Homemade Problem Collection

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Abstract

This document is a fairly comprehensive (though not exhaustive) list of problems I have proposed for various competitions. Exceptionally nice problems are marked with a star (★).

Contents

List of Common Acronyms

- AIME: American Invitational Mathematics Exam
- AMC: American Mathematics Competition
- AMSP: AwesomeMath Summer Program
- ARML: American Regional Mathematics League
- CMIMC: Carnegie Mellon Informatics and Mathematics Competition
- IMO: International Mathematical Olympiad
- NIMO: National Internet Math Olympiad *(now defunct)*

1 Algebra

1. (AMC 10A 2021) What is the value of
$$2^{1+2+3} - (2^1 + 2^2 + 2^3)?$$

2. (NIMO Summer Contest 2015) For all real numbers a and b, let
$$a \bowtie b = \frac{a+b}{a-b}.$$
Compute $1008 \bowtie 1007$.

3. (Various Contests)[1] Determine the value of $\dfrac{2011^2 - 1983^2}{2011+1983}$.

4. (AMC 10A 2018) Sangho uploaded a video to a website where viewers can vote that they like or dislike a video. Each video begins with a score of 0, and the score increases by 1 for each like vote and decreases by 1 for each dislike vote. At one point Sangho saw that his video had a score of 90, and that 65% of the votes cast on his video were like votes. How many votes had been cast on Sangho's video at that point?

5. (Mock AMC 12 2012) A set of 31 real numbers has a sum of 104. A new set is formed by taking each number in the old set and increasing it by 3. For instance, a 1 in the old set would become a 4 in the new set. What is the sum of the numbers in the new set?

6. (Mock AIME I 2015) David, Justin, Richard, and Palmer are demonstrating a "math magic" concept in front of an audience. There are four boxes, labeled A, B, C, and D, and each one contains a different number. First, David pulls out the numbers in boxes A and B and reports that their product is 14. Justin then claims that the product of the numbers in boxes B and C is 16, and Richard states the product of the numbers in boxes C and D to be 18. Finally, Palmer announces the product of the numbers in boxes D and A. If k is the number that Palmer says, what is $20k$?[2]

7. (NIMO Summer Contest 2015) On a 30 question test, Question 1 is worth one point, Question 2 is worth two points, and so on up to Question 30. David takes the test and afterward finds out he answered nine of the questions incorrectly. However, he was not told which nine were incorrect. What is the highest possible score he could have attained?[3]

8. (NIMO 23) For real numbers x and y, define
$$\nabla(x,y) = x - \frac{1}{y}.$$
If
$$\underbrace{\nabla(2,\nabla(2,\nabla(2,\ldots\nabla(2,\nabla(2,2))\ldots)))}_{2016\,\nabla\text{s}} = \frac{m}{n}$$
for relatively prime positive integers m, n, compute $100m+n$.

[1] Okay sure, this problem is definitely not original, but I've used it probably at least three times for three different contests and changed the year each time, so I figured I might as well put it here. (As a side note, the number 1983 was chosen because that was the year of the very first AIME!)

[2] This problem achieved a 100% success rate among users who submitted for grading. Hooray!

[3] The legendary 420 problem!

9. (NIMO April 2017) For rational numbers a and b with $a > b$, define the *fractional average* of a and b to be the unique rational number c with the following property: when c is written in lowest terms, there exists an integer N such that adding N to both the numerator and denominator of c gives a, and subtracting N from both the numerator and denominator of c gives b. Suppose the fractional average of $\frac{1}{7}$ and $\frac{1}{10}$ is $\frac{m}{n}$, where m, n are coprime positive integers. What is $100m + n$?

10. (CMIMC 2019) Points $A(0,0)$ and $B(1,1)$ are located on the parabola $y = x^2$. A third point C is positioned on this parabola between A and B such that $AC = CB = r$. What is r^2?

11. (CMIMC 2018) The solutions in z to the equation

$$\left(z + \frac{337}{z}\right)^2 = 1$$

form the vertices of a quadrilateral in the complex plane. Compute the area of this quadrilateral.[4]

12. (CMIMC 2018) Suppose $x > 1$ is a real number such that $x + \frac{1}{x} = \sqrt{22}$. What is $x^2 - \frac{1}{x^2}$?

13. (Brilliant.org) Find the positive integer x such that

$$\frac{1}{10x} = \frac{1}{x^2 + x} + \frac{1}{x^2 + 3x + 2} + \frac{1}{x^2 + 5x + 6}.$$

14. (Mock AMC 10/12 2013) Suppose a and b are real[5] numbers that satisfy the system of equations

$$a + b = 9,$$
$$a(a-2) + b(b-2) = 21.$$

What is ab?

15. (CMIMC 2019) Let $P(x)$ be a quadratic polynomial with real coefficients such that $P(3) = 7$ and

$$P(x) = P(0) + P(1)x + P(2)x^2$$

for all real x. What is $P(-1)$?

16. Two problems about the difference between roots of a quadratic.

    (a) (AMSP Team Contest 2015) A line passing through the point $(0,3)$ intersects the graph of $y = x^2$ in two distinct points. The positive difference in x-coordinates of these two points is 5. Compute the positive difference between the points' y-coordinates.

    (b) (RHS Guts Round 2015) There exists a positive real number a such that the equation

    $$ax^2 = x + a$$

    has two real solutions with the property that the larger one is 5 more than the smaller one. What is a^2?

[4]The choice of 337 came from the fact that this was the second half of a relay question.
[5]Note: they're not actually real

17. (CMIMC 2018) Let $P(x) = x^2 + 4x + 1$. What is the product of all real solutions to the equation $P(P(x)) = 0$?

18. (CMIMC 2016) Construction Mayhem University[6] has been on a mission to expand and improve its campus! The university has recently adopted a new construction schedule where a new project begins every two days. Each project will take exactly one more day than the previous one to complete (so the first project takes 3, the second takes 4, and so on.)

    Suppose the new schedule starts on Day 1. On which day will there first be at least 10 projects in place at the same time?

19. (NIMO April 2017) Nonzero real numbers a, b, c satisfy the equations $a^2 + b^2 + c^2 = 2915$ and $(a-1)(b-1)(c-1) = abc - 1$. Compute $a + b + c$.

20. (CMIMC 2017) Suppose $P(x)$ is a quadratic polynomial with integer coefficients satisfying the identity
$$P(P(x)) - P(x)^2 = x^2 + x + 2016$$
    for all real x. What is $P(1)$?

21. (NIMO 29) Let $\{a_n\}$ be a sequence of integers such that $a_1 = 2016$ and
$$\frac{a_{n-1} + a_n}{2} = n^2 - n + 1$$
    for all $n \geq 1$. Compute a_{100}.

★ 22. (CMIMC 2016) Let ℓ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2} = \frac{13}{37}$. Then
$$\frac{(1+\ell)^3}{1+\ell^3} = \frac{m}{n}$$
    where m and n are positive relatively prime integers. Find $m + n$.

23. (CMIMC 2018) 2018 little ducklings numbered 1 through 2018 are standing in a line, with each holding a slip of paper with a nonnegative number on it; it is given that ducklings 1 and 2018 have the number zero. At some point, ducklings 2 through 2017 simultaneously change their number to equal the average of the numbers of the ducklings to their immediate left and right. Suppose the new numbers on the ducklings sum to 1000. What is the maximum possible sum of the original numbers on all 2018 slips?[7]

24. (Mock AMC 10/12 2013) Let p and q be real numbers with $|p| < 1$ and $|q| < 1$ such that
$$p + pq + pq^2 + pq^3 + \cdots = 2 \qquad \text{and} \qquad q + qp + qp^2 + qp^3 + \cdots = 3.$$
    What is $100pq$?

25. (Mock AMC 12 2012) The polynomial $x^3 - Ax + 15$ has three real roots. Two of these roots sum to 5. What is $|A|$?

26. Two logarithm problems with a very similar flavor.

[6] Sorry CMU!

[7] The wording here is clarified to reflect the original intention of the problem. Apologies for any confusion!

(a) (Mock AMC 10/12 2013) Suppose x and y are real numbers such that $\log_x(y) = 6$ and $\log_{2x}(2y) = 5$. What is $\log_{4x}(4y)$?

(b) (Mock AIME I 2015) Suppose that x and y are real numbers such that $\log_x 3y = \frac{20}{13}$ and $\log_{3x} y = \frac{2}{3}$. The value of $\log_{3x} 3y$ can be expressed in the form $\frac{a}{b}$ where a and b are positive relatively prime integers. Find $a+b$.

27. (ARML Local 2019) Compute the maximum value of the function $f(x) = \sin x + \sin(x + \arccos \frac{4}{7})$ as x varies over all real numbers.

28. (NIMO 16) Let a and b be positive real numbers such that $ab = 2$ and

$$\frac{a}{a+b^2} + \frac{b}{b+a^2} = \frac{7}{8}.$$

Find $a^6 + b^6$.

29. (CMIMC 2018) For all real numbers r, denote by $\{r\}$ the fractional part of r, i.e. the unique real number $s \in [0,1)$ such that $r - s$ is an integer. How many real numbers $x \in [1,2)$ satisfy the equation $\left\{x^{2018}\right\} = \left\{x^{2017}\right\}$?

30. (CMIMC 2020) For all real numbers x, let $P(x) = 16x^3 - 21x$. What is the sum of all possible values of $\tan^2\theta$, given that θ is an angle satisfying

$$P(\sin\theta) = P(\cos\theta)?$$

31. (AMSP Team Contest 2015) There exists exactly one polynomial $g(z)$ with nonnegative coefficients such that

$$g(z)g(-z) = z^4 - 47z^2 + 1$$

for all z. Find this polynomial.[8]

32. (NIMO 29) Let $\{a_i\}_{i=0}^{\infty}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

for all $|x| < 1$. Find a_{1000}.

33. (ARML Local 2018) Suppose a, b, and c are real numbers such that a, b, c and $a, b+1, c+2$ are both geometric sequences in their respective orders. Find the smallest possible value of $(a+b+c)^2$.

34. (CMIMC 2017) Suppose x, y, and z are nonzero complex numbers such that $(x+y+z)(x^2+y^2+z^2) = x^3+y^3+z^3$. Compute

$$(x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$[9]

[8] The number of submissions during the contest which had the polynomial written in terms of x as opposed to in terms of z was quite astonishing.

[9] This may seem incredibly high placement given the fact this was problem 2 on the team round, but the latter placement was because I figured teams could easily fakesolve it by say plugging $x = y = 1$ and I didn't want to remove the beauty of the problem statement by adding in an unnecessary "maximum" clause.

35. (NIMO 18) There exists a unique strictly increasing arithmetic sequence $\{a_i\}_{i=1}^{100}$ of positive integers such that

$$a_1 + a_4 + a_9 + \cdots + a_{100} = 1000,$$

where the summation runs over all terms of the form a_{i^2} for $1 \le i \le 10$. Find a_{50}.[10]

36. (CMIMC 2016) A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$ respectively. What is the smallest possible value of $a + b$?[11]

37. (Mock AMC 10/12 2013) For each positive integer n, let $a_n = \dfrac{n^2}{2n+1}$. Furthermore, let P and Q be real numbers such that

$$P = a_1 a_2 \ldots a_{2013}, \qquad Q = (a_1 + 1)(a_2 + 1) \ldots (a_{2013} + 1).$$

What is the sum of the digits of $\lfloor \frac{Q}{P} \rfloor$? (Note: $\lfloor x \rfloor$ denotes the largest integer $\le x$.)

38. (NICE Spring 2021)[12] Suppose x and y are nonzero real numbers satisfying the system of equations

$$\begin{aligned} 3x^2 + y^2 &= 13x, \\ x^2 + 3y^2 &= 14y. \end{aligned}$$

Find $x + y$.

39. (NIMO 27) Suppose there exist constants A, B, C, and D such that

$$n^4 = A\binom{n}{4} + B\binom{n}{3} + C\binom{n}{2} + D\binom{n}{1}$$

holds true for all positive integers $n \ge 4$. What is $A + B + C + D$?[13]

40. (CMIMC 2018) Suppose a, b, and c are nonzero real numbers such that

$$bc + \frac{1}{a} = ca + \frac{2}{b} = ab + \frac{7}{c} = \frac{1}{a+b+c}.$$

Find $a + b + c$.

41. (Mock AMC 10/12 2013) Suppose A and B are angles such that

$$\begin{aligned} \cos(A) + \cos(B) &= \tfrac{5}{4}, \\ \cos(2A) + \cos(2B) &= \tfrac{1}{9}. \end{aligned}$$

If $\cos(3A) + \cos(3B)$ is written as a fraction in lowest terms, what is the sum of the numerator and denominator?[14]

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[10] Inspired from what eventually would become NIMO 24 #1, hence the dual authorship with Tony Kim

[11] Compare with 2016 Spring OMO #5. Fortunately, no harm was done, although I became very scared when I first saw the connection.

[12] An extensive account of the creation of this problem can be found here.

[13] This problem has interesting ties to Linear Algebra, since the sequence of polynomials $\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k}$ can be viewed as a basis of the set of all finite-degree polynomials in $\mathbb{R}$. (The Fibonacci polynomials question later in this section also revolves around this idea.)

[14] This is a bit computational. This was still relatively early on in my problem writing days, so I don't think I had mastered the art of picking nice numbers at this point....

42. (AMC 10A 2019) Let p, q, and r be the distinct roots of the polynomial $x^3 - 22x^2 + 80x - 67$. It is given that there exist real numbers A, B, and C such that

$$\frac{1}{s^3 - 22s^2 + 80s - 67} = \frac{A}{s-p} + \frac{B}{s-q} + \frac{C}{s-r}$$

for all $s \not\in \{p, q, r\}$. What is $\frac{1}{A} + \frac{1}{B} + \frac{1}{C}$?

43. (NIMO 24) Let f be the quadratic function with leading coefficient 1 whose graph is tangent to that of the lines $y = -5x + 6$ and $y = x - 1$. The sum of the coefficients of f is $\frac{p}{q}$, where p and q are positive relatively prime integers. Find $100p + q$.[15]

44. (NIMO 15) Let $S = \{1, 2, \ldots, 2014\}$. Suppose that

$$\sum_{T \subseteq S} i^{|T|} = p + qi$$

where p and q are integers, $i = \sqrt{-1}$, and the summation runs over all 2^{2014} subsets of S. Find the remainder when $|p| + |q|$ is divided by 1000. (Here $|X|$ denotes the number of elements in a set X.)

45. (CMIMC 2016) Determine the value of the sum

$$\left| \sum_{1 \le i < j \le 50} ij(-1)^{i+j} \right|.$$

46. (Problem Stash 2) Let a, b, and c be nonzero real numbers such that $a + b + c = 0$ and

$$28(a^4 + b^4 + c^4) = a^7 + b^7 + c^7.$$

Find $a^3 + b^3 + c^3$.

47. (CMIMC 2017) Suppose P is a quintic polynomial with real coefficients with $P(0) = 2$ and $P(1) = 3$ such that $|z| = 1$ whenever z is a complex number satisfying $P(z) = 0$. What is the smallest possible value of $P(2)$ over all such polynomials P?

48. (AMC 12A 2020) Let (a_n) and (b_n) be the sequences of real numbers such that

$$(2 + i)^n = a_n + b_n i$$

for all integers $n \ge 0$, where $i = \sqrt{-1}$. What is

$$\sum_{n=0}^{\infty} \frac{a_n b_n}{7^n}\,?$$

49. (NIMO 18) Find the sum of all positive integers $1 \le k \le 99$ such that there exist positive integers a and b with the property that

$$x^{100} - ax^k + b = (x^2 - 2x + 1)P(x)$$

for some polynomial P with integer coefficients.

[15] This was originally intended as a calculus problem. Then I realized the non-calculus solution was shorter.

50. (The CALT[16]) Suppose u and v are complex numbers satisfying the system of equations

$$(u-1)(v-1) = 9 \qquad \text{and} \qquad u^3 - u^2 = v^3 - v^2$$

Find the sum of all possible values of $|u|^2 + |v|^2$.

51. (ARML Local 2019)[17] Given that $\sum_{k=0}^{\infty} \binom{2k}{k} \frac{1}{5^k} = \sqrt{5}$, compute the value of the sum

$$\sum_{k=0}^{\infty} \binom{2k+1}{k} \frac{1}{5^k}.$$

52. (Problem Stash 1) Find the maximum possible value of $\sum_{j=1}^{n} |z_j|^2$ over all sequences $z_1, z_2, \ldots, z_n$ of complex numbers with nonnegative real parts satisfying

$$\sum_{j=1}^{n} z_j = 5 \quad \text{and} \quad \sum_{j=1}^{n} z_j^2 = 7.$$

53. (NIMO Summer Contest 2015) Suppose x and y are real numbers such that

$$x^2 + xy + y^2 = 2 \qquad \text{and} \qquad x^2 - y^2 = \sqrt{5}.$$

The sum of all possible distinct values of $|x|$ can be written in the form $\sum_{i=1}^{n} \sqrt{a_i}$, where each of the a_i is a rational number. If $\sum_{i=1}^{n} a_i = \frac{m}{n}$ where m and n are positive relatively prime integers, what is $100m + n$?

54. (Mock AIME I 2013) Let T_n denote the nth triangular number, i.e. $T_n = 1 + 2 + 3 + \cdots + n$. Let m and n be positive integers so that

$$\sum_{i=3}^{\infty} \sum_{k=1}^{\infty} \left(\frac{3}{T_i} \right)^k = \frac{m}{n}.$$

Find $m + n$.[18]

55. (Problem Stash 2) Suppose α, β, and γ are real numbers satisfying the system of equations

$$\begin{aligned} \cos\alpha \cos(\beta - \gamma) &= \tfrac{3}{8}, \\ \cos\beta \cos(\gamma - \alpha) &= \tfrac{1}{2}, \\ \cos\gamma \cos(\alpha - \beta) &= \tfrac{3}{4}. \end{aligned}$$

The minimum possible value of $\cos(2\gamma)$ can be written in the form $\frac{m - \sqrt{n}}{p}$, where m, n, and p are positive integers with m and p relatively prime. Find $m + n + p$.

[16] A mock competition run by two users on AoPS.

[17] An exercise toward the end of *A First Course in Complex Analysis* guides readers through a proof of the sum $\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$ using the identity

$$\binom{n}{k} = \frac{1}{2\pi i} \int_{\gamma} \frac{(z+1)^n}{z^{k+1}} \, dz,$$

where γ is any simple closed piecewise smooth path with 0 in its interior. This problem was born out of exploring this technique to try to evaluate different yet similar sums of binomial coefficients.

[18] I find it interesting how the 3 is so important here; replacing it with any other number will break the intended solution.

56. (Mock AIME II 2012) There exist real values of a and b such that $a+b=n$, $a^2+b^2=2n$, and $a^3+b^3=3n$ for some value of n. Let S be the sum of all possible values of a^4+b^4. Find S.

57. Two similar-looking Vieta problems.

    (a) (NICE Spring 2021) Let r_1, r_2, and r_3 be the roots of the polynomial x^3-x+1. Then

    $$\frac{1}{r_1^2+r_1+1}+\frac{1}{r_2^2+r_2+1}+\frac{1}{r_3^2+r_3+1}=\frac{m}{n},$$

    where m and n are positive relatively prime integers. Find $1000m+n$.

    (b) (CMIMC 2016) Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20}-7x^3+1$. If

    $$\frac{1}{r_1^2+1}+\frac{1}{r_2^2+1}+\cdots+\frac{1}{r_{20}^2+1}$$

    can be written in the form $\frac{m}{n}$ where m and n are positive integers, find $m+n$.

58. (ARML Local 2018) For all nonnegative integers c, let $f(c)$ denote the unique real number x satisfying the equation

    $$\frac{x^4+x^5-48}{x^9-1}=c.$$

    Compute $f(0)f(1)f(2)\cdots f(47)$.[19]

59. (AOIME[20] 2020) Let $P(x)=x^2-3x-7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of x^2 equal to 1. David computes each of the three sums $P+Q$, $P+R$, and $Q+R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0)=2$, then $R(0)=\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

60. (CMIMC 2019) Across all $x\in\mathbb{R}$, find the maximum value of the expression

    $$\sin x+\sin 3x+\sin 5x.$$

61. (CMIMC 2016) Suppose x and y are real numbers which satisfy the system of equations

    $$x^2-3y^2=\frac{17}{x} \qquad \text{and} \qquad 3x^2-y^2=\frac{23}{y}.$$

    Then x^2+y^2 can be written in the form $\sqrt[m]{n}$ where m and n are positive integers and m is as small as possible. Find $m+n$.

★ 62. (CMIMC 2016) Suppose a, b, c, and d are positive real numbers which satisfy the system of equations

$$\begin{aligned}(a+b)(c+d)&=143,\\(a+c)(b+d)&=150,\\(a+d)(b+c)&=169.\end{aligned}$$

[19] I originally submitted this to the NIMO database in June 2015! I think I intended for it to be a Summer Contest question, which is why it went unused for so long.

[20] a replacement for the AIME II, where the "O" stands for "online"

Find the smallest possible value of $a^2+b^2+c^2+d^2$.[21]

63. (CMIMC 2017) The parabola $\mathcal{P}$ given by equation $y=x^2$ is rotated some acute angle θ clockwise about the origin such that it hits both the x and y axes at two distinct points. Suppose the length of the segment $\mathcal{P}$ cuts the x-axis is 1. What is the length of the segment $\mathcal{P}$ cuts the y-axis?

64. (CMIMC 2018, with Keerthana Gurushankar) Let a be a complex number, and set α, β, and γ to be the roots of the polynomial x^3-x^2+ax-1. Suppose
$$(\alpha^3+1)(\beta^3+1)(\gamma^3+1)=2018.$$
Compute the product of all possible values of a.[22]

65. (NIMO 15) For all positive integers k, define $f(k)=k^2+k+1$. Compute the largest positive integer n such that
$$2015f(1^2)f(2^2)\cdots f(n^2)\geq\Big(f(1)f(2)\cdots f(n)\Big)^2.$$

66. (AIME 2020) Let $P(x)$ be a quadratic polynomial with complex coefficients whose x^2 coefficient is 1. Suppose the equation $P(P(x))=0$ has four distinct solutions, $x=3,4,a,b$. Find the sum of all possible values of $(a+b)^2$.

67. (CMIMC 2019) It is given that the roots of the polynomial $P(z)=z^{2019}-1$ can be written in the form $z_k=x_k+iy_k$ for $1\leq k\leq 2019$. Let Q denote the monic polynomial with roots equal to $2x_k+iy_k$ for $1\leq k\leq 2019$. Compute $Q(-2)$.

★ 68. (CMIMC 2018) Suppose P is a cubic polynomial satisfying $P(0)=3$ and
$$(x^3-2x+1-P(x))(2x^3-5x^2+4-P(x))\leq 0$$
for all $x\in\mathbb{R}$. Determine all possible values of $P(-1)$.[23]

69. (CMIMC 2018) Suppose $a_0,a_1,\ldots,a_{2018}$ are integers such that
$$(x^2-3x+1)^{1009}=\sum_{k=0}^{2018}a_kx^k$$
for all real numbers x. Compute the remainder when $a_0^2+a_1^2+\cdots+a_{2018}^2$ is divided by 2017.[24]

70. (Mock AIME I 2015) Suppose α, β, and γ are complex numbers that satisfy the system of equations
$$\alpha+\beta+\gamma=6,$$
$$\alpha^3+\beta^3+\gamma^3=87,$$
$$(\alpha+1)(\beta+1)(\gamma+1)=33.$$
If $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{m}{n}$ for positive relatively prime integers m and n, find $m+n$.

[21] Inspired simultaneously by NIMO 19 #7 and AMST 2016 Algebra #9. Also compare with 2016 Japan Mathematical Olympiad Preliminary #7 apparently??

[22] More specifically, Keerthana came up with the original setup of looking at $\prod(\alpha^3+1)$; I turned the problem in a new direction by adding another layer to the computation of the product.

[23] This was directly inspired by trying to generalize AIME 2010.I.6 to the case where the parabolas do not share a common vertex.

[24] This problem was the result of spending about a month trying to find a clever use of generating functions. While this result is nothing new (e.g. the OEIS pages for sequences A002426 and A082758 cross-reference each other), I've never seen this exact idea in a contest setting before, so it's all good.

★ 71. (CMIMC 2017) Let a, b, and c be complex numbers satisfying the system of equations

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=9,$$
$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}=32,$$
$$\frac{a^3}{b+c}+\frac{b^3}{c+a}+\frac{c^3}{a+b}=122.$$

Find abc.[25]

72. (Mock AIME I 2013) Let a, b, and c be the roots of the equation $x^3+2x-1=0$, and let X and Y be the two possible values of $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$. Find $(X+1)(Y+1)$.

73. (CMIMC 2017) Suppose $a_1, a_2, \ldots, a_{10}$ are nonnegative integers such that

$$\sum_{k=1}^{10} a_k = 15 \qquad \text{and} \qquad \sum_{k=1}^{10} k a_k = 80.$$

Let M and m denote the maximum and minimum respectively of $\sum_{k=1}^{10} k^2 a_k$. Compute $M-m$.[26]

74. (ARML Local 2018) It is given that there exist real numbers $a_0, \ldots, a_{18}$ such that

$$\frac{\sin^2(10x)}{\sin^2(x)} = a_0 + a_1\cos(x) + a_2\cos(2x) + \cdots + a_{18}\cos(18x)$$

for all x with $\sin x \neq 0$. Compute $a_0^2 + a_1^2 + \cdots + a_{18}^2$.[27]

75. (CMIMC 2017) The polynomial $P(x) = x^3 - 6x - 2$ has three real roots, α, β, and γ. Depending on the assignment of the roots, there exist two different quadratics Q such that the graph of $y = Q(x)$ pass through the points (α, β), (β, γ), and (γ, α). What is the larger of the two values of $Q(1)$?[28]

★ 76. (Mock AMC 10/12 2013) Let

$$P(x) = \prod_{n=1}^{10}\left(x^{2^n} - x^{2^{n-1}} + 1\right) = (x^2-x+1)(x^4-x^2+1)(x^8-x^4+1)\ldots(x^{1024}-x^{512}+1).$$

When $P(x)$ is expanded, how many terms does it contain?[29]

[25] This problem exemplifies perhaps what is one of the biggest consistent flaws over lots of my problems: interesting initial ideas but somewhat computational finishes.

[26] This was A6 until about a week or two before the contest. It seems that I heavily underestimated the difficulty of this problem.

[27] This problem is a small adaptation of a lemma from a **21-723 Advanced Real Analysis** homework assignment. More specifically, the calculation in this question can be used to show that the Fejér kernel forms an approximate identity, which in turn can be used to show convergence results of Fourier series in certain L^p spaces.

[28] Heavily inspired by Problem 8.63 in Intermediate Algebra.

[29] This is probably one of the only problems I've ever written for which I knew the answer but not the formal solution while the contest was active.

★ 77. (CMIMC 2017) Let c denote the largest possible real number such that there exists a nonconstant polynomial P with
$$P(z^2) = P(z-c)P(z+c)$$
for all z. Compute the sum of all values of $P(\frac{1}{3})$ over all nonconstant polynomials P satisfying the above constraint for this c.[30]

78. (CMIMC 2016) Let $\mathcal{P}$ be the unique parabola in the xy-plane which is tangent to the x-axis at $(5,0)$ and to the y-axis at $(0,12)$. We say a line ℓ is $\mathcal{P}$-friendly if the x-axis, y-axis, and $\mathcal{P}$ divide ℓ into three segments, each of which has equal length. If the sum of the slopes of all $\mathcal{P}$-friendly lines can be written in the form $-\frac{m}{n}$ for m and n positive relatively prime integers, find $m+n$.

★ 79. (CMIMC 2016) Denote by $F_0(x)$, $F_1(x)$, … the sequence of Fibonacci polynomials, which satisfy the recurrence $F_0(x) = 1$, $F_1(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for all $n \geq 2$.[31] It is given that there exist integers λ_0, λ_1, …, λ_{1000} such that
$$x^{1000} = \sum_{i=0}^{1000} \lambda_i F_i(x)$$
for all real x. For which integer k is $|\lambda_k|$ maximized?[32]

[30] I'm still surprised that a nonzero value of c exists. Perhaps one of my favorite algebra problems I've ever written.

[31] In reality, the indices are shifted up by one (so e.g. $F_1(x) = 1$), but this interpretation makes the problem statement easier to write since $\deg F_i(x) = i$ for all $i \geq 0$

[32] This, like all the other #10s which appeared on CMIMC in 2016, took me several days to solve. I was very much okay with putting problems on the test that did take me multiple days to solve because of the very high skill cap of the contestants coming. Indeed, a few people almost solved this one completely!

2 Combinatorics

1. (Problem Stash 2) A new disease dubbed sineavirus has arrived! Symptoms include a fear of trigonometry, and so we want to detect it as soon as possible. Unfortunately, these symptoms do not show as soon as the virus infects the body. Indeed, for every integer $n \geq 1$, the probability that symptoms show[33] on the n^{th} day is $\frac{1}{2^n}$. (Symptoms do not show on the 0^{th} day.)

   Suppose a patient is infected on a Sunday. The probability that this person's symptoms first show on a Thursday is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

2. (NIMO Summer Contest 2016) Evan writes a computer program that randomly rearranges the digits $0, 2, 4, 6$, and 8 to create a five-digit number with no leading zeroes. If he executes this program once, the probability the program outputs an integer divisible by 4 can be written in the form $\frac{m}{n}$ where m and n are positive integers which share no common factors. What is $m+n$?[34]

3. (ARML Local 2018) The ARML Local staff have ranked each of the 10 individual problems in order of difficulty from 1 to 10. They wish to order the problems such that the problem with difficulty i is before the problem with difficulty $i+2$ for all $1 \leq i \leq 8$. Compute the number of orderings of the problems that satisfy this condition.[35]

4. (Mock AMC 10/12 2013) A regular 2013-gon is placed in the plane in general position. A family of lines is drawn on this plane such that no two lines are parallel, and for every vertex A of the 2013-gon, there exists a line that passes through A. What is the smallest number of possible lines in the family?

5. (NIMO Summer Contest 2015, with Tony Kim) The NIMO problem writers have invented a new chess piece called the *Oriented Knight*. This new chess piece has a limited number of moves: it can either move two squares to the right and one square upward or two squares upward and one square to the right. How many ways can the knight move from the bottom-left square to the top-right square of a 16×16 chess board?[36]

6. (NIMO 30) David draws a 2×2 grid of squares in chalk on the sidewalk outside NIMO HQ. He then draws one arrow in each square, each pointing in one of the four cardinal directions (north, south, east, west) parallel to the sides of the grid. In how many ways can David draw his arrows such that no two of the arrows are pointing at each other?

7. (NIMO 21) In the *Fragmented Game of Spoons*, eight players sit in a row, each with a hand of four cards. Each round, the first player in the row selects the top card from the stack of unplayed cards and either passes it to the second player, which occurs with probability $\frac{1}{2}$, or swaps it with one of the four cards in his hand, each card having an equal chance of being chosen, and passes the new card to the second player. The second player then takes the card from the first player and chooses a card to pass to the third player in the same way. Play continues until the eighth player is passed a card, at which point the card he chooses to pass is removed from the game and the next round begins. To win, a player must hold four cards of the same number, one of each suit.

[33] To clarify: this is the probability symptoms show for the *first* time; these probabilities sum to 1 on purpose.

[34] This was a dropped CMIMC 2016 problem. I'm not sure why this didn't make the test; my best guess is that we had other problems which probably fit better in the early spots.

[35] Originally a proposal for NIMO Summer Contest 2017 (with the appropriate flavortext modifications of course)

[36] Heavily inspired by a NIMO proposal from Tony, hence the dual authorship

During a game, David is the eighth player in the row and needs an Ace of Clubs to win. At the start of the round, the dealer picks up a Ace of Clubs from the deck. Suppose that Justin, the fifth player, also has a Ace of Clubs, and that all other Ace of Clubs cards have been removed. The probability that David is passed an Ace of Clubs during the round is $\frac{m}{n}$, where m and n are positive integers with $\gcd(m,n) = 1$. Find $100m + n$.

8. (USAMTS 2020-2021) Infinitely many math beasts stand in a line, all six feet apart, wearing masks, and with clean hands. Grogg starts at the front of the line, holding n pieces of candy, $n \geq 1$, and everyone else has none. He passes his candy to the beasts behind him, one piece each to the next n beasts in line. Then, Grogg leaves the line. The other beasts repeat this process: the beast in front, who has k pieces of candy, passes one piece each to the next k beasts in line, and then leaves the line. For some values of n, another beast, besides Grogg, temporarily holds all the candy. For which values of n does this occur?

9. (CMIMC 2020) David is taking a true/false exam with 9 questions. Unfortunately, he doesn't know the answer to any of the questions, but he does know that exactly 5 of the answers are True. In accordance with this, David guesses the answers to all 9 questions, making sure that exactly 5 of his answers are True. What is the probability he answers at least 5 questions correctly?

10. (NIMO April 2017) Two neighboring towns, MWMTown and NIMOTown, have a strange relationship with regard to weather. On a certain day, the probability that it is sunny in either town is $\frac{1}{23}$ greater than the probability of MWMTown being sunny, and the probability that it is sunny in MWMTown, given that it is sunny in NIMOTown, is $\frac{12}{23}$. If the probability that it is sunny in NIMOTown is $\frac{p}{q}$, for coprime positive integers p, q, what is $100p + q$?[37]

11. (AMC 10A 2021) How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

★ 12. (NICE Spring 2021) Ethan draws a 2×6 grid of points on a piece of paper such that each pair of adjacent points is distance 1 apart. How many ways can Ethan divide the 12 points into six pairs such that the distance between the points in each pair is in the interval $[2, 4)$?

13. (Mock AMC 10/12 2013) Mr. Mebane's Preschool WOOT class is taking a class photograph. There are five boys in the class, and they are 48, 49, 50, 51, and 52 inches tall. In the class photograph, they are to sit themselves in five chairs positioned in a row. In how many ways can Mr. Mebane's students seat themselves such that no boy sits next to another boy who is exactly one inch taller than himself?

14. (AMSP Team Contest 2015) DG chooses a three-element subset of the set $\{-10, -9, -8, \ldots, 8, 9, 10\}$. What is the probability that the three elements in his subset sum to zero?

15. (NIMO Summer Contest 2015) On a blackboard lies 50 magnets in a line numbered from 1 to 50, with different magnets containing different numbers. David walks up to the blackboard and rearranges the magnets into some arbitrary order. He then writes underneath each pair of consecutive magnets the positive difference between the numbers on the magnets. If the expected number of times he writes the number[38] 1 can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n, compute $100m + n$.

[37] oops this is super contrived but whatever
[38] Not just the digit!

16. (CMIMC 2017) Emily draws six dots on a piece of paper such that no three lie on a straight line, then draws a line segment connecting each pair of dots. She then colors five of these segments red. Her coloring is said to be *red-triangle-free* if for every set of three points from her six drawn points there exists an uncolored segment connecting two of the three points. In how many ways can Emily color her drawing such that it is red-triangle-free?

17. (Problem Stash 1) Let $\mathcal{T}$ denote the set of tilings of a 1×12 board using only 1×1 and 1×2 pieces. For each tiling $T\in\mathcal{T}$, let $f(T)$ denote the number of times a 1×1 square immediately precedes a 1×2 domino. What is
$$\sum_{T\in\mathcal{T}} f(T)?$$

18. (NIMO April 2017) Twenty lamps are placed on the vertices of a regular 20-gon. The lights turn on and off according to the following rule: if, in second k the two lights adjacent to a light L are either both on or both off, L is on in second $k+1$; otherwise, L is off in second $k+1$. Let N denote the number of possible original settings of the lamps so that after four moves all lamps are on. What is the sum of the digits of N?

19. (CMIMC 2019) There are 100 lightbulbs $B_1,\dots,B_{100}$ spaced evenly around a circle in this order. Additionally, there are 100 switches $S_1,\dots,S_{100}$ such that for all $1\le i\le 100$, switch S_i toggles the states of lights B_{i-1} and B_{i+1} (where here $B_{101}=B_1$). Suppose David chooses whether to flick each switch with probability $\frac{1}{2}$. What is the expected number of lightbulbs which are on at the end of this process given that not all lightbulbs are off?

20. (AIME 2018) Find the number of functions f from $\{0,1,2,3,4,5,6\}$ to the integers such that $f(0)=0$, $f(6)=12$, and
$$|x-y|\le\left|f(x)-f(y)\right|\le 3|x-y|$$
for all x and y in $\{0,1,2,3,4,5,6\}$.

21. (CMIMC 2018) Compute the number of rearrangements $a_1,a_2,\dots,a_{2018}$ of the sequence $1,2,\dots,2018$ such that $a_k>k$ for *exactly* one value of k.

22. (Problem Stash 2) A new *new* disease dubbed *cosineavirus* has arrived! Symptoms include a fear of trigonometry, and so we want to detect it as soon as possible. Unfortunately, these symptoms do not show as soon as the virus infects the body. Indeed, for every integer $n\ge 1$, the probability that symptoms show on the n^{th} day is $\frac{1}{n(n+1)}$. (Symptoms do not show on the 0^{th} day.)

    Suppose patient zero is infected on Day 0, and each day thereafter one new person is infected. Let r be the expected number of days after Day 0 until *someone* first shows symptoms. What is $\lfloor 100r\rfloor$?[39]

23. (NIMO 18) In a 4×4 grid of unit squares, five squares are chosen at random. The probability that no two chosen squares share a side is $\frac{m}{n}$ for positive relatively prime integers m and n. Find $m+n$.[40]

[39] The inspiration should be fairly clear :). In particular, this was inspired from thinking about disease spread, especially from the 2-14 day incubation period of the novel coronavirus.

[40] Probably one of my most infamous problems to date. I actually liked it, in part because of the method that I found to enumerating all the cases. But it seems that this opinion was not universal.

24. (USAMTS 2017-2018) Let n be a positive integer. Aavid has a card deck consisting of 2n cards, each colored with one of n colors such that every color is on exactly two of the cards. The $2n$ cards are randomly ordered in a stack. Every second, he removes the top card from the stack and places the card into an area called the pit. If the other card of that color also happens to be in the pit, Aavid collects both cards of that color and discards them from the pit.

    Of the $(2n)!$ possible original orderings of the deck, determine how many have the following property: at every point, the pit contains cards of at most two distinct colors.

25. (CMIMC 2016) For how many permutations π of $\{1,2,\dots,9\}$ does there exist an integer N such that
$$N \equiv \pi(i) \pmod{i} \text{ for all integers } 1 \le i \le 9?$$[41]

26. (CMIMC 2018) Let $\mathcal{F}$ be a family of subsets of $\{1,2,\dots,2017\}$ with the following property: whenever S_1 and S_2 are two elements of $\mathcal{F}$ with $S_1 \subsetneq S_2$, the quantity $|S_2 \setminus S_1|$ is odd. Compute the largest number of subsets $\mathcal{F}$ may contain.[42]

★ 27. (CMIMC 2018) Consider an undirected, connected graph G with vertex set $\{v_1, v_2, \dots, v_6\}$. Starting at the vertex v_1, an ant uses a DFS algorithm to traverse through G under the condition that if there are multiple unvisited neighbors of some vertex, the ant chooses the v_i with smallest i. How many possible graphs G are there satisfying the following property: for each $1 \le i \le 6$, the vertex v_i is the i^{th} new vertex the ant traverses?[43]

28. (CMIMC 2016) For all positive integers $m \ge 1$, denote by $\mathcal{G}_m$ the set of simple graphs with exactly m edges. Find the number of pairs of integers (m,n) with $1 < 2n \le m \le 100$ such that there exists a simple graph $G \in \mathcal{G}_m$ satisfying the following property: it is possible to label the edges of G with labels E_1, E_2, …, E_m such that for all $i \ne j$, edges E_i and E_j are incident if and only if either $|i-j| \le n$ or $|i-j| \ge m-n$.[44]

29. (Problem Stash 2) David has 10 cards in a bowl, with each card containing a distinct integer between 1 and 10. He separates the cards into 5 bins, numbered 1 through 5, so that each bin contains exactly two cards; each possible distribution has an equal chance of occurring. Let p be the probability David may visit each of the bins in order and choose one number from each bin to obtain an increasing sequence. Then p can be written in the form $\frac{m}{n}$, where m and n are positive relatively prime integers. Find $m+n$.

    (For example, if the first bin houses cards 2 and 4 and the second bin houses cards 1 and 3, then David may pick the 2 and 3 to obtain the desired sequence; however, he can not do this if the first bin holds cards 3 and 4 while the second holds cards 1 and 2.)

★ 30. (IMO 2019)[45] The Bank of Bath issues coins with an H on one side and a T on the other. Harry has n of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing H, then he turns over the kth coin from the

[41] See here. Darn, why is almost all my hard combo just nasty casework? (If you want to get the spirit of this problem without worrying about lots of casework, replace 9 with 6.)

[42] Putting this here because proving the result is somewhat harder than arriving at the answer.

[43] Inspired by a 15-210 programming assignment during the F17 semester!

[44] This hints at the idea of a *line graph*, which is a transformation we studied in 15-251 while working with polynomial-time reductions.

[45] I'm amazed. This problem is at the end of the combo section not because it's the hardest problem I've written, but because for now it just feels right here.

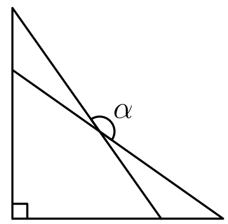
left; otherwise, all coins show T and he stops. For example, if $n = 3$ the process starting with the configuration THT would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration C, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all 2^n possible initial configurations C.

3 Geometry

1. (CMIMC 2019) The figure to the right depicts two congruent[46] triangles with angle measures $40°$, $50°$, and $90°$. What is the measure of the obtuse angle α formed by the hypotenuses of these two triangles?



2. (CMIMC 2016) Let $\triangle ABC$ be an equilateral triangle and P a point on $\overline{BC}$. If $PB = 50$ and $PC = 30$, compute PA.[47]

3. (CMIMC 2018) Let ABC be a triangle. Point P lies in the interior of ABC such that $\angle ABP = 20°$ and $\angle ACP = 15°$. Compute $\angle BPC - \angle BAC$.

4. (CMIMC 2017) Let ABC be a triangle with $\angle BAC = 117°$. The angle bisector of $\angle ABC$ intersects side AC at D. Suppose $\triangle ABD \sim \triangle ACB$. Compute the measure of $\angle ABC$, in degrees.

5. (AMSP Team Contest 2015) Suppose $ABCD$ is a convex quadrilateral with $\angle ABC = 144°$, $\angle ADC = 105°$, and $AB = BD = DC$. Compute $\angle BCD - \angle BAD$.

6. (CMIMC 2016) Let $ABCD$ be an isosceles trapezoid with $AD = BC = 15$ such that the distance between its bases AB and CD is 7. Suppose further that the circles with diameters $\overline{AD}$ and $\overline{BC}$ are tangent to each other. What is the area of the trapezoid?

7. (NICE Spring 2021) What is the smallest possible area of a rhombus $\mathcal{R}$ whose sides have length 5 and whose vertices are all points in the plane with integer coordinates?

8. (CMIMC 2019) Let ABC be an equilateral triangle with side length 2, and let M be the midpoint of $\overline{BC}$. Points X and Y are placed on AB and AC respectively such that $\triangle XMY$ is an isosceles right triangle with a right angle at M. What is the length of $\overline{XY}$?

9. (Problem Stash 2) Two congruent 3×11 rectangles fit into a triangle $\mathcal{T}$ as shown below. What is the area of $\mathcal{T}$?

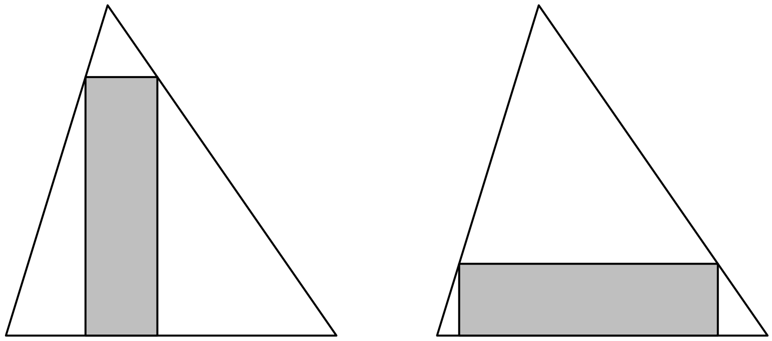
10. (CMIMC 2018) Let $ABCD$ be a square of side length 1, and let P be a variable point on $\overline{CD}$. Denote by Q the intersection point of the angle bisector of $\angle APB$ with $\overline{AB}$. The set of possible locations for Q as P varies along $\overline{CD}$ is a line segment; what is the length of this segment?

11. (NIMO 28) Trapezoid $ABCD$ is an isosceles trapezoid with $AD = BC$. Point P is the intersection of the diagonals AC and BD. If the area of $\triangle ABP$ is 50 and the area of $\triangle CDP$ is 72, what is the area of the entire trapezoid?[48]

[46] This part is not necessary, but I was worried about possible configuration loopholes that might occur otherwise.

[47] For some reason, I *really* like this problem. I think it is a very strong opening question.

[48] Also a dropped CMIMC problem. I think I took this off because I didn't feel it was original enough when compared with some of the other questions on the test.

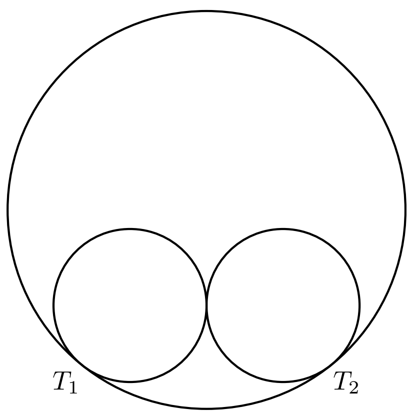


12. (CMIMC 2018) Let X and Y be points on semicircle AB with diameter 3. Suppose the distance from X to AB is $\frac{5}{4}$ and the distance from Y to AB is $\frac{1}{4}$. Compute

$$(AX+BX)^2-(AY+BY)^2.$$

13. (Mock AMC 10/12 2013) Rectangle $ABCD$ has $AB=5$ and $BC=12$. CD is extended to a point E such that $\triangle ACE$ is isosceles with base AC. What is the area of triangle ADE?[49]

14. (AMC 10A 2018) Two circles of radius 5 are externally tangent to each other and are internally tangent to a circle of radius 13 at points A and B, as shown in the diagram. The distance AB can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?



15. (CMIMC 2017) Triangle ABC has an obtuse angle at $\angle A$. Points D and E are placed on $\overline{BC}$ in the order B, D, E, C such that $\angle BAD=\angle BCA$ and $\angle CAE=\angle CBA$. If $AB=10$, $AC=11$, and $DE=4$, determine BC.

16. (TKMT[50]) Suppose $AOPS$ is an isosceles trapezoid with $AO\parallel PS$, $AO=26$, and $SP=36$. If the bisectors of $\angle ASP$ and $\angle OPS$ intersect on $\overline{AO}$, compute the area of $AOPS$.

17. (NIMO Summer Contest 2015) Let $\triangle ABC$ have $AB=3$, $AC=5$, and $\angle A=90^\circ$. Point D is the foot of the altitude from A to $\overline{BC}$, and X and Y are the feet of the altitudes from D to $\overline{AB}$ and $\overline{AC}$ respectively. If XY^2 can be written in the form $\frac{m}{n}$ where m and n are positive relatively prime integers, what is $100m+n$?

[49] This problem appeared on the Mock AMC 12 from 2012 as well; however, when submitting the problem, I accidentally miscopied and made the base EC instead. This makes the question much easier!

[50] Taimur Khalid Math Tournament, a contest on the Art of Problem Solving website I contributed a few questions to

18. (AMSP Team Contest 2015) Equilateral triangle ABC has side length 6. A circle ω is inscribed inside $\triangle ABC$. A point P is selected randomly on the circumference of ω. What is the probability that $\angle BPC$ is acute?

19. (CMIMC 2018) Let ABC be a triangle with side lengths 5, $4\sqrt{2}$, and 7. What is the area of the triangle with side lengths $\sin A$, $\sin B$, and $\sin C$?

20. (USAMTS 2020-2021) Find distinct points A, B, C, and D in the plane such that the length of the segment AB is an even integer, and the lengths of the segments AC, AD, BC, BD, and CD are all odd integers.[51]

21. (CMIMC 2019) Let $\triangle A_1B_1C_1$ be an equilateral triangle of area 60. Chloe constructs a new triangle $\triangle A_2B_2C_2$ as follows. First, she flips a coin. If it comes up heads, she constructs point A_2 such that B_1 is the midpoint of $\overline{A_2C_1}$. If it comes up tails, she instead constructs A_2 such that C_1 is the midpoint of $\overline{A_2B_1}$. She performs analogous operations on B_2 and C_2. What is the expected value of the area of $\triangle A_2B_2C_2$?

22. (CMIMC 2016) Let ABC be a triangle. The angle bisector of $\angle B$ intersects AC at point P, while the angle bisector of $\angle C$ intersects AB at a point Q. Suppose the area of $\triangle ABP$ is 27, the area of $\triangle ACQ$ is 32, and the area of $\triangle ABC$ is 72. The length of $\overline{BC}$ can be written in the form $m\sqrt{n}$ where m and n are positive integers with n as small as possible. What is $m+n$?[52]

23. (ARML Local 2018) Circles ω_1 and ω_2 have radii 5 and 12 respectively and have centers distance 13 apart. Let A and B denote the intersection points of ω_1 and ω_2. A line ℓ passing through A intersects ω_1 again at X and ω_2 again at Y such that $\angle ABY = 2\angle ABX$. Compute the length XY.

24. (CMIMC 2018) Let ABC be a triangle with $BC = 30$, $AC = 50$, and $AB = 60$. Circle ω_B is the circle passing through A and B tangent to BC at B; ω_C is defined similarly. Suppose the tangent to $\odot(ABC)$ at A intersects ω_B and ω_C for the second time at X and Y respectively. Compute XY.

25. (Mock AMC 10/12 2013) Let $ABCD$ be a rectangle such that $AB = 3$ and $BC = 4$. Suppose that M and N are the centers of the circles inscribed inside triangles $\triangle ABC$ and $\triangle ADC$ respectively. What is MN^2?[53]

26. (CMIMC 2016) Right isosceles triangle T is placed in the first quadrant of the coordinate plane. Suppose that the projection of T onto the x-axis has length 6, while the projection of T onto the y-axis has length 8. What is the sum of all possible areas of the triangle T?

27. (Farewell Mock Geometry AMC) Two concentric squares, one of side length 8 and the other of side length 6, are placed so that corresponding edges are parallel. A third square is placed so that it is inscribed inside the outer square but circumscribed around the inner square. What is the area of this third square?[54]

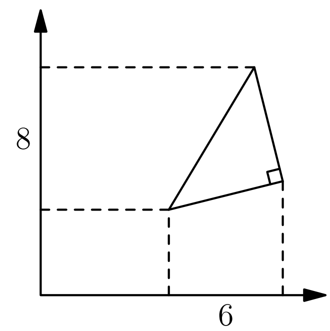
28. (Problem Stash 1, AMSP Team Contest 2019) Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Squares $ABDE$ and $ACGF$ are constructed externally to $\triangle ABC$, and lines BG and CD intersect at point P. If r is the distance from P to the line BC, compute $\lfloor 100r \rfloor$.

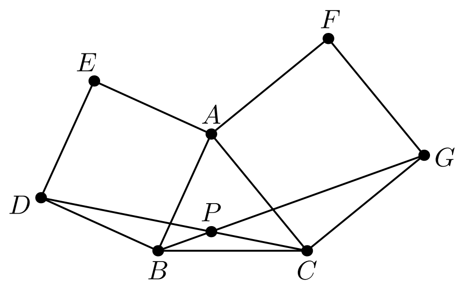
[51] This problem, written in 2017, was inspired by Miniature 6 of Jiři Matoušek's *Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra*, which in turn was based on a 1974 article in the American Mathematical Monthly. This miniature shows that, given six points in the plane, their distances cannot be all odd integers.

[52] Inspired by 2014 Purple Comet #20.

[53] I believe this is the oldest problem in this collection - it's dated all the way back to 2009!

[54] Compare with 2015 AMC 8 #25.





29. (CMIMC 2017) Let $ABCD$ be an isosceles trapezoid with $AD \parallel BC$. Points P and Q are placed on segments $\overline{CD}$ and $\overline{DA}$ respectively such that $AP \perp CD$ and $BQ \perp DA$, and point X is the intersection of these two altitudes. Suppose that $BX = 3$ and $XQ = 1$. Compute the largest possible area of $ABCD$.

30. (Mock AIME I 2015) Let A, B, C be points in the plane such that $AB = 25$, $AC = 29$, and $\angle BAC < 90^\circ$. Semicircles with diameters $\overline{AB}$ and $\overline{AC}$ intersect at a point P with $AP = 20$. Find the length of line segment $\overline{BC}$.[55]

31. (Mock AIME II 2012) Let $\triangle ABC$ be a triangle, and let I_A, I_B, and I_C be the points where the angle bisectors of A, B, and C, respectfully, intersect the sides opposite them. Given that $AI_B = 5$, $CI_B = 4$, and $CI_A = 3$, then the ratio $AI_C : BI_C$ can be written in the form m/n where m and n are positive relatively prime integers. Find $m + n$.

32. (Mock AMC 10/12 2013) A rhombus has height 15 and diagonals with lengths in a ratio of $3 : 4$. What is the length of the rhombus?

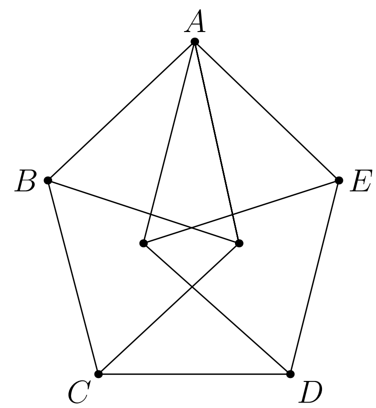
33. (AMC 10B/12B 2019) Points $A(6, 13)$ and $B(12, 11)$ lie on a circle Ω in the plane. Suppose that the tangents to Ω from A and B intersect at a point on the x-axis. What is the area of Ω?

34. (Mock AIME II 2012) A circle with radius 5 and center in the first quadrant is placed so that it is tangent to the y-axis. If the line passing through the origin that is tangent to the circle has slope $\frac{1}{2}$, then the y-coordinate of the center of the circle can be written in the form $\frac{m+\sqrt{n}}{p}$ where m, n, and p are positive integers, and $\gcd(m, p) = 1$. Find $m + n + p$.

35. (Farewell Mock Geometry AMC) Let $ABCD$ be a square with side length 4. A circle ω is drawn inscribed inside this square. Point E is placed on BC such that $BE = 1$, and line segment DE is drawn. This line segment intersects ω at distinct points F and G. Find the length FG.

[55] Compare with 2012 (2013?) Winter OMO #19. Oops! No harm done, though.

36. (AMC 10B/12B 2021) The figure below is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m}+\sqrt{n}$, where m and n are positive integers. What is $m+n$?



37. (CMIMC 2019) Let $MATH$ be a trapezoid with $MA = AT = TH = 5$ and $MH = 11$. Point S is the orthocenter of $\triangle ATH$. Compute the area of quadrilateral $MASH$.

★ 38. (CMIMC 2017) In acute triangle ABC, points D and E are the feet of the angle bisector and altitude from A respectively. Suppose that $AC - AB = 36$ and $DC - DB = 24$. Compute $EC - EB$.[56]

39. (Mock AMC 12 2012) Right triangle $\triangle ABC$ has $AB = 15$, $BC = 20$, and a right angle at B. Point D is placed on AC such that $\triangle DAB$ is isosceles with base AD. The circumcircle of $\triangle DAB$ intersects line CB at point $E \neq B$. What is BE?[57]

40. (CMIMC 2018) Let ABC be a triangle with $AB = 9$, $BC = 10$, $CA = 11$, and orthocenter H. Suppose point D is placed on $\overline{BC}$ such that $AH = HD$. Compute AD.

41. (AMC 12A 2021) Suppose that on a parabola with vertex V and a focus F there exists a point A such that $AF = 20$ and $AV = 21$. What is the sum of all possible values of the length FV?

42. (CMIMC 2017) Let $\mathcal{S}$ be the sphere with center $(0,0,1)$ and radius 1 in $\mathbb{R}^3$. A plane $\mathcal{P}$ is tangent to $\mathcal{S}$ at the point (x_0, y_0, z_0), where x_0, y_0, and z_0 are all positive. Suppose the intersection of plane $\mathcal{P}$ with the xy-plane is the line with equation $2x + y = 10$ in xy-space. What is z_0?[58]

43. (NIMO 16) Let $ABCD$ be a square with side length 2. Let M and N be the midpoints of $\overline{BC}$ and $\overline{CD}$ respectively, and let X and Y be the feet of the perpendiculars from A to $\overline{MD}$ and $\overline{NB}$, also respectively. The square of the length of segment $\overline{XY}$ can be written in the form $\frac{p}{q}$ where p and q are positive relatively prime integers. What is $100p + q$?

44. (Bridging the Gap) Triangle AEF is a right triangle with $AE = 4$ and $EF = 3$. The triangle is inscribed inside square $ABCD$ with E on BC and F on CD. What is the area of the square?[59]

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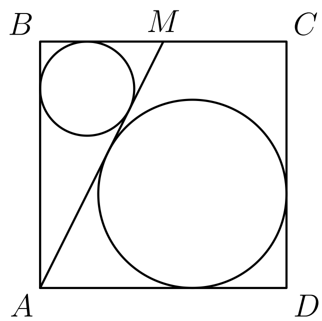
[56] Once again, this is an "easy" problem that I actually enjoy quite a bit. I'm surprised I've never seen this relationship exploited before.

[57] I remember when this was one of the problems I was most proud of writing. Oh, how times have changed....

[58] Oh my, it's a coordinate geometry question!

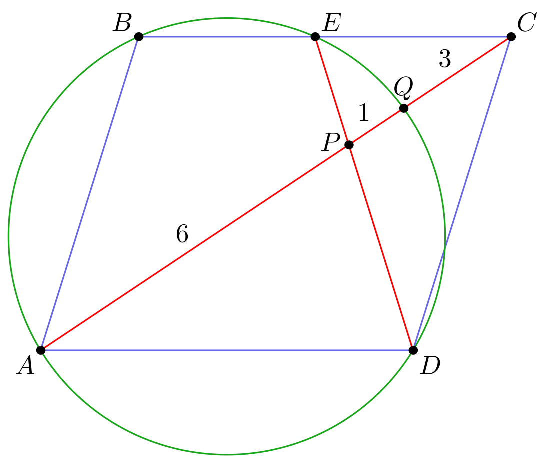
[59] Compare with 2015 MATHCOUNTS National Sprint #25. It seems that a lot of people who took the test actually worked on the problem set - yay!

45. (AMSP Team Contest 2015) Square $ABCD$ has side length 10, and point M is the midpoint of $\overline{BC}$. Two circles are inscribed inside the square as shown. What is the ratio of the area of the larger circle to the area of the smaller one?



46. (Mock AMC 12 2012) Two chords are constructed in a circle with radius 4 such that their intersection point lies on the circle, and the angle formed by the two chords has measure 30°. One of the chords has length 5. Given that the largest possible value for the length of the other chord is $\frac{\sqrt{m}+\sqrt{n}}{p}$ where m, n, and p are positive integers, what is $m+n+p$?

★ 47. (NICE Spring 2021) Let $ABCD$ be a parallelogram with $\angle BAD < 90°$. The circumcircle of $\triangle ABD$ intersects $\overline{BC}$ and $\overline{AC}$ again at E and Q, respectively. Let P be the intersection point of $\overline{ED}$ with $\overline{AC}$. If $AP = 6$, $PQ = 1$, and $QC = 3$, find the area of parallelogram $ABCD$.



48. (USAMTS 2020-2021) Find distinct points A, B, C, and D in the plane such that the length of the segment AB is an even integer, and the lengths of the segments AC, AD, BC, BD, and CD are all odd integers. In addition to stating the coordinates of the points and distances between points, please include a brief explanation of how you found the configuration of points and computed the distances.[60]

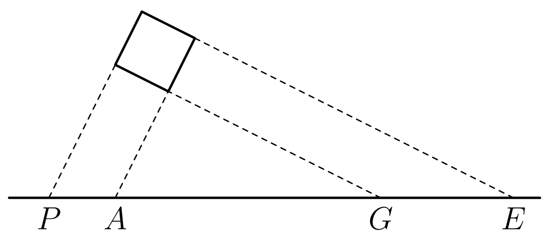
49. (ARML Local 2019) Let $\mathcal{T}$ be an isosceles trapezoid such that there exists a point P in the plane whose distances to the four vertices of $\mathcal{T}$ are 5, 6, 8, and 11. Compute the largest possible ratio of the length of the longer base of $\mathcal{T}$ to the length of its shorter base.

[60] It is known (see Putnam 1993 B5) that it is impossible for all six distances to be odd integers. This problem stemmed from looking at the next best thing.

50. (NIMO 19) Let Ω_1 and Ω_2 be two circles in the plane. Suppose the common external tangent to Ω_1 and Ω_2 has length 2017 while their common internal tangent has length 2009. Find the product of the radii of Ω_1 and Ω_2.[61]

51. (Mock AMC 10/12 2013) In $\triangle ABC$, $AB = 13$, $AC = 14$, and $BC = 15$. Let M denote the midpoint of $\overline{AC}$. Point P is placed on line segment $\overline{BM}$ such that $\overline{AP} \perp \overline{PC}$. Suppose that p, q, and r are positive integers with p and r relatively prime and q squarefree such that the area of $\triangle APC$ can be written in the form $\frac{p\sqrt{q}}{r}$. What is $p+q+r$?

52. (Mock AMC 10/12 2013) One deleted scene from the movie *Inception* involved a square hanging in mid-air at a certain angle as shown. Scientists tried to determine the dimensions of the square directly, but to no avail. However, working her magic, a passerby named Ellen was able to extend the sides of the square until they met the ground at points P, A, G, and E in that order. Simple measurements showed that $PA = 2$ and $GE = 3$. What was the area of the square?



53. (AMSP Team Contest 2015) Let $ABCD$ be an isosceles trapezoid with $AD = BC$, $AB = 6$, and $CD = 10$. Suppose the distance from A to the centroid of $\triangle BCD$ is 8. Compute the area of $ABCD$.[62]

54. (CMIMC 2019) Let ABC be a triangle with $AB = 209$, $AC = 243$, and $\angle BAC = 60^\circ$, and denote by N the midpoint of the major arc $\widehat{BAC}$ of circle $\odot(ABC)$. Suppose the parallel to AB through N intersects $\overline{BC}$ at a point X. Compute the ratio $\frac{BX}{XC}$.

55. (NIMO Summer Contest 2015) Let $\triangle ABC$ be a triangle with $AB = 85$, $BC = 125$, $CA = 140$, and incircle ω. Let D, E, F be the points of tangency of ω with $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively, and furthermore denote by X, Y, and Z the incenters of $\triangle AEF$, $\triangle BFD$, and $\triangle CDE$, also respectively. Find the circumradius of $\triangle XYZ$.

56. (AMC 12B 2021) Let $ABCD$ be a parallelogram with area 15. Points P and Q are the projections of A and C, respectively, onto the line BD; and points R and S are the projections of B and D, respectively, onto the line AC. See the figure, which also shows the relative locations of these points.

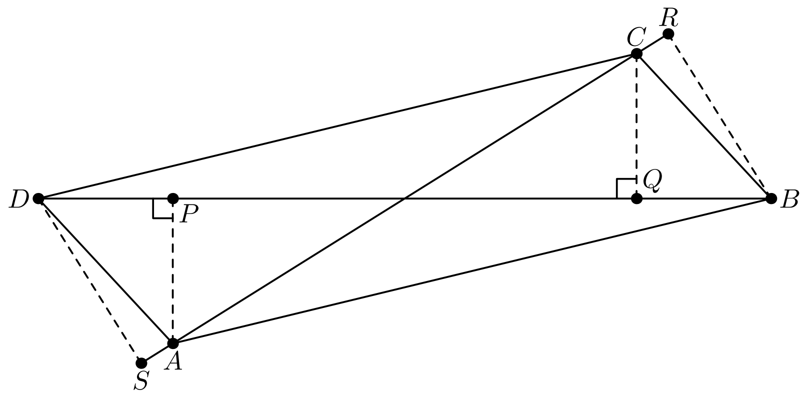
    Suppose $PQ = 6$ and $RS = 8$, and let d denote the length of $\overline{BD}$, the longer diagonal of $ABCD$. Then d^2 can be written in the form $m + n\sqrt{p}$, where m, n, and p are positive integers and p is not divisible by the square of any prime. What is $m+n+p$?

57. (CMIMC 2016, with Joshua Siktar) Let $\mathcal{P}$ be a parallelepiped with side lengths x, y, and z. Suppose that the four space diagonals of $\mathcal{P}$ have lengths 15, 17, 21, and 23. Compute $x^2+y^2+z^2$.[63]

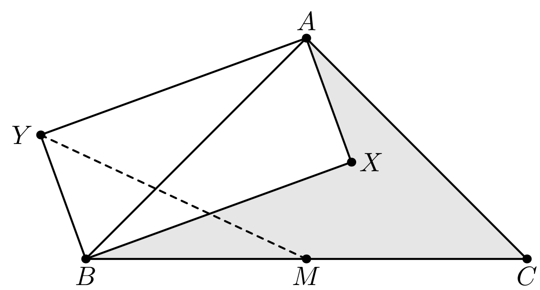
[61] Compare with HMMT Guts 2007 #11.

[62] This was written during Geo 2 lecture at AMSP 2015. Also, apparently this has connections to 2011 ISL G4?

[63] Joshua raised the question of extending the Parallelogram Law to three dimensions during Vector Analysis recitation. By the end of the recitation, I realized I had a CMIMC problem.



★ 58. (NICE Spring 2021, with Arul Kolla) Let ABC be an isosceles right triangle with $AB = AC$, and denote by M the midpoint of $\overline{BC}$. Construct rectangle $AXBY$, with X in the interior of $\triangle ABC$, such that $YM = 8$ and $AY^3 + BY^3 = 10^3$. The area of quadrilateral $AXBC$ can be expressed in the form $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers such that m and p are relatively prime and n is not divisible by the square of any prime. Find $1000000m + 1000n + p$.[64]



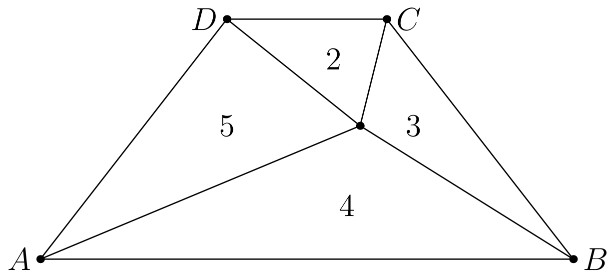
59. (AIME 2021) Let $ABCD$ be an isosceles trapezoid with $AD = BC$ and $AB < CD$. Suppose that the distances from A to the lines BC, CD, and BD are $15, 18$, and 10, respectively. Let K be the area of $ABCD$. Find $\sqrt{2} \cdot K$.

60. (CMIMC 2020) Points P and Q lie on a circle ω. The tangents to ω at P and Q intersect at point T, and point R is chosen on ω so that T and R lie on opposite sides of PQ and $\angle PQR = \angle PTQ$. Let RT meet ω for the second time at point S. Given that $PQ = 12$ and $TR = 28$, determine PS.

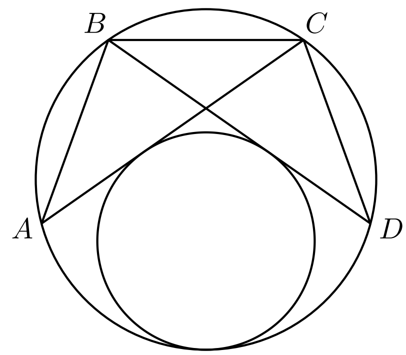
61. (CMIMC 2020) Two circles ω_A and ω_B have centers at points A and B respectively and intersect at points P and Q in such a way that A, B, P, and Q all lie on a common circle ω. The tangent to ω at P intersects ω_A and ω_B again at points X and Y respectively. Suppose $AB = 17$ and $XY = 20$. Compute the sum of the radii of ω_A and ω_B.

62. (AMC 12B 2021) Let $ABCD$ be an isoceles trapezoid having parallel bases $\overline{AB}$ and $\overline{CD}$ with $AB > CD$. Line segments from a point inside $ABCD$ to the vertices divide the trapezoid into four triangles whose areas are $2, 3, 4$, and 5 starting with the triangle with base $\overline{CD}$ and moving clockwise as shown in the diagram below. What is the ratio $\frac{AB}{CD}$?

[64]Arul originally wrote a different problem and put the cubic condition in as a bash deterrent. I saw the cubic condition and decided to transform the problem into something that used it more effectively.



63. (ARML Local 2019) Let Ω be a circle with radius 5, and suppose A, B, C, and D are points on Ω in that order such that $AB = BC = CD = 4$. Determine the radius of the circle below, which is tangent to AC, BD, and minor arc $\widehat{AD}$.



64. (CMIMC 2019) Let ABC be a triangle with $AB = 13$, $BC = 14$, and $AC = 15$. Denote by ω its incircle. A line ℓ tangent to ω intersects $\overline{AB}$ and $\overline{AC}$ at X and Y respectively. Suppose $XY = 5$. Compute the positive difference between the lengths of $\overline{AX}$ and $\overline{AY}$.

65. (Farewell Geometry Mock AMC) Let S be a square with side length 5. Square S is rotated at an angle of θ° clockwise about its center, $0 < \theta < 45$, to form square S'. If θ is chosen so that $[S \cup S'] = 29\frac{2}{7}$, what is $\lfloor 1000 \tan\theta \rfloor$? (Note: $[X]$ is defined as the area of region X.)[65]

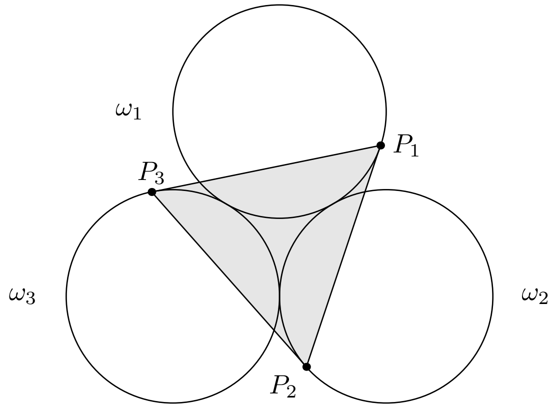
★ 66. (AMC 12B 2018) Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?[66]

67. (AMC 12B 2019) Let $ABCD$ be a convex quadrilateral with $BC = 2$ and $CD = 6$. Suppose that the centroids of $\triangle ABC$, $\triangle BCD$, and $\triangle CDA$ form the vertices of an equilateral triangle. What is the maximum possible area of $ABCD$?

68. (CMIMC 2019) Let $\triangle ABC$ be a triangle with side lengths a, b, and c. Circle ω_A is the A-excircle of $\triangle ABC$, defined as the circle tangent to BC and to the extensions of AB and AC past B and C respectively. Let $\mathcal{T}_A$ denote the triangle whose vertices are these three tangency points; denote

[65] Compare with 1999 AIME #4.

[66] This problem came about while I was waiting for my laundry during freshman year, since at the very beginning of my college experience I was worried that if I went back to my room I would forget that my clothes were still there. That might have been a terrific decision - this is one of my favorite problems of all time.



$\mathcal{T}_B$ and $\mathcal{T}_C$ similarly. Suppose the areas of $\mathcal{T}_A$, $\mathcal{T}_B$, and $\mathcal{T}_C$ are 4, 5, and 6 respectively. Find the ratio $a : b : c$.

69. (ARML Local 2018) Let ABC be a triangle with $AB = 5$, $BC = 12$, and an obtuse angle at B. It is given that there exists a point P in plane ABC for which $\angle PBC = \angle PCB = \angle PAB = \gamma$, where γ is an acute angle satisfying $\tan\gamma = \frac{3}{4}$. Compute $\sin\angle ABP$.[67]

★ 70. (CMIMC 2017) Two circles ω_1 and ω_2 are said to be *orthogonal* if they intersect each other at right angles. In other words, for any point P lying on both ω_1 and ω_2, if ℓ_1 is the line tangent to ω_1 at P and ℓ_2 is the line tangent to ω_2 at P, then $\ell_1 \perp \ell_2$. (Two circles which do not intersect are not orthogonal.)

    Let $\triangle ABC$ be a triangle with area 20. Orthogonal circles ω_B and ω_C are drawn with ω_B centered at B and ω_C centered at C. Points T_B and T_C are placed on ω_B and ω_C respectively such that AT_B is tangent to ω_B and AT_C is tangent to ω_C. If $AT_B = 7$ and $AT_C = 11$, what is $\tan\angle BAC$?

71. (USAMTS 2018-2019) Cyclic quadrilateral $ABCD$ has $AC \perp BD$, $AB + CD = 12$, and $BC + AD = 13$. Find the greatest possible area of $ABCD$.[68]

72. (NIMO 20) Let ABC be a triangle with $AB = 20$, $AC = 34$, and $BC = 42$. Let ω_1 and ω_2 be the semicircles with diameters $\overline{AB}$ and $\overline{AC}$ erected outwards of $\triangle ABC$ and denote by ℓ the common external tangent to ω_1 and ω_2. The line through A perpendicular to $\overline{BC}$ intersects ℓ at X and BC at Y. The length of $\overline{XY}$ can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.[69]

73. (CMIMC 2017) Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70^\circ$, $\angle ADB = 50^\circ$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$.[70]

[67] The original problem asked to compute BC, which I think is a much more natural question, but then I was reminded that I can't do Law of Sines correctly and so the answer to that problem became much messier. Darn. Also, this question was inspired by Waldemar Pompe and the very strange problems he would give to his Geo 3 class at AMSP Cornell 1 2017.

[68] This was written all the way back in the first semester of my freshman year with the intent to submit the problem to CMIMC 2016 (and later AMC 2018), but neither of those intentions materialized for reasons I don't recall.

[69] This was way more well received than I expected. In particular, I just expected everyone to straight up blast this problem with Cartesian. Fortunately, that was not the case.

[70] My original solution to this problem used Brokard oooooops

74. (ARML Local 2018) Let $ABCD$ be a rectangle with $AB = 15$ and $BC = 19$. Points E and F are located inside $ABCD$ such that quadrilaterals $AECF$ and $BEDF$ are both parallelograms with areas 107 and 88 respectively. Compute EF.

75. (NICE Spring 2021) David attempts to draw a regular hexagon $ABCDEF$ inscribed in a circle ω of radius 1. However, while the points A through E are accurately placed, his point F is slightly off, and so the diagonals AD, BE, and CF do not concur (Fortunately for him, the location of point F on ω is the only inaccurate part of the diagram). Undeterred by his artistic deficiencies, David pretends that the diagram is accurate[71] by shading in the circumcircle of the triangle bounded by these three line segments, which has radius $\frac{1}{24}$. He then continues to draw the rest of the diagram.

    The area of the hexagon $ABCDEF$ that David actually drew can be expressed in the form $a - \frac{b}{c}\sqrt{d}$, where a, b, c, and d are positive integers such that b and c are relatively prime and d is not divisible by the square of any prime. Find $a + b + c + d$.

76. (AMSP Team Contest 2015) Rectangles $ABCD$ and $APCQ$ share the same diagonal $\overline{AC}$. Suppose that $AB = 7$, $AD = 4$, and $AP = 1$. Compute the maximum possible area of quadrilateral $BPDQ$.

77. (CMIMC 2016) In parallelogram $ABCD$, angles B and D are acute while angles A and C are obtuse. The perpendicular from C to AB and the perpendicular from A to BC intersect at a point P inside the parallelogram. If $PB = 700$ and $PD = 821$, what is AC?

78. (NIMO April 2017) Let $ABCDE$ be a convex pentagon inscribed inside a circle of radius 38. Let I_1 and I_2 denote the incenters of $\triangle ABD$ and $\triangle EBD$ respectively. Suppose that $AE = 4\sqrt{37}$ and $BC = CD = 57$. Over all such pentagons, compute the square of the maximum possible value of I_1I_2.

79. (CMIMC 2016) In $\triangle ABC$, $AB = 17$, $AC = 25$, and $BC = 28$. Points M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively, and P is a point on $\overline{BC}$. Let Q be the second intersection point of the circumcircles of $\triangle BMP$ and $\triangle CNP$. It is known that as P moves along $\overline{BC}$, line PQ passes through some fixed point X. Compute the sum of the squares of the distances from X to each of A, B, and C.

★ 80. (AIME 2021) Circles ω_1 and ω_2 with radii 961 and 625, respectively, intersect at distinct points A and B. A third circle ω is externally tangent to both ω_1 and ω_2. Suppose line AB intersects ω at two points P and Q such that the measure of minor arc $\widehat{PQ}$ is 120°. Find the distance between the centers of ω_1 and ω_2.

81. (CMIMC 2019) Points A, B, and C lie in the plane such that $AB = 13$, $BC = 14$, and $CA = 15$. A peculiar laser is fired from A perpendicular to $\overline{BC}$. After bouncing off BC, it travels in a direction perpendicular to CA. When it hits CA, it travels in a direction perpendicular to AB, and after hitting AB its new direction is perpendicular to BC again. If this process is continued indefinitely, the laser path will eventually approach some finite polygonal shape T_∞. What is the ratio of the perimeter of T_∞ to the perimeter of $\triangle ABC$?[72]

82. (AIME 2018) Let $\triangle ABC$ have side lengths $AB = 30$, $BC = 32$, and $AC = 34$. Point X lies in the interior of $\overline{BC}$, and points I_1 and I_2 are the incenters of $\triangle ABX$ and $\triangle ACX$, respectively. Find the minimum possible area of $\triangle AI_1I_2$ as X varies along $\overline{BC}$.

[71] Big Point Theorem!

[72] This was actually written for the 2016 CMIMC contest, but for a long time my only solution used a nontrivial fact about Brocard points, and so I didn't feel comfortable putting the problem on the exam.

83. (CMIMC 2018) Circles ω_1 and ω_2 intersect at P and Q. The common external tangent ℓ to the two circles closer to Q touches ω_1 and ω_2 at A and B respectively. Line AQ intersects ω_2 at X while BQ intersects ω_1 again at Y. Let M and N denote the midpoints of $\overline{AY}$ and $\overline{BX}$, also respectively. If $AQ = 6$, $BQ = 7$, and $AB = 8$, then find the length of MN.[73]

84. (Problem Stash 2) Let $\mathcal{T}$ be a triangle with inradius 52 and circumradius 117. Suppose ω is the unique circle with the property that inversion about ω sends the incircle of $\mathcal{T}$ to the circumcircle of $\mathcal{T}$. The radius of ω can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m+n$.

85. (CMIMC 2017) Points A, B, and C lie on a circle Ω such that A and C are diametrically opposite each other. A line ℓ tangent to the incircle of $\triangle ABC$ at T intersects Ω at points X and Y. Suppose that $AB = 30$, $BC = 40$, and $XY = 48$. Compute $TX \cdot TY$.

86. (CMIMC 2018) Let ABC be a triangle with $AB = 10$, $AC = 11$, and circumradius 6. Points D and E are located on the circumcircle of $\triangle ABC$ such that $\triangle ADE$ is equilateral. Line segments $\overline{DE}$ and $\overline{BC}$ intersect at X. Find $\frac{BX}{XC}$.

87. (AMSP Team Contest 2016) Let $ABCD$ be a convex cyclic quadrilateral inscribed in a circle of radius 30 satisfying $DA = DC = 28$ and $DB = 49$. If P is the intersection of lines AC and BD, compute
$$AB \cdot BC - AP \cdot PC.$$[74]

88. (CMIMC 2016) Let ABC be a triangle with incenter I and incircle ω. It is given that there exist points X and Y on the circumference of ω such that $\angle BXC = \angle BYC = 90^\circ$. Suppose further that X, I, and Y are collinear. If $AB = 80$ and $AC = 97$, compute the length of BC.

89. (CMIMC 2017) Two non-intersecting circles, ω and Ω, have centers C_ω and C_Ω respectively. It is given that the radius of Ω is strictly larger than the radius of ω. The two common external tangents of Ω and ω intersect at a point P, and an internal tangent of the two circles intersects the common external tangents at X and Y. Suppose that the radius of ω is 4, the circumradius of $\triangle PXY$ is 9, and XY bisects $\overline{PC_\Omega}$. Compute XY.

90. (Northeastern WOOTers Mock AIME I) Let $\triangle ABC$ be a triangle with $AB = 39$, $BC = 42$, and $CA = 45$. A point D is placed on the extension of BC past C. Let O_1 and O_2 be the circumcenters of $\triangle ABD$ and $\triangle ACD$ respectively. If $O_1O_2 = 29$, then the ratio $\frac{BD}{CD}$ can be written in the form $\frac{m}{n}$ for relatively prime positive integers m and n. Find $m+n$.

91. (Mock AMC 10/12 2013) A *hexomino* is a geometric shape consisting of six congruent squares arranged in a pattern such that any two squares that share an edge also share the two endpoints of this edge. Suppose that a hexomino is inscribed inside a square as shown below.

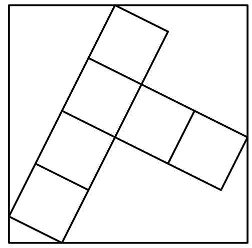
    What is the ratio of the area of the hexomino to the area of the square?[75]

92. (AMSP Team Contest 2015) A semicircle ω has radius 1 and diameter $\overline{AB}$. Two distinct circles ω_1 and ω_2 of a radius r are constructed internally tangent to ω and tangent to the segment $\overline{AB}$.

[73]This was going to be a 2019 AIME submission for a long time, but then I cut it off a few days before proposals were due in favor of a different problem. Perhaps this is a bit too meaty for Relay though....

[74]This was planned to be a CMIMC 2016 problem, but it turns out this is in a vague sense a mashup of 2016 AIME I #6 and 2016 AIME I #15, so my paranoid self dropped the problem.

[75]Protip: $6 \neq 7$.



Circle $\mathcal{A}$ is then constructed externally tangent to ω_1 and ω_2 as well as $\overline{AB}$, while circle $\mathcal{B}$ is constructed externally tangent to ω_1 and ω_2 and internally tangent to ω. Suppose circles $\mathcal{A}$ and $\mathcal{B}$ are externally tangent to each other. Compute r.

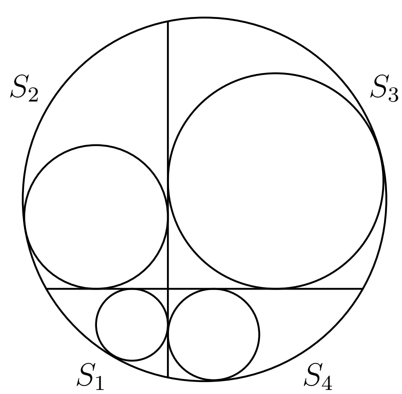
93. (NIMO 30) Let ABC be a triangle with $AB = 4$, $AC = 5$, $BC = 6$, and circumcircle Ω. Points E and F lie on AC and AB respectively such that $\angle ABE = \angle CBE$ and $\angle ACF = \angle BCF$. The second intersection point of the circumcircle of $\triangle AEF$ with Ω (other than A) is P. Suppose $AP^2 = \frac{m}{n}$ where m and n are positive relatively prime integers. Find $100m + n$.

94. (CMIMC 2016) Suppose $ABCD$ is a convex quadrilateral satisfying $AB = BC$, $AC = BD$, $\angle ABD = 80^\circ$, and $\angle CBD = 20^\circ$. What is $\angle BCD$ in degrees?

95. (Problem Stash 1) Let $\mathcal{E}$ be an ellipse passing through $(0,0)$, $(0,5)$, and $(6,0)$. The largest value of d such that the distance from $(0,0)$ to the center of $\mathcal{E}$ is at least d can be written in the form $\frac{m}{\sqrt{n}}$, where m and n are positive relatively prime integers. Find $m + n$.

96. (NIMO 16) Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $100m + n$.

97. (Problem Stash 1) Two perpendicular chords are drawn inside a circle Ω. These two chords divide the interior of Ω into four sectors, labeled S_1 through S_4 clockwise as shown below. Within each sector S_i a circle ω_i is inscribed, touching each of the two chords as well as the circle Ω internally. Suppose the radii of the circles ω_1, ω_2, and ω_3 are 1, 2, and 3 respectively. Then the radius of ω_4 can be expressed in the form $\frac{m-\sqrt{n}}{p}$, where m, n, and p are positive integers with m and p relatively prime. Find $m + n + p$.



98. (CMIMC 2020) Let $\mathcal{E}$ be an ellipse with foci F_1 and F_2. Parabola $\mathcal{P}$, having vertex F_1 and focus F_2, intersects $\mathcal{E}$ at two points X and Y. Suppose the tangents to $\mathcal{E}$ at X and Y intersect on the directrix of $\mathcal{P}$. Compute the eccentricity of $\mathcal{E}$.

    (A *parabola* $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$. An *ellipse* $\mathcal{E}$ is the set of points P such that the sum $PF_1 + PF_2$ is some constant d, where F_1 and F_2 are the *foci* of $\mathcal{E}$. The *eccentricity* of $\mathcal{E}$ is defined to be the ratio F_1F_2/d.)

99. (AIME 2018) David found four sticks of different lengths that can be used to form three non-congruent convex cyclic quadrilaterals, A, B, C, which can each be inscribed in a circle with radius 1. Let φ_A denote the measure of the acute angle made by the diagonals of quadrilateral A, and define φ_B and φ_C similarly. Suppose that $\sin\varphi_A = \frac{2}{3}$, $\sin\varphi_B = \frac{3}{5}$, and $\sin\varphi_C = \frac{6}{7}$. All three quadrilaterals have the same area K, which can be written in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

100. (TKMT) Let ABC be a triangle with $AB = 3$ and $AC = 4$. Points O and H are the circumcenter and orthocenter respectively of the triangle. If $OH \parallel BC$, then $\cos A$ can be expressed in the form $\frac{m-\sqrt{n}}{p}$ where m, n, and p are positive integers with m and p relatively prime. Find $m+n+p$.

★ 101. (CMIMC 2017, with Evan Chen) In triangle ABC with $AB = 23$, $AC = 27$, and $BC = 20$, let D be the foot of the A altitude. Let $\mathcal{P}$ be the parabola with focus A passing through B and C, and denote by T the intersection point of AD with the directrix of $\mathcal{P}$. Determine the value of $DT^2 - DA^2$. (Recall that a parabola $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$.)[76]

102. (NIMO 18) Let ABC be a non-degenerate triangle with incenter I and circumcircle Γ. Denote by M_a the midpoint of the arc $\widehat{BC}$ of Γ not containing A, and define M_b, M_c similarly. Suppose $\triangle ABC$ has inradius 4 and circumradius 9. Compute the maximum possible value of

$$IM_a^2 + IM_b^2 + IM_c^2.$$

103. (NIMO 23) Let $\triangle ABC$ be an equilateral triangle with side length s and P a point in the interior of this triangle. Suppose that PA, PB, and PC are the roots of the polynomial $t^3 - 18t^2 + 91t - 89$. Then s^2 can be written in the form $m + \sqrt{n}$ where m and n are positive integers. Find $100m + n$.[77]

104. (NIMO 21, with Evan Chen) Let $ABCD$ be an isosceles trapezoid with $AD \parallel BC$ and $BC > AD$ such that the distance between the incenters of $\triangle ABC$ and $\triangle DBC$ is 16. If the perimeters of $ABCD$ and ABC are 120 and 114 respectively, then the area of $ABCD$ can be written as $m\sqrt{n}$, where m and n are positive integers with n not divisible by the square of any prime. Find $100m + n$.[78]

105. (AIME 2020) Let ABC be an acute triangle with circumcircle ω and orthocenter H. Suppose the tangent to the circumcircle of $\triangle HBC$ at H intersects ω at points X and Y with $HA = 3$, $HX = 2$, $HY = 6$. The area of $\triangle ABC$ can be written as $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

[76] The original idea of looking at this parabola is mine, but Evan reworked the problem so that the main idea was disguised better.

[77] Finding a good polynomial to work with was quite hard. Not only did the numbers need to work out nicely, but I also needed the three roots to be side lengths of a non-degenerate triangle that didn't reduce the problem to a trivial case.

[78] Evan suggested the reformulation in terms of perimeters. (My original idea was to compute the distance between the incenters from the identity $OI^2 = R(R-2r)$.)

106. (CMIMC 2018) Let ABC be a triangle with incircle ω and incenter I. The circle ω is tangent to BC, CA, and AB at D, E, and F respectively. Point P is the foot of the angle bisector from A to BC, and point Q is the foot of the altitude from D to EF. Suppose $AI = 7$, $IP = 5$, and $DQ = 4$. Compute the radius of ω.

107. (CMIMC 2016) Triangle ABC satisfies $AB = 28$, $BC = 32$, and $CA = 36$, and M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Let point P be the unique point in the plane ABC such that $\triangle PBM \sim \triangle PNC$. What is AP?[79]

108. (CMIMC 2017) Triangle ABC satisfies $AB = 104$, $BC = 112$, and $CA = 120$. Let ω and ω_A denote the incircle and A-excircle of $\triangle ABC$, respectively. There exists a unique circle Ω passing through A which is internally tangent to ω and externally tangent to ω_A. Compute the radius of Ω.[80]

★ 109. (NIMO 21) Triangle ABC has $AB = 25$, $AC = 29$, and $BC = 36$. Additionally, Ω and ω are the circumcircle and incircle of $\triangle ABC$. Point D is situated on Ω such that AD is a diameter of Ω, and line AD intersects ω in two distinct points X and Y. Compute XY^2.[81]

110. (AIME 2019) In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$, where m and n are positive integers, and n is not divisible by the square of any prime. Find $m+n$.

111. (CMC[82] 2018) Let $\triangle ABC$ be a triangle with incenter I, and let P be the intersection of line AI with the circumcircle of $\triangle ABC$. Let points X and Y denote the incenters of $\triangle ABP$ and $\triangle APC$, respectively. Suppose that $XY = 2$ and $BI^2 + CI^2 = 15$. The value of $\cos\angle BAC$ can be written as $\frac{p}{q}$ for some relatively prime integers p and q. What is the value of $p+q$?

★ 112. (CMIMC 2017) Suppose $\triangle ABC$ is such that $AB = 13$, $AC = 15$, and $BC = 14$. It is given that there exists a unique point D on side $\overline{BC}$ such that the Euler lines of $\triangle ABD$ and $\triangle ACD$ are parallel. Determine the value of $\frac{BD}{CD}$. (The *Euler* line of a triangle ABC is the line connecting the centroid, circumcenter, and orthocenter of ABC.)[83]

113. (CMIMC 2017) Circles ω_1 and ω_2 are externally tangent to each other. Circle Ω is placed such that ω_1 is internally tangent to Ω at X while ω_2 is internally tangent to Ω at Y. Line ℓ is tangent to ω_1 at P and ω_2 at Q and furthermore intersects Ω at points A and B with $AP < AQ$. Suppose that $AP = 2$, $PQ = 4$, and $QB = 3$. Compute the length of line segment $\overline{XY}$.

114. (USAMTS 2018-2019) Acute scalene triangle $\triangle ABC$ has circumcenter O and orthocenter H. Points X and Y, distinct from B and C, lie on the circumcircle of $\triangle ABC$ such that $\angle BXH =$

[79] Compare with Balkan 2009 #2.

[80] Written during Geo 2 lecture at AMSP Puget Sound 2016.

[81] This is one of my favorite problems I've ever written - the key step is quite original, and writing this was probably the most productive use of an AP Economics class ever. Unfortunately, it was shadowed by, well, the rest of the mess that was Contest 21. Sigh.

[82] Christmas Mathematics Competitions, which was a series of mock exams I was asked to help out with. This problem was born out of an attempt to explore a configuration found in another proposal for the mock.

[83] This problem was inspired by misreading 2016 OMO Fall #24. It also turns out that this ended up replacing another CMIMC proposal, since I ended up using the main observation in that problem when solving this one. This might be the only reason why I was able to solve this synthetically. It's also very strange that the resulting equivalence is so nice.

$\angle CYH = 90°$. Show that if lines XY, AH, and BC are concurrent, then $\overline{OH}$ is parallel to $\overline{BC}$.[84]

115. (Mock AIME I 2015) Let $A_1A_2A_3A_4A_5A_6$ be a hexagon inscribed inside a circle of radius r. Furthermore, for each positive integer $1 \le i \le 6$ let M_i be the midpoint of the segment $\overline{A_iA_{i+1}}$, where $A_7 \equiv A_1$. Suppose that hexagon $M_1M_2M_3M_4M_5M_6$ can also be inscribed inside a circle. If $A_1A_2 = A_3A_4 = 5$ and $A_5A_6 = 23$, then r^2 can be written in the form $\frac{m}{n}$ where m and n are positive relatively prime integers. Find $m+n$.

116. (Mock AIME I 2015) Let $\triangle ABC$ be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let O denote its circumcenter and H its orthocenter. The circumcircle of $\triangle AOH$ intersects AB and AC at D and E respectively. Suppose $\frac{AD}{AE} = \frac{m}{n}$ where m and n are positive relatively prime integers. Find $m-n$.[85]

★ 117. (CMIMC 2017) Let $\triangle ABC$ be an acute triangle with circumcenter O, and let $Q \neq A$ denote the point on $\odot(ABC)$ for which $AQ \perp BC$. The circumcircle of $\triangle BOC$ intersects lines AC and AB for the second time at D and E respectively. Suppose that AQ, BC, and DE are concurrent. If $OD = 3$ and $OE = 7$, compute AQ.[86]

★ 118. (CMIMC 2016) Let $\triangle ABC$ be a triangle with $AB = 65$, $BC = 70$, and $CA = 75$. A semicircle Γ with diameter $\overline{BC}$ is erected outside the triangle. Suppose there exists a circle ω tangent to AB and AC and furthermore internally tangent to Γ at a point X. The length AX can be written in the form $m\sqrt{n}$ where m and n are positive integers with n not divisible by the square of any prime. Find $m+n$.[87]

★ 119. (CMIMC 2016) Let $\triangle ABC$ be a triangle with circumcircle Ω and let N be the midpoint of the major arc $\widehat{BC}$. The incircle ω of $\triangle ABC$ is tangent to AC and AB at points E and F respectively. Suppose point X is placed on the same side of EF as A such that $\triangle XEF \sim \triangle ABC$. Let NX intersect BC at a point P. If $AB = 15$, $BC = 16$, and $CA = 17$, then compute $\frac{PX}{XN}$.[88]

[84] This problem was originally considered for the 2018 TSTST but (from what I heard) it was rejected for being slightly too easy.

[85] I was exceedingly lucky with this problem. Everything - the points O and H, the 13-14-15 triangle, etc. - came so naturally and nicely from the original idea. In other words, this problem basically fell from the sky.

[86] This problem received the Michael Kural Seal of Approval! One of my favorite proposals of all time.

[87] This was probably my favorite problem out of all the ones I wrote for CMIMC 2016. The intended solution is very instructive.

[88] This problem was primarily written because I thought the then-current #10 (the one a few spots up with the 65-70-75 triangle) was too easy for the #10 spot. It turns out that this was not actually the case. Problem writer bias for the win!

4 Number Theory

1. (AMC 10B 2021) The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together give 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

2. (CMIMC 2016) David, when submitting a problem for CMIMC, wrote his answer as $100\tfrac{x}{y}$, where x and y are two positive integers with $x < y$. Andrew interpreted the expression as a product of two rational numbers, while Patrick interpreted the answer as a mixed fraction. In this case, Patrick's number was exactly double Andrew's! What is the smallest possible value of $x+y$?[89]

3. (CMIMC 2017) There exist two distinct positive integers, both of which are divisors of 10^{10}, with sum equal to 157. What are they?[90]

4. (CMIMC 2018) Suppose a, b, and c are relatively prime integers such that

$$\frac{a}{b+c} = 2 \qquad \text{and} \qquad \frac{b}{a+c} = 3.$$

What is the value of $|c|$?

5. (ARML Local 2019) Compute the number of ordered pairs of positive integers (x, y) that satisfy the equality $x^y = 2^{9!}$.[91]

6. (ARML Local 2018) Compute the sum of all possible values of $a+b$, where a and b are integers such that $a > b$ and $a^2 - b^2 = 2016$.

7. (NIMO Summer Contest 2017) If p, q, and r are nonzero integers satisfying

$$p^2 + q^2 = r^2,$$

compute the smallest possible value of $(p+q+r)^2$.

8. (NIMO Summer Contest 2015) A list of integers with average 89 is split into two disjoint groups. The average of the integers in the first group is 73 while the average of the integers in the second group is 111. What is the smallest possible number of integers in the original list?

9. (NIMO 16) For any interval $\mathcal{A}$ in the real number line not containing zero, define its *reciprocal* to be the set of numbers of the form $\frac{1}{x}$ where x is an element in $\mathcal{A}$. Compute the number of ordered pairs of positive integers (m, n) with $m < n$ such that the length of the interval $[m, n]$ is 10^{10} times the length of its reciprocal.

10. (CMIMC 2017) Determine all possible values of $m+n$, where m and n are positive integers satisfying

$$\operatorname{lcm}(m,n) - \gcd(m,n) = 103.$$

11. (NICE Spring 2021) How many positive integers $N \le 1000$ are there for which N^2 and $(N+1)^2$ both have at least five positive divisors? It is known there are 168 primes between 1 and 1000.

[89] Inspired by something I wrote in my solution to Team #10 from the same contest.

[90] Like the 2016 CMIMC G1, I actually really like this problem as a quirky yet simple opening question.

[91] I wrote this all the way back in 2013! I completely forgot about this question until I found it in an archived problem collection.

12. (The CALT) For all positive integer n, let $f(n)$ denote the closest prime to n. (If two such integers exist, take the larger of the two.) What is the smallest integer n such that

$$f(n+1)^2 - f(n)^2 = 120?$$

13. (AMSP Team Contest 2017) Let $\tau(n)$ denote the number of positive divisors of n. For example, $\tau(6) = 4$ since the divisors of 6 are 1, 2, 3, and 6. What is the value of

$$1\cdot(-1)^{\tau(1)} + 2\cdot(-1)^{\tau(2)} + 3\cdot(-1)^{\tau(3)} + \cdots + 100\cdot(-1)^{\tau(100)}?$$

14. (The CALT) For each pair (m,n) of non negative integers with $m \le n$, let $f(m,n)$ denote the remainder when $1+7+\ldots+7^n$ is divided by $1+7+\ldots+7^m$. Find, with proof, the maximum possible value of $f(m,n)$ over all pairs (m,n) with $1 \le m \le n \le 100$.

15. (CMIMC 2019) Determine the number of ordered pairs of positive integers (m,n) with $1 \le m \le 100$ and $1 \le n \le 100$ such that

$$\gcd(m+1, n+1) = 10\gcd(m,n).$$

16. (NIMO April 2017) Compute the least composite positive integer N such that the sum of all prime numbers that divide N is 37.

17. Two related problems.

    (a) (Mock AMC 12 2012) Let x_1, x_2, x_3, x_4 be a sequence of positive integers. Given that

    $$\sum_{i=1}^{n}(3^{x_i} + 2^{x_i}) = 930,$$

    determine $\sum_{i=1}^{4} x_i$.

    (b) (AMSP Team Contest 2015) What is the smallest positive integer n such that there exists a sequence of positive integers x_1, x_2, $\ldots$, x_n satisfying

    $$3^{x_1} + 3^{x_2} + \cdots + 3^{x_n} = 2^{x_1} + 2^{x_2} + \cdots + 2^{x_n} + 2015?$$

18. (Problem Stash 2) Let $\tau(n)$ denote the number of positive divisors of n. Compute the sum of the five smallest positive three-digit integers N such that the numbers $\tau(N)$, $\tau(2N)$, and $\tau(3N)$ form an increasing arithmetic progression.

19. (NIMO April 2017) Compute the least possible value of $m + 100n$, where m and n are positive integers such that

$$\frac{1^2 + 2^2 + \cdots + m^2}{1^2 + 2^2 + \cdots + n^2} = \frac{31}{254}.$$

20. (NIMO 30) Suppose a, b and c are positive integers with the property that ab, bc, and ac are pairwise distinct perfect squares. What is the smallest possible value of $a+b+c$?[92]

[92] "For some reason, in the test I thought that this problem was harder than 2-7 combined..." -Tristan Shin

21. (CMIMC 2016) Let a_1, a_2, ... be an infinite sequence of (positive) integers such that k divides $\gcd(a_{k-1}, a_k)$ for all $k \geq 2$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_{10}$.[93]

22. (Northeastern WOOTers Mock AIME I) It is given that 181^2 can be written as the difference of the cubes of two consecutive positive integers. Find the sum of these two integers.

23. (CMIMC 2017) For how many triples of positive integers (a, b, c) with $1 \leq a, b, c \leq 5$ is the quantity
$$(a+b)(a+c)(b+c)$$
not divisible by 4?

24. (RHS Guts Round 2015) Congratulations on making it halfway through the test! I wanted to reward you with an easy problem, but unfortunately some crazy physicist named Scott Pede[94] messed with my computer and now one of the digits on my keypad (not 0, thank goodness) outputs a ♮ instead. That's okay, though, I'll still put the problem here; after all, you should still be able to solve it, right?

    What is the quotient when 1♮4♮3 is divided by ♮1, given that there is no remainder?

25. (Mock AMC 10/12 2013) Define the sequence $\{a_k\}$ such that $a_k = 1! + 2! + 3! + \cdots + k!$ for each positive integer k. (Note: the factorial function $x!$ is defined as the product of the first x positive integers.) For how many positive integers $1 \leq n \leq 2013$ does the quantity $a_1 + a_2 + a_3 + \cdots + a_n$ end in the digit 3?

26. (Problem Stash 1) Let A denote the set of positive integers a such that
$$\frac{8!}{\gcd(a, 8!)} \text{ divides } \frac{7!}{\gcd(a, 7!)}.$$
Then the density of A, that is, the limit
$$\lim_{n \to \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n},$$
can be written in the form $\frac{p}{q}$ where p and q are positive relatively prime integers. Find $p + q$.

27. (CMIMC 2018) Let $a > 1$ be a positive integer. The sequence of natural numbers $\{a_n\}$ is defined as follows: $a_1 = a$ and for all $n \geq 1$, a_{n+1} is the largest prime factor of $a_n^2 - 1$. Determine the smallest possible value of a such that the numbers $a_1, a_2, \ldots, a_7$ are all distinct.

28. Two more related problems.

    (a) (Mock AMC 12 2012) How many positive integers n exist such that the sum of the first $2n$ positive perfect squares is an integer multiple of the sum of the first n positive odd integers?

    (b) (Brilliant.org) Compute the sum of all integers n with $1 \leq n \leq 100$ such that
    $$\frac{1^2 + 2^2 + \cdots + n^2}{1 + 2 + \cdots + n}$$
    is an integer.[95]

[93] Originally, the problem stated that $\gcd(a_{k-1}, a_k) = k$. It turns out that under that condition such a sequence does not exist. Somehow this got past all of the CMIMC staff and all but one of our testsolvers. Thanks Ray Li!

[94] An amalgamation of the names of two physics teachers in my high school

[95] This is actually not original (although I didn't realize this at first) - in fact it was a subpart of a Mock AIME problem from AwesomeMath Cornell 2013. But for some reason it grew super popular on Brilliant, which got me super confused. I believe this is my most popular problem on the website to date.

29. (CMIMC 2019) Determine the sum of all positive integers n between 1 and 100 inclusive such that
$$\gcd(n, 2^n - 1) = 3.$$

30. (AMC 10A 2021) Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

31. (Mock AMC 10/12 2013) An integer is said to be two-separable if it can be represented as the product of two positive integers that differ by two. (For example, since $8 = 2 \cdot 4$, 8 is two-separable.) Similarly, an integer is said to be nine-separable if it can be represented as the product of two positive integers that differ by 9. What is the sum of all positive integers that are both two-separable and nine-separable?[96]

32. (NIMO Summer Contest 2015) It is given that the number $4^{11} + 1$ is divisible by some prime greater than 1000. Determine this prime.

33. (Mock AIME I 2015) In an urn there are a certain number (at least two) of black marbles and a certain number of white marbles. Steven blindfolds himself and chooses two marbles from the urn at random. Suppose the probability that the two marbles are of opposite color is $\frac{1}{2}$. Let $k_1 < k_2 < \cdots < k_{100}$ be the 100 smallest possible values for the total number of marbles in the urn. Compute the remainder when
$$k_1 + k_2 + k_3 + \cdots + k_{100}$$
is divided by 1000.

34. (Mock AIME I 2015) For all points P in the coordinate plane, let P' denote the reflection of P across the line $y = x$. For example, if $P = (3,1)$, then $P' = (1,3)$. Define a function f such that for all points P, $f(P)$ denotes the area of the triangle with vertices $(0,0)$, P, and P'. Determine the number of lattice points Q in the first quadrant such that $f(Q) = 8!$.

35. (CMIMC 2018) It is given that there exist unique integers $m_1, \ldots, m_{100}$ such that
$$0 \le m_1 < m_2 < \cdots < m_{100} \quad \text{and} \quad 2018 = \binom{m_1}{1} + \binom{m_2}{2} + \cdots + \binom{m_{100}}{100}.$$
Find $m_1 + m_2 + \cdots + m_{100}$.[97]

36. (NICE Spring 2021) What is the sum of all two-digit primes p for which there exist positive integers x, y, and D such that
$$x^2 - Dy^2 = (x+1)^2 - D(y+1)^2 = p?$$

[96] When I submitted this to Brilliant.org, one of the main complaints about this problem was that it was made less interesting by the fact that only one number worked. Interestingly enough, one of the reasons I liked this problem was because I found it surprising that only one number worked. Huh.

[97] The general case of asking to prove that this representation is unique if 100 and 2018 are replaced with general positive integers is an exercise in the textbook for a course that I TA. I saw this while flipping through the textbook for review problems and realized that finding the specific representation in a specific case would make a decent contest problem.

37. (Problem Stash 1) Determine the number of nonzero polynomials $P(x)$ with nonnegative integer coefficients such that $P(1) \le 100$ and $P(n)$ divides n^{2020} for infinitely many positive integers n.

★ 38. (CMIMC 2017) One can define the greatest common divisor of two positive rational numbers as follows: for a, b, c, and d positive integers with $\gcd(a,b) = \gcd(c,d) = 1$, write

$$\gcd\left(\frac{a}{b}, \frac{c}{d}\right) = \frac{\gcd(ad, bc)}{bd}.$$

For all positive integers K, let $f(K)$ denote the number of ordered pairs of positive rational numbers (m,n) with $m < 1$ and $n < 1$ such that

$$\gcd(m,n) = \frac{1}{K}.$$

What is $f(2017) - f(2016)$?

39. (Mock AMC 10/12 2013) On a table are n marbles, each of which has a different integer weight in the set $1, 2, 3, \ldots, n$. The marbles are partitioned into three boxes, and the average weight of the marbles in each box is written on said box. Suppose that the numbers written on the boxes are 6, 10, and 11. How many possible integer values of n are there?[98]

40. (CMIMC 2016) Suppose integers $a < b < c$ satisfy

$$a + b + c = 95 \qquad \text{and} \qquad a^2 + b^2 + c^2 = 3083.$$

Find c.

41. (Problem Stash 2) Determine the number of ordered quadruples (p,q,r,s) of odd primes such that

$$\frac{pqrs}{(p-2)(q-2)(r-2)(s-2)}$$

is an integer.

42. (NICE Spring 2021)[99]Determine the number of ordered pairs (b,c) of integers with $0 \le b \le 2016$ and $0 \le c \le 2016$ such that 2017 divides $x^3 - bx^2 + c$ for exactly one integer x with $0 \le x \le 2016$.

43. (CMIMC 2019) For all positive integers n, let

$$f(n) = \sum_{k=1}^{n} \varphi(k) \left\lfloor \frac{n}{k} \right\rfloor^2.$$

Compute $f(2019) - f(2018)$. Here $\varphi(n)$ denotes the number of positive integers less than or equal to n which are relatively prime to n.

44. (CMIMC 2017) Find the largest positive integer N satisfying the following properties:

- N is divisible by 7;

[98]This problem definitely takes the cake for the problem which was intentionally the most similar to some other problem from a contest. See here. (It's worth noting that despite their similarity, the problems are not exactly the same!)

[99]Originally written for CMIMC but never got used.

- Swapping the i^{th} and j^{th} digits of N (for any i and j with $i \neq j$) gives an integer which is *not* divisible by 7.

45. (NIMO April 2017) Let N denote the number of positive integers n between 1 and 1000 inclusive such that $\binom{3n}{n}$ is odd. What are the last two digits of N?[100]

46. (Mock AIME I 2015) Let f be a function defined along the rational numbers such that $f(\frac{m}{n}) = \frac{1}{n}$ for all relatively prime positive integers m and n. The product of all rational numbers $0 < x < 1$ such that
$$f\left(\frac{x - f(x)}{1 - f(x)}\right) = f(x) + \frac{9}{52}$$
can be written in the form $\frac{p}{q}$ for positive relatively prime integers p and q. Find $p + q$.

47. (AIME 2021) Consider the sequence $(a_k)_{k\geq 1}$ of positive rational numbers defined by $a_1 = \frac{2020}{2021}$ and for $k \geq 1$, if $a_k = \frac{m}{n}$ for relatively prime positive integers m and n, then
$$a_{k+1} = \frac{m+18}{n+19}.$$
Determine the sum of all positive integers j such that the rational number a_j can be written in the form $\frac{t}{t+1}$ for some positive integer t.

48. (USEMO 2020) Which positive integers can be written in the form
$$\frac{\operatorname{lcm}(x,y) + \operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}$$
for positive integers x, y, z?

49. (CMIMC 2016) Recall that in any row of Pascal's Triangle, the first and last elements of the row are 1 and each other element in the row is the sum of the two elements above it from the previous row. With this in mind, define the *Pascal Squared Triangle* as follows:

- In the n^{th} row, where $n \geq 1$, the first and last elements of the row equal n^2;
- Each other element is the sum of the two elements directly above it.

The first few rows of the Pascal Squared Triangle are shown below.

Row 1:	1
Row 2:	4 4
Row 3:	9 8 9
Row 4:	16 17 17 16
Row 5:	25 33 34 33 25

Let S_n denote the sum of the entries in the n^{th} row. For how many integers $1 \leq n \leq 10^6$ is S_n divisible by 13?[101]

[100] This was written all the way back in 2015! It did take me a long time to figure out what to do with the problem, though; I wasn't sure if using Lucas killed it too much.

[101] My train of thought for constructing this problem was to go from Valentine's Day to chocolate assortment triangles to variants of Pascal's triangle. Might be one of the strangest origins for a problem I've written.

50. (CMIMC 2018) It is given that there exists a unique triple of positive primes (p,q,r) such that $p<q<r$ and
$$\frac{p^3+q^3+r^3}{p+q+r}=249.$$
Find r.

51. (CMIMC 2016) Determine the smallest positive prime p which satisfies the congruence
$$p+p^{-1}\equiv 25 \pmod{143}.$$
Here, p^{-1} as usual denotes multiplicative inverse.[102]

52. (NIMO Summer Contest 2017) For all positive integers n, denote by $\sigma(n)$ the sum of the positive divisors of n and $\nu_p(n)$ the largest power of p which divides n. Compute the largest positive integer k such that 5^k divides
$$\sum_{d\mid N}\nu_3(d!)(-1)^{\sigma(d)},$$
where $N=6^{1999}$.

53. (NIMO 23) Let $p=2017$ be a prime. Find the remainder when
$$\left\lfloor\frac{1^p}{p}\right\rfloor+\left\lfloor\frac{2^p}{p}\right\rfloor+\left\lfloor\frac{3^p}{p}\right\rfloor+\cdots+\left\lfloor\frac{2015^p}{p}\right\rfloor$$
is divided by p. Here $\lfloor\cdot\rfloor$ denotes the greatest integer function.[103]

54. (CMIMC 2016) Compute the number of positive integers $n\le 50$ such that there exist distinct positive integers a,b satisfying
$$\frac{a}{b}+\frac{b}{a}=n\left(\frac{1}{a}+\frac{1}{b}\right).$$

55. (AMM 12239) Determine all positive integers r such that there exist at least two pairs of positive integers (m,n) satisfying the equation $2^m=n!+r$.

★ 56. (CMIMC 2018) Let $a_1<a_2<\cdots<a_k$ denote the sequence of all positive integers between 1 and 91 which are relatively prime to 91, and set $\omega=e^{2\pi i/91}$. Define
$$S=\prod_{1\le q<p\le k}\left(\omega^{a_p}-\omega^{a_q}\right).$$
Given that S is a positive integer, compute the number of positive divisors of S.[104]

[102] I wrote this over a span of two days during vacation after my senior year of high school. Instead of relaxing on the beach, I did modular arithmetic computations in my head. I'm weird. I imagine some of the beach-goers around me thought so as well.

[103] Another example of a problem that ended up being way more well-received than I expected.

[104] I ended up writing this for the 2017 contest, but I could not solve it at the time; that honor would be bestowed on the problem about a year later.

5 Miscellaneous/Undergraduate

1. (NICE Spring 2021) In a hat, there are ten slips, each containing a different integer from 1 to 10, inclusive. David reaches into the hat and keeps for himself two numbers that differ by seven. Ankan then reaches into the hat and picks two numbers that differ by five. What is the largest possible sum of the four picked numbers?

2. (CMIMC 2019) David recently bought a large supply of letter tiles. One day he arrives back to his dorm to find that some of the tiles have been arranged to read CENTRAL MICHIGAN UNIVERSITY. What is the smallest number of tiles David must remove and/or replace so that he can rearrange them to read CARNEGIE MELLON UNIVERSITY?[105]

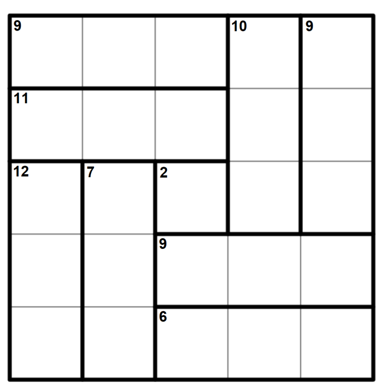
3. (CMIMC 2017) We are given the following function f, which takes a list of integers and outputs another list of integers. (Note that here the list is zero-indexed.)

```
FUNCTION f(A)
    FOR i = 1,...,length(A) - 1:
        A[i] ← A[A[i]]
        A[0] ← A[0] - 1
    RETURN A
```

   Suppose the list B is equal to $[0,1,2,8,2,0,1,7,0]$. In how many entries do B and $f(B)$ differ?

4. (Problem Stash 1) David is given the diagram shown below. He is able to write down the numbers 1 through 5 in each box such that the following two conditions hold:

   - Each row and column uses the numbers 1 through 5 exactly once;
   - The sum of the numbers in each "cage" (region bounded by a heavy border) is equal to the number written in the upper left corner of said cage.



   When he is finished, David reads from left to right along each long diagonal to obtain two five-digit numbers. Find their sum.

5. (CMIMC 2020) Find all sets of five positive integers whose mode, mean, median, and range are all equal to 5.

[105]This problem was based on an inside joke among CMU students where searching CMU on Google often gives the "wrong" university. It also was an attempt to give weaker teams something enjoyable to work on.

6. (Problem Stash 2) Compute the smallest possible value of

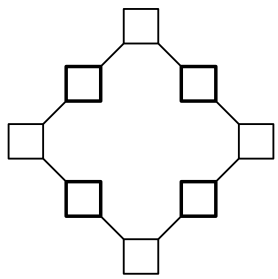
$$\int_0^{10} |x^2 - t|\, dx$$

as t ranges over all real numbers.[106]

7. (CMIMC 2018) How many ordered triples (a,b,c) of integers satisfy the inequality

$$a^2 + b^2 + c^2 \leq a + b + c + 2?^{107}$$

8. (RHS Guts Round 2015) Mr. Douglas, bored of scaring his students with pop trigonometry quizzes, decides one day to work on a logic puzzle. He is challenged to place each of the digits 1 through 8 in the eight boxes below, with one integer per box, such that each bold box contains an integer equal to the sum of the numbers in the squares immediately adjacent to it. After he is done, he takes each of the integers in the four bold boxes and multiplies them together. What is his result?



9. (CMIMC 2018) You are given the existence of an unsorted sequence $a_1, \ldots, a_5$ of five distinct real numbers. The Erdos-Szekeres theorem states that there exists a subsequence of length 3 which is either strictly increasing or strictly decreasing. You do not have access to the a_i, but you do have an oracle which, when given two indexes $1 \leq i < j \leq 5$, will tell you whether $a_i < a_j$ or $a_i > a_j$. What is the minimum number of calls to the oracle needed in order to identify one such requested subsequence?

10. (ARML Local 2018) Compute the number of positive integers N between 1 and 100 inclusive with the property that there exist distinct divisors a and b of N such that $a + b$ is also a divisor of N.

11. (CMIMC 2018) How many nonnegative integers with at most 40 digits consisting of entirely zeroes and ones are divisible by 11?[108]

12. (NICE Spring 2021) Suppose a and b are positive integers with $a < b$ such that the four intersection points of the lines $y = x + a$ and $y = x + b$ with the parabola $y = x^2$ are the vertices of a quadrilateral with area 720. Find $1000a + b$.

[106] This generalizes in the following way: if $f : [a,b] \to \mathbb{R}$ is increasing and differentiable, the quantity $\int_a^b |f(x) - t|\, dt$ is minimized when $t = f(\frac{a+b}{2})$. I encountered this fact while proving that $x \mapsto \log|x|$ is a Bounded Mean Oscillation (BMO) function for homework.

[107] This is in Miscellaneous simply because I'm not sure if it fits in either the Algebra or Number Theory sections. In any case, the problem is not terribly original anyway.

[108] Similarly, although this problem appeared in Combo Individual Finals, I can't tell whether this should belong in C or N.

13. (AMC 10A/12A 2020) There exists a unique strictly increasing sequence of nonnegative integers $a_1 < a_2 < \ldots < a_k$ such that
$$\frac{2^{289}+1}{2^{17}+1} = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_k}.$$
What is k?

14. (CMIMC 2017) Define $\{p_n\}_{n=0}^{\infty} \subset \mathbb{N}$ and $\{q_n\}_{n=0}^{\infty} \subset \mathbb{N}$ to be sequences of natural numbers as follows:

    - $p_0 = q_0 = 1$;
    - For all $n \in \mathbb{N}$, q_n is the smallest natural number such that there exists a natural number p_n with $\gcd(p_n, q_n) = 1$ satisfying
$$\frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} < \sqrt{2}.$$

    Find q_3.

15. (AMC 12A 2021) The five solutions to the equation
$$(z-1)(z^2+2z+4)(z^2+4z+6) = 0$$
may be written in the form $x_k + y_k i$ for $1 \le k \le 5$, where x_k and y_k are real. Let $\mathcal{E}$ be the unique ellipse that passes through the points $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$, and (x_5, y_5). The eccentricity of $\mathcal{E}$ can be written in the form $\sqrt{\frac{m}{n}}$ where m and n are relatively prime positive integers. What is $m+n$? (Recall that the eccentricity of an ellipse $\mathcal{E}$ is the ratio $\frac{c}{a}$, where $2a$ is the length of the major axis of $\mathcal{E}$ and $2c$ is the is the distence between its two foci.)

16. (Illinois Mock Putnam 2019) Find all locally integrable functions $g : \mathbb{R} \to \mathbb{R}$ such that
$$\int_0^1 g(f(x))\,dx = g\left(\int_0^1 f(x)\,dx\right)$$
for every integrable function $f : [0,1] \to \mathbb{R}$.

17. (Problem Stash 1/Illinois Undergraduate Contest 2020) For each positive integer n, let U_n denote the area enclosed by the graph of $|x|^n + |y|^n = 1$. The value of
$$\lim_{n\to\infty} n^2(4 - U_n)$$
can be written in the form $\frac{a\pi^b}{c}$, where a, b, and c are positive integers with a and c relatively prime. Compute $100a + 10b + c$.

18. (CMIMC 2016)[109] For any language $L \in \{0,1\}^*$, define a language L_0 which consists of the set of words of the form
$$a_1 s_{1,2} a_2 s_{2,3} a_3 \ldots a_{k-1} s_{k-1,k} a_k$$
where $a_1 a_2 a_3 \ldots a_k$ is a word in L. Here, $s_{i,j}$ is 1 when $a_i \neq a_j$ and zero otherwise. For example, the word 1101 in L corresponds to the word 1011011 in L_0.

    Suppose L is regular. Must L_0 also be regular?

[109] More specifically, the power round from that year; for information on the terminology used in this problem, see here.

6 Links to Solutions

Here are links to solutions to the majority of the problems organized by source. Some of these redirect to solutions manuals, while others redirect to various parts of the AoPS community.

- CMIMC: `http://cmimc.co/archive.html`
- Mock AIME I 2015: `https://www.dropbox.com/s/d66fcsu4hdfmotz/MockAIMEI2015Solutions.pdf?dl=0`
- Mock AMC 10/12 2013: No good source here, since a solutions manual was never written. Search `+Mock +AMC +Problem` with author `djmathman` in the AoPS Advanced search for links to problem threads.
- NIMO: `http://www.artofproblemsolving.com/community/c3423_nimo_problems`