

# **IMO from 1959 to 2003**

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**44th IMO 2003**



**A1.**  $S$  is the set  $\{1, 2, 3, \dots, 1000000\}$ . Show that for any subset  $A$  of  $S$  with 101 elements we can find 100 distinct elements  $x_i$  of  $S$ , such that the sets  $x_i + A$  are all pairwise disjoint. [Note that  $x_i + A$  is the set  $\{a + x_i \mid a \text{ is in } A\}$  ].



**A2.** Find all pairs  $(m, n)$  of positive integers such that  $m^2/(2mn^2 - n^3 + 1)$  is a positive integer.



**A3.** A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is  $(\sqrt{3})/2$  times the sum of their lengths. Show that all the hexagon's angles are equal.



**B1.**  $ABCD$  is cyclic. The feet of the perpendicular from  $D$  to the lines  $AB, BC, CA$  are  $P, Q, R$  respectively. Show that the angle bisectors of  $ABC$  and  $CDA$  meet on the line  $AC$  iff  $RP = RQ$ .



**B2.** Given  $n > 2$  and reals  $x_1 \leq x_2 \leq \dots \leq x_n$ , show that  $(\sum_{i,j} |x_i - x_j|)^2 \leq (2/3) (n^2 - 1) \sum_{i,j} (x_i - x_j)^2$ . Show that we have equality iff the sequence is an arithmetic progression.



**B3.** Show that for each prime  $p$ , there exists a prime  $q$  such that  $n^p - p$  is not divisible by  $q$  for any positive integer  $n$ .



### Problem A1

**S** is the set  $\{1, 2, 3, \dots, 1000000\}$ . Show that for any subset **A** of **S** with 101 elements we can find 100 distinct elements  $x_i$  of **S**, such that the sets  $x_i + A$  are all pairwise disjoint. [Note that  $x_i + A$  is the set  $\{a + x_i \mid a \text{ is in } A\}$  ].

### Solution

*Thanks to Li Yi*

Having found  $x_1, x_2, \dots, x_k$  there are  $k \cdot 101 \cdot 100$  forbidden values for  $x_{k+1}$  of the form  $x_i + a_m - a_n$  with  $m$  and  $n$  unequal and another  $k$  forbidden values with  $m = n$ . Since  $99 \cdot 101 \cdot 100 + 99 = 10^6 - 1$ , we can successively choose 100 distinct  $x_i$ .

*Gerhard Woeginger sent me a similar solution*

### IMO 2003



### Problem A2

Find all pairs  $(m, n)$  of positive integers such that  $m^2/(2mn^2 - n^3 + 1)$  is a positive integer.

**Answer :**  $(m, n) = (2k, 1), (k, 2k)$  or  $(8k^4 - k, 2k)$

**Solution :** *Thanks to Li Yi*

The denominator is  $2mn^2 - n^3 + 1 = n^2(2m - n) + 1$ , so  $2m \geq n > 0$ . If  $n = 1$ , then  $m$  must be even, in other words, we have the solution  $(m, n) = (2k, 1)$ .

So assume  $n > 1$ . Put  $h = m^2/(2mn^2 - n^3 + 1)$ . Then we have a quadratic equation for  $m$ , namely  $m^2 - 2hn^2m + (n^3 - 1)h = 0$ . This has solutions  $hn^2 \pm N$ , where  $N$  is the positive square root of  $h^2n^4 - hn^3 + h$ . Since  $n > 1, h \geq 1, N$  is certainly real. But the sum and product of the roots are both positive, so both roots must be positive. The sum is an integer, so if one root is a positive integer, then so is the other.

The larger root  $hn^2 + N$  is greater than  $hn^2$ , so the smaller root  $< h(n^3 - 1)/(hn^2) < n$ . But note that if  $2m - n > 0$ , then since  $h > 0$ , we must have the denominator  $(2m - n)n^2 + 1$  smaller than the numerator and hence  $m > n$ . So for the smaller root we cannot have  $2m - n > 0$ . But  $2m - n$  must be non-negative (since  $h$  is positive), so  $2m - n = 0$  for the smaller root. Hence  $hn^2 - N = n/2$ . Now  $N^2 = (hn^2 - n/2)^2 = h^2n^4 - hn^3 + h$ , so  $h = n^2/4$ . Thus  $n$  must be even. Put  $n = 2k$  and we get the solutions  $(m, n) = (k, 2k)$  and  $(8k^4 - k, 2k)$ . (*We have shown that any solution must be of one of the three forms given, but it is trivial to check that they are all indeed solutions.*)

### IMO 2003



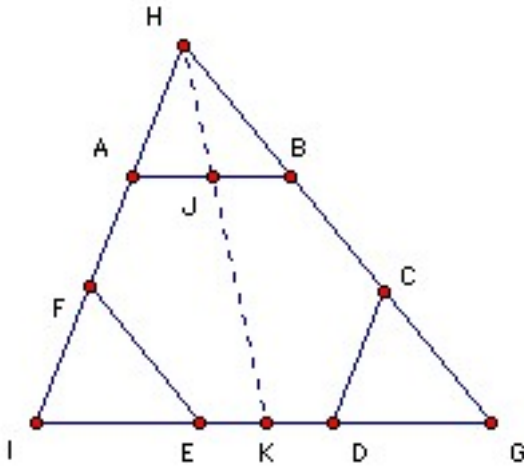
### Problem A3

A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is  $\frac{1}{2}\sqrt{3}$  times the sum of their lengths. Show that all the hexagon's angles are equal.

**Solution** Thanks to Li Yi

We use bold to denote vectors, so  $\mathbf{AB}$  means the vector from A to B. We take some arbitrary origin and write the vector  $\mathbf{OA}$  as  $\mathbf{A}$  for short. Note that the vector to the midpoint of AB is  $(\mathbf{A} + \mathbf{B})/2$ , so the vector from the midpoint of DE to the midpoint of AB is  $(\mathbf{A} + \mathbf{B} - \mathbf{D} - \mathbf{E})/2$ . So the starting point is  $|\mathbf{A} + \mathbf{B} - \mathbf{D} - \mathbf{E}| \geq \sqrt{3} (|\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}|)$  and two similar equations. The key is to notice that by the triangle inequality we have  $|\mathbf{A} - \mathbf{B}| + |\mathbf{D} - \mathbf{E}| \geq |\mathbf{A} - \mathbf{B} - \mathbf{D} + \mathbf{E}|$  with equality iff the opposite sides AB and DE are parallel. Thus we get  $|\mathbf{DA} + \mathbf{EB}| \geq \sqrt{3} |\mathbf{DA} - \mathbf{EB}|$ . Note that DA and EB are diagonals. Squaring, we get  $\mathbf{DA}^2 + 2 \mathbf{DA} \cdot \mathbf{EB} + \mathbf{EB}^2 \geq 3(\mathbf{DA}^2 - 2 \mathbf{DA} \cdot \mathbf{EB} + \mathbf{EB}^2)$ , or  $\mathbf{DA}^2 + \mathbf{EB}^2 \leq 4 \mathbf{DA} \cdot \mathbf{EB}$ . Similarly, we get  $\mathbf{EB}^2 + \mathbf{FC}^2 \leq 4 \mathbf{EB} \cdot \mathbf{FC}$  and  $\mathbf{FC}^2 + \mathbf{AD}^2 \leq 4 \mathbf{FC} \cdot \mathbf{AD} = -4 \mathbf{FC} \cdot \mathbf{DA}$ . Adding the three equations gives  $2(\mathbf{DA} - \mathbf{EB} + \mathbf{FC})^2 \leq 0$ . So it must be zero, and hence  $\mathbf{DA} - \mathbf{EB} + \mathbf{FC} = 0$  and opposite sides of the hexagon are parallel.

Note that  $\mathbf{DA} - \mathbf{EB} + \mathbf{FC} = \mathbf{A} - \mathbf{D} - \mathbf{B} + \mathbf{E} + \mathbf{C} - \mathbf{F} = \mathbf{BA} + \mathbf{DC} + \mathbf{FE}$ . So  $\mathbf{BA} + \mathbf{DC} + \mathbf{FE} = 0$ . In other words, the three vectors can form a triangle.

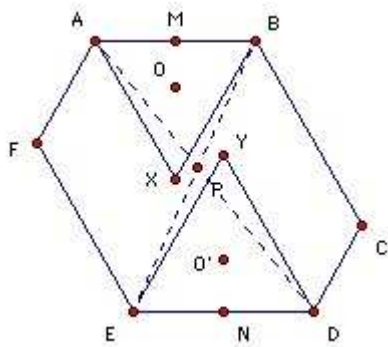


Since EF is parallel to BC, if we translate EF along the vector  $\mathbf{ED}$  we get CG, an extension of BC. Similarly, if we translate AB along the vector  $\mathbf{BC}$  we get an extension of ED. Since  $\mathbf{BA}$ ,  $\mathbf{DC}$  and  $\mathbf{FE}$  form a triangle, AB must translate to DG. Thus HAB and CDG are congruent. Similarly, if we take AF and DE to intersect at I, the triangle FIE is also congruent (and similarly oriented) to HAB and CDG. Take J, K as the midpoints of AB, ED. HIG and HAB are equiangular and hence similar.  $IE = DG$  and K is the midpoint of ED, so K is also the midpoint of IG. Hence HJ is parallel to HK, so H, J, K are collinear.

Hence  $HJ/AB = HK/IG = (HK - HJ)/(IG - AB) = JK/(AB + ED) = \frac{1}{2} \sqrt{3}$ . Similarly, each of the medians of the triangle HAB is  $\frac{1}{2} \sqrt{3}$  times the corresponding side. We will show that this implies it is equilateral. The required result then follows immediately.

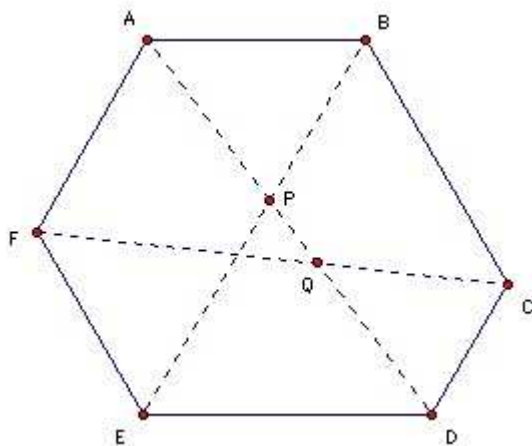
Suppose a triangle has side lengths  $a, b, c$  and the length of the median to the midpoint of side length  $c$  is  $m$ . Then applying the cosine rule twice we get  $m^2 = a^2/2 + b^2/2 - c^2/4$ . So if  $m^2 = \frac{3}{4} c^2$ , it follows that  $a^2 + b^2 = 2c^2$ . Similarly,  $b^2 + c^2 = 2a^2$ . Subtracting,  $a = c$ . Similarly for the other pairs of sides.

*An alternative (and rather more elegant) solution sent my some anonymous contestants at the IMO is as follows*



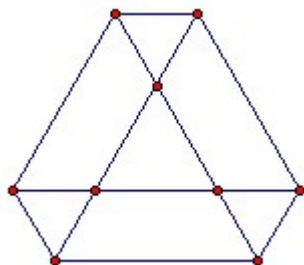
Let the diagonals AD and BE meet at P. We show that angle APB  $\leq 60^\circ$ . Suppose angle APB  $> 60^\circ$ . Take X and Y inside the hexagon so that ABX and DEY are equilateral (as shown). Then since angle APB  $>$  angle AXB, P lies inside the circumcircle of ABX (which we take to have center O, radius  $r$ ). Similarly, it lies inside the circumcircle of DEY (which we take to have center O' radius  $r'$ ), so these circles must meet and hence  $OO' < r + r'$ . Now  $\frac{\sqrt{3}}{2} (AB + DE) = MN$  (where M, N are the midpoints of AB, DE)  $\leq MO + OO' + O'N < r/2 + (r + r') + r'/2 = \frac{3}{2}(r + r') = \frac{\sqrt{3}}{2} (AB + DE)$ . Contradiction.

The same argument applies to any two long diagonals. Hence the angles must all be  $60^\circ$ . Also we must have  $MP \leq MX$  with equality iff  $P = X$ , and similarly  $NP \leq NY$  with equality iff  $P = Y$ . So  $MN \leq MP + PN \leq MX + NY = \frac{\sqrt{3}}{2} (AB + DE) = MN$ . Hence we have equality and so  $P = X = Y$ .



Hence angle  $APB = 60^\circ$ . Suppose  $AD$  and  $CF$  meet at  $Q$ . The same argument shows that angle  $AQF = 60^\circ$ . So the hexagon angle at  $A$  is angle  $APB +$  angle  $AQF = 120^\circ$ . Similarly for the other angles.

Finally, note that the only possible configuration is:



The ratio  $AB/BC$  is arbitrary, but the figure is symmetrical under rotations through  $120^\circ$ . That follows immediately from either of the two solutions above.

### IMO 2003

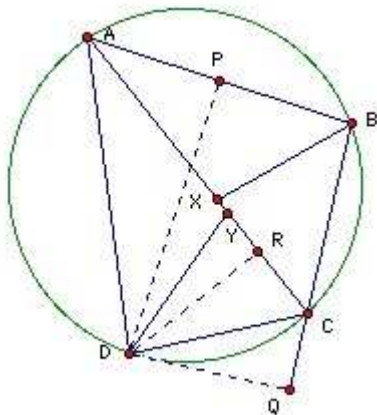


### Problem B1

$ABCD$  is cyclic. The feet of the perpendicular from  $D$  to the lines  $AB, BC, CA$  are  $P, Q, R$  respectively. Show that the angle bisectors of  $ABC$  and  $CDA$  meet on the line  $AC$  iff  $RP = RQ$ .

### Solution

Thanks to Li Yi



APRD is cyclic with diameter AD (because angle APD = angle ARD = 90°. Suppose its center is O and its radius r. Angle PAR = 1/2 angle POR, so PR = 2r sin 1/2POR = AD sin PAR. Similarly, RQ = CD sin RCQ. (Note that it makes no difference if R, P are on the same or opposite sides of the line AD.) But sin PAR = sin BAC, sin RCQ = sin ACB, so applying the sine rule to the triangle ABC, sin RCQ/sin PAR = AB/BC. Thus we have AD/CD = (PR/RQ) (AB/BC). Suppose the angle bisectors of B, D meet AD at X, Y. Then we have AB/BC = AX/CX and AD/CD = AY/CY. Hence (AY/CY)/(AX/CX) = PR/RQ. So PR = RQ iff X = Y, which is the required result.

*Note that ABCD does not need to be cyclic! Exercise: does it need to be convex?*

## IMO 2003



### Problem B2

**Given  $n > 2$  and reals  $x_1 \leq x_2 \leq \dots \leq x_n$ , show that  $(\sum_{i,j} |x_i - x_j|)^2 \leq (2/3)(n^2 - 1) \sum_{i,j} (x_i - x_j)^2$ . Show that we have equality iff the sequence is an arithmetic progression.**

**Solution** *Thanks to Li Yi*

Notice first that if we restrict the sums to  $i < j$ , then they are halved. The lhs sum is squared and the rhs sum is not, so the the desired inequality with sums restricted to  $i < j$  has (1/3) on the rhs instead of (2/3).

Consider the sum of all  $|x_i - x_j|$  with  $i < j$ .  $x_1$  occurs in  $(n-1)$  terms with a negative sign.  $x_2$  occurs in one term with a positive sign and  $(n-2)$  terms with a negative sign, and so on. So we get  $-(n-1)x_1 - (n-3)x_2 - (n-5)x_3 - \dots + (n-1)x_n = \sum (2i-1-n)x_i$ .

We can now apply Cauchy-Schwartz. The square of this sum is just  $\sum x_i^2 \sum (2i-1-n)^2$ .

Looking at the other side of the desired inequality, we see immediately that it is  $n \sum x_i^2 - (\sum x_i)^2$ . We would like to get rid of the second term, but that is easy because if we add  $h$  to every  $x_i$  the sums in the desired inequality are unaffected (since they use only differences of  $x_i$ ), so we can choose  $h$  so that  $\sum x_i$  is zero. Thus we are home if we can show that  $\sum (2i-1-n)^2 \leq n(n^2 - 1)/3$ . That is easy: lhs =  $4 \sum i^2 - 4(n+1) \sum i + n(n+1)^2 = (2/3)n(n+1)(2n+1) - 2n(n+1) + n(n+1)^2 = (1/3)n(n+1)(2(2n+1) - 6 + 3(n+1)) = (1/3)n(n^2 - 1) =$  rhs. That establishes the required inequality.

We have equality iff we have equality at the Cauchy-Schwartz step and hence iff  $x_i$  is proportional to  $(2i-1-n)$ . That implies that  $x_{i+1} - x_i$  is constant. So equality implies that the sequence is an AP. But if the sequence is an AP with difference  $d$  (so  $x_{i+1} = x_i + d$ ) and we take  $x_1 = -(d/2)(n-1)$ , then we get  $x_i = (d/2)(2i-1-n)$  and  $\sum x_i = 0$ , so we have equality.

**Problem B3**

**Show that for each prime  $p$ , there exists a prime  $q$  such that  $n^p - p$  is not divisible by  $q$  for any positive integer  $n$ .**

**Solution** *Thanks to Li Yi*

If  $p = 2$ , then we can take  $q = 3$ , since squares cannot be  $2 \pmod 3$ . So suppose  $p$  is odd.

Consider  $N = 1 + p + p^2 + \dots + p^{p-1}$ . There are  $p$  terms. Since  $p$  is odd, that means an odd number of odd terms, so  $N$  is odd. Also  $N = p + 1 \pmod{p^2}$ , which is not  $1 \pmod{p^2}$ , so  $N$  must have a prime factor  $q$  which is not  $1 \pmod{p^2}$ . We will show that  $q$  has the required property.

Since  $N = 1 \pmod p$ ,  $p$  does not divide  $N$ , so  $q$  cannot be  $p$ . If  $p = 1 \pmod q$ , then  $N = 1 + 1 + \dots + 1 = p \pmod q$ . Since  $N = 0 \pmod q$ , that implies  $q$  divides  $p$ . Contradiction. So  $q$  does not divide  $p - 1$ .

Now suppose  $n^p = p \pmod q$  (\*). We have just shown that  $n$  cannot be  $1 \pmod q$ . We have also shown that  $q$  is not  $p$ , so  $n$  cannot be a multiple of  $q$ . So assume  $n$  is not  $0$  or  $1 \pmod q$ . Take the  $p$ th power of both sides of (\*). Since  $(p - 1)N = p^p - 1$ , we have  $p^p = 1 \pmod q$ . So  $n$  to the power of  $p^2$  is  $1 \pmod q$ . But  $n^{q-1} = 1 \pmod q$  (the well-known Fermat little theorem). So if  $d = \gcd(q-1, p^2)$ , then  $n^d = 1 \pmod q$ . We chose  $q$  so that  $q-1$  is not divisible by  $p^2$ , so  $d$  must be  $1$  or  $p$ . But we are assuming  $n$  is not  $1 \pmod q$ , so  $d$  cannot be  $1$ . So it must be  $p$ . In other words,  $n^p = 1 \pmod q$ . But  $n^p = p \pmod q$ , so  $p = 1 \pmod q$ . Contradiction (we showed above that  $q$  does not divide  $p - 1$ ).

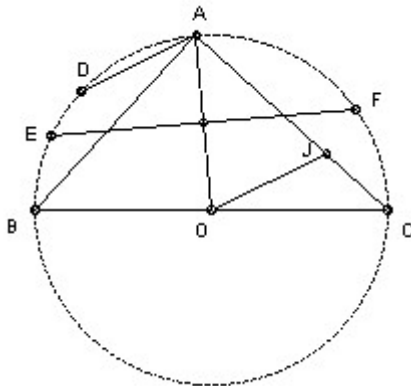




A1.  $S$  is the set of all  $(h, k)$  with  $h, k$  non-negative integers such that  $h + k < n$ . Each element of  $S$  is colored red or blue, so that if  $(h, k)$  is red and  $h' \leq h, k' \leq k$ , then  $(h', k')$  is also red. A type 1 subset of  $S$  has  $n$  blue elements with different first member and a type 2 subset of  $S$  has  $n$  blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.



A2.  $BC$  is a diameter of a circle center  $O$ .  $A$  is any point on the circle with angle  $AOC > 60^\circ$ .  $EF$  is the chord which is the perpendicular bisector of  $AO$ .  $D$  is the midpoint of the minor arc  $AB$ . The line through  $O$  parallel to  $AD$  meets  $AC$  at  $J$ . Show that  $J$  is the incenter of triangle  $CEF$ .



A3. Find all pairs of integers  $m > 2, n > 2$  such that there are infinitely many positive integers  $k$  for which  $(k^n + k^2 - 1)$  divides  $(k^m + k - 1)$ .



B1. The positive divisors of the integer  $n > 1$  are  $d_1 < d_2 < \dots < d_k$ , so that  $d_1 = 1, d_k = n$ . Let  $d = d_1d_2 + d_2d_3 + \dots + d_{k-1}d_k$ . Show that  $d < n^2$  and find all  $n$  for which  $d$  divides  $n^2$ .



B2. Find all real-valued functions  $f$  on the reals such that  $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$  for all  $x, y, u, v$ .



B3.  $n > 2$  circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are  $O_1, O_2, \dots, O_n$ . Show that  $\sum_{i < j} 1/O_iO_j \leq (n-1)\pi/4$ .





### Problem A1

$S$  is the set of all  $(h, k)$  with  $h, k$  non-negative integers such that  $h + k < n$ . Each element of  $S$  is colored red or blue, so that if  $(h, k)$  is red and  $h' \leq h, k' \leq k$ , then  $(h', k')$  is also red. A type 1 subset of  $S$  has  $n$  blue elements with different first member and a type 2 subset of  $S$  has  $n$  blue elements with different second member. Show that there are the same number of type 1 and type 2 subsets.

### Solution

Let  $a_i$  be the number of blue members  $(h, k)$  in  $S$  with  $h = i$ , and let  $b_i$  be the number of blue members  $(h, k)$  with  $k = i$ . It is sufficient to show that  $b_0, b_1, \dots, b_{n-1}$  is a rearrangement of  $a_0, a_1, \dots, a_{n-1}$  (because the number of type 1 subsets is the product of the  $a_i$  and the number of type 2 subsets is the product of the  $b_i$ ).

Let  $c_i$  be the largest  $k$  such that  $(i, k)$  is red. If  $(i, k)$  is blue for all  $k$  then we put  $c_i = -1$ . Note that if  $i < j$ , then  $c_i \geq c_j$ , since if  $(j, c_i)$  is red, then so is  $(i, c_i)$ . Note also that  $(i, k)$  is red for  $k \leq c_i$ , so the sequence  $c_0, c_1, \dots, c_{n-1}$  completely defines the coloring of  $S$ .

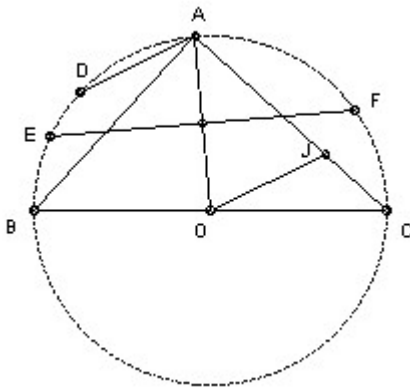
Let  $S_i$  be the set with the sequence  $c_0, c_1, \dots, c_i, -1, \dots, -1$ , so that  $S_{n-1} = S$ . We also take  $S_{-1}$  as the set with the sequence  $-1, -1, \dots, -1$ , so that all its members are blue. We show that the rearrangement result is true for  $S_{-1}$  and that if it is true for  $S_i$  then it is true for  $S_{i+1}$ . It is obvious for  $S_{-1}$ , because both  $a_i$  and  $b_i$  are  $n, n-1, \dots, 2, 1$ . So suppose it is true for  $S_i$  (where  $i < n-1$ ). The only difference between the  $a_j$  for  $S_i$  and for  $S_{i+1}$  is that  $a_{i+1} = n-i-1$  for  $S_i$  and  $(n-i-1)-(c_{i+1}+1)$  for  $S_{i+1}$ . In other words, the number  $n-i-1$  is replaced by the number  $n-i-c-2$ , where  $c = c_{i+1}$ . The difference in the  $b_j$  is that 1 is deducted from each of  $b_0, b_1, \dots, b_c$ . But these numbers are just  $n-i-1, n-i-1, n-i-2, \dots, n-i-c-1$ . So the effect of deducting 1 from each is to replace  $n-i-1$  by  $n-i-c-2$ , which is the same change as was made to the  $a_j$ . So the rearrangement result also holds for  $S_{i+1}$ . Hence it holds for  $S$ .



### Problem A2

**BC is a diameter of a circle center O. A is any point on the circle with angle  $\angle AOC > 60^\circ$ . EF is the chord which is the perpendicular bisector of AO. D is the midpoint of the minor arc AB. The line through O parallel to AD meets AC at J. Show that J is the incenter of triangle CEF.**

### Solution



F is equidistant from A and O. But  $OF = OA$ , so  $\triangle OFA$  is equilateral and hence  $\angle AOF = 60^\circ$ . Since  $\angle AOC > 60^\circ$ , F lies between A and C. Hence the ray CJ lies between CE and CF.

D is the midpoint of the arc AB, so  $\angle DOB = \frac{1}{2} \angle AOB = \angle ACB$ . Hence DO is parallel to AC. But OJ is parallel to AD, so AJOD is a parallelogram. Hence  $AJ = OD$ . So  $AJ = AE = AF$ , so J lies on the opposite side of EF to A and hence on the same side as C. So J must lie inside the triangle CEF.

Also, since EF is the perpendicular bisector of AO, we have  $AE = AF = OE$ , so A is the center of the circle through E, F and J. Hence  $\angle EFJ = \frac{1}{2} \angle EAJ$ . But  $\angle EAJ = \angle EAC$  (same angle) =  $\angle EFC$ . Hence J lies on the bisector of angle EFC.

Since EF is perpendicular to AO, A is the midpoint of the arc EF. Hence  $\angle ACE = \angle ACF$ , so J lies on the bisector of angle ECF. Hence J is the incenter



### Problem A3

Find all pairs of integers  $m > 2$ ,  $n > 2$  such that there are infinitely many positive integers  $k$  for which  $(k^n + k^2 - 1)$  divides  $(k^m + k - 1)$ .

### Solution

Answer:  $m = 5$ ,  $n = 3$ .

Obviously  $m > n$ . Take polynomials  $q(x)$ ,  $r(x)$  with integer coefficients and with degree  $r(x) < n$  such that  $x^m + x - 1 = q(x)(x^n + x^2 - 1) + r(x)$ . Then  $x^n + x^2 - 1$  divides  $r(x)$  for infinitely many positive integers  $x$ . But for sufficiently large  $x$ ,  $x^n + x^2 - 1 > r(x)$  since  $r(x)$  has smaller degree. So  $r(x)$  must be zero. So  $x^m + x - 1$  factorises as  $q(x)(x^n + x^2 - 1)$ , where  $q(x) = x^{m-n} + a_{m-n-1}x^{m-n-1} + \dots + a_0$ .

At this point I use an elegant approach provided by Jean-Pierre Ehrmann

We have  $(x^m + x - 1) = x^{m-n}(x^n + x^2 - 1) + (1 - x)(x^{m-n+1} + x^{m-n} - 1)$ , so  $(x^n + x^2 - 1)$  must divide  $(x^{m-n+1} + x^{m-n} - 1)$ . So, in particular,  $m \geq 2n-1$ . Also  $(x^n + x^2 - 1)$  must divide  $(x^{m-n+1} + x^{m-n} - 1) - x^{m-2n+1}(x^n + x^2 - 1) = x^{m-n} - x^{m-2n+3} + x^{m-2n+1} - 1$  (\*).

At this point there are several ways to go. The neatest is Bill Dubuque's:

(\*) can be written as  $x^{m-2n+3}(x^{n-3} - 1) + (x^{m-(2n-1)} - 1)$  which is  $< 0$  for all  $x$  in  $(0, 1)$  unless  $n - 3 = 0$  and  $m - (2n - 1) = 0$ . So unless  $n = 3$ ,  $m = 5$ , it has no roots in  $(0, 1)$ . But  $x^n + x^2 - 1$  (which divides it) has at least one because it is  $-1$  at  $x = 0$  and  $+1$  at  $x = 1$ . So we must have  $n = 3$ ,  $m = 5$ . It is easy to check that in this case we have an identity.

Two alternatives follow. Jean-Pierre Ehrmann continued:

If  $m = 2n-1$ , (\*) is  $x^{n-1} - x^2$ . If  $n = 3$ , this is 0 and indeed we find  $m = 5$ ,  $n = 3$  gives an identity. If  $n > 3$ , then it is  $x^2(x^{n-3} - 1)$ . But this has no roots in the interval  $(0, 1)$ , whereas  $x^n + x^2 - 1$  has at least one (because it is  $-1$  at  $x = 0$  and  $+1$  at  $x = 1$ ), so  $x^n + x^2 - 1$  cannot be a factor.

If  $m > 2n-1$ , then (\*) has four terms and factorises as  $(x - 1)(x^{m-n-1} + x^{m-n-2} + \dots + x^{m-2n+3} + x^{m-2n} + x^{m-2n-1} + \dots + 1)$ . Again, this has no roots in the interval  $(0, 1)$ , whereas  $x^n + x^2 - 1$  has at least one, so  $x^n + x^2 - 1$  cannot be a factor.

François Lo Jacomo, having got to  $x^n + x^2 - 1$  divides  $x^{m-n+1} + x^{m-n} - 1$  and looking at the case  $m - n + 1 > n$ , continues:

$x^n + x^2 - 1$  has a root  $r$  such that  $0 < r < 1$  (because it is  $-1$  at  $x = 0$  and  $+1$  at  $x = 1$ ). So  $r^n = 1 - r^2$ . It must also be a root of  $x^m + x - 1$ , so  $1 - r = r^m \leq r^{2n} = (1 - r^2)^2$ . Hence  $(1 - r^2)^2 - (1 - r) = (1 - r)r(1 - r - r^2) \geq 0$ , so  $1 - r - r^2 \geq 0$ . Hence  $r^n = 1 - r^2 \geq r$ , which is impossible.

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### Problem B1

The positive divisors of the integer  $n > 1$  are  $d_1 < d_2 < \dots < d_k$ , so that  $d_1 = 1$ ,  $d_k = n$ . Let  $d = d_1 d_2 + d_2 d_3 + \dots + d_{k-1} d_k$ . Show that  $d < n^2$  and find all  $n$  for which  $d$  divides  $n^2$ .

### Solution

$d_{k+1-m} \leq n/m$ . So  $d < n^2(1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + \dots)$ . The inequality is certainly strict because  $d$  has only finitely many terms. But  $1/(1 \cdot 2) + 1/(2 \cdot 3) + 1/(3 \cdot 4) + \dots = (1/1 - 1/2) + (1/2 - 1/3) + (1/3 - 1/4) + \dots = 1$ . So  $d < n^2$ .

Obviously  $d$  divides  $n^2$  for  $n$  prime. Suppose  $n$  is composite. Let  $p$  be the smallest prime dividing  $n$ . Then  $d > n^2/p$ . But the smallest divisor of  $n^2$  apart from 1 is  $p$ , so if  $d$  divides  $n^2$ , then  $d \leq n^2/p$ . So  $d$  cannot divide  $n^2$  for  $n$  composite.

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### Problem B2

Find all real-valued functions  $f$  on the reals such that  $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$  for all  $x, y, u, v$ .

### Solution

Answer: there are three possible functions: (1)  $f(x) = 0$  for all  $x$ ; (2)  $f(x) = 1/2$  for all  $x$ ; or (3)  $f(x) = x^2$ .

Put  $x = y = 0, u = v$ , then  $4f(0)f(u) = 2f(0)$ . So either  $f(u) = 1/2$  for all  $u$ , or  $f(0) = 0, f(u) = 1/2$  for all  $u$  is certainly a solution. So assume  $f(0) = 0$ .

Putting  $y = v = 0, f(x)f(u) = f(xu)$  (\*). In particular, taking  $x = u = 1, f(1)^2 = f(1)$ . So  $f(1) = 0$  or  $1$ . Suppose  $f(1) = 0$ . Putting  $x = y = 1, v = 0$ , we get  $0 = 2f(u)$ , so  $f(x) = 0$  or all  $x$ . That is certainly a solution. So assume  $f(1) = 1$ .

Putting  $x = 0, u = v = 1$  we get  $2f(y) = f(y) + f(-y)$ , so  $f(-y) = f(y)$ . So we need only consider  $f(x)$  for  $x$  positive. We show next that  $f(r) = r^2$  for  $r$  rational. The first step is to show that  $f(n) = n^2$  for  $n$  an integer. We use induction on  $n$ . It is true for  $n = 0$  and  $1$ . Suppose it is true for  $n-1$  and  $n$ . Then putting  $x = n, y = u = v = 1$ , we get  $2f(n) + 2 = f(n-1) + f(n+1)$ , so  $f(n+1) = 2n^2 + 2 - (n-1)^2 = (n+1)^2$  and it is true for  $n+1$ . Now (\*) implies that  $f(n)f(m/n) = f(m)$ , so  $f(m/n) = m^2/n^2$  for integers  $m, n$ . So we have established  $f(r) = r^2$  for all rational  $r$ .

From (\*) above, we have  $f(x^2) = f(x)^2 \geq 0$ , so  $f(x)$  is always non-negative for positive  $x$  and hence for all  $x$ . Putting  $u = y, v = x$ , we get  $(f(x) + f(y))^2 = f(x^2 + y^2)$ , so  $f(x^2 + y^2) = f(x)^2 + 2f(x)f(y) +$

$f(y)^2 \geq f(x)^2 = f(x^2)$ . For any  $u > v > 0$ , we may put  $u = x^2 + y^2$ ,  $v = x^2$  and hence  $f(u) \geq f(v)$ . In other words,  $f$  is an increasing function.

So for any  $x$  we may take a sequence of rationals  $r_n$  all less than  $x$  we converge to  $x$  and another sequence of rationals  $s_n$  all greater than  $x$  which converge to  $x$ . Then  $r_n^2 = f(r_n) \leq f(x) \leq f(s_n) = s_n^2$  for all  $x$  and hence  $f(x) = x^2$ .

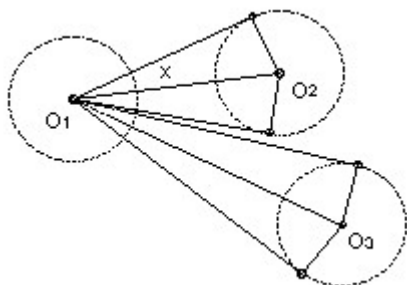
### 43rd IMO 2002



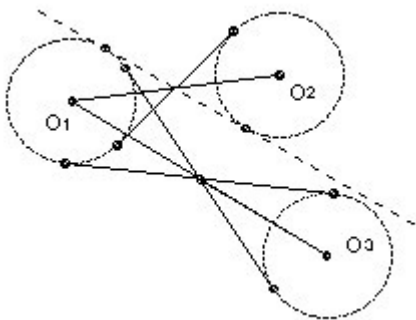
#### Problem B3

2 circles of radius 1 are drawn in the plane so that no line meets more than two of the circles. Their centers are  $O_1, O_2, \dots, O_n$ . Show that  $\sum_{i < j} 1/O_1O_j \leq (n-1)\pi/4$ .

#### Solution



Denote the circle center  $O_i$  by  $C_i$ . The tangents from  $O_1$  to  $C_i$  contain an angle  $2x$  where  $\sin x = 1/O_1O_i$ . So  $2x > 2/O_1O_i$ . These double sectors cannot overlap, so  $\sum 2/O_1O_i < \pi$ . Adding the equations derived from  $O_2, O_3, \dots$  we get  $4 \sum O_1O_j < n\pi$ , so  $\sum O_1O_j < n\pi/4$ , which is not quite good enough.



There are two key observations. The first is that it is better to consider the angle  $O_iO_1O_j$  than the angle between the tangents to a single circle. It is not hard to show that this angle must exceed both  $2/O_1O_i$  and  $2/O_1O_j$ . For consider the two common tangents to  $C_1$  and  $C_i$  which intersect at the midpoint of  $O_1O_i$ . The angle between the center line and one of the tangents is at least  $2/O_1O_i$ . No part of the circle  $C_j$  can cross this line, so its center  $O_j$  cannot cross the line parallel to the tangent through  $O_1$ . In other words, angle  $O_iO_1O_j$  is at least  $2/O_1O_i$ . A similar argument establishes it is at least  $2/O_1O_j$ .

Now consider the convex hull of the  $n$  points  $O_i$ .  $m \leq n$  of these points form the convex hull and the angles in the convex  $m$ -gon sum to  $(m-2)\pi$ . That is the second key observation. That gains us not one but two amounts  $\pi/4$ . However, we lose one back. Suppose  $O_1$  is a vertex of the convex hull and that its angle is  $\theta_1$ . Suppose for convenience that the rays  $O_1O_2, O_1O_3, \dots, O_1O_n$  occur in that order with  $O_2$  and  $O_n$  adjacent vertices to  $O_1$  in the convex hull. We have that the  $n-2$  angles between adjacent rays sum to  $\theta_1$ . So we have  $\sum 2/O_1O_i < \theta_1$ , where the sum is taken over only  $n-2$  of the  $i$ , not all  $n-1$ . But we can choose which  $i$  to drop, because of our freedom to choose either distance for each angle. So we drop the longest distance  $O_1O_i$ . [If  $O_1O_k$  is the longest, then we work outwards from that ray. Angle  $O_{k-1}O_1O_k > 2/O_1O_{k-1}$ , and angle  $O_kO_1O_{k+1} > 2/O_1O_{k+1}$  and so on.]

We now sum over all the vertices in the convex hull. For any centers  $O_i$  inside the hull we use the  $\sum_j 2/O_iO_j < \pi$  which we established in the first paragraph, where the sum has all  $n-1$  terms. Thus we get  $\sum_{i,j} 2/O_iO_j < (n-2)\pi$ , where for vertices  $i$  for which  $O_i$  is a vertex of the convex hull the sum is only over  $n-2$  values of  $j$  and excludes  $2/O_iO_{\max i}$  where  $O_{\max i}$  denotes the furthest center from  $O_i$ .

Now for  $O_i$  a vertex of the convex hull we have that the sum over all  $j$ ,  $\sum 2/O_iO_j$ , is the sum  $\Sigma'$  over all but  $j = \max i$  plus at most  $1/(n-2) \Sigma'$ . In other words we must increase the sum by at most a factor  $(n-1)/(n-2)$  to include the missing term. For  $O_i$  not a vertex of the hull, obviously no increase is needed. Thus the full sum  $\sum_{i,j} 2/O_iO_j < (n-1)\pi$ . Hence  $\sum_{i < j} 1/O_iO_j < (n-1)\pi/4$  as required.



**A1.**  $ABC$  is acute-angled.  $O$  is its circumcenter.  $X$  is the foot of the perpendicular from  $A$  to  $BC$ . Angle  $C \geq \text{angle } B + 30^\circ$ . Prove that angle  $A + \text{angle } COX < 90^\circ$ .



**A2.**  $a, b, c$  are positive reals. Let  $a' = \sqrt{a^2 + 8bc}$ ,  $b' = \sqrt{b^2 + 8ca}$ ,  $c' = \sqrt{c^2 + 8ab}$ . Prove that  $a/a' + b/b' + c/c' \geq 1$ .



**A3.** Integers are placed in each of the 441 cells of a  $21 \times 21$  array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.



**B1.** Let  $n_1, n_2, \dots, n_m$  be integers where  $m$  is odd. Let  $x = (x_1, \dots, x_m)$  denote a permutation of the integers  $1, 2, \dots, m$ . Let  $f(x) = x_1 n_1 + x_2 n_2 + \dots + x_m n_m$ . Show that for some distinct permutations  $a, b$  the difference  $f(a) - f(b)$  is a multiple of  $m!$ .



**B2.**  $ABC$  is a triangle.  $X$  lies on  $BC$  and  $AX$  bisects angle  $A$ .  $Y$  lies on  $CA$  and  $BY$  bisects angle  $B$ . Angle  $A$  is  $60^\circ$ .  $AB + BX = AY + YB$ . Find all possible values for angle  $B$ .



**B3.**  $K > L > M > N$  are positive integers such that  $KM + LN = (K + L - M + N)(-K + L + M + N)$ . Prove that  $KL + MN$  is composite.

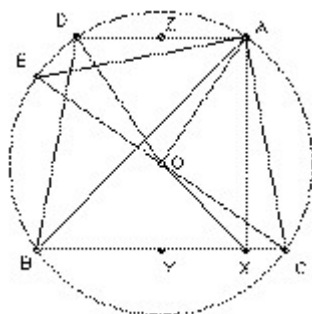




**Problem A1**

**ABC is acute-angled. O is its circumcenter. X is the foot of the perpendicular from A to BC. Angle C  $\geq$  angle B + 30°. Prove that angle A + angle COX  $<$  90°**

**Solution**



Take D on the circumcircle with AD parallel to BC. Angle CBD = angle BCA, so angle ABD  $\geq$  30°. Hence angle AOD  $\geq$  60°. Let Z be the midpoint of AD and Y the midpoint of BC. Then AZ  $\geq$  R/2, where R is the radius of the circumcircle. But AZ = YX (since AZYX is a rectangle).

Now O cannot coincide with Y (otherwise angle A would be 90° and the triangle would not be acute-angled). So OX  $>$  YX  $\geq$  R/2. But XC = YC - YX  $<$  R - YX  $\leq$  R/2. So OX  $>$  XC.

Hence angle COX  $<$  angle OCX. Let CE be a diameter of the circle, so that angle OCX = angle ECB. But angle ECB = angle EAB and angle EAB + angle BAC = angle EAC = 90°, since EC is a diameter. Hence angle COX + angle BAC  $<$  90°.

## IMO 2001



### Problem A2

**a, b, c are positive reals. Let  $a' = \sqrt{a^2 + 8bc}$ ,  $b' = \sqrt{b^2 + 8ca}$ ,  $c' = \sqrt{c^2 + 8ab}$ . Prove that  $a/a' + b/b' + c/c' \geq 1$ .**

### Solution

A not particularly elegant, but fairly easy, solution is to use Cauchy:  $(\sum xy)^2 \leq \sum x^2 \sum y^2$ .

To get the inequality the right way around we need to take  $x^2 = a/a'$  [to be precise, we are taking  $x_1^2 = a/a'$ ,  $x_2^2 = b/b'$ ,  $x_3^2 = c/c'$ ]. Take  $y^2 = a'$ , so that  $xy = a$ . Then we get  $\sum a/a' \geq (\sum a)^2 / \sum a a'$ .

Evidently we need to apply Cauchy again to deal with  $\sum a a'$ . This time we want  $\sum a a' \leq$  something. The obvious  $X=a$ ,  $Y=a'$  does not work, but if we put  $X=a^{1/2}$ ,  $Y=a^{1/2}a'$ , then we have  $\sum a a' \leq (\sum a)^{1/2} (\sum a a'^2)^{1/2}$ . So we get the required inequality provided that  $(\sum a)^3 \geq (\sum a a'^2)^{1/2}$  or  $(\sum a)^3 \geq \sum a a'^2$ .

Multiplying out, this is equivalent to:  $3(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2) \geq 18abc$ , or  $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0$ , which is clearly true.

## IMO 2001



### Problem A3

**Integers are placed in each of the 441 cells of a 21 x 21 array. Each row and each column has at most 6 different integers in it. Prove that some integer is in at least 3 rows and at least 3 columns.**

### Solution

Notice first that the result is not true for a 20 x 20 array. Make 20 rectangles each 2 x 10, labelled 1, 2, ..., 20. Divide the 20 x 20 array into four quadrants (each 10 x 10). In each of the top left and bottom right quadrants, place 5 rectangles horizontally. In each of the other two quadrants, place 5 rectangles vertically. Now each row intersects 5 vertical rectangles and 1 horizontal. In other words, it contains just 6 different numbers. Similarly each column. But any given number is in either 10 rows and 2 columns or vice versa, so no number is in 3 rows and 3 columns. [None of this is necessary for the solution, but it helps to show what is going on.]

Returning to the 21 x 21 array, assume that an arrangement is possible with no integer in at least 3 rows and at least 3 columns. Color a cell white if its integer appears in 3 or more rows and black if its integer appears in only 1 or 2 rows. We count the white and black squares.

Each row has 21 cells and at most 6 different integers.  $6 \times 2 < 21$ , so every row includes an integer which appears 3 or more times and hence in at most 2 rows. Thus at most 5 different integers in the row appear in 3 or more rows. Each such integer can appear at most 2 times in the row, so there are at most  $5 \times 2 = 10$  white cells in the row. This is true for every row, so there are at most 210 white cells in total.

Similarly, any given column has at most 6 different integers and hence at least one appears 3 or more times. So at most 5 different integers appear in 2 rows or less. Each such integer can occupy at most 2 cells in the column, so there are at most  $5 \times 2 = 10$  black cells in the column. This is true for every column, so there are at most 210 black cells in total.

This gives a contradiction since  $210 + 210 < 441$ .

*Comment. This looks easy, but (like question 6) I found it curiously difficult (it took me well over 2 hours). For a while I could not see how to do better than a  $12 \times 12$  array (with 2 rows of 1s, then 2 rows of 2s etc), which was disorienting. Then I got the argument almost right, but not quite right, which took more time.*

*The original question was phrased in terms of 21 boys and 21 girls in a competition with an unknown number of problems. Each boy, girl pair solved at least one problem. Each competitor solved at most 6 problems. One had to show that some problem was solved by at least 3 boys and at least 3 girls. The recasting in the terms above is almost immediate.*

*Equally, one can easily recast the solution above into the competition format. Take any boy  $B_0$ . At least one of the questions he attempts must be attempted by 3 or more girls (because he attempts at most 6 questions and there are more than  $6 \times 2$  girls). Hence he attempts at most 5 questions which are only attempted by less than 3 girls. So at most  $5 \times 2 = 10$  of the 21 pairs  $(B_0, G)$  attempt a question attempted by less than 3 girls. So at most 210 of the 441 pairs  $(B, G)$  attempt such a question. Similarly, at most 210 pairs  $(B, G)$  attempt a question attempted by less than 3 boys. Hence at least 21 pairs  $(B, G)$  attempt a question attempted by 3 or more girls and 3 or more boys. So there must be at least one such question.*

*Note that the arguments above generalise immediately to show that in a  $4N+1$  by  $4N+1$  array with at most  $N+1$  different integers in each row and column, there is some integer that appears in at least 3 rows and 3 columns, but this is not true for a  $4N$  by  $4N$  array.*

## IMO 2001



### Problem B1

Let  $n_1, n_2, \dots, n_m$  be integers, where  $m$  is odd. Let  $x = (x_1, \dots, x_m)$  denote a permutation of the integers  $1, 2, \dots, m$ . Let  $f(x) = x_1 n_1 + x_2 n_2 + \dots + x_m n_m$ . Show that for some distinct permutations  $a, b$  the difference  $f(a) - f(b)$  is a multiple of  $m!$ .

### Solution

This is a simple application of the pigeon hole principle.

The sum of all  $m!$  distinct residues mod  $m!$  is not divisible by  $m!$  because  $m!$  is even (since  $m > 1$ ). [The residues come in pairs  $a$  and  $m! - a$ , except for  $m!/2$ .]

However, the sum of all  $f(x)$  as  $x$  ranges over all  $m!$  permutations is  $1/2 (m+1)! \sum n_i$ , which is divisible by  $m!$  (since  $m+1$  is even). So at least one residue must occur more than once among the  $f(x)$ .

## IMO 2001

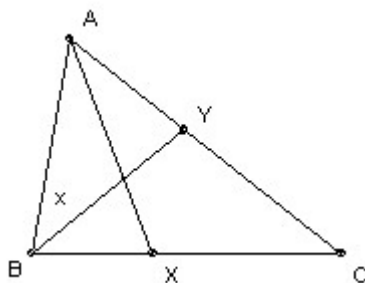


### Problem B2

$ABC$  is a triangle.  $X$  lies on  $BC$  and  $AX$  bisects angle  $A$ .  $Y$  lies on  $CA$  and  $BY$  bisects angle  $B$ . Angle  $A$  is  $60^\circ$ .  $AB + BX = AY + YB$ . Find all possible values for angle  $B$ .

**Answer** Answer:  $80^\circ$ .

### Solution

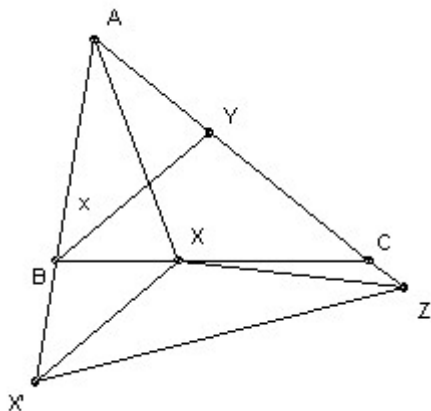


This is an inelegant solution, but I did get it fast! Without loss of generality we can take length  $AB = 1$ . Take angle  $ABY = x$ . Note that we can now solve the two triangles  $AXB$  and  $AYB$ . In particular, using the sine rule,  $BX = \sin 30^\circ / \sin(150^\circ - 2x)$ ,  $AY = \sin x / \sin(120^\circ - x)$ ,  $YB = \sin 60^\circ / \sin(120^\circ - x)$ . So we have an equation for  $x$ .

Using the usual formula for  $\sin(a + b)$  etc, and writing  $s = \sin x$ ,  $c = \cos x$ , we get:  $2\sqrt{3} s^2 c - 4sc - 2\sqrt{3} c^3 + 2\sqrt{3} c^2 + 6sc - 2s - \sqrt{3} = 0$  or  $-\sqrt{3} (4c^3 - 2c^2 - 2c + 1) = 2s(2c^2 - 3c + 1)$ . This has a common factor  $2c - 1$ . So  $c = 1/2$  or  $-\sqrt{3} (2c^2 - 1) = 2s(c - 1)$  (\*).

$c = 1/2$  means  $x = 60^\circ$  or angle  $B = 120^\circ$ . But in that case the sides opposite A and B are parallel and the triangle is degenerate (a case we assume is disallowed). So squaring (\*) and using  $s^2 = 1 - c^2$ , we get:  $16c^4 - 8c^3 - 12c^2 + 8c - 1 = 0$ . This has another factor  $2c - 1$ . Dividing that out we get:  $8c^3 - 6c + 1 = 0$ . But we remember that  $4c^3 - 3c = \cos 3x$ , so we conclude that  $\cos 3x = -1/2$ . That gives  $x = 40^\circ, 80^\circ, 160^\circ, 200^\circ, 280^\circ, 320^\circ$ . But we require that  $x < 60^\circ$  to avoid degeneracy. Hence the angle  $B = 2x = 80^\circ$ .

*I subsequently found this geometric solution on the official Wolfram site (Wolfram was one of the sponsors of IMO 2001). I cannot say it is much easier, but at least it is geometric.*



Extend AB to  $X'$  with  $BX' = BX$ . Extend AY to Z with  $YZ = YB$ . Then  $AZ = AY + YZ = AY + YB = AB + BX = AB + BX' = AX'$ . Angle  $A = 60^\circ$ , so  $AZX'$  is equilateral.

Use  $B$  also to denote the angle at B. Then angle  $YBX = B/2$ . Also angle  $BXX' +$  angle  $BX'X = B$ . The triangle is isosceles by construction, so angle  $BX'X = B/2$ . Hence angle  $XX'Z = 60^\circ - B/2$ .  $X$  lies on the bisector of A and  $AZ = AX'$ , so  $XZ = XX'$ . Hence  $XZX' = 60^\circ - B/2$  also. But angle  $Z = 60^\circ$ , so angle  $YZX = B/2 =$  angle  $YBX$ .

Now  $YZ = YB$ , so angle  $YZB =$  angle  $YBZ$ . Hence angle  $XZB =$  angle  $XBZ$  (they are the difference of pairs of equal angles). If  $X$  does not lie on  $BZ$ , then we can conclude that  $XB = XZ$ .

In that case, since  $XZ = XX'$ , we have  $XB = XX'$ . But already  $XB = BX'$  (by construction), so  $BXX'$  is equilateral and hence  $B/2 = 60^\circ$ . But then angle  $B +$  angle  $A = 180^\circ$ , so the triangle ABC is degenerate (with C at infinity), which we assume is disallowed. Hence X must lie on  $BZ$ , which means  $Z = C$  and angle  $B = 2$  angle  $C$ . Hence angle  $B = 80^\circ$ , angle  $C = 40^\circ$ .

**Problem B3**

**$K > L > M > N$  are positive integers such that  $KM + LN = (K + L - M + N)(-K + L + M + N)$ . Prove that  $KL + MN$  is composite.**

**Solution**

Note first that  $KL + MN > KM + LN > KN + LM$ , because  $(KL + MN) - (KM + LN) = (K - N)(L - M) > 0$  and  $(KM + LN) - (KN + LM) = (K - L)(M - N) > 0$ .

Multiplying out and rearranging, the relation in the question gives  $K^2 - KM + M^2 = L^2 + LN + N^2$ . Hence  $(KM + LN)(L^2 + LN + N^2) = KM(L^2 + LN + N^2) + LN(K^2 - KM + M^2) = KML^2 + KMN^2 + K^2LN + LM^2N = (KL + MN)(KN + LM)$ . In other words  $(KM + LN)$  divides  $(KL + MN)(KN + LM)$ .

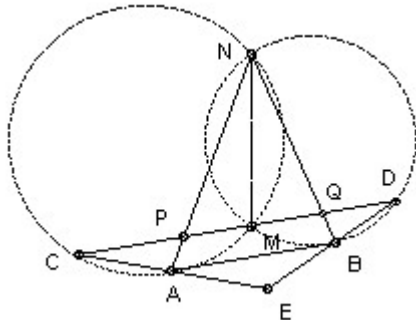
Now suppose  $KL + MN$  is prime. Since it is greater than  $KM + LN$ , it can have no common factors with  $KM + LN$ . Hence  $KM + LN$  must divide the smaller integer  $KN + LM$ . Contradiction.

*Comment. This looks easy, but in fact I found it curiously difficult. It is easy to go around in circles getting nowhere. Either I am getting older, or this is harder than it looks!*

*Note that it is not hard to find  $K, L, M, N$  satisfying the condition in the question. For example 11, 9, 5, 1.*



**A1.**  $AB$  is tangent to the circles  $CAMN$  and  $NMBD$ .  $M$  lies between  $C$  and  $D$  on the line  $CD$ , and  $CD$  is parallel to  $AB$ . The chords  $NA$  and  $CM$  meet at  $P$ ; the chords  $NB$  and  $MD$  meet at  $Q$ . The rays  $CA$  and  $DB$  meet at  $E$ . Prove that  $PE = QE$ .



**A2.**  $A, B, C$  are positive reals with product 1. Prove that  $(A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) \leq 1$ .



**A3.**  $k$  is a positive real.  $N$  is an integer greater than 1.  $N$  points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points  $A$  and  $B$  which are not coincident. Suppose that  $A$  lies to the right of  $B$ . Replace  $B$  by another point  $B'$  to the right of  $A$  such that  $AB' = k BA$ . For what values of  $k$  can we move the points arbitrarily far to the right by repeated moves?



**B1.** 100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?



**B2.** Can we find  $N$  divisible by just 2000 different primes, so that  $N$  divides  $2^N + 1$ ? [ $N$  may be divisible by a prime power.]



**B3.**  $A_1A_2A_3$  is an acute-angled triangle. The foot of the altitude from  $A_i$  is  $K_i$  and the incircle touches the side opposite  $A_i$  at  $L_i$ . The line  $K_1K_2$  is reflected in the line  $L_1L_2$ . Similarly, the line  $K_2K_3$  is reflected in  $L_2L_3$  and  $K_3K_1$  is reflected in  $L_3L_1$ . Show that the three new lines form a triangle with vertices on the incircle.

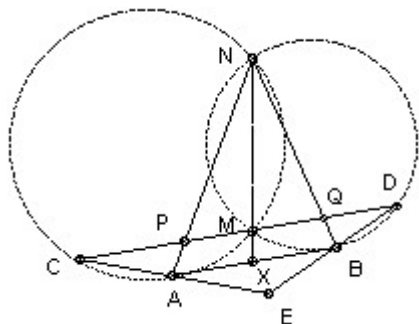




**Problem A1**

**AB is tangent to the circles CAMN and NMBD. M lies between C and D on the line CD, and CD is parallel to AB. The chords NA and CM meet at P; the chords NB and MD meet at Q. The rays CA and DB meet at E. Prove that PE = QE.**

**Solution**



Angle  $EBA = \text{angle } BDM$  (because  $CD$  is parallel to  $AB$ ) = angle  $ABM$  (because  $AB$  is tangent at  $B$ ). So  $AB$  bisects  $EBM$ . Similarly,  $BA$  bisects angle  $EAM$ . Hence  $E$  is the reflection of  $M$  in  $AB$ . So  $EM$  is perpendicular to  $AB$  and hence to  $CD$ . So it suffices to show that  $MP = MQ$ .

Let the ray  $NM$  meet  $AB$  at  $X$ .  $XA$  is a tangent so  $XA^2 = XM \cdot XN$ . Similarly,  $XB$  is a tangent, so  $XB^2 = XM \cdot XN$ . Hence  $XA = XB$ . But  $AB$  and  $PQ$  are parallel, so  $MP = MQ$ .

IMO 2000



**Problem A2**

**A, B, C are positive reals with product 1. Prove that  $(A - 1 + 1/B)(B - 1 + 1/C)(C - 1 + 1/A) \leq 1$ .**

**Solution**

*An elegant solution due to Robin Chapman is as follows:*

$$(B - 1 + 1/C) = B(1 - 1/B + 1/(BC)) = B(1 + A - 1/B). \text{ Hence, } (A - 1 + 1/B)(B - 1 + 1/C) = B(A^2 - (1 - 1/B)^2) \leq B A^2. \text{ So the square of the product of all three } \leq B A^2 C B^2 A C^2 = 1.$$

Actually, that is not quite true. The last sentence would not follow if we had some negative left hand sides, because then we could not multiply the inequalities. But it is easy to deal separately with the case where  $(A - 1 + 1/B)$ ,  $(B - 1 + 1/C)$ ,  $(C - 1 + 1/A)$  are not all positive. If one of the



three terms is negative, then the other two must be positive. For example, if  $A - 1 + 1/B < 0$ , then  $A < 1$ , so  $C - 1 + 1/A > 0$ , and  $B > 1$ , so  $B - 1 + 1/C > 0$ . But if one term is negative and two are positive, then their product is negative and hence less than 1.

*Few people would manage this under exam conditions, but there are plenty of longer and easier to find solutions!*

## IMO 2000



### Problem A3

**$k$  is a positive real.  $N$  is an integer greater than 1.  $N$  points are placed on a line, not all coincident. A *move* is carried out as follows. Pick any two points  $A$  and  $B$  which are not coincident. Suppose that  $A$  lies to the right of  $B$ . Replace  $B$  by another point  $B'$  to the right of  $A$  such that  $AB' = k BA$ . For what values of  $k$  can we move the points arbitrarily far to the right by repeated moves?**

**Answer :**  $k \geq 1/(N-1)$ .

### Solution

*An elegant solution by Gerhard Woeginger is as follows:*

Suppose  $k < 1/(N-1)$ , so that  $k_0 = 1/k - (N-1) > 0$ . Let  $X$  be the sum of the distances of the points from the rightmost point. If a move does not change the rightmost point, then it reduces  $X$ . If it moves the rightmost point a distance  $z$  to the right, then it reduces  $X$  by at least  $z/k - (N-1)z = k_0 z$ .  $X$  cannot be reduced below nil. So the total distance moved by the rightmost point is at most  $X_0/k_0$ , where  $X_0$  is the initial value of  $X$ .

Conversely, suppose  $k \geq 1/(N-1)$ , so that  $k_1 = (N-1) - 1/k \geq 0$ . We always move the leftmost point. This has the effect of moving the rightmost point  $z > 0$  and increasing  $X$  by  $(N-1)z - z/k = k_1 z \geq 0$ . So  $X$  is never decreased. But  $z \geq k X/(N-1) \geq k X_0/(N-1) > 0$ . So we can move the rightmost point arbitrarily far to the right (and hence all the points, since another  $N-1$  moves will move the other points to the right of the rightmost point).



### Problem B1

**100 cards are numbered 1 to 100 (each card different) and placed in 3 boxes (at least one card in each box). How many ways can this be done so that if two boxes are selected and a card is taken from each, then the knowledge of their sum alone is always sufficient to identify the third box?**

#### Answer

12. Place 1, 2, 3 in different boxes (6 possibilities) and then place  $n$  in the same box as its residue mod 3. Or place 1 and 100 in different boxes and 2 - 99 in the third box (6 possibilities).

#### Solution

*An elegant solution communicated (in outline) by both Mohd Suhaimi Ramly and Fokko J van de Bult is as follows:*

Let  $H_n$  be the corresponding result that for cards numbered 1 to  $n$  the only solutions are by residue mod 3, or 1 and  $n$  in separate boxes and 2 to  $n - 1$  in the third box. It is easy to check that they *are* solutions.  $H_n$  is the assertion that there are no others.  $H_3$  is obviously true (although the two cases coincide). We now use induction on  $n$ . So suppose that the result is true for  $n$  and consider the case  $n + 1$ .

Suppose  $n + 1$  is alone in its box. If 1 is not also alone, then let  $N$  be the sum of the largest cards in each of the boxes not containing  $n + 1$ . Since  $n + 2 \leq N \leq n + (n - 1) = 2n - 1$ , we can achieve the same sum  $N$  as from a different pair of boxes as  $(n + 1) + (N - n - 1)$ . Contradiction. So 1 must be alone and we have one of the solutions envisaged in  $H_{n+1}$ .

If  $n + 1$  is not alone, then if we remove it, we must have a solution for  $n$ . But that solution cannot be the  $n, 1, 2$  to  $n - 1$  solution. For we can easily check that none of the three boxes will then accommodate  $n + 1$ . So it must be the mod 3 solution. We can easily check that in this case  $n + 1$  must go in the box with matching residue, which makes the  $(n + 1)$  solution the other solution envisaged by  $H_{n+1}$ . That completes the induction.

*My much more plodding solution (which I was quite pleased with until I saw the more elegant solution above) follows. It took about half-an-hour and shows the kind of kludge one is likely to come up with under time pressure in an exam!*

With a suitable labeling of the boxes as A, B, C, there are 4 cases to consider:

Case 1: A contains 1; B contains 2; C contains 3

Case 2: A contains 1, 2

Case 3: A contains 1, 3; B contains 2

Case 4: A contains 1; B contains 2, 3.

We show that Cases 1 and 4 each yield just one possible arrangement and Cases 2 and 3 none.

In Case 1, it is an easy induction that  $n$  must be placed in the same box as its residue (in other words numbers with residue 1 mod 3 go into A, numbers with residue 2 go into B, and numbers with residue 0 go into C). For  $(n + 1) + (n - 2) = n + (n - 1)$ . Hence  $n + 1$  must go in the same box as  $n - 2$  (if they were in different boxes, then we would have two pairs from different pairs of boxes with the same sum). It is also clear that this is a possible arrangement. Given the sum of two numbers from different boxes, take its residue mod 3. A residue of 0 indicates that the third (unused) box was C, a residue of 1 indicates that the third box was A, and a residue of 2 indicates that the third box was B. Note that this unique arrangement gives 6 ways for the question, because there are 6 ways of arranging 1, 2 and 3 in the given boxes.

In Case 2, let  $n$  be the smallest number not in box A. Suppose it is in box B. Let  $m$  be the smallest number in the third box, C.  $m - 1$  cannot be in C. If it is in A, then  $m + (n - 1) = (m - 1) + n$ . Contradiction ( $m$  is in C,  $n - 1$  is in A, so that pair identifies B as the third box, but  $m - 1$  is in A and  $n$  is in B, identifying C). So  $m - 1$  must be in B. But  $(m - 1) + 2 = m + 1$ . Contradiction. So Case 2 is not possible.

In Case 3, let  $n$  be the smallest number in box C, so  $n - 1$  must be in A or B. If  $n - 1$  is in A, then  $(n - 1) + 2 = n + 2$ . Contradiction (a sum of numbers in A and B equals a sum from C and A). If  $n - 1$  is in B, then  $(n - 1) + 3 = n + 2$ . Contradiction (a sum from B and A equals a sum from C and B). So Case 3 is not possible.

In Case 4, let  $n$  be the smallest number in box C.  $n - 1$  cannot be in A, or  $(n - 1) + 2 = 3 + n$  (pair from A, B with same sum as pair from B, C), so  $n - 1$  must be in B. Now  $n + 1$  cannot be in A (or  $(n + 1) + 2 = 3 + n$ ), or in B or C (or  $1 + (n + 1) = 2 + n$ ). So  $n + 1$  cannot exist and hence  $n = 100$ . It is now an easy induction that all of 4, 5, ... 98 must be in B. For given that  $m$  is in B, if  $m + 1$  were in A, we would have  $100 + m = 99 + (m + 1)$ . But this arrangement (1 in A, 2 - 99 in B, 100 in C) is certainly possible: sums 3 - 100 identify C as the third box, sum 101 identifies B as the third box, and sums 102-199 identify A as the third box. Finally, as in Case 1, this unique arrangement corresponds to 6 ways of arranging the cards in the given boxes.

## IMO 2000



### Problem B2

**Can we find  $N$  divisible by just 2000 different primes, so that  $N$  divides  $2^N + 1$ ? [ $N$  may be divisible by a prime power.]**

**Answer :** Yes

### Solution

Note that for  $b$  odd we have  $2^{ab} + 1 = (2^a + 1)(2^{a(b-1)} - 2^{a(b-2)} + \dots + 1)$ , and so  $2^a + 1$  is a factor of  $2^{ab} + 1$ . It is sufficient therefore to find  $m$  such that (1)  $m$  has only a few distinct prime factors, (2)  $2^m + 1$  has a large number of distinct prime factors, (3)  $m$  divides  $2^m + 1$ . For then we can take  $k$ , a product of enough distinct primes dividing  $2^m + 1$  (but not  $m$ ), so that  $km$  has exactly 2000 factors. Then  $km$  still divides  $2^m + 1$  and hence  $2^{km} + 1$ .

The simplest case is where  $m$  has only one distinct prime factor  $p$ , in other words it is a power of  $p$ . But if  $p$  is a prime, then  $p$  divides  $2^p - 2$ , so the only  $p$  for which  $p$  divides  $2^p + 1$  is 3. So the questions are whether  $a_h = 2^m + 1$  is (1) divisible by  $m = 3^h$  and (2) has a large number of distinct prime factors.

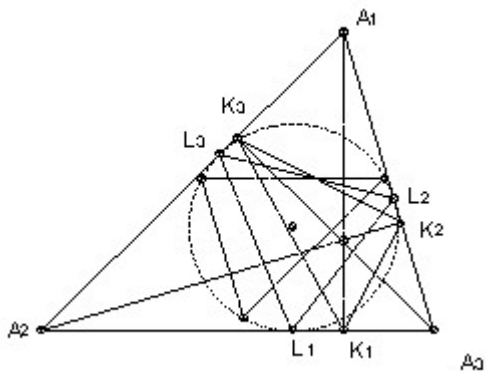
$a_{h+1} = a_h(2^{2^m} - 2^m + 1)$ , where  $m = 3^h$ . But  $2^m = (a_h - 1)$ , so  $a_{h+1} = a_h(a_h^2 - 3a_h + 3)$ . Now  $a_1 = 9$ , so an easy induction shows that  $3^{h+1}$  divides  $a_h$ , which answers (1) affirmatively. Also, since  $a_h$  is a factor of  $a_{h+1}$ , any prime dividing  $a_h$  also divides  $a_{h+1}$ . Put  $a_h = 3^{h+1}b_h$ . Then  $b_{h+1} = b_h(3^{2^{h+1}}b_h^2 - 3^{h+2}b_h + 1)$ . Now  $(3^{2^{h+1}}b_h^2 - 3^{h+2}b_h + 1) > 1$ , so it must have some prime factor  $p > 1$ . But  $p$  cannot be 3 or divide  $b_h$  (since  $(3^{2^{h+1}}b_h^2 - 3^{h+2}b_h + 1)$  is a multiple of  $3b_h$  plus 1), so  $b_{h+1}$  has at least one prime factor  $p > 3$  which does not divide  $b_h$ . So  $b_{h+1}$  has at least  $h$  distinct prime factors greater than 3, which answers (2) affirmatively. But that is all we need. We can take  $m$  in the first paragraph above to be  $3^{2000}$ : (1)  $m$  has only one distinct prime factor, (2)  $2^m + 1 = 3^{2001} b_{2000}$  has at least 1999 distinct prime factors other than 3, (3)  $m$  divides  $2^m + 1$ . Take  $k$  to be a product of 1999 distinct prime factors dividing  $b_{2000}$ . Then  $N = km$  is the required number with exactly 2000 distinct prime factors which divides  $2^N + 1$ .

### IMO 2000



#### Problem B3

$A_1A_2A_3$  is an acute-angled triangle. The foot of the altitude from  $A_i$  is  $K_i$  and the incircle touches the side opposite  $A_i$  at  $L_i$ . The line  $K_1K_2$  is reflected in the line  $L_1L_2$ . Similarly, the line  $K_2K_3$  is reflected in  $L_2L_3$  and  $K_3K_1$  is reflected in  $L_3L_1$ . Show that the three new lines form a triangle with vertices on the incircle.





Now the angle  $K_3K_2A_1 = \text{angle } A_1A_2A_3$ , because it is  $90^\circ - \text{angle } K_3K_2A_2 = 90^\circ - \text{angle } K_3A_3A_2$  ( $A_2A_3K_2K_3$  is cyclic with  $A_2A_3$  a diameter)  $= \text{angle } A_1A_2A_3$ . So the reflection of  $K_2K_3$  in  $L_2L_3$  is a line through  $X$  making an angle  $A_1A_2A_3$  with  $L_2X$ , in other words, it is the line through  $X$  parallel to  $A_2A_3$ .

Let  $M_1$  be the reflection of  $L_1$  in  $A_1O$ . The angle  $M_2XL_2 = 2 \text{ angle } OXL_2 = 2 \text{ angle } A_1A_2O$  (since  $A_1A_2$  is parallel to  $L_2X$ )  $= \text{angle } A_1A_2A_3$ , which is the angle between  $L_2X$  and  $A_2A_3$ . So  $M_2X$  is parallel to  $A_2A_3$ , in other words,  $M_2$  lies on the reflection of  $K_2K_3$  in  $L_2L_3$ .

It follows similarly that  $M_3$  lies on the reflection. Similarly, the line  $M_1M_3$  is the reflection of  $K_1K_3$  in  $L_1L_3$ , and the line  $M_1M_2$  is the reflection of  $K_1K_2$  in  $L_1L_2$  and hence the triangle formed by the intersections of the three reflections is just  $M_1M_2M_3$ .



**A1.** Find all finite sets  $S$  of at least three points in the plane such that for all distinct points  $A, B$  in  $S$ , the perpendicular bisector of  $AB$  is an axis of symmetry for  $S$ .



**A2.** Let  $n \geq 2$  be a fixed integer. Find the smallest constant  $C$  such that for all non-negative reals  $x_1, \dots, x_n$ :

$$\sum_{i < j} x_i x_j (x_i^2 + x_j^2) \leq C \left( \sum x_i \right)^4.$$



Determine when equality occurs.

**A3.** Given an  $n \times n$  square board, with  $n$  even. Two distinct squares of the board are said to be adjacent if they share a common side, but a square is not adjacent to itself. Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.



**B1.** Find all pairs  $(n, p)$  of positive integers, such that:  $p$  is prime;  $n \leq 2p$ ; and  $(p - 1)^n + 1$  is divisible by  $n^{p-1}$ .



**B2.** The circles  $C_1$  and  $C_2$  lie inside the circle  $C$ , and are tangent to it at  $M$  and  $N$ , respectively.  $C_1$  passes through the center of  $C_2$ . The common chord of  $C_1$  and  $C_2$ , when extended, meets  $C$  at  $A$  and  $B$ . The lines  $MA$  and  $MB$  meet  $C_1$  again at  $E$  and  $F$ . Prove that the line  $EF$  is tangent to  $C_2$ .



**B3.** Determine all functions  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that  $f(x - f(y)) = f(f(y)) + x f(y) + f(x) - 1$  for all  $x, y$  in  $\mathbb{R}$ . [ $\mathbb{R}$  is the reals.]

## IMO 1999



### Problem A1

Find all finite sets  $S$  of at least three points in the plane such that for all distinct points  $A, B$  in  $S$ , the perpendicular bisector of  $AB$  is an axis of symmetry for  $S$ .

**Solution** by Gerhard Woeginger

The possible sets are just the regular  $n$ -gons ( $n > 2$ ).

Let  $A_1, A_2, \dots, A_k$  denote the vertices of the convex hull of  $S$  (and take indices mod  $k$  as necessary). We show first that these form a regular  $k$ -gon.  $A_{i+1}$  must lie on the perpendicular bisector of  $A_i$  and  $A_{i+2}$  (otherwise its reflection would lie outside the hull). Hence the sides are all equal. Similarly,  $A_{i+1}$  and  $A_{i+2}$  must be reflections in the perpendicular bisector of  $A_i$  and  $A_{i+3}$  (otherwise one of the reflections would lie outside the hull). Hence all the angles are equal.

Any axis of symmetry for  $S$  must also be an axis of symmetry for the  $A_i$ , and hence must pass through the center  $C$  of the regular  $k$ -gon. Suppose  $X$  is a point of  $S$  in the interior of  $k$ -gon. Then it must lie inside or on some triangle  $A_i A_{i+1} C$ .  $C$  must be the circumcenter of  $A_i A_{i+1} X$  (since it lies on the three perpendicular bisectors, which must all be axes of symmetry of  $S$ ), so  $X$  must lie on the circle center  $C$ , through  $A_i$  and  $A_{i+1}$ . But all points of the triangle  $A_i A_{i+1} X$  lie strictly inside this circle, except  $A_i$  and  $A_{i+1}$ , so  $X$  cannot be in the interior of the  $k$ -gon.

## IMO 1999



### Problem A2

Let  $n \geq 2$  be a fixed integer. Find the smallest constant  $C$  such that for all non-negative reals  $x_1, \dots, x_n$ :

$$\sum_{i < j} x_i x_j (x_i^2 + x_j^2) \leq C \left( \sum x_i \right)^4.$$

Determine when equality occurs.

**Answer** Answer:  $C = 1/8$ . Equality iff two  $x_i$  are equal and the rest zero.

### Solution

By a member of the Chinese team at the IMO - does anyone know who?

$$\left( \sum x_i \right)^4 = \left( \sum x_i^2 + 2 \sum_{i < j} x_i x_j \right)^2 \geq 4 \left( \sum x_i^2 \right) \left( 2 \sum_{i < j} x_i x_j \right) = 8 \sum_{i < j} (x_i x_j \sum x_k^2) \geq 8 \sum_{i < j} x_i x_j (x_i^2 + x_j^2).$$



The second inequality is an equality only if  $n - 2$  of the  $x_i$  are zero. So assume that  $x_3 = x_4 = \dots = x_n = 0$ . Then for the first inequality to be an equality we require that  $(x_1^2 + x_2^2) = 2 x_1 x_2$  and hence that  $x_1 = x_2$ . However, that is clearly also sufficient for equality.

*Alternative solution by Gerhard Woeginger*

Setting  $x_1 = x_2 = 1$ ,  $x_i = 0$  for  $i > 2$  gives equality with  $C = 1/8$ , so,  $C$  cannot be smaller than  $1/8$ .

We now use induction on  $n$ . For  $n = 2$ , the inequality with  $C = 1/8$  is equivalent to  $(x_1 - x_2)^4 \geq 0$ , which is true, with equality iff  $x_1 = x_2$ . So the result is true for  $n = 2$ .

For  $n > 2$ , we may take  $x_1 + \dots + x_n = 1$ , and  $x_1 \leq x_2 \leq \dots \leq x_n$ . Now replace  $x_1$  and  $x_2$  by 0 and  $x_1 + x_2$ . The sum on the rhs is unchanged and the sum on the lhs is increased by  $(x_1 + x_2)^3 S - (x_1^3 + x_2^3) S - x_1 x_2 (x_1^2 + x_2^2)$ , where  $S = x_3 + x_4 + \dots + x_n$ . But  $S$  is at least  $1/3$  (the critical case is  $n = 3$ ,  $x_i = 1/3$ ), so this is at least  $x_1 x_2 (x_1 + x_2 - x_1^2 - x_2^2)$ . This is strictly greater than 0 unless  $x_1 = 0$  (when it equals 0), so the result follows by induction.

*Comment. The first solution is elegant and shows clearly why the inequality is true. The second solution is more plodding, but uses an approach which is more general and can be applied in many other cases. At least with hindsight, the first solution is not as impossible to find as one might think. A little playing around soon uncovers the fact that one can get  $C = 1/8$  with two  $x_i$  equal and the rest zero, and that this looks like the best possible. One just has to make the jump to replacing  $(x_i^2 + x_j^2)$  by  $\sum x_k^2$ . The solution is then fairly clear. Of course, that does not detract from the contestant's achievement, because I and almost everyone else who has looked at the problem failed to make that jump.*

## IMO 1999



### Problem A3

**Given an  $n \times n$  square board, with  $n$  even. Two distinct squares of the board are said to be adjacent if they share a common side, but a square is not adjacent to itself. Find the minimum number of squares that can be marked so that every square (marked or not) is adjacent to at least one marked square.**

**Answer :**  $n/2 (n/2 + 1) = n(n + 2)/4$ .

### Solution

Let  $n = 2m$ . Color alternate squares black and white (like a chess board). It is sufficient to show that  $m(m+1)/2$  white squares are necessary and sufficient to deal with all the black squares.

This is almost obvious if we look at the diagonals.

Look first at the odd-length white diagonals. In every other such diagonal, mark alternate squares (starting from the border each time, so that  $r+1$  squares are marked in a diagonal length  $2r+1$ ). Now each black diagonal is adjacent to a picked white diagonal and hence each black square on it

is adjacent to a marked white square. In all  $1 + 3 + 5 + \dots + m-1 + m + m-2 + \dots + 4 + 2 = 1 + 2 + 3 + \dots + m = m(m+1)/2$  white squares are marked. This proves sufficiency.

For necessity consider the alternate odd-length black diagonals. Rearranging, these have lengths  $1, 3, 5, \dots, 2m-1$ . A white square is only adjacent to squares in one of these alternate diagonals and is adjacent to at most 2 squares in it. So we need at least  $1 + 2 + 3 + \dots + m = m(m+1)/2$  white squares.

## IMO 1999



### Problem B1

**Find all pairs  $(n, p)$  of positive integers, such that:  $p$  is prime;  $n \leq 2p$ ; and  $(p - 1)^n + 1$  is divisible by  $n^{p-1}$ .**

**Answer**  $(1, p)$  for any prime  $p$ ;  $(2, 2)$ ;  $(3, 3)$ .

**Solution** by Gerhard Woeginger, Technical University, Graz

Evidently  $(1, p)$  is a solution for every prime  $p$ . Assume  $n > 1$  and take  $q$  to be the smallest prime divisor of  $n$ . We show first that  $q = p$ .

Let  $x$  be the smallest positive integer for which  $(p - 1)^x = -1 \pmod{q}$ , and  $y$  the smallest positive integer for which  $(p - 1)^y = 1 \pmod{q}$ . Certainly  $y$  exists and indeed  $y < q$ , since  $(p - 1)^{q-1} = 1 \pmod{q}$ . We know that  $(p - 1)^n = -1 \pmod{q}$ , so  $x$  exists also. Writing  $n = sy + r$ , with  $0 \leq r < y$ , we conclude that  $(p - 1)^r = -1 \pmod{q}$ , and hence  $x \leq r < y$  ( $r$  cannot be zero, since  $1$  is not  $-1 \pmod{q}$ ).

Now write  $n = hx + k$  with  $0 \leq k < x$ . Then  $-1 = (p - 1)^n = (-1)^h (p - 1)^k \pmod{q}$ .  $h$  cannot be even, because then  $(p - 1)^k = -1 \pmod{q}$ , contradicting the minimality of  $x$ . So  $h$  is odd and hence  $(p - 1)^k = 1 \pmod{q}$  with  $0 \leq k < x < y$ . This contradicts the minimality of  $y$  unless  $k = 0$ , so  $n = hx$ . But  $x < q$ , so  $x = 1$ . So  $(p - 1) = -1 \pmod{q}$ .  $p$  and  $q$  are primes, so  $q = p$ , as claimed.

So  $p$  is the smallest prime divisor of  $n$ . We are also given that  $n \leq 2p$ . So either  $p = n$ , or  $p = 2$ ,  $n = 4$ . The latter does not work, so we have shown that  $n = p$ . Evidently  $n = p = 2$  and  $n = p = 3$  work. Assume now that  $p > 3$ . We show that there are no solutions of this type.

Expand  $(p - 1)^p + 1$  by the binomial theorem, to get (since  $(-1)^p = -1$ ):  $1 + -1 + p^2 - 1/2 p(p - 1)p^2 + p(p - 1)(p - 2)/6 p^3 - \dots$

The terms of the form (bin coeff)  $p^i$  with  $i \geq 3$  are obviously divisible by  $p^3$ , since the binomial coefficients are all integral. Hence the sum is  $p^2 +$  a multiple of  $p^3$ . So the sum is not divisible by  $p^3$ . But for  $p > 3$ ,  $p^{p-1}$  is divisible by  $p^3$ , so it cannot divide  $(p - 1)^p + 1$ , and there are no more solutions.



### Problem B2

The circles  $C_1$  and  $C_2$  lie inside the circle  $C$ , and are tangent to it at  $M$  and  $N$ , respectively.  $C_1$  passes through the center of  $C_2$ . The common chord of  $C_1$  and  $C_2$ , when extended, meets  $C$  at  $A$  and  $B$ . The lines  $MA$  and  $MB$  meet  $C_1$  again at  $E$  and  $F$ . Prove that the line  $EF$  is tangent to  $C_2$ .

**Solution** by Jean-Pierre Ehrmann

Let  $O, O_1, O_2$  and  $r, r_1, r_2$  be the centers and radii of  $C, C_1, C_2$  respectively. Let  $EF$  meet the line  $O_1O_2$  at  $W$ , and let  $O_2W = x$ . We need to prove that  $x = r_2$ .

Take rectangular coordinates with origin  $O_2$ ,  $x$ -axis  $O_2O_1$ , and let  $O$  have coordinates  $(a, b)$ . Notice that  $O$  and  $M$  do not, in general, lie on  $O_1O_2$ . Let  $AB$  meet the line  $O_1O_2$  at  $V$ .

We observe first that  $O_2V = r_2^2/(2r_1)$ . [For example, let  $X$  be a point of intersection of  $C_1$  and  $C_2$  and let  $Y$  be the midpoint of  $O_2X$ . Then  $O_1YO_2$  and  $XVO_2$  are similar. Hence,  $O_2V/O_2X = O_2Y/O_2O_1$ .]

An expansion (or, to be technical, a *homothecy*) center  $M$ , factor  $r/r_1$  takes  $O_1$  to  $O$  and  $EF$  to  $AB$ . Hence  $EF$  is perpendicular to  $O_1O_2$ . Also the distance of  $O_1$  from  $EF$  is  $r_1/r$  times the distance of  $O$  from  $AB$ , so  $(r_1 - x) = r_1/r (a - r_2^2/(2r_1))$  (\*).

We now need to find  $a$ . We can get two equations for  $a$  and  $b$  by looking at the distances of  $O$  from  $O_1$  and  $O_2$ . We have:

$$(r - r_1)^2 = (r_1 - a)^2 + b^2, \text{ and}$$

$$(r - r_2)^2 = a^2 + b^2.$$

Subtracting to eliminate  $b$ , we get  $a = r_2^2/(2r_1) + r - r_2/r_1$ . Substituting back in (\*), we get  $x = r_2$ , as required.

*Alternative solution by Marcin Kuczma, communicated Arne Smeets*





**A1.** In the convex quadrilateral  $ABCD$ , the diagonals  $AC$  and  $BD$  are perpendicular and the opposite sides  $AB$  and  $DC$  are not parallel. The point  $P$ , where the perpendicular bisectors of  $AB$  and  $DC$  meet, is inside  $ABCD$ . Prove that  $ABCD$  is cyclic if and only if the triangles  $ABP$  and  $CDP$  have equal areas.



**A2.** In a competition there are  $a$  contestants and  $b$  judges, where  $b \geq 3$  is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose  $k$  is a number such that for any two judges their ratings coincide for at most  $k$  contestants. Prove  $k/a \geq (b-1)/2b$ .



**A3.** For any positive integer  $n$ , let  $d(n)$  denote the number of positive divisors of  $n$  (including 1 and  $n$ ). Determine all positive integers  $k$  such that  $d(n^2) = k d(n)$  for some  $n$ .



**B1.** Determine all pairs  $(a, b)$  of positive integers such that  $ab^2 + b + 7$  divides  $a^2b + a + b$ .



**B2.** Let  $I$  be the incenter of the triangle  $ABC$ . Let the incircle of  $ABC$  touch the sides  $BC, CA, AB$  at  $K, L, M$  respectively. The line through  $B$  parallel to  $MK$  meets the lines  $LM$  and  $LK$  at  $R$  and  $S$  respectively. Prove that the angle  $RIS$  is acute.



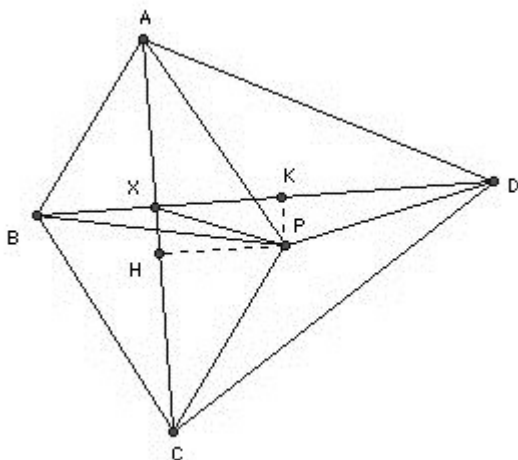
**B3.** Consider all functions  $f$  from the set of all positive integers into itself satisfying  $f(t^2f(s)) = s f(t)^2$  for all  $s$  and  $t$ . Determine the least possible value of  $f(1998)$ .



## Problem A1

In the convex quadrilateral  $ABCD$ , the diagonals  $AC$  and  $BD$  are perpendicular and the opposite sides  $AB$  and  $DC$  are not parallel. The point  $P$ , where the perpendicular bisectors of  $AB$  and  $DC$  meet, is inside  $ABCD$ . Prove that  $ABCD$  is cyclic if and only if the triangles  $ABP$  and  $CDP$  have equal areas.

### Solution



Let  $AC$  and  $BD$  meet at  $X$ . Let  $H, K$  be the feet of the perpendiculars from  $P$  to  $AC, BD$  respectively. We wish to express the areas of  $ABP$  and  $CDP$  in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have  $\text{area } PAB = \text{area } ABX + \text{area } PAX + \text{area } PBX$ , and  $\text{area } PCD = \text{area } CDX - \text{area } PCX - \text{area } PDX$ . So if the areas being equal is equivalent to:  $\text{area } ABX - \text{area } CDX + \text{area } PAX + \text{area } PCX + \text{area } PBX + \text{area } PDX = 0$ .  $ABX$  and  $CDX$  are right-angled, so we may write their areas as  $AX \cdot BX/2$  and  $CX \cdot DX/2$ . We may also put  $AX = AH - HX = AH - PK$ ,  $BX = BK - PH$ ,  $CX = CH + PK$ ,  $DX = DK + PH$ . The other triangles combine in pairs to give  $\text{area } ACP + \text{area } BDP = (AC \cdot PH + BD \cdot PK)/2$ . This leads, after some cancellation to  $AH \cdot BK = CH \cdot DK$ . There is a similar configuration with the roles of  $AB$  and  $CD$  reversed.

The second configuration is  $\text{area } PAB = \text{area } ABX + \text{area } PAX - \text{area } PBX$ ,  $\text{area } PCD = \text{area } CDX + \text{area } PDX - \text{area } PCX$ . In this case  $AX = AH + PK$ ,  $BX = BK - PH$ ,  $CX = CH - PK$ ,  $DX = DK + PH$ . But we end up with the same result:  $AH \cdot BK = CH \cdot DK$ .

Now if  $ABCD$  is cyclic, then it follows immediately that  $P$  is the center of the circumcircle and  $AH = CH, BK = DK$ . Hence the areas of  $PAB$  and  $PCD$  are equal.

Conversely, suppose the areas are equal. If  $PA > PC$ , then  $AH > CH$ . But since  $PA = PB$  and  $PC = PD$  (by construction),  $PB > PD$ , so  $BK > DK$ . So  $AH \cdot BK > CH \cdot DK$ . Contradiction. So  $PA$  is not greater than  $PC$ . Similarly it cannot be less. Hence  $PA = PC$ . But that implies  $PA = PB = PC = PD$ , so  $ABCD$  is cyclic.



### Problem A2

In a competition there are  $a$  contestants and  $b$  judges, where  $b \geq 3$  is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose  $k$  is a number such that for any two judges their ratings coincide for at most  $k$  contestants. Prove  $k/a \geq (b-1)/2b$ .

#### Solution

Let us count the number  $N$  of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There are  $b(b-1)/2$  pairs of judges in total and each pair rates at most  $k$  contestants the same, so  $N \leq kb(b-1)/2$ .

Now consider a fixed contestant  $X$  and count the number of pairs of judges rating  $X$  the same. Suppose  $x$  judges pass  $X$ , then there are  $x(x-1)/2$  pairs who pass  $X$  and  $(b-x)(b-x-1)/2$  who fail  $X$ , so a total of  $(x(x-1) + (b-x)(b-x-1))/2$  pairs rate  $X$  the same. But  $(x(x-1) + (b-x)(b-x-1))/2 = (2x^2 - 2bx + b^2 - b)/2 = (x - b/2)^2 + b^2/4 - b/2 \geq b^2/4 - b/2 = (b-1)^2/4 - 1/4$ . But  $(b-1)^2/4$  is an integer (since  $b$  is odd), so the number of pairs rating  $X$  the same is at least  $(b-1)^2/4$ . Hence  $N \geq a(b-1)^2/4$ . Putting the two inequalities together gives  $k/a \geq (b-1)/2b$ .

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### Problem A3

For any positive integer  $n$ , let  $d(n)$  denote the number of positive divisors of  $n$  (including 1 and  $n$ ). Determine all positive integers  $k$  such that  $d(n^2) = k d(n)$  for some  $n$ .

#### Solution

Let  $n = p_1^{a_1} \dots p_r^{a_r}$ . Then  $d(n) = (a_1 + 1)(a_2 + 1) \dots (a_r + 1)$ , and  $d(n^2) = (2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1)$ . So the  $a_i$  must be chosen so that  $(2a_1 + 1)(2a_2 + 1) \dots (2a_r + 1) = k(a_1 + 1)(a_2 + 1) \dots (a_r + 1)$ . Since all  $(2a_i + 1)$  are odd, this clearly implies that  $k$  must be odd. We show that conversely, given any odd  $k$ , we can find  $a_i$ .

We use a form of induction on  $k$ . First, it is true for  $k = 1$  (take  $n = 1$ ). Second, we show that if it is true for  $k$ , then it is true for  $2^m k - 1$ . That is sufficient, since any odd number has the form  $2^m k - 1$  for some smaller odd number  $k$ . Take  $a_i = 2^i((2^m - 1)k - 1)$  for  $i = 0, 1, \dots, m-1$ . Then  $2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$  and  $a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$ . So the product of the  $(2a_i + 1)$ 's divided by the product of the  $(a_i + 1)$ 's is  $2^m(2^m - 1)k - (2^m - 1)$  divided by  $(2^m - 1)k$ , or  $(2^m k - 1)/k$ . Thus if we take these  $a_i$ s together with those giving  $k$ , we get  $2^m k - 1$ , which completes the induction.

**Problem B1**

**Determine all pairs  $(a, b)$  of positive integers such that  $ab^2 + b + 7$  divides  $a^2b + a + b$ .**

**Answer**  $(a, b) = (11, 1), (49, 1)$  or  $(7k^2, 7k)$ .

**Solution**

If  $a < b$ , then  $b \geq a + 1$ , so  $ab^2 + b + 7 > ab^2 + b \geq (a + 1)(ab + 1) = a^2b + a + ab \geq a^2b + a + b$ . So there can be no solutions with  $a < b$ . Assume then that  $a \geq b$ .

Let  $k =$  the integer  $(a^2b + a + b)/(ab^2 + b + 7)$ . We have  $(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a + ab + 7a/b + 7/b + 1 > ab^2 + a + b$ . So  $k < a/b + 1/b$ . Now if  $b \geq 3$ , then  $(b - 7/b) > 0$  and hence  $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$ . Hence either  $b = 1$  or  $2$  or  $k > a/b - 1/b$ .

If  $a/b - 1/b < k < a/b + 1/b$ , then  $a - 1 < kb < a + 1$ . Hence  $a = kb$ . This gives the solution  $(a, b) = (7k^2, 7k)$ .

It remains to consider  $b = 1$  and  $2$ . If  $b = 1$ , then  $a + 8$  divides  $a^2 + a + 1$  and hence also  $a(a + 8) - (a^2 + a + 1) = 7a - 1$ , and hence also  $7(a + 8) - (7a - 1) = 57$ . The only factors bigger than  $8$  are  $19$  and  $57$ , so  $a = 11$  or  $49$ . It is easy to check that  $(a, b) = (11, 1)$  and  $(49, 1)$  are indeed solutions.

If  $b = 2$ , then  $4a + 9$  divides  $2a^2 + a + 2$ , and hence also  $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$ , and hence also  $7(4a + 9) - 4(7a - 4) = 79$ . The only factor greater than  $9$  is  $79$ , but that gives  $a = 35/2$  which is not integral. Hence there are no solutions for  $b = 2$ .

*A variant on this from Johannes Tang Lek Huo is as follows:*

We have  $ab^2 + b + 7$  divides  $a(ab^2 + b + 7) - b(a^2b + a + b) = 7a - b^2$ . If  $7a = b^2$ , then  $b$  must be a multiple of  $7$ , so  $b = 7k$  for some  $k$ . Then  $a = 7k^2$ , and it is easy to check that this is a solution. We cannot have  $7a < b^2$  for then  $0 < b^2 - 7a < ab^2 < ab^2 + b + 7$ . If  $7a > b^2$ , then we must have  $7a - b \leq ab^2 + b + 7 > ab^2$ , so  $7 > b^2$ , so  $b = 1$  or  $2$ .

We can then continue as above.





**Problem B2**

Let  $I$  be the incenter of the triangle  $ABC$ . Let the incircle of  $ABC$  touch the sides  $BC$ ,  $CA$ ,  $AB$  at  $K$ ,  $L$ ,  $M$  respectively. The line through  $B$  parallel to  $MK$  meets the lines  $LM$  and  $LK$  at  $R$  and  $S$  respectively. Prove that the angle  $RIS$  is acute.

**Solution**

We show that  $RI^2 + SI^2 - RS^2 > 0$ . The result then follows from the cosine rule.

$BI$  is perpendicular to  $MK$  and hence also to  $RS$ . So  $IR^2 = BR^2 + BI^2$  and  $IS^2 = BI^2 + BS^2$ . Obviously  $RS = RB + BS$ , so  $RS^2 = BR^2 + BS^2 + 2 BR \cdot BS$ . Hence  $RI^2 + SI^2 - RS^2 = 2 BI^2 - 2 BR \cdot BS$ . Consider the triangle  $BRS$ . The angles at  $B$  and  $M$  are  $90 - B/2$  and  $90 - A/2$ , so the angle at  $R$  is  $90 - C/2$ . Hence  $BR/BM = \cos A/2 / \cos C/2$  (using the sine rule). Similarly, considering the triangle  $BKS$ ,  $BS/BK = \cos C/2 / \cos A/2$ . So  $BR \cdot BS = BM \cdot BK = BK^2$ . Hence  $RI^2 + SI^2 - RS^2 = 2(BI^2 - BK^2) = 2 IK^2 > 0$ .

IMO 1998



**Problem B3**

Consider all functions  $f$  from the set of all positive integers into itself satisfying  $f(t^2 f(s)) = s f(t)^2$  for all  $s$  and  $t$ . Determine the least possible value of  $f(1998)$ .

**Answer :** 120

**Solution**

Let  $f(1) = k$ . Then  $f(kt^2) = f(t)^2$  and  $f(f(t)) = k^2 t$ . Also  $f(kt)^2 = 1 \cdot f(kt)^2 = f(k^3 t^2) = f(1^2 f(f(kt^2))) = k^2 f(kt^2) = k^2 f(t)^2$ . Hence  $f(kt) = k f(t)$ .

By an easy induction  $k^n f(t^{n+1}) = f(t)^{n+1}$ . But this implies that  $k$  divides  $f(t)$ . For suppose the highest power of a prime  $p$  dividing  $k$  is  $a > b$ , the highest power of  $p$  dividing  $f(t)$ . Then  $a > b(1 + 1/n)$  for some integer  $n$ . But then  $na > (n + 1)b$ , so  $k^n$  does not divide  $f(t)^{n+1}$ . Contradiction.

Let  $g(t) = f(t)/k$ . Then  $f(t^2 f(s)) = f(t^2 kg(s)) = k f(t^2 g(s)) = k^2 g(t^2 g(s))$ , whilst  $s f(t)^2 = k^2 s f(t)^2$ . So  $g(t^2 g(s)) = s g(t)^2$ . Hence  $g$  is also a function satisfying the conditions which evidently has smaller values than  $f$  (for  $k > 1$ ). It also satisfies  $g(1) = 1$ . Since we want the smallest possible value of  $f(1998)$  we may restrict attention to functions  $f$  satisfying  $f(1) = 1$ .

Thus we have  $f(f(t)) = t$  and  $f(t^2) = f(t)^2$ . Hence  $f(st)^2 = f(s^2 t^2) = f(s^2 f(f(t^2))) = f(s)^2 f(t)^2 = f(s)^2 f(t)^2$ . So  $f(st) = f(s) f(t)$ .

Suppose  $p$  is a prime and  $f(p) = m \cdot n$ . Then  $f(m)f(n) = f(mn) = f(f(p)) = p$ , so one of  $f(m), f(n) = 1$ . But if  $f(m) = 1$ , then  $m = f(f(m)) = f(1) = 1$ . So  $f(p)$  is prime. If  $f(p) = q$ , then  $f(q) = p$ .

Now we may define  $f$  arbitrarily on the primes subject only to the conditions that each  $f(\text{prime})$  is prime and that if  $f(p) = q$ , then  $f(q) = p$ . For suppose that  $s = p_1^{a_1} \dots p_r^{a_r}$  and that  $f(p_i) = q_i$ . If  $t$  has any additional prime factors not included in the  $q_i$ , then we may add additional  $p$ 's to the expression for  $s$  so that they are included (taking the additional  $a$ 's to be zero). So suppose  $t = q_1^{b_1} \dots q_r^{b_r}$ . Then  $t^2 f(s) = q_1^{2b_1+a_1} \dots q_r^{2b_r+a_r}$  and hence  $f(t^2 f(s)) = p_1^{2b_1+a_1} \dots p_r^{2b_r+a_r} = s f(t)^2$ .

We want the minimum possible value of  $f(1998)$ . Now  $1998 = 2 \cdot 3^3 \cdot 37$ , so we achieve the minimum value by taking  $f(2) = 3, f(3) = 2, f(37) = 5$  (and  $f(37) = 5$ ). This gives  $f(1998) = 3 \cdot 2^3 \cdot 5 = 120$ .



**A1.** In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white as on a chessboard. For any pair of positive integers  $m$  and  $n$ , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths  $m$  and  $n$ , lie along the edges of the squares. Let  $S_1$  be the total area of the black part of the triangle, and  $S_2$  be the total area of the white part. Let  $f(m, n) = |S_1 - S_2|$ .



- (a) Calculate  $f(m, n)$  for all positive integers  $m$  and  $n$  which are either both even or both odd.  
 (b) Prove that  $f(m, n) \leq \max(m, n)/2$  for all  $m, n$ .  
 (c) Show that there is no constant  $C$  such that  $f(m, n) < C$  for all  $m, n$ .

**A2.** The angle at  $A$  is the smallest angle in the triangle  $ABC$ . The points  $B$  and  $C$  divide the circumcircle of the triangle into two arcs. Let  $U$  be an interior point of the arc between  $B$  and  $C$  which does not contain  $A$ . The perpendicular bisectors of  $AB$  and  $AC$  meet the line  $AU$  at  $V$  and  $W$ , respectively. The lines  $BV$  and  $CW$  meet at  $T$ . Show that  $AU = TB + TC$ .



**A3.** Let  $x_1, x_2, \dots, x_n$  be real numbers satisfying  $|x_1 + x_2 + \dots + x_n| = 1$  and  $|x_i| \leq (n+1)/2$  for all  $i$ . Show that there exists a permutation  $y_i$  of  $x_i$  such that  $|y_1 + 2y_2 + \dots + ny_n| \leq (n+1)/2$ .



**B1.** An  $n \times n$  matrix whose entries come from the set  $S = \{1, 2, \dots, 2n-1\}$  is called a silver matrix if, for each  $i = 1, 2, \dots, n$ , the  $i$ th row and the  $i$ th column together contain all elements of  $S$ . Show that:



- (a) there is no silver matrix for  $n = 1997$ ;  
 (b) silver matrices exist for infinitely many values of  $n$ .

**B2.** Find all pairs  $(a, b)$  of positive integers that satisfy  $a^{b^2} = b^a$ .



**B3.** For each positive integer  $n$ , let  $f(n)$  denote the number of ways of representing  $n$  as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For example,  $f(4) = 4$ , because 4 can be represented as 4,  $2 + 2$ ,  $2 + 1 + 1$  or  $1 + 1 + 1 + 1$ . Prove that for any integer  $n \geq 3$ ,  $2^{n/4} < f(2^n) < 2^{n/2}$ .



### Problem A1

In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white as on a chessboard. For any pair of positive integers  $m$  and  $n$ , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths  $m$  and  $n$ , lie along the edges of the squares. Let  $S_1$  be the total area of the black part of the triangle, and  $S_2$  be the total area of the white part. Let  $f(m, n) = |S_1 - S_2|$ .

- (a) Calculate  $f(m, n)$  for all positive integers  $m$  and  $n$  which are either both even or both odd.  
 (b) Prove that  $f(m, n) \leq \max(m, n)/2$  for all  $m, n$ .  
 (c) Show that there is no constant  $C$  such that  $f(m, n) < C$  for all  $m, n$ .

### Solution

(a) If  $m$  and  $n$  are both even, then  $f(m, n) = 0$ . Let  $M$  be the midpoint of the hypotenuse. The critical point is that  $M$  is a lattice point. If we rotate the triangle through  $180^\circ$  to give the other half of the rectangle, we find that its coloring is the same. Hence  $S_1$  and  $S_2$  for the triangle are each half their values for the rectangle. But the values for the rectangle are equal, so they must also be equal for the triangle and hence  $f(m, n) = 0$ .

If  $m$  and  $n$  are both odd, then the midpoint of the hypotenuse is the center of a square and we may still find that the coloring of the two halves of the rectangle is the same. This time  $S_1$  and  $S_2$  differ by one for the rectangle, so  $f(m, n) = 1/2$ .

(b) The result is immediate from (a) for  $m$  and  $n$  of the same parity. The argument in (a) fails for  $m$  and  $n$  with opposite parity, because the two halves of the rectangle are oppositely colored. Let  $m$  be the odd side. Then if we extend the side length  $m$  by 1 we form a new triangle which contains the original triangle. But it has both sides even and hence  $S_1 = S_2$ . The area added is a triangle base 1 and height  $n$ , so area  $n/2$ . The worst case would be that all this area was the same color, in which case we would get  $f(m, n) = n/2$ . But  $n \leq \max(m, n)$ , so this establishes the result.

(c) Intuitively, it is clear that if the hypotenuse runs along the diagonal of a series of black squares, and we then extend one side, the extra area taken in will be mainly black. We need to make this rigorous. For the diagonal to run along the diagonal of black squares we must have  $n = m$ . It is easier to work out the white area added by extending a side. The white area takes the form of a series of triangles each similar to the new  $(n+1) \times n$  triangle. The biggest has sides 1 and  $n/(n+1)$ . The next biggest has sides  $(n-1)/n$  and  $(n-1)/(n+1)$ , the next biggest  $(n-2)/n$  and  $(n-2)/(n+1)$  and so on, down to the smallest which is  $1/n$  by  $1/(n+1)$ . Hence the additional white area is  $1/2 (n/(n+1) + (n-1)^2/(n(n+1)) + (n-2)^2/(n(n+1)) + \dots + 1/(n(n+1))) = 1/(2n(n+1)) (n^2 + \dots + 1^2) = (2n+1)/12$ . Hence the additional black area is  $n/2 - (2n+1)/12 = n/3 - 1/12$  and the black excess in the additional area is  $n/6 - 1/6$ . If  $n$  is even, then  $f(n, n) = 0$  for the original area, so for the new triangle  $f(n+1, n) = (n-1)/6$  which is unbounded.



### Problem A2

The angle at A is the smallest angle in the triangle ABC. The points B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respectively. The lines BV and CW meet at T. Show that  $AU = TB + TC$ .

### Solution

Extend BV to meet the circle again at X, and extend CW to meet the circle again at Y. Then by symmetry (since the perpendicular bisectors pass through the center of the circle)  $AU = BX$  and  $AU = CY$ . Also arc  $AX = \text{arc } BU$ , and arc  $AY = \text{arc } UC$ . Hence arc  $XY = \text{arc } BC$  and so angle  $BYC = \text{angle } XBY$  and hence  $TY = TB$ . So  $AU = CY = CT + TY = CT + TB$ .

### IMO 1997



### Problem A3

Let  $x_1, x_2, \dots, x_n$  be real numbers satisfying  $|x_1 + x_2 + \dots + x_n| = 1$  and  $|x_i| \leq (n+1)/2$  for all  $i$ . Show that there exists a permutation  $y_i$  of  $x_i$  such that  $|y_1 + 2y_2 + \dots + ny_n| \leq (n+1)/2$ .

### Solution

Without loss of generality we may assume  $x_1 + \dots + x_n = +1$ . [If not just reverse the sign of every  $x_i$ .] For any given arrangement  $x_i$  we use *sum* to mean  $x_1 + 2x_2 + 3x_3 + \dots + nx_n$ . Now if we add together the sums for  $x_1, x_2, \dots, x_n$  and the *reverse*  $x_n, x_{n-1}, \dots, x_1$ , we get  $(n+1)(x_1 + \dots + x_n) = n+1$ . So either we are home with the original arrangement or its reverse, or they have sums of opposite sign, one greater than  $(n+1)/2$  and one less than  $-(n+1)/2$ .

A transposition changes the sum from  $ka + (k+1)b + \text{other terms}$  to  $kb + (k+1)a + \text{other terms}$ . Hence it changes the sum by  $|a - b|$  (where  $a, b$  are two of the  $x_i$ ) which does not exceed  $n+1$ . Now we can get from the original arrangement to its reverse by a sequence of transpositions. Hence at some point in this sequence the sum must fall in the interval  $[-(n+1)/2, (n+1)/2]$  (because to get from a point below it to a point above it in a single step requires a jump of more than  $n+1$ ). That point gives us the required permutation.



### Problem B1

An  $n \times n$  matrix whose entries come from the set  $S = \{1, 2, \dots, 2n-1\}$  is called **silver matrix** if, for each  $i = 1, 2, \dots, n$ , the  $i$ th row and the  $i$ th column together contain all elements of  $S$ . Show that:

- (a) there is no silver matrix for  $n = 1997$ ;  
 (b) silver matrices exist for infinitely many values of  $n$ .

### Solution

(a) If we list all the elements in the rows followed by all the elements in the columns, then we have listed every element in the array twice, so each number in  $S$  must appear an even number of times. But considering the  $i$ th row with the  $i$ th column, we have also given  $n$  complete copies of  $S$  together with an additional copy of the numbers on the diagonal. If  $n$  is odd, then each of the  $2n-1$  numbers appears an odd number of times in the  $n$  complete copies, and at most  $n$  numbers can have this converted to an even number by an appearance on the diagonal. So there are no silver matrices for  $n$  odd. In particular, there is no silver matrix for  $n = 1997$ .

(b) Let  $A_{i,j}$  be an  $n \times n$  silver matrix with 1s down the main diagonal. Define the  $2n \times 2n$  matrix  $B_{i,j}$  with 1s down the main diagonal as follows:  $B_{i,j} = A_{i,j}$ ;  $B_{i+n,j+n} = A_{i,j}$ ;  $B_{i,j+n} = 2n + A_{i,j}$ ;  $B_{i+n,j} = 2n + A_{i,j}$  for  $i$  not equal  $j$  and  $B_{i+n,i} = 2n$ . We show that  $B_{i,j}$  is silver. Suppose  $i \leq n$ . Then the first half of the  $i$ th row is the  $i$ th row of  $A_{i,j}$ , and the top half of the  $i$ th column is the  $i$ th column of  $A_{i,j}$ , so between them those two parts comprise the numbers from 1 to  $2n - 1$ . The second half of the  $i$ th row is the  $i$ th row of  $A_{i,j}$  with each element increased by  $2n$ , and the bottom half of the  $i$ th column is the  $i$ th column of  $A_{i,j}$  with each element increased by  $2n$ , so between them they give the numbers from  $2n + 1$  to  $4n - 1$ . The only exception is that  $A_{i+n,i} = 2n$  instead of  $2n + A_{i,i}$ . We still get  $2n + A_{i,i}$  because it was in the second half of the  $i$ th row (these two parts do not have an element in common). The  $2n$  fills the gap so that in all we get all the numbers from 1 to  $4n - 1$ .

An exactly similar argument works for  $i > n$ . This time the second half of the row and the second half of the column (which overlap by one element) give us the numbers from 1 to  $2n - 1$ , and the first halves (which do not overlap) give us  $2n$  to  $4n - 1$ . So  $B_{i,j}$  is silver. Hence there are an infinite number of silver matrices.

## IMO 1997



### Problem B2

Find all pairs  $(a, b)$  of positive integers that satisfy  $a^{b^2} = b^a$ .

Answer : (1,1), (16,2), (27,3).

### Solution

Notice first that if we have  $a^m = b^n$ , then we must have  $a = c^e$ ,  $b = c^f$ , for some  $c$ , where  $m=fd$ ,  $n=ed$  and  $d$  is the greatest common divisor of  $m$  and  $n$ . [Proof: express  $a$  and  $b$  as products of primes in the usual way.]

In this case let  $d$  be the greatest common divisor of  $a$  and  $b^2$ , and put  $a = de$ ,  $b^2 = df$ . Then for some  $c$ ,  $a = c^e$ ,  $b = c^f$ . Hence  $f c^e = e c^{2f}$ . We cannot have  $e = 2f$ , for then the  $c$ 's cancel to give  $e = f$ . Contradiction. Suppose  $2f > e$ , then  $f = e c^{2f-e}$ . Hence  $e = 1$  and  $f = c^{2f-1}$ . If  $c = 1$ , then  $f = 1$  and we have the solution  $a = b = 1$ . If  $c \geq 2$ , then  $c^{2f-1} \geq 2^f > f$ , so there are no solutions.

Finally, suppose  $2f < e$ . Then  $e = f c^{e-2f}$ . Hence  $f = 1$  and  $e = c^{e-2}$ .  $c^{e-2} \geq 2^{e-2} \geq e$  for  $e \geq 5$ , so we must have  $e = 3$  or  $4$  ( $e > 2f = 2$ ).  $e = 3$  gives the solution  $a = 27$ ,  $b = 3$ .  $e = 4$  gives the solution  $a = 16$ ,  $b = 2$ .

## IMO 1997



### Problem 6

For each positive integer  $n$ , let  $f(n)$  denote the number of ways of representing  $n$  as a sum of powers of 2 with non-negative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For example,  $f(4) = 4$ , because 4 can be represented as 4,  $2 + 2$ ,  $2 + 1 + 1$  or  $1 + 1 + 1 + 1$ . Prove that for any integer  $n \geq 3$ ,  $2^{n^2/4} < f(2^n) < 2^{n^2/2}$ .

### Solution

The key is to derive a recurrence relation for  $f(n)$  [not for  $f(2^n)$ ]. If  $n$  is odd, then the sum must have a 1. In fact, there is a one-to-one correspondence between sums for  $n$  and sums for  $n-1$ . So:

$$f(2n+1) = f(2n)$$

Now consider  $n$  even. The same argument shows that there is a one-to-one correspondence between sums for  $n-1$  and sums for  $n$  which have a 1. Sums which do not have a 1 are in one-to-one correspondence with sums for  $n/2$  (just halve each term). So:

$$f(2n) = f(2n - 1) + f(n) = f(2n - 2) + f(n).$$

The upper limit is now almost immediate. First, the recurrence relations show that  $f$  is monotonic increasing. Now apply the second relation repeatedly to  $f(2^{n+1})$  to get:

$$f(2^{n+1}) = f(2^{n+1} - 2^n) + f(2^n - 2^{n-1} + 1) + \dots + f(2^n - 1) + f(2^n) = f(2^n) + f(2^n - 1) + \dots + f(2^{n-1} + 1) + f(2^n) \quad (*)$$

$$\text{and hence } f(2^{n+1}) \geq (2^{n-1} + 1)f(2^n)$$

We can now establish the upper limit by induction. It is false for  $n = 1$  and  $2$ , but almost true for  $n = 2$ , in that:  $f(2^2) = 2^{2^{2/2}}$ . Now if  $f(2^n) \leq 2^{n^{2/2}}$ , then the inequality just established shows that  $f(2^{n+1}) < 2^n 2^{n^{2/2}} < 2^{(n^2+2n+1)/2} = 2^{(n+1)^{2/2}}$ , so it is true for  $n + 1$ . Hence it is true for all  $n > 2$ .

Applying the same idea to the lower limit does not work. We need something stronger. We may continue (\*) inductively to obtain  $f(2^{n+1}) = f(2^n) + f(2^n - 1) + \dots + f(3) + f(2) + f(1) + 1$ . (\*\*)

We now use the following lemma:

$$f(1) + f(2) + \dots + f(2r) \geq 2r f(r)$$

We group the terms on the lhs into pairs and claim that  $f(1) + f(2r) \geq f(2) + f(2r-1) \geq f(3) + f(2r-2) \geq \dots \geq f(r) + f(r+1)$ . If  $k$  is even, then  $f(k) = f(k+1)$  and  $f(2r-k) = f(2r+1-k)$ , so  $f(k) + f(2r+1-k) = f(k+1) + f(2r-k)$ . If  $k$  is odd, then  $f(k+1) = f(k) + f((k+1)/2)$  and  $f(2r+1-k) = f(2r-k) + f((2r-k+1)/2)$ , but  $f$  is monotone so  $f((k+1)/2) \leq f((2r+1-k)/2)$  and hence  $f(k) + f(2r+1-k) \geq f(k+1) + f(2r-k)$ , as required.

Applying the lemma to (\*\*) gives  $f(2^{n+1}) > 2^{n+1}f(2^{n-1})$ . This is sufficient to prove the lower limit by induction. It is true for  $n = 1$ . Suppose it is true for  $n$ . Then  $f(2^{n+1}) > 2^{n+1}2^{(n-1)^{2/4}} = 2^{(n^2-2n+1+4n+4)/4} > 2^{(n+1)^{2/4}}$ , so it is true for  $n+1$ .





**A1.** We are given a positive integer  $r$  and a rectangular board divided into  $20 \times 12$  unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is  $\sqrt{r}$ . The task is to find a sequence of moves leading between two adjacent corners of the board which lie on the long side.



- (a) Show that the task cannot be done if  $r$  is divisible by 2 or 3.
- (b) Prove that the task is possible for  $r = 73$ .
- (c) Can the task be done for  $r = 97$ ?

**A2.** Let  $P$  be a point inside the triangle  $ABC$  such that  $\angle APB - \angle ACB = \angle APC - \angle ABC$ . Let  $D, E$  be the incenters of triangles  $APB, APC$  respectively. Show that  $AP, BD, CE$  meet at a point.



**A3.** Let  $S$  be the set of non-negative integers. Find all functions  $f: S \rightarrow S$  such that  $f(m + f(n)) = f(f(m)) + f(n)$  for all  $m, n$ .



**B1.** The positive integers  $a, b$  are such that  $15a + 16b$  and  $16a - 15b$  are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?



**B2.** Let  $ABCDEF$  be a convex hexagon such that  $AB$  is parallel to  $DE$ ,  $BC$  is parallel to  $EF$ , and  $CD$  is parallel to  $FA$ . Let  $R_A, R_C, R_E$  denote the circumradii of triangles  $FAB, BCD, DEF$  respectively, and let  $p$  denote the perimeter of the hexagon. Prove that:



$$R_A + R_C + R_E \geq p/2.$$

**B3.** Let  $p, q, n$  be three positive integers with  $p + q < n$ . Let  $x_0, x_1, \dots, x_n$  be integers such that  $x_0 = x_n = 0$ , and for each  $1 \leq i \leq n$ ,  $x_i - x_{i-1} = p$  or  $-q$ . Show that there exist indices  $i < j$  with  $(i, j) \neq (0, n)$  such that  $x_i = x_j$ .





### Problem A1

We are given a positive integer  $r$  and a rectangular board divided into  $20 \times 12$  unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is  $\sqrt{r}$ . The task is to find a sequence of moves leading between two adjacent corners of the board which lie on the long side.

- (a) Show that the task cannot be done if  $r$  is divisible by 2 or 3.  
 (b) Prove that the task is possible for  $r = 73$ .  
 (c) Can the task be done for  $r = 97$ ?

Answer No.

### Solution

(a) Suppose the move is  $a$  units in one direction and  $b$  in the orthogonal direction. So  $a^2 + b^2 = r$ . If  $r$  is divisible by 2, then  $a$  and  $b$  are both even or both odd. But that means that we can only access the black squares or the white squares (assuming the rectangle is colored like a chessboard). The two corners are of opposite color, so the task cannot be done. All squares are congruent to 0 or 1 mod 3, so if  $r$  is divisible by 3, then  $a$  and  $b$  must both be multiples of 3. That means that if the starting square has coordinates  $(0,0)$ , we can only move to squares of the form  $(3m,3n)$ . The required destination is  $(19,0)$  which is not of this form, so the task cannot be done.

(b) If  $r = 73$ , then we must have  $a = 8$ ,  $b = 3$  (or vice versa). There are 4 types of move:

- A:  $(x,y)$  to  $(x+8,y+3)$   
 B:  $(x,y)$  to  $(x+3,y+8)$   
 C:  $(x,y)$  to  $(x+8,y-3)$   
 D:  $(x,y)$  to  $(x+3,y-8)$

We regard  $(x,y)$  to  $(x-8,y-3)$  as a negative move of type A, and so on. Then if we have  $a$  moves of type A,  $b$  of type B and so on, then we require:

$$8(a + c) + 3(b + d) = 19; 3(a - c) + 8(b - d) = 0.$$

A simple solution is  $a = 5$ ,  $b = -1$ ,  $c = -3$ ,  $d = 2$ , so we start by looking for solutions of this type. After some fiddling we find:

$(0,0)$  to  $(8,3)$  to  $(16,6)$  to  $(8,9)$  to  $(11,1)$  to  $(19,4)$  to  $(11,7)$  to  $(19,10)$  to  $(16,2)$  to  $(8,5)$  to  $(16,8)$  to  $(19,0)$ .

(c) If  $r = 97$ , then we must have  $a = 9$ ,  $b = 4$ . As before, assume we start at  $(0,0)$ . A good deal of fiddling around fails to find a solution, so we look for reasons why one is impossible. Call moves which change  $y$  by 4 "toggle" moves. Consider the central strip  $y = 4, 5, 6$  or  $7$ . Toggle moves must toggle us in and out of the strip. Non-toggle moves cannot be made if we are in the strip and

keep us out of it if we are out of it. Toggle moves also change the parity of the x-coordinate, whereas non-toggle moves do not. Now we start and finish out of the strip, so we need an even number of toggle moves. On the other hand, we start with even x and end with odd x, so we need an odd number of toggle moves. Hence the task is impossible.

## IMO 1996

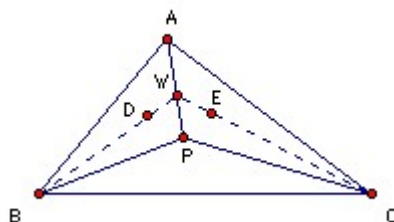


### Problem A2

Let  $P$  be a point inside the triangle  $ABC$  such that  $\angle APB - \angle ACB = \angle APC - \angle ABC$ . Let  $D, E$  be the incenters of triangles  $APB, APC$  respectively. Show that  $AP, BD, CE$  meet at a point.

### Solution

We need two general results: the angle bisector theorem; and the result about the feet of the perpendiculars from a general point inside a triangle. The second is not so well-known. Let  $P$  be a general point in the triangle  $ABC$  with  $X, Y, Z$  the feet of the perpendiculars to  $BC, CA, AB$ . Then  $PA = YZ/\sin A$  and  $\angle APB - \angle C = \angle XZY$ . To prove the first part:  $AP = AY/\sin APY = AY/\sin AZY$  (since  $AYPZ$  is cyclic)  $= YZ/\sin A$  (sine rule). To prove the second part:  $\angle XZY = \angle XZP + \angle YZP = \angle XBP + \angle YAP = 90^\circ - \angle XPB + 90^\circ - \angle YPA = 180^\circ - (360^\circ - \angle APB - \angle XPY) = -180^\circ + \angle APB + (180^\circ - \angle C) = \angle APB - \angle C$ .



So, returning to the problem,  $\angle APB - \angle C = \angle XZY$  and  $\angle APC - \angle B = \angle XYZ$ . Hence  $XYZ$  is isosceles:  $XY = XZ$ . Hence  $PC \sin C = PB \sin B$ . But  $AC \sin C = AB \sin B$ , so  $AB/PB = AC/PC$ . Let the angle bisector  $BD$  meet  $AP$  at  $W$ . Then, by the angle bisector theorem,  $AB/PB = AW/WP$ . Hence  $AW/WP = AC/PC$ , so, by the angle bisector theorem,  $CW$  is the bisector of angle  $ACP$ , as required.

**Problem A3**

Let  $S$  be the set of non-negative integers. Find all functions  $f: S \rightarrow S$  such that  $f(m + f(n)) = f(f(m)) + f(n)$  for all  $m, n$ .

**Solution**

Setting  $m = n = 0$ , the given relation becomes:  $f(f(0)) = f(f(0)) + f(0)$ . Hence  $f(0) = 0$ . Hence also  $f(f(0)) = 0$ . Setting  $m = 0$ , now gives  $f(f(n)) = f(n)$ , so we may write the original relation as  $f(m + f(n)) = f(m) + f(n)$ .

So  $f(n)$  is a fixed point. Let  $k$  be the smallest non-zero fixed point. If  $k$  does not exist, then  $f(n)$  is zero for all  $n$ , which is a possible solution. If  $k$  does exist, then an easy induction shows that  $f(qk) = qk$  for all non-negative integers  $q$ . Now if  $n$  is another fixed point, write  $n = kq + r$ , with  $0 \leq r < k$ . Then  $f(n) = f(r + f(kq)) = f(r) + f(kq) = kq + f(r)$ . Hence  $f(r) = r$ , so  $r$  must be zero. Hence the fixed points are precisely the multiples of  $k$ .

But  $f(n)$  is a fixed point for any  $n$ , so  $f(n)$  is a multiple of  $k$  for any  $n$ . Let us take  $n_1, n_2, \dots, n_{k-1}$  to be arbitrary non-negative integers and set  $n_0 = 0$ . Then the most general function satisfying the conditions we have established so far is:

$$f(qk + r) = qk + n_r k \text{ for } 0 \leq r < k.$$

We can check that this satisfies the functional equation. Let  $m = ak + r$ ,  $n = bk + s$ , with  $0 \leq r, s < k$ . Then  $f(f(m)) = f(m) = ak + n_r k$ , and  $f(n) = bk + n_s k$ , so  $f(m + f(n)) = ak + bk + n_r k + n_s k$ , and  $f(f(m)) + f(n) = ak + bk + n_r k + n_s k$ . So this is a solution and hence the most general solution.

**IMO 1996****Problem B1**

The positive integers  $a, b$  are such that  $15a + 16b$  and  $16a - 15b$  are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

**Answer**  $481^2$ .

**Solution**

Put  $15a + 16b = m^2$ ,  $16a - 15b = n^2$ . Then  $15m^2 + 16n^2 = 481a = 13 \cdot 37a$ . The quadratic residues mod 13 are  $0, \pm 1, \pm 3, \pm 4$ , so the residues of  $15m^2$  are  $0, \pm 2, \pm 5, \pm 6$ , and the residues of  $16n^2$  are  $0, \pm 1, \pm 3, \pm 4$ . Hence  $m$  and  $n$  must both be divisible by 13. Similarly, the quadratic residues of 37 are  $0, \pm 1, \pm 3, \pm 4, \pm 7, \pm 9, \pm 10, \pm 11, \pm 12, \pm 16$ , so the residues of  $15m^2$  are  $0, \pm 2, \pm 5, \pm 6, \pm 8, \pm 13$ ,

$\pm 14, \pm 15, \pm 17, \pm 18$ , and the residues of  $16n^2$  are  $0, \pm 1, \pm 3, \pm 4, \pm 7, \pm 9, \pm 10, \pm 11, \pm 12, \pm 16$ . Hence  $m$  and  $n$  must both be divisible by 37. Put  $m = 481m', n = 481n'$  and we get:  $a = 481(15m'^2 + 16n'^2)$ . We also have  $481b = 16m'^2 - 15n'^2$  and hence  $b = 481(16m'^2 - 15n'^2)$ . The smallest possible solution would come from putting  $m' = n' = 1$  and indeed that gives a solution.

This solution is straightforward, but something of a slog - all the residues have to be calculated. A more elegant variant is to notice that  $m^4 + n^4 = 481(a^2 + b^2)$ . Now if  $m$  and  $n$  are not divisible by 13 we have  $m^4 + n^4 = 0 \pmod{13}$ . Take  $k$  so that  $km = 1 \pmod{13}$ , then  $(nk)^4 = -(mk)^4 = -1 \pmod{13}$ . But that is impossible because then  $(nk)^{12} = -1 \pmod{13}$ , but  $x^{12} = 1 \pmod{13}$  for all non-zero residues. Hence  $m$  and  $n$  are both divisible by 13. The same argument shows that  $m$  and  $n$  are both divisible by 37.

## IMO 1996



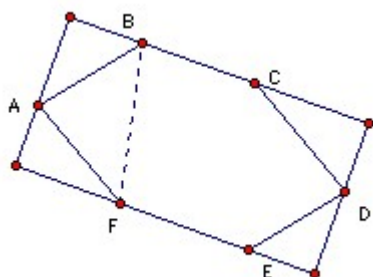
### Problem B2

Let  $ABCDEF$  be a convex hexagon such that  $AB$  is parallel to  $DE$ ,  $BC$  is parallel to  $EF$ , and  $CD$  is parallel to  $FA$ . Let  $R_A, R_C, R_E$  denote the circumradii of triangles  $FAB, BCD, DEF$  respectively, and let  $p$  denote the perimeter of the hexagon. Prove that:

$$R_A + R_C + R_E \geq p/2.$$

### Solution

The starting point is the formula for the circumradius  $R$  of a triangle  $ABC$ :  $2R = a/\sin A = b/\sin B = c/\sin C$ . [Proof: the side  $a$  subtends an angle  $2A$  at the center, so  $a = 2R \sin A$ .] This gives that  $2R_A = BF/\sin A$ ,  $2R_C = BD/\sin C$ ,  $2R_E = FD/\sin E$ . It is clearly not true in general that  $BF/\sin A > BA + AF$ , although it is true if angle  $FAB \geq 120^\circ$ , so we need some argument that involves the hexagon as a whole.



Extend sides  $BC$  and  $FE$  and take lines perpendicular to them through  $A$  and  $D$ , thus forming a rectangle. Then  $BF$  is greater than or equal to the side through  $A$  and the side through  $D$ . We may find the length of the side through  $A$  by taking the projections of  $BA$  and  $AF$  giving  $AB \sin B + AF \sin F$ . Similarly the side through  $D$  is  $CD \sin C + DE \sin E$ . Hence:

$$2BF \geq AB \sin B + AF \sin F + CD \sin C + DE \sin E. \quad \text{Similarly:}$$

$$2BD \geq BC \sin B + CD \sin D + AF \sin A + EF \sin E, \text{ and}$$

$$2FD \geq AB \sin A + BC \sin C + DE \sin D + EF \sin F.$$

Hence  $2BF/\sin A + 2BD/\sin C + 2FD/\sin E \geq AB(\sin A/\sin E + \sin B/\sin A) + BC(\sin B/\sin C + \sin C/\sin E) + CD(\sin C/\sin A + \sin D/\sin C) + DE(\sin E/\sin A + \sin D/\sin E) + EF(\sin E/\sin C + \sin F/\sin E) + AF(\sin F/\sin A + \sin A/\sin C)$ .

We now use the fact that opposite sides are parallel, which implies that opposite angles are equal:  $A = D$ ,  $B = E$ ,  $C = F$ . Each of the factors multiplying the sides in the last expression now has the form  $x + 1/x$  which has minimum value 2 when  $x = 1$ . Hence  $2(BF/\sin A + BD/\sin C + FD/\sin E) \geq 2p$  and the result is proved.

## IMO 1996



### Problem B3

Let  $p, q, n$  be three positive integers with  $p + q < n$ . Let  $x_0, x_1, \dots, x_n$  be integers such that  $x_0 = x_n = 0$ , and for each  $1 \leq i \leq n$ ,  $x_i - x_{i-1} = p$  or  $-q$ . Show that there exist indices  $i < j$  with  $(i, j)$  not  $(0, n)$  such that  $x_i = x_j$ .

### Solution

Let  $x_i - x_{i-1} = p$  occur  $r$  times and  $x_i - x_{i-1} = -q$  occur  $s$  times. Then  $r + s = n$  and  $pr = qs$ . If  $p$  and  $q$  have a common factor  $d$ , the  $y_i = x_i/d$  form a similar set with  $p/d$  and  $q/d$ . If the result is true for the  $y_i$  then it must also be true for the  $x_i$ . So we can assume that  $p$  and  $q$  are relatively prime. Hence  $p$  divides  $s$ . Let  $s = kp$ . If  $k = 1$ , then  $p = s$  and  $q = r$ , so  $p + q = r + s = n$ . But we are given  $p + q < n$ . Hence  $k > 1$ . Let  $p + q = n/k = h$ .

Up to this point everything is fairly obvious and the result looks as though it should be easy, but I did not find it so. Some fiddling around with examples suggested that we seem to get  $x_i = x_j$  for  $j = i + h$ . We observe first that  $x_{i+h} - x_i$  must be a multiple of  $h$ . For suppose  $e$  differences are  $p$ , and hence  $h-e$  are  $-q$ . Then  $x_{i+h} - x_i = ep - (h - e)q = (e - q)h$ .

The next step is not obvious. Let  $d_i = x_{i+h} - x_i$ . We know that all  $d_i$ s are multiples of  $h$ . We wish to show that at least one is zero. Now  $d_{i+1} - d_i = (x_{i+h+1} - x_{i+h}) - (x_{i+1} - x_i) = (p \text{ or } -q) - (p \text{ or } -q) = 0, h$  or  $-h$ . So if neither of  $d_i$  nor  $d_{i+1}$  are zero, then either both are positive or both are negative (a jump from positive to negative would require a difference of at least  $2h$ ). Hence if none of the  $d_i$ s are zero, then all of them are positive, or all of them are negative. But  $d_0 + d_h + \dots + d_{kh}$  is a concertina sum with value  $x_n - x_0 = 0$ . So this subset of the  $d_i$ s cannot



**A1.** Let  $A, B, C, D$  be four distinct points on a line, in that order. The circles with diameter  $AC$  and  $BD$  intersect at  $X$  and  $Y$ . The line  $XY$  meets  $BC$  at  $Z$ . Let  $P$  be a point on the line  $XY$  other than  $Z$ . The line  $CP$  intersects the circle with diameter  $AC$  at  $C$  and  $M$ , and the line  $BP$  intersects the circle with diameter  $BD$  at  $B$  and  $N$ . Prove that the lines  $AM, DN, XY$  are concurrent.



**A2.** Let  $a, b, c$  be positive real numbers with  $abc = 1$ . Prove that:

$$1/(a^3(b+c)) + 1/(b^3(c+a)) + 1/(c^3(a+b)) \geq 3/2.$$



**A3.** Determine all integers  $n > 3$  for which there exist  $n$  points  $A_1, \dots, A_n$  in the plane, no three collinear, and real numbers  $r_1, \dots, r_n$  such that for any distinct  $i, j, k$ , the area of the triangle  $A_i A_j A_k$  is  $r_i + r_j + r_k$ .



**B1.** Find the maximum value of  $x_0$  for which there exists a sequence  $x_0, x_1, \dots, x_{1995}$  of positive reals with  $x_0 = x_{1995}$  such that for  $i = 1, \dots, 1995$ :

$$x_{i-1} + 2/x_{i-1} = 2x_i + 1/x_i.$$



**B2.** Let  $ABCDEF$  be a convex hexagon with  $AB = BC = CD$  and  $DE = EF = FA$ , such that  $\angle BCD = \angle EFA = 60^\circ$ . Suppose that  $G$  and  $H$  are points in the interior of the hexagon such that  $\angle AGB = \angle DHE = 120^\circ$ . Prove that  $AG + GB + GH + DH + HE \geq CF$ .



**B3.** Let  $p$  be an odd prime number. How many  $p$ -element subsets  $A$  of  $\{1, 2, \dots, 2p\}$  are there, the sum of whose elements is divisible by  $p$ ?

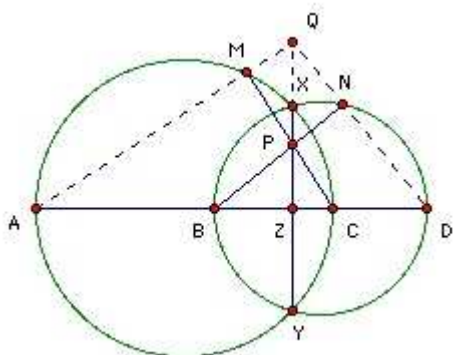




**Problem A1**

Let  $A, B, C, D$  be four distinct points on a line, in that order. The circles with diameter  $AC$  and  $BD$  intersect at  $X$  and  $Y$ . The line  $XY$  meets  $BC$  at  $Z$ . Let  $P$  be a point on the line  $XY$  other than  $Z$ . The line  $CP$  intersects the circle with diameter  $AC$  at  $C$  and  $M$ , and the line  $BP$  intersects the circle with diameter  $BD$  at  $B$  and  $N$ . Prove that the lines  $AM, DN, XY$  are concurrent.

**Solution**



Let  $DN$  meet  $XY$  at  $Q$ . Angle  $QDZ = 90^\circ - \text{angle } NBD = \text{angle } BPZ$ . So triangles  $QDZ$  and  $BPZ$  are similar. Hence  $QZ/DZ = BZ/PZ$ , or  $QZ = BZ \cdot DZ / PZ$ . Let  $AM$  meet  $XY$  at  $Q'$ . Then the same argument shows that  $Q'Z = AZ \cdot CZ / PZ$ . But  $BZ \cdot DZ = XZ \cdot YZ = AZ \cdot CZ$ , so  $QZ = Q'Z$ . Hence  $Q$  and  $Q'$  coincide.



**Problem A2**

Let  $a, b, c$  be positive real numbers with  $abc = 1$ . Prove that:

$$1/(a^3(b+c)) + 1/(b^3(c+a)) + 1/(c^3(a+b)) \geq 3/2.$$

**Solution**

Put  $a = 1/x, b = 1/y, c = 1/z$ . Then  $1/(a^3(b+c)) = x^3yz/(y+z) = x^2/(y+z)$ . Let the expression given be  $E$ . Then by Cauchy's inequality we have  $(y+z + z+x + x+y)E \geq (x+y+z)^2$ , so  $E \geq (x+y+z)/2$ . But applying the arithmetic/geometric mean result to  $x, y, z$  gives  $(x+y+z) \geq 3$ . Hence result.

*Thanks to Gerhard Woeginger for pointing out that the original solution was wrong.*





### Problem A3

Determine all integers  $n > 3$  for which there exist  $n$  points  $A_1, \dots, A_n$  in the plane, no three collinear, and real numbers  $r_1, \dots, r_n$  such that for any distinct  $i, j, k$ , the area of the triangle  $A_i A_j A_k$  is  $r_i + r_j + r_k$ .

**Answer**  $n = 4$ .

### Solution

The first point to notice is that if no arrangement is possible for  $n$ , then no arrangement is possible for any higher integer. Clearly the four points of a square work for  $n = 4$ , so we focus on  $n = 5$ .

If the 5 points form a convex pentagon, then considering the quadrilateral  $A_1 A_2 A_3 A_4$  as made up of two triangles in two ways, we have that  $r_1 + r_3 = r_2 + r_4$ . Similarly,  $A_5 A_1 A_2 A_3$  gives  $r_1 + r_3 = r_2 + r_5$ , so  $r_4 = r_5$ .

We show that we cannot have two  $r$ 's equal (whether or not the 4 points form a convex pentagon). For suppose  $r_4 = r_5$ . Then  $A_1 A_2 A_4$  and  $A_1 A_2 A_5$  have equal area. If  $A_4$  and  $A_5$  are on the same side of the line  $A_1 A_2$ , then since they must be equal distances from it,  $A_4 A_5$  is parallel to  $A_1 A_2$ . If they are on opposite sides, then the midpoint of  $A_4 A_5$  must lie on  $A_1 A_2$ . The same argument can be applied to  $A_1$  and  $A_3$ , and to  $A_2$  and  $A_3$ . But we cannot have two of  $A_1 A_2$ ,  $A_1 A_3$  and  $A_2 A_3$  parallel to  $A_4 A_5$ , because then  $A_1, A_2$  and  $A_3$  would be collinear. We also cannot have the midpoint of  $A_4 A_5$  lying on two of  $A_1 A_2$ ,  $A_1 A_3$  and  $A_2 A_3$  for the same reason. So we have established a contradiction. Hence no two of the  $r$ 's can be equal. In particular, this shows that the 5 points cannot form a convex pentagon.

Suppose the convex hull is a quadrilateral. Without loss of generality, we may take it to be  $A_1 A_2 A_3 A_4$ .  $A_5$  must lie inside one of  $A_1 A_2 A_4$  and  $A_2 A_3 A_4$ . Again without loss of generality we may take it to be the latter, so that  $A_1 A_2 A_5 A_4$  is also a convex quadrilateral. Then  $r_2 + r_4 = r_1 + r_3$  and also  $= r_1 + r_5$ . So  $r_3 = r_5$ , giving a contradiction as before.

The final case is the convex hull a triangle, which we may suppose to be  $A_1 A_2 A_3$ . Each of the other two points divides its area into three triangles, so we have:  $(r_1 + r_2 + r_4) + (r_2 + r_3 + r_4) + (r_3 + r_1 + r_4) = (r_1 + r_2 + r_5) + (r_2 + r_3 + r_5) + (r_3 + r_1 + r_5)$  and hence  $r_4 = r_5$ , giving a contradiction.

So the arrangement is not possible for 5 and hence not for any  $n > 5$ .



### Problem B1

Find the maximum value of  $x_0$  for which there exists a sequence  $x_0, x_1, \dots, x_{1995}$  of positive reals with  $x_0 = x_{1995}$  such that for  $i = 1, \dots, 1995$ :

$$x_{i-1} + 2/x_{i-1} = 2x_i + 1/x_i.$$

**Answer**  $2^{997}$ .

**Solution** : The relation given is a quadratic in  $x_i$ , so it has two solutions, and by inspection these are  $x_i = 1/x_{i-1}$  and  $x_i = x_{i-1}/2$ . For an even number of moves we can start with an arbitrary  $x_0$  and get back to it. Use  $n-1$  halvings, then take the inverse, that gets to  $2^{n-1}/x_0$  after  $n$  moves. Repeating brings you back to  $x_0$  after  $2n$  moves. However, 1995 is odd!

The sequence given above brings us back to  $x_0$  after  $n$  moves, provided that  $x_0 = 2^{(n-1)/2}$ . We show that this is the largest possible  $x_0$ . Suppose we have  $a$  halvings followed by an inverse followed by  $b$  halvings followed by an inverse. Then if the number of inverses is odd we end up with  $2^{a-b+c-\dots}/x_0$ , and if it is even we end up with  $x_0/2^{a-b+c-\dots}$ . In the first case, since the final number is  $x_0$  we must have  $x_0 = 2^{(a-b+\dots)/2}$ . All the  $a, b, \dots$  are non-negative and sum to the number of moves less the number of inverses, so we clearly maximise  $x_0$  by taking a single inverse and  $a = n-1$ . In the second case, we must have  $2^{a-b+c-\dots} = 1$  and hence  $a - b + c - \dots = 0$ . But that implies that  $a + b + c + \dots$  is even and hence the total number of moves is even, which it is not. So we must have an odd number of inverses and the maximum value of  $x_0$  is  $2^{(n-1)/2}$ .

### IMO 1995



### Problem B2

Let  $ABCDEF$  be convex hexagon with  $AB = BC = CD$  and  $DE = EF = FA$ , such that  $\angle BCD = \angle EFA = 60^\circ$ . Suppose that  $G$  and  $H$  are points in the interior of the hexagon such that  $\angle AGB = \angle DHE = 120^\circ$ . Prove that  $AG + GB + GH + DH + HE \geq CF$ .

#### Solution

$BCD$  is an equilateral triangle and  $AEF$  is an equilateral triangle. The presence of equilateral triangles and quadrilaterals suggests using Ptolemy's inequality. [If this is unfamiliar, see [ASU 61/6 solution](#)]. From  $CBGD$ , we get  $CG \cdot BD \leq BG \cdot CD + GD \cdot CB$ , so  $CG \leq BG + GD$ . Similarly from  $HAFE$  we get  $HF \leq HA + HE$ . Also  $CF$  is shorter than the indirect path  $C$  to  $G$  to  $H$  to  $F$ , so  $CF \leq CG + GH + HF$ . But we do not get quite what we want.

However, a slight modification of the argument does work.  $BAED$  is symmetrical about  $BE$  (because  $BA = BD$  and  $EA = ED$ ). So we may take  $C'$  the reflection of  $C$  in the line  $BE$  and  $F'$  the reflection of  $F$ . Now  $C'AB$  and  $F'ED$  are still equilateral, so the same argument gives  $C'G \geq AG + GB$  and  $HF' \leq DH + HE$ . So  $CF = C'F' \leq C'G + GH + HF' \leq AG + GB + GH + DH + HE$ .

**Problem B3**

Let  $p$  be an odd prime number. How many  $p$ -element subsets  $A$  of  $\{1, 2, \dots, 2p\}$  are there, the sum of whose elements is divisible by  $p$ ?

**Answer :**  $2 + (2^p C_p - 2)/p$ , where  $2^p C_p$  is the binomial coefficient  $(2p)!/(p!p!)$ .

**Solution**

Let  $A$  be a subset other than  $\{1, 2, \dots, p\}$  and  $\{p+1, p+2, \dots, 2p\}$ . Consider the elements of  $A$  in  $\{1, 2, \dots, p\}$ . The number  $r$  satisfies  $0 < r < p$ . We can change these elements to another set of  $r$  elements of  $\{1, 2, \dots, p\}$  by adding 1 to each element (and reducing mod  $p$  if necessary). We can repeat this process and get  $p$  sets in all. For example, if  $p = 7$  and the original subset of  $\{1, 2, \dots, 7\}$  was  $\{3, 5\}$ , we get:

$\{3, 5\}, \{4, 6\}, \{5, 7\}, \{6, 1\}, \{7, 2\}, \{1, 3\}, \{2, 4\}$ .

The sum of the elements in the set is increased by  $r$  each time. So, since  $p$  is prime, the sums must form a complete set of residues mod  $p$ . In particular, they must all be distinct and hence all the subsets must be different.

Now consider the sets  $A$  which have a given intersection with  $\{p+1, \dots, 2p\}$ . Suppose the elements in this intersection sum to  $k$  mod  $p$ . The sets can be partitioned into groups of  $p$  by the process described above, so that exactly one member of each group will have the sum  $-k$  mod  $p$  for its elements in  $\{1, 2, \dots, p\}$ . In other words, exactly one member of each group will have the sum of all its elements divisible by  $p$ .

There are  $2^p C_p$  subsets of  $\{1, 2, \dots, 2p\}$  of size  $p$ . Excluding  $\{1, 2, \dots, p\}$  and  $\{p+1, \dots, 2p\}$  leaves  $(2^p C_p - 2)$ . We have just shown that  $(2^p C_p - 2)/p$  of these have sum divisible by  $p$ . The two excluded subsets also have sum divisible by  $p$ , so there are  $2 + (2^p C_p - 2)/p$  subsets in all having sum divisible by  $p$ .



**A1.** Let  $m$  and  $n$  be positive integers. Let  $a_1, a_2, \dots, a_m$  be distinct elements of  $\{1, 2, \dots, n\}$  such that whenever  $a_i + a_j \leq n$  for some  $i, j$  (possibly the same) we have  $a_i + a_j = a_k$  for some  $k$ . Prove that:

$$(a_1 + \dots + a_m)/m \geq (n + 1)/2.$$



**A2.**  $ABC$  is an isosceles triangle with  $AB = AC$ .  $M$  is the midpoint of  $BC$  and  $O$  is the point on the line  $AM$  such that  $OB$  is perpendicular to  $AB$ .  $Q$  is an arbitrary point on  $BC$  different from  $B$  and  $C$ .  $E$  lies on the line  $AB$  and  $F$  lies on the line  $AC$  such that  $E, Q, F$  are distinct and collinear. Prove that  $OQ$  is perpendicular to  $EF$  if and only if  $QE = QF$ .



**A3.** For any positive integer  $k$ , let  $f(k)$  be the number of elements in the set  $\{k+1, k+2, \dots, 2k\}$  which have exactly three 1s when written in base 2. Prove that for each positive integer  $m$ , there is at least one  $k$  with  $f(k) = m$ , and determine all  $m$  for which there is exactly one  $k$ .



**B1.** Determine all ordered pairs  $(m, n)$  of positive integers for which  $(n^3 + 1)/(mn - 1)$  is an integer.



**B2.** Let  $S$  be the set of all real numbers greater than  $-1$ . Find all functions  $f$  from  $S$  into  $S$  such that  $f(x + f(y)) + xf(y) = y + f(x) + yf(x)$  for all  $x$  and  $y$ , and  $f(x)/x$  is strictly increasing on each of the intervals  $-1 < x < 0$  and  $0 < x$ .



**B3.** Show that there exists a set  $A$  of positive integers with the following property: for any infinite set  $S$  of primes, there exist two positive integers  $m$  in  $A$  and  $n$  not in  $A$ , each of which is a product of  $k$  distinct elements of  $S$  for some  $k \geq 2$ .



**Problem A1**

Let  $m$  and  $n$  be positive integers. Let  $a_1, a_2, \dots, a_m$  be distinct elements of  $\{1, 2, \dots, n\}$  such that whenever  $a_i + a_j \leq n$  for some  $i, j$  (possibly the same) we have  $a_i + a_j = a_k$  for some  $k$ . Prove that:

$$(a_1 + \dots + a_m)/m \geq (n + 1)/2.$$

**Solution**

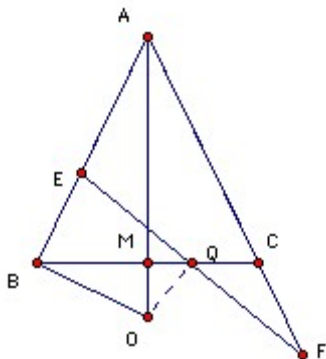
Take  $a_1 < a_2 < \dots < a_m$ . Take  $k \leq (m+1)/2$ . We show that  $a_k + a_{m-k+1} \geq n + 1$ . If not, then the  $k$  distinct numbers  $a_1 + a_{m-k+1}, a_2 + a_{m-k+1}, \dots, a_k + a_{m-k+1}$  are all  $\leq n$  and hence equal to some  $a_i$ . But they are all greater than  $a_{m-k+1}$ , so each  $i$  satisfies  $m-k+2 \leq i \leq m$ , which is impossible since there are only  $k-1$  available numbers in the range.



**Problem A2**

$ABC$  is an isosceles triangle with  $AB = AC$ .  $M$  is the midpoint of  $BC$  and  $O$  is the point on the line  $AM$  such that  $OB$  is perpendicular to  $AB$ .  $Q$  is an arbitrary point on  $BC$  different from  $B$  and  $C$ .  $E$  lies on the line  $AB$  and  $F$  lies on the line  $AC$  such that  $E, Q, F$  are distinct and collinear. Prove that  $OQ$  is perpendicular to  $EF$  if and only if  $QE = QF$ .

**Solution**



Assume  $OQ$  is perpendicular to  $EF$ . Then  $\angle EBO = \angle EQO = 90^\circ$ , so  $EBOQ$  is cyclic. Hence  $\angle OEQ = \angle OBQ$ . Also  $\angle OQF = \angle OCF = 90^\circ$ , so  $OQCF$  is cyclic. Hence  $\angle OFQ = \angle OCQ$ .

But  $\angle OCQ = \angle OBQ$  since ABC is isosceles. Hence  $\angle OEQ = \angle OFQ$ , so  $OE = OF$ , so triangles OEQ and OFQ are congruent and  $QE = QF$ .

Assume  $QE = QF$ . If OQ is not perpendicular to EF, then take E'F' through Q perpendicular to OQ with E' on AB and F' on AC. Then  $QE' = QF'$ , so triangles QEE' and QFF' are congruent. Hence  $\angle QEE' = \angle QFF'$ . So CA and AB make the same angles with EF and hence are parallel. Contradiction. So OQ is perpendicular to EF.

## IMO 1994



### Problem A3

For any positive integer  $k$ , let  $f(k)$  be the number of elements in the set  $\{k+1, k+2, \dots, 2k\}$  which have exactly three 1s when written in base 2. Prove that for each positive integer  $m$ , there is at least one  $k$  with  $f(k) = m$ , and determine all  $m$  for which there is exactly one  $k$ .

Answer : 2, 4, ... ,  $n(n-1)/2 + 1$ , ... .

### Solution

To get a feel, we calculate the first few values of  $f$  explicitly:

$f(2) = 0, f(3) = 0$   
 $f(4) = f(5) = 1, [7 = 111]$   
 $f(6) = 2, [7 = 111, 11 = 1011]$   
 $f(7) = f(8) = f(9) = 3 [11 = 1011, 13 = 1101, 14 = 1110]$   
 $f(10) = 4 [11, 13, 14, 19 = 10011]$   
 $f(11) = f(12) = 5 [13, 14, 19, 21 = 10101, 22 = 10110]$   
 $f(13) = 6 [14, 19, 21, 22, 25 = 11001, 26 = 11010]$

We show that  $f(k+1) = f(k)$  or  $f(k) + 1$ . The set for  $k+1$  has the additional elements  $2k+1$  and  $2k+2$  and it loses the element  $k+1$ . But the binary expression for  $2k+2$  is the same as that for  $k+1$  with the addition of a zero at the end, so  $2k+2$  and  $k+1$  have the same number of 1s. So if  $2k+1$  has three 1s, then  $f(k+1) = f(k) + 1$ , otherwise  $f(k+1) = f(k)$ . Now clearly an infinite number of numbers  $2k+1$  have three 1s, (all numbers  $2^r + 2^s + 1$  for  $r > s > 0$ ). So  $f(k)$  increases without limit, and since it only moves up in increments of 1, it never skips a number. In other words, given any positive integer  $m$  we can find  $k$  so that  $f(k) = m$ .

From the analysis in the last paragraph we can only have a single  $k$  with  $f(k) = m$  if both  $2k-1$  and  $2k+1$  have three 1s, or in other words if both  $k-1$  and  $k$  have two 1s. Evidently this happens when  $k-1$  has the form  $2^n + 1$ . This determines the  $k$ , namely  $2^n + 2$ , but we need to determine the corresponding  $m = f(k)$ . It is the number of elements of  $\{2^n+3, 2^n+4, \dots, 2^{n+1}+4\}$  which have three 1s. Elements with three 1s are either  $2^n+2^r+2^s$  with  $0 \leq r < s < n$ , or  $2^{n+1}+3$ . So there are  $m = n(n-1)/2 + 1$  of them. As a check, for  $n = 2$ , we have  $k = 2^2+2 = 6$ ,  $m = 2$ , and for  $n = 3$ , we have  $k = 2^3+2 = 10$ ,  $m = 4$ , which agrees with the  $f(6) = 2, f(10) = 4$  found earlier.

**Problem B1**

**Determine all ordered pairs  $(m, n)$  of positive integers for which  $(n^3 + 1)/(mn - 1)$  is an integer.**

**Answer :**  $(1, 2), (1, 3), (2, 1), (2, 2), (2, 5), (3, 1), (3, 5), (5, 2), (5, 3).$

**Solution**

We start by checking small values of  $n$ .  $n = 1$  gives  $n^3 + 1 = 2$ , so  $m = 2$  or  $3$ , giving the solutions  $(2, 1)$  and  $(3, 1)$ . Similarly,  $n = 2$  gives  $n^3 + 1 = 9$ , so  $2m - 1 = 1, 3$  or  $9$ , giving the solutions  $(1, 2), (2, 2), (5, 2)$ . Similarly,  $n = 3$  gives  $n^3 + 1 = 28$ , so  $3m - 1 = 2, 14$ , giving the solutions  $(1, 3), (5, 3)$ . So we assume hereafter that  $n > 3$ .

Let  $n^3 + 1 = (mn - 1)h$ . Then we must have  $h \equiv -1 \pmod{n}$ . Put  $h = kn - 1$ . Then  $n^3 + 1 = mkn^2 - (m + k)n + 1$ . Hence  $n^2 = mkn - (m + k)$ . (\*) Hence  $n$  divides  $m + k$ . If  $m + k \geq 3n$ , then since  $n > 3$  we have at least one of  $m, k \geq n + 2$ . But then  $(mn - 1)(kn - 1) \geq (n^2 + 2n - 1)(n - 1) = n^3 + n^2 - 3n + 1 = (n^3 + 1) + n(n - 3) > n^3 + 1$ . So we must have  $m + k = n$  or  $2n$ .

Consider first  $m + k = n$ . We may take  $m \geq k$  (provided that we remember that if  $m$  is a solution, then so is  $n - m$ ). So (\*) gives  $n = m(n - m) - 1$ . Clearly  $m = n - 1$  is not a solution. If  $m = n - 2$ , then  $n = 2(n - 2) - 1$ , so  $n = 5$ . This gives the two solutions  $(m, n) = (2, 5)$  and  $(3, 5)$ . If  $m < n - 2$  then  $n - m \geq 3$  and so  $m(n - m) - 1 \geq 3m - 1 \geq 3n/2 - 1 > n$  for  $n > 3$ .

Finally, take  $m + k = 2n$ . So (\*) gives  $n + 2 = m(2n - m)$ . Again we may take  $m \geq k$ .  $m = 2n - 1$  is not a solution (we are assuming  $n > 3$ ). So  $2n - m \geq 2$ , and hence  $m(2n - m) \geq 2m \geq 2n > n + 2$ .

**IMO 1994****Problem B2**

**Let  $S$  be the set of all real numbers greater than  $-1$ . Find all functions  $f : S \rightarrow S$  such that  $f(x + f(y)) + xf(y) = y + f(x) + yf(x)$  for all  $x$  and  $y$ , and  $f(x)/x$  is strictly increasing on each of the intervals  $-1 < x < 0$  and  $0 < x$ .**

**Answer :**  $f(x) = -x/(x+1)$ .

**Solution**

Suppose  $f(a) = a$ . Then putting  $x = y = a$  in the relation given, we get  $f(b) = b$ , where  $b = 2a + a^2$ . If  $-1 < a < 0$ , then  $-1 < b < a$ . But  $f(a)/a = f(b)/b$ . Contradiction. Similarly, if  $a > 0$ , then  $b > a$ , but  $f(a)/a = f(b)/b$ . Contradiction. So we must have  $a = 0$ .

But putting  $x = y$  in the relation given we get  $f(k) = k$  for  $k = x + f(x) + xf(x)$ . Hence for any  $x$  we have  $x + f(x) + xf(x) = 0$  and hence  $f(x) = -x/(x+1)$ .

Finally, it is straightforward to check that  $f(x) = -x/(x+1)$  satisfies the two conditions.

## IMO 1994



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### Problem B3

**Show that there exists a set  $A$  of positive integers with the following property: for any infinite set  $S$  of primes, there exist two positive integers  $m$  in  $A$  and  $n$  not in  $A$ , each of which is a product of  $k$  distinct elements of  $S$  for some  $k \geq 2$ .**

### Solution

Let the primes be  $p_1 < p_2 < p_3 < \dots$ . Let  $A$  consists of all products of  $n$  distinct primes such that the smallest is greater than  $p_n$ . For example: all primes except 2 are in  $A$ ; 21 is not in  $A$  because it is a product of two distinct primes and the smallest is greater than 3. Now let  $S = \{p_{i_1}, p_{i_2}, \dots\}$  be any infinite set of primes. Assume that  $p_{i_1} < p_{i_2} < \dots$ . Let  $n = i_1$ . Then  $p_{i_1}p_{i_2} \dots p_{i_n}$  is not in  $A$  because it is a product of  $n$  distinct primes, but the smallest is not greater than  $p_n$ . But  $p_{i_2}p_{i_3} \dots p_{i_{n+1}}$  is in  $A$ , because it is a product of  $n$  distinct primes and the smallest is greater than  $p_n$ . But both numbers are products of  $n$  distinct elements of  $S$ .





**A1.** Let  $f(x) = x^n + 5x^{n-1} + 3$ , where  $n > 1$  is an integer. Prove that  $f(x)$  cannot be expressed as the product of two non-constant polynomials with integer coefficients.



**A2.** Let  $D$  be a point inside the acute-angled triangle  $ABC$  such that  $\angle ADB = \angle ACB + 90^\circ$ , and  $AC \cdot BD = AD \cdot BC$ .

(a) Calculate the ratio  $AB \cdot CD / (AC \cdot BD)$ .



(b) Prove that the tangents at  $C$  to the circumcircles of  $ACD$  and  $BCD$  are perpendicular.

**A3.** On an infinite chessboard a game is played as follows. At the start  $n^2$  pieces are arranged in an  $n \times n$  block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of  $n$  for which the game can end with only one piece remaining on the board.



**B1.** For three points  $P, Q, R$  in the plane define  $m(PQR)$  as the minimum length of the three altitudes of the triangle  $PQR$  (or zero if the points are collinear). Prove that for any points  $A, B, C, X$ :



$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

**B2.** Does there exist a function  $f$  from the positive integers to the positive integers such that  $f(1) = 2$ ,  $f(f(n)) = f(n) + n$  for all  $n$ , and  $f(n) < f(n+1)$  for all  $n$ ?



**B3.** There are  $n > 1$  lamps  $L_0, L_1, \dots, L_{n-1}$  in a circle. We use  $L_{n+k}$  to mean  $L_k$ . A lamp is at all times either on or off. Initially they are all on. Perform steps  $s_0, s_1, \dots$  as follows: at step  $s_i$ , if  $L_{i-1}$  is lit, then switch  $L_i$  from on to off or vice versa, otherwise do nothing. Show that:

(a) There is a positive integer  $M(n)$  such that after  $M(n)$  steps all the lamps are on again;



(b) If  $n = 2^k$ , then we can take  $M(n) = n^2 - 1$ .

(c) If  $n = 2^k + 1$ , then we can take  $M(n) = n^2 - n + 1$ .

**Problem A1**

Let  $f(x) = x^n + 5x^{n-1} + 3$ , where  $n > 1$  is an integer. Prove that  $f(x)$  cannot be expressed as the product of two non-constant polynomials with integer coefficients.

**Solution**

Suppose  $f(x) = (x^r + a_{r-1}x^{r-1} + \dots + a_1x \pm 3)(x^s + b_{s-1}x^{s-1} + \dots + b_1x \pm 1)$ . We show that all the  $a$ 's are divisible by 3 and use that to establish a contradiction.

First,  $r$  and  $s$  must be greater than 1. For if  $r = 1$ , then  $\pm 3$  is a root, so if  $n$  is even, we would have  $0 = 3^n \pm 5 \cdot 3^{n-1} + 3 = 3^{n-1}(3 \pm 5) + 3$ , which is false since  $3 \pm 5 = 8$  or  $-2$ . Similarly if  $n$  is odd we would have  $0 = 3^{n-1}(\pm 3 + 5) + 3$ , which is false since  $\pm 3 + 5 = 8$  or  $2$ . If  $s = 1$ , then  $\pm 1$  is a root and we obtain a contradiction in the same way.

So  $r \leq n - 2$ , and hence the coefficients of  $x, x^2, \dots, x^r$  are all zero. Since the coefficient of  $x$  is zero, we have:  $a_1 \pm 3b_1 = 0$ , so  $a_1$  is divisible by 3. We can now proceed by induction. Assume  $a_1, \dots, a_t$  are all divisible by 3. Then consider the coefficient of  $x^{t+1}$ . If  $s-1 \geq t+1$ , then  $a_{t+1} =$  linear combination of  $a_1, \dots, a_t \pm 3b_{t+1}$ . If  $s-1 < t+1$ , then  $a_{t+1} =$  linear combination of some or all of  $a_1, \dots, a_t$ . Either way,  $a_{t+1}$  is divisible by 3. So considering the coefficients of  $x, x^2, \dots, x^{r-1}$  gives us that all the  $a$ 's are multiples of 3. Now consider the coefficient of  $x^r$ , which is also zero. It is a sum of terms which are multiples of 3 plus  $\pm 1$ , so it is not zero. Contradiction. Hence the factorization is not possible.

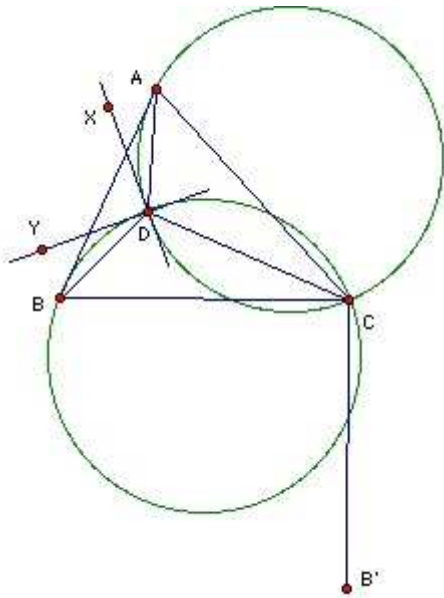
**IMO 1993****Problem A2**

Let  $D$  be a point inside the acute-angled triangle  $ABC$  such that  $\angle ADB = \angle ACB + 90^\circ$ , and  $AC \cdot BD = AD \cdot BC$ .

- Calculate the ratio  $AB \cdot CD / (AC \cdot BD)$ .
- Prove that the tangents at  $C$  to the circumcircles of  $ACD$  and  $BCD$  are perpendicular.

**Solution**

By Glen Ong, Oracle Corporation



Take  $B'$  so that  $CB = CB'$ ,  $\angle BCB' = 90^\circ$  and  $B'$  is on the opposite side of  $BC$  to  $A$ . It is easy to check that  $ADB$ ,  $ACB'$  are similar and  $DAC$ ,  $BAB'$  are similar. Hence  $AB/BD = AB'/B'C$  and  $CD/AC = BB'/AB'$ . It follows that the ratio given is  $BB'/B'C$  which is  $\sqrt{2}$ .

Take  $XD$  the tangent to the circumcircle of  $ADC$  at  $D$ , so that  $XD$  is in the  $\angle ADB$ . Similarly, take  $YD$  the tangent to the circumcircle  $BDC$  at  $D$ . Then  $\angle ADX = \angle ACD$ ,  $\angle BDY = \angle BCD$ , so  $\angle ADX + \angle BDY = \angle ACB$  and hence  $\angle XDY = \angle ADB - (\angle ADX + \angle BDY) = \angle ADB - \angle ACB = 90^\circ$ . In other words the tangents to the circumcircles at  $D$  are perpendicular. Hence, by symmetry (reflecting in the line of centers) the tangents at  $C$  are perpendicular.

*Theo Koupelis, University of Wisconsin, Marathon* provided a similar solution (about 10 minutes later!) taking the point  $B'$  so that  $\angle BDB' = 90^\circ$ ,  $BD = B'D$  and  $\angle B'DA = \angle ACB$ .  $DAC$ ,  $B'AB$  are similar; and  $ABC$ ,  $AB'D$  are similar.

*Marcin Mazur, University of Illinois at Urbana-Champaign* provided the first solution I received (about 10 minutes earlier!) using the generalized Ptolemy's equality (as opposed to the easier equality), but I do not know of a slick proof of this, so I prefer the proof above.



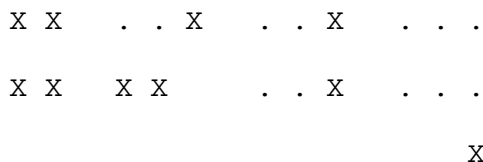
**Problem A3**

**On an infinite chessboard a game is played as follows. At the start  $n^2$  pieces are arranged in an  $n \times n$  block of adjoining squares, one piece on each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is removed. Find those values of  $n$  for which the game can end with only one piece remaining on the board.**

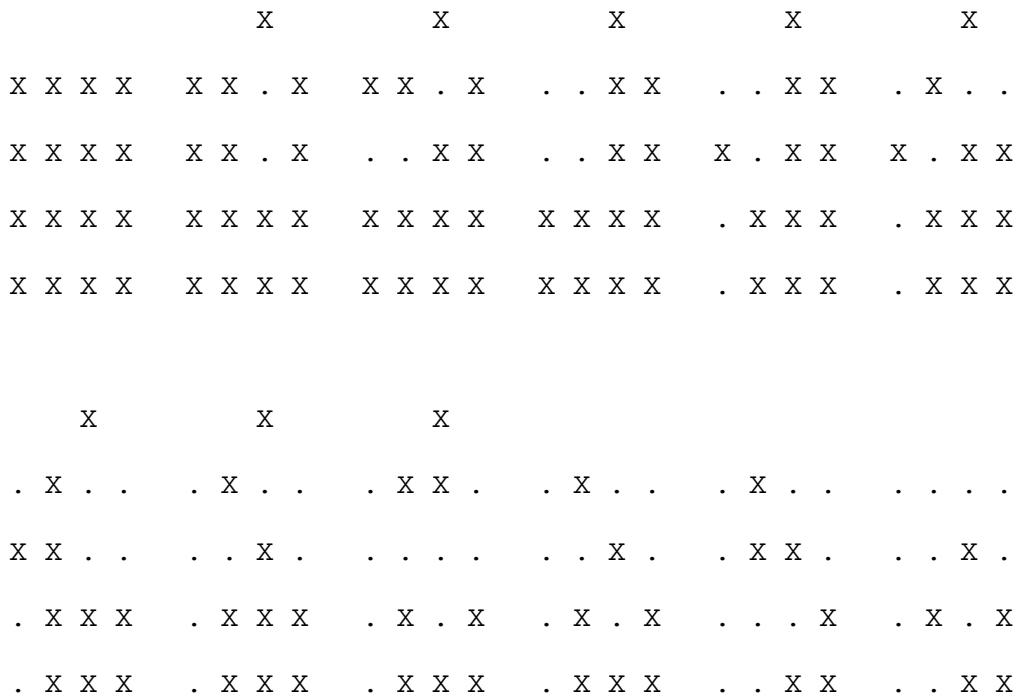
**Solution**

We show first that the game can end with only one piece if  $n$  is not a multiple of 3. Note first that the result is true for  $n = 2$  or  $4$ .

$n=2$



$n = 4$



```

. . . . .
. . X X . X . . . . .
. X . . . X . . . . .
. . X . . . X . . X X . . . X

```

The key technique is the following three moves which can be used to wipe out three adjacent pieces on the border provided there are pieces behind them:

```

X X X      X X .      X X .      X X X
X X X      X X .      . . X      . . .
           X          X

```

We can use this technique to reduce  $(r + 3) \times s$  rectangle to an  $r \times s$  rectangle. There is a slight wrinkle for the last two rows of three:

```

X X X X      X X . X      . . X X      . . X X      . . . X      . .
. X
X X X X      X X . X      . . X X      . . X X      . . . X      . .
. X
. . . X      . . X X      . . X X      . X . .      . X X .      . .
. X

```

Thus we can reduce a square side  $3n+2$  to a  $2 \times (3n+2)$  rectangle. We now show how to wipe out the rectangle. First, we change the  $2 \times 2$  rectangle at one end into a single piece alongside the (now)  $2 \times 3n$  rectangle:

```

X X      . .      . .
X X      . .      . .
           X X      X

```

Then we use the following technique to shorten the rectangle by 3:

```

X X X      X . X      X . X      . . .      . . .
X X X      X . X      X . X      . . .      . . .
X          X X      X .      X X . X      . X

```

That completes the case of the square side  $3n+2$ . For the square side  $3n+1$  we can use the technique for removing  $3 \times r$  rectangles to reduce it to a  $4 \times 4$  square and then use the technique above for the  $4 \times 4$  rectangle.

Finally, we use a parity argument to show that if  $n$  is a multiple of 3, then the square side  $n$  cannot be reduced to a single piece. Color the board with 3 colors, red, white and blue:

R W B R W B R W B . . .  
 W B R W B R W B R . . .  
 B R W B R W B R W . . .  
 R W B R W B R W B . . .  
 . . .

Let suppose that the single piece is on a red square. Let  $A$  be the number of moves onto a red square,  $B$  the number of moves onto a white square and  $C$  the number of moves onto a blue square. A move onto a red square increases the number of pieces on red squares by 1, reduces the number of pieces on white squares by 1, and reduces the number of pieces on blue squares by 1. Let  $n = 3m$ . Then there are initially  $m$  pieces on red squares,  $m$  on white and  $m$  on blue. Thus we have:

$$-A + B + C = m - 1; \quad A - B + C = m; \quad A + B - C = m.$$

Solving, we get  $A = m$ ,  $B = m - 1/2$ ,  $C = m - 1/2$ . But the number of moves of each type must be integral, so it is not possible to reduce the number of pieces to one if  $n$  is a multiple of 3.

### IMO 1993



#### Problem B1

For three points  $P, Q, R$  in the plane define  $m(PQR)$  as the minimum length of the three altitudes of the triangle  $PQR$  (or zero if the points are collinear). Prove that for any points  $A, B, C, X$ :

$$m(ABC) \leq m(ABX) + m(AXC) + m(XBC).$$

#### Solution

The length of an altitude is twice the area divided by the length of the corresponding side. Suppose that  $BC$  is the longest side of the triangle  $ABC$ . Then  $m(ABC) = \text{area } ABC/BC$ . [If  $A = B = C$ , so that  $BC = 0$ , then the result is trivially true.]

Consider first the case of  $X$  inside  $ABC$ . Then  $\text{area } ABC = \text{area } ABX + \text{area } AXC + \text{area } XBC$ , so  $m(ABC)/2 = \text{area } ABX/BC + \text{area } AXC/BC + \text{area } XBC/BC$ . We now claim that the longest side of  $ABX$  is at most  $BC$ , and similarly for  $AXC$  and  $XBC$ . It then follows at once that  $\text{area } ABX/BC \leq \text{area } ABX/\text{longest side of } ABX = m(ABX)/2$  and the result follows (for points  $X$  inside  $ABC$ ).

The claim follows from the following lemma. If  $Y$  lies between  $D$  and  $E$ , then  $FY$  is less than the greater of  $FD$  and  $FE$ . Proof: let  $H$  be the foot of the perpendicular from  $F$  to  $DE$ . One of  $D$  and  $E$  must lie on the opposite side of  $Y$  to  $H$ . Suppose it is  $D$ . Then  $FD = FH/\cos \angle HFD > FH/\cos \angle HFY = FY$ . Returning to  $ABCX$ , let  $CX$  meet  $AB$  at  $Y$ . Consider the three sides of  $ABX$ . By definition  $AB \leq BC$ . By the lemma  $AX$  is smaller than the larger of  $AC$  and  $AY$ , both of which do not exceed  $BC$ . Hence  $AX \leq BC$ . Similarly  $BX \leq BC$ .

It remains to consider  $X$  outside  $ABC$ . Let  $AX$  meet  $AC$  at  $O$ . We show that the sum of the smallest altitudes of  $ABY$  and  $BCY$  is at least the sum of the smallest altitudes of  $ABO$  and  $ACO$ . The result then follows, since we already have the result for  $X = O$ . The altitude from  $A$  in  $ABX$  is the same as the altitude from  $A$  in  $ABO$ . The altitude from  $X$  in  $ABX$  is clearly longer than the altitude from  $O$  in  $ABO$  (let the altitudes meet the line  $AB$  at  $Q$  and  $R$  respectively, then triangles  $BOR$  and  $BXQ$  are similar, so  $XQ = OR \cdot BX/BO > OR$ ). Finally, let  $k$  be the line through  $A$  parallel to  $BX$ , then the altitude from  $B$  in  $ABX$  either crosses  $k$  before it meets  $AX$ , or crosses  $AC$  before it crosses  $AX$ . If the former, then it is longer than the perpendicular from  $B$  to  $k$ , which equals the altitude from  $A$  to  $BO$ . If the latter, then it is longer than the altitude from  $B$  to  $AO$ . Thus each of the altitudes in  $ABX$  is longer than an altitude in  $ABO$ , so  $m(ABX) > m(ABO)$ .

## IMO 1993



### Problem B2

Does there exist a function  $f$  from the positive integers to the positive integers such that  $f(1) = 2$ ,  $f(f(n)) = f(n) + n$  for all  $n$ , and  $f(n) < f(n+1)$  for all  $n$ ?

#### Answer

Yes:  $f(n) = [g^*n + \frac{1}{2}]$ , where  $g = (1 + \sqrt{5})/2 = 1.618 \dots$ .

#### Solution

*This simple and elegant solution is due to Suengchur Pyun*

Let  $g(n) = [g^*n + \frac{1}{2}]$ . Obviously  $g(1) = 2$ . Also  $g(n+1) = g(n) + 1$  or  $g(n) + 2$ , so certainly  $g(n+1) > g(n)$ .

Consider  $d(n) = g^* [g^*n + \frac{1}{2}] + \frac{1}{2} - ([g^*n + \frac{1}{2}] + n)$ . We show that it is between 0 and 1. It follows immediately that  $g(g(n)) = g(n) + n$ , as required.

Certainly,  $[g^*n + \frac{1}{2}] > g^*n - \frac{1}{2}$ , so  $d(n) > 1 - g/2 > 0$  (the  $n$  term has coefficient  $g^2 - g - 1$  which is zero). Similarly,  $[g^*n + \frac{1}{2}] \leq g^*n + \frac{1}{2}$ , so  $d(n) \leq g/2 < 1$ , which completes the proof.

*I originally put up the much clumsier result following:*

Take  $n = b_r b_{r-1} \dots b_0$  in the Fibonacci base. Then  $f(n) = b_r b_{r-1} \dots b_0 0$ . This satisfies the required conditions.

Let  $u_0 = 1, u_1 = 2, \dots, u_n = u_{n-1} + u_{n-2}, \dots$  be the Fibonacci numbers. We say  $n = b_r b_{r-1} \dots b_0$  in the Fibonacci base if  $b_r = 1$ , every other  $b_i = 0$  or  $1$ , no two adjacent  $b_i$  are non-zero, and  $n = b_r u_r + \dots + b_0 u_0$ . For example,  $28 = 1001010$  because  $28 = 21 + 5 + 2$ .

We have to show that every  $n$  has a unique expression of this type. We show first by induction that it has at least one expression of this type. Clearly that is true for  $n = 1$ . Take  $u_r$  to be the largest Fibonacci number  $\leq n$ . Then by induction we have an expression for  $n - u_r$ . The leading term cannot be  $u_i$  for  $i > r - 2$ , for then we would have  $n \geq u_r + u_{r-1} = u_{r+1}$ . So adding  $u_r$  to the expression for  $n - u_r$  gives us an expression of the required type for  $n$ , which completes the induction.

We show that  $u_r + u_{r-2} + u_{r-4} + \dots = u_{r+1} - 1$ . Again we use induction. It is true for  $r = 1$  and  $2$ . Suppose it is true for  $r - 1$ , then  $u_{r+1} + u_{r-1} + \dots = u_{r+2} - u_r + u_{r-1} + u_{r-3} + \dots = u_{r+2} - u_r + u_r - 1 = u_{r+2} - 1$ . So it is true for  $r + 1$ . Hence it is true for all  $r$ . Now we can prove that the expression for  $n$  is unique. It is for  $n = 1$ . So assume it is for all numbers  $< n$ , but that the expression for  $n$  is not unique, so that we have  $n = u_r + \text{more terms} = u_s + \text{more terms}$ . If  $r = s$ , then the expression for  $n - u_r$  is not unique. Contradiction. So suppose  $r > s$ . But now the second expression is at most  $u_{s+1} - 1$  which is less than  $u_r$ . So the expression for  $n$  must be unique and the induction is complete.

It remains to show that  $f$  satisfies the required conditions. Evidently if  $n = 1 = u_0$ , then  $f(n) = u_1 = 2$ , as required. If  $n = u_{a_1} + \dots + u_{a_r}$ , then  $f(n) = u_{a_1+1} + \dots + u_{a_r+1}$  and  $f(f(n)) = u_{a_1+2} + \dots + u_{a_r+2}$ . So  $f(n) + n = (u_{a_1} + u_{a_1+1}) + \dots + (u_{a_r} + u_{a_r+1}) = f(f(n))$ .

## IMO 1993



### Problem B3

There are  $n > 1$  lamps  $L_0, L_1, \dots, L_{n-1}$  in a circle. We use  $L_{n+k}$  to mean  $L_k$ . A lamp is at all times either on or off. Initially they are all on. Perform steps  $s_0, s_1, \dots$  as follows: at step  $s_i$ , if  $L_{i-1}$  is lit, then switch  $L_i$  from on to off or vice versa, otherwise do nothing. Show that:

- There is a positive integer  $M(n)$  such that after  $M(n)$  steps all the lamps are on again;
- If  $n = 2^k$ , then we can take  $M(n) = n^2 - 1$ .
- If  $n = 2^k + 1$ , then we can take  $M(n) = n^2 - n + 1$ .

### Solution

(a) The process cannot terminate, because before the last move a single lamp would have been on. But the last move could not have turned it off, because the adjacent lamp was off. There are only finitely many states (each lamp is on or off and the next move can be at one of finitely many lamps), hence the process must repeat. The outcome of each step is uniquely determined by the state, so either the process moves around a single large loop, or there is an initial sequence of steps as far as state  $k$  and then the process goes around a loop back to  $k$ . However, the latter is not possible because then state  $k$  would have had two different precursors. But a state has only one possible precursor which can be found by toggling the lamp at the current position if the previous lamp is on and then moving the position back one. Hence the process must move around a single large loop, and hence it must return to the initial state.



(b) Represent a lamp by X when on, by - when not. For 4 lamps the starting situation and the situation after 4, 8, 12, 16 steps is as follows:

```

X X X X
- X - X
X - - X
- - - X
X X X -

```

On its first move lamp  $n-2$  is switched off and then remains off until each lamp has had  $n-1$  moves. Hence for each of its first  $n-1$  moves lamp  $n-1$  is not toggled and it retains its initial state. After each lamp has had  $n-1$  moves, all of lamps 1 to  $n-2$  are off. Finally over the next  $n-1$  moves, lamps 1 to  $n-2$  are turned on, so that all the lamps are on. We show by induction on  $k$  that these statements are all true for  $n = 2^k$ . By inspection, they are true for  $k = 2$ . So suppose they are true for  $k$  and consider  $2n = 2^{k+1}$  lamps. For the first  $n-1$  moves of each lamp the  $n$  left-hand and the  $n$  right-hand lamps are effectively insulated. Lamps  $n-1$  and  $2n-1$  remain on. Lamp  $2n-1$  being on means that lamps 0 to  $n-2$  are in just the same situation that they would be with a set of only  $n$  lamps. Similarly, lamp  $n-1$  being on means that lamps  $n$  to  $2n-2$  are in the same situation that they would be with a set of only  $n$  lamps. Hence after each lamp has had  $n-1$  moves, all the lamps are off except for  $n-1$  and  $2n-1$ . In the next  $n$  moves lamps 1 to  $n-2$  are turned on, lamp  $n-1$  is turned off, lamps  $n$  to  $2n-2$  remain off, and lamp  $2n-1$  remains on. For the next  $n-1$  moves for each lamp, lamp  $n-1$  is not toggled, so it remains off. Hence all of  $n$  to  $2n-2$  also remain off and  $2n-1$  remains on. Lamps 0 to  $n-2$  go through the same sequence as for a set of  $n$  lamps. Hence after these  $n-1$  moves for each lamp, all the lamps are off, except for  $2n-1$ . Finally, over the next  $2n-1$  moves, lamps 0 to  $2n-2$  are turned on. This completes the induction. Counting moves, we see that there are  $n-1$  sets of  $n$  moves, followed by  $n-1$  moves, a total of  $n^2 - 1$ .

(c) We show by induction on the number of moves that for  $n = 2^k + 1$  lamps after each lamp has had  $m$  moves, for  $i = 0, 1, \dots, 2^k - 2$ , lamp  $i+2$  is in the same state as lamp  $i$  is after each lamp has had  $m$  moves in a set of  $n - 1 = 2^k$  lamps (we refer to this as lamp  $i$  in the *reduced* case). Lamp 0 is off and lamp 1 is on. It is easy to see that this is true for  $m = 1$  (in both cases odd numbered lamps are on and even numbered lamps are off). Suppose it is true for  $m$ . Lamp 2 has the same state as lamp 0 in the reduced case and both toggle since their predecessor lamps are on. Hence lamps 3 to  $n - 1$  behave the same as lamps 1 to  $n - 3$  in the reduced case. That means that lamp  $n - 1$  remains off. Hence lamp 0 does not toggle on its  $m+1$ th move and remains off. Hence lamp 1 does not toggle on its  $m+1$ th move and remains on. The induction stops working when lamp  $n - 2$  toggles on its  $n$ th move in the reduced case, but it works up to and including  $m = n - 2$ . So after  $n - 2$  moves for each lamp all lamps are off except lamp 1. In the next two moves nothing happens, then in the following  $n - 1$  moves lamps 2 to  $n - 1$  and lamp 0 are turned on. So all the lamps are on after a total of  $(n - 2)n + n + 1 = n^2 + n + 1$  moves.



**A1.** Find all integers  $a, b, c$  satisfying  $1 < a < b < c$  such that  $(a - 1)(b - 1)(c - 1)$  is a divisor of  $abc - 1$ .



**A2.** Find all functions  $f$  defined on the set of all real numbers with real values, such that  $f(x^2 + f(y)) = y + f(x)^2$  for all  $x, y$ .



**A3.** Consider 9 points in space, no 4 coplanar. Each pair of points is joined by a line segment which is colored either blue or red or left uncolored. Find the smallest value of  $n$  such that whenever exactly  $n$  edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.



**B1.**  $L$  is a tangent to the circle  $C$  and  $M$  is a point on  $L$ . Find the locus of all points  $P$  such that there exist points  $Q$  and  $R$  on  $L$  equidistant from  $M$  with  $C$  the incircle of the triangle  $PQR$ .



**B2.** Let  $S$  be a finite set of points in three-dimensional space. Let  $S_x, S_y, S_z$  be the sets consisting of the orthogonal projections of the points of  $S$  onto the  $yz$ -plane,  $zx$ -plane,  $xy$ -plane respectively. Prove that:

$$|S|^2 \leq |S_x| |S_y| |S_z|, \text{ where } |A| \text{ denotes the number of points in the set } A.$$



[The orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane.]

**B3.** For each positive integer  $n$ ,  $S(n)$  is defined as the greatest integer such that for every positive integer  $k \leq S(n)$ ,  $n^2$  can be written as the sum of  $k$  positive squares.

(a) Prove that  $S(n) \leq n^2 - 14$  for each  $n \geq 4$ .

(b) Find an integer  $n$  such that  $S(n) = n^2 - 14$ .

(c) Prove that there are infinitely many integers  $n$  such that  $S(n) = n^2 - 14$ .





### Problem A1

Find all integers  $a, b, c$  satisfying  $1 < a < b < c$  such that  $(a - 1)(b - 1)(c - 1)$  is a divisor of  $abc - 1$ .

### Solution

**Answer:**  $a = 2, b = 4, c = 8$ ; or  $a = 3, b = 5, c = 15$ .

Let  $k = 2^{1/3}$ . If  $a \geq 5$ , then  $k(a - 1) > a$ . [Check:  $(k(a - 1))^3 - a^3 = a^3 - 6a^2 + 6a - 2$ . For  $a \geq 6$ ,  $a^3 \geq 6a^2$  and  $6a > 2$ , so we only need to check  $a = 5$ :  $125 - 150 + 30 - 2 = 3$ .] We know that  $c > b > a$ , so if  $a \geq 5$ , then  $2(a - 1)(b - 1)(c - 1) > abc > abc - 1$ . So we must have  $a = 2, 3$  or  $4$ .

Suppose  $abc - 1 = n(a - 1)(b - 1)(c - 1)$ . We consider separately the cases  $n = 1, n = 2$  and  $n \geq 3$ . If  $n = 1$ , then  $a + b + c = ab + bc + ca$ . But that is impossible, because  $a, b, c$  are all greater than 1 and so  $a < ab, b < bc$  and  $c < ca$ .

Suppose  $n = 2$ . Then  $abc - 1$  is even, so all  $a, b, c$  are odd. In particular,  $a = 3$ . So we have  $4(b - 1)(c - 1) = 3bc - 1$ , and hence  $bc + 5 = 4b + 4c$ . If  $b \geq 9$ , then  $bc \geq 9c > 4c + 4b$ . So we must have  $b = 5$  or  $7$ . If  $b = 5$ , then we find  $c = 15$ , which gives a solution. If  $b = 7$ , then we find  $c = 23/3$  which is not a solution.

The remaining case is  $n \geq 3$ . If  $a = 2$ , we have  $n(bc - b - c + 1) = 2bc - 1$ , or  $(n - 2)bc + (n + 1) = nb + nc$ . But  $b \geq 3$ , so  $(n - 2)bc \geq (3n - 6)c \geq 2nc$  for  $n \geq 6$ , so we must have  $n = 3, 4$  or  $5$ . If  $n = 3$ , then  $bc + 4 = 3b + 3c$ . If  $b \geq 6$ , then  $bc \geq 6c > 3b + 3c$ , so  $b = 3, 4$  or  $5$ . Checking we find only  $b = 4$  gives a solution:  $a = 2, b = 4, c = 8$ . If  $n = 4$ , then  $(n - 2)bc, nb$  and  $nc$  are all even, but  $(n + 1)$  is odd, so there is no solution. If  $n = 5$ , then  $3bc + 6 = 5b + 5c$ .  $b = 3$  gives  $c = 9/4$ , which is not a solution.  $b \geq 4$  gives  $3bc > 10c > 5b + 5c$ , so there are no solutions.

If  $a = 3$ , we have  $2n(bc - b - c + 1) = 3bc - 1$ , or  $(2n - 3)bc + (2n + 1) = 2nb + 2nc$ . But  $b \geq 4$ , so  $(2n - 3)bc \geq (8n - 12)c \geq 4nc > 2nc + 2nb$ . So there are no solutions. Similarly, if  $a = 4$ , we have  $(3n - 4)bc + (3n + 1) = 3nb + 3nc$ . But  $b \geq 4$ , so  $(3n - 4)bc \geq (12n - 16)c > 6nc > 3nb + 3nc$ , so there are no solutions.

**Problem A2**

**Find all functions  $f$  defined on the set of all real numbers with real values, such that  $f(x^2 + f(y)) = y + f(x)^2$  for all  $x, y$ .**

**Solution**

The first step is to establish that  $f(0) = 0$ . Putting  $x = y = 0$ , and  $f(0) = t$ , we get  $f(t) = t^2$ . Also,  $f(x^2 + t) = f(x)^2$ , and  $f(f(x)) = x + t^2$ . We now evaluate  $f(t^2 + f(1)^2)$  two ways. First, it is  $f(f(1)^2 + f(t)) = t + f(f(1))^2 = t + (1 + t^2)^2 = 1 + t + 2t^2 + t^4$ . Second, it is  $f(t^2 + f(1 + t)) = 1 + t + f(t)^2 = 1 + t + t^4$ . So  $t = 0$ , as required.

It follows immediately that  $f(f(x)) = x$ , and  $f(x^2) = f(x)^2$ . Given any  $y$ , let  $z = f(y)$ . Then  $y = f(z)$ , so  $f(x^2 + y) = z + f(x)^2 = f(y) + f(x)^2$ . Now given any positive  $x$ , take  $z$  so that  $x = z^2$ . Then  $f(x + y) = f(z^2 + y) = f(y) + f(z)^2 = f(y) + f(z^2) = f(x) + f(y)$ . Putting  $y = -x$ , we get  $0 = f(0) = f(x + -x) = f(x) + f(-x)$ . Hence  $f(-x) = -f(x)$ . It follows that  $f(x + y) = f(x) + f(y)$  and  $f(x - y) = f(x) - f(y)$  hold for all  $x, y$ .

Take any  $x$ . Let  $f(x) = y$ . If  $y > x$ , then let  $z = y - x$ .  $f(z) = f(y - x) = f(y) - f(x) = x - y = -z$ . If  $y < x$ , then let  $z = x - y$  and  $f(z) = f(x - y) = f(x) - f(y) = y - x$ . In either case we get some  $z > 0$  with  $f(z) = -z < 0$ . But now take  $w$  so that  $w^2 = z$ , then  $f(z) = f(w^2) = f(w)^2 \geq 0$ . Contradiction. So we must have  $f(x) = x$ .

**Problem A3**

**Consider 9 points in space, no 4 coplanar. Each pair of points is joined by a line segment which is colored either blue or red or left uncolored. Find the smallest value of  $n$  such that whenever exactly  $n$  edges are colored, the set of colored edges necessarily contains a triangle all of whose edges have the same color.**

**Solution**

by Gerhard Wöginger

We show that for  $n = 32$  we can find a coloring without a monochrome triangle. Take two squares  $R_1R_2R_3R_4$  and  $B_1B_2B_3B_4$ . Leave the diagonals of each square uncolored, color the remaining edges of  $R$  red and the remaining edges of  $B$  blue. Color blue all the edges from the ninth point  $X$  to the red square, and red all the edges from  $X$  to the blue square. Color  $R_iB_j$  red if  $i$  and  $j$  have the same parity and blue otherwise.

Clearly  $X$  is not the vertex of a monochrome square, because if  $XY$  and  $XZ$  are the same color then,  $YZ$  is either uncolored or the opposite color. There is no triangle within the red square or the blue square, and hence no monochrome triangle. It remains to consider triangles of the form  $R_iR_jB_k$  and  $B_iB_jR_k$ . But if  $i$  and  $j$  have the same parity, then  $R_iR_j$  is uncolored (and similarly  $B_iB_j$ ), whereas if they have opposite parity, then  $R_iB_k$  and  $R_jB_k$  have opposite colors (and similarly  $B_iR_k$  and  $B_jR_k$ ).

It remains to show that for  $n = 33$  we can always find a monochrome triangle. There are three uncolored edges. Take a point on each of the uncolored edges. The edges between the remaining 6 points must all be colored. Take one of these,  $X$ . At least 3 of the 5 edges to  $X$ , say  $XA$ ,  $XB$ ,  $XC$  must be the same color (say red). If  $AB$  is also red, then  $XAB$  is monochrome. Similarly, for  $BC$  and  $CA$ . But if  $AB$ ,  $BC$  and  $CA$  are all blue, then  $ABC$  is monochrome.

## IMO 1992



### Problem B1

**$L$  is a tangent to the circle  $C$  and  $M$  is a point on  $L$ . Find the locus of all points  $P$  such that there exist points  $Q$  and  $R$  on  $L$  equidistant from  $M$  with  $C$  the incircle of the triangle  $PQR$ .**

### Solution

Answer: Let  $X$  be the point where  $C$  meets  $L$ , let  $O$  be the center of  $C$ , let  $XO$  cut  $C$  again at  $Z$ , and take  $Y$  on  $QR$  so that  $M$  be the midpoint of  $XY$ . Let  $L'$  be the line  $YZ$ . The locus is the open ray from  $Z$  along  $L'$  on the opposite side to  $Y$ .

*mainly by Gerhard Wöginger, Technical University, Graz (I filled in a few details)*

Let  $C'$  be the circle on the other side of  $QR$  to  $C$  which also touches the segment  $QR$  and the lines  $PQ$  and  $PR$ . Let  $C'$  touch  $QR$  at  $Y'$ . If we take an expansion (technically, homothety) center  $P$ , factor  $PY'/PZ$ , then  $C$  goes to  $C'$ , the tangent to  $C$  at  $Z$  goes to the line  $QR$ , and hence  $Z$  goes to  $Y'$ . But it is easy to show that  $QX = RY'$ .

We focus on the  $QORO'$ . Evidently  $X, Y'$  are the feet of the perpendiculars from  $O, O'$  respectively to  $QR$ . Also,  $QO'O = ORO' = 90$ . So  $QY'O'$  and  $OXQ$  are similar, and hence  $QY'/Y'O' = OX/XQ$ . Also  $RXO$  and  $O'Y'R$  are similar, so  $RX/XO = O'Y'/Y'R$ . Hence  $QY' \cdot XQ = OX \cdot O'Y' = RX \cdot Y'R$ . Hence  $QX/RX = QX/(QR - QX) = RY'/(QR - RY') = RY'/QY'$ . Hence  $QX = RY'$ .

But  $QX = RY'$  by construction ( $M$  is the midpoint of  $XY$  and  $QR$ ), so  $Y = Y'$ . Hence  $P$  lies on the open ray as claimed. Conversely, if we take  $P$  on this ray, then by the same argument  $QX = RY'$ . But  $M$  is the midpoint of  $XY$ , so  $M$  must also be the midpoint of  $QR$ , so the locus is the entire (open) ray.

*Gerhard only found this after Theo Koupelis, University of Wisconsin, Marathon had already supplied the following analytic solution.*

Take Cartesian coordinates with origin X, so that M is (a, 0) and O is (0, R). Let R be the point (b, 0) (we take a, b >= 0). Then Q is the point (2a - b, 0), and Y is (2a, 0). Let angle XRO be  $\theta$ . Then  $\tan \theta = R/b$  and angle PRX =  $2\theta$ , so  $\tan PRX = 2 \tan \theta / (1 - \tan^2 \theta) = 2Rb/(b^2 - R^2)$ . Similarly,  $\tan PQX = 2R(b - 2a)/(b - 2a)^2 - R^2$ .

If P has coordinates (A, B), then  $B/(b - A) = \tan PRX$ , and  $B/(b - 2a + x) = \tan PQX$ . So we have two simultaneous equations for A and B. Solving, and simplifying slightly, we find  $A = -2aR^2/(b^2 - 2ab - R^2)$ ,  $B = 2b(b - 2a)R/(b^2 - 2ab - R^2)$ . (\*)

We may now check that  $B/(2a - A) = R/a$ , so P lies on YZ as claimed. So we have shown that the locus is a subset of the line YZ. But since  $b^2 - 2ab - R^2$  maps the open interval  $(a + \sqrt{a^2 + R^2}, \infty)$  onto the open interval  $(0, \infty)$ , (\*) shows that we can obtain any value A in the open interval  $(-\infty, 0)$  by a suitable choice of b, and hence any point P on the ray (except its endpoint Z) by a suitable choice of R.

## IMO 1992



### Problem B2

Let S be a finite set of points in three-dimensional space. Let  $S_x, S_y, S_z$  be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that:

$$|S|^2 \leq |S_x| |S_y| |S_z|, \text{ where } |A| \text{ denotes the number of points in the set } A.$$

[The orthogonal projection of a point onto a plane is the foot of the perpendicular from the point to the plane.]

### Solution

by Gerhard Wöginger

Induction on the number of different z-coordinates in S.

For 1, it is sufficient to note that  $S = S_z$  and  $|S| \leq |S_x| |S_y|$  (at most  $|S_x|$  points of S project onto each of the points of  $S_y$ ).

In the general case, take a horizontal (constant z) plane dividing S into two non-empty parts T and U. Clearly,  $|S| = |T| + |U|$ ,  $|S_x| = |T_x| + |U_x|$ , and  $|S_y| = |T_y| + |U_y|$ .

By induction,  $|S| = |T| + |U| \leq (|T_x| |T_y| |T_z|)^{1/2} + (|U_x| |U_y| |U_z|)^{1/2}$ . But  $|T_z|, |U_z| \leq |S_z|$ , and for any positive a, b, c, d we have  $(a b)^{1/2} + (c d)^{1/2} \leq ((a + c) (b + d))^{1/2}$  (square!).

Hence  $|S| \leq |S_z|^{1/2} ((|T_x| + |U_x|) (|T_y| + |U_y|))^{1/2} = (|S_x| |S_y| |S_z|)^{1/2}$ .

## IMO 1992



### Problem B3

For each positive integer  $n$ ,  $S(n)$  is defined as the greatest integer such that for every positive integer  $k \leq S(n)$ ,  $n^2$  can be written as the sum of  $k$  positive squares.

- (a) Prove that  $S(n) \leq n^2 - 14$  for each  $n \geq 4$ .
- (b) Find an integer  $n$  such that  $S(n) = n^2 - 14$ .
- (c) Prove that there are infinitely many integers  $n$  such that  $S(n) = n^2 - 14$ .

### Solution

(a) Let  $N = n^2$ . Suppose we could express  $N$  as a sum of  $N - 13$  squares. Let the number of 4s be  $a$ , the number of 9s be  $b$  and so on. Then we have  $13 = 3a + 8b + 15c + \dots$ . Hence  $c, d, \dots$  must all be zero. But neither 13 nor 8 is a multiple of 3, so there are no solutions. Hence  $S(n) \leq N - 14$ .

A little experimentation shows that the problem is getting started. Most squares cannot be expressed as a sum of two squares. For  $N = 13^2 = 169$ , we find:  $169 = 9 + 4 + 4 + 152$  1s, a sum of  $155 = N - 14$  squares. By grouping four 1s into a 4 repeatedly, we obtain all multiples of 3 plus 2 down to 41 ( $169 = 9 + 40$  4s). Then grouping four 4s into a 16 gives us 38, 35, ..., 11 ( $169 = 10$  16s + 9). Grouping four 16s into a 64 gives us 8 and 5. We obtain the last number congruent to 2 mod 3 by the decomposition:  $169 = 12^2 + 5^2$ .

For the numbers congruent to 1 mod 3, we start with  $N - 15 = 154$  squares:  $169 = 5$  4s + 149 1s. Grouping as before gives us all  $3m + 1$  down to 7:  $169 = 64 + 64 + 16 + 16 + 4 + 4 + 1$ . We may use  $169 = 10^2 + 8^2 + 2^2 + 1^2$  for 4.

For multiples of 3, we start with  $N - 16 = 153$  squares:  $169 = 9 + 9 + 151$  1s. Grouping as before gives us all multiples of 3 down to 9:  $169 = 64 + 64 + 16 + 9 + 9 + 4 + 1 + 1 + 1$ . Finally, we may take  $169 = 12^2 + 4^2 + 3^2$  for 3 and split the  $4^2$  to get  $169 = 12^2 + 3^2 + 2^2 + 2^2 + 2^2 + 2^2$  for 6. That completes the demonstration that we can write  $13^2$  as a sum of  $k$  positive squares for all  $k \leq S(13) = 13^2 - 14$ .

We now show how to use the expressions for  $13^2$  to derive further  $N$ . For any  $N$ , the grouping technique gives us the high  $k$ . Simply grouping 1s into 4s takes us down: from  $9 + 4 + 4 + (N-17)$  1s to  $(N-14)/4 + 6 < N/2$  or below; from  $4 + 4 + 4 + 4 + 4 + (N-20)$  1s to  $(N-23)/4 + 8 < N/2$  or below; from  $9 + 9 + (N-18)$  1s to  $(N-21)/4 + 5 < N/2$  or below. So we can certainly get all  $k$  in the range  $(N/2$  to  $N-14)$  by this approach. Now suppose that we already have a complete set of expressions for  $N_1$  and for  $N_2$  (where we may have  $N_1 = N_2$ ). Consider  $N_3 = N_1 N_2$ . Writing  $N_3 = N_1$  (an expression for  $N_2$  as a sum of  $k$  squares) gives  $N_3$  as a sum of 1 thru  $k_2$  squares, where  $k_2 = N_2 - 14$  squares (since  $N_1$  is a square). Now express  $N_1$  as a sum of two squares:  $n_1^2 + n_2^2$ . We have  $N_3 = n_1^2$  (a sum of  $k_2$  squares) +  $n_2^2$  (a sum of  $k$  squares). This gives  $N_3$  as a sum of  $k_2 + 1$  thru  $2k_2$  squares. Continuing in this way gives  $N_3$  as a sum of 1 thru  $k_1 k_2$  squares. But  $k_i = N_i - 14 > 2/3 N_i$ , so  $k_1 k_2 > N_3/2$ . So when combined with the top down grouping we get a complete set of expressions for  $N_3$ .

This shows that there are infinitely many squares  $N$  with a complete set of expressions, for example we may take  $N$  = the squares of 13,  $13^2$ ,  $13^3$ , ... .

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**A1.** Given a triangle  $ABC$ , let  $I$  be the incenter. The internal bisectors of angles  $A, B, C$  meet the opposite sides in  $A', B', C'$  respectively. Prove that:

$$1/4 < AI \cdot BI \cdot CI / (AA' \cdot BB' \cdot CC') \leq 8/27.$$



**A2.** Let  $n > 6$  be an integer and let  $a_1, a_2, \dots, a_k$  be all the positive integers less than  $n$  and relatively prime to  $n$ . If

$$a_2 - a_1 = a_3 - a_2 = \dots = a_k - a_{k-1} > 0,$$



prove that  $n$  must be either a prime number or a power of 2.

**A3.** Let  $S = \{1, 2, 3, \dots, 280\}$ . Find the smallest integer  $n$  such that each  $n$ -element subset of  $S$  contains five numbers which are pairwise relatively prime.



**B1.** Suppose  $G$  is a connected graph with  $k$  edges. Prove that it is possible to label the edges  $1, 2, \dots, k$  in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is 1.

[A graph is a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of edges belongs to at most one edge. The graph is connected if for each pair of distinct vertices  $x, y$  there is some sequence of vertices  $x = v_0, v_1, \dots, v_m = y$ , such that each pair  $v_i, v_{i+1}$  ( $0 \leq i < m$ ) is joined by an edge.]



**B2.** Let  $ABC$  be a triangle and  $X$  an interior point of  $ABC$ . Show that at least one of the angles  $XAB, XBC, XCA$  is less than or equal to  $30^\circ$ .



**B3.** Given any real number  $a > 1$  construct a bounded infinite sequence  $x_0, x_1, x_2, \dots$  such that  $|x_i - x_j| |i - j|^a \geq 1$  for every pair of distinct  $i, j$ .

[An infinite sequence  $x_0, x_1, x_2, \dots$  of real numbers is bounded if there is a constant  $C$  such that  $|x_i| < C$  for all  $i$ .]



## IMO 1991



### Problem A1

Given a triangle  $ABC$ , let  $I$  be the incenter. The internal bisectors of angles  $A, B, C$  meet the opposite sides in  $A', B', C'$  respectively. Prove that:

$$1/4 < AI \cdot BI \cdot CI / (AA' \cdot BB' \cdot CC') \leq 8/27.$$

### Solution

Consider the areas of the three triangles  $ABI, BCI, CAI$ . Taking base  $BC$  we conclude that  $(\text{area } ABI + \text{area } CAI) / \text{area } ABC = AI / AA'$ . On the other hand, if  $r$  is the radius of the in-circle, then  $\text{area } ABI = AB \cdot r / 2$  and similarly for the other two triangles. Hence  $AI / AA' = (CA + AB) / p$ , where  $p$  is the perimeter. Similarly  $BI / BB' = (AB + BC) / p$  and  $CI / CC' = (BC + CA) / p$ . But the arithmetic mean of  $(CA + AB) / p, (AB + BC) / p$  and  $(BC + CA) / p$  is  $2/3$ . Hence their product is at most  $(2/3)^3 = 8/27$ .

Let  $AB + BC - CA = 2z, BC + CA - AB = 2x, CA + AB - BC = 2y$ . Then  $x, y, z$  are all positive and we have  $AB = y + z, BC = z + x, CA = x + y$ . Hence  $(AI / AA')(BI / BB')(CI / CC') = (1/2 + y/p)(1/2 + z/p)(1/2 + x/p) > 1/8 + (x+y+z)/(4p) = 1/8 + 1/8 = 1/4$ .

## IMO 1991



### Problem A2

Let  $n > 6$  be an integer and let  $a_1, a_2, \dots, a_k$  be all the positive integers less than  $n$  and relatively prime to  $n$ . If  $a_2 - a_1 = a_3 - a_2 = \dots = a_k - a_{k-1} > 0$ , prove that  $n$  must be either a prime number or a power of 2.

**Solution** (by anon)

If  $n$  is odd, then 1 and 2 are prime to  $n$ , so all integers  $< n$  are prime to  $n$ , and hence  $n$  is prime.

If  $n = 4k$ , then  $2k-1$  and  $2k+1$  are prime to  $n$ , so all odd integers  $< n$  are prime to  $n$ , and hence  $n$  must be a power of 2.

If  $n = 4k+2$ , then  $2k+1$  divides  $n$ , but  $2k+3$  and  $2k+5$  are prime to  $n$ . But if  $n > 6$ , then  $2k+5 < n$ , so this cannot be a solution.

**Problem A3**

Let  $S = \{1, 2, 3, \dots, 280\}$ . Find the smallest integer  $n$  such that each  $n$ -element subset of  $S$  contains five numbers which are pairwise relatively prime.

**Solution**

**Answer:** 217.

Let  $A$  be the subset of all multiples of 2, 3, 5 or 7. Then  $A$  has 216 members and every 5-subset has 2 members with a common factor. [To show that  $|A| = 216$ , let  $a_n$  be the number of multiples of  $n$  in  $S$ . Then  $a_2 = 140$ ,  $a_3 = 93$ ,  $a_5 = 56$ ,  $a_6 = 46$ ,  $a_{10} = 28$ ,  $a_{15} = 18$ ,  $a_{30} = 9$ . Hence the number of multiples of 2, 3 or 5 is  $a_2 + a_3 + a_5 - a_6 - a_{10} - a_{15} + a_{30} = 206$ . There are ten additional multiples of 7: 7, 49, 77, 91, 119, 133, 161, 203, 217, 259.]

Let  $P$  be the set consisting of 1 and all the primes  $< 280$ . Define:

$$A_1 = \{2 \cdot 41, 3 \cdot 37, 5 \cdot 31, 7 \cdot 29, 11 \cdot 23, 13 \cdot 19\}$$

$$A_2 = \{2 \cdot 37, 3 \cdot 31, 5 \cdot 29, 7 \cdot 23, 11 \cdot 19, 13 \cdot 17\}$$

$$A_3 = \{2 \cdot 31, 3 \cdot 29, 5 \cdot 23, 7 \cdot 19, 11 \cdot 17, 13 \cdot 13\}$$

$$B_1 = \{2 \cdot 29, 3 \cdot 23, 5 \cdot 19, 7 \cdot 17, 11 \cdot 13\}$$

$$B_2 = \{2 \cdot 23, 3 \cdot 19, 5 \cdot 17, 7 \cdot 13, 11 \cdot 11\}$$

Note that these 6 sets are disjoint subsets of  $S$  and the members of any one set are relatively prime in pairs. But  $P$  has 60 members, the three  $A$ s have 6 each, and the two  $B$ s have 5 each, a total of 88. So any subset  $T$  of  $S$  with 217 elements must have at least 25 elements in common with their union. But  $6 \cdot 4 = 24 < 25$ , so  $T$  must have at least 5 elements in common with one of them. Those 5 elements are the required subset of elements relatively prime in pairs.

**Problem B1**

Suppose  $G$  is a connected graph with  $k$  edges. Prove that it is possible to label the edges  $1, 2, \dots, k$  in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is  $1$ .

[A graph is a set of points, called vertices, together with a set of edges joining certain pairs of distinct vertices. Each pair of edges belongs to at most one edge. The graph is connected if for each pair of distinct vertices  $x, y$  there is some sequence of vertices  $x = v_0, v_1, \dots, v_m = y$ , such that each pair  $v_i, v_{i+1}$  ( $0 \leq i < m$ ) is joined by an edge.]

**Solution**

The basic idea is that consecutive numbers are relatively prime.

We construct a labeling as follows. Pick any vertex  $A$  and take a path from  $A$  along unlabeled edges. Label the edges consecutively  $1, 2, 3, \dots$  as the path is constructed. Continue the path until it reaches a vertex with no unlabeled edges. Let  $B$  be the endpoint of the path.  $A$  is now guaranteed to have the gcd (= greatest common divisor) of its edges  $1$ , because one of its edges is labeled  $1$ . All the vertices between  $A$  and  $B$  are guaranteed to have gcd  $1$  because they have at least one pair of edges with consecutive numbers. Finally, either  $B$  has only one edge, in which case its gcd does not matter, or it is also one of the vertices between  $A$  and  $B$ , in which case its gcd is  $1$ .

Now take any vertex  $C$  with an unlabeled edge and repeat the process. The same argument shows that all the new vertices on the new path have gcd  $1$ . The endpoint is fine, because either it has only one edge (in which case its gcd does not matter) or it has already got gcd  $1$ .

Repeat until all the edges are labeled.



### Problem B2

Let  $ABC$  be a triangle and  $X$  an interior point of  $ABC$ . Show that at least one of the angles  $XAB$ ,  $XBC$ ,  $XCA$  is less than or equal to  $30^\circ$ .

### Solution

By Marcin Mazur, University of Illinois at Urbana-Champaign

Let  $P$ ,  $Q$ ,  $R$  be the feet of the perpendiculars from  $X$  to  $BC$ ,  $CA$ ,  $AB$  respectively. Use  $A$ ,  $B$ ,  $C$  to denote the interior angles of the triangle ( $BAC$ ,  $CBA$ ,  $ACB$ ). We have  $PX = BX \sin XBC = CX \sin(C - XCA)$ ,  $QX = CX \sin XCA = AX \sin(A - XAB)$ ,  $RX = AX \sin XAB = BX \sin(B - XBC)$ . Multiplying:  $\sin(A - XAB) \sin(B - XBC) \sin(C - XCA) = \sin A \sin B \sin C$ .

Now observe that  $\sin(A - x)/\sin x = \sin A \cot x - \cos A$  is a strictly decreasing function of  $x$  (over the range  $0$  to  $\pi$ ), so if  $XAB$ ,  $XBC$  and  $XCA$  are all greater than  $30^\circ$ , then  $\sin(A - 30^\circ) \sin(B - 30^\circ) \sin(C - 30^\circ) > \sin^3 30^\circ = 1/8$ .

But  $\sin(A - 30^\circ) \sin(B - 30^\circ) = (\cos(A - B) - \cos(A + B - 60^\circ))/2 \leq (1 - \cos(A + B - 60^\circ))/2 = (1 - \sin(C - 30^\circ))/2$ , since  $(A - 30^\circ) + (B - 30^\circ) + (C - 30^\circ) = 90^\circ$ . Hence  $\sin(A - 30^\circ) \sin(B - 30^\circ) \sin(C - 30^\circ) \leq 1/2 (1 - \sin(C - 30^\circ)) \sin(C - 30^\circ) = 1/2 (1/4 - (\sin(C - 30^\circ) - 1/2)^2) \leq 1/8$ . So  $XAB$ ,  $XBC$ ,  $XCA$  cannot all be greater than  $30^\circ$ .

By Jean-Pierre Ehrmann

$P$ ,  $Q$ ,  $R$  as above. Area  $ABX$  + area  $BCX$  + area  $CAX$  = area  $ABC$ , so  $AB \cdot XR + BC \cdot XP + CA \cdot XQ = 2$  area  $ABC \leq BC \cdot AP \leq BC(AX + XP)$ . Hence  $AB \cdot XR/AX + CA \cdot XQ/AX \leq BC$ .

Squaring and using  $(\lambda + \mu)^2 \geq 4\lambda\mu$ , we have:  $BC^2 \geq 4 AB \cdot CA \cdot XR \cdot XQ/AX^2$ . Similarly:  $CA^2 \geq 4 BC \cdot AB \cdot XP \cdot XR/BX^2$ , and  $AB^2 \geq 4 AB \cdot BC \cdot XQ \cdot XP/CX^2$ .

Multiplying these three inequalities together gives:  $1 \geq 64 (XR/AX)^2 (XP/BX)^2 (XQ/CX)^2$ , and hence:  $(XR/AX) (XP/BX) (XQ/CX) \leq 1/8$ , or  $\sin XAB \sin XBC \sin XCA \leq 1/8$ . So not all  $XAB$ ,  $XBC$ ,  $XCA$  are greater than  $30^\circ$ .

*Gerard Gjonej noted that the result follows almost immediately from the Erdos-Mordell inequality:  $XA + XB + XC \geq 2(XP + XQ + XR)$ . [For if all the angles are greater than  $30^\circ$ , then  $XR/XA$ ,  $XP/XB$ ,  $XQ/XC$  are all greater than  $\sin 30^\circ = 1/2$ ]. This result was notoriously hard to prove - Erdos hawked it around a large number of mathematicians before Mordell found a proof - but the proof now appears fairly innocuous, at least if you do not have to rediscover it:*

Let  $R_1$ ,  $Q_1$  be the feet of the perpendiculars from  $P$  to  $AB$ ,  $CA$  respectively. Similarly, let  $P_2$ ,  $R_2$  be the feet of the perpendiculars from  $Q$  to  $BC$ ,  $AB$ , and  $Q_3$ ,  $P_3$  the feet of the perpendiculars from

R to CA, BC. Then  $P_2P_3$  is the projection of QR onto BC, so  $P_2P_3/QR \leq 1$ . Similarly,  $Q_3Q_1/RP \leq 1$ , and  $R_1R_2/PQ \leq 1$ . Hence  $XA + XB + XC \geq XA \cdot P_2P_3/QR + XB \cdot Q_3Q_1/RP + XC \cdot R_1R_2/PQ$  (\*)

Now BPXR is cyclic, because BPX and XRB are both right angles. Hence angle BXR = angle BPR = angle RPP<sub>3</sub>, so triangles XBR and PRP<sub>3</sub> are similar. Hence  $PP_3 = PR \cdot XR/XB$ .

Similarly,  $QQ_1 = QP \cdot XP/XC$ ,  $RR_2 = RQ \cdot XQ/XA$ , and  $PP_2 = PQ \cdot XQ/XC$ ,  $QQ_3 = QR \cdot XR/XA$ ,  $RR_1 = RP \cdot XP/XB$ . Substituting into (\*), we obtain:

$$XA + XB + XC \geq XA \left( \frac{PQ}{QR} \frac{XQ}{XC} + \frac{PR}{QR} \frac{XR}{XB} \right) + XB \left( \frac{QR}{RP} \frac{XR}{XA} + \frac{QP}{RP} \frac{XP}{XC} \right) + XC \left( \frac{RP}{PQ} \frac{XP}{XB} + \frac{RQ}{PQ} \frac{XQ}{XA} \right).$$

On the right hand side, the terms involving XP are:  $XP \left( \frac{QP}{RP} \frac{XB}{XC} + \frac{RP}{PQ} \frac{XC}{XB} \right)$ , which has the form  $XP \left( x + \frac{1}{x} \right)$  and hence is at least  $2XP$ . Similarly for the terms involving XQ and XR.

## IMO 1991



### Problem B3

Given any real number  $a > 1$  construct a bounded infinite sequence  $x_0, x_1, x_2, \dots$  such that  $|x_n - x_m| |n - m|^a \geq 1$  for every pair of distinct  $n, m$ .

[An infinite sequence  $x_0, x_1, x_2, \dots$  of real numbers is bounded if there is a constant  $C$  such that  $|x_n| < C$  for all  $n$ .]

### Solution

By Marcin Mazur, University of Illinois at Urbana-Champaign

Let  $t = 1/2^a$ . Define  $c = 1 - t/(1 - t)$ . Since  $a > 1$ ,  $c > 0$ . Now given any integer  $n > 0$ , take the binary expansion  $n = \sum_i b_i 2^i$ , and define  $x_n = 1/c \sum_{b_i > 0} t^i$ . For example, taking  $n = 21 = 2^4 + 2^2 + 2^0$ , we have  $x_{21} = (t^4 + t^2 + t^0)/c$ . We show that for any unequal  $n, m$ ,  $|x_n - x_m| |n - m|^a \geq 1$ . This solves the problem, since the  $x_n$  are all positive and bounded by  $(\sum t^n)/c = 1/(1 - 2t)$ .

Take  $k$  to be the highest power of 2 dividing both  $n$  and  $m$ . Then  $|n - m| \geq 2^k$ . Also, in the binary expansions for  $n$  and  $m$ , the coefficients of  $2^0, 2^1, \dots, 2^{k-1}$  agree, but the coefficients for  $2^k$  are different. Hence  $c |x_n - x_m| = t^k + \sum_{i > k} y_i$ , where  $y_i = 0, t^i$  or  $-t^i$ . Certainly  $\sum_{i > k} y_i > -\sum_{i > k} t^i = t^{k+1}/(1 - t)$ , so  $c |x_n - x_m| > t^k(1 - t/(1 - t)) = c t^k$ . Hence  $|x_n - x_m| |n - m|^a > t^k 2^{ak} = 1$ .



**A1.** Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent at E to the circle through D, E and M intersects the lines BC and AC at F and G respectively. Find  $EF/EG$  in terms of  $t = AM/AB$ .



**A2.** Take  $n \geq 3$  and consider a set E of  $2n-1$  distinct points on a circle. Suppose that exactly  $k$  of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly  $n$  points from E. Find the smallest value of  $k$  so that every such coloring of  $k$  points of E is good.



**A3.** Determine all integers greater than 1 such that  $(2^n + 1)/n^2$  is an integer.



**B1.** Construct a function from the set of positive rational numbers into itself such that  $f(x f(y)) = f(x)/y$  for all  $x, y$ .



**B2.** Given an initial integer  $n_0 > 1$ , two players A and B choose integers  $n_1, n_2, n_3, \dots$  alternately according to the following rules:  
 Knowing  $n_{2k}$ , A chooses any integer  $n_{2k+1}$  such that  $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$ .  
 Knowing  $n_{2k+1}$ , B chooses any integer  $n_{2k+2}$  such that  $n_{2k+1}/n_{2k+2} = p^r$  for some prime  $p$  and integer  $r \geq 1$ .



Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which  $n_0$  does

- (a) A have a winning strategy?
- (b) B have a winning strategy?
- (c) Neither player have a winning strategy?

**B3.** Prove that there exists a convex 1990-gon such that all its angles are equal and the lengths of the sides are the numbers  $1^2, 2^2, \dots, 1990^2$  in some order.





**Problem A1**

**Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB. The tangent at E to the circle through D, E and M intersects the lines BC and AC at F and G respectively. Find EF/EG in terms of  $t = AM/AB$ .**

**Solution** *By Theo Koupelis, University of Wisconsin, Marathon*

$\angle ECF = \angle DCB$  (same angle)  $= \angle DAB$  (ACBD is cyclic)  $= \angle MAD$  (same angle). Also  $\angle CEF = \angle EMD$  (GE tangent to circle EMD)  $= \angle AMD$  (same angle). So triangles CEF and AMD are similar.

$\angle CEG = 180^\circ - \angle CEF = 180^\circ - \angle EMD = \angle BMD$ . Also  $\angle ECG = \angle ACD$  (same angle)  $= \angle ABD$  (BCAD is cyclic)  $= \angle MBD$  (same angle). So triangles CEG and BMD are similar.

Hence  $EF/CE = MD/AM$ ,  $EG/CE = MD/BM$ , and so dividing,  $EF/EG = BM/AM = (1-t)/t$ .

IMO 1990



**Problem A2**

**Take  $n \geq 3$  and consider a set E of  $2n-1$  distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is "good" if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E. Find the smallest value of k so that every such coloring of k points of E is good.**

**Solution** **Answer:**  $n$  for  $n \equiv 0$  or  $1 \pmod{3}$ ,  $n - 1$  for  $n \equiv 2 \pmod{3}$ .

Label the points 1 to  $2n - 1$ . Two points have exactly  $n$  points between them if their difference  $(\text{mod } 2n - 1)$  is  $n - 2$  or  $n + 1$ . We consider separately the three cases  $n = 3m$ ,  $3m + 1$  and  $3m + 2$ .

Let  $n = 3m$ . First, we exhibit a bad coloring with  $n - 1$  black points. Take the black points to be 1, 4, 7, ...,  $6m - 2$  ( $2m$  points) and 2, 5, 8, ...,  $3m - 4$  ( $m - 1$  points). It is easy to check that this is bad. The two points which could pair with  $r$  to give  $n$  points between are  $r + 3m - 2$  and  $r + 3m + 1$ . Considering the first of these, 1, 4, 7, ...,  $6m - 2$  would pair with  $3m - 1, 3m + 2, 3m + 5, \dots, 6m - 1, 3, 6, \dots, 3m - 6$ , none of which are black. Considering the second, they would pair with  $3m + 2, 3m + 5, \dots, 6m - 1, 3, \dots, 3m - 3$ , none of which are black. Similarly, 2, 5, 8, ...,  $3m - 4$  would pair with  $3m, 3m + 3, \dots, 6m - 3$ , none of which are black. So the set is bad.

Now if we start with 1 and keep adding  $3m - 2$ , reducing by  $6m - 1$  when necessary to keep the result in the range 1, ...,  $6m - 1$ , we eventually get back to 1: 1,  $3m - 1, 6m - 3, 3m - 4, 6m - 6, \dots, 2, 3m, 6m - 2, 3m - 3, 6m - 5, \dots, 3, 3m + 1, 6m - 1, \dots, 4, 3m + 2, 1$ . The sequence includes all  $6m - 1$  numbers. Moreover a bad coloring cannot have any two consecutive numbers colored

black. But this means that at most  $n - 1$  out of the  $2n - 1$  numbers in the sequence can be black. This establishes the result for  $n = 3m$ .

Take  $n = 3m + 1$ . A bad coloring with  $n - 1$  black points has the following black points:  $1, 4, 7, \dots, 3m - 2$  ( $m$  points) and  $2, 5, 8, \dots, 6m - 1$  ( $2m$  points). As before we add  $n - 2$  repeatedly starting with 1 to get:  $1, 3m, 6m - 1, 3m - 3, 6m - 4, \dots, 3, 3m + 2, 6m + 1, 3m - 1, \dots, 2, 3m + 1, 6m, 3m - 2, \dots, 1$ . No two consecutive numbers can be black in a bad set, so a bad set can have at most  $n - 1$  points.

Finally, take  $n = 3m + 2$ . A bad coloring with  $n - 2$  points is  $1, 2, \dots, n - 2$ . This time when we add  $n - 2 = 3m$  repeatedly starting with 1, we get back to 1 after including only one-third of the numbers:  $1, 3m + 1, 6m + 1, 3m - 2, \dots, 4, 3m + 4, 1$ . The usual argument shows that at most  $m$  of these  $2m + 1$  numbers can be colored black in a bad set. Similarly, we may add  $3m$  repeatedly starting with 2 to get another  $2m + 1$  numbers:  $2, 3m + 2, 6m + 2, 3m - 1, \dots, 3m + 5, 2$ . At most  $m$  of these can be black in a bad set. Similarly at most  $m$  of the  $2m + 1$  numbers:  $3, 3m + 3, 6m + 3, 3m, \dots, 3m + 6, 3$  can be black. So in total at most  $3m = n - 2$  can be black in a bad set.

## IMO 1990



### Problem A3

**Determine all integers greater than 1 such that  $(2^n + 1)/n^2$  is an integer.**

**Solution** by Gerhard Wöginger, Technical University, Graz

**Answer:**  $n = 3$ .

Since  $2^n + 1$  is odd,  $n$  must also be odd. Let  $p$  be its smallest prime divisor. Let  $x$  be the smallest positive integer such that  $2^x \equiv -1 \pmod{p}$ , and let  $y$  be the smallest positive integer such that  $2^y \equiv 1 \pmod{p}$ .  $y$  certainly exists and indeed  $y < p$ , since  $2^{p-1} \equiv 1 \pmod{p}$ .  $x$  exists since  $2^n \equiv -1 \pmod{p}$ . Write  $n = ys + r$ , with  $0 \leq r < y$ . Then  $-1 \equiv 2^n \equiv (2^y)^s 2^r \equiv 2^r \pmod{p}$ , so  $x \leq r < y$  ( $r$  cannot be 0, since  $-1$  is not  $1 \pmod{p}$ ).

Now write  $n = hx + k$ , with  $0 \leq k < x$ . Then  $-1 \equiv 2^n \equiv (-1)^h 2^k \pmod{p}$ . Suppose  $k > 0$ . Then if  $h$  is odd we contradict the minimality of  $y$ , and if  $h$  is even we contradict the minimality of  $x$ . So  $k = 0$  and  $x$  divides  $n$ . But  $x < p$  and  $p$  is the smallest prime dividing  $n$ , so  $x = 1$ . Hence  $2 \equiv -1 \pmod{p}$  and so  $p = 3$ .

Now suppose that  $3^m$  is the largest power of 3 dividing  $n$ . We show that  $m$  must be 1. Expand  $(3 - 1)^n + 1$  by the binomial theorem, to get (since  $n$  is odd):  $1 - 1 + n \cdot 3 - \frac{1}{2} n(n - 1) 3^2 + \dots = 3n - \frac{(n - 1)n}{2} 3^2 + \dots$ . Evidently  $3n$  is divisible by  $3^{m+1}$ , but not  $3^{m+2}$ . We show that the remaining terms are all divisible by  $3^{m+2}$ . It follows that  $3^{m+1}$  is the highest power 3 dividing  $2^n + 1$ . But  $2^n + 1$  is divisible by  $n^2$  and hence by  $3^{2m}$ , so  $m$  must be 1.

The general term is  $\binom{3^m a}{b} 3^b$ , for  $b \geq 3$ . The binomial coefficients are integral, so the term is certainly divisible by  $3^{m+2}$  for  $b \geq m+2$ . We may write the binomial coefficient as  $\frac{(3^m a)!}{b! (3^m a - b)!} \frac{1}{1} \frac{(3^m - 2)!}{2} \frac{(3^m - 3)!}{3} \dots \frac{(3^m - (b-1))!}{(b-1)!}$ . For  $b$  not a multiple of 3, the first term has the form  $3^m c/d$ , where 3 does not divide  $c$  or  $d$ , and the remaining terms have the form  $c/d$ , where 3



does not divide  $c$  or  $d$ . So if  $b$  is not a multiple of 3, then the binomial coefficient is divisible by  $3^m$ , since  $b > 3$ , this means that the whole term is divisible by at least  $3^{m+3}$ . Similarly, for  $b$  a multiple of 3, the whole term has the same maximum power of 3 dividing it as  $3^m 3^b/b$ . But  $b$  is at least 3, so  $3^b/b$  is divisible by at least 9, and hence the whole term is divisible by at least  $3^{m+2}$ .

We may check that  $n = 3$  is a solution. If  $n > 3$ , let  $n = 3t$  and let  $q$  be the smallest prime divisor of  $t$ . Let  $w$  be the smallest positive integer for which  $2^w = -1 \pmod{q}$ , and  $v$  the smallest positive integer for which  $2^v = 1 \pmod{q}$ .  $v$  certainly exists and  $< q$  since  $2^{q-1} = 1 \pmod{q}$ .  $2^n = -1 \pmod{q}$ , so  $w$  exists and, as before,  $w < v$ . Also as before, we conclude that  $w$  divides  $n$ . But  $w < q$ , the smallest prime divisor of  $n$ , except 3. So  $w = 1$  or 3. These do not work, because then  $2 = -1 \pmod{q}$  and so  $q = 3$ , or  $2^3 = -1 \pmod{q}$  and again  $q = 3$ , whereas we know that  $q > 3$ .

## IMO 1990



### Problem B1

**Construct a function from the set of positive rational numbers into itself such that  $f(x f(y)) = f(x)/y$  for all  $x, y$ .**

#### Solution

We show first that  $f(1) = 1$ . Taking  $x = y = 1$ , we have  $f(f(1)) = f(1)$ . Hence  $f(1) = f(f(1)) = f(1 f(1)) = f(1)/f(1) = 1$ .

Next we show that  $f(xy) = f(x)f(y)$ . For any  $y$  we have  $1 = f(1) = f(1/f(y) f(y)) = f(1/f(y))/y$ , so if  $z = 1/f(y)$  then  $f(z) = y$ . Hence  $f(xy) = f(xf(z)) = f(x)/z = f(x) f(y)$ .

Finally,  $f(f(x)) = f(1 f(x)) = f(1)/x = 1/x$ .

We are not required to find *all* functions, just one. So divide the primes into two infinite sets  $S = \{p_1, p_2, \dots\}$  and  $T = \{q_1, q_2, \dots\}$ . Define  $f(p_n) = q_n$ , and  $f(q_n) = 1/p_n$ . We extend this definition to all rationals using  $f(xy) = f(x) f(y)$ :  $f(p_{i_1} p_{i_2} \dots q_{j_1} q_{j_2} \dots / (p_{k_1} \dots q_{m_1} \dots)) = p_{m_1} \dots q_{i_1} \dots / (p_{j_1} \dots q_{k_1} \dots)$ . It is now trivial to verify that  $f(x f(y)) = f(x)/y$ .



### Problem B2

Given an initial integer  $n_0 > 1$ , two players A and B choose integers  $n_1, n_2, n_3, \dots$  alternately according to the following rules:

Knowing  $n_{2k}$ , A chooses any integer  $n_{2k+1}$  such that  $n_{2k} \leq n_{2k+1} \leq n_{2k}^2$ .

Knowing  $n_{2k+1}$ , B chooses any integer  $n_{2k+2}$  such that  $n_{2k+1}/n_{2k+2} = p^r$  for some prime  $p$  and integer  $r \geq 1$ .

Player A wins the game by choosing the number 1990; player B wins by choosing the number 1. For which  $n_0$  does

- A have a winning strategy?
- B have a winning strategy?
- Neither player have a winning strategy?

### Solution

Answer: if  $n_0 = 2, 3, 4$  or  $5$  then A loses; if  $n_0 \geq 8$ , then A wins; if  $n_0 = 6$  or  $7$ , then it is a draw.

A's strategy given a number  $n$  is as follows:

- if  $n \in [8, 11]$ , pick 60
- if  $n \in [12, 16]$ , pick 140
- if  $n \in [17, 22]$ , pick 280
- if  $n \in [23, 44]$ , pick 504
- if  $n \in [45, 1990]$ , pick 1990
- if  $n = 1991 = 11 \cdot 181$  (181 is prime), pick 1991
- if  $n \in [11^r \cdot 181 + 1, 11^{r+1} \cdot 181]$  for some  $r > 0$ , pick  $11^{r+1} \cdot 181$ .

Clearly (5) wins immediately for A. After (4) B has 7.8.9 so must pick 56, 63, 72 or 168, which gives A an immediate win by (5). After (3) B must pick 35, 40, 56, 70 or 140, so A wins by (4) and (5). After (2) B must pick 20, 28, 35 or 70, so A wins by (3) - (5). After (1) B must pick 12, 15, 20 or 30, so A wins by (2) - (5).

If B is given  $11^{r+1} \cdot 181$ , then B must pick 181,  $11 \cdot 181, \dots, 11^r \cdot 181$  or  $11^{r+1}$ , all of which are  $\leq 11^r \cdot 181$ . So if A is given a number  $n$  in (6) or (7) then after a turn each A is given a number  $< n$  (and  $\geq 11$ ), so after a finite number of turns A wins.

If B gets a number less than 6, then he can pick 1 and win. Hence if A is given 2, he loses, because he must pick a number less than 5. Now if B gets a number of 11 or less, he wins by picking 1 or 2. Hence if A is given 3, he loses, because he must pick a number less than 10. Now if B gets a number of 19 or less, he can win by picking 1, 2 or 3. So if A is given 4 he loses. Now if B is given 29 or less, he can pick 1, 2, 3 or 4 and win. So if A is given 5 he loses.

We now have to consider what happens if A gets 6 or 7. He must pick 30 or more, or B wins. If he picks 31, 32, 33, 34, 35 or 36, then B wins by picking (for example) 1, 1, 3, 2, 5, 4 respectively. So his only hope given 6 is to pick 30. B also wins given any of 37, 38, 39, 40, 41,

43, 44, 45, 46, 47, 48, 49 (winning moves, for example, 37, 1; 38, 2; 39, 3; 40, 5; 41, 1; 43, 1; 44, 4; 45, 5; 46, 3; 47, 1; 48, 3]. So A's only hope given 7 is to pick 30 or 42.

If B is faced with  $30=2\cdot3\cdot5$ , then he has a choice of 6, 10, 15. We have already established that 10 and 15 will lose, so he must pick 6. Thus 6 is a draw: A must pick 30 or lose, and then B must pick 6 or lose.

If B is faced with  $42=2\cdot3\cdot7$ , then he has a choice of 6, 14 or 21. We have already established that 14 and 21 lose, so he must pick 6. Thus 7 is also a draw: A must pick 30 or 42, and then B must pick 6.

## IMO 1990



### Problem B3

**Prove that there exists a convex 1990-gon such that all its angles are equal and the lengths of the sides are the numbers  $1^2, 2^2, \dots, 1990^2$  in some order.**

#### Solution

*By Robin Chapman, Dept of Maths, Macquarie University, Australia*

In the complex plane we can represent the sides as  $p_n^2 w^n$ , where  $p_n$  is a permutation of  $(1, 2, \dots, 1990)$  and  $w$  is a primitive 1990th root of unity.

The critical point is that 1990 is a product of more than 2 distinct primes:  $1990 = 2\cdot5\cdot199$ . So we can write  $w = -1\cdot a\cdot b$ , where  $-1$  is primitive 2nd root of unity,  $a$  is a primitive 5th root of unity, and  $b$  is a primitive 199th root of unity.

Now given one of the 1990th roots we may write it as  $(-1)^i a^j b^k$ , where  $0 < i < 2$ ,  $0 < j < 5$ ,  $0 < k < 199$  and hence associate it with the integer  $r(i,j,k) = 1 + 995i + 199j + k$ . This is a bijection onto  $(1, 2, \dots, 1990)$ . We have to show that the sum of  $r(i,j,k)^2 (-1)^i a^j b^k$  is zero.

We sum first over  $i$ . This gives  $-995^2 \times \text{sum of } a^j b^k$  which is zero, and  $-1990 \times \text{sum } s(j,k) a^j b^k$ , where  $s(j,k) = 1 + 199j + k$ . So it is sufficient to show that the sum of  $s(j,k) a^j b^k$  is zero. We now sum over  $j$ . The  $1 + k$  part of  $s(j,k)$  immediately gives zero. The  $199j$  part gives a constant times  $b^k$ , which gives zero when summed over  $k$ .



**A1.** Prove that the set  $\{1, 2, \dots, 1989\}$  can be expressed as the disjoint union of subsets  $A_1, A_2, \dots, A_{117}$  in such a way that each  $A_i$  contains 17 elements and the sum of the elements in each  $A_i$  is the same.



**A2.** In an acute-angled triangle  $ABC$ , the internal bisector of angle  $A$  meets the circumcircle again at  $A_1$ . Points  $B_1$  and  $C_1$  are defined similarly. Let  $A_0$  be the point of intersection of the line  $AA_1$  with the external bisectors of angles  $B$  and  $C$ . Points  $B_0$  and  $C_0$  are defined similarly. Prove that the area of the triangle  $A_0B_0C_0$  is twice the area of the hexagon  $AC_1BA_1CB_1$  and at least four times the area of the triangle  $ABC$ .



**A3.** Let  $n$  and  $k$  be positive integers, and let  $S$  be a set of  $n$  points in the plane such that no three points of  $S$  are collinear, and for any point  $P$  of  $S$  there are at least  $k$  points of  $S$  equidistant from  $P$ . Prove that  $k < 1/2 + \sqrt{2n}$ .



**B1.** Let  $ABCD$  be a convex quadrilateral such that the sides  $AB, AD, BC$  satisfy  $AB = AD + BC$ . There exists a point  $P$  inside the quadrilateral at a distance  $h$  from the line  $CD$  such that  $AP = h + AD$  and  $BP = h + BC$ . Show that:



$$1/\sqrt{h} \geq 1/\sqrt{AD} + 1/\sqrt{BC}.$$

**B2.** Prove that for each positive integer  $n$  there exist  $n$  consecutive positive integers none of which is a prime or a prime power.



**B3.** A permutation  $\{x_1, x_2, \dots, x_m\}$  of the set  $\{1, 2, \dots, 2n\}$  where  $n$  is a positive integer is said to have property  $P$  if  $|x_i - x_{i+1}| = n$  for at least one  $i$  in  $\{1, 2, \dots, 2n-1\}$ . Show that for each  $n$  there are more permutations with property  $P$  than without.

## IMO 1989



### Problem A1

Prove that the set  $\{1, 2, \dots, 1989\}$  can be expressed as the disjoint union of subsets  $A_1, A_2, \dots, A_{117}$  in such a way that each  $A_i$  contains 17 elements and the sum of the elements in each  $A_i$  is the same.

### Solution

We construct 116 sets of three numbers. Each set sums to  $3 \times 995 = 2985$ . The 348 numbers involved form 174 pairs  $\{r, 1990 - r\}$ . At this point we are essentially done. We take a 117th set which has one  $\{r, 1990 - r\}$  pair and 995. The original 1989 numbers comprise 995 and 994  $\{r, 1990 - r\}$  pairs. We have used up 995 and 175 pairs, leaving just 819 pairs. We now add 7 pairs to each of our 117 sets, bringing the total of each set up to  $2985 + 7 \cdot 1990 = 1990 \times 17/2$ .

It remains to exhibit the 116 sets. There are many possibilities. We start with:

301, 801, 1883 and the "complementary" set  $1990 - 301 = 1689$ ,  $1990 - 801 = 1189$ ,  $1990 - 1883 = 107$ . We then add one to each of the first two numbers to get:

302, 802, 1881 and 1688, 1188, 109, and so on:

303, 803, 1879 and 1687, 1187, 111,

...

358, 858, 1769 and 1632, 1132, 221.

We can immediately see that these triples are all disjoint. So the construction is complete.

## IMO 1989



### Problem A2

In an acute-angled triangle  $ABC$ , the internal bisector of angle  $A$  meets the circumcircle again at  $A_1$ . Points  $B_1$  and  $C_1$  are defined similarly. Let  $A_0$  be the point of intersection of the line  $AA_1$  with the external bisectors of angles  $B$  and  $C$ . Points  $B_0$  and  $C_0$  are defined similarly. Prove that the area of the triangle  $A_0B_0C_0$  is twice the area of the hexagon  $AC_1BA_1CB_1$  and at least four times the area of the triangle  $ABC$ .

**Solution** By Marcin Mazur, University of Illinois at Urbana-Champaign

Let  $I$  be the point of intersection of  $AA_0, BB_0, CC_0$  (the in-center).  $\angle BIC = 180 - 1/2 \angle ABC - 1/2 \angle BCA = 180 - 1/2 (180 - \angle CAB) = 90 + 1/2 \angle CAB$ . Hence  $\angle CA_1B = 180 - \angle CAB$  [ $BA_1CA$  is cyclic] =  $2(180 - \angle BIC) = 2\angle CA_0B$ . But  $\angle A_1B = \angle A_1C$ , so  $A_1$  is the center of the circumcircle of  $BCA_0$ . But  $I$  lies on this circumcircle ( $\angle IBA_0 = \angle ICA_0 = 90$ ), and hence  $A_1A_0 = A_1I$ .

Hence area  $IBA_1 = \text{area } A_0BA_1$  and area  $ICA_1 = \text{area } A_0CA_1$ . Hence area  $IBA_0C = 2 \text{ area } IBA_1C$ . Similarly, area  $ICB_0A = 2 \text{ area } ICB_1A$  and area  $IAC_0B = 2 \text{ area } IAC_1B$ . Hence area  $A_0B_0C_0 = 2 \text{ area hexagon } AB_1CA_1BC_1$ .

*A neat solution for the rest is as follows by Thomas Jäger*

Let  $H$  be the orthocentre of  $ABC$ . Let  $H_1$  be the reflection of  $H$  in  $BC$ , so  $H_1$  lies on the circumcircle. So area  $BCH = \text{area } BCH_1 \leq \text{area } BCA_1$ . Adding to the two similar inequalities gives area  $ABC \leq \text{area hexagon} - \text{area } ABC$ .

*Mazur's solution was as follows*

$CAB = 180^\circ - CA_1B$  and  $A_1B = A_1C$ , so  $A_1BC = 90^\circ - 1/2 CA_1B = 1/2 CAB$ . Hence the perpendicular from  $A_1$  to  $BC$  has length  $1/2 BC \tan(CAB/2)$  and area  $CA_1B = 1/4 BC^2 \tan(CAB/2)$ .

Put  $r = \text{radius of in-circle of } ABC$ ,  $x = \cot(CAB/2)$ ,  $y = \cot(ABC/2)$ ,  $z = \cot(BCA/2)$ . Then  $BC = r(y + z)$  and area  $CA_1B = r^2(y + z)^2/(4x)$ . Also area  $BIC = 1/2 r BC$ . Similarly for the other triangles, so area  $ABC = \text{area } BIC + \text{area } CIA + \text{area } AIB = r^2(x + y + z)$ . We have to show that area  $ABC \leq \text{area } CA_1B + \text{area } AB_1C + \text{area } BC_1A$ , or  $(x + y + z) \leq (y + z)^2/(4x) + (z + x)^2/(4y) + (x + y)^2/(4z)$ .

Putting  $s = x + y + z$ , this is equivalent to:  $4s \leq (s - x)^2/x + (s - y)^2/y + (s - z)^2/z$ , or  $9s \leq s^2(1/x + 1/y + 1/z)$ , but this is just the statement that the arithmetic mean of  $x, y, z$  is not less than the harmonic mean.

Note in passing that the requirement for  $ABC$  to be acute is unnecessary.

## IMO 1989



### Problem A3

**Let  $n$  and  $k$  be positive integers, and let  $S$  be a set of  $n$  points in the plane such that no three points of  $S$  are collinear, and for any point  $P$  of  $S$  there are at least  $k$  points of  $S$  equidistant from  $P$ . Prove that  $k < 1/2 + \sqrt{2n}$ .**

### Solution

Three variants on a theme, all kindly supplied by others (I spent 2 hours failing to solve it). My favorite first. *By Eli Bachmutsky*

Consider the pairs  $P, \{A, B\}$ , where  $P, A, B$  are points of  $S$ , and  $P$  lies on the perpendicular bisector of  $AB$ . There are at least  $n k(k - 1)/2$  such pairs, because for each point  $P$ , there are at least  $k$  points equidistant from  $P$  and hence at least  $k(k - 1)/2$  pairs of points equidistant from  $P$ .

If  $k \geq 1/2 + \sqrt{2n}$ , then  $k(k - 1) \geq 2n - 1/4 > 2(n - 1)$ , and so there are more than  $n(n - 1)$  pairs  $P, \{A, B\}$ . But there are only  $n(n - 1)/2$  possible pairs  $\{A, B\}$ , so for some  $\{A_0, B_0\}$  we must be able

to find at least 3 points P on the perpendicular bisector of  $A_0B_0$ . But these points are collinear, contradicting the assumption in the question.

*From an anonymous source*

Let the points be  $P_1, P_2, \dots, P_n$ . Let  $C_i$  be a circle center  $P_i$  containing at least  $k$  points of  $S$ . There are at least  $nk$  pairs  $(C_i, P_j)$ , where  $P_j$  lies on  $C_i$ . Hence there must a point P lying on at least  $k$  circles. Take  $k$  such circles  $C_\alpha$ . For each such circle  $C_\alpha$ , take a subset  $S_\alpha$  comprising exactly  $k$  points of  $S \cap C_\alpha$ .

We now count the points in  $\cup S_\alpha$ . Apart from P, there are  $k-1$  points in each  $S_\alpha$ . So we start with  $1 + k(k-1)$ . But this counts some points more than once. Each pair  $(S_\alpha, S_\beta)$  (with  $\alpha \neq \beta$ ) has at most one common point apart from P (because distinct circles have at most two common points). So we deduct 1 for each of the  $1/2 k(k-1)$  pairs  $(S_\alpha, S_\beta)$ , giving  $1 + k(k-1) - 1/2 k(k-1)$  (\*).

If  $Q (\neq P)$  is in exactly  $r$  sets  $S_\alpha$ , then it is counted  $r$  times in the second term  $k(k-1)$ , and subtracted  $1/2 r(r-1)$  times in the third term. So it is counted  $1/2 r(3-r)$  times in all. That is correct for  $r = 0, 1$  or  $2$  and too low for  $r > 2$ . So (\*) is  $\leq |\cup S_\alpha|$ . Clearly  $|\cup S_\alpha| \leq n$ , so  $n \geq 1 + k(k-1)/2 = (k-1/2)^2/2 + 7/8 > (k-1/2)^2/2$ . Hence  $\sqrt{(2n)+1/2} > k$ .

*By Thomas Jäger*

Define  $P_i$  and  $S_i$  as above. Let  $g(i)$  be the number of  $S_a$  containing  $P_i$ , and let  $f(i, j)$  be  $|S_i \cap S_j|$ . Let  $h(x) = x(x-1)/2$ . We count the number  $N$  of pairs  $i, \{a, b\}$ , where point  $i$  is in  $S_a$  and  $S_b$ .

Point  $i$  is in  $g(i)$  sets  $S_j$ , from which we can choose  $S_a, S_b$  in  $h(g(i))$  ways. Hence  $N = \sum h(g(i))$ . But  $f(a, b)$  points are in  $S_a$  and  $S_b$ , so  $N = \sum f(a, b)$ . But, since distinct circles intersect in at most 2 points,  $f(a, b) \leq 2$ , so  $\sum f(a, b) \leq 2 \sum h(g(i))$ . We conclude that  $2 \sum h(g(i)) \geq \sum h(g(i))$ .

$h$  is a convex function, so  $1/n \sum h(g(i)) \geq h(1/n \sum g(i)) = n h(1/n nk) = n h(k)$ . Hence  $n-1 \geq h(k)$ , which gives the result, as above.

## IMO 1989



### Problem B1

**Let ABCD be a convex quadrilateral such that the sides AB, AD, BC satisfy  $AB = AD + BC$ . There exists a point P inside the quadrilateral at a distance  $h$  from the line CD such that  $AP = h + AD$  and  $BP = h + BC$ . Show that:**

$$1/\sqrt{h} \geq 1/\sqrt{AD} + 1/\sqrt{BC}.$$

**Solution** by Gerhard Wöginger, Technical University, Graz

Let  $C_A$  be the circle center A, radius AD, and  $C_B$  the circle center B, radius BC. The circles touch on AB. Let  $C_P$  the the circle center P, radius  $h$ .  $C_P$  touches  $C_A$  and  $C_B$  and CD. Let  $t$  be the common tangent to  $C_A$  and  $C_B$  whose two points of contact are on the same side of AB as C and D. Then  $C_P$  is confined inside the curvilinear triangle whose sides are segments of  $t$ ,  $C_A$  and  $C_B$ . Evidently  $h$  attains its maximum value, for given lengths AB, AD, BC, when  $C_P$  touches  $t$ , in

which case D must be the point at which t touches  $C_A$ , and C the point at which it touches  $C_B$ . Suppose E is the point at which t touches  $C_P$ .

Angles ADC and BCD are right angles, so  $CD^2 = AB^2 - (AD - BC)^2 = 4 AD BC$ . Similarly,  $DE^2 = 4 h AD$ , and  $CE^2 = 4 h BC$ . But  $CD = DE + CE$ , so  $1/\sqrt{h} = 1/\sqrt{AD} + 1/\sqrt{BC}$ . This gives the maximum value of h, so in general we have the inequality stated.

## IMO 1989



### Problem B2

**Prove that for each positive integer n there exist n consecutive positive integers none of which is a prime or a prime power.**

#### Solution

Consider  $(N!)^2+2, (N!)^2+3, \dots, (N!)^2+N$ .  $(N!)^2+r$  is divisible by r, but  $((N!)^2+r)/r = N! (N!/r) + 1$ , which is greater than one, but relatively prime to r since  $N! (N!/r)$  is divisible by r. For each r we may take a prime  $p_r$  dividing r, so  $(N!)^2+r$  is divisible by  $p_r$ , but is not a power of  $p_r$ . Hence it is not a prime or a prime power. Taking  $N = n+1$  gives n consecutive numbers as required.

## IMO 1989



### Problem B3

**A permutation  $\{x_1, x_2, \dots, x_m\}$  of the set  $\{1, 2, \dots, 2n\}$  where n is a positive integer is said to have property P if  $|x_i - x_{i+1}| = n$  for at least one i in  $\{1, 2, \dots, 2n-1\}$ . Show that for each n there are more permutations with property P than without.**

#### Solution

*from Arthur Engel, Problem-Solving Strategies, Springer 1998 [Problem books in mathematics series], ISBN 0387982191. A rather good training book.*

Let  $A_k$  be the set of permutations with k and k+n in neighboring positions, and let A be the set of permutations with property P, so that A is the union of the  $A_k$ .

Then  $|A| = \sum_k |A_k| - \sum_{k<l} |A_k \cap A_l| + \sum_{k<l<m} |A_k \cap A_l \cap A_m| - \dots$ . But this is an alternating sequence of monotonically decreasing terms, hence  $|A| \geq \sum_k |A_k| - \sum_{k<l} |A_k \cap A_l|$ .

But  $|A_k| = 2 (2n - 1)!$  (two orders for k, k+n and then  $(2n - 1)!$  ways of arranging the  $2n - 1$  items, treating k, k+n as a single item). Similarly,  $|A_k \cap A_l| = 4 (2n - 2)!$  So  $|A| \geq (2n - 2)! [n \cdot 2(2n - 1) - n(n - 1)/2 \cdot 4] = 2n^2 (2n - 2)! > (2n)!/2$ .





**A1.** Consider two coplanar circles of radii  $R > r$  with the same center. Let  $P$  be a fixed point on the smaller circle and  $B$  a variable point on the larger circle. The line  $BP$  meets the larger circle again at  $C$ . The perpendicular to  $BP$  at  $P$  meets the smaller circle again at  $A$  (if it is tangent to the circle at  $P$ , then  $A = P$ ).



- (i) Find the set of values of  $AB^2 + BC^2 + CA^2$ .
- (ii) Find the locus of the midpoint of  $BC$ .

**A2.** Let  $n$  be a positive integer and let  $A_1, A_2, \dots, A_{2n+1}$  be subsets of a set  $B$ . Suppose that:

- (i) Each  $A_i$  has exactly  $2n$  elements,
- (ii) The intersection of every two distinct  $A_i$  contains exactly one element, and
- (iii) Every element of  $B$  belongs to at least two of the  $A_i$ .



For which values of  $n$  can one assign to every element of  $B$  one of the numbers  $0$  and  $1$  in such a way that each  $A_i$  has  $0$  assigned to exactly  $n$  of its elements?

**A3.** A function  $f$  is defined on the positive integers by:  $f(1) = 1$ ;  $f(3) = 3$ ;  $f(2n) = f(n)$ ,  $f(4n + 1) = 2f(2n + 1) - f(n)$ , and  $f(4n + 3) = 3f(2n + 1) - 2f(n)$  for all positive integers  $n$ . Determine the number of positive integers  $n$  less than or equal to  $1988$  for which  $f(n) = n$ .



**B1.** Show that the set of real numbers  $x$  which satisfy the inequality:

$$1/(x - 1) + 2/(x - 2) + 3/(x - 3) + \dots + 70/(x - 70) \geq 5/4$$



is a union of disjoint intervals, the sum of whose lengths is  $1988$ .

**B2.**  $ABC$  is a triangle, right-angled at  $A$ , and  $D$  is the foot of the altitude from  $A$ . The straight line joining the incenters of the triangles  $ABD$  and  $ACD$  intersects the sides  $AB, AC$  at  $K, L$  respectively. Show that the area of the triangle  $ABC$  is at least twice the area of the triangle  $AKL$ .



**B3.** Let  $a$  and  $b$  be positive integers such that  $ab + 1$  divides  $a^2 + b^2$ . Show that  $(a^2 + b^2)/(ab + 1)$  is a perfect square.



### Problem A1

Consider two coplanar circles of radii  $R > r$  with the same center. Let  $P$  be a fixed point on the smaller circle and  $B$  a variable point on the larger circle. The line  $BP$  meets the larger circle again at  $C$ . The perpendicular to  $BP$  at  $P$  meets the smaller circle again at  $A$  (if it is tangent to the circle at  $P$ , then  $A = P$ ).

- (i) Find the set of values of  $AB^2 + BC^2 + CA^2$ .
- (ii) Find the locus of the midpoint of  $BC$ .

### Solution

(i) Let  $M$  be the midpoint of  $BC$ . Let  $PM = x$ . Let  $BC$  meet the small circle again at  $Q$ . Let  $O$  be the center of the circles. Since angle  $APQ = 90$  degrees,  $AQ$  is a diameter of the small circle, so its length is  $2r$ . Hence  $AP^2 = 4r^2 - 4x^2$ .  $BM^2 = R^2 - OM^2 = R^2 - (r^2 - x^2)$ . That is essentially all we need, because we now have:  $AB^2 + AC^2 + BC^2 = (AP^2 + (BM - x)^2) + (AP^2 + (BM + x)^2) + 4BM^2 = 2AP^2 + 6BM^2 + 2x^2 = 2(4r^2 - 4x^2) + 6(R^2 - r^2 + x^2) + 2x^2 = 6R^2 + 2r^2$ , which is independent of  $x$ .

(ii)  $M$  is the midpoint of  $BC$  and  $PQ$  since the circles have a common center. If we shrink the small circle by a factor 2 with  $P$  as center, then  $Q$  moves to  $M$ , and hence the locus of  $M$  is the circle diameter  $OP$ .



### Problem A2

Let  $n$  be a positive integer and let  $A_1, A_2, \dots, A_{2n+1}$  be subsets of a set  $B$ . Suppose that:

- (i) Each  $A_i$  has exactly  $2n$  elements,
- (ii) The intersection of every two distinct  $A_i$  contains exactly one element, and
- (iii) Every element of  $B$  belongs to at least two of the  $A_i$ .

For which values of  $n$  can one assign to every element of  $B$  one of the numbers 0 and 1 in such a way that  $A_i$  has 0 assigned to exactly  $n$  of its elements?

### Solution

Answer:  $n$  even.

Each of the  $2n$  elements of  $A_i$  belongs to at least one other  $A_j$  because of (iii). But given another  $A_j$  it cannot contain more than one element of  $A_i$  because of (ii). There are just  $2n$  other  $A_j$  available, so each must contain exactly one element of  $A_i$ . Hence we can strengthen (iii) to every element of  $B$  belongs to exactly two of the  $A_s$ .

This shows that the arrangement is essentially unique. We may call the element of B which belongs to  $A_i$  and  $A_j$   $(i,j)$ . Then  $A_i$  contains the  $2n$  elements  $(i, j)$  with  $j$  not  $i$ .

$|B| = 1/2 \times \text{no. of } A\text{'s} \times \text{size of each } A = n(2n+1)$ . If the labeling with 0s and 1s is possible, then if we list all the elements in each  $A$ ,  $n(2n+1)$  out of the  $2n(2n+1)$  elements have value 0. But each element appears twice in this list, so  $n(2n+1)$  must be even. Hence  $n$  must be even.

*Next part thanks to Stan Dolan*

Label  $(i,j)$  0 if  $j = i-n/2, i-(n/2 - 1), \dots, i-1, i+1, i+2, \dots, i+n/2$  (working mod  $2n+1$  when necessary). This clearly has the required property.

*My original solution was a pedestrian induction:*

We show by induction that a labeling is always possible for  $n$  even. If  $n = 2$ , there is certainly a labeling. For example, we may assign 0 to  $(1,2), (1,3), (2,4), (3,5), (4,5)$ . Now suppose we have a labeling for  $n$ . For  $n + 2$ , we label  $(i, j)$  0 if it was labeled 0 for  $n$  or if it is:

- $(i, 2n+2)$  or  $(i, 2n+3)$  for  $i = 1, 2, \dots, n+1$
- $(i, 2n+4)$  or  $(i, 2n+5)$  for  $i = n+2, n+3, \dots, 2n+1$
- $(2n+2, 2n+4), (2n+3, 2n+5), (2n+4, 2n+5)$ .

For  $i = 1, 2, \dots, n+1$ ,  $A_i$  has  $n$  elements  $(i, j)$  labeled zero with  $j \leq 2n+1$  and also  $(i, 2n+2)$  and  $(i, 2n+3)$ , giving  $n+2$  in all. For  $i = n+2, n+3, \dots, 2n+1$ ,  $A_i$  has  $n$  elements  $(i, j)$  labeled zero with  $j \leq 2n+1$  and also  $(i, 2n+4)$  and  $(i, 2n+5)$ , giving  $n+2$  in all.  $A_{2n+2}$  has the  $n+1$  elements  $(i, 2n+2)$  with  $i \leq n+1$  and also  $(2n+2, 2n+4)$ , giving  $n+2$  in all.  $A_{2n+3}$  has the  $n+1$  elements  $(i, 2n+3)$  for  $i \leq n+1$  and also  $(2n+3, 2n+5)$ , giving  $n+2$  in all.  $A_{2n+4}$  has the  $n$  elements  $(i, 2n+4)$  with  $n+2 \leq i \leq 2n+1$  and also  $(2n+2, 2n+4)$  and  $(2n+4, 2n+5)$ , giving  $n+2$  in all. Finally  $A_{2n+5}$  has the  $n$  elements  $(i, 2n+5)$  with  $n+2 \leq i \leq 2n+1$  and also  $(2n+3, 2n+5)$  and  $(2n+4, 2n+5)$ , giving  $n+2$  in all.

## IMO 1988



### Problem A3

**A function  $f$  is defined on the positive integers by:  $f(1) = 1$ ;  $f(3) = 3$ ;  $f(2n) = f(n)$ ,  $f(4n + 1) = 2f(2n + 1) - f(n)$ , and  $f(4n + 3) = 3f(2n + 1) - 2f(n)$  for all positive integers  $n$ . Determine the number of positive integers  $n$  less than or equal to 1988 for which  $f(n) = n$ .**

### Solution

Answer: 92.

$f(n)$  is always odd. If  $n = b_{r+1}b_r \dots b_2b_1b_0$  in binary and  $n$  is odd, so that  $b_{r+1} = b_0 = 1$ , then  $f(n) = b_{r+1}b_1b_2 \dots b_r b_0$ . If  $n$  has  $r+2$  binary digits with  $r > 0$ , then there are  $2^{\lfloor (r+1)/2 \rfloor}$  numbers with the central  $r$  digits symmetrical, so that  $f(n) = n$  (because we can choose the central digit and those lying before it arbitrarily, the rest are then determined). Also there is one number with 1 digit (1) and one number with two digits (3) satisfying  $f(n) = 1$ . So we find a total of  $1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 = 62$  numbers in the range 1 to 1023 with  $f(n) = n$ .  $1988 = 11111000011$ . So we also have all 32 numbers in the range 1023 to 2047 except for 1111111111 and 1111101111, giving another 30, or 92 in total.

It remains to prove the assertions above.  $f(n)$  odd follows by an easy induction. Next we show that if  $2^m < 2n+1 < 2^{m+1}$ , then  $f(2n+1) = f(n) + 2^m$ . Again we use induction. It is true for  $m = 1$  ( $f(3) = f(1) + 2$ ). So suppose it is true for  $1, 2, \dots, m$ . Take  $4n+1$  so that  $2^{m+1} < 4n+1 < 2^{m+2}$ , then  $f(4n+1) = 2f(2n+1) - f(n) = 2(f(n) + 2^m) - f(n) = f(n) + 2^{m+1} = f(2n) + 2^{m+1}$ , so it is true for  $4n+1$ . Similarly, if  $4n+3$  satisfies,  $2^{m+1} < 4n+3 < 2^{m+2}$ , then  $f(4n+3) = 3f(2n+1) - 2f(n) = f(2n+1) + 2(f(n) + 2^m) - 2f(n) = f(2n+1) + 2^{m+1}$ , so it is true for  $4n+3$  and hence for  $m+1$ .

Finally, we prove the formula for  $f(2n+1)$ . Let  $2n+1 = b_{r+1}b_r \dots b_2b_1b_0$  with  $b_0 = b_{r+1} = 1$ . We use induction on  $r$ . So assume it is true for smaller values. Say  $b_1 = \dots = b_s = 0$  and  $b_{s+1} = 1$  (we may have  $s = 0$ , so that we have simply  $b_1 = 1$ ). Then  $n = b_{r+1} \dots b_1$  and  $f(n) = b_{r+1}b_{s+2}b_{s+3} \dots b_r b_{s+1}$  by induction. So  $f(n) + 2^{r+1} = b_{r+1}0 \dots 0b_{r+1}b_{s+2} \dots b_r b_{s+1}$ , where there are  $s$  zeros. But we may write this as  $b_{r+1}b_1 \dots b_s b_{s+1} \dots b_r b_{r+1}$ , since  $b_1 = \dots = b_s = 0$ , and  $b_{s+1} = b_{r+1} = 1$ . But that is the formula for  $f(2n+1)$ , so we have completed the induction

## IMO 1988



### Problem B1

Show that the set of real numbers  $x$  which satisfy the inequality:

$$1/(x-1) + 2/(x-2) + 3/(x-3) + \dots + 70/(x-70) \geq 5/4$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

### Solution

Let  $f(x) = 1/(x-1) + 2/(x-2) + 3/(x-3) + \dots + 70/(x-70)$ . For any integer  $n$ ,  $n/(x-n)$  is strictly monotonically decreasing except at  $x = n$ , where it is discontinuous. Hence  $f(x)$  is strictly monotonically decreasing except at  $x = 1, 2, \dots, 70$ . For  $n =$  any of  $1, 2, \dots, 70$ ,  $n/(x-n)$  tends to plus infinity as  $x$  tends to  $n$  from above, whilst the other terms  $m/(x-m)$  remain bounded. Hence  $f(x)$  tends to plus infinity as  $x$  tends to  $n$  from above. Similarly,  $f(x)$  tends to minus infinity as  $x$  tends to  $n$  from below. Thus in each of the intervals  $(n, n+1)$  for  $n = 1, \dots, 69$ ,  $f(x)$  decreases monotonically from plus infinity to minus infinity and hence  $f(x) = 5/4$  has a single root  $x_n$ . Also  $f(x) \geq 5/4$  for  $x$  in  $(n, x_n]$  and  $f(x) < 5/4$  for  $x$  in  $(x_n, n+1)$ . If  $x < 0$ , then every term is negative and hence  $f(x) < 0 < 5/4$ . Finally, as  $x$  tends to infinity, every term tends to zero, so  $f(x)$  tends to zero. Hence  $f(x)$  decreases monotonically from plus infinity to zero over the range  $[70, \text{infinity}]$ . Hence  $f(x) = 5/4$  has a single root  $x_{70}$  in this range and  $f(x) \geq 5/4$  for  $x$  in  $(70, x_{70}]$  and  $f(x) < 5/4$  for  $x > x_{70}$ . Thus we have established that  $f(x) \geq 5/4$  for  $x$  in any of the disjoint intervals  $(1, x_1], (2, x_2], \dots, (70, x_{70}]$  and  $f(x) < 5/4$  elsewhere.

The total length of these intervals is  $(x_1 - 1) + \dots + (x_{70} - 70) = (x_1 + \dots + x_{70}) - (1 + \dots + 70)$ . The  $x_i$  are the roots of the 70th order polynomial obtained from  $1/(x-1) + 2/(x-2) + 3/(x-3) + \dots + 70/(x-70) = 5/4$  by multiplying both sides by  $(x-1) \dots (x-70)$ . The sum of the roots is minus the coefficient of  $x^{69}$  divided by the coefficient of  $x^{70}$ . The coefficient of  $x^{70}$  is simply  $k$ , and the coefficient of  $x^{69}$  is  $-(1 + 2 + \dots + 70)k - (1 + \dots + 70)$ . Hence the sum of the roots is  $(1 + \dots + 70)(1 + k)/k$  and the total length of the intervals is  $(1 + \dots + 70)/k = 1/2 \cdot 70 \cdot 71 \cdot 4/5 = 28 \cdot 71 = 1988$ .



**Problem 5**

**ABC is a triangle, right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD and ACD intersects the sides AB, AC at K, L respectively. Show that the area of the triangle ABC is at least twice the area of the triangle AKL.**

**Solution**

The key is to show that  $AK = AL = AD$ . We do this indirectly. Take  $K'$  on  $AB$  and  $L'$  on  $AC$  so that  $AK' = AL' = AD$ . Let the perpendicular to  $AB$  at  $K'$  meet the line  $AD$  at  $X$ . Then the triangles  $AK'X$  and  $ADB$  are congruent. Let  $J$  be the incenter of  $ADB$  and let  $r$  be the in-radius of  $ADB$ . Then  $J$  lies on the angle bisector of angle  $BAD$  a distance  $r$  from the line  $AD$ . Hence it is also the incenter of  $AK'X$ . Hence  $JK'$  bisects the right angle  $AK'X$ , so  $\angle AK'J = 45^\circ$  and so  $J$  lies on  $K'L'$ . An exactly similar argument shows that  $I$ , the incenter of  $ADC$ , also lies on  $K'L'$ . Hence we can identify  $K$  and  $K'$ , and  $L$  and  $L'$ .

The area of  $AKL$  is  $AK \cdot AL / 2 = AD^2 / 2$ , and the area of  $ABC$  is  $BC \cdot AD / 2$ , so we wish to show that  $2AD \leq BC$ . Let  $M$  be the midpoint of  $BC$ . Then  $AM$  is the hypotenuse of  $AMD$ , so  $AM \geq AD$  with equality if and only if  $D = M$ . Hence  $2AD \leq 2AM = BC$  with equality if and only if  $AB = AC$ .



**Problem B3**

**Let a and b be positive integers such that  $ab + 1$  divides  $a^2 + b^2$ . Show that  $(a^2 + b^2)/(ab + 1)$  is a perfect square.**

**Solution**

A little experimentation reveals the following solutions:  $a, a^3$  giving  $a^2$ ;  $a^3, a^5 - a$  giving  $a^2$ ; and the recursive  $a_1 = 2, b_1 = 8, a_{n+1} = b_n, b_{n+1} = 4b_n - a_n$  giving 4. The latter may lead us to: if  $a^2 + b^2 = k(ab + 1)$ , then take  $A = b, B = kb - a$ , and then  $A^2 + B^2 = k(AB + 1)$ . Finally, we may notice that this can be used to go down as well as up.

So starting again suppose that  $a, b, k$  is a solution in positive integers to  $a^2 + b^2 = k(ab + 1)$ . If  $a = b$ , then  $2a^2 = k(a^2 + 1)$ . So  $a^2$  must divide  $k$ . But that implies that  $a = b = k = 1$ . Let us assume we do not have this trivial solution, so we may take  $a < b$ . We also show that  $a^3 > b$ . For  $(b/a - 1/a)(ab + 1) = b^2 + b/a - b - 1/a < b^2 < a^2 + b^2$ . So  $k > b/a - 1/a$ . But if  $a^3 < b$ , then  $b/a(ab + 1) > b^2 + a^2$ , so  $k < b/a$ . But now  $b > ak$  and  $< ak + 1$ , which is impossible. It follows that  $k \geq b/a$ .

Now define  $A = ka - b$ ,  $B = a$ . Then we can easily verify that  $A, B, k$  also satisfies  $a^2 + b^2 = k(ab + 1)$ , and  $B$  and  $k$  are positive integers. Also  $a < b$  implies  $a^2 + b^2 < ab + b^2 < ab + b^2 + 1 + b/a = (ab + 1)(1 + b/a)$ , and hence  $k < 1 + b/a$ , so  $ka - b < a$ . Finally, since  $k > b/a$ ,  $ka - b \geq 0$ . If  $ka - b > 0$ , then we have another smaller solution, in which case we can repeat the process. But we cannot have an infinite sequence of decreasing numbers all greater than zero, so we must eventually get  $A = ka - b = 0$ . But now  $A^2 + B^2 = k(AB + 1)$ , so  $k = B^2$ .  $k$  was unchanged during the descent, so  $k$  is a perfect square.

*A slightly neater variation on this is due to Stan Dolan*

As above take  $a^2 + b^2 = k(ab + 1)$ , so  $a, b$ , and  $k$  are all positive integers. Now fixing  $k$  take positive integers  $A, B$  such that  $A^2 + B^2 = k(AB + 1)$  (\*) and  $\min(A, B)$  is as small as possible. Assume  $B \leq A$ . Regarding (\*) as a quadratic for  $A$ , we see that the other root  $C$  satisfies  $A + C = kB$ ,  $AC = B^2 - k$ . The second equation implies that  $C = B^2/A - k/A < B$ . So  $C$  cannot be a positive integer (or the solution  $C, B$  would have  $\min(C, B) < \min(A, B)$ ). But we have  $(A+1)(C+1) = A+C + AC + 1 = B^2 + (B-1)k + 1 > 0$ , so  $C > -1$ .  $C = kB - A$  is an integer, so  $C = 0$ . Hence  $k = B^2$ .

*Note that jumping straight to the minimal without the infinite descent avoids some of the verification needed in the infinite descent.*



**A1.** Let  $p_n(k)$  be the number of permutations of the set  $\{1, 2, 3, \dots, n\}$  which have exactly  $k$  fixed points. Prove that the sum from  $k = 0$  to  $n$  of  $(k p_n(k))$  is  $n!$ .



[A permutation  $f$  of a set  $S$  is a one-to-one mapping of  $S$  onto itself. An element  $i$  of  $S$  is called a fixed point if  $f(i) = i$ .]

**A2.** In an acute-angled triangle  $ABC$  the interior bisector of angle  $A$  meets  $BC$  at  $L$  and meets the circumcircle of  $ABC$  again at  $N$ . From  $L$  perpendiculars are drawn to  $AB$  and  $AC$ , with feet  $K$  and  $M$  respectively. Prove that the quadrilateral  $AKNM$  and the triangle  $ABC$  have equal areas.



**A3.** Let  $x_1, x_2, \dots, x_n$  be real numbers satisfying  $x_1^2 + x_2^2 + \dots + x_n^2 = 1$ . Prove that for every integer  $k \geq 2$  there are integers  $a_1, a_2, \dots, a_n$ , not all zero, such that  $|a_i| \leq k - 1$  for all  $i$ , and  $|a_1x_1 + a_2x_2 + \dots + a_nx_n| \leq (k - 1)\sqrt{n/(k^n - 1)}$ .



**B1.** Prove that there is no function  $f$  from the set of non-negative integers into itself such that  $f(f(n)) = n + 1987$  for all  $n$ .



**B2.** Let  $n$  be an integer greater than or equal to 3. Prove that there is a set of  $n$  points in the plane such that the distance between any two points is irrational and each set of 3 points determines a non-degenerate triangle with rational area.



**B3.** Let  $n$  be an integer greater than or equal to 2. Prove that if  $k^2 + k + n$  is prime for all integers  $k$  such that  $0 \leq k \leq \sqrt{(n/3)}$ , then  $k^2 + k + n$  is prime for all integers  $k$  such that  $0 \leq k \leq n-2$ .





## IMO 1987



### Problem A2

In an acute-angled triangle  $ABC$  the interior bisector of angle  $A$  meets  $BC$  at  $L$  and meets the circumcircle of  $ABC$  again at  $N$ . From  $L$  perpendiculars are drawn to  $AB$  and  $AC$ , with feet  $K$  and  $M$  respectively. Prove that the quadrilateral  $AKNM$  and the triangle  $ABC$  have equal areas.

*Solution by Gerhard Wöginger*

$AKL$  and  $AML$  are congruent, so  $KM$  is perpendicular to  $AN$  and  $\text{area } AKNM = KM \cdot AN/2$ .

$AKLM$  is cyclic (2 opposite right angles), so  $\angle AKM = \angle ALM$  and hence  $KM/\sin BAC = AM/\sin AKM$  (sine rule)  $= AM/\sin ALM = AL$ .

$ABL$  and  $ANC$  are similar, so  $AB \cdot AC = AN \cdot AL$ . Hence  $\text{area } ABC = 1/2 AB \cdot AC \sin BAC = 1/2 AN \cdot AL \sin BAC = 1/2 AN \cdot KM = \text{area } AKNM$ .

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### Problem A3

Let  $x_1, x_2, \dots, x_n$  be real numbers satisfying  $x_1^2 + x_2^2 + \dots + x_n^2 = 1$ . Prove that for every integer  $k \geq 2$  there are integers  $a_1, a_2, \dots, a_n$ , not all zero, such that  $|a_i| \leq k - 1$  for all  $i$ , and  $|a_1 x_1 + a_2 x_2 + \dots + a_n x_n| \leq (k - 1) \sqrt{n}/(k^n - 1)$ .

### Solution

This is an application of the pigeon-hole principle.

Assume first that all  $x_i$  are non-negative. Observe that the sum of the  $x_i$  is at most  $\sqrt{n}$ . [This is a well-known variant,  $(\sum_{1 \leq i \leq n} x_i)^2 \leq n \sum_{1 \leq i \leq n} x_i^2$ , of the AM-GM result. See, for example, Arthur Engel, Problem Solving Strategies, Springer 1998, p163, ISBN 0387982191].

Consider the  $k^n$  possible values of  $\sum_{1 \leq i \leq n} b_i x_i$ , where each  $b_i$  is an integer in the range  $[0, k-1]$ . Each value must lie in the interval  $[0, k-1 \sqrt{n}]$ . Divide this into  $k^n - 1$  equal subintervals. Two values must lie in the same subinterval. Take their difference. Its coefficients are the required  $a_i$ . Finally, if any  $x_i$  are negative, solve for the absolute values and then flip signs in the  $a_i$ .

**Problem B1**

**Prove that there is no function  $f$  from the set of non-negative integers into itself such that  $f(f(n)) = n + 1987$  for all  $n$ .**

**Solution**

We prove that if  $f(f(n)) = n + k$  for all  $n$ , where  $k$  is a fixed positive integer, then  $k$  must be even. If  $k = 2h$ , then we may take  $f(n) = n + h$ .

Suppose  $f(m) = n$  with  $m = n \pmod{k}$ . Then by an easy induction on  $r$  we find  $f(m + kr) = n + kr$ ,  $f(n + kr) = m + k(r+1)$ . We show this leads to a contradiction. Suppose  $m < n$ , so  $n = m + ks$  for some  $s > 0$ . Then  $f(n) = f(m + ks) = n + ks$ . But  $f(n) = m + k$ , so  $m = n + k(s - 1) \geq n$ . Contradiction. So we must have  $m \geq n$ , so  $m = n + ks$  for some  $s \geq 0$ . But now  $f(m + k) = f(n + k(s+1)) = m + k(s + 2)$ . But  $f(m + k) = n + k$ , so  $n = m + k(s + 1) > n$ . Contradiction.

So if  $f(m) = n$ , then  $m$  and  $n$  have different residues mod  $k$ . Suppose they have  $r_1$  and  $r_2$  respectively. Then the same induction shows that all sufficiently large  $s = r_1 \pmod{k}$  have  $f(s) = r_2 \pmod{k}$ , and that all sufficiently large  $s = r_2 \pmod{k}$  have  $f(s) = r_1 \pmod{k}$ . Hence if  $m$  has a different residue  $r \pmod{k}$ , then  $f(m)$  cannot have residue  $r_1$  or  $r_2$ . For if  $f(m)$  had residue  $r_1$ , then the same argument would show that all sufficiently large numbers with residue  $r_1$  had  $f(m) = r \pmod{k}$ . Thus the residues form pairs, so that if a number is congruent to a particular residue, then  $f$  of the number is congruent to the pair of the residue. But this is impossible for  $k$  odd.

*A better solution by Sawa Pavlov is as follows*

Let  $N$  be the set of non-negative integers. Put  $A = N - f(N)$  (the set of all  $n$  such that we cannot find  $m$  with  $f(m) = n$ ). Put  $B = f(A)$ .

Note that  $f$  is injective because if  $f(n) = f(m)$ , then  $f(f(n)) = f(f(m))$  so  $m = n$ . We claim that  $B = f(N) - f(f(N))$ . Obviously  $B$  is a subset of  $f(N)$  and if  $k$  belongs to  $B$ , then it does not belong to  $f(f(N))$  since  $f$  is injective. Similarly, a member of  $f(f(N))$  cannot belong to  $B$ .

Clearly  $A$  and  $B$  are disjoint. They have union  $N - f(f(N))$  which is  $\{0, 1, 2, \dots, 1986\}$ . But since  $f$  is injective they have the same number of elements, which is impossible since  $\{0, 1, \dots, 1986\}$  has an odd number of elements.



### Problem B2

Let  $n$  be an integer greater than or equal to 3. Prove that there is a set of  $n$  points in the plane such that the distance between any two points is irrational and each set of 3 points determines a non-degenerate triangle with rational area.

### Solution

Let  $x_n$  be the point with coordinates  $(n, n^2)$  for  $n = 1, 2, 3, \dots$ . We show that the distance between any two points is irrational and that the triangle determined by any 3 points has non-zero rational area.

Take  $n > m$ .  $|x_n - x_m|$  is the hypotenuse of a triangle with sides  $n - m$  and  $n^2 - m^2 = (n - m)(n + m)$ . So  $|x_n - x_m| = (n - m)\sqrt{1 + (n+m)^2}$ . Now  $(n + m)^2 < (n + m)^2 + 1 < (n + m + 1)^2 = (n + m)^2 + 1 + 2(n + m)$ , so  $(n + m)^2 + 1$  is not a perfect square. Hence its square root is irrational. [For this we may use the classical argument. Let  $N'$  be a non-square and suppose  $\sqrt{N'}$  is rational. Since  $N'$  is a non-square we must be able to find a prime  $p$  such that  $p^{2a+1}$  divides  $N'$  but  $p^{2a+2}$  does not divide  $N'$  for some  $a \geq 0$ . Define  $N = N'/p^{2a}$ . Then  $\sqrt{N} = (\sqrt{N'})/p^a$ , which is also rational. So we have a prime  $p$  such that  $p$  divides  $N$ , but  $p^2$  does not divide  $N$ . Take  $\sqrt{N} = r/s$  with  $r$  and  $s$  relatively prime. So  $s^2N = r^2$ . Now  $p$  must divide  $r$ , hence  $p^2$  divides  $r^2$  and so  $p$  divides  $s^2$ . Hence  $p$  divides  $s$ . So  $r$  and  $s$  have a common factor. Contradiction. Hence non-squares have irrational square roots.]

Now take  $a < b < c$ . Let  $B$  be the point  $(b, a^2)$ ,  $C$  the point  $(c, a^2)$ , and  $D$  the point  $(c, b^2)$ . Area  $x_a x_b x_c = \text{area } x_a x_c C - \text{area } x_a x_b B - \text{area } x_b x_c D - \text{area } x_b D C B = (c - a)(c^2 - a^2)/2 - (b - a)(b^2 - a^2)/2 - (c - b)(c^2 - b^2)/2 - (c - b)(b^2 - a^2)$  which is rational.

**Problem B3**

Let  $n$  be an integer greater than or equal to 2. Prove that if  $k^2 + k + n$  is prime for all integers  $k$  such that  $0 \leq k \leq \sqrt{(n/3)}$ , then  $k^2 + k + n$  is prime for all integers  $k$  such that  $0 \leq k \leq n - 2$ .

**Solution**

First observe that if  $m$  is relatively prime to  $b + 1, b + 2, \dots, 2b - 1, 2b$ , then it is not divisible by any number less than  $2b$ . For if  $c \leq b$ , then take the largest  $j \geq 0$  such that  $2^j c \leq b$ . Then  $2^{j+1}c$  lies in the range  $b + 1, \dots, 2b$ , so it is relatively prime to  $m$ . Hence  $c$  is also. If we also have that  $(2b + 1)^2 > m$ , then we can conclude that  $m$  must be prime, since if it were composite it would have a factor  $\leq \sqrt{m}$ .

Let  $n = 3r^2 + h$ , where  $0 \leq h < 6r + 3$ , so that  $r$  is the greatest integer less than or equal to  $\sqrt{(n/3)}$ . We also take  $r \geq 1$ . That excludes the value  $n = 2$ , but for  $n = 2$ , the result is vacuous, so nothing is lost.

Assume that  $n + k(k+1)$  is prime for  $k = 0, 1, \dots, r$ . We show by induction that  $N = n + (r + s)(r + s + 1)$  is prime for  $s = 1, 2, \dots, n - r - 2$ . By the observation above, it is sufficient to show that  $(2r + 2s + 1)^2 > N$ , and that  $N$  is relatively prime to all of  $r + s + 1, r + s + 2, \dots, 2r + 2s$ . We have  $(2r + 2s + 1)^2 = 4r^2 + 8rs + 4s^2 + 4r + 4s + 1$ . Since  $r, s \geq 1$ , we have  $4s + 1 > s + 2$ ,  $4s^2 > s^2$ , and  $6rs > 3r$ . Hence  $(2r + 2s + 1)^2 > 4r^2 + 2rs + s^2 + 7r + s + 2 = 3r^2 + 6r + 2 + (r + s)(r + s + 1) > N$ .

Now if  $N$  has a factor which divides  $2r - i$  with  $i$  in the range  $-2s$  to  $r - s - 1$ , then so does  $N - (i + 2s + 1)(2r - i) = n + (r - i - s - 1)(r - i - s)$  which has the form  $n + s'(s'+1)$  with  $s'$  in the range  $0$  to  $r + s - 1$ . But  $n + s'(s' + 1)$  is prime by induction (or absolutely for  $s = 1$ ), so the only way it can have a factor in common with  $2r - i$  is if it divides  $2r - i$ . But  $2r - i \leq 2r + 2s \leq 2n - 4 < 2n$  and  $n + s'(s' + 1) \geq n$ , so if  $n + s'(s' + 1)$  has a factor in common with  $2r - i$ , then it equals  $2r - i = s + r + 1 + s'$ . Hence  $s'^2 = s - (n - r - 1) < 0$ , which is not possible. So we can conclude that  $N$  is relatively prime to all of  $r + s + 1, \dots, 2r + 2s$  and hence prime.



**A1.** Let  $d$  be any positive integer not equal to 2, 5 or 13. Show that one can find distinct  $a, b$  in the set  $\{2, 5, 13, d\}$  such that  $ab - 1$  is not a perfect square.



**A2.** Given a point  $P_0$  in the plane of the triangle  $A_1A_2A_3$ . Define  $A_s = A_{s-3}$  for all  $s \geq 4$ . Construct a set of points  $P_1, P_2, P_3, \dots$  such that  $P_{k+1}$  is the image of  $P_k$  under a rotation center  $A_{k+1}$  through an angle  $120^\circ$  clockwise for  $k = 0, 1, 2, \dots$ . Prove that if  $P_{1986} = P_0$ , then the triangle  $A_1A_2A_3$  is equilateral.



**A3.** To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers  $x, y, z$  respectively, and  $y < 0$ , then the following operation is allowed:  $x, y, z$  are replaced by  $x + y, -y, z + y$  respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.



**B1.** Let  $A, B$  be adjacent vertices of a regular  $n$ -gon ( $n \geq 5$ ) with center  $O$ . A triangle  $XYZ$ , which is congruent to and initially coincides with  $OAB$ , moves in the plane in such a way that  $Y$  and  $Z$  each trace out the whole boundary of the polygon, with  $X$  remaining inside the polygon. Find the locus of  $X$ .



**B2.** Find all functions  $f$  defined on the non-negative reals and taking non-negative real values such that:  $f(2) = 0$ ,  $f(x) \neq 0$  for  $0 \leq x < 2$ , and  $f(xf(y)) f(y) = f(x + y)$  for all  $x, y$ .



**B3.** Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line  $L$  parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on  $L$  is not greater than 1?

## IMO 1986



### Problem A1

Let  $d$  be any positive integer not equal to 2, 5 or 13. Show that one can find distinct  $a, b$  in the set  $\{2, 5, 13, d\}$  such that  $ab - 1$  is not a perfect square.

#### Solution

Consider residues mod 16. A perfect square must be 0, 1, 4 or 9 (mod 16).  $d$  must be 1, 5, 9, or 13 for  $2d - 1$  to have one of these values. However, if  $d$  is 1 or 13, then  $13d - 1$  is not one of these values. If  $d$  is 5 or 9, then  $5d - 1$  is not one of these values. So we cannot have all three of  $2d - 1$ ,  $5d - 1$ ,  $13d - 1$  perfect squares.

*Alternative solution from Marco Dalai*

Suppose  $2d-1$ ,  $5d-1$ ,  $13d-1$  are all squares. Squares mod 4 must be 0 or 1, considering  $2d-1$ , so  $d$  must be odd. Put  $d = 2k+1$ . Then  $10k+4 = b^2$ . So  $b$  must be even, so  $k$  must be even. Put  $k = 2h$ , then  $5k+1$  is a square. Similarly,  $52h+12$  is a square, so  $13h+3$  is a square. Hence  $(13h+3)-(5h+1) = 8h+2$  is a difference of two squares, which is impossible (a difference of two squares must be 0, 1, or 3 mod 4).

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### Problem A2

Given a point  $P_0$  in the plane of the triangle  $A_1A_2A_3$ . Define  $A_s = A_{s-3}$  for all  $s \geq 4$ . Construct a set of points  $P_1, P_2, P_3, \dots$  such that  $P_{k+1}$  is the image of  $P_k$  under a rotation center  $A_{k+1}$  through  $120^\circ$  clockwise for  $k = 0, 1, 2, \dots$ . Prove that if  $P_{1986} = P_0$ , then the triangle  $A_1A_2A_3$  is equilateral.

#### Solution

The product of three successive rotations about the three vertices of a triangle must be a translation (see below). But that means that  $P_{1986}$  (which is the result of 662 such operations, since  $1986 = 3 \times 662$ ) can only be  $P_0$  if it is the identity, for a translation by a non-zero amount would keep moving the point further away. It is now easy to show that it can only be the identity if the triangle is equilateral. Take a circle center  $A_1$ , radius  $A_1A_2$  and take  $P$  on the circle so that a  $120^\circ$  clockwise rotation about  $A_1$  brings  $P$  to  $A_2$ . Take a circle center  $A_3$ , radius  $A_3A_2$  and take  $Q$  on the circle so that a  $120^\circ$  clockwise rotation about  $A_3$  takes  $A_2$  to  $Q$ . Then successive  $120^\circ$  clockwise rotations about  $A_1, A_2, A_3$  take  $P$  to  $Q$ . So if these three are equivalent to the identity we must have  $P = Q$ . Hence  $\angle A_1A_2A_3 = \angle A_1A_2P + \angle A_3A_2P = 30^\circ + 30^\circ = 60^\circ$ . Also  $A_2P = 2A_1A_2 \cos 30^\circ$  and  $= 2A_2A_3 \cos 30^\circ$ . Hence  $A_1A_2 = A_2A_3$ . So  $A_1A_2A_3$  is equilateral. Note in passing that it is not sufficient for the triangle to be equilateral. We also have to take the rotations in the right order. If we move around the vertices the opposite way, then we get a net translation.

It remains to show that the three rotations give a translation. Define rectangular coordinates  $(x, y)$  by taking  $A_1$  to be the origin and  $A_2$  to be  $(a, b)$ . Let  $A_3$  be  $(c, d)$ . A clockwise rotation through 120 degrees about the origin takes  $(x, y)$  to  $(-x/2 + y\sqrt{3}/2, -x\sqrt{3}/2 - y/2)$ . A clockwise rotation through 120 degrees about some other point  $(e, f)$  is obtained by subtracting  $(e, f)$  to get  $(x - e, y - f)$ , the coordinates relative to  $(e, f)$ , then rotating, then adding  $(e, f)$  to get the coordinates relative to  $(0, 0)$ . Thus after the three rotations we will end up with a linear combination of  $x$ 's,  $y$ 's,  $a$ 's,  $b$ 's,  $c$ 's and  $d$ 's for each coordinate. But the linear combination of  $x$ 's and  $y$ 's must be just  $x$  for the  $x$ -coordinate and  $y$  for the  $y$ -coordinate, since three successive 120 degree rotations about the same point is the identity. Hence we end up with simply  $(x + \text{constant}, y + \text{constant})$ , in other words, a translation.

[Of course, there is nothing to stop you actually carrying out the computation. It makes things slightly easier to take the triangle to be  $(0, 0), (1, 0), (a, b)$ . The net result turns out to be  $(x, y)$  goes to  $(x + 3a/2 - b\sqrt{3}/2, y - \sqrt{3} + a\sqrt{3}/2 + 3b/2)$ . For this to be the identity requires  $a = 1/2, b = \sqrt{3}/2$ . So the third vertex must make the triangle equilateral (and it must be on the correct side of the line joining the other two). This approach avoids the need for the argument in the first paragraph above, but is rather harder work.]

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### Problem A3

**To each vertex of a regular pentagon an integer is assigned, so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers  $x, y, z$  respectively, and  $y < 0$ , then the following operation is allowed:  $x, y, z$  are replaced by  $x + y, -y, z + y$  respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.**

### Solution

Let  $S$  be the sum of the absolute value of each set of adjacent vertices, so if the integers are  $a, b, c, d, e$ , then  $S = |a| + |b| + |c| + |d| + |e| + |a + b| + |b + c| + |c + d| + |d + e| + |e + a| + |a + b + c| + |b + c + d| + |c + d + e| + |d + e + a| + |e + a + b| + |a + b + c + d| + |b + c + d + e| + |c + d + e + a| + |d + e + a + b| + |e + a + b + c| + |a + b + c + d + e|$ . Then the operation reduces  $S$ , but  $S$  is a greater than zero, so the process must terminate in a finite number of steps. So see that  $S$  is reduced, we can simply write out all the terms. Suppose the integers are  $a, b, c, d, e$  before the operation, and  $a+b, -b, b+c, d, e$  after it. We find that we mostly get the same terms before and after (although not in the same order), so that the sum  $S'$  after the operation is  $S - |a + c + d + e| + |a + 2b + c + d + e|$ . Certainly  $a + c + d + e > a + 2b + c + d + e$  since  $b$  is negative, and  $a + c + d + e > -(a + 2b + c + d + e)$  because  $a + b + c + d + e > 0$ .

*S is not the only expression we can use. If we take  $T = (a - c)^2 + (b - d)^2 + (c - e)^2 + (d - a)^2 + (e - b)^2$ , then after replacing  $a, b, c$  by  $a+b, -b, b+c$ , we get  $T' = T + 2b(a + b + c + d + e) < T$ . Thanks to Demetres Chrisofides for T*



**Problem B1**

Let  $A, B$  be adjacent vertices of a regular  $n$ -gon ( $n \geq 5$ ) with center  $O$ . A triangle  $XYZ$ , which is congruent to and initially coincides with  $OAB$ , moves in the plane in such a way that  $Y$  and  $Z$  each trace out the whole boundary of the polygon, with  $X$  remaining inside the polygon. Find the locus of  $X$ .

**Solution**

Take  $AB = 2$  and let  $M$  be the midpoint of  $AB$ . Take coordinates with origin at  $A$ ,  $x$ -axis as  $AB$  and  $y$ -axis directed inside the  $n$ -gon. Let  $Z$  move along  $AB$  from  $B$  towards  $A$ . Let  $\angle YZA$  be  $t$ . Let the coordinates of  $X$  be  $(x, y)$ .  $\angle YZX = \pi/2 - \pi/n$ , so  $XZ = 1/\sin \pi/n$  and  $y = XZ \sin(t + \pi/2 - \pi/n) = \sin t + \cot \pi/n \cos t$ .

$BY \sin 2\pi/n = YZ \sin t = 2 \sin t$ .  $MX = \cot \pi/n$ . So  $x = MY \cos t - BY \cos 2\pi/n + MX \sin t = \cos t + (\cot \pi/n - 2 \cot 2\pi/n) \sin t = \cos t + \tan \pi/n \sin t = y \tan \pi/n$ . Thus the locus of  $X$  is a star formed of  $n$  lines segments emanating from  $O$ .  $X$  moves out from  $O$  to the tip of a line segment and then back to  $O$ , then out along the next segment and so on.  $x^2 + y^2 = (1/\sin^2 \pi/n + 1/\cos^2 \pi/n) \cos^2(t + \pi/n)$ . Thus the length of each segment is  $(1 - \cos \pi/n)/(\sin \pi/n \cos \pi/n)$ .

IMO 1986



**Problem B2**

Find all functions  $f$  defined on the non-negative reals and taking non-negative real values such that:  $f(2) = 0$ ,  $f(x) \neq 0$  for  $0 \leq x < 2$ , and  $f(xf(y)) f(y) = f(x + y)$  for all  $x, y$ .

**Solution**

$f(x+2) = f(xf(2)) f(2) = 0$ . So  $f(x) = 0$  for all  $x \geq 2$ .

$f(y) f((2-y)f(y)) = f(2) = 0$ . So if  $y < 2$ , then  $f((2-y) f(y)) = 0$  and hence  $(2 - y) f(y) \geq 2$ , or  $f(y) \geq 2/(2 - y)$ .

Suppose that for some  $y_0$  we have  $f(y_0) > 2/(2 - y_0)$ , then we can find  $y_1 > y_0$  (and  $y_1 < 2$ ) so that  $f(y_0) = 2/(2 - y_1)$ . Now let  $x_1 = 2 - y_1$ . Then  $f(x_1 f(y_0)) = f(2) = 0$ , so  $f(x_1 + y_0) = 0$ . But  $x_1 + y_0 < 2$ . Contradiction. So we must have  $f(x) = 2/(2 - x)$  for all  $x < 2$ .

We have thus established that if a function  $f$  meets the conditions then it must be defined as above. It remains to prove that with this definition  $f$  does meet the conditions. Clearly  $f(2) = 0$  and  $f(x)$  is non-zero for  $0 \leq x < 2$ .  $f(xf(y)) = f(2x/(2 - y))$ . If  $2x/(2 - y) \geq 2$ , then  $f(xf(y)) = 0$ . But it also follows that  $x + y \geq 2$ , and so  $f(x + y) = 0$  and hence  $f(xf(y)) f(y) = f(x + y)$  as required. If  $2x/(2 - y) < 2$ , then  $f(xf(y)) f(y) = 2/(2 - 2x/(2-y)) 2/(2 - y) = 2/(2 - x - y) = f(x + y)$ . So the unique function satisfying the conditions is:  $f(x) = 0$  for  $x \geq 2$ , and  $2/(2 - x)$  for  $0 \leq x < 2$ .





### Problem B3

**Given a finite set of points in the plane, each with integer coordinates, is it always possible to color the points red or white so that for any straight line  $L$  parallel to one of the coordinate axes the difference (in absolute value) between the numbers of white and red points on  $L$  is not greater than 1?**

### Solution

Answer: yes.

We prove the result by induction on the number  $n$  of points. It is clearly true for  $n = 1$ . Suppose it is true for all numbers less than  $n$ . Pick an arbitrary point  $P$  and color it red. Now take a point in the same row and color it white. Take a point in the same column as the new point and color it red. Continue until either you run out of eligible points or you pick a point in the same column as  $P$ . The process must terminate because there are only finitely many points. Suppose the last point picked is  $Q$ . Let  $S$  be the set of points picked.

If  $Q$  is in the same column as  $P$ , then it is colored white (because the "same row" points are all white, and the "same column" points are all red). Now every row and column contains an equal number of red points of  $S$  and of white points of  $S$ . By induction we can color the points excluding those in  $S$ , then the difference between the numbers of red and white points in each row and column will be unaffected by adding the points in  $S$  and so we will have a coloring for the whole set. This completes the induction for the case where  $Q$  is in the same column as  $P$ .

If it is not, then continue the path backwards from  $P$ . In other words, pick a point in the same column as  $P$  and color it white. Then pick a point in the same row as the new point and color it red and so on. Continue until either you run out of eligible points or you pick a point to pair with  $Q$ . If  $Q$  was picked as being in the same row as its predecessor, this means a point in the same column as  $Q$ ; if  $Q$  was picked as being in the same column as its predecessor, this means a point in the same row as  $Q$ . Again the process must terminate. Suppose the last point picked is  $R$ . Let  $S$  be the set of all points picked.

If  $R$  pairs with  $Q$ , then we can complete the coloring by induction as before. Suppose  $S$  does not pair with  $Q$ . Then there is a line (meaning a row or column) containing  $Q$  and no uncolored points. There is also a line containing  $R$  and no uncolored points. These two lines have an excess of one red or one white. All other lines contain equal number of red and white points of  $S$ . Now color the points outside  $S$  by induction. This gives a coloring for the whole set, because no line with a color excess in  $S$  has any points outside  $S$ . So we have completed the induction.



**A1.** A circle has center on the side AB of the cyclic quadrilateral ABCD. The other three sides are tangent to the circle. Prove that  $AD + BC = AB$ .



**A2.** Let  $n$  and  $k$  be relatively prime positive integers with  $k < n$ . Each number in the set  $M = \{1, 2, 3, \dots, n-1\}$  is colored either blue or white. For each  $i$  in  $M$ , both  $i$  and  $n-i$  have the same color. For each  $i$  in  $M$  not equal to  $k$ , both  $i$  and  $|i-k|$  have the same color. Prove that all numbers in  $M$  must have the same color.



**A3.** For any polynomial  $P(x) = a_0 + a_1x + \dots + a_kx^k$  with integer coefficients, the number of odd coefficients is denoted by  $o(P)$ . For  $i = 0, 1, 2, \dots$  let  $Q_i(x) = (1+x)^i$ . Prove that if  $i_1, i_2, \dots, i_n$  are integers satisfying  $0 \leq i_1 < i_2 < \dots < i_n$ , then:



$$o(Q_{i_1} + Q_{i_2} + \dots + Q_{i_n}) \geq o(Q_{i_1}).$$

**B1.** Given a set  $M$  of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that  $M$  contains a subset of 4 elements whose product is the 4th power of an integer.



**B2.** A circle center  $O$  passes through the vertices  $A$  and  $C$  of the triangle  $ABC$  and intersects the segments  $AB$  and  $BC$  again at distinct points  $K$  and  $N$  respectively. The circumcircles of  $ABC$  and  $KBN$  intersect at exactly two distinct points  $B$  and  $M$ . Prove that  $\angle OMB$  is a right angle.



**B3.** For every real number  $x_1$ , construct the sequence  $x_1, x_2, \dots$  by setting:

$$x_{n+1} = x_n(x_n + 1/n).$$



Prove that there exists exactly one value of  $x_1$  which gives  $0 < x_n < x_{n+1} < 1$  for all  $n$ .

## IMO 1985



### Problem A1

A circle has center on the side  $AB$  of the cyclic quadrilateral  $ABCD$ . The other three sides are tangent to the circle. Prove that  $AD + BC = AB$ .

### Solution

Let the circle touch  $AD$ ,  $CD$ ,  $BC$  at  $L$ ,  $M$ ,  $N$  respectively. Take  $X$  on the line  $AD$  on the same side of  $A$  as  $D$ , so that  $AX = AO$ , where  $O$  is the center of the circle. Now the triangles  $OLX$  and  $OMC$  are congruent:  $OL = OM = \text{radius of circle}$ ,  $\angle OLX = \angle OMC = 90^\circ$ , and  $\angle OXL = 90^\circ - A/2 = (180^\circ - A)/2 = C/2$  (since  $ABCD$  is cyclic)  $= \angle OCM$ . Hence  $LX = MC$ . So  $OA = AL + MC$ . Similarly,  $OB = BN + MD$ . But  $MC = CN$  and  $MD = DL$  (tangents have equal length), so  $AB = OA + OB = AL + LD + CN + NB = AD + BC$ .

## IMO 1985



### Problem A2

Let  $n$  and  $k$  be relatively prime positive integers with  $k < n$ . Each number in the set  $M = \{1, 2, 3, \dots, n-1\}$  is colored either blue or white. For each  $i$  in  $M$ , both  $i$  and  $n-i$  have the same color. For each  $i$  in  $M$  not equal to  $k$ , both  $i$  and  $|i-k|$  have the same color. Prove that all numbers in  $M$  must have the same color.

### Solution

$n$  and  $k$  are relatively prime, so  $0, k, 2k, \dots, (n-1)k$  form a complete set of residues mod  $n$ . So  $k, 2k, \dots, (n-1)k$  are congruent to the numbers  $1, 2, \dots, n-1$  in some order. Suppose  $ik$  is congruent to  $r$  and  $(i+1)k$  is congruent to  $s$ . Then either  $s = r + k$ , or  $s = r + k - n$ . If  $s = r + k$ , then we have immediately that  $r = s - k$  and  $s$  have the same color. If  $s = r + k - n$ , then  $r = n - (k - s)$ , so  $r$  has the same color as  $k - s$ , and  $k - s$  has the same color as  $s$ . So in any case  $r$  and  $s$  have the same color. By giving  $i$  values from  $1$  to  $n-2$  this establishes that all the numbers have the same color.



### Problem A3

For any polynomial  $P(x) = a_0 + a_1x + \dots + a_kx^k$  with integer coefficients, the number of odd coefficients is denoted by  $o(P)$ . For  $i = 0, 1, 2, \dots$  let  $Q_i(x) = (1 + x)^i$ . Prove that if  $i_1, i_2, \dots, i_n$  are integers satisfying  $0 \leq i_1 < i_2 < \dots < i_n$ , then:

$$o(Q_{i_1} + Q_{i_2} + \dots + Q_{i_n}) \geq o(Q_{i_1}).$$

### Solution

If  $i$  is a power of 2, then all coefficients of  $Q_i$  are even except the first and last. [There are various ways to prove this. Let  $iCr$  denote the  $r$ th coefficient, so  $iCr = i!/(r!(i-r)!)$ . Suppose  $0 < r < i$ . Then  $iCr = i-1Cr-1 \cdot i/r$ , but  $i-1Cr-1$  is an integer and  $i$  is divisible by a higher power of 2 than  $r$ , hence  $iCr$  is even.]

Let  $Q = Q_{i_1} + \dots + Q_{i_n}$ . We use induction on  $i_n$ . If  $i_n = 1$ , then we must have  $n = 2$ ,  $i_1 = 0$ , and  $i_2 = 1$ , so  $Q = 2 + x$ , which has the same number of odd coefficients as  $Q_{i_1} = 1$ . So suppose it is true for smaller values of  $i_n$ . Take  $m$  a power of 2 so that  $m \leq i_n < 2m$ . We consider two cases  $i_1 \geq m$  and  $i_1 < m$ .

Consider first  $i_1 \geq m$ . Then  $Q_{i_1} = (1 + x)^m A$ ,  $Q = (1 + x)^m B$ , where  $A$  and  $B$  have degree less than  $m$ . Moreover,  $A$  and  $B$  are of the same form as  $Q_{i_1}$  and  $Q$ , (all the  $i_j$ s are reduced by  $m$ , so we have  $o(A) \leq o(B)$  by induction. Also  $o(Q_{i_1}) = o((1 + x)^m A) = o(A + x^m A) = 2o(A) \leq 2o(B) = o(B + x^m B) = o((1 + x)^m B) = o(Q)$ , which establishes the result for  $i_n$ .

It remains to consider the case  $i_1 < m$ . Take  $r$  so that  $i_r < m$ ,  $i_{r+1} > m$ . Set  $A = Q_{i_1} + \dots + Q_{i_r}$ ,  $(1 + x)^m B = Q_{i_{r+1}} + \dots + Q_{i_n}$ , so that  $A$  and  $B$  have degree  $< m$ . Then  $o(Q) = o(A + (1 + x)^m B) = o(A + B + x^m B) = o(A + B) + o(B)$ . Now  $o(A - B) + o(B) \geq o(A - B + B) = o(A)$ , because a coefficient of  $A$  is only odd if just one of the corresponding coefficients of  $A - B$  and  $B$  is odd. But  $o(A - B) = o(A + B)$ , because corresponding coefficients of  $A - B$  and  $A + B$  are either equal or of the same parity. Hence  $o(A + B) + o(B) \geq o(A)$ . But  $o(A) \geq o(Q_{i_1})$  by induction. So we have established the result for  $i_n$ .

**Problem B1**

**Given a set  $M$  of 1985 distinct positive integers, none of which has a prime divisor greater than 23, prove that  $M$  contains a subset of 4 elements whose product is the 4th power of an integer.**

**Solution**

Suppose we have a set of at least  $3 \cdot 2^n + 1$  numbers whose prime divisors are all taken from a set of  $n$ . So each number can be written as  $p_1^{r_1} \dots p_n^{r_n}$  for some non-negative integers  $r_i$ , where  $p_i$  is the set of prime factors common to all the numbers. We classify each  $r_i$  as even or odd. That gives  $2^n$  possibilities. But there are more than  $2^n + 1$  numbers, so two numbers have the same classification and hence their product is a square. Remove those two and look at the remaining numbers. There are still more than  $2^n + 1$ , so we can find another pair. We may repeat to find  $2^n + 1$  pairs with a square product. [After removing  $2^n$  pairs, there are still  $2^n + 1$  numbers left, which is just enough to find the final pair.] But we may now classify these pairs according to whether each exponent in the square root of their product is odd or even. We must find two pairs with the same classification. The product of these four numbers is now a fourth power.

Applying this to the case given, there are 9 primes less than or equal to 23 (2, 3, 5, 7, 11, 13, 17, 19, 23), so we need at least  $3 \cdot 512 + 1 = 1537$  numbers for the argument to work (and we have 1985).

The key is to find the 4th power in two stages, by first finding lots of squares. If we try to go directly to a 4th power, this type of argument does not work (we certainly need more than 5 numbers to be sure of finding four which sum to 0 mod 4, and  $5^9$  is far too big).

**IMO 1985****Problem B2**

**A circle center  $O$  passes through the vertices  $A$  and  $C$  of the triangle  $ABC$  and intersects the segments  $AB$  and  $BC$  again at distinct points  $K$  and  $N$  respectively. The circumcircles of  $ABC$  and  $KBN$  intersect at exactly two distinct points  $B$  and  $M$ . Prove that angle  $OMB$  is a right angle.**

**Solution**

The three radical axes of the three circles taken in pairs,  $BM$ ,  $NK$  and  $AC$  are concurrent. Let  $X$  be the point of intersection. [They cannot all be parallel or  $B$  and  $M$  would coincide.] The first step is to show that  $XMNC$  is cyclic. The argument depends slightly on how the points are arranged. We may have:  $\angle XMN = 180^\circ - \angle BMN = \angle BKN = 180^\circ - \angle AKN = \angle ACN = 180^\circ -$

$\angle XCN$ , or we may have  $\angle XMN = 180^\circ - \angle BMN = 180^\circ - \angle BKN = \angle AKN = 180^\circ - \angle ACN = 180^\circ - \angle XCN$ .

Now  $XM \cdot XB = XK \cdot XN = XO^2 - ON^2$ .  $BM \cdot BX = BN \cdot BC = BO^2 - ON^2$ , so  $XM \cdot XB - BM \cdot BX = XO^2 - BO^2$ . But  $XM \cdot XB - BM \cdot BX = XB(XM - BM) = (XM + BM)(XM - BM) = XM^2 - BM^2$ . So  $XO^2 - BO^2 = XM^2 - BM^2$ . Hence  $OM$  is perpendicular to  $XB$ , or  $\angle OMB = 90^\circ$ .

## IMO 1985



### Problem B3

For every real number  $x_1$ , construct the sequence  $x_1, x_2, \dots$  by setting:

$$x_{n+1} = x_n(x_n + 1/n).$$

Prove that there exists exactly one value of  $x_1$  which gives  $0 < x_n < x_{n+1} < 1$  for all  $n$ .

#### Solution

Define  $S_0(x) = x$ ,  $S_n(x) = S_{n-1}(x) (S_{n-1}(x) + 1/n)$ . The motivation for this is that  $x_n = S_{n-1}(x_1)$ .

$S_n(0) = 0$  and  $S_n(1) > 1$  for all  $n > 1$ . Also  $S_n(x)$  has non-negative coefficients, so it is strictly increasing in the range  $[0, 1]$ . Hence we can find (unique) solutions  $a_n, b_n$  to  $S_n(a_n) = 1 - 1/n$ ,  $S_n(b_n) = 1$ .

$S_{n+1}(a_n) = S_n(a_n) (S_n(a_n) + 1/n) = 1 - 1/n > 1 - 1/(n+1)$ , so  $a_n < a_{n+1}$ . Similarly,  $S_{n+1}(b_n) = S_n(b_n) (S_n(b_n) + 1/n) = 1 + 1/n > 1$ , so  $b_n > b_{n+1}$ . Thus  $a_n$  is an increasing sequence and  $b_n$  is a decreasing sequence with all  $a_n$  less than all  $b_n$ . So we can certainly find at least one point  $x_1$  which is greater than all the  $a_n$  and less than all the  $b_n$ . Hence  $1 - 1/n < S_n(x_1) < 1$  for all  $n$ . But  $S_n(x_1) = x_{n+1}$ . So  $x_{n+1} < 1$  for all  $n$ . Also  $x_n > 1 - 1/n$  implies that  $x_{n+1} = x_n(x_n + 1/n) > x_n$ . Finally, we obviously have  $x_n > 0$ . So the resulting series  $x_n$  satisfies all the required conditions.

It remains to consider uniqueness. Suppose that there is an  $x_1$  satisfying the conditions given. Then we must have  $S_n(x_1)$  lying in the range  $1 - 1/n, 1$  for all  $n$ . [The lower limit follows from  $x_{n+1} = x_n(x_n + 1/n)$ .] Hence we must have  $a_n < x_1 < b_n$  for all  $n$ . We show uniqueness by showing that  $b_n - a_n$  tends to zero as  $n$  tends to infinity. Since all the coefficients of  $S_n(x)$  are non-negative, it is has increasing derivative.  $S_n(0) = 0$  and  $S_n(b_n) = 1$ , so for any  $x$  in the range  $0, b_n$  we have  $S_n(x) \leq x/b_n$ . In particular,  $1 - 1/n < a_n/b_n$ . Hence  $b_n - a_n \leq b_n - b_n(1 - 1/n) = b_n/n < 1/n$ , which tends to zero.

25th IMO 1984



**A1.** Prove that  $0 \leq yz + zx + xy - 2xyz \leq 7/27$ , where  $x, y$  and  $z$  are non-negative real numbers satisfying  $x + y + z = 1$ .



**A2.** Find one pair of positive integers  $a, b$  such that  $ab(a+b)$  is not divisible by 7, but  $(a+b)^7 - a^7 - b^7$  is divisible by  $7^7$ .



**A3.** Given points  $O$  and  $A$  in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point  $X$  in the plane, the circle  $C(X)$  has center  $O$  and radius  $OX + \angle AOX/OX$ , where  $\angle AOX$  is measured in radians in the range  $[0, 2\pi)$ . Prove that we can find a point  $X$ , not on  $OA$ , such that its color appears on the circumference of the circle  $C(X)$ .



**B1.** Let  $ABCD$  be a convex quadrilateral with the line  $CD$  tangent to the circle on diameter  $AB$ . Prove that the line  $AB$  is tangent to the circle on diameter  $CD$  if and only if  $BC$  and  $AD$  are parallel.



**B2.** Let  $d$  be the sum of the lengths of all the diagonals of a plane convex polygon with  $n > 3$  vertices. Let  $p$  be its perimeter. Prove that:



$n - 3 < 2d/p < [n/2] [(n+1)/2] - 2$ , where  $[x]$  denotes the greatest integer not exceeding  $x$ .

**B3.** Let  $a, b, c, d$  be odd integers such that  $0 < a < b < c < d$  and  $ad = bc$ . Prove that if  $a + d = 2^k$  and  $b + c = 2^m$  for some integers  $k$  and  $m$ , then  $a = 1$ .

## IMO 1984



### Problem A1

Prove that  $0 \leq yz + zx + xy - 2xyz \leq 7/27$ , where  $x, y$  and  $z$  are non-negative real numbers satisfying  $x + y + z = 1$ .

### Solution

$(1 - 2x)(1 - 2y)(1 - 2z) = 1 - 2(x + y + z) + 4(yz + zx + xy) - 8xyz = 4(yz + zx + xy) - 8xyz - 1$ .  
Hence  $yz + zx + xy - 2xyz = 1/4 (1 - 2x)(1 - 2y)(1 - 2z) + 1/4$ . By the arithmetic/geometric mean theorem  $(1 - 2x)(1 - 2y)(1 - 2z) \leq ((1 - 2x + 1 - 2y + 1 - 2z)/3)^3 = 1/27$ . So  $yz + zx + xy - 2xyz \leq 1/4 \cdot 28/27 = 7/27$ .

## IMO 1984



### Problem A2

Find one pair of positive integers  $a, b$  such that  $ab(a+b)$  is not divisible by 7, but  $(a+b)^7 - a^7 - b^7$  is divisible by  $7^7$ .

### Solution

We find that  $(a + b)^7 - a^7 - b^7 = 7ab(a + b)(a^2 + ab + b^2)^2$ . So we must find  $a, b$  such that  $a^2 + ab + b^2$  is divisible by  $7^3$ .

At this point I found  $a = 18, b = 1$  by trial and error.

A more systematic argument turns on noticing that  $a^2 + ab + b^2 = (a^3 - b^3)/(a - b)$ , so we are looking for  $a, b$  with  $a^3 \equiv b^3 \pmod{7^3}$ . We now remember that  $a^{\varphi(m)} \equiv 1 \pmod{m}$ . But  $\varphi(7^3) = 2 \cdot 3 \cdot 49$ , so  $a^3 \equiv 1 \pmod{343}$  if  $a \equiv n^{98}$ . We find  $2^{98} \equiv 18 \pmod{343}$ , which gives the solution 18, 1.

This approach does not give a flood of solutions.  $n^{98} \equiv 0, 1, 18, \text{ or } 324$ . So the only solutions we get are 1, 18; 18, 324; 1, 324.



**Problem A3**

Given points  $O$  and  $A$  in the plane. Every point in the plane is colored with one of a finite number of colors. Given a point  $X$  in the plane, the circle  $C(X)$  has center  $O$  and radius  $OX + (\angle AOX)/OX$ , where  $\angle AOX$  is measured in radians in the range  $[0, 2\pi)$ . Prove that we can find a point  $X$ , not on  $OA$ , such that its color appears on the circumference of the circle  $C(X)$ .

**Solution**

Suppose the result is false. Let  $C_1$  be any circle center  $O$ . Then the locus of points  $X$  such that  $C(X) = C_1$  is a spiral from  $O$  to the point of intersection of  $OA$  and  $C_1$ . Every point of this spiral must be a different color from all points of the circle  $C_1$ . Hence every circle center  $O$  with radius smaller than  $C_1$  must include a point of different color to those on  $C_1$ . Suppose there are  $n$  colors. Then by taking successively smaller circles  $C_2, C_3, \dots, C_{n+1}$  we reach a contradiction, since each circle includes a point of different color to those on any of the larger circles.

**Problem B1**

Let  $ABCD$  be a convex quadrilateral with the line  $CD$  tangent to the circle on diameter  $AB$ . Prove that the line  $AB$  is tangent to the circle on diameter  $CD$  if and only if  $BC$  and  $AD$  are parallel.

**Solution**

If  $AB$  and  $CD$  are parallel, then  $AB$  is tangent to the circle on diameter  $CD$  if and only if  $AB = CD$  and hence if and only if  $ABCD$  is a parallelogram. So the result is true.

Suppose then that  $AB$  and  $DC$  meet at  $O$ . Let  $M$  be the midpoint of  $AB$  and  $N$  the midpoint of  $CD$ . Let  $S$  be the foot of the perpendicular from  $N$  to  $AB$ , and  $T$  the foot of the perpendicular from  $M$  to  $CD$ . We are given that  $MT = MA$ .  $OMT, ONS$  are similar, so  $OM/MT = ON/NS$  and hence  $OB/OA = (ON - NS)/(ON + NS)$ . So  $AB$  is tangent to the circle on diameter  $CD$  if and only if  $OB/OA = OC/OD$  which is the condition for  $BC$  to be parallel to  $AD$ .



**Problem B2**

Let  $d$  be the sum of the lengths of all the diagonals of a plane convex polygon with  $n > 3$  vertices. Let  $p$  be its perimeter. Prove that:  $n - 3 < 2d/p < [n/2] [(n+1)/2] - 2$ , where  $[x]$  denotes the greatest integer not exceeding  $x$ .

**Solution**

Given any diagonal  $AX$ , let  $B$  be the next vertex counterclockwise from  $A$ , and  $Y$  the next vertex counterclockwise from  $X$ . Then the diagonals  $AX$  and  $BY$  intersect at  $K$ .  $AK + KB > AB$  and  $XK + KY > XY$ , so  $AX + BY > AB + XY$ . Keeping  $A$  fixed and summing over  $X$  gives  $n - 3$  cases. So if we then sum over  $A$  we get every diagonal appearing four times on the lhs and every side appearing  $2(n-3)$  times on the rhs, giving  $4d > 2(n-3)p$ .

Denote the vertices as  $A_0, \dots, A_{n-1}$  and take subscripts mod  $n$ . The ends of a diagonal  $AX$  are connected by  $r$  sides and  $n-r$  sides. The idea of the upper limit is that its length is less than the sum of the shorter number of sides. Evaluating it is slightly awkward.

We consider  $n$  odd and  $n$  even separately. Let  $n = 2m+1$ . For the diagonal  $A_i A_{i+r}$  with  $r \leq m$ , we have  $A_i A_{i+r} \leq A_i A_{i+2} + \dots + A_i A_{i+r}$ . Summing from  $r = 2$  to  $m$  gives for the rhs  $(m-1)A_i A_{i+1} + (m-1)A_{i+1} A_{i+2} + (m-2)A_{i+2} A_{i+3} + (m-3)A_{i+3} A_{i+4} + \dots + 1 \cdot A_{i+m-1} A_{i+m}$ . Now summing over  $i$  gives  $d$  for the lhs and  $p((m-1) + (1 + 2 + \dots + m-1)) = p(m^2 + m - 2)/2$  for the rhs. So we get  $2d/p \leq m^2 + m - 2 = [n/2] [(n+1)/2] - 2$ .

Let  $n = 2m$ . As before we have  $A_i A_{i+r} \leq A_i A_{i+2} + \dots + A_i A_{i+r}$  for  $2 \leq r \leq m-1$ . We may also take  $A_i A_{i+m} \leq p/2$ . Summing as in the even case we get  $2d/p = m^2 - 2 = [n/2] [(n+1)/2] - 2$ .

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**Problem B3**

Let  $a, b, c, d$  be odd integers such that  $0 < a < b < c < d$  and  $ad = bc$ . Prove that if  $a + d = 2^k$  and  $b + c = 2^m$  for some integers  $k$  and  $m$ , then  $a = 1$ .

**Solution**

$a < c$ , so  $a(d - c) < c(d - c)$  and hence  $bc - ac < c(d - c)$ . So  $b - a < d - c$ , or  $a + d > b + c$ , so  $k > m$ .

$bc = ad$ , so  $b(2^m - b) = a(2^k - a)$ . Hence  $b^2 - a^2 = 2^m(b - 2^{k-m}a)$ . But  $b^2 - a^2 = (b + a)(b - a)$ , and  $(b + a)$  and  $(b - a)$  cannot both be divisible by 4 (since  $a$  and  $b$  are odd), so  $2^{m-1}$  must divide  $b + a$  or  $b - a$ . But if it divides  $b - a$ , then  $b - a \geq 2^{m-1}$ , so  $b$  and  $c > 2^{m-1}$  and  $b + c > 2^m$ . Contradiction. Hence  $2^{m-1}$  divides  $b + a$ . If  $b + a \geq 2^m = b + c$ , then  $a \geq c$ . Contradiction. Hence  $b + a = 2^{m-1}$ .

So we have  $b = 2^{m-1} - a$ ,  $c = 2^{m-1} + a$ ,  $d = 2^k - a$ . Now using  $bc = ad$  gives:  $2^k a = 2^{2m-2}$ . But  $a$  is odd, so  $a = 1$ .



**A1.** Find all functions  $f$  defined on the set of positive reals which take positive real values and satisfy:

$$f(x(f(y))) = yf(x) \text{ for all } x, y; \text{ and } f(x) \rightarrow 0 \text{ as } x \rightarrow \infty.$$



**A2.** Let  $A$  be one of the two distinct points of intersection of two unequal coplanar circles  $C_1$  and  $C_2$  with centers  $O_1$  and  $O_2$  respectively. One of the common tangents to the circles touches  $C_1$  at  $P_1$  and  $C_2$  at  $P_2$ , while the other touches  $C_1$  at  $Q_1$  and  $C_2$  at  $Q_2$ . Let  $M_1$  be the midpoint of  $P_1Q_1$  and  $M_2$  the midpoint of  $P_2Q_2$ . Prove that  $\angle O_1AO_2 = \angle M_1AM_2$ .



**A3.** Let  $a$ ,  $b$  and  $c$  be positive integers, no two of which have a common divisor greater than 1. Show that  $2abc - ab - bc - ca$  is the largest integer which cannot be expressed in the form  $xbc + yca + zab$ , where  $x, y, z$  are non-negative integers.



**B1.** Let  $ABC$  be an equilateral triangle and  $E$  the set of all points contained in the three segments  $AB$ ,  $BC$  and  $CA$  (including  $A$ ,  $B$  and  $C$ ). Determine whether, for every partition of  $E$  into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.



**B2.** Is it possible to choose 1983 distinct positive integers, all less than or equal to  $10^5$ , no three of which are consecutive terms of an arithmetic progression?



**B3.** Let  $a$ ,  $b$  and  $c$  be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0.$$

Determine when equality occurs.

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### Problem A1

Find all functions  $f$  defined on the set of positive reals which take positive real values and satisfy:  $f(xf(y)) = yf(x)$  for all  $x, y$ ; and  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ .

### Solution

If  $f(k) = 1$ , then  $f(x) = f(xf(k)) = kf(x)$ , so  $k = 1$ . Let  $y = 1/f(x)$  and set  $k = xf(y)$ , then  $f(k) = f(xf(y)) = yf(x) = 1$ . Hence  $f(1) = 1$  and  $f(1/f(x)) = 1/x$ . Also  $f(f(y)) = f(1f(y)) = y$ . Hence  $f(1/x) = 1/f(x)$ . Finally, let  $z = f(y)$ , so that  $f(z) = y$ . Then  $f(xy) = f(xf(z)) = zf(x) = f(x)f(y)$ .

Now notice that  $f(xf(x)) = xf(x)$ . Let  $k = xf(x)$ . We show that  $k = 1$ .  $f(k^2) = f(k)f(k) = k^2$  and by a simple induction  $f(k^n) = k^n$ , so we cannot have  $k > 1$ , or  $f(x)$  would not tend to 0 as  $x$  tends to infinity. But  $f(1/k) = 1/k$  and the same argument shows that we cannot have  $1/k > 1$ . Hence  $k = 1$ .

So the only such function  $f$  is  $f(x) = 1/x$ .

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### Problem A2

Let  $A$  be one of the two distinct points of intersection of two unequal coplanar circles  $C_1$  and  $C_2$  with centers  $O_1$  and  $O_2$  respectively. One of the common tangents to the circles touches  $C_1$  at  $P_1$  and  $C_2$  at  $P_2$ , while the other touches  $C_1$  at  $Q_1$  and  $C_2$  at  $Q_2$ . Let  $M_1$  be the midpoint of  $P_1Q_1$  and  $M_2$  the midpoint of  $P_2Q_2$ . Prove that  $\angle O_1AO_2 = \angle M_1AM_2$ .

### Solution

Let  $P_1P_2$  and  $O_1O_2$  meet at  $O$ . Let  $OA$  meet  $C_2$  again at  $A_2$ .  $O$  is the center of similitude for  $C_1$  and  $C_2$  so  $\angle M_1AO_1 = \angle M_2A_2O_2$ . Hence if we can show that  $\angle M_2AO_2 = \angle M_2A_2O_2$ , then we are home.

Let  $X$  be the other point of intersection of the two circles. The key is to show that  $A_2, M_2$  and  $X$  are collinear, for then  $\angle M_2AO_2 = \angle M_2XO_2$  (by reflection) and  $O_2A_2X$  is isosceles.

But since  $O$  is the center of similitude,  $M_2A_2$  is parallel to  $M_1A$ , and by reflection  $\angle XM_2O = \angle AM_2O$ , so we need to show that triangle  $AM_1M_2$  is isosceles. Extend  $XA$  to meet  $P_1P_2$  at  $Y$ . Then  $YP_1^2 = YA \cdot YX = YP_2^2$ , so  $YX$  is the perpendicular bisector of  $M_1M_2$ , and hence  $AM_1 = AM_2$  as required.

**Problem A3**

Let  $a$ ,  $b$  and  $c$  be positive integers, no two of which have a common divisor greater than 1. Show that  $2abc - ab - bc - ca$  is the largest integer which cannot be expressed in the form  $xbc + yca + zab$ , where  $x, y, z$  are non-negative integers.

**Solution**

We start with the lemma that  $bc - b - c$  is the largest number which cannot be written as  $mb + nc$  with  $m$  and  $n$  non-negative. [Proof:  $0, c, 2c, \dots, (b-1)c$  is a complete set of residues mod  $b$ . If  $r > (b-1)c - b$ , then  $r = nc \pmod{b}$  for some  $0 \leq n \leq b-1$ . But  $r > nc - b$ , so  $r = nc + mb$  for some  $m \geq 0$ . That proves that every number larger than  $bc - b - c$  can be written as  $mb + nc$  with  $m$  and  $n$  non-negative. Now consider  $bc - b - c$ . It is  $(b-1)c \pmod{b}$ , and not congruent to any  $nc$  with  $0 \leq n < b-1$ . So if  $bc - b - c = mb + nc$ , then  $n \geq b-1$ . Hence  $mb + nc \geq nc \geq (b-1)c > bc - b - c$ . Contradiction.]

$0, bc, 2bc, \dots, (a-1)bc$  is a complete set of residues mod  $a$ . So given  $N > 2abc - ab - bc - ca$  we may take  $xbc = N \pmod{a}$  with  $0 \leq x < a$ . But  $N - xbc > 2abc - ab - bc - ca - (a-1)bc = abc - ab - ca = a(bc - b - c)$ . So  $N - xbc = ka$ , with  $k > bc - b - c$ . Hence we can find non-negative  $y, z$  so that  $k = zb + yc$ . Hence  $N = xbc + yca + zab$ .

Finally, we show that for  $N = 2abc - ab - bc - ca$  we cannot find non-negative  $x, y, z$  so that  $N = xbc + yca + zab$ .  $N = -bc \pmod{a}$ , so we must have  $x = -1 \pmod{a}$  and hence  $x \geq a-1$ . Similarly,  $y \geq b-1$ , and  $z \geq c-1$ . Hence  $xbc + yca + zab \geq 3abc - ab - bc - ca > N$ . Contradiction.

**Problem B1**

Let  $ABC$  be an equilateral triangle and  $E$  the set of all points contained in the three segments  $AB, BC$  and  $CA$  (including  $A, B$  and  $C$ ). Determine whether, for every partition of  $E$  into two disjoint subsets, at least one of the two subsets contains the vertices of a right-angled triangle.

**Solution** Answer: It does.

Suppose otherwise, that  $E$  is the disjoint union of  $e$  and  $e'$  with no right-angled triangles in either set. Take points  $X, Y, Z$  two-thirds of the way along  $BC, CA, AB$  respectively (so that  $BX/BC = 2/3$  etc). Then two of  $X, Y, Z$  must be in the same set. Suppose  $X$  and  $Y$  are in  $e$ . Now  $YX$  is perpendicular to  $BC$ , so all points of  $BC$  apart from  $X$  must be in  $e'$ . Take  $W$  to be the foot of the perpendicular from  $Z$  to  $BC$ . Then  $B$  and  $W$  are in  $e'$ , so  $Z$  must be in  $e$ .  $ZY$  is perpendicular to  $AC$ , so all points of  $AC$  apart from  $Y$  must be in  $e'$ .  $e'$  is now far too big. For example let  $M$  be the midpoint of  $BC$ , then  $AMC$  is in  $e'$  and right-angled.

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### Problem B2

Is it possible to choose 1983 distinct positive integers, all less than or equal to  $10^5$ , no three of which are consecutive terms of an arithmetic progression?

### Solution

We may notice that an efficient way to build up a set with no APs length 3 is as follows. Having found  $2^n$  numbers in  $\{1, 2, \dots, u_n\}$  we add the same pattern starting at  $2u_n$ , thus giving  $2^{n+1}$  numbers in  $\{1, 2, \dots, 3u_n-1\}$ . If  $x$  is in the first part and  $y, z$  in the second part, then  $2y$  is at least  $4u_n$ , whereas  $x + z$  is less than  $4u_n$ , so  $x, y, z$  cannot be an AP length 3. If  $x$  and  $y$  are in the first part, and  $z$  in the second part, then  $2y$  is at most  $2u_n$ , but  $x + z$  is more than  $2u_n$ , so  $x, y, z$  cannot be an AP length 3. To start the process off, we have the 4 numbers 1, 2, 4, 5 in  $\{1, 2, 3, 4, 5\}$ . So  $u_2 = 5$ . This gives  $u_{11} = 88574$ , in other words we can find 2048 numbers in the first 88574 with no AP length 3.

If we are lucky, we may notice that if we reduce each number in the set we have constructed by 1 we get the numbers which have no 2 when written base 3. This provides a neater approach. Take  $x, y, z$  with no 2 when written in base 3. Then  $2y$  has only 0s and 2s when written base 3. But  $x + z$  only has no 1s if  $x = z$ . So  $x, y, z$  cannot form an AP length 3. Also there are  $2^{11} = 2048$  numbers of this type with 11 digits or less and hence  $\leq 11111111111_3 = 88573$ .

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### Problem B3

Let  $a, b$  and  $c$  be the lengths of the sides of a triangle. Prove that

$$a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0.$$

Determine when equality occurs.

### Solution

Put  $a = y + z, b = z + x, c = x + y$ . Then the triangle condition becomes simply  $x, y, z > 0$ . The inequality becomes (after some manipulation):

$$xy^3 + yz^3 + zx^3 \geq xyz(x + y + z).$$

Applying Cauchy's inequality we get  $(xy^3 + yz^3 + zx^3)(z + x + y) \geq xyz(y + z + x)^2$  with equality iff  $xy^3/z = yz^3/x = zx^3/y$ . So the inequality holds with equality iff  $x = y = z$ . Thus the original inequality holds with equality iff the triangle is equilateral.



**A1.** The function  $f(n)$  is defined on the positive integers and takes non-negative integer values.  $f(2) = 0$ ,  $f(3) > 0$ ,  $f(9999) = 3333$  and for all  $m, n$ :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$



Determine  $f(1982)$ .

**A2.** A non-isosceles triangle  $A_1A_2A_3$  has sides  $a_1, a_2, a_3$  with  $a_i$  opposite  $A_i$ .  $M_i$  is the midpoint of side  $a_i$  and  $T_i$  is the point where the incircle touches side  $a_i$ . Denote by  $S_i$  the reflection of  $T_i$  in the interior bisector of angle  $A_i$ . Prove that the lines  $M_1S_1, M_2S_2$  and  $M_3S_3$  are concurrent.



**A3.** Consider infinite sequences  $\{x_n\}$  of positive reals such that  $x_0 = 1$  and  $x_0 \geq x_1 \geq x_2 \geq \dots$ .

(a) Prove that for every such sequence there is an  $n \geq 1$  such that:

$$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n \geq 3.999.$$



(b) Find such a sequence for which:

$$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n < 4 \text{ for all } n.$$

**B1.** Prove that if  $n$  is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$



has a solution in integers  $x, y$ , then it has at least three such solutions. Show that the equation has no solutions in integers for  $n = 2891$ .

**B2.** The diagonals  $AC$  and  $CE$  of the regular hexagon  $ABCDEF$  are divided by inner points  $M$  and  $N$  respectively, so that:

$$AM/AC = CN/CE = r.$$



Determine  $r$  if  $B, M$  and  $N$  are collinear.

**B3.** Let  $S$  be a square with sides length 100. Let  $L$  be a path within  $S$  which does not meet itself and which is composed of line segments  $A_0A_1, A_1A_2, A_2A_3, \dots, A_{n-1}A_n$  with  $A_0 = A_n$ . Suppose that for every point  $P$  on the boundary of  $S$  there is a point of  $L$  at a distance from  $P$  no greater than  $1/2$ . Prove that there are two points  $X$  and  $Y$  of  $L$  such that the distance between  $X$  and  $Y$  is not greater than 1 and the length of the part of  $L$  which lies between  $X$  and  $Y$  is not smaller than 198.



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### Problem A1

The function  $f(n)$  is defined on the positive integers and takes non-negative integer values.  $f(2) = 0$ ,  $f(3) > 0$ ,  $f(9999) = 3333$  and for all  $m, n$ :

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1.$$

Determine  $f(1982)$ .

### Solution

We show that  $f(n) = [n/3]$  for  $n \leq 9999$ , where  $[ ]$  denotes the integral part.

We show first that  $f(3) = 1$ .  $f(1)$  must be 0, otherwise  $f(2) - f(1) - f(1)$  would be negative. Hence  $f(3) = f(2) + f(1) + 0$  or  $1 = 0$  or  $1$ . But we are told  $f(3) > 0$ , so  $f(3) = 1$ . It follows by induction that  $f(3n) \geq n$ . For  $f(3n+3) = f(3) + f(3n) + 0$  or  $1 = f(3n) + 1$  or  $2$ . Moreover if we ever get  $f(3n) > n$ , then the same argument shows that  $f(3m) > m$  for all  $m > n$ . But  $f(3.3333) = 3333$ , so  $f(3n) = n$  for all  $n \leq 3333$ .

Now  $f(3n+1) = f(3n) + f(1) + 0$  or  $1 = n$  or  $n + 1$ . But  $3n+1 = f(9n+3) \geq f(6n+2) + f(3n+1) \geq 3f(3n+1)$ , so  $f(3n+1) < n+1$ . Hence  $f(3n+1) = n$ . Similarly,  $f(3n+2) = n$ . In particular  $f(1982) = 660$ .

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### Problem A2

A non-isosceles triangle  $A_1A_2A_3$  has sides  $a_1, a_2, a_3$  with  $a_i$  opposite  $A_i$ .  $M_i$  is the midpoint of side  $a_i$  and  $T_i$  is the point where the incircle touches side  $a_i$ . Denote by  $S_i$  the reflection of  $T_i$  in the interior bisector of  $\angle A_i$ . Prove that the lines  $M_1S_1, M_2S_2$  and  $M_3S_3$  are concurrent.

### Solution

Let  $B_i$  be the point of intersection of the interior angle bisector of the angle at  $A_i$  with the opposite side. The first step is to figure out which side of  $B_i T_i$  lies. Let  $A_1$  be the largest angle, followed by  $A_2$ . Then  $T_2$  lies between  $A_1$  and  $B_2$ ,  $T_3$  lies between  $A_1$  and  $B_3$ , and  $T_1$  lies between  $A_2$  and  $B_1$ . For  $\angle OB_2A_1 = 180^\circ - A_1 - A_2/2 = A_3 + A_2/2$ . But  $A_3 + A_2/2 < A_1 + A_2/2$  and their sum is  $180^\circ$ , so  $A_3 + A_2/2 < 90^\circ$ . Hence  $T_2$  lies between  $A_1$  and  $B_2$ . Similarly for the others.

Let  $O$  be the center of the incircle. Then  $\angle T_1OS_2 = \angle T_1OT_2 - 2\angle T_2OB_2 = 180^\circ - A_3 - 2(90^\circ - \angle OB_2T_2) = 2(A_3 + A_2/2) - A_3 = A_2 + A_3$ . A similar argument shows  $\angle T_1OS_3 = A_2 + A_3$ . Hence  $S_2S_3$  is parallel to  $A_2A_3$ .



Now  $\angle T_3OS_2 = 360^\circ - \angle T_3OT_1 - \angle T_1OS_2 = 360^\circ - (180^\circ - A_2) - (A_2 + A_3) = 180^\circ - A_3 = A_1 + A_2$ .  $\angle T_3OS_1 = \angle T_3OT_1 + 2 \angle T_1OB_1 = (180^\circ - A_2) + 2(90^\circ - \angle OB_1T_1) = 360^\circ - A_2 - 2(A_3 + A_1/2) = 2(A_1 + A_2 + A_3) - A_2 - 2A_3 - A_1 = A_1 + A_2 = \angle T_3OS_2$ . So  $S_1S_2$  is parallel to  $A_1A_2$ . Similarly we can show that  $S_1S_3$  is parallel to  $A_1A_3$ .

So  $S_1S_2S_3$  is similar to  $A_1A_2A_3$  and turned through  $180^\circ$ . But  $M_1M_2M_3$  is also similar to  $A_1A_2A_3$  and turned through  $180^\circ$ . So  $S_1S_2S_3$  and  $M_1M_2M_3$  are similar and similarly oriented. Hence the lines through corresponding vertices are concurrent.

## IMO 1982



### Problem A3

Consider infinite sequences  $\{x_n\}$  of positive reals such that  $x_0 = 1$  and  $x_0 \geq x_1 \geq x_2 \geq \dots$ .

(a) Prove that for every such sequence there is an  $n \geq 1$  such that:

$$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n \geq 3.999.$$

(b) Find such a sequence for which:

$$x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n < 4 \text{ for all } n.$$

### Solution

(a) It is sufficient to show that the sum of the (infinite) sequence is at least 4. Let  $k$  be the greatest lower bound of the limits of all such sequences. Clearly  $k \geq 1$ . Given any  $\varepsilon > 0$ , we can find a sequence  $\{x_n\}$  with sum less than  $k + \varepsilon$ . But we may write the sum as:

$$x_0^2/x_1 + x_1 \left( (x_1/x_1)^2/(x_2/x_1) + (x_2/x_1)^2/(x_3/x_1) + \dots + (x_n/x_1)^2/(x_{n+1}/x_1) + \dots \right).$$

The term in brackets is another sum of the same type, so it is at least  $k$ . Hence  $k + \varepsilon > 1/x_1 + x_1k$ . This holds for all  $\varepsilon > 0$ , and so  $k \geq 1/x_1 + x_1k$ . But  $1/x_1 + x_1k \geq 2\sqrt{k}$ , so  $k \geq 4$ .

(b) Let  $x_n = 1/2^n$ . Then  $x_0^2/x_1 + x_1^2/x_2 + \dots + x_{n-1}^2/x_n = 2 + 1 + 1/2 + \dots + 1/2^{n-2} = 4 - 1/2^{n-2} < 4$ .

## IMO 1982



### Problem B1

Prove that if  $n$  is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers  $x, y$ , then it has at least three such solutions. Show that the equation has no solutions in integers for  $n = 2891$ .

### Solution

If  $x, y$  is a solution then so is  $y-x, -x$ . Hence also  $-y, x-y$ . If the first two are the same, then  $y = -x$ , and  $x = y-x = -2x$ , so  $x = y = 0$ , which is impossible, since  $n > 0$ . Similarly, if any other pair are the same.

$2891 = 2 \pmod{9}$  and there is no solution to  $x^3 - 3xy^2 + y^3 = 2 \pmod{9}$ . The two cubes are each  $-1, 0$  or  $1$ , and the other term is  $0, 3$  or  $6$ , so the only solution is to have the cubes congruent to  $1$  and  $-1$  and the other term congruent to  $0$ . But the other term cannot be congruent to  $0$ , unless one of  $x, y$  is a multiple of  $3$ , in which case its cube is congruent to  $0$ , not  $1$  or  $-1$ .

## IMO 1982



### Problem B2

The diagonals  $AC$  and  $CE$  of the regular hexagon  $ABCDEF$  are divided by inner points  $M$  and  $N$  respectively, so that:

$$AM/AC = CN/CE = r.$$

Determine  $r$  if  $B, M$  and  $N$  are collinear.

### Solution

For an inelegant solution one can use coordinates. The advantage of this type of approach is that it is quick and guaranteed to work! Take  $A$  as  $(0, \sqrt{3})$ ,  $B$  as  $(1, \sqrt{3})$ ,  $C$  as  $(3/2, \sqrt{3}/2)$ ,  $D$  as  $(1, 0)$ . Take the point  $X$ , coordinates  $(x, 0)$ , on  $ED$ . We find where the line  $BX$  cuts  $AC$  and  $CE$ . The general point on  $BX$  is  $(k + (1-k)x, k\sqrt{3})$ . If this is also the point  $M$  with  $AM/AC = r$  then we have:  $k + (1-k)x = 3r/2$ ,  $k\sqrt{3} = (1-r)\sqrt{3} + r\sqrt{3}/2$ . Hence  $k = 1 - r/2$ ,  $r = 2/(4-x)$ . Similarly, if it is the point  $N$  with  $CN/CE = r$ , then  $k + (1-k)x = 3(1-r)/2$ ,  $k\sqrt{3} = (1-r)\sqrt{3}/2$ . Hence  $k = (1-r)/2$  and  $r = (2-x)/(2+x)$ . Hence for the ratios to be equal we require  $2/(4-x) = (2-x)/(2+x)$ , so  $x^2 - 8x + 4 = 0$ . We also have  $x < 1$ , so  $x = 4 - \sqrt{12}$ . This gives  $r = 1/\sqrt{3}$ .

A more elegant solution uses the ratio theorem for the triangle  $EBC$ . We have  $CM/MX \cdot XB/BE \cdot EN/NC = -1$ . Hence  $(1-r)/(r - 1/2) \cdot (-1/4) \cdot (1-r)/r = -1$ . So  $r = 1/\sqrt{3}$ .



### Problem B3

Let  $S$  be a square with sides length 100. Let  $L$  be a path within  $S$  which does not meet itself and which is composed of line segments  $A_0A_1, A_1A_2, A_2A_3, \dots, A_{n-1}A_n$  with  $A_0 = A_n$ . Suppose that for every point  $P$  on the boundary of  $S$  there is a point of  $L$  at a distance from  $P$  no greater than  $1/2$ . Prove that there are two points  $X$  and  $Y$  of  $L$  such that the distance between  $X$  and  $Y$  is not greater than 1 and the length of the part of  $L$  which lies between  $X$  and  $Y$  is not smaller than 198.

### Solution

Let the square be  $A'B'C'D'$ . The idea is to find points of  $L$  close to a particular point of  $A'D'$  but either side of an excursion to  $B'$ .

We say  $L$  approaches a point  $P'$  on the boundary of the square if there is a point  $P$  on  $L$  with  $PP' \leq 1/2$ . We say  $L$  approaches  $P'$  before  $Q'$  if there is a point  $P$  on  $L$  which is nearer to  $A_0$  (the starting point of  $L$ ) than any point  $Q$  with  $QQ' \leq 1/2$ .

Let  $A'$  be the first vertex of the square approached by  $L$ .  $L$  must subsequently approach both  $B'$  and  $D'$ . Suppose it approaches  $B'$  first. Let  $B$  be the first point on  $L$  with  $BB' \leq 1/2$ . We can now divide  $L$  into two parts  $L_1$ , the path from  $A_0$  to  $B$ , and  $L_2$ , the path from  $B$  to  $A_n$ .

Take  $X'$  to be the point on  $A'D'$  closest to  $D'$  which is approached by  $L_1$ . Let  $X$  be the corresponding point on  $L_1$ . Now every point on  $X'D'$  must be approached by  $L_2$  (and  $X'D'$  is non-empty, because we know that  $D'$  is approached by  $L$  but not by  $L_1$ ). So by compactness  $X'$  itself must be approached by  $L_2$ . Take  $Y$  to be the corresponding point on  $L_2$ .  $XY \leq XX' + X'Y \leq 1/2 + 1/2 = 1$ . Also  $BB' \leq 1/2$ , so  $XB \geq X'B' - XX' - BB' \geq X'B' - 1 \geq A'B' - 1 = 99$ . Similarly  $YB \geq 99$ , so the path  $XY \geq 198$ .

## 22nd IMO 1981



**A1.** P is a point inside the triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB respectively. Find all P which minimise:

$$BC/PD + CA/PE + AB/PF.$$



**A2.** Take r such that  $1 \leq r \leq n$ , and consider all subsets of r elements of the set  $\{1, 2, \dots, n\}$ . Each subset has a smallest element. Let  $F(n,r)$  be the arithmetic mean of these smallest elements. Prove that:

$$F(n,r) = (n+1)/(r+1).$$



**A3.** Determine the maximum value of  $m^2 + n^2$ , where m and n are integers in the range 1, 2, ..., 1981 satisfying  $(n^2 - mn - m^2)^2 = 1$ .



**B1.** (a) For which  $n > 2$  is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining n - 1 numbers?



(b) For which  $n > 2$  is there exactly one set having this property?

**B2.** Three circles of equal radius have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point O.



**B3.** The function  $f(x,y)$  satisfies:  $f(0,y) = y + 1$ ,  $f(x+1,0) = f(x,1)$ ,  $f(x+1,y+1) = f(x,f(x+1,y))$  for all non-negative integers x, y. Find  $f(4, 1981)$ .

## IMO 1981



### Problem A1

**P** is a point inside the triangle **ABC**. **D**, **E**, **F** are the feet of the perpendiculars from **P** to the lines **BC**, **CA**, **AB** respectively. Find all **P** which minimise:

$$BC/PD + CA/PE + AB/PF.$$

### Solution

We have  $PD \cdot BC + PE \cdot CA + PF \cdot AB = 2 \times \text{area of triangle}$ . Now use Cauchy's inequality with  $x_1 = \sqrt{PD \cdot BC}$ ,  $x_2 = \sqrt{PE \cdot CA}$ ,  $x_3 = \sqrt{PF \cdot AB}$ , and  $y_1 = \sqrt{BC/PD}$ ,  $y_2 = \sqrt{CA/PE}$ ,  $y_3 = \sqrt{AB/PF}$ . We get that  $(BC + CA + AB)^2 < 2 \times \text{area of triangle} \times (BC/PD + CA/PE + AB/PF)$  with equality only if  $x_i/y_i = \text{const}$ , ie  $PD = PE = PF$ . So the unique minimum position for **P** is the incenter.

## IMO 1981



### Problem A2

Take  $r$  such that  $1 \leq r \leq n$ , and consider all subsets of  $r$  elements of the set  $\{1, 2, \dots, n\}$ . Each subset has a smallest element. Let  $F(n, r)$  be the arithmetic mean of these smallest elements. Prove that:

$$F(n, r) = (n+1)/(r+1).$$

### Solution

Denote the binomial coefficient  $n!/(r!(n-r)!)$  by  $nCr$ .

Evidently  $nCr F(n, r) = 1 \cdot (n-1)C(r-1) + 2 \cdot (n-2)C(r-1) + \dots + (n-r+1) \cdot (r-1)C(r-1)$ . [The first term denotes the contribution from subsets with smallest element 1, the second term smallest element 2 and so on.]

Let the rhs be  $g(n, r)$ . Then, using the relation  $(n-i)C(r-1) - (n-i-1)C(r-2) = (n-i-1)C(r-1)$ , we find that  $g(n, r) - g(n-1, r-1) = g(n-1, r)$ , and we can extend this relation to  $r=1$  by taking  $g(n, 0) = n+1 = (n+1)C1$ . But  $g(n, 1) = 1 + 2 + \dots + n = n(n+1)/2 = (n+1)C2$ . So it now follows by an easy induction that  $g(n, r) = (n+1)C(r+1) = nCr \cdot (n+1)/(r+1)$ . Hence  $F(n, r) = (n+1)/(r+1)$ .

*A more elegant solution by Oliver Nash is as follows*

Let  $k$  be the smallest element. We want to evaluate  $g(n, r) = \sum_{k=1 \text{ to } n-r+1} k \cdot (n-k)C(r-1)$ . Consider the subsets with  $r+1$  elements taken from  $1, 2, 3, \dots, n+1$ . Suppose  $k+1$  is the second smallest element. Then there are  $k \cdot (n-k)C(r-1)$  possible subsets. So  $g(n, r) = (n+1)C(r+1)$ . Hence  $F(n, r) = (n+1)C(r+1) / nCr = (n+1)/(r+1)$ , as required.

## IMO 1981



### Problem A3

Determine the maximum value of  $m^2 + n^2$ , where  $m$  and  $n$  are integers in the range  $1, 2, \dots, 1981$  satisfying  $(n^2 - mn - m^2)^2 = 1$ .

### Solution

Experimenting with small values suggests that the solutions of  $n^2 - mn - m^2 = 1$  or  $-1$  are successive Fibonacci numbers. So suppose  $n > m$  is a solution. This suggests trying  $m+n, n$ :  $(m+n)^2 - (m+n)n - n^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$  or  $-1$ . So if  $n > m$  is a solution, then  $m+n, n$  is another solution. Running this forward from  $2, 1$  gives  $3, 2; 5, 3; 8, 5; 13, 8; 21, 13; 34, 21; 55, 34; 89, 55; 144, 89; 233, 144; 377, 233; 610, 377; 987, 610; 1597, 987; 2584, 1597$ .

But how do we know that there are no other solutions? The trick is to run the recurrence the other way. For suppose  $n > m$  is a solution, then try  $m, n-m$ :  $m^2 - m(n-m) - (n-m)^2 = m^2 + mn - n^2 = -(n^2 - mn - m^2) = 1$  or  $-1$ , so that also satisfies the equation. Also if  $m > 1$ , then  $m > n-m$  (for if not, then  $n \geq 2m$ , so  $n(n-m) \geq 2m^2$ , so  $n^2 - nm - m^2 \geq m^2 > 1$ ). So given a solution  $n > m$  with  $m > 1$ , we have a smaller solution  $m > n-m$ . This process must eventually terminate, so it must finish at a solution  $n, 1$  with  $n > 1$ . But the only such solution is  $2, 1$ . Hence the starting solution must have been in the forward sequence from  $2, 1$ .

Hence the solution to the problem stated is  $1597^2 + 987^2$ .

## IMO 1981



### Problem B1

(a) For which  $n > 2$  is there a set of  $n$  consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining  $n - 1$  numbers?

(b) For which  $n > 2$  is there exactly one set having this property?

### Solution

(a)  $n = 3$  is not possible. For suppose  $x$  was the largest number in the set. Then  $x$  cannot be divisible by 3 or any larger prime, so it must be a power of 2. But it cannot be a power of 2, because  $2^m - 1$  is odd and  $2^m - 2$  is not a positive integer divisible by  $2^m$ .

For  $k \geq 2$ , the set  $2k-1, 2k, \dots, 4k-2$  gives  $n = 2k$ . For  $k \geq 3$ , so does the set  $2k-5, 2k-4, \dots, 4k-6$ . For  $k \geq 2$ , the set  $2k-2, 2k-3, \dots, 4k-2$  gives  $n = 2k+1$ . For  $k \geq 4$  so does the set  $2k-6, 2k-5, \dots, 4k-6$ . So we have at least one set for every  $n \geq 4$ , which answers (a).

(b) We also have at least two sets for every  $n \geq 4$  except possibly  $n = 4, 5, 7$ . For 5 we may take as a second set: 8, 9, 10, 11, 12, and for 7 we may take 6, 7, 8, 9, 10, 11, 12. That leaves  $n = 4$ . Suppose  $x$  is the largest number in a set with  $n = 4$ .  $x$  cannot be divisible by 5 or any larger prime, because  $x-1, x-2, x-3$  will not be. Moreover,  $x$  cannot be divisible by 4, because then  $x-1$  and  $x-3$  will be odd, and  $x-2$  only divisible by 2 (not 4). Similarly, it cannot be divisible by 9. So the only possibilities are 1, 2, 3, 6. But we also require  $x \geq 4$ , which eliminates the first three. So the only solution for  $n = 4$  is the one we have already found: 3, 4, 5, 6.

## IMO 1981

### Problem B2

**Three circles of equal radius have a common point  $O$  and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle are collinear with the point  $O$ .**

#### Solution

Let the triangle be  $ABC$ . Let the center of the circle touching  $AB$  and  $AC$  be  $D$ , the center of the circle touching  $AB$  and  $BC$  be  $E$ , and the center of the circle touching  $AC$  and  $BC$  be  $F$ . Because the circles center  $D$  and  $E$  have the same radius the perpendiculars from  $D$  and  $E$  to  $AB$  have the same length, so  $DE$  is parallel to  $AB$ . Similarly  $EF$  is parallel to  $BC$  and  $FD$  is parallel to  $CA$ . Hence  $DEF$  is similar and similarly oriented to  $ABC$ . Moreover  $D$  must lie on the angle bisector of  $A$  since the circle center  $D$  touches  $AB$  and  $AC$ . Similarly  $E$  lies on the angle bisector of  $B$  and  $F$  lies on the angle bisector of  $C$ . Hence the incenter  $I$  of  $ABC$  is also the incenter of  $DEF$  and acts as a center of symmetry so that corresponding points  $P$  of  $ABC$  and  $P'$  of  $DEF$  lie on a line through  $I$  with  $PI/P'I$  having a fixed ratio. But  $OD = OE = OF$  since the three circles have equal radii, so  $O$  is the circumcenter of  $DEF$ . Hence it lies on a line with  $I$  and the circumcenter of  $ABC$ .

## IMO 1981

### Problem B3

**The function  $f(x,y)$  satisfies:  $f(0,y) = y + 1$ ,  $f(x+1,0) = f(x,1)$ ,  $f(x+1,y+1) = f(x,f(x+1,y))$  for all non-negative integers  $x, y$ . Find  $f(4, 1981)$ .**

#### Solution

$f(1,n) = f(0,f(1,n-1)) = 1 + f(1,n-1)$ . So  $f(1,n) = n + f(1,0) = n + f(0,1) = n + 2$ .

$f(2,n) = f(1,f(2,n-1)) = f(2,n-1) + 2$ . So  $f(2,n) = 2n + f(2,0) = 2n + f(1,1) = 2n + 3$ .

$f(3,n) = f(2,f(3,n-1)) = 2f(3,n-1) + 3$ . Let  $u_n = f(3,n) + 3$ , then  $u_n = 2u_{n-1}$ . Also  $u_0 = f(3,0) + 3 = f(2,1) + 3 = 8$ . So  $u_n = 2^{n+3}$ , and  $f(3,n) = 2^{n+3} - 3$ .

$f(4,n) = f(3,f(4,n-1)) = 2^{f(4,n-1)+3} - 3$ .  $f(4,0) = f(3,1) = 2^4 - 3 = 13$ . We calculate two more terms to see the pattern:  $f(4,1) = 2^{24} - 3$ ,  $f(4,2) = 2^{2^{24}} - 3$ . In fact it looks neater if we replace 4 by  $2^2$ , so that  $f(4,n)$  is a tower of  $n+3$  2s less 3.

21st IMO 1979 (no IMO in 1980)



**A1.** Let  $m$  and  $n$  be positive integers such that:

$$m/n = 1 - 1/2 + 1/3 - 1/4 + \dots - 1/1318 + 1/1319.$$



Prove that  $m$  is divisible by 1979.

**A2.** A prism with pentagons  $A_1A_2A_3A_4A_5$  and  $B_1B_2B_3B_4B_5$  as the top and bottom faces is given. Each side of the two pentagons and each of the 25 segments  $A_iB_j$  is colored red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Prove that all 10 sides of the top and bottom faces have the same color.



**A3.** Two circles in a plane intersect.  $A$  is one of the points of intersection. Starting simultaneously from  $A$  two points move with constant speed, each traveling along its own circle in the same sense. The two points return to  $A$  simultaneously after one revolution. Prove that there is a fixed point  $P$  in the plane such that the two points are always equidistant from  $P$ .



**B1.** Given a plane  $k$ , a point  $P$  in the plane and a point  $Q$  not in the plane, find all points  $R$  in  $k$  such that the ratio  $(QP + PR)/QR$  is a maximum.



**B2.** Find all real numbers  $a$  for which there exist non-negative real numbers  $x_1, x_2, x_3, x_4, x_5$  satisfying:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= a, \\x_1 + 2^3x_2 + 3^3x_3 + 4^3x_4 + 5^3x_5 &= a^2, \\x_1 + 2^5x_2 + 3^5x_3 + 4^5x_4 + 5^5x_5 &= a^3.\end{aligned}$$



**B3.** Let  $A$  and  $E$  be opposite vertices of an octagon. A frog starts at vertex  $A$ . From any vertex except  $E$  it jumps to one of the two adjacent vertices. When it reaches  $E$  it stops. Let  $a_n$  be the number of distinct paths of exactly  $n$  jumps ending at  $E$ . Prove that:

$$\begin{aligned}a_{2n-1} &= 0 \\a_{2n} &= (2 + \sqrt{2})^{n-1}/\sqrt{2} - (2 - \sqrt{2})^{n-1}/\sqrt{2}.\end{aligned}$$



## IMO 1979



### Problem A1

Let  $m$  and  $n$  be positive integers such that:

$$m/m = 1 - 1/2 + 1/3 - 1/4 + \dots - 1/1318 + 1/1319.$$

Prove that  $m$  is divisible by 1979.

### Solution

This is difficult.

The obvious step of combining adjacent terms to give  $1/(n(n+1))$  is unhelpful. The trick is to separate out the negative terms:

$$1 - 1/2 + 1/3 - 1/4 + \dots - 1/1318 + 1/1319 = 1 + 1/2 + 1/3 + \dots + 1/1319 - 2(1/2 + 1/4 + \dots + 1/1318) = 1/660 + 1/661 + \dots + 1/1319.$$

and to notice that  $660 + 1319 = 1979$ . Combine terms in pairs from the outside:

$$1/660 + 1/1319 = 1979/(660 \cdot 1319); 1/661 + 1/1318 = 1979/(661 \cdot 1318) \text{ etc.}$$

There are an even number of terms, so this gives us a sum of terms  $1979/m$  with  $m$  not divisible by 1979 (since 1979 is prime and so does not divide any product of smaller numbers). Hence the sum of the  $1/m$  gives a rational number with denominator not divisible by 1979 and we are done.

## IMO 1979



### Problem A2

A prism with pentagons  $A_1A_2A_3A_4A_5$  and  $B_1B_2B_3B_4B_5$  as the top and bottom faces is given. Each side of the two pentagons and each of the 25 segments  $A_iB_j$  is colored red or green. Every triangle whose vertices are vertices of the prism and whose sides have all been colored has two sides of a different color. Prove that all 10 sides of the top and bottom faces have the same color.

### Solution

We show first that the  $A_i$  are all the same color. If not then, there is a vertex, call it  $A_1$ , with edges  $A_1A_2, A_1A_5$  of opposite color. Now consider the five edges  $A_1B_i$ . At least three of them must be the same color. Suppose it is green and that  $A_1A_2$  is also green. Take the three edges to be  $A_1B_i, A_1B_j, A_1B_k$ . Then considering the triangles  $A_1A_2B_i, A_1A_2B_j, A_1A_2B_k$ , the three edges  $A_2B_i, A_2B_j, A_2B_k$  must all be red. Two of  $B_i, B_j, B_k$  must be adjacent, but if the resulting edge is red then we

have an all red triangle with  $A_2$ , whilst if it is green we have an all green triangle with  $A_1$ . Contradiction. So the  $A_i$  are all the same color. Similarly, the  $B_i$  are all the same color. It remains to show that they are the same color. Suppose otherwise, so that the  $A_i$  are green and the  $B_i$  are red.

Now we argue as before that 3 of the 5 edges  $A_1B_i$  must be the same color. If it is red, then as before 2 of the 3  $B_i$  must be adjacent and that gives an all red triangle with  $A_1$ . So 3 of the 5 edges  $A_1B_i$  must be green. Similarly, 3 of the 5 edges  $A_2B_i$  must be green. But there must be a  $B_i$  featuring in both sets and it forms an all green triangle with  $A_1$  and  $A_2$ . Contradiction. So the  $A_i$  and the  $B_i$  are all the same color.

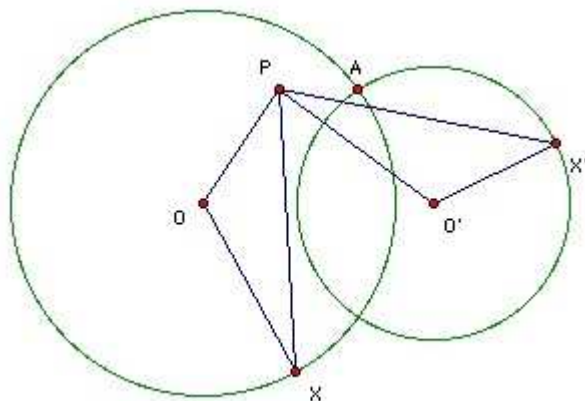
## IMO 1979



### Problem A3

**Two circles in a plane intersect.  $A$  is one of the points of intersection. Starting simultaneously from  $A$  two points move with constant speed, each traveling along its own circle in the same sense. The two points return to  $A$  simultaneously after one revolution. Prove that there is a fixed point  $P$  in the plane such that the two points are always equidistant from  $P$ .**

### Solution



Let the circles have centers  $O$ ,  $O'$  and let the moving points be  $X$ ,  $X'$ . Let  $P$  be the reflection of  $A$  in the perpendicular bisector of  $OO'$ . We show that triangles  $POX$ ,  $X'O'P$  are congruent. We have  $OX = OA$  (pts on circle) =  $O'P$  (reflection). Also  $OP = O'A$  (reflection) =  $O'X'$  (pts on circle). Also  $\angle AOX = \angle AO'X'$  ( $X$  and  $X'$  circle at same rate), and  $\angle AOP = \angle AO'P$  (reflection), so  $\angle POX = \angle PO'X'$ . So the triangles are congruent. Hence  $PX = PX'$ .

*Another approach is to show that  $XX'$  passes through the other point of intersection of the two circles, but that involves looking at many different cases depending on the relative positions of the moving points.*

**Problem B1**

Given a plane  $k$ , a point  $P$  in the plane and a point  $Q$  not in the plane, find all points  $R$  in  $k$  such that the ratio  $(QP + PR)/QR$  is a maximum.

**Solution**

Consider the points  $R$  on a circle center  $P$ . Let  $X$  be the foot of the perpendicular from  $Q$  to  $k$ . Assume  $P$  is distinct from  $X$ , then we minimise  $QR$  (and hence maximise  $(QP + PR)/QR$ ) for points  $R$  on the circle by taking  $R$  on the line  $PX$ . Moreover,  $R$  must lie on the same side of  $P$  as  $X$ . Hence if we allow  $R$  to vary over  $k$ , the points maximising  $(QP + PR)/QR$  must lie on the ray  $PX$ . Take  $S$  on the line  $PX$  on the opposite side of  $P$  from  $X$  so that  $PS = PQ$ . Then for points  $R$  on the ray  $PX$  we have  $(QP + PR)/QR = SR/QR = \sin RQS/\sin QSR$ . But  $\sin QSR$  is fixed for points on the ray, so we maximise the ratio by taking  $\angle RQS = 90^\circ$ . Thus there is a single point maximising the ratio.

If  $P = X$ , then we still require  $\angle RQS = 90^\circ$ , but  $R$  is no longer restricted to a line, so it can be anywhere on a circle center  $P$ .

**Problem B2**

Find all real numbers  $a$  for which there exist non-negative real numbers  $x_1, x_2, x_3, x_4, x_5$  satisfying:

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= a, \\x_1 + 2^3x_2 + 3^3x_3 + 4^3x_4 + 5^3x_5 &= a^2, \\x_1 + 2^5x_2 + 3^5x_3 + 4^5x_4 + 5^5x_5 &= a^3.\end{aligned}$$

**Solution**

Take  $a^2 \times$  1st equ -  $2a \times$  2nd equ + 3rd equ. The rhs is 0. On the lhs the coefficient of  $x_n$  is  $a^2n - 2an^3 + n^5 = n(a - n^2)^2$ . So the lhs is a sum of non-negative terms. Hence each term must be zero separately, so for each  $n$  either  $x_n = 0$  or  $a = n^2$ . So there are just 5 solutions, corresponding to  $a = 1, 4, 9, 16, 25$ . We can check that each of these gives a solution. [For  $a = n^2$ ,  $x_n = n$  and the other  $x_i$  are zero.]



### Problem B3

Let  $A$  and  $E$  be opposite vertices of an octagon. A frog starts at vertex  $A$ . From any vertex except  $E$  it jumps to one of the two adjacent vertices. When it reaches  $E$  it stops. Let  $a_n$  be the number of distinct paths of exactly  $n$  jumps ending at  $E$ . Prove that:

$$\begin{aligned} a_{2n-1} &= 0 \\ a_{2n} &= (2 + \sqrt{2})^{n-1}/\sqrt{2} - (2 - \sqrt{2})^{n-1}/\sqrt{2}. \end{aligned}$$

### Solution

Each jump changes the parity of the shortest distance to  $E$ . The parity is initially even, so an odd number of jumps cannot end at  $E$ . Hence  $a_{2n-1} = 0$ .

We derive a recurrence relation for  $a_{2n}$ . This is not easy to do directly, so we introduce  $b_n$  which is the number of paths length  $n$  from  $C$  to  $E$ . Then we have immediately:

$$\begin{aligned} a_{2n} &= 2a_{2n-2} + 2b_{2n-2} \text{ for } n > 1 \\ b_{2n} &= 2b_{2n-2} + a_{2n-2} \text{ for } n > 1 \end{aligned}$$

Hence, using the first equation:  $a_{2n} - 2a_{2n-2} = 2a_{2n-2} - 4a_{2n-4} + 2b_{2n-2} - 4b_{2n-4}$  for  $n > 2$ . Using the second equation, this leads to:  $a_{2n} = 4a_{2n-2} - 2a_{2n-4}$  for  $n > 2$ . This is a linear recurrence relation with the general solution:  $a_{2n} = a(2 + \sqrt{2})^{n-1} + b(2 - \sqrt{2})^{n-1}$ . But we easily see directly that  $a_4 = 2$ ,  $a_6 = 8$  and we can now solve for the coefficients to get the solution given.

## 20th IMO 1978



**A1.**  $m$  and  $n$  are positive integers with  $m < n$ . The last three decimal digits of  $1978^m$  are the same as the last three decimal digits of  $1978^n$ . Find  $m$  and  $n$  such that  $m + n$  has the least possible value.



**A2.**  $P$  is a point inside a sphere. Three mutually perpendicular rays from  $P$  intersect the sphere at points  $U$ ,  $V$  and  $W$ .  $Q$  denotes the vertex diagonally opposite  $P$  in the parallelepiped determined by  $PU$ ,  $PV$ ,  $PW$ . Find the locus of  $Q$  for all possible sets of such rays from  $P$ .



**A3.** The set of all positive integers is the union of two disjoint subsets  $\{f(1), f(2), f(3), \dots\}$ ,  $\{g(1), g(2), g(3), \dots\}$ , where  $f(1) < f(2) < f(3) < \dots$ , and  $g(1) < g(2) < g(3) < \dots$ , and  $g(n) = f(f(n)) + 1$  for  $n = 1, 2, 3, \dots$ . Determine  $f(240)$ .



**B1.** In the triangle  $ABC$ ,  $AB = AC$ . A circle is tangent internally to the circumcircle of the triangle and also to  $AB$ ,  $AC$  at  $P$ ,  $Q$  respectively. Prove that the midpoint of  $PQ$  is the center of the incircle of the triangle.



**B2.**  $\{a_k\}$  is a sequence of distinct positive integers. Prove that for all positive integers  $n$ ,  $\sum_{1 \leq k \leq n} a_k/k^2 \geq \sum_{1 \leq k \leq n} 1/k$ .



**B3.** An international society has its members from six different countries. The list of members has 1978 names, numbered  $1, 2, \dots, 1978$ . Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice the number of a member from his own country.

## IMO 1978



### Problem A1

**$m$  and  $n$  are positive integers with  $m < n$ . The last three decimal digits of  $1978^m$  are the same as the last three decimal digits of  $1978^n$ . Find  $m$  and  $n$  such that  $m + n$  has the least possible value.**

### Solution

We require  $1978^m(1978^{n-m} - 1)$  to be a multiple of  $1000=8 \cdot 125$ . So we must have 8 divides  $1978^m$ , and hence  $m \geq 3$ , and 125 divides  $1978^{n-m} - 1$ .

By Euler's theorem,  $1978^{\phi(125)} = 1 \pmod{125}$ .  $\phi(125) = 125 - 25 = 100$ , so  $1978^{100} = 1 \pmod{125}$ . Hence the smallest  $r$  such that  $1978^r = 1 \pmod{125}$  must be a divisor of 100 (because if it was not, then the remainder on dividing it into 100 would give a smaller  $r$ ). That leaves 9 possibilities to check: 1, 2, 4, 5, 10, 20, 25, 50, 100. To reduce the work we quickly find that the smallest  $s$  such that  $1978^s = 1 \pmod{5}$  is 4 and hence  $r$  must be a multiple of 4. That leaves 4, 20, 100 to examine.

We find  $1978^2 = 109 \pmod{125}$ , and hence  $1978^4 = 6 \pmod{125}$ . Hence  $1978^{20} = 6^5 = 36 \cdot 91 = 26 \pmod{125}$ . So the smallest  $r$  is 100 and hence the solution to the problem is 3, 103.

## IMO 1978



### Problem A2

**$P$  is a point inside a sphere. Three mutually perpendicular rays from  $P$  intersect the sphere at points  $U, V$  and  $W$ .  $Q$  denotes the vertex diagonally opposite  $P$  in the parallelepiped determined by  $PU, PV, PW$ . Find the locus of  $Q$  for all possible sets of such rays from  $P$ .**

### Solution

Suppose  $ABCD$  is a rectangle and  $X$  any point inside, then  $XA^2 + XC^2 = XB^2 + XD^2$ . This is most easily proved using coordinates. Take the origin  $O$  as the center of the rectangle and take  $OA$  to be the vector  $\underline{a}$ , and  $OB$  to be  $\underline{b}$ . Since it is a rectangle,  $|\underline{a}| = |\underline{b}|$ . Then  $OC$  is  $-\underline{a}$  and  $OD$  is  $-\underline{b}$ . Let  $OX$  be  $\underline{c}$ . Then  $XA^2 + XC^2 = (\underline{a} - \underline{c})^2 + (\underline{a} + \underline{c})^2 = 2\underline{a}^2 + 2\underline{c}^2 = 2\underline{b}^2 + 2\underline{c}^2 = XB^2 + XD^2$ .

Let us fix  $U$ . Then the plane  $k$  perpendicular to  $PU$  through  $P$  cuts the sphere in a circle center  $C$ .  $V$  and  $W$  must lie on this circle. Take  $R$  so that  $PVRW$  is a rectangle. By the result just proved  $CR^2 = 2CV^2 - CP^2$ .  $OC$  is also perpendicular to the plane  $k$ . Extend it to  $X$ , so that  $CX = PU$ . Then extend  $XU$  to  $Y$  so that  $YR$  is perpendicular to  $k$ . Now  $OY^2 = OX^2 + XY^2 = OX^2 + CR^2 = OX^2 + 2CV^2 - CP^2 = OU^2 - UX^2 + 2CV^2 - CP^2 = OU^2 - CP^2 + 2(OV^2 - OC^2) - CP^2 = 3OU^2 - 2OP^2$ . Thus the locus of  $Y$  is a sphere.

**Problem A3**

The set of all positive integers is the union of two disjoint subsets  $\{f(1), f(2), f(3), \dots\}$ ,  $\{g(1), g(2), g(3), \dots\}$ , where  $f(1) < f(2) < f(3) < \dots$ , and  $g(1) < g(2) < g(3) < \dots$ , and  $g(n) = f(f(n)) + 1$  for  $n = 1, 2, 3, \dots$ . Determine  $f(240)$ .

**Solution**

Let  $F = \{f(1), f(2), f(3), \dots\}$ ,  $G = \{g(1), g(2), g(3), \dots\}$ ,  $N_n = \{1, 2, 3, \dots, n\}$ .  $f(1) \geq 1$ , so  $f(f(1)) \geq 1$  and hence  $g(1) \geq 2$ . So 1 is not in  $G$ , and hence must be in  $F$ . It must be the smallest element of  $F$  and so  $f(1) = 1$ . Hence  $g(1) = 2$ . We can never have two successive integers  $n$  and  $n+1$  in  $G$ , because if  $g(m) = n+1$ , then  $f(\text{something}) = n$  and so  $n$  is in  $F$  and  $G$ . Contradiction. In particular, 3 must be in  $F$ , and so  $f(2) = 3$ .

Suppose  $f(n) = k$ . Then  $g(n) = f(k) + 1$ . So  $|N_{f(k)+1} \cap G| = n$ . But  $|N_{f(k)+1} \cap F| = k$ , so  $n + k = f(k) + 1$ , or  $f(k) = n + k - 1$ . Hence  $g(n) = n + k$ . So  $n + k + 1$  must be in  $F$  and hence  $f(k+1) = n + k + 1$ . This so given the value of  $f$  for  $n$  we can find it for  $k$  and  $k+1$ .

Using  $k+1$  each time, we get, successively,  $f(2) = 3, f(4) = 6, f(7) = 11, f(12) = 19, f(20) = 32, f(33) = 53, f(54) = 87, f(88) = 142, f(143) = 231, f(232) = 375$ , which is not much help. Trying again with  $k$ , we get:  $f(3) = 4, f(4) = 6, f(6) = 9, f(9) = 14, f(14) = 22, f(22) = 35, f(35) = 56, f(56) = 90, f(90) = 145, f(145) = 234$ . Still not right, but we can try backing up slightly and using  $k+1$ :  $f(146) = 236$ . Still not right, we need to back up further:  $f(91) = 147, f(148) = 239, f(240) = 388$ .

**Problem B1**

In the triangle  $ABC$ ,  $AB = AC$ . A circle is tangent internally to the circumcircle of the triangle and also to  $AB, AC$  at  $P, Q$  respectively. Prove that the midpoint of  $PQ$  is the center of the incircle of the triangle.

**Solution**

It is not a good idea to get bogged down in complicated formulae for the various radii. The solution is actually simple.

By symmetry the midpoint,  $M$ , is already on the angle bisector of  $A$ , so it is sufficient to show it is on the angle bisector of  $B$ . Let the angle bisector of  $A$  meet the circumcircle again at  $R$ .  $AP$  is a tangent to the circle touching  $AB$  at  $P$ , so  $\angle PRQ = \angle APQ = \angle ABC$ . Now the quadrilateral  $PBRM$  is cyclic because the angles  $PBR, PMR$  are both  $90^\circ$ . Hence  $\angle PBM = \angle PRM = (\angle PRQ)/2$ , so  $BM$  does indeed bisect angle  $B$  as claimed.

## IMO 1978



### Problem B2

$\{a_k\}$  is a sequence of distinct positive integers. Prove that for all positive integers  $n$ ,  $\sum_1^n a_k/k^2 \geq \sum_1^n 1/k$ .

### Solution

We use the general rearrangement result: given  $b_1 \geq b_2 \geq \dots \geq b_n$ , and  $c_1 \leq c_2 \leq \dots \leq c_n$ , if  $\{a_i\}$  is a permutation of  $\{c_i\}$ , then  $\sum a_i b_i \geq \sum c_i b_i$ . To prove it, suppose that  $i < j$ , but  $a_i > a_j$ . Then interchanging  $a_i$  and  $a_j$  does not increase the sum, because  $(a_i - a_j)(b_i - b_j) \geq 0$ , and hence  $a_i b_i + a_j b_j \geq a_j b_i + a_i b_j$ . By a series of such interchanges we transform  $\{a_i\}$  into  $\{c_i\}$  (for example, first swap  $c_1$  into first place, then  $c_2$  into second place and so on).

Hence we do not increase the sum by permuting  $\{a_i\}$  so that it is in increasing order. But now we have  $a_i > i$ , so we do not increase the sum by replacing  $a_i$  by  $i$  and that gives the sum from 1 to  $n$  of  $1/k$ .

## IMO 1978



### Problem B3

An international society has its members from six different countries. The list of members has 1978 names, numbered 1, 2, ..., 1978. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice the number of a member from his own country.

### Solution

The trick is to use differences.

At least  $6.329 = 1974$ , so at least 330 members come from the same country, call it C1. Let their numbers be  $a_1 < a_2 < \dots < a_{330}$ . Now take the 329 differences  $a_2 - a_1, a_3 - a_1, \dots, a_{330} - a_1$ . If any of them are in C1, then we are home, so suppose they are all in the other five countries.

At least 66 must come from the same country, call it C2. Write the 66 as  $b_1 < b_2 < \dots < b_{66}$ . Now form the 65 differences  $b_2 - b_1, b_3 - b_1, \dots, b_{66} - b_1$ . If any of them are in C2, then we are home. But each difference equals the difference of two of the original  $a_i$ s, so if it is in C1 we are also home.

So suppose they are all in the other four countries. At least 17 must come from the same country, call it C3. Write the 17 as  $c_1 < c_2 < \dots < c_{17}$ . Now form the 16 differences  $c_2 - c_1, c_3 - c_1, \dots, c_{17} - c_1$ . If any of them are in C3, we are home. Each difference equals the difference of two  $b_i$ s, so if any of them are in C2 we are home. [For example, consider  $c_i - c_1$ . Suppose  $c_i = b_n - b_1$  and  $c_1 = b_m - b_1$ , then  $c_i - c_1 = b_n - b_m$ , as claimed.] Each difference also equals the difference of two  $a_i$ s, so



if any of them are in C1, we are also home. [For example, consider  $c_i - c_1$ , as before. Suppose  $b_n = a_j - a_1$ ,  $b_m = a_k - a_1$ , then  $c_i - c_1 = b_n - b_m = a_j - a_k$ , as claimed.]

So suppose they are all in the other three countries. At least 6 must come from the same country, call it C4. We look at the 5 differences and conclude in the same way that at least 3 must come from C5. Now the 2 differences must both be in C6 and their difference must be in one of the C1, ... , C6 giving us the required sum.



**A1.** Construct equilateral triangles  $ABK$ ,  $BCL$ ,  $CDM$ ,  $DAN$  on the inside of the square  $ABCD$ . Show that the midpoints of  $KL$ ,  $LM$ ,  $MN$ ,  $NK$  and the midpoints of  $AK$ ,  $BK$ ,  $BL$ ,  $CL$ ,  $CM$ ,  $DM$ ,  $DN$ ,  $AN$  form a regular dodecahedron.



**A2.** In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.



**A3.** Given an integer  $n > 2$ , let  $V_n$  be the set of integers  $1 + kn$  for  $k$  a positive integer. A number  $m$  in  $V_n$  is called indecomposable if it cannot be expressed as the product of two members of  $V_n$ . Prove that there is a number in  $V_n$  which can be expressed as the product of indecomposable members of  $V_n$  in more than one way (decompositions which differ solely in the order of factors are not regarded as different).



**B1.** Define  $f(x) = 1 - a \cos x - b \sin x - A \cos 2x - B \sin 2x$ , where  $a, b, A, B$  are real constants. Suppose that  $f(x) \geq 0$  for all real  $x$ . Prove that  $a^2 + b^2 \leq 2$  and  $A^2 + B^2 \leq 1$ .



**B2.** Let  $a$  and  $b$  be positive integers. When  $a^2 + b^2$  is divided by  $a + b$ , the quotient is  $q$  and the remainder is  $r$ . Find all pairs  $a, b$  such that  $q^2 + r = 1977$ .



**B3.** The function  $f$  is defined on the set of positive integers and its values are positive integers. Given that  $f(n+1) > f(f(n))$  for all  $n$ , prove that  $f(n) = n$  for all  $n$ .



## IMO 1977



### Problem A1

Construct equilateral triangles  $ABK$ ,  $BCL$ ,  $CDM$ ,  $DAN$  on the inside of the square  $ABCD$ . Show that the midpoints of  $KL$ ,  $LM$ ,  $MN$ ,  $NK$  and the midpoints of  $AK$ ,  $BK$ ,  $BL$ ,  $CL$ ,  $CM$ ,  $DM$ ,  $DN$ ,  $AN$  form a regular dodecahedron.

### Solution

The most straightforward approach is to use coordinates. Take  $A, B, C, D$  to be  $(1,1), (-1,1), (-1,-1), (1,-1)$ . Then  $K, L, M, N$  are  $(0, -2k), (2k, 0), (0, 2k), (-2k, 0)$ , where  $k = (\sqrt{3} - 1)/2$ . The midpoints of  $KL, LM, MN, NK$  are  $(k, -k), (k, k), (-k, k), (-k, -k)$ . These are all a distance  $k\sqrt{2}$  from the origin, at angles  $315, 45, 135, 225$  respectively. The midpoints of  $AK, BK, BL, CL, CM, DM, DN, AN$  are  $(h, j), (-h, j), (-j, h), (-j, -h), (-h, -j), (h, -j), (j, -h), (j, h)$ , where  $h = 1/2, j = (1 - 1/2\sqrt{3})$ . These are also at a distance  $k\sqrt{2}$  from the origin, at angles  $15, 165, 105, 255, 195, 345, 285, 75$  respectively. For this we need to consider the right-angled triangle sides  $k, h, j$ . The angle  $x$  between  $h$  and  $k$  has  $\sin x = j/k$  and  $\cos x = h/k$ . So  $\sin 2x = 2 \sin x \cos x = 2hj/k^2 = 1/2$ . Hence  $x = 15$ .

So the 12 points are all at the same distance from the origin and at angles  $15 + 30n$ , for  $n = 0, 1, 2, \dots, 11$ . Hence they form a regular dodecagon.

## IMO 1977



### Problem A2

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

### Solution

Answer: 16.

$x_1 + \dots + x_7 < 0, x_8 + \dots + x_{14} < 0$ , so  $x_1 + \dots + x_{14} < 0$ . But  $x_4 + \dots + x_{14} > 0$ , so  $x_1 + x_2 + x_3 < 0$ . Also  $x_5 + \dots + x_{11} < 0$  and  $x_1 + \dots + x_{11} > 0$ , so  $x_4 > 0$ . If there are 17 or more elements then the same argument shows that  $x_5, x_6, x_7 > 0$ . But  $x_1 + \dots + x_7 < 0$ , and  $x_5 + \dots + x_{11} < 0$ , whereas  $x_1 + \dots + x_{11} > 0$ , so  $x_5 + x_6 + x_7 < 0$ . Contradiction.

If we assume that there is a solution for  $n = 16$  and that the sum of 7 consecutive terms is  $-1$  and that the sum of 11 consecutive terms is  $1$ , then we can easily solve the equations to get:  $5, 5, -13, 5, 5, 5, -13, 5, 5, -13, 5, 5, 5, -13, 5, 5$  and we can check that this works for 16.



**Problem A3**

Given an integer  $n > 2$ , let  $V_n$  be the set of integers  $1 + kn$  for  $k$  a positive integer. A number  $m$  in  $V_n$  is called **indecomposable** if it cannot be expressed as the product of two members of  $V_n$ . Prove that there is a number in  $V_n$  which can be expressed as the product of indecomposable members of  $V_n$  in more than one way (decompositions which differ solely in the order of factors are not regarded as different).

**Solution**

Take  $a, b, c, d \equiv -1 \pmod{n}$ . The idea is to take  $abcd$  which factorizes as  $ab \cdot cd$  or  $ac \cdot bd$ . The hope is that  $ab, cd, ac, bd$  will not factorize in  $V_n$ . But a little care is needed, since this is not necessarily true.

Try taking  $a = b = n - 1, c = d = 2n - 1$ .  $a^2$  must be indecomposable because it is less than the square of the smallest element in  $V_n$ . If  $ac = 2n^2 - 3n + 1$  is decomposable, then we have  $kk'n + k + k' = 2n - 3$  for some  $k, k' \geq 1$ . But neither of  $k$  or  $k'$  can be 2 or more, because then the lhs is too big, and  $k = k' = 1$  does not work unless  $n = 5$ . Similarly, if  $c^2$  is decomposable, then we have  $kk'n + k + k' = 4n - 4$ .  $k = k' = 1$  only works for  $n = 2$ , but we are told  $n > 2$ .  $k = 1, k' = 2$  does not work (it would require  $n = 7/2$ ).  $k = 1, k' = 3$  only works for  $n = 8$ . Other possibilities make the lhs too big.

So if  $n$  is not 5 or 8, then we can take the number to be  $(n - 1)^2(2n - 1)^2$ , which factors as  $(n - 1)^2 \times (2n - 1)^2$  or as  $(n - 1)(2n - 1) \times (n - 1)(2n - 1)$ . This does not work for 5 or 8:  $16 \cdot 81 = 36 \cdot 36$ , but 36 decomposes as  $6 \cdot 6$ ;  $49 \cdot 225 = 105 \cdot 105$ , but 225 decomposes as  $9 \cdot 25$ .

For  $n = 5$ , we can use  $3136 = 16 \cdot 196 = 56 \cdot 56$ . For  $n = 8$ , we can use  $25921 = 49 \cdot 529 = 161 \cdot 161$ .

IMO 1977



**Problem B1**

Define  $f(x) = 1 - a \cos x - b \sin x - A \cos 2x - B \sin 2x$ , where  $a, b, A, B$  are real constants. Suppose that  $f(x) \geq 0$  for all real  $x$ . Prove that  $a^2 + b^2 \leq 2$  and  $A^2 + B^2 \leq 1$ .

**Solution**

Take  $y$  so that  $\cos y = a/\sqrt{a^2 + b^2}$ ,  $\sin y = b/\sqrt{a^2 + b^2}$ , and  $z$  so that  $\cos 2z = A/\sqrt{A^2 + B^2}$ ,  $\sin 2z = B/\sqrt{A^2 + B^2}$ . Then  $f(x) = 1 - c \cos(x - y) - C \cos 2(x - z)$ , where  $c = \sqrt{a^2 + b^2}$ ,  $C = \sqrt{A^2 + B^2}$ .

$f(z) + f(\pi + z) \geq 0$  gives  $C \leq 1$ .  $f(y + \pi/4) + f(y - \pi/4) \geq 0$  gives  $c \leq \sqrt{2}$ .

## IMO 1977



### Problem B2

Let  $a$  and  $b$  be positive integers. When  $a^2 + b^2$  is divided by  $a + b$ , the quotient is  $q$  and the remainder is  $r$ . Find all pairs  $a, b$  such that  $q^2 + r = 1977$ .

### Solution

$a^2 + b^2 \geq (a + b)^2/2$ , so  $q \geq (a + b)/2$ . Hence  $r < 2q$ . The largest square less than 1977 is  $1936 = 44^2$ .  $1977 = 44^2 + 41$ . The next largest gives  $1977 = 43^2 + 128$ . But  $128 > 2 \cdot 43$ . So we must have  $q = 44$ ,  $r = 41$ . Hence  $a^2 + b^2 = 44(a + b) + 41$ , so  $(a - 22)^2 + (b - 22)^2 = 1009$ . By trial, we find that the only squares with sum 1009 are  $28^2$  and  $15^2$ . This gives two solutions  $50, 37$  or  $50, 7$ .

## IMO 1977



### Problem B3

The function  $f$  is defined on the set of positive integers and its values are positive integers. Given that  $f(n+1) > f(f(n))$  for all  $n$ , prove that  $f(n) = n$  for all  $n$ .

### Solution

The first step is to show that  $f(1) < f(2) < f(3) < \dots$ . We do this by induction on  $n$ . We take  $S_n$  to be the statement that  $f(n)$  is the unique smallest element of  $\{f(n), f(n+1), f(n+2), \dots\}$ .

For  $m > 1$ ,  $f(m) > f(s)$  where  $s = f(m-1)$ , so  $f(m)$  is not the smallest member of the set  $\{f(1), f(2), f(3), \dots\}$ . But the set is bounded below by zero, so it must have a smallest member. Hence the unique smallest member is  $f(1)$ . So  $S_1$  is true.

Suppose  $S_n$  is true. Take  $m > n+1$ . Then  $m-1 > n$ , so by  $S_n$ ,  $f(m-1) > f(n)$ . But  $S_n$  also tells us that  $f(n) > f(n-1) > \dots > f(1)$ , so  $f(n) \geq n - 1 + f(1) \geq n$ . Hence  $f(m-1) \geq n+1$ . So  $f(m-1)$  belongs to  $\{n+1, n+2, n+3, \dots\}$ . But we are given that  $f(m) > f(f(m-1))$ , so  $f(m)$  is not the smallest element of  $\{f(n+1), f(n+2), f(n+3), \dots\}$ . But there must be a smallest element, so  $f(n+1)$  must be the unique smallest member, which establishes  $S_{n+1}$ . So,  $S_n$  is true for all  $n$ .

So  $n \leq m$  implies  $f(n) \leq f(m)$ . Suppose for some  $m$ ,  $f(m) \geq m+1$ , then  $f(f(m)) \geq f(m+1)$ . Contradiction. Hence  $f(m) \leq m$  for all  $m$ . But since  $f(1) \geq 1$  and  $f(m) > f(m-1) > \dots > f(1)$ , we also have  $f(m) \geq m$ . Hence  $f(m) = m$  for all  $m$ .

18th IMO 1976



**A1.** A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.



**A2.** Let  $P_1(x) = x^2 - 2$ , and  $P_{i+1} = P_1(P_i(x))$  for  $i = 1, 2, 3, \dots$ . Show that the roots of  $P_n(x) = x$  are real and distinct for all  $n$ .



**A3.** A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.



**B1.** Determine the largest number which is the product of positive integers with sum 1976.



**B2.**  $n$  is a positive integer and  $m = 2n$ .  $a_{ij} = 0, 1$  or  $-1$  for  $1 \leq i \leq n, 1 \leq j \leq m$ . The  $m$  unknowns  $x_1, x_2, \dots, x_m$  satisfy the  $n$  equations:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m = 0,$$



for  $i = 1, 2, \dots, n$ . Prove that the system has a solution in integers of absolute value at most  $m$ , not all zero.

**B3.** The sequence  $u_0, u_1, u_2, \dots$  is defined by:  $u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$  for  $n = 1, 2, \dots$ . Prove that  $[u_n] = 2^{(2^n - (-1)^n)/3}$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ .



## IMO 1976



### Problem A1

A plane convex quadrilateral has area 32, and the sum of two opposite sides and a diagonal is 16. Determine all possible lengths for the other diagonal.

### Solution

At first sight, the length of the other diagonal appears unlikely to be significantly constrained. However, a little experimentation shows that it is hard to get such a low value as 16. This suggests that 16 may be the smallest possible value.

If the diagonal which is part of the 16 has length  $x$ , then the area is the sum of the areas of two triangles base  $x$ , which is  $xy/2$ , where  $y$  is the sum of the altitudes of the two triangles.  $y$  must be at most  $(16 - x)$ , with equality only if the two triangles are right-angled. But  $x(16 - x)/2 = (64 - (x - 8)^2)/2 \leq 32$  with equality only iff  $x = 8$ . Thus the only way we can achieve the values given is with one diagonal length 8 and two sides perpendicular to this diagonal with lengths totalling 8. But in this case the other diagonal has length  $8\sqrt{2}$ .

## IMO 1976



### Problem A2

Let  $P_1(x) = x^2 - 2$ , and  $P_{i+1} = P_1(P_i(x))$  for  $i = 1, 2, 3, \dots$ . Show that the roots of  $P_n(x) = x$  are real and distinct for all  $n$ .

### Solution

We show that the graph of  $P_n$  can be divided into  $2^n$  lines each joining the top and bottom edges of the square side 4 centered on the origin (vertices  $(2,2)$ ,  $(-2,2)$ ,  $(-2,-2)$ ,  $(2,-2)$ ). We are then home because the upward sloping diagonal of the square, which represents the graph of  $y = x$ , must cut each of these lines and hence give  $2^n$  distinct real roots of  $P_n(x) = x$  in the range  $[-2,2]$ . But  $P_n$  is a polynomial of degree  $2^n$ , so it has exactly  $2^n$  roots. Hence all its roots are real and distinct.

We prove the result about the graph by induction. It is true for  $n = 1$ : the first line is the graph from  $x = -2$  to 0, and the second line is the graph from 0 to 2. So suppose it is true for  $n$ . Then  $P_1$  turns each of the  $2^n$  lines for  $P_n$  into two lines for  $P_{n+1}$ , so the result is true for  $n+1$ .

Put  $x = 2 \cos t$  (so we are restricting attention to  $-2 \leq x \leq 2$ ). Then we find  $P_n(x) = 2 \cos 2^n t$ , so the equation  $P_n(x) = x$  becomes  $\cos 2^n t = \cos t$ . By inspection, has the  $2^n$  solutions  $t = 2k\pi/(2^n - 1)$  and  $t = 2k\pi/(2^n + 1)$ , giving  $2^n$  distinct solutions in  $x$ .

## IMO 1976



### Problem A3

A rectangular box can be completely filled with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, with their edges parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of the box.

### Solution

Answer:  $2 \times 3 \times 5$  or  $2 \times 5 \times 6$ .

This is somewhat messy. The basic idea is that the sides cannot be too long, because then the ratio becomes too big. Let  $k$  denote the (real) cube root of 2. Given any integer  $n$ , let  $n'$  denote the least integer such that  $n'k \leq n$ . Let the sides of the box be  $a \leq b \leq c$ . So we require  $5a'b'c' = abc$  (\*).

It is useful to derive  $n'$  for small  $n$ :  $1' = 0$ ,  $2' = 1$ ,  $3' = 2$ ,  $4' = 3$ ,  $5' = 3$ ,  $6' = 4$ ,  $7' = 5$ ,  $8' = 6$ ,  $9' = 7$ ,  $10' = 7$ .

Clearly  $n'k \geq n-2$ . But  $6^3 > 0.4 \cdot 8^3$ , and hence  $(n'k)^3 \geq (n-2)^3 > 0.4 n^3$  for all  $n \geq 8$ . We can check directly that  $(n'k)^3 > 0.4 n^3$  for  $n = 3, 4, 5, 6, 7$ . So we must have  $a = 2$  (we cannot have  $a = 1$ , because  $1' = 0$ ).

From (\*) we require  $b$  or  $c$  to be divisible by 5. Suppose we take it to be 5. Then since  $5' = 3$ , the third side  $n$  must satisfy:  $n' = 2/3 n$ . We can easily check that  $2k/3 < 6/7$  and hence  $(2/3 nk + 1) < n$  for  $n \geq 7$ , so  $n' > 2/3 n$  for  $n \geq 7$ . This just leaves the values  $n = 3$  and  $n = 6$  to check (since  $n' = 2/3 n$  is integral so  $n$  must be a multiple of 3). Referring to the values above, both these work. So this gives us two possible boxes:  $2 \times 3 \times 5$  and  $2 \times 5 \times 6$ .

The only remaining possibility is that the multiple of 5 is at least 10. But then it is easy to check that if it is  $m$  then  $m'/m \geq 7/10$ . It follows from (\*) that the third side  $r$  must satisfy  $r'/r \leq 4/7$ . But using the limit above and referring to the small values above, this implies that  $r$  must be 2. So  $a = b = 2$ . But now  $c$  must satisfy  $c' = 4/5 c$ . However, that is impossible because  $4/5 k > 1$ .

## IMO 1976



### Problem B1

Determine the largest number which is the product of positive integers with sum 1976.

### Solution

Answer:  $2 \cdot 3^{658}$ .

There cannot be any integers larger than 4 in the maximal product, because for  $n > 4$ , we can replace  $n$  by 3 and  $n - 3$  to get a larger product. There cannot be any 1s, because there must be an



integer  $r > 1$  (otherwise the product would be 1) and  $r + 1 > 1.r$ . We can also replace any 4s by two 2s leaving the product unchanged. Finally, there cannot be more than two 2s, because we can replace three 2s by two 3s to get a larger product. Thus the product must consist of 3s, and either zero, one or two 2s. The number of 2s is determined by the remainder on dividing the number 1976 by 3.

$1976 = 3 \cdot 658 + 2$ , so there must be just one 2, giving the product  $2 \cdot 3^{658}$ .

## IMO 1976



### Problem B2

$n$  is a positive integer and  $m = 2n$ .  $a_{ij} = 0, 1$  or  $-1$  for  $1 \leq i \leq n, 1 \leq j \leq m$ . The  $m$  unknowns  $x_1, x_2, \dots, x_m$  satisfy the  $n$  equations:

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{im}x_m = 0,$$

for  $i = 1, 2, \dots, n$ . Prove that the system has a solution in integers of absolute value at most  $n$ , not all zero.

### Solution

We use a counting argument. If the modulus of each  $x_i$  is at most  $n$ , then each of the linear combinations has a value between  $-2n^2$  and  $2n^2$ , so there are at most  $(4n^2 + 1)$  possible values for each linear combination and at most  $(2n^2 + 1)^n$  possible sets of values. But there are  $2n+1$  values for each  $x_i$  with modulus at most  $n$ , and hence  $(2n+1)^{2n} = (4n^2+4n+1)^n$  sets of values. So two distinct sets must give the same set of values for the linear combinations. But now if these sets are  $x_i$  and  $x_i'$ , then the values  $x_i - x_i'$  give zero for each linear combination, and have modulus at most  $2n$ . Moreover they are not all zero, since the two sets of values were distinct.

## IMO 1976



### Problem B3

The sequence  $u_0, u_1, u_2, \dots$  is defined by:  $u_0 = 2, u_1 = 5/2, u_{n+1} = u_n(u_{n-1}^2 - 2) - u_1$  for  $n = 1, 2, \dots$ . Prove that  $[u_n] = 2^{(2^n - (-1)^n)/3}$ , where  $[x]$  denotes the greatest integer less than or equal to  $x$ .

### Solution

Experience with recurrence relations suggests that the solution is probably the value given for  $[u_n]$  plus its inverse. It is straightforward to verify this guess by induction.

Squaring  $u_{n-1}$  gives the sum of positive power of 2, its inverse and 2. So  $u_{n-1} - 2 =$  the sum of a positive power of 2 and its inverse. Multiplying this by  $u_n$  gives a positive power of 2 + its inverse + 2 + 1/2, and we can check that the power of 2 is correct for  $u_{n+1}$ .

17th IMO 1975



**A1.** Let  $x_1 \geq x_2 \geq \dots \geq x_n$ , and  $y_1 \geq y_2 \geq \dots \geq y_n$  be real numbers. Prove that if  $z_i$  is any permutation of the  $y_i$ , then:

$$\sum_{1 \leq i \leq n} (x_i - y_i)^2 \leq \sum_{1 \leq i \leq n} (x_i - z_i)^2.$$



**A2.** Let  $a_1 < a_2 < a_3 < \dots$  be positive integers. Prove that for every  $i \geq 1$ , there are infinitely many  $a_n$  that can be written in the form  $a_n = ra_i + sa_j$ , with  $r, s$  positive integers and  $j > i$ .



**A3.** Given any triangle ABC, construct external triangles ABR, BCP, CAQ on the sides, so that  $\angle PBC = 45^\circ$ ,  $\angle PCB = 30^\circ$ ,  $\angle QAC = 45^\circ$ ,  $\angle QCA = 30^\circ$ ,  $\angle RAB = 15^\circ$ ,  $\angle RBA = 15^\circ$ . Prove that  $\angle QRP = 90^\circ$  and  $QR = RP$ .



**B1.** Let A be the sum of the decimal digits of  $4444^{4444}$ , and B be the sum of the decimal digits of A. Find the sum of the decimal digits of B.



**B2.** Find 1975 points on the circumference of a unit circle such that the distance between each pair is rational, or prove it impossible.



**B3.** Find all polynomials  $P(x, y)$  in two variables such that:

- (1)  $P(tx, ty) = t^n P(x, y)$  for some positive integer  $n$  and all real  $t, x, y$ ;
- (2) for all real  $x, y, z$ :  $P(y + z, x) + P(z + x, y) + P(x + y, z) = 0$ ;
- (3)  $P(1, 0) = 1$ .

## IMO 1975



### Problem A1

Let  $x_1 \geq x_2 \geq \dots \geq x_n$ , and  $y_1 \geq y_2 \geq \dots \geq y_n$  be real numbers. Prove that if  $z_i$  is any permutation of the  $y_i$ , then:

$$\sum_{i=1}^n (x_i - y_i)^2 \leq \sum_{i=1}^n (x_i - z_i)^2.$$

### Solution

If  $x \geq x'$  and  $y \geq y'$ , then  $(x - y)^2 + (x' - y')^2 \leq (x - y')^2 + (x' - y)^2$ . Hence if  $i < j$ , but  $z_i \leq z_j$ , then swapping  $z_i$  and  $z_j$  reduces the sum of the squares. But we can return the order of the  $z_i$  to  $y_i$  by a sequence of swaps of this type: first swap 1 to the 1st place, then 2 to the 2nd place and so on.

## IMO 1975



### Problem A2

Let  $a_1 < a_2 < a_3 < \dots$  be positive integers. Prove that for every  $i \geq 1$ , there are infinitely many  $a_n$  that can be written in the form  $a_n = ra_i + sa_j$ , with  $r, s$  positive integers and  $j > i$ .

### Solution

We must be able to find a set  $S$  of infinitely many  $a_n$  in some residue class mod  $a_i$ . Take  $a_j$  to be a member of  $S$ . Then for any  $a_n$  in  $S$  satisfying  $a_n > a_j$ , we have  $a_n = a_j + a$  multiple of  $a_i$ .

## IMO 1975



### Problem A3

Given any triangle  $ABC$ , construct external triangles  $ABR$ ,  $BCP$ ,  $CAQ$  on the sides, so that  $\angle PBC = 45^\circ$ ,  $\angle PCB = 30^\circ$ ,  $\angle QAC = 45^\circ$ ,  $\angle QCA = 30^\circ$ ,  $\angle RAB = 15^\circ$ ,  $\angle RBA = 15^\circ$ . Prove that  $\angle QRP = 90^\circ$  and  $QR = RP$ .

### Solution

Trigonometry provides a routine solution. Let  $BC = a$ ,  $CA = b$ ,  $AB = c$ . Then, by the sine rule applied to  $AQC$ ,  $AQ = b/(2 \sin 105^\circ) = b/(2 \cos 15^\circ)$ . Similarly,  $PB = a/(2 \cos 15^\circ)$ . Also  $AR = RB = c/(2 \cos 15^\circ)$ . So by the cosine rule  $RP^2 = (a^2 + c^2 - 2ac \cos(B+60^\circ))/(4 \cos^2 15^\circ)$ , and  $RQ^2 = (b^2 + c^2 - 2bc \cos(A+60^\circ))/(4 \cos^2 15^\circ)$ . So  $RP = RQ$  is equivalent to:  $a^2 - 2ac \cos(60^\circ+B) = b^2 - 2bc \cos(60^\circ+A)$  and hence to  $a^2 - ac \cos B + \sqrt{3} ac \sin B = b^2 - bc \cos A + \sqrt{3} bc \sin A$ . By the sine rule, the sine terms cancel. Also  $b - b \cos A = a \cos C$ , and  $a - c \cos B = b \cos C$ , so the last

equality is true and hence  $RP = RQ$ . We get an exactly similar expression for  $PQ^2$  and show that it equals  $2 RP^2$  in the same way.

A more elegant solution is to construct  $S$  on the outside of  $AB$  so that  $ABS$  is equilateral. Then we find that  $CAS$  and  $QAR$  are similar and that  $CBS$  and  $PBR$  are similar. So  $QR/CS = PR/CS$ . The ratio of the sides is the same in each case ( $CA/QA = CB/PB$  since  $CQA$  and  $CPB$  are similar), so  $QR = PR$ . Also there is a  $45^\circ$  rotation between  $QAR$  and  $CAS$  and another  $45^\circ$  rotation between  $CBS$  and  $PBR$ , hence  $QR$  and  $PR$  are at  $90^\circ$ .

## IMO 1975



### Problem B1

Let  $A$  be the sum of the decimal digits of  $4444^{4444}$ , and  $B$  be the sum of the decimal digits of  $A$ . Find the sum of the decimal digits of  $B$ .

### Solution

Let  $X = 4444^{4444}$ . Then  $X$  has less than  $4.4444 = 17776$  digits, so  $A$  is at most  $9.17776 = 159984$ . Hence  $B$  is at most  $6.9 = 54$ . But all these numbers are congruent mod 9.  $4444 \equiv -2 \pmod{9}$ , so  $X \equiv (-2)^{4444} \pmod{9}$ . But  $(-2)^3 \equiv 1 \pmod{9}$ , and  $4444 \equiv 1 \pmod{3}$ , so  $X \equiv -2 \equiv 7 \pmod{9}$ . But any number less than 55 and congruent to 7 has digit sum 7 (possibilities are 7, 16, 25, 34, 43, 52). Hence the answer is 7.

## IMO 1975



### Problem B2

Find 1975 points on the circumference of a unit circle such that the distance between each pair is rational, or prove it impossible.

### Solution

Let  $x$  be the angle  $\cos^{-1}4/5$ , so that  $\cos x = 4/5$ ,  $\sin x = 3/5$ . Take points on the unit circle at angles  $2nx$  for  $n$  integral. Then the distance between the points at angles  $2nx$  and  $2mx$  is  $2 \sin(n - m)x$ . The usual formula, giving  $\sin(n - m)x$  in terms of  $\sin x$  and  $\cos x$ , shows that  $\sin(n - m)x$  is rational. So it only remains to show that this process generates arbitrarily many distinct points, in other words that  $x$  is not a rational multiple of  $\pi$ .

This is quite hard. There is an elegant argument in sections 5 and 8 of Hadwiger et al, *Combinatorial geometry in the Plane*. But we can avoid it by observing that there are only finitely many numbers which are  $n$ th roots of unity for  $n \leq 2 \times 1975$ , whereas there are infinitely many Pythagorean triples, so we simply pick a triple which is not such a root of unity.



### Problem B3

Find all polynomials  $P(x, y)$  in two variables such that:

(1)  $P(tx, ty) = t^n P(x, y)$  for some positive integer  $n$  and all real  $t, x, y$ ;

(2) for all real  $x, y, z$ :  $P(y + z, x) + P(z + x, y) + P(x + y, z) = 0$ ;

(3)  $P(1, 0) = 1$ .

### Solution

(1) means that  $P$  is homogeneous of degree  $n$  for some  $n$ . Experimenting with low  $n$ , shows that the only solutions for  $n = 1, 2, 3$  are  $(x - 2y)$ ,  $(x + y)(x - 2y)$ ,  $(x + y)^2(x - 2y)$ . It then obvious by inspection that  $(x + y)^n(x - 2y)$  is a solution for any  $n$ . Taking  $x = y = z$  in (2) shows that  $P(2x, x) = 0$ , so  $(x - 2y)$  is always a factor. Taking  $x = y = 1, z = -2$  gives  $P(1, -1)(2^n - 2) = 0$ , so  $(x + y)$  is a factor for  $n > 1$ . All this suggests (but does not prove) that the general solution is  $(x + y)^n(x - 2y)$ .

Take  $y = 1 - x, z = 0$  in (2) and we get:  $P(x, 1-x) = -1 - P(1-x, x)$ . In particular,  $P(0, 1) = -2$ . Now take  $z = 1 - x - y$  and we get:  $P(1-x, x) + P(1-y, y) + P(x+y, 1-x-y) = 0$  and hence  $f(x+y) = f(x) + f(y)$ , where  $f(x) = P(1-x, x) - 1$ . By induction we conclude that, for any integer  $m$  and real  $x$ ,  $f(mx) = mf(x)$ . Hence  $f(1/s) = 1/s f(1)$  and  $f(r/s) = r/s f(1)$  for any integers  $r, s$ . But  $P(0, 1) = -2$ , so  $f(1) = -3$ . So  $f(x) = -3x$  for all rational  $x$ . But  $f$  is continuous, so  $f(x) = -3x$  for all  $x$ . So set  $x = b/(a+b)$ , where  $a$  and  $b$  are arbitrary reals (with  $a+b$  non-zero). Then  $P(a, b) = (a+b)^n P(1-x, x) = (a+b)^n (-3b/(a+b) + 1) = (a+b)^{n-1} (a-2b)$ , as claimed. [For  $a+b = 0$ , we appeal to continuity, or use the already derived fact that for  $n > 1$ ,  $P(a, b) = 0$ .]



**A1.** Three players play the following game. There are three cards each with a different positive integer. In each round the cards are randomly dealt to the players and each receives the number of counters on his card. After two or more rounds, one player has received 20, another 10 and the third 9 counters. In the last round the player with 10 received the largest number of counters. Who received the middle number on the first round?



**A2.** Prove that there is a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB if and only if  $\sin A \sin B \leq \sin^2(C/2)$ .



**A3.** Prove that the sum from  $k = 0$  to  $n$  of  $(2n+1)C(2k+1) 2^{3k}$  is not divisible by 5 for any non-negative integer  $n$ . [ $rCs$  denotes the binomial coefficient  $r!/(s!(r-s)!) .$ ]



**B1.** An  $8 \times 8$  chessboard is divided into  $p$  disjoint rectangles (along the lines between squares), so that each rectangle has the same number of white squares as black squares, and each rectangle has a different number of squares. Find the maximum possible value of  $p$  and all possible sets of rectangle sizes.



**B2.** Determine all possible values of  $a/(a+b+d) + b/(a+b+c) + c/(b+c+d) + d/(a+c+d)$  for positive reals  $a, b, c, d$ .



**B3.** Let  $P(x)$  be a polynomial with integer coefficients of degree  $d > 0$ . Let  $n$  be the number of distinct integer roots to  $P(x) = 1$  or  $-1$ . Prove that  $n \leq d + 2$ .

## IMO 1974



### Problem A1

Three players play the following game. There are three cards each with a different positive integer. In each round the cards are randomly dealt to the players and each receives the number of counters on his card. After two or more rounds, one player has received 20, another 10 and the third 9 counters. In the last round the player with 10 received the largest number of counters. Who received the middle number on the first round?

### Solution

The player with 9 counters.

The total of the scores, 39, must equal the number of rounds times the total of the cards. But 39 has no factors except 1, 3, 13 and 39, the total of the cards must be at least  $1 + 2 + 3 = 6$ , and the number of rounds is at least 2. Hence there were 3 rounds and the cards total 13.

The highest score was 20, so the highest card is at least 7. The score of 10 included at least one highest card, so the highest card is at most 8. The lowest card is at most 2, because if it was higher then the highest card would be at most  $13 - 3 - 4 = 6$ , whereas we know it is at least 7. Thus the possibilities for the cards are: 2, 3, 8; 2, 4, 7; 1, 4, 8; 1, 5, 7. But the only one of these that allows a score of 20 is 1, 4, 8. Thus the scores were made up:  $8 + 8 + 4 = 20$ ,  $8 + 1 + 1 = 10$ ,  $4 + 4 + 1 = 9$ . The last round must have been 4 to the player with 20, 8 to the player with 10 and 1 to the player with 9. Hence on each of the other two rounds the cards must have been 8 to the player with 20, 1 to the player with 10 and 4 to the player with 9.

## IMO 1974



### Problem A2

Prove that there is a point D on the side AB of the triangle ABC, such that CD is the geometric mean of AD and DB if and only if  $\sin A \sin B \leq \sin^2(C/2)$

### Solution

Extend CD to meet the circumcircle of ABC at E. Then  $CD \cdot DE = AD \cdot DB$ , so CD is the geometric mean of AD and DB iff  $CD = DE$ . So we can find such a point iff the distance of C from AB is less than the distance of AB from the furthest point of the arc AB on the opposite side of AB to C. The furthest point F is evidently the midpoint of the arc AB. F lies on the angle bisector of C. So  $\angle FAB = \angle FAC = \angle C/2$ . Hence distance of F from AB is  $c/2 \tan C/2$  (as usual we set  $c = AB$ ,  $b = CA$ ,  $a = BC$ ). The distance of C from AB is  $a \sin B$ . So a necessary and sufficient condition is  $c/2 \tan C/2 \geq a \sin B$ . But by the sine rule,  $a = c \sin A / \sin C$ , so the condition becomes  $(\sin C/2 \sin C) / (2 \cos C/2) \geq \sin A \sin B$ . But  $\sin C = 2 \sin C/2 \cos C/2$ , so we obtain the condition quoted in the question.

IMO 1974



**Problem A3**

Prove that the sum from  $k = 0$  to  $n$  of  $(2n+1)C(2k+1) 2^{3k}$  is not divisible by 5 for any non-negative integer  $n$ . [ $rC_s$  denotes the binomial coefficient  $r!/(s!(r-s)!) .$ ]

**Solution**

Let  $k = \sqrt{8}$ . Then  $(1 + k)^{2n+1} = a + bk$ , where  $b$  is the sum given in the question. Similarly,  $(1 - k)^{2n+1} = a - bk$ . This looks like a dead end, because eliminating  $a$  gives an unhelpful expression for  $b$ . The trick is to multiply the two expressions to get  $7^{2n+1} = 8b^2 - a^2$ . This still looks unhelpful, but happens to work, because we soon find that  $7^{2n+1} \not\equiv \pm 2 \pmod{5}$ . So if  $b$  was a multiple of 5 then we would have a square congruent to  $\pm 2 \pmod{5}$  which is impossible.

IMO 1974



**Problem B1**

An  $8 \times 8$  chessboard is divided into  $p$  disjoint rectangles (along the lines between squares), so that each rectangle has the same number of white squares as black squares, and each rectangle has a different number of squares. Find the maximum possible value of  $p$  and all possible sets of rectangle sizes.

**Solution**

The requirement that the number of black and white squares be equal is equivalent to requiring that the each rectangle has an even number of squares.  $2 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 72 > 64$ , so  $p < 8$ . There are 5 possible divisions of 64 into 7 unequal even numbers:  $2 + 4 + 6 + 8 + 10 + 12 + 22$ ;  $2 + 4 + 6 + 8 + 10 + 16 + 18$ ;  $2 + 4 + 6 + 8 + 12 + 14 + 18$ ;  $2 + 4 + 6 + 10 + 12 + 14 + 16$ . The first is ruled out because a rectangle with 22 squares would have more than 8 squares on its longest side. The others are all possible.

- 2 2 2 2 2 2 2 4      2 2 2 2 2 2 2 2
- 2 2 2 2 2 2 2 4      2 2 2 2 2 2 2 2
- 1 1 1 1 1 5 5 4      1 1 1 1 1 1 5 5
- 1 1 1 1 1 5 5 4      1 1 1 1 1 1 5 5
- 1 1 1 1 1 5 5 4      1 1 1 1 1 1 5 5



|                 |                 |
|-----------------|-----------------|
| 1 1 1 1 1 6 6 4 | 3 3 3 3 3 7 6 6 |
| 3 3 3 3 3 6 6 4 | 3 3 3 3 3 7 6 6 |
| 3 3 3 3 3 7 7 4 | 4 4 4 4 4 4 4 4 |
| 2 2 2 2 2 2 2 7 | 1 1 1 1 1 1 1 1 |
| 2 2 2 2 2 2 2 7 | 1 1 1 1 1 1 1 1 |
| 1 1 1 1 1 1 4 4 | 2 2 2 2 2 2 2 7 |
| 1 1 1 1 1 1 4 4 | 2 2 2 2 2 2 2 7 |
| 1 1 1 1 1 1 4 4 | 3 3 3 3 3 3 6 6 |
| 3 3 3 3 3 3 4 4 | 3 3 3 3 3 3 6 6 |
| 3 3 3 3 3 3 6 6 | 4 4 4 4 4 5 5 5 |
| 5 5 5 5 5 5 6 6 | 4 4 4 4 4 5 5 5 |

## IMO 1974



### Problem B2

Determine all possible values of  $\frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$  for positive reals  $a, b, c, d$ .

### Solution

We show first that the sum must lie between 1 and 2. If we replace each denominator by  $a+b+c+d$  then we reduce each term and get 1. Hence the sum is more than 1. Suppose  $a$  is the largest of the four reals. Then the first term is less than 1. The second and fourth terms have denominators greater than  $b+c+d$ , so the terms are increased if we replace the denominators by  $b+c+d$ . But then the last three terms sum to 1. Thus the sum of the last three terms is less than 1. Hence the sum is less than 2.

If we set  $a = c = 1$  and make  $b$  and  $d$  small, then the first and third terms can be made arbitrarily close to 1 and the other two terms arbitrarily close to 0, so we can make the sum arbitrarily close to 2. If we set  $a = 1, c = d$  and make  $b$  and  $c/b$  arbitrarily small, then the first term is arbitrarily close to 1 and the last three terms are all arbitrarily small, so we can make the sum arbitrarily close to 1. Hence, by continuity, we can achieve any value in the open interval  $(1,2)$ .

**Problem B3**

Let  $P(x)$  be a polynomial with integer coefficients of degree  $d > 0$ . Let  $n$  be the number of distinct integer roots to  $P(x) = 1$  or  $-1$ . Prove that  $n \leq d + 2$ .

**Solution**

Suppose that  $A(x)$  and  $B(x)$  are two polynomials with integer coefficients which are identical except for their constant terms, which differ by 2. Suppose  $A(r) = 0$ , and  $B(s) = 0$  with  $r$  and  $s$  integers. Then subtracting we get 2 plus a sum of terms  $a(r^i - s^i)$ . Each of these terms is divisible by  $(r - s)$ , so 2 must be divisible by  $(r - s)$ . Hence  $r$  and  $s$  differ by 0, 1 or 2.

Now let  $r$  be the smallest root of  $P(x) = 1$  and  $P(x) = -1$ . The polynomial with  $r$  as a root can have at most  $d$  distinct roots and hence at most  $d$  distinct integer roots. If  $s$  is a root of the other equation then  $s$  must differ from  $r$  by 0, 1, or 2. But  $s \geq r$ , so  $s = r, r+1$  or  $r+2$ . Hence the other equation adds at most 2 distinct integer roots.

15th IMO 1973



**A1.**  $OP_1, OP_2, \dots, OP_{2n+1}$  are unit vectors in a plane.  $P_1, P_2, \dots, P_{2n+1}$  all lie on the same side of a line through  $O$ . Prove that  $|OP_1 + \dots + OP_{2n+1}| \geq 1$ .



**A2.** Can we find a finite set of non-coplanar points, such that given any two points,  $A$  and  $B$ , there are two others,  $C$  and  $D$ , with the lines  $AB$  and  $CD$  parallel and distinct?



**A3.**  $a$  and  $b$  are real numbers for which the equation  $x^4 + ax^3 + bx^2 + ax + 1 = 0$  has at least one real solution. Find the least possible value of  $a^2 + b^2$ .



**B1.** A soldier needs to sweep a region with the shape of an equilateral triangle for mines. The detector has an effective radius equal to half the altitude of the triangle. He starts at a vertex of the triangle. What path should he follow in order to travel the least distance and still sweep the whole region?



**B2.**  $G$  is a set of non-constant functions  $f$ . Each  $f$  is defined on the real line and has the form  $f(x) = ax + b$  for some real  $a, b$ . If  $f$  and  $g$  are in  $G$ , then so is  $fg$ , where  $fg$  is defined by  $fg(x) = f(g(x))$ . If  $f$  is in  $G$ , then so is the inverse  $f^{-1}$ . If  $f(x) = ax + b$ , then  $f^{-1}(x) = x/a - b/a$ . Every  $f$  in  $G$  has a fixed point (in other words we can find  $x_f$  such that  $f(x_f) = x_f$ ). Prove that all the functions in  $G$  have a common fixed point.



**B3.**  $a_1, a_2, \dots, a_n$  are positive reals, and  $q$  satisfies  $0 < q < 1$ . Find  $b_1, b_2, \dots, b_n$  such that:

(a)  $a_i < b_i$  for  $i = 1, 2, \dots, n$ ,

(b)  $q < b_{i+1}/b_i < 1/q$  for  $i = 1, 2, \dots, n-1$ ,

(c)  $b_1 + b_2 + \dots + b_n < (a_1 + a_2 + \dots + a_n)(1 + q)/(1 - q)$ .

## IMO 1973



### Problem A1

$OP_1, OP_2, \dots, OP_{2n+1}$  are unit vectors in a plane.  $P_1, P_2, \dots, P_{2n+1}$  all lie on the same side of a line through  $O$ . Prove that  $|OP_1 + \dots + OP_{2n+1}| \geq 1$ .

### Solution

We proceed by induction on  $n$ . It is clearly true for  $n = 1$ . Assume it is true for  $2n-1$ . Given  $OP_i$  for  $2n+1$ , reorder them so that all  $OP_i$  lie between  $OP_{2n}$  and  $OP_{2n+1}$ . Then  $u = OP_{2n} + OP_{2n+1}$  lies along the angle bisector of angle  $P_{2n}OP_{2n+1}$  and hence makes an angle less than  $90^\circ$  with  $v = OP_1 + OP_2 + \dots + OP_{2n-1}$  (which must lie between  $OP_1$  and  $OP_{2n-1}$  and hence between  $OP_{2n}$  and  $OP_{2n+1}$ ). By induction  $|v| \geq 1$ . But  $|u + v| \geq |v|$  (use the cosine formula). Hence the result is true for  $2n+1$ .

It is clearly best possible: take  $OP_1 = \dots = OP_n = -OP_{n+1} = \dots = -OP_{2n}$ , and  $OP_{2n+1}$  in an arbitrary direction

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### Problem A2

Can we find a finite set of non-coplanar points, such that given any two points,  $A$  and  $B$ , there are two others,  $C$  and  $D$ , with the lines  $AB$  and  $CD$  parallel and distinct?

### Solution

To warm up, we may notice that a regular hexagon is a planar set satisfying the condition.

Take two regular hexagons with a common long diagonal and their planes perpendicular. Now if we take  $A, B$  in the same hexagon, then we can find  $C, D$  in the same hexagon. If we take  $A$  in one and  $B$  in the other, then we may take  $C$  at the opposite end of a long diagonal from  $A$ , and  $D$  at the opposite end of a long diagonal from  $B$ .

## IMO 1973



### Problem A3

**a and b are real numbers for which the equation  $x^4 + ax^3 + bx^2 + ax + 1 = 0$  has at least one real solution. Find the least possible value of  $a^2 + b^2$ .**

#### Solution

Put  $y = x + 1/x$  and the equation becomes  $y^2 + ay + b - 2 = 0$ , which has solutions  $y = -a/2 \pm \sqrt{(a^2 + 8 - 2b)/2}$ . We require  $|y| \geq 2$  for the original equation to have a real root and hence we need  $|a| + \sqrt{(a^2 + 8 - 4b)} \geq 4$ . Squaring gives  $2|a| - b \geq 2$ . Hence  $a^2 + b^2 \geq a^2 + (2 - 2|a|)^2 = 5a^2 - 8|a| + 4 = 5(|a| - 4/5)^2 + 4/5$ . So the least possible value of  $a^2 + b^2$  is  $4/5$ , achieved when  $a = 4/5$ ,  $b = -2/5$ . In this case, the original equation is  $x^4 + 4/5 x^3 - 2/5 x^2 + 4/5 x + 1 = (x + 1)^2(x^2 - 6/5 x + 1)$ .

## IMO 1973



### Problem B1

**A soldier needs to sweep a region with the shape of an equilateral triangle for mines. The detector has an effective radius equal to half the altitude of the triangle. He starts at a vertex of the triangle. What path should he follow in order to travel the least distance and still sweep the whole region?**

#### Solution

In particular he must sweep the other two vertices. Let us take the triangle to be ABC, with side 1 and assume the soldier starts at A. So the path must intersect the circles radius  $\sqrt{3}/4$  centered on the other two vertices. Let us look for the shortest path of this type. Suppose it intersects the circle center B at X and the circle center C at Y, and goes first to X and then to Y. Clearly the path from A to X must be a straight line and the path from X to Y must be a straight line. Moreover the shortest path from X to the circle center C follows the line XC and has length  $AX + XC - \sqrt{3}/4$ . So we are looking for the point X which minimises  $AX + XC$ .

Consider the point P where the altitude intersects the circle. By the usual reflection argument the distance  $AP + PC$  is shorter than the distance  $AP' + P'C$  for any other point P' on the line perpendicular to the altitude through P. Moreover for any point X on the circle, take AX to cut the line at P'. Then  $AX + XC > AP' + P'C > AP + PC$ .

It remains to check that the three circles center A, X, Y cover the triangle. In fact the circle center X covers the whole triangle except for a small portion near A and a small portion near C, which are covered by the triangles center A and Y.



### Problem B2

**G** is a set of non-constant functions  $f$ . Each  $f$  is defined on the real line and has the form  $f(x) = ax + b$  for some real  $a, b$ . If  $f$  and  $g$  are in  $G$ , then so is  $fg$ , where  $fg$  is defined by  $fg(x) = f(g(x))$ . If  $f$  is in  $G$ , then so is the inverse  $f^{-1}$ . If  $f(x) = ax + b$ , then  $f^{-1}(x) = x/a - b/a$ . Every  $f$  in  $G$  has a fixed point (in other words we can find  $x_f$  such that  $f(x_f) = x_f$ ). Prove that all the functions in  $G$  have a common fixed point.

### Solution

$f(x) = ax + b$  has fixed point  $b/(1-a)$ . If  $a = 1$ , then  $b$  must be 0, and any point is a fixed point. So suppose  $f(x) = ax + b$  and  $g(x) = cx + d$  are in  $G$ . Then  $h$  the inverse of  $f$  is given by  $h(x) = x/a - b/a$ , and  $hg(x) = x + b'/a - b/a$ . This is in  $G$ , so we must have  $b' = b$ .

Suppose  $f(x) = ax + b$ , and  $g(x) = cx + d$  are in  $G$ . Then  $fg(x) = acx + (ad + b)$ , and  $gf(x) = acx + (bc + d)$ . We must have  $ad + b = bc + d$  and hence  $b/(1-a) = d/(1-c)$ , in other words  $f$  and  $g$  have the same fixed point.

### IMO 1973



### Problem B3

$a_1, a_2, \dots, a_n$  are positive reals, and  $q$  satisfies  $0 < q < 1$ . Find  $b_1, b_2, \dots, b_n$  such that:

- (a)  $a_i < b_i$  for  $i = 1, 2, \dots, n$ ,
- (b)  $q < b_{i+1}/b_i < 1/q$  for  $i = 1, 2, \dots, n-1$ ,
- (c)  $b_1 + b_2 + \dots + b_n < (a_1 + a_2 + \dots + a_n)(1 + q)/(1 - q)$ .

### Solution

We notice that the constraints are linear, in the sense that if  $b_i$  is a solution for  $a_i, q$ , and  $b'_i$  is a solution for  $a'_i, q$ , then for any  $k, k' > 0$  a solution for  $ka_i + k'a'_i, q$  is  $kb_i + k'b'_i$ . Also a "near" solution for  $a_h = 1$ , other  $a_i = 0$  is  $b_1 = q^{h-1}, b_2 = q^{h-2}, \dots, b_{h-1} = q, b_h = 1, b_{h+1} = q, \dots, b_n = q^{n-h}$ . "Near" because the inequalities in (a) and (b) are not strict.

However, we might reasonably hope that the inequalities would become strict in the linear combination, and indeed that is true. Define  $b_r = q^{r-1}a_1 + q^{r-2}a_2 + \dots + qa_{r-1} + a_r + qa_{r+1} + \dots + q^{n-r}a_n$ . Then we may easily verify that (a) - (c) hold.

14th IMO 1972



**A1.** Given any set of ten distinct numbers in the range 10, 11, ..., 99, prove that we can always find two disjoint subsets with the same sum.



**A2.** Given  $n > 4$ , prove that every cyclic quadrilateral can be dissected into  $n$  cyclic quadrilaterals.



**A3.** Prove that  $(2m)!(2n)!$  is a multiple of  $m!n!(m+n)!$  for any non-negative integers  $m$  and  $n$ .



**B1.** Find all positive real solutions to:

$$\begin{aligned}(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 \\(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 \\(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 \\(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 \\(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0\end{aligned}$$



**B2.**  $f$  and  $g$  are real-valued functions defined on the real line. For all  $x$  and  $y$ ,  $f(x + y) + f(x - y) = 2f(x)g(y)$ .  $f$  is not identically zero and  $|f(x)| \leq 1$  for all  $x$ . Prove that  $|g(x)| \leq 1$  for all  $x$ .



**B3.** Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

## IMO 1972



### Problem A1

**Given any set of ten distinct numbers in the range 10, 11, ..., 99, prove that we can always find two disjoint subsets with the same sum.**

### Solution

The number of non-empty subsets is  $2^{10} - 1 = 1023$ . The sum of each subset is at most  $90 + \dots + 99 = 945$ , so there must be two distinct subsets  $A$  and  $B$  with the same sum.  $A \setminus B$  and  $B \setminus A$  are disjoint subsets, also with the same sum.

## IMO 1972



### Problem A2

**Given  $n > 4$ , prove that every cyclic quadrilateral can be dissected into  $n$  cyclic quadrilaterals.**

### Solution

A little tinkering soon shows that it is easy to divide a cyclic quadrilateral  $ABCD$  into 4 cyclic quadrilaterals. Take a point  $P$  inside the quadrilateral and take an arbitrary line  $PK$  joining it to  $AB$ . Now take  $L$  on  $BC$  so that  $\angle KPL = 180^\circ - \angle B$  (thus ensuring that  $KPLB$  is cyclic), then  $M$  on  $CD$  so that  $\angle LPM = 180^\circ - \angle C$ , then  $N$  on  $AD$  so that  $\angle MPN = 180^\circ - \angle D$ . Then  $\angle NPK = 180^\circ - \angle A$ . We may need to impose some restrictions on  $P$  and  $K$  to ensure that we can obtain the necessary angles. It is not clear, however, what to do next.

The trick is to notice that the problem is easy if two sides are parallel. For then we may take arbitrarily many lines parallel to the parallel sides and divide the original quadrilateral into any number of parts.

So we need to arrange our choice of  $P$  and  $K$  so that one of the new quadrilaterals has parallel sides. But that is easy, since  $K$  is arbitrary. So take  $PK$  parallel to  $AD$ , then we must also have  $PL$  parallel to  $CD$ .

It remains to consider how we ensure that the points lie on the correct sides. Consider first  $K$  and  $L$ .  $K$  cannot lie on  $AD$  since  $PK$  is parallel to  $AD$ , and we can avoid it lying on  $BC$  by taking  $P$  sufficiently close to  $D$ . Similarly, taking  $P$  sufficiently close to  $D$  ensures that  $L$  lies on  $BC$ . Now suppose that  $M$  and  $N$  are both on  $AD$ . Then if we keep  $K$  fixed and move  $P$  closer to  $CD$ ,  $N$  will move on to  $CD$ , leaving  $M$  on  $AD$ .



## IMO 1972



### Problem A3

Prove that  $(2m)!(2n)!$  is a multiple of  $m!n!(m+n)!$  for any non-negative integers  $m$  and  $n$ .

### Solution

The trick is to find a recurrence relation for  $f(m,n) = (2m)!(2n)!/(m!n!(m+n)!)$ . In fact,  $f(m,n) = 4f(m,n-1) - f(m+1,n-1)$ , which is sufficient to generate all the  $f(m,n)$ , given that  $f(m,0) = (2m)!/(m!m!)$ , which we know to be integral.

## IMO 1972



### Problem B1

Find all positive real solutions to:

$$\begin{aligned}(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 \\(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 \\(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 \\(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 \\(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0\end{aligned}$$

### Solution

Answer:  $x_1 = x_2 = x_3 = x_4 = x_5$ .

The difficulty with this problem is that it has more information than we need. There is a neat solution in Greitzer which shows that all we need is the sum of the 5 inequalities, because one can rewrite that as  $(x_1x_2 - x_1x_4)^2 + (x_2x_3 - x_2x_5)^2 + \dots + (x_5x_1 - x_5x_3)^2 + (x_1x_3 - x_1x_5)^2 + \dots + (x_5x_2 - x_5x_4)^2 \leq 0$ . The difficulty is how one ever dreams up such an idea!

The more plodding solution is to break the symmetry by taking  $x_1$  as the largest. If the second largest is  $x_2$ , then the first inequality tells us that  $x_1^2$  or  $x_2^2 = x_3x_5$ . But if  $x_3$  and  $x_5$  are unequal, then the larger would exceed  $x_1$  or  $x_2$ . Contradiction. Hence  $x_3 = x_5$  and also equals  $x_2$  or  $x_1$ . If they equal  $x_1$ , then they would also equal  $x_2$  (by definition of  $x_2$ ), so in any case they must equal  $x_2$ . Now the second inequality gives  $x_2 = x_1x_4$ . So either all the numbers are equal, or  $x_1 > x_2 = x_3 = x_5 > x_4$ . But in the second case the last inequality is violated. So the only solution is all numbers equal.

If the second largest is  $x_5$ , then we can use the last inequality to deduce that  $x_2 = x_4 = x_5$  and proceed as before.

If the second largest is  $x_3$ , then the fourth inequality gives that  $x_1 = x_3 = x_5$  or  $x_1 = x_3 = x_4$ . In the first case,  $x_5$  is the second largest and we are home already. In the second case, the third

inequality gives  $x_3^2 = x_2x_5$  and hence  $x_3 = x_2 = x_5$  (or one of  $x_2, x_5$  would be larger than the second largest). So  $x_5$  is the second largest and we are home.

Finally, if the second largest is  $x_4$ , then the second inequality gives  $x_1 = x_2 = x_4$  or  $x_1 = x_3 = x_4$ . Either way, we have a case already covered and so the numbers are

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### Problem B2

**$f$  and  $g$  are real-valued functions defined on the real line. For all  $x$  and  $y$ ,  $f(x + y) + f(x - y) = 2f(x)g(y)$ .  $f$  is not identically zero and  $|f(x)| \leq 1$  for all  $x$ . Prove that  $|g(x)| \leq 1$  for all  $x$ .**

#### Solution

Let  $k$  be the least upper bound for  $|f(x)|$ . Suppose  $|g(y)| > 1$ . Take any  $x$  with  $|f(x)| > 0$ , then  $2k \geq |f(x+y)| + |f(x-y)| \geq |f(x+y) + f(x-y)| = 2|g(y)||f(x)|$ , so  $|f(x)| < k/|g(y)|$ . In other words,  $k/|g(y)|$  is an upper bound for  $|f(x)|$  which is less than  $k$ . Contradiction.

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### Problem B3

**Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.**

#### Solution

Intuitively, we can place  $A$  and  $B$  on the two outer planes with  $AB$  perpendicular to the planes. Then tilt  $AB$  in one direction until we bring  $C$  onto one of the middle planes (keeping  $A$  and  $B$  on the outer planes), then tilt  $AB$  the other way (keeping  $A, B, C$  on their respective planes) until  $D$  gets onto the last plane.

Take  $A$  as the origin. Let the vectors  $AB, AC, AD$  be  $\mathbf{b}, \mathbf{c}, \mathbf{d}$ . Take  $p$  as one of the outer planes. Let the distances to the other planes be  $e, f, g$ . Now we find a vector  $\mathbf{n}$  satisfying:  $\mathbf{n} \cdot \mathbf{b} = e$ ,  $\mathbf{n} \cdot \mathbf{c} = f$ ,  $\mathbf{n} \cdot \mathbf{d} = g$ . This is a system of three equations in three unknowns with non-zero determinant (because  $\mathbf{b} \cdot \mathbf{c} \times \mathbf{d}$  is non-zero), so it has a solution  $\mathbf{n}$ . Scale the tetrahedron by  $|\mathbf{n}|$ , orient  $p$  perpendicular to  $\mathbf{n}/|\mathbf{n}|$ , then  $B, C, D$  will be on the other planes as required.

## 13th IMO 1971



**A1.** Let  $E_n = (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})$ . Let  $S_n$  be the proposition that  $E_n \geq 0$  for all real  $a_i$ .



Prove that  $S_n$  is true for  $n = 3$  and  $5$ , but for no other  $n > 2$ .

**A2.** Let  $P_1$  be a convex polyhedron with vertices  $A_1, A_2, \dots, A_9$ . Let  $P_i$  be the polyhedron obtained from  $P_1$  by a translation that moves  $A_1$  to  $A_i$ . Prove that at least two of the polyhedra  $P_1, P_2, \dots, P_9$  have an interior point in common.



**A3.** Prove that we can find an infinite set of positive integers of the form  $2^n - 3$  (where  $n$  is a positive integer) every pair of which are relatively prime.



**B1.** All faces of the tetrahedron  $ABCD$  are acute-angled. Take a point  $X$  in the interior of the segment  $AB$ , and similarly  $Y$  in  $BC$ ,  $Z$  in  $CD$  and  $T$  in  $AD$ .

(a) If  $\text{angle } DAB + \text{angle } BCD \neq \text{angle } CDA + \text{angle } ABC$ , then prove that none of the closed paths  $XYZTX$  has minimal length;



(b) If  $\text{angle } DAB + \text{angle } BCD = \text{angle } CDA + \text{angle } ABC$ , then there are infinitely many shortest paths  $XYZTX$ , each with length  $2 AC \sin k$ , where  $2k = \text{angle } BAC + \text{angle } CAD + \text{angle } DAB$ .

**B2.** Prove that for every positive integer  $m$  we can find a finite set  $S$  of points in the plane, such that given any point  $A$  of  $S$ , there are exactly  $m$  points in  $S$  at unit distance from  $A$ .



**B3.** Let  $A = (a_{ij})$ , where  $i, j = 1, 2, \dots, n$ , be a square matrix with all  $a_{ij}$  non-negative integers. For each  $i, j$  such that  $a_{ij} = 0$ , the sum of the elements in the  $i$ th row and the  $j$ th column is at least  $n$ . Prove that the sum of all the elements in the matrix is at least  $n^2/2$ .

**Problem A1**

Let  $E_n = (a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n) + \dots + (a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})$ . Let  $S_n$  be the proposition that  $E_n \geq 0$  for all real  $a_i$ .

Prove that  $S_n$  is true for  $n = 3$  and  $5$ , but for no other  $n > 2$ .

**Solution**

Take  $a_1 < 0$ , and the remaining  $a_i = 0$ . Then  $E_n = a_1^{n-1} < 0$  for  $n$  even, so the proposition is false for even  $n$ .

Suppose  $n \geq 7$  and odd. Take any  $c > a > b$ , and let  $a_1 = a$ ,  $a_2 = a_3 = a_4 = b$ , and  $a_5 = a_6 = \dots = a_n = c$ . Then  $E_n = (a - b)^3(a - c)^{n-4} < 0$ . So the proposition is false for odd  $n \geq 7$ .

Assume  $a_1 \geq a_2 \geq a_3$ . Then in  $E_3$  the sum of the first two terms is non-negative, because  $(a_1 - a_3) \geq (a_2 - a_3)$ . The last term is also non-negative. Hence  $E_3 \geq 0$ , and the proposition is true for  $n = 3$ .

It remains to prove  $S_5$ . Suppose  $a_1 \geq a_2 \geq a_3 \geq a_4 \geq a_5$ . Then the sum of the first two terms in  $E_5$  is  $(a_1 - a_2)\{(a_1 - a_3)(a_1 - a_4)(a_1 - a_5) - (a_2 - a_3)(a_2 - a_4)(a_2 - a_5)\} \geq 0$ . The third term is non-negative (the first two factors are non-positive and the last two non-negative). The sum of the last two terms is:  $(a_4 - a_5)\{(a_1 - a_5)(a_2 - a_5)(a_3 - a_5) - (a_1 - a_4)(a_2 - a_4)(a_3 - a_4)\} \geq 0$ . Hence  $E_5 \geq 0$ .

**Problem A2**

Let  $P_1$  be a convex polyhedron with vertices  $A_1, A_2, \dots, A_9$ . Let  $P_i$  be the polyhedron obtained from  $P_1$  by a translation that moves  $A_1$  to  $A_i$ . Prove that at least two of the polyhedra  $P_1, P_2, \dots, P_9$  have an interior point in common.

**Solution**

The result is false for 8 vertices - for example, the cube. We get 8 cubes, with only faces in common, forming a cube 8 times as large.

This suggests a trick. Each  $P_i$  is contained in  $D$ , the polyhedron formed from  $P_1$  by doubling the scale. Take  $A_1$  as the origin and take the vertex  $B_1$  to have twice the coordinates of  $A_1$ . Given a point  $X$  inside  $P_1$ , the midpoint of  $P_1X$  must lie in  $P_1$  by convexity. Hence the point with doubled coordinates, which is obtained by adding the coordinates of  $A_1$  to the coordinates of  $X$ , lies in  $D$ . In other words every point of  $P_i$  lies in  $D$ . But the volume of  $D$  is 8 times the volume of  $P_1$ , which is less than the sum of the volumes of  $P_1, \dots, P_9$ .

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### Problem A3

**Prove that we can find an infinite set of positive integers of the form  $2^n - 3$  (where  $n$  is a positive integer) every pair of which are relatively prime.**

### Solution

We show how to enlarge a set of  $r$  such integers to a set of  $r+1$ . So suppose  $2^{n_1} - 3, \dots, 2^{n_r} - 3$  are all relatively prime. The idea is to find  $2^n - 1$  divisible by  $m = (2^{n_1} - 3) \dots (2^{n_r} - 3)$ , because then  $2^n - 3$  must be relatively prime to all of the factors of  $m$ . At least two of  $2^0, 2^1, \dots, 2^m$  must be congruent mod  $m$ . So suppose  $m_1 > m_2$  and  $2^{m_1} = 2^{m_2} \pmod{m}$ , then we must have  $2^{m_1 - m_2} - 1 = 0 \pmod{m}$ , since  $m$  is odd. So we may take  $n_{r+1}$  to be  $m_1 - m_2$ .

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### Problem B1

**All faces of the tetrahedron  $ABCD$  are acute-angled. Take a point  $X$  in the interior of the segment  $AB$ , and similarly  $Y$  in  $BC$ ,  $Z$  in  $CD$  and  $T$  in  $AD$ .**

**(a) If  $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$ , then prove that none of the closed paths  $XYZTX$  has minimal length;**

**(b) If  $\angle DAB + \angle BCD = \angle CDA + \angle ABC$ , then there are infinitely many shortest paths  $XYZTX$ , each with length  $2 AC \sin k$ , where  $2k = \angle BAC + \angle CAD + \angle DAB$ .**

### Solution

The key is to pretend the tetrahedron is made of cardboard, cut it along three edges and unfold it. Suppose we do this to get the hexagon  $CAC'BDB'$ . Now the path is a line joining  $Y$  on  $B'C$  to  $Y'$  on the opposite side  $BC'$  of the hexagon. Clearly this line must be straight for a minimal path. If  $B'C$  and  $BC'$  are parallel, then we can take  $Y$  anywhere on the side and the minimal path length is the expression given.

But if they are not parallel, then the minimal path will come from an extreme position. Suppose  $CC' < BB'$ . If the interior angle  $CAC'$  is less than  $180^\circ$ , then the minimal path is obtained by taking  $Y$  at  $C$ . But this does not meet the requirement that  $Y$  be an interior point of the edge, so there is no minimal path in the permitted set. If the interior angle  $CAC'$  is greater than  $180^\circ$ , then the minimal path is obtained by taking  $X$  and  $T$  at  $A$ . Again this is not permitted.

The problem therefore reduces to finding the condition for  $B'C$  and  $BC'$  to be parallel. This is evidently angles  $BCD + DCA + CAD + BAD + BAC + ACB = 360^\circ$ . But  $DCA + CAD = 180^\circ - ADC$ , and  $BAC + ACB = 180^\circ - ABC$ , so we obtain the condition given.

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### Problem B2

Prove that for every positive integer  $m$  we can find a finite set  $S$  of points in the plane, such that given any point  $A$  of  $S$ , there are exactly  $m$  points in  $S$  at unit distance from  $A$ .

### Solution

Take  $a_1, a_2, \dots, a_m$  to be points a distance  $1/2$  from the origin  $O$ . Form the set of  $2^m$  points  $\pm a_1 \pm a_2 \pm \dots \pm a_m$ . Given such a point, it is at unit distance from the  $m$  points with just one coefficient different. So we are home, provided that we can choose the  $a_i$  to avoid any other pairs of points being at unit distance, and to avoid any degeneracy (where some of the  $2^m$  points coincide).

The distance between two points in the set is  $|c_1 a_1 + c_2 a_2 + \dots + c_m a_m|$ , where  $c_i = 0, 2$  or  $-2$ . So let us choose the  $a_i$  inductively. Suppose we have already chosen up to  $m$ . The constraints on  $a_{m+1}$  are that we do not have  $|c_1 a_1 + c_2 a_2 + \dots + c_m a_m + 2a_{m+1}|$  equal to 0 or 1 for any  $c_i = 0, 2$  or  $-2$ , apart from the trivial cases of all  $c_i = 0$ . Each  $|| = 0$  rules out a single point and each  $|| = 1$  rules out a circle which intersects the circle radius  $1/2$  about the origin at 2 points and hence rules out two points. So the effect of the constraints is to rule out a finite number of points, whereas we have uncountably many to choose from.

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### Problem B3

Let  $A = (a_{ij})$ , where  $i, j = 1, 2, \dots, n$ , be a square matrix with all  $a_{ij}$  non-negative integers. For each  $i, j$  such that  $a_{ij} = 0$ , the sum of the elements in the  $i$ th row and the  $j$ th column is at least  $n$ . Prove that the sum of all the elements in the matrix is at least  $n^2/2$ .

### Solution

Let  $x$  be the smallest row or column sum. If  $x \geq n/2$ , then we are done, so assume  $x < n/2$ . Suppose it is a row. (If not, interchange rows and columns.) The number of non-zero elements in the row,  $y$ , must also satisfy  $y < n/2$ , since each non-zero element is at least 1. Now move across this row summing the columns. The  $y$  columns with a non-zero element have sum at least  $x$  (by the definition of  $x$ ). The  $n - y$  columns with a zero have sum at least  $n - x$ . Hence the total sum is at least  $xy + (n - x)(n - y) = n^2/2 + (n - 2x)(n - 2y)/2 > n^2/2$ .

The result is evidently best possible, because we can fill the matrix alternately with zeros and ones (so that  $a_{ij} = 1$  if  $i$  and  $j$  are both odd or both even, 0 otherwise). For  $n$  even, every row and column has  $n/2$  1s, so the condition is certainly satisfied and the total sum is  $n^2/2$ . For  $n$  odd, odd numbered rows have  $(n+1)/2$  1s and even numbered one less. But the only zeros are in positions which have either the row or the column odd-numbered, so the sum in such cases is  $n$  as required. The total sum is  $n^2/2 + 1/2$ . Alternatively, for  $n$  even, we could place  $n/2$  down the main diagonal.



**A1.** M is any point on the side AB of the triangle ABC. r, r<sub>1</sub>, r<sub>2</sub> are the radii of the circles inscribed in ABC, AMC, BMC. q is the radius of the circle on the opposite side of AB to C, touching the three sides of AB and the extensions of CA and CB. Similarly, q<sub>1</sub> and q<sub>2</sub>. Prove that r<sub>1</sub>r<sub>2</sub>q = rq<sub>1</sub>q<sub>2</sub>.



**A2.** We have  $0 \leq x_i < b$  for  $i = 0, 1, \dots, n$  and  $x_n > 0, x_{n-1} > 0$ . If  $a > b$ , and  $x_n x_{n-1} \dots x_0$  represents the number A base a and B base b, whilst  $x_{n-1} x_{n-2} \dots x_0$  represents the number A' base a and B' base b, prove that  $A'B < AB'$ .



**A3.** The real numbers  $a_0, a_1, a_2, \dots$  satisfy  $1 = a_0 \leq a_1 \leq a_2 \leq \dots$ .  $b_1, b_2, b_3, \dots$  are defined by  $b_n = \sum_{1 \leq k \leq n} (1 - a_{k-1}/a_k) / \sqrt{a_k}$ .

(a) Prove that  $0 \leq b_n < 2$ .



(b) Given c satisfying  $0 \leq c < 2$ , prove that we can find  $a_n$  so that  $b_n > c$  for all sufficiently large n.

**B1.** Find all positive integers n such that the set  $\{n, n+1, n+2, n+3, n+4, n+5\}$  can be partitioned into two subsets so that the product of the numbers in each subset is equal.



**B2.** In the tetrahedron ABCD, angle BDC = 90° and the foot of the perpendicular from D to ABC is the intersection of the altitudes of ABC. Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$



When do we have equality?

**B3.** Given 100 coplanar points, no 3 collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

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### Problem A1

**M** is any point on the side **AB** of the triangle **ABC**.  $r, r_1, r_2$  are the radii of the circles inscribed in **ABC**, **AMC**, **BMC**.  $q$  is the radius of the circle on the opposite side of **AB** to **C**, touching the three sides of **AB** and the extensions of **CA** and **CB**. Similarly,  $q_1$  and  $q_2$ . Prove that  $r_1 r_2 q = r q_1 q_2$ .

### Solution

We need an expression for  $r/q$ . There are two expressions, one in terms of angles and the other in terms of sides. The latter is a poor choice, because it is both harder to derive and less useful. So we derive the angle expression.

Let **I** be the center of the in-circle for **ABC** and **X** the center of the external circle for **ABC**. **I** is the intersection of the two angle bisectors from **A** and **B**, so  $c = r(\cot A/2 + \cot B/2)$ . The **X** lies on the bisector of the external angle, so angle **XAB** is  $90^\circ - A/2$ . Similarly, angle **XBA** is  $90^\circ - B/2$ , so  $c = q(\tan A/2 + \tan B/2)$ . Hence  $r/q = (\tan A/2 + \tan B/2)/(\cot A/2 + \cot B/2) = \tan A/2 \tan B/2$ .

Applying this to the other two triangles, we get  $r_1/q_1 = \tan A/2 \tan CMA/2$ ,  $r_2/q_2 = \tan B/2 \tan CMB/2$ . But  $CMB/2 = 90^\circ - CMA/2$ , so  $\tan CMB/2 = 1/\tan CMA/2$ . Hence result.

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### Problem A2

We have  $0 \leq x_i < b$  for  $i = 0, 1, \dots, n$  and  $x_n > 0, x_{n-1} > 0$ . If  $a > b$ , and  $x_n x_{n-1} \dots x_0$  represents the number **A** base  $a$  and **B** base  $b$ , whilst  $x_{n-1} x_{n-2} \dots x_0$  represents the number **A'** base  $a$  and **B'** base  $b$ , prove that **A'B** < **AB'**.

### Solution

We have  $a^n b^m > b^n a^m$  for  $n > m$ . Hence  $a^n B' > b^n A'$ . Adding  $a^n b^n$  to both sides gives  $a^n B > b^n A$ . Hence  $x_n a^n B > x_n b^n A$ . But  $x_n a^n = A - A'$  and  $x_n b^n = B - B'$ , so  $(A - A')B > (B - B')A$ . Hence result.

Note that the only purpose of requiring  $x_{n-1} > 0$  is to prevent **A'** and **B'** being zero.



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### Problem A3

The real numbers  $a_0, a_1, a_2, \dots$  satisfy  $1 = a_0 \leq a_1 \leq a_2 \leq \dots$ .  $b_1, b_2, b_3, \dots$  are defined by  $b_n = \sum_{k=1}^n (1 - a_{k-1}/a_k)/\sqrt{a_k}$ .

(a) Prove that  $0 \leq b_n < 2$ .

(b) Given  $c$  satisfying  $0 \leq c < 2$ , prove that we can find  $a_n$  so that  $b_n > c$  for all sufficiently large  $n$ .

### Solution

(a) Each term of the sum is non-negative, so  $b_n$  is non-negative. Let  $c_k = \sqrt{a_k}$ . Then the  $k$ th term =  $(1 - a_{k-1}/a_k)/\sqrt{a_k} = c_{k-1}^2/c_k (1/a_{k-1} - 1/a_k) = c_{k-1}^2/c_k (1/c_{k-1} + 1/c_k)(1/c_{k-1} - 1/c_k)$ . But  $c_{k-1}^2/c_k (1/c_{k-1} + 1/c_k) \leq 2$ , so the  $k$ th term  $\leq 2(1/c_{k-1} - 1/c_k)$ . Hence  $b_n \leq 2 - 2/c_n < 2$ .

(b) Let  $c_k = d^k$ , where  $d$  is a constant  $> 1$ , which we will choose later. Then the  $k$ th term is  $(1 - 1/d^2)1/d^k$ , so  $b_n = (1 - 1/d^2)(1 - 1/d^{n+1})/(1 - 1/d) = (1 + 1/d)(1 - 1/d^{n+1})$ . Now take  $d$  sufficiently close to 1 that  $1 + 1/d > c$ , and then take  $n$  sufficiently large so that  $(1 + 1/d)(1 - 1/d^{n+1}) > c$ .

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### Problem B1

Find all positive integers  $n$  such that the set  $\{n, n+1, n+2, n+3, n+4, n+5\}$  can be partitioned into two subsets so that the product of the numbers in each subset is equal.

### Solution

The only primes dividing numbers in the set can be 2, 3 or 5, because if any larger prime was a factor, then it would only divide one number in the set and hence only one product. Three of the numbers must be odd. At most one of the odd numbers can be a multiple of 3 and at most one can be a multiple of 5. The other odd number cannot have any prime factors. The only such number is 1, so the set must be  $\{1, 2, 3, 4, 5, 6\}$ , but that does not work because only one of the numbers is a multiple of 5. So there are no such sets.

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### Problem B2

In the tetrahedron  $ABCD$ , angle  $BDC = 90^\circ$  and the foot of the perpendicular from  $D$  to  $ABC$  is the intersection of the altitudes of  $ABC$ . Prove that:

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

When do we have equality?

### Solution

The first step is to show that angles  $ADB$  and  $ADC$  are also  $90^\circ$ . Let  $H$  be the intersection of the altitudes of  $ABC$  and let  $CH$  meet  $AB$  at  $X$ . Planes  $CED$  and  $ABC$  are perpendicular and  $AB$  is perpendicular to the line of intersection  $CE$ . Hence  $AB$  is perpendicular to the plane  $CDE$  and hence to  $ED$ . So  $BD^2 = DE^2 + BE^2$ . Also  $CB^2 = CE^2 + BE^2$ . Subtracting:  $CB^2 - BD^2 = CE^2 - DE^2$ . But  $CB^2 - BD^2 = CD^2$ , so  $CE^2 = CD^2 + DE^2$ , so angle  $CDE = 90^\circ$ . But angle  $CDB = 90^\circ$ , so  $CD$  is perpendicular to the plane  $DAB$ , and hence angle  $CDA = 90^\circ$ . Similarly, angle  $ADB = 90^\circ$ .

Hence  $AB^2 + BC^2 + CA^2 = 2(DA^2 + DB^2 + DC^2)$ . But now we are done, because Cauchy's inequality gives  $(AB + BC + CA)^2 \leq 3(AB^2 + BC^2 + CA^2)$ .

We have equality iff we have equality in Cauchy's inequality, which means  $AB = BC = CA$ .

## IMO 1970



### Problem B3

Given 100 coplanar points, no 3 collinear, prove that at most 70% of the triangles formed by the points have all angles acute.

**Solution** *Improved and corrected by Gerhard Wöginger, Technical University Graz*

At most 3 of the triangles formed by 4 points can be acute. It follows that at most 7 out of the 10 triangles formed by any 5 points can be acute. For given 10 points, the maximum no. of acute triangles is: the no. of subsets of 4 points  $\times$  3 / the no. of subsets of 4 points containing 3 given points. The total no. of triangles is the same expression with the first 3 replaced by 4. Hence at most  $3/4$  of the 10, or 7.5, can be acute, and hence at most 7 can be acute.

The same argument now extends the result to 100 points. The maximum number of acute triangles formed by 100 points is: the no. of subsets of 5 points  $\times$  7 / the no. of subsets of 5 points containing 3 given points. The total no. of triangles is the same expression with 7 replaced by 10. Hence at most  $7/10$  of the triangles are acute

11th IMO 1969



**A1.** Prove that there are infinitely many positive integers  $m$ , such that  $n^4 + m$  is not prime for any positive integer  $n$ .



**A2.** Let  $f(x) = \cos(a_1 + x) + 1/2 \cos(a_2 + x) + 1/4 \cos(a_3 + x) + \dots + 1/2^{n-1} \cos(a_n + x)$ , where  $a_i$  are real constants and  $x$  is a real variable. If  $f(x_1) = f(x_2) = 0$ , prove that  $x_1 - x_2$  is a multiple of  $\pi$ .



**A3.** For each of  $k = 1, 2, 3, 4, 5$  find necessary and sufficient conditions on  $a > 0$  such that there exists a tetrahedron with  $k$  edges length  $a$  and the remainder length 1.



**B1.**  $C$  is a point on the semicircle diameter  $AB$ , between  $A$  and  $B$ .  $D$  is the foot of the perpendicular from  $C$  to  $AB$ . The circle  $K_1$  is the in-circle of  $ABC$ , the circle  $K_2$  touches  $CD$ ,  $DA$  and the semicircle, the circle  $K_3$  touches  $CD$ ,  $DB$  and the semicircle. Prove that  $K_1$ ,  $K_2$  and  $K_3$  have another common tangent apart from  $AB$ .



**B2.** Given  $n > 4$  points in the plane, no three collinear. Prove that there are at least  $(n-3)(n-4)/2$  convex quadrilaterals with vertices amongst the  $n$  points.



**B3.** Given real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$ , satisfying  $x_1 > 0, x_2 > 0, x_1 y_1 > z_1^2$ , and  $x_2 y_2 > z_2^2$ , prove that:

$$8/((x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2) \leq 1/(x_1 y_1 - z_1^2) + 1/(x_2 y_2 - z_2^2).$$

Give necessary and sufficient conditions for equality.

## IMO 1969



### Problem A1

Prove that there are infinitely many positive integers  $m$ , such that  $n^4 + m$  is not prime for any positive integer  $n$ .

#### Solution

$n^4 + 4r^4 = (n^2 + 2rn + 2r^2)(n^2 - 2rn + 2r^2)$ . Clearly the first factor is greater than 1, the second factor is  $(n - r)^2 + r^2$ , which is also greater than 1 for  $r$  greater than 1. So we may take  $m = 4r^4$  for any  $r$  greater than 1.

## IMO 1969



### Problem A2

Let  $f(x) = \cos(a_1 + x) + 1/2 \cos(a_2 + x) + 1/4 \cos(a_3 + x) + \dots + 1/2^{n-1} \cos(a_n + x)$ , where  $a_i$  are real constants and  $x$  is a real variable. If  $f(x_1) = f(x_2) = 0$ , prove that  $x_1 - x_2$  is a multiple of  $\pi$ .

#### Solution

$f$  is not identically zero, because  $f(-a_1) = 1 + 1/2 \cos(a_2 - a_1) + \dots > 1 - 1/2 - 1/4 - \dots - 1/2^{n-1} > 0$ .

Using the expression for  $\cos(x + y)$  we obtain  $f(x) = b \cos x + c \sin x$ , where  $b = \cos a_1 + 1/2 \cos a_2 + \dots + 1/2^{n-1} \cos a_n$ , and  $c = -\sin a_1 - 1/2 \sin a_2 - \dots - 1/2^{n-1} \sin a_n$ .  $b$  and  $c$  are not both zero, since  $f$  is not identically zero, so  $f(x) = \sqrt{(b^2 + c^2)} \cos(d + x)$ , where  $\cos d = b/\sqrt{(b^2 + c^2)}$ , and  $\sin d = c/\sqrt{(b^2 + c^2)}$ . Hence the roots of  $f(x) = 0$  are just  $m\pi + \pi/2 - d$ .

## IMO 1969



### Problem A3

For each of  $k = 1, 2, 3, 4, 5$  find necessary and sufficient conditions on  $a > 0$  such that there exists a tetrahedron with  $k$  edges length  $a$  and the remainder length 1.

#### Solution

A plodding question. Take the tetrahedron to be ABCD.

Take  $k = 1$  and AB to have length  $a$ , the other edges length 1. Then we can hinge triangles ACD and BCD about CD to vary AB. The extreme values evidently occur with A, B, C, D coplanar. The least value, 0, when A coincides with B, and the greatest value  $\sqrt{3}$ , when A and B are on

opposite sides of CD. We rule out the extreme values on the grounds that the tetrahedron is degenerate, thus obtaining  $0 < a < \sqrt{3}$ .

For  $k = 5$ , the same argument shows that  $0 < 1 < \sqrt{3} a$ , and hence  $a > 1/\sqrt{3}$ .

For  $k = 2$ , there are two possible configurations: the sides length  $a$  adjacent, or not. Consider first the adjacent case. Take the sides length  $a$  to be AC and AD. As before, the two extreme cases gave A, B, C, D coplanar. If A and B are on opposite sides of CD, then  $a = \sqrt{2 - \sqrt{3}}$ . If they are on the same side, then  $a = \sqrt{2 + \sqrt{3}}$ . So this configuration allows any  $a$  satisfying  $\sqrt{2 - \sqrt{3}} < a < \sqrt{2 + \sqrt{3}}$ .

The other configuration has  $AB = CD = a$ . One extreme case has  $a = 0$ . We can increase  $a$  until we reach the other extreme case with ADBC a square side 1, giving  $a = \sqrt{2}$ . So this configuration allows any  $a$  satisfying  $0 < a < \sqrt{2}$ . Together, the two configurations allow any  $a$  satisfying:  $0 < a < \sqrt{2 + \sqrt{3}}$ .

This also solves the case  $k = 4$ , and allows any  $a$  satisfying:  $a > 1/\sqrt{2 + \sqrt{3}} = \sqrt{2 - \sqrt{3}}$ .

For  $k = 3$ , any value of  $a > 0$  is allowed. For  $a \leq 1$ , we may take the edges length  $a$  to form a triangle. For  $a \geq 1$  we may take a triangle with unit edges and the edges joining the vertices to the fourth vertex to have length  $a$ .

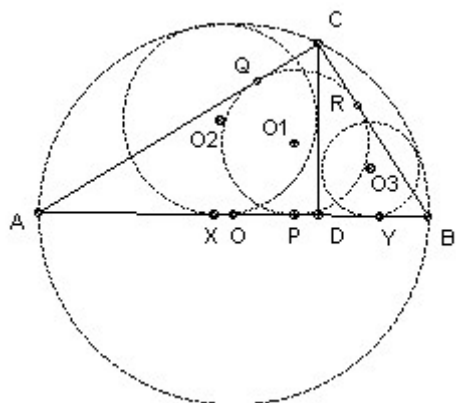
## IMO 1969



### Problem B1

**C is a point on the semicircle diameter AB, between A and B. D is the foot of the perpendicular from C to AB. The circle  $K_1$  is the in-circle of ABC, the circle  $K_2$  touches CD, DA and the semicircle, the circle  $K_3$  touches CD, DB and the semicircle. Prove that  $K_1$ ,  $K_2$  and  $K_3$  have another common tangent apart from AB.**

### Solution



Let the three centers be  $O_1, O_2$  and  $O_3$ . We show that  $O_1$  is the midpoint of  $O_2O_3$ . In fact it is sufficient to show that  $O_1$  lies on  $O_2O_3$ , because then we can reflect the known tangent  $AB$  in the line  $O_2O_3$ .

As usual, let  $AB = c, BC = a, CA = b$ . Let the in-circle touch  $AB$  at  $P, AC$  at  $Q$  and  $BC$  at  $R$ . Then since angle  $ACB = 90^\circ$ ,  $O_1QCR$  is a square. Also  $AQ = AP$  and  $BP = BR$ , so  $r_1 = b - AP$ , and  $r_1 = a - BP = a - (c - AP)$ . Adding:  $r_1 = (a + b - c)/2$ , and  $AP = (b + c - a)/2$ .

Let the circle center  $O_2$  touch  $AB$  at  $X$ , and the circle center  $O_3$  touch  $AB$  at  $Y$ . Let  $O$  be the midpoint of  $AB$ . Now consider the right-angled triangle  $OXO_2$ . Since the circle center  $O_2$  touches the semicircle,  $OO_2 = c/2 - r_2$ .  $OX = OD + DX = (c/2 - AD) + r_2$ . Also, by similar triangles,  $AD = b^2/c$ . So, using Pythagoras:  $(c/2 - r_2)^2 = r_2^2 + (c/2 - b^2/c + r_2)^2$ . Multiplying out and rearranging:  $r_2^2 - 2r_2(c - b^2/c) - (b^2 - b^4/c^2)$ . But  $ABC$  is right-angled, so  $c^2 = a^2 + b^2$ , and hence  $c - b^2/c = a^2/c$  and  $b^2 - b^4/c^2 = a^2b^2/c^2$ . So  $r_2^2 + 2r_2 a^2/c - a^2b^2/c^2 = 0$ , which has roots  $r_2 = a - a^2/c$  (positive) and  $-a + a^2/c$  (negative). So  $r_2 = a - a^2/c$ . Similarly,  $r_3 = b - b^2/c$ . So  $O_2X + O_3Y = XY = r_2 + r_3 = a + b - c = 2r_1$ .

$XP = AP - AX = AP - (AD - DX) = (b + c - a)/2 - (b^2/c - r_2) = (b + c - a)/2 - (c - a) = (a + b - c)/2 = r_1$ . We now have all we need:  $XP = PY = PO_1$ , and  $XO_2 + YO_3 = 2PO_1$ .

## IMO 1969



### Problem B2

**Given  $n > 4$  points in the plane, no three collinear. Prove that there are at least  $(n-3)(n-4)/2$  convex quadrilaterals with vertices amongst the  $n$  points.**

### Solution

$(n-3)(n-4)/2$  is a poor lower bound.

Observe first that any 5 points include 4 forming a convex quadrilateral. For take the convex hull. If it consists of more than 3 points, we are done. If not, it must consist of 3 points,  $A, B$  and  $C$ , with the other 2 points,  $D$  and  $E$ , inside the triangle  $ABC$ . Two vertices of the triangle must lie on the same side of the line  $DE$  and they form convex quadrilateral with  $D$  and  $E$ .

Given  $n$  points, we can choose 5 in  $n(n-1)(n-2)(n-3)(n-4)/120$  different ways. Each choice gives us a convex quadrilateral, but any given convex quadrilateral may arise from  $n-4$  different sets of 5 points, so we have at least  $n(n-1)(n-2)(n-3)/120$  different convex quadrilaterals. We now show that  $n(n-1)(n-2)(n-3)/120 \geq (n-3)(n-4)/2$  for all  $n \geq 5$ .

We wish to prove that  $n(n-1)(n-2) \geq 60(n-4)$ , or  $n(n-1)(n-2) - 60(n-4) \geq 0$ . Trial shows equality for  $n = 5$  and  $6$ , so we can factorise and get  $(n-5)(n-6)(n+8)$ , which is clearly at least 0 for  $n$  at least 5.



### Problem B3

Given real numbers  $x_1, x_2, y_1, y_2, z_1, z_2$ , satisfying  $x_1 > 0, x_2 > 0, x_1y_1 > z_1^2$ , and  $x_2y_2 > z_2^2$ , prove that:

$$8/((x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2) \leq 1/(x_1y_1 - z_1^2) + 1/(x_2y_2 - z_2^2).$$

Give necessary and sufficient conditions for equality.

### Solution

Let  $a_1 = x_1y_1 - z_1^2$  and  $a_2 = x_2y_2 - z_2^2$ . We apply the arithmetic/geometric mean result 3 times:

(1) to  $a_1^2, a_2^2$ , giving  $2a_1a_2 \leq a_1^2 + a_2^2$ ;

(2) to  $a_1, a_2$ , giving  $\sqrt{a_1a_2} \leq (a_1 + a_2)/2$ ;

(3) to  $a_1y_2/y_1, a_2y_1/y_2$ , giving  $\sqrt{a_1a_2} \leq (a_1y_2/y_1 + a_2y_1/y_2)/2$ ;

We also use  $(z_1/y_1 - z_2/y_2)^2 \geq 0$ . Now  $x_1y_1 > z_1^2 \geq 0$ , and  $x_1 > 0$ , so  $y_1 > 0$ . Similarly,  $y_2 > 0$ . So:

(4)  $y_1y_2(z_1/y_1 - z_2/y_2)^2 \geq 0$ , and hence  $z_1^2y_2/y_1 + z_2^2y_1/y_2 \geq 2z_1z_2$ .

Using (3) and (4) gives  $2\sqrt{a_1a_2} \leq (x_1y_2 + x_2y_1) - (z_1^2y_2/y_1 + z_2^2y_1/y_2) \leq (x_1y_2 + x_2y_1 - 2z_1z_2)$ .

Multiplying by (2) gives:  $4a_1a_2 \leq (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2)$ .

Adding (1) and  $2a_1a_2$  gives:  $8a_1a_2 \leq (a_1 + a_2)^2 + (a_1 + a_2)(x_1y_2 + x_2y_1 - 2z_1z_2) = a(a_1 + a_2)$ , where  $a = (x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2$ . Dividing by  $a_1a_2a$  gives the required inequality.

Equality requires  $a_1 = a_2$  from (1),  $y_1 = y_2$  from (2),  $z_1 = z_2$  from (3), and hence  $x_1 = x_2$ . Conversely, it is easy to see that these conditions are sufficient for equality.

10th IMO 1968



**A1.** Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.



**A2.** Find all natural numbers  $n$  the product of whose decimal digits is  $n^2 - 10n - 22$ .



**A3.**  $a, b, c$  are real with  $a$  non-zero.  $x_1, x_2, \dots, x_n$  satisfy the  $n$  equations:

$$ax_i^2 + bx_i + c = x_{i+1}, \text{ for } 1 \leq i < n$$

$$ax_n^2 + bx_n + c = x_1$$



Prove that the system has zero, 1 or  $>1$  real solutions according as  $(b - 1)^2 - 4ac$  is  $<0$ ,  $=0$  or  $>0$ .

**B1.** Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.



**B2.** Let  $f$  be a real-valued function defined for all real numbers, such that for some  $a > 0$  we have

$$f(x + a) = 1/2 + \sqrt{(f(x) - f(x)^2)} \text{ for all } x.$$



Prove that  $f$  is periodic, and give an example of such a non-constant  $f$  for  $a = 1$ .

**B3.** For every natural number  $n$  evaluate the sum

$[(n+1)/2] + [(n+2)/4] + [(n+4)/8] + \dots + [(n+2^k)/2^{k+1}] + \dots$ , where  $[x]$  denotes the greatest integer  $\leq x$ .



## IMO 1968



### Problem A1

**Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.**

### Solution

Let the sides be  $a$ ,  $a+1$ ,  $a+2$ , the angle opposite  $a$  be  $A$ , the angle opposite  $a+1$  be  $B$ , and the angle opposite  $a+2$  be  $C$ .

Using the cosine rule, we find  $\cos A = (a+5)/(2a+4)$ ,  $\cos B = (a+1)/2a$ ,  $\cos C = (a-3)/2a$ . Finally, using  $\cos 2x = 2 \cos^2 x - 1$ , we find solutions  $a = 4$  for  $C = 2A$ ,  $a = 1$  for  $B = 2A$ , and no solutions for  $C = 2B$ .

$a = 1$  is a degenerate solution (the triangle has the three vertices collinear). The other solution is 4, 5, 6.

## IMO 1968



### Problem A2

**Find all natural numbers  $n$  the product of whose decimal digits is  $n^2 - 10n - 22$ .**

### Solution

Suppose  $n$  has  $m > 1$  digits. Let the first digit be  $d$ . Then the product of the digits is at most  $d \cdot 9^{m-1} < d \cdot 10^{m-1} \leq n$ . But  $(n^2 - 10n - 22) - n = n(n - 11) - 22 > 0$  for  $n \geq 13$ . So there are no solutions for  $n \geq 13$ . But  $n^2 - 10n - 22 < 0$  for  $n \leq 11$ , so the only possible solution is  $n = 12$  and indeed that is a solution.

## IMO 1968

### Problem A3

**a, b, c are real with a non-zero.  $x_1, x_2, \dots, x_n$  satisfy the n equations:**

$$ax_i^2 + bx_i + c = x_{i+1}, \text{ for } 1 \leq i < n$$

$$ax_n^2 + bx_n + c = x_1$$

**Prove that the system has zero, 1 or >1 real solutions according as  $(b - 1)^2 - 4ac$  is  $<0, =0$  or  $>0$ .**

### Solution

Let  $f(x) = ax^2 + bx + c - x$ . Then  $f(x)/a = (x + (b-1)/2a)^2 + (4ac - (b-1)^2)/4a^2$ . Hence if  $4ac - (b-1)^2 > 0$ , then  $f(x)$  has the same sign for all  $x$ . But  $f(x) > 0$  means  $ax^2 + bx + c > x$ , so if  $\{x_i\}$  is a solution, then either  $x_1 < x_2 < \dots < x_n < x_1$ , or  $x_1 > x_2 > \dots > x_n > x_1$ . Either way we have a contradiction. So if  $4ac - (b-1)^2 > 0$  there cannot be any solutions.

If  $4ac - (b-1)^2 = 0$ , then we can argue in the same way that either  $x_1 \leq x_2 \leq \dots \leq x_n \leq x_1$ , or  $x_1 \geq x_2 \geq \dots \geq x_n \geq x_1$ . So we must have all  $x_i$  the single root of  $f(x) = 0$  (which clearly is a solution).

If  $4ac - (b-1)^2 < 0$ , then  $f(x) = 0$  has two distinct real roots  $y$  and  $z$  and so we have at least two solutions to the equations: all  $x_i = y$ , and all  $x_i = z$ . We may, however, have additional solutions. For example, if  $a = 1, b = 0, c = -1$  and  $n$  is even, then we have the additional solution  $x_1 = x_3 = x_5 = \dots = 0, x_2 = x_4 = \dots = -1$ .

## IMO 1968



### Problem B1

**Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.**

### Solution

The trick is to consider the longest side. That avoids getting into lots of different possible cases for which edge is longer than the sum of the other two.

So assume the result is false and let  $AB$  be the longest side. Then we have  $AB > AC + AD$  and  $BA > BC + BD$ . So  $2AB > AC + AD + BC + BD$ . But by the triangle inequality,  $AB < AC + CB$ ,  $AB < AD + DB$ , so  $2AB < AC + CB + AD + DB$ . Contradiction.

## IMO 1968



### Problem B2

Let  $f$  be a real-valued function defined for all real numbers, such that for some  $a > 0$  we have

$$f(x + a) = 1/2 + \sqrt{(f(x) - f(x)^2)} \text{ for all } x.$$

Prove that  $f$  is periodic, and give an example of such a non-constant  $f$  for  $a = 1$ .

### Solution

Directly from the equality given:  $f(x+a) \geq 1/2$  for all  $x$ , and hence  $f(x) \geq 1/2$  for all  $x$ .

So  $f(x+2a) = 1/2 + \sqrt{(f(x+a) - f(x+a)^2)} = 1/2 + \sqrt{f(x+a)} \sqrt{(1 - f(x+a))} = 1/2 + \sqrt{(1/4 - f(x) + f(x)^2)}$   
 $= 1/2 + (f(x) - 1/2) = f(x)$ . So  $f$  is periodic with period  $2a$ .

We may take  $f(x)$  to be arbitrary in the interval  $[0,1)$ . For example, let  $f(x) = 1$  for  $0 \leq x < 1$ ,  $f(x) = 1/2$  for  $1 \leq x < 2$ . Then use  $f(x+2) = f(x)$  to define  $f(x)$  for all other values of  $x$ .

## IMO 1968



### Problem B3

For every natural number  $n$  evaluate the sum

$[(n+1)/2] + [(n+2)/4] + [(n+4)/8] + \dots + [(n+2^k)/2^{k+1}] + \dots$ , where  $[x]$  denotes the greatest integer  $\leq x$ .

### Solution

For any real  $x$  we have  $[x] = [x/2] + [(x+1)/2]$ . For if  $x = 2n + 1 + k$ , where  $n$  is an integer and  $0 \leq k < 1$ , then lhs =  $2n + 1$ , and rhs =  $n + n + 1$ . Similarly, if  $x = 2n + k$ .

Hence for any integer  $n$ , we have:  $[n/2^k] - [n/2^{k+1}] = [(n/2^k + 1)/2] = [(n + 2^k)/2^{k+1}]$ . Hence summing over  $k$ , and using the fact that  $n < 2^k$  for sufficiently large  $k$ , so that  $[n/2^k] = 0$ , we have:  
 $n = [(n + 1)/2] + [(n + 2)/4] + [(n + 4)/8] + \dots$



**A1.** The parallelogram ABCD has  $AB = a$ ,  $AD = 1$ , angle  $BAD = A$ , and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram iff

$$a \leq \cos A + \sqrt{3} \sin A.$$



**A2.** Prove that a tetrahedron with just one edge length greater than 1 has volume at most  $1/8$ .



**A3.** Let  $k, m, n$  be natural numbers such that  $m + k + 1$  is a prime greater than  $n + 1$ . Let  $c_s = s(s+1)$ . Prove that:

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$$



is divisible by the product  $c_1 c_2 \dots c_n$ .

**B1.**  $A_0 B_0 C_0$  and  $A_1 B_1 C_1$  are acute-angled triangles. Construct the triangle ABC with the largest possible area which is circumscribed about  $A_0 B_0 C_0$  (BC contains  $A_0$ , CA contains  $B_0$ , and AB contains  $C_0$ ) and similar to  $A_1 B_1 C_1$ .



**B2.**  $a_1, \dots, a_8$  are reals, not all zero. Let  $c_n = a_1^n + a_2^n + \dots + a_8^n$  for  $n = 1, 2, 3, \dots$ . Given that an infinite number of  $c_n$  are zero, find all  $n$  for which  $c_n$  is zero.



**B3.** In a sports contest a total of  $m$  medals were awarded over  $n$  days. On the first day one medal and  $1/7$  of the remaining medals were awarded. On the second day two medals and  $1/7$  of the remaining medals were awarded, and so on. On the last day, the remaining  $n$  medals were awarded. How many medals were awarded, and over how many days?

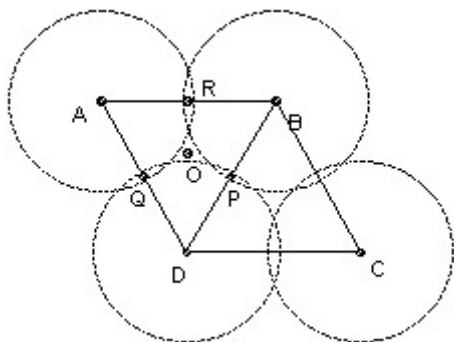


**Problem A1**

The parallelogram ABCD has  $AB = a$ ,  $AD = 1$ , angle  $BAD = A$ , and the triangle ABD has all angles acute. Prove that circles radius 1 and center A, B, C, D cover the parallelogram iff

$$a \leq \cos A + \sqrt{3} \sin A.$$

**Solution**



Evidently the parallelogram is a red herring, since the circles cover it iff and only if the three circles center A, B, D cover the triangle ABD.

The three circles radius  $x$  and centers the three vertices cover an acute-angled triangle ABD iff  $x$  is at least  $R$ , the circumradius. The circumcenter  $O$  is a distance  $R$  from each vertex, so the condition is clearly necessary. If the midpoints of  $BD$ ,  $DA$ ,  $AB$  are  $P$ ,  $Q$ ,  $R$ , then the circle center  $A$ , radius  $R$  covers the quadrilateral  $AQOR$ , the circle center  $B$ , radius  $R$  covers the quadrilateral  $BROP$ , and the circle center  $D$  radius  $R$  covers the quadrilateral  $DPOQ$ , so the condition is also sufficient.

We need an expression for  $R$  in terms of  $a$  and  $A$ . We can express  $BD$  two ways:  $2R \sin A$ , and  $\sqrt{a^2 + 1 - 2a \cos A}$ . So a necessary and sufficient condition for the covering is  $4 \sin^2 A \geq (a^2 + 1 - 2a \cos A)$ , which reduces to  $a \leq \cos A + \sqrt{3} \sin A$ , since  $\cos A \leq a$  (the foot of the perpendicular from  $D$  onto  $AB$  must lie between  $A$  and  $B$ ).

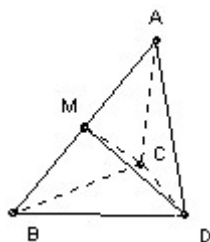
IMO 1967



**Problem 2**

**Prove that a tetrahedron with just one edge length greater than 1 has volume at most 1/8.**

**Solution**



Let the tetrahedron be ABCD and assume that all edges except AB have length at most 1. The volume is the  $\frac{1}{3}$  x area BCD x height of A above BCD. The height is at most the height of A above CD, so we maximise the volume by taking the planes ACD and BCD to be perpendicular. If AC or AD is less than 1, then we can increase the altitude from A to CD whilst keeping BCD fixed by taking AC = AD = 1. A similar argument shows that we must have BC = BD = 1.

But the volume is also the  $\frac{1}{3}$  x area ABC x height of D above ABC, so we must adjust CD to maximise this height. We want the angle between planes ABC and ABD to be as close as possible to  $90^\circ$ . The angle increases with increasing CD until it becomes  $90^\circ$ . CMD is then a right-angled triangle. Now the angle ACB must be less than the angle between the planes ACD and BCD and hence  $< 90^\circ$ , so angle ACM  $< 45^\circ$ , so  $CM > \frac{1}{\sqrt{2}}$ . Similarly DM. Hence when  $CMD = 90^\circ$  we have  $CD > 1$ . Thus we maximise the height of D above ABC by taking  $CD = 1$ .

So BCD is equilateral with area  $(\sqrt{3})/4$ . ACD is also equilateral with altitude  $(\sqrt{3})/2$ . Since the planes ACD and BCD are perpendicular, that is also the height of A above BCD. So the volume is  $\frac{1}{3} \times (\sqrt{3})/4 \times (\sqrt{3})/2 = 1/8$ .

IMO 1967



**Problem A3**

**Let  $k, m, n$  be natural numbers such that  $m + k + 1$  is a prime greater than  $n + 1$ . Let  $c_s = s(s+1)$ . Prove that:**

$$(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$$

**is divisible by the product  $c_1 c_2 \dots c_n$ .**

## Solution

The key is that  $c_a - c_b = (a - b)(a + b + 1)$ . Hence the product  $(c_{m+1} - c_k)(c_{m+2} - c_k) \dots (c_{m+n} - c_k)$  is the product of the  $n$  consecutive numbers  $(m - k + 1), \dots, (m - k + n)$ , times the product of the  $n$  consecutive numbers  $(m + k + 2), \dots, (m + k + n + 1)$ . The first product is just the binomial coefficient  $\binom{m-k+n}{n}$  times  $n!$ , so it is divisible by  $n!$ . The second product is  $1/(m + k + 1) \times (m + k + 1)(m + k + 2) \dots (m + k + n + 1) = 1/(m + k + 1) \times (m+k+n+1)C(n+1) \times (n+1)!$ . But  $m + k + 1$  is a prime greater than  $n + 1$ , so it has no factors in common with  $(n+1)!$ , hence the second product is divisible by  $(n+1)!$ . Finally note that  $c_1 c_2 \dots c_n = n! (n+1)!$ .

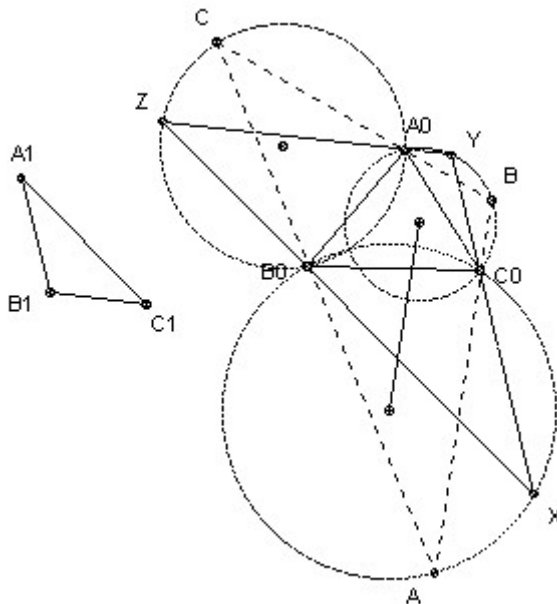
## IMO 1967



### Problem B1

$A_0B_0C_0$  and  $A_1B_1C_1$  are acute-angled triangles. Construct the triangle  $ABC$  with the largest possible area which is circumscribed about  $A_0B_0C_0$  ( $BC$  contains  $A_0$ ,  $CA$  contains  $B_0$ , and  $AB$  contains  $C_0$ ) and similar to  $A_1B_1C_1$ .

### Solution



Take any triangle similar to  $A_1B_1C_1$  and circumscribing  $A_0B_0C_0$ . For example, take an arbitrary line through  $A_0$  and then lines through  $B_0$  and  $C_0$  at the appropriate angles to the first line. Label the triangle's vertices  $X, Y, Z$  so that  $A_0$  lies on  $YZ$ ,  $B_0$  on  $ZX$ , and  $C_0$  on  $XY$ . Now any circumscribed  $ABC$  (labeled with the same convention) must have  $C$  on the circle through  $A_0, B_0$  and  $Z$ , because it has  $\angle C = \angle Z = \angle C_1$ . Similarly it must have  $B$  on the circle through  $C_0, A_0$  and  $Y$ , and it must have  $A$  on the circle through  $B_0, C_0$  and  $X$ .

Consider the side  $AB$ . It passes through  $C_0$ . Its length is twice the projection of the line joining the centers of the two circles onto  $AB$  (because each center projects onto the midpoint of the part

of AB that is a chord of its circle). But this projection is maximum when it is parallel to the line joining the two centers. The area is maximised when AB is maximised (because all the triangles are similar), so we take AB parallel to the line joining the centers. [Note, in passing, that this proves that the other sides must also be parallel to the lines joining the respective centers and hence that the three centers form a triangle similar to  $A_1B_1C_1$ .]

## IMO 1967



### Problem B2

$a_1, \dots, a_8$  are reals, not all zero. Let  $c_n = a_1^n + a_2^n + \dots + a_8^n$  for  $n = 1, 2, 3, \dots$ . Given that an infinite number of  $c_n$  are zero, find all  $n$  for which  $c_n$  is zero.

#### Solution

Take  $|a_1| \geq |a_2| \geq \dots \geq |a_8|$ . Suppose that  $|a_1|, \dots, |a_r|$  are all equal and greater than  $|a_{r+1}|$ . Then for sufficiently large  $n$ , we can ensure that  $|a_s|^n < 1/8 |a_1|^n$  for  $s > r$ , and hence the sum of  $|a_s|^n$  for all  $s > r$  is less than  $|a_1|^n$ . Hence  $r$  must be even with half of  $a_1, \dots, a_r$  positive and half negative.

If that does not exhaust the  $a_i$ , then in a similar way there must be an even number of  $a_i$  with the next largest value of  $|a_i|$ , with half positive and half negative, and so on. Thus we find that  $c_n = 0$  for all odd  $n$ .

## IMO 1967



### Problem B3

In a sports contest a total of  $m$  medals were awarded over  $n$  days. On the first day one medal and  $1/7$  of the remaining medals were awarded. On the second day two medals and  $1/7$  of the remaining medals were awarded, and so on. On the last day, the remaining  $n$  medals were awarded. How many medals were awarded, and over how many days?

#### Solution

Let the number of medals remaining at the start of day  $r$  be  $m_r$ . Then  $m_1 = m$ , and  $6(m_k - k)/7 = m_{k+1}$  for  $k < n$  with  $m_n = n$ .

After a little rearrangement, we find that  $m = 1 + 2(7/6) + 3(7/6)^2 + \dots + n(7/6)^{n-1}$ . Summing, we get  $m = 36(1 - (n+1)(7/6)^n + n(7/6)^{n+1}) = 36 + (n-6)7^n/6^{n-1}$ . 6 and 7 are coprime, so  $6^{n-1}$  must divide  $n-6$ . But  $6^{n-1} > n-6$ , so  $n=6$  and  $m=36$ .



## 8th IMO 1966



**A1.** Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B?



**A2.** Prove that if  $BC + AC = \tan C/2 (BC \tan A + AC \tan B)$ , then the triangle ABC is isosceles.



**A3.** Prove that a point in space has the smallest sum of the distances to the vertices of a regular tetrahedron iff it is the center of the tetrahedron.



**B1.** Prove that  $1/\sin 2x + 1/\sin 4x + \dots + 1/\sin 2^n x = \cot x - \cot 2^n x$  for any natural number  $n$  and any real  $x$  (with  $\sin 2^n x$  non-zero).



**B2.** Solve the equations:

$|a_i - a_1| x_1 + |a_i - a_2| x_2 + |a_i - a_3| x_3 + |a_i - a_4| x_4 = 1$ ,  $i = 1, 2, 3, 4$ , where  $a_i$  are distinct reals.



**B3.** Take any points K, L, M on the sides BC, CA, AB of the triangle ABC. Prove that at least one of the triangles AML, BKM, CLK has area  $\leq 1/4$  area ABC.

## IMO 1966



### Problem A1

Problems A, B and C were posed in a mathematical contest. 25 competitors solved at least one of the three. Amongst those who did not solve A, twice as many solved B as C. The number solving only A was one more than the number solving A and at least one other. The number solving just A equalled the number solving just B plus the number solving just C. How many solved just B?

Answer 6.

### Solution

Let  $a$  solve just A,  $b$  solve just B,  $c$  solve just C, and  $d$  solve B and C but not A. Then  $25 - a - b - c - d$  solve A and at least one of B or C. The conditions give:

$$b + d = 2(c + d); a = 1 + 25 - a - b - c - d; a = b + c.$$

Eliminating  $a$  and  $d$ , we get:  $4b + c = 26$ . But  $d = b - 2c \geq 0$ , so  $b = 6, c = 2$ .

## IMO 1966



### Problem A2

Prove that if  $BC + AC = \tan C/2 (BC \tan A + AC \tan B)$ , then the triangle ABC is isosceles.

### Solution

A straight slog works. Multiply up to get  $(a + b) \cos A \cos B \cos C/2 = a \sin A \cos B \sin C/2 + b \cos A \sin B \sin C/2$  (where  $a = BC, b = AC$ , as usual). Now use  $\cos(A + C/2) = \cos A \cos C/2 - \sin A \sin C/2$  and similar relation for  $\cos(B + C/2)$  to get:  $a \cos B \cos(A + C/2) + b \cos A \cos(B + C/2) = 0$ . Using  $C/2 = 90^\circ - A/2 - B/2$ , we find that  $\cos(A + C/2) = -\cos(B + C/2)$  (and = 0 only if  $A = B$ ). Result follows.

## IMO 1966



### Problem A3

Prove that a point in space has the smallest sum of the distances to the vertices of a regular tetrahedron iff it is the center of the tetrahedron.

#### Solution

Let the tetrahedron be ABCD and let P be a general point. Let X be the midpoint of CD. Let P' be the foot of the perpendicular from P to the plane ABX. We show that if P does not coincide with P', then  $PA + PB + PC + PD > P'A + P'B + P'C + P'D$ .

$PA > P'A$  (because angle  $PP'A = 90^\circ$ ) and  $PB > P'B$ .  $P'CD$  is isosceles and  $PCD$  is not but P is the same perpendicular distance from the line CD as P'. It follows that  $PC + PD > P'C + P'D$ . The easiest way to see this is to reflect C and D in the line  $PP'$  to give C' and D'. Then  $PC = P'C'$ , and  $PC' + PD > C'D = P'C' + P'D = P'C + P'D$ .

So if P has the smallest sum, it must lie in the plane ABX and similarly in the plane CDY, where Y is the midpoint of AB, and hence on the line XY. Similarly, it must lie on the line joining the midpoints of another pair of opposite sides and hence must be the center.

## IMO 1966



### Problem B1

Prove that  $1/\sin 2x + 1/\sin 4x + \dots + 1/\sin 2^n x = \cot x - \cot 2^n x$  for any natural number n and any real x (with  $\sin 2^n x$  non-zero).

#### Solution

$\cot y - \cot 2y = \cos y/\sin y - (2 \cos^2 y - 1)/(2 \sin y \cos y) = 1/(2 \sin y \cos y) = 1/\sin 2y$ . The result is now easy. Use induction. True for  $n = 1$  (just take  $y = x$ ). Suppose true for n, then taking  $y = 2^n x$ , we have  $1/\sin 2^{n+1} x = \cot 2^n x - \cot 2^{n+1} x$  and result follows for  $n + 1$ .

**Problem B2**

Solve the equations:

$$|a_i - a_1| x_1 + |a_i - a_2| x_2 + |a_i - a_3| x_3 + |a_i - a_4| x_4 = 1, \quad i = 1, 2, 3, 4, \text{ where } a_i \text{ are distinct reals.}$$

**Answer :**  $x_1 = 1/(a_1 - a_4), x_2 = x_3 = 0, x_4 = 1/(a_1 - a_4).$

**Solution**

Take  $a_1 > a_2 > a_3 > a_4$ . Subtracting the equation for  $i=2$  from that for  $i=1$  and dividing by  $(a_1 - a_2)$  we get:

$$-x_1 + x_2 + x_3 + x_4 = 0.$$

Subtracting the equation for  $i=4$  from that for  $i=3$  and dividing by  $(a_3 - a_4)$  we get:

$$-x_1 - x_2 - x_3 + x_4 = 0.$$

Hence  $x_1 = x_4$ . Subtracting the equation for  $i=3$  from that for  $i=2$  and dividing by  $(a_2 - a_3)$  we get:

$$-x_1 - x_2 + x_3 + x_4 = 0.$$

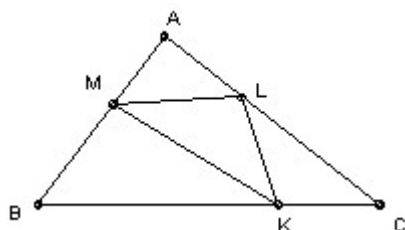
Hence  $x_2 = x_3 = 0$ , and  $x_1 = x_4 = 1/(a_1 - a_4)$ .



### Problem B3

Take any points  $K, L, M$  on the sides  $BC, CA, AB$  of the triangle  $ABC$ . Prove that at least one of the triangles  $AML, BKM, CLK$  has area  $\leq 1/4$  area  $ABC$ .

### Solution



If not, then considering  $ALM$  we have  $4 \cdot AL \cdot AM \cdot \sin A > AB \cdot AC \cdot \sin A$ , so  $4 \cdot AL \cdot AM > AB \cdot AC = (AM + BM)(AL + CL)$ , so  $3 \cdot AL \cdot AM > AM \cdot CL + BM \cdot AL + BM \cdot CL$ . Set  $k = BK/CK$ ,  $l = CL/AL$ ,  $m = AM/BM$ , and this inequality becomes:

$$3 > l + 1/m + l/m.$$

Similarly, considering the other two triangles we get:  $3 > k + 1/l + k/l$ , and  $3 > m + 1/k + m/k$ .

Adding gives:  $9 > k + l + m + 1/k + 1/l + 1/m + k/l + l/m + m/k$ , which is false by the arithmetic/geometric mean inequality.

## 7th IMO 1965



**A1.** Find all  $x$  in the interval  $[0, 2\pi]$  which satisfy:

$$2 \cos x \leq |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \leq \sqrt{2}.$$



**A2.** The coefficients  $a_{ij}$  of the following equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$



satisfy the following: (a)  $a_{11}, a_{22}, a_{33}$  are positive, (b) other  $a_{ij}$  are negative, (c) the sum of the coefficients in each equation is positive. Prove that the only solution is  $x_1 = x_2 = x_3 = 0$ .

**A3.** The tetrahedron  $ABCD$  is divided into two parts by a plane parallel to  $AB$  and  $CD$ . The distance of the plane from  $AB$  is  $k$  times its distance from  $CD$ . Find the ratio of the volumes of the two parts.



**B1.** Find all sets of four real numbers such that the sum of any one and the product of the other three is 2.



**B2.** The triangle  $OAB$  has  $\angle O$  acute.  $M$  is an arbitrary point on  $AB$ .  $P$  and  $Q$  are the feet of the perpendiculars from  $M$  to  $OA$  and  $OB$  respectively. What is the locus of  $H$ , the orthocenter of the triangle  $OPQ$  (the point where its altitudes meet)? What is the locus if  $M$  is allowed to vary over the interior of  $OAB$ ?



**B3.** Given  $n > 2$  points in the plane, prove that at most  $n$  pairs of points are the maximum distance apart (of any two points in the set).

### Problem A1

Find all  $x$  in the interval  $[0, 2\pi]$  which satisfy:

$$2 \cos x \leq |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \leq \sqrt{2}.$$

#### Solution

Let  $y = |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}|$ . Then  $y^2 = 2 - 2|\cos 2x|$ . If  $x$  belongs to  $[0, \pi/4]$  or  $[3\pi/4, 5\pi/4]$  or  $[7\pi/4, 2\pi]$ , then  $\cos 2x$  is non-negative, so  $y^2 = 2 - 2 \cos 2x = 4 \sin^2 x$ , so  $y = 2|\sin x|$ . We have  $\cos x \leq |\sin x|$  except for  $x$  in  $[0, \pi/4]$  and  $[7\pi/4, 2\pi]$ . So that leaves  $[3\pi/4, 5\pi/4]$  in which we certainly have  $|\sin x| \leq 1/\sqrt{2}$ .

If  $x$  belongs  $(\pi/4, 3\pi/4)$  or  $(5\pi/4, 7\pi/4)$ , then  $\cos 2x$  is negative, so  $y^2 = 2 + 2 \cos 2x = 4 \cos^2 x$ . So  $y = 2|\cos x|$ . So the first inequality certainly holds. The second also holds.

Thus the inequalities hold for all  $x$  in  $[\pi/4, 7\pi/4]$ .

### IMO 1965



### Problem A2

The coefficients  $a_{ij}$  of the following equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$$

satisfy the following: (a)  $a_{11}, a_{22}, a_{33}$  are positive, (b) other  $a_{ij}$  are negative, (c) the sum of the coefficients in each equation is positive. Prove that the only solution is  $x_1 = x_2 = x_3 = 0$ .

#### Solution

The slog solution is to multiply out the determinant and show it is non-zero. A slicker solution is to take the  $x_i$  with the largest absolute value. Say  $|x_1| \geq |x_2|, |x_3|$ . Then looking at the first equation we have an immediate contradiction, since the first term has larger absolute value than the sum of the absolute values of the second two terms.

## IMO 1965



### Problem A3

The tetrahedron  $ABCD$  is divided into two parts by a plane parallel to  $AB$  and  $CD$ . The distance of the plane from  $AB$  is  $k$  times its distance from  $CD$ . Find the ratio of the volumes of the two parts.

#### Solution

Let the plane meet  $AD$  at  $X$ ,  $BD$  at  $Y$ ,  $BC$  at  $Z$  and  $AC$  at  $W$ . Take plane parallel to  $BCD$  through  $WX$  and let it meet  $AB$  in  $P$ .

Since the distance of  $AB$  from  $WXYZ$  is  $k$  times the distance of  $CD$ , we have that  $AX = k \cdot XD$  and hence that  $AX/AD = k/(k+1)$ . Similarly  $AP/AB = AW/AC = AX/AD$ .  $XY$  is parallel to  $AB$ , so also  $AX/AD = BY/BD = BZ/BC$ .

$\text{vol } ABWXYZ = \text{vol } APWX + \text{vol } WXPBYZ$ .  $APWX$  is similar to the tetrahedron  $ABCD$ . The sides are  $k/(k+1)$  times smaller, so  $\text{vol } APWX = k^3/(k+1)^3 \text{ vol } ABCD$ . The base of the prism  $WXPBYZ$  is  $BYZ$  which is similar to  $BCD$  with sides  $k/(k+1)$  times smaller and hence area  $k^2/(k+1)^2$  times smaller. Its height is  $1/(k+1)$  times the height of  $A$  above  $ABCD$ , so  $\text{vol } \text{prism} = 3 k^2/(k+1)^3 \text{ vol } ABCD$ . Thus  $\text{vol } ABWXYZ = (k^3 + 3k^2)/(k+1)^3 \text{ vol } ABCD$ . We get the vol of the other piece as  $\text{vol } ABCD - \text{vol } ABWXYZ$  and hence the ratio is (after a little manipulation)  $k^2(k+3)/(3k+1)$ .

## IMO 1965



### Problem B1

Find all sets of four real numbers such that the sum of any one and the product of the other three is 2.

#### Answer

1,1,1,1 or 3,-1,-1,-1.

#### Solution

Let the numbers be  $x_1, \dots, x_4$ . Let  $t = x_1x_2x_3x_4$ . Then  $x_1 + t/x_1 = 2$ . So all the  $x_i$  are roots of the quadratic  $x^2 - 2x + t = 0$ . This has two roots, whose product is  $t$ .

If all  $x_i$  are equal to  $x$ , then  $x^3 + x = 2$ , and we must have  $x = 1$ . If not, then if  $x_1$  and  $x_2$  are unequal roots, we have  $x_1x_2 = t$  and  $x_1x_2x_3x_4 = t$ , so  $x_3x_4 = 1$ . But  $x_3$  and  $x_4$  are still roots of  $x^2 - 2x + t = 0$ . They cannot be unequal, otherwise  $x_3x_4 = t$ , which gives  $t = 1$  and hence all  $x_i = 1$ . Hence they are equal, and hence both 1 or both -1. Both 1 gives  $t = 1$  and all  $x_i = 1$ . Both -1 gives  $t = -3$  and hence  $x_i = 3, -1, -1, -1$  (in some order).

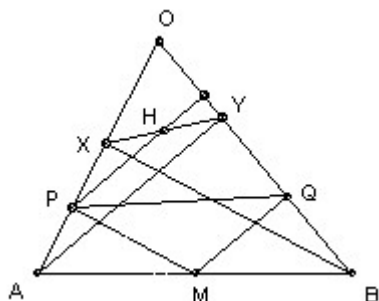




**Problem B2**

The triangle  $OAB$  has  $\angle O$  acute.  $M$  is an arbitrary point on  $AB$ .  $P$  and  $Q$  are the feet of the perpendiculars from  $M$  to  $OA$  and  $OB$  respectively. What is the locus of  $H$ , the orthocenter of the triangle  $OPQ$  (the point where its altitudes meet)? What is the locus if  $M$  is allowed to vary of the interior of  $OAB$ ?

**Solution**



Let  $X$  be the foot of the perpendicular from  $B$  to  $OA$ , and  $Y$  the foot of the perpendicular from  $A$  to  $OB$ . We show that the orthocenter of  $OPQ$  lies on  $XY$ .

$MP$  is parallel to  $BX$ , so  $AM/MB = AP/PX$ . Let  $H$  be the intersection of  $XY$  and the perpendicular from  $P$  to  $OB$ .  $PH$  is parallel to  $AY$ , so  $AP/PX = YH/HX$ .  $MQ$  is parallel to  $AY$ , so  $AM/MB = YQ/BQ$ . Hence  $YQ/BQ = YH/HX$  and so  $QH$  is parallel to  $BX$  and hence perpendicular to  $AO$ , so  $H$  is the orthocenter of  $OPQ$  as claimed.

If we restrict  $M$  to lie on a line  $A'B'$  parallel to  $AB$  (with  $A'$  on  $OA$ ,  $B'$  on  $OB$ ) then the locus is a line  $X'Y'$  parallel to  $XY$ , so as  $M$  moves over the whole interior, the locus is the interior of the triangle  $OXY$ .

## IMO 1965



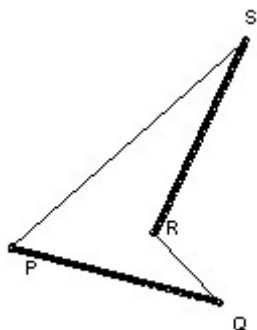
### Problem B3

Given  $n > 2$  points in the plane, prove that at most  $n$  pairs of points are the maximum distance apart (of any two points in the set).

### Solution

The key is that if two segments length  $d$  do not intersect then we can find an endpoint of one which is a distance  $> d$  from an endpoint of the other.

Given this, the result follows easily by induction. If false for  $n$ , then there is a point  $A$  in three pairs  $AB$ ,  $AC$  and  $AD$  of length  $d$  (the maximum distance). Take  $AC$  to lie between  $AB$  and  $AD$ . Now  $C$  cannot be in another pair. Suppose it was in  $CX$ . Then  $CX$  would have to cut both  $AB$  and  $AD$ , which is impossible.



To prove the result about the segments, suppose they are  $PQ$  and  $RS$ . We must have angle  $PQR$  less than  $90^\circ$ , otherwise  $PR > PQ = d$ . Similarly, the other angles of the quadrilateral must all be less than  $90^\circ$ . Contradiction.

6th IMO 1964



- A1.** (a) Find all natural numbers  $n$  for which 7 divides  $2^n - 1$ .  
(b) Prove that there is no natural number  $n$  for which 7 divides  $2^n + 1$ .



- A2.** Suppose that  $a, b, c$  are the sides of a triangle. Prove that:

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$



- A3.** Triangle  $ABC$  has sides  $a, b, c$ . Tangents to the inscribed circle are constructed parallel to the sides. Each tangent forms a triangle with the other two sides of the triangle and a circle is inscribed in each of these three triangles. Find the total area of all four inscribed circles.



- B1.** Each pair from 17 people exchange letters on one of three topics. Prove that there are at least 3 people who write to each other on the same topic. [In other words, if we color the edges of the complete graph  $K_{17}$  with three colors, then we can find a triangle all the same color.]



- B2.** 5 points in a plane are situated so that no two of the lines joining a pair of points are coincident, parallel or perpendicular. Through each point lines are drawn perpendicular to each of the lines through two of the other 4 points. Determine the maximum number of intersections these perpendiculars can have.



- B3.**  $ABCD$  is a tetrahedron and  $D_0$  is the centroid of  $ABC$ . Lines parallel to  $DD_0$  are drawn through  $A, B$  and  $C$  and meet the planes  $BCD, CAD$  and  $ABD$  in  $A_0, B_0,$  and  $C_0$  respectively. Prove that the volume of  $ABCD$  is one-third of the volume of  $A_0B_0C_0D_0$ . Is the result true if  $D_0$  is an arbitrary point inside  $ABC$ ?



## IMO 1964



### Problem A1

- (a) Find all natural numbers  $n$  for which 7 divides  $2^n - 1$ .  
(b) Prove that there is no natural number  $n$  for which 7 divides  $2^n + 1$ .

#### Solution

$2^3 = 1 \pmod{7}$ . Hence  $2^{3m} = 1 \pmod{7}$ ,  $2^{3m+1} = 2 \pmod{7}$ , and  $2^{3m+2} = 4 \pmod{7}$ . Hence we never have 7 dividing  $2^n + 1$ , and 7 divides  $2^n - 1$  iff 3 divides  $n$ .

## IMO 1964



### Problem A2

Suppose that  $a, b, c$  are the sides of a triangle. Prove that:

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

#### Solution

The condition that  $a, b, c$  be the sides of a triangle, together with the appearance of quantities like  $a + b - c$  is misleading. The inequality holds for any  $a, b, c \geq 0$ .

At most one of  $(b+c-a)$ ,  $(c+a-b)$ ,  $(a+b-c)$  can be negative. If one of them is negative, then certainly:

$$abc \geq (b + c - a)(c + a - b)(a + b - c) (*)$$

since the lhs is non-negative and the rhs is non-positive.

(\*) is also true if none of them is negative. For then the arithmetic/geometric mean on  $b + c - a$ ,  $c + a - b$  gives:

$$c^2 \geq (b + c - a)(c + a - b).$$

Similarly for  $a^2$  and  $b^2$ . Multiplying and taking the square root gives (\*). Multiplying out easily gives the required result.

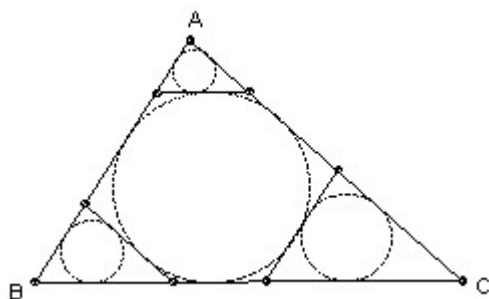
## IMO 1964



### Problem A3

Triangle ABC has sides  $a, b, c$ . Tangents to the inscribed circle are constructed parallel to the sides. Each tangent forms a triangle with the other two sides of the triangle and a circle is inscribed in each of these three triangles. Find the total area of all four inscribed circles

### Solution



This is easy once you realize that the answer is not nice and the derivation a slog. Use  $r = 2 \cdot \text{area} / \text{perimeter}$  and Heron's formula: area  $k$  is given by  $16k^2 = (a + b + c)(b + c - a)(c + a - b)(a + b - c)$ .

The small triangles at the vertices are similar to the main triangle and smaller by a factor  $(h - 2r)/h$ , where  $h$  is the relevant altitude. For the triangle opposite side  $a$ :  $(h - 2r)/h = 1 - 2(2k/p)/(2k/a) = 1 - 2a/p = (b + c - a)/(a + b + c)$ .

Hence the total area is  $((a + b + c)^2 + (b + c - a)^2 + (c + a - b)^2 + (a + b - c)^2)/(a + b + c)^2 \pi r^2 = (a^2 + b^2 + c^2) \cdot \pi \cdot (b + c - a)(c + a - b)(a + b - c)/(a + b + c)^3$ .

## IMO 1964



### Problem B1

Each pair from 17 people exchange letters on one of three topics. Prove that there are at least 3 people who write to each other on the same topic. [In other words, if we color the edges of the complete graph  $K_{17}$  with three colors, then we can find a triangle all the same color.]

### Solution

Take any person. He writes to 16 people, so he must write to at least 6 people on the same topic. If any of the 6 write to each other on that topic, then we have a group of three writing to each

other on the same topic. So assume they all write to each other on the other two topics. Take any of them, B. He must write to at least 3 of the other 5 on the same topic. If two of these write to each other on this topic, then they form a group of three with B. Otherwise, they must all write to each other on the third topic and so form a group of three.

## IMO 1964



### Problem B2

**5 points in a plane are situated so that no two of the lines joining a pair of points are coincident, parallel or perpendicular. Through each point lines are drawn perpendicular to each of the lines through two of the other 4 points. Determine the maximum number of intersections these perpendiculars can have.**

### Solution

It is not hard to see that the required number is at most 315. But it is not at all obvious how you prove it actually is 315, short of calculating the 315 points intersection for a specific example.

Call the points A, B, C, D, E. Given one of the points, the other 4 points determine 6 lines, so there are 6 perpendiculars through the given point and hence 30 perpendiculars in all. These determine at most  $30 \cdot 29/2 = 435$  points of intersection. But some of these necessarily coincide. There are three groups of coincidences. The first is that the 6 perpendiculars through A meet in one point (namely A), not the expected 15. So we lose  $5 \cdot 4 = 20$  points. Second, the lines through C, D and E perpendicular to AB are all parallel, and do not give the expected 3 points of intersection, so we lose another  $10 \cdot 3 = 30$  points. Third, the line through A perpendicular to BC is an altitude of the triangle ABC, as are the lines through B perpendicular to AC, and the through C perpendicular to AB. So we only get one point of intersection instead of three, thus losing another  $10 \cdot 2 = 20$  points. These coincidences are clearly all distinct (the categories do not overlap), so they bring us down to a maximum of  $435 - 120 = 315$ .

There is no obvious reason why there should be any further coincidences. But that is not quite the same as proving that there are no more. Indeed, for particular positions of the points A, B, C, D, E we can certainly arrange for additional coincidences (the constraints given in the problem are not sufficient to prevent additional coincidences). So we have to prove that it is possible to arrange the points so that there are no additional coincidences. I cannot see how to do this, short of exhibiting a particular set of points, which would be extremely tiresome. Apparently the contestants were instructed verbally that they did not have to do it.



### Problem B3

**ABCD is a tetrahedron and  $D_0$  is the centroid of ABC. Lines parallel to  $DD_0$  are drawn through A, B and C and meet the planes BCD, CAD and ABD in  $A_0$ ,  $B_0$ , and  $C_0$  respectively. Prove that the volume of ABCD is one-third of the volume of  $A_0B_0C_0D_0$ . Is the result true if  $D_0$  is an arbitrary point inside ABC?**

### Solution

Yes, indeed it is true for an arbitrary point in the plane of ABC not on any of the lines AB, BC, CA

Take D as the origin. Let A, B, C be the points  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  respectively. Then  $D_0$  is  $p\mathbf{a} + q\mathbf{b} + r\mathbf{c}$  with  $p + q + r = 1$  and  $p, q, r > 0$ . So a point on the line parallel to  $DD_0$  through A is  $\mathbf{a} + s(p\mathbf{a} + q\mathbf{b} + r\mathbf{c})$ . It is also in the plane DBC if  $s = -1/p$ , so  $A_0$  is the point  $-\frac{q}{p}\mathbf{b} - \frac{r}{p}\mathbf{c}$ . Similarly,  $B_0$  is  $-\frac{p}{q}\mathbf{a} - \frac{r}{q}\mathbf{c}$ , and  $C_0$  is  $-\frac{p}{r}\mathbf{a} - \frac{q}{r}\mathbf{b}$ .

The volume of ABCD is  $\frac{1}{6} |\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}|$  and the volume of  $A_0B_0C_0D_0$  is  $\frac{1}{6} |(p\mathbf{a} + (q + \frac{q}{p})\mathbf{b} + (r + \frac{r}{p})\mathbf{c}) \times ((\frac{p}{r} + \frac{p}{q})\mathbf{a} + q\mathbf{b} + (r + \frac{r}{q})\mathbf{c}) \cdot ((\frac{p}{r} + \frac{p}{r})\mathbf{a} + (q + \frac{q}{r})\mathbf{b} + r\mathbf{c})|$

Thus  $\text{vol } A_0B_0C_0D_0 / \text{vol } ABCD = \text{abs value of the determinant:}$

$$\begin{vmatrix} p & q + \frac{q}{p} & r + \frac{r}{p} \\ p + \frac{p}{q} & q & r + \frac{r}{q} \\ p + \frac{p}{r} & q + \frac{q}{r} & r \end{vmatrix}$$

$$\begin{vmatrix} p + \frac{p}{q} & q & r + \frac{r}{q} \\ p + \frac{p}{r} & q + \frac{q}{r} & r \end{vmatrix}$$

$$\begin{vmatrix} p + \frac{p}{r} & q + \frac{q}{r} & r \end{vmatrix}$$

which is easily found to be  $2 + p + q + r = 3$ .

## 5th IMO 1963



**A1.** For which real values of  $p$  does the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x$$
 have real roots? What are the roots?



**A2.** Given a point  $A$  and a segment  $BC$ , determine the locus of all points  $P$  in space for which  $\angle APX = 90^\circ$  for some  $X$  on the segment  $BC$ .



**A3.** An  $n$ -gon has all angles equal and the lengths of consecutive sides satisfy  $a_1 \geq a_2 \geq \dots \geq a_n$ . Prove that all the sides are equal.



**B1.** Find all solutions  $x_1, \dots, x_5$  to the five equations  $x_i + x_{i+2} = y x_{i+1}$  for  $i = 1, \dots, 5$ , where subscripts are reduced by 5 if necessary.



**B2.** Prove that  $\cos \pi/7 - \cos 2\pi/7 + \cos 3\pi/7 = 1/2$ .



**B3.** Five students  $A, B, C, D, E$  were placed 1 to 5 in a contest with no ties. One prediction was that the result would be the order  $A, B, C, D, E$ . But no student finished in the position predicted and no two students predicted to finish consecutively did so. For example, the outcome for  $C$  and  $D$  was not 1, 2 (respectively), or 2, 3, or 3, 4 or 4, 5. Another prediction was the order  $D, A, E, C, B$ . Exactly two students finished in the places predicted and two disjoint pairs predicted to finish consecutively did so. Determine the outcome.





## IMO 1963



### Problem A1

For which real values of  $p$  does the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x \text{ have real roots? What are the roots?}$$

### Solution

I must admit to having formed rather a dislike for this type of question which came up in almost every one of the early IMOs. Its sole purpose seems to be to teach you to be careful with one-way implications: the fact that  $a^2 = b^2$  does not imply  $a = b$ .

The lhs is non-negative, so  $x$  must be non-negative. Moreover  $2\sqrt{x^2 - 1} \leq x$ , so  $x \leq 2/\sqrt{3}$ . Also  $\sqrt{x^2 - p} \leq x$ , so  $p \geq 0$ .

Squaring etc gives that any solution must satisfy  $x^2 = (p - 4)^2 / (16 - 8p)$ . We require  $x \leq 2/\sqrt{3}$  and hence  $(3p - 4)(p + 4) \leq 0$ , so  $p \leq 4/3$ .

Substituting back in the original equality we get  $|3p - 4| + 2|p| = |p - 4|$ , which is indeed true for any  $p$  satisfying  $0 \leq p \leq 4/3$ .

## IMO 1963



### Problem A2

Given a point  $A$  and a segment  $BC$ , determine the locus of all points  $P$  in space for which  $\angle APX = 90^\circ$  for some  $X$  on the segment  $BC$ .

### Solution

Take the solid sphere on diameter  $AB$ , and the solid sphere on diameter  $AC$ . Then the locus is the points in one sphere but not the other (or on the surface of either sphere). Given  $P$ , consider the plane through  $P$  perpendicular to  $AP$  and the parallel planes through the other two points of intersection of  $AP$  with the two spheres (apart from  $A$ ) which pass through  $B$  and  $C$ .

## IMO 1963



### Problem A3

An  $n$ -gon has all angles equal and the lengths of consecutive sides satisfy  $a_1 \geq a_2 \geq \dots \geq a_n$ . Prove that all the sides are equal.

### Solution

For  $n$  odd consider the perpendicular distance of the shortest side from the opposite vertex. This is a sum of terms  $a_i \times \cos$  of some angle. We can go either way round. The angles are the same in both cases, so the inequalities give that  $a_1 = a_{n-1}$ , and hence  $a_1 = a_i$  for all  $i < n$ . We get  $a_1 = a_n$  by repeating the argument for the next shortest side. The case  $n$  even is easier, because we take a line through the vertex with sides  $a_1$  and  $a_n$  making equal angles with them and look at the perpendicular distance to the opposite vertex. This gives immediately that  $a_1 = a_n$ .

## IMO 1963



### Problem B1

Find all solutions  $x_1, \dots, x_5$  to the five equations  $x_i + x_{i+2} = y x_{i+1}$  for  $i = 1, \dots, 5$ , where subscripts are reduced by 5 if necessary.

### Solution

Successively eliminate variables to get  $x_1(y-2)(y^2+y-1)^2 = 0$ . We have the trivial solution  $x_i = 0$  for any  $y$ . For  $y = 2$ , we find  $x_i = s$  for all  $i$  (where  $s$  is arbitrary). Care is needed for the case  $y^2 + y - 1 = 0$ , because after eliminating three variables the two remaining equations have a factor  $y^2 + y - 1$ , and so they are automatically satisfied. In this case, we can take any two  $x_i$  arbitrary and still get a solution. For example,  $x_1 = s, x_2 = t, x_3 = -s + yt, x_4 = -ys - yt, x_5 = ys - t$ .

## IMO 1963



### Problem B2

Prove that  $\cos \pi/7 - \cos 2\pi/7 + \cos 3\pi/7 = 1/2$ .

### Solution

Consider the roots of  $x^7 + 1 = 0$ . They are  $e^{i\pi/7}$ ,  $e^{i3\pi/7}$ , ...,  $e^{i13\pi/7}$  and must have sum zero since there is no  $x^6$  term. Hence, in particular, their real parts sum to zero. But  $\cos 7\pi/7 = -1$  and the others are equal in pairs, because  $\cos(2\pi - x) = \cos x$ . So we get  $\cos \pi/7 + \cos 3\pi/7 + \cos 5\pi/7 = 1/2$ . Finally since  $\cos(\pi - x) = -\cos x$ ,  $\cos 5\pi/7 = -\cos 2\pi/7$ .

## IMO 1963



### Problem B3

Five students A, B, C, D, E were placed 1 to 5 in a contest with no ties. One prediction was that the result would be the order A, B, C, D, E. But no student finished in the position predicted and to two students predicted to finish consecutively did so. For example, the outcome for C and D was not 1, 2 (respectively), or 2, 3, or 3, 4 or 4, 5. Another prediction was the order D, A, E, C, B. Exactly two students finished in the places predicted and two disjoint pairs predicted to finish consecutively did so. Determine the outcome.

### Solution

Start from the second prediction. The disjoint pairs can only be: DA, EC; DC, CB; or AE, CB. The additional requirement of just two correct places means that the only possibilities (in the light of the information about the second prediction) are: DABEC, DACBE, EDACB, AEDCB. The first is ruled out because AB are consecutive. The second is ruled out because C is in the correct place. The fourth is ruled out because A is in the correct place. This leaves EDACB, which is indeed a solution.

## 4th IMO 1962



**A1.** Find the smallest natural number with 6 as the last digit, such that if the final 6 is moved to the front of the number it is multiplied by 4.



**A2.** Find all real  $x$  satisfying:  $\sqrt{3-x} - \sqrt{x+1} > 1/2$ .



**A3.** The cube  $ABCD A'B'C'D'$  has upper face  $ABCD$  and lower face  $A'B'C'D'$  with  $A$  directly above  $A'$  and so on. The point  $x$  moves at constant speed along the perimeter of  $ABCD$ , and the point  $Y$  moves at the same speed along the perimeter of  $B'C'CB$ .  $X$  leaves  $A$  towards  $B$  at the same moment as  $Y$  leaves  $B'$  towards  $C'$ . What is the locus of the midpoint of  $XY$ ?



**B1.** Find all real solutions to  $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$ .



**B2.** Given three distinct points  $A, B, C$  on a circle  $K$ , construct a point  $D$  on  $K$ , such that a circle can be inscribed in  $ABCD$ .



**B3.** The radius of the circumcircle of an isosceles triangle is  $R$  and the radius of its inscribed circle is  $r$ . Prove that the distance between the two centers is  $\sqrt{R(R-2r)}$ .



**B4.** Prove that a regular tetrahedron has five distinct spheres each tangent to its six extended edges. Conversely, prove that if a tetrahedron has five such spheres then it is regular.

## IMO 1962



### Problem A1

Find the smallest natural number with 6 as the last digit, such that if the final 6 is moved to the front of the number it is multiplied by 4.

### Solution

We have  $4(10n+6) = 6 \cdot 10^m + n$ , where  $n$  has  $m$  digits. So  $13n + 8 = 2 \cdot 10^m$ . Hence  $n = 2n'$  and  $13n' = 10^m - 4$ . Dividing, we quickly find that the smallest  $n'$ ,  $m$  satisfying this are:  $n' = 7692$ ,  $m = 5$ . Hence the answer is 153846.

## IMO 1962



### Problem A2

Find all real  $x$  satisfying:  $\sqrt{3-x} - \sqrt{x+1} > 1/2$ .

### Solution

It is easy to show that the inequality implies  $|x-1| > \sqrt{31}/8$ , so  $x > 1 + \sqrt{31}/8$ , or  $x < 1 - \sqrt{31}/8$ . But the converse is not true.

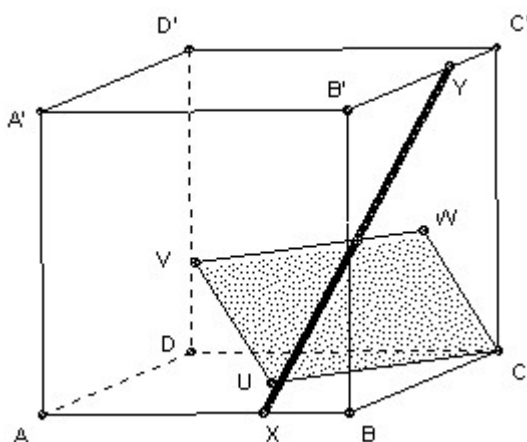
Indeed, we easily see that  $x > 1$  implies the lhs  $< 0$ . Also care is needed to ensure that the expressions under the root signs are not negative, which implies  $-1 \leq x \leq 3$ . Putting this together, suggests the solution is  $-1 \leq x < 1 - \sqrt{31}/8$ , which we can easily check.



**Problem A3**

The cube  $ABCD A'B'C'D'$  has upper face  $ABCD$  and lower face  $A'B'C'D'$  with  $A$  directly above  $A'$  and so on. The point  $x$  moves at constant speed along the perimeter of  $ABCD$ , and the point  $Y$  moves at the same speed along the perimeter of  $B'C'CB$ .  $X$  leaves  $A$  towards  $B$  at the same moment as  $Y$  leaves  $B'$  towards  $C'$ . What is the locus of the midpoint of  $XY$ ?

**Solution**



Answer: the rhombus  $CUVW$ , where  $U$  is the center of  $ABCD$ ,  $V$  is the center of  $ABB'A$ , and  $W$  is the center of  $BCC'B'$ .

Take rectangular coordinates with  $A$  as  $(0, 0, 0)$  and  $C'$  as  $(1, 1, 1)$ . Let  $M$  be the midpoint of  $XY$ . Whilst  $X$  is on  $AB$  and  $Y$  on  $B'C'$ ,  $X$  is  $(x, 0, 0)$  and  $Y$  is  $(1, x, 1)$ , so  $M$  is  $(x/2 + 1/2, x/2, 1/2) = x(1, 1/2, 1/2) + (1-x)(1/2, 0, 1/2) = xW + (1-x)V$ , so  $M$  traces out the line  $VW$ .

Whilst  $X$  is on  $BC$  and  $Y$  is on  $C'C$ ,  $X$  is  $(1, x, 0)$  and  $Y$  is  $(1, 1, 1-x)$ , so  $M$  is  $(1, x/2+1/2, 1/2 - x/2) = x(1, 1, 0) + (1-x)(1, 1/2, 1/2) = xC + (1-x)W$ , so  $M$  traces out the line  $WC$ .

Whilst  $X$  is on  $CD$  and  $Y$  is on  $CB$ ,  $X$  is  $(1-x, 1, 0)$  and  $Y$  is  $(1, 1-x, 0)$ , so  $M$  is  $(1-x/2, 1-x/2, 0) = x(1, 1, 0) + (1-x)(1/2, 1/2, 0) = xC + (1-x)U$ , so  $M$  traces out the line  $CU$ .

Whilst  $X$  is on  $DA$  and  $Y$  is on  $BB'$ ,  $X$  is  $(0, 1-x, 0)$  and  $Y$  is  $(1, 0, x)$ , so  $M$  is  $(1/2, 1/2 - x/2, x/2) = x(1/2, 0, 1/2) + (1-x)(1/2, 1/2, 0) = xV + (1-x)U$ , so  $M$  traces out the line  $UV$ .



**Problem B1**

Find all real solutions to  $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$ .

**Solution**

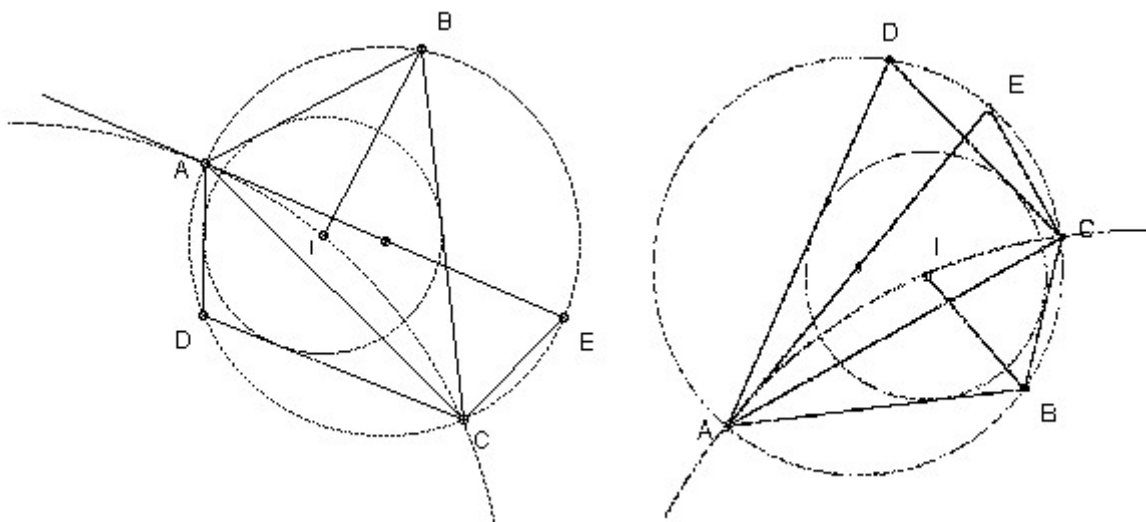
Put  $c = \cos x$ , and use  $\cos 3x = 4c^3 - 3c$ ,  $\cos 2x = 2c^2 - 1$ . We find the equation given is equivalent to  $c = 0$ ,  $c^2 = 1/2$  or  $c^2 = 3/4$ . Hence  $x = \pi/2, 3\pi/2, \pi/4, 3\pi/4, \pi/6, 5\pi/6$  or any multiple of  $\pi$  plus one of these.



**Problem B2**

Given three distinct points  $A, B, C$  on a circle  $K$ , construct a point  $D$  on  $K$ , such that a circle can be inscribed in  $ABCD$ .

**Solution**



$I$  be the center of the inscribed circle. Consider the quadrilateral  $ABCI$ .  $\angle BAI = 1/2 \angle BAD$  and  $\angle BCI = 1/2 \angle BCD$ , so  $\angle BAI + \angle BCI = 90^\circ$ , since  $ABCD$  is cyclic. Hence  $\angle AIC = 270^\circ - \angle ABC$ . So if we draw a circle through  $A$  and  $C$  such that for  $X$  points on the arc  $AC$   $\angle AXC =$

$90^\circ + \angle ABC$ , then the intersection of the circle with the angle bisector of  $\angle ABC$  gives the point I.

To draw this circle take the diameter AE. Then  $\angle CAE = 180^\circ - \angle ACE - \angle AEC = 90^\circ - \angle ABC$ . So we want AE to be tangent to the circle. Thus the center of the circle is on the perpendicular to AE through A and on the perpendicular bisector of AC.

To prove the construction possible we use the fact that a quadrilateral ABCD has an inscribed circle iff  $AB + CD = BC + AD$ . For D near C on the circumcircle of ABC we have  $AB + CD < BC + AD$ , whilst for D near A we have  $AB + CD > BC + AD$ , so as D moves continuously along the circumcircle there must be a point with equality. [Proof that the condition is sufficient: it is clearly necessary (use fact that tangents from a point are of equal length). So take a circle touching AB, BC and AD and let the other tangent from C (not BC) meet AD in D'. Then  $CD' - CD = AD' - AD$ , hence  $D' = D$ .]

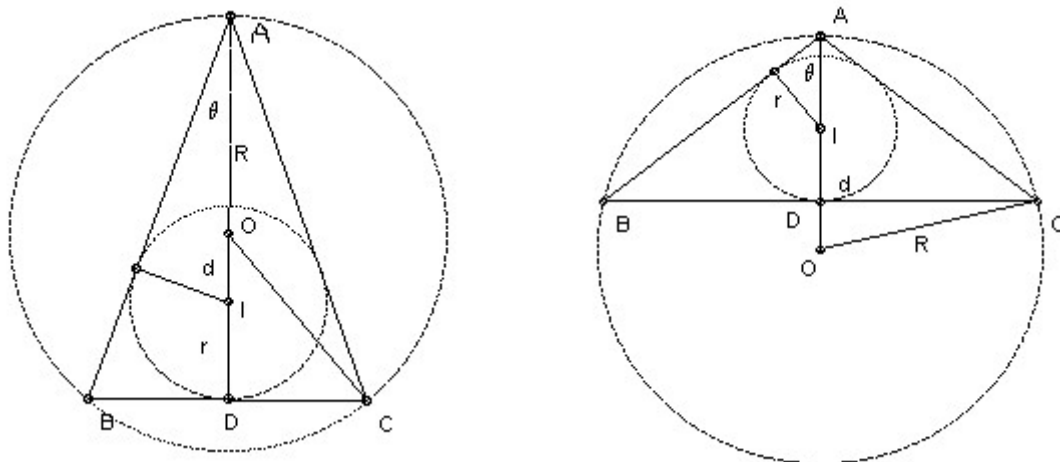
## IMO 1962



### Problem B3

The radius of the circumcircle of an isosceles triangle is R and the radius of its inscribed circle is r. Prove that the distance between the two centers is  $\sqrt{R(R - 2r)}$ .

### Solution

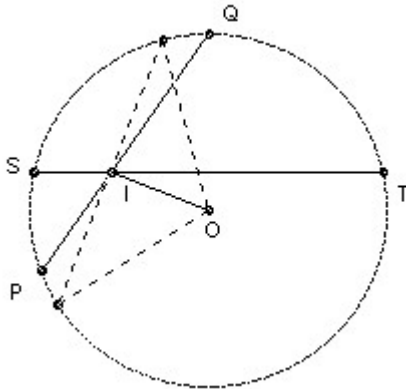


Let the triangle be ABC with  $AB = AC$ , let the incenter be I and the circumcenter O. Let the distance IO be d, taking d positive if O is closer to A than I, negative if I is closer. Let the  $\angle OAB$  be  $\theta$ .

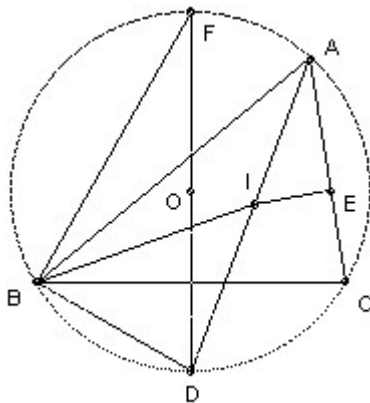


Then  $r = (R + d) \sin \theta$ , and  $r + d = R \cos 2\theta$ . It helps to draw a figure to check that this remains true for the various possible configurations. Using  $\cos 2\theta = 1 - 2 \sin^2\theta$ , we find that  $(d + R + r)(d^2 - R(R - 2r)) = 0$ . But  $OI < OA$ , so  $d$  is not  $-R - r$ . Hence result.

*Comment.*



*This result is known as Euler's formula and is true for any triangle. Suppose two chords  $PQ$  and  $ST$  of a circle intersect at  $I$ . Then  $PIS$  and  $TIQ$  are similar, so  $PI \cdot IQ = SI \cdot IT$ . Take the special case when  $ST$  is perpendicular to  $OI$ , where  $O$  is the center of the circle, then  $SI \cdot IT = SI^2 = R^2 - OI^2$ , where  $R$  is the radius of the circle, so  $PI \cdot IQ = R^2 - OI^2$ .*



*Now let  $O$  be the circumcenter,  $I$  the incenter of an arbitrary triangle  $ABC$ . Extend  $AI$  to meet the circumcircle again at  $D$ . Then by the above  $IO^2 = R^2 - AI \cdot ID$ . If  $E$  is the foot of the perpendicular from  $I$  to  $AC$ , then  $AI = r/\sin(A/2)$ . We show that  $DI = DB$ .  $\angle DBI = \angle DBC + \angle CBI = \angle DAC + \angle DBI = A/2 + B/2$ .  $\angle DIB = \angle IAB + \angle IBA = A/2 + B/2$ . Hence  $\angle DBI = \angle DIB$ , so  $DI = DB$ , as claimed. Take  $F$  on the circle so that  $DF$  is a diameter, then  $\angle DFB = \angle DAB = A/2$ , so  $DB = 2R \sin A/2$ . Thus  $IO^2 = R^2 - r/\sin(A/2) \cdot 2R \sin(A/2) = R^2 - 2Rr$ .*

### Problem B4

**Prove that a regular tetrahedron has five distinct spheres each tangent to its six extended edges. Conversely, prove that if a tetrahedron has five such spheres then it is regular.**

### Solution

First part is obvious. The wrong way to do the second part is to start looking for the locus of the center of a sphere which touches three edges. The key is to notice that the tangents to a sphere from a given point have the same length.

Let the tetrahedron be  $A_1A_2A_3A_4$ . Let  $S$  be the sphere inside the tetrahedron,  $S_1$  the tetrahedron opposite  $A_1$ , and so on. Let the tangents to  $S$  from  $A_i$  have length  $a_i$ . Then the side  $A_iA_j$  has length  $a_i+a_j$ . Now consider the tangents to  $S_1$  from  $A_1$ . Their lengths are  $a_1 + 2a_2$ ,  $a_1 + 2a_3$ , and  $a_1 + 2a_4$ . Hence  $a_2 = a_3 = a_4$ . Similarly, considering  $S_2$ , we have that  $a_1 = a_3 = a_4$ .

### 3rd IMO 1961



**A1.** Solve the following equations for  $x$ ,  $y$  and  $z$ :

$$x + y + z = a; \quad x^2 + y^2 + z^2 = b^2; \quad xy = z^2$$

What conditions must  $a$  and  $b$  satisfy for  $x$ ,  $y$  and  $z$  to be distinct positive numbers?



**A2.** Let  $a$ ,  $b$ ,  $c$  be the sides of a triangle and  $A$  its area. Prove that:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} A$$

When do we have equality?



**A3.** Solve the equation  $\cos^n x - \sin^n x = 1$ , where  $n$  is a natural number.



**B1.**  $P$  is inside the triangle  $ABC$ .  $PA$  intersects  $BC$  in  $D$ ,  $PB$  intersects  $AC$  in  $E$ , and  $PC$  intersects  $AB$  in  $F$ . Prove that at least one of  $AP/PD$ ,  $BP/PE$ ,  $CP/PF$  does not exceed 2, and at least one is not less than 2.



**B2.** Construct the triangle  $ABC$ , given the lengths  $AC=b$ ,  $AB=c$  and the acute  $\angle AMB = \alpha$ , where  $M$  is the midpoint of  $BC$ . Prove that the construction is possible if and only if

$$b \tan(\alpha/2) \leq c < b.$$

When does equality hold?



**B3.** Given 3 non-collinear points  $A$ ,  $B$ ,  $C$  and a plane  $p$  not parallel to  $ABC$  and such that  $A$ ,  $B$ ,  $C$  are all on the same side of  $p$ . Take three arbitrary points  $A'$ ,  $B'$ ,  $C'$  in  $p$ . Let  $A''$ ,  $B''$ ,  $C''$  be the midpoints of  $AA'$ ,  $BB'$ ,  $CC'$  respectively, and let  $O$  be the centroid of  $A''$ ,  $B''$ ,  $C''$ . What is the locus of  $O$  as  $A'$ ,  $B'$ ,  $C'$  vary?



### Problem A1

Solve the following equations for  $x$ ,  $y$  and  $z$ :

$$x + y + z = a; \quad x^2 + y^2 + z^2 = b^2; \quad xy = z^2$$

What conditions must  $a$  and  $b$  satisfy for  $x$ ,  $y$  and  $z$  to be distinct positive numbers?

#### Solution

A routine slog gives  $z = (a^2 - b^2)/2a$ ,  $x$  and  $y = (a^2 + b^2)/4a \pm \sqrt{(10a^2b^2 - 3a^4 - 3b^4)}/4a$ .

A little care is needed with the conditions. Clearly  $x, y, z$  positive implies  $a > 0$ , and then  $z$  positive implies  $|b| < a$ . The expression under the root must be positive. It helps if you notice that it factorizes as  $(3a^2 - b^2)(3b^2 - a^2)$ . The second factor is positive because  $|b| < a$ , so the first factor must also be positive and hence  $a < \sqrt{3} |b|$ . These conditions are also sufficient to ensure that  $x$  and  $y$  are distinct, but then  $z$  must also be distinct because  $z^2 = xy$ .

### IMO 1961

#### Problem A2

Let  $a, b, c$  be the sides of a triangle and  $A$  its area. Prove that:

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} A$$

When do we have equality?

#### Solution

One approach is a routine slog from Heron's formula. The inequality is quickly shown to be equivalent to  $a^2b^2 + b^2c^2 + c^2a^2 \leq a^4 + b^4 + c^4$ , which is true since  $a^2b^2 \leq (a^4 + b^4)/2$ . We get equality iff the triangle is equilateral.

Another approach is to take an altitude lying inside the triangle. If it has length  $h$  and divides the base into lengths  $r$  and  $s$ , then we quickly find that the inequality is equivalent to  $(h - (r + s)\sqrt{3}/2)^2 + (r - s)^2 \geq 0$ , which is true. We have equality iff  $r = s$  and  $h = (r + s)\sqrt{3}/2$ , which means the triangle is equilateral.

*A third solution is due to Jonathan Mizrahi (somewhat adapted):*

We have  $b^2 + c^2 \geq 2bc$  with equality iff  $b = c$ . Also for any angle  $x$  in the range  $0^\circ$  to  $180^\circ$  we have  $2bc \geq 2bc \sin(X + 30^\circ)$  with equality iff  $X = 60^\circ$ . So taking  $X$  to be the angle between the sides  $b$  and  $c$  (we cannot call it  $A$  because  $A$  is already used to mean the area in this question!) we have that  $b^2 + c^2 \geq bc \sin(X + 30^\circ)$  with equality iff the triangle is equilateral. Now  $2 \sin(X + 30^\circ) = \sqrt{3} \sin X + \cos X$ , so using the cosine rule  $a^2 = b^2 + c^2 - 2bc \cos X$ , we get the required inequality.

## IMO 1961



### Problem A3

Solve the equation  $\cos^n x - \sin^n x = 1$ , where  $n$  is a natural number.

#### Solution

Since  $\cos^2 x + \sin^2 x = 1$ , we cannot have solutions with  $n$  not 2 and  $0 < |\cos x|, |\sin x| < 1$ . Nor can we have solutions with  $n=2$ , because the sign is wrong. So the only solutions have  $\sin x = 0$  or  $\cos x = 0$ , and these are:  $x = \text{multiple of } \pi$ , and  $n$  even;  $x = \text{even multiple of } \pi$  and  $n$  odd;  $x = \text{even multiple of } \pi + 3\pi/2$  and  $n$  odd.

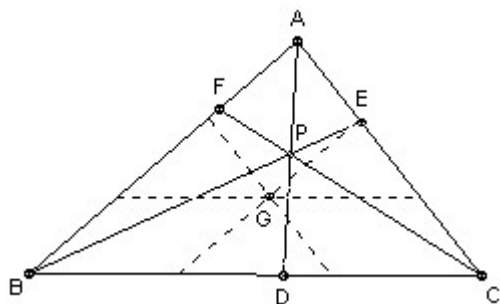
## IMO 1961



### Problem B1

$P$  is inside the triangle  $ABC$ .  $PA$  intersects  $BC$  in  $D$ ,  $PB$  intersects  $AC$  in  $E$ , and  $PC$  intersects  $AB$  in  $F$ . Prove that at least one of  $AP/PD$ ,  $BP/PE$ ,  $CP/PF$  does not exceed 2, and at least one is not less than 2.

#### Solution



Take lines through the centroid parallel to the sides of the triangle. The result is then obvious.



**Problem B2**

Construct the triangle  $ABC$ , given the lengths  $AC = b$ ,  $AB = c$  and the acute  $\angle AMB = \alpha$ , where  $M$  is the midpoint of  $BC$ . Prove that the construction is possible if and only if

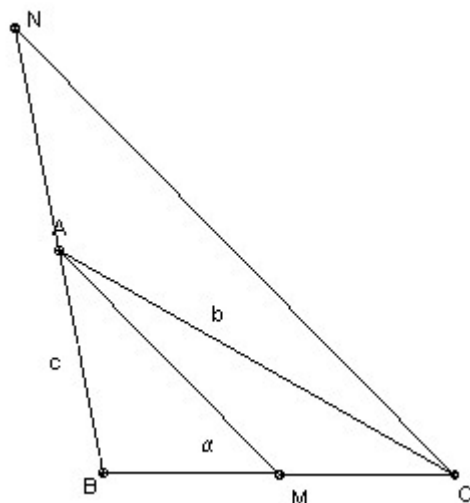
$$b \tan(\alpha/2) \leq c < b.$$

When does equality hold?

**Answer**

Equality holds if  $\angle BAC = 90^\circ$  and  $\angle ACB = \alpha/2$

**Solution**



The key is to take  $N$  so that  $A$  is the midpoint of  $NB$ , then  $\angle NCB = \alpha$ .

The construction is as follows: take  $BN$  length  $2AB$ . Take circle through  $B$  and  $N$  such that the  $\angle BPN = \alpha$  for points  $P$  on the arc  $BN$ . Take  $A$  as the midpoint of  $BN$  and let the circle center  $A$ , radius  $AC$  cut the arc  $BN$  at  $C$ . In general there are two possibilities for  $C$ .

Let  $X$  be the intersection of the arc  $BN$  and the perpendicular to the segment  $BN$  through  $A$ . For the construction to be possible we require  $AX \geq AC > AB$ . But  $AB/AX = \tan \alpha/2$ , so we get the condition in the question.

Equality corresponds to  $C = X$  and hence to  $\angle BAC = 90^\circ$  and  $\angle ACB = \alpha/2$ .

### IMO 1961



#### Problem B3

Given 3 non-collinear points  $A, B, C$  and a plane  $p$  not parallel to  $ABC$  and such that  $A, B, C$  are all on the same side of  $p$ . Take three arbitrary points  $A', B', C'$  in  $p$ . Let  $A'', B'', C''$  be the midpoints of  $AA', BB', CC'$  respectively, and let  $O$  be the centroid of  $A'', B'', C''$ . What is the locus of  $O$  as  $A', B', C'$  vary?

#### Solution

The key is to notice that  $O$  is the midpoint of the segment joining the centroids of  $ABC$  and  $A'B'C'$ . The centroid of  $ABC$  is fixed, so the locus is just the plane parallel to  $p$  and midway between  $p$  and the centroid of  $ABC$ .

## 2nd IMO 1960



**A1.** Determine all 3 digit numbers  $N$  which are divisible by 11 and where  $N/11$  is equal to the sum of the squares of the digits of  $N$ .



**A2.** For what real values of  $x$  does the following inequality hold:

$$4x^2/(1 - \sqrt{(1 + 2x)})^2 < 2x + 9 ?$$



**A3.** In a given right triangle  $ABC$ , the hypotenuse  $BC$ , length  $a$ , is divided into  $n$  equal parts with  $n$  an odd integer. The central part subtends an angle  $\alpha$  at  $A$ .  $h$  is the perpendicular distance from  $A$  to  $BC$ . Prove that:

$$\tan \alpha = 4nh/(an^2 - a).$$



**B1.** Construct a triangle  $ABC$  given the lengths of the altitudes from  $A$  and  $B$  and the length of the median from  $A$ .



**B2.** The cube  $ABCA'B'C'D'$  has  $A$  above  $A'$ ,  $B$  above  $B'$  and so on.  $X$  is any point of the face diagonal  $AC$  and  $Y$  is any point of  $B'D'$ .

(a) find the locus of the midpoint of  $XY$ ;

(b) find the locus of the point  $Z$  which lies one-third of the way along  $XY$ , so that  $ZY=2 \cdot XZ$ .



**B3.** A cone of revolution has an inscribed sphere tangent to the base of the cone (and to the sloping surface of the cone). A cylinder is circumscribed about the sphere so that its base lies in the base of the cone. The volume of the cone is  $V_1$  and the volume of the cylinder is  $V_2$ .

(a) Prove that  $V_1 \neq V_2$ ;

(b) Find the smallest possible value of  $V_1/V_2$ . For this case construct the half angle of the cone.



**B4.** In the isosceles trapezoid  $ABCD$  ( $AB$  parallel to  $DC$ , and  $BC = AD$ ), let  $AB = a$ ,  $CD = c$  and let the perpendicular distance from  $A$  to  $CD$  be  $h$ . Show how to construct all points  $X$  on the axis of symmetry such that  $\angle BXC = \angle AXD = 90^\circ$ . Find the distance of each such  $X$  from  $AB$  and from  $CD$ . What is the condition for such points to exist?





## IMO 1960



### Problem A1

Determine all 3 digit numbers  $N$  which are divisible by 11 and where  $N/11$  is equal to the sum of the squares of the digits of  $N$ .

### Answer

550, 803.

### Solution

So, put  $N/11 = 10a + b$ . If  $a + b \leq 9$ , we have  $2a^2 + 2ab + 2b^2 = 10a + b$  (\*), so  $b$  is even. Put  $b = 2B$ , then  $B = a(a-5) + 2aB + 4B^2$ , which is even. So  $b$  must be a multiple of 4, so  $b = 0, 4$  or  $8$ . If  $b = 0$ , then (\*) gives  $a = 5$  and we get the solution 550. If  $b = 4$ , then (\*) gives  $a^2 - a + 14 = 0$ , which has no integral solutions. If  $b = 8$ , then (since  $a + b \leq 9$  and  $a > 0$ )  $a$  must be 1, but that does not satisfy (\*).

If  $a + b > 9$ , we have  $(a+1)^2 + (a+b-10)^2 + b^2 = 10a + b$ , or  $2a^2 + 2ab + 2b^2 - 28a - 21b + 101 = 0$  (\*\*), so  $b$  is odd. Put  $b = 2B+1$ . Then  $a^2 + 2aB + 4B^2 - 13a - 17B + 41 = 0$ . But  $a(a-13)$  is even, so  $B$  is odd. Hence  $b = 3$  or  $7$ . If  $b = 3$ , then (\*\*) gives  $a^2 - 11a + 28 = 0$ , so  $a = 4$  or  $7$ . But  $a + b > 9$ , so  $a = 7$ . That gives the solution 803. If  $b = 7$ , then (\*\*) gives  $a^2 - 7a + 26 = 0$ , which has no integral solutions.

*Comment* : Personally, I hate this type of question. The fastest way to solve it is almost certainly to scan the 81 multiples of 11 from 110 to 990.

## IMO 1960



### Problem A2

For what real values of  $x$  does the following inequality hold:

$$4x^2/(1 - \sqrt{(1 + 2x)})^2 < 2x + 9 ?$$

**Answer** :  $-1/2 \leq x < 45/8$ .

### Solution

We require the first inequality to avoid imaginary numbers. Hence we may set  $x = -1/2 + a^2/2$ , where  $a \geq 0$ . The inequality now gives immediately  $a < 7/2$  and hence  $x < 45/8$ . It is a matter of taste whether to avoid  $x = 0$ . I would allow it because the limit as  $x$  tends to 0 of the lhs is 4, and the inequality holds.

*Thanks to Dave Arthur for the idea of putting  $x = -1/2 + a$ , which saves some tedious algebra.*

IMO 1960



**Problem A3**

In a given right triangle ABC, the hypotenuse BC, length a, is divided into n equal parts with n an odd integer. The central part subtends an angle  $\alpha$  at A. h is the perpendicular distance from A to BC. Prove that:

$$\tan \alpha = 4nh/(an^2 - a).$$

**Solution**

Let M be the midpoint of BC, and P and Q the two points  $a/2n$  either side of it, with P nearer B. Then  $\alpha = \angle PAQ = \angle QAH - \angle PAH$  (taking angles as negative if P (or Q) lies to the left of H). So  $\tan \alpha = (QH - PH)/(AH^2 + QH \cdot PH) = AH \cdot PQ/(AH^2 + (MH - a/2n)(MH + a/2n)) = (ah/n)/(a^2/4 - a^2/(4n^2)) = 4nh/(an^2 - a)$ .

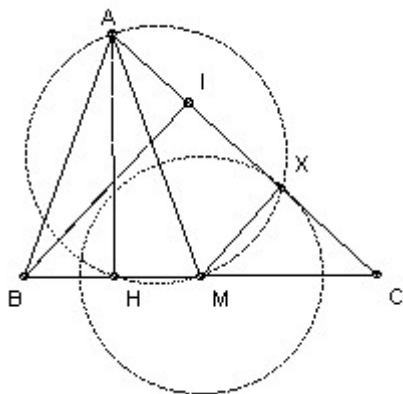
IMO 1960



**Problem B1**

Construct a triangle ABC given the lengths of the altitudes from A and B and the length of the median from A.

**Solution**



Let M be the midpoint of BC, AH the altitude from A, and BI the altitude from B. Start by constructing AHM. Take X on the circle diameter AM with  $MX = BI/2$ . Let the lines AX, HM meet at C and take B so that  $BM = MC$ . [This works because  $CMX$  and  $CBI$  are similar with  $MX = BI/2$  and hence  $CM = CB/2$ .]

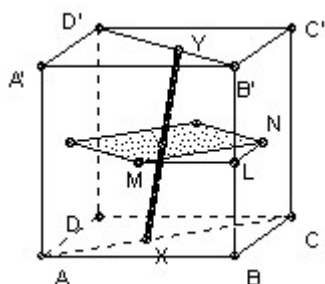


**Problem B2**

The cube  $ABCD A'B'C'D'$  has  $A$  above  $A'$ ,  $B$  above  $B'$  and so on.  $X$  is any point of the face diagonal  $AC$  and  $Y$  is any point of  $B'D'$ .

- (a) find the locus of the midpoint of  $XY$ ;
- (b) find the locus of the point  $Z$  which lies one-third of the way along  $XY$ , so that  $ZY=2 \cdot XZ$ .

**Solution**

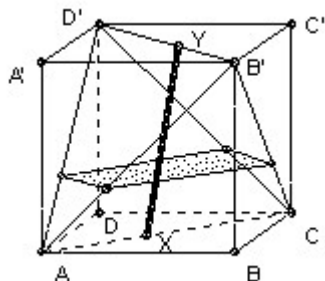


The key idea is that the midpoint must lie in the plane half-way between  $ABCD$  and  $A'B'C'D'$ . Similarly,  $Z$  must lie in the plane one-third of the way from  $ABCD$  to  $A'B'C'D'$ .

- (a) Regard  $ABCD$  as horizontal. Then the locus is the square with vertices the midpoints of the vertical faces (shown shaded in the diagram).

Take  $Y$  at  $B'$  and let  $X$  vary, then we trace out  $MN$ . Similarly, we can get the other sides. Now with  $Y$  at  $B'$ , take  $X$  in general position, so the midpoint of  $XY$  is on  $MN$ . Now move  $Y$  to  $D'$ , the midpoint traces out a line parallel to the other two sides of the square, so we can get any point inside the square. But equally, it is clear that any point inside the triangle  $LMN$  corresponds to a point  $Y$  on the ray  $D'B'$  not between  $B'$  and  $D'$ , so it does not lie in the locus. Similarly for the other three triangles. So the locus is the square.

- (b) A similar argument shows that the locus is the rectangle shown in the diagram below which is  $\sqrt{2}/3 \times 2\sqrt{2}/3$ .



**Problem B3**

A cone of revolution has an inscribed sphere tangent to the base of the cone (and to the sloping surface of the cone). A cylinder is circumscribed about the sphere so that its base lies in the base of the cone. The volume of the cone is  $V_1$  and the volume of the cylinder is  $V_2$ .

- (a) Prove that  $V_1 \neq V_2$ ;  
 (b) Find the smallest possible value of  $V_1/V_2$ . For this case construct the half angle of the cone.

**Solution**

Let the vertex of the cone be  $V$ , the center of the sphere be  $O$  and the center of the base be  $X$ . Let the radius of the sphere be  $r$  and the half-angle of the cone  $\theta$ .

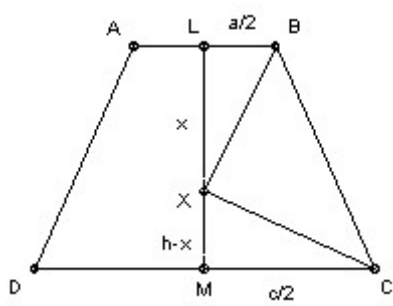
Then the cone's height is  $VO + OX = r(1 + 1/\sin \theta)$ , and the radius of its base is  $r(1 + 1/\sin \theta) \tan \theta$ . Hence  $V_1/V_2 = (1/6) (1 + 1/\sin \theta)^3 \tan^2 \theta = (1 + s)^3 (6s(1 - s^2))$ , where  $s = \sin \theta$ .

We claim that  $(1 + s)^3 (6s(1 - s^2)) \geq 4/3$ . This is equivalent to  $1 + 3s + 3s^2 + s^3 \geq 8s - 3s^3$  or  $1 - 5s + 3s^2 + 9s^3 \geq 0$ . But we can factorise the cubic as  $(1 - 3s)^2(1 + s)$ . So we have  $V_1/V_2 \geq 4/3$  with equality iff  $s = 1/3$ .



**Problem B4**

In the isosceles trapezoid  $ABCD$  ( $AB$  parallel to  $DC$ , and  $BC = AD$ ), let  $AB = a$ ,  $CD = c$  and let the perpendicular distance from  $A$  to  $CD$  be  $h$ . Show how to construct all points  $X$  on the axis of symmetry such that  $\angle BXC = \angle AXD = 90^\circ$ . Find the distance of each such  $X$  from  $AB$  and from  $CD$ . What is the condition for such points to exist?



**Solution**

Since angle  $BXC = 90^\circ$ ,  $X$  lies on the circle diameter  $BC$ . In general this will intersect the axis of symmetry in 0, 1 or 2 points. By symmetry any points of intersection  $X$  will also lie on the circle diameter  $AD$  and so will have angle  $AXD = 90^\circ$  also.

Let  $L$  be the midpoint of  $AB$ , and  $M$  the midpoint of  $CD$ . Let  $X$  lie on  $LM$  a distance  $x$  from  $L$ . We have  $LB = a/2$ ,  $MC = c/2$ , and  $XM = h - x$ . The triangles  $LBX$  and  $MXC$  are similar, so  $2x/a = c/(2(h-x))$ . Hence  $4x^2 - 4xh + ac = 0$ , so  $x = h/2 \pm (\sqrt{h^2 - ac})/2$ .

There are 0, 1, 2 points according as  $h^2 <, =, > ac$ .

## 1st IMO 1959



**A1.** Prove that  $(21n+4)/(14n+3)$  is irreducible for every natural number  $n$ .



**A2.** For what real values of  $x$  is  $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = A$  given (a)  $A = \sqrt{2}$ , (b)  $A = 1$ , (c)  $A = 2$ , where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?



**A3.** Let  $a, b, c$  be real numbers. Given the equation for  $\cos x$ :

$$a \cos^2 x + b \cos x + c = 0,$$



form a quadratic equation in  $\cos 2x$  whose roots are the same values of  $x$ . Compare the equations in  $\cos x$  and  $\cos 2x$  for  $a=4, b=2, c=-1$ .

**B1.** Given the length  $|AC|$ , construct a triangle  $ABC$  with  $\angle ABC = 90^\circ$ , and the median  $BM$  satisfying  $BM^2 = AB \cdot BC$ .



**B2.** An arbitrary point  $M$  is taken in the interior of the segment  $AB$ . Squares  $AMCD$  and  $MBEF$  are constructed on the same side of  $AB$ . The circles circumscribed about these squares, with centers  $P$  and  $Q$ , intersect at  $M$  and  $N$ .

- (a) prove that  $AF$  and  $BC$  intersect at  $N$ ;
- (b) prove that the lines  $MN$  pass through a fixed point  $S$  (independent of  $M$ );
- (c) find the locus of the midpoints of the segments  $PQ$  as  $M$  varies.



**B3.** The planes  $P$  and  $Q$  are not parallel. The point  $A$  lies in  $P$  but not  $Q$ , and the point  $C$  lies in  $Q$  but not  $P$ . Construct points  $B$  in  $P$  and  $D$  in  $Q$  such that the quadrilateral  $ABCD$  satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order  $A, B, C, D$ , (3) it is an isosceles trapezoid with  $AB$  parallel to  $CD$  (meaning that  $AD = BC$ , but  $AD$  is not parallel to  $BC$  unless it is a square), and (4) a circle can be inscribed in  $ABCD$  touching the sides.

## IMO 1959



### Problem A1

Prove that  $(21n+4)/(14n+3)$  is irreducible for every natural number  $n$ .

### Solution

$$3(14n+3) - 2(21n+4) = 1.$$

## IMO 1959



### Problem A2

For what real values of  $x$  is  $\sqrt{x + \sqrt{2x-1}} + \sqrt{x - \sqrt{2x-1}} = A$ , given (a)  $A = \sqrt{2}$ , (b)  $A = 1$ , (c)  $A = 2$ , where only non-negative real numbers are allowed in square roots and the root always denotes the non-negative root?

### Answer

(a) any  $x$  in the interval  $[1/2, 1]$ ; (b) no solutions; (c)  $x=3/2$ .

### Solution

Note that we require  $x \geq 1/2$  to avoid a negative sign under the inner square roots. Since  $(x-1)^2 \geq 0$ , we have  $x \geq \sqrt{2x-1}$ , so there is no difficulty with  $\sqrt{x - \sqrt{2x-1}}$ , provided that  $x \geq 1/2$ .

Squaring gives  $2x + 2\sqrt{(x^2-2x+1)} = A^2$ . Note that the square root is  $|x-1|$ , not simply  $(x-1)$ . So we get finally  $2x + 2|x-1| = A^2$ . It is now easy to see that we get the solutions above.

## IMO 1959



### Problem A3

Let  $a, b, c$  be real numbers. Given the equation for  $\cos x$ :

$$a \cos^2 x + b \cos x + c = 0,$$

form a quadratic equation in  $\cos 2x$  whose roots are the same values of  $x$ . Compare the equations in  $\cos x$  and  $\cos 2x$  for  $a=4, b=2, c=-1$ .

### Solution

You need that  $\cos 2x = 2 \cos^2 x - 1$ . Some easy manipulation then gives:

$$a^2 \cos^2 2x + (2a^2 + 4ac - 2b^2) \cos 2x + (4c^2 + 4ac - 2b^2 + a^2) = 0.$$

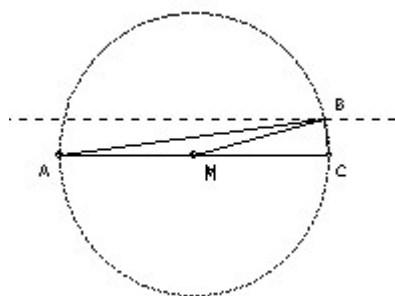
The equations are the same for the values of  $a, b, c$  given. The angles are  $2\pi/5$  (or  $8\pi/5$ ) and  $4\pi/5$  (or  $6\pi/5$ ).

## IMO 1959

### Problem B1

Given the length  $|AC|$ , construct a triangle  $ABC$  with  $\angle ABC = 90^\circ$ , and the median  $BM$  satisfying  $BM^2 = AB \cdot BC$ .

### Solution



Area =  $AB \cdot BC / 2$  (because  $\angle ABC = 90^\circ = BM^2 / 2$  (required) =  $AC^2 / 8$  (because  $BM = AM = MC$ ), so  $B$  lies a distance  $AC/4$  from  $AC$ . Take  $B$  as the intersection of a circle diameter  $AC$  with a line parallel to  $AC$  distance  $AC/4$ .



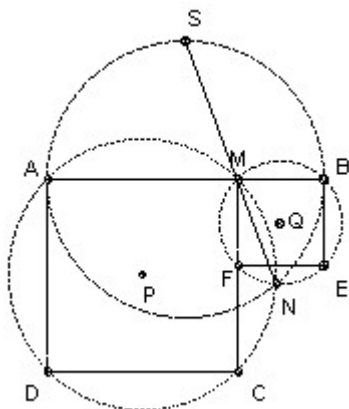


**Problem B2**

An arbitrary point  $M$  is taken in the interior of the segment  $AB$ . Squares  $AMCD$  and  $MBEF$  are constructed on the same side of  $AB$ . The circles circumscribed about these squares, with centers  $P$  and  $Q$ , intersect at  $M$  and  $N$ .

- (a) prove that  $AF$  and  $BC$  intersect at  $N$ ;
- (b) prove that the lines  $MN$  pass through a fixed point  $S$  (independent of  $M$ );
- (c) find the locus of the midpoints of the segments  $PQ$  as  $M$  varies.

**Solution**



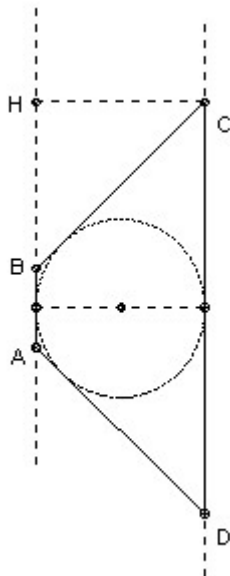
- (a)  $\angle ANM = \angle ACM = 45^\circ$ . But  $\angle FNM = \angle FEM = 45^\circ$ , so  $A, F, N$  are collinear. Similarly,  $\angle BNM = \angle BEM = 45^\circ$  and  $\angle CNM = 180^\circ - \angle CAM = 135^\circ$ , so  $B, N, C$  are collinear.
- (b) Since  $\angle ANM = \angle BNM = 45^\circ$ ,  $\angle ANB = 90^\circ$ , so  $N$  lies on the semicircle diameter  $AB$ . Let  $NM$  meet the circle diameter  $AB$  again at  $S$ .  $\angle ANS = \angle BNS$  implies  $AS = BS$  and hence  $S$  is a fixed point.
- (c) Clearly the distance of the midpoint of  $PQ$  from  $AB$  is  $AB/4$ . Since it varies continuously with  $M$ , it must be the interval between the two extreme positions, so the locus is a segment length  $AB/2$  centered over  $AB$ .



**Problem B3**

The planes  $P$  and  $Q$  are not parallel. The point  $A$  lies in  $P$  but not  $Q$ , and the point  $C$  lies in  $Q$  but not  $P$ . Construct points  $B$  in  $P$  and  $D$  in  $Q$  such that the quadrilateral  $ABCD$  satisfies the following conditions: (1) it lies in a plane, (2) the vertices are in the order  $A, B, C, D$ , (3) it is an isosceles trapezoid with  $AB$  parallel to  $CD$  (meaning that  $AD = BC$ , but  $AD$  is not parallel to  $BC$  unless it is a square), and (4) a circle can be inscribed in  $ABCD$  touching the sides.

**Solution**



Let the planes meet in the line  $L$ . Then  $AB$  and  $CD$  must be parallel to  $L$ . Let  $H$  be the foot of the perpendicular from  $C$  to  $AB$ . The fact that a circle can be inscribed implies  $AB + CD = BC + AD$  (equal tangents from  $A, B, C, D$  to the circle). Also  $CD = AB \pm 2BH$ . This leads to  $AH = AD = BC$ .

The construction is now easy. First construct the point  $H$ . Then using the circle center  $C$  radius  $AH$ , construct  $B$ . Using the circle center  $A$  radius  $AH$  construct  $D$ .

Note that if  $CH > AH$  then no construction is possible. If  $CH < AH$ , then there are two solutions, one with  $AB > CD$ , the other with  $AB < CD$ . If  $CH = AH$ , then there is a single solution, which is a square.

----- *The end* -----

