcal{S}_E) \leqslant 1$ by the inductive hypothesis.

Clearly, $r_{\mathcal{S}_O}(x) \leqslant r_{\mathcal{S}}(x)$ and $r_{\mathcal{S}_E}(x) \leqslant r_{\mathcal{S}}(x)$ for any $x \in R$, and thus

$$\begin{aligned}\sum_{x \in R} 2^{-r_{\mathcal{S}}(x)} &= \frac{1}{2} \sum_{x \in R} \left(2^{-r_{\mathcal{S}}(x)} + 2^{-r_{\mathcal{S}}(x)}\right) \\ &\leqslant \frac{1}{2} \sum_{x \in R} \left(2^{-r_{\mathcal{S}_O}(x)} + 2^{-r_{\mathcal{S}_E}(x)}\right).\end{aligned}$$

On the other hand, for every $x \in O$, there is no $y \in \mathcal{S}_O$ such that $D_{\mathcal{S}_O}(x,y) = d$ (as all candidates from $\mathcal{S}$ were in E). Hence, we have $r_{\mathcal{S}_O}(x) \leqslant r_{\mathcal{S}}(x) - 1$, and thus

$$\sum_{x \in O} 2^{-r_{\mathcal{S}}(x)} \leqslant \frac{1}{2} \sum_{x \in O} 2^{-r_{\mathcal{S}_O}(x)}.$$

Similarly, for every $x \in E$, we have

$$\sum_{x \in E} 2^{-r_{\mathcal{S}}(x)} \leqslant \frac{1}{2} \sum_{x \in E} 2^{-r_{\mathcal{S}_E}(x)}.$$

We can then combine these to give

$$\begin{aligned}
w(S) &= \sum_{x \in R} 2^{-r_{\mathcal{S}}(x)} + \sum_{x \in O} 2^{-r_{\mathcal{S}}(x)} + \sum_{x \in E} 2^{-r_{\mathcal{S}}(x)} \\
&\leqslant \frac{1}{2} \sum_{x \in R} (2^{-r_{\mathcal{S}_O}(x)} + 2^{-r_{\mathcal{S}_E}(x)}) + \frac{1}{2} \sum_{x \in O} 2^{-r_{\mathcal{S}_O}(x)} + \frac{1}{2} \sum_{x \in E} 2^{-r_{\mathcal{S}_E}(x)} \\
&= \frac{1}{2} \left(\sum_{x \in \mathcal{S}_O} 2^{-r_{\mathcal{S}_O}(x)} + \sum_{x \in \mathcal{S}_E} 2^{-r_{\mathcal{S}_E}(x)} \right) \qquad (\text{since } \mathcal{S}_O = O \cup R \text{ and } \mathcal{S}_E = E \cup R) \\
&= \frac{1}{2} \left(w(\mathcal{S}_O) + w(\mathcal{S}_E)) \right) \qquad (\text{by definition of } w(\cdot)) \\
&\leqslant 1 \qquad (\text{by the inductive hypothesis})
\end{aligned}$$

which completes the induction. □

Comment 1. The sets O and E above are not the only ones we could have chosen. Indeed, we could instead have used the following definitions:

Let d be the maximal scale between two distinct elements of $\mathcal{S}$; that is, $d = D(x_1, x_n)$. Let $O = \{x \in \mathcal{S} : D(x, x_n) = d\}$ (a 'left' part of the set) and let $E = \{x \in \mathcal{S} : D(x_1, x) = d\}$ (a 'right' part of the set). Note that these two sets are disjoint, and nonempty (since they contain x_1 and x_n respectively). The rest of the proof is then the same as in Solution 1.

Comment 2. Another possible set $\mathcal{F}$ containing 2^k members could arise from considering a binary tree of height k, allocating a real number to each leaf, and trying to make the scale between the values of two leaves dependent only on the (graph) distance between them. The following construction makes this more precise.

We build up sets $\mathcal{F}_k$ recursively. Let $\mathcal{F}_0 = \{0\}$, and then let $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{x + 3 \cdot 4^k : x \in \mathcal{F}_k\}$ (i.e. each half of $\mathcal{F}_{k+1}$ is a copy of F_k). We have that $\mathcal{F}_k$ is contained in the interval $[0, 4^{k+1})$, and so it follows by induction on k that every member of F_{k+1} has k different scales in its own half of F_{k+1} (by the inductive hypothesis), and only the single scale $2k + 1$ in the other half of F_{k+1}.

Both of the constructions presented here have the property that every member of $\mathcal{F}$ has exactly k different scales in $\mathcal{F}$. Indeed, it can be seen that this must hold (up to a slight perturbation) for any such maximal set. Suppose there were some element x with only $k - 1$ different scales in $\mathcal{F}$ (and every other element had at most k different scales). Then we take some positive real ϵ, and construct a new set $\mathcal{F}' = \{y : y \in \mathcal{F}, y \leqslant x\} \cup \{y + \epsilon : y \in \mathcal{F}, y \geqslant x\}$. We have $|\mathcal{F}'| = |\mathcal{F}| + 1$, and if ϵ is sufficiently small then $\mathcal{F}'$ will also satisfy the property that no member has more than k different scales in $\mathcal{F}'$.

This observation might be used to motivate the idea of weighting members of an arbitrary set $\mathcal{S}$ of reals according to how many different scales they have in $\mathcal{S}$.

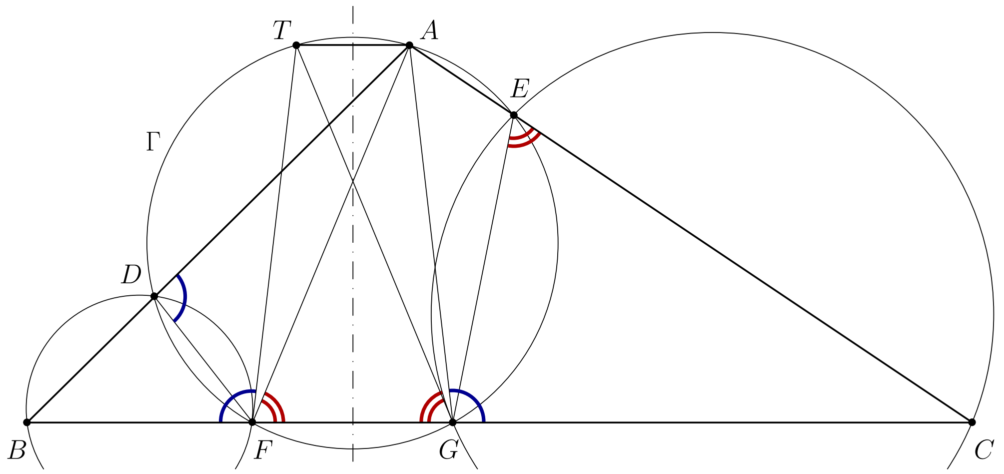
Geometry

G1. Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC.

(Nigeria)

Solution. Notice that $\angle TFB = \angle FDA$ because FT is tangent to circle BDF, and moreover $\angle FDA = \angle CGA$ because quadrilateral $ADFG$ is cyclic. Similarly, $\angle TGB = \angle GEC$ because GT is tangent to circle CEG, and $\angle GEC = \angle CFA$. Hence,

$$\angle TFB = \angle CGA \quad \text{and} \quad \angle TGB = \angle CFA. \tag{1}$$



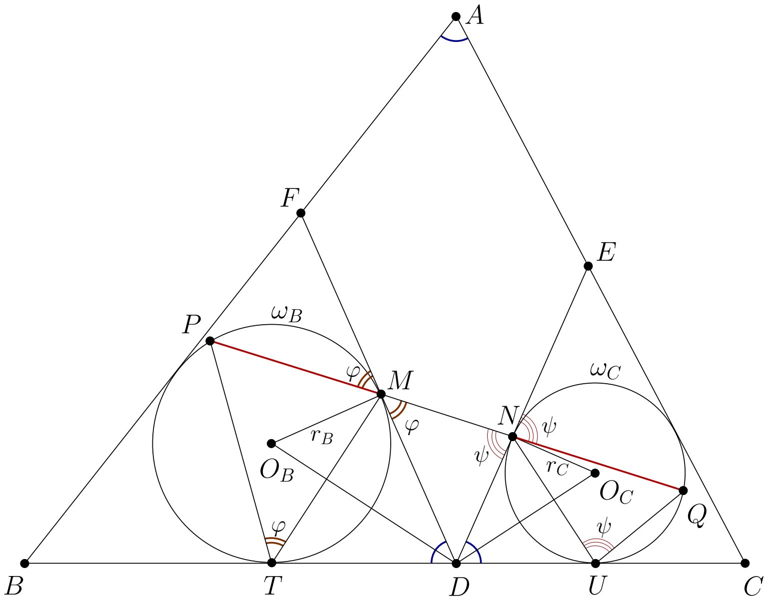
Triangles FGA and GFT have a common side FG, and by (1) their angles at F, G are the same. So, these triangles are congruent. So, their altitudes starting from A and T, respectively, are equal and hence AT is parallel to line $BFGC$.

Comment. Alternatively, we can prove first that T lies on Γ. For example, this can be done by showing that $\angle AFT = \angle AGT$ using (1). Then the statement follows as $\angle TAF = \angle TGF = \angle GFA$.

G2. Let ABC be an acute-angled triangle and let D, E, and F be the feet of altitudes from A, B, and C to sides BC, CA, and AB, respectively. Denote by ω_B and ω_C the incircles of triangles BDF and CDE, and let these circles be tangent to segments DF and DE at M and N, respectively. Let line MN meet circles ω_B and ω_C again at $P \neq M$ and $Q \neq N$, respectively. Prove that $MP = NQ$.

(Vietnam)

Solution. Denote the centres of ω_B and ω_C by O_B and O_C, let their radii be r_B and r_C, and let BC be tangent to the two circles at T and U, respectively.



From the cyclic quadrilaterals $AFDC$ and $ABDE$ we have

$$\angle MDO_B = \frac{1}{2}\angle FDB = \frac{1}{2}\angle BAC = \frac{1}{2}\angle CDE = \angle O_C DN,$$

so the right-angled triangles DMO_B and DNO_C are similar. The ratio of similarity between the two triangles is

$$\frac{DN}{DM} = \frac{O_C N}{O_B M} = \frac{r_C}{r_B}.$$

Let $\varphi = \angle DMN$ and $\psi = \angle MND$. The lines FM and EN are tangent to ω_B and ω_C, respectively, so

$$\angle MTP = \angle FMP = \angle DMN = \varphi \quad \text{and} \quad \angle QUN = \angle QNE = \angle MND = \psi.$$

(It is possible that P or Q coincides with T or U, or lie inside triangles DMT or DUN, respectively. To reduce case-sensitivity, we may use directed angles or simply ignore angles MTP and QUN.)

In the circles ω_B and ω_C the lengths of chords MP and NQ are

$$MP = 2r_B \cdot \sin\angle MTP = 2r_B \cdot \sin\varphi \quad \text{and} \quad NQ = 2r_C \cdot \sin\angle QUN = 2r_C \cdot \sin\psi.$$

By applying the sine rule to triangle DNM we get

$$\frac{DN}{DM} = \frac{\sin\angle DMN}{\sin\angle MND} = \frac{\sin\varphi}{\sin\psi}.$$

Finally, putting the above observations together, we get

$$\frac{MP}{NQ} = \frac{2r_B\sin\varphi}{2r_C\sin\psi} = \frac{r_B}{r_C}\cdot\frac{\sin\varphi}{\sin\psi} = \frac{DM}{DN}\cdot\frac{\sin\varphi}{\sin\psi} = \frac{\sin\psi}{\sin\varphi}\cdot\frac{\sin\varphi}{\sin\psi} = 1,$$

so $MP = NQ$ as required.

G3. In triangle ABC, let A_1 and B_1 be two points on sides BC and AC, and let P and Q be two points on segments AA_1 and BB_1, respectively, so that line PQ is parallel to AB. On ray PB_1, beyond B_1, let P_1 be a point so that $\angle PP_1C = \angle BAC$. Similarly, on ray QA_1, beyond A_1, let Q_1 be a point so that $\angle CQ_1Q = \angle CBA$. Show that points P, Q, P_1, and Q_1 are concyclic.

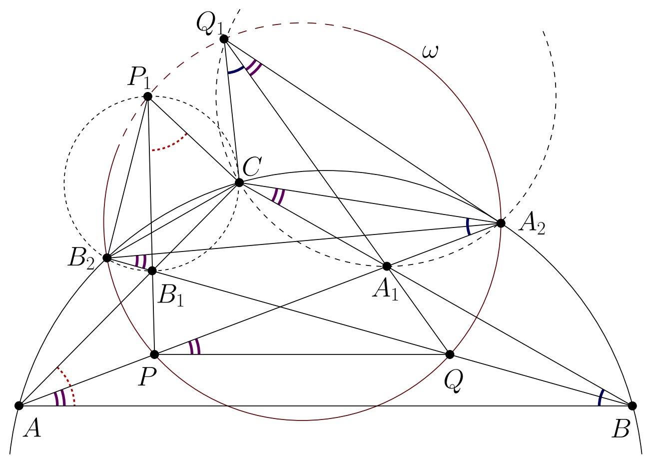
(Ukraine)

Solution 1. Throughout the solution we use oriented angles.

Let rays AA_1 and BB_1 intersect the circumcircle of $\triangle ACB$ at A_2 and B_2, respectively. By

$$\angle QPA_2 = \angle BAA_2 = \angle BB_2A_2 = \angle QB_2A_2,$$

points P, Q, A_2, B_2 are concyclic; denote the circle passing through these points by ω. We shall prove that P_1 and Q_1 also lie on ω.



By

$$\angle CA_2A_1 = \angle CA_2A = \angle CBA = \angle CQ_1Q = \angle CQ_1A_1,$$

points C, Q_1, A_2, A_1 are also concyclic. From that we get

$$\angle QQ_1A_2 = \angle A_1Q_1A_2 = \angle A_1CA_2 = \angle BCA_2 = \angle BAA_2 = \angle QPA_2,$$

so Q_1 lies on ω.

It follows similarly that P_1 lies on ω.

Solution 2. First consider the case when lines PP_1 and QQ_1 intersect each other at some point R.

Let line PQ meet the sides AC and BC at E and F, respectively. Then

$$\angle PP_1C = \angle BAC = \angle PEC,$$

so points C, E, P, P_1 lie on a circle; denote that circle by ω_P. It follows analogously that points C, F, Q, Q_1 lie on another circle; denote it by ω_Q.

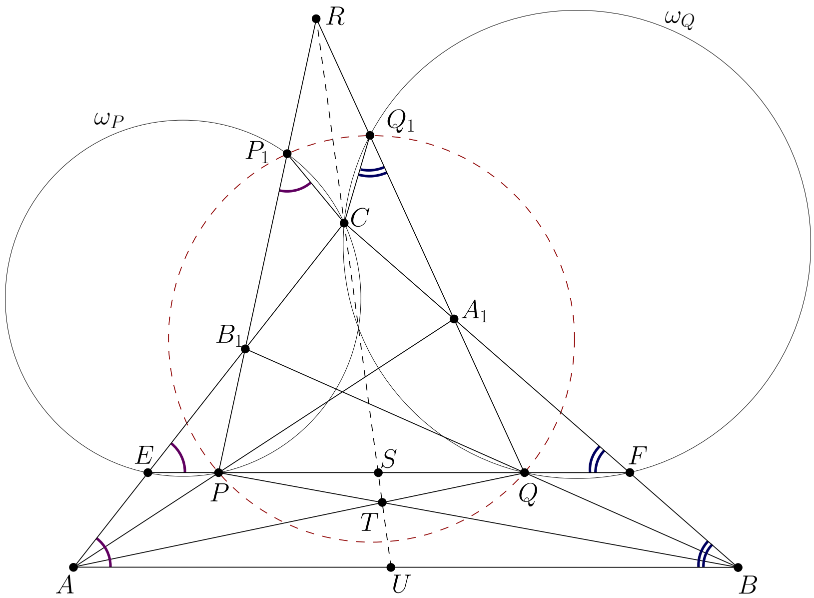
Let AQ and BP intersect at T. Applying Pappus' theorem to the lines AA_1P and BB_1Q provides that points $C = AB_1 \cap BA_1$, $R = A_1Q \cap B_1P$ and $T = AQ \cap BP$ are collinear.

Let line RCT meet PQ and AB at S and U, respectively. From $AB \parallel PQ$ we obtain

$$\frac{SP}{SQ} = \frac{UB}{UA} = \frac{SF}{SE},$$

so

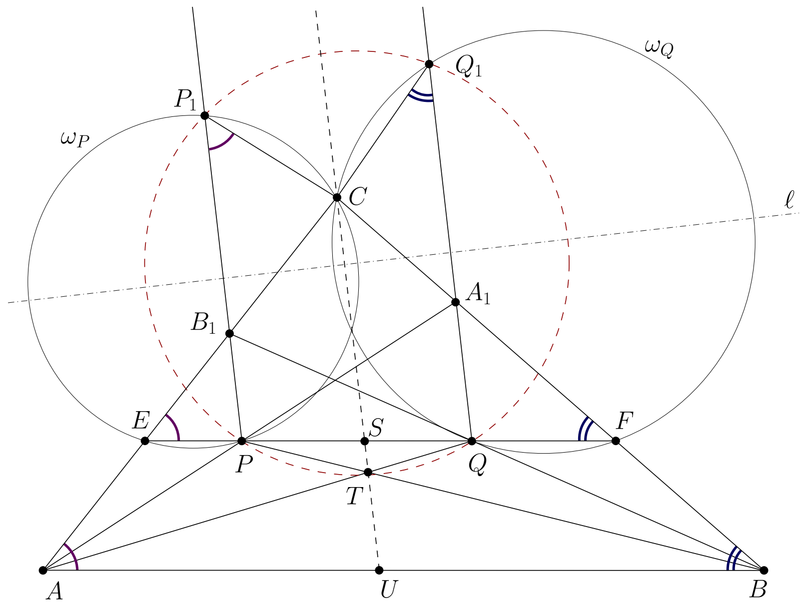
$$SP \cdot SE = SQ \cdot SF.$$



So, point S has equal powers with respect to ω_P and ω_Q, hence line RCS is their radical axis; then R also has equal powers to the circles, so $RP \cdot RP_1 = RQ \cdot RQ_1$, proving that points P, P_1, Q, Q_1 are indeed concyclic.

Now consider the case when PP_1 and QQ_1 are parallel. Like in the previous case, let AQ and BP intersect at T. Applying Pappus' theorem again to the lines AA_1P and BB_1Q, in this limit case it shows that line CT is parallel to PP_1 and QQ_1.

Let line CT meet PQ and AB at S and U, as before. The same calculation as in the previous case shows that $SP \cdot SE = SQ \cdot SF$, so S lies on the radical axis between ω_P and ω_Q.



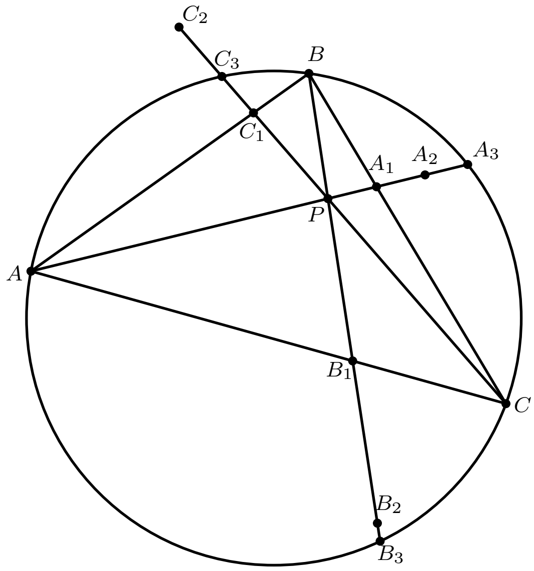
Line CST, that is the radical axis between ω_P and ω_Q, is perpendicular to the line ℓ of centres of ω_P and ω_Q. Hence, the chords PP_1 and QQ_1 are perpendicular to ℓ. So the quadrilateral PP_1Q_1Q is an isosceles trapezium with symmetry axis ℓ, and hence is cyclic.

Comment. There are several ways of solving the problem involving Pappus' theorem. For example, one may consider the points $K = PB_1 \cap BC$ and $L = QA_1 \cap AC$. Applying Pappus' theorem to the lines AA_1P and QB_1B we get that K, L, and $PQ \cap AB$ are collinear, i.e. that $KL \parallel AB$. Therefore, cyclicity of P, Q, P_1, and Q_1 is equivalent to that of K, L, P_1, and Q_1. The latter is easy after noticing that C also lies on that circle. Indeed, e.g. $\angle(LK, LC) = \angle(AB, AC) = \angle(P_1K, P_1C)$ shows that K lies on circle KLC.

This approach also has some possible degeneracy, as the points K and L may happen to be ideal.

G4. Let P be a point inside triangle ABC. Let AP meet BC at A_1, let BP meet CA at B_1, and let CP meet AB at C_1. Let A_2 be the point such that A_1 is the midpoint of PA_2, let B_2 be the point such that B_1 is the midpoint of PB_2, and let C_2 be the point such that C_1 is the midpoint of PC_2. Prove that points A_2, B_2, and C_2 cannot all lie strictly inside the circumcircle of triangle ABC.

(Australia)



Solution 1. Since

$$\angle APB + \angle BPC + \angle CPA = 2\pi = (\pi - \angle ACB) + (\pi - \angle BAC) + (\pi - \angle CBA),$$

at least one of the following inequalities holds:

$$\angle APB \geqslant \pi - \angle ACB, \quad \angle BPC \geqslant \pi - \angle BAC, \quad \angle CPA \geqslant \pi - \angle CBA.$$

Without loss of generality, we assume that $\angle BPC \geqslant \pi - \angle BAC$. We have $\angle BPC > \angle BAC$ because P is inside $\triangle ABC$. So $\angle BPC \geqslant \max(\angle BAC, \pi - \angle BAC)$ and hence

$$\sin \angle BPC \leqslant \sin \angle BAC. \tag{$*$}$$

Let the rays AP, BP, and CP cross the circumcircle Ω again at A_3, B_3, and C_3, respectively. We will prove that at least one of the ratios $\frac{PB_1}{B_1B_3}$ and $\frac{PC_1}{C_1C_3}$ is at least 1, which yields that one of the points B_2 and C_2 does not lie strictly inside Ω.

Because A, B, C, B_3 lie on a circle, the triangles CB_1B_3 and BB_1A are similar, so

$$\frac{CB_1}{B_1B_3} = \frac{BB_1}{B_1A}.$$

Applying the sine rule we obtain

$$\frac{PB_1}{B_1B_3} = \frac{PB_1}{CB_1} \cdot \frac{CB_1}{B_1B_3} = \frac{PB_1}{CB_1} \cdot \frac{BB_1}{B_1A} = \frac{\sin \angle ACP}{\sin \angle BPC} \cdot \frac{\sin \angle BAC}{\sin \angle PBA}.$$

Similarly,

$$\frac{PC_1}{C_1C_3} = \frac{\sin \angle PBA}{\sin \angle BPC} \cdot \frac{\sin \angle BAC}{\sin \angle ACP}.$$

Multiplying these two equations we get

$$\frac{PB_1}{B_1B_3} \cdot \frac{PC_1}{C_1C_3} = \frac{\sin^2 \angle BAC}{\sin^2 \angle BPC} \geqslant 1$$

using $(*)$, which yields the desired conclusion.

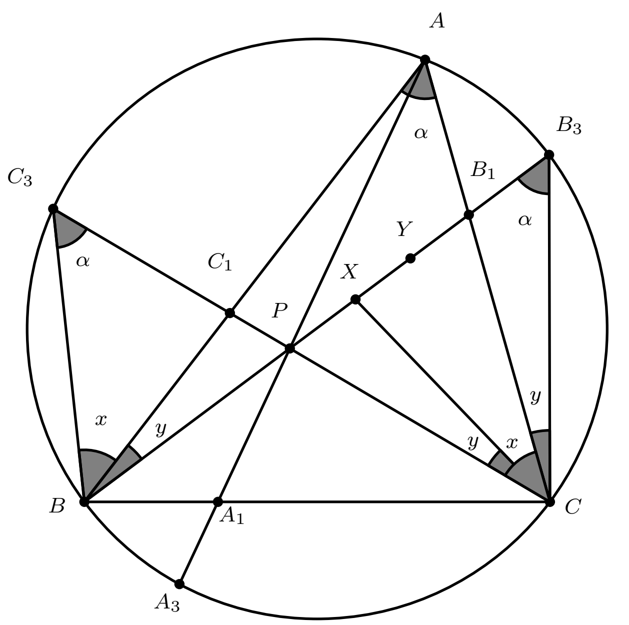
Comment. It also cannot happen that all three points A_2, B_2, and C_2 lie strictly *outside* Ω. The same proof works almost literally, starting by assuming without loss of generality that $\angle BPC \leqslant \pi - \angle BAC$ and using $\angle BPC > \angle BAC$ to deduce that $\sin\angle BPC \geqslant \sin\angle BAC$. It is possible for A_2, B_2, and C_2 all to lie on the circumcircle; from the above solution we may derive that this happens if and only if P is the orthocentre of the triangle ABC, (which lies strictly inside ABC if and only if ABC is acute).

Solution 2. Define points A_3, B_3, and C_3 as in Solution 1. Assume for the sake of contradiction that A_2, B_2, and C_2 all lie strictly inside circle ABC. It follows that $PA_1 < A_1A_3$, $PB_1 < B_1B_3$, and $PC_1 < C_1C_3$.

Observe that $\triangle PBC_3 \sim \triangle PCB_3$. Let X be the point on side PB_3 that corresponds to point C_1 on side PC_3 under this similarity. In other words, X lies on segment PB_3 and satisfies $PX : XB_3 = PC_1 : C_1C_3$. It follows that

$$\angle XCP = \angle PBC_1 = \angle B_3BA = \angle B_3CB_1 .$$

Hence lines CX and CB_1 are isogonal conjugates in $\triangle PCB_3$.



Let Y be the foot of the bisector of $\angle B_3CP$ in $\triangle PCB_3$. Since $PC_1 < C_1C_3$, we have $PX < XB_3$. Also, we have $PY < YB_3$ because $PB_1 < B_1B_3$ and Y lies between X and B_1. By the angle bisector theorem in $\triangle PCB_3$, we have $PY : YB_3 = PC : CB_3$. So $PC < CB_3$ and it follows that $\angle PB_3C < \angle CPB_3$. Now since $\angle PB_3C = \angle BB_3C = \angle BAC$, we have

$$\angle BAC < \angle CPB_3 .$$

Similarly, we have

$$\angle CBA < \angle APC_3 \quad \text{and} \quad \angle ACB < \angle BPA_3 = \angle B_3PA .$$

Adding these three inequalities yields $\pi < \pi$, and this contradiction concludes the proof.

Solution 3. Choose coordinates such that the circumcentre of $\triangle ABC$ is at the origin and the circumradius is 1. Then we may think of A, B, and C as vectors in $\mathbb{R}^2$ such that

$$|A|^2 = |B|^2 = |C|^2 = 1 .$$

P may be represented as a convex combination $\alpha A+\beta B+\gamma C$ where $\alpha, \beta, \gamma > 0$ and $\alpha+\beta+\gamma = 1$. Then

$$A_1 = \frac{\beta B + \gamma C}{\beta + \gamma} = \frac{1}{1-\alpha}P - \frac{\alpha}{1-\alpha}A,$$

so

$$A_2 = 2A_1 - P = \frac{1+\alpha}{1-\alpha}P - \frac{2\alpha}{1-\alpha}A .$$

Hence

$$|A_2|^2 = \left(\frac{1+\alpha}{1-\alpha}\right)^2 |P|^2 + \left(\frac{2\alpha}{1-\alpha}\right)^2 |A|^2 - \frac{4\alpha(1+\alpha)}{(1-\alpha)^2} A \cdot P .$$

Using $|A|^2 = 1$ we obtain

$$\frac{(1-\alpha)^2}{2(1+\alpha)}|A_2|^2 = \frac{1+\alpha}{2}|P|^2 + \frac{2\alpha^2}{1+\alpha} - 2\alpha A \cdot P. \tag{1}$$

Likewise

$$\frac{(1-\beta)^2}{2(1+\beta)}|B_2|^2 = \frac{1+\beta}{2}|P|^2 + \frac{2\beta^2}{1+\beta} - 2\beta B \cdot P \tag{2}$$

and

$$\frac{(1-\gamma)^2}{2(1+\gamma)}|C_2|^2 = \frac{1+\gamma}{2}|P|^2 + \frac{2\gamma^2}{1+\gamma} - 2\gamma C \cdot P. \tag{3}$$

Summing (1), (2) and (3) we obtain on the LHS the positive linear combination

$$\text{LHS} = \frac{(1-\alpha)^2}{2(1+\alpha)}|A_2|^2 + \frac{(1-\beta)^2}{2(1+\beta)}|B_2|^2 + \frac{(1-\gamma)^2}{2(1+\gamma)}|C_2|^2$$

and on the RHS the quantity

$$\left(\frac{1+\alpha}{2} + \frac{1+\beta}{2} + \frac{1+\gamma}{2}\right)|P|^2 + \left(\frac{2\alpha^2}{1+\alpha} + \frac{2\beta^2}{1+\beta} + \frac{2\gamma^2}{1+\gamma}\right) - 2(\alpha A \cdot P + \beta B \cdot P + \gamma C \cdot P) .$$

The first term is $2|P|^2$ and the last term is $-2P \cdot P$, so

$$\begin{aligned}
\text{RHS} &= \left(\frac{2\alpha^2}{1+\alpha} + \frac{2\beta^2}{1+\beta} + \frac{2\gamma^2}{1+\gamma}\right) \\
&= \frac{3\alpha-1}{2} + \frac{(1-\alpha)^2}{2(1+\alpha)} + \frac{3\beta-1}{2} + \frac{(1-\beta)^2}{2(1+\beta)} + \frac{3\gamma-1}{2} + \frac{(1-\gamma)^2}{2(1+\gamma)} \\
&= \frac{(1-\alpha)^2}{2(1+\alpha)} + \frac{(1-\beta)^2}{2(1+\beta)} + \frac{(1-\gamma)^2}{2(1+\gamma)} .
\end{aligned}$$

Here we used the fact that

$$\frac{3\alpha-1}{2} + \frac{3\beta-1}{2} + \frac{3\gamma-1}{2} = 0 .$$

We have shown that a linear combination of $|A_1|^2$, $|B_1|^2$, and $|C_1|^2$ with positive coefficients is equal to the sum of the coefficients. Therefore at least one of $|A_1|^2$, $|B_1|^2$, and $|C_1|^2$ must be at least 1, as required.

Comment. This proof also works when P is any point for which $\alpha, \beta, \gamma > -1$, $\alpha+\beta+\gamma = 1$, and $\alpha, \beta, \gamma \neq 1$. (In any cases where $\alpha = 1$ or $\beta = 1$ or $\gamma = 1$, some points in the construction are not defined.)

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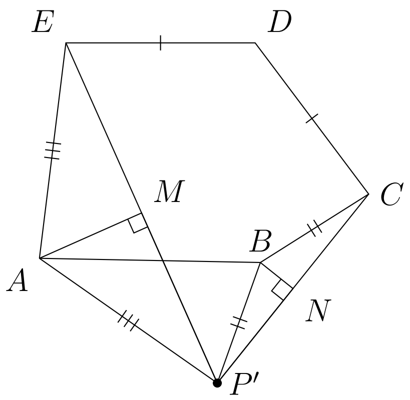
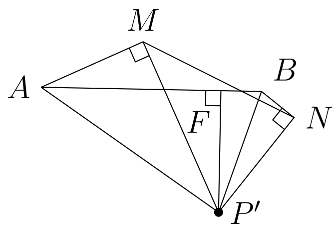
G5. Let $ABCDE$ be a convex pentagon with $CD = DE$ and $\angle EDC \neq 2 \cdot \angle ADB$. Suppose that a point P is located in the interior of the pentagon such that $AP = AE$ and $BP = BC$. Prove that P lies on the diagonal CE if and only if $\operatorname{area}(BCD) + \operatorname{area}(ADE) = \operatorname{area}(ABD) + \operatorname{area}(ABP)$.

(Hungary)

Solution 1. Let P' be the reflection of P across line AB, and let M and N be the midpoints of $P'E$ and $P'C$ respectively. Convexity ensures that P' is distinct from both E and C, and hence from both M and N. We claim that both the area condition and the collinearity condition in the problem are equivalent to the condition that the (possibly degenerate) right-angled triangles $AP'M$ and $BP'N$ are directly similar (equivalently, $AP'E$ and $BP'C$ are directly similar).



For the equivalence with the collinearity condition, let F denote the foot of the perpendicular from P' to AB, so that F is the midpoint of PP'. We have that P lies on CE if and only if F lies on MN, which occurs if and only if we have the equality $\angle AFM = \angle BFN$ of signed angles modulo π. By concyclicity of $AP'FM$ and $BFP'N$, this is equivalent to $\angle AP'M = \angle BP'N$, which occurs if and only if $AP'M$ and $BP'N$ are directly similar.



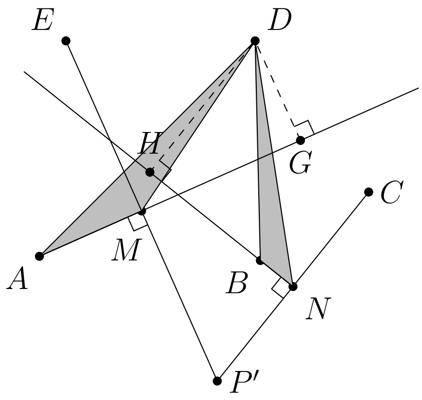
For the other equivalence with the area condition, we have the equality of signed areas $\operatorname{area}(ABD) + \operatorname{area}(ABP) = \operatorname{area}(AP'BD) = \operatorname{area}(AP'D) + \operatorname{area}(BDP')$. Using the identity $\operatorname{area}(ADE) - \operatorname{area}(AP'D) = \operatorname{area}(ADE) + \operatorname{area}(ADP') = 2\operatorname{area}(ADM)$, and similarly for B, we find that the area condition is equivalent to the equality

$$\operatorname{area}(DAM) = \operatorname{area}(DBN).$$

Now note that A and B lie on the perpendicular bisectors of $P'E$ and $P'C$, respectively. If we write G and H for the feet of the perpendiculars from D to these perpendicular bisectors respectively, then this area condition can be rewritten as

$$MA \cdot GD = NB \cdot HD.$$

(In this condition, we interpret all lengths as signed lengths according to suitable conventions: for instance, we orient $P'E$ from P' to E, orient the parallel line DH in the same direction, and orient the perpendicular bisector of $P'E$ at an angle $\pi/2$ clockwise from the oriented segment $P'E$ – we adopt the analogous conventions at B.)



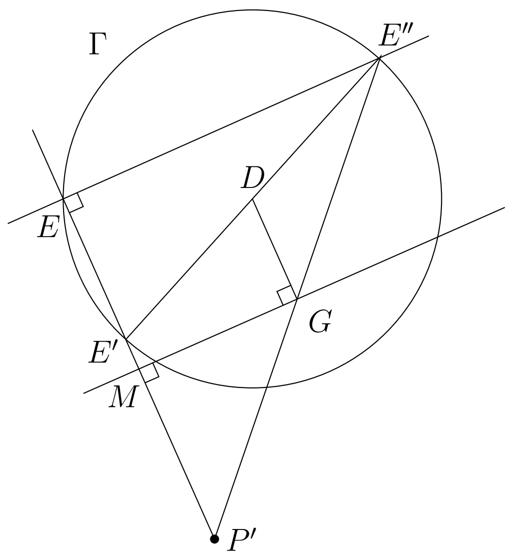
To relate the signed lengths GD and HD to the triangles $AP'M$ and $BP'N$, we use the following calculation.

Claim. Let Γ denote the circle centred on D with both E and C on the circumference, and h the power of P' with respect to Γ. Then we have the equality

$$GD \cdot P'M = HD \cdot P'N = \frac{1}{4}h \neq 0.$$

Proof. Firstly, we have $h \neq 0$, since otherwise P' would lie on Γ, and hence the internal angle bisectors of $\angle EDP'$ and $\angle P'DC$ would pass through A and B respectively. This would violate the angle inequality $\angle EDC \neq 2 \cdot \angle ADB$ given in the question.

Next, let E' denote the second point of intersection of $P'E$ with Γ, and let E'' denote the point on Γ diametrically opposite E', so that $E''E$ is perpendicular to $P'E$. The point G lies on the perpendicular bisectors of the sides $P'E$ and EE'' of the right-angled triangle $P'EE''$; it follows that G is the midpoint of $P'E''$. Since D is the midpoint of $E'E''$, we have that $GD = \frac{1}{2}P'E'$. Since $P'M = \frac{1}{2}P'E$, we have $GD \cdot P'M = \frac{1}{4}P'E' \cdot P'E = \frac{1}{4}h$. The other equality $HD \cdot P'N$ follows by exactly the same argument.



□

From this claim, we see that the area condition is equivalent to the equality

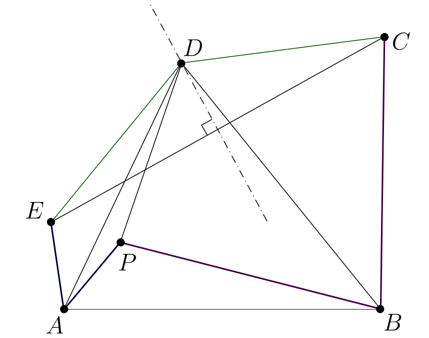
$$(MA : P'M) = (NB : P'N)$$

of ratios of signed lengths, which is equivalent to direct similarity of $AP'M$ and $BP'N$, as desired.

Solution 2. Along the perpendicular bisector of CE, define the linear function

$$f(X) = \text{area}(BCX) + \text{area}(AXE) - \text{area}(ABX) - \text{area}(ABP),$$

where, from now on, we always use signed areas. Thus, we want to show that C, P, E are collinear if and only if $f(D) = 0$.



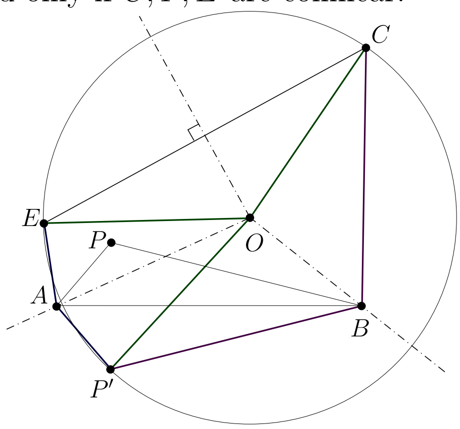
Let P' be the reflection of P across line AB. The point P' does not lie on the line CE. To see this, we let A'' and B'' be the points obtained from A and B by dilating with scale factor 2 about P', so that P is the orthogonal projection of P' onto $A''B''$. Since A lies on the perpendicular bisector of $P'E$, the triangle $A''EP'$ is right-angled at E (and $B''CP'$ similarly). If P' were to lie on CE, then the lines $A''E$ and $B''C$ would be perpendicular to CE and A'' and B'' would lie on the opposite side of CE to D. It follows that the line $A''B''$ does not meet triangle CDE, and hence point P does not lie inside CDE. But then P must lie inside $ABCE$, and it is clear that such a point cannot reflect to a point P' on CE.

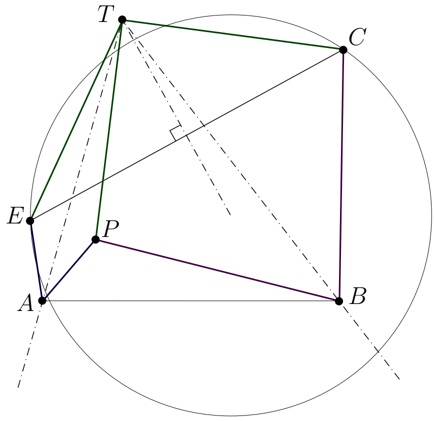
We thus let O be the centre of the circle CEP'. The lines AO and BO are the perpendicular bisectors of EP' and CP', respectively, so

$$\begin{aligned}\text{area}(BCO) + \text{area}(AOE) &= \text{area}(OP'B) + \text{area}(P'OA) = \text{area}(P'BOA) \\ &= \text{area}(ABO) + \text{area}(BAP') = \text{area}(ABO) + \text{area}(ABP),\end{aligned}$$

and hence $f(O) = 0$.

Notice that if point O coincides with D then points A, B lie in angle domain CDE and $\angle EOC = 2 \cdot \angle AOB$, which is not allowed. So, O and D must be distinct. Since f is linear and vanishes at O, it follows that $f(D) = 0$ if and only if f is constant zero – we want to show this occurs if and only if C, P, E are collinear.





In the one direction, suppose firstly that C, P, E are not collinear, and let T be the centre of the circle CEP. The same calculation as above provides

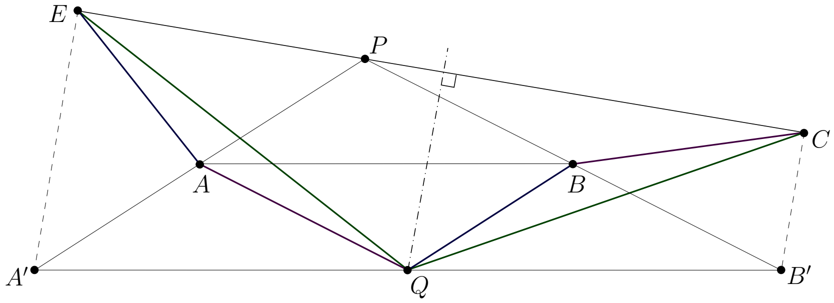
$$\text{area}(BCT) + \text{area}(ATE) = \text{area}(PBTA) = \text{area}(ABT) - \text{area}(ABP)$$

so

$$f(T) = -2\,\text{area}(ABP) \neq 0.$$

Hence, the linear function f is nonconstant with its zero is at O, so that $f(D) \neq 0$.

In the other direction, suppose that the points C, P, E are collinear. We will show that f is constant zero by finding a second point (other than O) at which it vanishes.



Let Q be the reflection of P across the midpoint of AB, so $PAQB$ is a parallelogram. It is easy to see that Q is on the perpendicular bisector of CE; for instance if A' and B' are the points produced from A and B by dilating about P with scale factor 2, then the projection of Q to CE is the midpoint of the projections of A' and B', which are E and C respectively. The triangles BCQ and AQE are indirectly congruent, so

$$f(Q) = \big(\operatorname{area}(BCQ) + \operatorname{area}(AQE)\big) - \big(\operatorname{area}(ABQ) - \operatorname{area}(BAP)\big) = 0 - 0 = 0.$$

The points O and Q are distinct. To see this, consider the circle ω centred on Q with P' on the circumference; since triangle $PP'Q$ is right-angled at P', it follows that P lies outside ω. On the other hand, P lies between C and E on the line CPE. It follows that C and E cannot both lie on ω, so that ω is not the circle CEP' and $Q \neq O$.

Since O and Q are distinct zeroes of the linear function f, we have $f(D) = 0$ as desired.

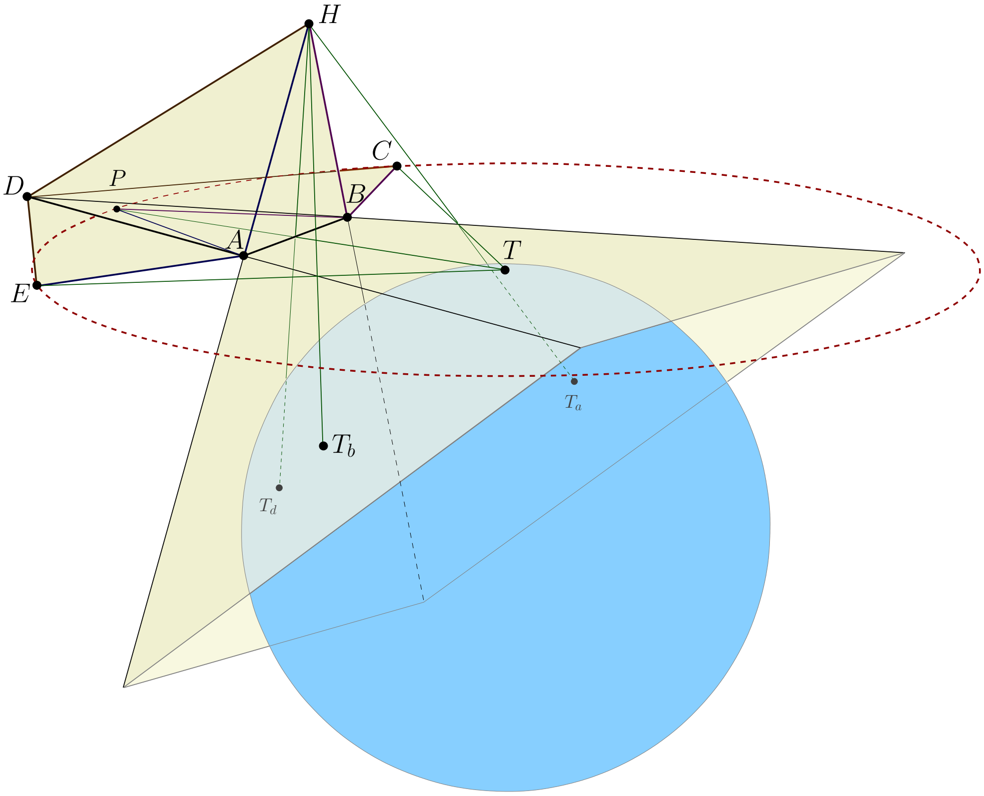
Comment 1. The condition $\angle EDC \neq 2\cdot\angle ADB$ cannot be omitted. If D is the centre of circle CEP', then the condition on triangle areas is satisfied automatically, without having P on line CE.

Comment 2. The "only if" part of this problem is easier than the "if" part. For example, in the second part of Solution 2, the triangles EAQ and QBC are indirectly congruent, so the sum of their areas is 0, and $DCQE$ is a kite. Now one can easily see that $\angle(AQ, DE) = \angle(CD, CB)$ and $\angle(BQ, DC) = \angle(ED, EA)$, whence $\operatorname{area}(BCD) = \operatorname{area}(AQD) + \operatorname{area}(EQA)$ and $\operatorname{area}(ADE) = \operatorname{area}(BDQ) + \operatorname{area}(BQC)$, which yields the result.

Comment 3. The origin of the problem is the following observation. Let $ABDH$ be a tetrahedron and consider the sphere $\mathcal{S}$ that is tangent to the four face planes, internally to planes ADH and BDH and externally to ABD and ABH (or vice versa). It is known that the sphere $\mathcal{S}$ exists if and only if $\operatorname{area}(ADH) + \operatorname{area}(BDH) \neq \operatorname{area}(ABH) + \operatorname{area}(ABD)$; this relation comes from the usual formula for the volume of the tetrahedron.

Let T, T_a, T_b, T_d be the points of tangency between the sphere and the four planes, as shown in the picture. Rotate the triangle ABH inward, the triangles BDH and ADH outward, into the triangles ABP, BDC and ADE, respectively, in the plane ABD. Notice that the points T_d, T_a, T_b are rotated to T, so we have $HT_a = HT_b = HT_d = PT = CT = ET$. Therefore, the point T is the centre of the circle CEP. Hence, if the sphere exists then C, E, P cannot be collinear.

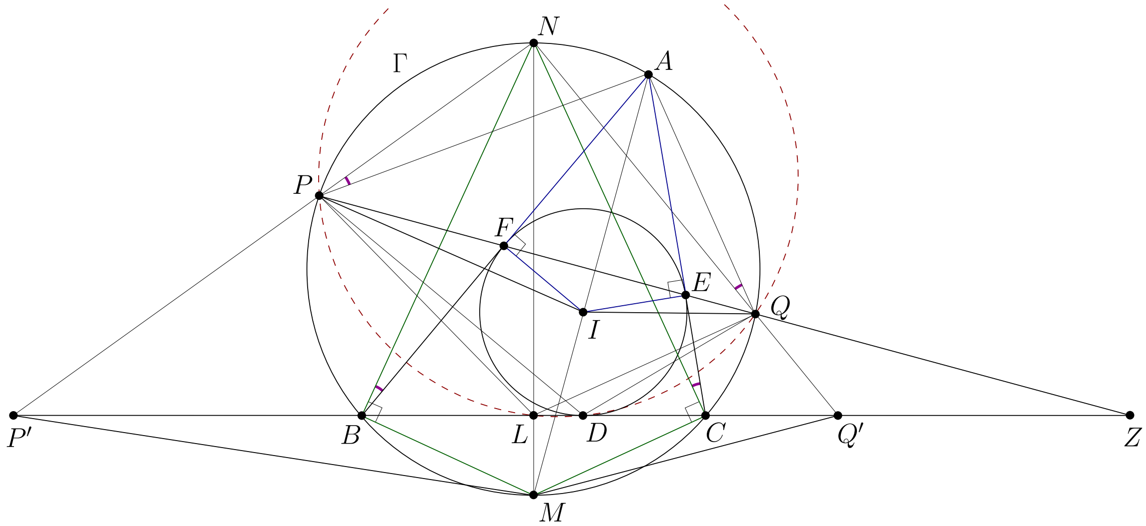
If the condition $\angle EDC \neq 2 \cdot \angle ADB$ is replaced by the constraint that the angles $\angle EDA$, $\angle ADB$ and $\angle BDC$ satisfy the triangle inequality, it enables reconstructing the argument with the tetrahedron and the tangent sphere.



G6. Let I be the incentre of acute-angled triangle ABC. Let the incircle meet BC, CA, and AB at D, E, and F, respectively. Let line EF intersect the circumcircle of the triangle at P and Q, such that F lies between E and P. Prove that $\angle DPA + \angle AQD = \angle QIP$.

(Slovakia)

Solution 1. Let N and M be the midpoints of the arcs $\overparen{BC}$ of the circumcircle, containing and opposite vertex A, respectively. By $\angle FAE = \angle BAC = \angle BNC$, the right-angled kites $AFIE$ and $NBMC$ are similar. Consider the spiral similarity φ (dilation in case of $AB = AC$) that moves $AFIE$ to $NBMC$. The directed angle in which φ changes directions is $\angle(AF, NB)$, same as $\angle(AP, NP)$ and $\angle(AQ, NQ)$; so lines AP and AQ are mapped to lines NP and NQ, respectively. Line EF is mapped to BC; we can see that the intersection points $P = EF \cap AP$ and $Q = EF \cap AQ$ are mapped to points $BC \cap NP$ and $BC \cap NQ$, respectively. Denote these points by P' and Q', respectively.



Let L be the midpoint of BC. We claim that points P, Q, D, L are concyclic (if $D = L$ then line BC is tangent to circle PQD). Let PQ and BC meet at Z. By applying Menelaus' theorem to triangle ABC and line EFZ, we have

$$\frac{BD}{DC} = \frac{BF}{FA} \cdot \frac{AE}{EC} = -\frac{BZ}{ZC},$$

so the pairs B, C and D, Z are harmonic. It is well-known that this implies $ZB \cdot ZC = ZD \cdot ZL$. (The inversion with pole Z that swaps B and C sends Z to infinity and D to the midpoint of BC, because the cross-ratio is preserved.) Hence, $ZD \cdot ZL = ZB \cdot ZC = ZP \cdot ZQ$ by the power of Z with respect to the circumcircle; this proves our claim.

By $\angle MPP' = \angle MQQ' = \angle MLP' = \angle MLQ' = 90°$, the quadrilaterals $MLPP'$ and $MLQQ'$ are cyclic. Then the problem statement follows by

$$\begin{aligned}\angle DPA + \angle AQD &= 360° - \angle PAQ - \angle QDP = 360° - \angle PNQ - \angle QLP \\ &= \angle LPN + \angle NQL = \angle P'ML + \angle LMQ' = \angle P'MQ' = \angle PIQ.\end{aligned}$$

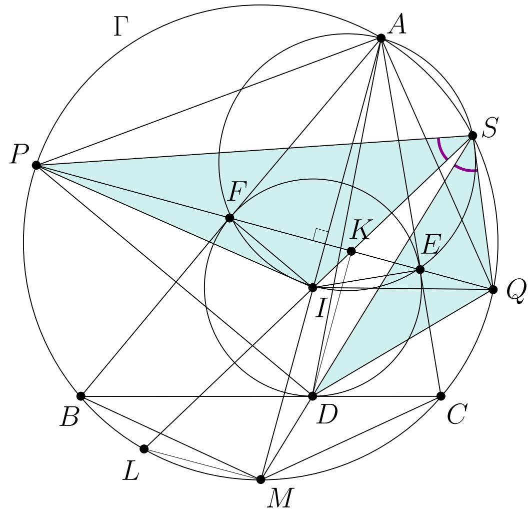
Solution 2. Define the point M and the same spiral similarity φ as in the previous solution. (The point N is not necessary.) It is well-known that the centre of the spiral similarity that maps F, E to B, C is the Miquel point of the lines FE, BC, BF and CE; that is, the second intersection of circles ABC and AEF. Denote that point by S.

By $\varphi(F) = B$ and $\varphi(E) = C$ the triangles SBF and SCE are similar, so we have

$$\frac{SB}{SC} = \frac{BF}{CE} = \frac{BD}{CD}.$$

By the converse of the angle bisector theorem, that indicates that line SD bisects $\angle BSC$ and hence passes through M.

Let K be the intersection point of lines EF and SI. Notice that φ sends points S, F, E, I to S, B, C, M, so $\varphi(K) = \varphi(FE \cap SI) = BC \cap SM = D$. By $\varphi(I) = M$, we have $KD \parallel IM$.



We claim that triangles SPI and SDQ are similar, and so are triangles SPD and SIQ. Let ray SI meet the circumcircle again at L. Note that the segment EF is perpendicular to the angle bisector AM. Then by $\angle AML = \angle ASL = \angle ASI = 90^\circ$, we have $ML \parallel PQ$. Hence, $\widehat{PL} = \widehat{MQ}$ and therefore $\angle PSL = \angle MSQ = \angle DSQ$. By $\angle QPS = \angle QMS$, the triangles SPK and SMQ are similar. Finally,

$$\frac{SP}{SI} = \frac{SP}{SK} \cdot \frac{SK}{SI} = \frac{SM}{SQ} \cdot \frac{SD}{SM} = \frac{SD}{SQ}$$

shows that triangles SPI and SDQ are similar. The second part of the claim can be proved analogously.

Now the problem statement can be proved by

$$\angle DPA + \angle AQD = \angle DPS + \angle SQD = \angle QIS + \angle SIP = \angle QIP.$$

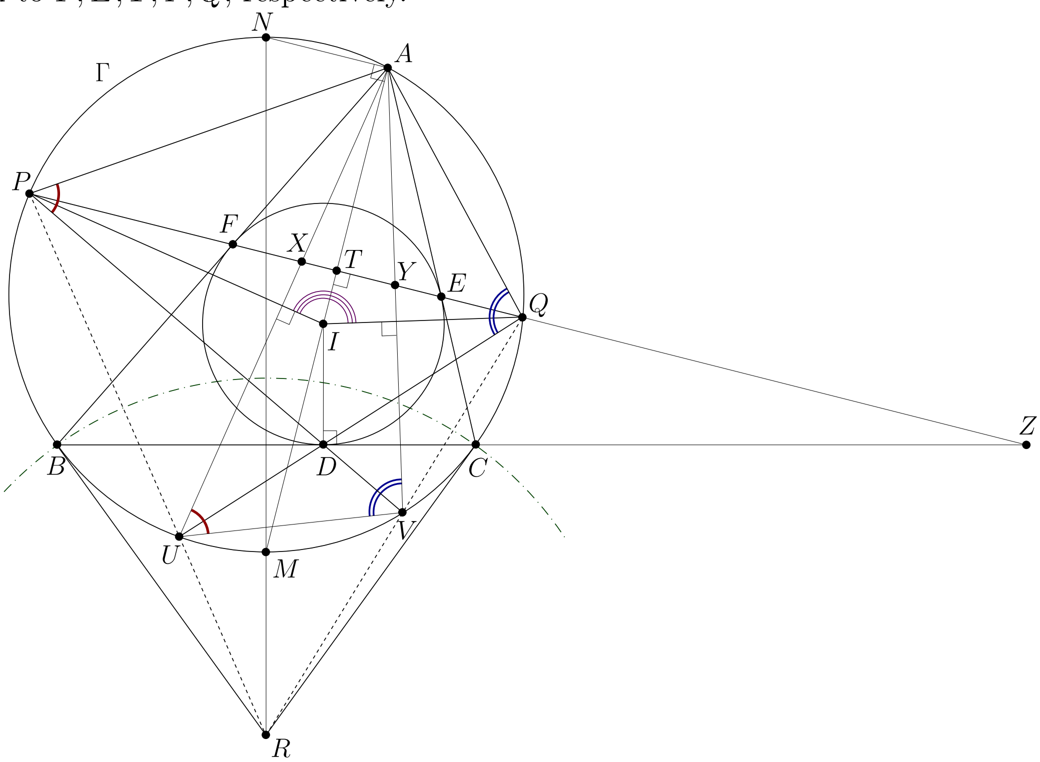
Solution 3. Denote the circumcircle of triangle ABC by Γ, and let rays PD and QD meet Γ again at V and U, respectively. We will show that $AU \perp IP$ and $AV \perp IQ$. Then the problem statement will follow as

$$\angle DPA + \angle AQD = \angle VUA + \angle AVU = 180^\circ - \angle UAV = \angle QIP.$$

Let M be the midpoint of arc $\overset{\frown}{BUVC}$ and let N be the midpoint of arc $\overset{\frown}{CAB}$; the lines AIM and AN being the internal and external bisectors of angle BAC, respectively, are perpendicular. Let the tangents drawn to Γ at B and C meet at R; let line PQ meet AU, AI, AV and BC at X, T, Y and Z, respectively.

As in Solution 1, we observe that the pairs B, C and D, Z are harmonic. Projecting these points from Q onto the circumcircle, we can see that B, C and U, P are also harmonic. Analogously, the pair V, Q is harmonic with B, C. Consider the inversion about the circle with centre R, passing through B and C. Points B and C are fixed points, so this inversion exchanges every point of Γ by its harmonic pair with respect to B, C. In particular, the inversion maps points B, C, N, U, V to points B, C, M, P, Q, respectively.

Combine the inversion with projecting Γ from A to line PQ; the points B, C, M, P, Q are projected to F, E, T, P, Q, respectively.



The combination of these two transformations is projective map from the lines AB, AC, AN, AU, AV to IF, IE, IT, IP, IQ, respectively. On the other hand, we have $AB \perp IF$, $AC \perp IE$ and $AN \perp AT$, so the corresponding lines in these two pencils are perpendicular. This proves $AU \perp IP$ and $AV \perp IQ$, and hence completes the solution.

G7. The incircle ω of acute-angled scalene triangle ABC has centre I and meets sides BC, CA, and AB at D, E, and F, respectively. The line through D perpendicular to EF meets ω again at R. Line AR meets ω again at P. The circumcircles of triangles PCE and PBF meet again at $Q \neq P$. Prove that lines DI and PQ meet on the external bisector of angle BAC.

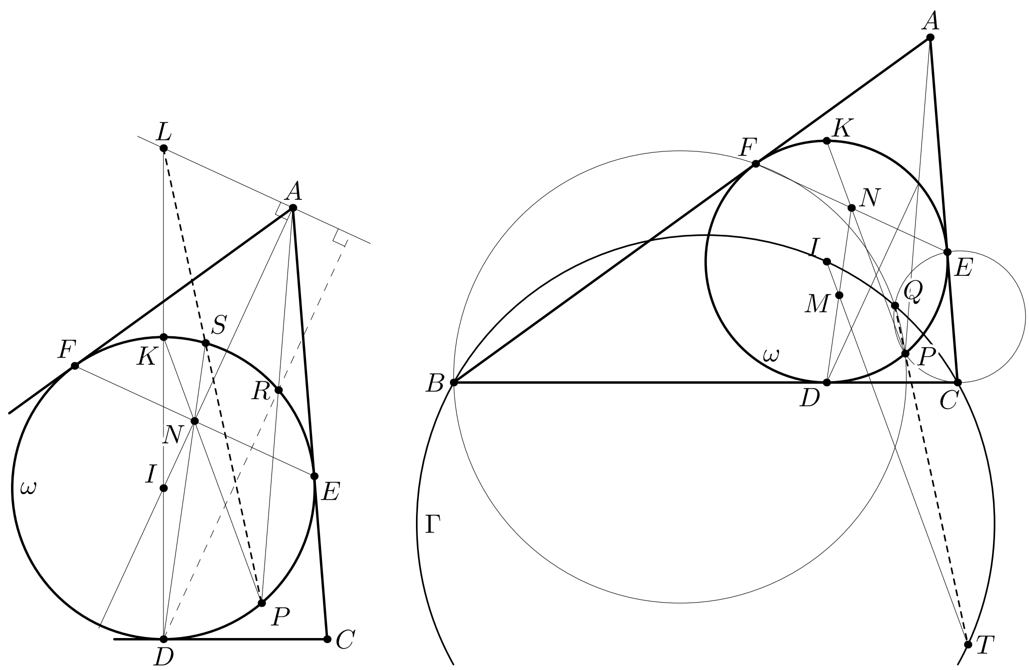
(India)

Common remarks. Throughout the solution, $\angle(a, b)$ denotes the directed angle between lines a and b, measured modulo π.

Solution 1.

Step 1. The external bisector of $\angle BAC$ is the line through A perpendicular to IA. Let DI meet this line at L and let DI meet ω at K. Let N be the midpoint of EF, which lies on IA and is the pole of line AL with respect to ω. Since $AN \cdot AI = AE^2 = AR \cdot AP$, the points R, N, I, and P are concyclic. As $IR = IP$, the line NI is the external bisector of $\angle PNR$, so PN meets ω again at the point symmetric to R with respect to AN – i.e. at K.

Let DN cross ω again at S. Opposite sides of any quadrilateral inscribed in the circle ω meet on the polar line of the intersection of the diagonals with respect to ω. Since L lies on the polar line AL of N with respect to ω, the line PS must pass through L. Thus it suffices to prove that the points S, Q, and P are collinear.

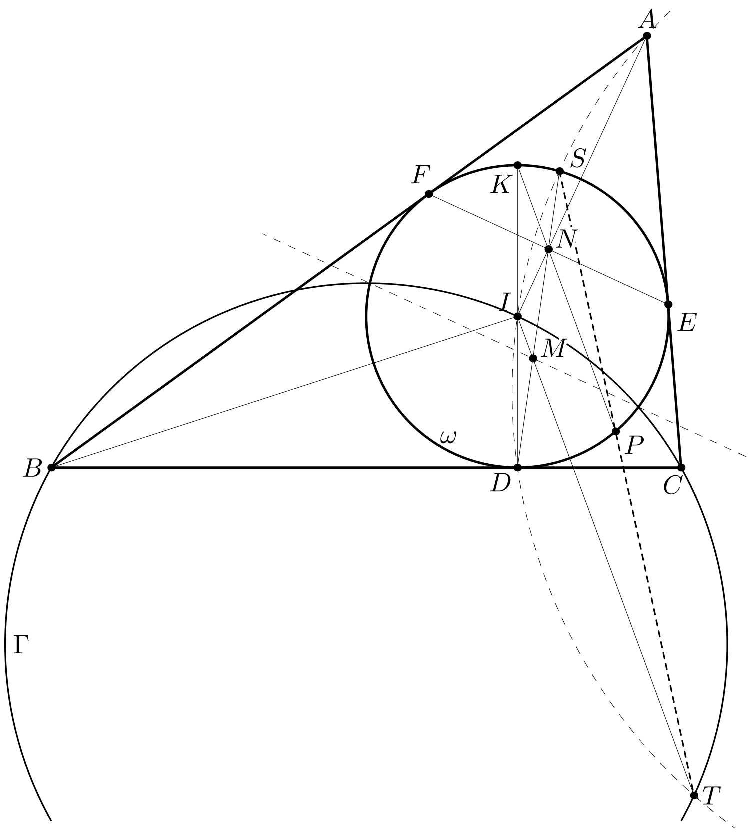


Step 2. Let Γ be the circumcircle of $\triangle BIC$. Notice that

$$\begin{aligned}\angle(BQ, QC) &= \angle(BQ, QP) + \angle(PQ, QC) = \angle(BF, FP) + \angle(PE, EC)\\ &= \angle(EF, EP) + \angle(FP, FE) = \angle(FP, EP) = \angle(DF, DE) = \angle(BI, IC),\end{aligned}$$

so Q lies on Γ. Let QP meet Γ again at T. It will now suffice to prove that S, P, and T are collinear. Notice that $\angle(BI, IT) = \angle(BQ, QT) = \angle(BF, FP) = \angle(FK, KP)$. Note $FD \perp FK$ and $FD \perp BI$ so $FK \parallel BI$ and hence IT is parallel to the line KNP. Since $DI = IK$, the line IT crosses DN at its midpoint M.

Step 3. Let F' and E' be the midpoints of DE and DF, respectively. Since $DE' \cdot E'F = DE'^2 = BE' \cdot E'I$, the point E' lies on the radical axis of ω and Γ; the same holds for F'. Therefore, this radical axis is $E'F'$, and it passes through M. Thus $IM \cdot MT = DM \cdot MS$, so S, I, D, and T are concyclic. This shows $\angle(DS, ST) = \angle(DI, IT) = \angle(DK, KP) = \angle(DS, SP)$, whence the points S, P, and T are collinear, as desired.



Comment. Here is a longer alternative proof in step 1 that P, S, and L are collinear, using a circular inversion instead of the fact that opposite sides of a quadrilateral inscribed in a circle ω meet on the polar line with respect to ω of the intersection of the diagonals. Let G be the foot of the altitude from N to the line $DIKL$. Observe that N, G, K, S are concyclic (opposite right angles) so

$$\angle DIP = 2\angle DKP = \angle GKN + \angle DSP = \angle GSN + \angle NSP = \angle GSP\,,$$

hence I, G, S, P are concyclic. We have $IG \cdot IL = IN \cdot IA = r^2$ since $\triangle IGN \sim \triangle IAL$. Inverting the circle $IGSP$ in circle ω, points P and S are fixed and G is taken to L so we find that P, S, and L are collinear.

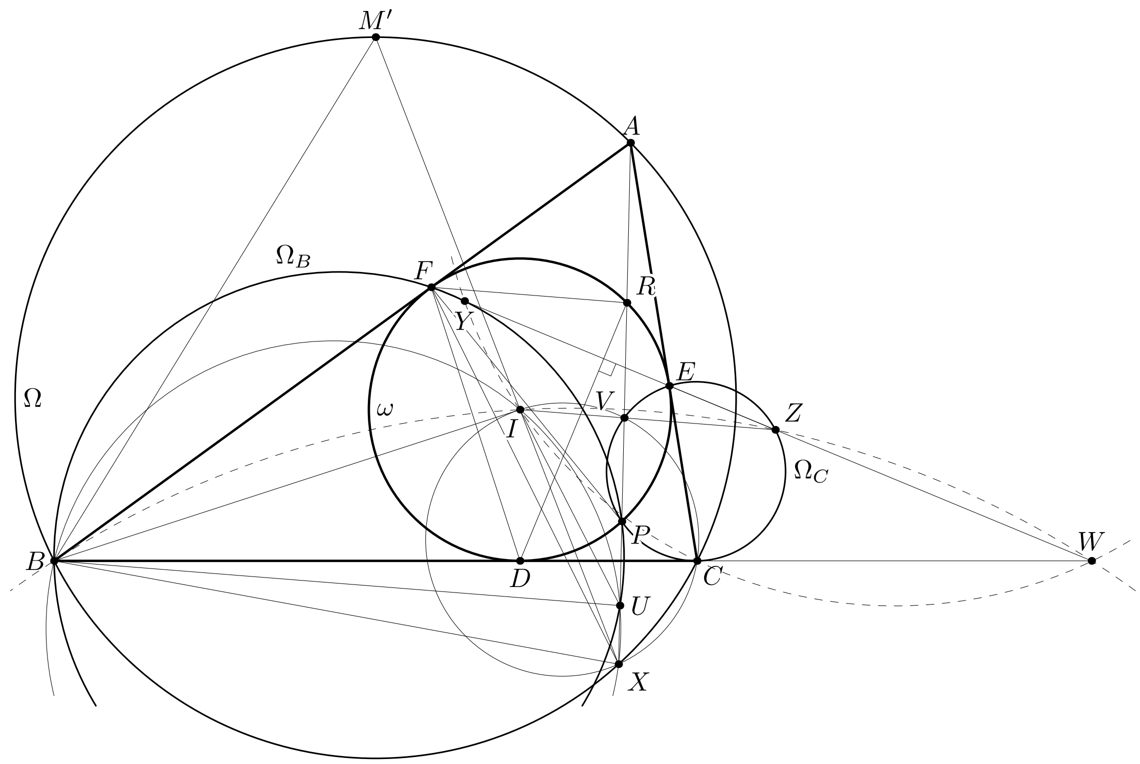
Solution 2. We start as in Solution 1. Namely, we introduce the same points K, L, N, and S, and show that the triples (P, N, K) and (P, S, L) are collinear. We conclude that K and R are symmetric in AI, and reduce the problem statement to showing that P, Q, and S are collinear.

Step 1. Let AR meet the circumcircle Ω of ABC again at X. The lines AR and AK are isogonal in the angle BAC; it is well known that in this case X is the tangency point of Ω with the A-mixtilinear circle. It is also well known that for this point X, the line XI crosses Ω again at the midpoint M' of arc BAC.

Step 2. Denote the circles BFP and CEP by Ω_B and Ω_C, respectively. Let Ω_B cross AR and EF again at U and Y, respectively. We have

$$\angle(UB, BF) = \angle(UP, PF) = \angle(RP, PF) = \angle(RF, FA),$$

so $UB \parallel RF$.



Next, we show that the points B, I, U, and X are concyclic. Since

$$\angle(UB, UX) = \angle(RF, RX) = \angle(AF, AR) + \angle(FR, FA) = \angle(M'B, M'X) + \angle(DR, DF),$$

it suffices to prove $\angle(IB, IX) = \angle(M'B, M'X) + \angle(DR, DF)$, or $\angle(IB, M'B) = \angle(DR, DF)$. But both angles equal $\angle(CI, CB)$, as desired. (This is where we used the fact that M' is the midpoint of arc BAC of Ω.)

It follows now from circles $BUIX$ and $BPUFY$ that

$$\angle(IU, UB) = \angle(IX, BX) = \angle(M'X, BX) = \frac{\pi - \angle A}{2}$$
$$= \angle(EF, AF) = \angle(YF, BF) = \angle(YU, BU),$$

so the points Y, U, and I are collinear.

Let EF meet BC at W. We have

$$\angle(IY, YW) = \angle(UY, FY) = \angle(UB, FB) = \angle(RF, AF) = \angle(CI, CW),$$

so the points W, Y, I, and C are concyclic.

Similarly, if V and Z are the second meeting points of Ω_C with AR and EF, we get that the 4-tuples (C, V, I, X) and (B, I, Z, W) are both concyclic.

Step 3. Let $Q' = CY \cap BZ$. We will show that $Q' = Q$.

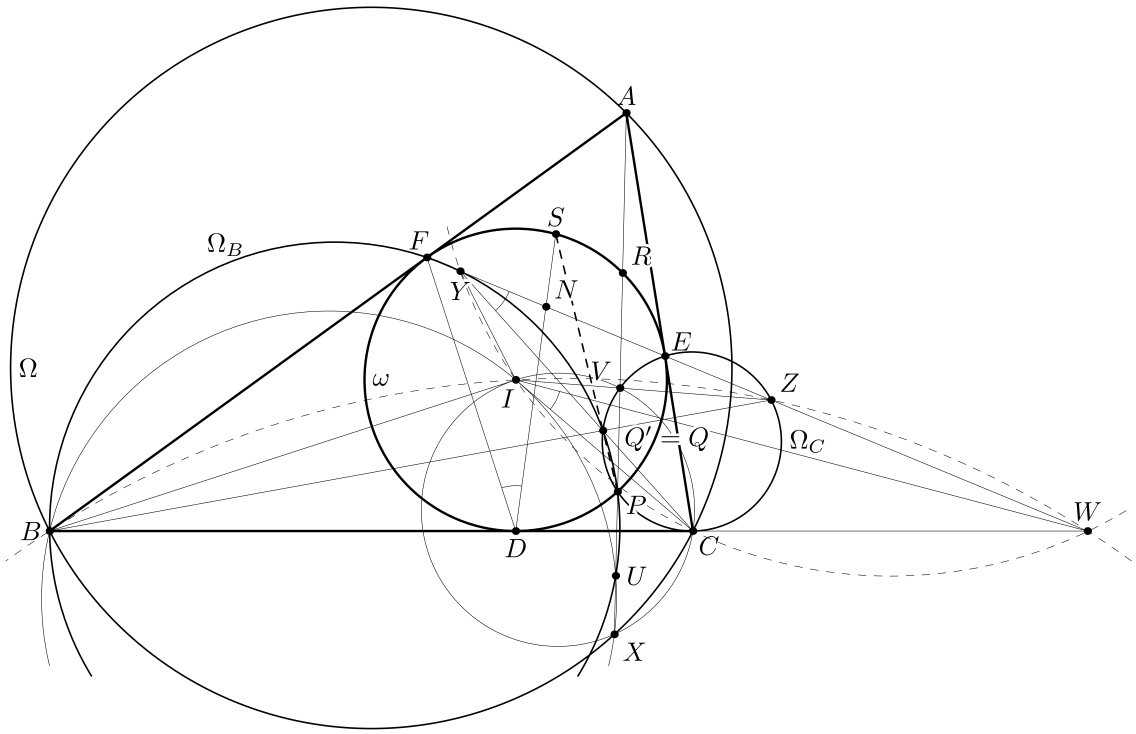
First of all, we have

$$\begin{aligned}\angle(Q'Y, Q'B) &= \angle(CY, ZB) = \angle(CY, ZY) + \angle(ZY, BZ)\\ &= \angle(CI, IW) + \angle(IW, IB) = \angle(CI, IB) = \frac{\pi - \angle A}{2} = \angle(FY, FB),\end{aligned}$$

so $Q' \in \Omega_B$. Similarly, $Q' \in \Omega_C$. Thus $Q' \in \Omega_B \cap \Omega_C = \{P, Q\}$ and it remains to prove that $Q' \neq P$. If we had $Q' = P$, we would have $\angle(PY, PZ) = \angle(Q'Y, Q'Z) = \angle(IC, IB)$. This would imply

$$\angle(PY, YF) + \angle(EZ, ZP) = \angle(PY, PZ) = \angle(IC, IB) = \angle(PE, PF),$$

so circles Ω_B and Ω_C would be tangent at P. That is excluded in the problem conditions, so $Q' = Q$.



Step 4. Now we are ready to show that P, Q, and S are collinear.

Notice that A and D are the poles of EW and DW with respect to ω, so W is the pole of AD. Hence, $WI \perp AD$. Since $CI \perp DE$, this yields $\angle(IC, WI) = \angle(DE, DA)$. On the other hand, DA is a symmedian in $\triangle DEF$, so $\angle(DE, DA) = \angle(DN, DF) = \angle(DS, DF)$. Therefore,

$$\begin{aligned}\angle(PS, PF) &= \angle(DS, DF) = \angle(DE, DA) = \angle(IC, IW)\\ &= \angle(YC, YW) = \angle(YQ, YF) = \angle(PQ, PF),\end{aligned}$$

which yields the desired collinearity.

G8. Let $\mathcal{L}$ be the set of all lines in the plane and let f be a function that assigns to each line $\ell \in \mathcal{L}$ a point $f(\ell)$ on ℓ. Suppose that for any point X, and for any three lines ℓ_1, ℓ_2, ℓ_3 passing through X, the points $f(\ell_1)$, $f(\ell_2)$, $f(\ell_3)$ and X lie on a circle.

Prove that there is a unique point P such that $f(\ell) = P$ for any line ℓ passing through P.

(Australia)

Common remarks. The condition on f is equivalent to the following: There is some function g that assigns to each point X a circle $g(X)$ passing through X such that for any line ℓ passing through X, the point $f(\ell)$ lies on $g(X)$. (The function g may not be uniquely defined for all points, if some points X have at most one value of $f(\ell)$ other than X; for such points, an arbitrary choice is made.)

If there were two points P and Q with the given property, $f(PQ)$ would have to be both P and Q, so there is at most one such point, and it will suffice to show that such a point exists.

Solution 1. We provide a complete characterisation of the functions satisfying the given condition.

Write $\angle(\ell_1, \ell_2)$ for the *directed* angle modulo 180° between the lines ℓ_1 and ℓ_2. Given a point P and an angle $\alpha \in (0, 180^\circ)$, for each line ℓ, let ℓ' be the line through P satisfying $\angle(\ell', \ell) = \alpha$, and let $h_{P,\alpha}(\ell)$ be the intersection point of ℓ and ℓ'. We will prove that there is some pair (P, α) such that f and $h_{P,\alpha}$ are the same function. Then P is the unique point in the problem statement.

Given an angle α and a point P, let a line ℓ be called (P, α)-*good* if $f(\ell) = h_{P,\alpha}(\ell)$. Let a point $X \neq P$ be called (P, α)-*good* if the circle $g(X)$ passes through P and some point $Y \neq P, X$ on $g(X)$ satisfies $\angle(PY, YX) = \alpha$. It follows from this definition that if X is (P, α)-good then every point $Y \neq P, X$ of $g(X)$ satisfies this angle condition, so $h_{P,\alpha}(XY) = Y$ for every $Y \in g(X)$. Equivalently, $f(\ell) \in \{X, h_{P,\alpha}(\ell)\}$ for each line ℓ passing through X. This shows the following lemma.

Lemma 1. If X is (P, α)-good and ℓ is a line passing through X then either $f(\ell) = X$ or ℓ is (P, α)-good.

Lemma 2. If X and Y are different (P, α)-good points, then line XY is (P, α)-good.

Proof. If XY is not (P, α)-good then by the previous Lemma, $f(XY) = X$ and similarly $f(XY) = Y$, but clearly this is impossible as $X \neq Y$. $\square$

Lemma 3. If ℓ_1 and ℓ_2 are different (P, α)-good lines which intersect at $X \neq P$, then either $f(\ell_1) = X$ or $f(\ell_2) = X$ or X is (P, α)-good.

Proof. If $f(\ell_1), f(\ell_2) \neq X$, then $g(X)$ is the circumcircle of X, $f(\ell_1)$ and $f(\ell_2)$. Since ℓ_1 and ℓ_2 are (P, α)-good lines, the angles

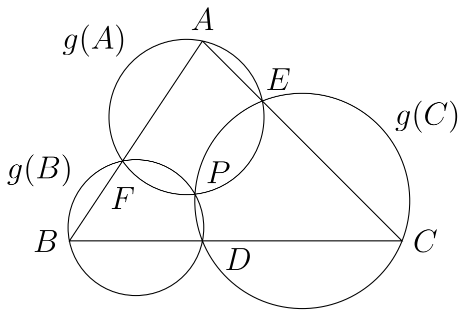
$$\angle(Pf(\ell_1), f(\ell_1)X) = \angle(Pf(\ell_2), f(\ell_2)X) = \alpha,$$

so P lies on $g(X)$. Hence, X is (P, α)-good. $\square$

Lemma 4. If ℓ_1, ℓ_2 and ℓ_3 are different (P, α)-good lines which intersect at $X \neq P$, then X is (P, α)-good.

Proof. This follows from the previous Lemma since at most one of the three lines ℓ_i can satisfy $f(\ell_i) = X$ as the three lines are all (P, α)-good. $\square$

Lemma 5. If ABC is a triangle such that A, B, C, $f(AB)$, $f(AC)$ and $f(BC)$ are all different points, then there is some point P and some angle α such that A, B and C are (P, α)-good points and AB, BC and CA are (P, α)-good lines.



Proof. Let D, E, F denote the points $f(BC)$, $f(AC)$, $f(AB)$, respectively. Then $g(A)$, $g(B)$ and $g(C)$ are the circumcircles of AEF, BDF and CDE, respectively. Let $P \neq F$ be the second intersection of circles $g(A)$ and $g(B)$ (or, if these circles are tangent at F, then $P = F$). By Miquel's theorem (or an easy angle chase), $g(C)$ also passes through P. Then by the cyclic quadrilaterals, the directed angles

$$\angle(PD, DC) = \angle(PF, FB) = \angle(PE, EA) = \alpha,$$

for some angle α. Hence, lines AB, BC and CA are all (P, α)-good, so by Lemma 3, A, B and C are (P, α)-good. (In the case where $P = D$, the line PD in the equation above denotes the line which is tangent to $g(B)$ at $P = D$. Similar definitions are used for PE and PF in the cases where $P = E$ or $P = F$.) □

Consider the set Ω of all points (x, y) with integer coordinates $1 \leqslant x, y \leqslant 1000$, and consider the set L_Ω of all horizontal, vertical and diagonal lines passing through at least one point in Ω. A simple counting argument shows that there are 5998 lines in L_Ω. For each line ℓ in L_Ω we colour the point $f(\ell)$ red. Then there are at most 5998 red points. Now we partition the points in Ω into 10000 ten by ten squares. Since there are at most 5998 red points, at least one of these squares Ω_{10} contains no red points. Let (m, n) be the bottom left point in Ω_{10}. Then the triangle with vertices (m, n), $(m + 1, n)$ and $(m, n + 1)$ satisfies the condition of Lemma 5, so these three vertices are all (P, α)-good for some point P and angle α, as are the lines joining them. From this point on, we will simply call a point or line *good* if it is (P, α)-good for this particular pair (P, α). Now by Lemma 1, the line $x = m + 1$ is good, as is the line $y = n + 1$. Then Lemma 3 implies that $(m+1, n+1)$ is good. By applying these two lemmas repeatedly, we can prove that the line $x+y = m+n+2$ is good, then the points $(m, n+2)$ and $(m+2, n)$ then the lines $x = m+2$ and $y = n+2$, then the points $(m+2, n+1)$, $(m+1, n+2)$ and $(m+2, n+2)$ and so on until we have prove that all points in Ω_{10} are good.

Now we will use this to prove that every point $S \neq P$ is good. Since $g(S)$ is a circle, it passes through at most two points of Ω_{10} on any vertical line, so at most 20 points in total. Moreover, any line ℓ through S intersects at most 10 points in Ω_{10}. Hence, there are at least eight lines ℓ through S which contain a point Q in Ω_{10} which is not on $g(S)$. Since Q is not on $g(S)$, the point $f(\ell) \neq Q$. Hence, by Lemma 1, the line ℓ is good. Hence, at least eight good lines pass through S, so by Lemma 4, the point S is good. Hence, every point $S \neq P$ is good, so by Lemma 2, every line is good. In particular, every line ℓ passing through P is good, and therefore satisfies $f(\ell) = P$, as required.

Solution 2. Note that for any distinct points X, Y, the circles $g(X)$ and $g(Y)$ meet on XY at the point $f(XY) \in g(X) \cap g(Y) \cap (XY)$. We write $s(X, Y)$ for the second intersection point of circles $g(X)$ and $g(Y)$.

Lemma 1. Suppose that X, Y and Z are not collinear, and that $f(XY) \notin \{X, Y\}$ and similarly for YZ and ZX. Then $s(X, Y) = s(Y, Z) = s(Z, X)$.

Proof. The circles $g(X)$, $g(Y)$ and $g(Z)$ through the vertices of triangle XYZ meet pairwise on the corresponding edges (produced). By Miquel's theorem, the second points of intersection of any two of the circles coincide. (See the diagram for Lemma 5 of Solution 1.) □

Now pick any line ℓ and any six different points $Y_1, \ldots, Y_6$ on $\ell \setminus \{f(\ell)\}$. Pick a point X not on ℓ or any of the circles $g(Y_i)$. Reordering the indices if necessary, we may suppose that $Y_1, \ldots, Y_4$ do not lie on $g(X)$, so that $f(XY_i) \notin \{X, Y_i\}$ for $1 \leqslant i \leqslant 4$. By applying the above lemma to triangles XY_iY_j for $1 \leqslant i < j \leqslant 4$, we find that the points $s(Y_i, Y_j)$ and $s(X, Y_i)$ are all equal, to point O say. Note that either O does not lie on ℓ, or $O = f(\ell)$, since $O \in g(Y_i)$.

Now consider an arbitrary point X' not on ℓ or any of the circles $g(Y_i)$ for $1 \leqslant i \leqslant 4$. As above, we see that there are two indices $1 \leqslant i < j \leqslant 4$ such that Y_i and Y_j do not lie on $g(X')$. By applying the above lemma to triangle $X'Y_iY_j$ we see that $s(X', Y_i) = O$, and in particular $g(X')$ passes through O.

We will now show that $f(\ell') = O$ for all lines ℓ' through O. By the above note, we may assume that $\ell' \neq \ell$. Consider a variable point $X' \in \ell' \setminus \{O\}$ not on ℓ or any of the circles $g(Y_i)$ for $1 \leqslant i \leqslant 4$. We know that $f(\ell') \in g(X') \cap \ell' = \{X', O\}$. Since X' was suitably arbitrary, we have $f(\ell') = O$ as desired.

Solution 3. Notice that, for any two different points X and Y, the point $f(XY)$ lies on both $g(X)$ and $g(Y)$, so any two such circles meet in at least one point. We refer to two circles as *cutting* only in the case where they cross, and so meet at exactly two points, thus excluding the cases where they are tangent or are the same circle.

Lemma 1. Suppose there is a point P such that all circles $g(X)$ pass through P. Then P has the given property.

Proof. Consider some line ℓ passing through P, and suppose that $f(\ell) \neq P$. Consider some $X \in \ell$ with $X \neq P$ and $X \neq f(\ell)$. Then $g(X)$ passes through all of P, $f(\ell)$ and X, but those three points are collinear, a contradiction. □

Lemma 2. Suppose that, for all $\epsilon > 0$, there is a point P_ϵ with $g(P_\epsilon)$ of radius at most ϵ. Then there is a point P with the given property.

Proof. Consider a sequence $\epsilon_i = 2^{-i}$ and corresponding points P_{ϵ_i}. Because the two circles $g(P_{\epsilon_i})$ and $g(P_{\epsilon_j})$ meet, the distance between P_{ϵ_i} and P_{ϵ_j} is at most $2^{1-i} + 2^{1-j}$. As $\sum_i \epsilon_i$ converges, these points converge to some point P. For all $\epsilon > 0$, the point P has distance at most 2ϵ from P_ϵ, and all circles $g(X)$ pass through a point with distance at most 2ϵ from P_ϵ, so distance at most 4ϵ from P. A circle that passes distance at most 4ϵ from P for all $\epsilon > 0$ must pass through P, so by Lemma 1 the point P has the given property. □

Lemma 3. Suppose no two of the circles $g(X)$ cut. Then there is a point P with the given property.

Proof. Consider a circle $g(X)$ with centre Y. The circle $g(Y)$ must meet $g(X)$ without cutting it, so has half the radius of $g(X)$. Repeating this argument, there are circles with arbitrarily small radius and the result follows by Lemma 2. □

Lemma 4. Suppose there are six different points A, B_1, B_2, B_3, B_4, B_5 such that no three are collinear, no four are concyclic, and all the circles $g(B_i)$ cut pairwise at A. Then there is a point P with the given property.

Proof. Consider some line ℓ through A that does not pass through any of the B_i and is not tangent to any of the $g(B_i)$. Fix some direction along that line, and let X_ϵ be the point on ℓ that has distance ϵ from A in that direction. In what follows we consider only those ϵ for which X_ϵ does not lie on any $g(B_i)$ (this restriction excludes only finitely many possible values of ϵ).

Consider the circle $g(X_\epsilon)$. Because no four of the B_i are concyclic, at most three of them lie on this circle, so at least two of them do not. There must be some sequence of $\epsilon \to 0$ such that it is the same two of the B_i for all ϵ in that sequence, so now restrict attention to that sequence, and suppose without loss of generality that B_1 and B_2 do not lie on $g(X_\epsilon)$ for any ϵ in that sequence.

Then $f(X_\epsilon B_1)$ is not B_1, so must be the other point of intersection of $X_\epsilon B_1$ with $g(B_1)$, and the same applies with B_2. Now consider the three points X_ϵ, $f(X_\epsilon B_1)$ and $f(X_\epsilon B_2)$. As $\epsilon \to 0$, the angle at X_ϵ tends to $\angle B_1AB_2$ or $180^\circ - \angle B_1AB_2$, which is not 0 or 180° because no three of the points were collinear. All three distances between those points are bounded above by constant multiples of ϵ (in fact, if the triangle is scaled by a factor of $1/\epsilon$, it tends to a fixed triangle). Thus the circumradius of those three points, which is the radius of $g(X_\epsilon)$, is also bounded above by a constant multiple of ϵ, and so the result follows by Lemma 2. □

Lemma 5. Suppose there are two points A and B such that $g(A)$ and $g(B)$ cut. Then there is a point P with the given property.

Proof. Suppose that $g(A)$ and $g(B)$ cut at C and D. One of those points, without loss of generality C, must be $f(AB)$, and so lie on the line AB. We now consider two cases, according to whether D also lies on that line.

Case 1: D does not lie on that line.

In this case, consider a sequence of X_ϵ at distance ϵ from D, tending to D along some line that is not a tangent to either circle, but perturbed slightly (by at most ϵ^2) to ensure that no three of the points A, B and X_ϵ are collinear and no four are concyclic.

Consider the points $f(X_\epsilon A)$ and $f(X_\epsilon B)$, and the circles $g(X_\epsilon)$ on which they lie. The point $f(X_\epsilon A)$ might be either A or the other intersection of $X_\epsilon A$ with the circle $g(A)$, and the same applies for B. If, for some sequence of $\epsilon \to 0$, both those points are the other point of intersection, the same argument as in the proof of Lemma 4 applies to find arbitrarily small circles. Otherwise, we have either infinitely many of those circles passing through A, or infinitely many passing through B; without loss of generality, suppose infinitely many through A.

We now show we can find five points B_i satisfying the conditions of Lemma 4 (together with A). Let B_1 be any of the X_ϵ for which $g(X_\epsilon)$ passes through A. Then repeat the following four times, for $2 \leqslant i \leqslant 5$.

Consider some line $\ell = X_\epsilon A$ (different from those considered for previous i) that is not tangent to any of the $g(B_j)$ for $j < i$, and is such that $f(\ell) = A$, so $g(Y)$ passes through A for all Y on that line. If there are arbitrarily small circles $g(Y)$ we are done by Lemma 2, so the radii of such circles must be bounded below. But as $Y \to A$, along any line not tangent to $g(B_j)$, the radius of a circle through Y and tangent to $g(B_j)$ at A tends to 0. So there must be some Y such that $g(Y)$ cuts $g(B_j)$ at A rather than being tangent to it there, for all of the previous B_j, and we may also pick it such that no three of the B_i and A are collinear and no four are concyclic. Let B_i be this Y. Now the result follows by Lemma 4.

Case 2: D does lie on that line.

In this case, we follow a similar argument, but the sequence of X_ϵ needs to be slightly different. C and D both lie on the line AB, so one must be A and the other must be B. Consider a sequence of X_ϵ tending to B. Rather than tending to B along a straight line (with small perturbations), let the sequence be such that all the points are inside the two circles, with the angle between $X_\epsilon B$ and the tangent to $g(B)$ at B tending to 0.

Again consider the points $f(X_\epsilon A)$ and $f(X_\epsilon B)$. If, for some sequence of $\epsilon \to 0$, both those points are the other point of intersection with the respective circles, we see that the angle at X_ϵ tends to the angle between AB and the tangent to $g(B)$ at B, which is not 0 or 180°, while the distances tend to 0 (although possibly slower than any multiple of ϵ), so we have arbitrarily small circumradii and the result follows by Lemma 2. Otherwise, we have either infinitely many of the circles $g(X_\epsilon)$ passing through A, or infinitely many passing through B, and the same argument as in the previous case enables us to reduce to Lemma 4. □

Lemmas 3 and 5 together cover all cases, and so the required result is proved.

Comment. From the property that all circles $g(X)$ pass through the same point P, it is possible to deduce that the function f has the form given in Solution 1. For any line ℓ not passing through P we may define a corresponding angle $\alpha(\ell)$, which we must show is the same for all such lines. For any point $X \neq P$, with at least one line ℓ through X and not through P, such that $f(\ell) \neq X$, this angle must be equal for all such lines through X (by (directed) angles in the same segment of $g(X)$).

Now consider all horizontal and all vertical lines not through P. For any pair consisting of a horizontal line ℓ_1 and a vertical line ℓ_2, we have $\alpha(\ell_1) = \alpha(\ell_2)$ unless $f(\ell_1)$ or $f(\ell_2)$ is the point of intersection of those lines. Consider the bipartite graph whose vertices are those lines and where an edge joins a horizontal and a vertical line with the same value of α. Considering a subgraph induced by n horizontal and n vertical lines, it must have at least $n^2 - 2n$ edges, so some horizontal line has edges to at least $n-2$ of the vertical lines. Thus, in the original graph, all but at most two of the vertical lines have the same value of α, and likewise all but at most two of the horizontal lines have the same value of α, and, restricting attention to suitable subsets of those lines, we see that this value must be the same for the vertical lines and for the horizontal lines.

But now we can extend this to *all* vertical and horizontal lines not through P (and thus to lines in other directions as well, since the only requirement for 'vertical' and 'horizontal' above is that they are any two nonparallel directions). Consider any horizontal line ℓ_1 not passing through P, and we wish to show that $\alpha(\ell_1)$ has the same value α it has for all but at most two lines not through P in any direction. Indeed, we can deduce this by considering the intersection with any but at most five of the vertical lines: the only ones to exclude are the one passing through P, the one passing through $f(\ell_1)$, at most two such that $\alpha(\ell) \neq \alpha$, and the one passing through $h_{P,\alpha}(\ell_1)$ (defined as in Solution 1). So all lines ℓ not passing through P have the same value of $\alpha(\ell)$.

Solution 4. For any point X, denote by $t(X)$ the line tangent to $g(X)$ at X; notice that $f(t(X)) = X$, so f is surjective.

Step 1: We find a point P for which there are at least two different lines p_1 and p_2 such that $f(p_i) = P$.

Choose any point X. If X does not have this property, take any $Y \in g(X) \setminus \{X\}$; then $f(XY) = Y$. If Y does not have the property, $t(Y) = XY$, and the circles $g(X)$ and $g(Y)$ meet again at some point Z. Then $f(XZ) = Z = f(YZ)$, so Z has the required property.

We will show that P is the desired point. From now on, we fix two different lines p_1 and p_2 with $f(p_1) = f(p_2) = P$. Assume for contradiction that $f(\ell) = Q \neq P$ for some line ℓ through P. We fix ℓ, and note that $Q \in g(P)$.

Step 2: We prove that $P \in g(Q)$.

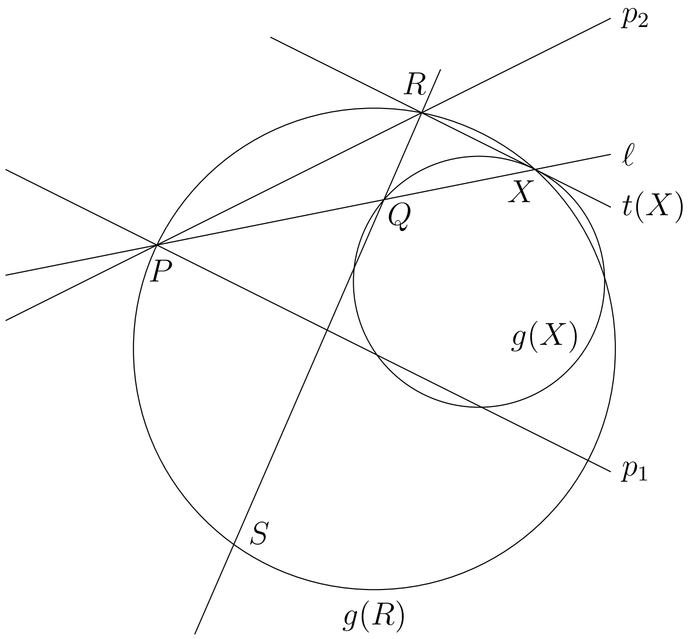
Take an arbitrary point $X \in \ell \setminus \{P, Q\}$. Two cases are possible for the position of $t(X)$ in relation to the p_i; we will show that each case (and subcase) occurs for only finitely many positions of X, yielding a contradiction.

Case 2.1: $t(X)$ is parallel to one of the p_i; say, to p_1.

Let $t(X)$ cross p_2 at R. Then $g(R)$ is the circle (PRX), as $f(RP) = P$ and $f(RX) = X$. Let RQ cross $g(R)$ again at S. Then $f(RQ) \in \{R, S\} \cap g(Q)$, so $g(Q)$ contains one of the points R and S.

If $R \in g(Q)$, then R is one of finitely many points in the intersection $g(Q) \cap p_2$, and each of them corresponds to a unique position of X, since RX is parallel to p_1.

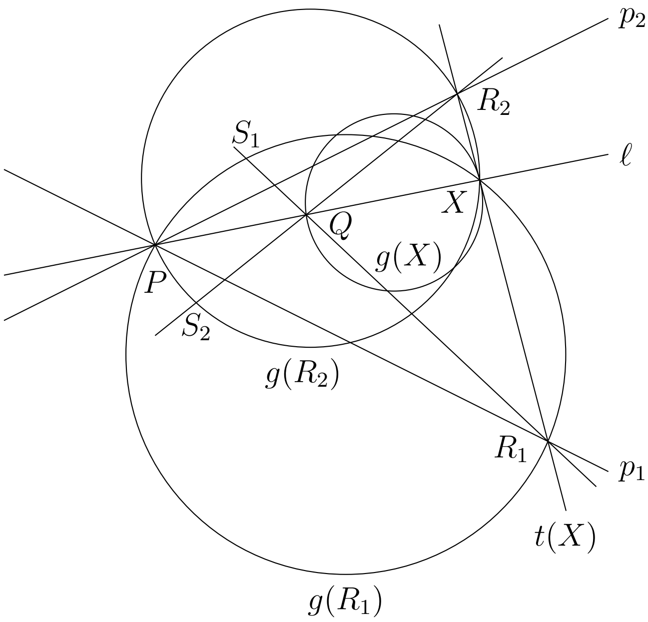
If $S \in g(Q)$, then $\angle(QS, SP) = \angle(RS, SP) = \angle(RX, XP) = \angle(p_1, \ell)$, so $\angle(QS, SP)$ is constant for all such points X, and all points S obtained in such a way lie on one circle γ passing through P and Q. Since $g(Q)$ does not contain P, it is different from γ, so there are only finitely many points S. Each of them uniquely determines R and thus X.



So, Case 2.1 can occur for only finitely many points X.

Case 2.2: $t(X)$ crosses p_1 and p_2 at R_1 and R_2, respectively.

Clearly, $R_1 \neq R_2$, as $t(X)$ is the tangent to $g(X)$ at X, and $g(X)$ meets ℓ only at X and Q. Notice that $g(R_i)$ is the circle (PXR_i). Let R_iQ meet $g(R_i)$ again at S_i; then $S_i \neq Q$, as $g(R_i)$ meets ℓ only at P and X. Then $f(R_iQ) \in \{R_i, S_i\}$, and we distinguish several subcases.



Subcase 2.2.1: $f(R_1Q) = S_1$, $f(R_2Q) = S_2$; so $S_1, S_2 \in g(Q)$.

In this case we have $0 = \angle(R_1X, XP) + \angle(XP, R_2X) = \angle(R_1S_1, S_1P) + \angle(S_2P, S_2R_2) = \angle(QS_1, S_1P) + \angle(S_2P, S_2Q)$, which shows $P \in g(Q)$.

Subcase 2.2.2: $f(R_1Q) = R_1$, $f(R_2Q) = R_2$; so $R_1, R_2 \in g(Q)$.

This can happen for at most four positions of X – namely, at the intersections of ℓ with a line of the form K_1K_2, where $K_i \in g(Q) \cap p_i$.

Subcase 2.2.3: $f(R_1Q) = S_1$, $f(R_2Q) = R_2$ *(the case* $f(R_1Q) = R_1$, $f(R_2Q) = S_2$ *is similar).*

In this case, there are at most two possible positions for R_2 – namely, the meeting points of $g(Q)$ with p_2. Consider one of them. Let X vary on ℓ. Then R_1 is the projection of X to p_1 via R_2, S_1 is the projection of R_1 to $g(Q)$ via Q. Finally, $\angle(QS_1, S_1X) = \angle(R_1S_1, S_1X) = \angle(R_1P, PX) = \angle(p_1, \ell) \neq 0$, so X is obtained by a fixed projective transform $g(Q) \to \ell$ from S_1. So, if there were three points X satisfying the conditions of this subcase, the composition of the three projective transforms would be the identity. But, if we apply it to $X = Q$, we successively get some point R_1', then R_2, and then some point different from Q, a contradiction.

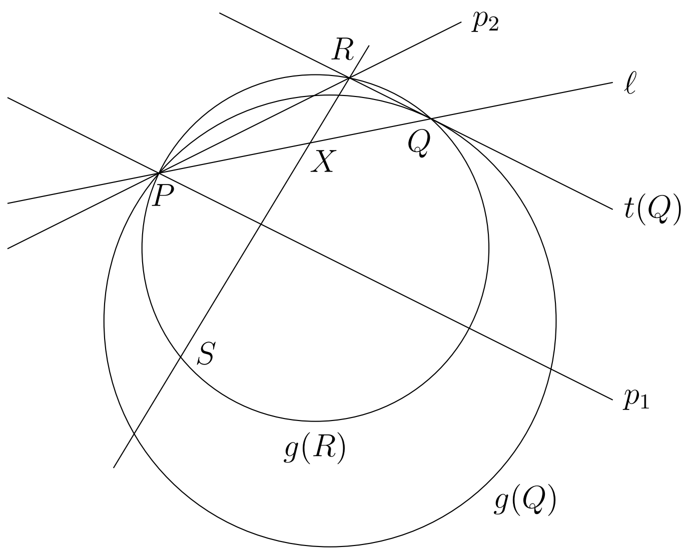
Thus Case 2.2 also occurs for only finitely many points X, as desired.

Step 3: We show that $f(PQ) = P$, *as desired.*

The argument is similar to that in Step 2, with the roles of Q and X swapped. Again, we show that there are only finitely many possible positions for a point $X \in \ell \setminus \{P, Q\}$, which is absurd.

Case 3.1: $t(Q)$ *is parallel to one of the* p_i*; say, to* p_1.

Let $t(Q)$ cross p_2 at R; then $g(R)$ is the circle (PRQ). Let RX cross $g(R)$ again at S. Then $f(RX) \in \{R, S\} \cap g(X)$, so $g(X)$ contains one of the points R and S.



Subcase 3.1.1: $S = f(RX) \in g(X)$.

We have $\angle(t(X), QX) = \angle(SX, SQ) = \angle(SR, SQ) = \angle(PR, PQ) = \angle(p_2, \ell)$. Hence $t(X) \parallel p_2$. Now we recall Case 2.1: we let $t(X)$ cross p_1 at R', so $g(R') = (PR'X)$, and let $R'Q$ meet $g(R')$ again at S'; notice that $S' \neq Q$. Excluding one position of X, we may assume that $R' \notin g(Q)$, so $R' \neq f(R'Q)$. Therefore, $S' = f(R'Q) \in g(Q)$. But then, as in Case 2.1, we get $\angle(t(Q), PQ) = \angle(QS', S'P) = \angle(R'X, XP) = \angle(p_2, \ell)$. This means that $t(Q)$ is parallel to p_2, which is impossible.

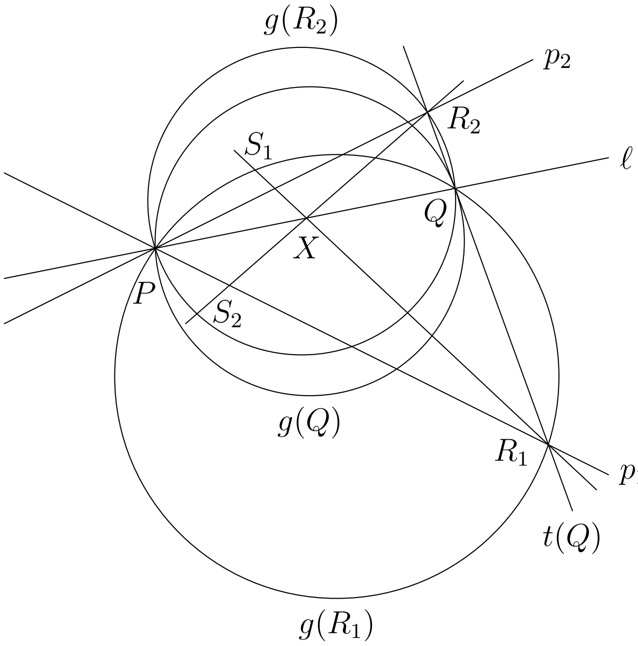
Subcase 3.1.2: $R = f(RX) \in g(X)$.

In this case, we have $\angle(t(X), \ell) = \angle(RX, RQ) = \angle(RX, p_1)$. Again, let $R' = t(X) \cap p_1$; this point exists for all but at most one position of X. Then $g(R') = (R'XP)$; let $R'Q$ meet $g(R')$ again at S'. Due to $\angle(R'X, XR) = \angle(QX, QR) = \angle(\ell, p_1)$, R' determines X in at most two ways, so for all but finitely many positions of X we have $R' \notin g(Q)$. Therefore, for those positions we have $S' = f(R'Q) \in g(Q)$. But then $\angle(RX, p_1) = \angle(R'X, XP) = \angle(R'S', S'P) = \angle(QS', S'P) = \angle(t(Q), QP)$ is fixed, so this case can hold only for one specific position of X as well.

Thus, in Case 3.1, there are only finitely many possible positions of X, yielding a contradiction.

Case 3.2: $t(Q)$ crosses p_1 and p_2 at R_1 and R_2, respectively.

By Step 2, $R_1 \neq R_2$. Notice that $g(R_i)$ is the circle (PQR_i). Let R_iX meet $g(R_i)$ at S_i; then $S_i \neq X$. Then $f(R_iX) \in \{R_i, S_i\}$, and we distinguish several subcases.



Subcase 3.2.1: $f(R_1X) = S_1$ and $f(R_2X) = S_2$, so $S_1, S_2 \in g(X)$.

As in Subcase 2.2.1, we have $0 = \angle(R_1Q, QP) + \angle(QP, R_2Q) = \angle(XS_1, S_1P) + \angle(S_2P, S_2X)$, which shows $P \in g(X)$. But $X, Q \in g(X)$ as well, so $g(X)$ meets ℓ at three distinct points, which is absurd.

Subcase 3.2.2: $f(R_1X) = R_1$, $f(R_2X) = R_2$, so $R_1, R_2 \in g(X)$.

Now three distinct collinear points R_1, R_2, and Q belong to $g(X)$, which is impossible.

Subcase 3.2.3: $f(R_1X) = S_1$, $f(R_2X) = R_2$ (the case $f(R_1X) = R_1$, $f(R_2X) = S_2$ is similar).

We have $\angle(XR_2, R_2Q) = \angle(XS_1, S_1Q) = \angle(R_1S_1, S_1Q) = \angle(R_1P, PQ) = \angle(p_1, \ell)$, so this case can occur for a unique position of X.

Thus, in Case 3.2, there is only a unique position of X, again yielding the required contradiction.

Number Theory

N1. Find all pairs (m, n) of positive integers satisfying the equation

$$(2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1}) = m! \tag{1}$$

(El Salvador)

Answer: The only such pairs are $(1, 1)$ and $(3, 2)$.

Common remarks. In all solutions, for any prime p and positive integer N, we will denote by $v_p(N)$ the exponent of the largest power of p that divides N. The left-hand side of (1) will be denoted by L_n; that is, $L_n = (2^n - 1)(2^n - 2)(2^n - 4) \cdots (2^n - 2^{n-1})$.

Solution 1. We will get an upper bound on n from the speed at which $v_2(L_n)$ grows.

From

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) = 2^{1+2+\cdots+(n-1)}(2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1)$$

we read

$$v_2(L_n) = 1 + 2 + \cdots + (n - 1) = \frac{n(n-1)}{2}.$$

On the other hand, $v_2(m!)$ is expressed by the Legendre formula as

$$v_2(m!) = \sum_{i=1}^{\infty} \left\lfloor \frac{m}{2^i} \right\rfloor.$$

As usual, by omitting the floor functions,

$$v_2(m!) < \sum_{i=1}^{\infty} \frac{m}{2^i} = m.$$

Thus, $L_n = m!$ implies the inequality

$$\frac{n(n-1)}{2} < m. \tag{2}$$

In order to obtain an opposite estimate, observe that

$$L_n = (2^n - 1)(2^n - 2) \cdots (2^n - 2^{n-1}) < (2^n)^n = 2^{n^2}.$$

We claim that

$$2^{n^2} < \left(\frac{n(n-1)}{2}\right)! \quad \text{for } n \geqslant 6. \tag{3}$$

For $n = 6$ the estimate (3) is true because $2^{6^2} < 6.9 \cdot 10^{10}$ and $\left(\frac{n(n-1)}{2}\right)! = 15! > 1.3 \cdot 10^{12}$.

For $n \geqslant 7$ we prove (3) by the following inequalities:

$$\begin{aligned}\left(\frac{n(n-1)}{2}\right)! &= 15! \cdot 16 \cdot 17 \cdots \frac{n(n-1)}{2} > 2^{36} \cdot 16^{\frac{n(n-1)}{2} - 15} \\ &= 2^{2n(n-1)-24} = 2^{n^2} \cdot 2^{n(n-2)-24} > 2^{n^2}.\end{aligned}$$

Putting together (2) and (3), for $n \geqslant 6$ we get a contradiction, since

$$L_n < 2^{n^2} < \left(\frac{n(n-1)}{2}\right)! < m! = L_n.$$

Hence $n \geqslant 6$ is not possible.

Checking manually the cases $n \leqslant 5$ we find

$$L_1 = 1 = 1!, \qquad L_2 = 6 = 3!, \qquad 5! < L_3 = 168 < 6!,$$
$$7! < L_4 = 20\,160 < 8! \qquad \text{and} \qquad 10! < L_5 = 9\,999\,360 < 11!.$$

So, there are two solutions:

$$(m, n) \in \{(1, 1), (3, 2)\}.$$

Solution 2. Like in the previous solution, the cases $n = 1, 2, 3, 4$ are checked manually. We will exclude $n \geqslant 5$ by considering the exponents of 3 and 31 in (1).

For odd primes p and distinct integers a, b, coprime to p, with $p \mid a - b$, the Lifting The Exponent lemma asserts that

$$v_p(a^k - b^k) = v_p(a - b) + v_p(k).$$

Notice that 3 divides $2^k - 1$ if only if k is even; moreover, by the Lifting The Exponent lemma we have

$$v_3(2^{2k} - 1) = v_3(4^k - 1) = 1 + v_3(k) = v_3(3k).$$

Hence,

$$v_3(L_n) = \sum_{2k \leqslant n} v_3(4^k - 1) = \sum_{k \leqslant \lfloor \frac{n}{2} \rfloor} v_3(3k).$$

Notice that the last expression is precisely the exponent of 3 in the prime factorisation of $\left(3\lfloor \frac{n}{2} \rfloor\right)!$. Therefore

$$v_3(m!) = v_3(L_n) = v_3\left(\left(3\lfloor \tfrac{n}{2} \rfloor\right)!\right)$$

$$3\left\lfloor \frac{n}{2} \right\rfloor \leqslant m \leqslant 3\left\lfloor \frac{n}{2} \right\rfloor + 2. \tag{4}$$

Suppose that $n \geqslant 5$. Note that every fifth factor in L_n is divisible by $31 = 2^5 - 1$, and hence we have $v_{31}(L_n) \geqslant \lfloor \frac{n}{5} \rfloor$. Then

$$\frac{n}{10} \leqslant \left\lfloor \frac{n}{5} \right\rfloor \leqslant v_{31}(L_n) = v_{31}(m!) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{31^k} \right\rfloor < \sum_{k=1}^{\infty} \frac{m}{31^k} = \frac{m}{30}. \tag{5}$$

By combining (4) and (5),

$$3n < m \leqslant \frac{3n}{2} + 2$$

so $n < \frac{4}{3}$ which is inconsistent with the inequality $n \geqslant 5$.

Comment 1. There are many combinations of the ideas above; for example combining (2) and (4) also provides $n < 5$. Obviously, considering the exponents of any two primes in (1), or considering one prime and the magnitude orders lead to an upper bound on n and m.

Comment 2. This problem has a connection to group theory. Indeed, the left-hand side is the order of the group $GL_n(\mathbb{F}_2)$ of invertible n-by-n matrices with entries modulo 2, while the right-hand side is the order of the symmetric group S_m on m elements. The result thus shows that the only possible isomorphisms between these groups are $GL_1(\mathbb{F}_2) \cong S_1$ and $GL_2(\mathbb{F}_2) \cong S_3$, and there are in fact isomorphisms in both cases. In general, $GL_n(\mathbb{F}_2)$ is a simple group for $n \geqslant 3$, as it is isomorphic to $PSL_n(\mathbb{F}_2)$.

There is also a near-solution of interest: the left-hand side for $n = 4$ is *half* of the right-hand side when $m = 8$; this turns out to correspond to an isomorphism $GL_4(\mathbb{F}_2) \cong A_8$ with the alternating group on eight elements.

However, while this indicates that the problem is a useful one, knowing group theory is of no use in solving it!

N2. Find all triples (a, b, c) of positive integers such that $a^3 + b^3 + c^3 = (abc)^2$.

(Nigeria)

Answer: The solutions are $(1, 2, 3)$ and its permutations.

Common remarks. Note that the equation is symmetric. In all solutions, we will assume without loss of generality that $a \geqslant b \geqslant c$, and prove that the only solution is $(a, b, c) = (3, 2, 1)$.

The first two solutions all start by proving that $c = 1$.

Solution 1. We will start by proving that $c = 1$. Note that

$$3a^3 \geqslant a^3 + b^3 + c^3 > a^3 .$$

So $3a^3 \geqslant (abc)^2 > a^3$ and hence $3a \geqslant b^2c^2 > a$. Now $b^3 + c^3 = a^2(b^2c^2 - a) \geqslant a^2$, and so

$$18b^3 \geqslant 9(b^3 + c^3) \geqslant 9a^2 \geqslant b^4c^4 \geqslant b^3c^5 ,$$

so $18 \geqslant c^5$ which yields $c = 1$.

Now, note that we must have $a > b$, as otherwise we would have $2b^3 + 1 = b^4$ which has no positive integer solutions. So

$$a^3 - b^3 \geqslant (b + 1)^3 - b^3 > 1$$

and

$$2a^3 > 1 + a^3 + b^3 > a^3 ,$$

which implies $2a^3 > a^2b^2 > a^3$ and so $2a > b^2 > a$. Therefore

$$4(1 + b^3) = 4a^2(b^2 - a) \geqslant 4a^2 > b^4 ,$$

so $4 > b^3(b - 4)$; that is, $b \leqslant 4$.

Now, for each possible value of b with $2 \leqslant b \leqslant 4$ we obtain a cubic equation for a with constant coefficients. These are as follows:

$$\begin{aligned} b = 2: &\qquad a^3 - 4a^2 + 9 = 0 \\ b = 3: &\qquad a^3 - 9a^2 + 28 = 0 \\ b = 4: &\qquad a^3 - 16a^2 + 65 = 0. \end{aligned}$$

The only case with an integer solution for a with $b \leqslant a$ is $b = 2$, leading to $(a, b, c) = (3, 2, 1)$.

Comment 1.1. Instead of writing down each cubic equation explicitly, we could have just observed that $a^2 \mid b^3 + 1$, and for each choice of b checked each square factor of $b^3 + 1$ for a^2.

We could also have observed that, with $c = 1$, the relation $18b^3 \geqslant b^4c^4$ becomes $b \leqslant 18$, and we can simply check all possibilities for b (instead of working to prove that $b \leqslant 4$). This check becomes easier after using the factorisation $b^3 + 1 = (b + 1)(b^2 - b + 1)$ and observing that no prime besides 3 can divide both of the factors.

Comment 1.2. Another approach to finish the problem after establishing that $c \leqslant 1$ is to set $k = b^2c^2 - a$, which is clearly an integer and must be positive as it is equal to $(b^3 + c^3)/a^2$. Then we divide into cases based on whether $k = 1$ or $k \geqslant 2$; in the first case, we have $b^3 + 1 = a^2 = (b^2 - 1)^2$ whose only positive root is $b = 2$, and in the second case we have $b^2 \leqslant 3a$, and so

$$b^4 \leqslant (3a)^2 \leqslant \frac{9}{2}(ka^2) = \frac{9}{2}(b^3 + 1),$$

which implies that $b \leqslant 4$.

Solution 2. Again, we will start by proving that $c = 1$. Suppose otherwise that $c \geqslant 2$. We have $a^3 + b^3 + c^3 \leqslant 3a^3$, so $b^2c^2 \leqslant 3a$. Since $c \geqslant 2$, this tells us that $b \leqslant \sqrt{3a/4}$. As the right-hand side of the original equation is a multiple of a^2, we have $a^2 \leqslant 2b^3 \leqslant 2(3a/4)^{3/2}$. In other words, $a \leqslant \frac{27}{16} < 2$, which contradicts the assertion that $a \geqslant c \geqslant 2$. So there are no solutions in this case, and so we must have $c = 1$.

Now, the original equation becomes $a^3 + b^3 + 1 = a^2b^2$. Observe that $a \geqslant 2$, since otherwise $a = b = 1$ as $a \geqslant b$.

The right-hand side is a multiple of a^2, so the left-hand side must be as well. Thus, $b^3 + 1 \geqslant a^2$. Since $a \geqslant b$, we also have

$$b^2 = a + \frac{b^3 + 1}{a^2} \leqslant 2a + \frac{1}{a^2}$$

and so $b^2 \leqslant 2a$ since b^2 is an integer. Thus $(2a)^{3/2} + 1 \geqslant b^3 + 1 \geqslant a^2$, from which we deduce $a \leqslant 8$.

Now, for each possible value of a with $2 \leqslant a \leqslant 8$ we obtain a cubic equation for b with constant coefficients. These are as follows:

$$\begin{aligned}
a = 2: &\qquad b^3 - 4b^2 + 9 = 0 \\
a = 3: &\qquad b^3 - 9b^2 + 28 = 0 \\
a = 4: &\qquad b^3 - 16b^2 + 65 = 0 \\
a = 5: &\qquad b^3 - 25b^2 + 126 = 0 \\
a = 6: &\qquad b^3 - 36b^2 + 217 = 0 \\
a = 7: &\qquad b^3 - 49b^2 + 344 = 0 \\
a = 8: &\qquad b^3 - 64b^2 + 513 = 0.
\end{aligned}$$

The only case with an integer solution for b with $a \geqslant b$ is $a = 3$, leading to $(a, b, c) = (3, 2, 1)$.

Comment 2.1. As in Solution 1, instead of writing down each cubic equation explicitly, we could have just observed that $b^2 \mid a^3 + 1$, and for each choice of a checked each square factor of $a^3 + 1$ for b^2.

Comment 2.2. This solution does not require initially proving that $c = 1$, in which case the bound would become $a \leqslant 108$. The resulting cases could, in principle, be checked by a particularly industrious student.

Solution 3. Set $k = (b^3 + c^3)/a^2 \leqslant 2a$, and rewrite the original equation as $a + k = (bc)^2$. Since b^3 and c^3 are positive integers, we have $(bc)^3 \geqslant b^3 + c^3 - 1 = ka^2 - 1$, so

$$a + k \geqslant (ka^2 - 1)^{2/3}.$$

As in Comment 1.2, k is a positive integer; for each value of $k \geqslant 1$, this gives us a polynomial inequality satisfied by a:

$$k^2a^4 - a^3 - 5ka^2 - 3k^2a - (k^3 - 1) \leqslant 0.$$

We now prove that $a \leqslant 3$. Indeed,

$$0 \geqslant \frac{k^2a^4 - a^3 - 5ka^2 - 3k^2a - (k^3 - 1)}{k^2} \geqslant a^4 - a^3 - 5a^2 - 3a - k \geqslant a^4 - a^3 - 5a^2 - 5a,$$

which fails when $a \geqslant 4$.

This leaves ten triples with $3 \geqslant a \geqslant b \geqslant c \geqslant 1$, which may be checked manually to give $(a, b, c) = (3, 2, 1)$.

Solution 4. Again, observe that $b^3 + c^3 = a^2(b^2c^2 - a)$, so $b \leqslant a \leqslant b^2c^2 - 1$.

We consider the function $f(x) = x^2(b^2c^2 - x)$. It can be seen that that on the interval $[0, b^2c^2 - 1]$ the function f is increasing if $x < \frac{2}{3}b^2c^2$ and decreasing if $x > \frac{2}{3}b^2c^2$. Consequently, it must be the case that

$$b^3 + c^3 = f(a) \geqslant \min\Big(f(b), f(b^2c^2 - 1)\Big).$$

First, suppose that $b^3 + c^3 \geqslant f(b^2c^2 - 1)$. This may be written $b^3 + c^3 \geqslant (b^2c^2 - 1)^2$, and so

$$2b^3 \geqslant b^3 + c^3 \geqslant (b^2c^2 - 1)^2 > b^4c^4 - 2b^2c^2 \geqslant b^4c^4 - 2b^3c^4.$$

Thus, $(b - 2)c^4 < 2$, and the only solutions to this inequality have $(b, c) = (2, 2)$ or $b \leqslant 3$ and $c = 1$. It is easy to verify that the only case giving a solution for $a \geqslant b$ is $(a, b, c) = (3, 2, 1)$.

Otherwise, suppose that $b^3 + c^3 = f(a) \geqslant f(b)$. Then, we have

$$2b^3 \geqslant b^3 + c^3 = a^2(b^2c^2 - a) \geqslant b^2(b^2c^2 - b).$$

Consequently $bc^2 \leqslant 3$, with strict inequality in the case that $b \neq c$. Hence $c = 1$ and $b \leqslant 2$. Both of these cases have been considered already, so we are done.

Comment 4.1. Instead of considering which of $f(b)$ and $f(b^2c^2 - 1)$ is less than $f(a)$, we may also proceed by explicitly dividing into cases based on whether $a \geqslant \frac{2}{3}b^2c^2$ or $a < \frac{2}{3}b^2c^2$. The first case may now be dealt with as follows. We have $b^3c^3 + 1 \geqslant b^3 + c^3$ as b^3 and c^3 are positive integers, so we have

$$b^3c^3 + 1 \geqslant b^3 + c^3 \geqslant a^2 \geqslant \frac{4}{9}b^4c^4.$$

This implies $bc \leqslant 2$, and hence $c = 1$ and $b \leqslant 2$.

N3. We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \ldots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

(Czech Republic)

Answer: The set $\mathbb{Z}$ of all integers is the only such rootiful set.

Solution 1. The set $\mathbb{Z}$ of all integers is clearly rootiful. We shall prove that any rootiful set S containing all the numbers of the form $2^a - 2^b$ for $a, b \in \mathbb{Z}_{>0}$ must be all of $\mathbb{Z}$.

First, note that $0 = 2^1 - 2^1 \in S$ and $2 = 2^2 - 2^1 \in S$. Now, $-1 \in S$, since it is a root of $2x + 2$, and $1 \in S$, since it is a root of $2x^2 - x - 1$. Also, if $n \in S$ then $-n$ is a root of $x + n$, so it suffices to prove that all positive integers must be in S.

Now, we claim that any positive integer n has a multiple in S. Indeed, suppose that $n = 2^\alpha \cdot t$ for $\alpha \in \mathbb{Z}_{\geqslant 0}$ and t odd. Then $t \mid 2^{\phi(t)} - 1$, so $n \mid 2^{\alpha+\phi(t)+1} - 2^{\alpha+1}$. Moreover, $2^{\alpha+\phi(t)+1} - 2^{\alpha+1} \in S$, and so S contains a multiple of every positive integer n.

We will now prove by induction that all positive integers are in S. Suppose that $0, 1, \ldots, n-1 \in S$; furthermore, let N be a multiple of n in S. Consider the base-n expansion of N, say $N = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$. Since $0 \leqslant a_i < n$ for each a_i, we have that all the a_i are in S. Furthermore, $a_0 = 0$ since N is a multiple of n. Therefore, $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n - N = 0$, so n is a root of a polynomial with coefficients in S. This tells us that $n \in S$, completing the induction.

Solution 2. As in the previous solution, we can prove that $0, 1$ and -1 must all be in any rootiful set S containing all numbers of the form $2^a - 2^b$ for $a, b \in \mathbb{Z}_{>0}$.

We show that, in fact, every integer k with $|k| > 2$ can be expressed as a root of a polynomial whose coefficients are of the form $2^a - 2^b$. Observe that it suffices to consider the case where k is positive, as if k is a root of $a_n x^n + \cdots + a_1 x + a_0 = 0$, then $-k$ is a root of $(-1)^n a_n x^n + \cdots - a_1 x + a_0 = 0$.

Note that

$$(2^{a_n} - 2^{b_n})k^n + \cdots + (2^{a_0} - 2^{b_0}) = 0$$

is equivalent to

$$2^{a_n} k^n + \cdots + 2^{a_0} = 2^{b_n} k^n + \cdots + 2^{b_0}.$$

Hence our aim is to show that two numbers of the form $2^{a_n} k^n + \cdots + 2^{a_0}$ are equal, for a fixed value of n. We consider such polynomials where every term $2^{a_i} k^i$ is at most $2k^n$; in other words, where $2 \leqslant 2^{a_i} \leqslant 2k^{n-i}$, or, equivalently, $1 \leqslant a_i \leqslant 1 + (n-i)\log_2 k$. Therefore, there must be $1 + \lfloor (n-i)\log_2 k \rfloor$ possible choices for a_i satisfying these constraints.

The number of possible polynomials is then

$$\prod_{i=0}^{n} (1 + \lfloor (n-i)\log_2 k \rfloor) \geqslant \prod_{i=0}^{n-1} (n-i)\log_2 k = n!(\log_2 k)^n$$

where the inequality holds as $1 + \lfloor x \rfloor \geqslant x$.

As there are $(n+1)$ such terms in the polynomial, each at most $2k^n$, such a polynomial must have value at most $2k^n(n+1)$. However, for large n, we have $n!(\log_2 k)^n > 2k^n(n+1)$. Therefore there are more polynomials than possible values, so some two must be equal, as required.

N4. Let $\mathbb{Z}_{>0}$ be the set of positive integers. A positive integer constant C is given. Find all functions $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ such that, for all positive integers a and b satisfying $a + b > C$,

$$a + f(b) \mid a^2 + b\,f(a). \qquad (*)$$

(Croatia)

Answer: The functions satisfying $(*)$ are exactly the functions $f(a) = ka$ for some constant $k \in \mathbb{Z}_{>0}$ (irrespective of the value of C).

Common remarks. It is easy to verify that the functions $f(a) = ka$ satisfy $(*)$. Thus, in the proofs below, we will only focus on the converse implication: that condition $(*)$ implies that $f = ka$.

A common minor part of these solutions is the derivation of some relatively easy bounds on the function f. An upper bound is easily obtained by setting $a = 1$ in $(*)$, giving the inequality

$$f(b) \leqslant b \cdot f(1)$$

for all sufficiently large b. The corresponding lower bound is only marginally more difficult to obtain: substituting $b = 1$ in the original equation shows that

$$a + f(1) \mid (a^2 + f(a)) - (a - f(1)) \cdot (a + f(1)) = f(1)^2 + f(a)$$

for all sufficiently large a. It follows from this that one has the lower bound

$$f(a) \geqslant a + f(1) \cdot (1 - f(1)),$$

again for all sufficiently large a.

Each of the following proofs makes use of at least one of these bounds.

Solution 1. First, we show that $b \mid f(b)^2$ for all b. To do this, we choose a large positive integer n so that $nb - f(b) \geqslant C$. Setting $a = nb - f(b)$ in $(*)$ then shows that

$$nb \mid (nb - f(b))^2 + bf(nb - f(b))$$

so that $b \mid f(b)^2$ as claimed.

Now in particular we have that $p \mid f(p)$ for every prime p. If we write $f(p) = k(p) \cdot p$, then the bound $f(p) \leqslant f(1) \cdot p$ (valid for p sufficiently large) shows that some value k of $k(p)$ must be attained for infinitely many p. We will show that $f(a) = ka$ for all positive integers a. To do this, we substitute $b = p$ in $(*)$, where p is any sufficiently large prime for which $k(p) = k$, obtaining

$$a + kp \mid (a^2 + pf(a)) - a(a + kp) = pf(a) - pka.$$

For suitably large p we have $\gcd(a + kp, p) = 1$, and hence we have

$$a + kp \mid f(a) - ka.$$

But the only way this can hold for arbitrarily large p is if $f(a) - ka = 0$. This concludes the proof.

Comment. There are other ways to obtain the divisibility $p \mid f(p)$ for primes p, which is all that is needed in this proof. For instance, if $f(p)$ were not divisible by p then the arithmetic progression $p^2 + bf(p)$ would attain prime values for infinitely many b by Dirichlet's Theorem: hence, for these pairs p, b, we would have $p + f(b) = p^2 + bf(p)$. Substituting $a \mapsto b$ and $b \mapsto p$ in $(*)$ then shows that $(f(p)^2 - p^2)(p - 1)$ is divisible by $b + f(p)$ and hence vanishes, which is impossible since $p \nmid f(p)$ by assumption.

Solution 2. First, we substitute $b = 1$ in $(*)$ and rearrange to find that

$$\frac{f(a) + f(1)^2}{a + f(1)} = f(1) - a + \frac{a^2 + f(a)}{a + f(1)}$$

is a positive integer for sufficiently large a. Since $f(a) \leqslant af(1)$, for all sufficiently large a, it follows that $\frac{f(a)+f(1)^2}{a+f(1)} \leqslant f(1)$ also and hence there is a positive integer k such that $\frac{f(a)+f(1)^2}{a+f(1)} = k$ for infinitely many values of a. In other words,

$$f(a) = ka + f(1) \cdot (k - f(1))$$

for infinitely many a.

Fixing an arbitrary choice of a in $(*)$, we have that

$$\frac{a^2 + bf(a)}{a + kb + f(1) \cdot (k - f(1))}$$

is an integer for infinitely many b (the same b as above, maybe with finitely many exceptions). On the other hand, for b taken sufficiently large, this quantity becomes arbitrarily close to $\frac{f(a)}{k}$; this is only possible if $\frac{f(a)}{k}$ is an integer and

$$\frac{a^2 + bf(a)}{a + kb + f(1) \cdot (k - f(1))} = \frac{f(a)}{k}$$

for infinitely many b. This rearranges to

$$\frac{f(a)}{k} \cdot \big(a + f(1) \cdot (k - f(1))\big) = a^2. \tag{$**$}$$

Hence a^2 is divisible by $a + f(1) \cdot (k - f(1))$, and hence so is $f(1)^2(k - f(1))^2$. The only way this can occur for all a is if $k = f(1)$, in which case $(**)$ provides that $f(a) = ka$ for all a, as desired.

Solution 3. Fix any two distinct positive integers a and b. From $(*)$ it follows that the two integers

$$(a^2 + cf(a)) \cdot (b + f(c)) \text{ and } (b^2 + cf(b)) \cdot (a + f(c))$$

are both multiples of $(a + f(c)) \cdot (b + f(c))$ for all sufficiently large c. Taking an appropriate linear combination to eliminate the $cf(c)$ term, we find after expanding out that the integer

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot f(c) + \big[(b - a)f(a)f(b)\big] \cdot c + \big[ab(af(b) - bf(a))\big] \tag{$\dagger$}$$

is also a multiple of $(a + f(c)) \cdot (b + f(c))$.

But as c varies, $(\dagger)$ is bounded above by a positive multiple of c while $(a + f(c)) \cdot (b + f(c))$ is bounded below by a positive multiple of c^2. The only way that such a divisibility can hold is if in fact

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot f(c) + \big[(b - a)f(a)f(b)\big] \cdot c + \big[ab(af(b) - bf(a))\big] = 0 \tag{$\dagger\dagger$}$$

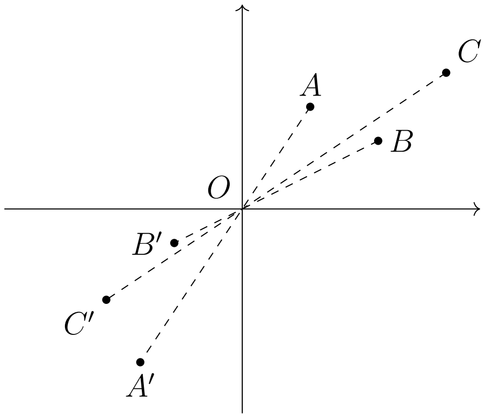
for sufficiently large c. Since the coefficient of c in this linear relation is nonzero, it follows that there are constants k, ℓ such that $f(c) = kc + \ell$ for all sufficiently large c; the constants k and ℓ are necessarily integers.

The value of ℓ satisfies

$$\big[a^2 f(b) - b^2 f(a)\big] \cdot \ell + \big[ab(af(b) - bf(a))\big] = 0 \tag{$\dagger\dagger\dagger$}$$

and hence $b \mid \ell a^2 f(b)$ for all a and b. Taking b sufficiently large so that $f(b) = kb + \ell$, we thus have that $b \mid \ell^2 a^2$ for all sufficiently large b; this implies that $\ell = 0$. From $(\dagger\dagger\dagger)$ it then follows that $\frac{f(a)}{a} = \frac{f(b)}{b}$ for all $a \neq b$, so that there is a constant k such that $f(a) = ka$ for all a (k is equal to the constant defined earlier).

Solution 4. Let Γ denote the set of all points $(a, f(a))$, so that Γ is an infinite subset of the upper-right quadrant of the plane. For a point $A = (a, f(a))$ in Γ, we define a point $A' = (-f(a), -f(a)^2/a)$ in the lower-left quadrant of the plane, and let Γ' denote the set of all such points A'.



Claim. For any point $A \in \Gamma$, the set Γ is contained in finitely many lines through the point A'.

Proof. Let $A = (a, f(a))$. The functional equation (with a and b interchanged) can be rewritten as $b + f(a) \mid af(b) - bf(a)$, so that all but finitely many points in Γ are contained in one of the lines with equation

$$ay - f(a)x = m(x + f(a))$$

for m an integer. Geometrically, these are the lines through $A' = (-f(a), -f(a)^2/a)$ with gradient $\frac{f(a)+m}{a}$. Since Γ is contained, with finitely many exceptions, in the region $0 \leqslant y \leqslant f(1) \cdot x$ and the point A' lies strictly in the lower-left quadrant of the plane, there are only finitely many values of m for which this line meets Γ. This concludes the proof of the claim. $\square$

Now consider any distinct points $A, B \in \Gamma$. It is clear that A' and B' are distinct. A line through A' and a line through B' only meet in more than one point if these two lines are equal to the line $A'B'$. It then follows from the above claim that the line $A'B'$ must contain all but finitely many points of Γ. If C is another point of Γ, then the line $A'C'$ also passes through all but finitely many points of Γ, which is only possible if $A'C' = A'B'$.

We have thus seen that there is a line ℓ passing through all points of Γ' and through all but finitely many points of Γ. We claim that this line passes through the origin O and passes through every point of Γ. To see this, note that by construction A, O, A' are collinear for every point $A \in \Gamma$. Since $\ell = AA'$ for all but finitely many points $A \in \Gamma$, it thus follows that $O \in \ell$. Thus any $A \in \Gamma$ lies on the line $\ell = A'O$.

Since Γ is contained in a line through O, it follows that there is a real constant k (the gradient of ℓ) such that $f(a) = ka$ for all a. The number k is, of course, a positive integer.

Comment. Without the $a + b > C$ condition, this problem is approachable by much more naive methods. For instance, using the given divisibility for $a, b \in \{1, 2, 3\}$ one can prove by a somewhat tedious case-check that $f(2) = 2f(1)$ and $f(3) = 3f(1)$; this then forms the basis of an induction establishing that $f(n) = nf(1)$ for all n.

N5. Let a be a positive integer. We say that a positive integer b is *a-good* if $\binom{an}{b}-1$ is divisible by $an+1$ for all positive integers n with $an \geqslant b$. Suppose b is a positive integer such that b is a-good, but $b+2$ is not a-good. Prove that $b+1$ is prime.

(Netherlands)

Solution 1. For p a prime and n a nonzero integer, we write $v_p(n)$ for the *p-adic valuation of n*: the largest integer t such that $p^t \mid n$.

We first show that b is a-good if and only if b is even, and $p \mid a$ for all primes $p \leqslant b$.

To start with, the condition that $an+1 \mid \binom{an}{b}-1$ can be rewritten as saying that

$$\frac{an(an-1)\cdots(an-b+1)}{b!} \equiv 1 \pmod{an+1}. \tag{1}$$

Suppose, on the one hand, there is a prime $p \leqslant b$ with $p \nmid a$. Take $t = v_p(b!)$. Then there exist positive integers c such that $ac \equiv 1 \pmod{p^{t+1}}$. If we take c big enough, and then take $n=(p-1)c$, then $an = a(p-1)c \equiv p-1 \pmod{p^{t+1}}$ and $an \geqslant b$. Since $p \leqslant b$, one of the terms of the numerator $an(an-1)\cdots(an-b+1)$ is $an-p+1$, which is divisible by p^{t+1}. Hence the p-adic valuation of the numerator is at least $t+1$, but that of the denominator is exactly t. This means that $p \mid \binom{an}{b}$, so $p \nmid \binom{an}{b}-1$. As $p \mid an+1$, we get that $an+1 \nmid \binom{an}{b}$, so b is not a-good.

On the other hand, if for all primes $p \leqslant b$ we have $p \mid a$, then every factor of $b!$ is coprime to $an+1$, and hence invertible modulo $an+1$: hence $b!$ is also invertible modulo $an+1$. Then equation (1) reduces to:

$$an(an-1)\cdots(an-b+1) \equiv b! \pmod{an+1}.$$

However, we can rewrite the left-hand side as follows:

$$an(an-1)\cdots(an-b+1) \equiv (-1)(-2)\cdots(-b) \equiv (-1)^b b! \pmod{an+1}.$$

Provided that $an>1$, if b is even we deduce $(-1)^b b! \equiv b!$ as needed. On the other hand, if b is odd, and we take $an+1 > 2(b!)$, then we will not have $(-1)^b b! \equiv b!$, so b is not a-good. This completes the claim.

To conclude from here, suppose that b is a-good, but $b+2$ is not. Then b is even, and $p \mid a$ for all primes $p \leqslant b$, but there is a prime $q \leqslant b+2$ for which $q \nmid a$: so $q=b+1$ or $q=b+2$. We cannot have $q=b+2$, as that is even too, so we have $q=b+1$: in other words, $b+1$ is prime.

Solution 2. We show only half of the claim of the previous solution: we show that if b is a-good, then $p \mid a$ for all primes $p \leqslant b$. We do this with Lucas' theorem.

Suppose that we have $p \leqslant b$ with $p \nmid a$. Then consider the expansion of b in base p; there will be some digit (not the final digit) which is nonzero, because $p \leqslant b$. Suppose it is the p^t digit for $t \geqslant 1$.

Now, as n varies over the integers, $an+1$ runs over all residue classes modulo p^{t+1}; in particular, there is a choice of n (with $an>b$) such that the p^0 digit of an is $p-1$ (so $p \mid an+1$) and the p^t digit of an is 0. Consequently, $p \mid an+1$ but $p \mid \binom{an}{b}$ (by Lucas' theorem) so $p \nmid \binom{an}{b}-1$. Thus b is not a-good.

Now we show directly that if b is a-good but $b+2$ fails to be so, then there must be a prime dividing $an+1$ for some n, which also divides $(b+1)(b+2)$. Indeed, the ratio between $\binom{an}{b+2}$ and $\binom{an}{b}$ is $(b+1)(b+2)/(an-b)(an-b-1)$. We know that there must be a choice of $an+1$ such that the former binomial coefficient is 1 modulo $an+1$ but the latter is not, which means that the given ratio must not be 1 mod $an+1$. If $b+1$ and $b+2$ are both coprime to $an+1$ then the ratio *is* 1, so that must not be the case. In particular, as any prime less than b divides a, it must be the case that either $b+1$ or $b+2$ is prime.

However, we can observe that b must be even by insisting that $an+1$ is prime (which is possible by Dirichlet's theorem) and hence $\binom{an}{b} \equiv (-1)^b = 1$. Thus $b+2$ cannot be prime, so $b+1$ must be prime.

N6. Let $H = \left\{ \lfloor i\sqrt{2} \rfloor : i \in \mathbb{Z}_{>0} \right\} = \{1, 2, 4, 5, 7, \ldots\}$, and let n be a positive integer. Prove that there exists a constant C such that, if $A \subset \{1, 2, \ldots, n\}$ satisfies $|A| \geqslant C\sqrt{n}$, then there exist $a, b \in A$ such that $a - b \in H$. (Here $\mathbb{Z}_{>0}$ is the set of positive integers, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

(Brazil)

Common remarks. In all solutions, we will assume that A is a set such that $\{a - b : a, b \in A\}$ is disjoint from H, and prove that $|A| < C\sqrt{n}$.

Solution 1. First, observe that if n is a positive integer, then $n \in H$ exactly when

$$\left\{ \frac{n}{\sqrt{2}} \right\} > 1 - \frac{1}{\sqrt{2}}. \tag{1}$$

To see why, observe that $n \in H$ if and only if $0 < i\sqrt{2} - n < 1$ for some $i \in \mathbb{Z}_{>0}$. In other words, $0 < i - n/\sqrt{2} < 1/\sqrt{2}$, which is equivalent to (1).

Now, write $A = \{a_1 < a_2 < \cdots < a_k\}$, where $k = |A|$. Observe that the set of differences is not altered by shifting A, so we may assume that $A \subseteq \{0, 1, \ldots, n-1\}$ with $a_1 = 0$.

From (1), we learn that $\{a_i/\sqrt{2}\} < 1 - 1/\sqrt{2}$ for each $i > 1$ since $a_i - a_1 \notin H$. Furthermore, we must have $\{a_i/\sqrt{2}\} < \{a_j/\sqrt{2}\}$ whenever $i < j$; otherwise, we would have

$$-\left(1 - \frac{1}{\sqrt{2}}\right) < \left\{ \frac{a_j}{\sqrt{2}} \right\} - \left\{ \frac{a_i}{\sqrt{2}} \right\} < 0.$$

Since $\{(a_j - a_i)/\sqrt{2}\} = \{a_j/\sqrt{2}\} - \{a_i/\sqrt{2}\} + 1$, this implies that $\{(a_j - a_i)/\sqrt{2}\} > 1/\sqrt{2} > 1 - 1/\sqrt{2}$, contradicting (1).

Now, we have a sequence $0 = a_1 < a_2 < \cdots < a_k < n$, with

$$0 = \left\{ \frac{a_1}{\sqrt{2}} \right\} < \left\{ \frac{a_2}{\sqrt{2}} \right\} < \cdots < \left\{ \frac{a_k}{\sqrt{2}} \right\} < 1 - \frac{1}{\sqrt{2}}.$$

We use the following fact: for any $d \in \mathbb{Z}$, we have

$$\left\{ \frac{d}{\sqrt{2}} \right\} > \frac{1}{2d\sqrt{2}}. \tag{2}$$

To see why this is the case, let $h = \lfloor d/\sqrt{2} \rfloor$, so $\{d/\sqrt{2}\} = d/\sqrt{2} - h$. Then

$$\left\{ \frac{d}{\sqrt{2}} \right\} \left(\frac{d}{\sqrt{2}} + h \right) = \frac{d^2 - 2h^2}{2} \geqslant \frac{1}{2},$$

since the numerator is a positive integer. Because $d/\sqrt{2} + h < 2d/\sqrt{2}$, inequality (2) follows.

Let $d_i = a_{i+1} - a_i$, for $1 \leqslant i < k$. Then $\{a_{i+1}/\sqrt{2}\} - \{a_i/\sqrt{2}\} = \{d_i/\sqrt{2}\}$, and we have

$$1 - \frac{1}{\sqrt{2}} > \sum_i \left\{ \frac{d_i}{\sqrt{2}} \right\} > \frac{1}{2\sqrt{2}} \sum_i \frac{1}{d_i} \geqslant \frac{(k-1)^2}{2\sqrt{2}} \frac{1}{\sum_i d_i} > \frac{(k-1)^2}{2\sqrt{2}} \cdot \frac{1}{n}. \tag{3}$$

Here, the first inequality holds because $\{a_k/\sqrt{2}\} < 1 - 1/\sqrt{2}$, the second follows from (2), the third follows from an easy application of the AM–HM inequality (or Cauchy–Schwarz), and the fourth follows from the fact that $\sum_i d_i = a_k < n$.

Rearranging this, we obtain

$$\sqrt{2\sqrt{2} - 2} \cdot \sqrt{n} > k - 1,$$

which provides the required bound on k.

Solution 2. Let $\alpha = 2 + \sqrt{2}$, so $(1/\alpha) + (1/\sqrt{2}) = 1$. Thus, $J = \{\lfloor i\alpha \rfloor : i \in \mathbb{Z}_{>0}\}$ is the complementary Beatty sequence to H (in other words, H and J are disjoint with $H \cup J = \mathbb{Z}_{>0}$). Write $A = \{a_1 < a_2 < \cdots < a_k\}$. Suppose that A has no differences in H, so all its differences are in J and we can set $a_i - a_1 = \lfloor \alpha b_i \rfloor$ for $b_i \in \mathbb{Z}_{>0}$.

For any $j > i$, we have $a_j - a_i = \lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor$. Because $a_j - a_i \in J$, we also have $a_j - a_i = \lfloor \alpha t \rfloor$ for some positive integer t. Thus, $\lfloor \alpha t \rfloor = \lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor$. The right hand side must equal either $\lfloor \alpha(b_j - b_i) \rfloor$ or $\lfloor \alpha(b_j - b_i) \rfloor - 1$, the latter of which is not a member of J as $\alpha > 2$. Therefore, $t = b_j - b_i$ and so we have $\lfloor \alpha b_j \rfloor - \lfloor \alpha b_i \rfloor = \lfloor \alpha(b_j - b_i) \rfloor$.

For $1 \leqslant i < k$ we now put $d_i = b_{i+1} - b_i$, and we have

$$\left\lfloor \alpha \sum_i d_i \right\rfloor = \lfloor \alpha b_k \rfloor = \sum_i \lfloor \alpha d_i \rfloor;$$

that is, $\sum_i \{\alpha d_i\} < 1$. We also have

$$1 + \left\lfloor \alpha \sum_i d_i \right\rfloor = 1 + a_k - a_1 \leqslant a_k \leqslant n$$

so $\sum_i d_i \leqslant n/\alpha$.

With the above inequalities, an argument similar to (3) (which uses the fact that $\{\alpha d\} = \{d\sqrt{2}\} > 1/(2d\sqrt{2})$ for positive integers d) proves that $1 > \big((k-1)^2/(2\sqrt{2})\big)(\alpha/n)$, which again rearranges to give

$$\sqrt{2\sqrt{2} - 2} \cdot \sqrt{n} > k - 1.$$

Comment. The use of Beatty sequences in Solution 2 is essentially a way to bypass (1). Both Solutions 1 and 2 use the fact that $\sqrt{2} < 2$; the statement in the question would still be true if $\sqrt{2}$ did not have this property (for instance, if it were replaced with α), but any argument along the lines of Solutions 1 or 2 would be more complicated.

Solution 3. Again, define $J = \mathbb{Z}_{>0} \setminus H$, so all differences between elements of A are in J. We start by making the following observation. Suppose we have a set $B \subseteq \{1, 2, \ldots, n\}$ such that all of the differences between elements of B are in H. Then $|A| \cdot |B| \leqslant 2n$.

To see why, observe that any two sums of the form $a + b$ with $a \in A, b \in B$ are different; otherwise, we would have $a_1 + b_1 = a_2 + b_2$, and so $|a_1 - a_2| = |b_2 - b_1|$. However, then the left hand side is in J whereas the right hand side is in H. Thus, $\{a + b : a \in A, b \in B\}$ is a set of size $|A| \cdot |B|$ all of whose elements are no greater than $2n$, yielding the claimed inequality.

With this in mind, it suffices to construct a set B, all of whose differences are in H and whose size is at least $C'\sqrt{n}$ for some constant $C' > 0$.

To do so, we will use well-known facts about the negative Pell equation $X^2 - 2Y^2 = -1$; in particular, that there are infinitely many solutions and the values of X are given by the recurrence $X_1 = 1, X_2 = 7$ and $X_m = 6X_{m-1} - X_{m-2}$. Therefore, we may choose X to be a solution with $\sqrt{n}/6 < X \leqslant \sqrt{n}$.

Now, we claim that we may choose $B = \{X, 2X, \ldots, \lfloor (1/3)\sqrt{n} \rfloor X\}$. Indeed, we have

$$\left(\frac{X}{\sqrt{2}} - Y\right)\left(\frac{X}{\sqrt{2}} + Y\right) = \frac{-1}{2}$$

and so

$$0 > \left(\frac{X}{\sqrt{2}} - Y\right) \geqslant \frac{-3}{\sqrt{2n}},$$

from which it follows that $\{X/\sqrt{2}\} > 1 - (3/\sqrt{2n})$. Combined with (1), this shows that all differences between elements of B are in H.

Comment. Some of the ideas behind Solution 3 may be used to prove that the constant $C = \sqrt{2\sqrt{2}-2}$ (from Solutions 1 and 2) is optimal, in the sense that there are arbitrarily large values of n and sets $A_n \subseteq \{1, 2, \ldots, n\}$ of size roughly $C\sqrt{n}$, all of whose differences are contained in J.

To see why, choose X to come from a sufficiently large solution to the Pell equation $X^2 - 2Y^2 = 1$, so $\{X/\sqrt{2}\} \approx 1/(2X\sqrt{2})$. In particular, $\{X\}, \{2X\}, \ldots, \{\lfloor 2X\sqrt{2}(1 - 1/\sqrt{2})\rfloor X\}$ are all less than $1 - 1/\sqrt{2}$. Thus, by (1) any positive integer of the form iX for $1 \leqslant i \leqslant \lfloor 2X\sqrt{2}(1 - 1/\sqrt{2})\rfloor$ lies in J.

Set $n \approx 2X^2\sqrt{2}(1 - 1/\sqrt{2})$. We now have a set $A = \{iX : i \leqslant \lfloor 2X\sqrt{2}(1 - 1/\sqrt{2})\rfloor\}$ containing roughly $2X\sqrt{2}(1 - 1/\sqrt{2})$ elements less than or equal to n such that all of the differences lie in J, and we can see that $|A| \approx C\sqrt{n}$ with $C = \sqrt{2\sqrt{2} - 2}$.

Solution 4. As in Solution 3, we will provide a construction of a large set $B \subseteq \{1, 2, \ldots, n\}$, all of whose differences are in H.

Choose Y to be a solution to the Pell-like equation $X^2 - 2Y^2 = \pm 1$; such solutions are given by the recurrence $Y_1 = 1, Y_2 = 2$ and $Y_m = 2Y_{m-1} + Y_{m-2}$, and so we can choose Y such that $n/(3\sqrt{2}) < Y \leqslant n/\sqrt{2}$. Furthermore, it is known that for such a Y and for $1 \leqslant x < Y$,

$$\{x\sqrt{2}\} + \{(Y - x)\sqrt{2}\} = \{Y/\sqrt{2}\} \tag{4}$$

if $X^2 - 2Y^2 = 1$, and

$$\{x\sqrt{2}\} + \{(Y - x)\sqrt{2}\} = 1 + \{Y/\sqrt{2}\} \tag{5}$$

if $X^2 - 2Y^2 = -1$. Indeed, this is a statement of the fact that X/Y is a best rational approximation to $\sqrt{2}$, from below in the first case and from above in the second.

Now, consider the sequence $\{\sqrt{2}\}, \{2\sqrt{2}\}, \ldots, \{(Y - 1)\sqrt{2}\}$. The Erdős–Szekeres theorem tells us that this sequence has a monotone subsequence with at least $\sqrt{Y - 2} + 1 > \sqrt{Y}$ elements; if that subsequence is decreasing, we may reflect (using (4) or (5)) to ensure that it is increasing. Call the subsequence $\{y_1\sqrt{2}\}, \{y_2\sqrt{2}\}, \ldots, \{y_t\sqrt{2}\}$ for $t > \sqrt{Y}$.

Now, set $B = \{\lfloor y_i\sqrt{2}\rfloor : 1 \leqslant i \leqslant t\}$. We have $\lfloor y_j\sqrt{2}\rfloor - \lfloor y_i\sqrt{2}\rfloor = \lfloor (y_j - y_i)\sqrt{2}\rfloor$ for $i < j$ (because the corresponding inequality for the fractional parts holds by the ordering assumption on the $\{y_i\sqrt{2}\}$), which means that all differences between elements of B are indeed in H. Since $|B| > \sqrt{Y} > \sqrt{n}/\sqrt{3\sqrt{2}}$, this is the required set.

Comment. Any solution to this problem will need to use the fact that $\sqrt{2}$ cannot be approximated well by rationals, either directly or implicitly (for example, by using facts about solutions to Pell-like equations). If $\sqrt{2}$ were replaced by a value of θ with very good rational approximations (from below), then an argument along the lines of Solution 3 would give long arithmetic progressions in $\{\lfloor i\theta\rfloor : 0 \leqslant i < n\}$ (with initial term 0) for certain values of n.

N7. Prove that there is a constant $c > 0$ and infinitely many positive integers n with the following property: there are infinitely many positive integers that cannot be expressed as the sum of fewer than $cn\log(n)$ pairwise coprime n^{th} powers.

(Canada)

Solution 1. Suppose, for an integer n, that we can find another integer N satisfying the following property:

$$n \text{ is divisible by } \varphi(p^e) \text{ for every prime power } p^e \text{ exactly dividing } N. \qquad (\dagger)$$

This property ensures that all n^{th} powers are congruent to 0 or 1 modulo each such prime power p^e, and hence that any sum of m pairwise coprime n^{th} powers is congruent to m or $m-1$ modulo p^e, since at most one of the n^{th} powers is divisible by p. Thus, if k denotes the number of distinct prime factors of N, we find by the Chinese Remainder Theorem at most $2^k m$ residue classes modulo N which are sums of at most m pairwise coprime n^{th} powers. In particular, if $N > 2^k m$ then there are infinitely many positive integers not expressible as a sum of at most m pairwise coprime n^{th} powers.

It thus suffices to prove that there are arbitrarily large pairs (n, N) of integers satisfying ($\dagger$) such that

$$N > c \cdot 2^k n \log(n)$$

for some positive constant c.

We construct such pairs as follows. Fix a positive integer t and choose (distinct) prime numbers $p \mid 2^{2^{t-1}} + 1$ and $q \mid 2^{2^t} + 1$; we set $N = pq$. It is well-known that $2^t \mid p - 1$ and $2^{t+1} \mid q - 1$, hence

$$n = \frac{(p-1)(q-1)}{2^t}$$

is an integer and the pair (n, N) satisfies ($\dagger$).

Estimating the size of N and n is now straightforward. We have

$$\log_2(n) \leqslant 2^{t-1} + 2^t - t < 2^{t+1} < 2 \cdot \frac{N}{n},$$

which rearranges to

$$N > \frac{1}{8} \cdot 2^2 n \log_2(n)$$

and so we are done if we choose $c < \frac{1}{8\log(2)} \approx 0.18$.

Comment 1. The trick in the above solution was to find prime numbers p and q congruent to 1 modulo some $d = 2^t$ and which are not too large. An alternative way to do this is via Linnik's Theorem, which says that there are absolute constants b and $L > 1$ such that for any coprime integers a and d, there is a prime congruent to a modulo d and of size $\leqslant bd^L$. If we choose some d not divisible by 3 and choose two distinct primes $p, q \leqslant b \cdot (3d)^L$ congruent to 1 modulo d (and, say, distinct modulo 3), then we obtain a pair (n, N) satisfying ($\dagger$) with $N = pq$ and $n = \frac{(p-1)(q-1)}{d}$. A straightforward computation shows that

$$N > Cn^{1+\frac{1}{2L-1}}$$

for some constant C, which is in particular larger than any $c \cdot 2^2 n \log(n)$ for p large. Thus, the statement of the problem is true for any constant c. More strongly, the statement of the problem is still true when $cn\log(n)$ is replaced by $n^{1+\delta}$ for a sufficiently small $\delta > 0$.

Solution 2, obtaining better bounds. As in the preceding solution, we seek arbitrarily large pairs of integers n and N satisfying (†) such that $N > c2^k n\log(n)$.

This time, to construct such pairs, we fix an integer $x \geqslant 4$, set N to be the lowest common multiple of $1, 2, \ldots, 2x$, and set n to be twice the lowest common multiple of $1, 2, \ldots, x$. The pair (n, N) does indeed satisfy the condition, since if p^e is a prime power divisor of N then $\frac{\varphi(p^e)}{2} \leqslant x$ is a factor of $\frac{n}{2} = \mathrm{lcm}_{r\leqslant x}(r)$.

Now $2N/n$ is the product of all primes having a power lying in the interval $(x, 2x]$, and hence $2N/n > x^{\pi(2x)-\pi(x)}$. Thus for sufficiently large x we have

$$\log\left(\frac{2N}{2^{\pi(2x)}n}\right) > (\pi(2x) - \pi(x))\log(x) - \log(2)\pi(2x) \sim x,$$

using the Prime Number Theorem $\pi(t) \sim t/\log(t)$.

On the other hand, n is a product of at most $\pi(x)$ prime powers less than or equal to x, and so we have the upper bound

$$\log(n) \leqslant \pi(x)\log(x) \sim x,$$

again by the Prime Number Theorem. Combined with the above inequality, we find that for any $\epsilon > 0$, the inequality

$$\log\left(\frac{N}{2^{\pi(2x)}n}\right) > (1-\epsilon)\log(n)$$

holds for sufficiently large x. Rearranging this shows that

$$N > 2^{\pi(2x)}n^{2-\epsilon} > 2^{\pi(2x)}n\log(n)$$

for all sufficiently large x and we are done.

Comment 2. The stronger bound $N > 2^{\pi(2x)}n^{2-\epsilon}$ obtained in the above proof of course shows that infinitely many positive integers cannot be written as a sum of at most $n^{2-\epsilon}$ pairwise coprime n^{th} powers.

By refining the method in Solution 2, these bounds can be improved further to show that infinitely many positive integers cannot be written as a sum of at most n^α pairwise coprime n^{th} powers for any positive $\alpha > 0$. To do this, one fixes a positive integer d, sets N equal to the product of the primes at most dx which are congruent to 1 modulo d, and $n = d\,\mathrm{lcm}_{r\leqslant x}(r)$. It follows as in Solution 2 that (n, N) satisfies (†).

Now the Prime Number Theorem in arithmetic progressions provides the estimates $\log(N) \sim \frac{d}{\varphi(d)}x$, $\log(n) \sim x$ and $\pi(dx) \sim \frac{dx}{\log(x)}$ for any fixed d. Combining these provides a bound

$$N > 2^{\pi(dx)}n^{d/\varphi(d)-\epsilon}$$

for any positive ϵ, valid for x sufficiently large. Since the ratio $\frac{d}{\varphi(d)}$ can be made arbitrarily large by a judicious choice of d, we obtain the n^α bound claimed.

Comment 3. While big results from analytic number theory such as the Prime Number Theorem or Linnik's Theorem certainly can be used in this problem, they do not seem to substantially simplify matters: all known solutions involve first reducing to condition (†), and even then analytic results do not make it clear how to proceed. For this reason, we regard this problem as suitable for the IMO.

Rather than simplifying the problem, what nonelementary results from analytic number theory allow one to achieve is a strengthening of the main bound, typically replacing the $n\log(n)$ growth with a power $n^{1+\delta}$. However, we believe that such stronger bounds are unlikely to be found by students in the exam.

The strongest bound we know how to achieve using purely elementary methods is a bound of the form $N > 2^k n\log(n)^M$ for any positive integer M. This is achieved by a variant of the argument in Solution 1, choosing primes $p_0, \ldots, p_M$ with $p_i \mid 2^{2^{t+i-1}} + 1$ and setting $N = \prod_i p_i$ and $n = 2^{-tM}\prod_i(p_i - 1)$.

N8. Let a and b be two positive integers. Prove that the integer

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil$$

is not a square. (Here $\lceil z \rceil$ denotes the least integer greater than or equal to z.)

(Russia)

Solution 1. Arguing indirectly, assume that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil = (a+k)^2, \quad \text{or} \quad \left\lceil \frac{(2a)^2}{b} \right\rceil = (2a+k)k.$$

Clearly, $k \geqslant 1$. In other words, the equation

$$\left\lceil \frac{c^2}{b} \right\rceil = (c+k)k \tag{1}$$

has a positive integer solution (c, k), with an even value of c.

Choose a positive integer solution of (1) with minimal possible value of k, without regard to the parity of c. From

$$\frac{c^2}{b} > \left\lceil \frac{c^2}{b} \right\rceil - 1 = ck + k^2 - 1 \geqslant ck$$

and

$$\frac{(c-k)(c+k)}{b} < \frac{c^2}{b} \leqslant \left\lceil \frac{c^2}{b} \right\rceil = (c+k)k$$

it can be seen that $c > bk > c - k$, so

$$c = kb + r \quad \text{with some } 0 < r < k.$$

By substituting this in (1) we get

$$\left\lceil \frac{c^2}{b} \right\rceil = \left\lceil \frac{(bk+r)^2}{b} \right\rceil = k^2 b + 2kr + \left\lceil \frac{r^2}{b} \right\rceil$$

and

$$(c+k)k = (kb + r + k)k = k^2 b + 2kr + k(k-r),$$

so

$$\left\lceil \frac{r^2}{b} \right\rceil = k(k-r). \tag{2}$$

Notice that relation (2) provides another positive integer solution of (1), namely $c' = r$ and $k' = k - r$, with $c' > 0$ and $0 < k' < k$. That contradicts the minimality of k, and hence finishes the solution.

Solution 2. Suppose that

$$a^2 + \left\lceil \frac{4a^2}{b} \right\rceil = c^2$$

with some positive integer $c > a$, so

$$\begin{aligned} c^2 - 1 < a^2 + \frac{4a^2}{b} &\leqslant c^2, \\ 0 \leqslant c^2 b - a^2(b+4) &< b. \end{aligned} \tag{3}$$

Let $d = c^2 b - a^2(b+4)$, $x = c + a$ and $y = c - a$; then we have $c = \frac{x+y}{2}$ and $a = \frac{x-y}{2}$, and (3) can be re-written as follows:

$$\begin{gathered} \left(\frac{x+y}{2}\right)^2 b - \left(\frac{x-y}{2}\right)^2 (b+4) = d, \\ x^2 - (b+2)xy + y^2 + d = 0; \qquad 0 \leqslant d < b. \end{gathered} \tag{4}$$

So, by the indirect assumption, the equation (4) has some positive integer solution (x, y).

Fix b and d, and take a pair (x, y) of positive integers, satisfying (4), such that $x + y$ is minimal. By the symmetry in (4) we may assume that $x \geqslant y \geqslant 1$.

Now we perform a usual "Vieta jump". Consider (4) as a quadratic equation in variable x, and let z be its second root. By the Vieta formulas,

$$x + z = (b+2)y, \quad \text{and} \quad zx = y^2 + d,$$

so

$$z = (b+2)y - x = \frac{y^2 + d}{x}.$$

The first formula shows that z is an integer, and by the second formula z is positive. Hence (z, y) is another positive integer solution of (4). From

$$\begin{aligned} (x-1)(z-1) &= xz - (x+z) + 1 = (y^2 + d) - (b+2)y + 1 \\ &< (y^2 + b) - (b+2)y + 1 = (y-1)^2 - b(y-1) \leqslant (y-1)^2 \leqslant (x-1)^2 \end{aligned}$$

we can see that $z < x$ and therefore $z + y < x + y$. But this contradicts the minimality of $x + y$ among the positive integer solutions of (4).

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