

HMMT February 2020

February 15, 2020

Algebra and Number Theory

1. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t . What is $P(1)$?

Proposed by: James Lin

Answer:

Solution 1: Plugging in $x = r$ gives $r^2 = 2020$. This means $P(1) = 2 - r^2 - 2020 = -4038$.

Solution 2: Vieta's formulas give the following equations:

$$\begin{aligned}r + s + t &= -1 \\rs + st + tr &= -r^2 \\rst &= 2020.\end{aligned}$$

The second equation is $(r+t)(r+s) = 0$. Without loss of generality, let $r+t = 0$. Then $s = r+s+t = -1$. Finally $r^2 = rst = 2020$, so $P(1) = -4038$.

2. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

Proposed by: James Lin

Answer:

Solution 1: If either a or b is larger than 2020, then both must be for the product to be positive. However, the resulting product would be less than 1, so this case is impossible. Now, we see that $(\frac{2020-a}{a}, \frac{2020-b}{b})$ must be in the form $(\frac{x}{y}, \frac{2y}{x})$, in some order, for relatively prime positive integers x and y . Then $\frac{2020-a}{a} = \frac{x+y}{y}$ and $\frac{2020-b}{b} = \frac{x+2y}{x}$, so $x+y$ and $x+2y$ are relatively prime factors of 2020. Since $x+y < x+2y < 2(x+y)$, the only possibility is $x+y = 4$, $x+2y = 5$. Thus, $(x, y) = (3, 1)$, and $(\frac{2020-a}{a}, \frac{2020-b}{b}) = (3, \frac{2}{3})$ because $a < b$. Solving gives $(a, b) = (505, 1212)$.

Solution 2: We rearrange to find that $(a+2020)(b+2020) = 2 \cdot 2020^2$. Note that $a+2020$ and $b+2020$ are both less than $4040 < 101^2$, so they must both be divisible by 101. Hence, we divide out a factor of 101^2 and solve the equivalent problem of $(a'+20)(b'+20) = 800$, where $a' = \frac{a}{101}$ and $b' = \frac{b}{101}$. Because each factor must be less than $\frac{4040}{101} = 40$, we get that $(a'+20, b'+20) = (25, 32)$, which yields $(a, b) = (505, 1212)$.

3. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

Proposed by: James Lin

Answer:

Solution: Let $y = \log_a x$ so $\log_a \log_a y = \log_{a^2} \log_{a^2} \frac{1}{2}y$. Setting $z = \log_a y$, we find $\log_a z = \log_{a^2}(\frac{1}{2}z - \frac{1}{16})$, or $z^2 - \frac{1}{2}z + \frac{1}{16} = 0$. Thus, we have $z = \frac{1}{4}$, so we can backsolve to get $y = 4$ and $x = 2^{32}$.

4. For positive integers n and k , let $\mathcal{U}(n, k)$ be the number of distinct prime divisors of n that are at least k . For example, $\mathcal{U}(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mathcal{U}(n, k)}{3^{n+k-7}}.$$

Proposed by: Daniel Zhu

Answer: 167

Solution: A prime p is counted in $\mathcal{U}(n, k)$ if $p \mid n$ and $k \leq p$. Thus, for a given prime p , the total contribution from p in the sum is

$$3^7 \sum_{m=1}^{\infty} \sum_{k=1}^p \frac{1}{3^{pm+k}} = 3^7 \sum_{i \geq p+1} \frac{1}{3^i} = \frac{3^{7-p}}{2}.$$

Therefore, if we consider $p \in \{2, 3, 5, 7, \dots\}$ we get

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mathcal{U}(n, k)}{3^{n+k-7}} = \frac{3^5}{2} + \frac{3^4}{2} + \frac{3^2}{2} + \frac{3^0}{2} + \varepsilon = 167 + \varepsilon,$$

where $\varepsilon < \sum_{i=11}^{\infty} \frac{3^{7-i}}{2} = \frac{1}{108} \ll \frac{1}{2}$. The closest integer to the sum is 167.

5. A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1 + n_2 + \dots + n_{10} = N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim_{M \rightarrow \infty} p_M$.

Proposed by: James Lin

Answer: $\frac{32}{55}$

Solution: For notation, let $n_i(m)$ denote the number of digits of m^i and $N(m) = n_1(m) + n_2(m) + \dots + n_{10}(m)$. Observe that $n_i(10m) = n_i(m) + i$ so $N(10m) = N(m) + 55$. We will determine, for $k \rightarrow \infty$, how many of the integers from $N(10^k)$ to $N(10^{k+1}) - 1$, inclusive, are piquant.

Increment m by 1 from 10^k to 10^{k+1} . The number of digits of m^i increases by one if $m^i < 10^h \leq (m+1)^i$, or $m < 10^{\frac{h}{i}} \leq m+1$ for some integer h . This means that, as we increment m by 1, the sum $n_1 + n_2 + \dots + n_{10}$ increases when m “jumps over” $10^{\frac{h}{i}}$ for $i \leq 10$. Furthermore, when m is big enough, all “jumps” are distinguishable, i.e. there does not exist two $\frac{h_1}{i_1} \neq \frac{h_2}{i_2}$ such that $m < 10^{h_1/i_1} < 10^{h_2/i_2} \leq m+1$.

Thus, for large k , the number of times $n_1(m) + n_2(m) + \dots + n_{10}(m)$ increases as m increments by 1 from 10^k to 10^{k+1} is the number of different $10^{\frac{h}{i}}$ in the range $(10^k, 10^{k+1}]$. If we take the fractional part of the exponent, this is equivalent to the number of distinct fractions $0 < \frac{j}{i} \leq 1$ where $1 \leq i \leq 10$. The number of such fractions with denominator i is $\varphi(i)$, so the total number of such fractions is $\varphi(1) + \varphi(2) + \dots + \varphi(10) = 32$.

We have shown that for sufficiently large k , $N(10^{k+1}) - N(10^k) = 55$ and exactly 32 integers in the range $[N(10^k), N(10^{k+1}))$ are piquant. This implies that $\lim_{M \rightarrow \infty} p_M = \frac{32}{55}$.

6. A polynomial $P(x)$ is a *base- n polynomial* if it is of the form $a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$, where each a_i is an integer between 0 and $n-1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c , there exists at most one base- n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

Proposed by: James Lin

Answer: 9

Solution: It is equivalent to determine the largest n such that we cannot find two distinct base- n polynomials P_1 and P_2 such that $P_1(\sqrt{2} + \sqrt{3}) = P_2(\sqrt{2} + \sqrt{3})$. The difference of two base- n polynomials is a polynomial with integer coefficients whose absolute values are less than n , and all such polynomials are the difference of two base- n polynomials. We compute the minimal polynomial of $x = \sqrt{2} + \sqrt{3}$ first: since $x^2 = 5 + 2\sqrt{6}$, we have $(x^2 - 5)^2 = 24$ so $x^4 - 10x^2 + 1 = 0$. Therefore $\sqrt{2} + \sqrt{3}$ is a root of $(x^2 + 1)(x^4 - 10x^2 + 1) = x^6 - 9x^4 - 9x^2 + 1$. The coefficients of this polynomial have magnitude at most 9, so $n < 10$.

In the other direction, observe that $(\sqrt{2} + \sqrt{3})^k$ is of the form $a + b\sqrt{6}$ for integers a and b if k is even, and $a\sqrt{2} + b\sqrt{3}$ if k is odd. As no integer linear combination of the first expression can equal the second, we can treat these cases separately. Suppose $Q(x) = c_d x^{2d} + c_{d-1} x^{2d-2} + \dots + c_0$ is an even polynomial with $|c_i| < 9$ for all i and $c_d \neq 0$. Let $y = (\sqrt{2} + \sqrt{3})^2 = 5 + 2\sqrt{6}$ and observe that $y > 9$. Then

$$\begin{aligned} |c_d y^d| &\geq y^d \\ &> \frac{8}{y-1}(y^d - 1) \\ &= 8y^{d-1} + 8y^{d-2} + \dots + 8y + 8 \\ &\geq |c_{d-1}y^{d-1} + c_{d-2}y^{d-2} + \dots + c_0|. \end{aligned}$$

Therefore $Q(\sqrt{2} + \sqrt{3}) = c_d y^d + c_{d-1} y^{d-1} + \dots + c_0 \neq 0$, so no two distinct base-9 polynomials coincide at $x = \sqrt{2} + \sqrt{3}$.

The same logic applies for the odd polynomial case after dividing out a factor of $\sqrt{2} + \sqrt{3}$, so $n = 9$ works.

7. Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n - 3}$$

is an integer.

Proposed by: Andrew Gu

Answer: 90

Solution: It is clear that $n = 1$ and $n = 2$ work so assume that $n > 2$. If $2n - 3$ is composite then its smallest prime factor is at most $\frac{2n-3}{2} < n$ so will be coprime to $15 \cdot n!^2 + 1$. Therefore assume that $2n - 3 = p$ is prime. We can rewrite the numerator as

$$(-1)^n \cdot 15 \cdot \left(1 \cdot 2 \cdot \dots \cdot \frac{p+3}{2}\right) \cdot \left(\frac{p-3}{2} \cdot \frac{p-1}{2} \cdot \dots \cdot (p-1)\right) + 1 \pmod{p}.$$

By Wilson's Theorem, $(p-1)! \equiv -1 \pmod{p}$, so the expression simplifies to

$$(-1)^{n+1} \cdot 15 \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \frac{p+3}{2} + 1 \equiv (-1)^{n+1} \cdot \frac{135}{16} + 1 \pmod{p}.$$

If $p \equiv 3 \pmod{4}$, then we have

$$\frac{135 + 16}{16} \equiv \frac{151}{16} \equiv 0 \pmod{p}.$$

If $p \equiv 1 \pmod{4}$, then we have

$$\frac{135 - 16}{16} \equiv \frac{119}{16} \equiv 0 \pmod{p}.$$

So p must be a prime divisor of 151 or 119, which means that $p \in \{7, 17, 151\}$. All of these numbers work aside from 7 (because $7 \equiv 3 \pmod{4}$) and the corresponding values of n are 10 and 77. The sum of the solutions is then $1 + 2 + 10 + 77 = 90$.

8. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \dots, 2020$. Compute $P(2021^2)$.

Proposed by: Milan Haiman

Answer: $2021 - \binom{4040}{2020}$

Solution 1: Since $P(0) = 0$, we see that P has no constant term. Let $Q(x) = \frac{P(x^2)-x}{x}$ be a polynomial with degree at most 4039. From the given values of P , we see that $Q(k) = 0$ and $Q(-k) = -2$ for $k = 1, 2, 3, \dots, 2020$.

Now, consider the polynomial $R(x) = Q(x+1) - Q(x)$, which has degree at most 4038. Then R has roots $-2020, -2019, \dots, -2, 1, 2, \dots, 2019$, so

$$R(x) = a(x+2020) \cdots (x+2)(x-1) \cdots (x-2019)$$

for some real number a . Using $R(0) + R(-1) = Q(1) - Q(-1) = 2$ yields $a = -\frac{1}{2020!2019!}$, so

$$Q(2021) = R(2020) + Q(2020) = -\frac{4040 \cdots 2022 \cdot 2019 \cdots 1}{2020!2019!} + 0 = -\frac{1}{2021} \binom{4040}{2020}.$$

It follows that $P(2021^2) = 2021Q(2021) + 2021 = 2021 - \binom{4040}{2020}$.

Solution 2: By Lagrange interpolation,

$$P(x) = \sum_{k=0}^{2020} k \prod_{\substack{0 \leq j \leq 2020 \\ j \neq k}} \frac{(x-j^2)}{(k^2-j^2)} = \sum_{k=0}^{2020} \frac{2(-1)^k k}{(2020-k)!(2020+k)!(x-k^2)} \prod_{j=0}^{2020} (x-j^2).$$

Therefore, by applying Pascal's identity multiple times, we get that

$$\begin{aligned} P(2021^2) &= \sum_{k=0}^{2020} \frac{4042!(-1)^k k}{(2021-k)!(2021+k)!} \\ &= \sum_{k=0}^{2020} \binom{4042}{2021-k} (-1)^k k \\ &= 2021 - \left(\sum_{k=0}^{2021} (-1)^{k+1} k \left(\binom{4041}{2021-k} + \binom{4041}{2020-k} \right) \right) \\ &= 2021 - \left(\sum_{k=1}^{2021} (-1)^{k+1} \binom{4041}{2021-k} \right) \\ &= 2021 - \left(\sum_{k=1}^{2021} (-1)^{k+1} \left(\binom{4040}{2021-k} + \binom{4040}{2020-k} \right) \right) \\ &= 2021 - \binom{4040}{2020}. \end{aligned}$$

9. Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

Proposed by: Milan Haiman

Answer: $\boxed{2020 \cdot 2^{2019}}$

Solution: Let $P(x)$ have degree $n = 2020$ with roots r_1, \dots, r_n . Let $R(x) = \prod_i (x - r_i^2)$. Then

$$\prod_i r_i^n P\left(\frac{x}{r_i}\right) = \prod_i \prod_j (x - r_i r_j) = Q(x)^2 R(x).$$

Using $R(x^2) = (-1)^n P(x)P(-x)$ and Vieta, we obtain

$$P(x)P(-x)Q(x^2)^2 = P(0)^n \prod_i P\left(\frac{x^2}{r_i}\right).$$

Plugging in $x = \alpha$, we use the facts that $P(\alpha) = 4$, $P(-\alpha) = 4 - 2\alpha$, and also

$$P\left(\frac{\alpha^2}{r_i}\right) = \frac{\alpha^{4040}}{r_i^{2020}} + \frac{\alpha^2}{r_i} + 2 = -\frac{(\alpha - 2)^2}{r_i + 2} + \frac{\alpha^2}{r_i} + 2 = \frac{2(-\alpha - r_i)^2}{r_i(r_i + 2)}.$$

This will give us

$$P(\alpha)P(-\alpha)Q(\alpha^2)^2 = 2^n \prod_i \frac{2(-\alpha - r_i)^2}{r_i(r_i + 2)} = 2^n \cdot \frac{2^n P(-\alpha)^2}{P(0)P(-2)}.$$

Therefore,

$$\begin{aligned} Q(\alpha^2)^2 &= \frac{4^n P(-\alpha)}{P(0)P(-2)P(\alpha)} \\ &= \frac{4^n (4 - 2\alpha)}{2 \cdot 2^{2020} \cdot 4} \\ &= \frac{2^{4041} (2 - \alpha)}{2^{2023}} \\ &= 2^{2018} (2 - \alpha). \end{aligned}$$

We can check that $P(x) - 4$ has no double roots (e.g. by checking that it shares no roots with its derivative), which means that all possible α are distinct. Therefore, adding over all α gives $2020 \cdot 2^{2019}$, because the sum of the roots of $P(x) - 4$ is 0.

10. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in \mathbb{F}_{101} (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in \mathbb{F}_{101} for each nonnegative integer k . For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.

Proposed by: Michael Ren

Answer: $\boxed{101^{954}}$

Solution 1: Let $p = 101$, $m = 1001$, and work in the ring $R := \mathbb{F}_p[x]/(x^m - 1)$. We want to find the number of elements a of this ring that are of the form $x^p - x$. We first solve this question for a field extension \mathbb{F}_{p^a} of \mathbb{F}_p . Note that $(x+n)^p - (x+n) = x^p - x$ for any $n \in \mathbb{F}_p$, and the polynomial $t^p - t = b$ has at most p solutions in \mathbb{F}_{p^a} for any $b \in \mathbb{F}_{p^a}$. Combining these implies that $t^p - t = b$ always has either p or 0 solutions in \mathbb{F}_{p^a} , so there are p^{d-1} elements of \mathbb{F}_{p^a} expressible in the form $x^p - x$. Now, note that we may factor R into a product of field extensions of \mathbb{F}_p , each corresponding to an irreducible factor of $x^m - 1$ in \mathbb{F}_p , as the polynomial $x^m - 1$ has no double roots in \mathbb{F}_p as $p \nmid m$. By the Chinese Remainder Theorem, we may multiply the number of lucky polynomials for each of the field extensions to find the final answer. A field extension of degree d will yield p^{d-1} lucky polynomials. Thus, the final answer is p^{m-q} , where q is the number of fields in the factorization of R into fields.

To do determine q , we first factor

$$x^m - 1 = \prod_{k|m} \Phi_k(x)$$

in $\mathbb{Z}[x]$ where $\Phi_k(x)$ are the cyclotomic polynomials. Then we compute the number of irreducible divisors of the cyclotomic polynomial $\Phi_k(x)$ in $\mathbb{F}_p[x]$. We claim that this is equal to $\frac{\varphi(k)}{\text{ord}_k(p)}$. Indeed, note that given a root ω of Φ_k in the algebraic closure of \mathbb{F}_p , the roots of its minimal polynomial are $\omega, \omega^p, \omega^{p^2}, \dots$, and this will cycle after the numerator repeats modulo k , from which it follows that the degree of the minimal polynomial of ω is $\text{ord}_k(p)$. Thus, $\Phi_k(x)$ factors into $\frac{\varphi(k)}{\text{ord}_k(p)}$ irreducible polynomials.

It remains to compute orders. We have that

$$\begin{aligned} \text{ord}_7(101) &= \text{ord}_7(3) = 6, \\ \text{ord}_{11}(101) &= \text{ord}_{11}(2) = 10, \\ \text{ord}_{13}(101) &= \text{ord}_{13}(10) = 6. \end{aligned}$$

Thus,

$$\begin{aligned} \text{ord}_{77}(101) &= 30, \\ \text{ord}_{91}(101) &= 6, \\ \text{ord}_{143}(101) &= 30, \\ \text{ord}_{1001}(101) &= 30, \\ \text{ord}_1(101) &= 1. \end{aligned}$$

The number of factors of $x^{1001} - 1$ in $\mathbb{F}_{101}[x]$ is thus

$$\frac{1}{1} + \frac{6}{6} + \frac{10}{10} + \frac{12}{6} + \frac{60}{30} + \frac{72}{6} + \frac{120}{30} + \frac{720}{30} = 1 + 1 + 1 + 2 + 2 + 12 + 4 + 24 = 47,$$

so the total number of lucky polynomials is $101^{1001-47} = 101^{954}$.

Solution 2: As in the previous solution, we work in the ring $R = \mathbb{F}_p/(x^m - 1)$, which we can treat as the set of polynomials in $\mathbb{F}_p[x]$ with degree less than m . The problem is asking us for the number of elements of the map $\alpha: h \mapsto h^p - h$ in R . Note that this map is linear because $(a+b)^p = a^p + b^p$ in any field where $p = 0$ (which R is an example of). Hence it suffices to determine the size of the kernel of α .

We can directly compute that if

$$h(x) = a_{m-1}x^{m-1} + a_{m-2}x^{m-2} + \dots + a_1x + a_0,$$

then

$$h(x)^p = a_{m-1}x^{p(m-1)} + a_{m-2}x^{p(m-2)} + \dots + a_1x^p + a_0,$$

where exponents are taken modulo m . Therefore h is in the kernel if and only if $a_k = a_{pk}$ for all k where indices are taken modulo m . Letting q denote the number of orbits of the map $x \mapsto px$ in $\mathbb{Z}/m\mathbb{Z}$,

the size of the kernel is then p^q so the size of the image is p^{m-q} . It remains to compute q , which will end up being the same computation as in the previous solution.

HMMT February 2020
February 15, 2020
Combinatorics

1. How many ways can the vertices of a cube be colored red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it?

Proposed by: Milan Haiman

Answer:

Solution: If all vertices of the cube are of the same color, then there are 2 ways. Otherwise, look at a red vertex. Since it must have at least 2 red neighbors, there is a face of the cube containing 3 red vertices. The last vertex on this face must also be red. Similarly, all vertices on the opposite face must be blue. Thus, all vertices on one face of the cube are red while the others are blue. Since a cube has 6 faces, the answer is $2 + 6 = 8$.

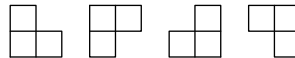
2. How many positive integers at most 420 leave different remainders when divided by each of 5, 6, and 7?

Proposed by: Milan Haiman

Answer:

Solution: Note that $210 = 5 \cdot 6 \cdot 7$ and 5, 6, 7 are pairwise relatively prime. So, by the Chinese Remainder Theorem, we can just consider the remainders n leaves when divided by each of 5, 6, 7. To construct an n that leaves distinct remainders, first choose its remainder modulo 5, then modulo 6, then modulo 7. We have $5 = 6 - 1 = 7 - 2$ choices for each remainder. Finally, we multiply by 2 because $420 = 2 \cdot 210$. The answer is $2 \cdot 5^3 = 250$.

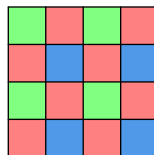
3. Each unit square of a 4×4 square grid is colored either red, green, or blue. Over all possible colorings of the grid, what is the maximum possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



Proposed by: Andrew Lin

Answer:

Solution: Notice that in each 2×2 square contained in the grid, we can form 4 L-trominoes. By the pigeonhole principle, some color appears twice among the four squares, and there are two trominoes which contain both. Therefore each 2×2 square contains at most 2 L-trominoes with distinct colors. Equality is achieved by coloring a square (x, y) red if $x + y$ is even, green if x is odd and y is even, and blue if x is even and y is odd. Since there are nine 2×2 squares in our 4×4 grid, the answer is $9 \times 2 = 18$.



4. Given an 8×8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

Proposed by: James Lin

Answer: 20736

Solution: Number both the rows and the columns from 1 to 8, and say that black squares are the ones where the rows and columns have the same parity. We will use, e.g. “even rows” to refer to rows 2, 4, 6, 8. Choosing 8 squares all in different rows and columns is equivalent to matching rows to columns.

For each of the 8 rows, we first decide whether they will be matched with a column of the same parity as itself (resulting in a black square) or with one of a different parity (resulting in a white square). Since we want to choose 4 squares of each color, the 4 rows matched to same-parity columns must contain 2 even rows and 2 odd rows. There are $\binom{4}{2}^2 = 6^2$ ways to choose 2 odd rows and 2 even rows to match with same-parity columns.

After choosing the above, we have fixed which 4 rows should be matched with odd columns (while the other 4 should be matched with even columns). Then there are $(4!)^2 = 24^2$ ways to assign the columns to the rows, so the answer is $(6 \cdot 24)^2 = 144^2 = 20736$.

5. Let S be a set of intervals defined recursively as follows:

- Initially, $[1, 1000]$ is the only interval in S .
- If $l \neq r$ and $[l, r] \in S$, then both $[l, \lfloor \frac{l+r}{2} \rfloor]$, $[\lfloor \frac{l+r}{2} \rfloor + 1, r] \in S$.

(Note that S can contain intervals such as $[1, 1]$, which contain a single integer.) An integer i is chosen uniformly at random from the range $[1, 1000]$. What is the expected number of intervals in S which contain i ?

Proposed by: Benjamin Qi

Answer: 10.976

Solution: The answer is given by computing the sum of the lengths of all intervals in S and dividing this value by 1000, where the length of an interval $[i, j]$ is given by $j - i + 1$. An interval may be categorized based on how many times $[1, 1000]$ must be split to attain it. An interval that is derived from splitting $[1, 1000]$ k times will be called a k -split.

The only 0-split is $[1, 1000]$, with a total length of 1000. The 1-splits are $[1, 500]$ and $[501, 1000]$, with a total length of 1000. As long as none of the k -splits have length 1, the $(k + 1)$ -splits will have the same total length. Since the length of the intervals is reduced by half each time (rounded down), we find that the sum of the lengths of the k -splits is 1000 for $0 \leq k \leq 9$.

Note that the 9-splits consist of $2^{10} - 1000$ intervals of length 1 and $1000 - 2^9$ intervals of length 2. Then the 10-splits consist of $2(1000 - 2^9)$ intervals of length 1, with total length $2(1000 - 2^9)$. The total interval length across all splits is equal to $12(1000) - 2^{10}$, so our answer is

$$12 - \frac{2^{10}}{1000} = 10.976.$$

6. Alice writes 1001 letters on a blackboard, each one chosen independently and uniformly at random from the set $S = \{a, b, c\}$. A move consists of erasing two distinct letters from the board and replacing them with the third letter in S . What is the probability that Alice can perform a sequence of moves which results in one letter remaining on the blackboard?

Proposed by: Daniel Zhu

Answer: $\boxed{\frac{3-3^{-999}}{4}}$

Solution: Let n_a , n_b , and n_c be the number of a 's, b 's, and c 's on the board, respectively. The key observation is that each move always changes the parity of all three of n_a , n_b , and n_c . Since the final configuration must have n_a , n_b , and n_c equal to 1, 0, 0 in some order, Alice cannot leave one letter on the board if n_a , n_b , and n_c start with the same parity (because then they will always have the same parity). Alice also cannot leave one letter on the board if all the letters are initially the same (because she will have no moves to make).

We claim that in all other cases, Alice can make a sequence of moves leaving one letter on the board. The proof is inductive: the base cases $n_a + n_b + n_c \leq 2$ are easy to verify, as the possible tuples are $(1, 0, 0)$, $(1, 1, 0)$, and permutations. If $n_a + n_b + n_c \geq 3$, assume without loss of generality that $n_a \geq n_b \geq n_c$. Then $n_b \geq 1$ (because otherwise all the letters are a) and $n_a \geq 2$ (because otherwise $(n_a, n_b, n_c) = (1, 1, 1)$, which all have the same parity). Then Alice will replace a and b by c , reducing to a smaller case.

We begin by computing the probability that n_a , n_b , and n_c start with the same parity. Suppose m letters are chosen at random in the same way (so that we are in the case $m = 1001$). Let x_m be the probability that n_a , n_b , and n_c all have the same parity. We have the recurrence $x_{m+1} = \frac{1}{3}(1 - x_m)$ because when choosing the $(m + 1)$ th letter, the n_i can only attain the same parity if they did not before, and the appropriate letter is drawn. Clearly $x_0 = 1$, which enables us to compute $x_m = \frac{1}{4}(1 + 3 \cdot (-3)^{-m})$. Then x_{1001} is the probability that n_a , n_b , and n_c have the same parity.

The probability that all the letters are initially the same is 3^{-1000} , as this occurs exactly when all the subsequent letters match the first. Thus our final answer is

$$1 - 3^{-1000} - \frac{1}{4}(1 + 3 \cdot (-3)^{-1001}) = \frac{3}{4} - \frac{1}{4 \cdot 3^{999}}.$$

7. Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

Proposed by: Michael Ren

Answer: $\boxed{\frac{837}{208}}$

Solution: Note that $2002 = 2 \cdot 7 \cdot 11 \cdot 13$, so that each positive factor of 2002 is included on exactly one card. Each card can be identified simply by whether or not it is divisible by each of the 4 primes, and we can uniquely achieve all of the 2^4 possibilities. Also, when considering the product of the values on many cards, we only care about the values of the exponents in the prime factorization modulo 2, as we have a perfect square exactly when each exponent is even.

Now suppose Anne-Marie has already drawn k cards. Then there are 2^k possible subsets of cards from those she has already drawn. Note that if any two of these subsets have products with the same four exponents modulo 2, then taking the symmetric difference yields a subset of cards in her hand where all four exponents are 0 (mod 2), which would cause her to stop. Now when she draws the $(k + 1)$ th card, she achieves a perfect square subset exactly when the exponents modulo 2 match those from a subset of the cards she already has. Thus if she has already drawn k cards, she will not stop if she draws one of $16 - 2^k$ cards that don't match a subset she already has.

Let p_k be the probability that Anne-Marie draws at least k cards. We have the recurrence

$$p_{k+2} = \frac{16 - 2^k}{16 - k} p_{k+1}$$

because in order to draw $k + 2$ cards, the $(k + 1)$ th card, which is drawn from the remaining $16 - k$ cards, must not be one of the $16 - 2^k$ cards that match a subset of Anne-Marie's first k cards. We now compute

$$\begin{aligned} p_1 &= 1, \\ p_2 &= \frac{15}{16}, \\ p_3 &= \frac{14}{15}p_2 = \frac{7}{8}, \\ p_4 &= \frac{12}{14}p_3 = \frac{3}{4}, \\ p_5 &= \frac{8}{13}p_4 = \frac{6}{13}, \\ p_6 &= 0. \end{aligned}$$

The expected number of cards that Anne-Marie draws is

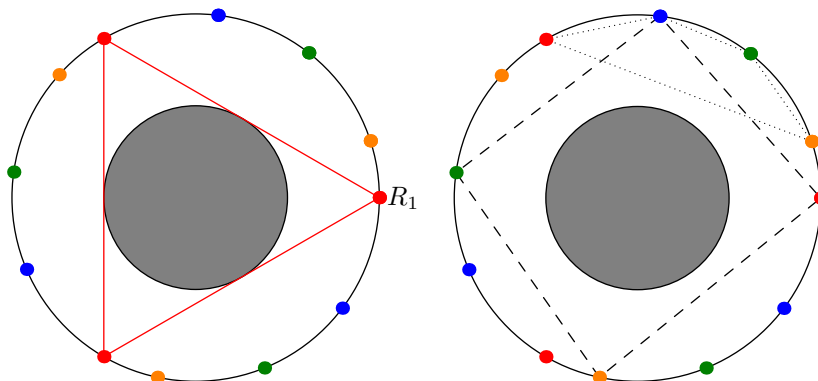
$$p_1 + p_2 + p_3 + p_4 + p_5 = 1 + \frac{15}{16} + \frac{7}{8} + \frac{3}{4} + \frac{6}{13} = \frac{837}{208}.$$

8. Let Γ_1 and Γ_2 be concentric circles with radii 1 and 2, respectively. Four points are chosen on the circumference of Γ_2 independently and uniformly at random, and are then connected to form a convex quadrilateral. What is the probability that the perimeter of this quadrilateral intersects Γ_1 ?

Proposed by: Yuan Yao

Answer:

Solution:



Define a *triplet* as three points on Γ_2 that form the vertices of an equilateral triangle. Note that due to the radii being 1 and 2, the sides of a triplet are all tangent to Γ_1 . Rather than choosing four points on Γ_2 uniformly at random, we will choose four triplets of Γ_2 uniformly at random and then choose a random point from each triplet. (This results in the same distribution.) Assume without loss of generality that the first step creates 12 distinct points, as this occurs with probability 1.

In the set of twelve points, a segment between two of those points does not intersect Γ_1 if and only if they are at most three vertices apart. (In the diagram shown above, the segments connecting R_1 to the other red vertices are tangent to Γ_1 , so the segments connecting R_1 to the six closer vertices do not intersect Γ_1 .) There are two possibilities for the perimeter of the convex quadrilateral to not intersect Γ_1 : either the convex quadrilateral contains Γ_1 or is disjoint from it.

Case 1: The quadrilateral contains Γ_1 .

Each of the four segments of the quadrilateral passes at most three vertices, so the only possibility is that every third vertex is chosen. This is shown by the dashed quadrilateral in the diagram, and there are 3 such quadrilaterals.

Case 2: The quadrilateral does not contain Γ_1 .

In this case, all of the chosen vertices are at most three apart. This is only possible if we choose four consecutive vertices, which is shown by the dotted quadrilateral in the diagram. There are 12 such quadrilaterals.

Regardless of how the triplets are chosen, there are 81 ways to pick four points and $12 + 3 = 15$ of these choices result in a quadrilateral whose perimeter does not intersect Γ_1 . The desired probability is $1 - \frac{5}{27} = \frac{22}{27}$.

Remark. The problem can easily be generalized for a larger number of vertices, where Γ_1 and Γ_2 are the inscribed and circumscribed circles of a regular n -gon and $n + 1$ points are chosen uniformly at random on Γ_2 . The probability that the perimeter of the convex $(n + 1)$ -gon formed by those vertices intersects Γ_1 is $1 - \frac{n+2}{n^n}$.

9. Farmer James wishes to cover a circle with circumference 10π with six different types of colored arcs. Each type of arc has radius 5, has length either π or 2π , and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:
- Any two adjacent arcs are of different colors.
 - Any three adjacent arcs where the middle arc has length π are of three different colors.

Find the number of distinct ways Farmer James can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

Proposed by: James Lin

Answer: 93

Solution: Fix an orientation of the circle, and observe that the the problem is equivalent to finding the number of ways to color ten equal arcs of the circle such that each arc is one of three different colors, and any two arcs which are separated by exactly one arc are of different colors. We can consider every other arc, so we are trying to color just five arcs so that no two adjacent arcs are of the same color. This is independent from the coloring of the other five arcs.

Let a_i be the number of ways to color i arcs in three colors so that no two adjacent arcs are the same color. Note that $a_1 = 3$ and $a_2 = 6$. We claim that $a_i + a_{i+1} = 3 \cdot 2^i$ for $i \geq 2$. To prove this, observe that a_i counts the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color, and the first and $(i + 1)$ th points are the same color. Meanwhile, a_{i+1} counts the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color, and the first and $(i + 1)$ th points are different colors. Then $a_i + a_{i+1}$ is the number of ways to color $i + 1$ points in a line so that no two adjacent points are the same color. There are clearly $3 \cdot 2^i$ ways to do this, as we pick the colors from left to right, with 3 choices for the first color and 2 for the rest. We then compute $a_3 = 6$, $a_4 = 18$, $a_5 = 30$. Then we can color the whole original circle by picking one of the 30 possible colorings for each of the two sets of 5 alternating arcs, for $30^2 = 900$ total.

Now, we must consider the rotational symmetry. If a configuration has no rotational symmetry, then we have counted it 10 times. If a configuration has 180° rotational symmetry, then we have counted it 5 times. This occurs exactly when we have picked the same coloring from our 30 for both choices, and in exactly one particular orientation, so there are 30 such cases. Having 72° or 36° rotational

symmetry is impossible, as arcs with exactly one arc between them must be different colors. Then after we correct for overcounting our answer is

$$\frac{900 - 30}{10} + \frac{30}{5} = 93.$$

10. Max repeatedly throws a fair coin in a hurricane. For each throw, there is a 4% chance that the coin gets blown away. He records the number of heads H and the number of tails T before the coin is lost. (If the coin is blown away on a toss, no result is recorded for that toss.) What is the expected value of $|H - T|$?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{\frac{24}{7}}$

Solution 1: (Dilhan Salgado) In all solutions, $p = \frac{1}{25}$ will denote the probability that the coin is blown away. Let $D = |H - T|$. Note that if $D \neq 0$, the expected value of D is not changed by a coin flip, whereas if $D = 0$, the expected value of D increases by 1. Therefore $\mathbf{E}(D)$ can be computed as the sum over all n of the probability that the n th coin flip occurs when $D = 0$. This only occurs when $n = 2k + 1$ is odd, where the probability that the first n coin flips occur is $(1 - p)^{2k+1}$ and the probability that $D = 0$ after the first $n - 1$ flips is $\frac{\binom{2k}{k}}{4^k}$. Therefore

$$\begin{aligned} \mathbf{E}(D) &= (1 - p) \sum_{k=0}^{\infty} \left(\frac{1 - p}{2}\right)^{2k} \binom{2k}{k} \\ &= \frac{1 - p}{\sqrt{1 - (1 - p)^2}} \end{aligned}$$

using the generating function

$$\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1 - 4x}}.$$

Plugging in $p = \frac{1}{25}$ yields $\mathbf{E}(D) = \frac{24}{7}$.

Solution 2: For each $n \geq 0$, the probability that Max made n successful throws (not counting the last throw) is $p(1 - p)^n$.

Claim: Assuming Max made $n \geq 1$ throws, the expected value of $|H - T|$ is given by

$$\prod_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{2k + 1}{2k}.$$

Proof. If n is odd then the expected value for $n + 1$ will be equal to that for n ; since $|H - T|$ will be nonzero, it will be equally likely to increase or decrease after the coin is flipped. Therefore, it suffices to compute the expected value for the n odd case. This is

$$\begin{aligned} \frac{\sum_{i=0}^{(n-1)/2} \binom{n}{i} \cdot (n - 2i)}{2^{n-1}} &= n - \frac{\sum_{i=0}^{(n-1)/2} \binom{n}{i} \cdot 2i}{2^{n-1}} \\ &= n \cdot \left(1 - \frac{2 \cdot \sum_{i=0}^{(n-3)/2} \binom{n-1}{i}}{2^{n-1}}\right) \\ &= n \cdot \frac{\binom{n-1}{(n-1)/2}}{2^{n-1}} \end{aligned}$$

$$\begin{aligned}
&= \frac{n!}{(n-1)!!^2} \\
&= \frac{n!!}{(n-1)!!} \\
&= \prod_{k=1}^{(n-1)/2} \frac{2k+1}{2k},
\end{aligned}$$

as desired. □

Using the claim, we have

$$\begin{aligned}
\mathbf{E}(|H - T|) &= p \left(\sum_{n=1}^{\infty} (1-p)^n \prod_{k=1}^{\lfloor (n-1)/2 \rfloor} \frac{2k+1}{2k} \right) \\
&= p(1-p)(2-p) \sum_{m=0}^{\infty} \left((1-p)^{2m} \prod_{k=1}^m \frac{2k+1}{2k} \right) \\
&= p(1-p)(2-p) (1 - (1-p)^2)^{-3/2} \\
&= \frac{1-p}{\sqrt{p(2-p)}}.
\end{aligned}$$

Plugging in $p = \frac{1}{25}$ gives

$$\mathbf{E}(|H - T|) = \frac{24}{25} \cdot 5 \cdot \frac{5}{7} = \frac{24}{7}.$$

Solution 3: Let E_n be the expected value of $|H - T + n|$. By symmetry, $E_{-n} = E_n$ for all n . Considering what happens in the next throw gives

$$2E_n = (1-p)E_{n-1} + (1-p)E_{n+1} + 2pn$$

for all $n \geq 0$. Now let $\alpha = \frac{1 - \sqrt{p(2-p)}}{1-p} < 1$ be the smaller root of $(1-p)x^2 - 2x + (1-p) = 0$. From

$$\begin{aligned}
\sum_{n=1}^{\infty} 2\alpha^n E_n &= \sum_{n=1}^{\infty} \alpha^n ((1-p)E_{n-1} + (1-p)E_{n+1} + 2pn) \\
&= \alpha(1-p)E_0 + \alpha^2(1-p)E_1 + \sum_{n=1}^{\infty} 2pn\alpha^n + \sum_{n=2}^{\infty} ((1-p)\alpha^{n-1} + (1-p)\alpha^{n+1}) E_n \\
&= \alpha(1-p)E_0 + (2\alpha - (1-p))E_1 + \sum_{n=1}^{\infty} 2pn\alpha^n + \sum_{n=2}^{\infty} 2\alpha^n E_n,
\end{aligned}$$

we have

$$(1-p)E_1 - \alpha(1-p)E_0 = \sum_{n=1}^{\infty} 2pn\alpha^n = \frac{2p\alpha}{(1-\alpha)^2}.$$

As $E_0 = (1-p)E_1$, this gives

$$E_0(1 - \alpha(1-p)) = \frac{2p\alpha}{(1-\alpha)^2}.$$

Plugging in $p = \frac{1}{25}$ and $\alpha = \frac{3}{4}$ gives $E_0 = \frac{24}{7}$.

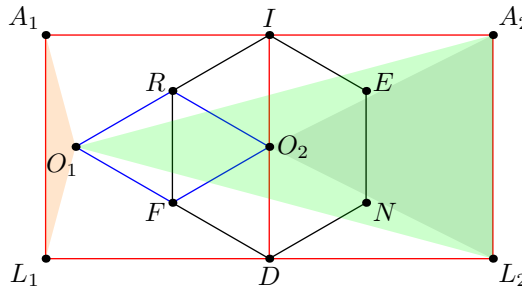
HMMT February 2020
February 15, 2020
Geometry

1. Let *DIAL*, *FOR*, and *FRIEND* be regular polygons in the plane. If $ID = 1$, find the product of all possible areas of *OLA*.

Proposed by: Andrew Gu

Answer: $\boxed{\frac{1}{32}}$

Solution: Focusing on *FRIEND* and *FOR* first, observe that either *DIO* is an equilateral triangle or *O* is the midpoint of *ID*. Next, *OLA* is always an isosceles triangle with base $LA = 1$. The possible distances of *O* from *LA* are 1 and $1 \pm \frac{\sqrt{3}}{2}$ as the distance from *O* to *ID* in the equilateral triangle case is $\frac{\sqrt{3}}{2}$.



The three possibilities are shown in the diagram as shaded triangles $\triangle O_1L_1A_1$, $\triangle O_2L_2A_2$, and $\triangle O_1L_2A_2$.

The product of all possible areas is thus

$$\frac{1 \cdot \left(1 - \frac{\sqrt{3}}{2}\right) \cdot \left(1 + \frac{\sqrt{3}}{2}\right)}{2^3} = \frac{1}{2^5} = \frac{1}{32}.$$

2. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $\angle BAC = 60^\circ$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside ABC such that U and V lie on side BA , W and X lie on side AC , and Z lies on side CB . What is the side length of hexagon $UVWXYZ$?

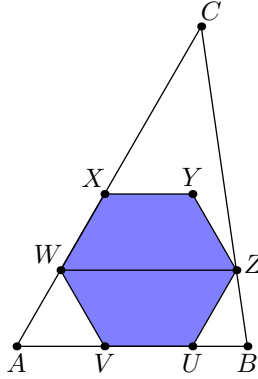
Proposed by: Ryan Kim

Answer: $\boxed{\frac{40}{21}}$

Solution: Let the side length of $UVWXYZ$ be s . We have $WZ = 2s$ and $WZ \parallel AB$ by properties of regular hexagons. Thus, triangles WCZ and ACB are similar. AWV is an equilateral triangle, so we have $AW = s$. Thus, using similar triangles, we have

$$\frac{WC}{WZ} = \frac{AC}{AB} \implies \frac{8-s}{2s} = \frac{8}{5},$$

so $5(8-s) = 8(2s) \implies s = \frac{40}{21}$.



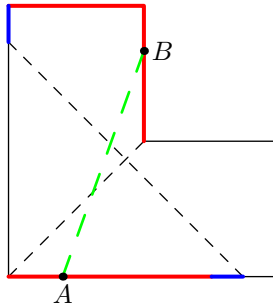
3. Consider the L-shaped tromino below with 3 attached unit squares. It is cut into exactly two pieces of equal area by a line segment whose endpoints lie on the perimeter of the tromino. What is the longest possible length of the line segment?



Proposed by: James Lin

Answer: $\boxed{\frac{5}{2}}$

Solution: Let the line segment have endpoints A and B . Without loss of generality, let A lie below the lines $x + y = \sqrt{3}$ (as this will cause B to be above the line $x + y = \sqrt{3}$) and $y = x$ (we can reflect about $y = x$ to get the rest of the cases):



Now, note that as A ranges from $(0, 0)$ to $(1.5, 0)$, B will range from $(1, 1)$ to $(1, 2)$ to $(0, 2)$, as indicated by the red line segments. Note that these line segments are contained in a rectangle bounded by $x = 0$, $y = 0$, $x = 1.5$, and $y = 2$, and so the longest line segment in this case has length $\sqrt{2^2 + 1.5^2} = \frac{5}{2}$.

As for the rest of the cases, as $A = (x, 0)$ ranges from $(1.5, 0)$ to $(\sqrt{3}, 0)$, B will be the point $(0, \frac{3}{x})$, so it suffices to maximize $\sqrt{x^2 + \frac{9}{x^2}}$ given $1.5 \leq x \leq \sqrt{3}$. Note that the further away x^2 is from 3, the larger $x^2 + \frac{9}{x^2}$ gets, and so the maximum is achieved when $x = 1.5$, which gives us the same length as before.

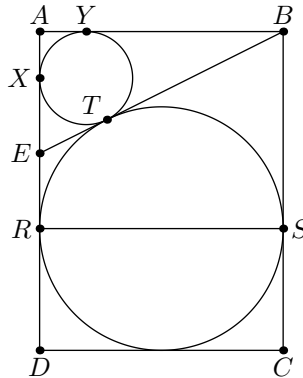
Thus, the maximum length is $\frac{5}{2}$.

4. Let $ABCD$ be a rectangle and E be a point on segment AD . We are given that quadrilateral $BCDE$ has an inscribed circle ω_1 that is tangent to BE at T . If the incircle ω_2 of ABE is also tangent to BE at T , then find the ratio of the radius of ω_1 to the radius of ω_2 .

Proposed by: James Lin

Answer: $\boxed{\frac{3+\sqrt{5}}{2}}$

Solution: Let ω_1 be tangent to AD , BC at R , S and ω_2 be tangent to AD , AB at X , Y . Let $AX = AY = r$, $EX = ET = ER = a$, $BY = BT = BS = b$. Then noting that $RS \parallel CD$, we see that $ABSR$ is a rectangle, so $r + 2a = b$. Therefore $AE = a + r$, $AB = b + r = 2(a + r)$, and so $BE = (a + r)\sqrt{5}$. On the other hand, $BE = b + a = r + 3a$. This implies that $a = \frac{1+\sqrt{5}}{2}r$. The desired ratio is then $\frac{RS}{2AY} = \frac{AB}{2r} = \frac{a+r}{r} = \frac{3+\sqrt{5}}{2}$.

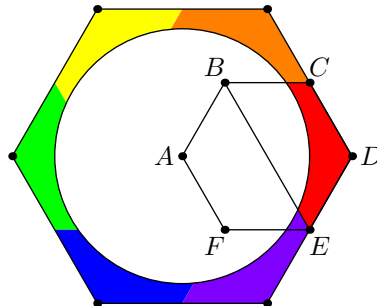


5. Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

Proposed by: Carl Joshua Quines

Answer: $\boxed{4\sqrt{3} - \frac{3}{2}\pi}$

Solution: Rotate the region 6 times about A to form a bigger hexagon with a circular hole. The larger hexagon has side length 4 and area $24\sqrt{3}$, so the area of the region is $\frac{1}{6}(24\sqrt{3} - 9\pi) = 4\sqrt{3} - \frac{3}{2}\pi$.

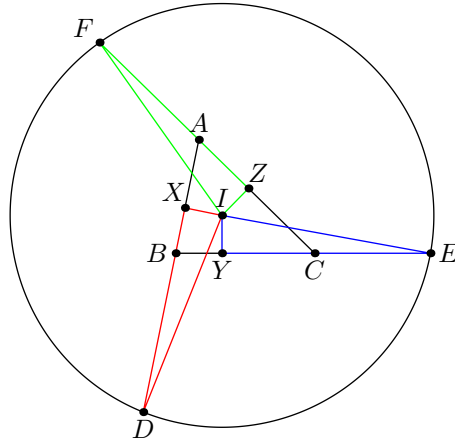


6. Let ABC be a triangle with $AB = 5$, $BC = 6$, $CA = 7$. Let D be a point on ray AB beyond B such that $BD = 7$, E be a point on ray BC beyond C such that $CE = 5$, and F be a point on ray CA beyond A such that $AF = 6$. Compute the area of the circumcircle of DEF .

Proposed by: James Lin

Answer: $\boxed{\frac{251}{3}\pi}$

Solution 1: Let I be the incenter of ABC . We claim that I is the circumcenter of DEF .

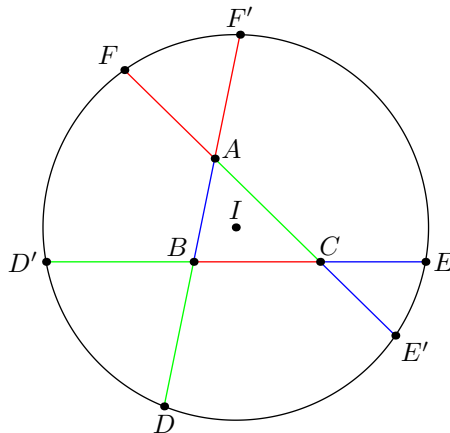


To prove this, let the incircle touch AB , BC , and AC at X , Y , and Z , respectively. Noting that $XB = BY = 2$, $YC = CZ = 4$, and $ZA = AX = 3$, we see that $XD = YE = ZF = 9$. Thus, since $IX = IY = IZ = r$ (where r is the inradius) and $\angle IXD = \angle IYE = \angle IZF = 90^\circ$, we have three congruent right triangles, and so $ID = IE = IF$, as desired.

Let $s = \frac{5+6+7}{2} = 9$ be the semiperimeter. By Heron's formula, $[ABC] = \sqrt{9(9-5)(9-6)(9-7)} = 6\sqrt{6}$, so $r = \frac{[ABC]}{s} = \frac{2\sqrt{6}}{3}$. Then the area of the circumcircle of DEF is

$$ID^2\pi = (IX^2 + XD^2)\pi = (r^2 + s^2)\pi = \frac{251}{3}\pi.$$

Solution 2: Let D' be a point on ray CB beyond B such that $BD' = 7$, and similarly define E' , F' . Noting that $DA = E'A$ and $AF = AF'$, we see that $DE'F'F$ is cyclic by power of a point. Similarly, $EF'D'D$ and $FD'E'E$ are cyclic. Now, note that the radical axes for the three circles circumscribing these quadrilaterals are the sides of ABC , which are not concurrent. Therefore, $DD'F'F'EE'$ is cyclic. We can deduce that the circumcenter of this circle is I in two ways: either by calculating that the midpoint of $D'E$ coincides with the foot from I to BC , or by noticing that the perpendicular bisector of FF' is AI . The area can then be calculated the same way as the previous solution.



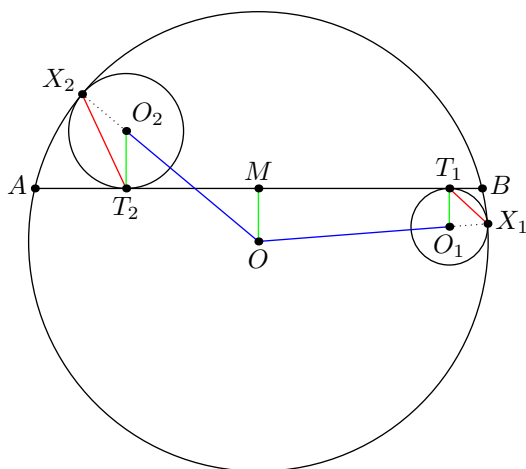
Remark. The circumcircle of DEF is the *Conway circle* of ABC .

7. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2 , respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2 , respectively, while intersecting Γ at two points A and B . Given that $2X_1T_1 = X_2T_2$ and that ω_1 , ω_2 , and Γ have radii 2, 3, and 12, respectively, compute the length of AB .

Proposed by: James Lin

Answer: $\boxed{\frac{96\sqrt{10}}{13}}$

Solution 1: Let ω_1 , ω_2 , Γ have centers O_1 , O_2 , O and radii r_1 , r_2 , R respectively. Let d be the distance from O to AB (signed so that it is positive if O and O_1 are on the same side of AB).



Note that

$$\begin{aligned} OO_i &= R - r_i, \\ \cos \angle T_1 O_1 O &= \frac{O_1 T_1 - OM}{OO_1} = \frac{r_1 - d}{R - r_1}, \\ \cos \angle T_2 O_2 O &= \frac{O_2 T_2 + OM}{OO_2} = \frac{r_2 + d}{R - r_2}. \end{aligned}$$

Then

$$\begin{aligned} X_1 T_1 &= r_1 \sqrt{2 - 2 \cos \angle X_1 O_1 T_1} \\ &= r_1 \sqrt{2 + 2 \cos \angle T_1 O_1 O} \\ &= r_1 \sqrt{2 + 2 \frac{r_1 - d}{R - r_1}} \\ &= r_1 \sqrt{2 \frac{R - d}{R - r_1}}. \end{aligned}$$

Likewise,

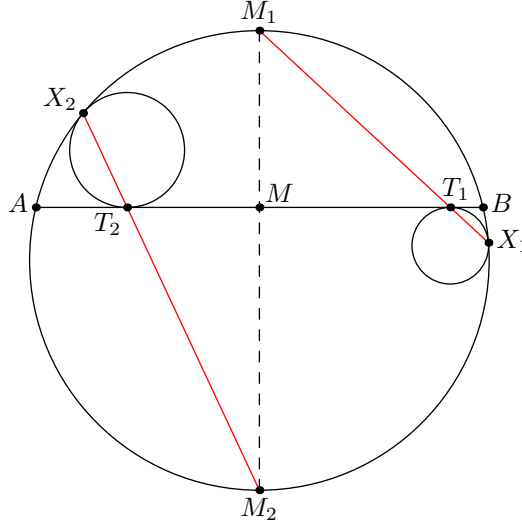
$$X_2 T_2 = r_2 \sqrt{2 \frac{R + d}{R - r_2}}.$$

From $2X_1T_1 = X_2T_2$ we have

$$8r_1^2 \left(\frac{R - d}{R - r_1} \right) = 4X_1T_1^2 = X_2T_2^2 = 2r_2^2 \left(\frac{R + d}{R - r_2} \right).$$

Plugging in $r_1 = 2$, $r_2 = 3$, $R = 12$ and solving yields $d = \frac{36}{13}$. Hence $AB = 2\sqrt{R^2 - d^2} = \frac{96\sqrt{10}}{13}$.

Solution 2: We borrow the notation from the previous solution. Let X_1T_1 and X_2T_2 intersect Γ again at M_1 and M_2 . Note that, if we orient AB to be horizontal, then the circles ω_1 and ω_2 are on opposite sides of AB . In addition, for $i \in \{1, 2\}$ there exist homotheties centered at X_i with ratio $\frac{R}{r_i}$ which send ω_i to Γ . Since T_1 and T_2 are points of tangencies and thus top/bottom points, we see that M_1 and M_2 are the top and bottom points of Γ , and so M_1M_2 is a diameter perpendicular to AB .



Now, note that through power of a point and the aforementioned homotheties,

$$P(M_1, \omega_1) = M_1T_1 \cdot M_1X_1 = X_1T_1^2 \left(\frac{R}{r_1}\right) \left(\frac{R}{r_1} - 1\right) = 30X_1T_1^2,$$

and similarly $P(M_2, \omega_2) = 12X_2T_2^2$. (Here P is the power of a point with respect to a circle). Then

$$\frac{P(M_1, \omega_1)}{P(M_2, \omega_2)} = \frac{30X_1T_1^2}{12X_2T_2^2} = \frac{30}{12(2)^2} = \frac{5}{8}.$$

Let M be the midpoint of AB , and suppose $M_1M = R + d$ (here d may be negative). Noting that M_1 and M_2 are arc bisectors, we have $\angle AX_1M_1 = \angle T_1AM_1$, so $\triangle M_1AT_1 \sim \triangle M_1X_1A$, meaning that $M_1A^2 = M_1T_1 \cdot M_1X_1 = P(M_1, \omega_1)$. Similarly, $\triangle M_2AT_2 \sim \triangle M_2X_2A$, so $M_2A^2 = P(M_2, \omega_2)$. Therefore,

$$\frac{P(M_1, \omega_1)}{P(M_2, \omega_2)} = \frac{M_1A^2}{M_2A^2} = \frac{(R^2 - d^2) + (R + d)^2}{(R^2 - d^2) + (R - d)^2} = \frac{2R^2 + 2Rd}{2R^2 - 2Rd} = \frac{R + d}{R - d} = \frac{5}{8},$$

giving $d = -\frac{3}{13}R$. Finally, we compute $AB = 2R\sqrt{1 - \left(\frac{3}{13}\right)^2} = \frac{8R\sqrt{10}}{13} = \frac{96\sqrt{10}}{13}$.

8. Let ABC be an acute triangle with circumcircle Γ . Let the internal angle bisector of $\angle BAC$ intersect BC and Γ at E and N , respectively. Let A' be the antipode of A on Γ and let V be the point where AA' intersects BC . Given that $EV = 6$, $VA' = 7$, and $A'N = 9$, compute the radius of Γ .

Proposed by: James Lin

Answer: $\boxed{\frac{15}{2}}$

Solution 1: Let H_a be the foot of the altitude from A to BC . Since AE bisects $\angle H_aAV$, by the angle bisector theorem $\frac{AH_a}{H_aE} = \frac{AV}{VE}$. Note that $\triangle AH_aE \sim \triangle ANA'$ are similar right triangles, so $\frac{AN}{NA'} = \frac{AH_a}{H_aE}$.

An inversion about γ preserves $\omega_a, \omega_b, \omega_c$ and sends ℓ to the circle with diameter IT . Since inversion preserves tangency, the circle with diameter IT must be ω . Therefore $IT = 98$ by the condition of the problem statement. Letting a, b, c be the radii of $\omega_a, \omega_b, \omega_c$ respectively and invoking Heron's formula as well as $A = rs$ for triangle ABC , we see that γ has radius

$$r = \sqrt{\frac{r_a r_b r_c}{r_a + r_b + r_c}}.$$

We will compute this quantity using Descartes' theorem. Note that there are two circles tangent to $\omega_a, \omega_b, \omega_c$, one with radius $IT/2$ and one with radius ∞ . By Descartes' circle theorem, we have (where $k_a := 1/a$ is the curvature)

$$k_a + k_b + k_c + 2\sqrt{k_a k_b + k_b k_c + k_a k_c} = \frac{1}{IT/2}$$

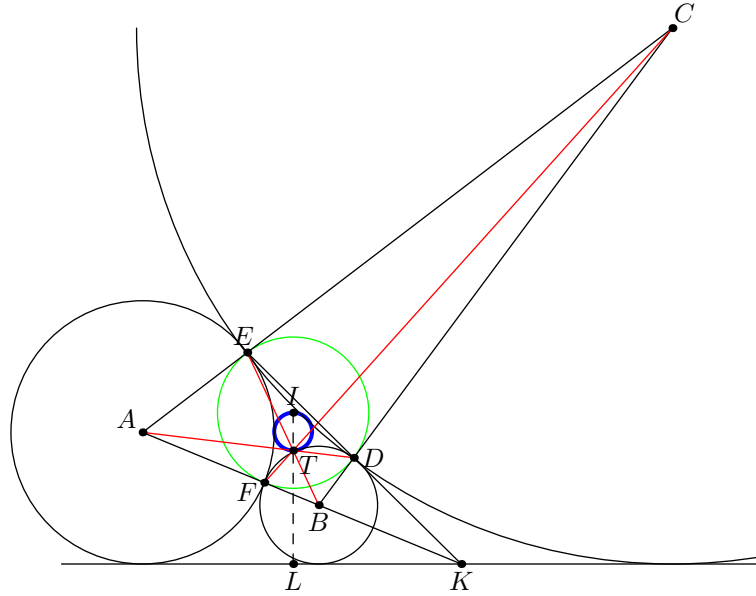
and

$$k_a + k_b + k_c - 2\sqrt{k_a k_b + k_b k_c + k_c k_a} = 0,$$

which implies

$$\sqrt{\frac{r_a + r_b + r_c}{r_a r_b r_c}} = \sqrt{k_a k_b + k_b k_c + k_c k_a} = \frac{1}{2IT}.$$

Therefore $r = 2IT$, which means $IL = \frac{r^2}{IT} = 4IT$ and $TL = 3IT = 294$.



Solution 2: Using the same notation as the previous solution, note that the point T can be expressed with un-normalized barycentric coordinates

$$\left(\frac{1}{r_a} : \frac{1}{r_b} : \frac{1}{r_c} \right)$$

with respect to ABC because T is the Gergonne point of triangle ABC . The distance from T to ℓ can be expressed as a weighted average of the distances from each of the points A, B, C , which is

$$\frac{1/r_a}{1/r_a + 1/r_b + 1/r_c} \cdot r_a + \frac{1/r_b}{1/r_a + 1/r_b + 1/r_c} \cdot r_b + \frac{1/r_c}{1/r_a + 1/r_b + 1/r_c} \cdot r_c = \frac{3}{1/r_a + 1/r_b + 1/r_c}.$$

Note that there are two circles tangent to $\omega_a, \omega_b, \omega_c$, one with radius 49 and one with radius ∞ . By Descartes' circle theorem, we have (where $k_a := 1/r_a$ is the curvature)

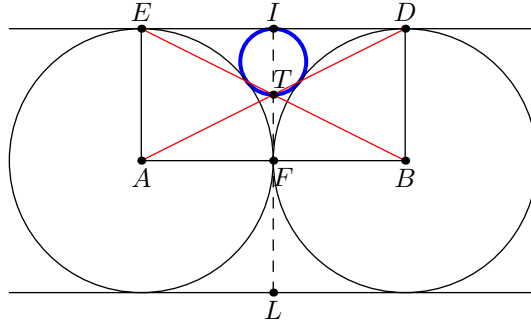
$$k_a + k_b + k_c + 2\sqrt{k_a k_b + k_b k_c + k_a k_c} = \frac{1}{49}$$

and

$$k_a + k_b + k_c - 2\sqrt{k_a k_b + k_b k_c + k_c k_a} = 0,$$

so $k_a + k_b + k_c = 1/98$. The distance from T to ℓ is then $3 \cdot 98 = 294$.

Solution 3: As in the first solution, we deduce that IT is a diameter of ω , with T being the point on ω closest to ℓ . By Steiner's porism, we can hold ω and ℓ fixed while making ω_c into a line parallel to ℓ , resulting in the following figure:



Let R be the common radius of ω_a and ω_b and r be the radius of ω . Notice that $ABDE$ is a rectangle with center T , so $R = AE = 2IT = 4r$. The distance from T to ℓ is $IL - IT = 2R - 2r = 6r = 294$.

10. Let Γ be a circle of radius 1 centered at O . A circle Ω is said to be *friendly* if there exist distinct circles $\omega_1, \omega_2, \dots, \omega_{2020}$, such that for all $1 \leq i \leq 2020$, ω_i is tangent to Γ , Ω , and ω_{i+1} . (Here, $\omega_{2021} = \omega_1$.) For each point P in the plane, let $f(P)$ denote the sum of the areas of all friendly circles centered at P . If A and B are points such that $OA = \frac{1}{2}$ and $OB = \frac{1}{3}$, determine $f(A) - f(B)$.

Proposed by: Michael Ren

Answer: $\boxed{\frac{1000}{9}\pi}$

Solution: Let P satisfy $OP = x$. (For now, we focus on $f(P)$ and ignore the A and B from the problem statement.) The key idea is that if we invert at some point along OP such that the images of Γ and Ω are concentric, then ω_i still exist. Suppose that this inversion fixes Γ and takes Ω to Ω' of radius r (and X to X' in general). If the inversion is centered at a point Q along ray OP such that $OQ = d$, then the radius of inversion is $\sqrt{d^2 - 1}$. Let the diameter of Ω meet OQ at A and B with A closer to Q than B . Then, $(AB; PP_\infty) = -1$ inverts to $(A'B'; P'Q) = -1$, where P_∞ is the point at infinity along line OP , so P' is the inverse of Q in Ω' . We can compute $OP' = \frac{r^2}{d}$ so $P'Q = d - \frac{r^2}{d}$ and $PQ = \frac{d^2 - 1}{d - \frac{r^2}{d}}$. Thus, we get the equation $\frac{d^2 - 1}{d - \frac{r^2}{d}} + x = d$, which rearranges to $\frac{1 - r^2}{d^2 - r^2}d = x$, or $d^2 - x^{-1}(1 - r^2)d - r^2 = 0$. Now, we note that the radius of Ω is

$$\frac{1}{2}AB = \frac{1}{2} \left(\frac{d^2 - 1}{d - r} - \frac{d^2 - 1}{d + r} \right) = \frac{r(d^2 - 1)}{d^2 - r^2} = r \left(1 + \frac{r^2 - 1}{d^2 - r^2} \right) = r \left(1 - \frac{x}{d} \right).$$

The quadratic formula gives us that $d = \frac{(1 - r^2) \pm \sqrt{r^4 - (2 - 4x^2)r^2 + 1}}{2x}$, so $\frac{x}{d} = -\frac{1 - r^2 \pm \sqrt{r^4 - (2 - 4x^2)r^2 + 1}}{2r^2}$, which means that the radius of Ω is

$$\frac{r^2 + 1 \pm \sqrt{r^4 - (2 - 4x^2)r^2 + 1}}{2r} = \frac{r + \frac{1}{r} \pm \sqrt{r^2 + \frac{1}{r^2} - 2 + 4x^2}}{2}.$$

Note that if r gives a valid chain of 2020 circles, so will $\frac{1}{r}$ by homothety/inversion. Thus, we can think of each pair of $r, \frac{1}{r}$ as giving rise to two possible values of the radius of Ω , which are $\frac{r + \frac{1}{r} \pm \sqrt{r^2 + \frac{1}{r^2} - 1}}{2}$. This means that the pairs have the same sum of radii as the circles centered at O , and the product of the radii is $1 - x^2$. (A simpler way to see this is to note that inversion at P with radius $\sqrt{1 - x^2}$ swaps the two circles.) From this, it follows that the difference between the sum of the areas for each pair is $2\pi \left(\frac{1}{2^2} - \frac{1}{3^2} \right) = \frac{5}{18}\pi$. There are $\frac{\varphi(2020)}{2} = 400$ such pairs, which can be explicitly computed as $\frac{1 - \sin \frac{\pi k}{2020}}{1 + \sin \frac{\pi k}{2020}}, \frac{1 + \sin \frac{\pi k}{2020}}{1 - \sin \frac{\pi k}{2020}}$ for positive integers $k < 1010$ relatively prime to 2020. Thus, the answer is $\frac{1000}{9}\pi$.

HMMT February 2020

February 15, 2020

Guts Round

1. [4] Submit an integer x as your answer to this problem. The number of points you receive will be $\max(0, 8 - |8x - 100|)$. (Non-integer answers will be given 0 points.)

Proposed by: Andrew Gu, Andrew Lin

Answer:

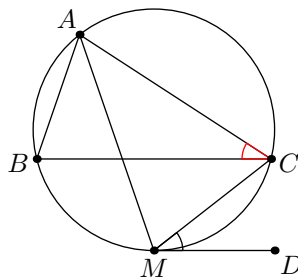
Solution: We want to minimize $|8x - 100|$, so x should equal either the floor or the ceiling of $\frac{100}{8} = 12.5$. Note that no other answers receive any points, while both 12 and 13 receive 4 points.

2. [4] Let ABC be a triangle and ω be its circumcircle. The point M is the midpoint of arc BC not containing A on ω and D is chosen so that DM is tangent to ω and is on the same side of AM as C . It is given that $AM = AC$ and $\angle DMC = 38^\circ$. Find the measure of angle $\angle ACB$.

Proposed by: Joseph Heerens

Answer:

Solution: By inscribed angles, we know that $\angle BAC = 38^\circ \cdot 2 = 76^\circ$ which means that $\angle C = 104^\circ - \angle B$. Since $AM = AC$, we have $\angle ACM = \angle AMC = 90^\circ - \frac{\angle MAC}{2} = 71^\circ$. Once again by inscribed angles, this means that $\angle B = 71^\circ$ which gives $\angle C = 33^\circ$.



3. [4] Let ABC be a triangle and D , E , and F be the midpoints of sides BC , CA , and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

Proposed by: Andrew Gu

Answer:

Solution: All $\binom{6}{3} = 20$ triples of points can produce distinct circles aside from the case where the three points are collinear (BDC , CEA , AFB).

4. [4] Compute the value of $\sqrt{105^3 - 104^3}$, given that it is a positive integer.

Proposed by: Andrew Gu

Answer:

Solution 1: First compute $105^3 - 104^3 = 105^2 + 105 \cdot 104 + 104^2 = 3 \cdot 105 \cdot 104 + 1 = 32761$. Note that $180^2 = 32400$, so $181^2 = 180^2 + 2 \cdot 180 + 1 = 32761$ as desired.

Solution 2: We have $105^3 - 104^3 = 105^2 + 105 \cdot 104 + 104^2$. Thus

$$104\sqrt{3} < \sqrt{105^3 - 104^3} < 105\sqrt{3}.$$

Now estimating gives the bounds $180 < 104\sqrt{3}$ and $105\sqrt{3} < 182$. So the answer is 181.

5. [5] Alice, Bob, and Charlie roll a 4, 5, and 6-sided die, respectively. What is the probability that a number comes up exactly twice out of the three rolls?

Proposed by: Andrew Lin

Answer: $\boxed{\frac{13}{30}}$

Solution 1: There are $4 \cdot 5 \cdot 6 = 120$ different ways that the dice can come up. The common number can be any of 1, 2, 3, 4, or 5: there are $3 + 4 + 5 = 12$ ways for it to be each of 1, 2, 3, or 4, because we pick one of the three people's rolls to disagree, and there are 3, 4, and 5 ways that roll can come up (for Alice, Bob, and Charlie respectively). Finally, there are 4 ways for Bob and Charlie to both roll a 5 and Alice to roll any number. Thus there are 52 different ways to satisfy the problem condition, and our answer is $\frac{52}{120} = \frac{13}{30}$.

Solution 2: If Bob rolls the same as Alice, Charlie must roll a different number. Otherwise Charlie must roll the same as either Alice or Bob. So the answer is

$$\frac{1}{5} \cdot \frac{5}{6} + \frac{4}{5} \cdot \frac{2}{6} = \frac{13}{30}.$$

Solution 3: By complementary counting, the answer is

$$1 - \frac{1}{5} \cdot \frac{1}{6} - \frac{4}{5} \cdot \frac{4}{6} = \frac{13}{30}.$$

The first and second products correspond to rolling the same number three times and rolling three distinct numbers, respectively.

6. [5] Two sides of a regular n -gon are extended to meet at a 28° angle. What is the smallest possible value for n ?

Proposed by: James Lin

Answer: $\boxed{45}$

Solution: We note that if we inscribe the n -gon in a circle, then according to the inscribed angle theorem, the angle between two sides is $\frac{1}{2}$ times some $x - y$, where x and y are integer multiples of the arc measure of one side of the n -gon. Thus, the angle is equal to $\frac{1}{2}$ times an integer multiple of $\frac{360}{n}$, so $\frac{1}{2} \cdot k \cdot \frac{360}{n} = 28$ for some integer k . Simplifying gives $7n = 45k$, and since all k are clearly attainable, the smallest possible value of n is 45.

7. [5] Ana and Banana are rolling a standard six-sided die. Ana rolls the die twice, obtaining a_1 and a_2 , then Banana rolls the die twice, obtaining b_1 and b_2 . After Ana's two rolls but before Banana's two rolls, they compute the probability p that $a_1b_1 + a_2b_2$ will be a multiple of 6. What is the probability that $p = \frac{1}{6}$?

Proposed by: James Lin

Answer: $\boxed{\frac{2}{3}}$

Solution: If either a_1 or a_2 is relatively prime to 6, then $p = \frac{1}{6}$. If one of them is a multiple of 2 but not 6, while the other is a multiple of 3 but not 6, we also have $p = \frac{1}{6}$. In other words, $p = \frac{1}{6}$ if $\gcd(a_1, a_2)$ is coprime to 6, and otherwise $p \neq \frac{1}{6}$. The probability that $p = \frac{1}{6}$ is $\frac{(3^2-1)(2^2-1)}{6^2} = \frac{2}{3}$ where $\frac{q^2-1}{q^2}$ corresponds to the probability that at least one of a_1 and a_2 is not divisible by q for $q = 2, 3$.

8. [5] Tessa picks three real numbers x, y, z and computes the values of the eight expressions of the form $\pm x \pm y \pm z$. She notices that the eight values are all distinct, so she writes the expressions down in increasing order. For example, if $x = 2, y = 3, z = 4$, then the order she writes them down is

$$-x - y - z, +x - y - z, -x + y - z, -x - y + z, +x + y - z, +x - y + z, -x + y + z, +x + y + z.$$

How many possible orders are there?

Proposed by: Yuan Yao

Answer:

Solution: There are $2^3 = 8$ ways to choose the sign for each of x, y , and z . Furthermore, we can order $|x|, |y|$, and $|z|$ in $3! = 6$ different ways. Now assume without loss of generality that $0 < x < y < z$. Then there are only two possible orders depending on the sign of $x + y - z$:

$$-x - y - z, +x - y - z, -x + y - z, -x - y + z, x + y - z, x - y + z, -x + y + z, x + y + z$$

$$-x - y - z, +x - y - z, -x + y - z, x + y - z, -x - y + z, x - y + z, -x + y + z, x + y + z$$

Thus, the answer is $8 \cdot 6 \cdot 2 = 96$.

9. [6] Let $P(x)$ be the monic polynomial with rational coefficients of minimal degree such that $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots, \frac{1}{\sqrt{1000}}$ are roots of P . What is the sum of the coefficients of P ?

Proposed by: Carl Joshua Quines

Answer:

Solution: For irrational $\frac{1}{\sqrt{r}}, -\frac{1}{\sqrt{r}}$ must also be a root of P . Therefore

$$P(x) = \frac{(x^2 - \frac{1}{2})(x^2 - \frac{1}{3}) \cdots (x^2 - \frac{1}{1000})}{(x + \frac{1}{2})(x + \frac{1}{3}) \cdots (x + \frac{1}{31})}.$$

We get the sum of the coefficients of P by setting $x = 1$, so we use telescoping to get

$$P(1) = \frac{\frac{1}{2} \cdot \frac{2}{3} \cdots \frac{999}{1000}}{\frac{3}{2} \cdot \frac{4}{3} \cdots \frac{32}{31}} = \frac{1}{16000}.$$

10. [6] Jarris is a weighted tetrahedral die with faces F_1, F_2, F_3, F_4 . He tosses himself onto a table, so that the probability he lands on a given face is proportional to the area of that face (i.e. the probability he lands on face F_i is $\frac{[F_i]}{[F_1]+[F_2]+[F_3]+[F_4]}$ where $[K]$ is the area of K). Let k be the maximum distance any part of Jarris is from the table after he rolls himself. Given that Jarris has an inscribed sphere of radius 3 and circumscribed sphere of radius 10, find the minimum possible value of the expected value of k .

Proposed by: James Lin

Answer:

Solution: Since the maximum distance to the table is just the height, the expected value is equal to $\frac{\sum_{i=1}^4 h_i[F_i]}{\sum_{i=1}^4 [F_i]}$. Let V be the volume of Jarris. Recall that $V = \frac{1}{3}h_i[F_i]$ for any i , but also $V = \frac{r}{3} \left(\sum_{i=1}^4 [F_i] \right)$ where r is the inradius (by decomposing into four tetrahedra with a vertex at the incenter). Therefore

$$\frac{\sum_{i=1}^4 h_i[F_i]}{\sum_{i=1}^4 [F_i]} = \frac{12V}{3V/r} = 4r = 12.$$

11. [6] Find the number of ordered pairs of positive integers (x, y) with $x, y \leq 2020$ such that $3x^2 + 10xy + 3y^2$ is the power of some prime.

Proposed by: James Lin

Answer:

Solution: We can factor as $(3x + y)(x + 3y)$. If $x \geq y$, we need $\frac{3x+y}{x+3y} \in \{1, 2\}$ to be an integer. So we get the case where $x = y$, in which we need both to be a power of 2, or the case $x = 5y$, in which case we need y to be a power of 2. This gives us $11 + 9 + 9 = 29$ solutions, where we account for $y = 5x$ as well.

12. [6] An 11×11 grid is labeled with consecutive rows $0, 1, 2, \dots, 10$ and columns $0, 1, 2, \dots, 10$ so that it is filled with integers from 1 to 2^{10} , inclusive, and the sum of all of the numbers in row n and in column n are both divisible by 2^n . Find the number of possible distinct grids.

Proposed by: Joseph Heerens

Answer:

Solution: We begin by filling the 10 by 10 grid formed by rows and columns 1 through 10 with any values, which we can do in $(2^{10})^{100} = 2^{1000}$ ways. Then in column 0, there is at most 1 way to fill in the square in row 10, 2 ways for the square in row 9, down to 2^{10} ways in row 0. Similarly, there is 1 way to fill in the square in row 0 and column 10, 2 ways to fill in the square in row 0 and column 9, etc. Overall, the number of ways to fill out the squares in row or column 0 is $2^1 \cdot 2^2 \cdot 2^3 \dots 2^9 \cdot 2^{10} \cdot 2^9 \cdot 2^8 \dots 2^1 = 2^{100}$, so the number of possible distinct grids $2^{1000} \cdot 2^{100} = 2^{1100}$.

13. [8] Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 1$, and $CA = 4\sqrt{3}$. The angle trisectors of C intersect \overline{AB} at D and E , and lines \overline{AC} and \overline{BC} intersect the circumcircle of $\triangle CDE$ again at X and Y , respectively. Find the length of XY .

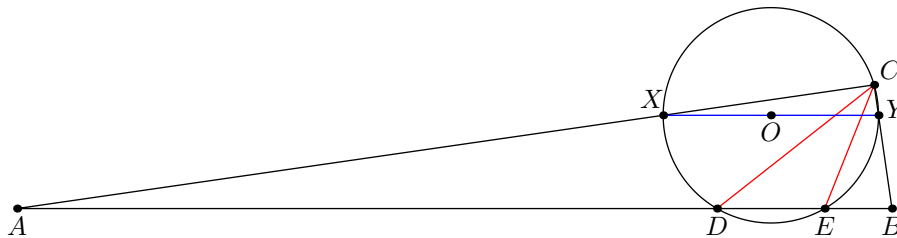
Proposed by: Hahn Lheem

Answer:

Solution: Let O be the circumcenter of $\triangle CDE$. Observe that $\triangle ABC \sim \triangle XYC$. Moreover, $\triangle ABC$ is a right triangle because $1^2 + (4\sqrt{3})^2 = 7^2$, so the length XY is just equal to $2r$, where r is the radius of the circumcircle of $\triangle CDE$. Since D and E are on the angle trisectors of angle C , we see that $\triangle ODE$, $\triangle XDO$, and $\triangle YEO$ are equilateral. The length of the altitude from C to AB is $\frac{4\sqrt{3}}{7}$. The distance from C to XY is $\frac{XY}{AB} \cdot \frac{4\sqrt{3}}{7} = \frac{2r}{7} \cdot \frac{4\sqrt{3}}{7}$, while the distance between lines XY and AB is $\frac{r\sqrt{3}}{2}$. Hence we have

$$\frac{4\sqrt{3}}{7} = \frac{2r}{7} \cdot \frac{4\sqrt{3}}{7} + \frac{r\sqrt{3}}{2}.$$

Solving for r gives that $r = \frac{56}{65}$, so $XY = \frac{112}{65}$.



14. [8] Let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n . Let S be the set of positive integers n such that $\frac{2n}{\varphi(n)}$ is an integer. Compute the sum

$$\sum_{n \in S} \frac{1}{n}.$$

Proposed by: Joseph Heerens, James Lin

Answer: $\boxed{\frac{10}{3}}$

Solution: Let T_n be the set of prime factors of n . Then

$$\frac{2n}{\varphi(n)} = 2 \prod_{p \in T} \frac{p}{p-1}.$$

We can check that this is an integer for the following possible sets:

$$\emptyset, \{2\}, \{3\}, \{2, 3\}, \{2, 5\}, \{2, 3, 7\}.$$

For each set T , the sum of the reciprocals of the positive integers having that set of prime factors is

$$\prod_{p \in T} \left(\sum_{m=1}^{\infty} \frac{1}{p^m} \right) = \prod_{p \in T} \frac{1}{p-1}.$$

Therefore the desired sum is

$$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{12} = \frac{10}{3}.$$

15. [8] You have six blocks in a row, labeled 1 through 6, each with weight 1. Call two blocks $x \leq y$ connected when, for all $x \leq z \leq y$, block z has not been removed. While there is still at least one block remaining, you choose a remaining block uniformly at random and remove it. The cost of this operation is the sum of the weights of the blocks that are connected to the block being removed, including itself. Compute the expected total cost of removing all the blocks.

Proposed by: Benjamin Qi

Answer: $\boxed{\frac{163}{10}}$

Solution: Note that the total cost is the total number of ordered pairs (x, y) with $1 \leq x, y \leq 6$ such that x and y are connected right before x gets removed.

The probability that blocks x and y are connected just before block x is removed is simply $\frac{1}{|x-y|+1}$, since all of the $|x-y|+1$ relevant blocks are equally likely to be removed first. Summing over $1 \leq x, y \leq 6$, combining terms with the same value of $|x-y|$, we get

$$\frac{2}{6} + \frac{4}{5} + \frac{6}{4} + \frac{8}{3} + \frac{10}{2} + 6 = \frac{163}{10}.$$

16. [8] Determine all triplets of real numbers (x, y, z) satisfying the system of equations

$$\begin{aligned}x^2y + y^2z &= 1040 \\x^2z + z^2y &= 260 \\(x-y)(y-z)(z-x) &= -540.\end{aligned}$$

Proposed by: Krit Boonsiriseth

Answer: $\boxed{(16, 4, 1), (1, 16, 4)}$

Solution: Call the three equations (1), (2), (3). (1)/(2) gives $y = 4z$. (3) + (1) - (2) gives

$$(y^2 - z^2)x = 15z^2x = 240,$$

so $z^2x = 16$. Therefore

$$\begin{aligned}z(x+2z)^2 &= x^2z + z^2y + 4z^2x = \frac{81}{5} \\z(x-2z)^2 &= x^2z + z^2y - 4z^2x = \frac{49}{5}\end{aligned}$$

so $\left| \frac{x+2z}{x-2z} \right| = \frac{9}{7}$. Thus either $x = 16z$ or $x = \frac{z}{4}$.

If $x = 16z$, then (1) becomes $1024z^3 + 16z^3 = 1040$, so $(x, y, z) = (16, 4, 1)$.

If $x = \frac{z}{4}$, then (1) becomes $\frac{1}{4}z^3 + 16z^3 = 1040$, so $(x, y, z) = (1, 16, 4)$.

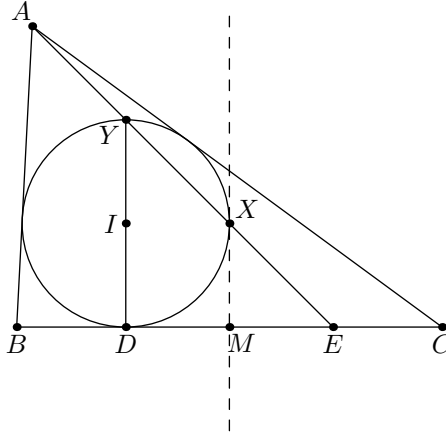
17. [10] Let ABC be a triangle with incircle tangent to the perpendicular bisector of BC . If $BC = AE = 20$, where E is the point where the A -excircle touches BC , then compute the area of $\triangle ABC$.

Proposed by: Tristan Shin

Answer: $\boxed{100\sqrt{2}}$

Solution: Let the incircle and BC touch at D , the incircle and perpendicular bisector touch at X , Y be the point opposite D on the incircle, and M be the midpoint of BC . Recall that A , Y , and E are collinear by homothety at A . Additionally, we have $MD = MX = ME$ so $\angle DXY = \angle DXE = 90^\circ$. Therefore E , X , and Y are collinear. Since $MX \perp BC$, we have $\angle AEB = 45^\circ$. The area of ABC is

$$\frac{1}{2}BC \cdot AE \cdot \sin \angle AEB = 100\sqrt{2}.$$



18. [10] A *vertex-induced* subgraph is a subset of the vertices of a graph together with any edges whose endpoints are both in this subset.

An undirected graph contains 10 nodes and m edges, with no loops or multiple edges. What is the minimum possible value of m such that this graph must contain a nonempty vertex-induced subgraph where all vertices have degree at least 5?

Proposed by: Benjamin Qi

Answer:

Solution: Suppose that we want to find the vertex-induced subgraph of maximum size where each vertex has degree at least 5. To do so, we start with the entire graph and repeatedly remove any vertex with degree less than 5.

If there are vertices left after this process terminates, then the subgraph induced by these vertices must have all degrees at least 5. Conversely, if there is a vertex-induced subgraph where all degrees are at least 5, then none of these vertices can be removed during the removing process. Thus, there are vertices remaining after this process if and only if such a vertex-induced subgraph exists.

If the process ends with an empty graph, the largest possible number of edges are removed when the first 5 removed vertices all have 4 edges at the time of removal, and the last 5 vertices are all connected to each other, resulting in $5 \times 4 + 4 + 3 + 2 + 1 + 0 = 30$ removed edges. The answer is $30 + 1 = 31$.

19. [10] The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exist unique positive integers $n_1, n_2, n_3, n_4, n_5, n_6$ such that

$$\sum_{i_1=0}^{100} \sum_{i_2=0}^{100} \sum_{i_3=0}^{100} \sum_{i_4=0}^{100} \sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} = F_{n_1} - 5F_{n_2} + 10F_{n_3} - 10F_{n_4} + 5F_{n_5} - F_{n_6}.$$

Find $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$.

Proposed by: Andrew Gu

Answer:

Solution: We make use of the identity

$$\sum_{i=0}^{\ell} F_i = F_{\ell+2} - 1,$$

(easily proven by induction) which implies

$$\sum_{i=k}^{\ell} F_i = F_{\ell+2} - F_{k+1}.$$

Applying this several times yields

$$\begin{aligned} & \sum_{i_1=0}^{100} \sum_{i_2=0}^{100} \sum_{i_3=0}^{100} \sum_{i_4=0}^{100} \sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} \\ &= \sum_{i_1=0}^{100} \sum_{i_2=0}^{100} \sum_{i_3=0}^{100} \sum_{i_4=0}^{100} (F_{i_1+i_2+i_3+i_4+102} - F_{i_1+i_2+i_3+i_4+1}) \\ &= \sum_{i_1=0}^{100} \sum_{i_2=0}^{100} \sum_{i_3=0}^{100} (F_{i_1+i_2+i_3+204} - 2F_{i_1+i_2+i_3+103} + F_{i_1+i_2+i_3+2}) \\ &= \sum_{i_1=0}^{100} \sum_{i_2=0}^{100} (F_{i_1+i_2+306} - 3F_{i_1+i_2+205} + 3F_{i_1+i_2+104} - F_{i_1+i_2+3}) \\ &= \sum_{i_1=0}^{100} (F_{i_1+408} - 4F_{i_1+307} + 6F_{i_1+206} - 4F_{i_1+105} + F_{i_1+4}) \\ &= F_{510} - 5F_{409} + 10F_{308} - 10F_{207} + 5F_{106} - F_5. \end{aligned}$$

This representation is unique because the Fibonacci terms grow exponentially quickly, so e.g. the F_{510} term dominates, forcing $n_1 = 510$ and similarly for the other terms. The final answer is

$$510 + 409 + 308 + 207 + 106 + 5 = 1545.$$

20. [10] There exist several solutions to the equation

$$1 + \frac{\sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x},$$

where x is expressed in degrees and $0^\circ < x < 180^\circ$. Find the sum of all such solutions.

Proposed by: Benjamin Qi

Answer:

Solution: We first apply sum-to-product and product-to-sum:

$$\frac{\sin 4x + \sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x}$$

$$2 \sin(2.5x) \cos(1.5x) \sin(2x) = \sin(4x) \sin(3x)$$

Factoring out $\sin(2x) = 0$,

$$\sin(2.5x) \cos(1.5x) = \cos(2x) \sin(3x)$$

Factoring out $\cos(1.5x) = 0$ (which gives us 60° as a solution),

$$\sin(2.5x) = 2 \cos(2x) \sin(1.5x)$$

$$\sin(2.5x) = \sin(3.5x) - \sin(0.5x)$$

Convert into complex numbers, we get

$$\begin{aligned}(x^{3.5} - x^{-3.5}) - (x^{0.5} - x^{-0.5}) &= (x^{2.5} - x^{-2.5}) \\ x^7 - x^6 - x^4 + x^3 + x - 1 &= 0 \\ (x - 1)(x^6 - x^3 + 1) &= 0\end{aligned}$$

We recognize the latter expression as $\frac{x^9+1}{x^3+1}$, giving us

$$x = 0^\circ, 20^\circ, 100^\circ, 140^\circ, 220^\circ, 260^\circ, 340^\circ.$$

The sum of the solutions is

$$20^\circ + 60^\circ + 100^\circ + 140^\circ = 320^\circ.$$

21. [12] We call a positive integer t *good* if there is a sequence a_0, a_1, \dots of positive integers satisfying $a_0 = 15, a_1 = t$, and

$$a_{n-1}a_{n+1} = (a_n - 1)(a_n + 1)$$

for all positive integers n . Find the sum of all good numbers.

Proposed by: Krit Boonsiriseth

Answer: 296

Solution: By the condition of the problem statement, we have

$$a_n^2 - a_{n-1}a_{n+1} = 1 = a_{n-1}^2 - a_{n-2}a_n.$$

This is equivalent to

$$\frac{a_{n-2} + a_n}{a_{n-1}} = \frac{a_{n-1} + a_{n+1}}{a_n}.$$

Let $k = \frac{a_0+a_2}{a_1}$. Then we have

$$\frac{a_{n-1} + a_{n+1}}{a_n} = \frac{a_{n-2} + a_n}{a_{n-1}} = \frac{a_{n-3} + a_{n-1}}{a_{n-2}} = \dots = \frac{a_0 + a_2}{a_1} = k.$$

Therefore we have $a_{n+1} = ka_n - a_{n-1}$ for all $n \geq 1$. We know that k is a positive rational number because a_0, a_1 , and a_2 are all positive integers. We claim that k must be an integer. Suppose that $k = \frac{p}{q}$ with $\gcd(p, q) = 1$. Since $ka_n = a_{n-1} + a_{n+1}$ is always an integer for $n \geq 1$, we must have $q \mid a_n$ for all $n \geq 1$. This contradicts $a_2^2 - a_1a_3 = 1$. Conversely, if k is an integer, inductively all a_i are integers.

Now we compute $a_2 = \frac{t^2-1}{15}$, so $k = \frac{t^2+224}{15t}$ is an integer. Therefore $15k - t = \frac{224}{t}$ is an integer. Combining with the condition that a_2 is an integer limits the possible values of t to 1, 4, 14, 16, 56, 224. The values $t < 15$ all lead to $a_n = 0$ for some n whereas $t > 15$ leads to a good sequence. The sum of the solutions is

$$16 + 56 + 224 = 296.$$

22. [12] Let A be a set of integers such that for each integer m , there exists an integer $a \in A$ and positive integer n such that $a^n \equiv m \pmod{100}$. What is the smallest possible value of $|A|$?

Proposed by: Daniel Zhu

Answer: 41

Solution: Work in $R = \mathbb{Z}/100\mathbb{Z} \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$.

Call an element $r \in R$ type (s, t) if $s = \nu_2(r) \leq 2$ and $t = \nu_5(r) \leq 2$. Also, define an element $r \in R$ to be *coprime* if it is of type $(0, 0)$, *powerful* if it is of types $(0, 2)$, $(2, 0)$, or $(2, 2)$, and *marginal* otherwise.

Then, note that if $r \in R$ is marginal, then any power of r is powerful. Therefore all marginal elements must be in A .

We claim that all powerful elements are the cube of some marginal element. To show this take a powerful element r . In modulo 4 or 25, if r is a unit, then since 3 is coprime to both the sizes of $(\mathbb{Z}/4\mathbb{Z})^\times$ and $(\mathbb{Z}/25\mathbb{Z})^\times$, it is the cube of some element. Otherwise, if r is zero then it is the cube of 2 or 5, respectively (since this case happens at least once this means that the constructed cube root is marginal).

We now claim that 4 additional elements are needed to generate the coprime elements. To see this, note that $R^\times \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/20\mathbb{Z}$ since there are primitive roots mod 4 and 25. Under this isomorphism, one can show that $(1, 1)$, $(1, 2)$, $(1, 4)$, and $(0, 1)$ generate anything, and that no element in R^\times has more than one of these as a multiple.

To wrap up, note that there are $100 - (20 + 1)(2 + 1) = 37$ marginal elements, so 41 elements are needed in total.

23. [12] A function $f: A \rightarrow A$ is called *idempotent* if $f(f(x)) = f(x)$ for all $x \in A$. Let I_n be the number of idempotent functions from $\{1, 2, \dots, n\}$ to itself. Compute

$$\sum_{n=1}^{\infty} \frac{I_n}{n!}.$$

Proposed by: Carl Schildkraut

Answer: $e^e - 1$

Solution: Let $A_{k,n}$ denote the number of idempotent functions on a set of size n with k fixed points. We have the formula

$$A_{k,n} = \binom{n}{k} k^{n-k}$$

for $1 \leq k \leq n$ because there are $\binom{n}{k}$ ways to choose the fixed points and all $n - k$ remaining elements must map to fixed points, which can happen in k^{n-k} ways. Hence

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{I_n}{n!} &= \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{A_{k,n}}{n!} \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^n \frac{k^{n-k}}{k!(n-k)!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=k}^{\infty} \frac{k^{n-k}}{(n-k)!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{n=0}^{\infty} \frac{k^n}{n!} \\ &= \sum_{k=1}^{\infty} \frac{1}{k!} e^k \\ &= e^e - 1. \end{aligned}$$

24. [12] In $\triangle ABC$, ω is the circumcircle, I is the incenter and I_A is the A -excenter. Let M be the midpoint of arc \widehat{BAC} on ω , and suppose that X, Y are the projections of I onto MI_A and I_A onto MI , respectively. If $\triangle XYI_A$ is an equilateral triangle with side length 1, compute the area of $\triangle ABC$.

Proposed by: Michael Diao

Answer: $\boxed{\frac{\sqrt{6}}{7}}$

Solution 1: Using Fact 5, we know that II_A intersects the circle (ABC) at M_A , which is the center of $(II_A BCXY)$. Let R be the radius of the latter circle. We have $R = \frac{1}{\sqrt{3}}$.

We have $\angle AIM = \angle YII_A = \angle YIX = \frac{\pi}{3}$. Also, $\angle IIA M = \angle IMI_A$ by calculating the angles from the equilateral triangle. Using 90-60-30 triangles, we have:

$$AI = \frac{1}{2}MI = \frac{1}{2}II_A = R$$

$$AM = \frac{\sqrt{3}}{2}MI = \sqrt{3}R$$

$$MM_A^2 = AM^2 + AM_A^2 = 7R^2$$

Now, let J and N be the feet of the altitudes from A and B respectively on MM_A . Note that as M is an arc midpoint of BC , N is actually the midpoint of BC .

$$M_A J = \frac{AM_A^2}{MM_A} = \frac{4}{\sqrt{7}}R$$

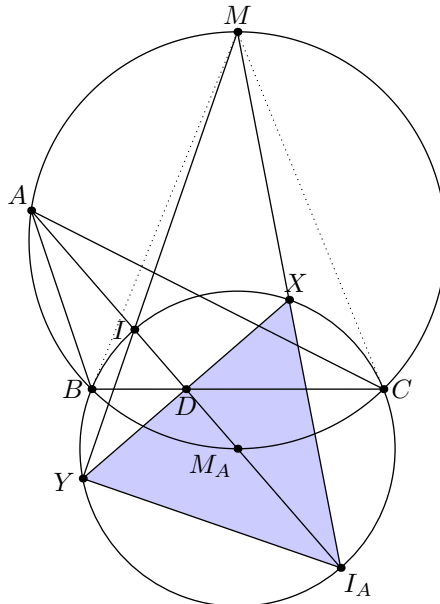
$$M_A N = \frac{BM_A^2}{MM_A} = \frac{1}{\sqrt{7}}R$$

Thus $JN = \frac{3}{\sqrt{7}}R$. Also, we have,

$$BN^2 = M_A N \cdot MN = \frac{6}{7}R^2$$

Now, $[ABC] = \frac{1}{2}JN \cdot BC = JN \cdot BN = \frac{3\sqrt{6}}{7}R^2 = \frac{\sqrt{6}}{7}$.

Solution 2: By Fact 5, we construct the diagram first with $\triangle XYI_A$ as the reference triangle.



Let M_A be the circumcenter of $\triangle XYI_A$ and let Ω be the circumcircle, which has circumradius $R = \frac{1}{\sqrt{3}}$. Then by Fact 5, M_A is the midpoint of minor arc \widehat{BC} , and $B, C \in \Omega$. Now we show the following result:

Claim. $b + c = 2a$.

Proof. Letting D be the intersection of II_A with BC , we have $M_AD = \frac{1}{2}R$. Then by the Shooting Lemma,

$$M_AA \cdot M_AD = R^2 \implies M_AA = 2R.$$

On the other hand, by Ptolemy,

$$M_AB \cdot AC + M_AC \cdot AB = M_AA \cdot BC \implies R \cdot (b + c) = 2R \cdot a,$$

whence the result follows. □

By the triangle area formula, we have

$$[ABC] = sr = \frac{1}{2}(a + b + c)r = \frac{3}{2}ar.$$

Therefore, we are left to compute a and r .

Since M_A is the antipode of M on (ABC) we get MB, MC are tangent to Ω ; in particular, MBM_AC is a kite with $MB = MC$ and $M_AB = M_AC$. Then by Power of a Point, we get

$$MB^2 = MC^2 = MI \cdot MY = 2R \cdot 3R = 2,$$

where $MI = II_A = 2R$ and $IY = R$.

Since $ID = DM_A$, we know that r is the length of the projection from M_A to BC . Finally, given our above information, we can compute

$$r = \frac{BM_A^2}{M_AM} = \frac{BM_A^2}{\sqrt{BM_A^2 + MB^2}} = \frac{1}{\sqrt{21}}$$

$$a = 2r \frac{BM_A}{MB} = \frac{2\sqrt{2}}{7}.$$

The area of ABC is then

$$\frac{3}{2} \left(\frac{2\sqrt{2}}{\sqrt{7}} \right) \left(\frac{1}{\sqrt{21}} \right) = \frac{\sqrt{6}}{7}.$$

25. [15] Let S be the set of 3^4 points in four-dimensional space where each coordinate is in $\{-1, 0, 1\}$. Let N be the number of sequences of points $P_1, P_2, \dots, P_{2020}$ in S such that $P_i P_{i+1} = 2$ for all $1 \leq i \leq 2020$ and $P_1 = (0, 0, 0, 0)$. (Here $P_{2021} = P_1$.) Find the largest integer n such that 2^n divides N .

Proposed by: James Lin

Answer: 4041

Solution: From $(0, 0, 0, 0)$ we have to go to $(\pm 1, \pm 1, \pm 1, \pm 1)$, and from $(1, 1, 1, 1)$ (or any of the other similar points), we have to go to $(0, 0, 0, 0)$ or $(-1, 1, 1, 1)$ and its cyclic shifts. If a_i is the number of ways to go from $(1, 1, 1, 1)$ to point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ in i steps, then we need to find $\nu_2(16a_{2018})$. To find a recurrence relation for a_i , note that to get to some point in $(\pm 1, \pm 1, \pm 1, \pm 1)$, we must either come from a previous point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ or the point $(0, 0, 0, 0)$. In order to go to one point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ through $(0, 0, 0, 0)$ from the point $(\pm 1, \pm 1, \pm 1, \pm 1)$, we have one way of going to the origin and 16 ways to pick which point we go to after the origin.

Additionally, if the previous point we visit is another point of the form $(\pm 1, \pm 1, \pm 1, \pm 1)$ then we have 4 possible directions to go in. Therefore the recurrence relation for a_i is $a_i = 4a_{i-1} + 16a_{i-2}$. Solving the linear recurrence yields

$$a_i = \frac{1}{\sqrt{5}}(2 + 2\sqrt{5})^i - \frac{1}{\sqrt{5}}(2 - 2\sqrt{5})^i = 4^i F_{i+1},$$

so it suffices to find $\nu_2(F_{2019})$. We have $F_n \equiv 0, 1, 1, 2, 3, 1 \pmod{4}$ for $n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$, so $\nu_2(F_{2019}) = 1$, and the answer is $4 + 2 \cdot 2018 + 1 = 4041$.

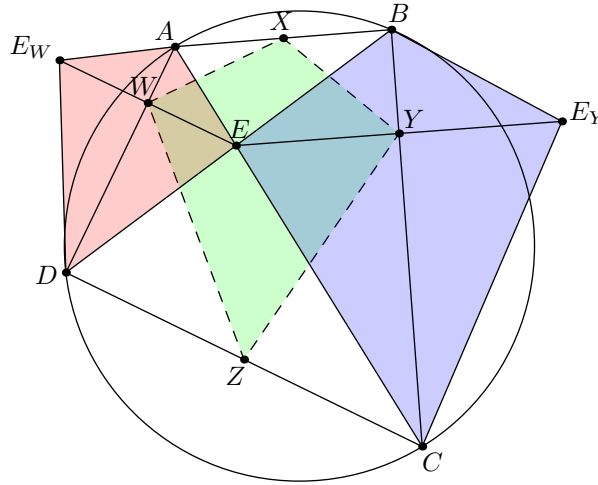
26. [15] Let $ABCD$ be a cyclic quadrilateral, and let segments AC and BD intersect at E . Let W and Y be the feet of the altitudes from E to sides DA and BC , respectively, and let X and Z be the midpoints of sides AB and CD , respectively. Given that the area of AED is 9, the area of BEC is 25, and $\angle EBC - \angle ECB = 30^\circ$, then compute the area of $WXYZ$.

Proposed by: James Lin

Answer: $17 + \frac{15}{2}\sqrt{3}$

Solution: Reflect E across DA to E_W , and across BC to E_Y . As $ABCD$ is cyclic, $\triangle AED$ and $\triangle BEC$ are similar. Thus $E_W AED$ and $E B E_Y C$ are similar too.

Now since W is the midpoint of $E_W E$, X is the midpoint of AB , Y is the midpoint of $E E_Y$, and Z is the midpoint of DC , we have that $WXYZ$ is similar to $E_W AED$ and $E B E_Y C$.



From the given conditions, we have $EW : EY = 3 : 5$ and $\angle WEY = 150^\circ$. Suppose $EW = 3x$ and $EY = 5x$. Then by the law of cosines, we have $WY = \sqrt{34 + 15\sqrt{3}}x$.

Thus, $E_W E : WY = 6 : \sqrt{34 + 15\sqrt{3}}$. So by the similarity ratio,

$$[WXYZ] = [E_W AED] \left(\frac{\sqrt{34 + 15\sqrt{3}}}{6} \right)^2 = 2 \cdot 9 \cdot \left(\frac{34 + 15\sqrt{3}}{36} \right) = 17 + \frac{15}{2}\sqrt{3}.$$

27. [15] Let $\{a_i\}_{i \geq 0}$ be a sequence of real numbers defined by

$$a_{n+1} = a_n^2 - \frac{1}{2^{2020 \cdot 2^n - 1}}$$

for $n \geq 0$. Determine the largest value for a_0 such that $\{a_i\}_{i \geq 0}$ is bounded.

Proposed by: Joshua Lee

Answer: $\boxed{1 + \frac{1}{2^{2020}}}$

Solution: Let $a_0 = \frac{1}{\sqrt{2}^{2020}} \left(t + \frac{1}{t}\right)$, with $t \geq 1$. (If $a_0 < \frac{1}{\sqrt{2}^{2018}}$ then no real t exists, but we ignore these values because a_0 is smaller.) Then, we can prove by induction that

$$a_n = \frac{1}{\sqrt{2}^{2020 \cdot 2^n}} \left(t^{2^n} + \frac{1}{t^{2^n}}\right).$$

For this to be bounded, it is easy to see that we just need

$$\frac{t^{2^n}}{\sqrt{2}^{2020 \cdot 2^n}} = \left(\frac{t}{\sqrt{2}^{2020}}\right)^{2^n}$$

to be bounded, since the second term approaches 0. We see that this is equivalent to $t \leq 2^{2020/2}$, which means

$$a_0 \leq \frac{1}{\sqrt{2}^{2020}} \left(\sqrt{2}^{2020} + \left(\frac{1}{\sqrt{2}}\right)^{2020}\right) = 1 + \frac{1}{2^{2020}}.$$

28. [15] Let $\triangle ABC$ be a triangle inscribed in a unit circle with center O . Let I be the incenter of $\triangle ABC$, and let D be the intersection of BC and the angle bisector of $\angle BAC$. Suppose that the circumcircle of $\triangle ADO$ intersects BC again at a point E such that E lies on IO . If $\cos A = \frac{12}{13}$, find the area of $\triangle ABC$.

Proposed by: Michael Diao

Answer: $\boxed{\frac{15}{169}}$

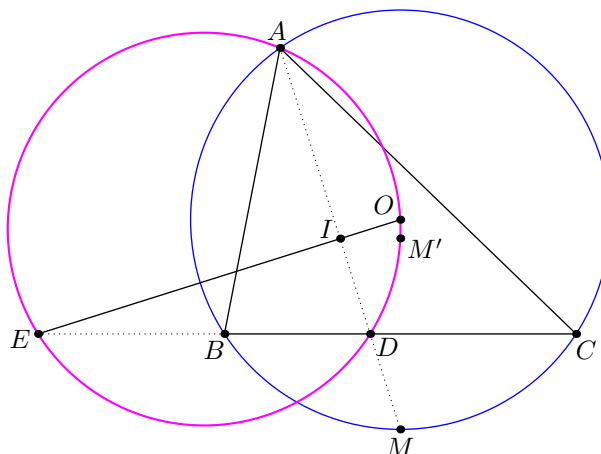
Solution: Consider the following lemma:

Lemma. $AD \perp EO$.

Proof. By the Shooting Lemma, the reflection of the midpoint M of arc BC not containing A over BC lies on (ADO) . Hence

$$\angle ADE + \angle DEO = \angle MDC + \angle DM'O = \angle MDC + \angle M'MD = 90^\circ.$$

This is enough to imply $AD \perp EO$. □



Thus I is the foot from O onto AD . Now

$$AI^2 + IO^2 = AO^2.$$

By Euler's formula,

$$\left(\frac{r}{\sin \frac{A}{2}}\right)^2 + R^2 - 2Rr = R^2.$$

Hence

$$r = 2R \sin^2 \frac{A}{2}.$$

Then

$$s = a + \frac{r}{\tan \frac{A}{2}} = a + R \sin A = 3R \sin A$$

and

$$[ABC] = rs = \left(2R \sin^2 \frac{A}{2}\right) (3R \sin A).$$

Since $R = 1$, we get

$$[ABC] = 3(1 - \cos A) \sin A.$$

Plugging in $\sin A = \frac{5}{13}$ and $\cos A = \frac{12}{13}$, we get

$$[ABC] = 3 \cdot \frac{1}{13} \cdot \frac{5}{13} = \frac{15}{169}.$$

Remark. On the contest, this problem stated that $\triangle ABC$ is an acute triangle and that $\sin A = \frac{5}{13}$, rather than $\cos A = \frac{12}{13}$. This is erroneous because there is no acute triangle satisfying the conditions of the problem statement (the given diagram is not accurate). We apologize for the mistake.

29. [18] Let $ABCD$ be a tetrahedron such that its circumscribed sphere of radius R and its inscribed sphere of radius r are concentric. Given that $AB = AC = 1 \leq BC$ and $R = 4r$, find BC^2 .

Proposed by: Michael Ren

Answer: $\boxed{1 + \sqrt{\frac{7}{15}}}$

Solution: Let O be the common center of the two spheres. Projecting O onto each face of the tetrahedron will divide it into three isosceles triangles. Unfolding the tetrahedron into its net, the reflection of any of these triangles about a side of the tetrahedron will coincide with another one of these triangles. Using this property, we can see that each of the faces is broken up into the same three triangles. It follows that the tetrahedron is isosceles, i.e. $AB = CD$, $AC = BD$, and $AD = BC$.

Let P be the projection of O onto ABC and $x = BC$. By the Pythagorean Theorem on triangle POA , P has distance $\sqrt{R^2 - r^2} = r\sqrt{15}$ from A , B , and C . Using the area-circumcenter formula, we compute

$$[ABC] = \frac{AB \cdot AC \cdot BC}{4PA} = \frac{x}{4r\sqrt{15}}.$$

However, by breaking up the volume of the tetrahedron into the four tetrahedra $OABC$, $OABD$, $OACD$, $OBCD$, we can write $[ABC] = \frac{V}{\frac{3}{4}r}$, where $V = [ABCD]$. Comparing these two expressions for $[ABC]$, we get $x = 3\sqrt{15}V$.

Using the formula for the volume of an isosceles tetrahedron (or some manual calculations), we can compute $V = x^2 \sqrt{\frac{1}{72}(2 - x^2)}$. Substituting into the previous equation (and taking the solution which is ≥ 1), we eventually get $x^2 = 1 + \sqrt{\frac{7}{15}}$.

30. [18] Let $S = \{(x, y) \mid x > 0, y > 0, x + y < 200, \text{ and } x, y \in \mathbb{Z}\}$. Find the number of parabolas \mathcal{P} with vertex V that satisfy the following conditions:

- \mathcal{P} goes through both $(100, 100)$ and at least one point in S ,
- V has integer coordinates, and
- \mathcal{P} is tangent to the line $x + y = 0$ at V .

Proposed by: James Lin

Answer: 264

Solution: We perform the linear transformation $(x, y) \rightarrow (x - y, x + y)$, which has the reverse transformation $(a, b) \rightarrow (\frac{a+b}{2}, \frac{b-a}{2})$. Then the equivalent problem has a parabola has a vertical axis of symmetry, goes through $A = (0, 200)$, a point $B = (u, v)$ in

$$S' = \{(x, y) \mid x + y > 0, x > y, y < 200, x, y \in \mathbb{Z}, \text{ and } x \equiv y \pmod{2}\},$$

and a new vertex $W = (w, 0)$ on $y = 0$ with w even. Then $(1 - \frac{u}{w})^2 = \frac{v}{200}$. The only way the RHS can be the square of a rational number is if $\frac{u}{w} = \frac{v'}{10}$ where $v = 2(10 - v')^2$. Since v is even, we can find conditions so that u, w are both even:

$$\begin{aligned} v' \in \{1, 3, 7, 9\} &\implies (2v') \mid u, 20 \mid w \\ v' \in \{2, 4, 6, 8\} &\implies v' \mid u, 10 \mid w \\ v' = 5 &\implies 2 \mid u, 4 \mid w \end{aligned}$$

It follows that any parabola that goes through $v' \in \{3, 7, 9\}$ has a point with $v' = 1$, and any parabola that goes through $v' \in \{4, 6, 8\}$ has a point with $v' = 2$. We then count the following parabolas:

- The number of parabolas going through $(2k, 162)$, where k is a nonzero integer with $|2k| < 162$.
- The number of parabolas going through $(2k, 128)$ not already counted, where k is a nonzero integer with $|2k| < 128$. (Note that this passes through $(k, 162)$.)
- The number of parabolas going through $(2k, 50)$ not already counted, where k is a nonzero integer with $|2k| < 50$. (Note that this passes through $(\frac{2k}{5}, 162)$, and any overlap must have been counted in the first case.)

The number of solutions is then

$$2 \left(80 + \frac{1}{2} \cdot 64 + \frac{4}{5} \cdot 25 \right) = 264.$$

31. [18] Anastasia is taking a walk in the plane, starting from $(1, 0)$. Each second, if she is at (x, y) , she moves to one of the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, and $(x, y + 1)$, each with $\frac{1}{4}$ probability. She stops as soon as she hits a point of the form (k, k) . What is the probability that k is divisible by 3 when she stops?

Proposed by: Michael Ren

Answer: $\frac{3-\sqrt{3}}{3}$ or $1 - \frac{1}{\sqrt{3}}$

Solution: The key idea is to consider $(a + b, a - b)$, where (a, b) is where Anastasia walks on. Then, the first and second coordinates are independent random walks starting at 1, and we want to find the probability that the first is divisible by 3 when the second reaches 0 for the first time. Let C_n be the n th Catalan number. The probability that the second random walk first reaches 0 after $2n - 1$ steps is

$\frac{C_{n-1}}{2^{2n-1}}$, and the probability that the first is divisible by 3 after $2n - 1$ steps is $\frac{1}{2^{2n-1}} \sum_{i \equiv n \pmod 3} \binom{2n-1}{i}$ (by letting i be the number of -1 steps). We then need to compute

$$\sum_{n=1}^{\infty} \left(\frac{C_{n-1}}{4^{2n-1}} \sum_{i \equiv n \pmod 3} \binom{2n-1}{i} \right).$$

By a standard root of unity filter,

$$\sum_{i \equiv n \pmod 3} \binom{2n-1}{i} = \frac{4^n + 2}{6}.$$

Letting

$$P(x) = \frac{2}{1 + \sqrt{1 - 4x}} = \sum_{n=0}^{\infty} C_n x^n$$

be the generating function for the Catalan numbers, we find that the answer is

$$\frac{1}{6} P\left(\frac{1}{4}\right) + \frac{1}{12} P\left(\frac{1}{16}\right) = \frac{1}{3} + \frac{1}{12} \cdot \frac{2}{1 + \sqrt{\frac{3}{4}}} = \frac{3 - \sqrt{3}}{3}.$$

32. [18] Find the smallest real constant α such that for all positive integers n and real numbers $0 = y_0 < y_1 < \dots < y_n$, the following inequality holds:

$$\alpha \sum_{k=1}^n \frac{(k+1)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^n \frac{k^2 + 3k + 3}{y_k}.$$

Proposed by: Andrew Gu

Answer: $\boxed{\frac{16\sqrt{2}}{9}}$

Solution: We first prove the following lemma:

Lemma. For positive reals a, b, c, d , the inequality

$$\frac{a^{3/2}}{c^{1/2}} + \frac{b^{3/2}}{d^{1/2}} \geq \frac{(a+b)^{3/2}}{(c+d)^{1/2}}$$

holds.

Proof. Apply Hölder's inequality in the form

$$\left(\frac{a^{3/2}}{c^{1/2}} + \frac{b^{3/2}}{d^{1/2}} \right)^2 (c+d) \geq (a+b)^3. \quad \square$$

For $k \geq 2$, applying the lemma to $a = (k-1)^2, b = 8k+8, c = y_{k-1}^2, d = y_k^2 - y_{k-1}^2$ yields

$$\frac{(k-1)^3}{y_{k-1}} + \frac{(8k+8)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \frac{(k+3)^3}{y_k}.$$

We also have the equality

$$\frac{(8 \cdot 1 + 8)^{3/2}}{\sqrt{y_1^2 - y_0^2}} = \frac{(1+3)^3}{y_1}.$$

Summing the inequality from $k = 2$ to $k = n$ with the equality yields

$$\sum_{k=1}^n \frac{(8k+8)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^n \frac{(k+3)^3 - k^3}{y_k} + \frac{n^3}{y_n} \geq \sum_{k=1}^n \frac{9(k^2 + 3k + 3)}{y_k}.$$

Hence the inequality holds for $\alpha = \frac{16\sqrt{2}}{9}$. In the reverse direction, this is sharp when $y_n = n(n+1)(n+2)(n+3)$ (so that $y_{k-1} = \frac{k-1}{k+3}y_k$ for $k = 2, \dots, n$) and $n \rightarrow \infty$.

33. [22] Estimate

$$N = \prod_{n=1}^{\infty} n^{n^{-1.25}}.$$

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N) \rfloor$ points.

Proposed by: Sujay Kazi

Answer: ≈ 8282580

Solution: We approximate

$$\ln N = \sum_{n=1}^{\infty} \frac{\ln n}{n^{5/4}}$$

with an integral as

$$\begin{aligned} \int_1^{\infty} \frac{\ln x}{x^{5/4}} dx &= \left(-4x^{-1/4} \ln x - 16x^{-1/4} \right) \Big|_1^{\infty} \\ &= 16. \end{aligned}$$

Therefore e^{16} is a good approximation. We can estimate e^{16} by repeated squaring:

$$\begin{aligned} e &\approx 2.72 \\ e^2 &\approx 7.4 \\ e^4 &\approx 55 \\ e^8 &\approx 3000 \\ e^{16} &\approx 9000000. \end{aligned}$$

The true value of e^{16} is around 8886111, which is reasonably close to the value of N . Both e^{16} and 9000000 would be worth 20 points.

34. [22] For odd primes p , let $f(p)$ denote the smallest positive integer a for which there does not exist an integer n satisfying $p \mid n^2 - a$. Estimate N , the sum of $f(p)^2$ over the first 10^5 odd primes p .

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N)^3 \rfloor$ points.

Proposed by: Michael Ren

Answer: 2266067

Solution: Note that the smallest quadratic nonresidue a is always a prime, because if $a = bc$ with $b, c > 1$ then one of b and c is also a quadratic nonresidue. We apply the following heuristic: if p_1, p_2, \dots are the primes in increasing order, then given a “uniform random prime” q , the values of $\left(\frac{p_1}{q}\right), \left(\frac{p_2}{q}\right), \dots$ are independent and are 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

Of course, there is no such thing as a uniform random prime. More rigorously, for any n , the joint distributions of $\left(\frac{p_1}{q}\right), \dots, \left(\frac{p_n}{q}\right)$ where q is a uniform random prime less than N converges in distribution

to n independent coin flips between 1 and -1 as $N \rightarrow \infty$. For ease of explanation, we won't adopt this more formal view, but it is possible to make the following argument rigorous by looking at primes $q < N$ and sending $N \rightarrow \infty$. Given any n , the residue of $q \bmod n$ is uniform over the $\varphi(n)$ residues mod n that are relatively prime to n . By quadratic reciprocity, conditioned on either $q \equiv 1 \pmod{4}$ or $q \equiv 3 \pmod{4}$, exactly half of the nonzero residues mod p_n satisfy $\left(\frac{p_n}{q}\right) = 1$ and exactly half satisfy $\left(\frac{p_n}{q}\right) = -1$ for odd p_n (the case of $p_n = 2$ is slightly different and one must look mod 8, but the result is the same). The residue of $q \bmod 8, p_2, p_3, \dots, p_n$ are independent as these are pairwise relatively prime, yielding our heuristic.

Thus, we may model our problem of finding the smallest quadratic nonresidue with the following process: independent fair coins are flipped for each prime, and we take the smallest prime that flipped heads. We can estimate the expected value of $f(p)^2$ as $\sum_{n=1}^{\infty} \frac{p_n^2}{2^n}$. Looking at the first few terms gives us

$$\frac{2^2}{2} + \frac{3^2}{4} + \frac{5^2}{8} + \frac{7^2}{16} + \frac{11^2}{32} + \frac{13^2}{64} + \frac{17^2}{128} + \frac{19^2}{256} + \frac{23^2}{512} + \frac{29^2}{1024} \approx 22.$$

The terms after this decay rapidly, so a good approximation is $E = 22 \cdot 10^5$, good enough for 20 points. The more inaccurate $E = 20 \cdot 10^5$ earns 15 points.

This Python code computes the exact answer:

```
def smallest_nqr(p):
    for a in range(1,p):
        if pow(a, (p-1)//2, p) == p-1:
            return a

import sympy
print(sum([smallest_nqr(p)**2 for p in sympy.ntheory.primerange(3, sympy.prime(10**5+2))]))
```

Remark. In 1961, Erdős showed that as $N \rightarrow \infty$, the average value of $f(p)$ over odd primes $p < N$ will converge to $\sum_{n=1}^{\infty} \frac{p_n}{2^n} \approx 3.675$.

35. [22] A collection \mathcal{S} of 10000 points is formed by picking each point uniformly at random inside a circle of radius 1. Let N be the expected number of points of \mathcal{S} which are vertices of the convex hull of the \mathcal{S} . (The convex hull is the smallest convex polygon containing every point of \mathcal{S} .) Estimate N .

An estimate of $E > 0$ will earn $\max(\lfloor 22 - |E - N| \rfloor, 0)$ points.

Proposed by: Shengtong Zhang

Answer: ≈ 72.8

Solution: Here is [C++ code](#) by Benjamin Qi to estimate the answer via simulation. It is known that the expected number of vertices of the convex hull of n points chosen uniformly at random inside a circle is $O(n^{1/3})$. See “On the Expected Complexity of Random Convex Hulls” by Har-Peled.

36. [22] A *snake of length k* is an animal which occupies an ordered k -tuple (s_1, \dots, s_k) of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k - 1$. If the snake is currently occupying (s_1, \dots, s_k) and s is an unoccupied cell sharing a side with s_1 , the snake can move to occupy (s, s_1, \dots, s_{k-1}) instead.

Initially, a snake of length 4 is in the grid $\{1, 2, \dots, 30\}^2$ occupying the positions $(1, 1), (1, 2), (1, 3), (1, 4)$ with $(1, 1)$ as its head. The snake repeatedly makes a move uniformly at random among moves it can legally make. Estimate N , the expected number of moves the snake makes before it has no legal moves remaining.

An estimate of $E > 0$ will earn $\lfloor 22 \min(N/E, E/N)^4 \rfloor$ points.

Proposed by: Andrew Gu

Answer: ≈ 4571.8706930

Solution: Let $n = 30$. The snake can get stuck in only 8 positions, while the total number of positions is about $n^2 \times 4 \times 3 \times 3 = 36n^2$. We can estimate the answer as $\frac{36n^2}{8} = 4050$, which is good enough for 13 points.

Let's try to compute the answer as precisely as possible. For each head position (a, b) and tail orientation $c \in [0, 36)$, let $x = 36(na + b) + c$ be an integer denoting the current state of the snake. Let E_x be the expected number of moves the snake makes if it starts at state x . If from state x the snake can transition to any of states y_1, y_2, \dots, y_k , then add an equation of the form

$$E_x - \frac{1}{k} \sum_{i=1}^k E_{y_i} = 1.$$

Otherwise, if there are no transitions out of state x then set $E_x = 0$.

It suffices to solve a system of $36n^2$ linear equations for $E_0, E_1, \dots, E_{36n^2-1}$. Then the answer will equal E_i , where i corresponds to the state described in the problem statement. Naively, using Gaussian elimination would require about $(36n^2)^3 \approx 3.4 \cdot 10^{13}$ operations, which is too slow. Also, it will require too much memory to store $(36n^2)^2$ real numbers at once.

We can use the observation that initially, the maximum difference between any two indices within the same equation is at most $\approx 72n$, so Gaussian elimination only needs to perform approximately $(36n^2) \cdot (72n)^2 \approx 1.5 \cdot 10^{11}$ operations. Furthermore, we'll only need to store $\approx (36n^2) \cdot (72n)$ real numbers at a time. Benjamin Qi's solution ends up finishing in less than two minutes for $n = 30$ ([C++ code](#)).

Here are some more examples of problems in which Gaussian elimination can be sped up by exploiting some sort of special structure:

- <https://codeforces.com/contest/963/problem/E>
- <https://codeforces.com/gym/102501/problem/E>

HMMT February 2020
February 15, 2020
Team Round

1. [20] Let n be a positive integer. Define a sequence by $a_0 = 1$, $a_{2i+1} = a_i$, and $a_{2i+2} = a_i + a_{i+1}$ for each $i \geq 0$. Determine, with proof, the value of $a_0 + a_1 + a_2 + \cdots + a_{2^n-1}$.

Proposed by: Kevin Ren

Answer: $\boxed{\frac{3^n+1}{2}}$

Solution 1: Note that $a_{2^n-1} = 1$ for all n by repeatedly applying $a_{2i+1} = a_i$. Now let $b_n = a_0 + a_1 + a_2 + \cdots + a_{2^n-1}$. Applying the given recursion to every term of b_n except a_0 gives

$$\begin{aligned} b_n &= a_0 + a_1 + a_2 + a_3 + \cdots + a_{2^n-1} \\ &= a_0 + a_2 + a_4 + \cdots + a_{2^n-2} + a_1 + a_3 + \cdots + a_{2^n-1} \\ &= a_0 + (a_0 + a_1) + (a_1 + a_2) + (a_2 + a_3) + \cdots + (a_{2^{n-1}-2} + a_{2^{n-1}-1}) \\ &\quad + a_0 + a_1 + a_2 + \cdots + a_{2^{n-1}-1} \\ &= 3a_0 + 3a_1 + 3a_2 + \cdots + 3a_{2^{n-1}-2} + 3a_{2^{n-1}-1} - a_{2^{n-1}-1} \\ &= 3b_{n-1} - 1. \end{aligned}$$

Now we easily obtain $b_n = \frac{3^n+1}{2}$ by induction.

Solution 2: Define a binary string to be good if it is the null string or of the form 101010...10. Let c_n be the number of good subsequences of n when written in binary form. We see $c_0 = 1$ and $c_{2n+1} = c_n$ because the trailing 1 in $2n+1$ cannot be part of a good subsequence. Furthermore, $c_{2n+2} - c_{n+1}$ equals the number of good subsequences of $2n+2$ that use the trailing 0 in $2n+2$. We will show that this number is exactly c_n .

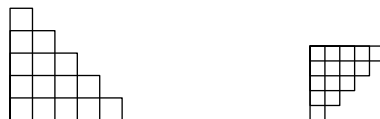
Let s be a good subsequence of $2n+2$ that contains the trailing 0. If s uses the last 1, remove both the last 1 and the trailing 0 from s ; the result s' will be a good subsequence of n . If s does not use the last 1, consider the sequence s' where the trailing 0 in $2n+2$ is replaced by the last 0 in n (which is at the same position as the last 1 in $2n+2$.) The map $s \mapsto s'$ can be seen to be a bijection, and thus $c_{2n+2} = c_n + c_{n+1}$.

Now it is clear that $a_n = c_n$ for all n . Consider choosing each binary string between 0 and $2^n - 1$ with equal probability. The probability that a given subsequence of length $2k$ is good is $\frac{1}{2^{2k}}$. There are $\binom{n}{2k}$ subsequences of length $2k$, so by linearity of expectation, the total expected number of good subsequences is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n}{2k}}{2^{2k}} = \frac{(1+1/2)^n + (1-1/2)^n}{2} = \frac{3^n+1}{2^{n+1}}.$$

This is equal to the average of a_0, \dots, a_{2^n-1} , therefore the sum $a_0 + \cdots + a_{2^n-1}$ is $\frac{3^n+1}{2}$.

2. [25] Let n be a fixed positive integer. An n -staircase is a polyomino with $\frac{n(n+1)}{2}$ cells arranged in the shape of a staircase, with arbitrary size. Here are two examples of 5-staircases:



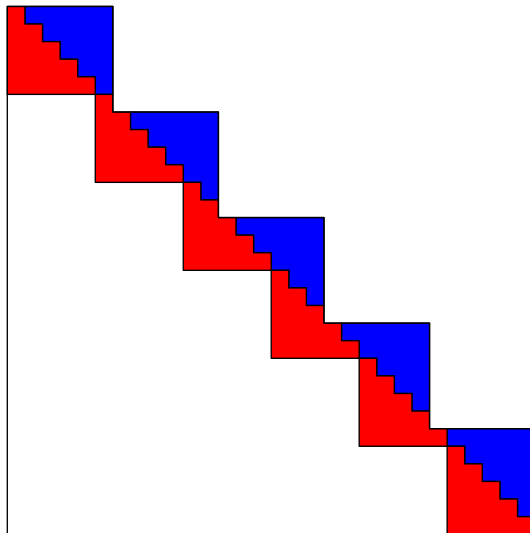
Prove that an n -staircase can be dissected into strictly smaller n -staircases.

Proposed by: James Lin

Solution 1: Viewing the problem in reverse, it is equivalent to show that we can use multiple n -staircases to make a single, larger n -staircase, because that larger n -staircase is made up of strictly smaller n -staircases, and is the example we need.

For the construction, we first attach two n -staircases of the same size together to make an $n \times (n + 1)$ rectangle. Then, we arrange $n(n + 1)$ of these rectangles in a $(n + 1) \times n$ grid, giving an $n(n + 1) \times n(n + 1)$ size square. Finally, we can use $\frac{n(n + 1)}{2}$ of these squares to create a larger n -staircase of $n^2(n + 1)^2$ smaller staircases, so we are done.

Solution 2: An alternative construction using only $2n + 2$ staircases was submitted by team Yeah Knights A. We provide a diagram for $n = 5$ and allow the interested reader to fill in the details.



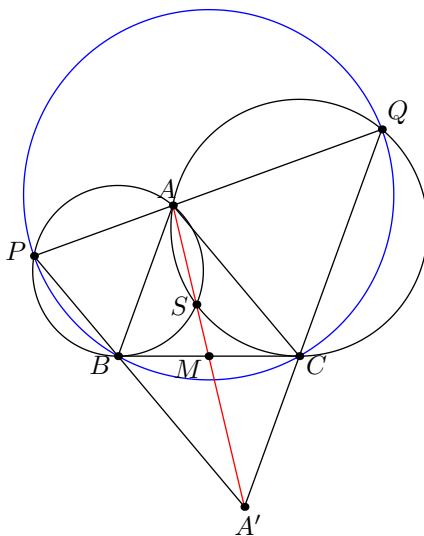
3. [25] Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A . The line through B parallel to AC meets ℓ at P , and the line through C parallel to AB meets ℓ at Q . The circumcircles of ABP and ACQ meet at $S \neq A$. Show that AS bisects BC .

Proposed by: Andrew Gu

Solution 1: In directed angles, we have

$$\angle CBP = \angle BCA = \angle BAP,$$

so BC is tangent to the circumcircle of ABP . Likewise, BC is tangent to the circumcircle of ACQ . Let M be the midpoint of BC . Then M has equal power $MB^2 = MC^2$ with respect to the circumcircles of ABP and ACQ , so the radical axis AS passes through M .



Solution 2: Since

$$\angle CBP = \angle BCA = \angle BAP = \angle CQP,$$

quadrilateral $BCQP$ is cyclic. Then AS , BP , and CQ concur at a point A' . Since $A'B \parallel AC$ and $A'C \parallel AB$, quadrilateral $ABA'C$ is a parallelogram so line ASA' bisects BC .

4. [35] Alan draws a convex 2020-gon $\mathcal{A} = A_1A_2 \cdots A_{2020}$ with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \dots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π . He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of \mathcal{A} with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases \mathcal{A} and the point B_1 . He then tells Jason the angles $\theta_1, \theta_2, \dots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

Proposed by: Andrew Gu, Colin Tang

Solution 1: For each i , let τ_i be the transformation of the plane which is rotation by θ_i counterclockwise about B_i . Recall that a composition of rotations is a rotation or translation, and that the angles of rotation add. Consider the composition $\tau_{2020} \circ \tau_{2019} \circ \cdots \circ \tau_1$, with total rotation angle 1010π . This must be a translation because $1010\pi = 505(2\pi)$. Also note that the composition sends A_1 to itself because $\tau_i(A_i) = A_{i+1}$. Therefore it is the identity. Now Jason can identify the map τ_1 as $\tau_2^{-1} \circ \tau_3^{-1} \circ \cdots \circ \tau_{2020}^{-1}$, and B_1 is the unique fixed point of this map.

Solution 2: Fix an arbitrary coordinate system. For $1 \leq k \leq 2020$, let a_k, b_k be the complex numbers corresponding to A_k, B_k . The given condition translates to

$$e^{i\theta_k}(b_k - a_k) = (b_k - a_{k+1}). \quad (1 \leq k \leq 2020)$$

In other words

$$(e^{i\theta_k} - 1)b_k = e^{i\theta_k}a_k - a_{k+1},$$

or

$$(e^{-i(\theta_{k-1} + \cdots + \theta_1)} - e^{-i(\theta_k + \cdots + \theta_1)})b_k = e^{-i(\theta_{k-1} + \cdots + \theta_1)}a_k - e^{-i(\theta_k + \cdots + \theta_1)}a_{k+1}.$$

Summing over all k , and using the fact that

$$e^{-i(\theta_1 + \cdots + \theta_{2020})} = 1,$$

we see that the right hand side cancels to 0, thus

$$\sum_{k=1}^{2020} (e^{-i(\theta_{k-1}+\dots+\theta_1)} - e^{-i(\theta_k+\dots+\theta_1)})b_k = 0.$$

Jason knows b_2, \dots, b_{2020} and all the θ_i , so the equation above is a linear equation in b_1 . We finish by noting that the coefficient of b_1 is $1 - e^{-i\theta_1}$ which is non-zero, as $\theta_1 \in (0, \pi)$. Thus Jason can solve for b_1 uniquely.

Solution 3: Let $A_1A_2 \cdots A_{2020}$ and $\tilde{A}_1\tilde{A}_2 \cdots \tilde{A}_{2020}$ be two 2020-gons that satisfy the conditions in the problem statement, and let B_k, \tilde{B}_k be the points Alan would construct with respect to these two polygons. It suffices to show that if $B_k = \tilde{B}_k$ for $k = 2, 3, \dots, 2020$, then $B_1 = \tilde{B}_1$.

For $2 \leq k \leq 2020$, we note that

$$A_kB_k = A_{k+1}B_k, \quad \tilde{A}_kB_k = \tilde{A}_{k+1}B_k$$

Furthermore, we have the equality of directed angles $\angle A_kB_kA_{k+1} = \angle \tilde{A}_kB_k\tilde{A}_{k+1} = \theta_k$, therefore $\angle A_kB_k\tilde{A}_k = \angle A_{k+1}B_k\tilde{A}_{k+1}$. This implies the congruence $\triangle A_kB_k\tilde{A}_k \cong \triangle A_{k+1}B_k\tilde{A}_{k+1}$.

The congruence shows that $A_k\tilde{A}_k = A_{k+1}\tilde{A}_{k+1}$; furthermore, the angle from the directed segment $\overrightarrow{A_k\tilde{A}_k}$ to $\overrightarrow{A_{k+1}\tilde{A}_{k+1}}$ is θ_k counterclockwise. This holds for $k = 2, 3, \dots, 2020$; we conclude that $A_1\tilde{A}_1 = A_2\tilde{A}_2$, and the angle from the directed segments $\overrightarrow{A_1\tilde{A}_1}$ to $\overrightarrow{A_2\tilde{A}_2}$ is

$$-\sum_{k=2}^{2020} \theta_k = \theta_1 - 1010\pi = \theta_1$$

counterclockwise.

Finally we observe that $A_1B_1 = A_2B_1$, and the angle from the directed segment $\overrightarrow{A_1B_1}$ to $\overrightarrow{A_2B_1}$ is θ_1 counterclockwise. This implies $\angle \overrightarrow{B_1A_1}\tilde{A}_1 = \angle B_1A_2\tilde{A}_2$, so $\triangle A_1B_1\tilde{A}_1 \cong \triangle A_2B_1\tilde{A}_2$. Thus $\tilde{A}_1B_1 = \tilde{A}_2B_1$, and the angle from $\overrightarrow{\tilde{A}_1B_1}$ to $\overrightarrow{\tilde{A}_2B_1}$ is θ_1 counterclockwise. We conclude that $B_1 = \tilde{B}_1$.

5. [40] Let a_0, b_0, c_0, a, b, c be integers such that $\gcd(a_0, b_0, c_0) = \gcd(a, b, c) = 1$. Prove that there exists a positive integer n and integers $a_1, a_2, \dots, a_n = a, b_1, b_2, \dots, b_n = b, c_1, c_2, \dots, c_n = c$ such that for all $1 \leq i \leq n$, $a_{i-1}a_i + b_{i-1}b_i + c_{i-1}c_i = 1$.

Proposed by: Michael Ren

Solution: The problem statement is equivalent to showing that we can find a sequence of vectors, each with 3 integer components, such that the first vector is (a_0, b_0, c_0) , the last vector is (a, b, c) , and every pair of adjacent vectors has dot product equal to 1.

We will show that any vector (a, b, c) can be sent to $(1, 0, 0)$. This is sufficient, because given vectors (a_0, b_0, c_0) and (a, b, c) , we take the sequence from (a_0, b_0, c_0) to $(1, 0, 0)$ and then add the reverse of the sequence from (a, b, c) to $(1, 0, 0)$.

First, suppose that some two of a, b, c are relatively prime. Here we will suppose that a and b are relatively prime; the other cases are similar. If neither of a or b is 0, then by Bezout's identity, there exist p, q such that $|p| + |q| < |a| + |b|$ and $ap + bq = 1$, so we can send (a, b, c) to $(p, q, 0)$. (Finding such numbers can be done using the extended Euclidean algorithm.) Clearly p and q must also be relatively prime, so we can apply Bezout's identity repeatedly until we eventually have $(1, 0, 0), (-1, 0, 0), (0, 1, 0)$, or $(0, -1, 0)$. Now, starting from $(0, -1, 0)$, we can do $(0, -1, 0) \rightarrow (1, -1, 0) \rightarrow (1, 0, 0)$, and we can do something similar to convert $(-1, 0, 0)$ to $(0, 1, 0)$.

Now suppose that no two of a, b, c are relatively prime. Let $f = \gcd(a, b)$. We claim that we can find x, y, z such that $axy + bx + cz = 1$. Notice that this is the same as $(ay + b)x + cz = 1$. Since

$\gcd(a, b, c) = 1$, there exists y such that $\gcd(ay + b, c) = 1$. Then by Bezout's identity, there exist x, z such that $(ay + b)x + cz = 1$. Therefore, we can send (a, b, c) to (xy, x, z) . Clearly x and z must be relatively prime, so we have reduced to the case above, and we can apply the process described above for that case.

At the end of this process, we will have $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$. The second of these can be converted into $(1, 0, 0)$ by doing $(0, 1, 0) \rightarrow (1, 1, 0) \rightarrow (1, 0, 0)$, and a similar sequence shows the same for the third. Therefore, (a, b, c) can be sent to $(1, 0, 0)$.

6. [40] Let $n > 1$ be a positive integer and S be a collection of $\frac{1}{2}\binom{2n}{n}$ distinct n -element subsets of $\{1, 2, \dots, 2n\}$. Show that there exists $A, B \in S$ such that $|A \cap B| \leq 1$.

Proposed by: Michael Ren

Solution 1: Assume for the sake of contradiction that there exist no such A, B . Pair up each subset with its complement, like so:

$$\begin{aligned} \{1, 2, 3, \dots, n\} &\leftrightarrow \{n+1, n+2, \dots, 2n\} \\ \{1, 2, 3, \dots, n-1, n+1\} &\leftrightarrow \{n, n+2, \dots, 2n\} \\ \{1, 2, 3, \dots, n-1, n+2\} &\leftrightarrow \{n, n+1, n+3, \dots, 2n\} \\ &\vdots \end{aligned}$$

Note that for each pair, we can have at most one of the two in S . Since S has $\frac{1}{2}$ of the total number of subsets with size n , it must be that we have exactly one element from each pair in S . For any $s_0 \in S$, none of the subsets that share exactly one element with s_0 can be in S , so their complements must be in S . This means that every subset with $n-1$ shared elements with s_0 must be in S . Without loss of generality, assume $\{1, 2, 3, \dots, n\} \in S$. Then, $\{2, 3, 4, \dots, n+1\} \in S$, so $\{3, 4, 5, \dots, n+2\} \in S$. Continuing on in this manner, we eventually reach $\{n+1, n+2, \dots, 2n\} \in S$, contradiction.

Solution 2: Let $[2n] = \{1, 2, \dots, 2n\}$. Consider the following cycle of $2n-1$ sets such that any two adjacent sets have an intersection of size 1:

$$\begin{aligned} &\{1, 2, 3, \dots, n\}, \\ &\{1, n+1, n+2, \dots, 2n-1\}, \\ &\{1, 2n, 2, 3, \dots, n-1\}, \\ &\vdots \\ &\{1, n+2, n+3, \dots, 2n\}. \end{aligned}$$

If S contains two adjacent elements of the cycle then we are done. For each permutation σ of $[2n]$, we can consider the cycle C_σ of sets obtained after applying σ , i.e. $\{\sigma(1), \sigma(2), \sigma(3), \dots, \sigma(n)\}$ and so on. In total, each subset of $[2n]$ with size n appears $(2n-1)(n!)^2$ times across all the cycles C_σ , so

$$\sum_{\sigma} |C_\sigma \cap S| = |S|(2n-1)(n!)^2 = \frac{2n-1}{2}(2n)!$$

where the sum is over all $(2n)!$ permutations of $[2n]$. This means that on average across possible cycles, $\frac{2n-1}{2}$ of its elements are in S . Thus, if we select a cycle C_σ uniformly at random, with positive probability we will have $|C_\sigma \cap S| \geq n$, so two adjacent elements in this cycle will be in S . Therefore, there must exist some two subsets in S that share at most one element.

This proof will work under the weaker condition $|S| > \frac{n-1}{2n-1}\binom{2n}{n}$.

Remark. A family of sets such that $|A \cap B| \geq t$ for every pair of distinct sets A, B is called t -intersecting. Ahlswede and Khachatrian solved the problem of determining the largest k -uniform t -intersecting family. See "Katona's Intersection Theorem: Four Proofs" or "The Complete Intersection

Theorem for Systems of Finite Sets” for exact results.

7. [50] Positive real numbers x and y satisfy

$$\left| \cdots \left| \left| |x| - y \right| - x \right| \cdots - y \right| - x \right| = \left| \cdots \left| \left| |y| - x \right| - y \right| \cdots - x \right| - y \right|$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of $\frac{x}{y}$.

Proposed by: Krit Boonsiriseth

Answer: $\boxed{\frac{1}{3}, 1, 3}$

Solution: Clearly $x = y$ works. Else WLOG $x < y$, define $d = y - x$, and define $f(z) := ||z - y| - x|$ so our expression reduces to

$$f^{1009}(x) = |f^{1009}(0) - y|.$$

Now note that for $z \in [0, y]$, $f(z)$ can be written as

$$f(z) = \begin{cases} d - z, & 0 \leq z \leq d \\ z - d, & d < z \leq y \end{cases}$$

Hence $f(f(z)) = f(d - z) = z$ for all $z \in [0, d]$. Therefore

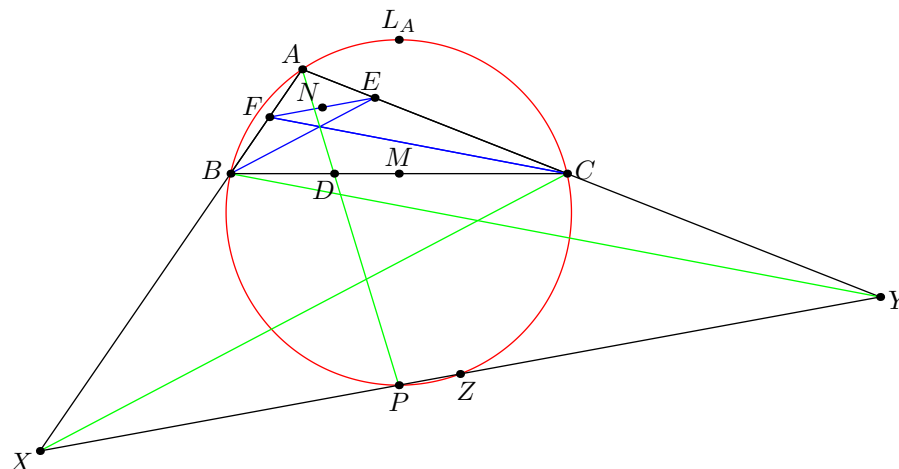
$$|f^{1009}(0) - y| = |f(0) - y| = x.$$

If $x > d$ then $f^{1009}(x) < x$ which is impossible (if $f^{1009}(x) \leq d$ then the conclusion trivially holds, and if $f^{1009}(x) > d$ we must have $f^{1009}(x) = x - 1009d < x$). Therefore $x \leq d$, so $f^{1009}(x) = f(x) = d - x$ and we must have $d - x = x$. Hence $y = 3x$ which is easily seen to work. To summarize, the possible values of $\frac{x}{y}$ are $\frac{1}{3}, 1, 3$.

8. [50] Let ABC be a scalene triangle with angle bisectors AD , BE , and CF so that D , E , and F lie on segments BC , CA , and AB respectively. Let M and N be the midpoints of BC and EF respectively. Prove that line AN and the line through M parallel to AD intersect on the circumcircle of ABC if and only if $DE = DF$.

Proposed by: Michael Ren

Solution 1:



Let X, Y be on AB, AC such that $CX \parallel BE$ and $BY \parallel CF$. Then $BX = BC = CY$. Let Z be the midpoint of XY . Then $\overrightarrow{MZ} = \frac{1}{2}(\overrightarrow{BX} + \overrightarrow{CY})$, which bisects the angle between BX and CY because they have the same length. Therefore $MZ \parallel AD$. Furthermore, by similar triangles we have

$$AE \cdot AX = AB \cdot AC = AF \cdot AY.$$

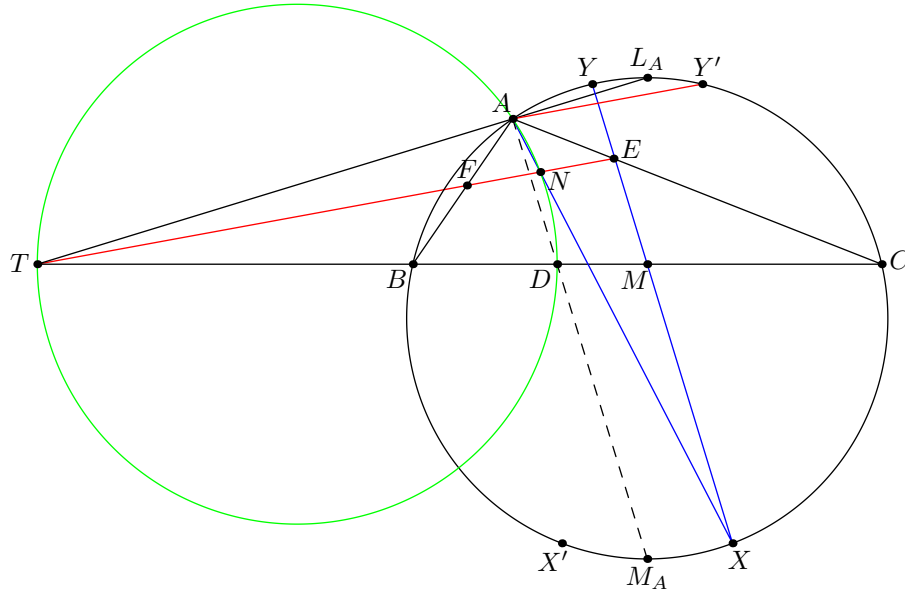
This rearranges to $\frac{AE}{AF} = \frac{AY}{AX}$, so $EF \parallel XY$. Therefore Z is the intersection of the lines in the problem statement. Then

$$\frac{\sin \angle BZX}{\sin \angle CZY} = \frac{BX \frac{\sin \angle ZBX}{XZ}}{CY \frac{\sin \angle ZCY}{YZ}} = 1$$

iff $Z \in (ABC)$, so XY is the external angle bisector of $\angle BZC$ iff $Z \in (ABC)$. Thus if $P = AD \cap XY$, $P \in (ABC)$ iff $Z \in (ABC)$. Additionally the spiral similarity from BX to CY gives $L_A Z \perp XY$ where L_A is the midpoint of arc BAC , so if $P \in (ABC)$ then Z must be on (ABC) because $\angle L_A Z P = 90^\circ$. Therefore $Z \in (ABC)$ iff $P \in (ABC)$.

From the previous length computation, we know that an inversion at A with radius $\sqrt{AB \cdot AC}$ composed with reflection about AD will send X and Y to E and F . We have $P \in (ABC)$ iff its image under the inversion is D , but since P was defined as $AD \cap XY$ this is true iff $(AEDF)$ is cyclic. Since ABC is scalene and $AE \neq AF$, this is true iff $DE = DF$.

Solution 2:



Let L_A be the midpoint of arc BAC and let M_A be diametrically opposite L_A . Let EF, AL_A , and BC meet at T so $\angle DAT = 90^\circ$; note that $DE = DF$ iff $DN \perp EF$, which is equivalent to $(TAND)$ being cyclic. Let $AN \cap (ABC) = X$ and $XM \cap (ABC) = Y$, and let Y' be the reflection of Y over $L_A M_A$ with similarly X' the reflection of X over $L_A M_A$. We wish to show $N \in (TAD)$ iff $XY \parallel AM_A$.

We claim $AY' \parallel EF$. By projecting $-1 = (B, C; M, \infty) \stackrel{X}{\cong} (B, C; Y, X')$ and reflecting over $L_A M_A$, we find $(X, Y'; B, C) = -1$. Then projecting through A gives $(N, AY' \cap EF; F, E) = -1$, and since N is the midpoint of EF we find $AY' \parallel EF$.

Now $(TAND)$ cyclic iff $\angle DAN = \angle DTN$, and $\angle DTN = \angle YY'A$ by the parallel lines. But we have $\angle DAN = \angle M_A A X$, so arcs $M_A X$ and $Y A$ are equal iff $(TAND)$ is cyclic. Thus $XY \parallel AM_A$ iff $DE = DF$ as desired.

9. [55] Let $p > 5$ be a prime number. Show that there exists a prime number $q < p$ and a positive integer n such that p divides $n^2 - q$.

Proposed by: Andrew Gu

Solution 1: Note that the condition $p \mid n^2 - q$ just means that q is a quadratic residue modulo p , or that the Legendre symbol $\left(\frac{q}{p}\right)$ is 1. We use these standard facts about the Legendre symbol:

- If $p \equiv \pm 1 \pmod{8}$, then $\left(\frac{2}{p}\right) = 1$.

- For an odd prime p ,

$$\left(\frac{-1}{p}\right) = \begin{cases} -1 & \text{if } p \equiv 3 \pmod{4} \\ +1 & \text{if } p \equiv 1 \pmod{4} \end{cases}.$$

- Quadratic reciprocity: for distinct odd primes p and q ,

$$\left(\frac{p}{q}\right) \left(\frac{q}{p}\right) = \begin{cases} -1 & \text{if } p \equiv q \equiv 3 \pmod{4} \\ +1 & \text{otherwise} \end{cases}.$$

If p is a Fermat prime or Mersenne prime, then p is congruent to 1 or 7 modulo 8 respectively, since $p > 5$. In that case $q = 2$ works. Otherwise assume p is not a Fermat prime or Mersenne prime, so that $p - 1$ and $p + 1$ are not powers of 2.

If $p \equiv 1 \pmod{4}$, then let q be an odd prime divisor of $p - 1$, so that $p \equiv 1 \pmod{q}$. Then by quadratic reciprocity $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = 1$.

If $p \equiv 3 \pmod{4}$, then let q be an odd prime divisor of $p + 1$, so that $p \equiv -1 \pmod{q}$. Either $q \equiv 1 \pmod{4}$ so that $\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = 1$ or $q \equiv 3 \pmod{4}$ so that $\left(\frac{q}{p}\right) = -\left(\frac{p}{q}\right) = -\left(\frac{-1}{q}\right) = 1$.

Solution 2: (Ankan Bhattacharya) We assume the same standard facts about quadratic residues as the previous solution.

If $p \equiv 1 \pmod{4}$, then since $p > 5$, there exists an odd prime divisor q of $p - 4$, which gives

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = \left(\frac{4}{q}\right) = 1.$$

If $p \equiv 7 \pmod{8}$, then we can take $q = 2$.

If $p \equiv 3 \pmod{8}$, then by Legendre's three square theorem there exist odd a, b, c satisfying $p = a^2 + b^2 + c^2$. Since $p > 3$, these are not all equal and we may assume without loss of generality that $b \neq c$. Then $p - a^2 = b^2 + c^2$ has a prime divisor $q \equiv 1 \pmod{4}$, which gives

$$\left(\frac{q}{p}\right) = \left(\frac{p}{q}\right) = \left(\frac{a^2}{q}\right) = 1.$$

Remark. For an odd prime number p , let $l(p)$ be the least prime number which is a quadratic residue modulo p and $h(-p)$ be the class number of the quadratic field $\mathbb{Q}[\sqrt{-p}]$. In the paper "The Least Prime Quadratic Residue and the Class Number" by Chowla, Cowles, and Cowles, the following results were proven:

- If $p > 5$ and $p \equiv 5 \pmod{8}$, then $l(p) < \sqrt{p}$.
- If $p > 3$, $p \equiv 3 \pmod{8}$, and $h(-p) > 1$, then $l(p) < \sqrt{p/3}$.
- If $p > 3$, $p \equiv 3 \pmod{8}$, and $h(-p) = 1$, then $l(p) = \frac{p+1}{4}$.

The proofs of the second and third results require knowledge of binary quadratic forms.

10. [60] Let n be a fixed positive integer, and choose n positive integers a_1, \dots, a_n . Given a permutation π on the first n positive integers, let $S_\pi = \{i \mid \frac{a_i}{\pi(i)} \text{ is an integer}\}$. Let N denote the number of distinct sets S_π as π ranges over all such permutations. Determine, in terms of n , the maximum value of N over all possible values of a_1, \dots, a_n .

Proposed by: James Lin

Answer: $2^n - n$

Solution: The answer is $2^n - n$.

Let $D = (d_{ij})$ be the matrix where d_{ij} is 1 if i is a divisor of a_j and 0 otherwise. For a subset S of $[n]$, let D_S be the matrix obtained from D by flipping ($0 \leftrightarrow 1$) every entry d_{ij} where $j \notin S$. Observe that $S = S_\pi$ if and only if $(D_S)_{\pi(i)i} = 1$ for all i .

To show that $N \leq 2^n - n$ we consider two cases. If all the rows of D are distinct, then there exist n different possibilities for S that set a row equal to zero. In this case, there is clearly no π so that $S_\pi = S$. Thus there are at most $2^n - n$ possible S_π . Otherwise, if two rows in D are the same, then choose an S_0 such that D_{S_0} has two zero rows. Then, the $n + 1$ sets S that are at most “one element away” from S_0 are such that D_S only has one column with nonzero entries in those two rows. This makes it impossible for $S_\pi = S$ as well, so $N \leq 2^n - n - 1$.

Now we construct $N = 2^n - n$ by setting $a_j = j$. By Hall’s marriage theorem, it suffices to prove the following:

Assuming that D_S has no completely-zero rows, given a set $I = \{i_1, i_2, \dots, i_k\}$ there exist at least k values of j so that there exists an $i \in I$ so that $(D_S)_{ij} = 1$. Call such j admissible.

Without loss of generality assume $i_1 < i_2 < \dots < i_k$.

Note that if $\{d_{ij} \mid i \in I\} = \{0, 1\}$, then j is admissible. Therefore the $k - 1$ numbers i_1, i_2, \dots, i_{k-1} are admissible, since for $\alpha < k$, i_α divides i_α but i_k does not. So we only need to find one more admissible j . Assume that i_k is not admissible; now it must be the case that all the i_α are divisors of i_k .

At this point we note that the $k = 1$ case is easy, since no row of D_S is zero. Moreover, if $k = 2$, $\{(D_S)_{i_1 i_1}, (D_S)_{i_2 i_1}\} = \{0, 1\}$, so in the row with the zero there must be 1 somewhere, yielding a second admissible column.

In the case where $k \geq 3$, note that $i_1 \leq i_k/3$. Therefore $i_k - i_1 \notin I$, but i_1 divides $i_k - i_1$ and i_k does not. Thus we have found the last admissible column. Having exhausted all cases, we are done.

HMIC 2020 Solutions

April 21, 2020

1. [6] Sir Alex is coaching a soccer team of n players of distinct heights. He wants to line them up so that for each player P , the total number of players that are either to the left of P and taller than P or to the right of P and shorter than P is even. In terms of n , how many possible orders are there?

Proposed by: Michael Ren

Answer: $\boxed{\lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!}$

Solution: We want to determine the number of permutations π of $1, \dots, n$ such that every number is involved in an even number of inversions. The key observation is that the number of inversions k is involved in has the same parity as $\pi(k) - k$. This is since $\pi(k) - k$ can be interpreted as the number of i with $\pi(i) < \pi(k)$ minus the number of i with $i < k$, meaning that i is counted with a weight of $\pm 1 \equiv 1 \pmod{2}$ if and only if i is in an inversion with k .

From this, it follows that such permutations are the ones that permute the even and odd numbers, so the answer is $\lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$.

2. [7] Some bishops and knights are placed on an infinite chessboard, where each square has side length 1 unit. Suppose that the following conditions hold:

- For each bishop, there exists a knight on the same diagonal as that bishop (there may be another piece between the bishop and the knight).
- For each knight, there exists a bishop that is exactly $\sqrt{5}$ units away from it.
- If any piece is removed from the board, then at least one of the two conditions above is no longer satisfied.

If n is the total number of pieces on the board, find all possible values of n .

Proposed by: Sheldon Kieren Tan

Answer: $\boxed{4 \mid n}$

Solution: Color the chessboard with the usual chessboard coloring. Note that bishops can only attack knights on the same colored squares, while knights can only attack bishops on different colored squares. Let B_B, W_B, B_N, W_N denote the number of bishops on black colored squares, bishops on white colored squares, knights on black colored squares and knights on white colored squares respectively.

Since removing a knight on a white colored square will cause a leave a bishop on a white colored square with no knight to attack, $W_N \leq W_B$. Similarly, $B_N \leq B_B$, $B_B \leq W_N$, $W_B \leq B_N$. Combining the inequalities, we find that $W_N = W_B = B_N = B_B$. Hence $4 \mid n$.

To construct $n = 4$, we can place bishops at $(1, 0), (2, 2)$ and knights at $(0, 1), (3, 1)$. We can easily extend this to any multiple of 4 by placing this configuration as many times as required across the grid, making sure that we avoid the (finitely many) diagonals that existing bishops occupy.

(A harder variant of the problem involves changing the third condition from "If any piece" to "If any proper subset of pieces". The proof remains unchanged but the construction is more tedious as the pieces must be "connected" without accidentally attacking one another. The question can also be phrased in a non-chess way to obscure the motivation behind the coloring.)

3. [9] Let $P_1P_2P_3P_4$ be a tetrahedron in \mathbb{R}^3 and let O be a point equidistant from each of its vertices. Suppose there exists a point H such that for each i , the line P_iH is perpendicular to the plane through the other three vertices. Line P_1H intersects the plane through P_2, P_3, P_4 at A , and contains a point $B \neq P_1$ such that $OP_1 = OB$. Show that $HB = 3HA$.

Proposed by: Michael Ren

Solution 1: Note that A is the orthocenter of triangle $P_2P_3P_4$ by projecting the givens about H . Next, let P_2A intersect P_3P_4 at C and $(P_2P_3P_4)$ at D . Then $AD = 2AC$.

Note that H is the orthocenter of triangle P_1P_2C , with A being the foot of the altitude from P_1 . Thus $AP_2 \cdot AC = AH \cdot AP_1$.

Finally, by Power of a Point at A ,

$$AP_1 \cdot AB = AP_2 \cdot AD = 2AP_2 \cdot AC = 2AH \cdot AP_1.$$

Thus $AB = 2AH$ which gives $HB = 3HA$, as desired.

Solution 2: We use vectors with H as the origin. Then $P_i \cdot (P_j - P_k) = 0$ from the definition of H . Thus $P_i \cdot P_j = P_i \cdot P_k$ for any distinct i, j, k . In particular, the quantity $P_i \cdot P_j$ is a constant over all distinct choices of i, j . Let this value be D . Let

$$O' = \frac{1}{2}(P_1 + P_2 + P_3 + P_4).$$

We claim $O' = O$. To see this, note that

$$(O' - P_1) \cdot (O' - P_1) = \frac{1}{4}(P_2 + P_3 + P_4 - P_1) \cdot (P_2 + P_3 + P_4 - P_1)$$

will be symmetric after expanding. Thus O' is equidistant from the P_i 's, which means $O' = O$.

Now, the projection of O onto P_1H is $\frac{O \cdot P_1}{P_1 \cdot P_1} P_1$, so

$$B = P_1 \left(\frac{(2O - P_1) \cdot P_1}{P_1 \cdot P_1} \right) = P_1 \left(\frac{3D}{P_1 \cdot P_1} \right).$$

Meanwhile, A is the projection of P_2 on P_1H , so

$$A = P_1 \left(\frac{P_1 \cdot P_2}{P_1 \cdot P_1} \right) = P_1 \left(\frac{D}{P_1 \cdot P_1} \right).$$

Thus $HB = 3HA$, as desired.

4. [9] Let $C_k = \frac{1}{k+1} \binom{2k}{k}$ denote the k^{th} Catalan number and p be an odd prime. Prove that exactly half of the numbers in the set

$$\left\{ \sum_{k=1}^{p-1} C_k n^k \mid n \in \{1, 2, \dots, p-1\} \right\}$$

are divisible by p .

Proposed by: Tristan Shin

Solution 1: We work in $\mathbb{F}_p[X]$.

We claim that

$$(1 - 4X)^{\frac{p+1}{2}} - 1 + 2X^p + 2X \sum_{k=0}^{p-1} C_k X^k = 0$$

The solution follows from this claim, as

$$\sum_{k=1}^{p-1} C_k n^k \equiv 0 \pmod{p} \iff (1 - 4n)^{\frac{p+1}{2}} - 1 + 2n^p + 2n \equiv 0 \pmod{p}.$$

Since $n^p \equiv n \pmod{p}$, this is equivalent to $n \equiv \frac{1}{4} \pmod{p}$ or $1 - 4n$ being a non-zero quadratic residue in \mathbb{F}_p . But we must omit the solution of $n = 0$, so we get $1 + (\frac{p-1}{2} - 1) = \frac{p-1}{2}$ values of n that work.

Now we prove the claim. Observe the following facts:

- $C_{p-1} = \frac{1}{2^{p-1}} \binom{2p-1}{p} = -1$ by Lucas' theorem.
- For $\frac{p-1}{2} < k < p-1$,

$$\nu_p(C_k) = \nu_p((2k)!) - \nu_p((k+1)!) - \nu_p(k!) = 1$$

since $k+1 < p \leq 2k$, so $C_k = 0$.

We show that the coefficient of X^k in the LHS is 0 for $k = 0, \dots, p$. This is obvious for $k > \frac{p+1}{2}$ and $k = 0$ by the above facts, as the coefficient is $2 + 2C_{p-1}$ for $k = p$, $2C_k$ for $\frac{p+1}{2} < k < p$, and 0 for $k = 0$.

Now for $1 \leq k \leq \frac{p+1}{2}$, the coefficient is $\binom{\frac{p+1}{2}}{k}(-4)^k + 2C_{k-1} = 0$. Write

$$\begin{aligned} \binom{\frac{p+1}{2}}{k} &= \frac{\frac{p+1}{2} \cdot \frac{p-1}{2} \cdot \frac{p-3}{2} \cdots \frac{p-2k+3}{2}}{k!} \\ &= \frac{\frac{1}{2} \cdot (-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdots (-\frac{2k-3}{2})}{k!} \\ &= \frac{(-1)^{k-1}}{2^k} \cdot \frac{1 \cdot 3 \cdots (2k-3)}{k!} \\ &= \frac{(-1)^{k-1}}{2^k} \cdot \frac{(2k-2)!}{2^{k-1}(k-1)!k!} \\ &= -\frac{2C_{k-1}}{(-4)^k} \end{aligned}$$

so the coefficient is 0.

Solution 2: We present an alternate proof of the key claim. Use the same starting facts as before.

$$\text{Let } Q(X) = -2 + \sum_{k=0}^{\frac{p-1}{2}} C_k X^k = -2 + X^{p-1} + \sum_{k=0}^{p-1} C_k X^k.$$

Square Q and multiply by X to get

$$\begin{aligned} XQ^2 &= 4X - 4X \sum_{k=0}^{\frac{p-1}{2}} C_k X^k + X \left(\sum_{k=0}^{\frac{p-1}{2}} C_k X^k \right)^2 \\ &= -4X - 4XQ + X \sum_{k=0}^{p-2} \sum_{j=0}^k C_j C_{k-j} X^k + C_{\frac{p-1}{2}}^2 X^p \\ &= -4X - 4XQ + \sum_{k=0}^{p-2} C_{k+1} X^{k+1} + 4X^p \\ &= -4X - 4XQ + Q + 1 - X^{p-1} + 4X^p \\ &= (1 - 4X)Q - (1 - 4X)(X^{p-1} - 1). \end{aligned}$$

It follows that

$$\begin{aligned} 0 &= 4X \cdot (XQ^2 - (1 - 4X)Q + (1 - 4X) \cdot (X^{p-1} - 1)) \\ &= 4X^2 Q^2 - 4X(1 - 4X)Q + 4(1 - 4X) \cdot (X^p - X) \\ &= 4X^2 Q^2 - 4X(1 - 4X)Q + (1 - 4X)^2 - (1 - 4X)^{p+1} \\ &= (2XQ + (1 - 4X)^{\frac{p+1}{2}} - (1 - 4X)) \cdot (2XQ - (1 - 4X)^{\frac{p+1}{2}} - (1 - 4X)). \end{aligned}$$

Evaluating the second factor at 0 gives -2 , so it is not the zero polynomial. Thus the first factor is the zero polynomial, from which the claim follows.

Solution 3: We prove the following generalization: Let q be a power of p . Then the polynomial $\sum_{k=1}^{q-1} C_k X^k$ has $\frac{q+1}{2}$ roots in \mathbb{F}_q and $\frac{p+1}{2}$ roots in \mathbb{F}_p . It once again suffices to prove the key claim, just with p replaced by q .

Work in $\mathbb{F}_q[[X]]$, the ring of formal power series over \mathbb{F}_q . Then

$$\begin{aligned} \sum_{k=0}^{\infty} \binom{2k}{k} X^k &= (1 - 4X)^{-\frac{1}{2}} \\ &= (1 - 4X)^{\frac{q-1}{2}} ((1 - 4X)^q)^{-\frac{1}{2}} \\ &= (1 - 4X)^{\frac{q-1}{2}} (1 - 4X^q)^{-\frac{1}{2}} \\ &= (1 - 4X)^{\frac{q-1}{2}} \sum_{k=0}^{\infty} \binom{2k}{k} X^{qk}. \end{aligned}$$

Taking both sides mod X^q ,

$$\sum_{k=0}^{q-1} \binom{2k}{k} X^k = (1 - 4X)^{\frac{q-1}{2}}.$$

Then using $\binom{2q}{q} = 2$, we have

$$\sqrt{1 - 4X} = (1 - 4X) \cdot (1 - 4X)^{-\frac{1}{2}} \equiv (1 - 4X) \cdot ((1 - 4X)^{\frac{q-1}{2}} + 2X^q) \pmod{X^{q+1}}$$

so

$$X \sum_{k=0}^{\infty} C_k X^k = \frac{1 - \sqrt{1 - 4X}}{2} \equiv \frac{1 - (1 - 4X)^{\frac{q+1}{2}} - 2X^q}{2} \pmod{X^{q+1}}$$

from which the claim follows.

Alternatively, one can also finish by integrating $(1 - 4X)^{\frac{p-1}{2}}$, noting that the “constant of integration” is no longer a constant but rather a polynomial of the form $a + bX^p$.

5. [11] A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.

Proposed by: Krit Boonsiriseth

Solution 1: (Ankan Bhattacharya¹) Let Δ denote the triangle, \circ denote the circle, and Δ' denote the reflection of the triangle in the center of the circle. Letting overline denote complement,

$$\begin{aligned} [\Delta \cap \circ] &\leq \frac{1}{3}[\Delta] + \frac{1}{2}[\circ] \\ \iff [\Delta \cap \Delta' \cap \circ] + [\Delta \cap \overline{\Delta'} \cap \circ] &\leq \frac{1}{3}[\Delta] + \frac{1}{2}[[\circ \cap \Delta \cap \Delta'] + 2[\circ \cap \Delta \cap \overline{\Delta'}] + [\circ \cap \overline{\Delta} \cap \overline{\Delta'}]] \\ \iff \frac{1}{2}[\Delta \cap \Delta' \cap \circ] &\leq \frac{1}{3}[\Delta] + \frac{1}{2}[\circ \cap \overline{\Delta} \cap \overline{\Delta'}] \\ \iff [\Delta \cap \Delta' \cap \circ] &\leq \frac{2}{3}[\Delta] + [\circ \cap \overline{\Delta} \cap \overline{\Delta'}]. \end{aligned}$$

Since $\Delta \cap \Delta' \cap \circ$ is centrally symmetric, it's enough to show the following well-known lemma.

Lemma. A centrally symmetric region \mathcal{R} of a triangle can have area at most $\frac{2}{3}$ that of the triangle.

Let the triangle be ABC with medial triangle DEF . We take two cases.

¹<https://artofproblemsolving.com/community/c6h2052367p14590886>

- **Case 1.** The center of symmetry is in one of the outer triangles, say $\triangle AEF$. Then the maximal possible \mathcal{R} is a parallelogram with one vertex A and two other vertices on \overline{AB} and \overline{AC} . By enlarging the parallelogram, we may assume its fourth vertex lies on \overline{BC} . If this vertex divides \overline{BC} into an $a : b$ ratio with $a + b = 1$, then the fraction of the area taken up by the parallelogram is $1 - a^2 - b^2 \leq \frac{1}{2}$.
- **Case 2.** The center of symmetry is in $\triangle DEF$. Then the maximal possible \mathcal{R} is a centrally symmetric hexagon with two vertices on each side. Then there are three little triangles similar to $\triangle ABC$ on the hexagon; by equal lengths, the similarity ratios a, b, c sum to 1. Then the fraction of the area taken up by the hexagon is $1 - a^2 - b^2 - c^2 \leq \frac{2}{3}$. \square

We are done.

Solution 2: It is also possible to approach this as an optimization problem. Fix a triangle $\Delta = ABC$ on the plane, vary the circle $\odot = \odot(O, r)$, and consider the objective function

$$f(O, r) := [\Delta \cap \odot] - \frac{1}{2}[\odot].$$

By a compactness argument we can show that f reaches a maximum, and at that maximum $\frac{\partial f}{\partial r} = 0$ and $\nabla_O f = \vec{0}$ must simultaneously hold.

- We have

$$\begin{aligned} \frac{\partial}{\partial r} f(O, r) &= \frac{\partial}{\partial r} [\Delta \cap \odot] - \frac{1}{2} \frac{\partial}{\partial r} [\odot] \\ &= r \cdot ((\text{total angle subtended by } AB \cap \odot, BC \cap \odot, CA \cap \odot \text{ at point } O) - \pi), \end{aligned}$$

therefore

$$\frac{\partial}{\partial r} f(O, r) = 0 \iff \text{the total angle subtended by the sides of } \Delta \text{ inside } \odot \text{ is } \pi. \quad (1)$$

- Let $\vec{n}_a, \vec{n}_b, \vec{n}_c$ denote the normal vectors of BC, CA and AB . Define k_a to be the length of $BC \cap \odot$, and define k_b, k_c similarly.

$$\begin{aligned} \nabla_O f(O, r) &= \nabla_O [\Delta \cap \odot] - \frac{1}{2} \nabla_O [\odot] \\ &= (k_a \vec{n}_a + k_b \vec{n}_b + k_c \vec{n}_c) - \vec{0} \end{aligned}$$

It is easy to see that $a\vec{n}_a + b\vec{n}_b + c\vec{n}_c = \vec{0}$ is the only linear relation between $\vec{n}_a, \vec{n}_b, \vec{n}_c$, so

$$\nabla_O f(O, r) = \vec{0} \iff k_a : k_b : k_c = a : b : c \quad (2)$$

Now suppose that $\odot(O, r)$ is chosen so that $\frac{\partial f}{\partial r} = 0$ and $\nabla_O f = \vec{0}$, so the angle condition in (1) holds, and $(k_a, k_b, k_c) = k(a, b, c)$ for some k . There are two possible cases:

- **Case 1.** All vertices of Δ do not lie inside \odot . In this case \odot intersects Δ at six points. The lines joining O and these six points divide $\Delta \cap \odot$ into three sectors and three triangles. By the angle condition in (1), the three sectors have total area $\frac{1}{2}[\odot]$. The three triangles have total area $k[\Delta]$, and it suffices to show that $k \leq \frac{1}{3}$.

As the total angle of the triangles at O is π , we may join two copies of each of the three triangles to form a cyclic hexagon. Therefore,

$$\begin{aligned} 2k \cdot [\Delta] &= 2 \cdot (\text{total area of three triangles}) \\ &= \text{area of cyclic hexagon with side lengths } ka, kb, kc, ka, kb, kc \\ &\geq \text{area of hexagon with side lengths } ka, kb, kc, ka, kb, kc \\ &\quad \text{created by six copies of a triangle with side lengths } ka, kb, kc \\ &= 6k^2 \cdot [\Delta], \end{aligned}$$

and our conclusion readily follows.

- **Case 2.** A vertex of \triangle lies in the interior of \odot . WLOG let that vertex be A . Let \odot intersect AB at X , AC at Y , and BC at D and E . From (2), $AX : AY : DE = AB : AC : BC$, $XY \parallel BC$ and $XY = DE$, therefore $\square XYED$ is a rectangle. From (1), $\angle XOD + \angle YOE = \pi$, so in fact $\square XYED$ is a square. Let x be the side length of $\square XYDE$, and let h be the height of the altitude from A to BC . Clearly $h > x$. As A lies inside $\odot = (XYDE)$, $\angle BAC > \frac{3\pi}{4}$, so $a > 3h > 3x$. Now we are done because $k = \frac{x}{a} < \frac{1}{3}$. ■

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Algebra and Number Theory

1. What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is *palindromic* if the sequence of decimal digits are the same when read backwards.)

Proposed by: Yuan Yao

Answer: 21

We need to first prove that every positive integer N less than 21 can be written as sum of two nonnegative palindromic integers. If N is in the interval $[1, 9]$, then it can be written as $0 + N$. If N is in the interval $[10, 18]$, it can be written as $9 + (N - 9)$. In addition, 19 and 20 can be written as $11 + 8$ and $11 + 9$, respectively.

Second, we need to show that 21 cannot be expressed in such a way. Let's suppose $21 = a + b$ with $a \leq b$. It follows that b has to be at least 11. Since $b \leq 21$, the only way for b to be palindromic is that $b = 11$. However, this leads to $a = 21 - b = 10$, which is not a palindrome. Therefore, 21 is the smallest number that satisfy the problem condition.

2. Let $N = 2^{(2^2)}$ and x be a real number such that $N^{(N^N)} = 2^{(2^x)}$. Find x .

Proposed by: Yuan Yao

Answer: 66

We compute

$$N^{(N^N)} = 16^{16^{16}} = 2^{4 \cdot 2^4 \cdot 2^4} = 2^{2^{26+2}} = 2^{2^{66}},$$

so $x = 66$.

3. Let x and y be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

Proposed by: Michael Tang

Answer: 50

Note that $a - 1 = \frac{x}{y}$ and $b - 1 = \frac{y}{x}$ are reciprocals. That is,

$$(a - 1)(b - 1) = 1 \implies ab - a - b + 1 = 1 \implies ab = a + b.$$

Let $t = ab = a + b$. Then we can write

$$a^2 + b^2 = (a + b)^2 - 2ab = t^2 - 2t,$$

so $t^2 - 2t = 15$, which factors as $(t - 5)(t + 3) = 0$. Since $a, b > 0$, we must have $t = 5$. Then, we compute

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 5^3 - 3 \cdot 5^2 = 50.$$

4. Let \mathbb{N} be the set of positive integers, and let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function satisfying

- $f(1) = 1$;
- for $n \in \mathbb{N}$, $f(2n) = 2f(n)$ and $f(2n + 1) = 2f(n) - 1$.

Determine the sum of all positive integer solutions to $f(x) = 19$ that do not exceed 2019.

Proposed by: Yuan Yao

Answer: 1889

For $n = 2^{a_0} + 2^{a_1} + \dots + 2^{a_k}$ where $a_0 > a_1 > \dots > a_k$, we can show that $f(n) = 2^{a_0} - 2^{a_1} - \dots - 2^{a_k} = 2^{a_0+1} - n$ by induction: the base case $f(1) = 1$ clearly holds; for the inductive step, when n is even

we note that $f(n) = 2f(\frac{n}{2}) = 2(2^{a_0} - \frac{n}{2}) = 2^{a_0+1} - n$ as desired, and when n is odd we also have $f(n) = 2f(\frac{n-1}{2}) - 1 = 2(2^{a_0} - \frac{n-1}{2}) - 1 = 2^{a_0+1} - n$, again as desired.

Since $19 = f(n) \leq 2^{a_0} \leq n$, we have $a_0 \geq 5$ and $n = 2^{a_0+1} - 19 \leq 2019$ gives $a_0 \leq 9$. So the answer is $\sum_{a=5}^9 (2^{a+1} - 19) = (2^{11} - 2^6) - 19 \cdot 5 = 1889$.

5. Let a_1, a_2, \dots be an arithmetic sequence and b_1, b_2, \dots be a geometric sequence. Suppose that $a_1 b_1 = 20$, $a_2 b_2 = 19$, and $a_3 b_3 = 14$. Find the greatest possible value of $a_4 b_4$.

Proposed by: Michael Tang

Answer: $\boxed{\frac{37}{4}}$

We present two solutions: the first more algebraic and computational, the second more conceptual.

Solution 1. Let $\{a_n\}$ have common difference d and $\{b_n\}$ have common ratio d ; for brevity, let $a_1 = a$ and $b_1 = b$. Then we have the equations $ab = 20$, $(a+d)br = 19$, and $(a+2d)br^2 = 14$, and we want to maximize $(a+3d)br^3$.

The equation $(a+d)br = 19$ expands as $abr + dbr = 19$, or $20r + bdr = 19$ since $ab = 20$. Similarly, $(20+2bd)r^2 = 14$, or $10r^2 + bdr^2 = 7$. Multiplying the first equation by r and subtracting the second, we get

$$10r^2 = 19r - 7 \implies (5r - 7)(2r - 1) = 0,$$

so either $r = \frac{7}{5}$ or $r = \frac{1}{2}$.

For each value of r , we have $bd = \frac{19-20r}{r} = \frac{19}{r} - 20$, so

$$(a+3d)br^3 = (20+3bd)r^3 = \left(\frac{57}{r} - 40\right)r^3 = r^2(57-40r).$$

The greater value of this expression is $\frac{37}{4}$, achieved when $r = \frac{1}{2}$.

Solution 2. The key is to find a (linear) recurrence relation that the sequence $c_n = a_n b_n$ satisfies. Some knowledge of theory helps here: c_n is of the form $snr^n + tr^n$ for some constants r, s, t , so $\{c_n\}$ satisfies a linear recurrence relation with characteristic polynomial $(x-r)^2 = x^2 - 2rx + r^2$. That is,

$$c_n = 2rc_{n-1} - r^2c_{n-2}$$

for some constant r .

Taking $n = 3$, we get $14 = 2r \cdot 19 - r^2 \cdot 20$, which factors as $(5r - 7)(2r - 1) = 0$, so either $r = \frac{7}{5}$ or $r = \frac{1}{2}$. Then

$$c_4 = 2rc_3 - r^2c_2 = 28r - 19r^2.$$

This expression is maximized at $r = \frac{14}{19}$, and strictly decreases on either side. Since $\frac{1}{2}$ is closer to $\frac{14}{19}$ than $\frac{7}{5}$, we should choose $r = \frac{1}{2}$, giving the answer $c_4 = 14 - \frac{19}{4} = \frac{37}{4}$.

6. For positive reals p and q , define the *remainder* when p is divided by q as the smallest nonnegative real r such that $\frac{p-r}{q}$ is an integer. For an ordered pair (a, b) of positive integers, let r_1 and r_2 be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs (a, b) such that $a, b \leq 20$ and $r_1 + r_2 = \sqrt{2}$.

Proposed by: Yuan Yao

Answer: $\boxed{16}$

The remainder when we divide $a\sqrt{2} + b\sqrt{3}$ by $\sqrt{2}$ is defined to be the smallest non-negative real r_1 such that $\frac{a\sqrt{2}+b\sqrt{3}-r_1}{\sqrt{2}}$ is integral. As $\frac{x}{\sqrt{2}}$ is integral iff x is an integral multiple of $\sqrt{2}$, it follows that $r_1 = b\sqrt{3} - c\sqrt{2}$, for some integer c . Furthermore given any real r such that $\frac{a\sqrt{2}+b\sqrt{3}-r}{\sqrt{2}}$ is integral, we may add or subtract $\sqrt{2}$ to r and the fraction remains an integer. Thus, the smallest non-negative real r_1 such that the fraction is an integer must satisfy $0 \leq r_1 < \sqrt{2}$.

Similarly, we find $r_2 = a\sqrt{2} - d\sqrt{3}$ for some integer d and $0 \leq r_2 < \sqrt{3}$. Since $r_1 + r_2 = \sqrt{2}$, then

$$(a - c)\sqrt{2} + (b - d)\sqrt{3} = \sqrt{2} \iff a - c = 1 \text{ and } b - d = 0.$$

Finally, substituting in $c = a - 1$ and $d = b$ plugging back into our bounds for r_1 and r_2 , we get

$$\begin{cases} 0 \leq b\sqrt{3} - (a - 1)\sqrt{2} < \sqrt{2} \\ 0 \leq a\sqrt{2} - b\sqrt{3} < \sqrt{3} \end{cases}$$

or

$$\begin{cases} (a - 1)\sqrt{2} \leq b\sqrt{3} \\ b\sqrt{3} < a\sqrt{2} \\ b\sqrt{3} \leq a\sqrt{2} \\ a\sqrt{2} < (b + 1)\sqrt{3} \end{cases}$$

Note that $b\sqrt{3} < a\sqrt{2} \implies b\sqrt{3} \leq a\sqrt{2}$ and

$$(a - 1)\sqrt{2} \leq b\sqrt{3} \implies a\sqrt{2} \leq b\sqrt{3} + \sqrt{2} < b\sqrt{3} + \sqrt{3} = (b + 1)\sqrt{3}$$

so the last two inequalities are redundant. We are left with

$$(a - 1)\sqrt{2} \leq b\sqrt{3} < a\sqrt{2}.$$

Since the non-negative number line is partitioned by intervals of the form $[(a - 1)\sqrt{2}, a\sqrt{2})$ for positive integers a , for any positive integer b , we can find a positive integer a that satisfies the inequalities. As clearly $a > b$, it remains to find the number of b such that $a \leq 20$. This is bounded by

$$b\sqrt{3} < a\sqrt{2} \leq 20\sqrt{2} \iff b < \frac{20\sqrt{2}}{\sqrt{3}} \implies b \leq 16$$

so there are 16 values of b and thus 16 ordered pairs of positive integers (a, b) that satisfy the problem.

7. Find the value of

$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{ab(3a + c)}{4^{a+b+c}(a + b)(b + c)(c + a)}.$$

Proposed by: Andrew Gu

Answer: $\boxed{\frac{1}{54}}$

Let S denote the given sum. By summing over all six permutations of the variables a, b, c we obtain

$$\begin{aligned} 6S &= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b) + 6abc}{4^{a+b+c}(a + b)(b + c)(c + a)} \\ &= \sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{3}{4^{a+b+c}} \\ &= 3 \left(\sum_{a=1}^{\infty} \frac{1}{4^a} \right) \left(\sum_{b=1}^{\infty} \frac{1}{4^b} \right) \left(\sum_{c=1}^{\infty} \frac{1}{4^c} \right) \\ &= 3 \left(\frac{1}{3} \right)^3 \\ &= \frac{1}{9}. \end{aligned}$$

Hence $S = \frac{1}{54}$.

8. There is a unique function $f : \mathbb{N} \rightarrow \mathbb{R}$ such that $f(1) > 0$ and such that

$$\sum_{d|n} f(d)f\left(\frac{n}{d}\right) = 1$$

for all $n \geq 1$. What is $f(2018^{2019})$?

Proposed by: Ashwin Sah

Answer: $\boxed{\frac{\binom{4038}{2019}^2}{2^{8076}} \text{ OR } \frac{(4038!)^2}{(2019!)^4 \cdot 2^{8076}}}$

Fix any prime p , and let $a_n = f(p^n)$ for $n \geq 0$. Notice that using the relation for p^n , we obtain

$$\sum_{i=0}^n a_i a_{n-i} = 1,$$

which means that if we let $g(x) = \sum_{n \geq 0} a_n x^n$, then $g(x)^2 = 1 + x + x^2 + \dots = \frac{1}{1-x}$ as a generating function. Thus $g(x) = (1-x)^{-\frac{1}{2}}$, and this is well-known to have generating function with coefficients $a_n = \frac{\binom{2n}{n}}{4^n}$. One way to see this is using the Taylor series and then reorganizing terms; it is also intimately related to the generating function for the Catalan numbers. In particular, a_n is independent of our choice of p .

Now if we define $f_0(n) = \prod_{p|n} a_{v_p(n)}$, then we see that $f = f_0$ on the prime powers.

If we define the Dirichlet convolution of two functions $\chi_1, \chi_2 : \mathbb{N} \rightarrow \mathbb{R}$ as χ_3 such that

$$\chi_3(n) = \sum_{d|n} \chi_1(d)\chi_2\left(\frac{n}{d}\right),$$

then it is well-known that multiplicative functions ($\chi(m)\chi(n) = \chi(mn)$ if $\gcd(m, n) = 1$, so e.g. $\phi(n)$, the Euler totient function) convolve to a multiplicative function.

In particular, f_0 is a multiplicative function by definition (it is equivalent to only define it at prime powers then multiply), so the convolution of f_0 with itself is multiplicative. By definition of a_n , the convolution of f_0 with itself equals 1 at all prime powers. Thus by multiplicativity, it equals the constant function 1 everywhere.

Two final things to note: $f_0(1) = a_0 = 1 > 0$, and f satisfying the conditions in the problem statement is indeed unique (proceed by induction on n that $f(n)$ is determined uniquely and that the resulting algorithm for computing f gives a well-defined function). Therefore f_0 , satisfying those same conditions, must equal f .

At last, we have

$$f(p^{2019}) = f_0(p^{2019}) = \frac{\binom{4038}{2019}}{4^{2019}}$$

so

$$f(2018^{2019}) = f(2^{2019})f(1009^{2019}) = \frac{\binom{4038}{2019}^2}{4^{4038}} = \frac{\binom{4038}{2019}^2}{2^{8076}}.$$

9. Tessa the hyper-ant has a 2019-dimensional hypercube. For a real number k , she calls a placement of nonzero real numbers on the 2^{2019} vertices of the hypercube k -harmonic if for any vertex, the sum of all 2019 numbers that are edge-adjacent to this vertex is equal to k times the number on this vertex. Let S be the set of all possible values of k such that there exists a k -harmonic placement. Find $\sum_{k \in S} |k|$.

Proposed by: Yuan Yao

Answer: $\boxed{2040200}$

By adding up all the equations on each vertex, we get $2019S = kS$ where S is the sum of all entries, so $k = 2019$ unless $S = 0$. In the latter case, by adding up all the equations on a half of the cube, we get

2018S - S = kS where S is the sum of all entries on that half of the cube, so k = 2017 unless S = 0. In the latter case (the sum of all entries of any half is zero), by adding up all the equations on a half of the half-cube, we get 2017S - 2S = kS, so k = 2015 unless S = 0. We continue this chain of casework until we get that the sum of every two vertices connected by unit segments is zero, in which case we have k = -2019. This means that k can take any odd value between -2019 and 2019 inclusive, so the sum of absolute values is 2 · 1010² = 2040200.

To achieve these values, suppose that the vertices of the hypercube are {0, 1}²⁰¹⁹ and that the label of (x₁, x₂, ..., x₂₀₁₉) is a₁^{x₁} a₂^{x₂} ... a₂₀₁₉^{x₂₀₁₉} for constants a₁, a₂, ..., a₂₀₁₉ ∈ {-1, 1}, then it is not difficult to see that this labeling is (a₁ + a₂ + ... + a₂₀₁₉)-harmonic for any choice of a_i's, so this can achieve all odd values between -2019 and 2019 inclusive.

10. The sequence of integers {a_i}_{i=0}[∞] satisfies a₀ = 3, a₁ = 4, and

$$a_{n+2} = a_{n+1}a_n + \left\lceil \sqrt{a_{n+1}^2 - 1} \sqrt{a_n^2 - 1} \right\rceil$$

for n ≥ 0. Evaluate the sum

$$\sum_{n=0}^{\infty} \left(\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} + \frac{a_{n+1}}{a_{n+3}} - \frac{a_n}{a_{n+1}} \right).$$

Proposed by: Ernest Chiu

Answer: $\boxed{\frac{14}{69}}$

The key idea is to note that a_{n+1}a_n + √(a_{n+1}² - 1)√(a_n² - 1) is the larger zero of the quadratic

$$f_n(x) = x^2 - (2a_{n+1}a_n)x + a_n^2 + a_{n+1}^2 - 1.$$

Since a_{n+2} is the smallest integer greater than or equal to this root, it follows that a_n² + a_{n+1}² + a_{n+2}² - 2a_na_{n+1}a_{n+2} - 1 is some small nonnegative integer. For these particular initial conditions (a₀ = 3, a₁ = 4, a₂ = 12 + ⌈√120⌉ = 23), this integer is (3² + 4²) + (23² - 2 · 3 · 4 · 23) - 1 = 25 - 23 - 1 = 1.

We now use induction to prove both

$$a_{n+3} = 2a_{n+2}a_{n+1} - a_n \text{ and } a_n^2 + a_{n+1}^2 + a_{n+2}^2 - 2a_n a_{n+1} a_{n+2} - 1 = 1$$

for n ≥ 0. The base case is not difficult to check: a₃ = 4 · 23 + ⌈√7920⌉ = 181 = 2 · 4 · 23 - 3, and the other equation has been checked above. Since the quadratic equation f_{n+1}(x) = 1 has a solution a_n by induction hypothesis. Then, using Vieta's theorem, 2a_{n+1}a_{n+2} - a_n is also a solution. Then, the two roots of f_{n+1}(x) = 0 must be in strictly between a_n and 2a_{n+1}a_{n+2} - a_n, so we have that a_{n+3} ≤ 2a_{n+1}a_{n+2} - a_n since a_{n+3} is the ceiling of the larger root. In fact, since a_{n+1}a_{n+2} is much larger than a_n for n ≥ 1 (it is not difficult to see that a_n grows faster than exponential), meaning that the two roots of f_{n+1} are more than 1 away from the minimum, and f(2a_{n+1}a_{n+2} - a_n) = 1, we have f(2a_{n+1}a_{n+2} - a_n - 1) < 0, which mean that we must have a_{n+3} = 2a_{n+1}a_{n+2} - a_n, which simultaneously proves both statements due to Vieta jumping.

To finish, note that the above recurrence gives

$$\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} = 2a_{n+1} - \frac{a_n}{a_{n+2}} - 2a_{n+1} + \frac{a_{n-1}}{a_n} = -\frac{a_n}{a_{n+2}} + \frac{a_{n-1}}{a_n},$$

which telescopes with the other two terms. (Convergence can be shown since the ratio of adjacent terms is bounded above by 1/3. In fact it goes to zero rapidly.) The only leftover terms after telescoping are $\frac{a_3}{a_2} - \frac{a_2}{a_0} = -\frac{a_0}{a_2} + \frac{a_{-1}}{a_0} = -\frac{3}{23} + \frac{1}{3} = \frac{14}{69}$, giving the answer. (Here, we use the backwards recurrence a_{n-1} = 2a_na_{n+1} - a_{n+2} to find a₋₁ = 2 · 3 · 4 - 23 = 1.)

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Combinatorics

1. How many distinct permutations of the letters of the word REDDER are there that do not contain a palindromic substring of length at least two? (A *substring* is a contiguous block of letters that is part of the string. A string is *palindromic* if it is the same when read backwards.)

Proposed by: Yuan Yao

Answer: $\boxed{6}$

If two identical letters are adjacent or have a single letter in between, there is clearly a palindromic substring of length (respectively) two or three. So there cannot be any such substrings.

Say we have a permutation of the word REDDER without any palindromic substrings. Let us call the first letter X. The second letter has to be different, let us call it Y. The third letter cannot be X or Y, let it be Z. Again, the fourth letter cannot be Y or Z, and we only have 3 letters to choose from, so it has to be X. Continuing analogously, the fifth letter has to be Y, and the sixth letter has to be Z. So any word satisfying the problem statement has to be of the form XYZXYZ. It is easy to check that such a word indeed does not have any palindromic substrings. X, Y, Z can be any permutation of R, E, D, giving a total of 6 possibilities.

2. Your math friend Steven rolls five fair icosahedral dice (each of which is labelled $1, 2, \dots, 20$ on its sides). He conceals the results but tells you that at least half of the rolls are 20. Suspicious, you examine the first two dice and find that they show 20 and 19 in that order. Assuming that Steven is truthful, what is the probability that all three remaining concealed dice show 20?

Proposed by: Evan Chen

Answer: $\boxed{\frac{1}{58}}$

The given information is equivalent to the first two dice being 20 and 19 and there being at least two 20's among the last three dice. Thus, we need to find the probability that given at least two of the last three dice are 20's, all three are. Since there is only one way to get all three 20's and $3 \cdot 19 = 57$ ways to get exactly two 20's, the probability is $\frac{1}{1+57} = \frac{1}{58}$.

3. Reimu and Sanae play a game using 4 fair coins. Initially both sides of each coin are white. Starting with Reimu, they take turns to color one of the white sides either red or green. After all sides are colored, the 4 coins are tossed. If there are more red sides showing up, then Reimu wins, and if there are more green sides showing up, then Sanae wins. However, if there is an equal number of red sides and green sides, then *neither* of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that Reimu wins?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{5}{16}}$

Clearly Reimu will always color a side red and Sanae will always color a side green, because their situation is never worse off when a side of a coin changes to their own color. Since the number of red-only coins is always equal to the number of green-only coins, no matter how Reimu and Sanae color the coins, they will have an equal probability of winning by symmetry, so instead they will cooperate to make sure that the probability of a tie is minimized, which is when all 4 coins have different colors on both sides (which can easily be achieved by Reimu coloring one side of a new coin red and Sanae immediately coloring the opposite side green). Therefore, the probability of Reimu winning is $\frac{\binom{4}{3} + \binom{4}{4}}{2^4} = \frac{5}{16}$.

4. Yannick is playing a game with 100 rounds, starting with 1 coin. During each round, there is a $n\%$ chance that he gains an extra coin, where n is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

Proposed by: Yuan Yao

Answer: $\boxed{1.01^{100}}$

Let X_i be the random variable which is the number of coins at the end of round i . Say that $X_0 = 1$ for convenience. Fix $i > 0$ and some positive integer x . Conditioning on the event $X_{i-1} = x$, there are only two cases with positive probability. In particular,

$$\Pr[X_i = x + 1 \mid X_{i-1} = x] = \frac{x}{100}$$

and

$$\Pr[X_i = x \mid X_{i-1} = x] = 1 - \frac{x}{100}.$$

Therefore

$$\begin{aligned} \mathbb{E}[X_i] &= \sum_{x>0} x \cdot \Pr[X_i = x] \\ &= \sum_{x>0} x \cdot \left(\left(1 - \frac{x}{100}\right) \Pr[X_{i-1} = x] + \frac{x-1}{100} \Pr[X_{i-1} = x-1] \right) \\ &= \sum_{x>0} x \Pr[X_{i-1} = x] - \frac{1}{100} \sum_{x>0} x \Pr[X_{i-1} = x-1] \\ &\quad + \frac{1}{100} \sum_{x>0} x^2 \Pr[X_{i-1} = x-1] - \frac{1}{100} \sum_{x>0} x^2 \Pr[X_i = x] \\ &= \frac{99}{100} \mathbb{E}[X_{i-1}] - \frac{1}{100} + \frac{1}{50} \mathbb{E}[X_{i-1}] + \frac{1}{100} \\ &= \frac{101}{100} \mathbb{E}[X_{i-1}]. \end{aligned}$$

(A different way to understand this is that no matter how many coins Yannick has currently (as long as he does not have more than 100 coins, which is guaranteed in this problem), the expected number of coins after one round is always 1.01 times the current number of coins, so the expected value is multiplied by 1.01 each round.)

Therefore

$$\mathbb{E}[X_{100}] = \left(\frac{101}{100}\right)^{100} \mathbb{E}[X_0] = 1.01^{100}.$$

5. Contessa is taking a random lattice walk in the plane, starting at $(1, 1)$. (In a random lattice walk, one moves up, down, left, or right 1 unit with equal probability at each step.) If she lands on a point of the form $(6m, 6n)$ for $m, n \in \mathbb{Z}$, she ascends to heaven, but if she lands on a point of the form $(6m+3, 6n+3)$ for $m, n \in \mathbb{Z}$, she descends to hell. What is the probability that she ascends to heaven?

Proposed by: John Michael Wu

Answer: $\boxed{\frac{13}{22}}$

Let $P(m, n)$ be the probability that she ascends to heaven from point (m, n) . Then $P(6m, 6n) = 1$ and $P(6m+3, 6n+3) = 0$ for all integers $m, n \in \mathbb{Z}$. At all other points,

$$4P(m, n) = P(m-1, n) + P(m+1, n) + P(m, n-1) + P(m, n+1) \quad (1)$$

This gives an infinite system of equations. However, we can apply symmetry arguments to cut down the number of variables to something more manageable. We have $P(m, n) = P(m+6a, n+6b)$ for $a, b \in \mathbb{Z}$, and $P(m, n) = P(n, m)$, and $P(m, n) = P(-m, n)$, and $P(m, n) = 1 - P(m+3, n+3)$ (since any path from the latter point to heaven corresponds with a path from the former point to hell, and vice versa).

Thus for example we have

$$P(1, 2) = P(-1, -2) = 1 - P(2, 1) = 1 - P(1, 2),$$

so $P(1, 2) = 1/2$.

Applying Equation (1) to points $(1, 1)$, $(0, 1)$, and $(0, 2)$, and using the above symmetries, we get the equations

$$4P(1, 1) = 2P(0, 1) + 1,$$

$$4P(0, 1) = P(0, 2) + 2P(1, 1) + 1,$$

$$4P(0, 2) = P(0, 1) + 3/2.$$

Solving yields $P(1, 1) = 13/22$.

6. A point P lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0 = P$, and given point P_i , point P_{i+1} is obtained by reflecting P_i over one of the four lines AB, BC, CD, DA , chosen uniformly at random and independently for each i . What is the probability that $P_8 = P$?

Proposed by: Yuan Yao

Answer:

$\frac{1225}{16384}$

Solution 1. WLOG, AB and CD are horizontal line segments and BC and DA are vertical. Then observe that we can consider the reflections over vertical lines separately from those over horizontal lines, as each reflection over a vertical line moves P_i horizontally to point P_{i+1} , and vice versa. Now consider only the reflections over horizontal segments AB and CD . Note that it is impossible for P_8 to be in the same location vertical location as P if there are an odd number of these reflections. Then we consider the reflections in pairs: let w denote reflecting twice over AB , let x denote reflecting over AB and then CD , let y denote reflecting over CD and then AB , and let z denote reflecting twice over CD . Note that both w and z preserve the position of our point. Also note that in order to end at the same vertical location as P , we must have an equal number of x 's and y 's. Now we count the number of sequences of length at most 4 with this property:

- Case 1: Length 0
There is just the empty sequence here, so 1.
- Case 2: Length 1
There are just the sequences w and z , so 2.
- Case 3: Length 2
We may either have an x and a y or two characters that are either w or z . There are 2 sequences of the former type and 4 of the latter, for 6 total.
- Case 4: Length 3
There are 12 sequences with an x , a y , and either a w or a z , and 8 sequences of only w 's and z 's, for 12 total.
- Case 5: Length 4
There are 6 sequences of 2 x 's and 2 y 's, 48 with one of each and two terms that are either w or z , and 16 of just w 's and z 's, for a total of 70.

Now let the number of such sequences of length k be a_k (so $a_3 = 20$). Note that these counts work also if we consider only reflections over vertical line segments BC and AD . Now to finish, we only need to count the number of ways to combine 2 valid sequences of total length 4. This is

$$\sum_{i=0}^4 a_i a_{4-i} \binom{8}{2i},$$

as there are a_i sequences of reflections over AB and CD , a_{4-i} sequences of reflections over BC and AD such that there are 8 total reflections, and $\binom{8}{2i}$ ways to choose which of the 8 reflections will be

over AB or CD . We compute that this sum is $1 \cdot 70 \cdot 1 + 2 \cdot 20 \cdot 28 + 6 \cdot 6 \cdot 70 + 20 \cdot 2 \cdot 28 + 70 \cdot 1 \cdot 1 = 4900$ total sequences of reflections that place P_8 at P . There are of course $4^8 = 65536$ total sequences of 8 reflections, each chosen uniformly at random, so our answer is $\frac{4900}{65536} = \frac{1225}{16384}$.

Solution 2. Suppose that P_0 is the origin and the four lines are $x = \pm 0.5$ and $y = \pm 0.5$. We consider a permutation of the lattice points on the coordinate plane, where all points with even x -coordinates are reflected across the y -axis and all points with even y -coordinates are reflected across the x -axis, so that the x - and y -coordinates are both rearranged in the following order:

$$\dots, 4, -3, 2, -1, 0, 1, -2, 3, -4, \dots$$

It is not difficult to see that a reflection across one of the lines corresponds to changing one of the coordinates from one number to either the previous number or the next number. Therefore, after the permutation, the question is equivalent to asking for the number of lattice walks of length 8 that returns to the origin. For such a lattice walk to return to origin, there needs to be the same number of up and down moves, and the same number of left and right moves. This condition is equivalent to having four moves that are left or up (LU), and four moves that are right or up (RU). Moreover, knowing whether a move is LU and whether it is RU uniquely determines what the move is, so it suffices to designate four LU moves and four RU moves, giving $\binom{8}{4}^2 = 4900$ possible walks. Hence the probability is $\frac{4900}{4^8} = \frac{1225}{16384}$.

7. In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

What is the probability that Alice wins the election (by having more votes than Celia)?

Proposed by: Yuan Yao

Answer: $\frac{1513}{2017}$

Let $P_n(m)$ be the probability that after n voters have voted, Alice gets m votes. We show by induction that for $n \geq 3$, the ratio $P_n(2) : P_n(3) : \dots : P_n(n-1)$ is equal to $1 : 2 : \dots : (n-2)$. We take a base case of $n = 3$, for which the claim is obvious. Then suppose the claim holds for $n = k$. Then $P_k(m) = \frac{2m-2}{(k-1)(k-2)}$. Then

$$P_{k+1}(i) = \frac{k-i}{k} P_k(i) + \frac{i-1}{k} P_k(i-1) = \frac{(k-i)(2i-2) + (i-1)(2i-4)}{k(k-1)(k-2)} = \frac{2i-2}{k(k-1)}.$$

Also, we can check $P_{k+1}(2) = \frac{2}{k(k-1)}$ and $P_{k+1}(k) = \frac{2}{k}$, so indeed the claim holds for $n = k+1$, and thus by induction our claim holds for all $n \geq 3$. The probability that Ceila wins the election is then

$$\frac{\sum_{m=2}^{1009} P_{2019}(m)}{\sum_{m=2}^{2018} P_{2019}(m)} = \frac{1008 \cdot (1+1008)/2}{2017 \cdot (1+2017)/2} = \frac{504}{2017},$$

and thus the probability that Alice wins is $\frac{1513}{2017}$.

8. For a positive integer N , we color the positive divisors of N (including 1 and N) with four colors. A coloring is called *multichromatic* if whenever a, b and $\gcd(a, b)$ are pairwise distinct divisors of N , then they have pairwise distinct colors. What is the maximum possible number of multichromatic colorings a positive integer can have if it is not the power of any prime?

Proposed by: Evan Chen

Answer: $\frac{192}{1}$

First, we show that N cannot have three distinct prime divisors. For the sake of contradiction, suppose $pqr|N$ for three distinct primes p, q, r . Then by the problem statement, $(p, q, 1)$, $(p, r, 1)$, and $(q, r, 1)$ have three distinct colors, so $(p, q, r, 1)$ has four distinct colors. In addition, $(pq, r, 1)$, (pq, pr, p) , and (pq, qr, q) have three distinct colors, so $(pq, p, q, r, 1)$ has five distinct colors, contradicting the fact that there are only four possible colors.

Similarly, if $p^3q|N$ for some distinct primes p and q , then $(p, q, 1)$, $(p^2, q, 1)$, $(p^3, q, 1)$, (p^2, pq, p) , (p^3, pq, p) , and (p^3, p^2q, p^2) are all triples with distinct colors, so $(1, q, p, p^2, p^3)$ must have five distinct colors, which is again a contradiction. In addition, if $p^2q^2|N$ for some distinct primes p and q , then $(p, q, 1)$, $(p^2, q^2, 1)$, $(p^2, q, 1)$, and $(p, q^2, 1)$ are all triples with pairwise distinct colors, so $(1, p, q, p^2, q^2)$ must have five distinct colors, another contradiction.

We are therefore left with two possibilities:

- Case 1: $N = pq$

In this case, the only triple of factors that must have pairwise distinct colors is $(p, q, 1)$. We have $4 \cdot 3 \cdot 2 = 24$ choices for these three, and 4 choices for pq itself, giving $4 \cdot 24 = 96$ multichromatic colorings.

- Case 2: $N = p^2q$

In this case, the triples of pairwise distinctly colored factors are $(p, q, 1)$, $(p^2, q, 1)$, and (p^2, pq, p) . From this, we see that $(1, p, q, p^2)$ must have four distinct colors, and the color of pq must be distinct from p and p^2 . There are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to assign the four distinct colors, 2 ways to assign the color of pq after that, and 4 ways to color p^2q after that, giving a total of $24 \cdot 2 \cdot 4 = 192$ monochromatic colorings.

Therefore, there can be at most 192 multichromatic colorings.

9. How many ways can one fill a 3×3 square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

Proposed by: Sam Korsky

Answer: 216

In what ways could we potentially fill a single row? The only possibilities are if it contains the numbers $(0, 0, 7)$ or $(0, 1, 6)$ or $(0, 2, 5)$ or $(0, 3, 4)$ or $(1, 2, 4)$. Notice that if we write these numbers in binary, in any choices for how to fill the row, there will be exactly one number with a 1 in its rightmost digit, exactly one number with a 1 in the second digit from the right, and exactly exactly one number with a 1 in the third digit from the right. Thus, consider the following operation: start with every unit square filled with the number 0. Add 1 to three unit squares, no two in the same row or column. Then add 2 to three unit squares, no two in the same row or column. Finally, add 4 to three unit squares, no two in the same row or column. There are clearly $6^3 = 216$ ways to perform this operation and every such operation results in a unique, suitably filled-in 3 by 3 square. Hence the answer is 216.

10. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

$$(x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 + (x_4 - a_4)^2 = 4 \quad \text{and} \quad |(x_1 + x_2 + x_3 + x_4) - (a_1 + a_2 + a_3 + a_4)| = 2.$$

In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

Proposed by: Brice Huang

Answer: $\binom{40}{10} \binom{40}{20}^3$

The possible moves correspond to the vectors $\pm\langle 2, 0, 0, 0 \rangle$, $\pm\langle 1, 1, 1, -1 \rangle$, and their permutations. It's not hard to see that these vectors form the vertices of a 4-dimensional hypercube, which motivates the change of coordinates

$$(x_1, x_2, x_3, x_4) \Rightarrow \left(\frac{x_1 + x_2 + x_3 + x_4}{2}, \frac{x_1 + x_2 - x_3 - x_4}{2}, \frac{x_1 - x_2 + x_3 - x_4}{2}, \frac{x_1 - x_2 - x_3 + x_4}{2} \right).$$

Under this change of coordinates, Fred must travel from $(0, 0, 0, 0)$ to $(20, 0, 0, 0)$, and the possible moves correspond to the sixteen vectors $\langle \pm 1, \pm 1, \pm 1, \pm 1 \rangle$. The new x_1 -coordinate must increase 30 times and decrease 10 times during Fred's walk, while the other coordinates must increase 20 times and decrease 20 times. Therefore, there are $\binom{40}{10} \binom{40}{20}^3$ possible walks.

HMMT February 2019

February 16, 2019

Geometry

1. Let d be a real number such that every non-degenerate quadrilateral has at least two interior angles with measure less than d degrees. What is the minimum possible value for d ?

Proposed by: James Lin

Answer:

The sum of the internal angles of a quadrilateral is 360° . To find the minimum d , we note the limiting case where three of the angles have measure d and the remaining angle has measure approaching zero. Hence, $d \geq 360^\circ/3 = 120$. It is not difficult to see that for any $0 < \alpha < 120$, a quadrilateral of which three angles have measure α degrees and fourth angle has measure $(360 - 3\alpha)$ degrees can be constructed.

2. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD . Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of BD .

Proposed by: Yuan Yao

Answer:

Observe that $AECF$ is a parallelogram. The equal area condition gives that $BE = DF = \frac{1}{3}AB$. Let $CE \cap BD = X$, then $\frac{EX}{CX} = \frac{BE}{CD} = \frac{1}{3}$, so that $BX^2 = EX \cdot CX = 3EX^2 \Rightarrow BX = \sqrt{3}EX \Rightarrow \angle EBX = 30^\circ$. Now, $CE = 2BE = CF$, so CEF is an equilateral triangle and $CD = \frac{3}{2}CF = \frac{3}{2}$. Hence, $BD = \frac{2}{\sqrt{3}} \cdot \frac{3}{2} = \sqrt{3}$.

3. Let AB be a line segment with length 2, and S be the set of points P on the plane such that there exists point X on segment AB with $AX = 2PX$. Find the area of S .

Proposed by: Yuan Yao

Answer:

Observe that for any X on segment AB , the locus of all points P such that $AX = 2PX$ is a circle centered at X with radius $\frac{1}{2}AX$. Note that the point P on this circle where PA forms the largest angle with AB is where PA is tangent to the circle at P , such that $\angle PAB = \arcsin(1/2) = 30^\circ$. Therefore, if we let Q and Q' be the tangent points of the tangents from A to the circle centered at B (call it ω) with radius $\frac{1}{2}AB$, we have that S comprises the two 30-60-90 triangles AQB and $AQ'B$, each with area $\frac{1}{2}\sqrt{3}$ and the 240° sector of ω bounded by BQ and BQ' with area $\frac{2}{3}\pi$. Therefore the total area is $\sqrt{3} + \frac{2\pi}{3}$.

4. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G . Given that the area of AGB is 10 more than the area of CGB , find the smallest possible area of hexagon $ABCDEF$.

Proposed by: Andrew Lin

Answer:

Since $ACDF$ and $ABDE$ have area 168, triangles ABD and ACD (which are each half a parallelogram) both have area 84. Thus, B and C are the same height away from AD , and since $ABCDEF$ is convex, B and C are on the same side of AD . Thus, BC is parallel to AD , and $ABCD$ is a trapezoid. In particular, we have that the area of ABG equals the area of CDG . Letting this quantity be x , we have that the area of BCG is $x - 10$, and the area of ADG is $84 - x$. Then notice that $\frac{[ABG]}{[CBG]} = \frac{AG}{GC} = \frac{[ADG]}{[CDG]}$. This means that $\frac{x}{x-10} = \frac{84-x}{x}$. Simplifying, we have $x^2 - 47x + 420 = 0$; this has solutions $x = 12$ and $x = 35$. The area of $ABCDEF$ is twice the area of trapezoid $ABCD$, or $2[x + (x - 10) + (84 - x) + x] = 4x + 148$; choosing $x = 12$, we get that the smallest possible area is $48 + 148 = 196$.

5. Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O . Point D is the reflection of C across AB . Given that $DO = \sqrt{3}$, find the area of triangle ABC .

Proposed by: Lillian Zhang

Answer: $\frac{\sqrt{2}+1}{2}$ OR $\frac{\sqrt{2}-1}{2}$

Solution 1. Observe that

$$\angle DBO = \angle DBA + \angle ABO = \angle CBA + \angle BAO = \frac{1}{2}(\angle CBA + \angle BCA) + \frac{1}{2}(\angle BAC) = \frac{1}{2}(180^\circ) = 90^\circ.$$

Thus $BC = BD = \sqrt{2}$ by the Pythagorean Theorem on $\triangle DBO$. Then $\angle BOC = 90^\circ$, and the distance from O to BC is $\frac{\sqrt{2}}{2}$. Depending on whether A is on the same side of BC as O , the height from A to BC is either $1 + \frac{\sqrt{2}}{2}$ or $1 - \frac{\sqrt{2}}{2}$, so the area is $(\sqrt{2} \cdot (1 \pm \frac{\sqrt{2}}{2}))/2 = \frac{\sqrt{2} \pm 1}{2}$.

Solution 2. One can observe that $\angle DBA = \angle CBA = \angle ACB$ by property of reflection and ABC being isosceles, hence DB is tangent to Ω and Power of a Point (and reflection property) gives $BC = BD = \sqrt{OD^2 - OB^2} = \sqrt{2}$. Proceed as in Solution 1.

Note. It was intended, but not specified in the problem statement that triangle ABC is acute, so we accepted either of the two possible answers.

6. Six unit disks $C_1, C_2, C_3, C_4, C_5, C_6$ are in the plane such that they don't intersect each other and C_i is tangent to C_{i+1} for $1 \leq i \leq 6$ (where $C_7 = C_1$). Let C be the smallest circle that contains all six disks. Let r be the smallest possible radius of C , and R the largest possible radius. Find $R - r$.

Proposed by: Daniel Liu

Answer: $\sqrt{3} - 1$

The minimal configuration occurs when the six circles are placed with their centers at the vertices of a regular hexagon of side length 2. This gives a radius of 3.

The maximal configuration occurs when four of the circles are placed at the vertices of a square of side length 2. Letting these circles be C_1, C_3, C_4, C_6 in order, we place the last two so that C_2 is tangent to C_1 and C_3 and C_5 is tangent to C_4 and C_6 . (Imagine pulling apart the last two circles on the plane; this is the configuration you end up with.) The resulting radius is $2 + \sqrt{3}$, so the answer is $\sqrt{3} - 1$.

Now we present the proofs for these configurations being optimal. First, we rephrase the problem: given an equilateral hexagon of side length 2, let r be the minimum radius of a circle completely containing the vertices of the hexagon. Find the difference between the minimum and maximum values in r . (Technically this r is off by one from the actual problem, but since we want $R - r$ in the actual problem, this difference doesn't matter.)

Proof of minimality. We claim the minimal configuration stated above cannot be covered by a circle with radius $r < 2$. If $r < 2$ and all six vertices O_1, O_2, \dots, O_6 are in the circle, then we have that $\angle O_1 O O_2 > 60^\circ$ since $O_1 O_2$ is the largest side of the triangle $O_1 O O_2$, and similar for other angles $\angle O_2 O O_3, \angle O_3 O O_4, \dots$, but we cannot have six angles greater than 60° into 360° , contradiction. Therefore $r \geq 2$.

Proof of maximality. Let $ABCDEF$ be the hexagon, and choose the covering circle to be centered at O , the midpoint of AD , and radius $\sqrt{3} + 1$. We claim the other vertices are inside this covering circle. First, we will show the claim for B . Let M be the midpoint of AC . Since ABC is isosceles and $AM \geq 1$, we must have $BM \leq \sqrt{4 - 1} = \sqrt{3}$. Furthermore, MO is a midline of ACD , so $MO = \frac{CD}{2} = 1$. Thus by the triangle inequality, $OB \leq MB + OM = \sqrt{3} + 1$, proving the claim. A similar argument proves the claim for C, E, F . Finally, an analogous argument to above shows if we define P as the midpoint of BE , then $AP \leq \sqrt{3} + 1$ and $DP \leq \sqrt{3} + 1$, so by triangle inequality $AD \leq 2(\sqrt{3} + 1)$. Hence $OA = OD \leq \sqrt{3} + 1$, proving the claim for A and D . Thus the covering circle contains all six vertices of $ABCDEF$.

7. Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let H be the orthocenter of ABC . Find the radius of the circle with nonzero radius tangent to the circumcircles of AHB, BHC, CHA .

Proposed by: Michael Ren

Answer: $\boxed{\frac{65}{4}}$

Solution 1. We claim that the circle in question is the circumcircle of the anticomplementary triangle of ABC , the triangle for which ABC is the medial triangle.

Let $A'B'C'$ be the anticomplementary triangle of ABC , such that A is the midpoint of $B'C'$, B is the midpoint of $A'C'$, and C is the midpoint of $A'B'$. Denote by ω the circumcircle of $A'B'C'$. Denote by ω_A the circumcircle of BHC , and similarly define ω_B, ω_C .

Since $\angle BA'C = \angle BAC = 180^\circ - \angle BHC$, we have that ω_A passes through A' . Thus, ω_A can be redefined as the circumcircle of $A'BC$. Since triangle $A'B'C'$ is triangle $A'BC$ dilated by a factor of 2 from point A' , ω is ω_A dilated by a factor of 2 from point A' . Thus, circles ω and ω_A are tangent at A' .

By a similar logic, ω is also tangent to ω_B and ω_C . Therefore, the circumcircle of the anticomplementary triangle of ABC is indeed the circle that the question is asking for.

Using the formula $R = \frac{abc}{4A}$, we can find that the circumradius of triangle ABC is $\frac{65}{8}$. The circumradius of the anticomplementary triangle is double of that, so the answer is $\frac{65}{4}$.

Solution 2. It is well-known that the circumcircle of AHB is the reflection of the circumcircle of ABC over AB . In particular, the circumcircle of AHB has radius equal to the circumradius $R = \frac{65}{8}$. Similarly, the circumcircles of BHC and CHA have radii R . Since H lies on all three circles (in the question), the circle centered at H with radius $2R = \frac{65}{4}$ is tangent to each circle at the antipode of H in that circle.

8. In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC , $BC = 1$, and the perimeter of ABC is 6, find the area of ABC .

Proposed by: Andrew Lin

Answer: $\boxed{\frac{6}{7}}$

Solution 1. Let $A'B'C'$ be the medial triangle of ABC , where A' is the midpoint of BC and so on. Notice that the midpoint of OH , which is the nine-point-center N of triangle ABC , is also the circumcenter of $A'B'C'$ (since the midpoints of the sides of ABC are on the nine-point circle). Thus, if N is on BC , then NA' is parallel to $B'C'$, so by similarity, we also know that OA is parallel to BC .

Next, $AB < AC$, so B is on the minor arc AC . This means that $\angle OAC = \angle OCA = \angle C$, so $\angle AOC = 180 - 2\angle C$. This gives us the other two angles of the triangle in terms of angle C : $\angle B = 90 + \angle C$ and $\angle A = 90 - 2\angle C$. To find the area, we now need to find the height of the triangle from A to BC , and this is easiest by finding the circumradius R of the triangle.

We do this by the Extended Law of Sines. Letting $AC = x$ and $AB = 5 - x$,

$$\frac{1}{\sin(90 - 2C)} = \frac{x}{\sin(90 + C)} = \frac{5 - x}{\sin C} = 2R,$$

which can be simplified to

$$\frac{1}{\cos 2C} = \frac{x}{\cos C} = \frac{5 - x}{\sin C} = 2R.$$

This means that

$$\frac{1}{\cos 2C} = \frac{(x) + (5 - x)}{(\cos C) + (\sin C)} = \frac{5}{\cos C + \sin C}$$

and the rest is an easy computation:

$$\cos C + \sin C = 5 \cos 2C = 5(\cos^2 C - \sin^2 C)$$

$$\frac{1}{5} = \cos C - \sin C$$

Squaring both sides,

$$\frac{1}{25} = \cos^2 C - 2 \sin C \cos C + \sin^2 C = 1 - \sin 2C$$

so $\sin 2C = \frac{24}{25}$, implying that $\cos 2C = \frac{7}{25}$. Therefore, since $\frac{1}{\cos 2C} = 2R$ from above, $R = \frac{25}{14}$. Finally, viewing triangle ABC with $BC = 1$ as the base, the height is

$$\sqrt{R^2 - \left(\frac{BC}{2}\right)^2} = \frac{12}{7}$$

by the Pythagorean Theorem, yielding an area of $\frac{1}{2} \cdot 1 \cdot \frac{12}{7} = \frac{6}{7}$.

Solution 2. The midpoint of OH is the nine-point center N . We are given N lies on BC , and we also know N lies on the perpendicular bisector of EF , where E is the midpoint of AC and F is the midpoint of AB . The main observation is that N is equidistant from M and F , where M is the midpoint of BC .

Translating this into coordinates, we pick $B(-0.5, 0)$ and $C(0.5, 0)$, and arbitrarily set $A(a, b)$ where (without loss of generality) $b > 0$. We get $E(\frac{a+0.5}{2}, \frac{b}{2})$, $F(\frac{a-0.5}{2}, \frac{b}{2})$, $M(0, 0)$. Thus N must have x -coordinate equal to the average of those of E and F , or $\frac{a}{2}$. Since N lies on BC , we have $N(\frac{a}{2}, 0)$.

Since $MN = EN$, we have $\frac{a^2}{4} = \frac{1}{16} + \frac{b^2}{4}$. Thus $a^2 = b^2 + \frac{1}{4}$. The other equation is $AB + AC = 5$, which is just

$$\sqrt{(a+0.5)^2 + b^2} + \sqrt{(a-0.5)^2 + b^2} = 5.$$

This is equivalent to

$$\begin{aligned} \sqrt{2a^2 + a} + \sqrt{2a^2 - a} &= 5 \\ \sqrt{2a^2 + a} &= 5 - \sqrt{2a^2 - a} \\ 2a^2 + a &= 25 - 10\sqrt{2a^2 - a} + 2a^2 - a \\ 25 - 2a &= 10\sqrt{2a^2 - a} \\ 625 - 100a + 4a^2 &= 200a^2 - 100a \\ 196a^2 &= 625 \end{aligned}$$

Thus $a^2 = \frac{625}{196}$, so $b^2 = \frac{576}{196}$. Thus $b = \frac{24}{14} = \frac{12}{7}$, so $[ABC] = \frac{b}{2} = \frac{6}{7}$.

9. In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges BF, FG, GH, HD at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{141\sqrt{11}}{2}}$

Let AD be the positive x -axis, AB be the positive y -axis, and AE be the positive z -axis, with A the origin. The plane, which passes through the origin, has equation $k_1x + k_2y = z$ for some undetermined parameters k_1, k_2 . Because $AP = AS$ and $AB = AD$, we get $PB = SD$, so P and S have the same z -coordinate. But $P(0, 6, 6k_2)$ and $S(6, 0, 6k_1)$, so $k_1 = k_2 = k$ for some k . Then Q and R both have z -coordinate 49, so $Q(\frac{49}{k} - 6, 6, 49)$ and $R(6, \frac{49}{k} - 6, 49)$. The equation $QR^2 = RS^2$ then gives

$$\left(\frac{49}{k} - 6\right)^2 + (49 - 6k)^2 = 2\left(12 - \frac{49}{k} - 12\right)^2.$$

This is equivalent to

$$(49 - 6k)^2(k^2 + 1) = 2(49 - 12k)^2,$$

which factors as

$$(k - 7)(36k^3 - 336k^2 - 203k + 343) = 0.$$

This gives $k = 7$ as a root. Note that for Q and R to actually lie on FG and GH respectively, we must have $\frac{49}{6} \geq k \geq \frac{49}{12}$. Via some estimation, one can show that the cubic factor has no roots in this range (for example, it's easy to see that when $k = 1$ and $k = \frac{336}{36} = \frac{28}{3}$, the cubic is negative, and it also remains negative between the two values), so we must have $k = 7$.

Now consider projecting $APQRS$ onto plane $ABCD$. The projection is $ABCD$ save for a triangle $Q'CR'$ with side length $12 - \frac{49}{k} = 5$. Thus the projection has area $36 - \frac{25}{2} = \frac{47}{2}$. Since the area of the projection equals $[APQRS] \cdot \cos \theta$, where θ is the (smaller) angle between planes $APQRS$ and $ABCD$, and since the planes have normal vectors $(k, k, -1)$ and $(0, 0, 1)$ respectively, we get $\cos \theta = \frac{(k, k, -1) \cdot (0, 0, 1)}{\sqrt{k^2 + k^2 + 1}} = \frac{1}{\sqrt{2k^2 + 1}} = \frac{1}{\sqrt{99}}$ and so

$$[APQRS] = \frac{47\sqrt{99}}{2} = \frac{141\sqrt{11}}{2}.$$

10. In triangle ABC , $AB = 13, BC = 14, CA = 15$. Squares $ABB_1A_2, BCC_1B_2, CAA_1C_2$ are constructed outside the triangle. Squares $A_1A_2A_3A_4, B_1B_2B_3B_4, C_1C_2C_3C_4$ are constructed outside the hexagon $A_1A_2B_1B_2C_1C_2$. Squares $A_3A_4B_3B_4A_6, B_3C_3C_4B_6, C_3A_4A_5C_6$ are constructed outside the hexagon $A_4A_3B_4B_3C_4C_3$. Find the area of the hexagon $A_5A_6B_5B_6C_5C_6$.

Proposed by: Yuan Yao

Answer: 19444

Solution 1.

We can use complex numbers to find synthetic observations. Let $A = a, B = b, C = c$. Notice that B_2 is a rotation by -90° (counter-clockwise) of C about B , and similarly C_1 is a rotation by 90° of B about C . Since rotation by 90° corresponds to multiplication by i , we have $B_2 = (c - b) \cdot (-i) + b = b(1 + i) - ci$ and $C_1 = (b - c) \cdot i + c = bi + c(1 - i)$. Similarly, we get $C_2 = c(1 + i) - ai, A_1 = ci + a(1 - i), A_2 = a(1 + i) - bi, B_1 = ai + b(1 - i)$. Repeating the same trick on $B_1B_2B_3B_4$ et. al, we get $C_4 = -a + b(-1 + i) + c(3 - i), C_3 = a(-1 - i) - b + c(3 + i), A_4 = -b + c(-1 + i) + a(3 - i), A_3 = b(-1 - i) - c + a(3 + i), B_4 = -c + a(-1 + i) + b(3 - i), B_3 = c(-1 - i) - a + b(3 + i)$. Finally, repeating the same trick on the outermost squares, we get $B_6 = -a + b(3 + 5i) + c(-3 - 3i), C_5 = -a + b(-3 + 3i) + c(3 - 5i), C_6 = -b + c(3 + 5i) + a(-3 - 3i), A_5 = -b + c(-3 + 3i) + a(3 - 5i), A_6 = -c + a(3 + 5i) + b(-3 - 3i), B_5 = -c + a(-3 + 3i) + b(3 - 5i)$.

From here, we observe the following synthetic observations.

- S1. $B_2C_1C_4B_3, C_2A_1A_4C_3, A_2B_1B_4A_3$ are trapezoids with bases of lengths $BC, 4BC; AC, 4AC; AB, 4AB$ and heights h_a, h_b, h_c respectively (where h_a is the length of the altitude from A to BC , and likewise for h_b, h_c)
- S2. If we extend B_5B_4 and B_6B_3 to intersect at B_7 , then $B_7B_4B_3 \cong BB_1B_2 \sim B_7B_5B_6$ with scale factor 1:5. Likewise when we replace all B 's with A 's or C 's.

Proof of S1. Observe $C_1 - B_2 = c - b$ and $C_4 - B_3 = 4(c - b)$, hence $B_2C_1 \parallel B_3C_4$ and $B_3C_4 = 4B_2C_1$. Furthermore, since translation preserves properties of trapezoids, we can translate $B_2C_1C_4B_3$ such that B_2 coincides with A . Being a translation of $a - B_2$, we see that B_3 maps to $B'_3 = 2b - c$ and C_4 maps to $C'_4 = -2b + 3c$. Both $2b - c$ and $-2b + 3c$ lie on the line determined by b and c (since $-2 + 3 = 2 - 1 = 1$), so the altitude from A to BC is also the altitude from A to $B'_3C'_4$. Thus h_a equals the length of the altitude from B_2 to B_3C_4 , which is the height of the trapezoid $B_2C_1C_4B_3$. This proves S1 for $B_2C_1C_4B_3$; the other trapezoids follow similarly.

Proof of S2. Notice a translation of $-a + 2b - c$ maps B_1 to B_4, B_2 to B_3 , and B to a point $B_8 = -a + 3b - c$. This means $B_8B_3B_4 \cong BB_1B_2$. We can also verify that $4B_8 + B_6 = 5B_3$ and $4B_8 + B_5 = 5B_4$, showing that $B_8B_5B_6$ is a dilation of $B_8B_4B_3$ with scale factor 5. We also get B_8 lies on B_3B_6 and B_5B_4 , so $B_8 = B_7$. This proves S2 for $B_3B_4B_5B_6$, and similar arguments prove the likewise part.

Now we are ready to attack the final computation. By S2, $[B_3B_4B_5B_6] + [BB_1B_2] = [B_7B_5B_6] = [BB_1B_2]$. But by the $\frac{1}{2}ac \sin B$ formula, $[BB_1B_2] = [ABC]$ (since $\angle B_1BB_2 = 180^\circ - \angle ABC$). Hence,

$[B_3B_4B_5B_6] + [BB_1B_2] = 25[ABC]$. Similarly, $[C_3C_4C_5C_6] + [CC_1C_2] = 25[ABC]$ and $[A_3A_4A_5A_6] + [AA_1A_2] = 25[ABC]$. Finally, the formula for area of a trapezoid shows $[B_2C_1C_4B_3] = \frac{5BC}{2} \cdot h_a = 5[ABC]$, and similarly the other small trapezoids have area $5[ABC]$. The trapezoids thus contribute area $(75 + 3 \cdot 5) = 90[ABC]$. Finally, ABC contributes area $[ABC] = 84$.

By S1, the outside squares have side lengths $4BC, 4CA, 4AB$, so the sum of areas of the outside squares is $16(AB^2 + AC^2 + BC^2)$. Furthermore, a Law of Cosines computation shows $A_1A_2^2 = AB^2 + AC^2 + 2 \cdot AB \cdot AC \cdot \cos \angle BAC = 2AB^2 + 2AC^2 - BC^2$, and similarly $B_1B_2^2 = 2AB^2 + 2BC^2 - AC^2$ and $C_1C_2^2 = 2BC^2 + 2AC^2 - AB^2$. Thus the sum of the areas of $A_1A_2A_3A_4$ et. al is $3(AB^2 + AC^2 + BC^2)$. Finally, the small squares have area add up to $AB^2 + AC^2 + BC^2$. Aggregating all contributions from trapezoids, squares, and triangle, we get

$$[A_5A_6B_5B_6C_5C_6] = 91[ABC] + 20(AB^2 + AC^2 + BC^2) = 7644 + 11800 = 19444.$$

Solution 2. Let $a = BC, b = CA, c = AB$. We can prove S1 and S2 using some trigonometry instead.

Proof of S1. The altitude from B_3 to B_2C_1 has length $B_2B_3 \sin \angle BB_2B_1 = B_1B_2 \sin \angle BB_2B_1 = BB_1 \sin \angle B_1BB_2 = AB \sin \angle ABC = h_a$ using Law of Sines. Similarly, we find the altitude from C_4 to B_2C_1 equals h_a , thus proving $B_2C_1C_4B_3$ is a trapezoid. Using $B_1B_2 = \sqrt{2a^2 + 2c^2 - b^2}$ from end of Solution 1, we get the length of the projection of B_2B_3 onto B_3C_4 is $B_2B_3 \cos \angle BB_2B_1 = \frac{(2a^2 + 2c^2 - b^2) + a^2 - c^2}{2a} = \frac{3a^2 + c^2 - b^2}{2a}$, and similarly the projection of C_1C_4 onto B_3C_4 has length $\frac{3a^2 + b^2 - c^2}{2a}$. It follows that $B_3C_4 = \frac{3a^2 + c^2 - b^2}{2a} + a + \frac{3a^2 + b^2 - c^2}{2a} = 4a$, proving S1 for $B_2C_1C_4B_3$; the other cases follow similarly.

Proof of S2. Define B_8 to be the image of B under the translation taking B_1B_2 to B_4B_3 . We claim B_8 lies on B_3B_6 . Indeed, $B_8B_4B_3 \cong BB_1B_2$, so $\angle B_8B_3B_4 = \angle BB_2B_1 = 180^\circ - \angle B_3B_2C_1 = \angle B_2B_3C_4$. Thus $\angle B_8B_3C_4 = \angle B_4B_3B_2 = 90^\circ$. But $\angle B_6B_3C_4 = 90^\circ$, hence B_8, B_3, B_6 are collinear. Similarly we can prove B_5B_4 passes through B_8 , so $B_8 = B_7$. Finally, $\frac{B_7B_3}{B_7B_6} = \frac{B_7B_4}{B_7B_5} = \frac{1}{5}$ (using $B_3B_6 = 4a, B_4B_5 = 4c, B_7B_3 = a, B_7B_4 = c$) shows $B_7B_4B_3 \sim B_7B_5B_6$ with scale factor 1:5, as desired. The likewise part follows similarly.

HMMT February 2019

February 16, 2019

Guts Round

1. [3] Find the sum of all real solutions to $x^2 + \cos x = 2019$.

Proposed by: Evan Chen

Answer:

The left-hand side is an even function, hence for each x that solves the equation, $-x$ will also be a solution. Pairing the solutions up in this way, we get that the sum must be 0.

2. [3] There are 100 people in a room with ages $1, 2, \dots, 100$. A pair of people is called *cute* if each of them is at least seven years older than half the age of the other person in the pair. At most how many pairwise disjoint cute pairs can be formed in this room?

Proposed by: Yuan Yao

Answer:

For a *cute* pair (a, b) we would have

$$a \geq \frac{b}{2} + 7, \quad b \geq \frac{a}{2} + 7.$$

Solving the system, we get that a and b must both be at least 14. However 14 could only be paired with itself or a smaller number; therefore, only people with age 15 or above can be paired with someone of different age. Pairing consecutive numbers $(15, 16), (17, 18), \dots, (99, 100)$ works, giving $\frac{100-14}{2} = 43$ pairs.

3. [3] Let $S(x)$ denote the sum of the digits of a positive integer x . Find the maximum possible value of $S(x + 2019) - S(x)$.

Proposed by: Alec Sun

Answer:

We note that $S(a + b) \leq S(a) + S(b)$ for all positive a and b , since carrying over will only decrease the sum of digits. (A bit more rigorously, one can show that $S(x + a \cdot 10^b) - S(x) \leq a$ for $0 \leq a \leq 9$.) Hence we have $S(x + 2019) - S(x) \leq S(2019) = 12$, and equality can be achieved with $x = 100000$ for example.

4. [3] Tessa has a figure created by adding a semicircle of radius 1 on each side of an equilateral triangle with side length 2, with semicircles oriented outwards. She then marks two points on the boundary of the figure. What is the greatest possible distance between the two points?

Proposed by: Yuan Yao

Answer:

Note that both points must be in different semicircles to reach the maximum distance. Let these points be M and N , and O_1 and O_2 be the centers of the two semicircles where they lie respectively. Then

$$MN \leq MO_1 + O_1O_2 + O_2N.$$

Note that the the right side will always be equal to 3 ($MO_1 = O_2N = 1$ from the radius condition, and $O_1O_2 = 1$ from being a midline of the equilateral triangle), hence MN can be at most 3. Finally, if the four points are collinear (when M and N are defined as the intersection of line O_1O_2 with the two semicircles), then equality will hold. Therefore, the greatest possible distance between M and N is 3.

5. [4] Call a positive integer n *weird* if n does not divide $(n-2)!$. Determine the number of weird numbers between 2 and 100 inclusive.

Proposed by: Yuan Yao

Answer: 26

We claim that all the weird numbers are all the prime numbers and 4. Since no numbers between 1 and $p-2$ divide prime p , $(p-2)!$ will not be divisible by p . We also have $2! = 2$ not being a multiple of 4.

Now we show that all other numbers are not weird. If $n = pq$ where $p \neq q$ and $p, q \geq 2$, then since p and q both appear in $1, 2, \dots, n-2$ and are distinct, we have $pq \mid (n-2)!$. This leaves the only case of $n = p^2$ for prime $p \geq 3$. In this case, we can note that p and $2p$ are both less than $p^2 - 2$, so $2p^2 \mid (n-2)!$ and we are similarly done.

Since there are 25 prime numbers not exceeding 100, there are $25 + 1 = 26$ weird numbers.

6. [4] The pairwise products ab, bc, cd , and da of positive integers a, b, c , and d are 64, 88, 120, and 165 in some order. Find $a + b + c + d$.

Proposed by: Anders Olsen

Answer: 42

The sum $ab + bc + cd + da = (a+c)(b+d) = 437 = 19 \cdot 23$, so $\{a+c, b+d\} = \{19, 23\}$ as having either pair sum to 1 is impossible. Then the sum of all 4 is $19 + 23 = 42$. (In fact, it is not difficult to see that the only possible solutions are $(a, b, c, d) = (8, 8, 11, 15)$ or its cyclic permutations and reflections.)

7. [4] For any real number α , define

$$\text{sign}(\alpha) = \begin{cases} +1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

How many triples $(x, y, z) \in \mathbb{R}^3$ satisfy the following system of equations

$$\begin{aligned} x &= 2018 - 2019 \cdot \text{sign}(y + z), \\ y &= 2018 - 2019 \cdot \text{sign}(z + x), \\ z &= 2018 - 2019 \cdot \text{sign}(x + y)? \end{aligned}$$

Proposed by: Pakawut Jiradilok

Answer: 3

Since $\text{sign}(x+y)$ can take one of 3 values, z can be one of 3 values: 4037, 2018, or -1 . The same is true of x and y . However, this shows that $x+y$ cannot be 0, so z can only be 4037 or -1 . The same is true of x and y . Now note that, if any two of x, y, z are -1 , then the third one must be 4037. Furthermore, if any one of x, y, z is 4037, then the other two must be -1 . Thus, the only possibility is to have exactly two of x, y, z be -1 and the third one be 4037. This means that the only remaining triples are $(-1, -1, 4037)$ and its permutations. These all work, so there are exactly 3 ordered triples.

8. [4] A regular hexagon *PROFIT* has area 1. Every minute, greedy George places the largest possible equilateral triangle that does not overlap with other already-placed triangles in the hexagon, with ties broken arbitrarily. How many triangles would George need to cover at least 90% of the hexagon's area?

Proposed by: Yuan Yao

Answer: 46

It's not difficult to see that the first triangle must connect three non-adjacent vertices (e.g. *POI*), which covers area $\frac{1}{2}$, and leaves three 30-30-120 triangles of area $\frac{1}{6}$ each. Then, the next three triangles cover $\frac{1}{3}$ of the respective small triangle they are in, and leave six 30-30-120 triangles of area $\frac{1}{18}$ each.

This process continues, doubling the number of 30-30-120 triangles each round and the area of each triangle is divided by 3 each round. After $1 + 3 + 6 + 12 + 24 = 46$ triangles, the remaining area is $\frac{3 \cdot 2^4}{6 \cdot 3^4} = \frac{48}{486} = \frac{8}{81} < 0.1$, and the last triangle removed triangle has area $\frac{1}{486}$, so this is the minimum number necessary.

9. [5] Define $P = \{S, T\}$ and let \mathcal{P} be the set of all proper subsets of P . (A *proper subset* is a subset that is not the set itself.) How many ordered pairs $(\mathcal{S}, \mathcal{T})$ of proper subsets of \mathcal{P} are there such that
- \mathcal{S} is not a proper subset of \mathcal{T} and \mathcal{T} is not a proper subset of \mathcal{S} ; and
 - for any sets $S \in \mathcal{S}$ and $T \in \mathcal{T}$, S is not a proper subset of T and T is not a proper subset of S ?

Proposed by: Yuan Yao

Answer: 7

For ease of notation, we let $0 = \emptyset, 1 = \{S\}, 2 = \{T\}$. Then both \mathcal{S} and \mathcal{T} are proper subsets of $\{0, 1, 2\}$. We consider the following cases:

Case 1. If $\mathcal{S} = \emptyset$, then \mathcal{S} is a proper subset of any set except the empty set, so we must have $\mathcal{T} = \emptyset$.

Case 2. If $\mathcal{S} = \{0\}$, then \mathcal{T} cannot be empty, nor can it contain either 1 or 2, so we must have $\mathcal{T} = \{0\}$. This also implies that if \mathcal{S} contains another element, then there would be no choice of \mathcal{T} because $\{0\}$ would be a proper subset.

Case 3. If $\mathcal{S} = \{1\}$, then \mathcal{T} cannot contain 0, and cannot contain both 1 and 2 (or it becomes a proper superset of \mathcal{S}), so it can only be $\{1\}$ or $\{2\}$, and both work. The similar apply when $\mathcal{S} = \{2\}$.

Case 4. If $\mathcal{S} = \{1, 2\}$, then since \mathcal{T} cannot contain 0, it must contain both 1 and 2 (or it becomes a proper subset of \mathcal{S}), so $\mathcal{T} = \{1, 2\}$.

Hence, all the possibilities are

$$(\mathcal{S}, \mathcal{T}) = (\emptyset, \emptyset), (\{0\}, \{0\}), (\{1\}, \{1\}), (\{1\}, \{2\}), (\{2\}, \{1\}), (\{2\}, \{2\}), (\{1, 2\}, \{1, 2\}),$$

for 7 possible pairs in total.

10. [5] Let

$$A = (1 + 2\sqrt{2} + 3\sqrt{3} + 6\sqrt{6})(2 + 6\sqrt{2} + \sqrt{3} + 3\sqrt{6})(3 + \sqrt{2} + 6\sqrt{3} + 2\sqrt{6})(6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}),$$

$$B = (1 + 3\sqrt{2} + 2\sqrt{3} + 6\sqrt{6})(2 + \sqrt{2} + 6\sqrt{3} + 3\sqrt{6})(3 + 6\sqrt{2} + \sqrt{3} + 2\sqrt{6})(6 + 2\sqrt{2} + 3\sqrt{3} + \sqrt{6}).$$

Compute the value of A/B .

Proposed by: Yuan Yao

Answer: 1

Note that

$$A = ((1 + 2\sqrt{2})(1 + 3\sqrt{3}))((2 + \sqrt{3})(1 + 3\sqrt{2}))((3 + \sqrt{2})(1 + 2\sqrt{3}))((3 + \sqrt{3})(2 + \sqrt{2})),$$

$$B = ((1 + 3\sqrt{2})(1 + 2\sqrt{3}))((2 + \sqrt{2})(1 + 3\sqrt{3}))((3 + \sqrt{3})(1 + 2\sqrt{2}))((2 + \sqrt{3})(3 + \sqrt{2})).$$

It is not difficult to check that they have the exact same set of factors, so $A = B$ and thus the ratio is 1.

11. [5] In the Year 0 of Cambridge there is one squirrel and one rabbit. Both animals multiply in numbers quickly. In particular, if there are m squirrels and n rabbits in Year k , then there will be $2m + 2019$ squirrels and $4n - 2$ rabbits in Year $k + 1$. What is the first year in which there will be strictly more rabbits than squirrels?

Proposed by: Yuan Yao

Answer: 13

In year k , the number of squirrels is

$$2(2(\cdots(2 \cdot 1 + 2019) + 2019) + \cdots) + 2019 = 2^k + 2019 \cdot (2^{k-1} + 2^{k-2} + \cdots + 1) = 2020 \cdot 2^k - 2019$$

and the number of rabbits is

$$4(4(\cdots(4 \cdot 1 - 2) - 2) - \cdots) - 2 = 4^k - 2 \cdot (4^{k-1} + 4^{k-2} + \cdots + 1) = \frac{4^k + 2}{3}.$$

For the number of rabbits to exceed that of squirrels, we need

$$4^k + 2 > 6060 \cdot 2^k - 6057 \Leftrightarrow 2^k > 6059.$$

Since $2^{13} > 6059 > 2^{12}$, $k = 13$ is the first year for which there are more rabbits than squirrels.

12. [5] Bob is coloring lattice points in the coordinate plane. Find the number of ways Bob can color five points in $\{(x, y) \mid 1 \leq x, y \leq 5\}$ blue such that the distance between any two blue points is *not* an integer.

Proposed by: Michael Ren

Answer: 80

We can see that no two blue points can have the same x or y coordinate. The blue points then must make a permutation of $1, 2, 3, 4, 5$ that avoid the pattern of 3-4-5 triangles. It is not hard to use complementary counting to get the answer from here.

There are 8 possible pairs of points that are a distance of 5 apart while not being in the same row or column (i.e. a pair that is in the 3-4-5 position). If such a pair of points is included in the choice of five points, then there are $3! = 6$ possibilities for the remaining three points, yielding $8 \times 6 = 48$ configurations that have violations. However, we now need to consider overcounting.

The only way to have more than one violation in one configuration is to have a corner point and then two points adjacent to the opposite corner, e.g. $(1, 1), (4, 5), (5, 4)$. In each such case, there are exactly $2! = 2$ possibilities for the other two points, and there are exactly two violations so there are a total of $2 \times 4 = 8$ configurations that are double-counted.

Therefore, there are $48 - 8 = 40$ permutations that violate the no-integer-condition, leaving $120 - 40 = 80$ good configurations.

13. [7] Reimu has 2019 coins $C_0, C_1, \dots, C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, \dots, 1009$, she puts C_0 and C_i into the machine once, and machine picks C_i . What is the probability that C_0 is fake?

Proposed by: Yuan Yao

Answer: $\frac{2^{1009}}{2^{1009} + 1009}$

Let E denote the event that C_0 is fake, and let F denote the event that the machine picks C_i over C_0 for all $i = 1, 2, \dots, 1009$. By the definition of conditional probability, $P(E|F) = \frac{P(E \cap F)}{P(F)}$. Since E implies F , $P(E \cap F) = P(E) = \frac{1}{2019}$. Now we want to compute $P(F)$. If C_0 is fake, F is guaranteed to happen. If C_i is fake for some $1 \leq i \leq 1009$, then F is impossible. Finally, if C_i is fake for some $1010 \leq i \leq 2018$, then F occurs with probability 2^{-1009} , since there is a $\frac{1}{2}$ probability for each machine decision. Therefore, $P(F) = \frac{1}{2019} \cdot 1 + \frac{1009}{2019} \cdot 0 + \frac{1009}{2019} \cdot 2^{-1009} = \frac{2^{1009} + 1009}{2019 \cdot 2^{1009}}$. Therefore, $P(E|F) = \frac{2^{1009}}{2^{1009} + 1009}$.

14. [7] Let ABC be a triangle where $AB = 9, BC = 10, CA = 17$. Let Ω be its circumcircle, and let A_1, B_1, C_1 be the diametrically opposite points from A, B, C , respectively, on Ω . Find the area of the convex hexagon with the vertices A, B, C, A_1, B_1, C_1 .

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1155}{4}}$

We first compute the circumradius of ABC : Since $\cos A = \frac{9^2 - 17^2 - 10^2}{2 \cdot 9 \cdot 17} = -\frac{15}{17}$, we have $\sin A = \frac{8}{17}$ and $R = \frac{a}{2 \sin A} = \frac{170}{16}$. Moreover, we get that the area of triangle ABC is $\frac{1}{2}bc \sin A = 36$.

Note that triangle ABC is obtuse. The area of the hexagon is equal to twice the area of triangle ABC (which is really $[ABC] + [A_1B_1C_1]$) plus the area of rectangle ACA_1C_1 . The dimensions of ACA_1C_1 are $AC = 17$ and $A_1C = \sqrt{(2R)^2 - AC^2} = \frac{51}{4}$, so the area of the hexagon is $36 \cdot 2 + 17 \cdot \frac{51}{4} = \frac{1155}{4}$.

15. [7] Five people are at a party. Each pair of them are *friends*, *enemies*, or *frenemies* (which is equivalent to being both *friends* and *enemies*). It is known that given any three people A, B, C :

- If A and B are friends and B and C are friends, then A and C are friends;
- If A and B are enemies and B and C are enemies, then A and C are friends;
- If A and B are friends and B and C are enemies, then A and C are enemies.

How many possible relationship configurations are there among the five people?

Proposed by: Yuan Yao

Answer: $\boxed{17}$

If A and B are frenemies, then regardless of whether another person C is friends or enemies with A , C will have to be frenemies with B and vice versa. Therefore, if there is one pair of frenemies then all of them are frenemies with each other, and there is only one possibility.

If there are no frenemies, then one can always separate the five people into two possibly “factions” (one of which may be empty) such that two people are friends if and only if they belong to the same faction. Since the factions are unordered, there are $2^5/2 = 16$ ways to assign the “alignments” that each gives a unique configuration of relations. So in total there are $16 + 1 = 17$ possibilities.

16. [7] Let \mathbb{R} be the set of real numbers. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function such that for all real numbers x and y , we have

$$f(x^2) + f(y^2) = f(x+y)^2 - 2xy.$$

Let $S = \sum_{n=-2019}^{2019} f(n)$. Determine the number of possible values of S .

Proposed by: Yuan Yao

Answer: $\boxed{2039191 \text{ OR } \binom{2020}{2} + 1}$

Letting $y = -x$ gives

$$f(x^2) + f(x^2) = f(0)^2 + 2x^2$$

for all x . When $x = 0$ the equation above gives $f(0) = 0$ or $f(0) = 2$.

If $f(0) = 2$, then $f(x) = x + 2$ for all nonnegative x , so the LHS becomes $x^2 + y^2 + 4$, and RHS becomes $x^2 + y^2 + 4x + 4y + 4$ for all $x + y \geq 0$, which cannot be equal to LHS if $x + y > 0$.

If $f(0) = 0$ then $f(x) = x$ for all nonnegative x . Moreover, letting $y = 0$ gives

$$f(x^2) = f(x)^2 \Rightarrow f(x) = \pm x$$

for all x . Since negative values are never used as inputs on the LHS and the output on the RHS is always squared, we may conclude that for all negative x , $f(x) = x$ and $f(x) = -x$ are both possible (and the values are independent). Therefore, the value of S can be written as

$$S = f(0) + (f(1) + f(-1)) + (f(2) + f(-2)) + \cdots + (f(2019) + f(-2019)) = 2 \sum_{i=1}^{2019} i \delta_i$$

for $\delta_1, \delta_2, \dots, \delta_{2019} \in \{0, 1\}$. It is not difficult to see that $\frac{S}{2}$ can take any integer value between 0 and $\frac{2020 \cdot 2019}{2} = 2039190$ inclusive, so there are 2039191 possible values of S .

17. [9] Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let A_1, A_2 be points on side BC , B_1, B_2 be points on side CA , and C_1, C_2 be points on side AB . Suppose that there exists a point P such that PA_1A_2 , PB_1B_2 , and PC_1C_2 are congruent equilateral triangles. Find the area of convex hexagon $A_1A_2B_1B_2C_1C_2$.

Proposed by: Michael Ren

Answer: $\boxed{\frac{12+22\sqrt{3}}{15}}$

Since P is the shared vertex between the three equilateral triangles, we note that P is the incenter of ABC since it is equidistant to all three sides. Since the area is 6 and the semiperimeter is also 6, we can calculate the inradius, i.e. the altitude, as 1, which in turn implies that the side length of the equilateral triangle is $\frac{2}{\sqrt{3}}$. Furthermore, since the incenter is the intersection of angle bisectors, it is easy to see that $AB_2 = AC_1$, $BC_2 = BA_1$, and $CA_2 = CB_1$. Using the fact that the altitudes from P to AB and CB form a square with the sides, we use the side lengths of the equilateral triangle to compute that $AB_2 = AC_1 = 2 - \frac{1}{\sqrt{3}}$, $BA_1 = BC_2 = 1 - \frac{1}{\sqrt{3}}$, and $CB_1 = CA_2 = 3 - \frac{1}{\sqrt{3}}$. We have that the area of the hexagon is therefore

$$6 - \left(\frac{1}{2} \left(2 - \frac{1}{\sqrt{3}} \right)^2 \cdot \frac{4}{5} + \frac{1}{2} \left(1 - \frac{1}{\sqrt{3}} \right)^2 + \frac{1}{2} \left(3 - \frac{1}{\sqrt{3}} \right)^2 \cdot \frac{3}{5} \right) = \frac{12 + 22\sqrt{3}}{15}.$$

18. [9] 2019 points are chosen independently and uniformly at random on the interval $[0, 1]$. Tairitsu picks 1000 of them randomly and colors them black, leaving the remaining ones white. Hikari then computes the sum of the positions of the leftmost white point and the rightmost black point. What is the probability that this sum is at most 1?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1019}{2019}}$

Note that each point is chosen uniformly and independently from 0 to 1, so we can apply symmetry. Given any coloring, suppose that we flip all the positions of the black points: then the problem becomes computing the probability that the leftmost white point is to the left of the leftmost black point, which is a necessary and sufficient condition for the sum of the original leftmost white point and the original rightmost black point being at most 1. This condition, however, is equivalent to the leftmost point of all 2019 points being white. Since there are 1019 white points and 1000 black points and each point is equally likely to be the leftmost, this happens with probability $\frac{1019}{2019}$.

19. [9] Complex numbers a, b, c form an equilateral triangle with side length 18 in the complex plane. If $|a + b + c| = 36$, find $|bc + ca + ab|$.

Proposed by: Henrik Boecken

Answer: $\boxed{432}$

Using basic properties of vectors, we see that the complex number $d = \frac{a+b+c}{3}$ is the center of the triangle. From the given, $|a + b + c| = 36 \implies |d| = 12$. Then, let $a' = a - d$, $b' = b - d$, and $c' = c - d$. Due to symmetry, $|a' + b' + c'| = 0$ and $|b'c' + c'a' + a'b'| = 0$.

Finally, we compute

$$\begin{aligned} |bc + ca + ab| &= |(b' + d)(c' + d) + (c' + d)(a' + d) + (a' + d)(b' + d)| \\ &= |b'c' + c'a' + a'b' + 2d(a' + b' + c') + 3d^2| \\ &= |3d^2| = 3 \cdot 12^2 = 432. \end{aligned}$$

20. [9] On floor 0 of a weird-looking building, you enter an elevator that only has one button. You press the button twice and end up on floor 1. Thereafter, every time you press the button, you go up by one floor with probability $\frac{X}{Y}$, where X is your current floor, and Y is the total number of times you have pressed the button thus far (not including the current one); otherwise, the elevator does nothing.

Between the third and the 100th press inclusive, what is the expected number of pairs of consecutive presses that both take you up a floor?

Proposed by: Kevin Yang

Answer: $\boxed{\frac{97}{3}}$

By induction, we can determine that after n total button presses, your current floor is uniformly distributed from 1 to $n - 1$: the base case $n = 2$ is trivial to check, and for the $n + 1$ th press, the probability that you are now on floor i is $\frac{1}{n-1}(1 - \frac{i}{n}) + \frac{1}{n-1}(\frac{i-1}{n}) = \frac{1}{n}$ for $i = 1, 2, \dots, n$, finishing the inductive step.

Hence, the probability that the $(n + 1)$ -th and $(n + 2)$ -th press both take you up a floor is

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{i}{n} \cdot \frac{i+1}{n+1} = \frac{\sum_{i=1}^{n-1} i^2 + i}{(n-1)n(n+1)} = \frac{\frac{(n-1)n(2n-1)}{6} + \frac{n(n-1)}{2}}{(n-1)n(n+1)} = \frac{1}{3}.$$

Since there are $100 - 3 = 97$ possible pairs of consecutive presses, the expected value is $\frac{97}{3}$.

21. [12] A regular hexagon $ABCDEF$ has side length 1 and center O . Parabolas P_1, P_2, \dots, P_6 are constructed with common focus O and directrices AB, BC, CD, DE, EF, FA respectively. Let χ be the set of all distinct points on the plane that lie on at least two of the six parabolas. Compute

$$\sum_{X \in \chi} |OX|.$$

(Recall that the focus is the point and the directrix is the line such that the parabola is the locus of points that are equidistant from the focus and the directrix.)

Proposed by: Yuan Yao

Answer: $\boxed{35\sqrt{3}}$

Recall the focus and the directrix are such that the parabola is the locus of points equidistant from the focus and the directrix. We will consider pairs of parabolas and find their points of intersections (we label counterclockwise):

(1): $P_1 \cap P_2$, two parabolas with directrices adjacent edges on the hexagon (sharing vertex A). The intersection inside the hexagon can be found by using similar triangles: by symmetry this X must lie on OA and must have that its distance from AB and FA are equal to $|OX| = x$, which is to say

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{x}{|OA| - x} = \frac{x}{1 - x} \implies x = 2\sqrt{3} - 3.$$

By symmetry also, the second intersection point, outside the hexagon, must lie on OD . Furthermore, X must have that its distance AB and FA are equal to $|OX|$. Then again by similar triangles

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{x}{|OA| + x} = \frac{x}{1 + x} \implies x = 2\sqrt{3} + 3.$$

(2): $P_1 \cap P_3$, two parabolas with directrices edges one apart on the hexagon, say AB and CD . The intersection inside the hexagon is clearly immediately the circumcenter of triangle BOC (equidistance condition), which gives

$$x = \frac{\sqrt{3}}{3}.$$

Again by symmetry the X outside the hexagon must lie on the line through O and the midpoint of EF ; then one can either observe immediately that $x = \sqrt{3}$ or set up

$$\sin 30^\circ = \frac{1}{2} = \frac{x}{x + \sqrt{3}} \implies x = \sqrt{3}$$

where we notice $\sqrt{3}$ is the distance from O to the intersection of AB with the line through O and the midpoint of BC .

(3): $P_1 \cap P_4$, two parabolas with directrices edges opposite on the hexagon, say AB and DE . Clearly the two intersection points are both inside the hexagon and must lie on CF , which gives

$$x = \frac{\sqrt{3}}{2}.$$

These together give that the sum desired is

$$6(2\sqrt{3} - 3) + 6(2\sqrt{3} + 3) + 6\left(\frac{\sqrt{3}}{3}\right) + 6(\sqrt{3}) + 6\left(\frac{\sqrt{3}}{2}\right) = 35\sqrt{3}.$$

22. [12] Determine the number of subsets S of $\{1, 2, \dots, 1000\}$ that satisfy the following conditions:

- S has 19 elements, and
- the sum of the elements in any non-empty subset of S is not divisible by 20.

Proposed by: Alec Sun

Answer: $\boxed{8 \cdot \binom{50}{19}}$

First we prove that each subset must consist of elements that have the same residue mod 20. Let a subset consist of elements a_1, \dots, a_{19} , and consider two lists of partial sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_{19}$$

$$a_2, a_1 + a_2, a_1 + a_2 + a_3, \dots, a_1 + a_2 + \dots + a_{19}.$$

The residues mod 20 of the partial sums in each list must be pairwise distinct, otherwise subtracting the sum with less terms from the sum with more terms yields a subset whose sum of elements is 0 (mod 20). Since the residues must also be nonzero, each list forms a complete nonzero residue class mod 20. Since the latter 18 sums in the two lists are identical, $a_1 \equiv a_2 \pmod{20}$. By symmetric arguments, $a_i \equiv a_j \pmod{20}$ for any i, j .

Furthermore this residue $1 \leq r \leq 20$ must be relatively prime to 20, because if $d = \gcd(r, 20) > 1$ then any $20/d$ elements of the subset will sum to a multiple of 20. Hence there are $\varphi(20) = 8$ possible residues. Since there are 50 elements in each residue class, the answer is $\binom{50}{19}$. We can see that any such subset whose elements are a relatively prime residue $r \pmod{20}$ works because the sum of any $1 \leq k \leq 19$ elements will be $kr \not\equiv 0 \pmod{20}$.

23. [12] Find the smallest positive integer n such that

$$\underbrace{2^{2^{2^{\dots^2}}}}_{n \text{ 2's}} > \underbrace{(((\dots((100!)!) \dots)!)!)^2}_{100 \text{ factorials}}.$$

Proposed by: Zack Chroman

Answer: $\boxed{104}$

Note that $2^{2^{2^2}} > 100^2$. We claim that $a > b^2 \implies 2^a > (b!)^2$, for $b > 2$. This is because

$$2^a > b^{2b} \iff a > 2b \log_2(b),$$

and $\log_2(b) < b^2/2$ for $b > 2$. Then since $b^b > b!$ this bound works. Then

$$\underbrace{(2^{2^{2^{\dots^2}}})^2}_{m \text{ 2's}} > \underbrace{(((\dots((100!)!) \dots)!)!)^2}_{m-4 \text{ factorials}}$$

for all $m \geq 4$ by induction. So $n = 104$ works. The lower bound follows from the fact that $n! > 2^n$ for $n > 3$, and since $100 > 2^{2^2}$, we have

$$\underbrace{(((100!)!)!) \dots}_{100 \text{ factorials}} > \underbrace{2^{2^{\dots 2^{100}}}}_{100 \text{ 2's}} > \underbrace{2^{2^{\dots 2}}}_{103 \text{ 2's}}.$$

24. [12] Let S be the set of all positive factors of 6000. What is the probability of a random quadruple $(a, b, c, d) \in S^4$ satisfies

$$\text{lcm}(\text{gcd}(a, b), \text{gcd}(c, d)) = \text{gcd}(\text{lcm}(a, b), \text{lcm}(c, d))?$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{41}{512}}$

For each prime factor, let the greatest power that divides a, b, c, d be p, q, r, s . WLOG assume that $p \leq q$ and $r \leq s$, and further WLOG assume that $p \leq r$. Then we need $r = \min(q, s)$. If $q = r$ then we have $p \leq q = r \leq s$, and if $r = s$ then we have $p \leq r = s \leq q$, and in either case the condition reduces to the two “medians” among p, q, r, s are equal. (It is not difficult to see that this condition is also sufficient.)

Now we compute the number of quadruples (p, q, r, s) of integers between 0 and n inclusive that satisfy the above condition. If there are three distinct numbers then there are $\binom{n+1}{3}$ ways to choose the three numbers and $4!/2 = 12$ ways to assign them (it must be a 1-2-1 split). If there are two distinct numbers then there are $\binom{n+1}{2}$ ways to choose the numbers and $4 + 4 = 8$ ways to assign them (it must be a 3-1 or a 1-3 split). If there is one distinct number then there are $n + 1$ ways to assign. Together we have

$$12 \binom{n+1}{3} + 8 \binom{n+1}{2} + (n+1) = 2(n+1)n(n-1) + 4(n+1)n + (n+1) = (n+1)(2n(n+1) + 1)$$

possible quadruples. So if we choose a random quadruple then the probability that it satisfies the condition is $\frac{(n+1)(2n(n+1)+1)}{(n+1)^4} = \frac{2n(n+1)+1}{(n+1)^3}$.

Since $6000 = 2^4 \cdot 5^3 \cdot 3^1$ and the power of different primes are independent, we plug in $n = 4, 3, 1$ to get the overall probability to be

$$\frac{41}{125} \cdot \frac{25}{64} \cdot \frac{5}{8} = \frac{41}{512}.$$

25. [15] A 5 by 5 grid of unit squares is partitioned into 5 pairwise incongruent rectangles with sides lying on the gridlines. Find the maximum possible value of the product of their areas.

Proposed by: Yuan Yao

Answer: $\boxed{2304}$

The greatest possible value for the product is $3 \cdot 4 \cdot 4 \cdot 6 \cdot 8 = 2304$, achieved when the rectangles are $3 \times 1, 1 \times 4, 2 \times 2, 2 \times 3, 4 \times 2$. To see that this is possible, orient these rectangles so that the first number is the horizontal dimension and the second number is the vertical dimension. Then, place the bottom-left corners of these rectangles at $(2, 4), (4, 0), (2, 2), (0, 2), (0, 0)$ respectively on the grid.

We will now prove that no larger product can be achieved. Suppose that there is at least one rectangle of area at most 2. Then the product is at most $2 \cdot 5 \cdot 75^4 = 2 \cdot 33.0625^2 < 2 \cdot 1100 = 2200$ by AM-GM. Now suppose that there is at least one rectangle of area at least 9. Then the product is at most $9 \cdot 4^4 = 2304$ by AM-GM. (Neither of these is tight, since you cannot have non-integer areas, nor can you have four rectangles all of area 4.)

Now consider the last possibility that is not covered by any of the above: that there are no rectangles of size at most 2 and no rectangles of area at least 9. There can be at most one rectangle of area 3, 5, 6, 8 each, at most two rectangles of area 4, and no rectangles of area 7. The only way to achieve a sum of 25 with these constraints is 3, 4, 4, 6, 8, which produces a product of 2304. We have shown through the earlier cases that a larger product cannot be achieved, so this is indeed the maximum.

26. [15] Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let I_A, I_B, I_C be the A, B, C excenters of this triangle, and let O be the circumcenter of the triangle. Let $\gamma_A, \gamma_B, \gamma_C$ be the corresponding excircles and ω be the circumcircle. X is one of the intersections between γ_A and ω . Likewise, Y is an intersection of γ_B and ω , and Z is an intersection of γ_C and ω . Compute

$$\cos \angle OXI_A + \cos \angle OYI_B + \cos \angle OZI_C.$$

Proposed by: Andrew Gu

Answer: $\boxed{-\frac{49}{65}}$

Let r_A, r_B, r_C be the exradii. Using $OX = R, XI_A = r_A, OI_A = \sqrt{R(R + 2r_A)}$ (Euler's theorem for excircles), and the Law of Cosines, we obtain

$$\cos \angle OXI_A = \frac{R^2 + r_A^2 - R(R + 2r_A)}{2Rr_A} = \frac{r_A}{2R} - 1.$$

Therefore it suffices to compute $\frac{r_A + r_B + r_C}{2R} - 3$. Since

$$r_A + r_B + r_C - r = 2K \left(\frac{1}{-a + b + c} + \frac{1}{a - b + c} + \frac{1}{a + b - c} - \frac{1}{a + b + c} \right) = 2K \frac{8abc}{(4K)^2} = \frac{abc}{K} = 4R$$

where $K = [ABC]$, this desired quantity the same as $\frac{r}{2R} - 1$. For this triangle, $r = 4$ and $R = \frac{65}{8}$, so the answer is $\frac{4}{65/4} - 1 = -\frac{49}{65}$.

27. [15] Consider the eighth-sphere $\{(x, y, z) \mid x, y, z \geq 0, x^2 + y^2 + z^2 = 1\}$. What is the area of its projection onto the plane $x + y + z = 1$?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{\pi\sqrt{3}}{4}}$

Consider the three flat faces of the eighth-ball. Each of these is a quarter-circle of radius 1, so each has area $\frac{\pi}{4}$. Furthermore, the projections of these faces cover the desired area without overlap. To find the projection factor one can find the cosine of the angle θ between the planes, which is the same as the angle between their normal vectors. Using the dot product formula for the cosine of the angle between two vectors, $\cos \theta = \frac{(1,0,0) \cdot (1,1,1)}{|(1,0,0)| |(1,1,1)|} = \frac{1}{\sqrt{3}}$. Therefore, each area is multiplied by $\frac{1}{\sqrt{3}}$ by the projection, so the area of the projection is $3 \cdot \frac{\pi}{4} \cdot \frac{1}{\sqrt{3}} = \frac{\pi\sqrt{3}}{4}$.

28. [15] How many positive integers $2 \leq a \leq 101$ have the property that there exists a positive integer N for which the last two digits in the decimal representation of a^{2^n} is the same for all $n \geq N$?

Proposed by: Pakawut Jiradilok

Answer: $\boxed{36}$

Solution 1. It suffices to consider the remainder mod 100. We start with the four numbers that have the same last two digits when squared: 0, 1, 25, 76.

We can now go backwards, repeatedly solving equations of the form $x^2 \equiv n \pmod{100}$ where n is a number that already satisfies the condition.

0 and 25 together gives all multiples of 5, for 20 numbers in total.

1 gives 1, 49, 51, 99, and 49 then gives 7, 43, 57, 93. Similarly 76 gives 24, 26, 74, 76, and 24 then gives 18, 32, 68, 82, for 16 numbers in total.

Hence there are $20 + 16 = 36$ such numbers in total.

Solution 2. An equivalent formulation of the problem is to ask for how many elements of \mathbb{Z}_{100} the map $x \mapsto x^2$ reaches a fixed point. We may separately solve this modulo 4 and modulo 25.

Modulo 4, it is easy to see that all four elements work.

Modulo 25, all multiples of 5 will work, of which there are 5. For the remaining 25 elements that are coprime to 5, we may use the existence of a primitive root to equivalently ask for how many elements of \mathbb{Z}_{20} the map $y \mapsto 2y$ reaches a fixed point. The only fixed point is 0, so the only valid choices are the multiples of 5 again. There are $5 + 4 = 9$ solutions here.

Finally, the number of solutions modulo 100 is $4 \times 9 = 36$.

29. [20] Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n . He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N , but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{2 \ln 2 - 1}}$

Let S denote the value drawn from the hat. The probability that 2 is picked is $\frac{1}{n}$ if $n \geq 2$ and 0 if $n = 1$. Thus, the total probability X that 2 is picked is

$$P(S = 2) = \sum_{k=2}^{\infty} \frac{2^{-k}}{k}.$$

By the definition of conditional probability, $P(N = n | S = 2) = \frac{P(N=n, S=2)}{P(S=2)} = \frac{2^{-n}/n}{X}$ if $n \geq 2$ and 0 if $n = 1$. Thus the conditional expectation of N is

$$\mathbb{E}[N | S = 2] = \sum_{n=1}^{\infty} n \cdot P(N = n | S = 2) = \sum_{n=2}^{\infty} n \cdot \frac{2^{-n}/n}{X} = \frac{1}{X} \sum_{n=2}^{\infty} 2^{-n} = \frac{1}{2X}.$$

It remains to compute X . Note that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$. Integrating both sides with respect to x yields

$$\sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x) + C$$

for some constant C , and plugging in $x = 0$ shows that $C = 0$. Plugging in $x = \frac{1}{2}$ shows that $\sum_{k=1}^{\infty} \frac{2^{-k}}{k} = \ln 2$. Note that X is exactly this summation but without the first term. Thus, $X = \ln 2 - \frac{1}{2}$, so $\frac{1}{2X} = \frac{1}{2 \ln 2 - 1}$.

30. [20] Three points are chosen inside a unit cube uniformly and independently at random. What is the probability that there exists a cube with side length $\frac{1}{2}$ and edges parallel to those of the unit cube that contains all three points?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{8}}$

Let the unit cube be placed on a xyz -coordinate system, with edges parallel to the x, y, z axes. Suppose the three points are labeled A, B, C . If there exists a cube with side length $\frac{1}{2}$ and edges parallel to the edges of the unit cube that contain all three points, then there must exist a segment of length $\frac{1}{2}$ that contains all three projections of A, B, C onto the x -axis. The same is true for the y - and z - axes. Likewise, if there exists segments of length $\frac{1}{2}$ that contains each of the projections of A, B, C onto the x, y , and z axes, then there must exist a unit cube of side length $\frac{1}{2}$ that contains A, B, C . It is easy to see that the projection of a point onto the x -axis is uniform across a segment of length 1, and that each of the dimensions are independent. The problem is therefore equivalent to finding the cube of the probability that a segment of length $\frac{1}{2}$ can cover three points chosen randomly on a segment of length 1.

Note that selecting three numbers $p < q < r$ uniformly and independently at random from 0 to 1 splits the number line into four intervals. That is, we can equivalently sample four positive numbers a, b, c, d

uniformly satisfying $a+b+c+d=1$ (here, we set $a=p, b=q-p, c=r-q, d=1-r$). The probability that the points p, q, r all lie on a segment of length $\frac{1}{2}$ is the probability that $r-q \leq \frac{1}{2}$, or $b+c \leq \frac{1}{2}$. Since $a+d$ and $b+c$ are symmetric, we have that this probability is $\frac{1}{2}$ and our final answer is $(\frac{1}{2})^3 = \frac{1}{8}$.

31. [20] Let ABC be a triangle with $AB=6, AC=7, BC=8$. Let I be the incenter of ABC . Points Z and Y lie on the interior of segments AB and AC respectively such that YZ is tangent to the incircle. Given point P such that

$$\angle ZPC = \angle YPB = 90^\circ,$$

find the length of IP .

Proposed by: Zack Chroman

Answer: $\boxed{\frac{\sqrt{30}}{2}}$

Solution 1. Let PU, PV tangent from P to the incircle. We will invoke the *dual of the Desargues Involution Theorem*, which states the following:

Given a point P in the plane and four lines $\ell_1, \ell_2, \ell_3, \ell_4$, consider the set of conics tangent to all four lines. Then we define a function on the pencil of lines through P by mapping one tangent from P to each conic to the other. This map is well defined and is a projective involution, and in particular maps $PA \rightarrow PD, PB \rightarrow PE, PC \rightarrow PF$, where $ABCDEF$ is the complete quadrilateral given by the pairwise intersections of $\ell_1, \ell_2, \ell_3, \ell_4$. ■

An overview of the projective background behind the (Dual) Desargues Involution Theorem can be found here: <https://www.scribd.com/document/384321704/Desargues-Involution-Theorem>, and a proof can be found at <https://www2.washjeff.edu/users/mwoltermann/Dorrie/63.pdf>.

Now, we apply this to the point P and the lines AB, AC, BC, YZ , to get that the pairs

$$(PU, PV), (PY, PB), (PZ, PC)$$

are swapped by some involution. But we know that the involution on lines through P which rotates by 90° swaps the latter two pairs, thus it must also swap the first one and $\angle UPV = 90$. It follows by equal tangents that $IUPV$ is a square, thus $IP = r\sqrt{2}$ where r is the inradius of ABC . Since $r = \frac{2K}{a+b+c} = \frac{21\sqrt{15}/2}{21} = \frac{\sqrt{15}}{2}$, we have $IP = \frac{\sqrt{30}}{2}$.

Solution 2. Let H be the orthocenter of ABC .

Lemma. $HI^2 = 2r^2 - 4R^2 \cos(A) \cos(B) \cos(C)$, where r is the inradius and R is the circumradius.

Proof. This follows from barycentric coordinates or the general result that for a point X in the plane,

$$aXA^2 + bXB^2 + cXC^2 = (a+b+c)XI^2 + aAI^2 + bBI^2 + cCI^2,$$

which itself is a fact about vectors that follows from barycentric coordinates. This can also be computed directly using trigonometry.

Let $E = BH \cap AC, F = CH \cap AB$, then note that B, P, E, Y are concyclic on the circle of diameter BY , and C, P, F, Z are concyclic on the circle of diameter CZ . Let Q be the second intersection of these circles. Since $BCYZ$ is a tangential quadrilateral, the midpoints of BY and CZ are collinear with I (this is known as *Newton's theorem*), which implies that $IP = IQ$ by symmetry. Note that as $BH \cdot HE = CH \cdot HF$, H lies on the radical axis of the two circles, which is PQ . Thus, if $IP = IQ = x$, $BH \cdot HE$ is the power of H with respect to the circle centered at I with radius x , which implies $BH \cdot HE = x^2 - HI^2$.

As with the first solution, we claim that $x = r\sqrt{2}$, which by the lemma is equivalent to $BH \cdot HE = 4R^2 \cos(A) \cos(B) \cos(C)$. Then note that

$$BH \cdot HE = BH \cdot CH \cos(A) = (2R \cos(B))(2R \cos(C)) \cos(A),$$

so our claim holds and we finish as with the first solution.

Note. Under the assumption that the problem is well-posed (the answer does not depend on the choice of Y, Z , or P), then here is an alternative method to obtain $IP = r\sqrt{2}$ by making convenient choices. Let U be the point where YZ is tangent to the incircle, and choose U so that $IU \parallel BC$ (and therefore $YZ \perp BC$). Note that $YZ \cap BC$ is a valid choice for P , so assume that P is the foot from U to BC . If D is the point where BC is tangent to the incircle, then $IUPD$ is a square so $IP = r\sqrt{2}$. (This disregards the condition that Y and Z are in the interior of segments AC and AB , but there is no reason to expect that this condition is important.)

32. [20] For positive integers a and b such that a is coprime to b , define $\text{ord}_b(a)$ as the least positive integer k such that $b \mid a^k - 1$, and define $\varphi(a)$ to be the number of positive integers less than or equal to a which are coprime to a . Find the least positive integer n such that

$$\text{ord}_n(m) < \frac{\varphi(n)}{10}$$

for all positive integers m coprime to n .

Proposed by: Andrew Gu

Answer: 240

The maximum order of an element modulo n is the Carmichael function, denoted $\lambda(n)$. The following properties of the Carmichael function are established:

- For primes $p > 2$ and positive integers k , $\lambda(p^k) = (p-1)p^{k-1}$.
- For a positive integer k ,

$$\lambda(2^k) = \begin{cases} 2^{k-2} & \text{if } k \geq 3 \\ 2^{k-1} & \text{if } k \leq 2 \end{cases}.$$

- For a positive integer n with prime factorization $n = \prod p_i^{k_i}$,

$$\lambda(n) = \text{lcm} \left(\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots \right)$$

Meanwhile, for $n = \prod p_i^{k_i}$, we have $\varphi(n) = \prod (p_i - 1)p_i^{k_i-1}$. Hence the intuition is roughly that the $(p_i - 1)p_i^{k_i-1}$ terms must share divisors in order to reach a high value of $\frac{\varphi(n)}{\lambda(n)}$.

We will now show that $n \geq 240$ by doing casework on the prime divisors of $z = \frac{\varphi(n)}{\lambda(n)}$. Suppose $p \mid z$ and $p > 2$. This requires two terms among $\lambda(p_1^{k_1}), \lambda(p_2^{k_2}), \dots$ to be multiples of p because $\lambda(n)$ is the lcm of the terms whereas the product of these numbers has the same number of factors of p as $\varphi(n)$ (note that this does not hold for $p = 2$ because $\lambda(2^k) \neq 2^{k-1}$ in general). These correspond to either $p^2 \mid n$ or $q \mid n$ with $q \equiv 1 \pmod{p}$. Therefore

$$n \geq \max(p^2(2p+1), (2p+1)(4p+1))$$

because the smallest primes congruent to 1 (mod p) are at least $2p+1$ and $4p+1$. For $p \geq 5$ this gives $n > 240$, so we may assume $p \leq 3$.

First we address the case $p = 3$. This means that two numbers among $9, 7, 13, 19, 31, 37, \dots$ divide n . As $7 \times 37 > 240$, we discard primes greater than 31. Of the remaining numbers, we have

$$\lambda(9) = 6, \lambda(7) = 6, \lambda(13) = 12, \lambda(19) = 18, \lambda(31) = 30.$$

No candidate value of n is the product of just two of these numbers as the gcd of any two of the associated λ values is at most 6. Furthermore, multiplying by just 2 will not affect $\varphi(n)$ or $\lambda(n)$, so we must multiply at least two of these numbers by a number greater than 2. Throwing out numbers greater than 240, this leaves only $3 \times 9 \times 7$, which does not work. (A close candidate is $3 \times 7 \times 13 = 273$, for which $\varphi(n) = 144, \lambda(n) = 12$.)

The remaining case is when the only prime divisors of $\frac{\varphi(n)}{\lambda(n)}$ are 2. It is not hard to see that $\lambda(n) \geq 4$ when $n \nmid 24$ (and when $n \mid 24$ it's clear that $\phi(n) \leq 8$, so we do not need to consider them). When $\lambda(n) = 4$, we need $\varphi(n) \geq 4 \cdot 2^4 = 64$ and $v_2(n) \leq 4$, so the smallest such integer is $n = 2^4 \cdot 3 \cdot 5 = 240$, which we can check does indeed satisfy $\frac{\varphi(n)}{\lambda(n)} > 10$. It is not difficult to check that higher values of $\lambda(n)$ will not yield any n below 240, so 240 is indeed the smallest possible n .

Note: The sequence $\frac{\varphi(n)}{\lambda(n)}$ is given by A034380 in the OEIS.

H_1 . Let $r = H_1$ be the answer to this problem. Given that r is a nonzero real number, what is the value of $r^4 + 4r^3 + 6r^2 + 4r$?

Answer: -1

Since H_1 is the answer, we know $r^4 + 4r^3 + 6r^2 + 4r = r \Rightarrow (r + 1)^4 = r + 1$. Either $r + 1 = 0$, or $(r + 1)^3 = 1 \Rightarrow r = 0$. Since r is nonzero, $r = -1$.

H_2 . Given two distinct points A, B and line ℓ that is not perpendicular to AB , what is the maximum possible number of points P on ℓ such that ABP is an isosceles triangle?

Answer: 5

In an isosceles triangle, one vertex lies on the perpendicular bisector of the opposite side. Thus, either P is the intersection of AB and ℓ , or P lies on the circle centered at A with radius AB , or P lies on the circle centered at B with radius AB . Each circle-line intersection has at most two solutions, and the line-line intersection has at most one, giving 5. This can be easily constructed by taking any \overline{AB} , and taking ℓ that isn't a diameter but intersects both relevant circles twice.

H_3 . Let $A = H_1, B = H_6 + 1$. A real number x is chosen randomly and uniformly in the interval $[A, B]$. Find the probability that $x^2 > x^3 > x$.

Answer: $\frac{1}{4}$

$A = -1, B = 3$. For $x^3 > x$, either $x > 1$ or $-1 < x < 0$. However, for $x > 1$, $x^2 < x^3$, so there are no solutions. $-1 < x < 0$ also satisfies $x^2 > x^3$, so our answer is $1/4$.

H_4 . Let $A = \lceil 1/H_3 \rceil, B = \lceil H_5/2 \rceil$. How many ways are there to partition the set $\{1, 2, \dots, A + B\}$ into two sets U and V with size A and B respectively such that the probability that a number chosen from U uniformly at random is greater than a number chosen from V uniformly at random is exactly $\frac{1}{2}$?

Answer: 24

$A = 4, B = 7$. There are 28 total ways of choosing an element from U and V , so there must be 14 ways where U 's is larger. If we relabel the elements to be $0, 1, \dots, 10$, then element i is greater than exactly i elements in the set. However, we overcount other elements in U , so the four elements in $U = \{a, b, c, d\}$ must satisfy

$$(a - 0) + (b - 1) + (c - 2) + (d - 3) = 14 \Rightarrow a + b + c + d = 20.$$

To remove the uniqueness condition, we subtract 1 from b , 2 from c , and 3 from d , so we wish to find solutions $a \leq b \leq c \leq d \leq 7$ to $a + b + c + d = 14$. From here, we do casework. If $a = 0$, $b = 0, 1, 2, 3, 4$ give 1, 1, 2, 2, 3 solutions, respectively. If $a = 1$, $b = 1, 2, 3, 4$ give 2, 2, 3, 1 solutions, respectively. If $a = 2$, $b = 2, 3, 4$ give 3, 2, 1 solutions, respectively. If $a = 3$, the only solution is 3, 3, 4, 4. Thus, the answer is $(1 + 1 + 2 + 2 + 3) + (2 + 2 + 3 + 1) + (3 + 2 + 1) + 1 = 24$.

H_5 . Let $A = H_2, B = H_7$. Two circles with radii A and B respectively are given in the plane. If the length of their common external tangent is twice the length of their common internal tangent (where both tangents are considered as segments with endpoints being the points of tangency), find the distance between the two centers.

Answer: $\frac{2\sqrt{429}}{3}$

Let the distance between the centers be d . The length of the common external tangent is $E = \sqrt{d^2 - (7 - 5)^2} = \sqrt{d^2 - 4}$, and the length of the internal tangent is $I = \frac{12}{5} \sqrt{\left(\frac{5}{12}d\right)^2 - 5^2}$. Solving the equation $E = 2I$ gives $d = \frac{2\sqrt{429}}{3} (\approx 13.8)$.

H_6 . How many ways are there to arrange the numbers 21, 22, 33, 35 in a row such that any two adjacent numbers are relatively prime?

Answer:

21 cannot be adjacent to 33 or 35, so it must be on one end bordering 22. 33 cannot be adjacent to 21 or 22, so it must be on the other end bordering 35. Thus, there are only 2 orderings: 21, 22, 35, 33, and 33, 35, 22, 21.

H_7 . How many pairs of integers (x, y) are there such that $|x^2 - 2y^2| \leq 1$ and $|3x - 4y| \leq 1$?

Answer:

Note that if (x, y) is a solution, so is $(-x, -y)$. Thus, we consider $x \geq 0$.

When $x \equiv 0 \pmod{4}$, $y = 3x/4$ by inequality 2. Inequality 1 gives $|x^2/9| \leq 1$, so $x \leq 3$, so $x = 0$.

When $x \equiv 1 \pmod{4}$, $y = (3x + 1)/4$ by inequality 2. Beyond $x = 1$, $2y^2 - x^2 > 1$, so there are no more solutions.

When $x \equiv 2 \pmod{4}$, there are no solutions for y .

When $x \equiv 3 \pmod{4}$, $y = (3x - 1)/4$ by inequality 2. Beyond $x = 7$, $2y^2 - x^2 > 1$, so there are no more solutions.

Thus, the solutions are $(0, 0)$, $(1, 1)$, $(3, 2)$, $(7, 5)$, and the negations of the latter three, giving 7 solutions.

M_1 . Let $S = M_{10}$. Determine the number of ordered triples (a, b, c) of nonnegative integers such that $a + 2b + 4c = S$.

Answer:

$S = 53$. Firstly, the number of solutions is the same as the number of solutions to $a + 2b + 4c = 52$, since $2b, 4c$ are both even. Then, $a + 2b = 2x$ has $x + 1$ solutions in nonnegative integers, so we wish to find $27 + 25 + \dots + 1$. This is the sum of the first 14 odd numbers, which is $14^2 = 196$.

M_2 . Let $S = \lfloor M_5 \rfloor$. Two integers m and n are chosen between 1 and S inclusive uniformly and independently at random. What is the probability that $m^n = n^m$?

Answer:

$S = 12$. The solutions are (x, x) for all x , and $(2, 4), (4, 2)$. Thus, there are $S + 2 = 14$ solutions out of $S^2 = 196$ possibilities, so the answer is $14/196 = 7/98$.

M_3 . Let $S = \lceil M_7 \rceil$. In right triangle ABC , $\angle C = 90^\circ$, $AC = 27, BC = 36$. A circle with radius S is tangent to both AC and BC and intersects AB at X and Y . Find the length of XY .

Answer:

$S = 14$. We first note that the distance from the center of the circle to AB (which has length 45 by Pythagorean theorem) is $\frac{27 \cdot 36 - 27 \cdot 14 - 36 \cdot 14}{45} = 2$, so the length of the chord XY is equal to $2\sqrt{14^2 - 2^2} = 16\sqrt{3}$.

M_4 . Let $S = M_{13} + 5$. Compute the product of all positive divisors of S .

Answer:

$S = 30 = 2 \cdot 3 \cdot 5$. The divisors of S are 1, 2, 3, 5, 6, 10, 15, 30. Each prime factor appears 4 times, so the product is $2^4 3^4 5^4 = 30^4 = 810000$.

M_5 . Let $A = \sqrt{M_1}, B = \lceil M_{11} \rceil$. Given complex numbers x and y such that $x + \frac{1}{y} = A, \frac{1}{x} + y = B$, compute the value of $xy + \frac{1}{xy}$.

Answer: 12

$A = 14, B = 1$. Multiplying the two given equations gives $xy + 1/(xy) + 2 = 14$, so the answer is $14 - 2 = 12$.

M_6 . Let $A = \lfloor 1/M_2 \rfloor, B = \lfloor M_3^2/100 \rfloor$. Let P and Q both be quadratic polynomials. Given that the real roots of $P(Q(x)) = 0$ are $0, A, B, C$ in some order, find the sum of all possible values of C .

Answer: 17

$A = 10, B = 7$. Let the roots of $P(x) = 0$ be p, q . Then, the roots of $P(Q(x)) = 0$ are when $Q(x) = p$ or $Q(x) = q$. If these are r, s, t, u in order, then note that $(r + s)/2 = (t + u)/2$ must both be the center of the parabola Q . Thus, $0, 10, 7, C$ must divide into two pairs with equal sum. When we consider the three possible groupings, we get that the possible values are $10+7-0 = 17, 10+0-7 = 3, 7+0-10 = -3$. Therefore the sum is $17 + (-3) + 3 = 17$.

M_7 . Let $A = \lceil \log_2 M_4 \rceil, B = M_{12} + 1$. A 5-term sequence of positive reals satisfy that the first three terms and the last three terms both form an arithmetic sequence and the middle three terms form a geometric sequence. If the first term is A and the fifth term is B , determine the third term of the sequence.

Answer: $\frac{40}{3}$

$A = 20, B = 8$. If the middle term is x , then $(x + 20)/2, x, (x + 8)/2$ forms a geometric series. This means that $(x + 20)/2 \cdot (x + 8)/2 = x^2$, which upon solving gives $x = 40/3$ or $x = -4$ (which we discard because $x > 0$).

M_8 . Let $A = \lfloor M_5^2 \rfloor, B = \lfloor M_6^2 \rfloor$. A regular A -gon, a regular B -gon, and a circle are given in the plane. What is the greatest possible number of regions that these shapes divide the plane into?

Answer: 1156

$A = 144, B = 289$. First, note that with only the circle, there are 2 regions. If the three shapes never coincide at a point, then each intersection adds precisely one region. Optimistically, we wish to have the maximal number of intersections where all intersections have both shapes. The maximum number of intersections between the 289-gon and the circle is 578, since each side can only intersect the circle twice. Similarly, the 144-gon and the circle add at most 288. Finally, each side of the 144-gon can only intersect the 289-gon twice, so this adds another 288. This maximum can be achieved when all three shapes have the same circumcenter and circumradius, and are rotated slightly. The answer is $2 + 578 + 288 + 288 = 1156$.

M_9 . Let A and B be the unit digits of $\lceil 7M_6 \rceil$ and $\lfloor 6M_7 \rfloor$ respectively. When all the positive integers not containing digit A or B are written in increasing order, what is the 2019th number in the list?

Answer: 3743

$A = 9, B = 0$. First, there are 8 numbers with 1 digit, 64 with two digits, and 512 with three digits. This leaves $2019 - 512 - 64 - 8 = 1435$ of four-digit numbers we have to go through, starting with 1111. Since we don't have two digits 0 and 9, we are basically counting in base 8, where the digits 01234567 in base 8 are actually 12345678, and $1111_{10} = 0000_8$. Converting 1435 to base 8, we get 2633_8 , and mapping this back, we get 3744. However, we must remember that 1111 is 0_8 under our mapping, so the 1435th four-digit number is actually 3743.

M_{10} . What is the smallest positive integer with remainder 2, 3, 4 when divided by 3, 5, 7 respectively?

Answer: 53

We note that if we double the number then it leaves a remainder of 1 when divided by all of 3, 5, and 7. The smallest even number satisfying this is $3 \cdot 5 \cdot 7 + 1 = 106$, so the smallest possible number is $106/2 = 53$.

M_{11} . An equiangular hexagon has side lengths 1, 2, 3, 4, 5, 6 in some order. Find the nonnegative difference between the largest and the smallest possible area of this hexagon.

Answer: $\boxed{\frac{\sqrt{3}}{2}}$

Extending three sides of the equiangular hexagon gives an equilateral triangle. Thus, if the sides are a, b, c, d, e, f , in order, then $a + b + c = a + f + e \Rightarrow b + c = e + f \Rightarrow f - c = b - e$. By a symmetric argument, we see that $d - a = f - c = b - e$ holds, which means that they must be separated into three groups of two with equal differences. If the grouping is (1, 2), (3, 4), (5, 6), then we have 1, 4, 5, 2, 3, 6 around the hexagon. If the grouping is (1, 4), (2, 5), (3, 6), then we get 1, 5, 3, 4, 2, 6 as the other possibility. Finally, we can use our equilateral triangle trick to find the areas. For the first, we get a big triangle of side $1 + 4 + 5 = 10$, and must subtract smaller triangles of sides 1, 5, 3. This gives $(100 - 1 - 25 - 9)\sqrt{3}/4 = 65\sqrt{3}/4$. For the other, we get $(81 - 1 - 9 - 4)\sqrt{3}/4 = 67\sqrt{3}/4$. The positive difference between these is $\sqrt{3}/2$.

M_{12} . Determine the second smallest positive integer n such that $n^3 + n^2 + n + 1$ is a perfect square.

Answer: $\boxed{7}$

$n^3 + n^2 + n + 1 = (n + 1)(n^2 + 1)$. Note that $\gcd(n^2 + 1, n + 1) = \gcd(2, n + 1) = 1$ or 2 , and since $n^2 + 1$ is not a perfect square for $n \geq 1$, we must have $n^2 + 1 = 2p^2$ and $n + 1 = 2q^2$ for some integers p and q . The first equation is a variant of Pell's equation, which (either by brute-forcing small cases or using the known recurrence) gives solutions $(n, p) = (1, 1), (7, 5), \dots$. Incidentally, both smallest solutions $n = 1$ and $n = 7$ allows an integer solution to the second equation, so $n = 7$ is the second smallest integer that satisfy the condition.

M_{13} . Given that A, B are nonzero base-10 digits such that $A \cdot \overline{AB} + B = \overline{BB}$, find \overline{AB} .

Answer: $\boxed{25}$

We know $A \cdot \overline{AB}$ ends in 0. Since neither is 0, they must be 2, 5 in some order. We easily find that $A = 2, B = 5$ works while the opposite doesn't, so $\overline{AB} = 25$.

T_1 . Let S, P, A, C, E be (not necessarily distinct) decimal digits where $E \neq 0$. Given that $N = \sqrt{\overline{ESCAPE}}$ is a positive integer, find the minimum possible value of N .

Answer: $\boxed{319}$

Since $E \neq 0$, the 6-digit number \overline{ESCAPE} is at least 10^5 , so $N \geq 317$. If N were 317 or 318, the last digit of N^2 would not match the first digit of N^2 , which contradicts the condition. However, $N = 319$ will work, since the first and last digit of N^2 are both 1.

T_2 . Let $X = \lfloor T_1/8 \rfloor, Y = T_3 - 1, Z = T_4 - 2$. A point P lies inside the triangle ABC such that $PA = X, PB = Y, PC = Z$. Find the largest possible area of the triangle.

Answer: $\boxed{1344}$

$X = 39, Y = 33, Z = 25$. Fix some position for P, A , and B , and we shall find the optimal position for C . Letting \overline{AB} be the base of the triangle, we wish to maximize the height. The legal positions for C are a subset of the circle with center P and radius PC , so the height is maximized when \overline{PC} is orthogonal to \overline{AB} . Symmetrically, we deduce that P is the orthocenter of ABC when the area is maximized (moreover, P must be inside the triangle). If ray BP intersects AC at E , then since AEB is similar to PEC , we have

$$\frac{\sqrt{39^2 - x^2}}{33 + x} = \frac{\sqrt{25^2 - x^2}}{x} \Rightarrow x^2(x + 33)^2 = (39^2 - x^2)(25^2 - x^2) \Rightarrow 66x^3 + 3235x^2 - 950625 = 0.$$

The LHS factors to $(x - 15)(66x^2 + 4225x + 63375)$, meaning that $x = 15$ is the only positive solution, giving $AE = 36, BE = 20$, and therefore the maximum area of triangle ABC is $(33 + 15)(36 + 20)/2 = 1344$.

T_3 . How many ways can one tile a 2×8 board with 1×1 and 2×2 tiles? Rotations and reflections of the same configuration are considered distinct.

Answer:

Let $f(n)$ denote the number of ways to fill a $2 \times n$ board. One can fill the leftmost column with two 1×1 tiles, leaving $f(n-1)$ ways, or one can fill the leftmost two columns with one 2×2 tile, leaving $f(n-2)$ ways. Therefore, $f(n) = f(n-1) + f(n-2)$. One can also directly verify that $f(0) = f(1) = 1$. Therefore, $f(n) = F_{n+1}$, where F_n is the n^{th} Fibonacci number. Easy calculation shows that the desired quantity is $f(8) = F_9 = 34$.

T_4 . Let $S = T_5$. Given real numbers a, b, c such that $a^2 + b^2 + c^2 + (a + b + c)^2 = S$, find the maximum possible value of $(a + b)(b + c)(c + a)$.

Answer:

Notice that $S = 27 = a^2 + b^2 + c^2 + (a + b + c)^2 = (a + b)^2 + (b + c)^2 + (c + a)^2$. By AM-GM, $\frac{S}{3} \geq ((a + b)(b + c)(c + a))^{2/3}$ with equality if and only if $a + b = b + c = c + a$, i.e. $a = b = c$. Thus, the maximum possible value is $(\frac{S}{3})^{3/2} = 27$, achieved at $a = b = c = \frac{3}{2}$.

T_5 . A regular tetrahedron has volume 8. What is the volume of the set of all the points in the space (*not necessarily inside the tetrahedron*) that are closer to the center of the tetrahedron than any of the four vertices?

Answer:

Let h denote the height of the tetrahedron. The center of the tetrahedron is a distance $\frac{h}{4}$ from each face. Therefore, the perpendicular bisector plane of the segment connecting the center to a vertex lies a distance $\frac{3}{8}h$ away from both the vertex and the center. Symmetrical considerations with the other three vertices will thus show that the desired region is also a regular tetrahedron, with the center of the original tetrahedron a distance $\frac{3}{8}h$ away from each face. Based on the distance from the center to a face, one can see that the scale factor of this tetrahedron is $\frac{3h}{8} : \frac{h}{4} = 3 : 2$ relative to the original tetrahedron, so its volume is $8 \cdot (\frac{3}{2})^3 = 27$.

33. [25] Determine the value of H_4 .

Proposed by: Yuan Yao

Answer:

34. [25] Determine the value of M_8 .

Proposed by: Yuan Yao

Answer:

35. [25] Determine the value of M_9 .

Proposed by: Yuan Yao

Answer:

36. [25] Determine the value of T_2 .

Proposed by: Yuan Yao

Answer:

HMMT February 2019

February 16, 2019

Team Round

1. [20] Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z . Prove that

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

Proposed by: Yuan Yao

Solution 1. (Similar Triangles)

Let X' and Y' lie on segments AB and AD respectively such that $ZX' \parallel AD$ and $ZY' \parallel AB$. We note that triangles AXY and $Y'YZ$ are similar, and that triangles $AY'Z$ and ADC are similar. Thus, we have

$$\frac{AC}{AZ} = \frac{AD}{AY'} \text{ and } \frac{AY'}{AY} = \frac{XZ}{XY}.$$

This means that

$$\frac{AD}{AY} = \frac{AD}{AY'} \cdot \frac{AY'}{AY} = \frac{XZ}{XY} \cdot \frac{AC}{AZ},$$

and similarly,

$$\frac{AB}{AX} = \frac{ZY}{XY} \cdot \frac{AC}{AZ}.$$

Therefore we have

$$\frac{AB}{AX} + \frac{AD}{AY} = \left(\frac{XZ}{XY} + \frac{ZY}{XY} \right) \cdot \frac{AC}{AZ} = \frac{AC}{AZ},$$

as desired.

Solution 2. (Affine Transformations)

We recall that affine transformations preserve both parallel lines and ratios between distances of collinear points. It thus suffices to show the desired result when $ABCD$ is a square. This can be done in a variety of ways. For instance, a coordinate bash can be applied by setting A to be the origin. Let the length of the square be 1 and set X and Y as $(a, 0)$ and $(0, b)$ respectively, so the line XY has equation $bx + ay = ab$. Then, we note that Z is the point $(\frac{ab}{a+b}, \frac{ab}{a+b})$, so

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} = \frac{AC}{AZ}.$$

2. [20] Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of all positive integers, and let f be a bijection from \mathbb{N} to \mathbb{N} . Must there exist some positive integer n such that $(f(1), f(2), \dots, f(n))$ is a permutation of $(1, 2, \dots, n)$?

Proposed by: Michael Tang

Answer: No

Consider the bijection f defined by

$$(f(1), f(2), f(3), f(4), \dots) = (2, 4, 6, 1, 8, 3, 10, 5, 12, \dots),$$

which alternates between even and odd numbers after the second entry. (More formally, we define $f(n) = 2n$ for $n = 1, 2$, $f(n) = n + 3$ for odd $n \geq 3$ and $f(n) = n - 3$ for even $n \geq 4$.) No such n can exist for this f as the largest number among $f(1), f(2), \dots, f(n)$ is more than n for all n : for $k \geq 2$, the maximum of the first $2k - 1$ or $2k$ values is achieved by $f(2k - 1) = 2k + 2$. (Checking $n = 1$ and $n = 2$ is trivial.)

3. [25] For any angle $0 < \theta < \pi/2$, show that

$$0 < \sin \theta + \cos \theta + \tan \theta + \cot \theta - \sec \theta - \csc \theta < 1.$$

Proposed by: Yuan Yao

We use the following geometric construction, which follows from the geometric definition of the trigonometric functions: Let Z be a point on the unit circle in the coordinate plane with origin O . Let X_1, Y_1 be the projections of Z onto the x - and y -axis respectively, and let X_2, Y_2 lie on x - and y -axis respectively such that X_2Y_2 is tangent to the unit circle at Z . Then we have

$$OZ = X_1Y_1 = 1, X_1Z = \sin \theta, Y_1Z = \cos \theta, X_2Z = \tan \theta, Y_2Z = \cot \theta, OX_2 = \sec \theta, OY_2 = \csc \theta.$$

It then suffices to show that $0 < X_2Y_2 - X_1X_2 - Y_1Y_2 < 1 = X_1Y_1$. The left inequality is true because X_1X_2 and Y_1Y_2 are the projections of ZX_2 and ZY_2 onto x - and y -axis respectively. The right inequality is true because $X_1X_2 + X_1Y_1 + Y_1Y_2 > X_2Y_2$ by triangle inequality. Therefore we are done.

4. [35] Find all positive integers n for which there do not exist n consecutive composite positive integers less than $n!$.

Proposed by: Brian Reinhart

Answer: 1, 2, 3, 4

Solution 1. First, note that clearly there are no composite positive integers less than $2!$, and no 3 consecutive composite positive integers less than $3!$. The only composite integers less than $4!$ are

$$4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22,$$

and it is easy to see that there are no 4 consecutive composite positive integers among them. Therefore, all $n \leq 4$ works.

Define $M = \text{lcm}(1, 2, \dots, n+1)$. To see that there are no other such positive integers, we first show that for all $n \geq 5$, $n! > M$. Let $k = \lfloor \log_2(n+1) \rfloor$. Note that $v_2(M) = k$, while

$$v_2((n+1)!) = \sum_{i=1}^k \left\lfloor \frac{n+1}{2^i} \right\rfloor \geq \sum_{i=1}^k \left(\frac{n+1}{2^i} - 1 \right) = \left(n+1 - \frac{n+1}{2^k} \right) - k \geq (n+1-2) - k = n-k-1.$$

This means that at least $(n-k-1) - k = n-2k-1$ powers of 2 are lost when going from $(n+1)!$ to M . Since $M \mid (n+1)!$, when $n-2k-1 \geq k+1 \iff n \geq 3k+2$, we have

$$M \leq \frac{(n+1)!}{2^{k+1}} \leq \frac{(n+1)!}{2(n+1)} < n!,$$

as desired. Since $n \geq 2^k - 1$, we can rule out all k such that $2^k \geq 3k+3$, which happens when $k \geq 4$ or $n \geq 15$. Moreover, when $k=3$, we may also rule out all $n \geq 3k+2 = 11$.

We thus need only check values of n between 5 and 10:

$$n = 5: n! = 120, M = 60;$$

$$n = 6: n! = 720, M = 420;$$

$$n = 7: n! = 5040, M = 840;$$

$$n \in \{8, 9, 10\}: n! \geq 40320, M \leq 27720.$$

In all cases, $n! > M$, as desired.

To finish, note that $M-2, M-3, \dots, M-(n+1)$ are all composite (divisible by $2, 3, \dots, n+1$ respectively), which gives the desired n consecutive numbers. Therefore, all integers $n \geq 5$ do not satisfy the problem condition, and we are done.

Solution 2. Here is a different way to show that constructions exist for $n \geq 5$. Note that when $n+1$ is not prime, the numbers $n! - 2, n! - 3, \dots, n! - (n+1)$ are all composite (the first $n-1$ are clearly

composite, the last one is composite because $n + 1 \mid n!$ and $n! > 2(n + 1)$). Otherwise, if $n = p - 1$ for prime $p \geq 7$, then the numbers $(n - 1)!, (n - 1)! - 1, (n - 1)! - 2, \dots, (n - 1)! - (n - 1)$ are all composite (the first one and the last $n - 2$ are clearly composite since $(n - 1)! > 2(n - 1)$, the second one is composite since $p \mid (p - 2)! - 1 = (n - 1)! - 1$ by Wilson's theorem).

5. [40] Find all positive integers n such that the unit segments of an $n \times n$ grid of unit squares can be partitioned into groups of three such that the segments of each group share a common vertex.

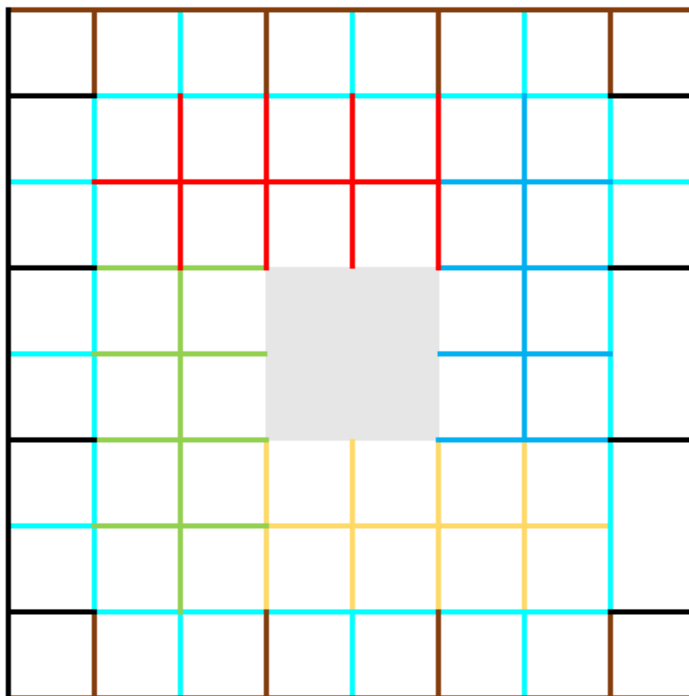
Proposed by: Yuan Yao

Answer: $n \equiv 0, 2 \pmod{6}$

We first prove that $n \equiv 0, 2 \pmod{6}$ is necessary for there to be such a partitioning. We break this down into proving that n has to be even and that $n \equiv 0, 2 \pmod{3}$.

The only way a segment on a side of the square can be part of such a T-shape is as one of the two consecutive segments along the longer side of the T-shape, so they must come in pairs and therefore, the length of each side has to be even. On the other hand, the total number of segments, which is $2n(n + 1)$, has to be a multiple of three as each T-shape consists of three segments, hence either n or $n + 1$ is a multiple of 3, implying that $n \equiv 0, 2 \pmod{3}$.

We can then show that these two conditions is sufficient by showing that $n = 2$ and $n = 6$ works and $n = k + 6$ works whenever $n = k$ works. The construction for $n = 2$ is simple; just put a T-shape with the longer side on each of the four sides. For $n = 6$ and to go from $n = k$ to $n = k + 6$, consider the following diagram:



There are two main parts – the cycle of stacks of T's in all four orientation (see the red, blue, yellow, and green stacks), and the border (seen here by the cyan, brown, and black T-shapes). The case $n = 6$ can be considered as a special case where the middle square is a single point.

6. [45] Scalene triangle ABC satisfies $\angle A = 60^\circ$. Let the circumcenter of ABC be O , the orthocenter be H , and the incenter be I . Let D, T be the points where line BC intersects the internal and external

angle bisectors of $\angle A$, respectively. Choose point X on the circumcircle of $\triangle IHO$ such that $HX \parallel AI$. Prove that $OD \perp TX$.

Proposed by: Wanlin Li

Let I_A denote the A -excenter. Because $\angle A = 60^\circ$, AI is the perpendicular bisector of OH and B, H, O, C all lie on the circle with diameter II_A . We are given that X is on this circle as well, and since $HI = OI$, $XIOI_A$ is also an isosceles trapezoid. But II_A is a diameter, so this means X must be diametrically opposite O on (BOC) and is actually the intersection of the tangents to (ABC) from B and C .

Now $(T, D; B, C) = -1$, so T is on the polar of D with respect to (ABC) . BC is the polar of X and D lies on BC , so X must also lie on the polar of D . Therefore TX is the polar of D with respect to (ABC) , and $OD \perp TX$ as desired.

7. [50] A convex polygon on the plane is called *wide* if the projection of the polygon onto any line in the same plane is a segment with length at least 1. Prove that a circle of radius $\frac{1}{3}$ can be placed completely inside any wide polygon.

Proposed by: Shengtong Zhang

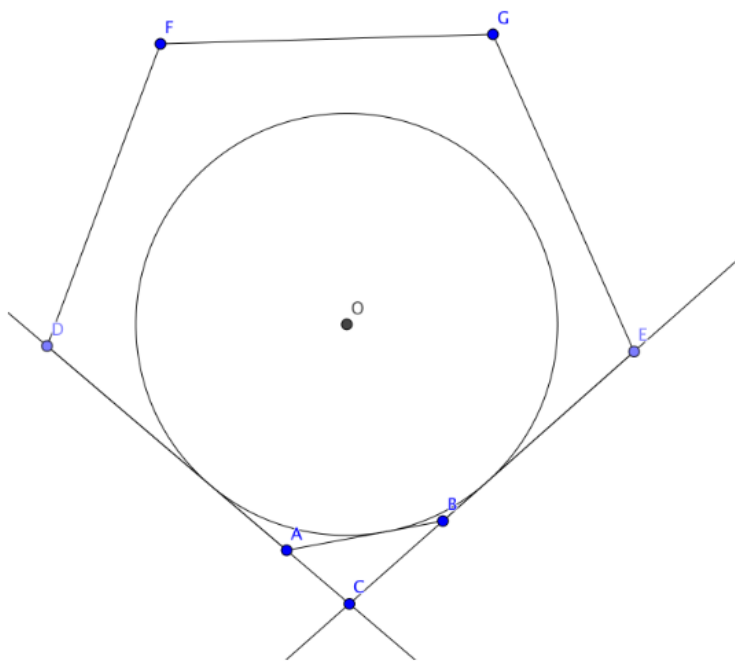
Solution 1.

Lemma. for any polygon including its boundary, there exists a largest circle contained inside it.

Proof. Its easy to see that for any circle inside the polygon, it can be increased in size until it is tangent to at least three sides of the polygon. Then for any three sides of the polygon, there is only one circle tangent to all three, so there are only finitely many possibilities. Therefore there exists a largest one.

Alternatively, one can show that the space of valid (x, y, r) such that the circle with center (x, y) and radius r is compact, e.g. by showing the complement is open and that the complement is open. Then the map $(x, y, r) \rightarrow r$ is continuous and therefore has a maximum.

Now, take the largest circle. It clearly must be tangent to three sides. If the circle lies inside the triangle made by the three lines, we can expand the polygon to that triangle and solve it for the triangle instead. Otherwise, we have the following diagram:



Here the circle is an excircle of the triangle ABC made by the lines AB, AD , and BE . (Note that AD and AB dont have to be consecutive sides of the polygon, but the ones in between dont really matter.)

Then since the circle is an excircle, we can consider a homothety at C with power $1 + \epsilon$, which sends the circle to a slightly larger circle which does not touch line AB . If this homothety causes the circle to leave the polygon for small enough ϵ , it must be because the circle was initially tangent to another line ℓ , for which it would be an incircle of the triangle made by ℓ and lines AD , BE , bringing us back to the first case.

Thus we can reduce to a case where we have a triangle with each height at least 1, and we want to show the inradius is at least $1/3$. Let K be the area of the triangle, so the heights $\frac{2K}{a}$, $\frac{2K}{b}$, $\frac{2K}{c}$ are all at least 1. Then the inradius r satisfies

$$r = \frac{K}{s} = \frac{2K}{a+b+c} \geq \frac{2K}{2K+2K+2K} = \frac{1}{3},$$

as desired.

Solution 2. Consider the center of mass G . We will use the notion of *support lines* for convex shapes. (*Support lines* are the lines that touches the shape but does not cut through it.) If a circle centered at G with radius $1/3$ cannot be contained inside the polygon, then there exist a point P on the boundary that $GP < 1/3$. Let ℓ_1 be the support line passing through p , ℓ_2 be the line parallel to ℓ_1 and passing through G , and ℓ_3 be the other support line that is parallel to ℓ_1 , touching the polygon at P' . Suppose ℓ_2 intersects the polygon at A and B . Extend $P'A$ and $P'B$, intersecting ℓ_1 at A' and B' . Then, if we consider the two parts of the polygon that ℓ_2 divides the polygon into, we have:

- the part of the polygon that contains P' contains the triangle $P'AB$;
- the part of the polygon that contains P is contained in the quadrilateral $AA'B'B$.

Then we conclude that the center of mass G' of the triangle $P'A'B'$ lies between ℓ_2 and ℓ_1 , which by assumption is less than $\frac{1}{3}$ away from ℓ_1 . However, because the height from P' to ℓ_1 is at least 1, the distance from G' to ℓ_1 is at least $\frac{1}{3}$, so we have a contradiction. Therefore no such P exists and the circle can be placed inside the polygon.

8. [50] Can the set of lattice points $\{(x, y) | x, y \in \mathbb{Z}, 1 \leq x, y \leq 252, x \neq y\}$ be colored using 10 distinct colors such that for all $a \neq b, b \neq c$, the colors of (a, b) and (b, c) are distinct?

Proposed by: Franklyn Wang

Answer: Yes

Associate to each number from 1 to 252 a distinct 5-element subset of $S = \{1, 2, \dots, 10\}$. Then assign to (a, b) an element of S that is in the subset associated to a but not in that associated to b . It's not difficult to see that this numerical assignment is a valid coloring: the color assigned to (a, b) is not in b , while the color assigned to (b, c) is in b , so they must be distinct.

9. [55] Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of all polynomials in x with coefficients in \mathbb{F}_p (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in \mathbb{F}_p for each nonnegative integer k . For example, $(x+2)(2x+3) = 2x^2 + 2x + 1$ in $\mathbb{F}_5[x]$ because the corresponding coefficients are equal modulo 5.

Let $f, g \in \mathbb{F}_p[x]$. The pair (f, g) is called *compositional* if

$$f(g(x)) \equiv x^{p^2} - x$$

in $\mathbb{F}_p[x]$. Find, with proof, the number of compositional pairs (in terms of p).

Proposed by: Ashwin Sah

Answer: $4p(p-1)$

Solution 1. First, notice that $(\deg f)(\deg g) = p^2$ and both polynomials are clearly nonconstant. Therefore there are three possibilities for the ordered pair $(\deg f, \deg g)$, which are $(1, p^2)$, $(p^2, 1)$, and (p, p) .

In the subsequent parts of the solution, equalities are modulo p . If $f(x) = ax + b, a \neq 0$ is linear, then it is invertible so then g is uniquely determined as $g(x) = f^{-1}(f(g(x))) = \frac{x^p - ax - b}{a}$. Similarly, if $g(x) = cx + d, c \neq 0 \pmod{p}$ is linear then f is uniquely determined as $f(x) = f(g(g^{-1}(x))) = \left(\frac{x-d}{c}\right)^p - \left(\frac{x-d}{c}\right)$. In each case there are $p(p-1)$ compositional pairs.

The last case is $\deg f = \deg g = p$. We take the derivative of both sides (we use the formal derivative $D_x f(x) = \sum_{n \geq 1} n f_n x^{n-1}$, which satisfies the usual chain and product rules but can be used on arbitrary polynomials, including those in $\mathbb{F}_p[x]$).

Thus

$$f'(g(x))g'(x) = p^2 x^{p^2-1} - 1 = -1,$$

using that $p = 0$ in \mathbb{F}_p . Now $g'(x)$ and $f'(g(x))$ must both be constant polynomials. Since g is nonconstant, this means that $f'(x)$ is also a constant polynomial. We must be careful here, as unlike in \mathbb{R} , nonlinear polynomials can have constant derivatives. From the formula of derivative, we see that $h'(x) = 0$ as a polynomial exactly when $h(x)$ is a linear combination of $1, x^p, x^{2p}, \dots$ (remember that $p = 0$). Thus f', g' both being constant and f, g being of degree p tells us

$$f(x) = ax^p + bx + c, g(x) = dx^p + ex + f$$

where a, b, c, d, e, f are some elements of \mathbb{F}_p . Now we must have

$$a(dx^p + ex + f)^p + b(dx^p + ex + f) + c = x^{p^2} - x$$

over $\mathbb{F}_p[x]$. We use the fact that $(x+y)^p = x^p + y^p$ as polynomials in \mathbb{F}_p , since the binomial coefficients $\binom{p}{j} \equiv 0 \pmod{p}$ for $1 \leq j \leq p-1$. This implies $(x+y+z)^p = x^p + y^p + z^p$. Therefore we can expand the previous equation as

$$a(d^p x^{p^2} + e^p x^p + f^p) + b(dx^p + ex + f) + c = x^{p^2} - x.$$

Equating coefficients, we see that

$$\begin{aligned} ad^p &= 1, \\ ae^p + bd &= 0, \\ be &= -1, \\ af^p + bf + c &= 0. \end{aligned}$$

The first and third equations imply that a, d, b, e are nonzero \pmod{p} and $a = d^{-p}, b = -e^{-1}$. Then $ae^p + bd = 0$ gives

$$d^{-p}e^p - e^{-1}d = 0$$

or $e^{p+1} = d^{p+1}$. Recalling that $e^{p-1} = d^{p-1} = 1 \pmod{p}$, this tells us $d^2 = e^2$ so $d = \pm e$. Furthermore, any choice of such (d, e) give unique (a, b) which satisfy the first three equations. Finally, once we have determined a, b, d, e , any choice of f gives a unique valid choice of c .

Thus we have $p-1$ choices for d , two choices for e after choosing d (n.b. for $p=2$ there is only one choice for e , so the assumption $p > 2$ is used here), and then p choices for f , for a total of $2p(p-1)$ compositional pairs in this case.

Finally, adding the number of compositional pairs from all three cases, we obtain $4p(p-1)$ compositional pairs in total.

Solution 2. The key step is obtaining

$$f(x) = ax^p + bx + c, g(x) = dx^p + ex + f$$

in the case where $\deg f = \deg g = p$. We present an alternative method of obtaining this, with the rest of the solution being the same as the first solution. Let

$$\begin{aligned} f(x) &= f_p x^p + f_{p-1} x^{p-1} + \dots + f_0 \\ g(x) &= g_p x^p + g_{p-1} x^{p-1} + \dots + g_0 \end{aligned}$$

where f_p, g_p are nonzero. Like before, we have $g(x)^p = g(x^p)$ in $\mathbb{F}_p[x]$, so

$$x^{p^2} - x = f_p g(x^p) + f_{p-1} g(x)^{p-1} + \cdots + f_0.$$

Consider the maximal $k < p$ for which $f_k \neq 0$. (It is not hard to see that in fact $k \geq 1$, as $f_p g(x^p) + f_0$ cannot be $x^{p^2} - x$.) First assume that $k > 1$. We look at the x^{kp-1} coefficient, which is affected only by the $f_k g(x)^k$ term. By expanding, the coefficient is $k f_k g_p^{k-1} g_{p-1}$. Therefore $g_{p-1} = 0$. Then we look at the x^{kp-2} coefficient, then the x^{kp-3} coefficient, etc. down to the x^{kp-p+1} coefficient to conclude that $g_{p-1} = g_{p-2} = \cdots = g_1 = 0$. However, then the x coefficient of $f(g(x))$ is zero, contradiction.

Therefore we must have $k = 1$, so f is of the form $ax^p + bx + c$. Using the same method as we used when $k > 1$, we get $g_{p-1} = g_{p-2} = \cdots = g_2 = 0$, though the x^{kp-p+1} coefficient is now the x coefficient which we want to be nonzero. Hence we do not obtain $g_1 = 0$ anymore and we find that g is of the form $dx^p + ex + f$.

10. [60] Prove that for all positive integers n , all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

Proposed by: Faraz Masroor

Note that neither 0 nor 1 are roots of the polynomial. Consider the function

$$Q(x) = P(x)/x^n = (2n)x^n + (2n)x^{-n} + (2n-1)x^{n-1} + (2n-1)x^{-n+1} + \cdots + (n+1)x^1 + (n+1)x^{-1} + n.$$

All $2n$ of the complex roots of $P(x)$ will be roots of $Q(x)$.

If $|x| = 1$, then $x = e^{i\theta}$, and

$$\begin{aligned} Q(x) &= (2n)(x^n + x^{-n}) + (2n-1)(x^{n-1} + x^{-n+1}) + \cdots + (n+1)(x + x^{-1}) + n \\ &= (2n)(e^{in\theta} + e^{-in\theta}) + (2n-1)(e^{i(n-1)\theta} + e^{-i(n-1)\theta}) + \cdots + (n+1)(e^{i\theta} + e^{-i\theta}) + n \\ &= (2n)(2\cos(n\theta)) + (2n-1)(2\cos((n-1)\theta)) + \cdots + (n+1)(\cos(\theta)) + n, \end{aligned}$$

which is real. Thus on the unit circle, we have $Q(x)$ is real, and we want to show it has $2n$ roots there.

Rewrite

$$\begin{aligned} P(x) &= (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + 2n \\ &= (2n)(x^{2n} + x^{2n-1} + \cdots + 1) \\ &\quad - (x^{2n-1} + 2x^{2n-2} + \cdots + (n-1)x^n + nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x^2 + x) \\ &= 2n \frac{x^{2n+1} - 1}{x-1} - x(x^{2n-2} + 2x^{2n-3} + \cdots + (n-1)x^n + nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1) \\ &= 2n \frac{x^{2n+1} - 1}{x-1} - x(x^{n-1} + x^{n-2} + \cdots + x + 1)^2 \\ &= 2n \frac{x^{2n+1} - 1}{x-1} - x \left(\frac{x^n - 1}{x-1} \right)^2, \end{aligned}$$

and thus

$$Q(x) = \frac{2n}{x^n} \frac{x^{2n+1} - 1}{x-1} - x \left(\frac{x^n - 1}{x-1} \right)^2.$$

Consider the roots of unity $r_j = e^{i\frac{2\pi}{2n}j}$, for $j = 0$ to $2n-1$. There are $2n$ such roots of unity: they all have $r_j^{2n} = 1$, and they alternate between those which satisfy $r_j^n = 1$ or $r_j^n = -1$. At those $x = r_j$, if

$r_j^n = 1$ but $x \neq 1$, then

$$\begin{aligned} Q(x) &= \frac{2n}{x^n} \frac{x^{2n+1} - 1}{x - 1} - \frac{x}{x^n} \left(\frac{x^n - 1}{x - 1} \right)^2 \\ &= 2n \frac{x^1 - 1}{x - 1} - x \left(\frac{1 - 1}{x - 1} \right)^2 = 2n > 0. \end{aligned}$$

At $x = 1$, we can easily see $Q(1) > 0$.

If $r_j^n = -1$, then

$$\begin{aligned} Q(x) &= \frac{2n}{x^n} \frac{x^{2n+1} - 1}{x - 1} - \frac{x}{x^n} \left(\frac{x^n - 1}{x - 1} \right)^2 \\ &= -2n \frac{x^1 - 1}{x - 1} + x \left(\frac{-1 - 1}{x - 1} \right)^2 \\ &= -2n + \frac{4x}{(x - 1)^2} \\ &= -2n + \frac{4}{x - 2 + 1/x} \\ &= -2n + \frac{4}{2 \cos(\frac{2\pi}{2n}j) - 2} < -2n - 4 < 0 \end{aligned}$$

since the denominator of this second term is strictly negative ($j \neq 0$).

Thus at each of the $2n$ -roots of unity, $Q(x)$ alternates in sign, and because $Q(x)$ is real and continuous on the unit circle, it has at least one root between every pair of consecutive roots of unity. Since there are $2n$ of these pairs, and we know that $Q(x)$ has exactly $2n$ roots (by the Fundamental Theorem of Algebra), we have found all of Q 's roots, and therefore those of P .

HMMT November 2019
November 9, 2019
General Round

1. Dylan has a 100×100 square, and wants to cut it into pieces of area at least 1. Each cut must be a straight line (not a line segment) and must intersect the interior of the square. What is the largest number of cuts he can make?

Proposed by: Carl Schildkraut

Answer: 9999

Since each piece has area at least 1 and the original square has area 10000, Dylan can end up with at most 10000 pieces. There is initially 1 piece, so the number of pieces can increase by at most 9999. Each cut increases the number of pieces by at least 1, so Dylan can make at most 9999 cuts. Notice that this is achievable if Dylan makes 9999 vertical cuts spaced at increments of $\frac{1}{100}$ units.

2. Meghana writes two (not necessarily distinct) primes q and r in base 10 next to each other on a blackboard, resulting in the concatenation of q and r (for example, if $q = 13$ and $r = 5$, the number on the blackboard is now 135). She notices that three more than the resulting number is the square of a prime p . Find all possible values of p .

Proposed by: Carl Schildkraut

Answer: 5

Trying $p = 2$, we see that $p^2 - 3 = 1$ is not the concatenation of two primes, so p must be odd. Then $p^2 - 3$ is even. Since r is prime and determines the units digit of the concatenation of q and r , r must be 2. Then p^2 will have units digit 5, which means that p will have units digit 5. Since p is prime, we find that p can only be 5, and in this case, $p^2 - 3 = 22$ allows us to set $q = r = 2$ to satisfy the problem statement. So there is a valid solution when $p = 5$, and this is the only possibility.

3. Katie has a fair 2019-sided die with sides labeled $1, 2, \dots, 2019$. After each roll, she replaces her n -sided die with an $(n+1)$ -sided die having the n sides of her previous die and an additional side with the number she just rolled. What is the probability that Katie's 2019th roll is a 2019?

Proposed by: Freddie Zhao

Answer: $\frac{1}{2019}$

Since Katie's original die is fair, the problem is perfectly symmetric. So on the 2019th roll, each number is equally probable as any other. Therefore, the probability of rolling a 2019 is just $\frac{1}{2019}$.

4. In $\triangle ABC$, $AB = 2019$, $BC = 2020$, and $CA = 2021$. Yannick draws three regular n -gons in the plane of $\triangle ABC$ so that each n -gon shares a side with a distinct side of $\triangle ABC$ and no two of the n -gons overlap. What is the maximum possible value of n ?

Proposed by: Carl Schildkraut

Answer: 11

If any n -gon is drawn on the same side of one side of $\triangle ABC$ as $\triangle ABC$ itself, it will necessarily overlap with another triangle whenever $n > 3$. Thus either $n = 3$ or the triangles are all outside ABC . The interior angle of a regular n -gon is $180^\circ \cdot \frac{n-2}{n}$, so we require

$$360^\circ \cdot \frac{n-2}{n} + \max(\angle A, \angle B, \angle C) < 360^\circ.$$

As $\triangle ABC$ is almost equilateral (in fact the largest angle is less than 60.1°), each angle is approximately 60° , so we require

$$360 \cdot \frac{n-2}{n} < 300 \implies n < 12.$$

Hence the answer is $n = 11$.

5. Let a, b, c be positive real numbers such that $a \leq b \leq c \leq 2a$. Find the maximum possible value of

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

Proposed by: Carl Schildkraut

Answer: $\boxed{\frac{7}{2}}$

Fix the values of b, c . By inspecting the graph of

$$f(x) = \frac{b}{x} + \frac{x}{c},$$

we see that on any interval the graph attains its maximum at an endpoint. This argument applies when we fix any two variables, so it suffices to check boundary cases in which $b = a$ or $b = c$, and $c = b$ or $c = 2a$. All pairs of these conditions determine the ratio between a, b, c , except $b = c$ and $c = b$, in which case the boundary condition on a tells us that $a = b$ or $2a = b = c$. In summary, these cases are

$$(a, b, c) \in \{(a, a, a), (a, a, 2a), (a, 2a, 2a)\}.$$

The largest value achieved from any of these three is $\frac{7}{2}$.

6. Find all ordered pairs (a, b) of positive integers such that $2a + 1$ divides $3b - 1$ and $2b + 1$ divides $3a - 1$.

Proposed by: Milan Haiman

Answer: $\boxed{(2, 2), (12, 17), (17, 12)}$

This is equivalent to the existence of nonnegative integers c and d such that $3b - 1 = c(2a + 1)$ and $3a - 1 = d(2b + 1)$. Then

$$cd = \frac{(3b - 1)(3a - 1)}{(2a + 1)(2b + 1)} = \frac{3a - 1}{2a + 1} \cdot \frac{3b - 1}{2b + 1} < \frac{3}{2} \cdot \frac{3}{2} = 2.25.$$

Neither c nor d can equal 0 since that would give $a = \frac{1}{3}$ or $b = \frac{1}{3}$, so $cd \leq 2.25$ implies $(c, d) \in \{(1, 1), (2, 1), (1, 2)\}$. Substituting (c, d) back in gives three systems of equations and the three solutions: $(2, 2), (12, 17), (17, 12)$.

7. In Middle-Earth, nine cities form a 3 by 3 grid. The top left city is the capital of Gondor and the bottom right city is the capital of Mordor. How many ways can the remaining cities be divided among the two nations such that all cities in a country can be reached from its capital via the grid-lines without passing through a city of the other country?

Proposed by: Shengtong Zhang

Answer: $\boxed{30}$

For convenience, we will center the grid on the origin of the coordinate plane and align the outer corners of the grid with the points $(\pm 1, \pm 1)$, so that $(-1, 1)$ is the capital of Gondor and $(1, -1)$ is the capital of Mordor.

We will use casework on which nation the city at $(0, 0)$ is part of. Assume that it belongs to Gondor. Then consider the sequence of cities at $(1, 0), (1, 1), (0, 1)$. If one of these belongs to Mordor, then all of the previous cities belong to Mordor, since Mordor must be connected. So we have 4 choices for which cities belong to Mordor. Note that this also makes all the other cities in the sequence connected to Gondor. Similarly, we have 4 (independent) choices for the sequence of cities $(0, -1), (-1 - 1), (-1, 0)$. All of these choices keep $(0, 0)$ connected to Gondor except the choice that assigns all cities in both sequences to Mordor. Putting this together, the answer is $2(4 \cdot 4 - 1) = 30$.

8. Compute the number of ordered pairs of integers (x, y) such that $x^2 + y^2 < 2019$ and

$$x^2 + \min(x, y) = y^2 + \max(x, y).$$

Proposed by: Milan Haiman

Answer: 127

We have

$$x^2 - y^2 = \max(x, y) - \min(x, y) = |x - y|$$

Now if $x \neq y$, we can divide by $x - y$ to obtain $x + y = \pm 1$. Thus $x = y$ or $x + y = \pm 1$.

If $x = y$, we see that $2019 > x^2 + y^2 = 2x^2$, so we see that $-31 \leq x \leq 31$. There are 63 ordered pairs in this case.

In the second case, note that $|x| \geq |y|$ since $x^2 - y^2 = |x - y| \geq 0$. Since $x + y = \pm 1$, we cannot have $xy > 0$, so either $x \geq 0, y \leq 0$, or $x \leq 0, y \geq 0$. In the first case, $x + y = 1$; in the second case, $x + y = -1$. Thus, the solutions for (x, y) are of the form $(k, 1 - k)$ or $(-k, k - 1)$ for some $k > 0$. In either case, we must have $k^2 + (k - 1)^2 < 2019$, which holds true for any $1 \leq k \leq 32$ but fails for $k = 33$. There are a total of $32 \cdot 2 = 64$ solutions in this case.

In summary, there are a total of $63 + 64 = 127$ integer solutions to the equation $x^2 + \min(x, y) = y^2 + \max(x, y)$ with $x^2 + y^2 < 2019$.

9. Let $ABCD$ be an isosceles trapezoid with $AD = BC = 255$ and $AB = 128$. Let M be the midpoint of CD and let N be the foot of the perpendicular from A to CD . If $\angle MBC = 90^\circ$, compute $\tan \angle NBM$.

Proposed by: Milan Haiman

Answer: $\frac{120}{353}$

Construct P , the reflection of A over CD . Note that P, M , and B are collinear. As $\angle PNC = \angle PBC = 90^\circ$, $PNBC$ is cyclic. Thus, $\angle NBM = \angle NCP$, so our desired tangent is $\tan \angle ACN = \frac{AN}{CN}$. Note that $NM = \frac{1}{2}AB = 64$. Since $\triangle AND \sim \triangle MAD$,

$$\frac{255}{64 + ND} = \frac{ND}{255}.$$

Solving, we find $ND = 225$, which gives $AN = 120$. Then we calculate $\frac{AN}{CN} = \frac{120}{128 + 225} = \frac{120}{353}$.

10. An *up-right path* between two lattice points P and Q is a path from P to Q that takes steps of 1 unit either up or to the right. A lattice point (x, y) with $0 \leq x, y \leq 5$ is chosen uniformly at random. Compute the expected number of up-right paths from $(0, 0)$ to $(5, 5)$ not passing through (x, y) .

Proposed by: Mehtaab Sawhney

Answer: 175

For a lattice point (x, y) , let $F(x, y)$ denote the number of up-right paths from $(0, 0)$ to $(5, 5)$ that don't pass through (x, y) , and let

$$S = \sum_{0 \leq x \leq 5} \sum_{0 \leq y \leq 5} F(x, y).$$

Our answer is $\frac{S}{36}$, as there are 36 lattice points (x, y) with $0 \leq x, y \leq 5$.

Notice that the number of up-right paths from $(0, 0)$ to $(5, 5)$ is $\binom{10}{5} = 252$ because each path consists of 10 steps, of which we can choose 5 to be to the right. Each of these paths passes through 11 lattice points (x, y) with $0 \leq x, y \leq 5$, so each path contributes $36 - 11 = 25$ to the quantity we are counting in S . Then $S = 25 \cdot 252$, so our answer is $\frac{25 \cdot 252}{36} = 175$.

HMMT November 2019

November 9, 2019

Guts Round

1. [5] A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

A polynomial is called *teeny* if it has degree at most 1 and integer coefficients between -7 and 7 , inclusive.

How many nonzero tricky teeny polynomials are there?

Proposed by: Carl Schildkraut

Answer: $\boxed{2}$

If a degree 0 polynomial has 4 as a root, then it must be the constant zero polynomial. Thus, we will only consider polynomials of degree 1.

If P has degree 1, integer coefficients, and 4 as a root, then it must be of the form $P(x) = a(x - 4) = ax - 4a$ for some nonzero integer a . Since all integer coefficients are between -7 and 7 , inclusive, we require $-7 \leq 4a \leq 7$, which gives us $a = -1, 1$. Note that for both values, the coefficient of x is also between -7 and 7 , so there are 2 tricky teeny polynomials.

2. [5] You are trying to cross a 6 foot wide river. You can jump at most 4 feet, but you have one stone you can throw into the river; after it is placed, you may jump to that stone and, if possible, from there to the other side of the river. However, you are not very accurate and the stone ends up landing uniformly at random in the river. What is the probability that you can get across?

Proposed by: Carl Schildkraut and Milan Haiman

Answer: $\boxed{\frac{1}{3}}$

To be able to cross, the stone must land between 2 and 4 feet from the river bank you are standing on. Therefore the probability is $\frac{2}{6} = \frac{1}{3}$.

3. [5] For how many positive integers a does the polynomial

$$x^2 - ax + a$$

have an integer root?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{1}$

Let r, s be the roots of $x^2 - ax + a = 0$. By Vieta's, we have $r + s = rs = a$. Note that if one root is an integer, then both roots must be integers, as they sum to an integer a . Then,

$$rs - (r + s) + 1 = 1 \implies (r - 1)(s - 1) = 1.$$

Because we require r, s to be both integers, we have $r - 1 = s - 1 = \pm 1$, which yields $r = s = 0, 2$. If $r = 0$ or $s = 0$, then $a = 0$, but we want a to be a positive integer. Therefore, our only possibility is when $r = s = 2$, which yields $a = 4$, so there is exactly 1 value of a (namely, $a = 4$) such that $x^2 - ax + a$ has an integer root.

4. [6] In 2019, a team, including professor Andrew Sutherland of MIT, found three cubes of integers which sum to 42:

$$42 = (-8053873881207597_)^3 + (80435758145817515)^3 + (12602123297335631)^3$$

One of the digits, labeled by an underscore, is missing. What is that digit?

Proposed by: Andrew Gu

Answer: $\boxed{4}$

Let the missing digit be x . Then, taking the equation modulo 10, we see that $2 \equiv -x^3 + 5^3 + 1^3$. This simplifies to $x^3 \equiv 4 \pmod{10}$, which gives a unique solution of $x = 4$.

5. [6] A point P is chosen uniformly at random inside a square of side length 2. If P_1, P_2, P_3 , and P_4 are the reflections of P over each of the four sides of the square, find the expected value of the area of quadrilateral $P_1P_2P_3P_4$.

Proposed by: Carl Schildkraut

Answer: $\boxed{8}$

Let $ABCD$ denote the square defined in the problem. We see that if P_1 is the reflection of P over \overline{AB} , then the area of P_1AB is the same as the area of PAB . Furthermore, if P_4 is the reflection of P over \overline{DA} , P_1, A , and P_4 are collinear, as A is the midpoint of $\overline{P_1P_4}$. Repeating this argument for the other points gives us that the desired area is

$$[P_1AB] + [P_2BC] + [P_3CD] + [P_4DA] + [ABCD] = [PAB] + [PBC] + [PCD] + [PDA] + [ABCD] = 2[ABCD] = 8.$$

6. [6] Compute the sum of all positive integers $n < 2048$ such that n has an even number of 1's in its binary representation.

Proposed by: Milan Haiman

Answer: $\boxed{1048064}$

Note that the positive integers less than 2047 are those with at most 11 binary digits. Consider the contribution from any one of those digits. If we set that digit to 1, then the remaining 10 digits can be set in $2^9 = 512$ ways so that the number of 1's is even. Therefore the answer is

$$512(2^0 + \dots + 2^{10}) = 512 \cdot 2047 = 1048064.$$

7. [7] Let S be the set of all nondegenerate triangles formed from the vertices of a regular octagon with side length 1. Find the ratio of the largest area of any triangle in S to the smallest area of any triangle in S .

Proposed by: Carl Schildkraut

Answer: $\boxed{3 + 2\sqrt{2}}$

By a smoothing argument, the largest triangle is that where the sides span 3, 3, and 2 sides of the octagon respectively (i.e. it has angles 45° , 67.5° , and 67.5°), and the smallest triangle is that formed by three adjacent vertices of the octagon. Scaling so that the circumradius of the octagon is 1, our answer is

$$\frac{\sin(90^\circ) + 2\sin(135^\circ)}{2\sin(45^\circ) - \sin(90^\circ)} = \frac{1 + \sqrt{2}}{\sqrt{2} - 1} = 3 + 2\sqrt{2},$$

where the numerator is derived from splitting the large triangle by the circumradii, and the denominator is derived from adding the areas of the two triangles formed by the circumradii, then subtracting the area not in the small triangle.

8. [7] There are 36 students at the Multiples Obfuscation Program, including a singleton, a pair of identical twins, a set of identical triplets, a set of identical quadruplets, and so on, up to a set of identical octuplets. Two students look the same if and only if they are from the same identical multiple. Nithya the teaching assistant encounters a random student in the morning and a random student in the afternoon (both chosen uniformly and independently), and the two look the same. What is the probability that they are actually the same person?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{17}}$

Let X and Y be the students Nithya encounters during the day. The number of pairs (X, Y) for which X and Y look the same is $1 \cdot 1 + 2 \cdot 2 + \dots + 8 \cdot 8 = 204$, and these pairs include all the ones in which X and Y are identical. As X and Y are chosen uniformly and independently, all 204 pairs are equally likely to be chosen, thus the problem reduces to choosing one of the 36 pairs in 204, the probability for which is $\frac{3}{17}$.

9. [7] Let p be a real number between 0 and 1. Jocelin has a coin that lands heads with probability p and tails with probability $1 - p$; she also has a number written on a blackboard. Each minute, she flips the coin, and if it lands heads, she replaces the number x on the blackboard with $3x + 1$; if it lands tails she replaces it with $x/2$. Given that there are constants a, b such that the expected value of the value written on the blackboard after t minutes can be written as $at + b$ for all positive integers t , compute p .

Proposed by: Carl Schildkraut

Answer: $\boxed{\frac{1}{5}}$

If the blackboard has the value x written on it, then the expected value of the value after one flip is

$$f(x) = p(3x - 1) + (1 - p)x/2.$$

Because this expression is linear, we can say the same even if we only know the blackboard's initial expected value is x . Therefore, if the blackboard value is x_0 at time 0, then after t minutes, the expected blackboard value is $f^t(x_0)$. We are given that $x_0, f(x_0), f^2(x_0), \dots$ is an arithmetic sequence, so for there to be a constant difference, we must have $f(x) = x + c$ for some c .

This only occurs when $3p + (1 - p)/2 = 1$, so $p = 1/5$.

10. [8] Let $ABCD$ be a square of side length 5, and let E be the midpoint of side AB . Let P and Q be the feet of perpendiculars from B and D to CE , respectively, and let R be the foot of the perpendicular from A to DQ . The segments CE, BP, DQ , and AR partition $ABCD$ into five regions. What is the median of the areas of these five regions?

Proposed by: Carl Schildkraut

Answer: $\boxed{5}$

We have $DQ \perp CE$ and $AR \perp DQ$, so $AR \parallel CE$. Thus, we can show that $\triangle ARD \cong \triangle DQC \cong \triangle CPB$, so the median of the areas of the five regions is equal to the area of one of the three triangles listed above.

Now, note that $\triangle EBC \sim \triangle BPC$, so $\frac{BP}{BC} = \frac{EB}{EC} = \frac{1}{\sqrt{5}}$. This means that $BP = \sqrt{5}$, so $CP = 2\sqrt{5}$. Therefore, the area of $\triangle BPC$, the median area, is 5.

11. [8] Let a, b, c, d be real numbers such that

$$\min(20x + 19, 19x + 20) = (ax + b) - |cx + d|$$

for all real numbers x . Find $ab + cd$.

Proposed by: Yuan Yao

Answer: $\boxed{380}$

In general, $\min(p, q) = \frac{p+q}{2} - \left| \frac{p-q}{2} \right|$. Letting $p = 20x + 19$ and $q = 19x + 20$ gives $a = b = 19.5$ and $c = d = \pm 0.5$. Then the answer is $19.5^2 - 0.5^2 = 19 \cdot 20 = 380$.

12. [8] Four players stand at distinct vertices of a square. They each independently choose a vertex of the square (which might be the vertex they are standing on). Then, they each, at the same time, begin running in a straight line to their chosen vertex at 10mph, stopping when they reach the vertex. If at any time two players, whether moving or not, occupy the same space (whether a vertex or a point inside the square), they collide and fall over. How many different ways are there for the players to choose vertices to go to so that none of them fall over?

Proposed by: Carl Schildkraut

Answer: $\boxed{11}$

Observe that no two players can choose the same vertex, and no two players can choose each others vertices. Thus, if two players choose their own vertices, then the remaining two also must choose their

own vertices (because they can't choose each other's vertices), thus all 4 players must choose their own vertices. There is 1 way to choose the vertices in this case.

Name the players top left, top right, bottom left, and bottom right, based on their initial positions. Assume exactly one player (without loss of generality, say the top left) chooses their own vertex. Then, the remaining 3 players have to form a triangle (recall no two players can choose each other's vertices). There are 4 ways to choose which player chooses their own vertex, and 2 ways to choose which direction the players move in the triangle, thus there are 8 ways to choose the vertices in this case.

Lastly, assume no one chooses their own vertex. We will first prove that no player can choose the vertex across them. Assume the contrary, without loss of generality, let the top left player choose the bottom right vertex. Then, neither of the bottom left and the top right players can choose the other's vertex, because they would meet the top left player at the center of the square. As they can't choose bottom right (it is chosen by the top left player), and can't choose their own vertex (by assumption), they both have to choose the top left vertex, which is an immediate contradiction.

Now, the top left player has to choose either the top right vertex or the bottom left. Without loss of generality, let the player choose the top right vertex. Then, the top right player has to choose the bottom right vertex (as they can neither go across nor back to top left), the bottom right player has to choose the bottom left vertex, and the bottom left player has to choose the top left vertex, and all the choices are determined by the first player's choice. There are 2 ways to choose where the first player goes, thus there are 2 ways to choose the vertices in this case.

In total, there are $1 + 8 + 2 = 11$ ways to choose the vertices.

13. [9] In $\triangle ABC$, the incircle centered at I touches sides AB and BC at X and Y , respectively. Additionally, the area of quadrilateral $BXIY$ is $\frac{2}{5}$ of the area of ABC . Let p be the smallest possible perimeter of a $\triangle ABC$ that meets these conditions and has integer side lengths. Find the smallest possible area of such a triangle with perimeter p .

Proposed by: Joey Heerens

Answer: $\boxed{2\sqrt{5}}$

Note that $\angle BXI = \angle BYI = 90^\circ$, which means that AB and BC are tangent to the incircle of ABC at X and Y respectively. So $BX = BY = \frac{AB+BC-AC}{2}$, which means that $\frac{2}{5} = \frac{[BXIY]}{[ABC]} = \frac{AB+BC-AC}{AB+BC+AC}$. The smallest perimeter is achieved when $AB = AC = 3$ and $BC = 4$. The area of this triangle ABC is $2\sqrt{5}$.

14. [9] Compute the sum of all positive integers n for which

$$9\sqrt{n} + 4\sqrt{n+2} - 3\sqrt{n+16}$$

is an integer.

Proposed by: Milan Haiman

Answer: $\boxed{18}$

For the expression to be an integer at least one of n and $n+2$ must be a perfect square. We also note that at most one of n and $n+2$ can be a square, so exactly one of them is a square.

Case 1: n is a perfect square. By our previous observation, it must be that $4\sqrt{n+2} = 3\sqrt{n+16} \Rightarrow n = 16$.

Case 2: $n+2$ is a perfect square. By our previous observation, it must be that $9\sqrt{n} = 3\sqrt{n+16} \Rightarrow n = 2$.

Consequently, the answer is $16 + 2 = 18$.

15. [9] Let a, b, c be positive integers such that

$$\frac{a}{77} + \frac{b}{91} + \frac{c}{143} = 1.$$

What is the smallest possible value of $a + b + c$?

Proposed by: James Lin

Answer: $\boxed{79}$

We need $13a + 11b + 7c = 1001$, which implies $13(a + b + c - 77) = 2b + 6c$. Then $2b + 6c$ must be divisible by both 2 and 13, so it is minimized at 26 (e.g. with $b = 10, c = 1$). This gives $a + b + c = 79$.

16. [10] Equilateral $\triangle ABC$ has side length 6. Let ω be the circle through A and B such that CA and CB are both tangent to ω . A point D on ω satisfies $CD = 4$. Let E be the intersection of line CD with segment AB . What is the length of segment DE ?

Proposed by: Benjamin Qi

Answer: $\boxed{\frac{20}{13}}$

Let F be the second intersection of line CD with ω . By power of a point, we have $CF = 9$, so $DF = 5$. This means that $\frac{[ADB]}{[AFB]} = \frac{DE}{EF} = \frac{DE}{5-DE}$. Now, note that triangle CAD is similar to triangle CFA , so $\frac{FA}{AD} = \frac{CA}{CD} = \frac{3}{2}$. Likewise, $\frac{FB}{BD} = \frac{CB}{CD} = \frac{3}{2}$. Also, note that $\angle ADB = 180 - \angle DAB - \angle DBA = 180 - \angle CAB = 120$, and $\angle AFB = 180 - \angle ADB = 60$. This means that $\frac{[ADB]}{[AFB]} = \frac{AD \cdot BD \cdot \sin 120}{FA \cdot FB \cdot \sin 60} = \frac{4}{9}$. Therefore, we have that $\frac{DE}{5-DE} = \frac{4}{9}$. Solving yields $DE = \frac{20}{13}$.

17. [10] Kelvin the frog lives in a pond with an infinite number of lily pads, numbered 0, 1, 2, 3, and so forth. Kelvin starts on lily pad 0 and jumps from pad to pad in the following manner: when on lily pad i , he will jump to lily pad $(i + k)$ with probability $\frac{1}{2^k}$ for $k > 0$. What is the probability that Kelvin lands on lily pad 2019 at some point in his journey?

Proposed by: Nikhil Reddy

Answer: $\boxed{\frac{1}{2}}$

Suppose we combine all of the lily pads with numbers greater than 2019 into one lily pad labeled ∞ . Also, let Kelvin stop once he reaches one of these lily pads.

Now at every leap, Kelvin has an equal chance of landing on 2019 as landing on ∞ . Furthermore, Kelvin is guaranteed to reach 2019 or ∞ within 2020 leaps. Therefore the chance of landing on 2019 is the same as missing it, so our answer is just $\frac{1}{2}$.

18. [10] The polynomial $x^3 - 3x^2 + 1$ has three real roots r_1, r_2 , and r_3 . Compute

$$\sqrt[3]{3r_1 - 2} + \sqrt[3]{3r_2 - 2} + \sqrt[3]{3r_3 - 2}.$$

Proposed by: Milan Haiman

Answer: $\boxed{0}$

Let r be a root of the given polynomial. Then

$$r^3 - 3r^2 + 1 = 0 \implies r^3 - 3r^2 + 3r - 1 = 3r - 2 \implies r - 1 = \sqrt[3]{3r - 2}.$$

Now by Vieta the desired value is $r_1 + r_2 + r_3 - 3 = 3 - 3 = 0$.

19. [11] Let ABC be a triangle with $AB = 5, BC = 8, CA = 11$. The incircle ω and A -excircle¹ Γ are centered at I_1 and I_2 , respectively, and are tangent to BC at D_1 and D_2 , respectively. Find the ratio of the area of $\triangle AI_1D_1$ to the area of $\triangle AI_2D_2$.

Proposed by: Carl Schildkraut

Answer: $\boxed{\frac{1}{9}}$

Let D'_1 and D'_2 be the points diametrically opposite D_1 and D_2 on the incircle and A -excircle, respectively. As I_x is the midpoint of D_x and D'_x , we have

$$\frac{[AI_1D_1]}{[AI_2D_2]} = \frac{[AD_1D'_1]}{[AD_2D'_2]}.$$

Now, $\triangle AD_1D'_1$ and $\triangle AD_2D'_2$ are homothetic with ratio $\frac{r}{r_A} = \frac{s-a}{s}$, where r is the inradius, r_A is the A -exradius, and s is the semiperimeter. Our answer is thus

$$\left(\frac{s-a}{s}\right)^2 = \left(\frac{4}{12}\right) = \frac{1}{9}.$$

20. [11] Consider an equilateral triangle T of side length 12. Matthew cuts T into N smaller equilateral triangles, each of which has side length 1, 3, or 8. Compute the minimum possible value of N .

Proposed by: Matthew Cho

Answer: 16

Matthew can cut T into 16 equilateral triangles with side length 3. If he instead included a triangle of side 8, then let him include a triangles of side length 3. He must include $12^2 - 8^2 - 3^2a = 80 - 9a$ triangles of side length 1. Thus $a \leq 8$, giving that he includes at least

$$(80 - 9a) + (a) + 1 = 81 - 8a \geq 17$$

total triangles, so 16 is minimal.

21. [11] A positive integer n is *infallible* if it is possible to select n vertices of a regular 100-gon so that they form a convex, non-self-intersecting n -gon having all equal angles. Find the sum of all infallible integers n between 3 and 100, inclusive.

Proposed by: Benjamin Qi

Answer: 262

Suppose $A_1A_2 \dots A_n$ is an equiangular n -gon formed from the vertices of a regular 100-gon. Note that the angle $\angle A_1A_2A_3$ is determined only by the number of vertices of the 100-gon between A_1 and A_3 . Thus in order for $A_1A_2 \dots A_n$ to be equiangular, we require exactly that A_1, A_3, \dots are equally spaced and A_2, A_4, \dots are equally spaced. If n is odd, then all the vertices must be equally spaced, meaning $n \mid 100$. If n is even, we only need to be able to make a regular $(\frac{n}{2})$ -gon from the vertices of a 100-gon, which we can do if $n \mid 200$. Thus the possible values of n are 4, 5, 8, 10, 20, 25, 40, 50, and 100, for a total of 262.

22. [12] Let $f(n)$ be the number of distinct digits of n when written in base 10. Compute the sum of $f(n)$ as n ranges over all positive 2019-digit integers.

Proposed by: Milan Haiman

Answer: 9(10²⁰¹⁹ - 9²⁰¹⁹)

Write

$$f(n) = f_0(n) + \dots + f_9(n),$$

where $f_d(n) = 1$ if n contains the digit d and 0 otherwise. The sum of $f_d(n)$ over all 2019-digit positive integers n is just the number of 2019-digit positive integers that contain the digit d . For $1 \leq d \leq 9$,

$$\sum_n f_d(n) = 9 \cdot 10^{2018} - 8 \cdot 9^{2018}.$$

Also,

$$\sum_n f_0(n) = 9 \cdot 10^{2018} - 9^{2019}.$$

Summing over all possible values of d , we compute

$$\sum_n f(n) = \sum_{d=0}^9 \sum_n f_d(n) = 9(9 \cdot 10^{2018} - 8 \cdot 9^{2018}) + 9 \cdot 10^{2018} - 9^{2019} = 9(10^{2019} - 9^{2019}).$$

23. [12] For a positive integer n , let, $\tau(n)$ be the number of positive integer divisors of n . How many integers $1 \leq n \leq 50$ are there such that $\tau(\tau(n))$ is odd?

Proposed by: Kevin Liu

Answer: 17

Note that $\tau(n)$ is odd if and only if n is a perfect square. Thus, it suffices to find the number of integers n in the given range such that $\tau(n) = k^2$ for some positive integer k .

If $k = 1$, then we obtain $n = 1$ as our only solution. If $k = 2$, we see that n is either in the form pq or p^3 , where p and q are distinct primes. The first subcase gives $8 + 4 + 1 = 13$ solutions, while the second subcase gives 2 solutions. $k = 3$ implies that n is a perfect square, and it is easy to see that only $6^2 = 36$ works. Finally, $k \geq 4$ implies that n is greater than 50, so we've exhausted all possible cases. Our final answer is $1 + 13 + 2 + 1 = 17$.

24. [12] Let P be a point inside regular pentagon $ABCDE$ such that $\angle PAB = 48^\circ$ and $\angle PDC = 42^\circ$. Find $\angle BPC$, in degrees.

Proposed by: Dylan Liu

Answer: 84°

Since a regular pentagon has interior angles 108° , we can compute $\angle PDE = 66^\circ$, $\angle PAE = 60^\circ$, and $\angle APD = 360^\circ - \angle AED - \angle PDE - \angle PAE = 126^\circ$. Now observe that drawing PE divides quadrilateral $PAED$ into equilateral triangle PAE and isosceles triangle PED , where $\angle DPE = \angle EDP = 66^\circ$. That is, we get $PA = PE = s$, where s is the side length of the pentagon.

Now triangles PAB and PED are congruent (with angles $48^\circ - 66^\circ - 66^\circ$), so $PD = PB$ and $\angle PDC = \angle PBC = 42^\circ$. This means that triangles PDC and PBC are congruent (side-angle-side), so $\angle BPC = \angle DPC$.

Finally, we compute $\angle BPC + \angle DPC = 2\angle BPC = 360^\circ - \angle APB - \angle EPA - \angle DPE = 168^\circ$, meaning $\angle BPC = 84^\circ$.

25. [13] In acute $\triangle ABC$ with centroid G , $AB = 22$ and $AC = 19$. Let E and F be the feet of the altitudes from B and C to AC and AB respectively. Let G' be the reflection of G over BC . If E, F, G , and G' lie on a circle, compute BC .

Proposed by: Milan Haiman

Answer: 13

Note that B, C, E, F lie on a circle. Moreover, since BC bisects GG' , the center of the circle that goes through E, F, G, G' must lie on BC . Therefore, B, C, E, F, G, G' lie on a circle. Specifically, the center of this circle is M , the midpoint of BC , as $ME = MF$ because M is the center of the circumcircle of $BCEF$. So we have $GM = \frac{BC}{2}$, which gives $AM = \frac{3BC}{2}$. Then, by Apollonius's theorem, we have $AB^2 + AC^2 = 2(AM^2 + BM^2)$. Thus $845 = 5BC^2$ and $BC = 13$.

26. [13] Dan is walking down the left side of a street in New York City and must cross to the right side at one of 10 crosswalks he will pass. Each time he arrives at a crosswalk, however, he must wait t seconds, where t is selected uniformly at random from the real interval $[0, 60]$ (t can be different at different crosswalks). Because the wait time is conveniently displayed on the signal across the street, Dan employs the following strategy: if the wait time when he arrives at the crosswalk is no more than k seconds, he crosses. Otherwise, he immediately moves on to the next crosswalk. If he arrives at the last crosswalk and has not crossed yet, then he crosses regardless of the wait time. Find the value of k which minimizes his expected wait time.

Answer: $\boxed{60 \left(1 - \left(\frac{1}{10} \right)^{\frac{1}{9}} \right)}$

With probability $\left(1 - \frac{k}{60}\right)^9$, Dan reaches the last crosswalk without crossing at any previous site, in which case the expected value of his wait time is 30 seconds. Otherwise, with probability $1 - \left(1 - \frac{k}{60}\right)^9$, Dan crosses at an earlier crosswalk, in which case the expected value of his wait time is $\frac{k}{2}$. We want to find the k that minimizes

$$30 \left(1 - \frac{k}{60}\right)^9 + \frac{k}{2} \left(1 - \left(1 - \frac{k}{60}\right)^9\right) = 30 - \left(30 - \frac{k}{2}\right) \left(1 - \left(1 - \frac{k}{60}\right)^9\right)$$

Letting $a = 1 - \frac{k}{60}$, we can use weighted AM-GM:

$$9^{\frac{1}{10}} (a(1 - a^9))^{\frac{9}{10}} = (9a^9)^{\frac{1}{10}} (1 - a^9)^{\frac{9}{10}} \leq \frac{9}{10}$$

where equality occurs when $9a^9 = 1 - a^9$, or $a = \left(\frac{1}{10}\right)^{\frac{1}{9}}$, meaning that $k = 60 \left(1 - \left(\frac{1}{10}\right)^{\frac{1}{9}}\right)$. Because our original expression can be written as

$$30 - 30a(1 - a^9),$$

the minimum occurs at the same value, $k = 60 \left(1 - \left(\frac{1}{10}\right)^{\frac{1}{9}}\right)$.

27. [13] For a given positive integer n , we define $\varphi(n)$ to be the number of positive integers less than or equal to n which share no common prime factors with n . Find all positive integers n for which

$$\varphi(2019n) = \varphi(n^2).$$

Proposed by: Carl Schildkraut

Answer: $\boxed{1346, 2016, 2019}$

Let p_1, p_2, \dots, p_k be the prime divisors of n . Then it is known that $\varphi(n) = n \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k}$. As n^2 and n has the same set of prime divisors, it also holds that $\varphi(n^2) = n^2 \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k}$. We will examine the equality in four cases.

- $\gcd(n, 2019) = 1$ In this case, $2019 \cdot n$ has also 3 and 673 as prime divisors, thus $\varphi(2019 \cdot n) = 2019 \cdot n \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k} \cdot \frac{2}{3} \cdot \frac{672}{673}$, and the equality implies $n = 1342$, however $\gcd(1342, 3) \neq 1$, contradiction. Thus, there is no answer in this case.
- $\gcd(n, 2019) = 3$ In this case, $2019 \cdot n$ has also 673 as a prime divisor, thus $\varphi(2019 \cdot n) = 2019 \cdot n \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k} \cdot \frac{672}{673}$, and the equality implies $n = 2016$, which satisfies the equation. Thus, the only answer in this case is $n = 2016$.
- $\gcd(n, 2019) = 673$ In this case, $2019 \cdot n$ has also 3 as a prime divisor, thus $\varphi(2019 \cdot n) = 2019 \cdot n \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k} \cdot \frac{2}{3}$, and the equality implies $n = 1346$, which satisfies the equation. Thus, the only answer in this case is $n = 1346$.
- $\gcd(n, 2019) = 2019$ In this case, $2019 \cdot n$ has the same set of prime divisors, thus $\varphi(2019 \cdot n) = 2019 \cdot n \cdot \frac{p_1-1}{p_1} \dots \frac{p_k-1}{p_k}$, and the equality implies $n = 2019$, which satisfies the equation. Thus, the only answer in this case is $n = 2019$.

Thus, all the answers are $n = 1346, 2016, 2019$.

28. [15] A palindrome is a string that does not change when its characters are written in reverse order. Let S be a 40-digit string consisting only of 0's and 1's, chosen uniformly at random out of all such strings. Let E be the expected number of nonempty contiguous substrings of S which are palindromes. Compute the value of $\lfloor E \rfloor$.

Proposed by: Benjamin Qi

Answer: $\boxed{113}$

Note that S has $41 - n$ contiguous substrings of length n , so we see that the expected number of palindromic substrings of length n is just $(41 - n) \cdot 2^{-\lfloor n/2 \rfloor}$. By linearity of expectation, E is just the sum of this over all n from 1 to 40. However, it is much easier to just compute

$$\sum_{n=1}^{\infty} (41 - n) \cdot 2^{-\lfloor n/2 \rfloor}.$$

The only difference here is that we have added some insignificant negative terms in the cases where $n > 41$, so E is in fact slightly greater than this value (in fact, the difference between E and this sum is $\frac{7}{1048576}$). To make our infinite sum easier to compute, we can remove the floor function by pairing up consecutive terms. Then our sum becomes

$$40 + \sum_{n=1}^{\infty} \frac{81 - 4n}{2^n},$$

which is just $40 + 81 - 8 = 113$. E is only slightly larger than this value, so our final answer is $\lfloor E \rfloor = \boxed{113}$.

29. [15] In isosceles $\triangle ABC$, $AB = AC$ and P is a point on side BC . If $\angle BAP = 2\angle CAP$, $BP = \sqrt{3}$, and $CP = 1$, compute AP .

Proposed by: Milan Haiman

Answer: $\boxed{\sqrt{2}}$

Let $\angle CAP = \alpha$. By the Law of Sines, $\frac{\sqrt{3}}{\sin 2\alpha} = \frac{1}{\sin \alpha}$ which rearranges to $\cos \alpha = \frac{\sqrt{3}}{2} \Rightarrow \alpha = \frac{\pi}{6}$. This implies that $\angle BAC = \frac{\pi}{2}$. By the Pythagorean Theorem, $2AB^2 = (\sqrt{3} + 1)^2$, so $AB^2 = 2 + \sqrt{3}$. Applying Stewart's Theorem, it follows that $AP^2 = \frac{(\sqrt{3}+1)(2+\sqrt{3})}{\sqrt{3}+1} - \sqrt{3} \Rightarrow AP = \sqrt{2}$.

30. [15] A function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfies: $f(0) = 0$ and

$$|f((n+1)2^k) - f(n2^k)| \leq 1$$

for all integers $k \geq 0$ and n . What is the maximum possible value of $f(2019)$?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{4}$

Consider a graph on \mathbb{Z} with an edge between $(n+1)2^k$ and $n2^k$ for all integers $k \geq 0$ and n . Each vertex m is given the value $f(m)$. The inequality $|f((n+1)2^k) - f(n2^k)| \leq 1$ means that any two adjacent vertices of this graph must have values which differ by at most 1. Then it follows that for all m ,

$$f(m) \leq \text{number of edges in shortest path from } 0 \text{ to } m$$

because if we follow a path from 0 to m , along each edge the value increases by at most 1. Conversely, if we define $f(m)$ to be the number of edges in the shortest path between 0 and m , then this is a valid function because for any two adjacent vertices, the lengths of their respective shortest paths to 0 differ by at most 1. Hence it suffices to compute the distance from 0 to 2019 in the graph.

There exists a path with 4 edges, given by

$$0 \rightarrow 2048 \rightarrow 2016 \rightarrow 2018 \rightarrow 2019.$$

Suppose there existed a path with three edges. In each step, the number changes by a power of 2, so we have $2019 = \pm 2^{k_1} \pm 2^{k_2} \pm 2^{k_3}$ for some nonnegative integers k_1, k_2, k_3 and choice of signs. Since 2019 is odd, we must have 2^0 somewhere. Then we have $\pm 2^{k_1} \pm 2^{k_2} \in \{2018, 2020\}$. Without loss of

generality assume that $k_1 \geq k_2$. Then we can write this as $\pm 2^{k_2}(2^{k_1 k_2} \pm 1) \in \{2018, 2020\}$. It is easy to check that $k_1 = k_2$ is impossible, so the factorization $2^{k_2}(2^{k_1 k_2} \pm 1)$ is a product of a power of two and an odd number. Now compute $2018 = 2 \times 1009$ and $2020 = 4 \times 505$. Neither of the odd parts are of the form $2^{k_1 - k_2} \pm 1$, so there is no path of three steps.

We conclude that the maximum value of $f(2019)$ is 4.

31. [17] James is standing at the point $(0, 1)$ on the coordinate plane and wants to eat a hamburger. For each integer $n \geq 0$, the point $(n, 0)$ has a hamburger with n patties. There is also a wall at $y = 2.1$ which James cannot cross. In each move, James can go either up, right, or down 1 unit as long as he does not cross the wall or visit a point he has already visited.

Every second, James chooses a valid move uniformly at random, until he reaches a point with a hamburger. Then he eats the hamburger and stops moving. Find the expected number of patties that James eats on his burger.

Proposed by: Joey Heerens

Answer: $\boxed{\frac{7}{3}}$

Note that we desire to compute the number of times James moves to the right before moving down to the line $y = 0$. Note also that we can describe James's current state based on whether his y -coordinate is 0 or 1 and whether or not the other vertically adjacent point has been visited. Let $E(1, N)$ be the expected number of times James will go right before stopping if he starts at a point with y -coordinate 1 and the other available point with the same x -coordinate has not been visited. Define $E(1, Y), E(2, N)$, and $E(2, Y)$ similarly. Then we can construct equations relating the four variables:

$$E(1, N) = \frac{1}{3}E(2, Y) + \frac{1}{3}(E(1, N) + 1),$$

as James can either go up, right, or down with probability $1/3$ each if he starts in the state $(1, N)$. Similarly, we have

$$E(2, N) = \frac{1}{2}E(1, Y) + \frac{1}{2}(E(2, N) + 1), \quad E(1, Y) = \frac{1}{2}(E(1, N) + 1),$$

and $E(2, Y) = E(2, N) + 1$. Solving these equations, we get $E(1, N) = \frac{7}{3}$, which is our answer, as James starts in that state having gone left 0 times.

32. [17] A sequence of real numbers a_0, a_1, \dots, a_9 with $a_0 = 0$, $a_1 = 1$, and $a_2 > 0$ satisfies

$$a_{n+2}a_n a_{n-1} = a_{n+2} + a_n + a_{n-1}$$

for all $1 \leq n \leq 7$, but cannot be extended to a_{10} . In other words, no values of $a_{10} \in \mathbb{R}$ satisfy

$$a_{10}a_8a_7 = a_{10} + a_8 + a_7.$$

Compute the smallest possible value of a_2 .

Proposed by: Dylan Liu

Answer: $\boxed{\sqrt{2} - 1}$

Say $a_2 = a$. Then using the recursion equation, we have $a_3 = -1$, $a_4 = \frac{a+1}{a-1}$, $a_5 = \frac{-a+1}{a+1}$, $a_6 = -\frac{1}{a}$, $a_7 = -\frac{2a}{a^2-1}$, and $a_8 = 1$.

Now we have $a_{10}a_8a_7 = a_{10} + a_8 + a_7$. No value of a_{10} can satisfy this equation iff $a_8a_7 = 1$ and $a_8 + a_7 \neq 0$. Since a_8 is 1, we want $1 = a_7 = -\frac{2a}{a^2-1}$, which gives $a^2 + 2a - 1 = 0$. The only positive root of this equation is $\sqrt{2} - 1$.

This problem can also be solved by a tangent substitution. Write $a_n = \tan \alpha_n$. The given condition becomes

$$\alpha_{n+2} + \alpha_n + \alpha_{n-1} = 0.$$

We are given $\alpha_0 = 0$, $\alpha_1 = \pi/4$, and $\alpha_2 \in (0, \pi/2)$. Using this, we can recursively compute $\alpha_3, \alpha_4, \dots$ in terms of α_2 until we get to $\alpha_{10} = \frac{3\pi}{4} - 2\alpha_2$. For a_{10} not to exist, we need $\alpha_{10} \equiv \pi/2 \pmod{\pi}$. The only possible value of $\alpha_2 \in (0, \pi/2)$ is $\alpha_2 = \pi/8$, which gives $a_2 = \tan \pi/8 = \sqrt{2} - 1$.

33. [17] A circle Γ with center O has radius 1. Consider pairs (A, B) of points so that A is inside the circle and B is on its boundary. The circumcircle Ω of OAB intersects Γ again at $C \neq B$, and line AC intersects Γ again at $X \neq C$. The pair (A, B) is called *techy* if line OX is tangent to Ω . Find the area of the region of points A so that there exists a B for which (A, B) is techy.

Proposed by: Carl Schildkraut and Milan Haiman

Answer: $\boxed{\frac{3\pi}{4}}$

We claim that (A, B) is techy if and only if $OA = AB$.

Note that OX is tangent to the circle (OBC) if and only if OX is perpendicular to the angle bisector of $\angle BOC$, since $OB = OC$. Thus (A, B) is techy if and only if OX is parallel to BC . Now since $OC = OX$,

$$OX \parallel BC \iff \angle BCA = \angle OXA \iff \angle BCA = \angle ACO \iff OA = AB.$$

From the claim, the desired region of points A is an annulus between the circles centered at O with radii $\frac{1}{2}$ and 1. So the answer is $\frac{3\pi}{4}$.

34. [20] A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

A polynomial is called *k-tiny* if it has degree at most 7 and integer coefficients between $-k$ and k , inclusive.

A polynomial is called *nearly tricky* if it is the sum of a tricky polynomial and a 1-tiny polynomial.

Let N be the number of nearly tricky 7-tiny polynomials. Estimate N .

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^4 \right\rfloor$ points.

Proposed by: Carl Schildkraut

Answer: $\boxed{64912347}$

A tricky 7-tiny polynomial takes the form

$$(c_6x^6 + \dots + c_1x + c_0)(x - 4).$$

For each fixed value of k , $c_k - 4c_{k+1}$ should lie in $[-7, 7]$, so if we fix c_k , there are around $15/4$ ways of choosing c_{k+1} . Therefore if we pick c_0, \dots, c_6 in this order, there should be around $(15/4)^7$ tricky 7-tiny polynomials.

A 1-tiny polynomial takes the form $\varepsilon_6x^7 + \dots + \varepsilon_1x + \varepsilon_0$ with $\varepsilon_i \in \{-1, 0, +1\}$, so there are 3^8 1-tiny polynomials.

A nearly tricky 7-tiny polynomial P takes the form $Q+T$ where Q is roughly a tricky 7-tiny polynomial, and T is 1-tiny. Furthermore, there is a unique decomposition $Q+T$ because $T(4) = P(4)$ and each integer n can be written in the form $\sum \varepsilon_k 4^k$ in at most one way. Therefore the number of nearly tricky 7-tiny is around $(15/4)^7 \cdot 3^8 \approx 68420920$, which is worth 16 points.

The exact answer can be found by setting up recurrences. Let $t(d, \ell)$ be the number of polynomials of degree at most i of the form

$$(\ell x^{d-1} + c_{d-2}x^{d-2} + \dots + c_0)(x - 4) + (\varepsilon_{d-1}x^{d-1} + \dots + \varepsilon_1x + \varepsilon_0).$$

which has integer coefficients between -7 and 7 except the leading term ℓx^d . It follows that $t(0, 0) = 1$, $t(0, k) = 0$ for all $k \neq 0$, and $t(d+1, \ell)$ can be computed as follows: for each value of c_{d-1} , there are

$t(d, c_{d-1})$ ways to pick $c_{d-2}, \dots, c_0, \varepsilon_{d-1}, \dots, \varepsilon_0$, and exactly $w(c_{d-1} - 4\ell)$ ways of picking ε_d , where $w(k) = \min(9 - |k|, 3)$ for $|k| \leq 8$ and 0 otherwise. Therefore setting $c = c_{d-1} - 4\ell$ we have

$$t(d+1, \ell) = \sum_{c=-8}^8 t(d, c+4\ell)w(c).$$

The number of nearly tricky 7-tiny polynomials is simply $t(8, 0)$, which can be computed to be 64912347 using the following C code.

```
int w(int a){
    if(a < -9 || a > 9) return 0;
    else if(a == -8 || a == 8) return 1;
    else if(a == -7 || a == 7) return 2;
    else return 3;
}

int main()
{
    int m=8,n=7,r=4,d,l,c,c4l;
    int mid = 2 + n/r;
    int b = 2*mid+1;
    long int t[500][500];
    for(l=0; l<b; l++){
        t[0][l] = (l == mid) ? 1 : 0;
    }
    for(d=0; d<m+1; d++){
        for(l=0; l<b; l++){
            t[d+1][l] = 0;
            for(c=-8; c<9; c++){
                c4l = c + 4*(l-mid) + mid;
                t[d+1][l] += (c4l >= 0 && c4l <= 2*mid) ? t[d][c4l]*w(c) : 0;
            }
        }
    }
    printf("%ld",t[8][mid]);
}
```

35. [20] You are trying to cross a 400 foot wide river. You can jump at most 4 feet, but you have many stones you can throw into the river. You will stop throwing stones and cross the river once you have placed enough stones to be able to do so. You can throw straight, but you can't judge distance very well, so each stone ends up being placed uniformly at random along the width of the river. Estimate the expected number N of stones you must throw before you can get across the river.

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^3 \right\rfloor$ points.

Proposed by: Carl Schildkraut and Milan Haiman

Answer: 712.811

If we divide the river into 100 4-foot sections, then to be able to cross we need to get at least one stone into each section. On average, this takes

$$\frac{100}{100} + \frac{100}{99} + \dots + \frac{100}{1} \approx 100 \ln 100$$

stone throws (it takes $\frac{100}{100-k}$ moves on average to get a stone into a new section if k sections already have a stone). So the answer is at least $100 \ln 100 \approx 450$.

On the other hand, if we divide the river into 200 2-foot sections, then once we have a stone in each section we are guaranteed to be able to cross. By a similar argument, we obtain that the answer is at most $200 \ln 200 \approx 1050$.

Estimates near these bounds earn about 5 to 7 points. An estimate in between can earn close to 20 points.

To compute the answer (almost) exactly, we use the following argument.

Scale the problem so the river is of size 1, and the jumps are of size 0.01. Suppose that after n throws, the stones thrown are located at positions $0 < x_1 < x_2 < \dots < x_n < 1$. Let $x_0 = 0, x_{n+1} = 1, r = 0.01$. Define $P(n)$ to be the probability that you still cannot cross the river after n throws. In other words, there exists i such that $x_{i+1} - x_i > r$. Then our answer is $\sum_{n=0}^{\infty} P(n)$.

By PIE we can write

$$P(n) = \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+1}{i} \max(1 - ir, 0)^n.$$

based on which intervals $x_{i+1} - x_i$ have length greater than r . Now we switch the order of summation:

$$\sum_{n=0}^{\infty} P(n) = \sum_{n=0}^{\infty} \sum_{i=1}^{\infty} (-1)^{i-1} \binom{n+1}{i} \max(1 - ir, 0)^n = \sum_{i=1}^{\infty} (-1)^{i-1} \sum_{n=0}^{\infty} \binom{n+1}{i} \max(1 - ir, 0)^n.$$

Let $x = \max(1 - ir, 0)$. Then

$$\sum_{n=0}^{\infty} \binom{n+1}{i} x^n = x^{i-1} \sum_{j=0}^{\infty} \binom{i+j}{i} x^j = \frac{x^{i-1}}{(1-x)^{i+1}}.$$

Thus, our answer is

$$\sum_{i=1}^{\lfloor 1/r \rfloor} (-1)^{i-1} \frac{(1-ir)^{i-1}}{(ir)^{i+1}} \approx 712.811,$$

where the last approximation uses the C++ code below.

```
#include <bits/stdc++.h>

using namespace std;

typedef long double ld;

int main() {
    ld sum = 0, r = 0.01;
    for (int i = 1; ; ++i) {
        ld x = 1-r*i; if (x <= 0) break;
        ld ex = pow(x/(1-x), i-1)/(1-x)/(1-x);
        if (i&1) sum += ex;
        else sum -= ex;
    }
    cout << fixed << setprecision(8) << sum;
}
```

36. [20] Let N be the number of sequences of positive integers $(a_1, a_2, a_3, \dots, a_{15})$ for which the polynomials

$$x^2 - a_i x + a_{i+1}$$

each have an integer root for every $1 \leq i \leq 15$, setting $a_{16} = a_1$. Estimate N .

An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \right\rfloor$ points.

Proposed by: Krit Boonsiriseth

Answer: 1409

We note that $a_{i+1} = x(a_i - x)$ for some positive integer x , so $a_{i+1} \geq a_i - 1$. So, the only way a_i can decrease is decreasing by 1.

As it cannot decrease that quickly, we will make the assumption that if $a_i \geq 10$, $a_{i+1} = a_i - 1$, as otherwise it will increase at least above 16 at which point it will take many moves to go back down below 10. Write that $a \rightarrow b$ if b is a possible value of a_{i+1} given $a = a_i$. We have

$$5 \rightarrow 6, 6 \rightarrow 5, 8, 9, 7 \rightarrow 6, 8 \rightarrow 7, 9 \rightarrow 8,$$

and in addition by going to 10 and above, 7 can go to 9 in 2 or 4 steps, 8 can in 4, 7, 8 steps, and 9 can in 6, 10, 12 steps. We see from this that the vast majority of sequences should pass through 8. By looking at cycles from 8, we can determine exactly when a sequence can start at 8 and return to 8 (there is one way in 3 steps, two in 4 steps, etc.), and from there we can generate a list of types of sequences by when 8s occur. By dividing by the number of 8s and multiplying by 15, we can get the number of sequences that include 8, which gives us an estimate of 1235, giving us 15 points. As we note that this is a lower estimate, we may round up slightly to get better results.

To find the exact answer, we will first show that no element larger than 32 can occur in the sequence. Reorder the sequence to make a_1 maximal; we have

$$a_{i+1} \geq a_i - 1 \implies a_{15} \geq a_1 - 14.$$

Also, since $a_1 > a_{15}$, $a_1 \geq 2a_{15} - 4$, giving

$$a_1 - 14 \leq \frac{a_1 + 4}{2} \implies a_1 \leq 32.$$

We then construct the following Python code:

```
def p36(max_val,length):
L=[[i] for i in range(1,max_val+1)]
for j in range(length):
newL=[]
for k in L:
poss=[x*(k[-1]-x) for x in range(1,k[-1]//2+1)]
for t in poss:
if 1<=t<=max_val:
newL.append(k+[t])
L=newL
return len(L)

print(p36(32,15))
```

This gives the exact answer of 1409.

HMMT November 2019

November 9, 2019

Team Round

1. [20] Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice, where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $\frac{1}{6}$ of the total amount of water drunk and $\frac{1}{8}$ of the total amount of apple juice drunk. How many people are in Cambridge?

Proposed by: Alec Sun

Answer:

The total amount of liquid drunk must be more than 6 times of the amount that Marc drinks but less than 8 times of the amount that Marc drinks. Therefore, the number of people in Cambridge is more than 6 but less than 8, so it must be 7.

2. [20] 2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

Proposed by: Milan Haiman

Answer:

To get an item, a student must receive at least 2 votes on that item. Since each student receives at most 2019 votes, the number of items one student can receive does not exceed $\frac{2019}{2} = 1009.5$. So, the answer is at most 1009. This occurs when $N = 2018$ and item i was voted to student $1, 1, 2, 3, \dots, 2018$ by student $2i - 1, 2i - 2, \dots, 2019, 1, \dots, 2i - 2$ respectively for $i = 1, 2, \dots, 2018$. Thus, the maximum possible number of items one student can receive is 1009.

3. [30] The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

Proposed by: Carl Joshua Quines

Answer:

Let

$$P(x) = a_0 + a_1x + a_2x^2 + \dots$$

Then

$$\frac{1}{2}(P(10) + P(-10)) = a_0 + 100a_2 + \dots$$

and

$$\frac{1}{2}(P(10) - P(-10)) = 10a_1 + 1000a_3 + \dots$$

Since all the coefficients are nonnegative integers, these expressions give us each of the coefficients by just taking two digits in succession. Thus we have $a_0 = 3$, $a_1 = 13$, $a_2 = 25$, $a_3 = 29$, $a_4 = 30$ and $a_n = 0$ for $n > 4$. Thus

$$P(1) = a_0 + a_1 + a_2 + \dots = 100.$$

4. [35] Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n . If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

Proposed by: Milan Haiman

Answer:

The first player has a winning strategy if and only if N is not a multiple of 3. We show this by induction on N . If $N = 0$, then the first player loses.

If N is a multiple of 3, then $N - 2^n$ is never a multiple of 3 for any n , so the second player has a winning strategy.

If N is not a multiple of 3, the first player can remove either 1 or 2 coins to get the number of coins in the pile down to a multiple of 3, so the first player will always win.

5. [40] Compute the sum of all positive real numbers $x \leq 5$ satisfying

$$x = \frac{[x^2] + [x] \cdot [x]}{[x] + [x]}.$$

Proposed by: Milan Haiman

Answer: $\boxed{85}$

Note that all integer x work. If x is not an integer then suppose $n < x < n + 1$. Then $x = n + \frac{k}{2n+1}$, where n is an integer and $1 \leq k \leq 2n$ is also an integer, since the denominator of the fraction on the right hand side is $2n + 1$. We now show that all x of this form work.

Note that

$$x^2 = n^2 + \frac{2nk}{2n+1} + \left(\frac{k}{2n+1}\right)^2 = n^2 + k - \frac{k}{2n+1} + \left(\frac{k}{2n+1}\right)^2.$$

For $\frac{k}{2n+1}$ between 0 and 1, $-\frac{k}{2n+1} + \left(\frac{k}{2n+1}\right)^2$ is between $-\frac{1}{4}$ and 0, so we have $n^2 + k - 1 < x^2 \leq n^2 + k$, and $[x^2] = n^2 + k$.

Then,

$$\frac{[x^2] + [x] \cdot [x]}{[x] + [x]} = \frac{n^2 + k + n \cdot (n + 1)}{2n + 1} = n + \frac{k}{2n + 1} = x,$$

so all x of this form work.

Now, note that the $2n$ solutions in the interval $(n, n + 1)$, together with the solution $n + 1$, form an arithmetic progression with $2n + 1$ terms and average value $n + \frac{n+1}{2n+1}$. Thus, the sum of the solutions in the interval $(n, n + 1]$ is $2n^2 + 2n + 1 = n^2 + (n + 1)^2$. Summing this for n from 0 to 4, we get that the answer is

$$0^2 + 2(1^2 + 2^2 + 3^2 + 4^2) + 5^2 = 85.$$

6. [45] Let $ABCD$ be an isosceles trapezoid with $AB = 1$, $BC = DA = 5$, $CD = 7$. Let P be the intersection of diagonals AC and BD , and let Q be the foot of the altitude from D to BC . Let PQ intersect AB at R . Compute $\sin \angle RPD$.

Proposed by: Milan Haiman

Answer: $\boxed{\frac{4}{5}}$

Let M be the foot of the altitude from B to CD . Then $2CM + AB = CD \implies CM = 3$. Then $DM = 4$ and by the Pythagorean theorem, $BM = 4$. Thus BMD is a right isosceles triangle i.e. $\angle BDM = \angle PDC = \frac{\pi}{4}$. Similarly, $\angle PCD = \frac{\pi}{4}$. Thus $\angle DPC = \frac{\pi}{2}$, which means quadrilateral $PQDC$ is cyclic. Now, $\sin \angle RPD = \sin \angle DCQ = \sin \angle MCB = \frac{4}{5}$.

An alternate solution is also possible:

Note that $AC \perp BD$ since $AB^2 + CD^2 = 1^2 + 7^2 = 5^2 + 5^2 = BC^2 + DA^2$. Thus P is the foot of the altitude from D to AC . Since D is on the circumcircle of $\triangle ABC$, line PQR is the Simson line of D . Thus R is the foot from D to AB . Then from quadrilateral $RAPD$ being cyclic we have $\angle RPD = \angle RAD$. So $\sin \angle RPD = \frac{4}{5}$.

7. [55] Consider sequences a of the form $a = (a_1, a_2, \dots, a_{20})$ such that each term a_i is either 0 or 1. For each such sequence a , we can produce a sequence $b = (b_1, b_2, \dots, b_{20})$, where

$$b_i = \begin{cases} a_i + a_{i+1} & i = 1 \\ a_{i-1} + a_i + a_{i+1} & 1 < i < 20 \\ a_{i-1} + a_i & i = 20. \end{cases}$$

How many sequences b are there that can be produced by more than one distinct sequence a ?

Proposed by: Benjamin Qi

Answer: $\boxed{64}$

Let the two sequences be b and \hat{b} . Then, observe that given a , if $b_1 = \hat{b}_1$ and $b_2 = \hat{b}_2$, then $b = \hat{b}$ (since a will uniquely determine the remaining elements in b and \hat{b}). Thus, b and \hat{b} must start with $(1, 0, \dots)$ and $(0, 1, \dots)$, respectively (without loss of generality).

Note that a_3 is either 1 (in which case $b_3 = \hat{b}_3 = 0$) or 2 (in which case $b_3 = \hat{b}_3 = 1$). Moreover, b_4, b_5 must be the same as b_1, b_2 (and the same for \hat{b}) for the sequences to generate the same a_3, a_4 . We can then pick a_6, a_9, \dots

Observe, that the last elements also have to be $(\dots, 1, 0)$ for b and $(\dots, 0, 1)$ for \hat{b} . Thus, the answer is nonzero only for sequence lengths of $2 \pmod{3}$, in which case, our answer is 2^k , where the length is $3k + 2$ (since we have two choices for every third element).

Here, since $N = 20 = 3(6) + 2$, the answer is $2^6 = 64$.

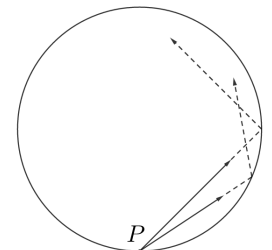
8. [60] In $\triangle ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D . E is a point on ray \overrightarrow{AC} such that $\angle BDE = 2\angle ADB$. If $AB = 10$, $AC = 12$, and $CE = 33$, compute $\frac{DB}{DE}$.

Proposed by: Milan Haiman

Answer: $\boxed{\frac{2}{3}}$

Let F be a point on ray \overrightarrow{CA} such that $\angle ADF = \angle ADB$. $\triangle ADF$ and $\triangle ADB$ are congruent, so $AF = 10$ and $DF = DB$. So, $CF = CA + AF = 22$. Since $\angle FDC = 2\angle ADB = \angle EDC$, by the angle bisector theorem we compute $\frac{DF}{DE} = \frac{CF}{CE} = \frac{22}{33} = \frac{2}{3}$.

9. [65] Will stands at a point P on the edge of a circular room with perfectly reflective walls. He shines two laser pointers into the room, forming angles of n° and $(n + 1)^\circ$ with the tangent at P , where n is a positive integer less than 90. The lasers reflect off of the walls, illuminating the points they hit on the walls, until they reach P again. (P is also illuminated at the end.) What is the minimum possible number of illuminated points on the walls of the room?



Proposed by: Handong Wang

Answer: $\boxed{28}$

Note that we want the path drawn out by the lasers to come back to P in as few steps as possible. Observe that if a laser is fired with an angle of n degrees from the tangent, then the number of points it creates on the circle is $\frac{180}{\gcd(180, n)}$. (Consider the regular polygon created by linking all the points that show up on the circle—if the center of the circle is O , and the vertices are numbered V_1, V_2, \dots, V_k , the angle $\angle V_1OV_2$ is equal to $2\gcd(180, n)$, so there are a total of $\frac{360}{2\gcd(180, n)}$ sides).

Now, we consider the case with both n and $n + 1$. Note that we wish to minimize the value $\frac{180}{\gcd(180, n)} + \frac{180}{\gcd(180, n+1)}$, or maximize both $\gcd(180, n)$ and $\gcd(180, n + 1)$. Note that since n and $n + 1$ are relatively prime and $180 = (4)(9)(5)$, the expression is maximized when $\gcd(180, n) = 20$ and $\gcd(180, n + 1) = 9$ (or vice versa). This occurs when $n = 80$. Plugging this into our expression, we have that the number of points that show up from the laser fired at 80 degrees is $\frac{180}{20} = 9$ and the number of points that appear from the laser fired at 81 degrees is $\frac{180}{9} = 20$. However, since both have a point that shows up at P (and no other overlapping points since $\gcd(9, 20) = 1$), we see that the answer is $20 + 9 - 1 = 28$.

10. [70] A convex 2019-gon $A_1A_2 \dots A_{2019}$ is cut into smaller pieces along its 2019 diagonals of the form A_iA_{i+3} for $1 \leq i \leq 2019$, where $A_{2020} = A_1$, $A_{2021} = A_2$, and $A_{2022} = A_3$. What is the least possible number of resulting pieces?

Proposed by: Krit Boonsiriseth

Answer: 5049

Each time we draw in a diagonal, we create one new region, plus one new region for each intersection on that diagonal. So, the number of regions will be

$$1 + (\text{number of diagonals}) + (\text{number of intersections}),$$

where (number of intersections) counts an intersection of three diagonals twice. Since no four diagonals can pass through a point, the only nonconstant term in our expression is the last one. To minimize this term, we want to maximize the number of triples of diagonals passing through the same point. Consider the set S of triples of diagonals A_nA_{n+3} that intersect at a single point. Each triple in S must come from three consecutive diagonals, and two different triples can only have one diagonal in common, so S has at most $\lfloor \frac{2019}{2} \rfloor = 1009$ triples. Hence the number of resulting pieces is at least

$$1 + (2019) + (2 \cdot 2019 - 1009) = 5049.$$

To show that 5049 is attainable, we use the following construction. Let $B_1 \dots B_{1010}$ be a regular 1010-gon, and let ℓ_n denote the external angle bisector of $\angle B_{n-1}B_nB_{n+1}$. Let $A_1 = \overleftrightarrow{B_{1009}B_{1010}} \cap \overleftrightarrow{B_1B_2}$, $A_{2018} = \overleftrightarrow{B_{1008}B_{1009}} \cap \overleftrightarrow{B_{1010}B_1}$, $A_{2019} = \ell_1 \cap \ell_{1009}$, and for $n = 1, \dots, 1008$, define $A_{2n} = \ell_{n+1} \cap \overleftrightarrow{B_{n-1}B_n}$ and $A_{2n+1} = \ell_n \cap \overleftrightarrow{B_{n+1}B_{n+2}}$. It follows that, for all $n = 0, \dots, 1008$, $\overleftrightarrow{A_{2n-1}A_{2n+2}}$, $\overleftrightarrow{A_{2n}A_{2n+3}}$, and $\overleftrightarrow{A_{2n+1}A_{2n+4}}$ intersect at B_{n+1} .

HMMT November 2019

November 9, 2019

Theme Round

Illustrations by Hanna Yang



1. For breakfast, Mihir always eats a bowl of Lucky Charms cereal, which consists of oat pieces and marshmallow pieces. He defines the *luckiness* of a bowl of cereal to be the ratio of the number of marshmallow pieces to the total number of pieces. One day, Mihir notices that his breakfast cereal has exactly 90 oat pieces and 9 marshmallow pieces, and exclaims, “This is such an unlucky bowl!” How many marshmallow pieces does Mihir need to add to his bowl to double its luckiness?

Proposed by: Krit Boonsiriseth

Answer: $\boxed{11}$

Let x be the number of marshmallows to add. We are given that

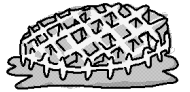
$$2 \cdot \frac{9}{99} = \frac{9+x}{99+x}.$$

Rearranging this gives

$$2(99+x) = 11(9+x).$$

Thus $9x = 99$ and $x = 11$.

2. Sandy likes to eat waffles for breakfast. To make them, she centers a circle of waffle batter of radius 3cm at the origin of the coordinate plane and her waffle iron imprints non-overlapping unit-square holes **centered** at each lattice point. How many of these holes are contained entirely within the area of the waffle?

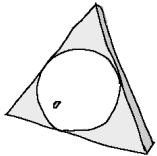


Proposed by: Carl Schildkraut

Answer: $\boxed{21}$

First, note that each divet must have its sides parallel to the coordinate axes; if the divet centered at the lattice point (a, b) does not have this orientation, then it contains the point $(a + 1/2, b)$ in its interior, so it necessarily overlaps with the divet centered at $(a + 1, b)$.

If we restrict our attention to one quadrant, we see geometrically that the divets centered at $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, and $(2, 1)$ are completely contained in the waffle, and no others are. We can make this more rigorous by considering the set of points (x, y) such that $x^2 + y^2 < 9$. We count 1 divet centered at the origin, 8 divets centered on the axes that are not centered at the origin, and 12 divets not centered on the axes, for a total of 21 divets.



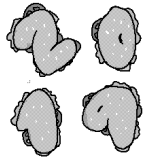
3. For breakfast, Milan is eating a piece of toast shaped like an equilateral triangle. On the piece of toast rests a single sesame seed that is one inch away from one side, two inches away from another side, and four inches away from the third side. He places a circular piece of cheese on top of the toast that is tangent to each side of the triangle. What is the area of this piece of cheese?

Proposed by: Carl Schildkraut

Answer: $\boxed{\frac{49\pi}{9}}$

Suppose the toast has side length s . If we draw the three line segments from the sesame seed to the three vertices of the triangle, we partition the triangle into three smaller triangles, with areas $\frac{s}{2}$, s , and $2s$, so the entire piece of toast has area $\frac{7s}{2}$. Suppose the cheese has radius r . We similarly see that the toast has area $\frac{3rs}{2}$. Equating these, we see that $r = \frac{7}{3}$, so the area of the cheese is $\pi \left(\frac{7}{3}\right)^2 = \frac{49\pi}{9}$.

4. To celebrate 2019, Faraz gets four sandwiches shaped in the digits 2, 0, 1, and 9 at lunch. However, the four digits get reordered (but not flipped or rotated) on his plate and he notices that they form a 4-digit multiple of 7. What is the greatest possible number that could have been formed?



Proposed by: Milan Haiman

Answer: 1092

Note that 2 and 9 are equivalent mod 7. So we will replace the 9 with a 2 for now. Since 7 is a divisor of 21, a four digit multiple of 7 consisting of 2, 0, 1, and 2 cannot have a 2 followed by a 1 (otherwise we could subtract a multiple of 21 to obtain a number of the form $2 \cdot 10^k$). Thus our number either starts with a 1 or has a 0 followed by a 1. We can check that 2201 and 2012 are not divisible by 7. Thus our number starts with a 1. Checking 1220, 1202, and 1022 gives that 1022 is the only possibility. So the answer is 1092.



5. Alison is eating 2401 grains of rice for lunch. She eats the rice in a very peculiar manner: every step, if she has only one grain of rice remaining, she eats it. Otherwise, she finds the smallest positive integer $d > 1$ for which she can group the rice into equal groups of size d with none left over. She then groups the rice into groups of size d , eats one grain from each group, and puts the rice back into a single pile. How many steps does it take her to finish all her rice?

Proposed by: Carl Schilkraut

Answer: 17

Note that $2401 = 7^4$. Also, note that the operation is equivalent to replacing n grains of rice with $n \cdot \frac{p-1}{p}$ grains of rice, where p is the smallest prime factor of n .

Now, suppose that at some moment Alison has 7^k grains of rice. After each of the next four steps, she will have $6 \cdot 7^{k-1}$, $3 \cdot 7^{k-1}$, $2 \cdot 7^{k-1}$, and 7^{k-1} grains of rice, respectively. Thus, it takes her 4 steps to decrease the number of grains of rice by a factor of 7 given that she starts at a power of 7.

Thus, it will take $4 \cdot 4 = 16$ steps to reduce everything to $7^0 = 1$ grain of rice, after which it will take one step to eat it. Thus, it takes a total of 17 steps for Alison to eat all of the rice.

6. Wendy eats sushi for lunch. She wants to eat six pieces of sushi arranged in a 2×3 rectangular grid, but sushi is sticky, and Wendy can only eat a piece if it is adjacent to (not counting diagonally) at most two other pieces. In how many orders can Wendy eat the six pieces of sushi, assuming that the pieces of sushi are distinguishable?



Proposed by: Milan Haiman

Answer: 360

Call the sushi pieces A, B, C in the top row and D, E, F in the bottom row of the grid. Note that Wendy must first eat either A, C, D , or F . Due to the symmetry of the grid, all of these choices are equivalent. Without loss of generality, suppose Wendy eats piece A .

Now, note that Wendy cannot eat piece E , but can eat all other pieces. If Wendy eats piece B, D , or F , then in the resulting configuration, all pieces of sushi are adjacent to at most 2 pieces, so she will have $4!$ ways to eat the sushi. Thus, the total number of possibilities in this case is $4 \cdot 3 \cdot 4! = 288$.

If Wendy eats A and then C , then Wendy will only have 3 choices for her next piece of sushi, after which she will have $3!$ ways to eat the remaining 3 pieces of sushi. Thus, the total number of possibilities in this case is $4 \cdot 1 \cdot 3 \cdot 3! = 72$.

Thus, the total number of ways for Wendy to eat the sushi is $288 + 72 = 360$.



7. Carl only eats food in the shape of equilateral pentagons. Unfortunately, for dinner he receives a piece of steak in the shape of an equilateral triangle. So that he can eat it, he cuts off two corners with straight cuts to form an equilateral pentagon. The set of possible perimeters of the pentagon he obtains is exactly the interval $[a, b]$, where a and b are positive real numbers. Compute $\frac{a}{b}$.

Proposed by: Krit Boonsiriseth and Milan Haiman

Answer: $4\sqrt{3} - 6$

Assume that the triangle has side length 1. We will show the pentagon side length x is in $[2\sqrt{3} - 3, \frac{1}{2})$. Call the triangle ABC and let corners B, C be cut. Choose P on AB , Q, R on BC , and S on AC such that $APQRS$ is equilateral. If $x \geq \frac{1}{2}$ then Q is to the right of R , causing self-intersection. Also the distance from P to BC is at most x , so

$$x = PQ \geq PB \sin 60^\circ = (1 - x) \cdot \frac{\sqrt{3}}{2}.$$

Solving gives $(2 + \sqrt{3})x \geq \sqrt{3}$, or $x \geq \sqrt{3}(2 - \sqrt{3}) = 2\sqrt{3} - 3$. Finally, these are attainable if we choose P such that $AP = x$, then Q such that $PQ = x$, and so on. Therefore $\frac{a}{b} = 4\sqrt{3} - 6$.

8. Omkar, Krit₁, Krit₂, and Krit₃ are sharing $x > 0$ pints of soup for dinner. Omkar always takes 1 pint of soup (unless the amount left is less than one pint, in which case he simply takes all the remaining soup). Krit₁ always takes $\frac{1}{6}$ of what is left, Krit₂ always takes $\frac{1}{5}$ of what is left, and Krit₃ always takes $\frac{1}{4}$ of what is left. They take soup in the order of Omkar, Krit₁, Krit₂, Krit₃, and then cycle through this order until no soup remains. Find all x for which everyone gets the same amount of soup.



Proposed by: Krit Boonsiriseth and Milan Haiman

Answer: $\frac{49}{3}$

The main observation is that if $x > 1$ pints of soup are left, then in one round, Omkar gets 1 and each Krit _{n} gets $\frac{x-1}{6}$, with $\frac{x-1}{2}$ soup left. Thus it is evident that each Krit _{n} gets the same amount of soup, which means it suffices to find x for which Omkar gets $\frac{x}{4}$.

Omkar gets 1 for each cycle and then all the remaining soup when there is less than one pint remaining. The amount of soup becomes (after each cycle)

$$x \rightarrow \frac{x-1}{2} \rightarrow \frac{x-3}{4} \rightarrow \dots \rightarrow \frac{x+1}{2^n} - 1,$$

so if n is the number of cycles, then Omkar's soup is $n + \frac{x+1}{2^n} - 1$. Setting this equal to $\frac{x}{4}$, we obtain

$$x = \frac{n + 1/2^n - 1}{1/4 - 1/2^n} = \frac{(n-1)2^n + 1}{2^{n-2} - 1}.$$

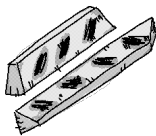
This immediately implies $n > 2$. On the other hand, we necessarily have $0 \leq \frac{x+1}{2^n} - 1 \leq 1$, so $2^n \leq x + 1 \leq 2^{n+1}$. But

$$x + 1 = \frac{(n-1)2^n + 2^{n-2}}{2^{n-2} - 1} \leq \frac{(n-1)2^n + 2^n}{2^{n-3}} = 8n,$$

So $2^n \leq 8n \implies n \leq 5$. Testing $n = 3, 4, 5$:

- For $n = 3$ we get $x = 17$ which is greater than 2^4 .
- For $n = 4$ we get $x = \frac{49}{3}$ which works.
- For $n = 5$ we get $x = \frac{129}{7}$ which is less than 2^5 .

We see that only $n = 4$ and $x = \frac{49}{3}$ works.



9. For dinner, Priya is eating grilled pineapple spears. Each spear is in the shape of the quadrilateral $PINE$, with $PI = 6\text{cm}$, $IN = 15\text{cm}$, $NE = 6\text{cm}$, $EP = 25\text{cm}$, and $\angle NEP + \angle EPI = 60^\circ$. What is the area of each spear, in cm^2 ?

Proposed by: Milan Haiman

Answer: $\boxed{\frac{100\sqrt{3}}{3}}$

We consider a configuration composed of 2 more quadrilaterals congruent to $PINE$. Let them be $P'I'N'E'$, with $E' = P$ and $N' = I$, and $P''I''N''E''$ with $P'' = E, E'' = P', N'' = I'$, and $I'' = N$. Notice that this forms an equilateral triangle of side length 25 since $\angle PP'P'' = \angle PP''P' = \angle P'PP'' = 60^\circ$. Also, we see that the inner triangle $NN'N''$ forms an equilateral triangle of side length 15 since all the side lengths are equal. So the area inside the big equilateral triangle and outside the small one is $\frac{625\sqrt{3}}{4} - \frac{225\sqrt{3}}{4} = 100\sqrt{3}$. Since there are two other congruent quadrilaterals to $PINE$, we have that the area of one of them is $\frac{100\sqrt{3}}{3}$.

10. For dessert, Melinda eats a spherical scoop of ice cream with diameter 2 inches. She prefers to eat her ice cream in cube-like shapes, however. She has a special machine which, given a sphere placed in space, cuts it through the planes $x = n, y = n,$ and $z = n$ for every integer n (not necessarily positive). Melinda centers the scoop of ice cream uniformly at random inside the cube $0 \leq x, y, z \leq 1$, and then cuts it into pieces using her machine. What is the expected number of pieces she cuts the ice cream into?



Proposed by: Carl Schildkraut

Answer: $\boxed{7 + \frac{13\pi}{3}}$

Note that if we consider the division of \mathbb{R}^3 into unit cubes by the given planes, we only need to compute the sum of the probabilities that the ice cream scoop intersects each cube. There are three types of cubes that can be intersected:

- The cube $0 \leq x, y, z \leq 1$ in which the center lies, as well as the 6 face-adjacent cubes are always intersected, for a total of 7.
- The cubes edge-adjacent to the center cube are intersected if the center of the ice cream lies within 1 unit of the connecting edge, which happens with probability $\frac{\pi}{4}$. There are 12 such cubes, for a total of 3π .
- The cubes corner-adjacent to the center cube are intersected if the center of the ice cream lies within 1 unit of the connecting corner, which happens with probability $\frac{\pi}{6}$. There are 8 such cubes, for a total of $\frac{4\pi}{3}$.

Adding these all up gives our answer of $7 + \frac{13\pi}{3}$.

An alternate solution is possible:

We compute the number of regions into which a convex region S in \mathbb{R}^3 is divided by planes: Let a be the number of planes intersecting S . Let b be the number of lines (intersections of two planes) passing through S . Let c be the number of points (intersections of three planes) lying inside S . Then S is divided into $a + b + c + 1$ regions. Then the computation for the problem is fairly straight forward. Note that the only planes, lines, and points that can intersect the ice cream scoop I are the faces, edges, and vertices of the cube $0 \leq x, y, z \leq 1$. The computation is essentially the same as in the first solution. The scoop intersects each of the 6 faces with probability 1, each of the 12 edges with probability $\frac{\pi}{4}$, and each of the 8 vertices with probability $\frac{\pi}{6}$, for a total expected number of regions $1 + 6 + 3\pi + \frac{4\pi}{3} = 7 + \frac{13\pi}{3}$.

HMIC 2019
March 30, 2019 – April 5, 2019

1. [5] Let ABC be an acute scalene triangle with incenter I . Show that the circumcircle of BIC intersects the Euler line of ABC in two distinct points.

(Recall that the *Euler line* of a scalene triangle is the line that passes through its circumcenter, centroid, orthocenter, and the nine-point center.)

Proposed by: Andrew Gu

Let O and H be the circumcenter and orthocenter of ABC . Recall that

$$\begin{aligned}\angle BOC &= 2\angle A, \\ \angle BHC &= 180^\circ - \angle A, \\ \angle BIC &= 90^\circ + \frac{1}{2}\angle A.\end{aligned}$$

As ABC is acute, A, I, O, H all lie on the same side of BC .

- If $\angle A > 60^\circ$, then $\angle BOC > \angle BIC$, so O lies inside (BIC) .
- If $\angle A < 60^\circ$, then $\angle BHC > \angle BIC$, so H lies inside (BIC) .
- If $\angle A = 60^\circ$, then O and H are on (BIC) .

In all cases, two intersections are guaranteed.

Note: For an obtuse triangle, the statement is not necessarily true.

2. [7] Annie has a permutation $(a_1, a_2, \dots, a_{2019})$ of $S = \{1, 2, \dots, 2019\}$, and Yannick wants to guess her permutation. With each guess Yannick gives Annie an n -tuple $(y_1, y_2, \dots, y_{2019})$ of integers in S , and then Annie gives the number of indices $i \in S$ such that $a_i = y_i$.

- (a) Show that Yannick can always guess Annie's permutation with at most 1200000 guesses.
 (b) Show that Yannick can always guess Annie's permutation with at most 24000 guesses.

Proposed by: Yuan Yao

Part (a) uses the idea that for $x \neq y$, the guess (x, y, y, \dots, y) returns 0 if the first number is y , 2 if the first number is x , and 1 if the number is something else. If he tests the pairs $(x, y) = (1, 2), \dots, (2017, 2018)$, he will get a 0 or 2 (and therefore find out what the first number is) or get only 1 (so the first number is 2019). In general, with n numbers unknown, Yannick needs $\lfloor \frac{n}{2} \rfloor$ guesses to find out the next number in the permutation. Using this strategy, he can guess the permutation within at most

$$\sum_{n=1}^{2019} \left\lfloor \frac{n}{2} \right\rfloor + 1 = 1009 \cdot 1010 + 1 = 1019091$$

guesses.

Part (b) uses the idea that after the first number has been determined and n numbers remain undetermined, it takes $\lceil \log_2 n \rceil$ guesses to determine the index of another number, by doing a binary search to halve the candidate positions each time (using the first number as a filler). Using this Yannick can guess the permutation within

$$1009 + \sum_{n=1}^{2018} \lceil \log_2 n \rceil + 1 = 1009 + 1 \cdot 1 + 2 \cdot 2 + 4 \cdot 3 + \dots + 512 \cdot 10 + 994 \cdot 11 + 1 = 21161$$

guesses.

3. [8] Do there exist four points $P_i = (x_i, y_i) \in \mathbb{R}^2$ ($1 \leq i \leq 4$) on the plane such that:

- for all $i = 1, 2, 3, 4$, the inequality $x_i^4 + y_i^4 \leq x_i^3 + y_i^3$ holds, and
- for all $i \neq j$, the distance between P_i and P_j is greater than 1?

Proposed by: Pakawut Jiradilok

In fact, there are! One might think that the region

$$\{(x, y) \in \mathbb{R}^2 \mid x^4 + y^4 \leq x^3 + y^3\}$$

is inside the ball defined by $x^2 + y^2 \leq x + y$, which is a ball of radius $1/\sqrt{2}$. It turns out that it is not the case.

We claim that for all $\epsilon > 0$ small enough, we can choose $(0, 0)$, $(1, 1)$, $(1 - \epsilon^2, -2\epsilon)$, and $(-2\epsilon, 1 - \epsilon^2)$. First we check the condition $x_i^4 + y_i^4 \leq x_i^3 + y_i^3$. This is obviously satisfied for the first two points, and for the other two points this translates to

$$\begin{aligned} (1 - \epsilon^2)^4 + (-2\epsilon)^4 &\leq (1 - \epsilon^2)^3 + (-2\epsilon)^3 \\ (2\epsilon)^3(1 + 2\epsilon) &\leq (1 - \epsilon^2)^3(\epsilon^2) \\ 8\epsilon(1 + 2\epsilon) &\leq (1 - \epsilon^2)^3. \end{aligned}$$

This is satisfied for small ϵ . Next we check that the distances are greater than 1. Clearly the third and fourth points are at distance greater than 1 from $(1, 1)$ as they have negative y and x coordinates respectively. They are also at distance greater than 1 from each other for small ϵ as their distance tends to $\sqrt{2}$ as $\epsilon \rightarrow 0$. Also, $(0, 0)$ and $(1, 1)$ are at distance greater than 1 as well. The only remaining distances to check are the distance from the first point to the third and fourth points (which are the same distance). This distance is

$$\sqrt{(1 - \epsilon^2)^2 + (-2\epsilon)^2} = \sqrt{(1 + \epsilon^2)^2} = 1 + \epsilon^2 > 0$$

so all conditions are satisfied (for sufficiently small ϵ).

For example, one may choose $\epsilon = 1/20$ to obtain the four points $(0, 0)$, $(1, 1)$, $(-\frac{1}{10}, \frac{399}{400})$ and $(\frac{399}{400}, -\frac{1}{10})$.

4. [10] A *cactus* is a finite simple connected graph where no two cycles share an edge. Show that in a nonempty cactus, there must exist a vertex which is part of at most one cycle.

Proposed by: Kevin Yang

Let C be the original cactus. For every cycle in C , arbitrarily remove one of its edges, yielding a new graph T . Observe that since the cycles are edge-disjoint, we removed *exactly* one edge from every cycle, meaning the graph stays connected. However, there are no longer any cycles, so T is a tree.

Now consider any leaf v of T (note that any nonempty tree must have a leaf). If v is the only vertex in the graph we're trivially done, since then v was the only vertex in C . Otherwise v has degree 1. If v was originally a leaf of C , we're done. If not, observe that in the process of turning C into T , a vertex's degree cannot decrease by more than half, because for every cycle that a vertex is part of in C , it gains a degree of 2, but can only lose 1 degree from an edge of that cycle being removed. Therefore, the original degree of v in C was at most 2, meaning it could have been part of at most 1 cycle, as desired.

Remark. The problem originally did not state that the graph is finite. We apologize for the omission.

5. [12] Let $p = 2017$ be a prime and \mathbb{F}_p be the integers modulo p . A function $f : \mathbb{Z} \rightarrow \mathbb{F}_p$ is called *good* if there is $\alpha \in \mathbb{F}_p$ with $\alpha \not\equiv 0 \pmod{p}$ such that

$$f(x)f(y) = f(x + y) + \alpha^y f(x - y) \pmod{p}$$

for all $x, y \in \mathbb{Z}$. How many good functions are there that are periodic with minimal period 2016?

Proposed by: Ashwin Sah

Answer: 1327392

We will classify all good functions with parameter $\alpha \in \mathbb{F}_p \setminus \{0\}$. We will use $=$ to denote equality modulo p when appropriate. Let the given statement be $P(x, y)$. Then $P(x, 0)$ gives $f(x)f(0) = 2f(x)$ for all $x \in \mathbb{Z}$. f cannot be the zero function because it does not have minimal period 2016. Therefore $f(0) = 2$.

$P(x, 1)$ gives $f(x+1) - f(1)f(x) + \alpha f(x-1) = 0$. This means that f satisfies a recurrence of depth two.

We can interpret f as a function $f: \mathbb{Z} \rightarrow \mathbb{F}_{p^2}$, embedding the codomain \mathbb{F}_p inside of \mathbb{F}_{p^2} . Let τ, ρ be the unique roots of $t^2 - f(1)t + \alpha = 0$ in \mathbb{F}_{p^2} , so that $\tau + \rho = f(1), \tau\rho = \alpha$. Since α is nonzero, so are τ and ρ . Notice that $f(0) = \tau^0 + \rho^0$ and $f(1) = \tau^1 + \rho^1$. By induction up and down, we see that $f(n) = \tau^n + \rho^n$ for all $n \in \mathbb{Z}$.

We can also easily check that given any value of $f(1)$ and α with $\alpha \not\equiv 0 \pmod{p}$, we get a unique such function. Thus there are $p(p-1)$ total such functions, one corresponding to each quadratic $t^2 - at + b$, where $b \neq 0$. If τ, ρ are the roots of the quadratic, the associated function is $f(n) = \tau^n + \rho^n$.

Now if f is q -periodic then $f(n) = f(n+q)$ for all $n \in \mathbb{Z}$, so $\tau^n + \rho^n = \tau^{n+q} + \rho^{n+q}$ for all $n \in \mathbb{Z}$ so

$$\tau^n(1 - \tau^q) = \rho^n(\rho^q - 1).$$

We have two cases.

Case 1. $\tau \neq \rho$.

Then $n = 0$ and $n = 1$ above give $1 = \tau^q = \rho^q$, and this is furthermore sufficient to be q -periodic. Thus the period of f is the minimal q such that $\tau^q = \rho^q = 1$, so the lcm of the orders of τ, ρ as elements of \mathbb{F}_{p^2} .

Case 2. $\tau = \rho$.

Then we see that $\tau = \rho = \frac{f(1)}{2}$, so they are in \mathbb{F}_p . This automatically means that $f(n) = 2 \left(\frac{f(1)}{2}\right)^n$ for all $n \in \mathbb{Z}$, where $f(1) \neq 0$ is forced (as $\tau\rho = \alpha$). This has period dividing $p-1$. More specifically, the period is the order of $\frac{f(1)}{2}$ as an element of \mathbb{F}_p .

In the first case, note that the orders of τ, ρ dividing $p-1$ implies that τ, ρ are actually in \mathbb{F}_p , so that we can restrict to counting recurrences $f(n) = a^n + b^n$ where a, b are nonzero elements of \mathbb{F}_p .

The unordered pair $\{a, b\}$ determines $a + b = f(1)$ and $ab = \alpha$, so it determines a unique f by our earlier observation. We need to count the number of such pairs with $\text{lcm}(\text{ord}_p(a), \text{ord}_p(b)) = p-1$.

Recall that for $d \mid p-1$, there are $\phi(d)$ elements of order d in \mathbb{F}_p^\times . If $a = b$ we need $\text{ord}_p(a) = p-1$, so there are $\phi(p-1)$ functions. If S is the number of ordered pairs (a, b) with $\text{lcm}(\text{ord}_p(a), \text{ord}_p(b)) = p-1$, then we see that the desired count is $\frac{\phi(p-1)+S}{2}$ (as each unordered pair $\{a, b\}$ is counted twice as an ordered pair except for those with $a = b$).

Now let $n = p-1$. We are counting pairs with the lcm equal to n . The number of pairs with respective orders d_1, d_2 is $\phi(d_1)\phi(d_2)$ hence our desired sum is $\sum_{d_1, d_2 \mid n, \text{lcm}(d_1, d_2) = n} \phi(d_1)\phi(d_2)$. Now notice that ϕ is a multiplicative function, and the condition on the sum splits among the prime powers dividing n , so we can choose how d_1, d_2 behave on the prime factors of n independently. Therefore if we define

$$\chi(n) = \sum_{d_1, d_2 \mid n, \text{lcm}(d_1, d_2) = n} \phi(d_1)\phi(d_2),$$

then χ is multiplicative. Finally, if q is prime and $e \geq 1$ we have

$$\begin{aligned} \chi(q^e) &= \sum_{d_1, d_2 \mid q^e, \text{lcm}(d_1, d_2) = q^e} \phi(d_1)\phi(d_2) \\ &= 2\phi(q^e)(\phi(1) + \phi(q) + \cdots + \phi(q^{e-1})) + \phi(q^e)^2 \\ &= 2q^{e-1}(q-1)q^{e-1} + q^{2e-2}(q-1)^2 \\ &= q^{2e-2}(q^2 - 1) = q^{2e} \left(1 - \frac{1}{q^2}\right). \end{aligned}$$

This implies that $\chi(n) = n^2 \prod_{q|n} \left(1 - \frac{1}{q^2}\right)$ by multiplicativity. Finally, our original answer is

$$\frac{\phi(p-1) + (p-1)^2 \prod_{q|p-1} \left(1 - \frac{1}{q^2}\right)}{2},$$

which we now compute for $p = 2017$. We have $p-1 = 2016 = 2^5 \cdot 3^2 \cdot 7$. Thus $\phi(p-1) = 2^4 \cdot 2 \cdot 3 \cdot 6 = 576$ and $\prod_{q|2016} \left(1 - \frac{1}{q^2}\right) = \frac{3}{4} \cdot \frac{8}{9} \cdot \frac{48}{49} = \frac{32}{49}$. Thus the number of functions is

$$\frac{576 + \frac{32}{49} \cdot 2016^2}{2} = \frac{576 + 2^{15} \cdot 3^4}{2} = 288 + 2^{14} \cdot 3^4 = \boxed{1327392}.$$

HMMT February 2018

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Algebra and Number Theory

1. For some real number c , the graphs of the equation $y = |x - 20| + |x + 18|$ and the line $y = x + c$ intersect at exactly one point. What is c ?

Proposed by: Henrik Boecken

Answer:

We want to know the value of c so that the graph $|x - 20| + |x + 18| - x = c$ has one solution. The graph of the function $|x - 20| + |x + 18| - x$ consists of an infinite section of slope -3 for $x \in (-\infty, -18]$, then a finite section of slope -1 for $x \in [-18, 20]$, then an infinite section of slope 1 for $x \in [20, \infty)$. Notice that this graph is strictly decreasing on $(-\infty, 20]$ and strictly increasing on $[20, \infty)$. Therefore any horizontal line will intersect this graph 0 or 2 times, except the one that passes through the “vertex” $(20, |20 - 20| + |20 + 18| - 20) = (20, 18)$, giving a value of $c = 18$.

2. Compute the positive real number x satisfying

$$x^{(2x^6)} = 3.$$

Proposed by: Henrik Boecken

Answer:

Let $t = x^6$, so $x^{2t} = 3$. Taking this to the third power gives $x^{6t} = 27$, or equivalently $t^t = 3^3$. We can see by inspection that $t = 3$, and this is the only solution as for $t > 1$, the function t^t is monotonically increasing, and if $0 < t < 1$, $t^t < 1$. Solving for x gives $x^6 = 3$, or $x = \sqrt[6]{3}$.

3. There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \frac{n^2}{5} \right\rfloor$ for some positive integer n . What is the sum of these two prime numbers?

Proposed by: Kevin Sun

Answer:

Note that the remainder when n^2 is divided by 5 must be 0, 1, or 4. Then we have that $25p = n^2$ or $25p = n^2 - 1$ or $25p = n^2 - 4$. In the first case there are no solutions. In the second case, if $25p = (n - 1)(n + 1)$, then we must have $n - 1 = 25$ or $n + 1 = 25$ as $n - 1$ and $n + 1$ cannot both be divisible by 5, and also cannot both have a factor besides 25. Similarly, in the third case, $25p = (n - 2)(n + 2)$, so we must have $n - 2 = 25$ or $n + 2 = 25$.

Therefore the n we have to check are 23, 24, 26, 27. These give values of $p = 21$, $p = 23$, $p = 27$, and $p = 29$, of which only 23 and 29 are prime, so the answer is $23 + 29 = 52$.

4. Distinct prime numbers p, q, r satisfy the equation

$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$

for some positive integer A . What is A ?

Proposed by: Kevin Sun

Answer:

Note that A is a multiple of p, q , and r , so $K = \frac{A}{pqr}$ is an integer. Dividing through, we have that

$$K = 8 + \frac{12}{p} = 7 + \frac{55}{q} = 2 + \frac{50}{r}.$$

Then $p \in \{2, 3\}$, $q \in \{5, 11\}$, and $r \in \{2, 5\}$. These values give $K \in \{14, 12\}$, $K \in \{18, 12\}$, and $K \in \{27, 12\}$, giving $K = 12$ and $(p, q, r) = (3, 11, 5)$. We can then compute $A = pqr \cdot K = 3 \cdot 11 \cdot 5 \cdot 12 = 1980$.

5. Let $\omega_1, \omega_2, \dots, \omega_{100}$ be the roots of $\frac{x^{101}-1}{x-1}$ (in some order). Consider the set

$$S = \{\omega_1^1, \omega_2^2, \omega_3^3, \dots, \omega_{100}^{100}\}.$$

Let M be the maximum possible number of unique values in S , and let N be the minimum possible number of unique values in S . Find $M - N$.

Proposed by: Henrik Boecken

Answer: 98

Throughout this solution, assume we're working modulo 101.

First, $N = 1$. Let ω be a primitive 101st root of unity. We then let $\omega_n = \omega^{1/n}$, which we can do because 101 is prime, so $1/n$ exists for all nonzero n and $1/n = 1/m \implies m = n$. Thus the set contains only one distinct element, ω .

$M = 100$ is impossible. Fix ζ , a primitive 101st root of unity, and let $\omega_n = \zeta^{\pi(n)}$ for each n . Suppose that there are 100 distinct such $n\pi(n)$ exponents; then π permutes the set $\{1, 2, \dots, 100\}$. Fix g , a primitive root of 101; write $n = g^{e_n}$ and $\pi(n) = g^{\tau(e_n)}$. Then $\{e_n\} = \{0, 1, 2, \dots, 100\}$ and τ is a permutation of this set, as is $e_n + \tau(e_n)$. However, this is impossible: $\sum_{n=1}^{100} e_n + \tau(e_n) = 5050 + 5050 \equiv 5050 \pmod{100}$, which is a contradiction. Thus there cannot be 100 distinct exponents.

$M = 99$ is possible. Again, let ζ be a primitive root of unity and let $\omega_n = \zeta^{1/(n+1)}$, except when $n = 100$, in which case let ω_{100} be the last possible root. Notice that $\frac{n}{n+1} = \frac{m}{m+1}$ if and only if $n = m$, so this will produce 99 different elements in the set.

Thus $M - N = 99 - 1 = 98$.

6. Let α, β , and γ be three real numbers. Suppose that

$$\begin{aligned}\cos \alpha + \cos \beta + \cos \gamma &= 1 \\ \sin \alpha + \sin \beta + \sin \gamma &= 1.\end{aligned}$$

Find the smallest possible value of $\cos \alpha$.

Proposed by: Henrik Boecken

Answer: $\frac{-1-\sqrt{7}}{4}$

Let $a = \cos \alpha + i \sin \alpha$, $b = \cos \beta + i \sin \beta$, and $c = \cos \gamma + i \sin \gamma$.

We then have

$$a + b + c = 1 + i$$

where a, b, c are complex numbers on the unit circle. Now, to minimize $\cos \alpha = \operatorname{Re}[a]$, consider a triangle with vertices $a, 1 + i$, and the origin. We want a as far away from $1 + i$ as possible while maintaining a nonnegative imaginary part. This is achieved when b and c have the same argument, so $|b + c| = |1 + i - a| = 2$. Now $a, 0$, and $1 + i$ form a $1 - 2 - \sqrt{2}$ triangle. The value of $\cos \alpha$ is now the cosine of the angle between the 1 and $\sqrt{2}$ sides plus the $\frac{\pi}{4}$ angle from $1 + i$. Call the first angle δ . Then

$$\begin{aligned}\cos \delta &= \frac{1^2 + (\sqrt{2})^2 - 2^2}{2 \cdot 1 \cdot \sqrt{2}} \\ &= \frac{-1}{2\sqrt{2}}\end{aligned}$$

and

$$\begin{aligned}
 \cos \alpha &= \cos \left(\frac{\pi}{4} + \delta \right) \\
 &= \cos \frac{\pi}{4} \cos \delta - \sin \frac{\pi}{4} \sin \delta \\
 &= \frac{\sqrt{2}}{2} \cdot \frac{-1}{2\sqrt{2}} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{7}}{2\sqrt{2}} \\
 &= \boxed{\frac{-1 - \sqrt{7}}{4}}
 \end{aligned}$$

7. Rachel has the number 1000 in her hands. When she puts the number x in her left pocket, the number changes to $x + 1$. When she puts the number x in her right pocket, the number changes to x^{-1} . Each minute, she flips a fair coin. If it lands heads, she puts the number into her left pocket, and if it lands tails, she puts it into her right pocket. She then takes the new number out of her pocket. If the expected value of the number in Rachel's hands after eight minutes is E , then compute $\lfloor \frac{E}{10} \rfloor$.

Proposed by: Kevin Sun

Answer: 13

Call a real number very large if $x \in [1000, 1008]$, very small if $x \in [0, \frac{1}{1000}]$, and medium-sized if $x \in [\frac{1}{8}, 8]$. Every number Rachel is ever holding after at most 8 steps will fall under one of these categories. Therefore the main contribution to E will come from the probability that Rachel is holding a number at least 1000 at the end.

Note that if her number ever becomes medium-sized, it will never become very large or very small again. Therefore the only way her number ends up above 1000 is if the sequence of moves consists of $x \rightarrow x + 1$ moves and consecutive pairs of $x \rightarrow x^{-1}$ moves. Out of the 256 possible move sequences, the number of ways for the number to stay above 1000 is the number of ways of partitioning 8 into an ordered sum of 1 and 2, or the ninth Fibonacci number $F_9 = 34$.

Therefore

$$\frac{34}{256} \cdot 1000 \leq E \leq \frac{34}{256} \cdot 1000 + 8,$$

where $\frac{34}{256} \cdot 1000 \approx 132.8$. Furthermore, the extra contribution will certainly not exceed 7, so we get that $\lfloor \frac{E}{10} \rfloor = 13$.

(The actual value of E is

$$\frac{1538545594943410132524842390483285519695831541468827074238984121209064525621}{11415831910281261197289931074429782903650103348754306523894286954489856000},$$

which is approximately equal to 134.77297. We can see that the extra contribution is about 2 and is very insignificant.)

8. For how many pairs of sequences of nonnegative integers $(b_1, b_2, \dots, b_{2018})$ and $(c_1, c_2, \dots, c_{2018})$ does there exist a sequence of nonnegative integers (a_0, \dots, a_{2018}) with the following properties:
- For $0 \leq i \leq 2018$, $a_i < 2^{2018}$;
 - For $1 \leq i \leq 2018$, $b_i = a_{i-1} + a_i$ and $c_i = a_{i-1} | a_i$;

where $|$ denotes the bitwise or operation?

(The *bitwise or* of two nonnegative integers $x = \dots x_3 x_2 x_1 x_0$ and $y = \dots y_3 y_2 y_1 y_0$ expressed in binary is defined as $x | y = \dots z_3 z_2 z_1 z_0$, where $z_i = 1$ if at least one of x_i and y_i is 1, and 0 otherwise.)

Proposed by: Kevin Sun

Answer: $(2^{2019} - 1)^{2018}$

Define the *bitwise and* of two nonnegative integers $x = \cdots x_3x_2x_1x_0$ and $y = \cdots y_3y_2y_1y_0$ expressed in binary to be $x\&y = \cdots z_3z_2z_1z_0$, where $z_i = 1$ if both x_i and y_i are 1, and 0 otherwise.

Now, we can prove that from the definitions of $|$ and $\&$ that $x + y = (x|y) + (x\&y)$. Therefore it suffices to count pairs of sequences $(c_1, c_2, \dots, c_{2018})$ and $(d_1, d_2, \dots, d_{2018})$ such that $c_i = a_{i-1}|a_i$ and $d_i = a_{i-1}\&a_i$ for $0 \leq a_i < 2^{2018}$.

Since both $|$ and $\&$ are bitwise operations, it suffices to count the number of sequences $\{c_i\}$ and $\{d_i\}$ restricting each a_i to $\{0, 2^k\}$ for each $k \in [0, 2017]$ and multiply these counts together. Each sequence (a_0, \dots, a_{2018}) leads to a unique $\{c_i\}$ and $\{d_i\}$ except for the sequences $(2^k, 0, 2^k, 0, \dots, 2^k)$ and the sequences $(0, 2^k, 0, 2^k, \dots, 0)$, which lead to the same $\{c_i\}$ and $\{d_i\}$.

Therefore for each k , there are $2^{2019} - 1$ ways to determine the k -th bits of each c_i and d_i . Multiplying this over all k gives a final count of $(2^{2019} - 1)^{2018}$.

9. Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \leq x_1, x_2, x_3, x_4 \leq 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$ (over all valid choices of a, b, d).

Proposed by: Allen Liu

Answer: $\boxed{\frac{5}{4}}$

We can rewrite the expression as

$$\frac{x_4^2}{x_1^2} \cdot \frac{(x_1+x_1)(x_1+x_2)(x_1+x_3)(x_1+x_4)}{(x_4+x_1)(x_4+x_2)(x_4+x_3)(x_4+x_4)}$$

$$\frac{x_4^2}{x_1^2} \cdot \frac{f(-x_1)}{f(-x_4)}$$

where $f(x)$ is the quartic. We attempt to find a simple expression for $f(-x_1)$. We know that

$$f(-x_1) - f(x_1) = 2a \cdot x_1^3 + 2a \cdot x_1$$

Since x_1 is a root, we have

$$f(-x_1) = 2a \cdot x_1^3 + 2a \cdot x_1$$

Plugging this into our previous expression:

$$\frac{x_4^2}{x_1^2} \cdot \frac{x_1^3 + x_1}{x_4^3 + x_4}$$

$$\frac{x_1 + \frac{1}{x_1}}{x_4 + \frac{1}{x_4}}$$

The expression $x + \frac{1}{x}$ is maximized at $x = 2, \frac{1}{2}$ and minimized at $x = 1$. We can therefore maximize the numerator with $x_1 = 2$ and minimize the denominator with $x_4 = 1$ to achieve the answer of $\frac{5}{4}$. It can be confirmed that such an answer can be achieved such as with $x_2 = x_3 = \frac{\sqrt{10}-1}{3}$.

10. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \dots, n\}$. Consider the polynomial $P(x) = \sum_{i \in S} x^i$. Let X_n be the probability that $P(x)$ is divisible by some nonconstant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \neq 0$. Find the limit of X_n as n goes to infinity.

Proposed by: Allen Liu

Answer: $\boxed{\frac{10015}{20736}}$

We begin with the following claims:

Claim 1: There are finitely many $Q(x)$ that divide some $P(x)$ of the given form.

Proof: First of all the leading coefficient of Q must be 1, because if Q divides P then P/Q must have integer coefficients too. Note that if $S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$ with elements in increasing order, then

$$|P(x)| \geq |x^{s_6}| - |x^{s_5}| - |x^{s_4}| - \dots - |x^{s_1}| = |x|^{s_6} - |x|^{s_5} - |x|^{s_4} - \dots - |x|^{s_1},$$

so all the roots of P must have magnitude less than 2, and so do all the roots of Q . Therefore, all the symmetric expressions involving the roots of Q are also bounded, so by Vieta's Theorem all the coefficients of Q of a given degree are bounded, and the number of such Q is therefore finite.

Claim 2: If Q has a nonzero root that does not have magnitude 1, then the probability that it divides a randomly chosen P vanishes as n goes to infinity.

Proof: WLOG suppose that Q has a root r with $|r| > 1$ (similar argument will apply for $|r| < 1$). Then from the bound given in the proof of Claim 1, it is not difficult to see that $s_6 - s_5$ is bounded since

$$|P(r)| > |r|^{s_6} - 5|r|^{s_5} > |r|^{s_6 - s_5} - 5,$$

which approaches infinity as $s_6 - s_5$ goes to infinity. By similar argument we can show that $s_5 - s_4, s_4 - s_3, \dots$ are all bounded. Therefore, the probability of choosing the correct coefficients is bounded above by the product of five fixed numbers divided by $n^5/5!$, which vanishes as n goes to infinity.

From the claims above, we see that we only need to consider polynomials with roots of magnitude 1, since the sum of all other possibilities vanishes as n goes to infinity. Moreover, this implies that we only need to consider roots of unity. Since Q has degree at most 3, the only possible roots are $-1, \pm i, \frac{-1 \pm i\sqrt{3}}{2}, \frac{1 \pm i\sqrt{3}}{2}$, corresponding to $x + 1, x^2 + 1, x^2 + x + 1, x^2 - x + 1$ (note that eighth root of unity is impossible because $x^4 + 1$ cannot be factored in the rationals).

Now we compute the probability of $P(r) = 0$ for each possible root r . Since the value of x^s cycles with s , and we only care about $n \rightarrow \infty$, we may even assume that the exponents are chosen independently at random, with repetition allowed.

Case 1: When $r = -1$, the number of odd exponents need to be equal to the number of even exponents, which happens with probability $\frac{\binom{6}{3}}{2^6} = \frac{5}{16}$.

Case 2: When $r = \pm i$, the number of exponents that are 0 modulo 4 need to be equal to those that are 2 modulo 4, and same for 1 modulo 4 and 3 modulo 4, which happens with probability $\frac{\binom{6}{0}}{2^6} \cdot \frac{\binom{0}{0}\binom{6}{3}}{2^6} + \frac{\binom{6}{2}}{2^6} \cdot \frac{\binom{2}{1}\binom{4}{2}}{2^6} + \frac{\binom{6}{4}}{2^6} \cdot \frac{\binom{4}{2}\binom{2}{1}}{2^6} + \frac{\binom{6}{6}}{2^6} \cdot \frac{\binom{6}{3}\binom{0}{0}}{2^6} = \frac{25}{256}$. Note that Case 1 and Case 2 have no overlaps, since the former requires 3 even exponents, and the latter requires 0, 2, 4, or 6 even exponents.

Case 3: When $r = \frac{-1 \pm i\sqrt{3}}{2}$, the number of exponents that are 0, 1, 2 modulo 3 need to be equal to each other, so the probability is $\frac{\binom{6}{2,2,2}}{3^6} = \frac{10}{81}$.

Case 4: When $r = \frac{1 \pm i\sqrt{3}}{2}$, then if n_i is the number of exponents that are i modulo 6 ($i = 0, 1, 2, 3, 4, 5$), then $n_0 - n_3 = n_2 - n_5 = n_4 - n_1 = k$ for some k . Since $3k \equiv n_0 + n_1 + \dots + n_5 = 6 \equiv 0 \pmod{2}$, k must be one of $-2, 0, 2$. When $k = 0$, we have $n_0 + n_2 + n_4 = n_1 + n_3 + n_5$, which is the same as Case 1. When $k = 2$, we have $n_0 = n_2 = n_4 = 2$, which is covered in Case 3, and similar for $k = -2$. Therefore we do not need to consider this case.

Now we deal with over-counting. Since Case 1 and 2 deal with the exponents modulo 4 and Case 3 deal with exponents modulo 3, the probabilities are independent from each other. So by complementary counting, we compute the final probability as

$$1 - \left(1 - \frac{5}{16} - \frac{25}{256}\right)\left(1 - \frac{10}{81}\right) = 1 - \frac{151}{256} \cdot \frac{71}{81} = \frac{10015}{20736}.$$

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Combinatorics

1. Consider a 2×3 grid where each entry is one of 0, 1, and 2. For how many such grids is the sum of the numbers in every row and in every column a multiple of 3? One valid grid is shown below.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Proposed by: Henrik Boecken

Answer:

Any two elements in the same row fix the rest of the grid, so $3^2 = 9$.

2. Let a and b be five-digit palindromes (without leading zeroes) such that $a < b$ and there are no other five-digit palindromes strictly between a and b . What are all possible values of $b - a$? (A number is a palindrome if it reads the same forwards and backwards in base 10.)

Proposed by: Kevin Sun

Answer:

Let \overline{xyzyx} be the digits of the palindrome a . There are three cases. If $z < 9$, then the next palindrome greater than \overline{xyzyx} is $\overline{xy(z+1)yx}$, which differs by 100. If $z = 9$ but $y < 9$, then the next palindrome up is $\overline{x(y+1)0(y+1)x}$, which differs from $\overline{xy9yx}$ by 110. Finally, if $y = z = 9$, then the next palindrome after $\overline{x999x}$ is $\overline{(x+1)000(x+1)}$, which gives a difference of 11. Thus, the possible differences are 11, 100, 110.

3. A 4×4 window is made out of 16 square windowpanes. How many ways are there to stain each of the windowpanes, red, pink, or magenta, such that each windowpane is the same color as exactly two of its neighbors? Two different windowpanes are neighbors if they share a side.

Proposed by: Kevin Sun

Answer:

For the purpose of explaining this solution, let's label the squares as

$$\begin{array}{cccc} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \\ 41 & 42 & 43 & 44 \end{array}$$

Note that since the corner squares 11, 14, 41, 44 each only have two neighbors, each corner square is the same color as both of its neighbors (for example, 11, 12, and 21 are the same color, 31, 41, and 42 are the same color, etc.). This corner square constraint heavily limits the possible colorings. We will now use casework.

Case 1: Suppose two corner squares on the same side (without loss of generality, let them be 11 and 14) have the same color (without loss of generality, red). Then 21, 11, 12, 13, 14, 24 are all red, and 12 has two red neighbors (11 and 13) so its third neighbor (22) is a color different from red (without loss of generality, magenta). But 22 has two red neighbors (12 and 21), so its other two neighbors (23 and 32) must be magenta. Applying the same logic symmetrically, we find that all four interior squares (22, 23, 32, 33) have the same color. Furthermore, 21 has one magenta neighbor 22, so 31 must be red. Symmetrically, 34 is red, and by the corner square constraint we have that all the exterior squares are

the same color. Thus in general, this case is equivalent to a window taking the following form (with distinct colors A and B)

$$\begin{array}{cccc} A & A & A & A \\ A & B & B & A \\ A & B & B & A \\ A & A & A & A \end{array}$$

The number of choices of A and B is $3 \cdot 2 = 6$.

Case 2: No two corner squares on the same side have the same color.

Then from the corner square constraint 12 has neighbor 11 of the same color and neighbor 13 of a different color, so its neighbor 22 must be the same color as 12. Therefore, this case is equivalent to coloring each quadrant entirely in one color such two quadrants sharing a side have different colors. (A quadrant refers to the four squares on one vertical half and one horizontal half, e.g. 13, 14, 23, 24).

If only two colors are used, the window will take the form (with distinct colors A and B):

$$\begin{array}{cccc} A & A & B & B \\ A & A & B & B \\ B & B & A & A \\ B & B & A & A \end{array}$$

Again there are $3 \cdot 2 = 6$ ways to chose A and B .

If all three colors are used, the window will take the form (with distinct colors A , B and C)

$$\begin{array}{cccc} A & A & B & B \\ A & A & B & B \\ C & C & A & A \\ C & C & A & A \end{array}$$

or

$$\begin{array}{cccc} A & A & B & B \\ A & A & B & B \\ B & B & C & C \\ B & B & C & C \end{array}$$

There are $3 \cdot 2 \cdot 1 = 6$ ways to select colors for each of these forms.

Therefore, there are 6 colorings in Case 1 and $6 + 6 + 6$ in Case 2, for a total of 24 colorings.

4. How many ways are there for Nick to travel from $(0,0)$ to $(16,16)$ in the coordinate plane by moving one unit in the positive x or y direction at a time, such that Nick changes direction an odd number of times?

Proposed by: Huaiyu Wu

Answer:
$$2 \cdot \binom{30}{15} = 310235040$$

This condition is equivalent to the first and last step being in different directions, as if you switch directions an odd number of times, you must end in a different direction than you started. If the first step is in the x direction and the last step is in the y direction, it suffices to count the number of paths from $(1,0)$ to $(16,15)$, of which there are $\binom{30}{15}$. Similarly, in the other case, it suffices to count the

number of paths from $(0, 1)$ to $(15, 16)$, of which there are also $\binom{30}{15}$. Therefore the total number of paths is $2 \cdot \binom{30}{15}$.

5. A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew?

Proposed by: Kevin Sun

Answer: $\boxed{\frac{20}{11}}$

The probability of drawing k marbles is the probability of drawing $k - 1$ blue marbles and then the special marble, which is $p_k = \left(\frac{9}{20}\right)^{k-1} \times \frac{1}{20}$. The probability of drawing no ugly marbles is therefore $\sum_{k=1}^{\infty} p_k = \frac{1}{11}$.

Then given that no ugly marbles were drawn, the probability that k marbles were drawn is $11p_k$. The expected number of marbles Ryan drew is

$$\sum_{k=1}^{\infty} k(11p_k) = \frac{11}{20} \sum_{k=1}^{\infty} k \left(\frac{9}{20}\right)^{k-1} = \frac{11}{20} \times \frac{400}{121} = \frac{20}{11}.$$

(To compute the sum in the last step, let $S = \sum_{k=1}^{\infty} k \left(\frac{9}{20}\right)^{k-1}$ and note that $\frac{9}{20}S = S - \sum_{k=1}^{\infty} \left(\frac{9}{20}\right)^{k-1} = S - \frac{20}{11}$).

6. Sarah stands at $(0, 0)$ and Rachel stands at $(6, 8)$ in the Euclidean plane. Sarah can only move 1 unit in the positive x or y direction, and Rachel can only move 1 unit in the negative x or y direction. Each second, Sarah and Rachel see each other, independently pick a direction to move at the same time, and move to their new position. Sarah catches Rachel if Sarah and Rachel are ever at the same point. Rachel wins if she is able to get to $(0, 0)$ without being caught; otherwise, Sarah wins. Given that both of them play optimally to maximize their probability of winning, what is the probability that Rachel wins?

Proposed by: Rachel Zhang

Answer: $\boxed{\frac{63}{64}}$

We make the following claim: In a game with $n \times m$ grid where $n \leq m$ and $n \equiv m \pmod{2}$, the probability that Sarah wins is $\frac{1}{2^n}$ under optimal play.

Proof: We induct on n . First consider the base case $n = 0$. In this case Rachel is confined on a line, so Sarah is guaranteed to win.

We then consider the case where $n = m$ (a square grid). If Rachel and Sarah move in parallel directions at first, then Rachel can win if she keep moving in this direction, since Sarah will not be able to catch Rachel no matter what. Otherwise, the problem is reduced to a $(n - 1) \times (n - 1)$ grid. Therefore, the optimal strategy for both players is to choose a direction completely randomly, since any bias can be abused by the other player. So the reduction happens with probability $\frac{1}{2}$, and by induction hypothesis Sarah will with probability $\frac{1}{2^{n-1}}$, so on a $n \times n$ grid Sarah wins with probability $\frac{1}{2^n}$.

Now we use induction to show that when $n < m$, both player will move in the longer (m) direction until they are at corners of a square grid (in which case Sarah wins with probability $\frac{1}{2^n}$). If Sarah moves in the n direction and Rachel moves in the m (or n) direction, then Rachel can just move in the n direction until she reaches the other side of the grid and Sarah will not be able to catch her. If Rachel moves in the n direction and Sarah moves in the m direction, then the problem is reduced to a $(n - 1) \times (m - 1)$ grid, which means that Sarah's winning probability is now doubled to $\frac{1}{2^{n-1}}$ by induction hypothesis. Therefore it is suboptimal for either player to move in the shorter (n) direction. This shows that the game will be reduced to $n \times n$ with optimal play, and thus the claim is proved.

From the claim, we can conclude that the probability that Rachel wins is $1 - \frac{1}{2^6} = \frac{63}{64}$.

7. A tourist is learning an incorrect way to sort a permutation (p_1, \dots, p_n) of the integers $(1, \dots, n)$. We define a *fix* on two adjacent elements p_i and p_{i+1} , to be an operation which swaps the two elements if $p_i > p_{i+1}$, and does nothing otherwise. The tourist performs $n - 1$ rounds of fixes, numbered $a = 1, 2, \dots, n - 1$. In round a of fixes, the tourist fixes p_a and p_{a+1} , then p_{a+1} and p_{a+2} , and so on, up to p_{n-1} and p_n . In this process, there are $(n - 1) + (n - 2) + \dots + 1 = \frac{n(n-1)}{2}$ total fixes performed. How many permutations of $(1, \dots, 2018)$ can the tourist start with to obtain $(1, \dots, 2018)$ after performing these steps?

Proposed by: Kevin Sun

Answer: $\boxed{1009! \cdot 1010!}$

Note that the given algorithm is very similar to the well-known Bubble Sort algorithm for sorting an array. The exception is that in the i -th round through the array, the first $i - 1$ pairs are not checked.

We claim a necessary and sufficient condition for the array to be sorted after the tourist's process is: for all i , after i rounds, the numbers $1, \dots, i$ are in the correct position. Firstly, this is necessary because these indices of the array are not touched in future rounds - so if a number was incorrect, then it would stay incorrect. On the other hand, suppose this condition holds. Then, we can "add" the additional fixes during each round (of the first $i - 1$ pairs during the i -th round) to make the process identical to bubble sort. The tourist's final result won't change because by our assumption these swaps won't do anything. However, this process is now identical to bubble sort, so the resulting array will be sorted. Thus, our condition is sufficient.

Now, there are two positions the 1 can be in (p_1, p_2) . There are three positions the 2 can be in (p_1, \dots, p_4) except for the position of 1). Similarly, for $1 \leq i \leq 1009$ there are $2i - (i - 1) = i + 1$ positions i can be in, and after that the remaining 1009 numbers can be arranged arbitrarily. Thus, the answer is $1010! \cdot 1009!$.

8. A permutation of $\{1, 2, \dots, 7\}$ is chosen uniformly at random. A partition of the permutation into contiguous blocks is correct if, when each block is sorted independently, the entire permutation becomes sorted. For example, the permutation $(3, 4, 2, 1, 6, 5, 7)$ can be partitioned correctly into the blocks $[3, 4, 2, 1]$ and $[6, 5, 7]$, since when these blocks are sorted, the permutation becomes $(1, 2, 3, 4, 5, 6, 7)$.

Find the expected value of the maximum number of blocks into which the permutation can be partitioned correctly.

Proposed by: Mehtaab Sawhney

Answer: $\boxed{\frac{151}{105}}$

Let σ be a permutation on $\{1, \dots, n\}$. Call $m \in \{1, \dots, n\}$ a breakpoint of σ if $\{\sigma(1), \dots, \sigma(m)\} = \{1, \dots, m\}$. Notice that the maximum partition is into k blocks, where k is the number of breakpoints: if our breakpoints are m_1, \dots, m_k , then we take $\{1, \dots, m_1\}, \{m_1 + 1, \dots, m_2\}, \dots, \{m_{k-1} + 1, \dots, m_k\}$ as our contiguous blocks.

Now we just want to find

$$\mathbb{E}[k] = \mathbb{E}[X_1 + \dots + X_n],$$

where $X_i = 1$ if i is a breakpoint, and $X_i = 0$ otherwise. We use linearity of expectation and notice that

$$\mathbb{E}[X_i] = \frac{i!(n-i)!}{n!},$$

since this is the probability that the first i numbers are just $1, \dots, i$ in some order. Thus,

$$\mathbb{E}[k] = \sum_{i=1}^n \frac{i!(n-i)!}{n!} = \sum_{i=1}^n \binom{n}{i}^{-1}.$$

We can compute for $n = 7$ that the answer is $\boxed{\frac{151}{105}}$.

9. How many ordered sequences of 36 digits have the property that summing the digits to get a number and taking the last digit of the sum results in a digit which is not in our original sequence? (Digits range from 0 to 9.)

Proposed by: Kevin Sun

Answer: $\boxed{9^{36} + 4}$

We will solve this problem for 36 replaced by n . We use $[n]$ to denote $\{1, 2, \dots, n\}$ and σ_s to denote the last digit of the sum of the digits of s .

Let D be the set of all sequences of n digits and let S_i be the set of digit sequences s such that $s_i = \sigma_s$, the i^{th} digit of s . The quantity we are asked to compute is equal to $\left| D \setminus \bigcup_{i=1}^n S_i \right|$. We use the principle of inclusion-exclusion to compute this:

$$\left| D \setminus \bigcup_{i=1}^n S_i \right| = \sum_{J \subseteq [n]} (-1)^{|J|} \left| \bigcap_{j \in J} S_j \right|$$

Note that a digit sequence is in S_i if and only if the $n - 1$ digits which are not i sum to a multiple of 10. This gives that $|S_i| = 10 \cdot 10^{n-2} = 10^{n-1}$ as there are 10 ways to pick the i^{th} digit and 10^{n-2} ways to pick the other digits.

Similarly, given a subset $J \subseteq [n]$, we can perform a similar analysis. If a string s is in $\bigcap_{j \in J} S_j$, we must have that $s_j = \sigma_s$ for all $j \in J$. There are 10 ways to pick σ_s , which determines s_j for all $j \in J$. From there, there are $10^{(n-|J|)-1}$ ways to pick the remaining digits as if we fix all but one, the last digit is uniquely determined. This gives $10^{n-|J|}$ choices.

However, this breaks down when $|J| = n$, as not all choices of σ_s lead to any valid solutions. When $|J| = n$, $J = [n]$ and we require that the last digit of $n\sigma_s$ is σ_s , which happens for $\gcd(n - 1, 10)$ values of σ_s .

We now compare our expression from the principle of inclusion-exclusion to the binomial expansion of $(10 - 1)^n$. By the binomial theorem,

$$9^n = (10 - 1)^n = \sum_{J \subseteq [n]} (-1)^{|J|} 10^{n-|J|}.$$

These agree on every term except for the term where $J = [n]$. In this case, we need to add an extra $(-1)^n \gcd(n - 1, 10)$ and subtract $(-1)^n$.

Thus our final value for $\left| D \setminus \bigcup_{i=1}^n S_i \right|$ is $9^n + (-1)^n (\gcd(n - 1, 10) - 1)$, which is $9^{36} + 4$ for $n = 36$.

10. Lily has a 300×300 grid of squares. She now removes 100×100 squares from each of the four corners and colors each of the remaining 50000 squares black and white. Given that no 2×2 square is colored in a checkerboard pattern, find the maximum possible number of (unordered) pairs of squares such that one is black, one is white and the squares share an edge.

Proposed by: Allen Liu

Answer: $\boxed{49998}$

First we show an upper bound. Define a grid point as a vertex of one of the squares in the figure. Construct a graph as follows. Place a vertex at each grid point and draw an edge between two adjacent points if that edge forms a black-white boundary. The condition of there being no 2×2 checkerboard is equivalent to no vertex having degree more than 2. There are $101^2 + 4 \cdot 99^2 = 49405$ vertices that are allowed to have degree 2 and $12 \cdot 99 = 1188$ vertices (on the boundary) that can have degree 1.

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Geometry

1. Triangle GRT has $GR = 5$, $RT = 12$, and $GT = 13$. The perpendicular bisector of GT intersects the extension of GR at O . Find TO .

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{169}{10}}$

First, note that $TO = GO$ as O lies on the perpendicular bisector of GT . Then if M is the midpoint of GT , we have that $\triangle GRT \sim \triangle GMO$, so we can compute $TO = GO = GM \cdot \frac{GT}{GR} = \frac{13}{2} \cdot \frac{13}{5} = \frac{169}{10}$.

2. Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths 2, 7, 5, 12, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC . What is the ordered pair (m, M) ?

Proposed by: Kevin Sun

Answer: $\boxed{(7, 9)}$

By the triangle inequality on triangle ACD , $AC + CD \geq AD$, or $AC \geq 7$. The minimum of 7 can be achieved when A, C, D lie on a line in that order. By the triangle inequality on triangle ABC , $AB + BC \geq AC$, or $AC \leq 9$. The maximum of 9 can be achieved when A, B, C lie on a line in that order. This gives the answer $(7, 9)$.

3. How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

Proposed by: Dai Yang

Answer: $\boxed{2}$

There are 3 possible vertices that can have an angle of 60° , we will name them. Call the vertex where the sides of length 20 and 17 meet α , denote the vertex where 17 doesn't meet 20 by β , and the final vertex, which meets 20 but not 17, we denote by γ .

The law of cosines states that if we have a triangle, then we have the equation $c^2 = a^2 + b^2 - 2ab \cos C$ where C is the angle between a and b . But $\cos 60^\circ = \frac{1}{2}$ so this becomes $c^2 = a^2 + b^2 - ab$. We then try satisfying this equation for the 3 possible vertices and find that, for α the equation reads $c^2 = 400 + 289 - 340 = 349$ so that $c = \sqrt{349}$. For β we find that $400 = 289 + b^2 - 17b$ or rather $b^2 - 17b - 111 = 0$ this is a quadratic, solving we find that it has two roots $b = \frac{17 \pm \sqrt{289 + 444}}{2}$, but since $\sqrt{733} > 17$ only one of these roots is positive. We can also see that this isn't congruent to the other triangle we had, as for both the triangles the shortest side has length 17, and so if they were congruent the lengths of all sides would need to be equal, but $18 < \sqrt{349} < 19$ and since $23^2 < 733$ clearly $\frac{17 \pm \sqrt{289 + 444}}{2} > \frac{17 + 23}{2} = 20$ and so the triangles aren't congruent. If we try applying the law of cosines to γ however, we get the equation $289 = a^2 + 400 - 20a$ which we can rewrite as $a^2 - 20a + 111 = 0$ which has no real solutions, as the discriminant $400 - 4 * 111 = -44$ is negative. Thus, γ cannot be 60° , and there are exactly two non congruent triangles with side lengths 20 and 17 with an angle being 60° .

4. A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C . Let M be the point on the sheet of paper halfway between A and C . Over time, point M is lifted upwards, folding the triangle along segment BM , while A, B , and C remain on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.

Proposed by: Dhruv Rohatgi

Answer: $\boxed{\frac{\sqrt{3}}{6}}$

View triangle ABM as a base of this tetrahedron. Then relative to triangle ABM , triangle CBM rotates around segment BM on a hinge. Therefore the volume is maximized when C is farthest from triangle ABM , which is when triangles ABM and CBM are perpendicular. The volume in this case can be calculated using the formula for the volume of a tetrahedron as $\frac{1}{6} \cdot 1 \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{6}$.

5. In the quadrilateral $MARE$ inscribed in a unit circle ω , AM is a diameter of ω , and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{8\sqrt{2}}{9}}$

Since AE bisects $\angle RAM$, we have $RE = EM$, and E, A lie on different sides of RM . Since AM is a diameter, $\angle ARM = 90^\circ$. If the midpoint of RM is N , then from $[RAM] = [REM]$ and $\angle ARM = 90^\circ$, we find $AR = NE$. Note that O , the center of ω , N , and E are collinear, and by similarity of triangles NOM and RAM , $ON = \frac{1}{2}AR = \frac{1}{2}NE$. Therefore, $ON = \frac{1}{3}$ and $NE = \frac{2}{3}$. By the Pythagorean theorem on triangle RAM , $RM = \frac{4\sqrt{2}}{3}$. Therefore, the area of $MARE$ is $2 \cdot \frac{1}{2} \cdot \frac{4\sqrt{2}}{3} \cdot \frac{2}{3} = \frac{8\sqrt{2}}{9}$.

6. Let ABC be an equilateral triangle of side length 1. For a real number $0 < x < 0.5$, let A_1 and A_2 be the points on side BC such that $A_1B = A_2C = x$, and let $T_A = \triangle AA_1A_2$. Construct triangles $T_B = \triangle BB_1B_2$ and $T_C = \triangle CC_1C_2$ similarly.

There exist positive rational numbers b, c such that the region of points inside all three triangles T_A, T_B, T_C is a hexagon with area

$$\frac{8x^2 - bx + c}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4}.$$

Find (b, c) .

Proposed by: Kevin Sun

Answer: $\boxed{(8, 2)}$

Solution 1:

Notice that the given expression is defined and continuous not only on $0 < x < 0.5$, but also on $0 \leq x \leq 0.5$. Let $f(x)$ be the function representing the area of the (possibly degenerate) hexagon for $x \in [0, 0.5]$. Since $f(x)$ is equal to the given expression over $(0, 0.5)$, we can conclude that $f(0)$ and $f(0.5)$ will also be equal to the expression when $x = 0$ and $x = 0.5$ respectively. (In other words, $f(x)$ is equal to the expression over $[0, 0.5]$.) In each of the cases, we can compute easily that $f(0) = \frac{\sqrt{3}}{4}$ and $f(0.5) = 0$, so by plugging them in, we get $\frac{c}{2 \cdot 1} = 1$ and $\frac{2-b/2+c}{(3/2) \cdot (3/2)} = 0$, which gives $b = 8$ and $c = 2$.

Solution 2:

Let $P = AA_1 \cap CC_2$, $Q = AA_2 \cap BB_1$, $R = BB_1 \cap CC_2$. These three points are the points on the hexagon farthest away from A . For reasons of symmetry, the area of the hexagon (call it H for convenience) is:

$$[H] = [ABC] - 3[BPRQC].$$

Also, $[BPC] = [BQC]$ by symmetry, so:

$$[BPRQC] = [BPC] + [BQC] - [BRC]$$

$$[BPRQC] = 2[BPC] - [BRC].$$

From this, one can see that all we need to do is calculate the A -level of the points P and R in barycentric coordinates. Ultimately, the A -level of P is $\frac{x}{x+1}$, and the A -level of R is $\frac{x}{2-x}$. From this, straightforward calculation shows that:

$$[H] = \frac{8x^2 - 8x + 2}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4},$$

thus giving us the answer $(b, c) = (8, 2)$.

7. Triangle ABC has sidelengths $AB = 14$, $AC = 13$, and $BC = 15$. Point D is chosen in the interior of \overline{AB} and point E is selected uniformly at random from \overline{AD} . Point F is then defined to be the intersection point of the perpendicular to \overline{AB} at E and the union of segments \overline{AC} and \overline{BC} . Suppose that D is chosen such that the expected value of the length of \overline{EF} is maximized. Find AD .

Proposed by: Gabriel Mintzer

Answer: $\boxed{\sqrt{70}}$

Let G be the intersection of the altitude to \overline{AB} at point D with $\overline{AC} \cup \overline{BC}$. We first note that the maximal expected value is obtained when $DG = \frac{[ADGC]}{AD}$, where $[P]$ denotes the area of polygon P . Note that if DG were not equal to this value, we could move D either closer or further from A and increase the value of the fraction, which is the expected value of EF . We note that this equality can only occur if D is on the side of the altitude to \overline{AB} nearest point B . Multiplying both sides of this equation by AD yields $AD \cdot DG = [ADGC]$, which can be interpreted as meaning that the area of the rectangle with base \overline{AD} and height \overline{DG} must have area equal to that of quadrilateral $ADGC$. We can now solve this problem with algebra.

Let $x = BD$. We first compute the area of the rectangle with base \overline{AD} and height \overline{DG} . We have that $AD = AB - BD = 14 - x$. By decomposing the $13 - 14 - 15$ triangle into a $5 - 12 - 13$ triangle and a $9 - 12 - 15$ triangle, and using a similarity argument, we find that $DG = \frac{4}{3}x$. Thus, the area of this rectangle is $\frac{4}{3}x(14 - x) = \frac{56}{3}x - \frac{4}{3}x^2$.

We next compute the area of quadrilateral $ADGC$. We note that $[ADGC] = [ABC] - [BDG]$. We have that $[ABC] = \frac{1}{2}(12)(14) = 84$. We have $BD = x$ and $DG = \frac{4}{3}x$, so $[BDG] = \frac{1}{2}(x)(\frac{4}{3}x) = \frac{2}{3}x^2$. Therefore, we have $[ADGC] = [ABC] - [BDG] = 84 - \frac{2}{3}x^2$.

Equating these two areas, we have

$$\frac{56}{3}x - \frac{4}{3}x^2 = 84 - \frac{2}{3}x^2,$$

or, simplifying,

$$x^2 - 28x + 126 = 0.$$

Solving yields $x = 14 \pm \sqrt{70}$, but $14 + \sqrt{70}$ exceeds AB , so we discard it as an extraneous root. Thus, $BD = 14 - \sqrt{70}$ and

$$AD = AB - BD = 14 - (14 - \sqrt{70}) = \boxed{\sqrt{70}}.$$

Remark: if the altitude to point C meets \overline{AB} at point H , then the general answer to this problem is $\sqrt{AH \cdot AB}$. This result can be derived by considering the effects of dilation in the \overline{AB} direction and dilation in the \overline{CH} direction then performing dilations such that $\angle C$ is right and carrying out the calculation described above while considering congruent triangles.

8. Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K . Compute $KX \cdot KQ$.

Proposed by: Allen Liu

Answer: $\boxed{304}$

Let BY and CX meet at O . O is on the circumcircle of AXY , since $\triangle AXC \cong \triangle CYB$.

We claim that KA and KO are tangent to the circumcircle of AXY . Let XY and BC meet at L . Then, $LBZC$ is harmonic. A perspectivity at X gives $AYOP$ is harmonic. Similarly, a perspectivity at Y gives $AXOQ$ is harmonic. Thus, K is the pole of chord AO .

Now we compute. Denote r as the radius and θ as $\angle AXO$. Then,

$$r = \frac{XY}{\sqrt{3}} = \frac{\sqrt{5^2 + 3^2 - 3 \cdot 5}}{\sqrt{3}} = \sqrt{\frac{19}{3}};$$

$$\sin \theta = \sin 60^\circ \cdot \frac{AC}{XC} = \frac{\sqrt{3}}{2} \cdot \frac{8}{\sqrt{5^2 + 8^2 - 5 \cdot 8}} = \frac{4}{7}\sqrt{3};$$

$$KX \cdot KQ = KA^2 = (r \cdot \tan \theta)^2 = \left(\sqrt{\frac{19}{3}} \cdot 4\sqrt{3}\right)^2 = 304.$$

9. Po picks 100 points P_1, P_2, \dots, P_{100} on a circle independently and uniformly at random. He then draws the line segments connecting $P_1P_2, P_2P_3, \dots, P_{100}P_1$. When all of the line segments are drawn, the circle is divided into a number of regions. Find the expected number of regions that have all sides bounded by straight lines.

Proposed by: Allen Liu

Answer: $\boxed{\frac{4853}{3}}$

If the 100 segments do not intersect on the interior, then the circle will be cut into 101 regions. By Euler's formula, each additional intersection cuts two edges into two each, and adds one more vertex, so since $V - E + F$ is constant, there will be one more region as well. It then suffices to compute the expected number of intersections, where two segments that share a vertex are not counted as intersections.

We use linearity of expectation to compute this value. It suffices to compute the expected number of segments that each segment intersects. Consider one such segment P_1P_2 . It cannot possibly intersect a segment that shares an endpoint, so that leaves 97 possible other segments. Again, by linearity of expectation, it suffices to compute the probability that P_1P_2 intersects P_iP_{i+1} . However, since each of the points was chosen uniformly at random, this is equal to the probability that AC intersects BD , where A, B, C, D are chosen uniformly at random from the circle. Since this probability is $1/3$, each segment intersects with $\frac{97}{3}$ segments on average.

Now, we can sum over all segments and divide by two to get $(100 \cdot \frac{97}{3})/2 = \frac{4850}{3}$ intersections, since each intersection is counted twice. Accounting for the fact that there are 101 regions to begin with, and exactly 100 of them have an arc on the boundary, we get $\frac{4850}{3} + 101 - 100 = \frac{4853}{3}$ as the answer.

10. Let ABC be a triangle such that $AB = 6, BC = 5, AC = 7$. Let the tangents to the circumcircle of ABC at B and C meet at X . Let Z be a point on the circumcircle of ABC . Let Y be the foot of the perpendicular from X to CZ . Let K be the intersection of the circumcircle of BCY with line AB . Given that Y is on the interior of segment CZ and $YZ = 3CY$, compute AK .

Proposed by: Allen Liu

Answer: $\boxed{\frac{147}{10}}$

Let ω_1 denote the circumcircle of ABC and ω_2 denote the circle centered at X through B and C . Let ω_2 intersect AB, AC again at B', C' . The (signed) power of Y with respect to ω_1 is $-CY \cdot YZ$. The power of Y with respect to ω_2 is $XY^2 - CX^2 = -CY^2$. Thus the ratio of the powers of Y with respect to the two circles is $3 : 1$. The circumcircle of BCY passes through the intersection points of ω_1 and ω_2 (B and C) and thus contains exactly the set of points such that the ratio of their powers with respect to ω_1 and ω_2 is $3 : 1$ (this fact can be verified in a variety of ways). We conclude that K must be the point on line AB such that $\frac{KB'}{KA} = 3$. It now suffices to compute AB' . Note $AB' = \frac{AC \cdot B'C'}{BC}$ by similar triangles. Also an angle chase gives that $B'XC'$ are collinear. We compute

$$BX = \frac{BC}{2 \cos A} = \frac{5}{2 \cdot \frac{5}{7}} = \frac{7}{2}$$

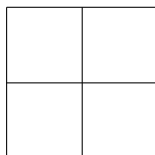
and thus $B'C' = 7$ so $AB' = \frac{49}{5}$ and $AK = \frac{3}{2}AB' = \frac{147}{10}$.

HMMT February 2018

February 10, 2018

Guts Round

1. [4] A square can be divided into four congruent figures as shown:



If each of the congruent figures has area 1, what is the area of the square?

Proposed by: Kevin Sun

Answer:

There are four congruent figures with area 1, so the area of the square is 4.

2. [4] John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

Proposed by: Farrell Eldrian Wu

Answer:

All the liquid was poured out eventually. 5 liters of water was poured in, and he started with 1 liter of orange juice, so the fraction is $\frac{5}{1+5} = \frac{5}{6}$.

3. [4] Allen and Yang want to share the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. How many ways are there to split all ten numbers among Allen and Yang so that each person gets at least one number, and either Allen's numbers or Yang's numbers sum to an even number?

Proposed by: Kevin Sun

Answer:

Since the sum of all of the numbers is odd, exactly one of Allen's sum and Yang's sum must be odd. Therefore any way of splitting the numbers up where each person receives at least one number is valid, so the answer is $2^{10} - 2 = 1022$.

4. [4] Find the sum of the digits of $11 \cdot 101 \cdot 111 \cdot 110011$.

Proposed by: Evan Chen

Answer:

There is no regrouping, so the answer is $2 \cdot 2 \cdot 3 \cdot 4 = 48$. The actual product is 13566666531.

5. [6] Randall proposes a new temperature system called *Felsius* temperature with the following conversion between Felsius $^{\circ}E$, Celsius $^{\circ}C$, and Fahrenheit $^{\circ}F$:

$$^{\circ}E = \frac{7 \times ^{\circ}C}{5} + 16 = \frac{7 \times ^{\circ}F - 80}{9}.$$

For example, $0^{\circ}C = 16^{\circ}E$. Let x, y, z be real numbers such that $x^{\circ}C = x^{\circ}E$, $y^{\circ}E = y^{\circ}F$, $z^{\circ}C = z^{\circ}F$. Find $x + y + z$.

Proposed by: Yuan Yao

Answer:

Notice that $(5k)^{\circ}C = (7k + 16)^{\circ}E = (9k + 32)^{\circ}F$, so Felsius is an exact average of Celsius and Fahrenheit at the same temperature. Therefore we conclude that $x = y = z$, and it is not difficult to compute that they are all equal to -40 .

6. [6] A bug is on a corner of a cube. A *healthy* path for the bug is a path along the edges of the cube that starts and ends where the bug is located, uses no edge multiple times, and uses at most two of the edges adjacent to any particular face. Find the number of healthy paths.

Proposed by: Dhruv Rohatgi

Answer:

There are 6 symmetric ways to choose the first two edges on the path. After these are chosen, all subsequent edges are determined, until the starting corner is reached once again.

7. [6] A triple of integers (a, b, c) satisfies $a + bc = 2017$ and $b + ca = 8$. Find all possible values of c .

Proposed by: Ashwin Sah

Answer:

Add and subtract the two equations to find

$$(b + a)(c + 1) = 8 + 2017,$$

$$(b - a)(c - 1) = 2017 - 8.$$

We see that c is even and then that every integer c with $c + 1 \mid 2025, c - 1 \mid 2009$ works. We factor and solve.

The full solutions are $(2017, 8, 0), (-667, 1342, 2), (-59, -346, -6), (-31, 256, 8)$.

8. [6] Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

Proposed by: Michael Tang

Answer:

Let A and B be these sums, respectively. Then

$$\begin{aligned} B - A &= \log_2\left(\frac{\log_{16} x}{\log_4 x}\right) + \log_4\left(\frac{\log_2 x}{\log_{16} x}\right) + \log_{16}\left(\frac{\log_4 x}{\log_2 x}\right) \\ &= \log_2(\log_{16} 4) + \log_4(\log_2 16) + \log_{16}(\log_4 2) \\ &= \log_2\left(\frac{1}{2}\right) + \log_4 4 + \log_{16}\left(\frac{1}{2}\right) \\ &= (-1) + 1 + \left(-\frac{1}{4}\right) \\ &= -\frac{1}{4}. \end{aligned}$$

Since $A = 0$, we have the answer $B = -\frac{1}{4}$.

9. [7] In a game, N people are in a room. Each of them simultaneously writes down an integer between 0 and 100 inclusive. A person wins the game if their number is exactly two-thirds of the average of all the numbers written down. There can be multiple winners or no winners in this game. Let m be the maximum possible number such that it is possible to win the game by writing down m . Find the smallest possible value of N for which it is possible to win the game by writing down m in a room of N people.

Proposed by: Kevin Sun

Answer:

Since the average of the numbers is at most 100, the winning number is an integer which is at most two-thirds of 100, or at most 66. This is achieved in a room with 34 people, in which 33 people pick 100 and one person picks 66, so the average number is 99.

Furthermore, this cannot happen with less than 34 people. If the winning number is 66 and there are N people, the sum of the numbers must be 99. then we must have that $99N \leq 66 + 100(N - 1)$, which reduces to $N \geq 34$.

10. [7] Let a positive integer n be called a *cubic square* if there exist positive integers a, b with $n = \gcd(a^2, b^3)$. Count the number of cubic squares between 1 and 100 inclusive.

Proposed by: Ashwin Sah

Answer: 13

This is easily equivalent to $v_p(n) \not\equiv 1, 5 \pmod{6}$ for all primes p . We just count: $p \geq 11 \implies v_p(n) = 1$ is clear, so we only look at the prime factorizations with primes from $\{2, 3, 5, 7\}$. This is easy to compute: we obtain 13.

11. [7] Find the value of

$$\sum_{k=1}^{60} \sum_{n=1}^k \frac{n^2}{61 - 2n}.$$

Proposed by: Henrik Boecken

Answer: -18910

Change the order of summation and simplify the inner sum:

$$\begin{aligned} \sum_{k=1}^{60} \sum_{n=1}^k \frac{n^2}{61 - 2n} &= \sum_{n=1}^{60} \sum_{k=n}^{60} \frac{n^2}{61 - 2n} \\ &= \sum_{n=1}^{60} \frac{n^2(61 - n)}{61 - 2n} \end{aligned}$$

Then, we rearrange the sum to add the terms corresponding to n and $61 - n$:

$$\begin{aligned} \sum_{n=1}^{60} \frac{n^2(61 - n)}{61 - 2n} &= \sum_{n=1}^{30} \left(\frac{n^2(61 - n)}{61 - 2n} + \frac{(61 - n)^2(61 - (61 - n))}{61 - 2(61 - n)} \right) \\ &= \sum_{n=1}^{30} \frac{n^2(61 - n) - n(61 - n)^2}{61 - 2n} \\ &= \sum_{n=1}^{30} \frac{n(61 - n)(n - (61 - n))}{61 - 2n} \\ &= \sum_{n=1}^{30} -n(61 - n) \\ &= \sum_{n=1}^{30} n^2 - 61n \end{aligned}$$

Finally, using the formulas for the sum of the first k squares and sum of the first k positive integers, we conclude that this last sum is

$$\frac{30(31)(61)}{6} - 61 \frac{30(31)}{2} = -18910$$

So, the original sum evaluates to -18910 .

12. [7] $\triangle PNR$ has side lengths $PN = 20$, $NR = 18$, and $PR = 19$. Consider a point A on PN . $\triangle NRA$ is rotated about R to $\triangle N'RA'$ so that R , N' , and P lie on the same line and AA' is perpendicular to PR . Find $\frac{PA}{AN}$.

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{19}{18}}$

Denote the intersection of PR and AA' be D . Note $RA' = RA$, so D , being the altitude of an isosceles triangle, is the midpoint of AA' . Thus,

$$\angle ARD = \angle A'RD = \angle NRA$$

so RA is the angle bisector of PNR through R . By the angle bisector theorem, we have $\frac{PA}{AN} = \frac{PR}{RN} = \frac{19}{18}$

13. [9] Suppose $\triangle ABC$ has lengths $AB = 5$, $BC = 8$, and $CA = 7$, and let ω be the circumcircle of $\triangle ABC$. Let X be the second intersection of the external angle bisector of $\angle B$ with ω , and let Y be the foot of the perpendicular from X to BC . Find the length of YC .

Proposed by: Caleb He

Answer: $\boxed{\frac{13}{2}}$

Extend ray \overrightarrow{AB} to a point D . Since BX is an angle bisector, we have $\angle XBC = \angle XBD = 180^\circ - \angle XBA = \angle XCA$, so $XC = XD$ by the inscribed angle theorem. Now, construct a point E on BC so that $CE = AB$. Since $\angle BAX \cong \angle BCX$, we have $\triangle BAX \cong \triangle ECX$ by SAS congruence. Thus, $XB = XE$, so Y bisects segment BE . Since $BE = BC - EC = 8 - 5 = 3$, we have $YC = EC + YE = 5 + \frac{1}{2} \cdot 3 = \frac{13}{2}$.

(Archimedes Broken Chord Theorem).

14. [9] Given that x is a positive real, find the maximum possible value of

$$\sin\left(\tan^{-1}\left(\frac{x}{9}\right) - \tan^{-1}\left(\frac{x}{16}\right)\right).$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{7}{25}}$

Consider a right triangle AOC with right angle at O , $AO = 16$ and $CO = x$. Moreover, let B be on AO such that $BO = 9$. Then $\tan^{-1}\frac{x}{9} = \angle CBO$ and $\tan^{-1}\frac{x}{16} = \angle CAO$, so their difference is equal to $\angle ACB$. Note that the locus of all possible points C given the value of $\angle ACB$ is part of a circle that passes through A and B , and if we want to maximize this angle then we need to make this circle as small as possible. This happens when OC is tangent to the circumcircle of ABC , so $OC^2 = OA \cdot OB = 144 = 12^2$, thus $x = 12$, and it suffices to compute $\sin(\alpha - \beta)$ where $\sin \alpha = \cos \beta = \frac{4}{5}$ and $\cos \alpha = \sin \beta = \frac{3}{5}$. By angle subtraction formula we get $\sin(\alpha - \beta) = \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$.

15. [9] Michael picks a random subset of the complex numbers $\{1, \omega, \omega^2, \dots, \omega^{2017}\}$ where ω is a primitive 2018th root of unity and all subsets are equally likely to be chosen. If the sum of the elements in his subset is S , what is the expected value of $|S|^2$? (The sum of the elements of the empty set is 0.)

Proposed by: Nikhil Reddy

Answer: $\boxed{\frac{1009}{2}}$

Consider a and $-a$ of the set of complex numbers. If x is the sum of some subset of the other complex numbers, then expected magnitude squared of the sum including a and $-a$ is

$$\frac{(x+a)(\overline{x+a}) + x\bar{x} + x\bar{x} + (x-a)(\overline{x-a})}{4}$$

$$x\bar{x} + \frac{a\bar{a}}{2}$$

$$x\bar{x} + \frac{1}{2}$$

By repeating this process on the remaining 2016 elements of the set, we can obtain a factor of $\frac{1}{2}$ every time. In total, the answer is

$$\frac{1009}{2}$$

16. [9] Solve for x :

$$x[x[x[x[x]]]] = 122.$$

Proposed by: John Michael Wu

Answer: $\boxed{\frac{122}{41}}$

This problem can be done without needless casework.

(For negative values of x , the left hand side will be negative, so we only need to consider positive values of x .)

The key observation is that for $x \in [2, 3)$, 122 is an extremely large value for the expression. Indeed, we observe that:

$$\begin{aligned} [x] &= 2 \\ [x[x]] &\leq 2(3) - 1 &&= 5 \\ [x[x[x]]] &\leq 3(5) - 1 &&= 14 \\ [x[x[x[x]]]] &\leq 3(14) - 1 &&= 41 \\ x[x[x[x[x]]]] &< 3(41) &&= 123 \end{aligned}$$

So the expression can only be as large as 122 if ALL of those equalities hold (the the fourth line equaling 40 isn't good enough), and $x = \frac{122}{41}$. Note that this value is extremely close to 3. We may check that this value of x indeed works. Note that the expression is strictly increasing in x , so $x = \frac{122}{41}$ is the only value that works.

17. [10] Compute the value of

$$\frac{\cos 30.5^\circ + \cos 31.5^\circ + \dots + \cos 44.5^\circ}{\sin 30.5^\circ + \sin 31.5^\circ + \dots + \sin 44.5^\circ}.$$

Proposed by: Sujay Kazi

Answer: $\boxed{(\sqrt{2} - 1)(\sqrt{3} + \sqrt{2}) = 2 - \sqrt{2} - \sqrt{3} + \sqrt{6}}$

Consider a 360-sided regular polygon with side length 1, rotated so that its sides are at half-degree inclinations (that is, its sides all have inclinations of $0.5^\circ, 1.5^\circ, 2.5^\circ$, and so on. Go to the bottom point on this polygon and then move clockwise, numbering the sides 1, 2, 3, ..., 360 as you go. Then, take the section of 15 sides from side 31 to side 45. These sides have inclinations of $30.5^\circ, 31.5^\circ, 32.5^\circ$, and so on, up to 44.5° . Therefore, over this section, the horizontal and vertical displacements are, respectively:

$$H = \cos 30.5^\circ + \cos 31.5^\circ + \dots + \cos 44.5^\circ$$

$$V = \sin 30.5^\circ + \sin 31.5^\circ + \dots + \sin 44.5^\circ$$

However, we can also see that, letting R be the circumradius of this polygon:

$$H = R(\sin 45^\circ - \sin 30^\circ)$$

$$V = R[(1 - \cos 45^\circ) - (1 - \cos 30^\circ)]$$

From these, we can easily compute that our desired answer is $\frac{H}{V} = (\sqrt{2} - 1)(\sqrt{3} + \sqrt{2}) = 2 - \sqrt{2} - \sqrt{3} + \sqrt{6}$.

18. [10] Compute the number of integers $n \in \{1, 2, \dots, 300\}$ such that n is the product of two distinct primes, and is also the length of the longest leg of some nondegenerate right triangle with integer side lengths.

Proposed by:

Answer: 13

Let $n = p \cdot q$ for primes $p < q$. If n is the second largest side of a right triangle there exist integers c, a such that $a < pq$ and $(pq)^2 = c^2 - a^2 = (c - a)(c + a)$. Since $c - a < c + a$ there are three cases for the values of $c - a, c + a$, and in each case we determine when $a < pq$.

- (a) $c - a = 1$ and $c + a = p^2 q^2$: Then $a = \frac{p^2 q^2 - 1}{2} > pq$, so there are no solutions.
- (b) $c - a = p$ and $c + a = pq^2$: Then $a = \frac{pq^2 - p}{2} > pq$.
- (c) $c - a = p^2$ and $c + a = q^2$. Then $a = \frac{q^2 - p^2}{2}$ which we require to be less than pq . This is equivalent to

$$\begin{aligned} \frac{q^2 - p^2}{2} &< pq \\ q^2 &< 2pq + p^2 \\ 2q^2 &< (q + p)^2 \\ \sqrt{2}q &< q + p \\ (\sqrt{2} - 1)q &< p < q \end{aligned}$$

So the problem is equivalent to finding the number of distinct prime pairs (p, q) such that $pq < 300$ and $(\sqrt{2} - 1)q < p < q$. There are 13 such pairs:

$\{(3, 5), (3, 7), (5, 7), (5, 11), (7, 11), (7, 13), (11, 13), (11, 17), (11, 19), (11, 23), (13, 17), (13, 19), (13, 23)\}$

and $13 \cdot 23 = 299$ which is the biggest such pair.

The most interesting borderline case are $(3, 7) : \frac{3}{7} \approx .42 > \sqrt{2} - 1$, which leads to the $(20, 21, 29)$ triangle, $(5, 13) : \frac{5}{13} \approx .385 < \sqrt{2} - 1$, which leads to the $(65, 72, 97)$ triangle, and $(7, 17) : \frac{7}{17} \approx .411 < \sqrt{2} - 1$ which leads to the $(119, 120, 169)$ right triangle.

19. [10] Suppose there are 100 cookies arranged in a circle, and 53 of them are chocolate chip, with the remainder being oatmeal. Pearl wants to choose a contiguous subsegment of exactly 67 cookies and wants this subsegment to have exactly k chocolate chip cookies. Find the sum of the k for which Pearl is guaranteed to succeed regardless of how the cookies are arranged.

Proposed by: Alexander Wei

Answer: 71

We claim that the only values of k are 35 and 36.

WLOG assume that the cookies are labelled 0 through 99 around the circle. Consider the following arrangement: cookies 0 through 17, 34 through 50, and 67 through 84 are chocolate chip, and the remaining are oatmeal. (The cookies form six alternating blocks around the circle of length 18, 16, 17, 16, 18, 15.) Consider the block of 33 cookies that are not chosen. It is not difficult to see that since the sum of the lengths of each two adjacent block is always at least 33 and at most 34, this block of unchosen cookies always contains at least one complete block of cookies of the same type (and no other cookies of this type). So this block contains 17 or 18 or $33 - 16 = 17$ or $33 - 15 = 18$ chocolate chip cookies. Therefore, the block of 67 chosen cookies can only have $53 - 17 = 36$ or $53 - 18 = 35$ chocolate chip cookies.

Now we show that 35 and 36 can always be obtained. Consider all possible ways to choose 67 cookies: cookies 0 through 66, 1 through 67, ..., 99 through 65. It is not difficult to see that the number of chocolate chip cookies in the block changes by at most 1 as we advance from one way to the next.

Moreover, each cookie will be chosen 67 times, so on average there will be $\frac{67 \cdot 53}{100} = 35.51$ chocolate chip cookies in each block. Since not all blocks are below average and not all blocks are above average, there must be a point where a block below average transition into a block above average. The difference of these two blocks is at most 1, so one must be 35 and one must be 36.

Therefore, the sum of all possible values of k is $35 + 36 = 71$.

20. [10] Triangle $\triangle ABC$ has $AB = 21$, $BC = 55$, and $CA = 56$. There are two points P in the plane of $\triangle ABC$ for which $\angle BAP = \angle CAP$ and $\angle BPC = 90^\circ$. Find the distance between them.

Proposed by: Michael Tang

Answer: $\boxed{\frac{5}{2}\sqrt{409}}$

Let P_1 and P_2 be the two possible points P , with $AP_1 < AP_2$. Both lie on the $\angle A$ -bisector and the circle γ with diameter BC . Let D be the point where the $\angle A$ -bisector intersects BC , let M be the midpoint of BC , and let X be the foot of the perpendicular from M onto the $\angle A$ -bisector.

Since we know the radius of γ , to compute P_1P_2 it suffices to compute MX . By the angle bisector theorem we find $BD = 15$ and $DC = 40$, so Stewart's theorem gives

$$15 \cdot 40 \cdot 55 + 55 \cdot AD^2 = 21^2 \cdot 40 + 56^2 \cdot 15 \implies AD = 24.$$

Then $\cos \angle ADB = \frac{-21^2 + 15^2 + 24^2}{2 \cdot 15 \cdot 24} = \frac{1}{2}$, so $\angle ADB = \angle MDX = 60^\circ$. Since $DM = BM - BD = \frac{55}{2} - 15 = \frac{25}{2}$, we get $MX = DM \sin \angle MDX = \frac{25\sqrt{3}}{4}$. Hence

$$P_1P_2 = 2\sqrt{\left(\frac{55}{2}\right)^2 - \left(\frac{25\sqrt{3}}{4}\right)^2} = \frac{5\sqrt{409}}{2}.$$

21. [12] You are the first lucky player to play in a slightly modified episode of Deal or No Deal! Initially, there are sixteen cases marked 1 through 16. The dollar amounts in the cases are the powers of 2 from $2^1 = 2$ to $2^{16} = 65536$, in some random order. The game has eight turns. In each turn, you choose a case and claim it, without opening it. Afterwards, a random remaining case is opened and revealed to you, then removed from the game.

At the end of the game, all eight of your cases are revealed and you win all of the money inside them.

However, the hosts do not realize you have X-ray vision and can see the amount of money inside each case! What is the expected amount of money you will make, given that you play optimally?

Proposed by: Kevin Sun

Answer: $\boxed{\frac{7 \cdot 2^{18} + 4}{15} \text{ (or } \frac{1835012}{15})}$

Firstly, note that it is always optimal for you to take the case with the largest amount of money. To prove this rigorously, consider a strategy where you don't - then change the first move where you deviate to taking the maximal case. This can only increase your return.

We calculate the probability $f(n, k)$ that, if there are n cases numbered $1, \dots, n$ in increasing order of value, that you will take case k in the course of play. We claim that $f(n, k) = \frac{k-1}{n-1}$ and prove this by induction on $n/2$ (n always even). The base case $n = 2$ is true because you will always take case 2 and leave case 1. Then, for the general n , you will always take case n (so $f(n, n) = 1$). Afterward, one case at random will be removed. When calculating $f(n, k)$ there is a $\frac{n-1-k}{n-1}$ probability a case numbered greater than k is removed, which inductively gives a probability $f(n-2, k)$. Also, there is a $\frac{k-1}{n-1}$ probability a case numbered less than k is removed, which inductively gives a probability

$f(n-2, k-1)$. We can compute

$$\begin{aligned}
 f(n, k) &= \frac{n-1-k}{n-1} f(n-2, k) + \frac{k-1}{n-1} f(n-2, k-1) \\
 &= \frac{n-1-k}{n-1} \cdot \frac{k-1}{n-3} + \frac{k-1}{n-1} \frac{k-2}{n-3} \\
 &= \frac{k-1}{(n-1)(n-3)} (n-1-k+k-2) \\
 &= \frac{k-1}{n-1}
 \end{aligned}$$

as desired.

Finally, we must find $\sum_{i=1}^{16} f(16, i)2^{i-1}$. Using standard procedures, we get

$$\begin{aligned}
 \sum_{i=1}^{16} f(16, i)2^{i-1} &= \sum_{i=1}^{15} \frac{i}{15} 2^i \\
 &= \sum_{i=1}^{15} \frac{i}{15} (2^{17} - 2^{i+1}) \\
 &= \frac{1}{15} (15)(2^{17}) - \frac{1}{15} \left(\sum_{i=1}^{15} 2^{i+1} \right) \\
 &= 2^{17} - \frac{1}{15} (2^{17} - 4) \\
 &= \boxed{\frac{14 \cdot 2^{17} + 4}{15}}
 \end{aligned}$$

22. [12] How many graphs are there on 10 vertices labeled $1, 2, \dots, 10$ such that there are exactly 23 edges and no triangles?

Proposed by: Allen Liu

Answer: 42840

Note that the sum of the degrees of the graph is $23 \cdot 2 = 46$, so at least one vertex has degree 5 or more. We casework on the maximal degree n .

Case 1: $n \geq 7$, then none of the n neighbors can have an edge between each other, for $\binom{n}{2}$ edges unusable, and the vertex with maximal degree cannot connect to the $9-n$ other vertices. Then we have $\binom{n}{2} + 9 - n > \binom{10}{2} - 23 = 22$ when $n \geq 7$, so there cannot be any graph in this case.

Case 2: $n = 6$. WLOG suppose that 1 is connected to 2, 3, 4, 5, 6, 7, then none of 2, 3, 4, 5, 6, 7 can connect to each other.

Case 2.1: There is at least one edge between 8, 9, 10, then each of 2, 3, 4, 5, 6, 7 can connect to at most two of 8, 9, 10, for at most $6 \cdot 2 + \binom{3}{2} = 15$ additional edges. Along with the 6 original edges, it is not enough to each 23 edges.

Case 2.2: There are no edges between 8, 9, 10, then this graph is a bipartite graph between 1, 8, 9, 10 and 2, 3, 4, 5, 6, 7. There can be at most $4 \cdot 6 = 24$ edges in this graph, so exactly one edge is removed from this graph. There are $\binom{10}{4} \cdot 24 = 5040$ possible graphs in this case.

Case 3: $n = 5$. WLOG suppose that 1 is connected to 2, 3, 4, 5, 6, then none of 2, 3, 4, 5, 6 can connect to each other.

Case 3.1: There is at least one edge between 7, 8, 9, 10. Then each of 2, 3, 4, 5, 6 can connect to at most three of 7, 8, 9, 10, for $5 \cdot 3 = 15$ edges. In this case at least three of 7, 8, 9, 10 must not be connected to each other, so there can be at most three edges, for $5 + 15 + 3 = 23$ edges at most. However, this requires the three disconnected vertices of 7, 8, 9, 10 to be connected to all of 2, 3, 4, 5, 6 and the other

vertex of 7, 8, 9, 10, causing them to have degree 6. We can therefore ignore this case. (The case where 2, 3, 4, 5, 6 can connect to two or less of 7, 8, 9, 10 can be easily ruled out.)

Case 3.2: There are no edges between 7, 8, 9, 10, then this graph is a bipartite graph between 1, 7, 8, 9, 10 and 2, 3, 4, 5, 6. This is a $K_{5,5}$ with two edges removed, which accounts for $\binom{10}{5}/2 \cdot \binom{25}{2} = 126 \cdot 300 = 37800$ graphs.

It is not difficult to see that Case 2.2 and Case 3.2 are disjoint (by considering max degree), so there are $5040 + 37800 = 42840$ graphs in total.

23. [12] Kevin starts with the vectors $(1, 0)$ and $(0, 1)$ and at each time step, he replaces one of the vectors with their sum. Find the cotangent of the minimum possible angle between the vectors after 8 time steps.

Proposed by: Allen Liu

Answer: 987

Say that the vectors Kevin has at some step are (a, b) and (c, d) . Notice that regardless of which vector he replaces with $(a + c, b + d)$, the area of the triangle with vertices $(0, 0)$, (a, b) , and (c, d) is preserved with the new coordinates. We can see this geometrically: the parallelogram with vertices $(0, 0)$, (a, b) , (c, d) , and $(a + c, b + d)$ can be cut in half by looking at the triangle formed by any 3 of the vertices, which include the original triangle, and both possible triangles that might arise in the next step.

Because the area is preserved, the minimum possible angle then arises when the two vectors, our sides, are as long as possible. This occurs when we alternate which vector is getting replaced for the sum. Given two vectors (a, b) and (c, d) , with $\sqrt{a^2 + b^2} > \sqrt{c^2 + d^2}$, we would rather replace (c, d) than (a, b) , and $(a + c, b + d)$ has a larger norm than (a, b) . Then at the n th step, Kevin has the vectors (F_n, F_{n-1}) and (F_{n+1}, F_n) , where $F_0 = 0$ and $F_1 = 1$. The tangent of the angle between them is the tangent of the difference of the angles they make with the x-axis, which is just their slope. We can then compute the cotangent as

$$\left| \frac{1 + \frac{F_{n-1}}{F_n} \cdot \frac{F_n}{F_{n+1}}}{\frac{F_n}{F_{n+1}} - \frac{F_{n-1}}{F_n}} \right| = \left| \frac{F_n(F_{n+1} + F_{n-1})}{F_n^2 - F_{n-1}F_{n+1}} \right|.$$

We can show (by induction) that $F_n(F_{n+1} + F_{n-1}) = F_{2n}$ and $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$. Thus at the 8th step, the cotangent of the angle is $F_{16} = 987$.

24. [12] Find the largest positive integer n for which there exist n finite sets X_1, X_2, \dots, X_n with the property that for every $1 \leq a < b < c \leq n$, the equation

$$|X_a \cup X_b \cup X_c| = \lceil \sqrt{abc} \rceil$$

holds.

Proposed by: Pakawut Jiradilok

Answer: 4

First, we construct an example for $N = 4$. Let X_1, X_2, X_3, X_4 be pairwise disjoint sets such that $X_1 = \emptyset$, $|X_2| = 1$, $|X_3| = 2$, and $|X_4| = 2$. It is straightforward to verify the condition.

We claim that there are no five sets X_1, X_2, \dots, X_5 for which $\#(X_a \cup X_b \cup X_c) = \lceil \sqrt{abc} \rceil$, for $1 \leq a < b < c \leq 5$. Note that showing the non-existence of five such sets implies that there are no n sets with the desired property for $n \geq 5$ as well.

Suppose, for sake of contradiction, that there are such X_1, \dots, X_5 . Then, note that $|X_1 \cup X_2 \cup X_4| = 3$, $|X_1 \cup X_2 \cup X_5| = 4$, and $|X_2 \cup X_4 \cup X_5| = 7$. Note that

$$|X_1 \cup X_2 \cup X_4| + |X_1 \cup X_2 \cup X_5| = |X_2 \cup X_4 \cup X_5|.$$

For any sets A, B, C, D , we have the following two inequalities:

$$|A \cup B \cup C| + |A \cup B \cup D| \geq |A \cup B \cup C \cup D| \geq |B \cup C \cup D|.$$

For $A = X_1$, $B = X_2$, $C = X_4$, and $D = X_5$ in the situation above, we conclude that the equalities must both hold in both inequalities. The first equality shows that $X_1 \cup X_2 = \emptyset$, and therefore both X_1 and X_2 are empty.

Now observe that $|X_1 \cup X_4 \cup X_5| = 5 \neq 7 = |X_2 \cup X_4 \cup X_5|$. This gives a contradiction.

25. [15] Fran writes the numbers $1, 2, 3, \dots, 20$ on a chalkboard. Then she erases all the numbers by making a series of moves; in each move, she chooses a number n uniformly at random from the set of all numbers still on the chalkboard, and then erases all of the divisors of n that are still on the chalkboard (including n itself). What is the expected number of moves that Fran must make to erase all the numbers?

Proposed by: Michael Tang

Answer: $\boxed{\frac{131}{10}}$

For each n , $1 \leq n \leq 20$, consider the first time that Fran chooses one of the multiples of n . It is in this move that n is erased, and all the multiples of n at most 20 are equally likely to be chosen for this move. Hence this is the only move in which Fran could possibly choose n ; since there are $\lfloor 20/n \rfloor$ multiples of n at most 20, this means that the probability that n is ever chosen is $1/\lfloor 20/n \rfloor$. Therefore the expected number of moves is

$$\begin{aligned} E &= \sum_{n=1}^{20} \frac{1}{\lfloor 20/n \rfloor} \\ &= \frac{1}{20} + \frac{1}{10} + \frac{1}{6} + \frac{1}{5} + \frac{1}{4} + \frac{1}{3} + 4 \left(\frac{1}{2} \right) + 10(1) = \frac{131}{10}. \end{aligned}$$

(This sum is easier to compute than it may seem, if one notes that $1/20 + 1/5 + 1/4 = 1/2$ and $1/6 + 1/3 = 1/2$).

26. [15] Let ABC be a triangle with $\angle A = 18^\circ$, $\angle B = 36^\circ$. Let M be the midpoint of AB , D a point on ray CM such that $AB = AD$; E a point on ray BC such that $AB = BE$, and F a point on ray AC such that $AB = AF$. Find $\angle FDE$.

Proposed by: Faraz Masroor

Answer: $\boxed{27}$

Let $\angle ABD = \angle ADB = x$, and $\angle DAB = 180 - 2x$. In triangle ACD , by the law of sines, $CD = \frac{AD}{\sin \angle ACM} \cdot \sin 198 - 2x$, and by the law of sines in triangle BCD , $CD = \frac{BD}{\sin \angle BCM} \cdot \sin x + 36$. Combining the two, we have $2 \cos x = \frac{BD}{AD} = \frac{\sin 198 - 2x}{\sin x + 36} \cdot \frac{\sin \angle BCM}{\sin \angle ACM}$. But by the ratio lemma, $1 = \frac{MB}{MA} = \frac{CB \sin \angle BCM}{CA \sin \angle ACM}$, meaning that $\frac{\sin \angle BCM}{\sin \angle ACM} = \frac{CA}{CB} = \frac{\sin 36}{\sin 18} = 2 \cos 18$. Plugging this in and simplifying, we have $2 \cos x = \frac{\sin 198 - 2x}{\sin x + 36} \cdot 2 \cos 18 = \frac{\cos 108 - 2x}{\cos 54 - x} \cdot 2 \cos 18$, so that $\frac{\cos x}{\cos 18} = \frac{\cos 108 - 2x}{\cos 54 - x}$. We see that $x = 36^\circ$ is a solution to this equation, and by carefully making rough sketches of both functions, we can convince ourselves that this is the only solution where x is between 0 and 90 degrees. Therefore $\angle ABD = \angle ADB = 36^\circ$, $\angle DAB = 108^\circ$. Simple angle chasing yields $\angle AEB = 72^\circ$, $\angle ECA = 54^\circ$, $\angle EAC = 54^\circ$, $\angle EAB = 72^\circ$, making D, A , and E collinear, and so $\angle BDE = 36^\circ$. And because $AF = AB = AD$, $\angle FDB = 1/2 \angle FAB = 9^\circ$, so $\angle FDE = 36 - 9 = 27^\circ$.

27. [15] There are 2018 frogs in a pool and there is 1 frog on the shore. In each time-step thereafter, one random frog moves position. If it was in the pool, it jumps to the shore, and vice versa. Find the expected number of time-steps before all frogs are in the pool for the first time.

Proposed by: Dhruv Rohatgi

Answer: $\boxed{2^{2018} - 1}$

Consider the general case of n frogs. Let E_i be the expected time for all frogs to enter the pool when i frogs are on the shore and $n - i$ frogs are in the pool. We have $E_0 = 0$, $E_n = 1 + E_{n-1}$, and

$$E_i = \frac{i}{n}E_{i-1} + \frac{n-i}{n}E_{i+1} + 1$$

for $0 < i < n$. Define f_i so that

$$E_i = \frac{f_i}{(n-1)(n-2)\cdots(i)} + E_{i-1}.$$

Then by plugging this equation into the first equation, we can show that

$$f_i = n(n-1)\cdots(i+1) + (n-i)f_{i+1}.$$

Furthermore, we know that $f_n = 1$. Therefore

$$\begin{aligned} f_1 &= \sum_{i=1}^n \frac{n!}{i!} \frac{(n-1)!}{(n-i)!} \\ &= (n-1)! \sum_{i=1}^n \binom{n}{i} \\ &= (n-1)!(2^n - 1). \end{aligned}$$

Therefore

$$E_1 = \frac{(n-1)!(2^n - 1)}{(n-1)!} + E_0 = 2^n - 1.$$

Plugging in $n = 2018$ yields $E_1 = 2^{2018} - 1$.

28. [15] Arnold and Kevin are playing a game in which Kevin picks an integer $1 \leq m \leq 1001$, and Arnold is trying to guess it. On each turn, Arnold first pays Kevin 1 dollar in order to guess a number k of Arnold's choice. If $m \geq k$, the game ends and he pays Kevin an additional $m - k$ dollars (possibly zero). Otherwise, Arnold pays Kevin an additional 10 dollars and continues guessing.

Which number should Arnold guess first to ensure that his worst-case payment is minimized?

Proposed by:

Answer: 859

We let $f(n)$ denote the smallest amount we can guarantee to pay at most if Arnold's first choice is n . For each $k < n$, if Arnold's first choice is $k + 1$, in both worst case scenarios, he could end up paying either $n - k$ or $11 + f(k)$. It is then clear that $f(n) = \min_{k+1 < n} \max\{n - k, 11 + f(k)\}$.

Now clearly $f(k)$ is a non-decreasing function of k , and $n - k$ is a strictly decreasing function of k . Therefore if there exists k such that $n - k = 11 + f(k)$, we have $f(n) = n - k = 11 + f(k)$ with picking $k + 1$ as an optimal play (and picking $K + 1$ also optimal iff $K \geq k$ and $f(K) = f(k)$).

Now note that $f(k) = k$ for $k \leq 12$ (but $f(13) = 12$ though it's not relevant to the solution). Let $a_1 = 11$. Now recursively define a_i such that $a_i - a_{i-1} = 11 + f(a_{i-1})$. Thus $f(a_i) = a_i - a_{i-1}$ with the optimal move to pick $a_{i-1} + 1$.

$$a_1 = 11$$

$$a_2 - 11 = 11 + 11 : a_2 = 33, f(a_2) = 22$$

$$a_3 - 33 = 11 + f(33) : a_3 = 66, f(a_3) = 33$$

It is clear by induction that a_i is 11 times the i th triangular number. 1001 is $11 \times 91 = \frac{14 \times 13}{2}$, so the optimal strategy is to pick 1 more than $11 \times \frac{12 \times 13}{2} = 858$. So the answer is 859.

29. [17] Let a, b, c be positive integers. All the roots of each of the quadratics

$$ax^2 + bx + c, ax^2 + bx - c, ax^2 - bx + c, ax^2 - bx - c$$

are integers. Over all triples (a, b, c) , find the triple with the third smallest value of $a + b + c$.

Proposed by: *Henrik Boecken*

Answer: (1, 10, 24)

The quadratic formula yields that the answers to these four quadratics are $\frac{\pm b \pm \sqrt{b^2 \pm 4ac}}{2a}$. Given that all eight of these expressions are integers, we can add or subtract appropriate pairs to get that $\frac{b}{a}$ and $\frac{\sqrt{b^2 \pm 4ac}}{a}$ are integers. Let $b' = \frac{b}{a}$ and $c' = \frac{4c}{a}$. We can rewrite the expressions to get that b' and $\sqrt{b'^2 \pm c'}$ are positive integers, which also tells us that c' is a positive integer. Let $b'^2 + c' = n^2$, $b'^2 - c' = m^2$.

Notice that $a + b + c = a(1 + b' + \frac{c'}{4})$, so to find the third smallest value of $a + b + c$, we first find small solutions to (b', c') . To do this, we find triples (m, b', n) such that m^2, b'^2, n^2 form an arithmetic sequence. Because odd squares are 1 mod 4 and even squares are 0 mod 4, if any of these three terms is odd, then all three terms must be odd. By dividing these terms by the largest possible power of 2 then applying the same logic, we can extend our result to conclude that $v_2(m) = v_2(b') = v_2(n)$. Thus, we only need to look at (m, b', n) all odd, then multiply them by powers of 2 to get even solutions.

We then plug in $b' = 3, 5, 7, 9$, and find that out of these options, only $(n, b', m) = (1, 5, 7)$ works, giving $(b', c') = (5, 24)$, $a + b + c = 12a$. Multiplying by 2 yields that $(n, b', m) = (2, 10, 14)$ also works, giving $(b', c') = (10, 96)$, $a + b + c = 35a$. For $11 \leq b \leq 17$, we can check that $m = b + 2$ fails to give an integer n . For $11 \leq b \leq 17$, $m \neq b + 2$, $a + b + c = a(1 + b' + \frac{c'}{4}) \geq a(1 + 11 + \frac{15^2 - 11^2}{4}) = 38a$, the smallest possible value of which is greater than $12a$ with $a = 1$, $12a$ with $a = 2$, and $35a$ with $a = 1$. Thus, it cannot correspond to the solution with the third smallest $a + b + c$. For $b \geq 19$, $a + b + c = a(1 + b' + \frac{c'}{4}) \geq a(1 + 19 + \frac{21^2 + 19^2}{4}) = 40a$, which, similar as before, can't correspond to the solution with the third smallest $a + b + c$.

Thus the smallest solution is $(a, b', c') = (1, 5, 24)$, $(a, b, c) = (1, 5, 6)$, the second smallest solution is $(a, b', c') = (2, 5, 24)$, $(a, b, c) = (2, 10, 12)$, and the third smallest solution that the problem asks for is $(a, b', c') = (1, 10, 96)$, $(a, b, c) = (1, 10, 24)$.

30. [17] Find the number of *unordered* pairs $\{a, b\}$, where $a, b \in \{0, 1, 2, \dots, 108\}$ such that 109 divides $a^3 + b^3 - ab$.

Proposed by: *Henrik Boecken*

Answer: 54

We start with the equation

$$a^3 + b^3 \equiv ab \pmod{109}.$$

If either a or b are 0, then we get $a^3 \equiv 0$, implying that both are 0. Thus, $\{0, 0\}$ is a pair. For the rest of the problem, let's assume that neither a nor b are 0. Multiplying both sides by $a^{-1}b^{-2}$ yields

$$(ab^{-1})^2 + a^{-1}b \equiv b^{-1}$$

from which we make the substitution

$$\begin{aligned} a &= xy^{-1} \\ b &= y^{-1} \end{aligned}$$

to get the equation

$$y \equiv x^2 + x^{-1}.$$

Plugging this value back into (a, b) , we get that all solutions must be of the form

$$(a, b) = ((x + x^{-2})^{-1}, (x^2 + x^{-1})^{-1}),$$

where $1 \leq x \leq 108$. It now suffices to find all nonzero unordered pairs $\{m, n\}$ of the form $\{x+x^{-2}, x^2+x^{-1}\}$, where $1 \leq x \leq 108$. There are four values of x for which $x+x^{-2} \equiv x^2+x^{-1}$, and of these values, three of them give $x+x^{-2} \equiv 0$. This is because we can re-arrange the equation at hand to get

$$x^4 - x^3 + x - 1 \equiv 0,$$

which factors into

$$(x-1)(x^3+1) \equiv 0.$$

If $x = 1$, then $\{m, n\} = \{2, 2\}$, and if $x^3 + 1 \equiv 0$ (which has three solutions: 46, 64 and 108), then

$$\{m, n\} = \{x^{-1}(x^3+1), x^{-2}(x^3+1)\} = \{0, 0\}.$$

Therefore, we keep $x = 1$ and discard $x = 46, 64, 108$. Of the remaining 104 values of x , $m \neq n$, and neither are 0. We have to worry about collisions between distinct values of x . There are two ways a collision can occur: if there exists $x \neq y$ such that

$$(x+x^{-2}, x^2+x^{-1}) = (y+y^{-2}, y^2+y^{-1}),$$

or if there exists $x \neq y$ such that

$$(x+x^{-2}, x^2+x^{-1}) = (y^2+y^{-1}, y+y^{-2}).$$

The first case cannot occur: if $x+x^{-2} \equiv y+y^{-2}$, we have that $x^2+x^{-1} = x(x+x^{-2}) \neq y(x+x^{-2}) = y(y+y^{-2}) = y^2+y^{-1}$. As a consequence of this, the second case only occurs if $y = x^{-1}$. Therefore, the remaining 104 values of x can be partitioned into 52 pairs of (x, x^{-1}) , which ends up producing 52 distinct unordered pairs $\{m, n\}$. Adding this to the $x = 1$ case and $\{0, 0\}$, we get a total of

$$52 + 1 + 1 = \boxed{54}$$

unordered pairs.

31. [17] In triangle ABC , $AB = 6$, $BC = 7$ and $CA = 8$. Let D, E, F be the midpoints of sides BC, AC, AB , respectively. Also let O_A, O_B, O_C be the circumcenters of triangles AFD, BDE , and CEF , respectively. Find the area of triangle $O_A O_B O_C$.

Proposed by: Henrik Boecken

Let $AB = z, BC = x, CA = y$. Let X, Y, Z, O, N be the circumcenter of AEF, BFD, CDE, ABC, DEF respectively. Note that N is the nine-point center of ABC , and X, Y, Z are the midpoints of OA, OB, OC respectively, and thus XYZ is the image of homothety of ABC with center O and ratio $\frac{1}{2}$, so this triangle has side lengths $\frac{x}{2}, \frac{y}{2}, \frac{z}{2}$. Since NX perpendicularly bisects EF , which is parallel to BC and thus YZ , we see that N is the orthocenter of XYZ . Moreover, O_1 lies on YN and O_1X is perpendicular to XY .

To compute the area of $O_1 O_2 O_3$, it suffices to compute $[NO_1 O_2] + [NO_2 O_3] + [NO_3 O_1]$. Note that $O_1 X$ is parallel to NO_2 , and $O_2 Y$ is parallel to XN , so $[NO_1 O_2] = [NXO_2] = [NXY]$. Similarly the other two triangles have equal area as $[NYZ]$ and $[NZX]$ respectively, so the desired area is simply the area of $[XYZ]$, which is

$$\frac{1}{4} \frac{\sqrt{(x+y+z)(x+y-z)(x-y+z)(-x+y+z)}}{4} = \frac{\sqrt{21 \cdot 9 \cdot 5 \cdot 7}}{16} = \frac{21\sqrt{15}}{16}.$$

32. [17] How many 48-tuples of positive integers $(a_1, a_2, \dots, a_{48})$ between 0 and 100 inclusive have the property that for all $1 \leq i < j \leq 48$, $a_i \notin \{a_j, a_j + 1\}$?

Proposed by: Mehtaab Sawhney

Answer: $\boxed{54^{48}}$

(With Ashwin Sah) The key idea is write the elements of the sequence in increasing order. These sets are in bijection with solutions to $d_1 + \dots + d_k = 48$ and $a_1 + \dots + a_{k+1} = 53$ with $d_i \geq 1, a_i \geq 1$ for

$2 \leq I \leq k$, and $a_1, a_{k+1} \geq 0$. Notice that there are $\binom{54}{k}$ solutions to the second equation and then there are $\frac{48!}{d_1! \dots d_k!}$ solutions for each $\{d_i\}$ set. Then this gives that the answer is

$$\begin{aligned} & \sum_{1 \leq k \leq 48} \binom{54}{k} \sum_{d_1 + \dots + d_k = 48} \frac{48!}{\prod_{i=1}^k d_i!} \\ &= 48! [x^{48}] \sum_{1 \leq k \leq 48} (e^x - 1)^k \binom{54}{k} \\ &= 48! [x^{48}] \sum_{0 \leq k \leq 54} (e^x - 1)^k \binom{54}{k} \\ &= 48! [x^{48}] (e^x)^{54} \\ &= 54^{48}. \end{aligned}$$

33. [20] 679 contestants participated in HMMT February 2017. Let N be the number of these contestants who performed at or above the median score in at least one of the three individual tests. Estimate N . An estimate of E earns $\lfloor 20 - \frac{|E-N|}{2} \rfloor$ or 0 points, whichever is greater.

Proposed by: Henrik Boecken

Answer:

Out of the 679 total contestants at HMMT February 2017, 188 contestants scored at least the median on all three tests, 159 contestants scored at least the median on two tests, and 169 contestants scored at least the median on one test, giving a total of 516 contestants

34. [20] The integers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 are written on a blackboard. Each day, a teacher chooses one of the integers uniformly at random and decreases it by 1. Let X be the expected value of the number of days which elapse before there are no longer positive integers on the board. Estimate X . An estimate of E earns $\lfloor 20 \cdot 2^{-|X-E|/8} \rfloor$ points.

Proposed by: Mehtaab Sawhney

Answer:

2967376198031036029043518754502337892568216567911415193776896659976433643147319195286268925688566342694

35. [20] In a wooden block shaped like a cube, all the vertices and edge midpoints are marked. The cube is cut along *all* possible planes that pass through at least four marked points. Let N be the number of pieces the cube is cut into. Estimate N . An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N) \rfloor$ points.

Proposed by: Yuan Yao

Answer:

36. [20] In the game of Connect Four, there are seven vertical columns which have spaces for six tokens. These form a 7×6 grid of spaces. Two players White and Black move alternately. A player takes a turn by picking a column which is not already full and dropping a token of their color into the lowest unoccupied space in that column. The game ends when there are four consecutive tokens of the same color in a line, either horizontally, vertically, or diagonally. The player who has four tokens in a row of their color wins.

Assume two players play this game randomly. Each player, on their turn, picks a random column which is not full and drops a token of their color into that column. This happens until one player wins or all of the columns are filled. Let P be the probability that all of the columns are filled without any player obtaining four tokens in a row of their color. Estimate P .

An estimate of $E > 0$ earns $\lfloor 20 \min(P/E, E/P) \rfloor$ points.

Proposed by: Allen Liu

Answer:

HMMT February 2018

February 10, 2018

Team Round

1. [20] In an $n \times n$ square array of 1×1 cells, at least one cell is colored pink. Show that you can always divide the square into rectangles along cell borders such that each rectangle contains exactly one pink cell.

Proposed by: Kevin Sun

We claim that the statement is true for arbitrary rectangles. We proceed by induction on the number of marked cells. Our base case is $k = 1$ marked cell, in which case the original rectangle works.

To prove it for k marked cells, we split the rectangle into two smaller rectangles, both of which contains at least one marked cell. By induction, we can divide the two smaller rectangles into rectangles with exactly one marked cell. Combining these two sets of rectangles gives a way to divide our original rectangle into rectangles with exactly one marked cell, completing the induction.

2. [25] Is the number

$$\left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{6}\right) \cdots \left(1 + \frac{1}{2018}\right)$$

greater than, less than, or equal to 50?

Proposed by: Henrik Boecken

Call the expression S . Note that

$$\left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{6}\right) \cdots \left(1 + \frac{1}{2018}\right) < \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{5}\right) \cdots \left(1 + \frac{1}{2017}\right)$$

Multiplying these two products together, we get

$$\begin{aligned} & \left(1 + \frac{1}{1}\right) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \cdots \left(1 + \frac{1}{2018}\right) \\ &= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2019}{2018} \\ &= 2019 \end{aligned}$$

This shows that

$$S^2 < 2019 \implies S < \sqrt{2019} < 50$$

as desired.

3. [30] Michelle has a word with 2^n letters, where a word can consist of letters from any alphabet. Michelle performs a *switcheroo* on the word as follows: for each $k = 0, 1, \dots, n-1$, she switches the first 2^k letters of the word with the next 2^k letters of the word. For example, for $n = 3$, Michelle changes

$$ABCDEFGH \rightarrow BACDEFGH \rightarrow CDBAEFGH \rightarrow EFGHCDBA$$

in one switcheroo.

In terms of n , what is the minimum positive integer m such that after Michelle performs the switcheroo operation m times on any word of length 2^n , she will receive her original word?

Proposed by: Mehtaab Sawhney

Let $m(n)$ denote the number of switcheroos needed to take a word of length 2^n back to itself. Consider a word of length 2^n for some $n > 1$. After 2 switcheroos, one has separately performed a switcheroo on the first half of the word and on the second half of the word, while returning the (jumbled) first half of the word to the beginning and the (jumbled) second half of the word to the end.

After $2 \cdot m(n-1)$ switcheroos, one has performed a switcheroo on each half of the word $m(n-1)$ times while returning the halves to their proper order. Therefore, the word is in its proper order. However, it is never in its proper order before this, either because the second half precedes the first half (i.e. after an odd number of switcheroos) or because the halves are still jumbled (because each half has had fewer than $m(n-1)$ switcheroos performed on it).

It follows that $m(n) = 2m(n-1)$ for all $n > 1$. We can easily see that $m(1) = 2$, and a straightforward proof by induction shows that $m = 2^n$.

4. [30] In acute triangle ABC , let D, E , and F be the feet of the altitudes from A, B , and C respectively, and let L, M , and N be the midpoints of BC, CA , and AB , respectively. Lines DE and NL intersect at X , lines DF and LM intersect at Y , and lines XY and BC intersect at Z . Find $\frac{ZB}{ZC}$ in terms of AB, AC , and BC .

Proposed by: Faraz Masroor

Because $NL \parallel AC$ we have triangles DXL and DEC are similar. From angle chasing, we also have that triangle DEC is similar to triangle ABC . We have $\angle XNA = 180^\circ - \angle XNB = 180^\circ - \angle LNB = 180^\circ - \angle CAB = \angle LMA$. In addition, we have $\frac{NX}{NA} = \frac{XD \cdot XE}{XL \cdot NA} = \frac{AB}{BC} \frac{XE}{LC} \frac{NM}{NA} = \frac{AB}{BC} \frac{ED}{DC} \frac{BC}{AB} = \frac{ED}{DC} = \frac{AB}{AC} = \frac{ML}{MA}$. These two statements mean that triangles ANX and AML are similar, and $\angle XAB = \angle XAN = \angle LAM = \angle LAC$. Similarly, $\angle XAY = \angle LAC$, making A, X , and Y collinear, with $\angle YAB = \angle XAB = \angle LAC$; i.e. line AXY is a symmedian of triangle ABC .

Then $\frac{ZB}{ZC} = \frac{AB \sin \angle ZAB}{AC \sin \angle ZAC} = \frac{AB \sin \angle LAC}{AC \sin \angle LAB}$, by the ratio lemma. But using the ratio lemma, $1 = \frac{LB}{LC} = \frac{AB \sin \angle LAB}{AC \sin \angle LAC}$, so $\frac{\sin \angle LAC}{\sin \angle LAB} = \frac{AB}{AC}$, so $\frac{ZB}{ZC} = \frac{AB^2}{AC^2}$.

5. [30] Is it possible for the projection of the set of points (x, y, z) with $0 \leq x, y, z \leq 1$ onto some two-dimensional plane to be a simple convex pentagon?

Proposed by: Yuan Yao

It is not possible. Consider P , the projection of $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ onto the plane. Since for any point (x, y, z) in the cube, $(1-x, 1-y, 1-z)$ is also in the cube, and the midpoint of their projections will be the projection of their midpoint, which is P , the projection of the cube onto this plane will be a centrally symmetric region around P , and thus cannot be a pentagon.

6. [35] Let $n \geq 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \leq x < n$, there is a subset of S whose sum has remainder x when divided by n . Note that the empty set has sum 0. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

Proposed by: Allen Liu

We will show that if $|S| \geq n$, we can remove one element from S and still have a comprehensive set. Doing this repeatedly will always allow us to find a comprehensive subset of size at most $n-1$.

Write $S = \{s_1, s_2, \dots, s_k\}$ for some $k \geq n$. Now start with the empty set and add in the elements s_i in order. During this process, we will keep track of all possible remainders of sums of any subset.

If T is the set of current remainders at any time, and we add an element s_i , the set of remainders will be $T \cup \{t + s_i | t \in T\}$. In particular, the set of remainders only depends on the previous set of remainders and the element we add in.

At the beginning of our process, the set of possible remainders is $\{0\}$ for the empty set. Since we assumed that S is comprehensive, the final set is $\{0, 1, \dots, n\}$. The number of elements changes from 1 to $n-1$. However, since we added $k \geq n$ elements, at least one element did not change the size of our remainder set. This implies that adding this element did not contribute to making any new remainders and S is still comprehensive without this element, proving our claim.

7. [50] Let $[n]$ denote the set of integers $\{1, 2, \dots, n\}$. We randomly choose a function $f : [n] \rightarrow [n]$, out of the n^n possible functions. We also choose an integer a uniformly at random from $[n]$. Find the probability that there exist positive integers $b, c \geq 1$ such that $f^b(1) = a$ and $f^c(a) = 1$. ($f^k(x)$ denotes the result of applying f to x k times).

Proposed by: Allen Liu

Answer: $\boxed{\frac{1}{n}}$

Given a function f , define $N(f)$ to be the number of numbers that are in the same cycle as 1 (including 1 itself), if there is one, and zero if there is no such cycle. The problem is equivalent to finding $\mathbb{E}(N(f))/n$. Note that

$$P(N(f) = k) = \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{n-k+1}{n} \cdot \frac{1}{n},$$

and it suffices to compute $\sum_{k=1}^n P_k$ where $P_k = \frac{k}{n} P(N(f) = k)$. Observe that

$$\begin{aligned} P_n &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \cdot \frac{1}{n} \right) \cdot \frac{n}{n} && \frac{1}{n} \\ P_{n-1} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \right) \cdot \frac{n-1}{n} && \frac{1}{n} \\ \Rightarrow P_n + P_{n-1} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \cdot \frac{2}{n} \right) && \frac{1}{n} \\ P_{n-2} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \right) \cdot \frac{n-2}{n} && \frac{1}{n} \\ \Rightarrow P_n + P_{n-1} + P_{n-2} &= \left(\frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{3}{n} \right) && \frac{1}{n} \\ &\dots && \\ \Rightarrow \sum_{k=1}^n P_k &= 1 && \frac{1}{n} \end{aligned}$$

Therefore the answer is $\frac{1}{n}$.

8. [60] Allen plays a game on a tree with $2n$ vertices, each of whose vertices can be red or blue. Initially, all of the vertices of the tree are colored red. In one move, Allen is allowed to take two vertices of the same color which are connected by an edge and change both of them to the opposite color. He wins if at any time, all of the vertices of the tree are colored blue.
- (a) (20) Show that Allen can win if and only if the vertices can be split up into two groups V_1 and V_2 of size n , such that each edge in the tree has one endpoint in V_1 and one endpoint in V_2 .
- (b) (40) Let $V_1 = \{a_1, \dots, a_n\}$ and $V_2 = \{b_1, \dots, b_n\}$ from part (a). Let M be the minimum over all permutations σ of $\{1, \dots, n\}$ of the quantity

$$\sum_{i=1}^n d(a_i, b_{\sigma(i)}),$$

where $d(v, w)$ denotes the number of edges along the shortest path between vertices v and w in the tree.

Show that if Allen can win, then the minimum number of moves that it can take for Allen to win is equal to M .

(A *graph* consists of a set of vertices and some edges between distinct pairs of vertices. It is *connected* if every pair of vertices are connected by some path of one or more edges. A *tree* is a graph which is connected, in which the number of edges is one less than the number of vertices.)

Proposed by: Kevin Sun

Part (a): First we show that if we can't split the vertices in the desired way then Allen cannot win. To do so, observe that there is a unique way to split the vertices into two groups so that all edges cross between the two groups, since trees are bipartite. Each of Allen's moves either adds one more blue vertex to each group or removes a blue vertex from each group, so the difference in the number of blue vertices between the two groups is invariant. Both groups initially have no blue vertices, so Allen can only possibly arrive at a state where both groups have all blue vertices if both groups were initially of equal size.

To show that Allen can win when the splitting is possible, we induct on n . It's trivial for $n = 1$. A tree on $2n$ vertices has $2n - 1$ edges, so the total degree for each of the two n -vertex groups is $2n - 1$. Therefore, at least one vertex in each group is a leaf, say L_1 in one group and L_2 in the other. Allen can then flip all the vertices from L_1 to L_2 since there is a path of even length (in terms of number of vertices) from one to the other, including flipping L_1 and L_2 , and then flip back all of the middle vertices in the path, with the end result being that L_1 and L_2 are blue while all other vertices in the tree remain red. This lets us remove L_1 and L_2 from consideration and apply the inductive hypothesis on the remaining graph.

Part (b): For each edge e in the graph, we can determine a minimum number of times Allen must perform an operation on the endpoints of that edge (henceforth, "flip this edge"), as follows. Observe that removing e disconnects the graph into two components. Each component is still a tree which can be split into two groups as before, though not necessarily with an equal number of vertices. In particular, suppose that the two groups differ by d vertices. Flipping any edge other than e does not change the difference in number of blue vertices between the two groups of the component. So e must flip at least d times if Allen is to win. Let m_e be the minimum number of times that edge e must flip, as determined in this way, and let M' be the total sum of all of the m_e . And let T' be the original tree with every edge duplicated m_e times (or removed if $m_e = 0$).

Observe that any permutation σ that gives us n paths must have total length at least M' for the same reasons as in the above paragraph, so $M' \leq M$.

In fact, we will show that $M' = M$. To show that $M' = M$, it suffices to show that $M' \geq M$, which we will show by finding a partitioning of T' into n odd-length paths for which every vertex is the endpoint of exactly one path. This will show that M' is the sum of path lengths for some permutation, meaning it is at least the sum of path lengths for the optimal permutation. We do this by strong induction. The base case is trivial, and if any m_e is 0 then we're immediately done because we can split the T' into two smaller trees. Otherwise we repeat the argument from part (a) where we use a path between two leaves, since each group of n vertices must contain a leaf, thereby reducing the number of vertices in our tree by 2 and allowing us to use our inductive hypothesis.

It now suffices to show that Allen can win the game in M' moves, but this is exactly the same induction as in the previous paragraph. If any m_e is 0 then we're immediately done by splitting T' into two smaller trees, otherwise we use the path between the two leaves to use our inductive hypothesis.

9. [60] Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

Proposed by: Allen Liu

Fix v . We use strong induction on the number of edges e . If $e \leq v - 1$, the result trivially holds by removing 0 edges. Now take $e > v - 1$ and assume the result has been shown for all smaller values of e . Consider a graph G with v vertices and e edges.

Suppose G contains a cycle C of even length $2k$, where vertices (but not edges) may be repeated in the cycle. Let G' be the subgraph of G with the edges of C removed. Then G' has v vertices and $e - 2k$ edges. By the inductive hypothesis, it is possible to remove $\frac{e-2k-v+1}{2}$ edges from G' so that each vertex still has at least half its original degree. In the original graph G , remove these same edges,

and also remove every other edge of C (so, if the vertices of C are v_1, \dots, v_{2k} in order, we remove the edges between v_{2i-1} and v_{2i} for $1 \leq i \leq k$). In total, we have removed $\frac{e-2k-v+1}{2} + k = \frac{e-v+1}{2}$ edges. Furthermore, all vertices in G still have at least half their original degrees, as desired.

The remaining case to consider is if G has no cycles of even length. Then no two cycles in G can have any vertices or edges in common. Suppose the contrary; then two odd cycles overlap, so their union is connected and has an even number of edges. This union has an Eulerian tour, which is a cycle with an even number of edges, contradicting our assumption.

The number of edges in G is at most $v + c - 1$, where c is the number of cycles. So, we must remove at least $\frac{e-v+1}{2} = \frac{c}{2}$ edges from G . But we can remove c edges from G , one from each cycle. No vertex has its degree decreased by more than 1, and each vertex whose degree is decreased is in a cycle and so has degree at least 2. Therefore each vertex still has at least half of its original degree, and we have removed at least $\frac{e-v+1}{2}$ edges, as desired.

Thus our claim holds for a graph with e edges, and thus by induction holds for any number of edges, as needed.

10. **[60]** Let n and m be positive integers which are at most 10^{10} . Let R be the rectangle with corners at $(0, 0), (n, 0), (n, m), (0, m)$ in the coordinate plane. A simple non-self-intersecting quadrilateral with vertices at integer coordinates is called *far-reaching* if each of its vertices lie on or inside R , but each side of R contains at least one vertex of the quadrilateral. Show that there is a far-reaching quadrilateral with area at most 10^6 .

(A side of a rectangle includes the two endpoints.)

Proposed by: Kevin Sun

Let $g = \gcd(n, m)$, with $n = g \cdot a$ and $m = g \cdot b$. Note that the number of points on the diagonal of R connecting $(0, 0)$ and (n, m) is $g + 1$. We construct two far-reaching quadrilaterals and show that at least one of them has small area.

For our first quadrilateral, let (x_1, y_1) and (x_2, y_2) be the points with the shortest nonzero distances to the diagonal between $(0, 0)$ and (n, m) which lie above and below the diagonal, respectively. Now consider the quadrilateral with vertices $(0, 0), (x_1, y_1), (n, m), (x_2, y_2)$. Note that the only lattice points which can lie on or inside this quadrilateral are $(x_1, y_1), (x_2, y_2)$, and points on the diagonal of R , as otherwise we could find a closer point to the diagonal than (x_1, y_1) and (x_2, y_2) . Thus by Pick's Theorem, the area of this quadrilateral is at most g .

For our second quadrilateral, we will take as our vertices the points $(0, 0), (n - 1, m), (a, b)$, and $(n, m - 1)$. This is a concave quadrilateral which can be split into two triangles of areas $\frac{a}{2}$ and $\frac{b}{2}$, so the area of this quadrilateral is at most $\frac{1}{2}(a + b)$.

We have therefore shown that there is always a far-reaching quadrilateral with area at most $\min(g, \frac{1}{2}(a + b))$. Since $g \cdot \frac{1}{2}(a + b) = \frac{1}{2}(n + m) \leq 10^{10}$, we have that $\min(g, \frac{1}{2}(a + b)) \leq 10^5$, so we can always find a far-reaching quadrilateral with area at most 10^5 as desired.

HMMT November 2018

November 10, 2018

General Round

1. What is the largest factor of 130000 that does not contain the digit 0 or 5?

Proposed by: Farrell Eldrian Wu

Answer:

If the number is a multiple of 5, then its units digit will be either 0 or 5. Hence, the largest such number must have no factors of 5.

We have $130000 = 2^4 \cdot 5^4 \cdot 13$. Removing every factor of 5, we get that our number must be a factor of $2^4 \cdot 13 = 208$.

If our number contains a factor of 13, we cancel the factor of 2 from 208, 104, and 52 until we get 26. Otherwise, the largest number we can have is $2^4 = 16$. We conclude that the answer is 26.

2. Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team?

Proposed by: Yuan Yao

Answer:

Once we have assigned Zack and Mihir teams, there are 8 spots for more players on Zack's team and 9 for more players on the third team. Andrew is equally likely to occupy any of these spots, so our answer is $\frac{8}{17}$.

3. A square in the xy -plane has area A , and three of its vertices have x -coordinates 2, 0, and 18 in some order. Find the sum of all possible values of A .

Proposed by: Yuan Yao

Answer:

More generally, suppose three vertices of the square lie on lines $y = y_1, y = y_2, y = y_3$. One of these vertices must be adjacent to two others. If that vertex is on $y = y_1$ and the other two are on $y = y_2$ and $y = y_3$, then we can use the Pythagorean theorem to get that the square of the side length is $(y_2 - y_1)^2 + (y_3 - y_1)^2$.

For $(y_1, y_2, y_3) = (2, 0, 18)$, the possibilities are $2^2 + 16^2, 2^2 + 18^2, 16^2 + 18^2$, so the sum is $2(2^2 + 16^2 + 18^2) = 2(4 + 256 + 324) = 1168$.

4. Find the number of eight-digit positive integers that are multiples of 9 and have all distinct digits.

Proposed by: James Lin

Answer:

Note that $0 + 1 + \dots + 9 = 45$. Consider the two unused digits, which must then add up to 9. If it's 0 and 9, there are $8 \cdot 7!$ ways to finish; otherwise, each of the other four pairs give $7 \cdot 7!$ ways to finish, since 0 cannot be the first digit. This gives a total of $36 \cdot 7! = 181440$.

5. Compute the smallest positive integer n for which

$$\sqrt{100 + \sqrt{n}} + \sqrt{100 - \sqrt{n}}$$

is an integer.

Proposed by: Michael Tang

Answer:

The number $\sqrt{100 + \sqrt{n}} + \sqrt{100 - \sqrt{n}}$ is a positive integer if and only if its square is a perfect square. We have

$$\begin{aligned} \left(\sqrt{100 + \sqrt{n}} + \sqrt{100 - \sqrt{n}}\right)^2 &= (100 + \sqrt{n}) + (100 - \sqrt{n}) + 2\sqrt{(100 + \sqrt{n})(100 - \sqrt{n})} \\ &= 200 + 2\sqrt{10000 - n}. \end{aligned}$$

To minimize n , we should maximize the value of this expression, given that it is a perfect square. For this expression to be a perfect square, $\sqrt{10000 - n}$ must be an integer. Then $200 + 2\sqrt{10000 - n}$ is even, and it is less than $200 + 2\sqrt{10000} = 400 = 20^2$. Therefore, the greatest possible perfect square value of $200 + 2\sqrt{10000 - n}$ is $18^2 = 324$. Solving

$$200 + 2\sqrt{10000 - n} = 324$$

for n gives the answer, $n = \boxed{6156}$.

6. Call a polygon *normal* if it can be inscribed in a unit circle. How many non-congruent normal polygons are there such that the square of each side length is a positive integer?

Proposed by: Yuan Yao

Answer: $\boxed{14}$

The side lengths of the polygon can only be from the set $\{1, \sqrt{2}, \sqrt{3}, 2\}$, which take up $60^\circ, 90^\circ, 120^\circ, 180^\circ$ of the circle respectively. By working modulo 60 degrees we see that $\sqrt{2}$ must be used an even number of times. We now proceed to casework on the longest side of the polygon.

Case 1: If the longest side has length 2, then the remaining sides must contribute the remaining 180 degrees. There are 3 possibilities: $(1, 1, 1, 2), (1, \sqrt{3}, 2), (\sqrt{2}, \sqrt{2}, 2)$.

Case 2: If the longest side has length $\sqrt{3}$, then it takes up either 120° or 240° of the circle. In the former case we have 6 possibilities: $(1, 1, 1, 1, \sqrt{3}), (1, \sqrt{2}, \sqrt{2}, \sqrt{3}), (\sqrt{2}, 1, \sqrt{2}, \sqrt{3}), (1, 1, \sqrt{3}, \sqrt{3}), (1, \sqrt{3}, 1, \sqrt{3}), (\sqrt{3}, \sqrt{3}, \sqrt{3})$. In the latter case there is only 1 possibility: $(1, 1, \sqrt{3})$.

Case 3: If the longest side has length $\sqrt{2}$, then it shows up either twice or four times. In the former case we have 2 possibilities: $(1, 1, 1, \sqrt{2}, \sqrt{2}), (1, 1, \sqrt{2}, 1, \sqrt{2})$. In the latter case there is only 1 possibility: $(\sqrt{2}, \sqrt{2}, \sqrt{2}, \sqrt{2})$.

Case 4: If all sides have length 1, then there is 1 possibility: $(1, 1, 1, 1, 1, 1)$.

Adding up all cases, we have $3 + 6 + 1 + 2 + 1 + 1 = 14$ polygons.

7. Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical by expressing it as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible distinct values of ab can be expressed in the form $q \cdot 15!$ for some rational number q . Find q .

Proposed by: Nikhil Reddy

Answer: $\boxed{4}$

Note that $15! = 2^{11} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11^1 \cdot 13^1$. The possible a are thus precisely the factors of $2^5 \cdot 3^3 \cdot 5^1 \cdot 7^1 = 30240$. Since $\frac{ab}{15!} = \frac{ab}{a^2b} = \frac{1}{a}$, we have

$$\begin{aligned}
q &= \frac{1}{15!} \sum_{\substack{a, b: \\ a\sqrt{b}=\sqrt{15!}}} ab \\
&= \sum_{a|30420} \frac{ab}{15!} \\
&= \sum_{a|30420} \frac{1}{a} \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32}\right) \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7}\right) \\
&= \left(\frac{63}{32}\right) \left(\frac{40}{27}\right) \left(\frac{6}{5}\right) \left(\frac{8}{7}\right) \\
&= 4.
\end{aligned}$$

8. Equilateral triangle ABC has circumcircle Ω . Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC .

Proposed by: Yuan Yao

Answer: $\boxed{\frac{37}{7}}$

A rotation by 120° about the center of the circle will take ABE to BCD , so BCD has area 3. Let $AD = x, BD = y$, and observe that $\angle ADC = \angle CDB = 60^\circ$. By Ptolemy's Theorem, $CD = x + y$. We have

$$\begin{aligned}
4 &= [ACD] = \frac{1}{2}AD \cdot CD \cdot \sin 60^\circ = \frac{\sqrt{3}}{4}x(x+y) \\
3 &= [BCD] = \frac{1}{2}BD \cdot CD \cdot \sin 60^\circ = \frac{\sqrt{3}}{4}y(x+y)
\end{aligned}$$

By dividing these equations find $x : y = 4 : 3$. Let $x = 4t, y = 3t$. Substitute this into the first equation to get $1 = \frac{\sqrt{3}}{4} \cdot 7t^2$. By the Law of Cosines,

$$AB^2 = x^2 + xy + y^2 = 37t^2.$$

The area of ABC is then

$$\frac{AB^2\sqrt{3}}{4} = \frac{37}{7}.$$

9. 20 players are playing in a Super Smash Bros. Melee tournament. They are ranked 1 – 20, and player n will always beat player m if $n < m$. Out of all possible tournaments where each player plays 18 distinct other players exactly once, one is chosen uniformly at random. Find the expected number of pairs of players that win the same number of games.

Proposed by: Anders Olsen

Answer: $\boxed{4}$

Consider instead the complement of the tournament: The 10 possible matches that are not played. In order for each player to play 18 games in the tournament, each must appear once in these 10 unplayed matches. Players n and $n + 1$ will win the same number of games if, in the matching, they are matched with each other, or n plays a player $a > n + 1$ and $n + 1$ plays a player $b < n$. (Note no other pairs of players can possibly win the same number of games.) The first happens with probability $\frac{1}{19}$ (as there are 19 players for player n to be paired with), and the second happens with probability $\frac{(n-1)(20-n-1)}{19 \cdot 17}$.

By linearity of expectation, the expected number of pairs of players winning the same number of games is the sum of these probabilities. We compute

$$\sum_{n=1}^{19} \left(\frac{1}{19} + \frac{(n-1)(20-n-1)}{323} \right) = \sum_{n=0}^{18} \left(\frac{1}{19} + \frac{n(18-n)}{323} \right) = 1 + \frac{\binom{19}{3}}{323} = 4.$$

10. Real numbers x, y , and z are chosen from the interval $[-1, 1]$ independently and uniformly at random. What is the probability that

$$|x| + |y| + |z| + |x + y + z| = |x + y| + |y + z| + |z + x|?$$

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{8}}$

We assume that x, y, z are all nonzero, since the other case contributes zero to the total probability.

If x, y, z are all positive or all negative then the equation is obviously true. Otherwise, since flipping the signs of all three variables or permuting them does not change the equality, we assume WLOG that $x, y > 0$ and $z < 0$.

If $x + y + z > 0$, then the LHS of the original equation becomes $x + y - z + x + y = z = 2x + 2y$, and the RHS becomes $x + y + |x + z| + |y + z|$, so we need $|x + z| + |y + z| = x + y$. But this is impossible when $-x - y < z < 0$, since the equality is achieved only at the endpoints and all the values in between make the LHS smaller than the RHS. (This can be verified via simple casework.)

If $x + y + z < 0$, then $x + z, y + z < 0$ as well, so the LHS of the original equation becomes $x + y - z - x - y - z = -2z$ and the RHS becomes $x + y - x - z - y - z = -2z$. In this case, the equality holds true.

Thus, we seek the volume of all points $(x, y, z) \in [0, 1]^3$ that satisfy $x + y - z < 0$ (we flip the sign of z here for convenience). The equation $x + y - z = 0$ represents a plane through the vertices $(1, 0, 1), (0, 0, 0), (0, 1, 1)$, and the desired region is the triangular pyramid, above the plane inside the unit cube, which has vertices $(1, 0, 1), (0, 0, 0), (0, 1, 1), (0, 0, 1)$. This pyramid has volume $\frac{1}{6}$.

So the total volume of all points in $[-1, 1]^3$ that satisfy the equation is $2 \cdot 1 + 6 \cdot \frac{1}{6} = 3$, out of $2^3 = 8$, so the probability is $\frac{3}{8}$.

Note: A more compact way to express the equality condition is that the equation holds true if and only if $xyz(x + y + z) \geq 0$.

HMMT November 2018

November 10, 2018

Guts Round

1. [5] A positive integer is called *primer* if it has a prime number of distinct prime factors. Find the smallest primer number.

Proposed by: Yuan Yao

Answer:

A primer number must have at least two distinct prime factors, and 6 will work.

2. [5] Pascal has a triangle. In the n th row, there are $n + 1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \dots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n - 1$, $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$. What is the sum of all numbers in the 2018th row?

Proposed by: Michael Ren

Answer:

In general, the sum of the numbers on the n th row will be

$$\sum_{k=0}^n a_{n,k} = a_{n,0} + \sum_{k=1}^{n-1} (a_{n-1,k} + a_{n-1,k-1}) + a_{n,n} = a_{n,0} + (a_{n-1,n-1} + a_{n-1,0}) + a_{n,n} = 2.$$

3. [5] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×2 mazes.

Proposed by: James Lin

Answer:

We must have both top-left and bottom-right cells blank, and we cannot have both top-right and bottom-left cells with walls. As long as those conditions are satisfied, the maze is solvable, so the answer is 3.

4. [6] Let a, b, c, n be positive real numbers such that $\frac{a+b}{a} = 3$, $\frac{b+c}{b} = 4$, and $\frac{c+a}{c} = n$. Find n .

Proposed by: Evan Chen

Answer:

We have

$$1 = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = (3-1)(4-1)(n-1).$$

Solving for n yields $n = \frac{7}{6}$.

5. [6] Jerry has ten distinguishable coins, each of which currently has heads facing up. He chooses one coin and flips it over, so it now has tails facing up. Then he picks another coin (possibly the same one as before) and flips it over. How many configurations of heads and tails are possible after these two flips?

Proposed by: Andrew Gu

Answer:

We have two cases:

Case 1: Jerry picks the same coin twice. Then, the first time he flips the coin, it becomes tails, and then the second time, it becomes heads again, giving us the original state of all heads.

Case 2: Jerry picks two different coins. In this case, there are two coins with tails face up, and the rest are heads face up. There are $\binom{10}{2} = \frac{10 \cdot 9}{2} = 45$ ways to pick which two coins have tails.

Adding up the possibilities from both cases, we have a total of $1 + 45 = 46$ possible configurations.

6. [6] An equilateral hexagon with side length 1 has interior angles $90^\circ, 120^\circ, 150^\circ, 90^\circ, 120^\circ, 150^\circ$ in that order. Find its area.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3+\sqrt{3}}{2}}$

The area of this hexagon is the area of a $\frac{3}{2} \times \left(1 + \frac{\sqrt{3}}{2}\right)$ rectangle (with the 90° angles of the hexagon at opposite vertices) minus the area of an equilateral triangle with side length 1. Then this is

$$\frac{6 + 3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = \frac{3 + \sqrt{3}}{2}.$$

7. [7] At Easter-Egg Academy, each student has two eyes, each of which can be eggshell, cream, or cornsilk. It is known that 30% of the students have at least one eggshell eye, 40% of the students have at least one cream eye, and 50% of the students have at least one cornsilk eye. What percentage of the students at Easter-Egg Academy have two eyes of the same color?

Proposed by: Yuan Yao

Answer: $\boxed{80\%}$

For the purposes of this solution, we abbreviate “eggshell” by “egg”, and “cornsilk” by “corn”. We know that there are only six combinations of eye color possible: egg-cream, egg-corn, egg-egg, cream-corn, cream-cream, corn-corn. If we let the proportions for each of these be represented by $a, b, c, d, e,$ and f respectively, we have the following four equalities:

$$\begin{aligned} a + b + c &= .3 \\ a + d + e &= .4 \\ b + d + f &= .5 \\ a + b + c + d + e + f &= 1 \end{aligned}$$

where the first three equalities come from the given conditions. Adding the first three equations and subtracting the fourth, we obtain that

$$a + b + d = .2$$

which is the proportion of people with different colored eyes. The proportion of people with the same eye color is thus $1 - .2 = .8$.

8. [7] Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $\angle J = \angle A = \angle E = 90^\circ$, and $\angle M = \angle S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.

Proposed by: James Lin

Answer: $\boxed{\frac{1}{4}}$

Since $\angle J = \angle A = 90^\circ$ and $AM = JS$, $JAMS$ must be a rectangle. In addition, $\angle M + \angle S = 270^\circ$, so $\angle M = \angle S = 135^\circ$. Therefore, $\angle ESM = \angle EMS = 45^\circ$, which means MES is an isosceles right triangle. Note that AME and JSE are congruent, which means that $[JAES] = [JAE] + [JSE] = [JAE] + [AME] > [AME]$, so AE cannot be our diagonal. Similarly, JE cannot be our diagonal. Diagonals SA and JM bisect rectangle $JAMS$, so they also cannot bisect the pentagon. Thus, the only diagonal that can bisect $[JAMES]$ is MS , which implies $[JAMS] = [MES]$. We know $[JAMS] = JA \cdot AM$ and $[MES] = \frac{ME \cdot ES}{2}$, and $ME = ES = \frac{JA}{\sqrt{2}}$, which implies

$$JA \cdot AM = \frac{JA^2}{4} \implies \frac{AM}{JA} = \frac{1}{4}$$

Finally, EM and MS are both $\frac{1}{\sqrt{2}}$ the length of $SM = JA$. This means that AM is our shortest side and JA is our longest side, so $\frac{1}{4}$ is our answer.

9. [7] Farmer James has some strange animals. His hens have 2 heads and 8 legs, his peacocks have 3 heads and 9 legs, and his zombie hens have 6 heads and 12 legs. Farmer James counts 800 heads and 2018 legs on his farm. What is the number of animals that Farmer James has on his farm?

Proposed by: James Lin

Answer: 203

Note that each animal has 6 more legs than heads. Thus, if there are n animals, then there are $6n$ more legs than heads in total. There are $2018 - 800 = 1218$ more legs than heads in total, so there are $\frac{1218}{6} = 203$ animals.

10. [8] Abbot writes the letter A on the board. Every minute, he replaces every occurrence of A with AB and every occurrence of B with BA , hence creating a string that is twice as long. After 10 minutes, there are $2^{10} = 1024$ letters on the board. How many adjacent pairs are the same letter?

Proposed by: Yuan Yao

Answer: 341

Let a_n denote the number of adjacent pairs of letters that are the same after n minutes, and b_n the number of adjacent pairs that are different.

Lemma 1. $a_n = b_{n-1}$ for all $n \geq 0$.

Proof. Any adjacent pair of identical letters XX at stage n either came from the same letter of stage $n - 1$ ($W \rightarrow XX$), or two adjacent letters of stage $n - 1$ ($VW \rightarrow MXXN$). Because $A \rightarrow AB$ and $B \rightarrow BA$, they cannot have come from the same letter.

If they came from a pair of adjacent letters, then observing what each adjacent pair of letters results in in the next minute,

$$\begin{aligned} AA &\rightarrow ABAB \\ AB &\rightarrow ABBA \\ BA &\rightarrow BAAB \\ BB &\rightarrow BABA \end{aligned}$$

we see that our adjacent pair VW must have been AB or BA . The number of such pairs is precisely b_{n-1} .

From the relation $a_n + b_n = 2^n - 1$ for all $n \geq 0$, we obtain the recurrence relation

$$a_n = 2^{n-1} - 1 - a_{n-1}$$

from which we obtain values $a_0 = 0$, $a_1 = 0$, $a_2 = 1$, $a_3 = 2$, $a_4 = 5$, $a_5 = 10$, $a_6 = 21$, $a_7 = 42$, $a_8 = 85$, $a_9 = 170$, and $a_{10} = \boxed{341}$.

11. [8] Let $\triangle ABC$ be an acute triangle, with M being the midpoint of \overline{BC} , such that $AM = BC$. Let D and E be the intersection of the internal angle bisectors of $\angle AMB$ and $\angle AMC$ with AB and AC , respectively. Find the ratio of the area of $\triangle DME$ to the area of $\triangle ABC$.

Proposed by: Handong Wang

Answer: $\frac{2}{9}$

Let $[XYZ]$ denote the area of $\triangle XYZ$.

Solution 1: Let $AM = \ell$, let $DE = d$, and let the midpoint of \overline{DE} be F . Since $\frac{AD}{AB} = \frac{AE}{AC} = \frac{2}{3}$ by the angle bisector theorem, F lies on \overline{AM} and $\triangle ADE$ is similar to $\triangle ABC$. Note that $\angle DME$ is

formed by angle bisectors of $\angle AMB$ and $\angle AMC$, which add up to 180° . Thus $\angle DME$ is right, so both $\triangle DMF$ and $\triangle EMF$ are isosceles. This implies that $FM = \frac{d}{2}$. Applying the similarity between $\triangle ADE$ and $\triangle ABC$, we get $\frac{d}{\ell} = \frac{AD}{AB} = \frac{2}{3}$. Since $AF = 2FM$, $[ADE] = 2[DME]$. Finally, since

$$\frac{[ADE]}{[ABC]} = \frac{4}{9}, \quad \frac{[DME]}{[ABC]} = \frac{1}{2} \cdot \frac{4}{9} = \boxed{\frac{2}{9}}.$$

Solution 2: We compute the ratio $\frac{[DME]}{[ABC]}$ by finding $1 - \frac{[ADE]}{[ABC]} - \frac{[DBM]}{[ABC]} - \frac{[EMC]}{[ABC]}$. Since $\frac{AD}{DB} = \frac{AE}{EC} = \frac{AM}{\frac{1}{2}BC} = 2$ by the angle bisector theorem, we see that $\frac{[ADE]}{[ABC]} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$. Also, since $BM = CM = \frac{1}{2}BC$, $\frac{[DBM]}{[ABC]} = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right) = \frac{1}{6}$, and $\frac{[EMC]}{[ABC]} = \left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{1}{6}$. Thus, $\frac{[DME]}{[ABC]} = 1 - \frac{[ADE]}{[ABC]} -$

$$\frac{[DBM]}{[ABC]} - \frac{[EMC]}{[ABC]} = 1 - \frac{4}{9} - \frac{1}{6} - \frac{1}{6} = \boxed{\frac{2}{9}}.$$

12. [8] Consider an unusual biased coin, with probability p of landing heads, probability $q \leq p$ of landing tails, and probability $\frac{1}{6}$ of landing on its side (i.e. on neither face). It is known that if this coin is flipped twice, the likelihood that both flips will have the same result is $\frac{1}{2}$. Find p .

Proposed by: Brian Reinhart

Answer: $\boxed{\frac{2}{3}}$

The probability that both flips are the same is $p^2 + q^2 + \frac{1}{36}$. For this to be $\frac{1}{2}$, we must have

$$p^2 + q^2 + \frac{1}{36} = p^2 + \left(\frac{5}{6} - p\right)^2 + \frac{1}{36} = \frac{1}{2}$$

Using the quadratic formula, $p = \frac{2}{3}$ or $\frac{1}{6}$. Since $p \geq q$, we have that $p = \frac{2}{3}$.

13. [9] Find the smallest positive integer n for which

$$1!2! \cdots (n-1)! > n!^2.$$

Proposed by: Andrew Gu

Answer: $\boxed{8}$

Dividing both sides by $n!^2$, we obtain

$$\begin{aligned} \frac{1!2! \cdots (n-3)!(n-2)!(n-1)!}{[n(n-1)!][n(n-1)(n-2)!]} &> 1 \\ \frac{1!2! \cdots (n-3)!}{n^2(n-1)} &> 1 \\ 1!2! \cdots (n-3)! &> n^2(n-1) \end{aligned}$$

Factorials are small at first, so we can rule out some small cases: when $n = 6$, the left hand side is $1!2!3! = 12$, which is much smaller than $6^2 \cdot 5$. (Similar calculations show that $n = 1$ through $n = 5$ do not work. either.)

Setting $n = 7$, the left-hand side is 288, which is still smaller than $7^2 \cdot 6$. However, $n = 8$ gives $34560 > 448$, so 8 is the smallest integer for which the inequality holds.

14. [9] Call a triangle *nice* if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?

Proposed by: Yuan Yao

Answer: $\boxed{4}$

The triangles are 60-60-60, 45-45-90, 30-60-90, and 30-30-120.

We make two observations.

- By reflecting "around" the same point, any angle of the triangle must be an integer divisor of 360° .
- if any angle is an odd divisor of 360° , i.e equals $\frac{360}{k}$ for odd k , then the two adjacent sides must be equal.

We do casework on the largest angle.

- 60° . We are forced into a $\boxed{60-60-60}$ triangle, which works.
- 72° . By observation 2, this triangle's other two angles are 54° . This is not an integer divisor of 360° .
- 90° . The second largest angle is at least 45° . If it is 45° , it is the valid $\boxed{90-45-45}$ triangle. If it is $\frac{360^\circ}{7}$, the triangle is invalid by observation 2. If it is 60° , it is the valid $\boxed{90-60-30}$ triangle. If it is 72° , the triangle is invalid by observation 2.
- 120° . By observation 2, the other angles are 30° , meaning it is the valid $\boxed{120-30-30}$ triangle.

The conclusion follows.

15. [9] On a computer screen is the single character **a**. The computer has two keys: **c** (copy) and **p** (paste), which may be pressed in any sequence.

Pressing **p** increases the number of **a**'s on screen by the number that were there the last time **c** was pressed. **c** doesn't change the number of **a**'s on screen. Determine the fewest number of keystrokes required to attain at least 2018 **a**'s on screen. (**Note:** pressing **p** before the first press of **c** does nothing).

Proposed by: John Michael Wu

Answer: $\boxed{21}$

The first keystroke must be **c** and the last keystroke must be **p**. If there are k **c**'s pressed in total, let n_i denote one more than the number of **p**'s pressed immediately following the i 'th **c**, for $1 \leq i \leq k$.

Then, we have that the total number of keystrokes is

$$s := \sum_{i=1}^k n_i$$

and the total number of **a**'s is

$$r := \prod_{i=1}^k n_i$$

We desire to minimize s with the constraint that $r \geq 2018$. We claim that the minimum possible s is $s = 21$.

This value of s is achieved by $k = 7$ and $n_1 = n_2 = n_3 = n_4 = n_5 = n_6 = n_7 = 3$, so it remains to show that $s = 20$ is not possible.

Suppose it were for some k and n_i . By the AM-GM inequality,

$$\left(\frac{n_1 + n_2 + \cdots + n_k}{k} \right) \geq \sqrt[k]{n_1 n_2 \cdots n_k}$$

implying that

$$\begin{aligned}
2018 &\leq n_1 n_2 \cdots n_k \\
&\leq \left(\frac{n_1 + n_2 + \cdots + n_k}{k} \right)^k \\
&= \left(\frac{20}{k} \right)^k
\end{aligned}$$

which is satisfied by no positive integers k . More rigorously, the function $f(x) = x^{\frac{1}{x}}$ is well known to have a maximum at $x = e$. Making the substitution $u = \frac{20}{k}$, we obtain

$$\begin{aligned}
\left(\frac{20}{k} \right)^k &= u^{\frac{20}{u}} \\
&= \left(u^{\frac{1}{u}} \right)^{20}
\end{aligned}$$

which is maximized by setting $u = e$. However, $e^{\frac{20}{e}} \approx 1568.05$, meaning that $s = 20$ is not possible.

16. [10] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. Find the smallest primest number.

Proposed by: Yuan Yao

Answer: 72

We claim the answer is 72, as it has 6 primer factors: 6, 12, 24, 18, 36, 72, and 6 is a primer.

We now prove that there is no smaller primer. Suppose there were a smaller primer $r < 72$. We do casework on the number of distinct prime factors of r .

- r has ≥ 4 distinct prime factors. Then $r \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$, which is larger than 72.
- r has 3 distinct prime factors. If each of these factors has multiplicity 1, i.e $r = pqs$ for distinct primes p, q, s , then r has precisely 4 primer factors: pq, qs, sp, pqs , and 4 is not a primer. Thus, r contains at least one factor of multiplicity at least 2. If r is p^2qs for distinct primes p, q, s , then r has 7 distinct primer factors: $pq, qs, sp, pqs, p^2q, sp^2, p^2qs$, and 7 is not a primer. Thus, if $r = p^a q^b s^c$, $a + b + c \geq 5$, and $r \geq 2^3 \cdot 3 \cdot 5 = 120$, which is ≥ 72 .
- r has 2 distinct prime factors. If $r = p^a q^b$, for distinct primes p, q , then r 's primer factors are precisely its divisors of the form $p^i q^j$, where $1 \leq i \leq a$ and $1 \leq j \leq b$, meaning that it has ab primer factors. Thus, ab is a primer, meaning that $ab \geq 6$. Thus $r \geq 2^3 \cdot 3^2 = 72$, where the other possibilities can be ruled out through easy casework.
- r has 1 distinct prime factor. Then it doesn't have any primer factors, and thus cannot possibly have a primer number of them.

We conclude that 72 is the smallest primer number.

17. [10] Pascal has a triangle. In the n th row, there are $n + 1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \dots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n - 1$, $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$. What is the sum of the absolute values of all numbers in the 2018th row?

Proposed by: Michael Ren

Answer: $\frac{2^{2018} + 2}{3}$

Let s_n be the sum of the absolute values of numbers in the n th row. For odd n , we have that $a_{n,1}, \dots, a_{n,n-1}$ alternate in sign as $-, +, -, +, \dots, +$, with the last term being $a_{n,n-1} = 1$. For even

n , we have that $a_{n,1}, \dots, a_{n,n-2}$ alternate in sign as $-, +, -, +, \dots, +$, and $a_{n,n-1} = 0$. These facts can be proven by induction. Thus, $s_n = 1 - a_{n,1} + a_{n,2} - \dots + (-1)^{n-1}a_{n,n-1} + 1$. Applying the recursion, for $n > 0$ this becomes $s_n = 1 - (a_{n-1,1} - a_{n-1,0}) + (a_{n-1,2} - a_{n-1,1}) - \dots + (-1)^{n-1}(a_{n-1,n-1} - a_{n-1,n-2}) + 1 = 2(1 - a_{n-1,1} + a_{n-1,2} - \dots + (-1)^{n-2}a_{n-1,n-2} + 1) - 1 + (-1)^{n-1}$. In other words, if n is even then $s_n = 2s_{n-1} - 2$ and if n is odd then $s_n = 2s_{n-1}$. This means that $s_{2n} = 4s_{2n-2} - 2$. Since 2018 is even, we can write $s_{2018} = 4s_{2016} - 2 = 2^{2018} - 2^{2017} - 2^{2015} - \dots - 2$. Applying the formula for the sum of a geometric series, we get $s_{2018} = 2^{2018} - \frac{2^{2019} - 2}{4 - 1} = \frac{2^{2018} + 2}{3}$.

18. [10] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×5 mazes.

Proposed by: John Michael Wu

Answer: 49

Solution 1: Replace 5 by an arbitrary n . Label the cells of the maze by (x, y) where $1 \leq x \leq n$ and $1 \leq y \leq 2$. Let a_n denote the number of solvable $2 \times n$ mazes, and let b_n denote the number of $2 \times n$ mazes where there exists a sequence of adjacent blank cells from the $(1, 1)$ to $(n, 1)$. We observe the relations

$$\begin{aligned} a_n &= 2a_{n-1} + b_{n-2} \\ b_n &= 2b_{n-1} + a_{n-2}. \end{aligned}$$

The first relation follows from dividing into cases depending on if $(n-1, 2)$ is blank. If this cell is blank, then the maze is solvable if and only if there is a path to $(n-1, 2)$ and the cell $(n, 2)$ is blank. The cell $(n, 1)$ is arbitrary so we get $2a_{n-1}$. If $(n-1, 2)$ is a wall, then the maze is solvable if and only if there is a path to $(n-2, 1)$ and each of the cells $(n-1, 1), (n, 1), (n, 2)$ are blank. This gives the term b_{n-2} .

The second relation follows similarly dividing into cases based on whether the cell $(n-1, 1)$ is blank or not.

The base cases are $a_1 = 1, a_2 = 3, b_1 = 2, b_2 = 4$. We thus obtain:

n	a_n	b_n
1	1	2
2	3	4
3	8	9
4	20	21
5	49	50

The answer is then 49.

Solution 2: Call a maze *reachable* if there exists a sequence of adjacent blank cells from $(1, 1)$ to any cell in column n . Let x_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a blank in the top cell and a wall in the bottom cell, let y_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a wall in the top cell and a blank in the bottom cell, and let z_n denote the number of reachable $2 \times n$ mazes where the rightmost column has a blank in both of its cells. Then, observe the relations

$$\begin{aligned} x_n &= z_{n-1} + x_{n-1} \\ y_n &= z_{n-1} + y_{n-1} \\ z_n &= z_{n-1} + x_{n-1} + y_{n-1}, \end{aligned}$$

as well as base cases $x_1 = 1, y_1 = 0, z_1 = 1$. We thus obtain:

n	x_n	y_n	z_n
1	1	0	1
2	2	1	2
3	4	3	5
4	9	8	12
5	21	20	29

Note that the answer is $y_5 + z_5$, which is $20 + 29 = 49$.

19. [11] Let A be the number of unordered pairs of ordered pairs of integers between 1 and 6 inclusive, and let B be the number of ordered pairs of unordered pairs of integers between 1 and 6 inclusive. (Repetitions are allowed in both ordered and unordered pairs.) Find $A - B$.

Proposed by: Yuan Yao

Answer: 225

There are $6 \cdot 6$ ordered pairs of integers between 1 and 6 inclusive and 21 unordered pairs of integers ($\binom{6}{2} = 15$ different pairs and 6 doubles). Then, $A = \binom{36}{2} + 36 = 666$ and $B = 21 \cdot 21 = 441$. Therefore $A - B = 225$.

For general n , there are n^2 ordered pairs of integers and $\frac{n(n+1)}{2}$ unordered pairs of integers. Then $A = \frac{n^2(n^2+1)}{2}$ and $B = \frac{n^2(n+1)^2}{4}$ so

$$A - B = \frac{n^2(2(n^2 + 1) - (n + 1)^2)}{4} = \left(\frac{n(n - 1)}{2}\right)^2.$$

20. [11] Let z be a complex number. In the complex plane, the distance from z to 1 is 2, and the distance from z^2 to 1 is 6. What is the real part of z ?

Proposed by: James Lin

Answer: $\frac{5}{4}$

Note that we must have $|z - 1| = 2$ and $|z^2 - 1| = 6$, so $|z + 1| = \frac{|z^2 - 1|}{|z - 1|} = 3$. Thus, the distance from z to 1 in the complex plane is 2 and the distance from z to -1 in the complex plane is 3. Thus, $z, 1, -1$ form a triangle with side lengths 2, 3, 3. The area of a triangle with sides 2, 2, 3 can be computed to be $\frac{3\sqrt{7}}{4}$ by standard techniques, so the length of the altitude from z to the real axis is $\frac{3\sqrt{7}}{4} \cdot \frac{2}{2} = \frac{3\sqrt{7}}{4}$.

The distance between 1 and the foot from z to the real axis is $\sqrt{2^2 - \left(\frac{3\sqrt{7}}{4}\right)^2} = \frac{1}{4}$ by the Pythagorean Theorem. It is clear that z has positive imaginary part as the distance from z to -1 is greater than the distance from z to 1, so the distance from 0 to the foot from z to the real axis is $1 + \frac{1}{4} = \frac{5}{4}$. This is exactly the real part of z that we are trying to compute.

21. [11] A function $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ is said to be *nasty* if there do not exist distinct $a, b \in \{1, 2, 3, 4, 5\}$ satisfying $f(a) = b$ and $f(b) = a$. How many nasty functions are there?

Proposed by: Michael Ren

Answer: 1950

We use complementary counting. There are $5^5 = 3125$ total functions. If there is at least one pair of numbers which map to each other, there are $\binom{5}{2} = 10$ ways to choose the pair and $5^3 = 125$ ways to assign the other values of the function for a total of 1250. But we overcount each time there are two such pairs, so we must subtract off $5 \cdot 3 \cdot 5 = 75$, where there are 5 options for which number is not in a pair, 3 options for how the other four numbers are paired up, and 5 options for where the function outputs when the unpaired number is inputted. This results in a final answer of $3125 - (1250 - 75) = 1950$.

22. [12] In a square of side length 4, a point on the interior of the square is randomly chosen and a circle of radius 1 is drawn centered at the point. What is the probability that the circle intersects the square exactly twice?

Proposed by: Jason Lu

Answer: $\frac{\pi+8}{16}$

Consider the two intersection points of the circle and the square, which are either on the same side of the square or adjacent sides of the square. In order for the circle to intersect a side of the square twice, it must be at distance at most 1 from that side and at least 1 from all other sides. The region of points where the center could be forms a 2×1 rectangle.

In the other case, a square intersects a pair of adjacent sides once each if it is at distance at most one from the corner, so that the circle contains the corner. The region of points where the center could be is a quarter-circle of radius 1.

The total area of the regions where the center could be is $\pi + 8$, so the probability is $\frac{\pi+8}{16}$.

23. [12] Let S be a subset with four elements chosen from $\{1, 2, \dots, 10\}$. Michael notes that there is a way to label the vertices of a square with elements from S such that no two vertices have the same label, and the labels adjacent to any side of the square differ by at least 4. How many possibilities are there for the subset S ?

Proposed by: James Lin

Answer: 36

Let the four numbers be a, b, c, d around the square. Assume without loss of generality that a is the largest number, so that $a > b$ and $a > d$. Note that c cannot be simultaneously smaller than one of b, d and larger than the other because, e.g. if $b > c > d$, then $a > b > c > d$ and $a \geq d + 12$. Hence c is either smaller than b and d or larger than b and d .

Case 1: c is smaller than b and d . Then we have $a - c \geq 8$, but when $a - c = 8$, we have $b = c + 4 = d$, so we need $a - c = 9$, giving the only set $\{1, 5, 6, 10\}$.

Case 2: c is larger than b and d . Since $a > c$ and b, d are both at most $c - 4$, the range of possible values for c is $\{6, 7, 8, 9\}$. When $c = 9, 8, 7, 6$, there are 1, 2, 3, 4 choices for a respectively and $\binom{5}{2}, \binom{4}{2}, \binom{3}{2}, \binom{2}{2}$ for b and d respectively (remember that order of b and d does not matter). So there are $1 \cdot 10 + 2 \cdot 6 + 3 \cdot 3 + 4 \cdot 1 = 35$ sets in this case.

Therefore we have $1 + 35 = 36$ possible sets in total.

24. [12] Let $ABCD$ be a convex quadrilateral so that all of its sides and diagonals have integer lengths. Given that $\angle ABC = \angle ADC = 90^\circ$, $AB = BD$, and $CD = 41$, find the length of BC .

Proposed by: Anders Olsen

Answer: 580

Let the midpoint of AC be O which is the center of the circumcircle of $ABCD$. ADC is a right triangle with a leg of length 41, and $41^2 = AC^2 - AD^2 = (AC - AD)(AC + AD)$. As AC, AD are integers and 41 is prime, we must have $AC = 840, AD = 841$. Let M be the midpoint of AD . $\triangle AOM \sim \triangle ACD$, so $BM = BO + OM = 841/2 + 41/2 = 441$. Then $AB = \sqrt{420^2 + 441^2} = 609$ (this is a 20-21-29 triangle scaled up by a factor of 21). Finally, $BC^2 = AC^2 - AB^2$ so $BC = \sqrt{841^2 - 609^2} = 580$.

25. [13] Let a_0, a_1, \dots and b_0, b_1, \dots be geometric sequences with common ratios r_a and r_b , respectively, such that

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i = 1 \quad \text{and} \quad \left(\sum_{i=0}^{\infty} a_i^2 \right) \left(\sum_{i=0}^{\infty} b_i^2 \right) = \sum_{i=0}^{\infty} a_i b_i.$$

Find the smallest real number c such that $a_0 < c$ must be true.

Proposed by: Handong Wang

Answer: $\frac{4}{3}$

Let $a_0 = a$ and $b_0 = b$. From $\sum_{i=0}^{\infty} a_i = \frac{a_0}{1-r_a} = 1$ we have $a_0 = 1 - r_a$ and similarly $b_0 = 1 - r_b$. This means $\sum_{i=0}^{\infty} a_i^2 = \frac{a_0^2}{1-r_a^2} = \frac{a^2}{(1-r_a)(1+r_a)} = \frac{a^2}{a(2-a)} = \frac{a}{2-a}$, so $\sum_{i=0}^{\infty} a_i^2 \sum_{i=0}^{\infty} b_i^2 = \sum_{i=0}^{\infty} a_i b_i$ yields

$$\frac{a}{2-a} \cdot \frac{b}{2-b} = \frac{ab}{1-(1-a)(1-b)}.$$

Since the numerators are equal, the denominators must be equal, which when expanded gives $2ab - 3a - 3b + 4 = 0$, which is equivalent to $(2a - 3)(2b - 3) = 1$. But note that $0 < a, b < 2$ since we need the sequences to converge (or $|r_a|, |r_b| < 1$), so then $-3 < 2b - 3 < 1$, and thus $2a - 3 > 1$ (impossible) or $2a - 3 < -\frac{1}{3}$. Hence $a < \frac{4}{3}$, with equality when b approaches 0.

26. [13] Points E, F, G, H are chosen on segments AB, BC, CD, DA , respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of 30° , then compute the area of square $ABCD$.

Proposed by: Kevin Sun

Answer: $\boxed{\frac{784}{19}}$

Rotate EG by 90° about the center of the square to $E'G'$ with $E' \in AD$ and $G' \in BC$. Now $E'G'$ and FH intersect at an angle of 60° . Then consider the translation which takes E' to H and G' to I . Triangle FHI has $FH = 8, HI = 7$ and $\angle FHI = 60^\circ$. Furthermore, the height of this triangle is the side length of the square. Using the Law of Cosines,

$$FI = \sqrt{7^2 - 7 \cdot 8 + 8^2} = \sqrt{57}.$$

By computing the area of FHI in two ways, if h is the height then

$$\frac{1}{2} \times \sqrt{57} \times h = \frac{1}{2} \times \frac{\sqrt{3}}{2} \times 7 \times 8.$$

Then $h = \frac{28}{\sqrt{19}}$ and the area of the square is $h^2 = \frac{784}{19}$.

27. [13] At lunch, Abby, Bart, Carl, Dana, and Evan share a pizza divided radially into 16 slices. Each one takes one slice of pizza uniformly at random, leaving 11 slices. The remaining slices of pizza form "sectors" broken up by the taken slices, e.g. if they take five consecutive slices then there is one sector, but if none of them take adjacent slices then there will be five sectors. What is the expected number of sectors formed?

Proposed by: Andrew Gu

Answer: $\boxed{\frac{11}{3}}$

Consider the more general case where there are N slices and $M > 0$ slices are taken. Let S denote the number of adjacent pairs of slices of pizza which still remain. There are $N - M$ slices and a sector of k slices contributes $k - 1$ pairs to S . Hence the number of sectors is $N - M - S$. We compute the expected value of S by looking at each adjacent pair in the original pizza:

$$\mathbb{E}(S) = N \frac{\binom{N-2}{M}}{\binom{N}{M}} = N \frac{(N-M)(N-M-1)}{N(N-1)} = \frac{(N-M)(N-M-1)}{N-1}$$

The expected number of sectors is then

$$N - M - \frac{(N-M)(N-M-1)}{N-1} = \frac{(N-M)M}{N-1}.$$

For $N = 16, M = 5$ this yields $\frac{11}{3}$.

28. [15] What is the 3-digit number formed by the 9998th through 10000th digits after the decimal point in the decimal expansion of $\frac{1}{998}$?

Note: Make sure your answer has exactly three digits, so please include any leading zeroes if necessary.

Proposed by: Sujay Kazi

Answer: 042

Note that $\frac{1}{998} + \frac{1}{2} = \frac{250}{499}$ repeats every 498 digits because 499 is prime, so $\frac{1}{998}$ does as well (after the first 498 block). Now we need to find 38th to 40th digits. We expand this as a geometric series

$$\frac{1}{998} = \frac{\frac{1}{1000}}{1 - \frac{2}{1000}} = .001 + .001 \times .002 + .001 \times .002^2 + \dots$$

The contribution to the 36th through 39th digits is 4096, the 39th through 42nd digits is 8192, and 41st through 45th digits is 16384. We add these together:

$$\begin{array}{r} 4 \ 0 \ 9 \ 6 \\ \ 8 \ 1 \ 9 \ 2 \\ \ 1 \ 6 \ 8 \ 3 \ 4 \\ \hline 4 \ 1 \ 0 \ 4 \ 2 \ 0 \ \dots \end{array}$$

The remaining terms decrease too fast to have effect on the digits we are looking at, so the 38th to 40th digits are 042.

29. [15] An isosceles right triangle ABC has area 1. Points D, E, F are chosen on BC, CA, AB respectively such that DEF is also an isosceles right triangle. Find the smallest possible area of DEF .

Proposed by: Yuan Yao

Answer: $\frac{1}{5}$

Without loss of generality, suppose that AB is the hypotenuse.

If F is the right angle, then F must be the midpoint of AB . To prove this, let X and Y be the feet from F to BC and AC . Since $\angle XFY = \angle DFE = 90^\circ$, we have $\angle XFD = \angle YFE$ so

$$XF = DF \cos \angle XFD = EF \cos \angle YFE = YF.$$

Hence F is equidistant from AC and BC so it is the midpoint of AB . Then the minimum area is achieved by minimizing DF ; this occurs when DF is perpendicular to BC . The triangle DEF then becomes the medial triangle of ABC , so its area is $\frac{1}{4}$.

If F is not the right angle, without loss of generality, let the right angle be D . Place this triangle in the complex plane such that C is the origin, $B = \sqrt{2}$, and $A = \sqrt{2}i$.

Now since D is on the real axis and E is on the imaginary axis, $D = x$ and $E = yi$, and we can obtain F by a 90 degree counterclockwise rotation of D around E : this evaluates to $F = y + (x + y)i$. For F to be on AB , the only constraint is to have $y + (x + y) = \sqrt{2} \implies x = \sqrt{2} - 2y$.

To minimize the area, we minimize

$$\frac{DE^2}{2} = \frac{x^2 + y^2}{2} = \frac{(\sqrt{2} - 2y)^2 + y^2}{2} = \frac{5y^2 - 4\sqrt{2}y + 2}{2}$$

which has a minimum of $\frac{1}{5}$ at $y = \frac{2\sqrt{2}}{5}$. Since this is between 0 and $\sqrt{2}$, this is indeed a valid configuration.

Finally, we take the smallest area of $\frac{1}{5}$.

30. [15] Let n be a positive integer. Let there be P_n ways for Pretty Penny to make exactly n dollars out of quarters, dimes, nickels, and pennies. Also, let there be B_n ways for Beautiful Bill to make exactly n dollars out of one dollar bills, quarters, dimes, and nickels. As n goes to infinity, the sequence of fractions $\frac{P_n}{B_n}$ approaches a real number c . Find c .

Note: Assume both Pretty Penny and Beautiful Bill each have an unlimited number of each type of coin. Pennies, nickels, dimes, quarters, and dollar bills are worth 1, 5, 10, 25, 100 cents respectively.

Proposed by: James Lin

Answer: 20

Let d_x be the number ways to make exactly x cents using only dimes and nickels. It is easy to see that when x is a multiple of 5,

$$d_x = \left\lfloor \frac{x}{10} \right\rfloor + 1.$$

Now, let c_x be the number of ways to make exactly x cents using only quarters, dimes and nickels. Again, it is easy to see that when x is a multiple of 5,

$$c_x = c_{x-25} + d_x.$$

(We can either use 1 or more quarters, which corresponds to the c_{x-25} term, or we can use 0 quarters, which corresponds to the d_x term.) Combining these two equations, we see that c_x can be approximated by a polynomial of degree 2. (In fact, we get five different approximations of c_x , depending on the value of $x \pmod{25}$, but they all only differ by a constant, which will not affect the limit case.) We also see that

$$B_n = c_{100n} + c_{100(n-1)} + \dots + c_0$$

and

$$P_n = c_{100n} + c_{100n-5} + \dots + c_0.$$

Suppose a is the value such that $\lim_{n \rightarrow \infty} \frac{c_n}{an^2} = 1$. Then

$$\lim_{n \rightarrow \infty} \frac{B_n}{P_n} = \lim_{n \rightarrow \infty} \frac{\sum_{k=0}^{\lfloor n/100 \rfloor} a(100k)^2}{\sum_{k=0}^{\lfloor n/5 \rfloor} a(5k)^2} = \lim_{n \rightarrow \infty} \frac{400 \cdot \frac{n}{100} \left(\frac{n}{100} + 1 \right) \left(2 \cdot \frac{n}{100} + 1 \right)}{\frac{n}{5} \left(\frac{n}{5} + 1 \right) \left(2 \cdot \frac{n}{5} + 1 \right)} = 20.$$

31. [17] David and Evan each repeatedly flip a fair coin. David will stop when he flips a tail, and Evan will stop once he flips 2 consecutive tails. Find the probability that David flips more total heads than Evan.

Proposed by: Anders Olsen

Answer: $\frac{1}{5}$

Solution 1: We can find the values of the functions $D(h)$ and $E(h)$, the probabilities that David and Evan, respectively, flip exactly h heads. It is easy to see that $D(h) = 2^{-h-1}$. In order to find $E(h)$, we note that each sequence must end with the flips HTT (unless Evan flips only 2 heads). We disregard these flips for now. Then there are h prior places we can include an extra tails in the sequence, one between each pair of heads. There is a 2^{-h+1} probability of this happening with no extra tails, $h2^{-h}$ probability with 1 extra tail, $\binom{h}{2}2^{-h-1}$ probability with 2 extra tails, and so on. This sum is

$$2^{-h+1} \sum_{n=0}^h 2^{-n} \binom{h}{n} = 2 \left(\frac{3}{4} \right)^h.$$

We divide by 8 to account for the probability of getting HTT to finish our sequence to get that

$$E(h) = \frac{3^h}{4^{h+1}}.$$

Our answer is

$$\sum_{n=0}^{\infty} \left(E(n) \sum_{m=n+1}^{\infty} D(m) \right) = \sum_{n=0}^{\infty} \frac{3^n}{8^{n+1}} = \frac{1}{5}.$$

Solution 2: Since we only care about the number of heads, we think of this as a “survival” game where they flip a single head each round, such that David has a $\frac{1}{2}$ chance of flipping another head and Evan has a $\frac{3}{4}$ chance of flipping another head. (If they don’t get to flip another head, they lose.) David wins if and only if when at least one of David and Evan loses, David does not lose but Evan loses. The probability that at least one of them lose each round is $1 - \frac{1}{2} \cdot \frac{3}{4} = \frac{5}{8}$, and David wins this round with probability $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$, so the overall probability is $\frac{1}{5}$.

32. [17] Over all real numbers x and y , find the minimum possible value of

$$(xy)^2 + (x + 7)^2 + (2y + 7)^2.$$

Proposed by: James Lin

Answer: 45

Solution 1: Rewrite the given expression as $(x^2 + 4)(1 + y^2) + 14(x + 2y) + 94$. By Cauchy-Schwartz, this is at least $(x + 2y)^2 + 14(x + 2y) + 94 = (x + 2y + 7)^2 + 45$. The minimum is 45, attained when $xy = 2, x + 2y = -7$.

Solution 2: Let $z = 2y, s = x + z, p = xz$. We seek to minimize

$$\begin{aligned} \left(\frac{xz}{2}\right)^2 + (x + 7)^2 + (z + 7)^2 &= \frac{p^2}{4} + (x^2 + z^2) + 14(x + z) + 98 \\ &= \frac{p^2}{4} + s^2 - 2p + 14s + 98 \\ &= \left(\frac{p}{2} - 2\right)^2 + (s + 7)^2 + 45 \\ &\geq 45. \end{aligned}$$

Equality holds when $s = -7, p = 4$. Since $s^2 \geq 4p$, this system has a real solution for x and z .

33. [17] Let ABC be a triangle with $AB = 20, BC = 10, CA = 15$. Let I be the incenter of ABC , and let BI meet AC at E and CI meet AB at F . Suppose that the circumcircles of BIF and CIE meet at a point D different from I . Find the length of the tangent from A to the circumcircle of DEF .

Proposed by: Michael Ren

Answer: $2\sqrt{30}$

Solution 1: Let $O = AI \cap (AEF)$. We claim that O is the circumcenter of DEF . Indeed, note that $\angle EDF = \angle ECI + \angle FBI = \frac{\angle B + \angle C}{2} = \frac{\angle EOF}{2}$, and $OE = OF$, so the claim is proven.

Now note that the circumcircle of DEF passes through the incenter of AEF , so power of A with respect to (DEF) is $\sqrt{AE \cdot AF}$. We can compute that $AE = 10, AF = 12$ by the angle bisector theorem, so the length of the tangent is $\sqrt{10 \cdot 12} = 2\sqrt{30}$.

Solution 2: Let E' be the reflection of E across AI . Note that E' lies on AB . We claim that E' lies on the circumcircle of DEF , which will imply that the power of A with respect to (DEF) is $AE' \cdot AF = AE \cdot AF$ and we proceed as in Solution 1. We can easily compute

$$\angle EE'F = 90^\circ + \frac{\angle A}{2}$$

and

$$\angle EDF = \angle EDI + \angle IDF = \angle ECI + \angle IBF = 90^\circ - \frac{\angle A}{2}.$$

Therefore $\angle EDF + \angle EE'F = 180^\circ$, so E' lies on the circumcircle of DEF as desired.

34. [20] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. A positive integer is called *prime-minister* if it has a primest number of distinct primest factors. Let N be the smallest prime-minister number. Estimate N .

An estimate of $E > 0$ earns $\lfloor 20 \min(\frac{N}{E}, \frac{E}{N}) \rfloor$ points.

Proposed by: Yuan Yao

Answer: $2^4 \cdot 3^3 \cdot 5^3 \cdot 7 = 378000$

One heuristic for estimating the answer is that numbers of the form $p^q r^s$ for primes p, q, r, s with $p \neq r, q \neq s$ are primest. Thus, primest numbers are not very rare, so we can expect the answer to be relatively small with only a few distinct prime factors.

```

from operator import *

primes = []

primers = []

primests = []

for i in range(2,3000):
    prime_factors = 0
    primer_factors = 0
    temp = i
    for x in primes:
        if x > temp:
            break
        if temp % x == 0:
            prime_factors += 1
            while temp % x == 0:
                temp = temp // x
    if (prime_factors == 0):
        primes.append(i)
        continue
    elif (prime_factors in primers):
        primers.append(i)

    for x in primers:
        if i % x == 0:
            primer_factors += 1

    if (primer_factors in primests):
        primests.append(i)

def product(L):
    ans = 1
    for i in L:
        ans *= i
    return ans

def sum_prime_product(L, curr = ()):
    if (L == ()):
        #print(curr)
        if len(curr) in primests:

```

```

        return product(curr)
    return 0
return sum_prime_product(L[1:], curr + (L[0],)) + sum_prime_product(L[1:], curr)

def count_primests(L, curr = ()):
    if (L == ()):
        if (sum_prime_product(curr) in primers):
            return 1
        return 0
    ans = 0
    for i in range(0,L[0]+1):
        ans += count_primests(L[1:], curr+(i,))
    return ans

def compute(L):
    ans = 1
    for i in range(len(L)):
        ans *= (primers[i]**L[i])
    return ans

def find_best(M, best = 2**20 * 3**5, curr = ()):
    num = compute(curr)
    if (num > best):
        return False
    if (count_primests(curr) in primests):
        print(num, curr)
        return num
    for i in range(1,M):
        result = find_best(M, best, curr + (i,))
        if (result == False):
            break
        elif (result < best):
            best = result
    return best

print("Answer:", find_best(30))

```

35. [20] Pascal has a triangle. In the n th row, there are $n + 1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \dots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n - 1$, $a_{n,k} = a_{n-1,k} + a_{n-1,k-1}$. Let N be the value of the sum

$$\sum_{k=0}^{2018} \frac{|a_{2018,k}|}{\binom{2018}{k}}.$$

Estimate N .

An estimate of $E > 0$ earns $\lfloor 20 \cdot 2^{-|N-E|/70} \rfloor$ points.

Proposed by: Michael Ren

Answer:

A good estimate for this question is to use the fact that

$$\sum_{k=0}^{2018} |a_{2018,k}| = \frac{2^{2018} + 2}{3},$$

the answer to Guts 17. This suggests that each $|a_{2018,k}|$ is roughly $\frac{1}{3}$ of its corresponding entry $\binom{2018}{k}$ in the usual Pascal's triangle, as the sum of the terms in the 2018th row of Pascal's triangle is 2^{2018} . This then gives an estimate of $\frac{2^{2018}}{3}$, which earns 6 points. Code for computing answer in Python 3:

```
import math
lists=[[1]]
for i in range(2018):
    newlist=[]
    for j in range(i):
        newlist.append(lists[-1][j+1]-lists[-1][j])
    lists.append([1]+newlist+[1])
big=math.factorial(2018)
sum=0
for i in range(2019):
    sum+=abs(lists[-1][i])/(big//math.factorial(i)//math.factorial(2018-i))
print(sum)
```

36. [20] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Let N be the number of solvable 5×5 mazes. Estimate N .

An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right) \rfloor$ points.

Proposed by: John Michael Wu

Answer:

The following code solves the problem in Python 3.

```
# dfs that returns all paths with no adjacent vertices other than those consecutive in the path
def dfs(graph,start,end,path):
    if start==end:
        return [path]
    paths=[]
    for child in graph[start]:
        skip=False
        if child in path:
            continue
        for vert in graph[child]:
            if vert in path[:-1]:
                skip=True
                break
        if not skip:
            paths=paths+dfs(graph,child,end,path[:] + [child])
    return paths
# construct graph representing 5x5 grid
graph={}
for a in range(5):
    for b in range(5):
        graph[(a,b)]=[]
for a in range(4):
    for b in range(5):
        graph[(a,b)].append((a+1,b));graph[(a+1,b)].append((a,b))
        graph[(b,a)].append((b,a+1));graph[(b,a+1)].append((b,a))
paths=dfs(graph,(0,0),(4,4),[(0,0)])
```

```

paths.sort(key=len)
intpaths=[0]*len(paths)
# convert paths to 25-bit binary integers
for i in range(len(paths)):
    for j in paths[i]:
        intpaths[i]+=2**(5*j[0]+j[1])
mazes=0
for j in range(2**23):
    k=2*j
    # disregard cases that are common and never valid
    if k&8912896==8912896 or k&34==34 or k&4472832==4472832 or k&1092==1092:
        continue
    for path in intpaths:
        # check if case has empty spaces along whole path
        if path&k==0:
            mazes+=1
            break
print(mazes)

```

Alternatively, the following code solves the problem in Java SE 8.

```

import java.util.*;

public class guts36 {

    static int M = 5;
    static int N = 5;

    static long[] pow2 = new long[M * N];
    static int[][] dir = new int[][] {new int[] {0, 1}, new int[] {1, 0}, new int[] {0, -1}, new int[]
    {1, -1}};

    public static void main(String[] args) {

        pow2[0] = 1;
        for (int i = 1; i < pow2.length; i++) {
            pow2[i] = pow2[i - 1] * 2;
        }

        boolean[][] grid = new boolean[M][N];
        grid[0][0] = true;
        grid[M - 1][N - 1] = true;

        int ans = 0;

        for (long c = 0; c < pow2[M * N - 2]; c++) {

            long d = c;
            for (int b = 0; b < M * N - 2; b++) {
                int i = (b + 1) / N;
                int j = (b + 1) % N;
                grid[i][j] = ((d & 1) > 0);
                d >>= 1;
            }
        }
    }
}

```

```

if (check(grid)) {
    ans++;
}

}

System.out.println("answer: " + ans);

}

static int[] add(int[] a, int[] b) {
    return new int[] {a[0] + b[0], a[1] + b[1]};
}

static boolean get(boolean[][] g, int[] a) {
    return g[a[0]][a[1]];
}

static void set(boolean[][] g, int[] a, boolean v) {
    g[a[0]][a[1]] = v;
}

static boolean valid(int[] a) {
    return (a[0] >= 0) && (a[1] >= 0) && (a[0] < M) && (a[1] < N);
}

static boolean check(boolean[][] grid) {

    Stack<int[]> q = new Stack<int[]>();
    q.add(new int[] {0, 0});

    boolean[][] reached = new boolean[M][N];
    reached[0][0] = true;

    while (!q.isEmpty()) {

        int[] a = q.pop();
        for (int[] d: dir) {
            int[] b = add(a, d);
            if (valid(b) && get(grid, b) && !get(reached, b)) {
                if (b[0] == M - 1 && b[1] == N - 1) {
                    return true;
                }
                set(reached, b, true);
                q.add(b);
            }
        }

    }

    return false;

}

```


HMMT November 2018

November 10, 2018

Team Round

1. [15] Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

Proposed by: Michael Tang

Answer: $\boxed{\frac{31}{432}}$

If any two of the dice show an odd number, then this is impossible, so at most one of the dice can show an odd number. We take two cases:

Case 1: If exactly one of the dice shows an odd number, then all three other dice must show a multiple of 4, which can only be the number 4. The probability that this occurs is $4 \cdot \frac{1}{2} \cdot \left(\frac{1}{6}\right)^3 = \frac{1}{108}$.

Case 2: If all of the dice show even numbers, then the condition is satisfied. The probability that this occurs is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

The total probability is $\frac{1}{108} + \frac{1}{16} = \frac{31}{432}$.

2. [20] Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

Proposed by: Dhruv Rohatgi

Answer: $\boxed{94}$

Note that after applying the squaring operation, Alice's number will be a perfect square, so she can maximize her score by having a large number of adding operations at the end. However, her scores needs to be large enough that the many additions to not bring her close to a larger square. Hence the strategy is as follows: 2 additions to get to 2, 4 consecutive squares to get to 65536, and 94 more additions for a score of 94.

3. [25] For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

Proposed by: James Lin

Answer: $\boxed{28}$

Let $n = 2^a 5^b c$, where $2, 5 \nmid c$. Then, the ratio of the number of divisors of $10n$ to the number of divisors of n is $\frac{a+2}{a+1} \frac{b+2}{b+1} = 3$. Solving for b , we find that $b = \frac{1-a}{2a+1}$. This forces $(a, b) = (0, 1), (1, 0)$. Therefore, the answers are of the form $2k$ and $5k$ whenever $\gcd(k, 10) = 1$. There are 50 positive numbers of the form $2k$ and 20 positive numbers of the form $5k$ less than or equal to 100. Of those 70 numbers, only $\frac{1}{2} \cdot \frac{4}{5}$ have k relatively prime to 10, so the answer is $70 \cdot \frac{1}{2} \cdot \frac{4}{5} = \boxed{28}$.

4. [30] Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x . Find x .

Proposed by: James Lin

Answer: $\boxed{\frac{32}{27}}$

Let $p = \log_{10} a, q = \log_{10} b$. Since $a, b > 1$, p and q are positive. The condition $ab = 100$ translates to $p + q = 2$. We wish to maximize

$$x = \log_{10} a^{(\log_{10} b)^2} = (\log_{10} a)(\log_{10} b)^2 = pq^2.$$

By AM-GM,

$$\frac{27}{4} pq^2 \leq \left(p + \frac{q}{2} + \frac{q}{2}\right)^3 = 8.$$

Hence $pq^2 \leq \frac{32}{27}$ with equality when $p = \frac{2}{3}, q = \frac{4}{3}$.

5. [35] Find the sum of all positive integers n such that $1 + 2 + \dots + n$ divides

$$15[(n+1)^2 + (n+2)^2 + \dots + (2n)^2].$$

Proposed by: Michael Ren

Answer: 64

We can compute that $1 + 2 + \dots + n = \frac{n(n+1)}{2}$ and $(n+1)^2 + (n+2)^2 + \dots + (2n)^2 = \frac{2n(2n+1)(4n+1)}{6} - \frac{n(n+1)(2n+1)}{6} = \frac{n(2n+1)(7n+1)}{3(n+1)}$, so we need $\frac{15(2n+1)(7n+1)}{3(n+1)} = \frac{5(2n+1)(7n+1)}{n+1}$ to be an integer. The remainder when $(2n+1)(7n+1)$ is divided by $(n+1)$ is 6, so after long division we need $\frac{30}{n+1}$ to be an integer. The solutions are one less than a divisor of 30 so the answer is

$$1 + 2 + 4 + 5 + 9 + 14 + 29 = 64.$$

6. [45] Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω . Compute the radius of the circle with center on \overline{QR} which is tangent to both ω and \overline{PQ} .

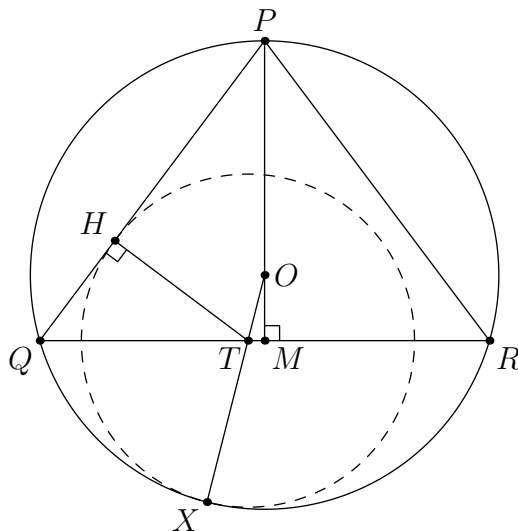
Proposed by: Michael Tang

Answer: $\frac{20}{9}$

Solution 1: Denote the second circle by γ . Let T and r be the center and radius of γ , respectively, and let X and H be the tangency points of γ with ω and \overline{PQ} , respectively. Let O be the center of ω , and let M be the midpoint of \overline{QR} . Note that $QM = MR = \frac{1}{2}QR = 3$, so $\triangle PMQ$ and $\triangle PMR$ are 3-4-5 triangles. Since $\triangle QHT \sim \triangle QMP$ and $HT = r$, we get $QT = \frac{5}{4}r$. Then $TM = QM - QT = 3 - \frac{5}{4}r$. By the extended law of sines, the circumradius of $\triangle PQR$ is $OP = \frac{PR}{2 \sin \angle PQR} = \frac{5}{2(4/5)} = \frac{25}{8}$, so $OM = MP - OP = 4 - \frac{25}{8} = \frac{7}{8}$. Also, we have $OT = OX - XT = \frac{25}{8} - r$. Therefore, by the Pythagorean theorem,

$$\left(3 - \frac{5}{4}r\right)^2 + \left(\frac{7}{8}\right)^2 = \left(\frac{25}{8} - r\right)^2.$$

This simplifies to $\frac{9}{16}r^2 - \frac{5}{4}r = 0$, so $r = \frac{5}{4} \cdot \frac{16}{9} = \frac{20}{9}$.



Solution 2: Following the notation of the previous solution, we compute the power of T with respect to ω in two ways. One of these is $-QT \cdot TR = -\frac{5}{4}r(6 - \frac{5}{4}r)$. The other way is $-(OX - OT)(OX + OT) = -r(\frac{25}{4} - r)$. Equating these two yields $r = \frac{20}{9}$.

7. [50] A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

Note: A rectangle must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k .

Proposed by: Henrik Boecken

Answer: 60

Consider any two rows and the five numbers obtained by adding the two numbers which share a given column. Suppose a of these are odd and b of these are even. The number of corner-odd rectangles with their sides contained in these two rows is ab . Since $a + b = 5$, we have $ab \leq 6$. Therefore every pair of rows contains at most 6 corner-odd rectangles.

There are $\binom{5}{2} = 10$ pairs of rows, so there are at most 60 corner-odd rectangles. Equality holds when we place 1 along one diagonal and 0 everywhere else.

8. [55] Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and
- if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from this vertex;
 - if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from this vertex.

When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

Proposed by: Yuan Yao

Answer: 42648

The order of the deck does not matter as black cards and white cards commute, therefore we can assume that the cards are alternating black and white, and only worry about the arrangement of the numbers. After each pair of black and white cards, each number is replaced by the sum of two times the edge neighbors and three times the diagonally opposite number. We can compute that after four pairs of operations, the number at vertex V will be $1641v + 1640(d_1 + d_2 + d_3)$, where v is the number originally at v and d_1, d_2, d_3 are the numbers at diagonally adjacent vertices. Set $v = 8$ and $d_1, d_2, d_3 = 5, 6, 7$ in any order to obtain the maximum number 42648.

9. [60] Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B , and let ℓ_1 and ℓ_2 be tangents to ω through A and C , respectively. Let K be the intersection of ℓ_1 and ℓ_2 . Let X lie on segment \overline{KA} and Y lie on segment \overline{KC} such that $XY \parallel BC$ and XY is tangent to ω . What is the largest possible integer length for XY ?

Proposed by: James Lin

Answer: 35

Note that B is the K -excenter of KXY , so XB is the angle bisector of $\angle AKY$. As AB and XY are parallel, $\angle XAB + 2\angle AXB = 180^\circ$, so $\angle XBA = 180^\circ - \angle AXB - \angle XAB$. This means that AXB is isosceles with $AX = AB = 20$. Similarly, $YC = BC = 18$. As KXY is similar to KAC , we have that $\frac{KX}{KY} = \frac{XA}{YC} = \frac{20}{18}$. Let $KA = 20x, KC = 18x$, so the Triangle Inequality applied to triangle KAC gives $KA < KC + AC \implies 20x < 18x + 38 \implies x < 19$. Then, $XY = AC \cdot \frac{KX}{KA} = 38 \cdot \frac{x-1}{x} = 38 - \frac{38}{x} < 36$, so the maximum possible integer length of XY is 35. The optimal configuration is achieved when the radius of ω becomes arbitrarily small and ℓ_1 and ℓ_2 are on opposite sides of AC .

10. [65] David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N . David wants to devise a strategy that will guarantee

that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.

Proposed by: Anders Olsen

Answer: 36440

We can represent each strategy as a binary tree labeled with the integers from 1 to 59, where David starts at the root and moves to the right child if he is too low and to the left child if he is too high. Our tree must have at most 6 layers as David must guess at most 5 times. Once David has been told that he guessed correctly or if the node he is at has no children, he will be sure of Evan's number. Consider the unique strategy for David when 59 is replaced with 63. This is a tree where every node in the first 5 layers has two children, and it can only be labeled in one way such that the strategy satisfies the given conditions. In order to get a valid strategy for 59, we only need to delete 4 of the vertices from this tree and relabel the vertices as necessary. Conversely, every valid strategy tree for 59 can be completed to the strategy tree for 63. If we delete a parent we must also delete its children. Thus, we can just count the number of ways to delete four nodes from the tree for 63 so that if a parent is deleted then so are its children. We cannot delete a node in the fourth layer, as that means we delete at least $1 + 2 + 4 = 7$ nodes. If we delete a node in the fifth layer, then we delete its two children as well, so in total we delete three nodes. There are now two cases: if we delete all four nodes from the sixth layer or if we delete one node in the fifth layer along with its children and another node in the sixth layer. There are $\binom{32}{4}$ ways to pick 4 from the sixth layer and $16 \cdot 30$ to pick one from the fifth layer along with its children and another node that is from the sixth layer, for a total of 36440.

HMMT November 2018

November 10, 2018

Theme Round

1. Square $CASH$ and regular pentagon $MONEY$ are both inscribed in a circle. Given that they do not share a vertex, how many intersections do these two polygons have?

Proposed by: Yuan Yao

Answer:

Pentagon $MONEY$ divides the circumference into 5 circular arcs, and each vertex of $CASH$ lies in a different arc. Then each side of $CASH$ will intersect two sides of $MONEY$, for a total of 8 intersections.

2. Consider the addition problem:

$$\begin{array}{rcccc} & C & A & S & H \\ + & & & & M & E \\ \hline O & S & I & D & E \end{array}$$

where each letter represents a base-ten digit, and $C, M, O \neq 0$. (Distinct letters are allowed to represent the same digit) How many ways are there to assign values to the letters so that the addition problem is true?

Proposed by: James Lin

Answer:

Clearly, $CASH$ and ME cannot add up to 11000 or more, so $O = 1$ and $S = 0$. By examining the units digit, we find that $H = 0$. Then $CASH + ME < 9900 + 99 < 10000$, so there are no solutions.

3. HOW , BOW , and DAH are equilateral triangles in a plane such that $WO = 7$ and $AH = 2$. Given that D, A, B are collinear in that order, find the length of BA .

Proposed by: James Lin

Answer:

Note that $H \neq B$ since otherwise DAB is an equilateral triangle. Let M be the midpoint of DA , so $HB = 7\sqrt{3}$ and $HM = \sqrt{3}$, and $\angle HMB = 90^\circ$. By the Pythagorean theorem,

$$BM = \sqrt{(7\sqrt{3})^2 - (\sqrt{3})^2} = 12.$$

Then $BA = BM - AM = 11$.

4. I have two cents and Bill has n cents. Bill wants to buy some pencils, which come in two different packages. One package of pencils costs 6 cents for 7 pencils, and the other package of pencils costs a *dime for a dozen* pencils (i.e. 10 cents for 12 pencils). Bill notes that he can spend **all** n of his cents on some combination of pencil packages to get P pencils. However, if I *give my two cents* to Bill, he then notes that he can instead spend **all** $n + 2$ of his cents on some combination of pencil packages to get fewer than P pencils. What is the smallest value of n for which this is possible?

Note: Both times Bill must spend **all** of his cents on pencil packages, i.e. have zero cents after either purchase.

Proposed by: James Lin

Answer:

Suppose that Bill buys a packages of 7 and b packages of 12 in the first scenario and c packages of 7 and d packages of 12 in the second scenario. Then we have the following system:

$$\begin{aligned} 6a + 10b &= n \\ 6c + 10d &= n + 2 \\ 7a + 12b &> 7c + 12d. \end{aligned}$$

Since the packages of 12 give more pencils per cent, we must have $b > d$. Subtract the first two equations and divide by 2 to get

$$3(c - a) - 5(b - d) = 1.$$

Note that the last inequality is $12(b - d) > 7(c - a)$. The minimal solutions to the equation with $b - d > 0$ are

$$(c - a, b - d) = (2, 1), (7, 4), (12, 7), (17, 10).$$

$(17, 10)$ is the first pair for which $12(b - d) > 7(c - a)$. Hence $b \geq 10$ so $n \geq 100$. We can easily verify that $(a, b, c, d, n) = (0, 10, 17, 0, 100)$ satisfies the system of equations.

5. Lil Wayne, the rain god, determines the weather. If Lil Wayne makes it rain on any given day, the probability that he makes it rain the next day is 75%. If Lil Wayne doesn't make it rain on one day, the probability that he makes it rain the next day is 25%. He decides not to make it rain today. Find the smallest positive integer n such that the probability that Lil Wayne *makes it rain* n days from today is greater than 49.9%.

Proposed by: Anders Olsen

Answer:

Let p_n denote the probability that Lil Wayne makes it rain n days from today. We have $p_0 = 0$ and

$$p_{n+1} = \frac{3}{4}p_n + \frac{1}{4}(1 - p_n) = \frac{1}{4} + \frac{1}{2}p_n.$$

This can be written as

$$p_{n+1} - \frac{1}{2} = \frac{1}{2} \left(p_n - \frac{1}{2} \right)$$

and we can check that the solution of this recurrence is

$$p_n = \frac{1}{2} - \frac{1}{2^{n+1}}$$

We want $\frac{1}{2^{n+1}} < \frac{1}{1000}$, which first occurs when $n = 9$.

6. Farmer James invents a new currency, such that for every positive integer $n \leq 6$, there exists an n -coin worth $n!$ cents. Furthermore, he has exactly n copies of each n -coin. An integer k is said to be *nice* if Farmer James can make k cents using at least one copy of each type of coin. How many positive integers less than 2018 are nice?

Proposed by: Nikhil Reddy

Answer:

We use the factorial base, where we denote

$$(d_n \dots d_1)_* = d_n \times n! + \dots + d_1 \times 1!$$

The representation of 2018_{10} is 244002_* and the representation of 720_{10} is 100000_* . The largest nice number less than 244002_* is 243321_* . Notice that for the digit d_i of a nice number, we can vary its value from 1 to i , while for a generic number in the factorial base, d_{i-1} can vary from 0 to $i - 1$. Hence we can map nice numbers to all numbers by truncating the last digit and reducing each previous digit by 1, and likewise reverse the procedure by increasing all digits by 1 and adding 1 at the end. Furthermore, this procedure preserves the ordering of numbers. Applying this procedure to 243321_* gives 13221_* . We count from 0_* to 13221_* (since the first nice number is 1_*), to get an answer of

$$13221_* + 1 = 210.$$

7. Ben "One Hunna Dolla" Franklin is flying a kite $KITE$ such that IE is the perpendicular bisector of KT . Let IE meet KT at R . The midpoints of KI, IT, TE, EK are A, N, M, D , respectively. Given that $[MAKE] = 18, IT = 10, [RAIN] = 4$, find $[DIME]$.

Note: $[X]$ denotes the area of the figure X .

Proposed by: Michael Ren

Answer: $\boxed{16}$

Let $[KIR] = [RIT] = a$ and $[KER] = [TER] = b$. We will relate all areas to a and b . First,

$$[RAIN] = [RAI] + [INR] = \frac{1}{2}a + \frac{1}{2}a = a.$$

Next, we break up $[MAKE] = [MAD] + [AKD] + [DEM]$. We have

$$\begin{aligned} [MAD] &= \frac{AD \cdot DM}{2} = \frac{1}{2} \cdot \frac{IE}{2} \cdot \frac{KT}{2} = \frac{[KITE]}{4} = \frac{a+b}{2} \\ [AKD] &= \frac{[KIE]}{4} = \frac{a+b}{4} \\ [DEM] &= \frac{[KTE]}{4} = \frac{b}{2}. \end{aligned}$$

After adding these we get $[MAKE] = \frac{3a+5b}{4}$. We want to find

$$[DIME] = 2[IME] = [ITE] = a+b = \frac{4}{5} \left(\frac{3a+5b}{4} \right) + \frac{2}{5}a = \frac{4}{5} \cdot 18 + \frac{2}{5} \cdot 4 = 16.$$

8. Crisp All, a basketball player, is *dropping dimes* and nickels on a number line. Crisp drops a dime on every positive multiple of 10, and a nickel on every multiple of 5 that is not a multiple of 10. Crisp then starts at 0. Every second, he has a $\frac{2}{3}$ chance of jumping from his current location x to $x+3$, and a $\frac{1}{3}$ chance of jumping from his current location x to $x+7$. When Crisp jumps on either a dime or a nickel, he stops jumping. What is the probability that Crisp *stops on a dime*?

Proposed by: James Lin

Answer: $\boxed{\frac{20}{31}}$

Let “a 3” mean a move in which Crisp moves from x to $x+3$, and “a 7” mean a move in which Crisp moves from x to $x+7$. Note that Crisp stops precisely the first time his number of 3’s and number of 7’s differs by a multiple of 5, and that he’ll *stop on a dime* if they differ by 0, and stop on a nickel if they differ by 5. This fact will be used without justification.

We split into two cases:

- (a) Crisp begins with a 3. Rather than consider the integer Crisp is on, we’ll count the difference, n , between his number of 3’s and his number of 7’s. Each 3 increases n by 1, and each 7 decreases n by 1. Currently, n is 1. The probability of stopping on a dime, then, is the probability n reaches 0 before n reaches 5, where n starts at 1. Let a_i be the probability n reaches 0 first, given a current position of i , for $i = 1, 2, 3, 4$. We desire a_1 . We have the system of linear equations

$$\begin{aligned} a_1 &= \frac{2}{3}a_2 + \frac{1}{3} \cdot 1 \\ a_2 &= \frac{2}{3}a_3 + \frac{1}{3}a_1 \\ a_3 &= \frac{2}{3}a_4 + \frac{1}{3}a_2 \\ a_4 &= \frac{2}{3} \cdot 0 + \frac{1}{3}a_3 \end{aligned}$$

From which we determine that $a_1 = \frac{15}{31}$.

- (b) Crisp begins with a 7. Now, let m be the difference between his number of 7's and his number of 3's. Let b_i denote his probability of stopping on a dime, given his current position of $m = i$. We desire b_1 . We have the system of linear equations

$$\begin{aligned} b_1 &= \frac{1}{3}b_2 + \frac{2}{3} \cdot 1 \\ b_2 &= \frac{1}{3}b_3 + \frac{2}{3}b_1 \\ b_3 &= \frac{1}{3}b_4 + \frac{2}{3}b_2 \\ b_4 &= \frac{1}{3} \cdot 0 + \frac{2}{3}b_3 \end{aligned}$$

From which we determine that $b_1 = \frac{30}{31}$.

We conclude that the answer is $\frac{2}{3}a_1 + \frac{1}{3}b_1 = \frac{2}{3} \cdot \frac{15}{31} + \frac{1}{3} \cdot \frac{30}{31} = \frac{20}{31}$.

9. Circle ω_1 of radius 1 and circle ω_2 of radius 2 are concentric. Godzilla inscribes square *CASH* in ω_1 and regular pentagon *MONEY* in ω_2 . It then writes down all 20 (not necessarily distinct) distances between a vertex of *CASH* and a vertex of *MONEY* and multiplies them all together. What is the maximum possible value of his result?

Proposed by: Michael Ren

Answer: 1048577 or $2^{20} + 1$

We represent the vertices with complex numbers. Place the vertices of *CASH* at $1, i, -1, -i$ and the vertices of *MONEY* at $2\alpha, 2\alpha\omega, 2\alpha\omega^2, 2\alpha\omega^3, 2\alpha\omega^4$ with $|\alpha| = 1$ and $\omega = e^{\frac{2\pi i}{5}}$. We have that the product of distances from a point z to the vertices of *CASH* is $|(z-1)(z-i)(z+1)(z+i)| = |z^4 - 1|$, so we want to maximize $|(16\alpha^4 - 1)(16\alpha^4\omega^4 - 1)(16\alpha^4\omega^3 - 1)(16\alpha^4\omega^2 - 1)(16\alpha^4\omega - 1)|$, which just comes out to be $|2^{20}\alpha^{20} - 1|$. By the triangle inequality, this is at most $2^{20} + 1$, and it is clear that some α makes equality hold.

10. *One million bucks* (i.e. one million male deer) are in different cells of a 1000×1000 grid. The left and right edges of the grid are then glued together, and the top and bottom edges of the grid are glued together, so that the grid forms a doughnut-shaped torus. Furthermore, some of the bucks are *honest bucks*, who always tell the truth, and the remaining bucks are *dishonest bucks*, who never tell the truth. Each of the million bucks claims that "at most one of my neighboring bucks is an *honest buck*." A pair of neighboring bucks is said to be *buckaroo* if exactly one of them is an *honest buck*. What is the minimum possible number of *buckaroo* pairs in the grid?

Note: Two bucks are considered to be *neighboring* if their cells (x_1, y_1) and (x_2, y_2) satisfy either: $x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod{1000}$, or $x_1 - x_2 \equiv \pm 1 \pmod{1000}$ and $y_1 = y_2$.

Proposed by: James Lin

Answer: 1200000

Note that each honest buck has at most one honest neighbor, and each dishonest buck has at least two honest neighbors. The connected components of honest bucks are singles and pairs. Then if there are K honest bucks and B buckaroo pairs, we get $B \geq 3K$. From the dishonest buck condition we get $B \geq 2(1000000 - K)$, so we conclude that $B \geq 1200000$. To find equality, partition the grid into five different parts with side $\sqrt{5}$, and put honest bucks on every cell in two of the parts.

HMIC 2018

April 09, 2018

HMIC 2018

1. [6] Let $m > 1$ be a fixed positive integer. For a nonempty string of base-ten digits S , let $c(S)$ be the number of ways to split S into contiguous nonempty strings of digits such that the base-ten number represented by each string is divisible by m . These strings are allowed to have leading zeroes.

In terms of m , what are the possible values that $c(S)$ can take?

For example, if $m = 2$, then $c(1234) = 2$ as the splits 1234 and $12|34$ are valid, while the other six splits are invalid.

Proposed by: Kevin Sun

Answer: 0 and 2^n for all nonnegative integer n

First, we note that $c(1) = 0$ and $c(00\dots 0) = 2^{n-1}$ if there are n zeroes in the string. Now we show that these are the only possibilities. Note that a split can be added if and only if the string before this split (ignoring all other splits) represent a multiple of m (if there is a split before it, then removing the digits before this preceding split is equivalent to subtracting the removed number times a power of 10, which will also be a multiple of m , so the remaining number between the two splits remain a multiple of m). Thus, whether we can add a split or not depends only on the string itself and no other splits, so $c(S)$ is 0 if the number is not divisible by m and a power of two otherwise.

2. [7] Consider a finite set of points $T \in \mathbb{R}^n$ contained in the n -dimensional unit ball centered at the origin, and let X be the convex hull of T . Prove that for all positive integers k and all points $x \in X$, there exist points $t_1, t_2, \dots, t_k \in T$, not necessarily distinct, such that their centroid

$$\frac{t_1 + t_2 + \dots + t_k}{k}$$

has Euclidean distance at most $\frac{1}{\sqrt{k}}$ from x .

(The n -dimensional unit ball centered at the origin is the set of points in \mathbb{R}^n with Euclidean distance at most 1 from the origin. The convex hull of a set of points $T \in \mathbb{R}^n$ is the smallest set of points X containing T such that each line segment between two points in X lies completely inside X .)

Proposed by: Henrik Boecken

By the definition of convex hull, we can write $x = \sum_{i=1}^m \lambda_i z_i$, where each $z_i \in T$, each $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$. Consider then a random variable Z that takes on value z_i with probability λ_i . We have $\mathbb{E}[Z] = x$. Let $\bar{Z} = \frac{1}{k} \sum_{i=1}^k Z_i$, where each Z_i is an independent copy of Z . Then we wish to compute

$$\text{Var}[\bar{Z}] = \frac{1}{k^2} \sum_{i=1}^k \text{Var}[Z_i]$$

Finally, we have

$$\text{Var}[Z_i] = \mathbb{E}[|Z_i - x|^2] = \mathbb{E}[|Z_i|^2] - x^2 \leq \mathbb{E}[|Z_i|^2] \leq 1$$

The second equality follows from the identity $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$. Now, we know that

$$\mathbb{E}[|x - \frac{1}{k} \sum_{i=1}^k Z_i|^2] = \text{Var}[\bar{Z}] \leq \frac{1}{k}$$

Thus, there must exist some realization of x_i of the Z_i such that

$$\|x - \frac{1}{k} \sum_{i=1}^k x_i\|^2 \leq \frac{1}{k}$$

and we are done.

3. [8] A polygon in the plane (with no self-intersections) is called *equitable* if every line passing through the origin divides the polygon into two (possibly disconnected) regions of equal area.

Does there exist an equitable polygon which is not centrally symmetric about the origin?

(A polygon is centrally symmetric about the origin if a 180-degree rotation about the origin sends the polygon to itself.)

Proposed by: Kevin Sun

Consider the polygon with vertices

$$\begin{aligned} &A(1, 0), B(0, 1), C(0, 5), D(5, 0), E(7, 0), \\ &F(0, -7), G(-7, 0), H(-\sqrt{84}, 0), I(0, \sqrt{84}), J(0, 6), \\ &K(-6, 0), L(-5, 0), M(0, -5), N(0, -1). \end{aligned}$$

Notice that its intersection with each of the four quadrants are all trapezoids with angles $45^\circ, 45^\circ, 135^\circ, 135^\circ$, and the trapezoids in quadrant 1 and 3 both have area $\frac{5^2-1^2}{2} = \frac{7^2-5^2}{2} = 12$, and the trapezoids in quadrant 2 and 4 both have area $\frac{7^2-1^2}{2} = \frac{84-6^2}{2} = 24$, so by similar triangles, we can show that this polygon is indeed equitable. It is also apparent that this polygon is not centrally symmetric.

4. [10] Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$$

for all $x, y \in \mathbb{R}^+$.

(\mathbb{R}^+ denotes the set of positive real numbers.)

Proposed by: Ashwin Sah

Let $P(x, y)$ denote the assertion that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1.$$

Claim 1. f is injective. Proof: If $f(a) = f(b)$ then $P\left(x, \frac{a}{x+1}\right), P\left(x, \frac{b}{x+1}\right)$ yields $a = b$, since $f(x + 1) \in \mathbb{R}^+$ so in particular is nonzero. \circ Now $P\left(x, \frac{1}{f(x+1)}\right)$ yields

$$f\left(x + f\left(\frac{x+1}{f(x+1)}\right)\right) = f(x+1)$$

hence by injectivity

$$x + f\left(\frac{x+1}{f(x+1)}\right) = x + 1$$

so that

$$f\left(\frac{x+1}{f(x+1)}\right) = 1.$$

By injectivity, this equals some constant c , so that

$$\frac{x+1}{f(x+1)} = c$$

for all $x \in \mathbb{R}^+$. Now letting $x, y > 1$ in $P(x, y)$ automatically yields

$$\frac{x}{c} + \frac{y+xy}{c^2} = (y+1)\left(\frac{x+1}{c}\right) - 1,$$

which immediately yields $c = 1$ if we take x, y large. Finally, we have $f(x+1) = x+1 \forall x \in \mathbb{R}^+$. Finally, $P\left(x, \frac{y}{x+1}\right)$ yields

$$f(x + f(y)) = \left(\frac{y}{x+1} + 1\right) f(x+1) - 1 = x + y,$$

so that fixing y and letting $x > 1$ yields

$$x + f(y) = x + y,$$

so that $f(y) = y$. This was for arbitrary positive y , so that $f(x) = x \forall x \in \mathbb{R}^+$, which clearly works.

5. [11] Let G be an undirected simple graph. Let $f(G)$ be the number of ways to orient all of the edges of G in one of the two possible directions so that the resulting directed graph has no directed cycles. Show that $f(G)$ is a multiple of 3 if and only if G has a cycle of odd length.

Proposed by: Yang Liu

Let $f_G(q)$ be the number of ways to color G with q colors. This is the chromatic polynomial of G , and turns out to be polynomial in q . Indeed, choose an edge e and let $G \setminus e$ be the graph G with e removed and let G/e be graph G with the vertices on either side of e merged together (multiple edges are removed). It is not hard to see that

$$f_G(q) = f_{G \setminus e}(q) - f_{G/e}(q)$$

and that for an empty graph E_n with n vertices, we have $f_{E_n}(q) = q^n$. Induction finishes.

Now we can also show that the desired number of ways to direct G to avoid directed cycles is $c_G = (-1)^n f_G(-1)$ where n is the number of vertices of G . The way to do it is to show that

$$c_G = c_{G \setminus e} + c_{G/e},$$

and $c_{E_n} = 1$. Thus $(-1)^n c_G$ will satisfy the desired recurrence, after multiplying the above by $(-1)^n$, and satisfies the correct initial conditions.

Now all we need to know is that $f_G(q)$ has integer coefficients, so that $(-1)^n c_G \equiv f_G(-1) \equiv f_G(2) \pmod{3}$, and $f_G(2) = 0$ when G is not bipartite and is $2^{c(G)}$ where $c(G)$ is the number of connected components of G otherwise. The result follows.

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Algebra and Number Theory

1. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \leq a_i < 3$ for all $0 \leq i \leq n$.

Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

Proposed by: Yang Liu

Answer: 86

One can evaluate

$$Q(\sqrt{3}) = (a_0 + 3a_2 + 3^2a_4 + \dots) + (a_1 + 3a_3 + 3^2a_5 + \dots)\sqrt{3}.$$

Therefore, we have that

$$(a_0 + 3a_2 + 3^2a_4 + \dots) = 20 \text{ and } (a_1 + 3a_3 + 3^2a_5 + \dots) = 17.$$

This corresponds to the base-3 expansions of 20 and 17. This gives us that $Q(x) = 2 + 2x + 2x^3 + 2x^4 + x^5$, so $Q(2) = 86$.

2. Find the value of

$$\sum_{1 \leq a < b < c} \frac{1}{2^a 3^b 5^c}$$

(i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

Proposed by: Alexander Katz

Answer: 1/1624

Let $x = b - a$ and $y = c - b$ so that $b = a + x$ and $c = a + x + y$. Then

$$2^a 3^b 5^c = 2^a 3^{a+x} 5^{a+x+y} = 30^a 15^x 5^y$$

and a, x, y are any positive integers. Thus

$$\begin{aligned} \sum_{1 \leq a < b < c} \frac{1}{2^a 3^b 5^c} &= \sum_{1 \leq a, x, y} \frac{1}{30^a 15^x 5^y} \\ &= \sum_{1 \leq a} \frac{1}{30^a} \sum_{1 \leq x} \frac{1}{15^x} \sum_{1 \leq y} \frac{1}{5^y} \\ &= \frac{1}{29} \cdot \frac{1}{14} \cdot \frac{1}{4} \\ &= \frac{1}{1624} \end{aligned}$$

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

Proposed by: Alexander Katz

Let $P(x, y)$ be the given assertion. From $P(0, 0)$ we get $f(0)^2 = f(0) \implies f(0) = 0, 1$.

From $P(x, x)$ we get $f(x)^2 = f(0)$. Thus, if $f(0) = 0$, we have $f(x) = 0$ for all x , which satisfies the given constraints. Thus $f(2017) = 0$ is one possibility.

Now suppose $f(0) = 1$. We then have $P(0, y) \implies f(-y) = f(y)$, so that $P(x, -y) \implies f(x)f(y) = f(x-y) = f(x)f(-y) = f(x+y)$. Thus $f(x-y) = f(x+y)$, and in particular $f(0) = f\left(\frac{x}{2} - \frac{x}{2}\right) = f\left(\frac{x}{2} + \frac{x}{2}\right) = f(x)$. It follows that $f(x) = 1$ for all x , which also satisfies all given constraints.

Thus the two possibilities are $f(2017) = \boxed{0, 1}$.

4. Find all pairs (a, b) of positive integers such that $a^{2017} + b$ is a multiple of ab .

Proposed by: Yang Liu

Answer: $\boxed{(1, 1) \text{ and } (2, 2^{2017})}$.

We want $ab|a^{2017} + b$. This gives that $a|b$. Therefore, we can set $b = b_{2017}a$. Substituting this gives $b_{2017}a^2|a^{2017} + b_{2017}a$, so $b_{2017}a|a^{2016} + b_{2017}$. Once again, we get $a|b_{2017}$, so we can set $b_{2017} = b_{2016}a$. Continuing this way, if we have $b_{i+1}a|a^i + b_{i+1}$, then $a|b_{i+1}$, so we can set $b_{i+1} = b_i a$ and derive $b_i a|a^{i-1} + b_i$. Continuing down to $i = 1$, we would have $b = b_1 a^{2017}$ so $ab_1|1 + b_1$. If $a \geq 3$, then $ab_1 > 1 + b_1$ for all $b_1 \geq 1$, so we need either $a = 1$ or $a = 2$. If $a = 1$, then $b|b + 1$, so $b = 1$. This gives the pair $(1, 1)$. If $a = 2$, we need $2b|b + 2^{2017}$. Therefore, we get $b|2^{2017}$, so we can write $b = 2^k$ for $0 \leq k \leq 2017$. Then we need $2^{k+1}|2^k + 2^{2017}$. As $k \leq 2017$, we need $2|1 + 2^{2017-k}$. This can only happen is $k = 2017$. This gives the pair $(2, 2^{2017})$.

5. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a, b, c \in \mathbb{N}}} (a + b + c),$$

where \mathbb{N} denotes the positive integers.

Proposed by: Yang Liu

Answer: $\boxed{49140}$

Note that $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$. The given sum clearly equals $3 \sum_{abc=2310} a$ by symmetry. The inner sum can be rewritten as

$$\sum_{a|2310} a \cdot \tau\left(\frac{2310}{a}\right),$$

as for any fixed a , there are $\tau\left(\frac{2310}{a}\right)$ choices for the integers b, c .

Now consider the function $f(n) = \sum_{a|n} a \cdot \tau\left(\frac{n}{a}\right)$. Therefore, $f = n * \tau$, where n denotes the function $g(n) = n$ and $*$ denotes Dirichlet convolution. As both n and τ are multiplicative, f is also multiplicative.

It is easy to compute that $f(p) = p + 2$ for primes p . Therefore, our final answer is $3(2+2)(3+2)(5+2)(7+2)(11+2) = 49140$.

6. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \dots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

Proposed by: Alexander Katz

Answer: $\boxed{-9}$

Let $Q(x) = x^2 P(x) - 1$. Then $Q(n) = n^2 P(n) - 1 = 0$ for $n = 1, 2, \dots, 2016$, and Q has degree 2017. Thus we may write

$$Q(x) = x^2 P(x) - 1 = (x-1)(x-2)\dots(x-2016)L(x)$$

where $L(x)$ is some linear polynomial. Then $Q(0) = -1 = (-1)(-2)\dots(-2016)L(0)$, so $L(0) = -\frac{1}{2016!}$.

Now note that

$$Q'(x) = x^2 P'(x) + 2xP(x) = \sum_{i=1}^{2016} (x-1)\dots(x-(i-1))(x-(i+1))\dots(x-2016)L(x) + (x-1)(x-2)\dots(x-2016)L'(x)$$

Thus

$$Q'(0) = 0 = L(0) \left(\frac{2016!}{-1} + \frac{2016!}{-2} + \dots + \frac{2016!}{-2016} \right) + 2016!L'(0)$$

whence $L'(0) = L(0) \left(\frac{1}{1} + \frac{1}{2} + \dots + \frac{1}{2016} \right) = -\frac{H_{2016}}{2016!}$, where H_n denotes the n th harmonic number.

As a result, we have $L(x) = -\frac{H_{2016}x+1}{2016!}$. Then

$$Q(2017) = 2017^2 P(2017) - 1 = 2016! \left(-\frac{2017H_{2016} + 1}{2016!} \right)$$

which is $-2017H_{2016} - 1$. Thus

$$P(2017) = \frac{-H_{2016}}{2017}.$$

From which we get $2017P(2017) = -H_{2016}$. It remains to approximate H_{2016} . We alter the well known approximation

$$H_n \approx \int_1^n \frac{1}{x} dx = \log x$$

to

$$H_n \approx 1 + \frac{1}{2} + \int_3^n \frac{1}{x} dx = 1 + \frac{1}{2} + \log(2016) - \log(3) \approx \log(2016) + \frac{1}{2}$$

so that it suffices to lower bound $\log(2016)$. Note that $e^3 \approx 20$, which is close enough for our purposes. Then $e^6 \approx 400 \implies e^7 \approx 1080$, and $e^3 \approx 20 < 2^5 \implies e^{0.6} \ll 2 \implies e^{7.6} < 2016$, so that $\log(2016) > 7.6$. It follows that $H_{2016} \approx \log(2016) + 0.5 = 7.6 + 0.5 > 8$ (of course these are loose estimates, but more than good enough for our purposes). Thus $-9 < 2017P(2017) < -8$, making our answer $\boxed{-9}$.

Alternatively, a well-read contestant might know that $H_n \approx \log n + \gamma$, where $\gamma \approx .577$ is the Euler-Mascheroni constant. The above solution essentially approximates γ as 0.5 which is good enough for our purposes.

7. Determine the largest real number c such that for any 2017 real numbers $x_1, x_2, \dots, x_{2017}$, the inequality

$$\sum_{i=1}^{2016} x_i(x_i + x_{i+1}) \geq c \cdot x_{2017}^2$$

holds.

Proposed by: Pakawut Jiradilok

Answer: $\boxed{-\frac{1008}{2017}}$

Let $n = 2016$. Define a sequence of real numbers $\{p_k\}$ by $p_1 = 0$, and for all $k \geq 1$,

$$p_{k+1} = \frac{1}{4(1-p_k)}.$$

Note that, for every $i \geq 1$,

$$(1 - p_i) \cdot x_i^2 + x_i x_{i+1} + p_{i+1} x_{i+1}^2 = \left(\frac{x_i}{2\sqrt{p_{i+1}}} + \sqrt{p_{i+1}} x_{i+1} \right)^2 \geq 0.$$

Summing from $i = 1$ to n gives

$$\sum_{i=1}^n x_i(x_i + x_{i+1}) \geq -p_{n+1} x_{n+1}^2.$$

One can show by induction that $p_k = \frac{k-1}{2k}$. Therefore, our answer is $-p_{2017} = -\frac{1008}{2017}$.

8. Consider all ordered pairs of integers (a, b) such that $1 \leq a \leq b \leq 100$ and

$$\frac{(a+b)(a+b+1)}{ab}$$

is an integer.

Among these pairs, find the one with largest value of b . If multiple pairs have this maximal value of b , choose the one with largest a . For example choose $(3, 85)$ over $(2, 85)$ over $(4, 84)$. Note that your answer should be an ordered pair.

Proposed by: Alexander Katz

Answer: $\boxed{(35, 90)}$

Firstly note that $\frac{(a+b)(a+b+1)}{ab} = 2 + \frac{a^2+b^2+a+b}{ab}$. Let c be this fraction so that $(a+b)(a+b+1) = ab(c+2)$ for some integers a, b, c . Suppose (a, b) with $a \geq b$ is a solution for some c . Consider the quadratic

$$x^2 - (bc - 1)x + b^2 + b = 0$$

It has one root a , and the other root is therefore $bc - a - 1$. Furthermore the other root can also be expressed as $\frac{b^2+b}{a} \leq \frac{b^2+b}{b+1} = b$, so that $0 < bc - a - 1 \leq b$. In particular, $(b, bc - a - 1)$ is a solution as well.

Thus all solutions (a, b) reduce to a solution where $a = b$, at which point $c = 2 + \frac{2}{a}$. Since a, c are positive integers we thus have $a = 1, 2$, and so $c = \boxed{3, 4}$.

Through this jumping process, we iteratively find the solutions for $c = 3$:

$$(2, 2) \rightarrow (2, 3) \rightarrow (3, 6) \rightarrow (6, 14) \rightarrow (14, 35) \rightarrow (35, 90)$$

and $c = 4$:

$$(1, 2) \rightarrow (2, 6) \rightarrow (6, 21) \rightarrow (21, 77)$$

so that the desired pair is $\boxed{(35, 90)}$.

9. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \geq 2$. Find the smallest positive integer m such that $F_m \equiv 0 \pmod{127}$ and $F_{m+1} \equiv 1 \pmod{127}$.

Proposed by: Sam Korsky

Answer: $\boxed{256}$

First, note that 5 is not a quadratic residue modulo 127. We are looking for the period of the Fibonacci numbers mod 127. Let $p = 127$. We work in \mathbb{F}_{p^2} for the remainder of this proof. Let α and β be the roots of $x^2 - x - 1$. Then we know that $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$. Note that since $x \rightarrow x^p$ is an automorphism and since automorphisms cycle the roots of a polynomial we have that $\alpha^p = \beta$ and $\beta^p = \alpha$. Then $F_p = \frac{\alpha^p - \beta^p}{\alpha - \beta} = -1$ and $F_{p+1} = \frac{\alpha^p \beta - \beta^p \alpha}{\alpha - \beta} = 0$ and similarly we obtain $F_{2p+1} = 1$ and $F_{2p+2} = 0$. Thus since $2p + 2$ is a power of 2 and since the period does not divide $p + 1$, we must have the answer is $2p + 2 = \boxed{256}$.

10. Let \mathbb{N} denote the natural numbers. Compute the number of functions $f : \mathbb{N} \rightarrow \{0, 1, \dots, 16\}$ such that

$$f(x + 17) = f(x) \quad \text{and} \quad f(x^2) \equiv f(x)^2 + 15 \pmod{17}$$

for all integers $x \geq 1$.

Proposed by: Yang Liu

Answer:

By plugging in $x = 0$, we get that $f(0)$ can be either $-1, 2$. As $f(0)$ is unrelated to all other values, we need to remember to multiply our answer by 2 at the end. Similarly, $f(1) = -1$ or 2 .

Consider the graph $x \rightarrow x^2$. It is a binary tree rooted at -1 , and there is an edge $-1 \rightarrow 1$, and a loop $1 \rightarrow 1$. Our first case is $f(1) = -1$. Note that if x, y satisfy $x^2 = y$, then $f(y) \neq 1$. Otherwise, we would have $f(x)^2 = 3 \pmod{17}$, a contradiction as 3 is a nonresidue. So only the 8 leaves can take the value 1. This contributes 2^8 .

For $f(1) = 2$, we can once again propagate down the tree. While it looks like we have 2 choices at each node (for the square roots), this is wrong, as if $f(x) = -2$ and $y^2 = x$, then $f(y) = 0$ is forced.

Given this intuition, let a_n denote the answer for a binary tree of height n where the top is either -2 or 2 . Therefore, $a_1 = 2, a_2 = 5$. You can show the recurrence $a_n = a_{n-1}^2 + 2^{2^n - 4}$. This is because if the top is 2, then we get a contribution of a_{n-1}^2 . If the top is -2 , then both entries below it must be 0. After that, you can show that each of the remaining $2^n - 4$ vertices can be either of 2 possible square roots. Therefore, we get the recurrence as claimed. One can compute that $a_4 = 5777$, so we get the final answer $2(256 + 5777) = 12066$.

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Combinatorics

1. Kelvin the Frog is going to roll three fair ten-sided dice with faces labelled $0, 1, 2, \dots, 9$. First he rolls two dice, and finds the sum of the two rolls. Then he rolls the third die. What is the probability that the sum of the first two rolls equals the third roll?

Proposed by: Yang Liu

Answer: $\boxed{\frac{11}{200}}$

First, there are $10^3 = 1000$ triples (a, b, c) . Now, we should count how many of these triples satisfy $a + b = c$. If $c = 0$, we get 1 triple $(0, 0, 0)$. If $c = 1$, we get two triples $(1, 0, 1)$ and $(0, 1, 1)$. Continuing, this gives that the total number of triples is $1 + 2 + \dots + 10 = 55$. Therefore, our final answer is $\frac{55}{1000} = \frac{11}{200}$.

2. How many ways are there to insert $+$'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

Proposed by: Yang Liu

Answer: $\boxed{2002}$

Note that because there are 15 1's, no matter how we insert $+$'s, the result will always be a multiple of 3. Therefore, it suffices to consider adding $+$'s to get a multiple of 10. By looking at the units digit, we need the number of summands to be a multiple of 10. Because there are only 15 digits in our number, we have to have exactly 10 summands. Therefore, we need to insert 9 $+$'s in 14 possible positions, giving an answer of $\binom{14}{9} = 2002$.

3. There are 2017 jars in a row on a table, initially empty. Each day, a nice man picks ten consecutive jars and deposits one coin in each of the ten jars. Later, Kelvin the Frog comes back to see that N of the jars all contain the same positive integer number of coins (i.e. there is an integer $d > 0$ such that N of the jars have exactly d coins). What is the maximum possible value of N ?

Proposed by: Kevin Sun

Answer: $\boxed{2014}$

Label the jars $1, 2, \dots, 2017$. I claim that the answer is 2014. To show this, we need both a construction and an upper bound. For the construction, for $1 \leq i \leq 201$, put a coin in the jars $10i + 1, 10i + 2, \dots, 10i + 10$. After this, each of the jars $1, 2, \dots, 2010$ has exactly one coin. Now, put a coin in each of the jars $2008, 2009, \dots, 2017$. Now, the jars $1, 2, \dots, 2007, 2011, 2012, \dots, 2017$ all have exactly one coin. This gives a construction for $N = 2014$ (where $d = 1$).

Now, we show that this is optimal. Let $c_1, c_2, \dots, c_{2017}$ denote the number of coins in each of the jars. For $1 \leq j \leq 10$, define

$$s_j = c_j + c_{j+10} + c_{j+20} + \dots$$

Note that throughout the process, $s_1 = s_2 = \dots = s_j$. It is also easy to check that the sums s_1, s_2, \dots, s_7 each involve 202 jars, while the sums s_8, s_9, s_{10} each involve 201 jars.

Call a jar *good* if it has exactly d coins. If there are at least 2015 good jars, then one can check that it is forced that at least one of s_1, s_2, \dots, s_7 only involves good jars, and similarly, at least one of s_8, s_9, s_{10} only involves good jars. But this would mean that $202d = 201d$ as all s_i are equal, contradiction.

4. Sam spends his days walking around the following 2×2 grid of squares.

1	2
4	3

Say that two squares are adjacent if they share a side. He starts at the square labeled 1 and every second walks to an adjacent square. How many paths can Sam take so that the sum of the numbers on every square he visits in his path is equal to 20 (not counting the square he started on)?

Proposed by: Sam Korsky

Answer: 167

Note that on the first step, Sam can either step on 2 or 4. On the second step, Sam can either step on 1 or 3, regardless of whether he is on 2 or 4. Now, for example, say that Sam takes 8 steps. His total sum will be $2 + 1 + 2 + 1 + 2 + 1 + 2 + 1 + 2a$, where a is the number of times that he decides to step on the larger number of his two choices. Solving gives $a = 4$. As he took 8 steps, this gives him $\binom{8}{4} = 70$ ways in this case.

We can follow a similar approach by doing casework on the number of steps he takes. I will simply list them out here for brevity. For 8 steps, we get $\binom{8}{4} = 70$. For 9 steps, we get $\binom{9}{3} = 84$. For 12 steps, we get a contribution on $\binom{12}{1} = 12$. For 13 steps, we get a contribution of $\binom{13}{0} = 1$. Therefore, the final answer is $70 + 84 + 12 + 1 = 167$.

5. Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if d_i denotes the i th digit, then $d_i \leq d_{i+1}$ for at most one value of i . For example, Kelvin likes the numbers 43210, 132, and 3, but not the numbers 1337 and 123. How many 5-digit numbers does Kelvin like?

Proposed by: Alexander Katz

Answer: 14034

Suppose first that no digit violates the constraint; i.e. the digits are in strictly decreasing order. There are $\binom{10}{5}$ ways to choose the digits of the number, and each set of digits can be arranged in exactly one way, so there are $\binom{10}{5}$ such numbers.

We now perform casework on which digit violates the constraint. If it is the final digit, the first four digits must be arranged in decreasing order, which there are $\binom{10}{4}$ ways to do. The final digit can then be any digit, but we have overcounted the ones in which the number is in fully decreasing order (this can happen, for example, if the first 4 digits we chose were 5, 4, 3, and 2 – a last digit of 1 was already counted in the first case). Therefore, there are $\binom{10}{4}\binom{10}{1} - 252$ new numbers in this case.

If the offending digit is second from the right, the first 3 digits must be decreasing, as must the last 2 digits. There are $\binom{10}{3}\binom{10}{2}$ ways to do this. As before, we overcount the case where the second digit from the right is not actually an offender, so we again overcount the case where all 5 digits decrease. Hence there are $\binom{10}{3}\binom{10}{2} - 252$ new numbers in this case.

The case where the third digit is the offender is identical to the previous case, so there are another $\binom{10}{3}\binom{10}{2} - 252$ numbers to account for. The final case is when the second digit is the offending digit, in which case there are $\binom{10}{4}$ ways to choose the final 4 digits, but only 9 ways to choose the opening digit (as 0 cannot be a leading digit). Accounting for the usual overcounting, our final answer is

$$252 + \left[\binom{10}{4} \binom{10}{1} - 252 \right] + 2 \left[\binom{10}{3} \binom{10}{2} - 252 \right] + \left[\binom{10}{4} \cdot 9 - 252 \right]$$

which is easily calculated as 14034.

6. Emily starts with an empty bucket. Every second, she either adds a stone to the bucket or removes a stone from the bucket, each with probability $\frac{1}{2}$. If she wants to remove a stone from the bucket and the bucket is currently empty, she merely does nothing for that second (still with probability $\frac{1}{2}$). What is the probability that after 2017 seconds her bucket contains exactly 1337 stones?

Proposed by: Sam Korsky

Answer: $\boxed{\frac{\binom{2017}{340}}{2^{2017}}}$

Replace 2017 with n and 1337 with k and denote the general answer by $f(n, k)$. I claim that $f(n, k) = \frac{\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}}{2^n}$. We proceed by induction on n .

The claim is obviously true for $n = 0$ since $f(0, 0) = 1$. Moreover, we have that $f(n, 0) = \frac{1}{2}f(n-1, 0) + \frac{1}{2}f(n-1, 1)$ and $f(n, k) = \frac{1}{2}f(n-1, k-1) + \frac{1}{2}f(n-1, k+1)$ for $k > 0$ so the inductive step is immediate by Pascal's identity. This concludes the proof.

7. There are 2017 frogs and 2017 toads in a room. Each frog is friends with exactly 2 distinct toads. Let N be the number of ways to pair every frog with a toad who is its friend, so that no toad is paired with more than one frog. Let D be the number of distinct possible values of N , and let S be the sum of all possible values of N . Find the ordered pair (D, S) .

Proposed by: Yang Liu

Answer: $\boxed{(1009, 2^{1009} - 2)}$

I claim that N can equal 0 or 2^i for $1 \leq i \leq 1008$. We prove this now. Note that the average number of friends a toad has is also 2. If there is a toad with 0 friends, then clearly $N = 0$. If a toad has 1 friend, then it must be paired with its only friend, so we have reduced to a smaller case. Otherwise, all toads and frogs have exactly degree 2, so the graph is a union of cycles. Each cycle can be paired off in exactly two ways. The number of cycles can range anywhere from 1 to 1008, and this completes the proof.

To construct all $N = 2^1, 2^2, \dots, 2^{1008}$, we can simply let our graph be a union of i cycles, which would have 2^i matchings. Clearly we can choose any $i = 1, 2, \dots, 1008$.

Therefore, $D = 1009$ and $S = 2^1 + 2^2 + \dots + 2^{1008} = 2^{1009} - 2$.

8. Kelvin and 15 other frogs are in a meeting, for a total of 16 frogs. During the meeting, each pair of distinct frogs becomes friends with probability $\frac{1}{2}$. Kelvin thinks the situation after the meeting is *cool* if for each of the 16 frogs, the number of friends they made during the meeting is a multiple of 4. Say that the probability of the situation being cool can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime. Find a .

Proposed by: Yang Liu

Consider the multivariate polynomial

$$\prod_{1 \leq i < j \leq 16} (1 + x_i x_j)$$

We're going to filter this by summing over all 4^{16} 16-tuples $(x_1, x_2, \dots, x_{16})$ such that $x_j = \pm 1, \pm i$. Most of these evaluate to 0 because $i^2 = (-i)^2 = -1$, and $1 \cdot -1 = -1$. If you do this filtering, you get the following 4 cases:

Case 1: Neither of i or $-i$ appears. Then the only cases we get are when all the x_j are 1, or they're all -1 . Total is 2^{121} . ($120 = \binom{16}{2}$.)

Case 2: i appears, but $-i$ does not. Then all the remaining x_j must be all 1 or all -1 . This contributes a sum of $(1+i)^{15} \cdot 2^{105} + (1-i)^{15} \cdot 2^{105} = 2^{113}$. i can be at any position, so we get $16 \cdot 2^{113}$.

Case 3: $-i$ appears, but i does not. Same contribution as above. $16 \cdot 2^{113}$.

Case 4: Both i and $-i$ appear. Then all the rest of the x_j must be all 1 or all -1 . This contributes a sum of $2 \cdot (1 + i(-i)) \cdot (1 + i)^{14} \cdot (1 - i)^{14} \cdot 2^{91} = 2^{107}$. i and $-i$ can appear in $16 \cdot 15$ places, so we get $240 \cdot 2^{107}$.

So the final answer is this divided a factor for our filter. ($4^{16} = 2^{32}$.) So our final answer is $\frac{2^{89} + 16 \cdot 2^{82} + 240 \cdot 2^{75}}{2^{120}} = \frac{1167}{2^{41}}$.

Therefore, the answer is 1167.

9. Let m be a positive integer, and let T denote the set of all subsets of $\{1, 2, \dots, m\}$. Call a subset S of T δ -good if for all $s_1, s_2 \in S$, $s_1 \neq s_2$, $|\Delta(s_1, s_2)| \geq \delta m$, where Δ denotes symmetric difference (the symmetric difference of two sets is the set of elements that is in exactly one of the two sets). Find the largest possible integer s such that there exists an integer m and a $\frac{1024}{2047}$ -good set of size s .

Proposed by: Yang Liu

Answer: 2048

Let $n = |S|$. Let the sets in S be s_1, s_2, \dots, s_n . We bound the sum $\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)|$ in two ways. On one hand, by the condition we have the obvious bound

$$\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)| \geq \binom{n}{2} \delta m.$$

On the other hand, for $1 \leq i \leq m$, let $t_i = |\{1 \leq j \leq n : i \in s_j\}|$. Then it is clear that

$$\sum_{1 \leq i < j \leq n} |\Delta(s_i, s_j)| = \sum_{k=1}^m t_k(n - t_k) \leq \frac{n^2}{4} m$$

by AM-GM. Therefore, we get the bound

$$\binom{n}{2} \delta m \leq \frac{n^2}{4} m \Rightarrow n \leq \frac{2\delta}{2\delta - 1} = 2048.$$

To give a construction with $n = 2048$, take $m = 2047$. For the rest of this construction, we will be interpreting the integers $1, 2, \dots, m$ as 11-digit integers in binary. Given this interpretation, define a *dot product* $x \odot y$ of two positive integers $0 \leq x, y \leq m$ the following way. If $x = (x_1 x_2 \dots x_{11})_2, y = (y_1 y_2 \dots y_{11})_2$ in binary, then

$$x \odot y = \sum x_i y_i \pmod{2}.$$

Now we can define the sets $s_1, s_2, \dots, s_{2048}$. Define

$$s_i = \{1 \leq j \leq m : (i - 1) \odot j = 1\}.$$

A computation shows that this construction works.

Some notes: here is the motivation behind the construction. We are treating the integers $0, 1, \dots, m$ as the vector space $V = \mathbb{F}_2^{11}$, and the sets s_i correspond to linear functionals $f_i : V \rightarrow \mathbb{F}_2$. In particular, the function $f_i : V \rightarrow \mathbb{F}_2$ is simply defined as $f_i(x) = (i - 1) \odot x$, which one can easily check to be linear. This construction corresponds to Hadamard matrices of size 2^{11} .

10. Compute the number of possible words $w = w_1 w_2 \dots w_{100}$ satisfying:

- w has exactly 50 A 's and 50 B 's (and no other letters).
- For $i = 1, 2, \dots, 100$, the number of A 's among w_1, w_2, \dots, w_i is at most the number of B 's among w_1, w_2, \dots, w_i .
- For all $i = 44, 45, \dots, 57$, if w_i is an B , then w_{i+1} must be an B .

Proposed by: Allen Liu

Call the last property in the problem statement $P(i, j)$ where in the statement $i = 44, j = 57$. We show that the number of words satisfying the first two conditions and $P(m, m+k)$ is the same independent of m (assuming k is fixed). It suffices to show that the number of words satisfying $P(m-1, m+k-1)$ is the same as the number of words satisfying $P(m, m+k)$. Construct a bijection as follows: for a word satisfying $P(m-1, m+k-1)$, if it satisfies $P(m, m+k)$, leave it as is. Otherwise, the character in position $m+k$ must be B and the character in position $m+k+1$ must be A . In this case, move these two characters to positions $m-1, m$ (shifting all other characters back). It is not difficult to verify that this is indeed a bijection.

Thus, the problem is now equivalent to computing the number of words satisfying the first two conditions and $P(1, 14)$. However, this condition simply means that the first 15 characters must be B . Now we are essentially counting the number of paths from $(15, 0)$ to $(50, 50)$ that don't go above $y = x$. There is a bijection between paths from $(15, 0)$ to $(50, 50)$ that do cross $y = x$ and paths from $(15, 0)$ to $(49, 51)$ (using the standard reflection argument). Thus the answer is $\binom{85}{35} - \binom{85}{34}$.

February 2017

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Geometry

1. Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagonals AC and BD intersect at P . Compute $\frac{AP}{CP}$.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{2}{5}}$

Note that $\triangle APB \sim \triangle DPC$ so $\frac{AP}{AB} = \frac{DP}{CD}$. Similarly, $\triangle BPC \sim \triangle APD$ so $\frac{CP}{BC} = \frac{DP}{DA}$. Dividing these two equations yields

$$\frac{AP}{CP} = \frac{AB \cdot DA}{BC \cdot CD} = \boxed{\frac{2}{5}}$$

2. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC . Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

Proposed by: Sam Korsky

Answer: $\boxed{\frac{1323}{26}}$

There are three cases: ℓ intersects AB, AC , ℓ intersects AB, BC , and ℓ intersects AC, BC . These cases are essentially identical, so let ℓ intersect segment AB at M and segment AC at N .

Then the condition is equivalent to

$$AM + MN + AN = MB + BC + CN + MN$$

$$AM + AN = MB + CN + 15$$

but $AN + CN = 14$ and $AM + BM = 13$, so that

$$BM + CN = 27 - AM - AN = AM - AN - 15$$

implying that $AM + AN = 21$.

Now let $\angle BAC = \theta$ for convenience, so that

$$[AMN] = \frac{1}{2} AM \cdot AN \cdot \sin \theta$$

which is maximized when $AM = AN = \frac{21}{2}$. Further we can easily calculate $\sin \theta = \frac{12}{13}$ (e.g. by LOC); note that this is why the area is maximized in this case (we want to maximize $\sin \theta$, which is equivalent to maximizing θ , so θ should be opposite the largest side). Our answer is thus

$$\frac{1}{2} \cdot \frac{21}{2} \cdot \frac{21}{2} \cdot \frac{12}{13} = \frac{1323}{26}$$

Alternatively we could also calculate

$$\begin{aligned} [AMN] &= [ABC] \cdot \frac{AM}{AB} \cdot \frac{AN}{AC} \\ &= 84 \cdot \frac{21}{13} \cdot \frac{21}{14} \end{aligned}$$

which gives the same answer.

3. Let S be a set of 2017 distinct points in the plane. Let R be the radius of the smallest circle containing all points in S on either the interior or boundary. Also, let D be the longest distance between two of the points in S . Let a, b are real numbers such that $a \leq \frac{D}{R} \leq b$ for all possible sets S , where a is as large as possible and b is as small as possible. Find the pair (a, b) .

Proposed by: Yang Liu

Answer: $\boxed{(\sqrt{3}, 2)}$

It is easy to verify that the smallest circle enclosing all the points will either have some 2 points in S as its diameter, or will be the circumcircle of some 3 points in S who form an acute triangle.

Now, clearly $\frac{D}{R} \leq 2$. Indeed consider the two farthest pair of points S_1, S_2 . Then $D = |S_1S_2| \leq 2R$, as both points S_1, S_2 are inside a circle of radius R . We can achieve this upper bound by taking S to have essentially only 2 points, and the remaining 2015 points in S are at the same place as these 2 points.

For the other direction, I claim $\frac{D}{R} \geq \sqrt{3}$. Recall that the smallest circle is either the circumcircle of 3 points, or has some 2 points as the diameter. In the latter case, say the diameter is S_1S_2 . Then $D \geq |S_1S_2| = 2R$, so $\frac{D}{R} \geq 2$ in that case. Now say the points S_1, S_2, S_3 are the circumcircle. WLOG, say that S_1S_2 is the longest side of the triangle. As remarked above, we can assume this triangle is acute. Therefore, $\frac{\pi}{3} \leq \angle S_1S_3S_2 \leq \frac{\pi}{2}$. By the Law of Sines we have that

$$D \geq |S_1S_2| = 2R \sin \angle S_1S_3S_2 \geq 2R \sin \frac{\pi}{3} = R\sqrt{3}.$$

This completes the proof. To achieve equality, we can take S to have 3 points in the shape of an equilateral triangle.

4. Let $ABCD$ be a convex quadrilateral with $AB = 5, BC = 6, CD = 7$, and $DA = 8$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 - PQ^2$.

Proposed by: Sam Korsky

Answer: $\boxed{13}$

Draw in the diagonals of the quad and use the median formula three times to get MN^2 in terms of the diagonals. Do the same for PQ^2 and subtract, the diagonal length terms disappear and the answer is

$$\frac{BC^2 + DA^2 - AB^2 - CD^2}{2} = \boxed{13}$$

5. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD . Let R_1, R_2, R_3, R_4 be the circumradii of triangles APB, BPC, CPD, DPA respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4 .

Proposed by: Sam Korsky

Answer: $\boxed{19}$

Note that $\angle APB = 180^\circ - \angle BPC = \angle CPD = 180^\circ - \angle DPA$ so $\sin APB = \sin BPC = \sin CPD = \sin DPA$. Now let ω touch sides AB, BC, CD, DA at E, F, G, H respectively. Then $AB + CD = AE + BF + CG + DH = BC + DA$ so

$$\frac{AB}{\sin APB} + \frac{CD}{\sin CPD} = \frac{BC}{\sin BPC} + \frac{DA}{\sin DPA}$$

and by the Extended Law of Sines this implies

$$2R_1 + 2R_3 = 2R_2 + 2R_4$$

which immediately yields $R_4 = R_1 + R_3 - R_2 = \boxed{19}$.

6. In convex quadrilateral $ABCD$ we have $AB = 15$, $BC = 16$, $CD = 12$, $DA = 25$, and $BD = 20$. Let M and γ denote the circumcenter and circumcircle of $\triangle ABD$. Line CB meets γ again at F , line AF meets MC at G , and line GD meets γ again at E . Determine the area of pentagon $ABCDE$.

Proposed by: Evan Chen

Answer: $\boxed{396}$

Note that $\angle ADB = \angle DCB = 90^\circ$ and $BC \parallel AD$. Now by Pascal theorem on $DDEBFA$ implies that B, M, E are collinear. So $[ADE] = [ABD] = 150$ and $[BCD] = 96$, so the total area is 396.

7. Let ω and Γ be circles such that ω is internally tangent to Γ at a point P . Let AB be a chord of Γ tangent to ω at a point Q . Let $R \neq P$ be the second intersection of line PQ with Γ . If the radius of Γ is 17, the radius of ω is 7, and $\frac{AQ}{BQ} = 3$, find the circumradius of triangle AQR .

Proposed by: Sam Korsky

Answer: $\boxed{\sqrt{170}}$

Let r denote the circumradius of triangle AQR . By Archimedes Lemma, R is the midpoint of arc AB of Γ . Therefore $\angle RAQ = \angle RPB = \angle RPA$ so $\triangle RAQ \sim \triangle RPA$. By looking at the similarity ratio between the two triangles we have

$$\frac{r}{17} = \frac{AQ}{AP}$$

Now, let AP intersect ω again at $X \neq P$. By homothety we have $XQ \parallel AR$ so

$$\frac{AX}{AP} = 1 - \frac{PQ}{PR} = 1 - \frac{7}{17} = \frac{10}{17}$$

But we also know

$$AX \cdot AP = AQ^2$$

so

$$\frac{10}{17}AP^2 = AQ^2$$

Thus

$$\frac{r}{17} = \frac{AQ}{AP} = \sqrt{\frac{10}{17}}$$

so we compute $r = \sqrt{170}$ as desired.

8. Let ABC be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{17+\sqrt{51}}{34}}$

Letting I and O denote the incenter and circumcenter of triangle ABC we have by the triangle inequality that

$$AO \leq AI + OI \implies R \leq \frac{r}{\sin \frac{A}{2}} + \sqrt{R(R-2r)}$$

and by plugging in our values for r and R we get

$$\sin \frac{A}{2} \leq \frac{17 + \sqrt{51}}{34}$$

as desired. Equality holds when ABC is isosceles and I lies between A and O .

9. Let ABC be a triangle, and let $BCDE, CAFG, ABHI$ be squares that do not overlap the triangle with centers X, Y, Z respectively. Given that $AX = 6, BY = 7,$ and $CZ = 8,$ find the area of triangle $XYZ.$

Proposed by: Sam Korsky

Answer: $\boxed{\frac{21\sqrt{15}}{4}}$

By the degenerate case of Von Aubel's Theorem we have that $YZ = AX = 6$ and $ZX = BY = 7$ and $XY = CZ = 8$ so it suffices to find the area of a $6 - 7 - 8$ triangle which is given by $\boxed{\frac{21\sqrt{15}}{4}}.$

To prove that $AX = YZ,$ note that by LoC we get

$$YX^2 = \frac{b^2}{2} + \frac{c^2}{2} + bc \sin \angle A$$

and

$$\begin{aligned} AX^2 &= b^2 + \frac{a^2}{2} - ab(\cos \angle C - \sin \angle C) \\ &= c^2 + \frac{a^2}{2} - ac(\cos \angle B - \sin \angle B) \\ &= \frac{b^2 + c^2 + a(b \sin \angle C + c \sin \angle B)}{2} \\ &= \frac{b^2}{2} + \frac{c^2}{2} + ah \end{aligned}$$

where h is the length of the A -altitude of triangle $ABC.$ In these calculations we used the well-known fact that $b \cos \angle C + c \cos \angle B = a$ which can be easily seen by drawing in the A -altitude. Then since $bc \sin \angle A$ and ah both equal twice the area of triangle $ABC,$ we are done.

10. Let $ABCD$ be a quadrilateral with an inscribed circle $\omega.$ Let I be the center of ω let $IA = 12, IB = 16, IC = 14,$ and $ID = 11.$ Let M be the midpoint of segment $AC.$ Compute $\frac{IM}{IN},$ where N is the midpoint of segment $BD.$

Proposed by: Sam Korsky

Answer: $\boxed{\frac{21}{22}}$

Let points W, X, Y, Z be the tangency points between ω and lines AB, BC, CD, DA respectively. Now invert about $\omega.$ Then A', B', C', D' are the midpoints of segments ZW, WX, XY, YZ respectively. Thus by Varignon's Theorem $A'B'C'D'$ is a parallelogram. Then the midpoints of segments $A'C'$ and $B'D'$ coincide at a point $P.$ Note that figure $IA'PC'$ is similar to figure $ICMA$ with similitude ratio $\frac{r^2}{IA \cdot IC}$ where r is the radius of $\omega.$ Similarly figure $IB'PD'$ is similar to figure $IDMB$ with similitude ratio $\frac{r^2}{IB \cdot ID}.$ Therefore

$$IP = \frac{r^2}{IA \cdot IC} \cdot IM = \frac{r^2}{IB \cdot ID} \cdot IN$$

which yields

$$\frac{IM}{IN} = \frac{IA \cdot IC}{IB \cdot ID} = \frac{12 \cdot 14}{16 \cdot 11} = \boxed{\frac{21}{22}}$$

February 2017

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Guts

1. [4] A random number generator will always output 7. Sam uses this random number generator once. What is the expected value of the output?

Proposed by: Sam Korsky

Answer:

The only output is 7, so the expected value is 7.

2. [4] Let A, B, C, D, E, F be 6 points on a circle in that order. Let X be the intersection of AD and BE , Y is the intersection of AD and CF , and Z is the intersection of CF and BE . X lies on segments BZ and AY and Y lies on segment CZ . Given that $AX = 3$, $BX = 2$, $CY = 4$, $DY = 10$, $EZ = 16$, and $FZ = 12$, find the perimeter of triangle XYZ .

Proposed by: Sam Korsky

Answer:

Let $XY = z$, $YZ = x$, and $ZX = y$. By Power of a Point, we have that

$$3(z + 10) = 2(y + 16), 4(x + 12) = 10(z + 3), \text{ and } 12(x + 4) = 16(y + 2).$$

Solving this system gives $XY = \frac{11}{3}$ and $YZ = \frac{14}{3}$ and $ZX = \frac{9}{2}$. Therefore, our answer is $XY + YZ + ZX = \frac{77}{6}$.

3. [4] Find the number of pairs of integers (x, y) such that $x^2 + 2y^2 < 25$.

Proposed by: Allen Liu

Answer:

We do casework on y .

If $y = 0$, we have $x^2 < 25$, so we get 9 values of x . If $y = \pm 1$, then $x^2 < 23$, so we still have 9 values of x . If $y = \pm 2$, we have $x^2 < 17$, so we have 9 values of x . If $y = \pm 3$, we have $x^2 < 7$, we get 5 values of x .

Therefore, the final answer is $9 + 2(9 + 9 + 5) = 55$.

4. [4] Find the number of ordered triples of nonnegative integers (a, b, c) that satisfy

$$(ab + 1)(bc + 1)(ca + 1) = 84.$$

Proposed by: Evan Chen

Answer:

The solutions are $(0, 1, 83)$ and $(1, 2, 3)$ up to permutation. First, we do the case where at least one of a, b, c is 0. WLOG, say $a = 0$. Then we have $1 + bc = 84 \implies bc = 83$. As 83 is prime, the only solution is $(0, 1, 83)$ up to permutation.

Otherwise, we claim that at least one of a, b, c is equal to 1. Otherwise, all are at least 2, so $(1 + ab)(1 + bc)(1 + ac) \geq 5^3 > 84$. So WLOG, set $a = 1$. We now need $(b + 1)(c + 1)(bc + 1) = 84$. Now, WLOG, say $b \leq c$. If $b = 1$, then $(c + 1)^2 = 42$, which has no solution. If $b \geq 3$, then $(b + 1)(c + 1)(bc + 1) \geq 4^2 \cdot 10 = 160 > 84$. So we need $b = 2$. Then we need $(c + 1)(2c + 1) = 21$. Solving this gives $c = 3$, for the solution $(1, 2, 3)$.

Therefore, the answer is $6 + 6 = 12$.

5. [6] Find the number of ordered triples of positive integers (a, b, c) such that

$$6a + 10b + 15c = 3000.$$

Proposed by: Yang Liu

Answer:

Note that $6a$ must be a multiple of 5, so a must be a multiple of 5. Similarly, b must be a multiple of 3, and c must be a multiple of 2.

Set $a = 5A, b = 3B, c = 2C$. Then the equation reduces to $A + B + C = 100$. This has $\binom{99}{2} = 4851$ solutions.

6. [6] Let $ABCD$ be a convex quadrilateral with $AC = 7$ and $BD = 17$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 + PQ^2$

Proposed by: Sam Korsky

Answer:

$MPNQ$ is a parallelogram whose side lengths are 3.5 and 8.5 so the sum of squares of its diagonals is $\frac{7^2+17^2}{2} = \text{169}$

7. [6] An ordered pair of sets (A, B) is *good* if A is not a subset of B and B is not a subset of A . How many ordered pairs of subsets of $\{1, 2, \dots, 2017\}$ are good?

Proposed by: Alexander Katz

Answer:

Firstly, there are 4^{2017} possible pairs of subsets, as each of the 2017 elements can be in neither subset, in A only, in B only, or in both.

Now let us count the number of pairs of subsets for which A is a subset of B . Under these conditions, each of the 2017 elements could be in neither subset, in B only, or in both A and B . So there are 3^{2017} such pairs.

By symmetry, there are also 3^{2017} pairs of subsets where B is a subset of A . But this overcounts the pairs in which A is a subset of B and B is a subset of A , i.e. $A = B$. There are 2^{2017} such subsets.

Thus, in total, there are $4^{2017} - 2 \cdot 3^{2017} + 2^{2017}$ good pairs of subsets.

8. [6] You have 128 teams in a single elimination tournament. The Engineers and the Crimson are two of these teams. Each of the 128 teams in the tournament is equally strong, so during each match, each team has an equal probability of winning.

Now, the 128 teams are randomly put into the bracket.

What is the probability that the Engineers play the Crimson sometime during the tournament?

Proposed by: Yang Liu

Answer:

There are $\binom{128}{2} = 127 \cdot 64$ pairs of teams. In each tournament, 127 of these pairs play.

By symmetry, the answer is $\frac{127}{127 \cdot 64} = \frac{1}{64}$.

9. [7] Jeffrey writes the numbers 1 and $100000000 = 10^8$ on the blackboard. Every minute, if x, y are on the board, Jeffrey replaces them with

$$\frac{x+y}{2} \quad \text{and} \quad 2 \left(\frac{1}{x} + \frac{1}{y} \right)^{-1}.$$

After 2017 minutes the two numbers are a and b . Find $\min(a, b)$ to the nearest integer.

Proposed by: Yang Liu

Answer: 10000

Note that the product of the integers on the board is a constant. Indeed, we have that

$$\frac{x+y}{2} \cdot 2 \left(\frac{1}{x} + \frac{1}{y} \right)^{-1} = xy.$$

Therefore, we expect that the answer to the problem is approximately $\sqrt{1 \cdot 10^8} = 10^4$.

To be more rigorous, we have to show that the process indeed converges quickly enough. To show this, we bound the difference between the integers on the board at time i . Say that at time i , the integers on the board are $a_i < b_i$. Note that

$$\begin{aligned} d_{i+1} = b_{i+1} - a_{i+1} &= \frac{a_i + b_i}{2} - 2 \left(\frac{1}{a_i} + \frac{1}{b_i} \right)^{-1} = \frac{(a_i - b_i)^2}{2(a_i + b_i)} \\ &< \frac{b_i - a_i}{2} = \frac{d_i}{2}. \end{aligned}$$

The inequality at the end follows from that obvious fact that $b_i - a_i < b_i + a_i$. Therefore, $d_{i+1} \leq \frac{d_i}{2}$, so $d_{2017} < \frac{10^8}{2^{2017}}$, which is extremely small. So the difference is essentially 0 at time 2017, which completes the argument.

10. [7] Let ABC be a triangle in the plane with $AB = 13, BC = 14, AC = 15$. Let M_n denote the smallest possible value of $(AP^n + BP^n + CP^n)^{\frac{1}{n}}$ over all points P in the plane. Find $\lim_{n \rightarrow \infty} M_n$.

Proposed by: Yang Liu

Answer: 65/8

Let R denote the circumradius of triangle ABC . As ABC is an acute triangle, it isn't hard to check that for any point P , we have either $AP \geq R, BP \geq R$, or $CP \geq R$. Also, note that if we choose $P = O$ (the circumcenter) then $(AP^n + BP^n + CP^n) = 3 \cdot R^n$. Therefore, we have the inequality

$$R \leq \min_{P \in \mathbb{R}^2} (AP^n + BP^n + CP^n)^{\frac{1}{n}} \leq (3R^n)^{\frac{1}{n}} = R \cdot 3^{\frac{1}{n}}.$$

Taking $n \rightarrow \infty$ yields

$$R \leq \lim_{n \rightarrow \infty} M_n \leq R$$

(as $\lim_{n \rightarrow \infty} 3^{\frac{1}{n}} = 1$), so the answer is $R = \frac{65}{8}$.

11. [7] Consider the graph in 3-space of

$$0 = xyz(x+y)(y+z)(z+x)(x-y)(y-z)(z-x).$$

This graph divides 3-space into N connected regions. What is N ?

Proposed by: Evan Chen

Answer: 48

Note that reflecting for each choice of sign for x, y, z , we get new regions. Therefore, we can restrict to the case where $x, y, z > 0$. In this case, the sign of the expression only depends on $(x-y)(y-z)(z-x)$. It is easy to see that for this expression, every one of the $3! = 6$ orderings for $\{x, y, z\}$ contributes a region.

Therefore, our answer is $2^3 \cdot 3! = 48$.

12. [7] In a certain college containing 1000 students, students may choose to major in exactly one of math, computer science, finance, or English. The *diversity ratio* $d(s)$ of a student s is defined as number of students in a different major from s divided by the number of students in the same major as s (including s). The *diversity* D of the college is the sum of all the diversity ratios $d(s)$.

Determine all possible values of D .

Proposed by: Evan Chen

Answer: $\boxed{\{0, 1000, 2000, 3000\}}$

It is easy to check that if n majors are present, the diversity is $1000(n - 1)$. Therefore, taking $n = 1, 2, 3, 4$ gives us all possible answers.

13. [9] The game of Penta is played with teams of five players each, and there are five roles the players can play. Each of the five players chooses two of five roles they wish to play. If each player chooses their roles randomly, what is the probability that each role will have exactly two players?

Proposed by: Alexander Katz

Consider a graph with five vertices corresponding to the roles, and draw an edge between two vertices if a player picks both roles. Thus there are exactly 5 edges in the graph, and we want to find the probability that each vertex has degree 2. In particular, we want to find the probability that the graph is composed entirely of cycles.

Thus there are two cases. The first case is when the graph is itself a 5-cycle. There are $4!$ ways to choose such a directed cycle (pick an arbitrary vertex A and consider a vertex it connects to, etc.), and thus $\frac{4!}{2} = 12$ ways for the undirected graph to be a 5-cycle. Now, there are $5!$ ways to assign the edges in this cycle to people, giving a total contribution of $12 \cdot 5!$.

The second case is when the graph is composed of a 2-cycle and a 3-cycle, which only requires choosing the two vertices to be the 2-cycle, and so there are $\binom{5}{2} = 10$ ways. To assign the players to edges, there are $\binom{5}{2} = 10$ ways to assign the players to the 2-cycle. For the 3-cycle, any of the $3! = 6$ permutations of the remaining 3 players work. The total contribution is $10 \cdot 10 \cdot 6$.

Therefore, our answer is

$$\frac{12 \cdot 120 + 10 \cdot 10 \cdot 6}{10^5} = \boxed{\frac{51}{2500}}.$$

14. [9] Mrs. Toad has a class of 2017 students, with unhappiness levels $1, 2, \dots, 2017$ respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly 15 groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all 15 groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into 15 groups?

Proposed by: Yang Liu

One can show that the optimal configuration is $\{1\}, \{2\}, \dots, \{14\}, \{15, \dots, 2017\}$. This would give us an answer of $1 + 2 + \dots + 14 + \frac{15+2017}{2} = 105 + 1016 = 1121$.

15. [9] Start by writing the integers $1, 2, 4, 6$ on the blackboard. At each step, write the smallest positive integer n that satisfies both of the following properties on the board.

- n is larger than any integer on the board currently.
- n cannot be written as the sum of 2 distinct integers on the board.

Find the 100-th integer that you write on the board. Recall that at the beginning, there are already 4 integers on the board.

Proposed by: Yang Liu

Answer: 388

The sequence goes

$$1, 2, 4, 6, 9, 12, 17, 20, 25, \dots$$

Common differences are 5, 3, 5, 3, 5, 3, \dots , starting from 12. Therefore, the answer is $12 + 47 \times 8 = 388$.

16. [9] Let a and b be complex numbers satisfying the two equations

$$\begin{aligned}a^3 - 3ab^2 &= 36 \\ b^3 - 3ba^2 &= 28i.\end{aligned}$$

Let M be the maximum possible magnitude of a . Find all a such that $|a| = M$.

Proposed by: Alexander Katz

Answer: $3, -\frac{3}{2} + \frac{3i\sqrt{3}}{2}, -\frac{3}{2} - \frac{3i\sqrt{3}}{2}$

Notice that

$$\begin{aligned}(a - bi)^3 &= a^3 - 3a^2bi - 3ab^2 + b^3i \\ &= (a^3 - 3ab^2) + (b^3 - 3ba^2)i \\ &= 36 + i(28i) \\ &= 8\end{aligned}$$

so that $a - bi = 2 + i$. Additionally

$$\begin{aligned}(a + bi)^3 &= a^3 + 3a^2bi - 3ab^2 - b^3i \\ &= (a^3 - 3ab^2) - (b^3 - 3ba^2)i \\ &= 36 - i(28i) \\ &= 64\end{aligned}$$

It follows that $a - bi = 2\omega$ and $a + bi = 4\omega'$ where ω, ω' are third roots of unity. So $a = \omega + 2\omega'$. From the triangle inequality $|a| \leq |\omega| + |2\omega'| = 3$, with equality when ω and ω' point in the same direction (and thus $\omega = \omega'$). It follows that $a = 3, 3\omega, 3\omega^2$, and so

$$a = 3, -\frac{3}{2} + \frac{3i\sqrt{3}}{2}, -\frac{3}{2} - \frac{3i\sqrt{3}}{2}$$

17. [10] Sean is a biologist, and is looking at a string of length 66 composed of the letters A, T, C, G . A *substring* of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has 10 substrings: $A, G, T, C, AG, GT, TC, AGT, GTC, AGTC$. What is the maximum number of distinct substrings of the string Sean is looking at?

Proposed by: Yang Liu

Answer: 2100

Let's consider the number of distinct substrings of length ℓ . On one hand, there are obviously at most 4^ℓ distinct substrings. On the other hand, there are $67 - \ell$ substrings of length ℓ in a length 66 string. Therefore, the number of distinct substrings is at most

$$\sum_{\ell=1}^{66} \min(4^\ell, 67 - \ell) = 2100.$$

To show that this bound is achievable, one can do a construction using deBruijn sequences that we won't elaborate on here.

18. [10] Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $DA = 4$. What is the maximum possible radius of a circle inscribed in quadrilateral $ABCD$?

Proposed by: Sam Korsky

Answer: $\boxed{\frac{2\sqrt{30}}{7}}$

Let the tangent lengths be a, b, c, d so that

$$a + b = 2$$

$$b + c = 3$$

$$c + d = 5$$

$$d + a = 4$$

Then $b = 2 - a$ and $c = 1 + a$ and $d = 4 - a$. The radius of the inscribed circle of quadrilateral $ABCD$ is given by

$$\sqrt{\frac{abc + abd + acd + bcd}{a + b + c + d}} = \sqrt{\frac{-7a^2 + 16a + 8}{7}}$$

This is clearly maximized when $a = \frac{8}{7}$ which leads to a radius of $\sqrt{\frac{120}{49}} = \boxed{\frac{2\sqrt{30}}{7}}$.

19. [10] Find (in terms of $n \geq 1$) the number of terms with odd coefficients after expanding the product:

$$\prod_{1 \leq i < j \leq n} (x_i + x_j)$$

e.g., for $n = 3$ the expanded product is given by $x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2^2x_1 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3$ and so the answer would be 6.

Proposed by: Sam Korsky

Answer: $\boxed{n!}$

Note that if we take $(\text{mod } 2)$, we get that

$$\prod_{1 \leq i < j \leq n} (x_i + x_j) \equiv \prod_{1 \leq i < j \leq n} (x_j - x_i) = \det(M),$$

where M is the matrix with $M_{ij} = x_i^{j-1}$. This is called a *Vandermonde determinant*. Expanding this determinant using the formula

$$\det(M) = \sum_{\sigma} \prod_{i=1}^n x_{\sigma(i)}^{i-1},$$

where the sum is over all $n!$ permutations σ , gives the result.

20. [10] For positive integers a and N , let $r(a, N) \in \{0, 1, \dots, N - 1\}$ denote the remainder of a when divided by N . Determine the number of positive integers $n \leq 1000000$ for which

$$r(n, 1000) > r(n, 1001).$$

Proposed by: Pakawut Jiradilok

Answer: $\boxed{499500}$

Note that $0 \leq r(n, 1000) \leq 999$ and $0 \leq r(n, 1001) \leq 1000$. Consider the $\binom{1000}{2} = 499500$ ways to choose pairs (i, j) such that $i > j$. By the Chinese Remainder Theorem, there is exactly one n such that $1 \leq n \leq 1000 \cdot 1001$ such that $n \equiv i \pmod{1000}$ and $n \equiv j \pmod{1001}$. Finally, it is easy to check that none of the n in the range 1000001 to 1001000 satisfy the condition, so the answer is exactly 499500.

21. [12] Let P and A denote the perimeter and area respectively of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$

Proposed by: Sam Korsky

Assume WLOG that the side lengths of the triangle are pairwise coprime. Then they can be written as $m^2 - n^2, 2mn, m^2 + n^2$ for some coprime integers m and n where $m > n$ and mn is even. Then we obtain

$$\frac{P^2}{A} = \frac{4m(m+n)}{n(m-n)}$$

But $n, m-n, m, m+n$ are all pairwise coprime so for this to be an integer we need $n(m-n)|4$ and by checking each case we find that $(m, n) = (5, 4)$ yields the maximum ratio of $\boxed{45}$.

22. [12] Kelvin the Frog and 10 of his relatives are at a party. Every pair of frogs is either *friendly* or *unfriendly*. When 3 pairwise friendly frogs meet up, they will gossip about one another and end up in a *fight* (but stay *friendly* anyway). When 3 pairwise unfriendly frogs meet up, they will also end up in a *fight*. In all other cases, common ground is found and there is no fight. If all $\binom{11}{3}$ triples of frogs meet up exactly once, what is the minimum possible number of fights?

Proposed by: Alexander Katz

Answer: $\boxed{28}$

Consider a graph G with 11 vertices – one for each of the frogs at the party – where two vertices are connected by an edge if and only if they are friendly. Denote by $d(v)$ the number of edges emanating from v ; i.e. the number of friends frog v has. Note that $d(1) + d(2) + \dots + d(11) = 2e$, where e is the number of edges in this graph.

Focus on a single vertex v , and choose two other vertices u, w such that uv is an edge but wv is not. There are then $d(v)$ choices for u and $10 - d(v)$ choices for w , so there are $d(v)(10 - d(v))$ sets of three frogs that include v and do not result in a fight. Each set, however, is counted twice – if uw is an edge, then we count this set both when we focus on v and when we focus on w , and otherwise we count it when we focus on v and when we focus on u . As such, there are a total of

$$\frac{1}{2} \sum_v d(v)(10 - d(v))$$

sets of 3 frogs that do not result in a fight.

Note that $\frac{d(v)+10-d(v)}{2} = 5 \geq \sqrt{d(v)(10-d(v))} \implies d(v)(10-d(v)) \leq 25$ by AM-GM. Thus there are a maximum of

$$\frac{1}{2} \sum_v d(v)(10 - d(v)) \leq \frac{1}{2}(25 \cdot 11) = \frac{275}{2}$$

sets of three frogs that do not result in a fight; since this number must be an integer, there are a maximum of 137 such sets. As there are a total of $\binom{11}{3} = 165$ sets of 3 frogs, this results in a minimum $165 - 137 = \boxed{28}$ number of fights.

It remains to show that such an arrangement can be constructed. Set $d(1) = d(2) = \dots = d(10) = 5$ and $d(11) = 4$. Arrange these in a circle, and connect each to the nearest two clockwise neighbors; this gives each vertex 4 edges. To get the final edge for the first ten vertices, connect 1 to 10, 2 to 9, 3 to 8, 4 to 7, and 5 to 6. Thus $\boxed{28}$ is constructable, and is thus the true minimum.

23. [12] Five points are chosen uniformly at random on a segment of length 1. What is the expected distance between the closest pair of points?

Proposed by: Meghal Gupta

Answer: $\boxed{\frac{1}{24}}$

Choose five points arbitrarily at a_1, a_2, a_3, a_4, a_5 in increasing order. Then the intervals $(a_2 - x, a_2), (a_3 - x, a_3), (a_4 - x, a_4), (a_5 - x, a_5)$ must all be unoccupied. The probability that this happens is the same

as doing the process in reverse: first defining these intervals, then choosing five random points none of which lie in the four intervals. This transformed process clearly has a $(1 - 4x)^5$ probability of success. It follows that the desired probability is

$$\int_0^{1/4} (1 - 4x)^5 dx = \boxed{\frac{1}{24}}$$

24. [12] At a recent math contest, Evan was asked to find $2^{2016} \pmod{p}$ for a given prime number p with $100 < p < 500$. Evan has forgotten what the prime p was, but still remembers how he solved it:

- Evan first tried taking 2016 modulo $p - 1$, but got a value e larger than 100.
- However, Evan noted that $e - \frac{1}{2}(p - 1) = 21$, and then realized the answer was $-2^{21} \pmod{p}$.

What was the prime p ?

Proposed by: Evan Chen

Answer: $\boxed{211}$

Answer is $p = 211$. Let $p = 2d + 1$, $50 < d < 250$. The information in the problem boils down to

$$2016 = d + 21 \pmod{2d}.$$

From this we can at least read off $d \mid 1995$.

Now factor $1995 = 3 \cdot 5 \cdot 7 \cdot 19$. The values of d in this interval are 57, 95, 105, 133. The prime values of $2d + 1$ are then 191 and 211. Of these, we take 211 since $(2/191) = +1$ while $(2/211) = -1$.

Also, this is (almost) a true story: the contest in question was the PUMaC 2016 Live Round.

25. [15] Find all real numbers x satisfying the equation $x^3 - 8 = 16\sqrt[3]{x + 1}$.

Proposed by: Alexander Katz

Answer: $\boxed{-2, 1 \pm \sqrt{5}}$

Let $f(x) = \frac{x^3 - 8}{8}$. Then $f^{-1}(x) = \sqrt[3]{8x + 8} = 2\sqrt[3]{x + 1}$, and so the given equation is equivalent to $f(x) = f^{-1}(x)$. This implies $f(f(x)) = x$. However, as f is monotonically increasing, this implies that $f(x) = x$. As a result, we have $\frac{x^3 - 8}{8} = x \implies x^3 - 8x - 8 = 0 \implies (x + 2)(x^2 - 2x - 4) = 0$, and so $x = -2, 1 \pm \sqrt{5}$.

26. [15] Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before getting eaten.

Proposed by: Allen Liu

First we compute probability that the mouse returns to 0 before being eaten. Then probability that it is at 0 in $2n$ minutes without being eaten is given by $\frac{1}{3^{2n}} \binom{2n}{n}$. Therefore, the overall expectation is given by

$$\begin{aligned} \sum_{n \geq 1} \binom{2n}{n} 9^{-n} &= -1 + \sum_{n \geq 0} \binom{2n}{n} 9^{-n} \\ &= -1 + \frac{1}{\sqrt{1 - 4/9}} = -1 + \frac{3}{\sqrt{5}} = \frac{3\sqrt{5} - 5}{5} \end{aligned}$$

where we use the well known fact that

$$\sum_{n \geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1 - 4x}}.$$

for $x = \frac{1}{9}$.

27. [15] Find the smallest possible value of $x + y$ where $x, y \geq 1$ and x and y are integers that satisfy $x^2 - 29y^2 = 1$

Proposed by: Sam Korsky

Answer: 11621

Continued fraction convergents to $\sqrt{29}$ are $5, \frac{11}{2}, \frac{16}{3}, \frac{27}{5}, \frac{70}{13}$ and you get $70^2 - 29 \cdot 13^2 = -1$ so since $(70 + 13\sqrt{29})^2 = 9801 + 1820\sqrt{29}$ the answer is $9801 + 1820 = 11621$

28. [15] Let $\dots, a_{-1}, a_0, a_1, a_2, \dots$ be a sequence of positive integers satisfying the following relations: $a_n = 0$ for $n < 0$, $a_0 = 1$, and for $n \geq 1$,

$$a_n = a_{n-1} + 2(n-1)a_{n-2} + 9(n-1)(n-2)a_{n-3} + 8(n-1)(n-2)(n-3)a_{n-4}.$$

Compute

$$\sum_{n \geq 0} \frac{10^n a_n}{n!}.$$

Proposed by: Yang Liu

Answer: e^{23110} .

Let $y = \sum_{n \geq 0} \frac{x^n a_n}{n!}$. Then $y' = (1 + 2x + 9x^2 + 8x^3)y$ by definition. So $y = C \exp(x + x^2 + 3x^3 + 2x^4)$. Take $x = 0$ to get $C = 1$. Take $x = 10$ to get the answer.

29. [17] Yang has the sequence of integers $1, 2, \dots, 2017$. He makes 2016 *swaps* in order, where a swap changes the positions of two integers in the sequence. His goal is to end with $2, 3, \dots, 2017, 1$. How many different sequences of swaps can Yang do to achieve his goal?

Proposed by: Yang Liu

Answer: 2017^{2015}

Let $n = 2017$. The problem is asking to write a cycle permutation of n integers as the product of $n - 1$ transpositions. Say that the transpositions Yang uses are (a_i, b_i) (i.e. swapping the a_i -th integer in the sequence with the b_i -th integer in the sequence). Draw the graph with edges (a_i, b_i) . One can show that the result is a cycle if and only if the resulting graph is acyclic, so it must be a tree. There are n^{n-2} trees by Cayley's formula, and for each tree, it can be made in $(n - 1)!$ ways (any ordering of the edges). So the total number of ways to end with a cycle is $n^{n-2} \cdot (n - 1)!$. By symmetry, each cycle can be made in the same number of ways, so in particular the cycle $2, 3, \dots, n, 1$ can be made in $\frac{n^{n-2} \cdot (n-1)!}{(n-1)!} = n^{n-2}$ ways.

30. [17] Consider an equilateral triangular grid G with 20 points on a side, where each row consists of points spaced 1 unit apart. More specifically, there is a single point in the first row, two points in the second row, \dots , and 20 points in the last row, for a total of 210 points. Let S be a closed non-self-intersecting polygon which has 210 vertices, using each point in G exactly once. Find the sum of all possible values of the area of S .

Proposed by: Sammy Luo

Answer: $52\sqrt{3}$

Imagine deforming the triangle lattice such that now it looks like a lattice of 45-45-90 right triangles with legs of length 1. Note that by doing this, the area has multiplied by $\frac{2}{\sqrt{3}}$, so we need to readjust our answer on the isosceles triangle lattice by a factor of $\frac{\sqrt{3}}{2}$ at the end. By Pick's Theorem, the area in the new lattice is given by $I + \frac{P}{2} - 1 = 0 + 105 - 1 = 104$. Therefore, the answer is $104 \cdot \frac{\sqrt{3}}{2} = 52\sqrt{3}$.

31. [17] A baseball league has 6 teams. To decide the schedule for the league, for each pair of teams, a coin is flipped. If it lands head, they will play a game this season, in which one team wins and one team loses. If it lands tails, they don't play a game this season. Define the *imbalance* of this schedule

to be the minimum number of teams that will end up undefeated, i.e. lose 0 games. Find the expected value of the imbalance in this league.

Proposed by: Yang Liu

Let n denote the number of teams.

Lemma: Given a connected graph G , the imbalance of G is 1 iff G is a tree. Let's just talk in terms of directed graphs and indegree/outdegree.

Proof. If there is a cycle, direct the cycle such that it is a directed cycle. Then from this cycle, point all remaining edges outwards. If G is a tree, induct on the size. Take any leaf. If it wins its game, it is undefeated. Otherwise, it must lose to its neighbor. Then induct on the tree resulting after deleting the leaf. \square

Now the finish is a simple counting argument using expected values. Using Cayley's formula, for each subset of vertices, we compute the probability that it is a maximal connected component and is a tree. This ends up being

$$2^{-\binom{n}{2}} \sum_{i=1}^n \binom{n}{i} \cdot i^{i-2} \cdot 2^{\binom{n-i}{2}}.$$

This evaluates to $\frac{5055}{2^{15}}$ for $n = 6$.

32. [17] Let a, b, c be non-negative real numbers such that $ab + bc + ca = 3$. Suppose that

$$a^3b + b^3c + c^3a + 2abc(a + b + c) = \frac{9}{2}.$$

What is the minimum possible value of $ab^3 + bc^3 + ca^3$?

Proposed by: Pakawut Jiradilok

Answer: 18

Expanding the inequality $\sum_{\text{cyc}} ab(b + c - 2a)^2 \geq 0$ gives

$$\left(\sum_{\text{cyc}} ab^3\right) + 4\left(\sum_{\text{cyc}} a^3b\right) - 4\left(\sum_{\text{cyc}} a^2b^2\right) - abc(a + b + c) \geq 0$$

Using $\left(\sum_{\text{cyc}} a^3b\right) + 2abc(a + b + c) = \frac{9}{2}$ in the inequality above yields

$$\left(\sum_{\text{cyc}} ab^3\right) - 4(ab + bc + ca)^2 \geq \left(\sum_{\text{cyc}} ab^3\right) - 4\left(\sum_{\text{cyc}} a^2b^2\right) - 9abc(a + b + c) \geq -18$$

Since $ab + bc + ca = 3$, we have $\sum_{\text{cyc}} ab^3 \geq 18$ as desired.

The equality occurs when $(a, b, c) \sim \left(\sqrt{\frac{3}{2}}, \sqrt{6}, 0\right)$.

33. [20] Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer n problems and get them **all** correct, you will receive $\max(0, (n - 1)(n - 2))$ points. If any of them are wrong, you will receive 0 points.

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you would answer YYNNBY (and receive 12 points if all five answers are correct, 0 points if any are wrong).

- (a) $a, b, c, d, A, B, C,$ and D are positive real numbers such that $\frac{a}{b} > \frac{A}{B}$ and $\frac{c}{d} > \frac{C}{D}$. Is it necessarily true that $\frac{a+c}{b+d} > \frac{A+C}{B+D}$?
- (b) Do there exist irrational numbers α and β such that the sequence $[\alpha] + [\beta], [2\alpha] + [2\beta], [3\alpha] + [3\beta], \dots$ is arithmetic?
- (c) For any set of primes \mathbb{P} , let $S_{\mathbb{P}}$ denote the set of integers whose prime divisors all lie in \mathbb{P} . For instance $S_{\{2,3\}} = \{2^a 3^b \mid a, b \geq 0\} = \{1, 2, 3, 4, 6, 8, 9, 12, \dots\}$. Does there exist a finite set of primes \mathbb{P} and integer polynomials P and Q such that $\gcd(P(x), Q(y)) \in S_{\mathbb{P}}$ for all x, y ?
- (d) A function f is called **P-recursive** if there exists a positive integer m and real polynomials $p_0(n), p_1(n), \dots, p_m(n)$ satisfying

$$p_m(n)f(n+m) = p_{m-1}(n)f(n+m-1) + \dots + p_0(n)f(n)$$

for all n . Does there exist a P-recursive function f satisfying $\lim_{n \rightarrow \infty} \frac{f(n)}{n\sqrt{2}} = 1$?

- (e) Does there exist a **nonpolynomial** function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $a - b$ divides $f(a) - f(b)$ for all integers $a \neq b$?
- (f) Do there exist periodic functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x) + g(x) = x$ for all x ?

Proposed by: Alexander Katz

Answer: NNYYYY

34. [20]

- (a) Can 1000 queens be placed on a 2017×2017 chessboard such that every square is attacked by some queen? A square is attacked by a queen if it lies in the same row, column, or diagonal as the queen.
- (b) A 2017×2017 grid of squares originally contains a 0 in each square. At any step, Kelvin the Frog chooses two adjacent squares (two squares are adjacent if they share a side) and increments the numbers in both of them by 1. Can Kelvin make every square contain a different power of 2?
- (c) A *tournament* consists of single games between every pair of players, where each game has a winner and loser with no ties. A set of people is *dominated* if there exists a player who beats all of them. Does there exist a tournament in which every set of 2017 people is dominated?
- (d) Every cell of a 19×19 grid is colored either red, yellow, green, or blue. Does there necessarily exist a rectangle whose sides are parallel to the grid, all of whose vertices are the same color?
- (e) Does there exist a $c \in \mathbb{R}^+$ such that $\max(|A \cdot A|, |A + A|) \geq c|A| \log^2 |A|$ for all finite sets $A \subset \mathbb{Z}$?
- (f) Can the set $\{1, 2, \dots, 1093\}$ be partitioned into 7 subsets such that each subset is sum-free (i.e. no subset contains a, b, c with $a + b = c$)?

Proposed by: Alexander Katz

Answer: NNYYYY

35. [20]

- (a) Does there exist a finite set of points, not all collinear, such that a line between any two points in the set passes through a third point in the set?
- (b) Let ABC be a triangle and P be a point. The *isogonal conjugate* of P is the intersection of the reflection of line AP over the A -angle bisector, the reflection of line BP over the B -angle bisector, and the reflection of line CP over the C -angle bisector. Clearly the incenter is its own isogonal conjugate. Does there exist another point that is its own isogonal conjugate?
- (c) Let F be a convex figure in a plane, and let P be the largest pentagon that can be inscribed in F . Is it necessarily true that the area of P is at least $\frac{3}{4}$ the area of F ?
- (d) Is it possible to cut an equilateral triangle into 2017 pieces, and rearrange the pieces into a square?

- (e) Let ABC be an acute triangle and P be a point in its interior. Let D, E, F lie on BC, CA, AB respectively so that PD bisects $\angle BPC$, PE bisects $\angle CPA$, and PF bisects $\angle APB$. Is it necessarily true that $AP + BP + CP \geq 2(PD + PE + PF)$?
- (f) Let P_{2018} be the surface area of the 2018-dimensional unit sphere, and let P_{2017} be the surface area of the 2017-dimensional unit sphere. Is $P_{2018} > P_{2017}$?

Proposed by: Alexander Katz

Answer:

36. [20]

- (a) Does $\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}$ for all odd prime numbers p ? (Note that $\frac{1}{i}$ denotes the number such that $i \cdot \frac{1}{i} \equiv 1 \pmod{p^2}$)
- (b) Do there exist 2017 positive perfect cubes that sum to a perfect cube?
- (c) Does there exist a right triangle with rational side lengths and area 5?
- (d) A *magic square* is a 3×3 grid of numbers, all of whose rows, columns, and major diagonals sum to the same value. Does there exist a magic square whose entries are all prime numbers?
- (e) Is $\prod_p \frac{p^2+1}{p^2-1} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \dots$ a rational number?
- (f) Do there exist an infinite number of pairs of *distinct* integers (a, b) such that a and b have the same set of prime divisors, and $a + 1$ and $b + 1$ also have the same set of prime divisors?

Proposed by: Alexander Katz

Answer:

February 2017

February 18, 2017

Team

1. [15] Let $P(x), Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y , then $P(x) = Q(x)$ for all real numbers x .

Proposed by: Alexander Katz

Answer:

By the condition, we know that $|P(x) - Q(x)| \leq 1$ for all x . This can only hold if $P(x) - Q(x)$ is a constant polynomial. Now take a constant c such that $P(x) = Q(x) + c$. Without loss of generality, we can assume that $c \geq 0$. Assume that $c > 0$. By continuity, if $\deg P = \deg Q > 0$, we can select an integer r and a real number x_0 such that $Q(x_0) + c = r$. Then $\lfloor P(x_0) \rfloor = \lfloor Q(x_0) + c \rfloor = r$. On the other hand, $\lfloor Q(x_0) \rfloor = \lfloor r - c \rfloor < r$ as r was an integer. This is a contradiction. Therefore, $c = 0$ as desired.

2. [25] Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

Proposed by: Alexander Katz

Answer: No

For any ϵ and positive x , $P(x, \epsilon) > 0$ and $P(x, -\epsilon) \leq 0$. Thus by continuity/IVT, $P(x, 0) = 0$ for all positive x . Similarly $P(0, y) = 0$ for all positive y . This implies $xy \mid P(x, y)$, and so we can write $P(x, y) = xyQ(x, y)$. But then this same logic holds for Q , and this cannot continue infinitely unless P is identically 0 – in which case the conditions do not hold. So no such polynomial exists.

3. [30] A polyhedron has $7n$ faces. Show that there exist $n + 1$ of the polyhedron's faces that all have the same number of edges.

Proposed by: Alexander Katz

Let V, E , and F denote the number of vertices, edges, and faces respectively. Let a_k denote the number of faces with k sides, and let M be the maximum number of sides any face has.

Suppose that $a_k \leq n$ for all k and that $M > 8$. Note that each edge is part of exactly two faces, and each vertex is part of at least three faces. It follows that

$$\begin{aligned} \sum_{k=3}^M a_k &= F \\ \sum_{k=3}^M \frac{ka_k}{2} &= E \\ \sum_{k=3}^M \frac{ka_k}{3} &\geq V \end{aligned}$$

and in particular

$$\sum_{k=3}^M a_k \left(1 - \frac{k}{2} + \frac{k}{3}\right) \geq F - E + V = 2$$

by Euler's formula. But on the other hand, $a_k \leq n$ by assumption, so

$$\begin{aligned}
2 &\leq \sum_{k=3}^M a_k \left(1 - \frac{k}{6}\right) \\
&\leq \sum_{k=3}^M n \left(1 - \frac{k}{6}\right) \\
&= \sum_{k=3}^8 n \left(1 - \frac{k}{6}\right) + \sum_{k=9}^M n \left(1 - \frac{k}{6}\right) \\
&\leq \frac{1}{2}n - \frac{1}{2}n(M-8)
\end{aligned}$$

where the last step follows from the fact that $1 - \frac{k}{6} \leq -\frac{1}{2}$ for $k \geq 9$. Thus

$$2 \leq \frac{9}{2}n - \frac{1}{2}nM \implies M \leq \frac{\frac{9}{2}n - 2}{\frac{1}{2}n} < 9$$

contradicting the fact that $M > 8$. It follows that $M \leq 8$, and as each face has at least 3 edges, the result follows directly from Pigeonhole.

4. [35] Let $w = w_1 w_2 \dots w_n$ be a word. Define a *substring* of w to be a word of the form $w_i w_{i+1} \dots w_{j-1} w_j$, for some pair of positive integers $1 \leq i \leq j \leq n$. Show that w has at most n distinct palindromic substrings.

For example, $aaaaa$ has 5 distinct palindromic substrings, and $abcata$ has 5 (a, b, c, t, ata).

Proposed by: Yang Liu

For each palindrome substring appearing in w , consider only the leftmost position in which it appears. I claim that now, no two substrings share the same right endpoint. If some two do, then you can reflect the smaller one about the center of the larger one to move the smaller one left.

5. [35] Let ABC be an acute triangle. The altitudes BE and CF intersect at the orthocenter H , and point O denotes the circumcenter. Point P is chosen so that $\angle APH = \angle OPE = 90^\circ$, and point Q is chosen so that $\angle AQH = \angle OQF = 90^\circ$. Lines EP and FQ meet at point T . Prove that points A, T, O are collinear.

Proposed by: Evan Chen

Observe that T is the radical center of the circles with diameter OE, OF, AH . So T lies on the radical axis of $(OE), (OF)$ which is the altitude from O to EF , hence passing through A .

So ATO are collinear, done.

6. [40] Let r be a positive integer. Show that if a graph G has no cycles of length at most $2r$, then it has at most $|V|^{2016}$ cycles of length exactly $2016r$, where $|V|$ denotes the number of vertices in the graph G .

Proposed by: Yang Liu

The key idea is that there is at most 1 path of length r between any pair of vertices, or else you get a cycle of length $\leq 2r$. Now, start at any vertex ($|V|$ choices) and walk 2015 times. There's at most $|V|^{2016}$ ways to do this by the previous argument. Now you have to go from the end to the start, and there's only one way to do this. So we're done.

7. [45] Let p be a prime. A *complete residue class modulo p* is a set containing at least one element equivalent to $k \pmod{p}$ for all k .

- (a) **(20)** Show that there exists an n such that the n th row of Pascal's triangle forms a complete residue class modulo p .
- (b) **(25)** Show that there exists an $n \leq p^2$ such that the n th row of Pascal's triangle forms a complete residue class modulo p .

Proposed by: Alexander Katz

We use the following theorem of Lucas:

Theorem. Given a prime p and nonnegative integers a, b written in base p as $a = \overline{a_n a_{n-1} \dots a_0}$ and $b = \overline{b_n b_{n-1} \dots b_0}$ respectively, where $0 \leq a_i, b_i \leq p-1$ for $0 \leq i \leq n$, we have

$$\binom{a}{b} = \prod_{i=0}^n \binom{a_i}{b_i} \pmod{p}.$$

Now, let $n = (p-1) \times p + (p-2) = p^2 - 2$. For $k = pq + r$ with $0 \leq q, r \leq p-1$, applying Lucas's theorem gives

$$\binom{n}{k} \equiv \binom{p-1}{q} \binom{p-2}{r} \pmod{p}.$$

Note that

$$\binom{p-1}{q} = \prod_{i=1}^q \frac{p-i}{i} \equiv (-1)^q \pmod{p},$$

and

$$\binom{p-2}{r} = \prod_{i=1}^r \frac{p-1-i}{i} \equiv (-1)^r \frac{(r+1)!}{r!} = (-1)^r (r+1) \pmod{p}.$$

So for $2 \leq i \leq p$ we can take $k = (p+1)(i-1)$ and obtain $\binom{n}{k} \equiv i \pmod{p}$, while for $i = 1$ we can take $k = 0$. Thus this row satisfies the desired property.

8. **[45]** Does there exist an irrational number $\alpha > 1$ such that

$$[\alpha^n] \equiv 0 \pmod{2017}$$

for all integers $n \geq 1$?

Proposed by: Sam Korsky

Answer: Yes

Let $\alpha > 1$ and $0 < \beta < 1$ be the roots of $x^2 - 4035x + 2017$. Then note that $[\alpha^n] = \alpha^n + \beta^n - 1$. Let $x_n = \alpha^n + \beta^n$ for all nonnegative integers n . It's easy to verify that $x_n = 4035x_{n-1} - 2017x_{n-2} \equiv x_{n-1} \pmod{2017}$ so since $x_1 = 4035 \equiv 1 \pmod{2017}$ we have that $x_n \equiv 1 \pmod{2017}$ for all n . Thus α satisfies the problem.

9. **[65]** Let n be a positive odd integer greater than 2, and consider a regular n -gon \mathcal{G} in the plane centered at the origin. Let a *subpolygon* \mathcal{G}' be a polygon with at least 3 vertices whose vertex set is a subset of that of \mathcal{G} . Say \mathcal{G}' is *well-centered* if its centroid is the origin. Also, say \mathcal{G}' is *decomposable* if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

Proposed by: Yang Liu

\Rightarrow , i.e. n has ≥ 3 prime divisors: Let $n = \prod p_i^{e_i}$. Note it suffices to only consider regular p_i -gons. Label the vertices of the n -gon $0, 1, \dots, n-1$. Let $S = \{\frac{xn}{p_1} : 0 \leq x \leq p_1 - 1\}$, and let $S_j = S + \frac{jn}{p_3}$ for $0 \leq j \leq p_3 - 2$. ($S + a = \{s + a : s \in S\}$.) Then let $S_{p_3-1} = \{\frac{xn}{p_2} : 0 \leq x \leq p_2 - 1\} + \frac{(p_3-1)n}{p_3}$. Finally, let $S' = \{\frac{xn}{p_3} : 0 \leq x \leq p_3 - 1\}$. Then I claim

$$\left(\bigcup_{i=0}^{p_3-1} S_i \right) \setminus S'$$

is well-centered but not decomposable. Well-centered follows from the construction: I only added and subtracted off regular polygons. To show that its decomposable, consider $\frac{n}{p_1}$. Clearly this is in the set, but isn't in S' . I claim that $\frac{n}{p_1}$ isn't in any more regular p_i -gons. For $i \geq 4$, this means that $\frac{n}{p_1} + \frac{n}{p_i}$ is in some set. But this is a contradiction, as we can easily check that all points we added in are multiples of $p_i^{e_i}$, while $\frac{n}{p_i}$ isn't.

For $i = 1$, note that 0 was removed by S' . For $i = 2$, note that the only multiples of $p_3^{e_3}$ that are in some S_j are $0, \frac{n}{p_1}, \dots, \frac{(p_1-1)n}{p_1}$. In particular, $\frac{n}{p_1} + \frac{n}{p_2}$ isn't in any S_j . So it suffices to consider the case $i = 3$, but it is easy to show that $\frac{n}{p_1} + \frac{(p_3-1)n}{p_3}$ isn't in any S_i . So we're done.

\Leftarrow , i.e. n has ≤ 2 prime divisors: This part seems to require knowledge of cyclotomic polynomials. These will easily give a solution in the case $n = p^a$. Now, instead turn to the case $n = p^a q^b$. The next lemma is the key ingredient to the solution.

Lemma: Every well-centered subpolygon can be gotten by adding in and subtracting off regular polygons.

Note that this is weaker than the problem claim, as the problem claims that adding in polygons is enough.

Proof. It is easy to verify that $\phi_n(x) = \frac{(x^n-1)(x^{\frac{n}{pq}}-1)}{(x^{\frac{n}{p}}-1)(x^{\frac{n}{q}}-1)}$. Therefore, it suffices to check that there exist integer polynomials $c(x), d(x)$ such that

$$\frac{x^n - 1}{x^{\frac{n}{p}} - 1} \cdot c(x) + \frac{x^n - 1}{x^{\frac{n}{q}} - 1} \cdot d(x) = \frac{(x^n - 1)(x^{\frac{n}{pq}} - 1)}{(x^{\frac{n}{p}} - 1)(x^{\frac{n}{q}} - 1)}.$$

Rearranging means that we want

$$(x^{\frac{n}{q}} - 1) \cdot c(x) + (x^{\frac{n}{p}} - 1) \cdot d(x) = x^{\frac{n}{pq}} - 1.$$

But now, since $\gcd(n/p, n/q) = n/pq$, there exist positive integers s, t such that $\frac{sn}{q} - \frac{tn}{p} = \frac{n}{pq}$. Now choose $c(x) = \frac{x^{\frac{sn}{q}} - 1}{x^{\frac{n}{q}} - 1}, d(x) = \frac{x^{\frac{sn}{q}} - x^{\frac{n}{pq}}}{x^{\frac{n}{p}} - 1}$ to finish. \square

Now we can finish combinatorially. Say we need subtraction, and at some point we subtract off a p -gon. All the points in the p -gon must have been added at some point. If any of them was added from a p -gon, we could just cancel both p -gons. If they all came from a q -gon, then the sum of those p q -gons would be a pq -gon, which could have been instead written as the sum of q p -gons. So we don't need subtraction either way. This completes the proof.

10. [65] Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and \overline{AK} the altitude from A . The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \neq K$ and $V \neq K$. Finally, let T be the projection of I onto line UV . Prove that the line through T and the midpoint of \overline{IK} passes through a fixed point as A varies.

Proposed by: Sam Korsky

Answer: \square

Let M be the midpoint of arc BC not containing L and let D be the point where the incircle of triangle ABC touches BC . Also let N be the projection from I to AK . We claim that M is the desired fixed point.

By Simson's Theorem on triangle KUV and point I we have that points T, D, N are collinear and since quadrilateral $NKDI$ is a rectangle we have that line DN passes through the midpoint of IK . Thus it suffices to show that M lies on line DN .

Now, let I_a, I_b, I_c be the A, B, C -excenters of triangle ABC respectively. Then I is the orthocenter of triangle $I_a I_b I_c$ and ABC is the Cevian triangle of I with respect to triangle $I_a I_b I_c$. It's also well-known that M is the midpoint of II_a .

Let D' be the reflection of I over BC and let N' be the reflection of I over AK . Clearly K is the midpoint of $D'N'$. If we could prove that I_a, D', K, N' were collinear then by taking a homothety centered at I with ratio $\frac{1}{2}$ we would have that points M, D, N were collinear as desired. Thus it suffices to show that points I_a, D', K are collinear.

Let lines BC and $I_b I_c$ intersect at R and let lines AI and BC intersect at S . Then it's well-known that $(I_b, I_c; A, R)$ is harmonic and projecting from C we have that $(I_a, I; S, A)$ is harmonic. But $KS \perp KA$ which means that KS bisects angle $\angle IKI_a$. But it's clear by the definition of D' that KS bisects angle $\angle IKD'$ which implies that points I_a, K, D' are collinear as desired. This completes the proof.

HMMT November 2017

November 11, 2017

General Round

1. Find the sum of all positive integers whose largest proper divisor is 55. (A proper divisor of n is a divisor that is strictly less than n .)

Proposed by: Michael Tang

Answer:

The largest proper divisor of an integer n is $\frac{n}{p}$, where p is the smallest prime divisor of n . So $n = 55p$ for some prime p . Since $55 = 5 \cdot 11$, we must have $p \leq 5$, so $p = 2, 3, 5$ gives all solutions. The sum of these solutions is $55(2 + 3 + 5) = 550$.

2. Determine the sum of all distinct real values of x such that

$$||| \cdots ||x| + x| \cdots | + x| + x| = 1,$$

where there are 2017 x 's in the equation.

Proposed by: Yuan Yao

Answer:

Note that $|x + |x|| = 2x$ when x is nonnegative, and is equal to 0 otherwise. Thus, when there are 2017 x 's, the expression equals $2017x$ when $x \geq 0$ and $-x$ otherwise, so the two solutions to the equation are $x = -1$ and $\frac{1}{2017}$, and their sum is $-\frac{2016}{2017}$.

3. Find the number of integers n with $1 \leq n \leq 2017$ so that $(n - 2)(n - 0)(n - 1)(n - 7)$ is an integer multiple of 1001.

Proposed by: Dhruv Rohatgi

Answer:

Note that $1001 = 7 \cdot 11 \cdot 13$, so the stated product must be a multiple of 7, as well as a multiple of 11, as well as a multiple of 13. There are 4 possible residues of n modulo 11 for which the product is a multiple of 11; similarly, there are 4 possible residues of n modulo 13 for which the product is a multiple of 13. However, there are only 3 possible residues of n modulo 7 for which the product is a multiple of 7.

Consider each of these $4 \cdot 4 \cdot 3 = 48$ possible triples of remainders. By the Chinese Remainder Theorem there is exactly one value of n with $1 \leq n \leq 1001$ achieving those remainders, and exactly one value of n with $16 \leq n \leq 1016$ achieving those remainders. Similarly, there is exactly one value of n with $1017 \leq n \leq 2017$ with those same remainders. Hence there are 96 values of n with $16 \leq n \leq 2017$ such that $(n - 2)(n - 0)(n - 1)(n - 7)$ is a multiple of 1001.

It remains to check $n \in \{1, 2, 3, \dots, 15\}$. Since the product must be a multiple of 7, we can narrow the set to $\{1, 2, 7, 8, 9, 14\}$. The first 3 values work trivially, since the product is 0. It can be easily checked that none of the remaining values of n yield a product which is a multiple of 11. Hence, the final answer is $96 + 3 = 99$.

4. Triangle ABC has $AB = 10$, $BC = 17$, and $CA = 21$. Point P lies on the circle with diameter AB . What is the greatest possible area of APC ?

Proposed by: Michael Tang

Answer:

To maximize $[APC]$, point P should be the farthest point on the circle from AC . Let M be the midpoint of AB and Q be the projection of M onto AC . Then $PQ = PM + MQ = \frac{1}{2}AB + \frac{1}{2}h_B$, where h_B is the length of the altitude from B to AC . By Heron's formula, one finds that the area of ABC is $\sqrt{24 \cdot 14 \cdot 7 \cdot 3} = 84$, so $h_B = \frac{2 \cdot 84}{AC} = 8$. Then $PQ = \frac{1}{2}(10 + 8) = 9$, so the area of APC is $\frac{1}{2} \cdot 21 \cdot 9 = \frac{189}{2}$.

5. Given that a, b, c are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of $|a + b\omega + c\omega^2|$.

Proposed by: Ashwin Sah

Answer: $\boxed{\sqrt{3}}$

Since $\omega^3 = 1$, and $\omega \neq 1$, ω is a third root of unity. For any complex number z , $|z|^2 = z \cdot \bar{z}$. Letting $z = a + b\omega + c\omega^2$, we find that $\bar{z} = a + c\omega + b\omega^2$, and

$$\begin{aligned} |z|^2 &= a^2 + ab\omega + ac\omega^2 + ab\omega^2 + b^2 + bc\omega + ac\omega + bc\omega^2 + c^2 \\ &= (a^2 + b^2 + c^2) + (ab + bc + ca)(\omega) + (ab + bc + ca)(\omega^2) \\ &= (a^2 + b^2 + c^2) - (ab + bc + ca) \\ &= \frac{1}{2}((a - b)^2 + (b - c)^2 + (c - a)^2), \end{aligned}$$

where we have used the fact that $\omega^3 = 1$ and that $\omega + \omega^2 = -1$. This quantity is minimized when a, b , and c are as close to each other as possible, making $a = 3, b = 4, c = 5$ the optimal choice, giving $|z|^2 = 3$. (A smaller value of $|z|$ requires two of a, b, c to be equal and the third differing from them by at most 2, which is impossible.) So $|z|_{\min} = \sqrt{3}$.

6. A positive integer n is *magical* if

$$\left\lfloor \sqrt{\lceil \sqrt{n} \rceil} \right\rfloor = \left\lceil \sqrt{\lfloor \sqrt{n} \rfloor} \right\rceil,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling function respectively. Find the number of magical integers between 1 and 10,000, inclusive.

Proposed by: Yuan Yao

Answer: $\boxed{1330}$

First of all, we have $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil$ when n is a perfect square and $\lfloor \sqrt{n} \rfloor = \lceil \sqrt{n} \rceil - 1$ otherwise. Therefore, in the first case, the original equation holds if and only if \sqrt{n} is a perfect square itself, i.e., n is a fourth power. In the second case, we need $m = \lfloor \sqrt{n} \rfloor$ to satisfy the equation $\lfloor \sqrt{m+1} \rfloor = \lceil \sqrt{m} \rceil$, which happens if and only if either m or $m+1$ is a perfect square k^2 . Therefore, n is magical if and only if $(k^2 - 1)^2 < n < (k^2 + 1)^2$ for some (positive) integer k . There are $(k^2 + 1)^2 - (k^2 - 1)^2 = 4k^2 - 1$ integers in this range. The range in the problem statement includes $k = 1, 2, \dots, 9$ and the interval $(99^2, 100^2]$, so the total number of magical numbers is

$$4(1^2 + 2^2 + \dots + 9^2) - 9 + (100^2 - 99^2) = 4 \cdot \frac{9 \cdot (9 + 1) \cdot (18 + 1)}{6} + 190 = 1330.$$

7. Reimu has a wooden cube. In each step, she creates a new polyhedron from the previous one by cutting off a pyramid from each vertex of the polyhedron along a plane through the trisection point on each adjacent edge that is closer to the vertex. For example, the polyhedron after the first step has six octagonal faces and eight equilateral triangular faces. How many faces are on the polyhedron after the fifth step?

Proposed by: Qi Qi

Answer: $\boxed{974}$

Notice that the number of vertices and edges triple with each step. We always have 3 edges meeting at one vertex, and slicing off a pyramid doesn't change this (we make new vertices from which one edge from the previous step and two of the pyramid edges emanate). So at each step we replace the sliced-off vertex with three new vertices, and to each edge we create four new "half-edges" (two from the pyramid at each endpoint), which is equivalent to tripling the number of vertices and edges. Then, by Euler's Theorem the number of faces is $E - V + 2 = 12 \cdot 3^5 - 8 \cdot 3^5 + 2 = 974$.

8. Marisa has a collection of $2^8 - 1 = 255$ distinct nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8 - 2 = 254$ times until there is only one set left in the collection. What is the expected size of this set?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1024}{255}}$

It suffices to compute the probability of each number appearing in the final subset. For any given integer $n \in [1, 8]$, there are $2^7 = 128$ subsets with n and $2^7 - 1 = 127$ without. When we focus on only this element, each operation is equivalent to taking two random sets and discarding one of them randomly. Therefore there is a $\frac{128}{255}$ probability that n is in the final subset, and the expected value of its size is $8 \cdot \frac{128}{255} = \frac{1024}{255}$.

Alternatively, since $|A| + |B| = |A \cup B| + |A \cap B|$, the expected value of the *average* size of all remaining subsets at a given step is constant, so the answer is simply the average size of all 255 subsets, which is $\frac{8 \cdot 128}{255} = \frac{1024}{255}$.

9. Find the minimum possible value of

$$\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$$

where $-1 \leq x \leq 1$.

Proposed by: Serina Hu

Answer: $\boxed{\sqrt{109}}$

Substitute $x = \cos \theta$ and $\sqrt{1 - x^2} = \sin \theta$, and notice that $58 = 3^2 + 7^2$, $42 = 2 \cdot 3 \cdot 7$, $149 = 7^2 + 10^2$, and $140 = 2 \cdot 7 \cdot 10$. Therefore the first term is an application of Law of Cosines on a triangle that has two sides 3 and 7 with an angle measuring θ between them to find the length of the third side; similarly, the second is for a triangle with two sides 7 and 10 that have an angle measuring $90 - \theta$ between them. "Gluing" these two triangles together along their sides of length 7 so that the merged triangles form a right angle, we see that the minimum length of the sum of their third sides occurs when the glued triangles form a right triangle. This right triangle has legs of length 3 and 10, so its hypotenuse has length $\sqrt{109}$.

10. Five equally skilled tennis players named Allen, Bob, Catheryn, David, and Evan play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50% chance of winning, and the results of the games are independent. Compute the probability that there exist four distinct players P_1, P_2, P_3, P_4 such that P_i beats P_{i+1} for $i = 1, 2, 3, 4$. (We denote $P_5 = P_1$).

Proposed by: Steven Hao

Answer: $\boxed{\frac{49}{64}}$

We make the following claim: if there is a 5-cycle (a directed cycle involving 5 players) in the tournament, then there is a 4-cycle.

Proof: Assume that A beats B , B beats C , C beats D , D beats E and E beats A . If A beats C then A, C, D, E forms a 4-cycle, and similar if B beats D , C beats E , and so on. However, if all five reversed matches occur, then A, D, B, C is a 4-cycle.

Therefore, if there are no 4-cycles, then there can be only 3-cycles or no cycles at all.

Case 1: There is a 3-cycle. Assume that A beats B , B beats C , and C beats A . (There are $\binom{5}{3} = 10$ ways to choose the cycle and 2 ways to orient the cycle.) Then D either beats all three or is beaten by all three, because otherwise there exists two people X and Y in these three people such that X beats Y , and D beats Y but is beaten by X , and then X, D, Y, Z will form a 4-cycle (Z is the remaining

person of the three). The same goes for E . If D and E both beat all three or are beaten by all three, then there is no restriction on the match between D and E . However, if D beats all three and E loses to all three, then E cannot beat D because otherwise E, D, A, B forms a 4-cycle. This means that A, B, C is the only 3-cycle in the tournament, and once the cycle is chosen there are $2 \cdot 2 + 2 \cdot 1 = 6$ ways to choose the results of remaining matches, for $10 \cdot 2 \cdot 6 = 120$ ways in total.

Case 2: There are no cycles. This means that the tournament is a complete ordering (the person with a higher rank always beats the person with a lower rank). There are $5! = 120$ ways in this case as well.

Therefore, the probability of *not* having a 4-cycle is $\frac{120+120}{2^{10}} = \frac{15}{64}$, and thus the answer is $1 - \frac{15}{64} = \frac{49}{64}$.

HMMT November 2017

November 11, 2017

Guts Round

1. [2] Suppose x is a rational number such that $x\sqrt{2}$ is also rational. Find x .

Proposed by: Michael Kural

Answer: $\boxed{0}$

Suppose $x \neq 0$. Then $\frac{x\sqrt{2}}{x} = \sqrt{2}$ is the quotient of two nonzero rationals, and so is rational. However, it is well-known that $\sqrt{2}$ is irrational. Therefore any solution x must satisfy $x = 0$. We can see that 0 is rational and $0\sqrt{2} = 0$ is rational, so the answer is indeed $x = \boxed{0}$.

2. [2] Regular octagon *CHILDREN* has area 1. Find the area of pentagon *CHILD*.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1}{2}}$

The pentagon *CHILD* is congruent to the pentagon *NERDC*, as their corresponding angles and sides are congruent. Moreover, the two pentagons together compose the entire octagon, so each pentagon must have area one-half of the area of the octagon, or $\frac{1}{2}$.

3. [2] The length of a rectangle is three times its width. Given that its perimeter and area are both numerically equal to $k > 0$, find k .

Proposed by: Yuan Yao

Answer: $\boxed{\frac{64}{3}}$

Let a be the width of the rectangle. Then the length of the rectangle is $3a$, so the perimeter is $2(a + 3a) = 8a$, and the area is $3a^2$. Since the length is numerically equal to the width, we know that

$$8a = 3a^2 = k.$$

Because $k > 0$, the rectangle is non-degenerate. It follows that $8 = 3a$, so $a = \frac{8}{3}$. Therefore, $k = \boxed{\frac{64}{3}}$.

4. [3] Alec wishes to construct a string of 6 letters using the letters A, C, G, and N, such that:

- The first three letters are pairwise distinct, and so are the last three letters;
- The first, second, fourth, and fifth letters are pairwise distinct.

In how many ways can he construct the string?

Proposed by: Yuan Yao

There are $4! = 24$ ways to decide the first, second, fourth, and fifth letters because these letters can be selected sequentially without replacement from the four possible letters. Once these four letters are selected, there are 2 ways to select the third letter because two distinct letters have already been selected for the first and second letters, leaving two possibilities. The same analysis applies to the sixth letter. Thus, there are $24 \cdot 2^2 = \boxed{96}$ total ways to construct the string.

5. [3] Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_n = (a_{n-1})! + 1$ for every $n > 1$. Find the least n for which $a_n > 10^{10}$.

Proposed by: Michael Tang

Answer: $\boxed{6}$

We have $a_2 = 2$, $a_3 = 3$, $a_4 = 7$, $a_5 = 7! + 1 = 5041$, and $a_6 = 5041! + 1$. But

$$5041! + 1 \gg 5041 \cdot 5040 \cdot 5039 > 10^{10}.$$

Hence, the answer is $\boxed{6}$.

6. [3] Lunasa, Merlin, and Lyrica each have a distinct hat. Every day, two of these three people, selected randomly, switch their hats. What is the probability that, after 2017 days, every person has their own hat back?

Proposed by: Yuan Yao

Answer:

Imagine that the three hats are the vertices of an equilateral triangle. Then each day the exchange is equivalent to reflecting the triangle along one of its three symmetry axes, which changes the orientation of the triangle (from clockwise to counterclockwise or vice versa). Thus, an even number of such exchanges must be performed if the orientation is to be preserved. Since the triangle is reflected 2017 times, it is impossible for the final triangle to have the same orientation as the original triangle, so the desired probability is .

7. [3] Compute

$$100^2 + 99^2 - 98^2 - 97^2 + 96^2 + 95^2 - 94^2 - 93^2 + \dots + 4^2 + 3^2 - 2^2 - 1^2.$$

Proposed by: Michael Tang

Answer:

Note that $(n+3)^2 - (n+2)^2 - (n+1)^2 + n^2 = 4$ for every n . Therefore, adding 0^2 to the end of the given sum and applying this identity for every four consecutive terms after 100^2 , we see that the given sum is equivalent to $100^2 + 25 \cdot 4 = \text{input type="text" value="10100"}$.

Alternatively, we can apply the difference-of-squares factorization to rewrite $100^2 - 98^2 = (100 - 98)(100 + 98) = 2(100 + 98)$, $99^2 - 97^2 = (99 - 97)(99 + 97) = 2(99 + 97)$, etc. Thus, the given sum is equivalent to $2(100 + 99 + \dots + 2 + 1) = 2 \cdot \frac{100 \cdot 101}{2} = 10100$.

8. [3] Regular octagon *CHILDREN* has area 1. Determine the area of quadrilateral *LINE*.

Proposed by: Yuan Yao

Answer:

Suppose that the side length $CH = \sqrt{2}a$, then the area of the octagon is $((2 + \sqrt{2})a)^2 - 4 \cdot \frac{1}{2}a^2 = (4 + 4\sqrt{2})a^2$, and the area of *LINE* is $(\sqrt{2}a)((2 + \sqrt{2})a) = (2 + 2\sqrt{2})a^2$, which is exactly one-half of the area of the octagon. Therefore the area of *LINE* is $\frac{1}{2}$.

9. [3] A malfunctioning digital clock shows the time 9 : 57 AM; however, the correct time is 10 : 10 AM. There are two buttons on the clock, one of which increases the time displayed by 9 minutes, and another which decreases the time by 20 minutes. What is the minimum number of button presses necessary to correctly set the clock to the correct time?

Proposed by: Kevin Sun

Answer:

We need to increase the time by 13 minutes. If we click the 9 minute button a times and the 20 minute button b times, then we must have $9a - 20b = 13$. Note that if this equation is satisfied, then b increases as a increases, so it suffices to minimize a . This means that a must end in a 7. However, since $63 - 20b = 13$ has no integer solution, the next smallest possible value of a is 17, which gives the solution $(a, b) = (17, 7)$, or 24 button presses.

10. [4] Compute $\frac{x}{w}$ if $w \neq 0$ and $\frac{x + 6y - 3z}{-3x + 4w} = \frac{-2y + z}{x - w} = \frac{2}{3}$.

Proposed by: Angela Deng

Answer:

We have $x + 6y - 3z = \frac{2}{3}(-3x + 4w)$ and $-2y + z = \frac{2}{3}(x - w)$, so

$$\frac{x}{w} = \frac{(x + 6y - 3z) + 3(-2y + z)}{(-3x + 4w) + 3(x - w)} = \frac{\frac{2}{3}(-3x + 4w) + 3 \cdot \frac{2}{3}(x - w)}{(-3x + 4w) + 3(x - w)} = \frac{\frac{2}{3}[(-3x + 4w) + 3(x - w)]}{(-3x + 4w) + 3(x - w)} = \boxed{\frac{2}{3}}$$

11. [4] Find the sum of all real numbers x for which

$$[\dots [[[x] + x] + x] \dots] + x = 2017 \text{ and } \{\{\dots \{\{x\} + x\} + x\} \dots\} + x = \frac{1}{2017}$$

where there are 2017 x 's in both equations. ($[x]$ is the integer part of x , and $\{x\}$ is the fractional part of x .) Express your sum as a mixed number.

Proposed by: Yuan Yao

Answer: $\boxed{3025\frac{1}{2017} \text{ or } \frac{6101426}{2017}}$

The two equations are equivalent to $2017[x] = 2017$ and $\{2017x\} = \frac{1}{2017}$, respectively. The first equation reduces to $[x] = 1$, so we must have $x = 1 + r$ for some real r satisfying $0 \leq r < 1$. From the second equation, we deduce that $\{2017x\} = \{2017 + 2017r\} = \{2017r\} = \frac{1}{2017}$, so $2017r = n + \frac{1}{2017}$, where n is an integer. Dividing both sides of this equation by 2017 yields $r = \frac{n}{2017} + \frac{1}{2017^2}$, where $n = 0, 1, 2, \dots, 2016$ so that we have $0 \leq r < 1$. Thus, we have $x = 1 + r = 1 + \frac{n}{2017} + \frac{1}{2017^2}$ for $n = 0, 1, 2, \dots, 2016$. The sum of these solutions is $2017 \cdot 1 + \frac{2016 \cdot 2017}{2} \cdot \frac{1}{2017} + 2017 \cdot \frac{1}{2017^2} =$

$$2017 + \frac{2016}{2} + \frac{1}{2017} = \boxed{3025\frac{1}{2017}}.$$

12. [4] Trapezoid $ABCD$, with bases AB and CD , has side lengths $AB = 28$, $BC = 13$, $CD = 14$, and $DA = 15$. Let diagonals AC and BD intersect at P , and let E and F be the midpoints of AP and BP , respectively. Find the area of quadrilateral $CDEF$.

Proposed by: Christopher Shao

Answer: $\boxed{112}$

Note that EF is a midline of triangle APB , so EF is parallel to AB and $EF = \frac{1}{2}AB = 14 = CD$. We also have that EF is parallel to CD , and so $CDEF$ is a parallelogram. From this, we have $EP = PC$ as well, so $\frac{CE}{CA} = \frac{2}{3}$. It follows that the height from C to EF is $\frac{2}{3}$ of the height from C to AB . We can calculate that the height from C to AB is 12, so the height from C to EF is 8. Therefore $CDEF$ is a parallelogram with base 14 and height 8, and its area is $14 \cdot 8 = 112$.

13. [5] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of V cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of 2017 cubic centimeters. Find the positive difference between the least and greatest possible positive values for V .

Proposed by: Yuan Yao

Answer: $\boxed{4035}$

The only possible way for Ritmo to get 2017 cubic centimeters is to have his measurements rounded to 1, 1, 2017 centimeters respectively. Therefore the largest value of V is achieved when the dimensions are $(1.5 - \epsilon)(1.5 - \epsilon)(2017.5 - \epsilon) = 4539.375 - \epsilon'$ for some very small positive real ϵ, ϵ' , and the smallest value of V is achieved when the dimensions are $(0.5 + \epsilon)(0.5 + \epsilon)(2016.5 + \epsilon) = 504.125 + \epsilon'$ for some very small positive real ϵ, ϵ' . Therefore the positive difference is $4539 - 504 = 4035$.

14. [5] Points A, B, C , and D lie on a line in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = 4$, find AD .

Proposed by: Yuan Yao

Answer: 6

Let $BC = x$, then the equation becomes $\frac{3-x}{x} = \frac{7-x}{4-x}$. This simplifies to a quadratic equation with solutions $x = 1$ and $x = 6$. Since $x < 3$, we have $x = 1$ and $AD = 7 - x = 6$.

15. [5] On a 3×3 chessboard, each square contains a knight with $\frac{1}{2}$ probability. What is the probability that there are two knights that can attack each other? (In chess, a knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction.)

Proposed by: Yuan Yao

Answer: $\frac{209}{256}$

Notice that a knight on the center square cannot attack any other square on the chessboard, so whether it contains a knight or not is irrelevant.

For ease of reference, we label the other eight squares as follows:

0	5	2
3	X	7
6	1	4

Notice that a knight in square i attacks both square $i + 1$ and $i - 1$ (where square numbers are reduced modulo 8). We now consider the number of ways such that *no* two knights attack each other.

- 0 knights: 1 way.
- 1 knights: 8 ways.
- 2 knights: $\binom{8}{2} - 8 = 20$ ways.
- 3 knights: $8 + 8 = 16$ ways, where the two 8s represent the number of ways such that the “distances” between the knights (index-wise) are 2, 2, 4 and 2, 3, 3 respectively.
- 4 knights: 2 ways.

Therefore, out of $2^8 = 256$ ways, $1 + 8 + 20 + 16 + 2 = 47$ of them doesn't have a pair of attacking knights. Thus the answer is $\frac{256-47}{256} = \frac{209}{256}$.

16. [7] A *repunit* is a positive integer, all of whose digits are 1s. Let $a_1 < a_2 < a_3 < \dots$ be a list of all the positive integers that can be expressed as the sum of distinct repunits. Compute a_{111} .

Proposed by: Michael Tang

Answer: 1223456

Let $\{r_n\}_{n \geq 0}$ be the repunits (so $r_0 = 1$, $r_1 = 11$, and so on). We see that for any n , there is

$$r_{n-1} + r_{n-2} + \dots + r_0 < \frac{r_n}{10} + \frac{r_n}{100} + \dots < \frac{r_n}{9} < r_n,$$

so r_n is only needed when all possible combinations of the first n repunits are exhausted (after 2^n terms), which shows that there is a bijection between the sequence $\{a_n\}$ and the binary numbers. In particular, if $k = 2^{n_1} + 2^{n_2} + \dots + 2^{n_s}$ for distinct n_i 's, then $a_k = r_{n_1} + r_{n_2} + \dots + r_{n_s}$. Since $111 = 1101111_2 = 2^0 + 2^1 + 2^2 + 2^3 + 2^5 + 2^6$, we have

$$a_{111} = r_0 + r_1 + r_2 + r_3 + r_5 + r_6 = 1223456.$$

17. [7] A standard deck of 54 playing cards (with four cards of each of thirteen ranks, as well as two Jokers) is shuffled randomly. Cards are drawn one at a time until the first queen is reached. What is the probability that the next card is also a queen?

Proposed by: Serina Hu

Answer: $\frac{2}{27}$

Since the four queens are equivalent, we can compute the probability that a specific queen, say the queen of hearts, is right after the first queen. Remove the queen of hearts; then for every ordering of the 53 other cards, there are 54 locations for the queen of hearts, and exactly one of those is after the first queen. Therefore the probability that the queen of hearts immediately follows the first queen is $\frac{1}{54}$, and the probability any queen follows the first queen is $\frac{1}{54} \cdot 4 = \frac{2}{27}$.

18. [7] Mr. Taf takes his 12 students on a road trip. Since it takes two hours to walk from the school to the destination, he plans to use his car to expedite the journey. His car can take at most 4 students at a time, and travels 15 times as fast as traveling on foot. If they plan their trip optimally, what is the shortest amount of time it takes for them to all reach the destination, in minutes?

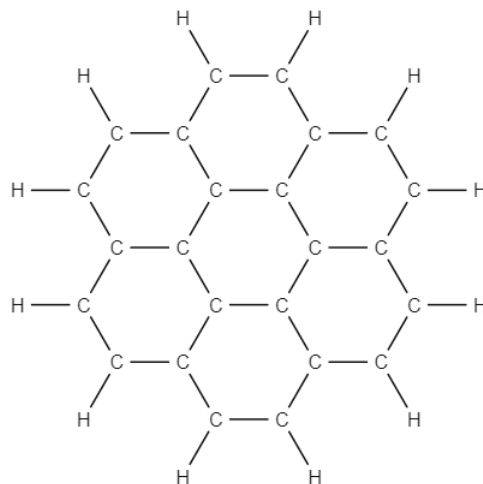
Proposed by: Yuan Yao

Answer: 30.4 OR $\frac{152}{5}$

A way to plan the trip is to have Mr. Taf drive 4 students to the 80% mark, then drive back to the 10% mark to pick up another 4 students to the 90% mark, and finally drive back to the 20% mark to pick up the last 4 students to the destination. All students will reach the destination at the same time, and Mr. Taf would have driven for $(0.8 + 0.7 + 0.8 + 0.7 + 0.8) \cdot \frac{120}{15} = 30.4$ minutes.

Now we show that 30.4 minutes is necessary. First of all, for a trip to be optimal, Mr. Taf must not carry students when he was driving away from the destination, and all student not on the car must keep walking towards the destination. Suppose that among all the students, the student that walked for the longest walked for $15m$ minutes, where $0 \leq m \leq 8$, then he spent $\frac{120-15m}{15} = 8 - m$ minutes on the car, so it took them exactly $14m + 8$ minutes to get to the destination. Moreover, all students must have spent at least $12(8 - m)$ minutes on the car total, and Mr. Taf would need to spend at least $3(8 - m) = 24 - 3m$ minutes driving students *towards* the destination. Since it takes Mr. Taf 8 minutes to drive the entire trip, he would need to drive $3(8 - m) - 8 = 16 - 3m$ minutes *away* from the destination, so Mr. Fat drove for at least $40 - 6m$ minutes. From this we derive the inequality $40 - 6m \geq 14m + 8$, which comes out to $m \geq 1.6$, so the journey is at least $14(1.6) + 8 = 30.4$ minutes.

19. [9] The skeletal structure of coronene, a hydrocarbon with the chemical formula $C_{24}H_{12}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied?

Proposed by: Yuan Yao

Answer: 20

Note that each carbon needs exactly one double bond. Label the six carbons in the center 1, 2, 3, 4, 5, 6 clockwise. We consider how these six carbons are double-bonded. If a carbon in the center is not double-bonded to another carbon in the center, it must double-bond to the corresponding carbon on the outer ring. This will result in the outer ring broken up into (some number of) strings instead of a loop, which means that there will be at most one way to pair off the outer carbons through double-bonds. (In fact, as we will demonstrate later, there will be exactly one way.)

Now we consider how many double bonds are on the center ring.

- 3 bonds. There are 2 ways to pair of the six carbons, and 2 ways to pair of the outer ring as well, for 4 ways in total.
- 2 bonds. Then either two adjacent carbons (6 ways) or two diametrically opposite carbons (3 ways) are not double-bonded, and in the former case the outer ring will be broken up into two “strands” with 2 and 14 carbons each, while in the latter case it will be broken up into two strands both with 8 carbons each, and each produce one valid way of double-bonding, for 9 ways in total.
- 1 bond. There are 6 ways to choose the two double-bonded center carbon, and the outer ring will be broken up into four strands with 2, 2, 2, 8 carbons each, which gives one valid way of double-bonding, for 6 ways in total.
- 0 bonds. Then the outer ring is broken up into six strands of 2 carbons each, giving 1 way.

Therefore, the number of possible arrangements is $4 + 9 + 6 + 1 = 20$.

Note: each arrangement of single/double bonds is also called a *resonance structure* of coronene.

20. [9] Rebecca has four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R ?

Proposed by: Yuan Yao

Answer: 15

Let R_n be the set of all possible resistances using exactly n 1-ohm circuit segments (without shorting any of them), then we get $R_n = \bigcup_{i=1}^{n-1} \left(\{a + b \mid a \in R_i, b \in R_{n-i}\} \cup \left\{ \frac{ab}{a+b} \mid a \in R_i, b \in R_{n-i} \right\} \right)$, starting with $R_1 = \{1\}$, we get:

$$\begin{aligned} R_2 &= \left\{ \frac{1}{2}, 2 \right\} \\ R_3 &= \left\{ \frac{1}{3}, \frac{2}{3}, \frac{3}{2}, 3 \right\} \\ R_4 &= \left\{ \frac{1}{4}, \frac{2}{5}, \frac{3}{5}, \frac{3}{4}, 1, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4 \right\} \end{aligned}$$

Their union is the set of all possible effective resistances we can get, which contains $2 + 4 + 9 = 15$ values. (Note that $R_1 \subset R_4$ and the sets R_2, R_3, R_4 are disjoint.)

21. [9] A box contains three balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box*. Consider the following two conditions:

- (1) Some ball has been drawn at least three times (not necessarily consecutively).
- (2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{13}{27}}$

At any time, we describe the current state by the number of times each ball is drawn, sorted in nonincreasing order. For example, if the red ball has been drawn twice and green ball once, then the state would be $(2, 1, 0)$. Given state S , let P_S be the probability that the state was achieved at some point of time before one of the two conditions are satisfied. Starting with $P_{(0,0,0)} = 1$, we compute:

$$\begin{aligned} P_{(1,0,0)} &= 1; \\ P_{(1,1,0)} &= \frac{2}{3}, \quad P_{(2,0,0)} = \frac{1}{3}; \\ P_{(1,1,1)} &= \frac{1}{3}P_{(1,1,0)} = \frac{2}{9}, \quad P_{(2,1,0)} = \frac{2}{3}P_{(1,1,0)} + \frac{2}{3}P_{(2,0,0)} = \frac{2}{3}, \quad P_{(3,0,0)} = \frac{1}{3}P_{(2,0,0)} = \frac{1}{9}; \\ P_{(2,1,1)} &= P_{(2,2,0)} = P_{(3,1,0)} = \frac{1}{3}P_{(2,1,0)} = \frac{2}{9}; \\ P_{(2,2,1)} &= \frac{1}{3}P_{(2,2,0)} = \frac{2}{27}, \quad P_{(3,2,0)} = \frac{2}{3}P_{(2,2,0)} = \frac{4}{27}. \end{aligned}$$

Therefore, the probability that the first condition is satisfied first is $P_{(3,0,0)} + P_{(3,1,0)} + P_{(3,2,0)} = \frac{1}{9} + \frac{2}{9} + \frac{4}{27} = \frac{13}{27}$.

22. [12] A sequence of positive integers $a_1, a_2, \dots, a_{2017}$ has the property that for all integers m where $1 \leq m \leq 2017$, $3(\sum_{i=1}^m a_i)^2 = \sum_{i=1}^m a_i^3$. Compute a_{1337} .

Proposed by: Serina Hu

Answer: $\boxed{4011}$

I claim that $a_i = 3i$ for all i . We can conjecture that the sequence should just be the positive multiples of three because the natural numbers satisfy the property that the square of their sum is the sum of their cubes, and prove this by induction. At $i = 1$, we have that $3a_1^2 = a_1^3$, so $a_1 = 3$. Now assuming this holds for $i = m$, we see that

$$\begin{aligned} 3 \left(\sum_{i=1}^{m+1} a_i \right)^2 &= 3 \left(a_{m+1} + \sum_{i=1}^m a_i \right)^2 \\ &= 3a_{m+1}^2 + \sum_{i=1}^m a_i^3 + 6a_{m+1} \sum_{i=1}^m a_i \\ &= 3a_{m+1}^2 + \sum_{i=1}^m a_i^3 + 6a_{m+1} \cdot 3 \left(\frac{m(m+1)}{2} \right) \\ &= \sum_{i=1}^{m+1} a_i^3. \end{aligned}$$

Therefore,

$$\begin{aligned} a_{m+1}^3 &= 3a_{m+1}^2 + a_{m+1}(9m^2 + 9m) \\ 0 &= a_{m+1}^2 - 3a_{m+1} - (9m^2 + 9m) \\ 0 &= (a_{m+1} - (3m + 3))(a_{m+1} + 3m) \end{aligned}$$

and because the sequence is positive, $a_{m+1} = 3m + 3$, which completes the induction. Then $a_{1337} = 1337 \cdot 3 = 4011$.

23. [12] A string of digits is defined to be *similar* to another string of digits if it can be obtained by reversing some contiguous substring of the original string. For example, the strings 101 and 110 are similar, but the strings 3443 and 4334 are not. (Note that a string is always similar to itself.) Consider the string of digits

$$S = 01234567890123456789012345678901234567890123456789,$$

consisting of the digits from 0 to 9 repeated five times. How many distinct strings are similar to S ?

Proposed by: Kevin Sun

Answer: 1126

We first count the number of substrings that one could pick to reverse to yield a new substring. If we insert two dividers into the sequence of 50 digits, each arrangement of 2 dividers among the 52 total objects specifies a substring that is contained between the two dividers, for a total of $\binom{52}{2}$ substrings. Next, we account for overcounting. Every substring of length 0 or 1 will give the identity string when reversed, so we are overcounting here by $51 + 50 - 1 = 100$ substrings. Next, for any longer substring s that starts and ends with the same digit, removing the digit from both ends results in a substring s' , such that reversing s would give the same string as reversing s' . Therefore, we are overcounting by $10 \cdot \binom{5}{2}$ substrings. Our total number of strings similar to S is therefore $\binom{52}{2} - 100 - 10 \cdot \binom{5}{2} = \boxed{1126}$.

24. [12] Triangle ABC has side lengths $AB = 15, BC = 18, CA = 20$. Extend CA and CB to points D and E respectively such that $DA = AB = BE$. Line AB intersects the circumcircle of CDE at P and Q . Find the length of PQ .

Proposed by: Yuan Yao

Answer: 37

WLOG suppose that P is closer to A than to B . Let $DA = AB = BE = c = 15, BC = a = 18, CA = b = 20, PA = x$, and $QB = y$. By Power of a Point on B and A , we get $ac = (x+c)y$ and $bc = (y+c)x$, respectively. Subtracting the two equations gives $cy - cx = ac - bc \Rightarrow y - x = a - b$. Substituting $y = x + a - b$ into the first equation gives $ac = (x+c)(x+a-b) = x^2 + (a-b+c)x + ac - bc$, which is a quadratic with unique positive solution $x = \frac{(b-a-c) + \sqrt{(a-b+c)^2 + 4bc}}{2}$. Thus,

$$PQ = x + y + c = (y - x) + 2x + c = (a - b + c) + (b - a - c) + \sqrt{(a - b + c)^2 + 4bc} = \sqrt{13^2 + 4 \cdot 20 \cdot 15} = 37.$$

25. [15] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of 2017 cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of V cubic centimeters. Find the positive difference between the least and greatest possible positive values for V .

Proposed by: Yuan Yao

Answer: 7174

It is not difficult to see that the maximum possible value of V can be achieved when the dimensions are $(0.5 + \epsilon) \times (0.5 + \epsilon) \times (8070 - \epsilon') = 2017.5 - \epsilon''$ for some very small reals $\epsilon, \epsilon', \epsilon'' > 0$, which when measured by Ritmo, gives $V = 1 \cdot 1 \cdot 8070 = 8070$. Similarly, the minimum possible *positive* value of V can be achieved when the dimensions are $(1.5 - \epsilon) \times (1.5 - \epsilon) \times (\frac{8066}{9} + \epsilon') = 2016.5 + \epsilon''$ for some very small reals $\epsilon, \epsilon', \epsilon'' > 0$, which when measured by Ritmo, gives $V = 1 \cdot 1 \cdot 896 = 896$. Therefore, the difference between the maximum and minimum is $8070 - 896 = 7174$.

26. [15] Points A, B, C, D lie on a circle in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = BC = 4$, find AD .

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{2}}$

By Ptolemy's theorem, we have $AB \cdot CD + BC \cdot DA = AC \cdot BD = 3 \cdot 4 = 12$. Since the condition implies $AB \cdot CD = BC \cdot DA$, we have $DA = \frac{6}{BC} = \frac{3}{2}$.

27. [15] On a 3×3 chessboard, each square contains a Chinese knight with $\frac{1}{2}$ probability. What is the probability that there are two Chinese knights that can attack each other? (In Chinese chess, a Chinese knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction, under the condition that there is no other piece immediately adjacent to it in the first direction.)

Proposed by: Yuan Yao

Answer: $\boxed{\frac{79}{256}}$

Suppose the 3×3 square is

A	B	C
D	E	F
G	H	I

 We count the number of ways a board could have two knights

attack each other using PIE. First notice that in any setup with two knights attack each other, the center square must be empty. Also, for any pair of knights that attack each other, one must be in a corner, and the other at the center of a nonadjacent side. There are $8 \cdot 2^5$ ways for one pair of knights to attack each other. Next, we count the number of ways two pairs of knights attack each other: up to symmetry, there are four cases: knights at A, B, G, H, and D and E empty; knights at A, H, F, and B, D, E empty; knights at A, B, H, I, and D, E, F empty; and knights at A, C, H, and D, E, F empty. Four each of these cases, there are four symmetries, so there are a total of $4 \cdot (2^3 + 2^3 + 2^2 + 2^3)$ ways to have two pairs of knights attack each other. Next, there's only one way for three pairs of knights to attack each other, discounting symmetry: A, B, G, H, I have knights, and D, E, F empty. Then there are $4 \cdot 2 \cdot 2$ ways for three knights to attack. Finally, there is only one way for four knights to attack: knights at A, B, C, G, H, I and empty squares at D, E, F, for a total of 2 ways after counting symmetries.

Applying PIE, we get that the total number of boards with at least one pair of knights attacking each other is

$$8 \cdot 2^5 - 4 \cdot (2^3 + 2^3 + 2^2 + 2^3) + 4 \cdot 2 \cdot 2 - 2 = 158.$$

Then the probability the 3×3 board has a pair of knights attacking each other is $\frac{158}{2^9} = \boxed{\frac{79}{256}}$.

28. [19] Compute the number of functions $f : \{1, 2, \dots, 9\} \rightarrow \{1, 2, \dots, 9\}$ which satisfy $f(f(f(f(f(x)))))) = x$ for each $x \in \{1, 2, \dots, 9\}$.

Proposed by: Evan Chen

Answer: $\boxed{3025}$

All cycles lengths in the permutation must divide 5, which is a prime number. Either $f(x) = x$ for all x , or there exists exactly one permutation cycle of length 5. In the latter case, there are $\binom{9}{5}$ ways to choose which numbers are in the cycle and $4!$ ways to create the cycle. The answer is thus $1 + \binom{9}{5} \cdot 4! = 3025$.

29. [19] Consider a sequence x_n such that $x_1 = x_2 = 1$, $x_3 = \frac{2}{3}$. Suppose that $x_n = \frac{x_{n-1}^2 x_{n-2}}{2x_{n-2}^2 - x_{n-1} x_{n-3}}$ for all $n \geq 4$. Find the least n such that $x_n \leq \frac{1}{10^6}$.

Proposed by: Mehtaab Sawhney

Answer: $\boxed{13}$

The recursion simplifies to $\frac{x_{n-1}}{x_n} + \frac{x_{n-3}}{x_{n-2}} = 2\frac{x_{n-2}}{x_{n-1}}$. So if we set $y_n = \frac{x_{n-1}}{x_n}$ for $n \geq 2$ then we have $y_n - y_{n-1} = y_{n-1} - y_{n-2}$ for $n \geq 3$, which means that $\{y_n\}$ is an arithmetic sequence. From the

starting values we have $y_2 = 1, y_3 = \frac{3}{2}$, so $y_n = \frac{n}{2}$ for all n . (This means that $x_n = \frac{2^{n-1}}{n!}$.) Since $\frac{x_1}{x_n} = y_2 y_3 \cdots y_n$, it suffices to find the minimal n such that the RHS is at least 10^6 . Note that

$$y_2 y_3 \cdots y_{12} = 1 \cdot (1.5 \cdot 2 \cdot 2.5 \cdot 3 \cdot 3.5) \cdot (4 \cdot 4.5 \cdot 5 \cdot 5.5 \cdot 6) < 2.5^5 \cdot 5^5 = 12.5^5 < 200^2 \cdot 12.5 = 500000 < 10^6,$$

while

$$y_2 y_3 \cdots y_{13} = 1 \cdot (1.5 \cdot 2 \cdot 2.5 \cdot 3) \cdot (3.5 \cdot 4 \cdot 4.5) \cdot (5 \cdot 5.5 \cdot 6 \cdot 6.5) > 20 \cdot 60 \cdot 900 = 1080000 > 10^6,$$

so the answer is 13.

30. [19] Given complex number z , define sequence z_0, z_1, z_2, \dots as $z_0 = z$ and $z_{n+1} = 2z_n^2 + 2z_n$ for $n \geq 0$. Given that $z_{10} = 2017$, find the minimum possible value of $|z|$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{1024\sqrt[4]{4035}-1}{2}}$

Define $w_n = z_n + \frac{1}{2}$, so $z_n = w_n - \frac{1}{2}$, and the original equation becomes

$$w_{n+1} - \frac{1}{2} = 2(w_n - \frac{1}{2})^2 + 2(w_n - \frac{1}{2}) = 2w_n^2 - \frac{1}{2},$$

which reduces to $w_{n+1} = 2w_n^2$. it is not difficult to show that

$$z_{10} + \frac{1}{2} = 2017 + \frac{1}{2} = \frac{4035}{2} = w_{10} = 2^{1023} w_0^{1024},$$

and thus $w_0 = \frac{1024\sqrt[4]{4035}}{2} \omega_{1024}$, where ω_{1024} is one of the 1024th roots of unity. Since $|w_0| = \frac{1024\sqrt[4]{4035}}{2} > \frac{1}{2}$, to minimize the magnitude of $z = w_0 - \frac{1}{2}$, we need $\omega_{1024} = -1$, which gives $|z| = \frac{1024\sqrt[4]{4035}-1}{2}$.

31. [24] In unit square $ABCD$, points E, F, G are chosen on side BC, CD, DA respectively such that AE is perpendicular to EF and EF is perpendicular to FG . Given that $GA = \frac{404}{1331}$, find all possible values of the length of BE .

Proposed by: Yuan Yao

Answer: $\boxed{\frac{9}{11}}$

Let $BE = x$, then since triangles ABE, ECF, FDG are all similar, we have $CE = 1 - x, CF = x(1 - x), FD = 1 - x(1 - x), DG = x - x^2(1 - x), GA = 1 - x + x^2(1 - x) = (1 - x)(x^2 + 1)$, therefore it remains to solve the equation

$$(1 - x)(x^2 + 1) = \frac{404}{1331}.$$

We first seek rational solutions $x = \frac{p}{q}$ for relatively prime positive integers p, q . Therefore we have $\frac{(q-p)(p^2+q^2)}{q^3} = \frac{404}{1331}$. Since both $q-p$ and p^2+q^2 are relatively prime to q^3 , we have $q^3 = 1331 \Rightarrow q = 11$, so $(11 - p)(p^2 + 121) = 404 = 2^2 \cdot 101$, and it is not difficult to see that $p = 9$ is the only integral solution. We can therefore rewrite the original equation as

$$(x - \frac{9}{11})(x^2 - \frac{2}{11}x + \frac{103}{121}) = 0.$$

It is not difficult to check that the quadratic factor has no zeroes, therefore $BE = x = \frac{9}{11}$ is the only solution.

32. [24] Let P be a polynomial with integer coefficients such that $P(0) + P(90) = 2018$. Find the least possible value for $|P(20) + P(70)|$.

Proposed by: Michael Tang

Answer: $\boxed{782}$

First, note that $P(x) = x^2 - 3041$ satisfy the condition and gives $|P(70) + P(20)| = |4900 + 400 - 6082| = 782$. To show that 782 is the minimum, we show $2800 \mid P(90) - P(70) - P(20) + P(0)$ for every P , since -782 is the only number in the range $[-782, 782]$ that is congruent to 2018 modulo 2800.

Proof: It suffices to show that $2800 \mid 90^n - 70^n - 20^n + 0^n$ for every $n \geq 0$ (having $0^0 = 1$).

Let $Q(n) = 90^n - 70^n - 20^n + 0^n$, then we note that $Q(0) = Q(1) = 0$, $Q(2) = 2800$, and $Q(3) = (9^3 - 7^3 - 2^3) \cdot 10^3 = 378000 = 135 \cdot 2800$. For $n \geq 4$, we note that 400 divides 10^4 , and $90^n + 0^n \equiv 70^n + 20^n \pmod{7}$. Therefore $2800 \mid Q(n)$ for all n .

33. [24] Tetrahedron $ABCD$ with volume 1 is inscribed in circumsphere ω such that $AB = AC = AD = 2$ and $BC \cdot CD \cdot DB = 16$. Find the radius of ω .

Proposed by: Caleb He

Answer: $\boxed{\frac{5}{3}}$

Let X be the foot of the perpendicular from A to $\triangle BCD$. Since $AB = AC = AD$, it follows that X is the circumcenter of $\triangle BCD$. Denote $XB = XC = XD = r$. By the Pythagorean Theorem on $\triangle ABX$, we have $AX = \sqrt{4 - r^2}$. Now, from the extended law of sines on $\triangle BCD$, we have the well-known identity

$$\frac{BC \cdot CD \cdot DB}{4r} = [BCD],$$

where $[BCD]$ denotes the area of $\triangle BCD$. However, we have

$$V = \frac{1}{3} AX \cdot [BCD],$$

where V is the volume of $ABCD$, which yields the expression

$$[BCD] = \frac{3}{\sqrt{4 - r^2}}.$$

Now, given that $BC \cdot CD \cdot DB = 16$, we have

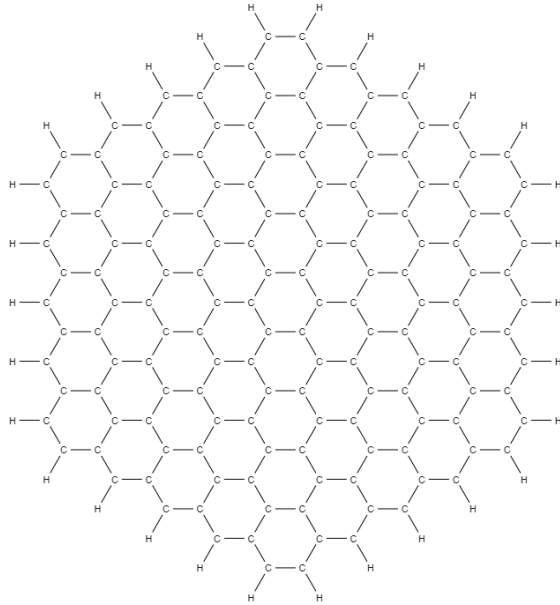
$$\frac{4}{r} = \frac{3}{\sqrt{4 - r^2}}.$$

Solving, we get $r = \frac{8}{5}$. Now, let O be the center of ω . Since $OB = OC = OD$, it follows that the foot of the perpendicular from O to $\triangle BCD$ must also be the circumcenter of $\triangle BCD$, which is X . Thus, A, X, O are collinear. Let R be the radius of ω . Then we have

$$\begin{aligned} R &= OA \\ &= OX + XA \\ &= \sqrt{R^2 - r^2} + \sqrt{4 - r^2} \\ &= \sqrt{R^2 - \frac{64}{25}} + \frac{6}{5}. \end{aligned}$$

Solving, we get $R = \frac{5}{3}$. (Note: solving for R from $OA = OX - XA$ gives a negative value for R .)

34. [30] The skeletal structure of circumcircumcircumcoronene, a hydrocarbon with the chemical formula $C_{150}H_{30}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? If the correct answer is C and your answer is A , you get $\max\left(\left\lfloor 30\left(1 - \left\lceil \log_{\log_2 C} \frac{A}{C} \right\rceil\right)\right\rfloor, 0\right)$ points.

Proposed by: Yuan Yao

Answer: 267227532

The problem is equivalent to the one in OEIS A008793, a.k.a. "number of ways to tile hexagon of edge n with diamonds of side 1." Notice that there is a bijection between such a tiling and the number of ways to stack some unit cubes alongside a corner of an $n \times n \times n$ box (see the Art of Problem Solving logo as an example, also known as 3-dimensional Young diagrams), where this problem $n = 5$. It is known that there are $\binom{2n}{n} = 252$ ways to stack one layer (since each way correspond a way to walk from a corner of a 5 by 5 grid to the opposite one), so $\frac{252^5}{5!} \approx 8 \times 10^9$ gives a somewhat loose upper bound (generate five layers and sort them by size, and hope that it will be a valid stack in general). This result can be improved by dividing out a reasonable constant factor after considering the probability that sorting by size indeed gives a valid stack (for example, it would be fair to guess that there is about a $\frac{1}{4!}$ chance that the first row of each layer will be in the right order, given that each row has a small role in determining the final size of the layer; dividing 24 from the previous result gives a very close guess). In general, a guess anywhere between 10^8 and 10^9 is a fair guess.

35. [30] Rebecca has twenty-four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after twenty-three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R ? If the correct answer is C and your answer is A , you get $\max\left(\left\lfloor 30\left(1 - \left\lceil \log_{\log_2 C} \frac{A}{C} \right\rceil\right)\right\rfloor, 0\right)$ points.

Proposed by: Yuan Yao

Answer: 1015080877

This is the same problem as in OEIS A153588. It is helpful to see (or guess) that neither the numerator or the denominator of the final resistance exceed the $(n + 1)$ -th Fibonacci number, which in this case is $F_{25} = 75025$, using concepts on the line of continued fractions. So $75025^2 \approx 5.6 \times 10^9$ is an upper bound for the total number, which is already close to the final answer. Multiplying by some constant factor to remove non-reduced fractions (such as $\frac{3}{4}$ to deal with parity) will improve this result.

36. [30] A box contains twelve balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box*. Consider the following two conditions:

- (1) Some ball has been drawn at least twelve times (not necessarily consecutively).
- (2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)? If the correct answer is C and your answer is A , you get $\max(\lfloor 30(1 - \frac{1}{2} |\log_2 C - \log_2 A| \rfloor, 0)$ points.

Proposed by: Yuan Yao

Answer: 0.02236412255...

Below is a python implementation to compute the probability, using the same method as the solution to the easier version (with three balls).

```
from fractions import Fraction

N = 12
probs = [{} for i in range((N-1)*(N-1)+2)]
prob1 = Fraction()
prob2 = Fraction()
init = tuple(0 for i in range(N))
probs[0][init] = Fraction(1,1)

for i in range((N-1)*(N-1)+1):
    for t in probs[i]:
        for j in range(N):
            val = probs[i][t] * Fraction(1,N)
            l = list(t)
            l[j] += 1
            l.sort()
            l = tuple(l)
            if (l[-1] == N):
                prob1 = prob1 + val
            elif (l[0] == 1):
                prob2 = prob2 + val
            else:
                probs[i+1][l] = probs[i+1].setdefault(l, Fraction()) + val

print(prob1)
```

Intuitively the probability should be quite small, since the distribution tends towards the second condition instead of the first. Indeed, the exact fraction is $p = \frac{M}{N}$, where

$$\begin{aligned} M &= 663659309086473387879121984765654681548533307869748367531 \\ &\quad 919050571107782711246694886954585701687513519369602069583, \\ N &= 2967517762021717138065641019865112420616209349876886946382 \\ &\quad 1672067789922444492392280614561539198623553884143178743808. \end{aligned}$$

Note: This is a simplified variant of the Bingo Paradox, which is a phenomenon where horizontal bingos are significantly more frequent than vertical bingos. For more information, see https://www.maa.org/sites/default/files/pdf/Mathhorizons/pdfs/The_Bingo_Paradox_MH_Sept17.pdf.

HMMT November 2017

November 11, 2017

Team Round

1. [15] A positive integer k is called *powerful* if there are distinct positive integers p, q, r, s, t such that $p^2, q^3, r^5, s^7, t^{11}$ all divide k . Find the smallest powerful integer.

Proposed by: Yuan Yao

Answer: 1024

First of all, 1024 is powerful because it can be divided evenly by $16^2 = 256, 8^3 = 512, 4^5 = 1024, 2^7 = 128$, and $1^{11} = 1$.

Now we show that 1024 is the smallest powerful number. Since $s \neq t$, at least one of them is at least 2. If $t \geq 2$ or $s \geq 3$, then we need the number to be divisible by at least $2^{11} = 2048$ or $3^7 = 2187$, which both exceed 1024, so we must have $s = 2$ and $t = 1$. If $r = 3$, then the number must be divisible by $3^5 = 243$ and $2^7 = 128$, which means that the number is at least $243 \cdot 128 > 1024$, so $r \geq 4$, and the number is at least $4^5 = 1024$. Therefore the smallest powerful number is indeed 1024.

2. [20] How many sequences of integers (a_1, \dots, a_7) are there for which $-1 \leq a_i \leq 1$ for every i , and

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 = 4?$$

Proposed by: mendel keller

Answer: 38

For $i = 1, 2, \dots, 6$, let $b_i = a_i a_{i+1}$. From the problem condition each of b_1, b_2, \dots, b_6 can only be $-1, 0$, or 1 . Since the sum of these six numbers is 4, either there are five 1s and a -1 or there are four 1s and two 0s.

In the first case, there are 6 ways to choose i such that $b_i = -1$. Once that is fixed, determining the value of a_1 (one of 1 and -1) will determine the value of all the remaining a_i 's, so there are $6 \cdot 2 = 12$ possible ways in this case.

In the second case, since if one of b_2, b_3, b_4, b_5 is zero, then one of the adjacent term to this zero term must also be zero. Therefore the two zeroes must be next to each other or be b_1 and b_6 .

If $b_1 = b_2 = 0$, then a_2 must be zero. a_1 's value doesn't matter, and a_3, a_4, \dots, a_7 must have the same sign. The same goes for $b_5 = b_6 = 0$, giving $3 \cdot 2 \cdot 2 = 12$ possibilities in these two cases.

If $b_i = b_{i+1} = 0$ for $i = 2, 3, 4$, then a_{i+1} must be zero. Moreover, a_1, a_2, \dots, a_i must have the same sign, and so do a_{i+2}, \dots, a_7 . this gives $2 \cdot 2 \cdot 3 = 12$ possibilities in these three cases.

If $b_1 = b_6 = 0$, then $a_1 = a_7 = 0$. Also, a_2, a_3, \dots, a_6 must have the same sign so there are 2 possibilities.

Combining these cases gives $12 + 12 + 12 + 2 = 38$ possible sequences in total.

3. [25] Michael writes down all the integers between 1 and N inclusive on a piece of paper and discovers that exactly 40% of them have leftmost digit 1. Given that $N > 2017$, find the smallest possible value of N .

Proposed by: Michael Tang

Answer: 1,481,480

Let d be the number of digits of N . Suppose that N does not itself have leftmost digit 1. Then the number of integers $1, 2, \dots, N$ which have leftmost digit 1 is

$$1 + 10 + 10^2 + \dots + 10^{d-1} = \frac{10^d - 1}{9},$$

so we must have $\frac{10^d - 1}{9} = \frac{2N}{5}$, or $5(10^d - 1) = 18N$. But the left-hand side is odd, so this is impossible.

Thus N must have leftmost digit 1. In this case, the number of integers $1, 2, \dots, N$ which have leftmost digit 1 is

$$\begin{aligned} & 1 + 10 + 10^2 + \dots + 10^{d-2} + (N - 10^{d-1} + 1) \\ &= \frac{10^{d-1} - 1}{9} + N - 10^{d-1} + 1 \\ &= N - 8 \left(\frac{10^{d-1} - 1}{9} \right). \end{aligned}$$

Therefore we need $N - 8 \left(\frac{10^{d-1} - 1}{9} \right) = \frac{2}{5}N$, or $N = \frac{40}{3} \left(\frac{10^{d-1} - 1}{9} \right)$. Then, $\frac{10^{d-1} - 1}{9}$ must be divisible by 3. The base-ten representation of $\frac{10^{d-1} - 1}{9}$ has $d - 1$ ones, so $d - 1$ must be divisible by 3. Both $d = 1$ and $d = 4$ make N less than 2017, but $d = 7$ gives the answer $N = \frac{40}{3}(111111) = \boxed{1481480}$.

4. [30] An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N . Determine the minimum possible value of the ratio $\frac{N}{M}$.

Proposed by: Yuan Yao

Answer: $\boxed{\frac{3}{\sqrt{3}-1} \text{ OR } \frac{3\sqrt{3}+3}{2}}$

We claim that the greatest possible value of M is $\sqrt{3} - 1$, whereas the least possible value of N is 3.

To begin, note that the condition requires the circle to intersect each side of the hexagon at two points *on its interior*. This implies that the center must be inside the hexagon as its projection onto all six sides must be on their interior. Suppose that the hexagon is $ABCDEF$, with $AB = BC = DE = EF = 1$, $CD = FA = a$, and the center O .

When $a \leq \sqrt{3} - 1$, we note that the distance from O to CD (which is $\frac{\sqrt{3}}{2}$) is greater than or equal to the distance from O to B or E (which is $\frac{a+1}{2}$). However, for the circle to intersect all six sides at two points each, the distance from the center of the circle to CD and to FA must be strictly less than that from the center to B and to E , because otherwise any circle that intersects CD and FA at two points each must include B or E on its boundary or interior, which will not satisfy the condition. WLOG assume that the center of the circle is closer to FA than to CD , including equality (in other words, the center is on the same side of BE as FA , possibly on BE itself), then note that the parabola with foci B and E and common directrix CD intersects on point O , which means that there does not exist a point in the hexagon on the same side of BE as FA that lies on the same side of both parabola as CD . This means that the center of the circle cannot be chosen.

When $a = \sqrt{3} - 1 + \epsilon$ for some very small real number $\epsilon > 0$, the circle with center O and radius $r = \frac{\sqrt{3}}{2}$ intersects sides AB, BC, DE, EF at two points each and is tangent to CD and FA on their interior. Therefore, there exists a real number $\epsilon' > 0$ such that the circle with center O and radius $r' = r + \epsilon'$ satisfy the requirement.

When $a \geq 3$, we note that the projection of BF onto BC has length $|\frac{1}{2} - \frac{a}{2}| \geq 1$, which means that the projection of F onto side BC is not on its interior, and the same goes for side EF onto BC . However, for a circle to intersect both BC and EF at two points, the projection of center of the circle onto the two sides must be on their interior, which cannot happen in this case.

When $a = 3 - \epsilon$ for some very small real number $\epsilon > 0$, a circle with center O and radius $r = \frac{\sqrt{3}}{4}(a+1)$ intersects AF and CD at two points each and is tangent to all four other sides on their interior. Therefore, there exists a real number $\epsilon' > 0$ such that the circle with center O and radius $r' = r + \epsilon'$ satisfy the requirement.

With $M \leq \sqrt{3} - 1$ and $N \geq 3$, we have $\frac{N}{M} \geq \frac{3}{\sqrt{3}-1} = \frac{3\sqrt{3}+3}{2}$, which is our answer.

5. [35] Ashwin the frog is traveling on the xy -plane in a series of $2^{2017} - 1$ steps, starting at the origin. At the n^{th} step, if n is odd, then Ashwin jumps one unit to the right. If n is even, then Ashwin jumps m units up, where m is the greatest integer such that 2^m divides n . If Ashwin begins at the origin, what is the area of the polygon bounded by Ashwin's path, the line $x = 2^{2016}$, and the x -axis?

Proposed by: Nikhil Reddy

Answer: $2^{2015} \cdot (2^{2017} - 2018)$

Notice that since $v_2(x) = v_2(2^{2017} - x)$, the path divides the rectangle bonded by the coordinate axes and the two lines passing through Ashwin's final location parallel to the axes. The answer is therefore half of the product of the coordinates of Ashwin's final coordinates. The x -coordinate is the number of odd number steps (which is 2^{2016}). The y -coordinate is the number of total powers of 2 in $(2^{2017} - 1)!$. The final answer is therefore $2^{2015} \cdot (2^{2017} - 2018)$.

6. [40] Consider five-dimensional Cartesian space

$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{R}\},$$

and consider the hyperplanes with the following equations:

- $x_i = x_j$ for every $1 \leq i < j \leq 5$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = -1$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 0$;
- $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

Into how many regions do these hyperplanes divide \mathbb{R}^5 ?

Proposed by: Mehtaab Sawhney

Answer: 480

(Joint with Junyao Peng)

Note that given a set of plane equations $P_i(x_1, x_2, x_3, x_4, x_5) = 0$, for $i = 1, 2, \dots, n$, each region that the planes separate the space into correspond to a n -tuple of -1 and 1 , representing the sign of P_1, P_2, \dots, P_n for all points in that region.

Therefore, the first set of planes separate the space into $5! = 120$ regions, with each region representing an ordering of the five coordinates by numerical size. Moreover, the next three planes are parallel to each other and perpendicular to all planes in the first set, so these three planes separate each region into 4. Therefore, a total of $4 \cdot 120 = 480$ regions is created.

7. [50] There are 12 students in a classroom; 6 of them are Democrats and 6 of them are Republicans. Every hour the students are randomly separated into four groups of three for political debates. If a group contains students from both parties, the minority in the group will change his/her political alignment to that of the majority at the end of the debate. What is the expected amount of time needed for all 12 students to have the same political alignment, in hours?

Proposed by: Yuan Yao

Answer: $\frac{341}{54}$

When the party distribution is 6 – 6, the situation can change (to 3 – 9) only when a group of three contains three people from the same party, and the remaining three are distributed evenly across the other three groups (to be converted).

To compute the probability, we assume that the groups and the members of the group are ordered (so there are $12!$ ways of grouping). There are 2 ways to choose the party, 4 ways to choose the group, $6 \cdot 5 \cdot 4$ ways to choose the three members of the group, $9 \cdot 6 \cdot 3$ ways to place the other three members of the party, and $6!$ ways to fill in the members of the other party. The probability is then

$$\frac{2 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 9 \cdot 6 \cdot 3 \cdot 6!}{12!} = \frac{2 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 9 \cdot 6 \cdot 3}{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7} = \frac{18}{77}.$$

This means that the shift in distribution will happen in $\frac{77}{18}$ hours on average.

When the distribution is 3 – 9, the situation can change (to 0 – 12) only when the three members of the minority party are all in different groups. Using the similar method as above, there are $12 \cdot 9 \cdot 6$ ways to place the three members and $9!$ ways to place the rest, so the probability is

$$\frac{12 \cdot 9 \cdot 6 \cdot 9!}{12!} = \frac{12 \cdot 9 \cdot 6}{12 \cdot 11 \cdot 10} = \frac{27}{55}.$$

This means that the shift in distribution will happen in $\frac{55}{27}$ hours on average.

By linearity of expectation, we can add up the two results and get that the expected value is $\frac{77}{18} + \frac{55}{27} = \frac{341}{55}$ hours.

8. [55] Find the number of quadruples (a, b, c, d) of integers with absolute value at most 5 such that

$$(a^2 + b^2 + c^2 + d^2)^2 = (a + b + c + d)(a - b + c - d)((a - c)^2 + (b - d)^2).$$

Proposed by: Mehtaab Sawhney

Answer: 49

Let $x = a + c$, $y = a - c$, $w = b + d$, and $z = b - d$. Then

$$(w^2 + x^2 + y^2 + z^2)^2 = 4(x^2 - w^2)(y^2 + z^2)$$

and since $|x^2 + w^2| \geq |x^2 - w^2|$ it follows that $w = 0$ or $y = z = 0$. Now $y = z = 0$ implies $a = b = c = d = 0$. Now $w = 0$ gives $b = -d$. Then for equality to hold $x^2 = y^2 + z^2$. This is equivalent to $ac = b^2$, which includes the previous case. It suffices to count the number of triples (a, b, c) that satisfy the equation.

When $b = 0$, either a or c is zero, which gives $11 + 11 - 1 = 21$ triples.

When $b = \pm 1$, we have $|a| = |c| = 1$ and a, c have the same sign, for $2 \cdot 2 = 4$ triples.

When $b = \pm 2$, we have $(a, c) = (1, 4), (2, 2), (4, 1)$ or their negatives, for $2 \cdot 6 = 12$ triples.

When $b = \pm 3, \pm 4, \pm 5$, we have $|a| = |b| = |c|$ and a, c have the same sign, for $6 \cdot 2 = 12$ triples.

So in total there are $21 + 4 + 12 + 12 = 49$ solutions.

9. [60] Let A, B, C, D be points chosen on a circle, in that order. Line BD is reflected over lines AB and DA to obtain lines ℓ_1 and ℓ_2 respectively. If lines ℓ_1, ℓ_2 , and AC meet at a common point and if $AB = 4, BC = 3, CD = 2$, compute the length DA .

Proposed by: Ashwin Sah

Answer: $\sqrt{21}$

Let the common point be E . Then since lines BE and BD are symmetric about line BA , BA is an exterior bisector of $\angle DBE$, and similarly DA is also an exterior bisector of $\angle BDE$. Therefore A is the E -excenter of triangle BDE and thus lie on the interior bisector of $\angle BED$. Since C lies on line AE , and by the fact that A, B, C, D are concyclic, we get that $\angle ABC + \angle ADC = 180^\circ$, which implies $\angle DBC + \angle BDC = \frac{1}{2}(\angle DBE + \angle BDE)$, so C is the incenter of triangle BDE . Thus $\angle ABC = \angle CDA = \frac{\pi}{2}$, and thus $DA^2 = AC^2 - CD^2 = AB^2 + BC^2 - CD^2 = 3^2 + 4^2 - 2^2 = 21$. The length of DA is then $\sqrt{21}$.

10. [70] Yannick has a bicycle lock with a 4-digit passcode whose digits are between 0 and 9 inclusive. (Leading zeroes are allowed.) The dials on the lock is currently set at 0000. To unlock the lock, every second he picks a contiguous set of dials, and increases or decreases all of them by one, until the dials are set to the passcode. For example, after the first second the dials could be set to 1100, 0010, or 9999, but not 0909 or 0190. (The digits on each dial are cyclic, so increasing 9 gives 0, and decreasing 0 gives 9.) Let the *complexity* of a passcode be the minimum number of seconds he needs to unlock the lock. What is the maximum possible complexity of a passcode, and how many passcodes have this maximum complexity? Express the two answers as an ordered pair.

Proposed by: Yuan Yao

Answer: $(12, 2)$

To simplify the solution, we instead consider the equivalent problem of reducing a passcode to 0000 using the given move.

Given a passcode $a_1a_2a_3a_4$, define a *differential* of the passcode to be a quintuple $(d_1, d_2, d_3, d_4, d_5)$ such that $d_i \equiv a_i - a_{i-1} \pmod{10}$ for $i = 1, 2, 3, 4, 5$, where we define $a_0 = a_5 = 0$.

Claim 1: For any passcode, there exists a differential that satisfies the following two conditions:

- $d_1 + d_2 + d_3 + d_4 + d_5 = 0$;
- The range (difference between the maximum and minimum) of these five numbers is at most 10.

Proof: We first see that the differential defined by $d_i = a_i - a_{i-1}$ satisfy the first condition since the sum of the five numbers is $a_5 - a_0 = 0$. Suppose that a differential satisfying the first condition has a range greater than 10, then we take one of the largest number d_m and one of the smallest number d_n (where $d_m - d_n > 10$), and replace the former by $d_m - 10$ and the latter by $d_n + 10$. This will either reduce the range or reduce the number of maximal and minimal numbers, so the process will terminate after finitely many iterations. Thus we can find a differential that satisfies both conditions.

(Note: we call such a differential a *standard differential* from now on, although it is important to remember that there may be more than one standard differential for one passcode. As a corollary, all of the numbers in a standard differential must be in the range $[-9, 9]$, as a number greater than 9 will be more than 10 away from a negative number, and similar for a number smaller than -9 .)

Given a passcode, we define the *magnitude* of one of its standard differentials to be the sum of all the positive values in the differential (which is also the absolute value of the sum all the negative values).

Claim 2: The magnitude of the a passcode's standard differential is equal to the passcode's complexity.

Proof: Obviously 0000 is the only passcode with complexity 0 whose standard differential has magnitude 0. Suppose that the magnitude of a standard differential is M , then it suffices show that the magnitude can be reduced to 0 in M moves, and that the magnitude can only decrease by at most 1 with each move.

The first part can be shown via the following algorithm. When the magnitude is not zero, there must be a positive number d_i and a negative number d_j . WLOG assume that $i < j$, then after taking the dials $a_i, a_{i+1}, \dots, a_{j-1}$ and decreases them all by 1, d_i decreases by 1 and d_j increases by 1. This will decrease the magnitude by 1 (and the differential remains standard), so by repeating this process M times we can bring the magnitude down to 0.

For the second part, we assume WLOG that we take the dials $a_i, a_{i+1}, \dots, a_{j-1}$ and decrease them all by 1, and then d_i is replaced by $d'_i = d_i - 1$ and $d'_j = d_j + 1$. If the differential remains standard, then the magnitude decreases by 1 (when $d_i > 0$ and $d_j < 0$), remains the same (when either $d_i \leq 0$ and $d_j < 0$ or $d_i > 0$ and $d_j \geq 0$), or increases by 1 (when $d_i \leq 0$ and $d_j \geq 0$).

In the latter two cases, it is possible that the differential is no longer standard.

If the magnitude previously remained the same (WLOG suppose that $d_i > 0$ and $d_j \geq 0$), then there exists a negative d_k that is minimal such that $d_j - d_k = 10$ and now d'_j is the unique maximum. Replacing d_k by $d_k + 10 = d_j$ and d'_j by $d'_j - 10 = d_k + 1$ will reduce the magnitude by 1, and the new differential will be standard because the unique maximum d'_j is no longer present and the minimum is now either d_k or $d_k + 1$. This means that the magnitude decreases by at most 1.

If the magnitude previously increased by 1 (when $d_i \leq 0$ and $d_j \geq 0$), then there exists either a negative d_k that is (previously) minimal such that $d_j - d_k = 10$, or a positive d_l that is (previously) maximal such that $d_l - d_i = 10$, or both. By similar logic as the previous case, replacing d_k by $d_k + 10$ and d'_j by $d'_j - 10$, or replacing d_l by $d_l - 10$ and d'_i by $d'_i + 10$ (or both, if both d_k and d_l exist). will decrease the magnitude by 1, and ensure that the new differential is standard. The replacement will decrease the current magnitude by at most 2, this means that the original magnitude decreases by at most 1 in total.

These considerations finishes the second part and therefore the proof.

With this claim, we also see that the magnitudes of all possible standard differentials of a given passcode are the same, so the choice of the differential is irrelevant.

We can now proceed to find the maximum possible complexity. Suppose that there are m positive numbers and n negative numbers in the differential, and suppose that the maximum and the minimum are x and $-y$ respectively. Since the sum of all positive numbers is at most mx and the absolute value of the sum of all negative numbers is at most ny , the complexity is at most $C = \min(mx, ny)$. It suffices to maximize C under the condition that $m + n \leq 5$ and $x + y = x - (-y) \leq 10$. It is not difficult to see (via casework) that the maximal C is 12, achieved by $m = 2, n = 3, x = 6, y = 4$ or $m = 3, n = 2, x = 4, y = 6$. In the first case, the digits must increase from 0 by 6 twice and decreases by 4 three times (and reduced modulo 10), which gives the passcode 6284; in the second case the digits increase by 4 three times and decreases by 6 twice instead, which gives the passcode 4826. Since all inequalities are tight, these two passcodes are the only ones that has the maximal complexity of 12.

HMMT November 2017

November 11, 2017

Theme Round

1. Two ordered pairs (a, b) and (c, d) , where a, b, c, d are real numbers, form a basis of the coordinate plane if $ad \neq bc$. Determine the number of ordered quadruples (a, b, c, d) of integers between 1 and 3 inclusive for which (a, b) and (c, d) form a basis for the coordinate plane.

Proposed by: Ashwin Sah

Answer:

Any pair of distinct points will form a basis except when (a, b) and (c, d) are both from $\{(1, 1), (2, 2), (3, 3)\}$, so the answer is $9 \cdot 8 - 3 \cdot 2 = 66$.

2. Horizontal parallel segments $AB = 10$ and $CD = 15$ are the bases of trapezoid $ABCD$. Circle γ of radius 6 has center within the trapezoid and is tangent to sides AB , BC , and DA . If side CD cuts out an arc of γ measuring 120° , find the area of $ABCD$.

Proposed by: Michael Tang

Answer:

Suppose that the center of the circle is O and the circle intersects CD at X and Y . Since $\angle XOY = 120^\circ$ and triangle XOY is isosceles, the distance from O to XY is $6 \cdot \sin(30^\circ) = 3$. On the other hand, the distance from O to AB is 6 as the circle is tangent to AB , and O is between AB and CD , so the height of the trapezoid is $6 + 3 = 9$ and its area is $\frac{9 \cdot (10+15)}{2} = \frac{225}{2}$.

3. Emilia wishes to create a basic solution with 7% hydroxide (OH) ions. She has three solutions of different bases available: 10% rubidium hydroxide (Rb(OH)), 8% cesium hydroxide (Cs(OH)), and 5% francium hydroxide (Fr(OH)). (The Rb(OH) solution has both 10% Rb ions and 10% OH ions, and similar for the other solutions.) Since francium is highly radioactive, its concentration in the final solution should not exceed 2%. What is the highest possible concentration of rubidium in her solution?

Proposed by: Yuan Yao

Answer:

Suppose that Emilia uses R liters of Rb(OH), C liters of Cs(OH), and F liters of Fr(OH), then we have

$$\frac{10\% \cdot R + 8\% \cdot C + 5\% \cdot F}{R + C + F} = 7\% \text{ and } \frac{5\% \cdot F}{R + C + F} \leq 2\%.$$

The equations simplify to $3R + C = 2F$ and $3F \leq 2R + 2C$, which gives

$$\frac{9R + 3C}{2} \leq 2R + 2C \Rightarrow 5R \leq C.$$

Therefore the concentration of rubidium is maximized when $5R = C$, so $F = 4R$, and the concentration of rubidium is

$$\frac{10\% \cdot R}{R + C + F} = 1\%.$$

4. Mary has a sequence m_2, m_3, m_4, \dots , such that for each $b \geq 2$, m_b is the least positive integer m for which none of the base- b logarithms $\log_b(m)$, $\log_b(m+1)$, \dots , $\log_b(m+2017)$ are integers. Find the largest number in her sequence.

Proposed by: Michael Tang

Answer:

It is not difficult to see that for all of the logarithms to be non-integers, they must lie strictly between n and $n+1$ for some integer n . Therefore, we require $b^{n+1} - b^n > 2018$, and so $m_b = b^n + 1$ where n is the smallest integer that satisfies the inequality. In particular, this means that $b^n - b^{n-1} \leq 2018$.

Note that $m_2 = 2^{11} + 1 = 2049$ (since $2^{12} - 2^{11} = 2048 > 2018$) and $m_3 = 3^7 + 1 = 2188$ (since $3^8 - 3^7 = 4374 > 2018$). we now show that 2188 is the maximum possible value for m_b .

If $n = 0$, then $m_b = 1 + 1 = 2$.

If $n = 1$, then $b - 1 \leq 2018$ and thus $m_b = b + 1 \leq 2020$.

If $n = 2$, then $b^2 - b \leq 2018$, which gives $b \leq 45$, and thus $m_b = b^2 + 1 \leq 2018 + b + 1 \leq 2065$.

If $n = 3$, then $b^3 - b^2 \leq 2018$, which gives $b \leq 12$, and thus $m_b = b^3 + 1 \leq 12^3 + 1 = 1729$.

If $n = 4$, then $b^4 - b^3 \leq 2018$, which gives $b \leq 6$, and thus $m_b = b^4 + 1 \leq 6^4 + 1 = 1297$.

It then remains to check the value of m_4 and m_5 . Indeed, $m_4 = 4^5 + 1 = 1025$ and $m_5 = 5^4 + 1 = 626$, so no values of m_b exceeds 2188.

5. Each of the integers $1, 2, \dots, 729$ is written in its base-3 representation without leading zeroes. The numbers are then joined together in that order to form a continuous string of digits: $12101112202122 \dots$. How many times in this string does the substring 012 appear?

Proposed by: Michael Tang

Answer: 148

Ignore $729 = 3^6 = 1000000_3$ since it will not contribute to a 012 substring. Break into cases on how 012 appears: (i) when an individual integer contains the string 012; (ii) when 01 are the last two digits of an integer and 2 is the first digit of the next integer; and (iii) when 0 is the last digit of an integer and 12 are the first two digits of the next integer.

For case (i), we want to find the total number of appearances of the string 012 in $1, 2, 3, \dots, N$. Since each number has at most six digits, 012 appears at most once per number. If such a number has d digits, $4 \leq d \leq 6$, then there are $d - 3$ possible positions for the substring 012 and $2 \cdot 3^{d-4}$ possible choices for the remaining digits (since the leftmost digit must be nonzero). Thus there are

$$\sum_{d=4}^6 (d-3) \cdot (2 \cdot 3^{d-4}) = 1 \cdot 2 + 2 \cdot 6 + 3 \cdot 18 = 68$$

appearances of 012 in case (i).

For case (ii), we have an integer n for which n ends in 01 and $n + 1$ starts with 2. Then n must also start with 2. Hence it suffices to count the number of integers $1 \leq n < N$ which start with 2 and end with 01 in base three. If n has d digits, $3 \leq d \leq 6$, then there are 3^{d-3} possibilities for the middle digits, so we have

$$\sum_{d=3}^6 3^{d-3} = 1 + 3 + 9 + 27 = 40$$

appearances of 012 in case (ii).

For case (iii), we have an integer n for which n ends in 0 and $n + 1$ starts with 12. Then n must also start with 12, so we want to count the number of $1 \leq n < N$ starting with 12 and ending in 0. Like case (ii), there are also 40 appearances of 012 in case (iii).

In total, the substring 012 appears $68 + 40 + 40 = 148$ times.

6. Rthea, a distant planet, is home to creatures whose DNA consists of two (distinguishable) strands of bases with a fixed orientation. Each base is one of the letters H, M, N, T, and each strand consists of a sequence of five bases, thus forming five pairs. Due to the chemical properties of the bases, each pair must consist of distinct bases. Also, the bases H and M cannot appear next to each other on the same strand; the same is true for N and T. How many possible DNA sequences are there on Rthea?

Proposed by: Yuan Yao

Answer: $12 \cdot 7^4$ or 28812

There are $4 \cdot 3 = 12$ ways to choose the first base pairs, and regardless of which base pair it is, there are 3 possibilities for the next base on one strand and 3 possibilities for the next base on the other

strand. Among these possibilities, exactly 2 of them have identical bases forming a base pair (using one of the base not in the previous base pair if the previous pair is H-M or N-T, or one of the base in the previous pair otherwise), which is not allowed. Therefore there are $3 \cdot 3 - 2 = 7$ ways to choose each of the following base pairs. Thus in total there are $12 \cdot 7^4 = 28812$ possible DNA (which is also the maximum number of species).

7. On a blackboard a stranger writes the values of $s_7(n)^2$ for $n = 0, 1, \dots, 7^{20} - 1$, where $s_7(n)$ denotes the sum of digits of n in base 7. Compute the average value of all the numbers on the board.

Proposed by: Ashwin Sah

Answer: 3680

Solution 1: We solve for 0 to $b^n - 1$ and $s_b(n)^2$ (i.e. base b).

Let $n = d_1 \dots d_n$ in base b , where there may be leading zeros. Then $s_b(n) = d_1 + \dots + d_n$, regardless of the leading zeros.

$$\mathbb{E}[s_d(n)^2] = \mathbb{E}[(d_1 + \dots + d_n)^2] = \sum_{1 \leq i \leq n} \mathbb{E}[d_i^2] + 2 \sum_{1 \leq i < j \leq n} \mathbb{E}[d_i d_j],$$

and now notice that we can treat choosing n uniformly as choosing the d_i uniformly independently from $\{0, \dots, b-1\}$. Thus this simplifies to

$$\mathbb{E}[s_d(n)^2] = n\mathbb{E}[d_1^2] + n(n-1)\mathbb{E}[d_1]^2.$$

Now

$$\begin{aligned} \mathbb{E}[d_1^2] &= \frac{0^2 + \dots + (b-1)^2}{b} = \frac{(b-1)(2b-1)}{6}, \\ \mathbb{E}[d_1] &= \frac{0 + \dots + (b-1)}{b} = \frac{b-1}{2}, \end{aligned}$$

so the answer is

$$n \cdot \frac{(b-1)(2b-1)}{6} + n(n-1) \cdot \left(\frac{b-1}{2}\right)^2.$$

Plugging in $b = 7, n = 20$ yields the result.

Solution 2: There are two theorems we will cite regarding variance and expected value. The first is that, for any variable X ,

$$\text{Var}(X) = E[X^2] - E[X]^2.$$

The second is that, for two independent variables X and Y ,

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y).$$

Let X be the sum of all of the digits. We want to find $E[X^2]$. The expected of a single digit is $\frac{1}{7}(0+1+2+3+4+5+6) = 3$. Thus, the expected value of the sum of the digits is $E[X] = 20 \times 3 = 60$, so $E[X]^2 = 3600$. The variance of a single digit is $\frac{1}{7}[(0-3)^2 + (1-3)^2 + \dots + (6-3)^2] = \frac{9+4+1+0+1+4+9}{7} = 4$. Since the digits are independent, their variances add by the second theorem above. Therefore, the variance of the sum of all of the digits is $\text{Var}(X) = 20 \times 4 = 80$. Finally, using the first theorem, we have $E[X^2] = E[X]^2 + \text{Var}(X) = 3680$.

8. Undecillion years ago in a galaxy far, far away, there were four space stations in the three-dimensional space, each pair spaced 1 light year away from each other. Admiral Ackbar wanted to establish a base somewhere in space such that the sum of squares of the distances from the base to each of the stations does not exceed 15 square light years. (The sizes of the space stations and the base are negligible.) Determine the volume, in cubic light years, of the set of all possible locations for the Admiral's base.

Proposed by: Yuan Yao

Answer: $\frac{27\sqrt{6}}{8}\pi$

Solution 1: Set up a coordinate system where the coordinates of the stations are $(\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$, $(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}})$, $(\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$, and $(-\frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}})$. The sum of squares of the distances is then

$$\sum_{\text{sym}} 2(x - \frac{1}{2\sqrt{2}})^2 + 2(x + \frac{1}{2\sqrt{2}})^2 = 4(x^2 + y^2 + z^2) + \frac{3}{2} = 4r^2 + \frac{3}{2},$$

where r is the distance from the center of the tetrahedron. It follows from $4r^2 + \frac{3}{2} \leq 15$ that $r \leq \frac{3\sqrt{6}}{4}$, so the set is a ball with radius $R = \frac{3\sqrt{6}}{4}$, and the volume is $\frac{4\pi}{3}R^3 = \frac{27\sqrt{6}\pi}{8}$.

Solution 2: Let P be the location of the base; S_1, S_2, S_3, S_4 be the stations; and G be the center of the tetrahedron. We have:

$$\begin{aligned} \sum_{i=1}^4 PS_i^2 &= \sum_{i=1}^4 \vec{PS}_i \cdot \vec{PS}_i \\ \sum_{i=1}^4 PS_i^2 &= \sum_{i=1}^4 (\vec{PG} + \vec{GS}_i) \cdot (\vec{PG} + \vec{GS}_i) \\ \sum_{i=1}^4 PS_i^2 &= \sum_{i=1}^4 (\vec{PG} \cdot \vec{PG} + 2\vec{PG} \cdot \vec{GS}_i + \vec{GS}_i \cdot \vec{GS}_i) \end{aligned}$$

Since $GS_1 = GS_2 = GS_3 = GS_4$, we have:

$$\begin{aligned} \sum_{i=1}^4 PS_i^2 &= 4PG^2 + 4GS_1^2 + 2 \sum_{i=1}^4 (\vec{PG} \cdot \vec{GS}_i) \\ \sum_{i=1}^4 PS_i^2 &= 4PG^2 + 4GS_1^2 + 2\vec{PG} \cdot \left(\sum_{i=1}^4 \vec{GS}_i \right) \end{aligned}$$

Since G is the center of the tetrahedron, $\vec{GS}_1 + \vec{GS}_2 + \vec{GS}_3 + \vec{GS}_4 = \vec{0}$. Thus:

$$\sum_{i=1}^4 PS_i^2 = 4PG^2 + 4GS_1^2$$

Since $GS_1^2 = \frac{3}{8}$, we have $PG^2 \leq \frac{27}{8}$. Thus, the locus of all good points is a ball centered at G with radius $r = \sqrt{\frac{27}{8}}$. Then, the volume is $V = \frac{4}{3}\pi r^3 = \frac{27\sqrt{6}\pi}{8}$.

9. New this year at HMNT: the exciting game of *RNG baseball*! In RNG baseball, a team of infinitely many people play on a square field, with a base at each vertex; in particular, one of the bases is called the *home base*. Every turn, a new player stands at home base and chooses a number n uniformly at random from $\{0, 1, 2, 3, 4\}$. Then, the following occurs:

- If $n > 0$, then the player and everyone else currently on the field moves (counterclockwise) around the square by n bases. However, if in doing so a player returns to or moves past the home base, he/she leaves the field immediately and the team scores one point.
- If $n = 0$ (a strikeout), then the game ends immediately; the team does not score any more points.

What is the expected number of points that a given team will score in this game?

Proposed by: Yuan Yao

Answer: $\boxed{\frac{409}{125}}$

For $i = 0, 1, 2, 3$, let P_i be the probability that a player on the i -th base scores a point before strikeout (with zeroth base being the home base). We have the following equations:

$$\begin{aligned}
P_0 &= \frac{1}{5}(P_1 + P_2 + P_3 + 1) \\
P_1 &= \frac{1}{5}(P_2 + P_3 + 1 + 1) \\
P_2 &= \frac{1}{5}(P_3 + 1 + 1 + 1) \\
P_3 &= \frac{1}{5}(1 + 1 + 1 + 1)
\end{aligned}$$

Solving the system of equations gives $P_3 = \frac{4}{5}, P_2 = \frac{19}{25}, P_1 = \frac{89}{125}, P_0 = \frac{409}{625}$, so the probability that a batter scores a point himself is $\frac{409}{625}$, given that he is able to enter the game before the game is over. Since the probability that the n th player will be able to stand on the home base is $(\frac{4}{5})^{n-1}$ (none of the previous $n-1$ players receive a strikeout), the expected value is $\frac{409}{625}(1 + \frac{4}{5} + (\frac{4}{5})^2 + \dots) = \frac{409}{625} \cdot \frac{1}{1-\frac{4}{5}} = \frac{409}{125}$.

10. Denote $\phi = \frac{1+\sqrt{5}}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string S has a "base- ϕ " value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n , let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number c as n goes to infinity. Determine the value of c .

Proposed by: Ashwin Sah

Answer: $\boxed{\frac{25+3\sqrt{69}}{2}}$

We write everything in base ϕ . Notice that

$$\frac{\phi^{48n}-1}{\phi^{48}-1} = 10\dots 010\dots 01\dots 10\dots 01,$$

where there are $n-1$ blocks of 47 zeros each. We can prove that every valid base- ϕ representation comes from replacing a consecutive string 100 with a 011 repeatedly. Using this, we can easily classify what base- ϕ representations are counted by $f(n)$.

Notice that 10000000 = 01100000 = 01011000 = 01010110 and similar, so that in each block of zeros we can choose how many times to perform a replacement. It turns out that we can do anywhere from 0 to 24 such replacements, but that if we choose to do 24 then the next block cannot have chosen 0 replacements. (An analogy with lower numbers is 10001000 = 01101000 = 01100110 = 01011110, with the first block "replaced twice," which was only allowed since the second block had "replaced once," opening up the slot which was filled by the last 1 in the final replacement 011).

Thus we have a bijection from $f(n)$ to sequences in $\{0, \dots, 24\}^{n-1}$ such that (a) the sequence does not end in 24 and (b) the sequence never has a 24 followed by a 0.

We let a_n denote the number of length- n sequences starting with a 0, b_n for the number of such sequences starting with any of 1 to 23, and c_n for the number of such sequences starting with 24. We know $a_1 = 1, b_1 = 23, c_0 = 0$ and that $f(n) = a_{n-1} + b_{n-1} + c_{n-1}$.

Now,

$$\begin{aligned}
a_n &= a_{n-1} + b_{n-1} + c_{n-1} \\
b_n &= 23(a_{n-1} + b_{n-1} + c_{n-1}) \\
c_n &= b_{n-1} + c_{n-1}
\end{aligned}$$

so $b_n = 23a_n$ for all n . Substituting gives $a_n = 24a_{n-1} + c_{n-1}, c_n = 23a_{n-1} + c_{n-1}$. Solving for $c_n = a_{n+1} - 24a_{n-1}$ and plugging in gives

$$a_{n+1} - 24a_n = a_n - a_{n-1},$$

which gives a characteristic polynomial of $\lambda^2 - 25\lambda + 1 = 0$. We easily find that a_n grows as λ^n (where λ is the larger solution to the quadratic equation) and thus b_n, c_n do as well, implying that $f(n)$ grows as λ^n , where

$$\lambda = \frac{25 + \sqrt{25^2 - 4}}{2} = \frac{25 + 3\sqrt{69}}{2},$$

which is our answer.

HMIC 2017

April 10, 2017

HMIC

1. [6] Kevin and Yang are playing a game. Yang has $2017 + \binom{2017}{2}$ cards with their front sides face down on the table. The cards are constructed as follows:

- For each $1 \leq n \leq 2017$, there is a blue card with n written on the back, and a fraction $\frac{a_n}{b_n}$ written on the front, where $\gcd(a_n, b_n) = 1$ and $a_n, b_n > 0$.
- For each $1 \leq i < j \leq 2017$, there is a red card with (i, j) written on the back, and a fraction $\frac{a_i + a_j}{b_i + b_j}$ written on the front.

It is given no two cards have equal fractions. In a turn Kevin can pick any two cards and Yang tells Kevin which card has the larger fraction on the front. Show that, in fewer than 10000 turns, Kevin can determine which red card has the largest fraction out of all of the red cards.

Proposed by: Evan Chen

We will provide an algorithm to determine which red card has the highest value in $2016 + 2015$ turns.

We start with a lemma: If the blue card with the largest fraction is labeled k and the red card with the largest fraction is labeled (i, j) , then $i = k$ or $j = k$.

To prove this lemma, we assume for contradiction that $i \neq k$ and $j \neq k$. Assume without loss of generality that $\frac{a_i}{b_i} > \frac{a_j}{b_j}$. Then $\frac{a_i + a_k}{b_i + b_k} - \frac{a_i + a_j}{b_i + b_j} = \frac{(a_i + a_k)(b_i + b_j) - (a_i + a_j)(b_i + b_k)}{(b_i + b_k)(b_i + b_j)} = \frac{a_k b_i + a_i b_j + a_k b_j - a_i b_k - a_j b_i - a_j b_k}{(b_i + b_k)(b_i + b_j)}$. However, we know since $\frac{a_k}{b_k} > \frac{a_i}{b_i} > \frac{a_j}{b_j}$, we have that $a_k b_i - a_i b_k > 0$, $a_k b_j - a_j b_k > 0$, $a_i b_j - a_j b_i > 0$, so $\frac{a_i + a_k}{b_i + b_k} - \frac{a_i + a_j}{b_i + b_j} > 0$.

Now, we can first find the blue card with the maximum fraction in 2016 turns. Afterwards, we know k , so we only need to consider pairs (i, j) with $i = k$ or $j = k$. There are only 2016 of these pairs, so we can find the maximum of these in 2015 turns. By the lemma, the maximum of the cards with $i = k$ or $j = k$ is the same as the global maximum of all the red cards.

Therefore we have provided an algorithm to determine the (i, j) with the maximum fraction in $2016 + 2015$ turns.

2. [7] Let $S = \{1, 2, \dots, n\}$ for some positive integer n , and let A be an n -by- n matrix having as entries only ones and zeros. Define an infinite sequence $\{x_i\}_{i \geq 0}$ to be *strange* if:

- $x_i \in S$ for all i ,
- $a_{x_k x_{k+1}} = 1$ for all k , where a_{ij} denotes the element in the i^{th} row and j^{th} column of A .

Prove that the set of strange sequences is empty if and only if A is nilpotent, i.e. $A^m = 0$ for some integer m .

Proposed by: Henrik Boecken

Answer: \square

Consider the directed graph G on n labeled vertices whose adjacency matrix is A . Then, observe that a strange sequence is simply an infinite path on G . Since powers of the adjacency matrix count paths, if A is nilpotent, there exists no infinite path. If A is not nilpotent, a cycle must exist, so we're done.

3. [9] Let v_1, v_2, \dots, v_m be vectors in \mathbb{R}^n , such that each has a strictly positive first coordinate. Consider the following process. Start with the zero vector $w = (0, 0, \dots, 0) \in \mathbb{R}^n$. Every round, choose an i such that $1 \leq i \leq m$ and $w \cdot v_i \leq 0$, and then replace w with $w + v_i$.

Show that there exists a constant C such that regardless of your choice of i at each step, the process is guaranteed to terminate in C rounds. The constant C may depend on the vectors v_1, \dots, v_m .

Proposed by: Allen Liu

Let $w_0 = 0$, let w_r be the vector w after r rounds. Also, let e_1 denote the vector $(1, 0, \dots, 0)$. Note that

$$w_{r+1}^2 = (w_r + v_i)^2 = w_r^2 + v_i^2 + 2w_r \cdot v_i \leq w_r^2 + v_i^2,$$

as we have by the condition that $w_r \cdot v_i \leq 0$. Therefore, we have by induction that $|w_r| \leq M\sqrt{r}$, where $M = \max_i |v_i|$. Also, we have that

$$w_{r+1} \cdot e_1 = w_r \cdot e_1 + v_i \cdot e_1,$$

so $w_r \cdot e_1 \geq mr$, where $m = \min_i (v_i \cdot e_1) > 0$.

Now using the obvious inequality

$$M\sqrt{r} \geq |w_r| \geq w_r \cdot e_1 \geq mr,$$

we get that $r \leq \frac{M^2}{m^2}$, as desired.

4. [9] Let G be a weighted bipartite graph $A \cup B$, with $|A| = |B| = n$. In other words, each edge in the graph is assigned a positive integer value, called its *weight*. Also, define the weight of a perfect matching in G to be the sum of the weights of the edges in the matching.

Let G' be the graph with vertex set $A \cup B$, and contains the edge e if and only if e is part of some minimum weight perfect matching in G .

Show that all perfect matchings in G' have the same weight.

Proposed by: Yang Liu

Let m denote the minimum weight of a matching in G . Let G'' be the (multi)graph formed by taking the union of all minimum weight perfect matchings, but keeping edges multiple times. Note that G'' is regular. Now, assume that some perfect matching in G' (equivalently G'') has weight $M > m$. Delete this matching from G'' , and call the resulting graph H . H is still regular, so it can be decomposed into a union of perfect matchings. Now using pigeonhole directly gives us a contradiction.

To show that all regular bipartite graphs can be decomposed into a union of perfect matchings, use Hall's marriage lemma to take out one perfect matching and use induction.

5. [11] Let S be the set $\{-1, 1\}^n$, that is, n -tuples such that each coordinate is either -1 or 1 . For

$$s = (s_1, s_2, \dots, s_n), t = (t_1, t_2, \dots, t_n) \in \{-1, 1\}^n,$$

define $s \odot t = (s_1 t_1, s_2 t_2, \dots, s_n t_n)$.

Let c be a positive constant, and let $f : S \rightarrow \{-1, 1\}$ be a function such that there are at least $(1 - c) \cdot 2^{2n}$ pairs (s, t) with $s, t \in S$ such that $f(s \odot t) = f(s)f(t)$. Show that there exists a function f' such that $f'(s \odot t) = f'(s)f'(t)$ for all $s, t \in S$ and $f(s) = f'(s)$ for at least $(1 - 10c) \cdot 2^n$ values of $s \in S$.

Proposed by: Yang Liu

We use finite Fourier analysis, which we explain the setup for below. For a subset $T \subseteq S$, define the function $\chi_T : S \rightarrow \{-1, 1\}$ to satisfy $\chi_T(s) = \prod_{t \in T} s_t$. Note that $\chi_T(s \odot t) = \chi_T(s)\chi_T(t)$ for all $s, t \in S$. Therefore, we should show that there exists a subset T such that taking $f' = \chi_T$ satisfies the constraint. This subset T also *has* to exist, as the χ_T are all the multiplicative functions from $S \rightarrow \{-1, 1\}$. Also, for two functions $f, g : S \rightarrow \mathbb{R}$ define

$$\langle f, g \rangle = \frac{1}{2^n} \sum_{s \in S} f(s)g(s).$$

Now we claim a few facts below that are easy to verify. One, for any subsets $T_1, T_2 \subseteq S$ we have that

$$\langle \chi_{T_1}, \chi_{T_2} \rangle = \begin{cases} 0 & \text{if } T_1 \neq T_2 \\ 1 & \text{if } T_1 = T_2 \end{cases}.$$

We will refer to this claim as *orthogonality*. Also, note that for any function $f : S \rightarrow \mathbb{R}$, we can write

$$f = \sum_{T \subseteq S} \langle f, \chi_T \rangle \chi_T.$$

This is called expressing f by its *Fourier basis*. By the given condition, we have that

$$\sum_{s, t \in S} f(s \odot t) f(s) f(t) = 1 - 2c.$$

Expanding the left-hand side in terms of the Fourier basis, we get that

$$\begin{aligned} 1 - 2c &= \frac{1}{2^{2n}} \sum_{s, t \in S} f(s \odot t) f(s) f(t) = \frac{1}{2^{2n}} \sum_{T_1, T_2, T_3 \subseteq S} \sum_{s, t \in S} \langle f, \chi_{T_1} \rangle \langle f, \chi_{T_2} \rangle \langle f, \chi_{T_3} \rangle \chi_{T_1}(s) \chi_{T_1}(t) \chi_{T_2}(s) \chi_{T_3}(t) \\ &= \sum_{T_1, T_2, T_3 \subseteq S} \langle f, \chi_{T_1} \rangle \langle f, \chi_{T_2} \rangle \langle f, \chi_{T_3} \rangle \langle \chi_{T_1}, \chi_{T_2} \rangle \langle \chi_{T_1}, \chi_{T_3} \rangle = \sum_{T \subseteq S} \langle f, \chi_T \rangle^3 \end{aligned}$$

by orthogonality. Also note by orthogonality that

$$1 = \langle f, f \rangle = \left\langle \sum_{T_1 \subseteq S} \langle f, \chi_{T_1} \rangle \chi_{T_1}(s), \sum_{T_2 \subseteq S} \langle f, \chi_{T_2} \rangle \chi_{T_2}(s) \right\rangle = \sum_{T \subseteq S} \langle f, \chi_T \rangle^2.$$

Finally, from above we have that

$$1 - 2c = \sum_{T \subseteq S} \langle f, \chi_T \rangle^3 \leq \left(\max_{T \subseteq S} \langle f, \chi_T \rangle \right) \sum_{T \subseteq S} \langle f, \chi_T \rangle^2 = \max_{T \subseteq S} \langle f, \chi_T \rangle.$$

This last equation implies that some χ_T and f agree on $(1 - c) \cdot 2^n$ values, as desired.

HMMT February 2016

February 20, 2016

Algebra

1. Let z be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of z .

Proposed by: Evan Chen

Answer: $\boxed{\frac{20}{29}}$

From the problem, let A denote the point z on the unit circle, B denote the point 1.45 on the real axis, and O the origin. Let AH be the height of the triangle OAH and H lies on the segment OB . The real part of z is OH . Now we have $OA = 1$, $OB = 1.45$, and $AB = 1.05$. Thus

$$OH = OA \cos \angle AOB = \cos \angle AOB = \frac{1^2 + 1.45^2 - 1.05^2}{2 \cdot 1 \cdot 1.45} = \frac{20}{29}.$$

2. For which integers $n \in \{1, 2, \dots, 15\}$ is $n^n + 1$ a prime number?

Proposed by: Evan Chen

Answer: $\boxed{1, 2, 4}$

$n = 1$ works. If n has an odd prime factor, you can factor, and this is simulated also by $n = 8$:

$$a^{2k+1} + 1 = (a + 1) \left(\sum_{i=0}^{2k} (-a)^i \right)$$

with both parts larger than one when $a > 1$ and $k > 0$. So it remains to check 2 and 4, which work. Thus the answers are 1, 2, 4.

3. Let A denote the set of all integers n such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of n is 2. Find the sum of the squares of the elements of A .

Proposed by: Evan Chen

Answer: $\boxed{7294927}$

From the given conditions, we want to calculate

$$\sum_{i=0}^3 \sum_{j=i}^3 (10^i + 10^j)^2.$$

By observing the formula, we notice that each term is an exponent of 10. 10^6 shows up 7 times, 10^5 shows up 2 times, 10^4 shows up 9 times, 10^3 shows up 4 times, 10^2 shows up 9 times, 10 shows up 2 times, 1 shows up 7 times. Thus the answer is 7294927.

4. Determine the remainder when

$$\sum_{i=0}^{2015} \left\lfloor \frac{2^i}{25} \right\rfloor$$

is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than x .

Proposed by: Alexander Katz

Answer: $\boxed{14}$

Let r_i denote the remainder when 2^i is divided by 25. Note that because $2^{\phi(25)} \equiv 2^{20} \equiv 1 \pmod{25}$, r is periodic with length 20. In addition, we find that 20 is the order of 2 mod 25. Since 2^i is never a multiple of 5, all possible integers from 1 to 24 are represented by r_1, r_2, \dots, r_{20} with the exceptions of 5, 10, 15, and 20. Hence, $\sum_{i=1}^{20} r_i = \sum_{i=1}^{24} i - (5 + 10 + 15 + 20) = 250$.

We also have

$$\begin{aligned}
 \sum_{i=0}^{2015} \left\lfloor \frac{2^i}{25} \right\rfloor &= \sum_{i=0}^{2015} \frac{2^i - r_i}{25} \\
 &= \sum_{i=0}^{2015} \frac{2^i}{25} - \sum_{i=0}^{2015} \frac{r_i}{25} \\
 &= \frac{2^{2016} - 1}{25} - \sum_{i=0}^{1999} \frac{r_i}{25} - \sum_{i=0}^{15} \frac{r_i}{25} \\
 &= \frac{2^{2016} - 1}{25} - 100 \binom{250}{25} - \sum_{i=0}^{15} \frac{r_i}{25} \\
 &\equiv \frac{2^{2016} - 1}{25} - \sum_{i=0}^{15} \frac{r_i}{25} \pmod{100}
 \end{aligned}$$

We can calculate $\sum_{i=0}^{15} r_i = 185$, so

$$\sum_{i=0}^{2015} \left\lfloor \frac{2^i}{25} \right\rfloor \equiv \frac{2^{2016} - 186}{25} \pmod{100}$$

Now $2^{\phi(625)} \equiv 2^{500} \equiv 1 \pmod{625}$, so $2^{2016} \equiv 2^{16} \equiv 536 \pmod{625}$. Hence $2^{2016} - 186 \equiv 350 \pmod{625}$, and $2^{2016} - 186 \equiv 2 \pmod{4}$. This implies that $2^{2016} - 186 \equiv 350 \pmod{2500}$, and so $\frac{2^{2016} - 186}{25} \equiv \boxed{14} \pmod{100}$.

5. An infinite sequence of real numbers a_1, a_2, \dots satisfies the recurrence

$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$

for every positive integer n . Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \dots + a_{100}$.

Proposed by: Evan Chen

Answer: $\boxed{3}$

A quick telescope gives that $a_1 + \dots + a_n = 2a_1 + a_3 + a_{n-1} - a_{n-2}$ for all $n \geq 3$:

$$\begin{aligned}
 \sum_{k=1}^n a_k &= a_1 + a_2 + a_3 + \sum_{k=1}^{n-3} (a_k - 2a_{k+1} + 2a_{k+2}) \\
 &= a_1 + a_2 + a_3 + \sum_{k=1}^{n-3} a_k - 2 \sum_{k=2}^{n-2} a_k + \sum_{k=3}^{n-1} a_k \\
 &= 2a_1 + a_3 - a_{n-2} + a_{n-1}.
 \end{aligned}$$

Putting $n = 100$ gives the answer.

One actual value of a_2 which yields the sequence is $a_2 = \frac{742745601954}{597303450449}$.

6. Call a positive integer $N \geq 2$ “special” if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). How many special integers are there less than 100?

Proposed by: Casey Fu

Answer: $\boxed{50}$

We claim that all odd numbers are special, and the only special even number is 2. For any even $N > 2$, the numbers relatively prime to N must be odd. When we consider $k = 3$, we see that N can't be expressed as a sum of 3 odd numbers.

Now suppose that N is odd, and we look at the binary decomposition of N , so write $N = 2^{a_1} + 2^{a_2} + \dots + 2^{a_j}$ as a sum of distinct powers of 2. Note that all these numbers only have factors of 2 and are therefore relatively prime to N . We see that $j < \log_2 N + 1$.

We claim that for any $k \geq j$, we can write N as a sum of k powers of 2. Suppose that we have N written as $N = 2^{a_1} + 2^{a_2} + \dots + 2^{a_k}$. Suppose we have at least one of these powers of 2 even, say 2^{a_1} . We can then write $N = 2^{a_1-1} + 2^{a_1-1} + 2^{a_2} + \dots + 2^{a_k}$, which is $k + 1$ powers of 2. The only way this process cannot be carried out is if we write N as a sum of ones, which corresponds to $k = N$. Therefore, this gives us all $k > \log_2 N$.

Now we consider the case $k = 2$. Let 2^a be the largest power of 2 such that $2^a < N$. We can write $N = 2^a + (N - 2^a)$. Note that since 2^a and N are relatively prime, so are $N - 2^a$ and N . Note that $a < \log_2 N$. Now similar to the previous argument, we can write 2^a as a sum of k powers of 2 for $1 < k < 2^a$, and since $2^a > \frac{N}{2}$, we can achieve all k such that $2 \leq k < \frac{N}{2} + 1$.

Putting these together, we see that since $\frac{N}{2} + 1 > \log_2 N$ for $N \geq 3$, we can achieve all k from 2 through N , where N is odd.

7. Determine the smallest positive integer $n \geq 3$ for which

$$A \equiv 2^{10n} \pmod{2^{170}}$$

where A denotes the result when the numbers $2^{10}, 2^{20}, \dots, 2^{10n}$ are written in decimal notation and concatenated (for example, if $n = 2$ we have $A = 10241048576$).

Proposed by: Evan Chen

Answer: 14

Note that

$$2^{10n} = 1024^n = 1.024^n \times 10^{3n}.$$

So 2^{10n} has roughly $3n + 1$ digits for relatively small n 's. (Actually we have that for $0 < x < 1$,

$$(1 + x)^2 = 1 + 2x + x^2 < 1 + 3x.$$

Therefore, $1.024^2 < 1.03^2 < 1.09$, $1.09^2 < 1.27$, $1.27^2 < 1.81 < 2$, and $2^2 = 4$, so $1.024^{16} < 4$. Thus the conclusion holds for $n \leq 16$.)

For any positive integer $n \leq 16$,

$$A = \sum_{i=1}^n 2^{10i} \times 10^{\sum_{j=i+1}^n (3j+1)}.$$

Let

$$A_i = 2^{10i} \times 10^{\sum_{j=i+1}^n (3j+1)}$$

for $1 \leq i \leq n$, then we know that

$$A - 2^{10n} = \sum_{i=1}^{n-1} A_i$$

and

$$A_i = 2^{10i + \sum_{j=i+1}^n (3j+1)} \times 5^{\sum_{j=i+1}^n (3j+1)} = 2^{u_i} \times 5^{v_i}$$

where $u_i = 10i + \sum_{j=i+1}^n (3j + 1)$, $v_i = \sum_{j=i+1}^n (3j + 1)$. We have that

$$u_i - u_{i-1} = 10 - (3i + 1) = 3(3 - i).$$

Thus, for $1 \leq i \leq n - 1$, u_i is minimized when $i = 1$ or $i = n - 1$, with $u_1 = \frac{3n^2 + 5n + 12}{2}$ and $u_{n-1} = 13n - 9$. When $n = 5$,

$$A - 2^{10n} = A_1 + A_2 + A_3 + A_4 = 2^{10} \times 10^{46} + 2^{20} \times 10^{39} + 2^{30} \times 10^{29} + 2^{40} \times 10^{16}$$

is at most divisible by 2^{57} instead of 2^{170} . For all other n 's, we have that $u_1 \neq u_{n-1}$, so we should have that both $170 \leq u_1$ and $170 \leq u_{n-1}$. Therefore, since $170 \leq u_{n-1}$, we have that $14 \leq n$. We can see that $u_1 > 170$ and $14 < 16$ in this case. Therefore, the minimum of n is 14.

8. Define $\phi^1(n)$ as the product of all positive integers less than or equal to n and relatively prime to n . Compute the number of integers $2 \leq n \leq 50$ such that n divides $\phi^1(n) + 1$.

Proposed by: Alexander Katz

Answer: 30

Note that, if k is relatively prime to n , there exists a unique $0 < k^{-1} < n$ such that $kk^{-1} \equiv 1 \pmod{n}$. Hence, if $k^2 \not\equiv 1 \pmod{n}$, we can pair k with its inverse to get a product of 1.

If $k^2 \equiv 1 \pmod{n}$, then $(n-k)^2 \equiv 1 \pmod{n}$ as well, and $k(n-k) \equiv -k^2 \equiv -1 \pmod{n}$. Hence these k can be paired up as well, giving products of -1. When $n \neq 2$, there is no k such that $k^2 \equiv 1 \pmod{n}$ and $k \equiv n-k \pmod{n}$, so the total product \pmod{n} is $(-1)^{\frac{m}{2}}$, where m is the number of k such that $k^2 \equiv 1 \pmod{n}$.

For prime p and positive integer i , the number of solutions to $k^2 \equiv 1 \pmod{p^i}$ is 2 if p is odd, 4 if $p = 2$ and $i \geq 3$, and 2 if $p = i = 2$. So, by Chinese remainder theorem, if we want the product to be -1 , we need $n = p^k, 2p^k$, or 4. We can also manually check the $n = 2$ case to work.

Counting the number of integers in the allowed range that are of one of these forms (or, easier, doing complementary counting), we get an answer of 30.

(Note that this complicated argument basically reduces to wanting a primitive root.)

9. For any positive integer n , S_n be the set of all permutations of $\{1, 2, 3, \dots, n\}$. For each permutation $\pi \in S_n$, let $f(\pi)$ be the number of ordered pairs (j, k) for which $\pi(j) > \pi(k)$ and $1 \leq j < k \leq n$. Further define $g(\pi)$ to be the number of positive integers $k \leq n$ such that $\pi(k) \equiv k \pm 1 \pmod{n}$. Compute

$$\sum_{\pi \in S_{999}} (-1)^{f(\pi)+g(\pi)}.$$

Proposed by: Ritesh Ragavender

Answer: 995×2^{998}

Define an $n \times n$ matrix $A_n(x)$ with entries $a_{i,j} = x$ if $i \equiv j \pm 1 \pmod{n}$ and 1 otherwise. Let $F(x) = \sum_{\pi \in S_n} (-1)^{f(\pi)} x^{g(\pi)}$ (here $(-1)^{f(\pi)}$ gives the sign $\prod \frac{\pi(u)-\pi(v)}{u-v}$ of the permutation π). Note by construction that $F(x) = \det(A_n(x))$.

We find that the eigenvalues of $A_n(x)$ are $2x + n - 2$ (eigenvector of all ones) and $(x-1)(\omega_j + \omega_j^{-1})$, where $\omega_j = e^{\frac{2\pi j i}{n}}$, for $1 \leq j \leq n-1$. Since the determinant is the product of the eigenvalues,

$$F(x) = (2x + n - 2)2^{n-1}(x-1)^{n-1} \prod_{k=1}^{n-1} \cos\left(\frac{2\pi k}{n}\right).$$

Evaluate the product and plug in $x = -1$ to finish. (As an aside, this approach also tells us that the sum is 0 whenever n is a multiple of 4.)

10. Let a, b and c be positive real numbers such that

$$\begin{aligned} a^2 + ab + b^2 &= 9 \\ b^2 + bc + c^2 &= 52 \\ c^2 + ca + a^2 &= 49. \end{aligned}$$

Compute the value of $\frac{49b^2 - 33bc + 9c^2}{a^2}$.

Proposed by: Alexander Katz

Answer: 52

Consider a triangle ABC with Fermat point P such that $AP = a, BP = b, CP = c$. Then

$$AB^2 = AP^2 + BP^2 - 2AP \cdot BP \cos(120^\circ)$$

by the Law of Cosines, which becomes

$$AB^2 = a^2 + ab + b^2$$

and hence $AB = 3$. Similarly, $BC = \sqrt{52}$ and $AC = 7$.

Furthermore, we have

$$\begin{aligned} BC^2 = 52 &= AB^2 + AC^2 - 2AB \cdot AC \cos \angle BAC \\ &= 3^2 + 7^2 - 2 \cdot 3 \cdot 7 \cos \angle BAC \\ &= 58 - 42 \cos \angle BAC \end{aligned}$$

And so $\cos \angle BAC = \frac{1}{7}$.

Invert about A with arbitrary radius r . Let B', P', C' be the images of B, P, C respectively. Since $\angle APB = \angle AB'P' = 120^\circ$ and $\angle APC = \angle AC'P' = 120^\circ$, we note that $\angle B'P'C' = 120^\circ - \angle BAC$, and so

$$\begin{aligned} \cos \angle B'P'C' &= \cos(120^\circ - \angle BAC) \\ &= \cos 120^\circ \cos \angle BAC - \sin 120^\circ \sin \angle BAC \\ &= -\frac{1}{2} \left(\frac{1}{7} \right) + \frac{\sqrt{3}}{2} \left(\frac{4\sqrt{3}}{7} \right) \\ &= \frac{11}{14} \end{aligned}$$

Furthermore, using the well-known result

$$B'C' = \frac{r^2 BC}{AB \cdot AC}$$

for an inversion about A , we have

$$\begin{aligned} B'P' &= \frac{BP r^2}{AB \cdot AP} \\ &= \frac{br^2}{a \cdot 3} \\ &= \frac{br^2}{3a} \end{aligned}$$

and similarly $P'C' = \frac{cr^2}{7a}$, $B'C' = \frac{r^2 \sqrt{52}}{21}$. Applying the Law of Cosines to $B'P'C'$ gives us

$$\begin{aligned} B'C'^2 &= B'P'^2 + P'C'^2 - 2B'P' \cdot P'C' \cos(120^\circ - \angle BAC) \\ \implies \frac{52r^4}{21^2} &= \frac{b^2 r^4}{9a^2} + \frac{c^2 r^4}{49a^2} - \frac{11bcr^4}{147a^2} \\ \implies \frac{52}{21^2} &= \frac{b^2}{9a^2} + \frac{c^2}{49a^2} - \frac{11bc}{147a^2} \\ \implies \frac{52}{21^2} &= \frac{49b^2 - 33bc + 9c^2}{21^2 a^2} \end{aligned}$$

and so $\frac{49b^2 - 33bc + 9c^2}{a^2} = \boxed{52}$.

Motivation: the desired sum looks suspiciously like the result of some Law of Cosines, so we should try building a triangle with sides $\frac{7b}{a}$ and $\frac{3c}{a}$. Getting the $-\frac{33bc}{a}$ term is then a matter of setting $\cos \theta = \frac{11}{14}$. Now there are two possible leaps: noticing that $\cos \theta = \cos(120 - \angle BAC)$, or realizing that it's pretty difficult to contrive a side of $\frac{7b}{a}$ - but it's much easier to contrive a side of $\frac{b}{3a}$. Either way leads to the natural inversion idea, and the rest is a matter of computation.

HMMT February 2016
February 20, 2016
Combinatorics

1. For positive integers n , let S_n be the set of integers x such that n distinct lines, no three concurrent, can divide a plane into x regions (for example, $S_2 = \{3, 4\}$, because the plane is divided into 3 regions if the two lines are parallel, and 4 regions otherwise). What is the minimum i such that S_i contains at least 4 elements?

Proposed by:

Answer:

For S_3 , either all three lines are parallel (4 regions), exactly two are parallel (6 regions), or none are parallel (6 or seven regions, depending on whether they all meet at one point), so $|S_3| = 3$. Then, for S_4 , either all lines are parallel (5 regions), exactly three are parallel (8 regions), there are two sets of parallel pairs (9 regions), exactly two are parallel (9 or 10 regions), or none are parallel (8, 9, 10, or 11 regions), so $|S_4| = 4$.

2. Starting with an empty string, we create a string by repeatedly appending one of the letters H , M , T with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, respectively, until the letter M appears twice consecutively. What is the expected value of the length of the resulting string?

Proposed by:

Answer:

Let E be the expected value of the resulting string. Starting from the empty string,

- We have a $\frac{1}{2}$ chance of not selecting the letter M ; from here the length of the resulting string is $1 + E$.
- We have a $\frac{1}{4}$ chance of selecting the letter M followed by a letter other than M , which gives a string of length $2 + E$.
- We have a $\frac{1}{4}$ chance of selecting M twice, for a string of length 2.

Thus, $E = \frac{1}{2}(1 + E) + \frac{1}{4}(2 + E) + \frac{1}{4}(2)$. Solving gives $E = 6$.

3. Find the number of ordered pairs of integers (a, b) such that a, b are divisors of 720 but ab is not.

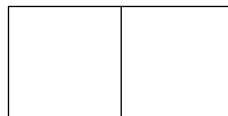
Proposed by: Casey Fu

Answer:

First consider the case $a, b > 0$. We have $720 = 2^4 \cdot 3^2 \cdot 5$, so the number of divisors of 720 is $5 \cdot 3 \cdot 2 = 30$. We consider the number of ways to select an ordered pair (a, b) such that a, b, ab all divide 720. Using the balls and urns method on each of the prime factors, we find the number of ways to distribute the factors of 2 across a and b is $\binom{6}{2}$, the factors of 3 is $\binom{4}{2}$, the factors of 5 is $\binom{3}{2}$. So the total number of ways to select (a, b) with a, b, ab all dividing 720 is $15 \cdot 6 \cdot 3 = 270$. The number of ways to select any (a, b) with a and b dividing 720 is $30 \cdot 30 = 900$, so there are $900 - 270 = 630$ ways to select a and b such that a, b divide 720 but ab doesn't.

Now, each $a, b > 0$ corresponds to four solutions $(\pm a, \pm b)$ giving the final answer of 2520. (Note that $ab \neq 0$.)

4. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



How many subsets of these seven edges form a connected figure?

Proposed by: Joy Zheng

Answer: $\boxed{81}$

We break this into cases. First, if the middle edge is not included, then there are $6 * 5 = 30$ ways to choose two distinct points for the figure to begin and end at. We could also allow the figure to include all or none of the six remaining edges, for a total of 32 connected figures not including the middle edge. Now let's assume we are including the middle edge. Of the three edges to the left of the middle edge, there are 7 possible subsets we can include (8 total subsets, but we subtract off the subset consisting of only the edge parallel to the middle edge since it's not connected). Similarly, of the three edges to the right of the middle edge, there are 7 possible subsets we can include. In total, there are 49 possible connected figures that include the middle edge. Therefore, there are $32 + 49 = 81$ possible connected figures.

5. Let a, b, c, d, e, f be integers selected from the set $\{1, 2, \dots, 100\}$, uniformly and at random with replacement. Set

$$M = a + 2b + 4c + 8d + 16e + 32f.$$

What is the expected value of the remainder when M is divided by 64?

Proposed by: Evan Chen

Answer: $\boxed{\frac{63}{2}}$

Consider M in binary. Assume we start with $M = 0$, then add a to M , then add $2b$ to M , then add $4c$ to M , and so on. After the first addition, the first bit (defined as the rightmost bit) of M is toggled with probability $\frac{1}{2}$. After the second addition, the second bit of M is toggled with probability $\frac{1}{2}$. After the third addition, the third bit is toggled with probability $\frac{1}{2}$, and so on for the remaining three additions. As such, the six bits of M are each toggled with probability $\frac{1}{2}$ - specifically, the k^{th} bit is toggled with probability $\frac{1}{2}$ at the k^{th} addition, and is never toggled afterwards. Therefore, each residue from 0 to 63 has probability $\frac{1}{64}$ of occurring, so they are all equally likely. The expected value is then just $\frac{63}{2}$.

6. Define the sequence $a_1, a_2 \dots$ as follows: $a_1 = 1$ and for every $n \geq 2$,

$$a_n = \begin{cases} n - 2 & \text{if } a_{n-1} = 0 \\ a_{n-1} - 1 & \text{if } a_{n-1} \neq 0 \end{cases}$$

A non-negative integer d is said to be *jet-lagged* if there are non-negative integers r, s and a positive integer n such that $d = r + s$ and that $a_{n+r} = a_n + s$. How many integers in $\{1, 2, \dots, 2016\}$ are jet-lagged?

Proposed by: Pakawut Jiradilok

Let $N = n + r$, and $M = n$. Then $r = N - M$, and $s = a_N - a_M$, and $d = r + s = (a_N + N) - (a_M + M)$. So we are trying to find the number of possible values of $(a_N + N) - (a_M + M)$, subject to $N \geq M$ and $a_N \geq a_M$.

Divide the a_i into the following "blocks":

- $a_1 = 1, a_2 = 0,$
- $a_3 = 1, a_4 = 0,$
- $a_5 = 3, a_6 = 2, a_7 = 1, a_8 = 0,$
- $a_9 = 7, a_{10} = 6, \dots, a_{16} = 0,$

and so on. The k^{th} block contains a_i for $2^{k-1} < i \leq 2^k$. It's easy to see by induction that $a_{2^k} = 0$ and thus $a_{2^{k+1}} = 2^k - 1$ for all $k \geq 1$. Within each block, the value $a_n + n$ is constant, and for the k^{th} block ($k \geq 1$) it equals 2^k . Therefore, $d = (a_N + N) - (a_M + M)$ is the difference of two powers of 2, say $2^n - 2^m$. For any $n \geq 1$, it is clear there exists an N such that $a_N + N = 2^n$ (consider the n^{th}

block). We can guarantee $a_N \geq a_M$ by setting $M = 2^m$. Therefore, we are searching for the number of integers between 1 and 2016 that can be written as $2^n - 2^m$ with $n \geq m \geq 1$. The pairs (n, m) with $n > m \geq 1$ and $n \leq 10$ all satisfy $1 \leq 2^n - 2^m \leq 2016$ (45 possibilities). In the case that $n = 11$, we have that $2^n - 2^m \leq 2016$ so $2^m \geq 32$, so $m \geq 5$ (6 possibilities). There are therefore $45 + 6 = 51$ jetlagged numbers between 1 and 2016.

7. Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal!

Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$.

Proposed by: Alexander Katz

Answer: 24, 28, 32

Ed. note: I'm probably horribly abusing notation

Define the *generating function* of an event A as the polynomial

$$g(A, x) = \sum p_i x^i$$

where p_i denotes the probability that i occurs during event A . We note that the generating is multiplicative; i.e.

$$g(A \text{ AND } B, x) = g(A)g(B) = \sum p_i q_j x^{i+j}$$

where q_j denotes the probability that j occurs during event B .

In our case, events A and B are the rolling of the first and second dice, respectively, so the generating functions are the same:

$$g(\text{die}, x) = \frac{1}{8}x^1 + \frac{1}{8}x^2 + \frac{1}{8}x^3 + \frac{1}{8}x^4 + \frac{1}{8}x^5 + \frac{1}{8}x^6 + \frac{1}{8}x^7 + \frac{1}{8}x^8$$

and so

$$g(\text{both dice rolled}, x) = g(\text{die}, x)^2 = \frac{1}{64}(x^1 + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8)^2$$

where the coefficient of x^i denotes the probability of rolling a sum of i .

We wish to find two alternate dice, C and D , satisfying the following conditions:

- C and D are both 8-sided dice; i.e. the sum of the coefficients of $g(C, x)$ and $g(D, x)$ are both 8 (or $g(C, 1) = g(D, 1) = 8$).
- The faces of C and D are all labeled with a positive integer; i.e. the powers of each term of $g(C, x)$ and $g(D, x)$ are positive integer (or $g(C, 0) = g(D, 0) = 0$).
- The probability of rolling any given sum upon rolling C and D is equal to the probability of rolling any given sum upon rolling A and B ; i.e. $g(C, x)g(D, x) = g(A, x)g(B, x)$.

Because the dice are "fair" – i.e. the probability of rolling any face is $\frac{1}{8}$ – we can multiply $g(A, x)$, $g(B, x)$, $g(C, x)$ and $g(D, x)$ by 8 to get integer polynomials; as this does not affect any of the conditions, we can assume $g(C, x)$ and $g(D, x)$ are integer polynomials multiplying to $(x^1 + x^2 + \dots + x^8)^2$ (and subject to the other two conditions as well). Since \mathbb{Z} is a UFD (i.e. integer polynomials can be expressed as the product of integer polynomials in exactly one way, up to order and scaling by a constant), all factors of $g(C, x)$ and $g(D, x)$ must also be factors of $x^1 + x^2 + \dots + x^8$. Hence it is useful to factor $x^1 + x^2 + \dots + x^8 = x(x+1)(x^2+1)(x^4+1)$.

We thus have $g(C, x)g(D, x) = x^2(x+1)^2(x^2+1)^2(x^4+1)^2$. We know that $g(C, 0) = g(D, 0) = 0$, so $x \mid g(C, x), g(D, x)$. It remains to distribute the remaining term $(x+1)^2(x^2+1)^2(x^4+1)^2$; we can view each of these 6 factors as being “assigned” to either C or D . Note that since $g(C, 1) = g(D, 1) = 8$, and each of the factors $x+1, x^2+1, x^4+1$ evaluates to 2 when $x=1$, exactly three factors must be assigned to C and exactly three to D . Finally, assigning $x+1, x^2+1$, and x^4+1 to C results in the standard die, with $a = b = 28$. This gives us the three cases (and their permutations):

- $g(C, x) = x(x+1)^2(x^2+1)$, $g(D, x) = x(x^2+1)(x^4+1)^2$. In this case we get $g(C, x) = x^5 + 2x^4 + 2x^3 + 2x^2 + x$ and $g(D, x) = x^{11} + x^9 + 2x^7 + 2x^5 + x^3 + x$, so the “smaller” die has faces 5, 4, 4, 3, 3, 2, and 1 which sum to 24.
- $g(C, x) = x(x+1)(x^2+1)^2$, $g(D, x) = x(x+1)(x^4+1)^2$. In this case we have $g(C, x) = x^6 + x^5 + 2x^4 + 2x^3 + x^2 + x$ and $g(D, x) = x^{10} + x^9 + 2x^6 + 2x^5 + x^2 + x$, so the “smaller” die has faces 6, 5, 4, 4, 3, 3, 2 and 1 which sum to 28.
- $g(C, x) = x(x^2+1)^2(x^4+1)$, $g(D, x) = x(x+1)^2(x^4+1)$. In this case we have $g(C, x) = x^9 + 2x^7 + 2x^5 + 2x^3 + x$ and $g(D, x) = x^7 + 2x^6 + x^5 + x^3 + 2x^2 + x$, so the “smaller die” has faces 7, 6, 6, 5, 3, 2, 2, 1 which sum to 32.

Therefore, $\min\{a, b\}$ is equal to $\boxed{24, 28, \text{ or } 32}$.

8. Let X be the collection of all functions $f : \{0, 1, \dots, 2016\} \rightarrow \{0, 1, \dots, 2016\}$. Compute the number of functions $f \in X$ such that

$$\max_{g \in X} \left(\min_{0 \leq i \leq 2016} (\max(f(i), g(i))) - \max_{0 \leq i \leq 2016} (\min(f(i), g(i))) \right) = 2015.$$

Proposed by:

Answer: $\boxed{2 \cdot (3^{2017} - 2^{2017})}$

For each $f, g \in X$, we define

$$d(f, g) := \min_{0 \leq i \leq 2016} (\max(f(i), g(i))) - \max_{0 \leq i \leq 2016} (\min(f(i), g(i)))$$

Thus we desire $\max_{g \in X} d(f, g) = 2015$.

First, we count the number of functions $f \in X$ such that

$$\exists g : \min_i \max\{f(i), g(i)\} \geq 2015 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} = 0.$$

That means for every value of i , either $f(i) = 0$ (then we pick $g(i) = 2015$) or $f(i) \geq 2015$ (then we pick $g(i) = 0$). So there are $A = 3^{2017}$ functions in this case.

Similarly, the number of functions such that

$$\exists g : \min_i \max\{f(i), g(i)\} = 2016 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} \leq 1$$

is also $B = 3^{2017}$.

Finally, the number of functions such that

$$\exists g : \min_i \max\{f(i), g(i)\} = 2016 \text{ and } \exists g : \min_i \max\{f(i), g(i)\} = 0$$

is $C = 2^{2017}$.

Now $A + B - C$ counts the number of functions with $\max_{g \in X} d(f, g) \geq 2015$ and C counts the number of functions with $\max_{g \in X} d(f, g) \geq 2016$, so the answer is $A + B - 2C = 2 \cdot (3^{2017} - 2^{2017})$.

9. Let $V = \{1, \dots, 8\}$. How many permutations $\sigma : V \rightarrow V$ are automorphisms of some tree?

(A *graph* consists of a some set of vertices and some edges between pairs of distinct vertices. It is *connected* if every two vertices in it are connected by some path of one or more edges. A *tree* G on V is a connected graph with vertex set V and exactly $|V| - 1$ edges, and an *automorphism* of G is a permutation $\sigma : V \rightarrow V$ such that vertices $i, j \in V$ are connected by an edge if and only if $\sigma(i)$ and $\sigma(j)$ are.)

Proposed by: Mitchell Lee

Answer: 30212

We decompose into cycle types of σ . Note that within each cycle, all vertices have the same degree; also note that the tree has total degree 14 across its vertices (by all its seven edges).

For any permutation that has a 1 in its cycle type (i.e it has a fixed point), let $1 \leq a \leq 8$ be a fixed point. Consider the tree that consists of the seven edges from a to the seven other vertices - this permutation (with a as a fixed point) is an automorphism of this tree.

For any permutation that has cycle type $2 + 6$, let a and b be the two elements in the 2-cycle. If the 6-cycle consists of c, d, e, f, g, h in that order, consider the tree with edges between a and b, c, e, g and between b and d, f, h . It's easy to see σ is an automorphism of this tree.

For any permutation that has cycle type $2 + 2 + 4$, let a and b be the two elements of the first two-cycle. Let the other two cycle consist of c and d , and the four cycle be e, f, g, h in that order. Then consider the tree with edges between a and b, a and c, b and d, a and e, b and f, a and g, b and h . It's easy to see σ is an automorphism of this tree.

For any permutation that has cycle type $2 + 3 + 3$, let a and b be the vertices in the 2-cycle. One of a and b must be connected to a vertex distinct from a, b (follows from connectedness), so there must be an edge between a vertex in the 2-cycle and a vertex in a 3-cycle. Repeatedly applying σ to this edge leads to a cycle of length 4 in the tree, which is impossible (a tree has no cycles). Therefore, these permutations cannot be automorphisms of any tree.

For any permutation that has cycle type $3 + 5$, similarly, there must be an edge between a vertex in the 3-cycle and a vertex in the 5-cycle. Repeatedly applying σ to this edge once again leads to a cycle in the tree, which is not possible. So these permutations cannot be automorphisms of any tree.

The only remaining possible cycle types of σ are $4 + 4$ and 8 . In the first case, if we let x and y be the degrees of the vertices in each of the cycles, then $4x + 4y = 14$, which is impossible for integer x, y . In the second case, if we let x be the degree of the vertices in the 8-cycle, then $8x = 14$, which is not possible either.

So we are looking for the number of permutations whose cycle type is not $2 + 2 + 3, 8, 4 + 4, 3 + 5$. The number of permutations with cycle type $2 + 2 + 3$ is $\binom{8}{2} \frac{1}{2} \binom{6}{3} (2!)^2 = 1120$, with cycle type 8 is $7! = 5040$, with cycle type $4 + 4$ is $\frac{1}{2} \binom{8}{4} (3!)^2 = 1260$, with cycle type $3 + 5$ is $\binom{8}{3} (2!)(4!) = 2688$. Therefore, by complementary counting, the number of permutations that ARE automorphisms of some tree is $8! - 1120 - 1260 - 2688 - 5040 = 30212$.

10. Kristoff is planning to transport a number of indivisible ice blocks with positive integer weights from the north mountain to Arendelle. He knows that when he reaches Arendelle, Princess Anna and Queen Elsa will name an ordered pair (p, q) of nonnegative integers satisfying $p + q \leq 2016$. Kristoff must then give Princess Anna *exactly* p kilograms of ice. Afterward, he must give Queen Elsa *exactly* q kilograms of ice.

What is the minimum number of blocks of ice Kristoff must carry to guarantee that he can always meet Anna and Elsa's demands, regardless of which p and q are chosen?

Proposed by: Pakawut Jiradilok

Answer: 18

The answer is 18.

First, we will show that Kristoff must carry at least 18 ice blocks. Let

$$0 < x_1 \leq x_2 \leq \cdots \leq x_n$$

be the weights of ice blocks he carries which satisfy the condition that for any $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \leq 2016$, there are disjoint subsets I, J of $\{1, \dots, n\}$ such that $\sum_{\alpha \in I} x_\alpha = p$ and $\sum_{\alpha \in J} x_\alpha = q$.

Claim: For any i , if $x_1 + \cdots + x_i \leq 2014$, then

$$x_{i+1} \leq \left\lfloor \frac{x_1 + \cdots + x_i}{2} \right\rfloor + 1.$$

Proof. Suppose to the contrary that $x_{i+1} \geq \lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 2$. Consider when Anna and Elsa both demand $\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1$ kilograms of ice (which is possible as $2 \times (\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1) \leq x_1 + \cdots + x_i + 2 \leq 2016$). Kristoff cannot give any ice x_j with $j \geq i + 1$ (which is too heavy), so he has to use from x_1, \dots, x_i . Since he is always able to satisfy Anna's and Elsa's demands, $x_1 + \cdots + x_i \geq 2 \times (\lfloor \frac{x_1 + \cdots + x_i}{2} \rfloor + 1) \geq x_1 + \cdots + x_i + 1$. A contradiction. \square

It is easy to see $x_1 = 1$, so by hand we compute obtain the inequalities $x_2 \leq 1, x_3 \leq 2, x_4 \leq 3, x_5 \leq 4, x_6 \leq 6, x_7 \leq 9, x_8 \leq 14, x_9 \leq 21, x_{10} \leq 31, x_{11} \leq 47, x_{12} \leq 70, x_{13} \leq 105, x_{14} \leq 158, x_{15} \leq 237, x_{16} \leq 355, x_{17} \leq 533, x_{18} \leq 799$. And we know $n \geq 18$; otherwise the sum $x_1 + \cdots + x_n$ would not reach 2016.

Now we will prove that $n = 18$ works. Consider the 18 numbers named above, say $a_1 = 1, a_2 = 1, a_3 = 2, a_4 = 3, \dots, a_{18} = 799$. We claim that with a_1, \dots, a_k , for any $p, q \in \mathbb{Z}_{\geq 0}$ such that $p + q \leq a_1 + \cdots + a_k$, there are two disjoint subsets I, J of $\{1, \dots, k\}$ such that $\sum_{\alpha \in I} a_\alpha = p$ and $\sum_{\alpha \in J} a_\alpha = q$. We prove this by induction on k . It is clear for small $k = 1, 2, 3$. Now suppose this is true for a certain k , and we add in a_{k+1} . When Kristoff meets Anna first and she demands p kilograms of ice, there are two cases.

Case I: if $p \geq a_{k+1}$, then Kristoff gives the a_{k+1} block to Anna first, then he consider $p' = p - a_{k+1}$ and the same unknown q . Now $p' + q \leq a_1 + \cdots + a_k$ and he has a_1, \dots, a_k , so by induction he can successfully complete his task.

Case II: if $p < a_{k+1}$, regardless of the value of q , he uses the same strategy as if $p + q \leq a_1 + \cdots + a_k$ and he uses ice from a_1, \dots, a_k without touching a_{k+1} . Then, when he meets Elsa, if $q \leq a_1 + \cdots + a_k - p$, he is safe. If $q \geq a_1 + \cdots + a_k - p + 1$, we know $q - a_{k+1} \geq a_1 + \cdots + a_k - p + 1 - (\lfloor \frac{a_1 + \cdots + a_k}{2} \rfloor + 1) \geq 0$. So he can give the a_{k+1} to Elsa first then do as if $q' = q - a_{k+1}$ is the new demand by Elsa. He can now supply the ice to Elsa because $p + q' \leq a_1 + \cdots + a_k$. Thus, we finish our induction.

Therefore, Kristoff can carry those 18 blocks of ice and be certain that for any $p + q \leq a_1 + \cdots + a_{18} = 2396$, there are two disjoint subsets $I, J \subseteq \{1, \dots, 18\}$ such that $\sum_{\alpha \in I} a_\alpha = p$ and $\sum_{\alpha \in J} a_\alpha = q$. In other words, he can deliver the amount of ice both Anna and Elsa demand.

HMMT February 2016

February 20, 2016

Geometry

1. Dodecagon $QWARTZSPHINX$ has all side lengths equal to 2, is not self-intersecting (in particular, the twelve vertices are all distinct), and moreover each interior angle is either 90° or 270° . What are all possible values of the area of $\triangle SIX$?

Proposed by: Evan Chen

Answer: 2, 6

The dodecagon has to be a "plus shape" of area 20, then just try the three non-congruent possibilities.

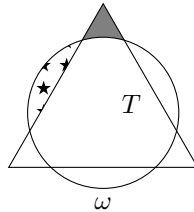
2. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC . Find the distance between the circumcenters of triangles AHB and AHC .

Proposed by: Evan Chen

Answer: 14

Let H_B be the reflection of H over AC and let H_C be the reflection of H over AB . The reflections of H over AB, AC lie on the circumcircle of triangle ABC . Since the circumcenters of triangles $AH_C B, AH_B C$ are both O , the circumcenters of AHB, AHC are reflections of O over AB, AC respectively. Moreover, the lines from O to the circumcenters in question are the perpendicular bisectors of AB and AC . Now we see that the distance between the two circumcenters is simply twice the length of the midline of triangle ABC that is parallel to BC , meaning the distance is $2(\frac{1}{2}BC) = 14$.

3. In the below picture, T is an equilateral triangle with a side length of 5 and ω is a circle with a radius of 2. The triangle and the circle have the same center. Let X be the area of the shaded region, and let Y be the area of the starred region. What is $X - Y$?



Proposed by:

Answer: $\frac{25\sqrt{3}}{12} - \frac{4\pi}{3}$

Let $K(P)$ denote the area of P . Note that $K(T) - K(\omega) = 3(X - Y)$, which gives our answer.

4. Let ABC be a triangle with $AB = 3$, $AC = 8$, $BC = 7$ and let M and N be the midpoints of \overline{AB} and \overline{AC} , respectively. Point T is selected on side BC so that $AT = TC$. The circumcircles of triangles BAT, MAN intersect at D . Compute DC .

Proposed by: Evan Chen

Answer: $\frac{7\sqrt{3}}{3}$

We note that D is the circumcenter O of ABC , since $2\angle C = \angle ATB = \angle AOB$. So we are merely looking for the circumradius of triangle ABC . By Heron's Formula, the area of the triangle is $\sqrt{9 \cdot 6 \cdot 1 \cdot 2} = 6\sqrt{3}$, so using the formula $\frac{abc}{4R} = K$, we get an answer of $\frac{3 \cdot 8 \cdot 7}{4 \cdot 6\sqrt{3}} = \frac{7\sqrt{3}}{3}$. Alternatively, one can compute the circumradius using trigonometric methods or the fact that $\angle A = 60^\circ$.

5. Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

Proposed by: Evan Chen

Answer: 462

The lines in question are the radical axes of the 9 circles. Three circles with noncollinear centers have a radical center where their three pairwise radical axes concur, but all other intersections between two of the $\binom{9}{2}$ lines can be made to be distinct. So the answer is

$$\binom{\binom{9}{2}}{2} - 2\binom{9}{3} = 462$$

by just counting pairs of lines, and then subtracting off double counts due to radical centers (each counted three times).

6. Let ABC be a triangle with incenter I , incircle γ and circumcircle Γ . Let M, N, P be the midpoints of sides $\overline{BC}, \overline{CA}, \overline{AB}$ and let E, F be the tangency points of γ with \overline{CA} and \overline{AB} , respectively. Let U, V be the intersections of line EF with line MN and line MP , respectively, and let X be the midpoint of arc \widehat{BAC} of Γ . Given that $AB = 5, AC = 8$, and $\angle A = 60^\circ$, compute the area of triangle XUV .

Proposed by: Evan Chen

Answer: $\frac{21\sqrt{3}}{8}$

Let segments AI and EF meet at K . Extending AK to meet the circumcircle again at Y , we see that X and Y are diametrically opposite, and it follows that AX and EF are parallel. Therefore the height from X to \overline{UV} is merely AK . Observe that $AE = AF$, so $\triangle AEF$ is equilateral; since MN, MP are parallel to AF, AE respectively, it follows that $\triangle MVU, \triangle UEN, \triangle FPV$ are equilateral as well. Then $MV = MP - PV = \frac{1}{2}AC - FP = \frac{1}{2}AC - AF + AP = \frac{1}{2}AC - AF + \frac{1}{2}AB = \frac{1}{2}BC$, since E, F are the tangency points of the incircle. Since $\triangle MVU$ is equilateral, we have $UV = MU = MV = \frac{1}{2}BC$.

Now we can compute $BC = 7$, whence $UV = \frac{7}{2}$ and

$$AK = \frac{AB + AC - BC}{2} \cdot \cos 30^\circ = \frac{3\sqrt{3}}{2}.$$

Hence, the answer is $\frac{21\sqrt{3}}{8}$.

7. Let $S = \{(x, y) | x, y \in \mathbb{Z}, 0 \leq x, y, \leq 2016\}$. Given points $A = (x_1, y_1), B = (x_2, y_2)$ in S , define

$$d_{2017}(A, B) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \pmod{2017}.$$

The points $A = (5, 5), B = (2, 6), C = (7, 11)$ all lie in S . There is also a point $O \in S$ that satisfies

$$d_{2017}(O, A) = d_{2017}(O, B) = d_{2017}(O, C).$$

Find $d_{2017}(O, A)$.

Proposed by: Yang Liu

Answer: 1021

Note that the triangle is a right triangle with right angle at A . Therefore, $R^2 = \frac{(7-2)^2 + (11-6)^2}{4} = \frac{25}{2} = (25)(2^{-1}) \equiv 1021 \pmod{2017}$. (An equivalent approach works for general triangles; the fact that the triangle is right simply makes the circumradius slightly easier to compute.)

8. For $i = 0, 1, \dots, 5$ let l_i be the ray on the Cartesian plane starting at the origin, an angle $\theta = i\frac{\pi}{3}$ counterclockwise from the positive x -axis. For each i , point P_i is chosen uniformly at random from the intersection of l_i with the unit disk. Consider the convex hull of the points P_i , which will (with probability 1) be a convex polygon with n vertices for some n . What is the expected value of n ?

Proposed by:

Answer: $\boxed{2 + 4\ln(2)}$

A vertex P_i is part of the convex hull if and only if it is not contained in the triangle formed by the origin and the two adjacent vertices. Let the probability that a given vertex is contained in the aforementioned triangle be p . By linearity of expectation, our answer is simply $6(1 - p)$. Say $|P_0| = a, |P_2| = b$. Stewart's Theorem and the Law of Cosines give that p is equal to the probability that $|P_1| < \sqrt{ab - ab\frac{a^2+b^2+ab}{(a+b)^2}} = \frac{ab}{a+b}$; alternatively this is easy to derive using coordinate methods. The corresponding double integral evaluates to $p = \frac{2}{3}(1 - \ln(2))$, thus telling us our answer.

9. In cyclic quadrilateral $ABCD$ with $AB = AD = 49$ and $AC = 73$, let I and J denote the incenters of triangles ABD and CBD . If diagonal \overline{BD} bisects \overline{IJ} , find the length of IJ .

Proposed by: Evan Chen

Answer: $\boxed{\frac{28}{5}\sqrt{69}}$

Let O be circumcenter, R the circumradius and r the common inradius. We have $IO^2 = JO^2 = R(R - 2r)$ by a result of Euler; denote x for the common value of IO and JO . Additionally, we know $AJ = AB = AD = 49$ (angle chase to find that $\angle BJA = \angle JBA$). Since A is the midpoint of the arc \widehat{BD} not containing C , both J and A lie on the angle bisector of angle $\angle BCD$, so C, J, A are collinear. So by Power of a Point we have

$$R^2 - x^2 = 2Rr = AJ \cdot JC = 49 \cdot 24.$$

Next, observe that the angle bisector of angle BAD contains both I and O , so A, I, O are collinear. Let M be the midpoint of IJ , lying on \overline{BD} . Let K be the intersection of IO and BD . Observing that the right triangles $\triangle IMO$ and $\triangle IKM$ are similar, we find $IM^2 = IK \cdot IO = rx$, so $IJ^2 = 4rx$. Now apply Stewart's Theorem to $\triangle AOJ$ to derive

$$R(x(R - x) + 4rx) = 49^2x + x^2(R - x).$$

Eliminating the common factor of x and rearranging gives

$$49^2 - (R - x)^2 = 4Rr = 48 \cdot 49$$

so $R - x = 7$. Hence $R + x = \frac{49 \cdot 24}{7} = 168$, and thus $2R = 175$, $2x = 161$. Thus $r = \frac{49 \cdot 24}{175} = \frac{168}{25}$.

Finally, $IJ = 2\sqrt{rx} = 2\sqrt{\frac{84 \cdot 161}{25}} = \frac{28\sqrt{69}}{5}$.

10. The incircle of a triangle ABC is tangent to BC at D . Let H and Γ denote the orthocenter and circumcircle of $\triangle ABC$. The B -mixtilinear incircle, centered at O_B , is tangent to lines BA and BC and internally tangent to Γ . The C -mixtilinear incircle, centered at O_C , is defined similarly. Suppose that $\overline{DH} \perp \overline{O_B O_C}$, $AB = \sqrt{3}$ and $AC = 2$. Find BC .

Proposed by: Evan Chen

Answer: $\boxed{\sqrt{\frac{1}{3}(7 + 2\sqrt{13})}}$

Let the B -mixtilinear incircle ω_B touch Γ at T_B , BA at B_1 and BC at B_2 . Define $T_C \in \Gamma$, $C_1 \in CB$, $C_2 \in CA$, and ω_C similarly. Call I the incenter of triangle ABC , and γ the incircle.

We first identify two points on the radical axis of the B and C mixtilinear incircles:

- The midpoint M of arc BC of the circumcircle of ABC . This follows from the fact that M , B_1 , T_B are collinear with

$$MB^2 = MC^2 = MB_1 \cdot MT_B$$

and similarly for C .

- The midpoint N of ID . To see this, first recall that I is the midpoint of segments B_1B_2 and C_1C_2 . From this, we can see that the radical axis of ω_B and γ contains N (since it is the line through the midpoints of the common external tangents of ω_B , γ). A similar argument for C shows that the midpoint of ID is actually the radical center of the ω_B , ω_C , γ .

Now consider a homothety with ratio 2 at I . It sends line MN to the line through D and the A -excenter I_A (since M is the midpoint of II_A , by “Fact 5”). Since DH was supposed to be parallel to line MN , it follows that line DH passes through I_A ; however a homothety at D implies that this occurs only if H is the midpoint of the A -altitude.

Let $a = BC$, $b = CA = 2$ and $c = AB = \sqrt{3}$. So, we have to just find the value of a such that the orthocenter of ABC lies on the midpoint of the A -altitude. This is a direct computation with the Law of Cosines, but a more elegant solution is possible using the fact that H has barycentric coordinates $(S_B S_C : S_C S_A : S_A S_B)$, where $S_A = \frac{1}{2}(b^2 + c^2 - a^2)$ and so on. Indeed, as H is on the A -midline we deduce directly that

$$S_B S_C = S_A(S_B + S_C) = a^2 S_A \implies \frac{1}{4}(a^2 - 1)(a^2 + 1) = \frac{1}{2}a^2(7 - a^2).$$

Solving as a quadratic in a^2 and taking the square roots gives

$$3a^4 - 14a^2 - 1 = 0 \implies a = \sqrt{\frac{1}{3}(7 + 2\sqrt{13})}$$

as desired. 1

HMMT February 2016

February 20, 2016

Guts

1. [5] Let x and y be complex numbers such that $x + y = \sqrt{20}$ and $x^2 + y^2 = 15$. Compute $|x - y|$.

Proposed by: Evan Chen

Answer: $\boxed{\sqrt{10}}$

We have $(x - y)^2 + (x + y)^2 = 2(x^2 + y^2)$, so $(x - y)^2 = 10$, hence $|x - y| = \sqrt{10}$.

2. [5] Sherry is waiting for a train. Every minute, there is a 75% chance that a train will arrive. However, she is engrossed in her game of sudoku, so even if a train arrives she has a 75% chance of not noticing it (and hence missing the train). What is the probability that Sherry catches the train in the next five minutes?

Proposed by:

Answer: $\boxed{1 - \left(\frac{13}{16}\right)^5}$

During any given minute, the probability that Sherry doesn't catch the train is $\frac{1}{4} + \left(\frac{3}{4}\right)^2 = \frac{13}{16}$. The desired probability is thus one minus the probability that she doesn't catch the train for the next five minutes: $1 - \left(\frac{13}{16}\right)^5$.

3. [5] Let *PROBLEMZ* be a regular octagon inscribed in a circle of unit radius. Diagonals *MR*, *OZ* meet at *I*. Compute *LI*.

Proposed by: Evan Chen

Answer: $\boxed{\sqrt{2}}$

If *W* is the center of the circle then *I* is the incenter of $\triangle RWZ$. Moreover, *PRIZ* is a rhombus. It follows that *PI* is twice the inradius of a $1-1-\sqrt{2}$ triangle, hence the answer of $2 - \sqrt{2}$. So $LI = \sqrt{2}$.

Alternatively, one can show (note, really) that the triangle *OIL* is isosceles.

4. [5] Consider a three-person game involving the following three types of fair six-sided dice.

- Dice of type *A* have faces labelled 2, 2, 4, 4, 9, 9.
- Dice of type *B* have faces labelled 1, 1, 6, 6, 8, 8.
- Dice of type *C* have faces labelled 3, 3, 5, 5, 7, 7.

All three players simultaneously choose a die (more than one person can choose the same type of die, and the players don't know one another's choices) and roll it. Then the score of a player *P* is the number of players whose roll is less than *P*'s roll (and hence is either 0, 1, or 2). Assuming all three players play optimally, what is the expected score of a particular player?

Proposed by:

Answer: $\boxed{\frac{8}{9}}$

Short version: third player doesn't matter; against 1 opponent, by symmetry, you'd both play the same strategy. Type *A* beats *B*, *B* beats *C*, and *C* beats *A* all with probability $5/9$. It can be determined that choosing each die with probability $1/3$ is the best strategy. Then, whatever you pick, there is a $1/3$ of dominating, a $1/3$ chance of getting dominated, and a $1/3$ chance of picking the same die (which gives a $1/3 \cdot 2/3 + 1/3 \cdot 1/3 = 1/3$ chance of rolling a higher number). Fix your selection; then the expected payout is then $1/3 \cdot 5/9 + 1/3 \cdot 4/9 + 1/3 \cdot 1/3 = 1/3 + 1/9 = 4/9$. Against 2 players, your EV is just $E(p1) + E(p2) = 2E(p1) = 8/9$

5. [6] Patrick and Anderson are having a snowball fight. Patrick throws a snowball at Anderson which is shaped like a sphere with a radius of 10 centimeters. Anderson catches the snowball and uses the snow from the snowball to construct snowballs with radii of 4 centimeters. Given that the total volume of the snowballs that Anderson constructs cannot exceed the volume of the snowball that Patrick threw, how many snowballs can Anderson construct?

Proposed by:

Answer: 15

$$\left\lfloor \left(\frac{10}{4} \right)^3 \right\rfloor = \left\lfloor \frac{125}{8} \right\rfloor = 15.$$

6. [6] Consider a $2 \times n$ grid of points and a path consisting of $2n - 1$ straight line segments connecting all these $2n$ points, starting from the bottom left corner and ending at the upper right corner. Such a path is called *efficient* if each point is only passed through once and no two line segments intersect. How many efficient paths are there when $n = 2016$?

Proposed by: Casey Fu

Answer: $\binom{4030}{2015}$

The general answer is $\binom{2(n-1)}{n-1}$: Simply note that the points in each column must be taken in order, and anything satisfying this avoids intersections, so just choose the steps during which to be in the first column.

7. [6] A contest has six problems worth seven points each. On any given problem, a contestant can score either 0, 1, or 7 points. How many possible total scores can a contestant achieve over all six problems?

Proposed by: Evan Chen

Answer: 28

For $0 \leq k \leq 6$, to obtain a score that is $k \pmod{6}$ exactly k problems must get a score of 1. The remaining $6 - k$ problems can generate any multiple of 7 from 0 to $7(6 - k)$, of which there are $7 - k$. So the total number of possible scores is $\sum_{k=0}^6 (7 - k) = 28$.

8. [6] For each positive integer n and non-negative integer k , define $W(n, k)$ recursively by

$$W(n, k) = \begin{cases} n^n & k = 0 \\ W(W(n, k - 1), k - 1) & k > 0. \end{cases}$$

Find the last three digits in the decimal representation of $W(555, 2)$.

Proposed by:

Answer: 875

For any n , we have

$$W(n, 1) = W(W(n, 0), 0) = (n^n)^{n^n} = n^{n^{n+1}}.$$

Thus,

$$W(555, 1) = 555^{555^{556}}.$$

Let $N = W(555, 1)$ for brevity, and note that $N \equiv 0 \pmod{125}$, and $N \equiv 3 \pmod{8}$. Then,

$$W(555, 2) = W(N, 1) = N^{N^{N+1}}$$

is $0 \pmod{125}$ and $3 \pmod{8}$.

From this we can conclude (by the Chinese Remainder Theorem) that the answer is 875.

9. [7] Victor has a drawer with two red socks, two green socks, two blue socks, two magenta socks, two lavender socks, two neon socks, two mauve socks, two wisteria socks, and 2000 copper socks, for a total of 2016 socks. He repeatedly draws two socks at a time from the drawer at random, and stops if the socks are of the same color. However, Victor is red-green colorblind, so he also stops if he sees a red and green sock.

What is the probability that Victor stops with two socks of the same color? Assume Victor returns both socks to the drawer at each step.

Proposed by: Evan Chen

Answer: $\boxed{\frac{1999008}{1999012}}$

There are $\binom{2000}{2} + 8\binom{2}{2} = 1999008$ ways to get socks which are matching colors, and four extra ways to get a red-green pair, hence the answer.

10. [7] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let O be the circumcenter of ABC . Find the distance between the circumcenters of triangles AOB and AOC .

Proposed by: Evan Chen

Answer: $\boxed{\frac{91}{6}}$

Let S, T be the intersections of the tangents to the circumcircle of ABC at A, C and at A, B respectively. Note that $ASCO$ is cyclic with diameter SO , so the circumcenter of AOC is the midpoint of OS , and similarly for the other side. So the length we want is $\frac{1}{2}ST$. The circumradius R of ABC can be computed by Heron's formula and $K = \frac{abc}{4R}$, giving $R = \frac{65}{8}$. A few applications of the Pythagorean theorem and similar triangles gives $AT = \frac{65}{6}$, $AS = \frac{39}{2}$, so the answer is $\boxed{\frac{91}{6}}$.

11. [7] Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n . Compute the remainder when

$$\sum_{\substack{2 \leq n \leq 50 \\ \gcd(n, 50) = 1}} \phi^!(n)$$

is divided by 50.

Proposed by: Evan Chen

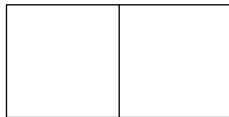
Answer: $\boxed{12}$

First, $\phi^!(n)$ is even for all odd n , so it vanishes modulo 2.

To compute the remainder modulo 25, we first evaluate $\phi^!(3) + \phi^!(7) + \phi^!(9) \equiv 2 + 5 \cdot 4 + 5 \cdot 3 \equiv 12 \pmod{25}$. Now, for $n \geq 11$ the contribution modulo 25 vanishes as long as $5 \nmid n$.

We conclude the answer is 12.

12. [7] Let R be the rectangle in the Cartesian plane with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



Compute the number of ways to choose one or more of the seven edges such that the resulting figure is traceable without lifting a pencil. (Rotations and reflections are considered distinct.)

Proposed by: Joy Zheng

Answer: $\boxed{61}$

We have two cases, depending on whether we choose the middle edge. If so, then either all the remaining edges are either to the left of or to the right of this edge, or there are edges on both sides, or neither; in the first two cases there are 6 ways each, in the third there are $16 + 1 = 17$ ways, and in the last there is 1 way. Meanwhile, if we do not choose the middle edge, then we have to choose a beginning and endpoint, plus the case where we have a loop, for a total of $6 \cdot 5 + 1 = 31$ cases. This gives a total of $6 + 6 + 17 + 1 + 31 = 61$ possible cases.

13. [9] A right triangle has side lengths a , b , and $\sqrt{2016}$ in some order, where a and b are positive integers. Determine the smallest possible perimeter of the triangle.

Proposed by: Evan Chen

Answer: $\boxed{48 + \sqrt{2016}}$

There are no integer solutions to $a^2 + b^2 = 2016$ due to the presence of the prime 7 on the right-hand side (by Fermat's Christmas Theorem). Assuming $a < b$, the minimal solution $(a, b) = (3, 45)$ which gives the answer above.

14. [9] Let ABC be a triangle such that $AB = 13$, $BC = 14$, $CA = 15$ and let E, F be the feet of the altitudes from B and C , respectively. Let the circumcircle of triangle AEF be ω . We draw three lines, tangent to the circumcircle of triangle AEF at A, E , and F . Compute the area of the triangle these three lines determine.

Proposed by: Christopher Shao

Answer: $\boxed{\frac{462}{5}}$

Note that $AEF \sim ABC$. Let the vertices of the triangle whose area we wish to compute be P, Q, R , opposite A, E, F respectively. Since H, O are isogonal conjugates, line AH passes through the circumcenter of AEF , so $QR \parallel BC$.

Let M be the midpoint of BC . We claim that $M = P$. This can be seen by angle chasing at E, F to find that $\angle PFB = \angle ABC$, $\angle PEC = \angle ACB$, and noting that M is the circumcenter of $BFEC$. So, the height from P to QR is the height from A to BC , and thus if K is the area of ABC , the area we want is $\frac{QR}{BC}K$.

Heron's formula gives $K = 84$, and similar triangles QAF, MBF and RAE, MCE give $QA = \frac{BC \tan B}{2 \tan A}$, $RA = \frac{BC \tan C}{2 \tan A}$, so that $\frac{QR}{BC} = \frac{\tan B + \tan C}{2 \tan A} = \frac{\tan B \tan C - 1}{2} = \frac{11}{10}$,

since the height from A to BC is 12. So our answer is $\boxed{\frac{462}{5}}$.

15. [9] Compute $\tan\left(\frac{\pi}{7}\right) \tan\left(\frac{2\pi}{7}\right) \tan\left(\frac{3\pi}{7}\right)$.

Proposed by: Alexander Katz

Answer: $\boxed{\sqrt{7}}$

Consider the polynomial $P(z) = z^7 - 1$. Let $z = e^{ix} = \cos x + i \sin x$. Then

$$\begin{aligned} z^7 - 1 &= \left(\cos^7 x - \binom{7}{2} \cos^5 x \sin^2 x + \binom{7}{4} \cos^3 x \sin^4 x - \binom{7}{6} \cos x \sin^6 x - 1 \right) \\ &\quad + i \left(-\sin^7 x + \binom{7}{2} \sin^5 x \cos^2 x - \binom{7}{4} \sin^3 x \cos^4 x + \binom{7}{6} \sin x \cos^6 x \right) \end{aligned}$$

Consider the real part of this equation. We may simplify it to $64 \cos^7 x - \dots - 1$, where the middle terms are irrelevant. The roots of P are $x = \frac{2\pi}{7}, \frac{4\pi}{7}, \dots$, so $\prod_{k=1}^7 \cos\left(\frac{2\pi k}{7}\right) = \frac{1}{64}$. But

$$\prod_{k=1}^7 \cos\left(\frac{2\pi k}{7}\right) = \left(\prod_{k=1}^3 \cos\left(\frac{k\pi}{7}\right) \right)^2$$

so $\prod_{k=1}^3 \cos\left(\frac{k\pi}{7}\right) = \frac{1}{8}$.

Now consider the imaginary part of this equation. We may simplify it to $-64\sin^{11}x + \dots + 7\sin x$, where again the middle terms are irrelevant. We can factor out $\sin x$ to get $-64\sin^{10}x + \dots + 7$, and this polynomial has roots $x = \frac{2\pi}{7}, \dots, \frac{12\pi}{7}$ (but not 0). Hence $\prod_{k=1}^6 \sin\left(\frac{2\pi k}{7}\right) = -\frac{7}{64}$. But, like before, we have

$$\prod_{k=1}^6 \sin\left(\frac{2\pi k}{7}\right) = -\left(\prod_{k=1}^3 \sin\left(\frac{2\pi k}{7}\right)\right)^2$$

hence $\prod_{k=1}^3 \sin\left(\frac{k\pi}{7}\right) = \frac{\sqrt{7}}{8}$. As a result, our final answer is $\frac{\sqrt{7}}{\frac{1}{8}} = \boxed{\sqrt{7}}$.

16. [9] Determine the number of integers $2 \leq n \leq 2016$ such that $n^n - 1$ is divisible by 2, 3, 5, 7.

Proposed by: Evan Chen

Answer: $\boxed{9}$

Only $n \equiv 1 \pmod{210}$ work. Proof: we require $\gcd(n, 210) = 1$. Note that $\forall p \leq 7$ the order of $n \pmod{p}$ divides $p-1$, hence is relatively prime to any $p \leq 7$. So $n^n \equiv 1 \pmod{p} \iff n \equiv 1 \pmod{p}$ for each of these p .

17. [11] Compute the sum of all integers $1 \leq a \leq 10$ with the following property: there exist integers p and q such that $p, q, p^2 + a$ and $q^2 + a$ are all distinct prime numbers.

Proposed by: Evan Chen

Answer: $\boxed{20}$

Odd a fail for parity reasons and $a \equiv 2 \pmod{3}$ fail for mod 3 reasons. This leaves $a \in \{4, 6, 10\}$. It is easy to construct p and q for each of these, take $(p, q) = (3, 5), (5, 11), (3, 7)$, respectively.

18. [11] Alice and Bob play a game on a circle with 8 marked points. Alice places an apple beneath one of the points, then picks five of the other seven points and reveals that none of them are hiding the apple. Bob then drops a bomb on any of the points, and destroys the apple if he drops the bomb either on the point containing the apple or on an adjacent point. Bob wins if he destroys the apple, and Alice wins if he fails. If both players play optimally, what is the probability that Bob destroys the apple?

Proposed by:

Answer: $\boxed{\frac{1}{2}}$

Let the points be $0, \dots, 7 \pmod{8}$, and view Alice's reveal as revealing the three possible locations of the apple. If Alice always picks 0, 2, 4 and puts the apple randomly at 0 or 4, by symmetry Bob cannot achieve more than $\frac{1}{2}$. Here's a proof that $\frac{1}{2}$ is always possible.

Among the three revealed indices a, b, c , positioned on a circle, two must (in the direction in which they're adjacent) have distance at least 3, so without loss of generality the three are $0, b, c$ where $1 \leq b < c \leq 5$. Modulo reflection and rotation, the cases are: (0, 1, 2): Bob places at 1 and wins. (0, 1, 3): Bob places at 1 half the time and 3 half the time, so wherever the apple is Bob wins with probability $\frac{1}{2}$. (0, 1, 4): Bob places at 1 or 4, same as above. (0, 2, 4): Bob places at 1 or 3, same as above. (0, 2, 5): Bob places at 1 or 5, same as above.

These cover all cases, so we're done.

19. [11] Let

$$A = \lim_{n \rightarrow \infty} \sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i} \binom{n}{i+2}}{\binom{n}{i+1}^2}$$

Find the largest integer less than or equal to $\frac{1}{A}$.

The following decimal approximation might be useful: $0.6931 < \ln(2) < 0.6932$, where \ln denotes the natural logarithm function.

Proposed by: Pakawut Jiradilok

Answer: 1

Note

$$\sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i} \binom{n}{i+2}}{\binom{n}{i+1}^2} = \sum_{i=0}^{2016} (-1)^i \cdot \frac{(i+1)(n-i-1)}{(i+2)(n-i)},$$

So

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i} \binom{n}{i+2}}{\binom{n}{i+1}^2} = \sum_{i=0}^{2016} (-1)^i \cdot \frac{(i+1)}{(i+2)} = 1 - \sum_{i=2}^{2016} \frac{(-1)^i}{i} \approx \ln(2).$$

Then $\frac{1}{A} \approx \frac{1}{\ln(2)} \approx 1.44$, so the answer is 1.

20. [11] Let ABC be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$. Let G be the point on AC such that the reflection of BG over the angle bisector of $\angle B$ passes through the midpoint of AC . Let Y be the midpoint of GC and X be a point on segment AG such that $\frac{AX}{XG} = 3$. Construct F and H on AB and BC , respectively, such that $FX \parallel BG \parallel HY$. If AH and CF concur at Z and W is on AC such that $WZ \parallel BG$, find WZ .

Proposed by: Ritesh Ragavender

Answer: $\frac{1170\sqrt{37}}{1379}$

Observe that BG is the B -symmedian, and thus $\frac{AG}{GC} = \frac{c^2}{a^2}$. Stewart's theorem gives us

$$BG = \sqrt{\frac{2a^2c^2b}{b(a^2+c^2)} - \frac{a^2b^2c^2}{a^2+c^2}} = \frac{ac}{a^2+c^2} \sqrt{2(a^2+c^2)-b^2} = \frac{390\sqrt{37}}{197}.$$

Then by similar triangles,

$$ZW = HY \frac{ZA}{HA} = BG \frac{YC}{GC} \frac{ZA}{HA} = BG \frac{1}{2} \frac{6}{7} = \frac{1170\sqrt{37}}{1379}$$

where $\frac{ZA}{HA}$ is found with mass points or Ceva.

21. [12] Tim starts with a number n , then repeatedly flips a fair coin. If it lands heads he subtracts 1 from his number and if it lands tails he subtracts 2. Let E_n be the expected number of flips Tim does before his number is zero or negative. Find the pair (a, b) such that

$$\lim_{n \rightarrow \infty} (E_n - an - b) = 0.$$

Proposed by:

Answer: $(\frac{2}{3}, \frac{2}{9})$

We have the recurrence $E_n = \frac{1}{2}(E_{n-1} + 1) + \frac{1}{2}(E_{n-2} + 1)$, or $E_n = 1 + \frac{1}{2}(E_{n-1} + E_{n-2})$, for $n \geq 2$.

Let $F_n = E_n - \frac{2}{3}n$. By directly plugging this into the recurrence for E_n , we get the recurrence $F_n = \frac{1}{2}(F_{n-1} + F_{n-1})$. The roots of the characteristic polynomial of this recurrence are 1 and $-\frac{1}{2}$, so $F_n = A + B(-\frac{1}{2})^n$ for some A and B depending on the initial conditions. But clearly we have $E_0 = 0$ and $E_1 = 1$ so $F_0 = 0$ and $F_1 = \frac{1}{3}$ so $A = \frac{2}{9}$ and $B = -\frac{2}{9}$.

Hence, $E_n = \frac{2}{3}n + \frac{2}{9} - \frac{2}{9}(-\frac{1}{2})^n$, so $\lim_{n \rightarrow \infty} (E_n - \frac{2}{3}n - \frac{2}{9}) = 0$. Hence $(\frac{2}{3}, \frac{2}{9})$ is the desired pair.

22. [12] On the Cartesian plane \mathbb{R}^2 , a circle is said to be *nice* if its center is at the origin $(0, 0)$ and it passes through at least one lattice point (i.e. a point with integer coordinates). Define the points $A = (20, 15)$ and $B = (20, 16)$. How many nice circles intersect the open segment AB ?

For reference, the numbers 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691 are the only prime numbers between 600 and 700.

Proposed by:

Answer: 10

The square of the radius of a nice circle is the sum of the square of two integers.

The nice circle of radius r intersects (the open segment) \overline{AB} if and only if a point on \overline{AB} is a distance r from the origin. \overline{AB} consists of the points $(20, t)$ where t ranges over $(15, 16)$. The distance from the origin is $\sqrt{20^2 + t^2} = \sqrt{400 + t^2}$. As t ranges over $(15, 16)$, $\sqrt{400 + t^2}$ ranges over $(\sqrt{625}, \sqrt{656})$, so the nice circle of radius r intersects \overline{AB} if and only if $625 < r^2 < 656$.

The possible values of r^2 are those in this range that are the sum of two perfect squares, and each such value corresponds to a unique nice circle. By Fermat's Christmas theorem, an integer is the sum of two squares if and only if in its prime factorization, each prime that is 3 mod 4 appears with an even exponent (possibly 0). In addition, since squares are 0, 1, or 4 mod 8, we can quickly eliminate integers that are 3, 6, or 7 mod 8.

Now I will list all the integers that aren't 3, 6, or 7 mod 8 in the range and either supply the bad prime factor or write "nice" with the prime factorization.

626: nice $(2 \cdot 313)$

628: nice $(2^2 \cdot 157)$

629: nice $(17 \cdot 37)$

632: 79

633: 3

634: nice $(2 \cdot 317)$

636: 3

637: nice $(7^2 \cdot 13)$

640: nice $(2^7 \cdot 5)$

641: nice (641)

642: 3

644: 7

645: 3

648: nice $(2^3 \cdot 3^4)$

649: 11

650: nice $(2 \cdot 5^2 \cdot 13)$

652: 163

653: nice (653). There are 10 nice circles that intersect \overline{AB} .

23. [12] Let $t = 2016$ and $p = \ln 2$. Evaluate in closed form the sum

$$\sum_{k=1}^{\infty} \left(1 - \sum_{n=0}^{k-1} \frac{e^{-t} t^n}{n!} \right) (1-p)^{k-1} p.$$

Proposed by: Aaron Landesman

Answer: $1 - \left(\frac{1}{2}\right)^{2016}$

Let $q = 1 - p$. Then

$$\begin{aligned}
\sum_{k=1}^{\infty} \left(1 - \sum_{n=0}^{k-1} \frac{e^{-t} t^n}{n!} \right) q^{k-1} p &= \sum_{k=1}^{\infty} q^{k-1} p - \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \frac{e^{-t} t^n}{n!} q^{k-1} p \\
&= 1 - \sum_{k=1}^{\infty} \sum_{n=0}^{k-1} \frac{e^{-t} t^n}{n!} q^{k-1} p \\
&= 1 - \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \frac{e^{-t} t^n}{n!} q^{k-1} p \\
&= 1 - \sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} q^n \\
&= 1 - \sum_{n=0}^{\infty} \frac{e^{-t} (qt)^n}{n!} = 1 - e^{-t} e^{qt} = 1 - e^{-pt}.
\end{aligned}$$

Thus the answer is $1 - \left(\frac{1}{2}\right)^{2016}$.

24. [12] Let $\triangle A_1 B_1 C$ be a triangle with $\angle A_1 B_1 C = 90^\circ$ and $\frac{CA_1}{CB_1} = \sqrt{5} + 2$. For any $i \geq 2$, define A_i to be the point on the line $A_1 C$ such that $A_i B_{i-1} \perp A_1 C$ and define B_i to be the point on the line $B_1 C$ such that $A_i B_i \perp B_1 C$. Let Γ_1 be the incircle of $\triangle A_1 B_1 C$ and for $i \geq 2$, Γ_i be the circle tangent to $\Gamma_{i-1}, A_1 C, B_1 C$ which is smaller than Γ_{i-1} .

How many integers k are there such that the line $A_1 B_{2016}$ intersects Γ_k ?

Proposed by:

Answer: 4030

We claim that Γ_2 is the incircle of $\triangle B_1 A_2 C$. This is because $\triangle B_1 A_2 C$ is similar to $A_1 B_1 C$ with dilation factor $\sqrt{5} - 2$, and by simple trigonometry, one can prove that Γ_2 is similar to Γ_1 with the same dilation factor. By similarities, we can see that for every k , the incircle of $\triangle A_k B_k C$ is Γ_{2k-1} , and the incircle of $\triangle B_k A_{k+1} C$ is Γ_{2k} . Therefore, $A_1 B_{2016}$ intersects all $\Gamma_1, \dots, \Gamma_{4030}$ but not Γ_k for any $k \geq 4031$.

25. [14] A particular coin can land on heads (H), on tails (T), or in the middle (M), each with probability $\frac{1}{3}$. Find the expected number of flips necessary to observe the contiguous sequence HMMTHMMT...HMMT, where the sequence HMMT is repeated 2016 times.

Proposed by: Ritesh Ragavender

Answer: $\frac{3^{8068} - 81}{80}$

Let E_0 be the expected number of flips needed. Let E_1 be the expected number more of flips needed if the first flip landed on H. Let E_2 be the expected number more if the first two landed on HM. In general, let E_k be the expected number more of flips needed if the first k flips landed on the first k values of the sequence HMMTHMMT...HMMT.

We have

$$E_i = \begin{cases} 1 + \frac{1}{3}E_{i+1} + \frac{1}{3}E_1 + \frac{1}{3}E_0 & i \not\equiv 0 \pmod{4} \\ 1 + \frac{1}{3}E_{i+1} + \frac{2}{3}E_0 & i \equiv 0 \pmod{4} \end{cases}$$

Using this relation for $i = 0$ gives us $E_1 = E_0 - 3$. Let $F_i = \frac{1}{3^i} E_i$. By simple algebraic manipulations we have

$$F_{i+1} - F_i = \begin{cases} -\frac{2}{3^{i+1}} \cdot E_0 & i \not\equiv 0 \pmod{4} \\ -\frac{1}{3^i} - \frac{2}{3^{i+1}} \cdot E_0 & i \equiv 0 \pmod{4} \end{cases}$$

We clearly have $F_{2016 \cdot 4} = 0$ and $F_0 = E_0$. So adding up the above relations for $i = 0$ to $i = 2016 \cdot 4 - 1$ gives

$$\begin{aligned}
-E_0 &= -2E_0 \sum_{i=1}^{2016 \cdot 4} \frac{1}{3^i} - \sum_{k=0}^{2015} \frac{1}{3^{4k}} \\
&= E_0 \left(\frac{1}{3^{2016 \cdot 4}} - 1 \right) - \frac{1 - \frac{1}{3^{2016 \cdot 4}}}{\frac{80}{81}}
\end{aligned}$$

so $E_0 = \frac{3^{8068} - 81}{80}$.

26. [14] For positive integers a, b , $a \uparrow\uparrow b$ is defined as follows: $a \uparrow\uparrow 1 = a$, and $a \uparrow\uparrow b = a^{a \uparrow\uparrow (b-1)}$ if $b > 1$. Find the smallest positive integer n for which there exists a positive integer a such that $a \uparrow\uparrow 6 \not\equiv a \uparrow\uparrow 7 \pmod n$.

Proposed by: Sammy Luo

Answer: 283

We see that the smallest such n must be a prime power, because if two numbers are distinct mod n , they must be distinct mod at least one of the prime powers that divide n . For $k \geq 2$, if $a \uparrow\uparrow k$ and $a \uparrow\uparrow (k+1)$ are distinct mod p^r , then $a \uparrow\uparrow (k-1)$ and $a \uparrow\uparrow k$ must be distinct mod $\phi(p^r)$. In fact they need to be distinct mod $\frac{\phi(p^r)}{2}$ if $p = 2$ and $r \geq 3$ because then there are no primitive roots mod p^r .

Using this, for $1 \leq k \leq 5$ we find the smallest prime p such that there exists a such that $a \uparrow\uparrow k$ and $a \uparrow\uparrow (k+1)$ are distinct mod p . The list is: 3, 5, 11, 23, 47. We can easily check that the next largest prime for $k = 5$ is 139, and also any prime power other than 121 for which $a \uparrow\uparrow 5$ and $a \uparrow\uparrow 6$ are distinct is also larger than 139.

Now if $a \uparrow\uparrow 6$ and $a \uparrow\uparrow 7$ are distinct mod p , then $p - 1$ must be a multiple of 47 or something that is either 121 or at least 139. It is easy to see that 283 is the smallest prime that satisfies this.

If n is a prime power less than 283 such that $a \uparrow\uparrow 6$ and $a \uparrow\uparrow 7$ are distinct mod n , then the prime can be at most 13 and clearly this doesn't work because $\phi(p^r) = p^{r-1}(p-1)$.

To show that 283 works, choose a so that a is a primitive root mod 283, 47, 23, 11, 5 and 3. This is possible by the Chinese Remainder theorem, and it is easy to see that this a works by induction.

27. [14] Find the smallest possible area of an ellipse passing through $(2, 0)$, $(0, 3)$, $(0, 7)$, and $(6, 0)$.

Proposed by: Calvin Deng

Answer: $\frac{56\pi\sqrt{3}}{9}$

Let Γ be an ellipse passing through $A = (2, 0)$, $B = (0, 3)$, $C = (0, 7)$, $D = (6, 0)$, and let $P = (0, 0)$ be the intersection of AD and BC . $\frac{\text{Area of } \Gamma}{\text{Area of } ABCD}$ is unchanged under an affine transformation, so we just have to minimize this quantity over situations where Γ is a circle and $\frac{PA}{PD} = \frac{1}{3}$ and $\frac{PB}{BC} = \frac{3}{7}$. In fact, we may assume that $PA = \sqrt{7}$, $PB = 3$, $PC = 7$, $PD = 3\sqrt{7}$. If $\angle P = \theta$, then we can compute lengths to get

$$r = \frac{\text{Area of } \Gamma}{\text{Area of } ABCD} = \pi \frac{32 - 20\sqrt{7} \cos \theta + 21 \cos^2 \theta}{9\sqrt{7} \cdot \sin^3 \theta}$$

Let $x = \cos \theta$. Then if we treat r as a function of x ,

$$0 = \frac{r'}{r} = \frac{3x}{1-x^2} + \frac{42x - 20\sqrt{7}}{32 - 20x\sqrt{7} + 21x^2}$$

which means that $21x^3 - 40x\sqrt{7} + 138x - 20\sqrt{7} = 0$. Letting $y = x\sqrt{7}$ gives

$$0 = 3y^3 - 40y^2 + 138y - 140 = (y-2)(3y^2 - 34y + 70)$$

The other quadratic has roots that are greater than $\sqrt{7}$, which means that the minimum ratio is attained when $\cos \theta = x = \frac{y}{\sqrt{7}} = \frac{2}{\sqrt{7}}$. Plugging that back in gives that the optimum $\frac{\text{Area of } \Gamma}{\text{Area of } ABCD}$ is

$\frac{28\pi\sqrt{3}}{81}$, so putting this back into the original configuration gives Area of $\Gamma \geq \frac{56\pi\sqrt{3}}{9}$. If you want to check on Geogebra, this minimum occurs when the center of Γ is $(\frac{8}{3}, \frac{7}{3})$.

28. [14] Among citizens of Cambridge there exist 8 different types of blood antigens. In a crowded lecture hall are 256 students, each of whom has a blood type corresponding to a distinct subset of the antigens; the remaining of the antigens are foreign to them.

Quito the Mosquito flies around the lecture hall, picks a subset of the students uniformly at random, and bites the chosen students in a random order. After biting a student, Quito stores a bit of any antigens that student had. A student bitten while Quito had k blood antigen foreign to him/her will suffer for k hours. What is the expected total suffering of all 256 students, in hours?

Proposed by: Sammy Luo

Answer: $\frac{2^{135} - 2^{128} + 1}{2^{119} \cdot 129}$

Let $n = 8$.

First, consider any given student S and an antigen a foreign to him/her. Assuming S has been bitten, we claim the probability S will suffer due to a is

$$1 - \frac{2^{2^{n-1}+1} - 1}{2^{2^{n-1}}(2^{n-1} + 1)}.$$

Indeed, let $N = 2^{n-1}$ denote the number of students with a . So considering just these students and summing over the number bitten, we obtain a probability

$$\frac{1}{2^N} \sum_{t=0}^N \binom{N}{t} \binom{N}{t} \frac{t}{t+1} = \frac{1}{2^N} \frac{2^N N - 2^N + 1}{N + 1}.$$

We now use linearity over all pairs (S, a) of students S and antigens a foreign to them. Noting that each student is bitten with probability $\frac{1}{2}$, and retaining the notation $N = 2^{n-1}$, we get

$$\frac{1}{2} \sum_{k=0}^n \left[\binom{n}{k} \cdot k \left(\frac{2^N N - 2^N + 1}{2^N (N + 1)} \right) \right] = \frac{nN(2^N N - 2^N + 1)}{2^{N+1}(N + 1)}.$$

Finally, setting $n = 8 = 2^3$ and $N = 2^{n-1} = 2^7 = 128$, we get the claimed answer.

29. [16] Katherine has a piece of string that is 2016 millimeters long. She cuts the string at a location chosen uniformly at random, and takes the left half. She continues this process until the remaining string is less than one millimeter long. What is the expected number of cuts that she makes?

Proposed by:

Answer: $1 + \log(2016)$

Letting $f(x)$ be the expected number of cuts if the initial length of the string is x , we get the integral equation $f(x) = 1 + \frac{1}{x} \int_1^x f(y) dy$. Letting $g(x) = \int_1^x f(y) dy$, we get $dg/dx = 1 + \frac{1}{x} g(x)$. Using integrating factors, we see that this has as its solution $g(x) = x \log(x)$, and thus $f(x) = 1 + \log(x)$.

30. [16] Determine the number of triples $0 \leq k, m, n \leq 100$ of integers such that

$$2^m n - 2^n m = 2^k.$$

Proposed by: Casey Fu

Answer: 22

First consider when $n \geq m$, so let $n = m + d$ where $d \geq 0$. Then we have $2^m(m + d - 2^d m) = 2^m(m(1 - 2^d) + d)$, which is non-positive unless $m = 0$. So our first set of solutions is $m = 0, n = 2^j$.

Now, we can assume that $m > n$, so let $m = n + d$ where $d > 0$. Rewrite $2^m n - 2^n m = 2^{n+d} n - 2^n(n + d) = 2^n((2^d - 1)n - d)$. In order for this to be a power of 2, $(2^d - 1)n - d$ must be a power of 2. This implies that for some j , $2^j \equiv -d \pmod{2^d - 1}$. But notice that the powers of 2 $\pmod{2^d - 1}$ are $1, 2, 4, \dots, 2^{d-1}$ ($2^d \equiv 1$ so the cycle repeats).

In order for the residues to match, we need $2^j + d = c(2^d - 1)$, where $0 \leq j \leq d - 1$ and $c \geq 1$. In order for this to be true, we must have $2^{d-1} + d \geq 2^d - 1 \iff d + 1 \geq 2^{d-1}$. This inequality is only true for $d = 1, 2, 3$. We plug each of these into the original expression $(2^d - 1)n - d$.

For $d = 1$: $n - 1$ is a power of 2. This yields the set of solutions $(2^j + 2, 2^j + 1)$ for $j \geq 0$.

For $d = 2$: $3n - 2$ is a power of 2. Note that powers of 2 are $-2 \pmod{3}$ if and only if it is an even power, so $n = \frac{2^{2j+2}}{3}$. This yields the solution set $(\frac{2^{2j+8}}{3}, \frac{2^{2j+2}}{3}), j \geq 0$.

For $d = 3$: $7n - 3$ is a power of 2. Powers of 2 have a period of 3 when taken $\pmod{7}$, so inspection tells us $7n - 3 = 2^{3j+2}$, yielding the solution set $(\frac{2^{3j+2}+24}{7}, \frac{2^{3j+2}+3}{7}), j \geq 0$.

Therefore, all the solutions are of the form

$$(m, n) = (0, 2^j), (2^j + 2, 2^j + 1) \\ (\frac{2^{2j} + 8}{3}, \frac{2^{2j} + 2}{3}), (\frac{2^{3j+2} + 24}{7}, \frac{2^{3j+2} + 3}{7})$$

for $j \geq 0$.

Restricting this family to $m, n \leq 100$ gives $7 + 7 + 5 + 3 = 22$.

31. [16] For a positive integer n , denote by $\tau(n)$ the number of positive integer divisors of n , and denote by $\phi(n)$ the number of positive integers that are less than or equal to n and relatively prime to n . Call a positive integer n *good* if $\phi(n) + 4\tau(n) = n$. For example, the number 44 is good because $\phi(44) + 4\tau(44) = 44$.

Find the sum of all good positive integers n .

Proposed by: Lawrence Sun

Answer: 172

We claim that 44, 56, 72 are the only good numbers. It is easy to check that these numbers work.

Now we prove none others work. First, remark that as $n = 1, 2$ fail so we have $\phi(n)$ is even, thus n is even. This gives us $\phi(n) \leq n/2$. Now remark that $\tau(n) < 2\sqrt{n}$, so it follows we need $n/2 + 8\sqrt{n} > n \implies n \leq 256$. This gives us a preliminary bound. Note that in addition we have $8\tau(n) > n$.

Now, it is easy to see that powers of 2 fail. Thus let $n = 2^a p_1^b$ where p_1 is an odd prime. From $8\tau(n) > n$ we get $8(a + 1)(b + 1) > 2^a p_1^b \geq 2^a 3^b$ from which we get that (a, b) is one of

$$(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 2), (4, 1).$$

Remark that $p_1 \leq \sqrt[b]{\frac{8(a+1)(b+1)}{2^a}}$. From this we can perform some casework:

- If $a = 1, b = 1$ then $p_1 - 1 + 16 = 2p_1$ but then $p = 15$, absurd.
- If $a = 1, b = 2$ then we have $p_1 \leq 5$ which is obviously impossible.
- If $a = 1, b = 3$ then $p_1 \leq 4$ which is impossible.
- If $a = 2, b = 1$ then $p_1 \leq 12$ and it is easy to check that $p_1 = 11$ and thus $n = 44$ is the only solution.
- If $a = 2, b = 2$ then $p_1 \leq 4$ which is impossible.
- If $a = 3, b = 1$ then $p_1 \leq 8$ and only $p_1 = 7$ or $n = 56$ works.
- If $a = 3, b = 2$ then $p_1 \leq 3$ and $p_1 = 3, n = 72$ works.
- If $a = 4, b = 1$ then $p_1 \leq 1$ which is absurd.

Now suppose n is the product of 3 distinct primes, so $n = 2^a p_1^b p_2^c$ so we have $8(a+1)(b+1)(c+1) > 2^a 3^b 5^c$ then we must have (a, b, c) equal to one of

$$(1, 1, 1), (1, 2, 1), (2, 1, 1), (3, 1, 1).$$

Again, we can do some casework:

- If $a = b = c = 1$ then $8\tau(n) = 64 > 2p_1 p_2$ but then $p_1 = 3, p_2 = 5$ or $p_1 = 3, p_2 = 7$ is forced neither of which work.
- If $a = 1, b = 2, c = 1$ then $8\tau(n) = 96 > 2p_1^2 p_2$ but then $p_1 = 3, p_2 = 5$ is forced which does not work.
- If $a = 2, b = 1, c = 1$ then $8\tau(n) = 96 > 4p_1 p_2$ forces $p_1 = 3, p_2 = 5$ or $p_1 = 3, p_2 = 7$ neither of which work.
- If $a = 3, b = 1, c = 1$ then $8\tau(n) = 108 > 8p_1 p_2$ which has no solutions for p_1, p_2 .

Finally, take the case where n is the product of at least 4 distinct primes. But then $n \geq 2 \cdot 3 \cdot 5 \cdot 7 = 210$ and as $2 \cdot 3 \cdot 5 \cdot 11 > 256$, it suffices to check only the case of 210. But 210 clearly fails, so it follows that 44, 56, 72 are the only good numbers so we are done.

32. [16] How many equilateral hexagons of side length $\sqrt{13}$ have one vertex at $(0, 0)$ and the other five vertices at lattice points?

(A lattice point is a point whose Cartesian coordinates are both integers. A hexagon may be concave but not self-intersecting.)

Proposed by: Casey Fu

Answer: 216

We perform casework on the point three vertices away from $(0, 0)$. By inspection, that point can be $(\pm 8, \pm 3)$, $(\pm 7, \pm 2)$, $(\pm 4, \pm 3)$, $(\pm 3, \pm 2)$, $(\pm 2, \pm 1)$ or their reflections across the line $y = x$. The cases are as follows:

If the third vertex is at any of $(\pm 8, \pm 3)$ or $(\pm 3, \pm 8)$, then there are 7 possible hexagons. There are 8 points of this form, contributing 56 hexagons.

If the third vertex is at any of $(\pm 7, \pm 2)$ or $(\pm 2, \pm 7)$, there are 6 possible hexagons, contributing 48 hexagons.

If the third vertex is at any of $(\pm 4, \pm 3)$ or $(\pm 3, \pm 4)$, there are again 6 possible hexagons, contributing 48 more hexagons.

If the third vertex is at any of $(\pm 3, \pm 2)$ or $(\pm 2, \pm 3)$, then there are again 6 possible hexagons, contributing 48 more hexagons.

Finally, if the third vertex is at any of $(\pm 2, \pm 1)$, then there are 2 possible hexagons only, contributing 16 hexagons.

Adding up, we get our answer of 216.

33. [20] (**Lucas Numbers**) The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. There are N integers $1 \leq n \leq 2016$ such that L_n contains the digit 1. Estimate N .

An estimate of E earns $\lfloor 20 - 2|N - E| \rfloor$ or 0 points, whichever is greater.

Proposed by: Evan Chen

Answer: 1984

```
lucas_ones n = length . filter (elem '1') $ take (n + 1) lucas_strs
  where
    lucas = 2 : 1 : zipWith (+) lucas (tail lucas)
    lucas_strs = map show lucas

main = putStrLn . show $ lucas_ones 2016
```

34. [20] (Caos) A cao [sic] has 6 legs, 3 on each side. A walking pattern for the cao is defined as an ordered sequence of raising and lowering each of the legs exactly once (altogether 12 actions), starting and ending with all legs on the ground. The pattern is safe if at any point, he has at least 3 legs on the ground and not all three legs are on the same side. Estimate N , the number of safe patterns.

An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N)^4 \rfloor$ points.

Proposed by:

Answer:

```
# 1 = on ground, 0 = raised, 2 = back on ground
cache = {}
```

```
def pangzi(legs):
    if legs == (2,2,2,2,2,2): return 1
    elif legs.count(0) > 3: return 0
    elif legs[0] + legs[1] + legs[2] == 0: return 0
    elif legs[3] + legs[4] + legs[5] == 0: return 0
    elif cache.has_key(legs): return cache[legs]

    cache[legs] = 0
    for i in xrange(6): # raise a leg
        if legs[i] == 1:
            new = list(legs)
            new[i] = 0
            cache[legs] += pangzi(tuple(new))
        elif legs[i] == 0: # lower a leg
            new = list(legs)
            new[i] = 2
            cache[legs] += pangzi(tuple(new))
    return cache[legs]

print pangzi((1,1,1,1,1,1))
```

35. [20] (Maximal Determinant) In a 17×17 matrix M , all entries are ± 1 . The maximum possible value of $|\det M|$ is N . Estimate N .

An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N)^2 \rfloor$ points.

Proposed by: *Evan Chen*

Answer:

This is Hadamard's maximal determinant problem. There's an upper bound of $n^{\frac{1}{2}n}$ which empirically seems to give reasonably good estimates, but in fact this is open for general n .

36. [20] (Self-Isogonal Cubics) Let ABC be a triangle with $AB = 2$, $AC = 3$, $BC = 4$. The *isogonal conjugate* of a point P , denoted P^* , is the point obtained by intersecting the reflection of lines PA , PB , PC across the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively.

Given a point Q , let $\mathfrak{R}(Q)$ denote the unique cubic plane curve which passes through all points P such that line PP^* contains Q . Consider:

- the M'Cay cubic $\mathfrak{R}(O)$, where O is the circumcenter of $\triangle ABC$,
- the Thomson cubic $\mathfrak{R}(G)$, where G is the centroid of $\triangle ABC$,
- the Napoleon-Feurerbach cubic $\mathfrak{R}(N)$, where N is the nine-point center of $\triangle ABC$,
- the Darboux cubic $\mathfrak{R}(L)$, where L is the de Longchamps point (the reflection of the orthocenter across point O),

- (e) the Neuberg cubic $\mathfrak{K}(X_{30})$, where X_{30} is the point at infinity along line OG ,
- (f) the nine-point circle of $\triangle ABC$,
- (g) the incircle of $\triangle ABC$, and
- (h) the circumcircle of $\triangle ABC$.

Estimate N , the number of points lying on at least two of these eight curves. An estimate of E earns $\lfloor 20 \cdot 2^{-|N-E|/6} \rfloor$ points.

Proposed by: *Evan Chen*

Answer: 49

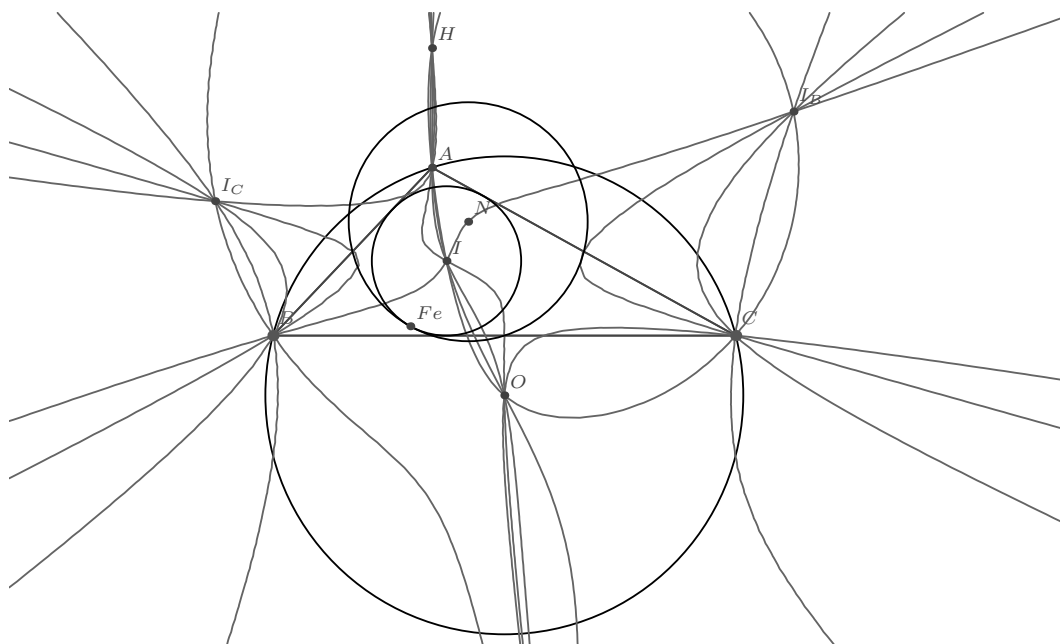
The first main insight is that all the cubics pass through the points A, B, C, H (orthocenter), O , and the incenter and three excenters. Since two cubics intersect in at most nine points, this is all the intersections of a cubic with a cubic.

On the other hand, it is easy to see that among intersections of circles with circles, there are exactly 3 points; the incircle is tangent to the nine-point circle at the Feuerbach point while being contained completely in the circumcircle; on the other hand for this obtuse triangle the nine-point circle and the circumcircle intersect exactly twice.

All computations up until now are exact, so it remains to estimate:

- Intersection of the circumcircle with cubics. Each cubic intersects the circumcircle at an even number of points, and moreover we already know that A, B, C are among these, so the number of additional intersections contributed is either 1 or 3; it is the former only for the Neuberg cubic which has a “loop”. Hence the actual answer in this case is $1 + 3 + 3 + 3 + 3 = 13$ (but an estimate of $3 \cdot 5 = 15$ is very reasonable).
- Intersection of the incircle with cubics. Since $\angle A$ is large the incircle is small, but on the other hand we know I lies on each cubic. Hence it’s very likely that each cubic intersects the incircle twice (once “coming in” and once “coming out”). This is the case, giving $2 \cdot 5 = 10$ new points.
- Intersection of the nine-point with cubics. We guess this is close to the 10 points of the incircle, as we know the nine-point circle and the incircle are tangent to each other. In fact, the exact count is 14 points; just two additional branches appear.

In total, $N = 9 + 3 + 13 + 10 + 14 = 49$.



HMMT February 2016

February 20, 2016

Team

1. [25] Let a and b be integers (not necessarily positive). Prove that $a^3 + 5b^3 \neq 2016$.

Proposed by: Evan Chen

Since cubes are 0 or ± 1 modulo 9, by inspection we see that we must have $a^3 \equiv b^3 \equiv 0 \pmod{3}$ for this to be possible. Thus a, b are divisible by 3. But then we get $3^3 \mid 2016$, which is a contradiction.

One can also solve the problem in the same manner by taking modulo 7 exist, since all cubes are 0 or ± 1 modulo 7. The proof can be copied literally, noting that $7 \mid 2016$ but $7^3 \nmid 2016$.

2. [25] For positive integers n , let c_n be the smallest positive integer for which $n^{c_n} - 1$ is divisible by 210, if such a positive integer exists, and $c_n = 0$ otherwise. What is $c_1 + c_2 + \dots + c_{210}$?

Proposed by: Joy Zheng

Answer: 329

In order for $c_n \neq 0$, we must have $\gcd(n, 210) = 1$, so we need only consider such n . The number $n^{c_n} - 1$ is divisible by 210 iff it is divisible by each of 2, 3, 5, and 7, and we can consider the order of n modulo each modulus separately; c_n will simply be the LCM of these orders. We can ignore the modulus 2 because order is always 1. For the other moduli, the sets of orders are

$$\begin{aligned} a &\in \{1, 2\} \pmod{3} \\ b &\in \{1, 2, 4, 4\} \pmod{5} \\ c &\in \{1, 2, 3, 3, 6, 6\} \pmod{7}. \end{aligned}$$

By the Chinese Remainder Theorem, each triplet of choices from these three multisets occurs for exactly one n in the range $\{1, 2, \dots, 210\}$, so the answer we seek is the sum of $\text{lcm}(a, b, c)$ over a, b, c in the Cartesian product of these multisets. For $a = 1$ this table of LCMs is as follows:

	1	2	3	3	6	6
1	1	2	3	3	6	6
2	2	2	6	6	6	6
4	4	4	12	12	12	12
4	4	4	12	12	12	12

which has a sum of $21 + 56 + 28 + 56 = 161$. The table for $a = 2$ is identical except for the top row, where 1, 3, 3 are replaced by 2, 6, 6, and thus has a total sum of 7 more, or 168. So our answer is $161 + 168 = \boxed{329}$.

This can also be computed by counting how many times each LCM occurs:

- 12 appears 16 times when $b = 4$ and $c \in \{3, 6\}$, for a contribution of $12 \times 16 = 192$;
- 6 appears 14 times, 8 times when $c = 6$ and $b \leq 2$ and 6 times when $c = 3$ and $(a, b) \in \{(1, 2), (2, 1), (2, 2)\}$, for a contribution of $6 \times 14 = 84$;
- 4 appears 8 times when $b = 4$ and $a, c \in \{1, 2\}$, for a contribution of $4 \times 8 = 32$;
- 3 appears 2 times when $c = 3$ and $a = b = 1$, for a contribution of $3 \times 2 = 6$;
- 2 appears 7 times when $a, b, c \in \{1, 2\}$ and $(a, b, c) \neq (1, 1, 1)$, for a contribution of $2 \times 7 = 14$;
- 1 appears 1 time when $a = b = c = 1$, for a contribution of $1 \times 1 = 1$.

The result is again $192 + 84 + 32 + 6 + 14 + 1 = 329$.

3. [30] Let ABC be an acute triangle with incenter I and circumcenter O . Assume that $\angle OIA = 90^\circ$. Given that $AI = 97$ and $BC = 144$, compute the area of $\triangle ABC$.

Proposed by: *Evan Chen*

Answer: 14040

We present five different solutions and outline a sixth and seventh one. In what follows, let $a = BC$, $b = CA$, $c = AB$ as usual, and denote by r and R the inradius and circumradius. Let $s = \frac{1}{2}(a + b + c)$.

In the first five solutions we will only prove that

$$\angle AIO = 90^\circ \implies b + c = 2a.$$

Let us see how this solves the problem. This lemma implies that $s = 216$. If we let E be the foot of I on AB , then $AE = s - BC = 72$, consequently the inradius is $r = \sqrt{97^2 - 72^2} = 65$. Finally, the area is $sr = 216 \cdot 65 = \boxed{14040}$.

First Solution. Since $OI \perp DA$, $AI = DI$. Now, it is a well-known fact that $DI = DB = DC$ (this is occasionally called “Fact 5”). Then by Ptolemy’s Theorem,

$$DB \cdot AC + DC \cdot AB = DA \cdot BC \implies AC + AB = 2BC. \quad \square$$

Second Solution. As before note that I is the midpoint of AD . Let M and N be the midpoints of AB and AC , and let the reflection of M across BI be P ; thus $BM = BP$. Also, $MI = PI$, but we know $MI = NI$ as I lies on the circumcircle of triangle AMN . Consequently, we get $PI = NI$; moreover by angle chasing we have

$$\angle INC = \angle AMI = 180^\circ - \angle BPI = \angle IPC.$$

Thus triangles INC and PIC are congruent (CI is a bisector) so we deduce $PC = NC$. Thus,

$$BC = BP + PC = BM + CN = \frac{1}{2}(AB + AC). \quad \square$$

Third Solution. We appeal to Euler’s Theorem, which states that $IO^2 = R(R - 2r)$.

Thus by the Pythagorean Theorem on $\triangle AIO$ (or by Power of a Point) we may write

$$(s - a)^2 + r^2 = AI^2 = R^2 - IO^2 = 2Rr = \frac{abc}{2s}$$

with the same notations as before. Thus, we derive that

$$\begin{aligned} abc &= 2s((s - a)^2 + r^2) \\ &= 2(s - a)(s(s - a) + (s - b)(s - c)) \\ &= \frac{1}{2}(s - a)((b + c)^2 - a^2 + a^2 - (b - c)^2) \\ &= 2bc(s - a). \end{aligned}$$

From this we deduce that $2a = b + c$, and we can proceed as in the previous solution. □

Fourth Solution. From Fact 5 again ($DB = DI = DC$), drop perpendicular from I to AB at E ; call M the midpoint of BC . Then, by AAS congruency on AIE and CDM , we immediately get that $CM = AE$. As $AE = \frac{1}{2}(AB + AC - BC)$, this gives the desired conclusion. □

Fifth Solution. This solution avoids angle-chasing and using the fact that BI and CI are angle-bisectors. Recall the perpendicularity lemma, where

$$WX \perp YZ \iff WY^2 - WZ^2 = XY^2 - XZ^2.$$

Let B' be on the extension of ray CA such that $AB' = AB$. Of course, as in the proof of the angle bisector theorem, $BB' \parallel AI$, meaning that $BB' \perp IO$. Let I' be the reflection of I across A ; of course, I' is then the incenter of triangle $AB'C'$. Now, we have $B'I^2 - BI^2 = B'O^2 - BO^2$ by the perpendicularity and by power of a point $B'O^2 - BO^2 = B'A \cdot B'C$. Moreover $BI^2 + B'I^2 = BI^2 + BI'^2 = 2BA^2 + 2AI^2$ by the median formula. Subtracting, we get $BI^2 = AI^2 + \frac{1}{2}(AB)(AB - AC)$. We have a similar expression for CI , and subtracting the two results in $BI^2 - CI^2 = \frac{1}{2}(AB^2 - AC^2)$. Finally,

$$BI^2 - CI^2 = \frac{1}{4}[(BC + AB - AC)^2 - (BC - AB + AC)^2]$$

from which again, the result $2BC = AB + AC$ follows. \square

Sixth Solution, outline. Use complex numbers, setting $I = ab + bc + ca$, $A = -a^2$, etc. on the unit circle (scale the picture to fit in a unit circle; we calculate scaling factor later). Set $a = 1$, and let $u = b + c$ and $v = bc$. Write every condition in terms of u and v , and the area in terms of u and v too. There should be two equations relating u and v : $2u + v + 1 = 0$ and $u^2 = \frac{130}{97}v$ from the right angle and the 144 to 97 ratio, respectively. The square area can be computed in terms of u and v , because the area itself is antisymmetric so squaring it suffices. Use the first condition to homogenize (not coincidentally the factor $(1 - b^2)(1 - c^2) = (1 + bc)^2 - (b + c)^2 = (1 + v)^2 - u^2$ from the area homogenizes perfectly... because $AB \cdot AC = AI \cdot AI_A$, where I_A is the A -excenter, and of course the way the problem is set up $AI_A = 3AI$), and then we find the area of the scaled down version. To find the scaling factor simply determine $|b - c|$ by squaring it, writing in terms again of u and v , and comparing this to the value of 144. \square

Seventh Solution, outline. Trigonometric solutions are also possible. One can write everything in terms of the angles and solve the equations; for instance, the $\angle AIO = 90^\circ$ condition can be rewritten as $\frac{1}{2} \cos \frac{B-C}{2} = 2 \sin \frac{B}{2} \sin \frac{C}{2}$ and the 97 to 144 ratio condition can be rewritten as $\frac{2 \sin \frac{B}{2} 2 \sin \frac{C}{2}}{\sin A} = \frac{97}{144}$. The first equation implies $\sin \frac{A}{2} = 2 \sin \frac{B}{2} \sin \frac{C}{2}$, which we can plug into the second equation to get $\cos \frac{A}{2}$. \square

4. [30] Let $n > 1$ be an odd integer. On an $n \times n$ chessboard the center square and four corners are deleted. We wish to group the remaining $n^2 - 5$ squares into $\frac{1}{2}(n^2 - 5)$ pairs, such that the two squares in each pair intersect at exactly one point (i.e. they are diagonally adjacent, sharing a single corner).

For which odd integers $n > 1$ is this possible?

Proposed by: Evan Chen

Answer: 3,5

Constructions for $n = 3$ and $n = 5$ are easy. For $n > 5$, color the odd rows black and the even rows white. If the squares can be paired in the way desired, each pair we choose must have one black cell and one white cell, so the numbers of black cells and white cells are the same. The number of black cells is $\frac{n+1}{2}n - 4$ or $\frac{n+1}{2}n - 5$ depending on whether the removed center cell is in an odd row. The number of white cells is $\frac{n-1}{2}n$ or $\frac{n-1}{2}n - 1$. But

$$\left(\frac{n+1}{2}n - 5\right) - \frac{n-1}{2}n = n - 5$$

so for $n > 5$ this pairing is impossible. Thus the answer is $n = 3$ and $n = 5$.

5. [35] Find all prime numbers p such that $y^2 = x^3 + 4x$ has exactly p solutions in integers modulo p .

In other words, determine all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y) such that $x, y \in \{0, 1, \dots, p-1\}$ and

$$p \text{ divides } y^2 - x^3 - 4x.$$

Proposed by: *Evan Chen*

Answer: $p = 2$ and $p \equiv 3 \pmod{4}$

Clearly $p = 2$ works with solutions $(0, 0)$ and $(1, 1)$ and not $(0, 1)$ or $(1, 0)$.

If $p \equiv 3 \pmod{4}$ then -1 is not a quadratic residue, so for $x^3 + 4x \neq 0$, exactly one of $x^3 + 4x$ and $-x^3 - 4x$ is a square and gives two solutions (for positive and negative y), so there's exactly two solutions for each such pair $\{x, -x\}$. If x is such that $x^3 + 4x = 0$, there's exactly one solution.

If $p \equiv 1 \pmod{4}$, let i be a square root of $-1 \pmod{p}$. The right hand side factors as $x(x+2i)(x-2i)$. For $x = 0, 2i, -2i$ this is zero, there is one choice of y , namely zero. Otherwise, the right hand side is nonzero. For any fixed x , there are either 0 or 2 choices for y . Replacing x by $-x$ negates the right hand side, again producing two choices for y since -1 is a quadratic residue. So the total number of solutions (x, y) is $3 \pmod{4}$, and thus there cannot be exactly p solutions.

Remark: This is a conductor 36 elliptic curve with complex multiplication, and the exact formula for the number of solutions is given in <http://www.mathcs.emory.edu/~ono/publications-cv/pdfs/026.pdf>.

6. [35] A nonempty set S is called *well-filled* if for every $m \in S$, there are fewer than $\frac{1}{2}m$ elements of S which are less than m . Determine the number of well-filled subsets of $\{1, 2, \dots, 42\}$.

Proposed by: *Casey Fu*

Answer: $\binom{43}{21} - 1$

Let a_n be the number of well-filled subsets whose maximum element is n (setting $a_0 = 1$). Then it's easy to see that

$$\begin{aligned} a_{2k+1} &= a_{2k} + a_{2k-1} + \dots + a_0 \\ a_{2k+2} &= (a_{2k+1} - C_k) + a_{2k} + \dots + a_0. \end{aligned}$$

where C_k is the number of well-filled subsets of size $k+1$ with maximal element $2k+1$.

We proceed to compute C_k . One can think of such a subset as a sequence of numbers $1 \leq s_1 < \dots < s_{k+1} \leq 2k+1$ such that $s_i \geq 2i-1$ for every $1 \leq i \leq k+1$. Equivalently, letting $s_i = i+1+t_i$ it's the number of sequences $0 \leq t_1 \leq \dots \leq t_{k+1} \leq k+1$ such that $t_i \geq i$ for every i . This gives the list of x -coordinates of steps up in a Catalan path from $(0, 0)$ to $(k+1, k+1)$, so

$$C_k = \frac{1}{k+2} \binom{2(k+1)}{(k+1)}$$

is equal to the $(k+1)$ th Catalan number.

From this we can solve the above recursion to derive that

$$a_n = \binom{n}{\lfloor (n-1)/2 \rfloor}.$$

Consequently, for even n ,

$$a_0 + \dots + a_n = a_{n+1} = \binom{n+1}{\lfloor n/2 \rfloor}.$$

Putting $n = 42$ gives the answer, after subtracting off the empty set (counted in a_0).

7. [40] Let $q(x) = q^1(x) = 2x^2 + 2x - 1$, and let $q^n(x) = q(q^{n-1}(x))$ for $n > 1$. How many negative real roots does $q^{2016}(x)$ have?

Proposed by: Ernest Chiu

Answer: $\boxed{\frac{2^{2017}+1}{3}}$

Define $g(x) = 2x^2 - 1$, so that $q(x) = -\frac{1}{2} + g\left(x + \frac{1}{2}\right)$. Thus

$$q^N(x) = 0 \iff \frac{1}{2} = g^N\left(x + \frac{1}{2}\right)$$

where $N = 2016$.

But, viewed as function $g : [-1, 1] \rightarrow [-1, 1]$ we have that $g(x) = \cos(2 \arccos(x))$. Thus, the equation $q^N(x) = 0$ is equivalent to

$$\cos\left(2^{2016} \arccos\left(x + \frac{1}{2}\right)\right) = \frac{1}{2}.$$

Thus, the solutions for x are

$$x = -\frac{1}{2} + \cos\left(\frac{\pi/3 + 2\pi n}{2^{2016}}\right) \quad n = 0, 1, \dots, 2^{2016} - 1.$$

So, the roots are negative for the values of n such that

$$\frac{1}{3}\pi < \frac{\pi/3 + 2\pi n}{2^{2016}} < \frac{5}{3}\pi$$

which is to say

$$\frac{1}{6}(2^{2016} - 1) < n < \frac{1}{6}(5 \cdot 2^{2016} - 1).$$

The number of values of n that fall in this range is $\frac{1}{6}(5 \cdot 2^{2016} - 2) - \frac{1}{6}(2^{2016} + 2) + 1 = \frac{1}{6}(4 \cdot 2^{2016} + 2) = \frac{1}{3}(2^{2017} + 1)$.

8. [40] Compute

$$\int_0^\pi \frac{2 \sin \theta + 3 \cos \theta - 3}{13 \cos \theta - 5} d\theta.$$

Proposed by: Carl Lian

Answer: $\boxed{\frac{3\pi}{13} - \frac{4}{13} \log \frac{3}{2}}$

We have

$$\begin{aligned} \int_0^\pi \frac{2 \sin \theta + 3 \cos \theta - 3}{13 \cos \theta - 5} d\theta &= 2 \int_0^{\pi/2} \frac{2 \sin 2x + 3 \cos 2x - 3}{13 \cos 2x - 5} dx \\ &= 2 \int_0^{\pi/2} \frac{4 \sin x \cos x - 6 \sin^2 x}{8 \cos^2 x - 18 \sin^2 x} dx \\ &= 2 \int_0^{\pi/2} \frac{\sin x (2 \cos x - 3 \sin x)}{(2 \cos x + 3 \sin x)(2 \cos x - 3 \sin x)} dx \\ &= 2 \int_0^{\pi/2} \frac{\sin x}{2 \cos x + 3 \sin x}. \end{aligned}$$

To compute the above integral we want to write $\sin x$ as a linear combination of the denominator and its derivative:

$$\begin{aligned}
2 \int_0^{\pi/2} \frac{\sin x}{2 \cos x + 3 \sin x} &= 2 \int_0^{\pi/2} \frac{-\frac{1}{13}[-3(2 \cos x + 3 \sin x) + 2(3 \cos x - 2 \sin x)]}{2 \cos x + 3 \sin x} \\
&= -\frac{2}{13} \left[\int_0^{\pi/2} (-3) + 2 \int_0^{\pi/2} \frac{-2 \sin x + 3 \cos x}{2 \cos x + 3 \sin x} \right] \\
&= -\frac{2}{13} \left[-\frac{3\pi}{2} + 2 \log(3 \sin x + 2 \cos x) \Big|_0^{\pi/2} \right] \\
&= -\frac{2}{13} \left[-\frac{3\pi}{2} + 2 \log \frac{3}{2} \right] \\
&= \frac{3\pi}{13} - \frac{4}{13} \log \frac{3}{2}.
\end{aligned}$$

9. [40] Fix positive integers $r > s$, and let F be an infinite family of sets, each of size r , no two of which share fewer than s elements. Prove that there exists a set of size $r - 1$ that shares at least s elements with each set in F .

Proposed by: Victor Wang

This is a generalization of 2002 ISL C5.

Solution 1. Say a set S s -meets F if it shares at least s elements with each set in F . Suppose no such set of size (at most) $r - 1$ exists. (Each $S \in F$ s -meets F by the problem hypothesis.)

Let T be a maximal set such that $T \subseteq S$ for infinitely many $S \in F$, which form $F' \subseteq F$ (such T exists, since the empty set works). Clearly $|T| < r$, so by assumption, T does not s -meet F , and there exists $U \in F$ with $|U \cap T| \leq s - 1$. But U s -meets F' , so by pigeonhole, there must exist $u \in U \setminus T$ belonging to infinitely many $S \in F'$, contradicting the maximality of T .

Comment. Let X be an infinite set, and a_1, \dots, a_{2r-2-s} elements not in X . Then $F = \{B \cup \{x\} : B \subseteq \{a_1, \dots, a_{2r-2-s}\}, |B| = r - 1, x \in X\}$ shows we cannot replace $r - 1$ with any smaller number.

Solution 2. We can also use a more indirect approach (where the use of contradiction is actually essential).

Fix $S \in F$ and $a \in S$. By assumption, $S \setminus \{a\}$ does not s -meet F , so there exists $S' \in F$ such that S' contains at most $s - 1$ elements of $S \setminus \{a\}$, whence $S \cap S'$ is an s -set containing a . We will derive a contradiction from the following lemma:

Lemma. Let F, G be families of r -sets such that any $f \in F$ and $g \in G$ share at least s elements. Then there exists a finite set H such that for any $f \in F$ and $g \in G$, $|f \cap g \cap H| \geq s$.

Proof. Suppose not, and take a counterexample with $r + s$ minimal; then F, G must be infinite and $r > s > 0$.

Take arbitrary $f_0 \in F$ and $g_0 \in G$; then the finite set $X = f_0 \cup g_0$ meets F, G . For every subset $Y \subseteq X$, let $F_Y = \{S \in F : S \cap X = Y\}$; analogously define G_Y . Then the F_Y, G_Y partition F, G , respectively. For any F_Y and $y \in Y$, define $F_Y(y) = \{S \setminus \{y\} : S \in F_Y\}$.

Now fix subsets $Y, Z \subseteq X$. If one of F_Y, G_Z is empty, define $H_{Y,Z} = \emptyset$.

Otherwise, if Y, Z are disjoint, take arbitrary $y \in Y, z \in Z$. By the minimality assumption, there exists finite $H_{Y,Z}$ such that for any $f \in F_Y(y)$ and $g \in G_Z(z)$, $|f \cap g \cap H_{Y,Z}| \geq s$.

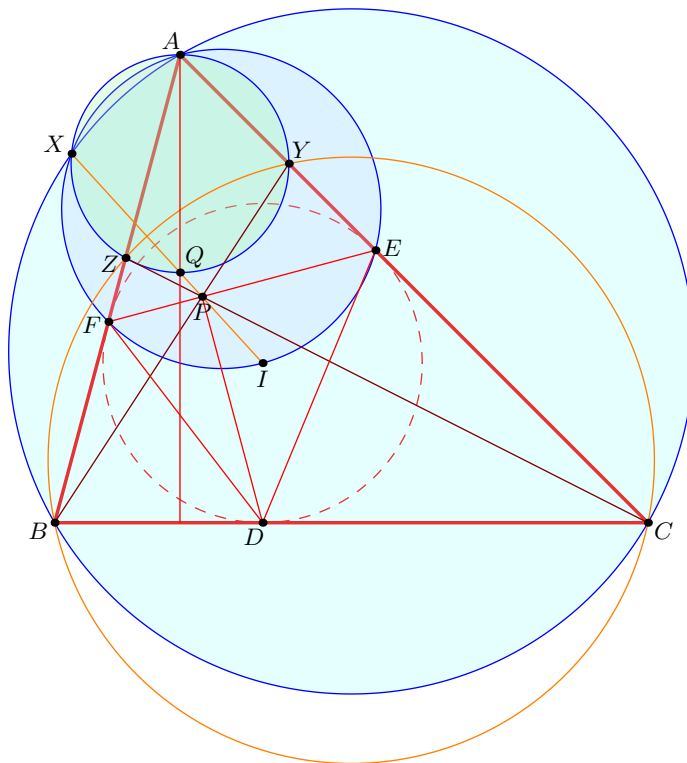
If Y, Z share an element a , and $s = 1$, take $H_{Y,Z} = \{a\}$. Otherwise, if $s \geq 2$, we find again by minimality a finite $H_{Y,Z}(a)$ such that for $f \in F_Y(a)$ and $g \in G_Z(a)$, $|f \cap g \cap H_{Y,Z}| \geq s - 1$; then take $H_{Y,Z} = H_{Y,Z}(a) \cup \{a\}$.

Finally, we see that $H = \bigcup_{Y,Z \subseteq X} H_{Y,Z}$ shares at least s elements with each $f \cap g$ (by construction), contradicting our assumption.

10. [50] Let ABC be a triangle with incenter I whose incircle is tangent to \overline{BC} , \overline{CA} , \overline{AB} at D , E , F . Point P lies on \overline{EF} such that $\overline{DP} \perp \overline{EF}$. Ray BP meets \overline{AC} at Y and ray CP meets \overline{AB} at Z . Point Q is selected on the circumcircle of $\triangle AYZ$ so that $\overline{AQ} \perp \overline{BC}$.

Prove that P , I , Q are collinear.

Proposed by: Evan Chen



The proof proceeds through a series of seven lemmas.

Lemma 1. Lines DP and EF are the internal and external angle bisectors of $\angle BPC$.

Proof. Since DEF the cevian triangle of ABC with respect to its Gregonne point, we have that

$$-1 = (\overline{EF} \cap \overline{BC}, D; B, C).$$

Then since $\angle DPF = 90^\circ$ we see P is on the Apollonian circle of BC through D . So the conclusion follows. \square

Lemma 2. Triangles BPF and CEP are similar.

Proof. Invoking the angle bisector theorem with the previous lemma gives

$$\frac{BP}{BF} = \frac{BP}{BD} = \frac{CP}{CD} = \frac{CP}{CE}.$$

But $\angle BFP = \angle CEP$, so $\triangle BFP \sim \triangle CEP$. \square

Lemma 3. Quadrilateral $BZYC$ is cyclic; in particular, line YZ is the antiparallel of line BC through $\angle BAC$.

Proof. Remark that $\angle YBZ = \angle PBF = \angle ECP = \angle YCZ$. □

Lemma 4. The circumcircles of triangles AYZ , AEF , ABC are concurrent at a point X such that $\triangle XBF \sim \triangle XCE$.

Proof. Note that line EF is the angle bisector of $\angle BPZ = \angle CPY$. Thus

$$\frac{ZF}{FB} = \frac{ZP}{PB} = \frac{YP}{PC} = \frac{YE}{EC}.$$

Then, if we let X be the Miquel point of quadrilateral $ZYCB$, it follows that the spiral similarity mapping segment BZ to segment CY maps E to F ; therefore the circumcircle of $\triangle AEF$ must pass through X too. □

Lemma 5. Ray XP bisects $\angle FXE$.

Proof. The assertion amounts to

$$\frac{XF}{XE} = \frac{BF}{EC} = \frac{FP}{PE}.$$

The first equality follows from the spiral similarity $\triangle BFX \sim \triangle CEX$, while the second is from $\triangle BFP \sim \triangle CEP$. So the proof is complete by the converse of angle bisector theorem. □

Lemma 6. Points X , P , I are collinear.

Proof. On one hand, $\angle FXI = \angle FAI = \frac{1}{2}\angle A$. On the other hand, $\angle FXP = \frac{1}{2}\angle FXE = \frac{1}{2}\angle A$. Hence, X , Y , I collinear. □

Lemma 7. Points X , Q , I are collinear.

Proof. On one hand, $\angle AXQ = 90^\circ$, because we established earlier that line YZ was antiparallel to line BC through $\angle A$, hence $AQ \perp BC$ means exactly that $\angle AZQ = \angle AYQ = 90^\circ$. On the other hand, $\angle AXI = 90^\circ$ according to the fact that X lies on the circle with diameter AI . This completes the proof of the lemma. □

Finally, combining the final two lemmas solves the problem.

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November 12, 2016

General

1. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab .

Proposed by: Eshaan Nichani

Answer:

We multiply $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$ to get $2 + ab + \frac{1}{ab} = \frac{64}{15} \iff (ab)^2 - \frac{34}{15}(ab) + 1 = 0$. Since $(\frac{34}{15})^2 - 4 \cdot 1 \cdot 1 > 0$, there are two roots ab , so the product of both possible values of ab is by Vieta's.

2. I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

Proposed by: Kevin Yang

Answer:

I get a matching pair on the third day with probability $\frac{1}{9}$ because there is a $\frac{1}{9}$ probability of the second sock matching the first. Given that I already removed a matching pair of the third day, I get a matching pair on the fifth day with probability $\frac{1}{7}$. We multiply these probabilities to get .

3. Let V be a rectangular prism with integer side lengths. The largest face has area 240 and the smallest face has area 48. A third face has area x , where x is not equal to 48 or 240. What is the sum of all possible values of x ?

Proposed by: Eshaan Nichani

Answer:

Let the length, width, and height of the prism be s_1, s_2, s_3 . Without loss of generality, assume that $s_1 \leq s_2 \leq s_3$. Then, we have that $s_1 s_2 = 48$ and $s_2 s_3 = 240$. Noting that $s_1 \leq s_2$, we must have $(s_1, s_2) = (1, 48), (2, 24), (3, 16), (4, 12), (6, 8)$. We must also have $s_2 s_3 = 240$ and $s_2 \leq s_3$, and the only possibilities for (s_1, s_2) that yield integral s_3 that satisfy these conditions are $(4, 12)$, which gives $s_3 = 20$, and $(6, 8)$, which gives $s_3 = 30$. Thus, the only valid (s_1, s_2, s_3) are $(4, 12, 20)$ and $(6, 8, 30)$. It follows that the only possible areas of the third face are $4(20) = 80$ and $6(30) = 180$, so the desired answer is $80 + 180 =$.

4. A rectangular pool table has vertices at $(0, 0)$, $(12, 0)$, $(0, 10)$, and $(12, 10)$. There are pockets only in the four corners. A ball is hit from $(0, 0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. Find the number of walls that the ball bounces off of before entering a pocket.

Proposed by: Allen Liu

Answer:

Consider the tiling of the plane with the 12×10 rectangle to form a grid. Then the reflection of the ball off a wall is equivalent to traveling along the straight line $y = x$ into another 12×10 rectangle. Hence we want to find the number of walls of the grid that the line $y = x$ hits before it reaches the first corner it hits, $(60, 60)$.

The line $y = x$ hits each of the horizontal lines $y = 10, 20, 30, 40, 50$ and each of the vertical lines $x = 12, 24, 36, 48$. This gives a total of walls hit before entering a pocket.

5. Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. Find the product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \dots$$

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{2}{3}}$

Let $\{b_i\}_{i=0}^{\infty}$ be defined by $b_n = a_n - 1$ and note that $b_n = b_{n-1}^2$. The infinite product is then

$$(1 + b_0)(1 + b_0^2)(1 + b_0^4) \dots (1 + b_0^{2^k}) \dots$$

By the polynomial identity

$$(1 + x)(1 + x^2)(1 + x^4) \dots (1 + x^{2^k}) \dots = 1 + x + x^2 + x^3 + \dots = \frac{1}{1 - x}$$

Our desired product is then simply

$$\frac{1}{1 - (a_0 - 1)} = \boxed{\frac{2}{3}}$$

6. The numbers $1, 2, \dots, 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

Proposed by: Allen Liu

Answer: $\boxed{\frac{10}{33}}$

Suppose the middle number is k . Then there are $k - 1$ ways to pick the number smaller than k to its left and $\binom{11-k}{4}$ ways to pick the 4 numbers larger than k to its right. Hence there is a total of $\sum_{k=2}^7 (k - 1) \cdot \binom{11-k}{4}$ ways for there to be exactly one number smaller than k to its left. We calculate this total:

$$\begin{aligned} \sum_{k=2}^7 (k - 1) \cdot \binom{11 - k}{4} &= \sum_{j=4}^9 \sum_{i=4}^j \binom{i}{4} \\ &= \sum_{j=4}^9 \binom{j + 1}{5} \\ &= \binom{11}{6}. \end{aligned}$$

The only way k can be larger than exactly one number to its right is if $k = 3$. Then the probability of this happening is $\frac{2 \cdot \binom{8}{4}}{\binom{11}{6}} = \boxed{\frac{10}{33}}$.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. The altitude from A intersects BC at D . Let ω_1 and ω_2 be the incircles of ABD and ACD , and let the common external tangent of ω_1 and ω_2 (other than BC) intersect AD at E . Compute the length of AE .

Proposed by: Eshaan Nichani

Answer: $\boxed{7}$

Solution 1:

Let I_1, I_2 be the centers of ω_1, ω_2 , respectively, X_1, X_2 be the tangency points of ω_1, ω_2 with BC , respectively, and Y_1, Y_2 be the tangency points of ω_1, ω_2 with AD , respectively. Let the two common external tangents of ω_1, ω_2 meet at P . Note that line I_1I_2 also passes through P .

By Heron's formula, the area of triangle ABC is 84. Thus, $\frac{1}{2}AD \cdot BC = 84$, and so $AD = 12$. By the Pythagorean Theorem on right triangles ABD and ACD , $BD = 5$ and $CD = 9$.

The inradius of ABD , r_{ABD} , is $\frac{[ABD]}{s_{ABD}}$, where $[ABD]$ is the area of ABD and s_{ABD} is its semiperimeter. ABD is a 5-12-13 right triangle, so $[ABD] = 30$ and $s_{ABD} = 15$. Thus, $r_{ABD} = 2$. Similarly, we get

that ACD 's inradius is $r_{ACD} = 3$. $I_1Y_1DX_1$ is a square, so $X_1D = I_1X_1 = r_{ABD} = 2$, and similarly $X_2D = 3$. X_1 and X_2 are on opposite sides of D , so $X_1X_2 = X_1D + X_2D = 5$.

Since P lies on lines I_1I_2, X_1X_2 , and I_1X_1, I_2X_2 are parallel, triangles PX_1I_1 and PX_2I_2 are similar. Thus, $\frac{X_1I_1}{X_2I_2} = \frac{2}{3} = \frac{PX_1}{PX_2} = \frac{PX_1}{PX_1 + X_1X_2} = \frac{PX_1}{PX_1 + 5}$. Solving gives $PX_1 = 10$. Letting $\angle I_1PX_1 = \theta$, since I_1X_1P is a right angle, we have $\tan \theta = \frac{X_1I_1}{X_1P_1} = \frac{1}{5}$. D and E lie on different common external tangents, so PI_1 bisects $\angle EPD$, and thus $\angle EPD = 2\theta$. Thus, $\tan \angle EPD = \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{5}{12}$.

ED is perpendicular to BC , so triangle EDP is a right triangle with right angle at D . Thus, $\frac{5}{12} = \tan \angle EPD = \frac{ED}{PD}$. $PD = PX_1 + X_1D = 12$, so $ED = 5$. $AD = 12$, so it follows that $AE = \boxed{7}$.

Solution 2:

Lemma: Let Ω_1, Ω_2 be two non-intersecting circles. Let ℓ_A, ℓ_B be their common external tangents, and ℓ_C be one of their common internal tangents. Ω_1 intersects ℓ_A, ℓ_B at points A_1, B_1 , respectively, and Ω_2 intersects ℓ_A, ℓ_B at points A_2, B_2 . If ℓ_C intersects ℓ_A at X , and ℓ_B at Y , then $XY = A_1A_2 = B_1B_2$.

Proof: Let ℓ_C intersect Ω_1, Ω_2 at C_1, C_2 , respectively. Then, examining the tangents to Ω_1, Ω_2 from X, Y , we have $A_1X = C_1X, A_2X = C_2X, B_1Y = C_1Y, B_2Y = C_2Y$. Thus, $2A_1A_2 = 2B_1B_2 = A_1A_2 + B_1B_2 = A_1X + A_2X + B_1Y + B_2Y = C_1X + C_2X + C_1Y + C_2Y = 2XY$, and the conclusion follows.

Using the notation from above, we apply the lemma to circles ω_1, ω_2 , and conclude that $ED = X_1X_2$. Then, we proceed as above to compute $X_1X_2 = 5 = ED$. Thus, $AD = \boxed{7}$.

8. Let $S = \{1, 2, \dots, 2016\}$, and let f be a randomly chosen bijection from S to itself. Let n be the smallest positive integer such that $f^{(n)}(1) = 1$, where $f^{(i)}(x) = f(f^{(i-1)}(x))$. What is the expected value of n ?

Proposed by: Eshaan Nichani

Answer: $\boxed{\frac{2017}{2}}$

Say that $n = k$. Then $1, f(1), f^2(1), \dots, f^{(k-1)}(1)$ are all distinct, which means there are $2015 \cdot 2014 \cdot \dots \cdot (2016 - k + 1)$ ways to assign these values. There is 1 possible value of $f^k(1)$, and $(2016 - k)!$ ways to assign the image of the $2016 - k$ remaining values. Thus the probability that $n = k$ is $\frac{1}{2016}$.

Therefore the expected value of n is $\frac{1}{2016}(1 + 2 + \dots + 2016) = \frac{2017}{2}$

9. Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \leq n \leq 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

Proposed by: Allen Liu

Answer: $\boxed{50}$

We claim that a_{1000} is constant mod 100.

a_{997} is divisible by 2. This means that a_{998} is divisible by 4. Thus a_{999} is constant mod 5. Since it is also divisible by 4, it is constant mod 20. Thus a_{1000} is constant mod 25, since $\phi(25) = 20$. Since a_{1000} is also divisible by 4, it is constant mod 100.

We know that a_{1000} is divisible by 4, and let it be congruent to k mod 25.

Then 2^n is divisible by 4 ($n \geq 2$) and $2^n \equiv k$ mod 25. We can also show that 2 is a primitive root mod 25, so there is one unique value of n mod 20. It suffices to show this value isn't 1. But $2^{2^{0 \bmod 4}} \equiv 2^{16 \bmod 20} \pmod{25}$, so $n \equiv 16 \pmod{20}$. Thus there are $1000/20 = 50$ values of n .

10. Quadrilateral $ABCD$ satisfies $AB = 8, BC = 5, CD = 17, DA = 10$. Let E be the intersection of AC and BD . Suppose $BE : ED = 1 : 2$. Find the area of $ABCD$.

Proposed by: Brice Huang

Answer: $\boxed{60}$

Since $BE : ED = 1 : 2$, we have $[ABC] : [ACD] = 1 : 2$.

Suppose we cut off triangle ACD , reflect it across the perpendicular bisector of AC , and re-attach it as triangle $A'C'D'$ (so $A' = C, C' = A$).

Triangles ABC and $C'A'D'$ have vertex $A = C'$ and bases BC and $A'D'$. Their areas and bases are both in the ratio $1 : 2$. Thus in fact BC and $A'D'$ are collinear.

Hence the union of ABC and $C'A'D'$ is the $8 - 15 - 17$ triangle ABD' , which has area $\frac{1}{2} \cdot 8 \cdot 15 = 60$.

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Guts

1. [5] If five fair coins are flipped simultaneously, what is the probability that at least three of them show heads?

Proposed by: Allen Liu

Answer:

The coin either lands heads-up at least three times or lands heads-down at least three times. These scenarios are symmetric so the probability is just $\frac{1}{2}$.

Alternatively, we can explicitly compute the probability, which is just $\frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{2^5} = \frac{16}{32} = \frac{1}{2}$

2. [5] How many perfect squares divide 10^{10} ?

Proposed by: Evan Chen

Answer:

A perfect square s divides 10^{10} if and only if $s = 2^a \cdot 5^b$ where $a, b \in \{0, 2, 4, 6, 8, 10\}$. There are 36 choices, giving 36 different s 's.

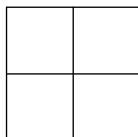
3. [5] Evaluate $\frac{2016!^2}{2015!2017!}$. Here $n!$ denotes $1 \times 2 \times \cdots \times n$.

Proposed by: Evan Chen

Answer:

$$\frac{2016!^2}{2015!2017!} = \frac{2016!}{2015!} \frac{2016!}{2017!} = \frac{2016}{1} \frac{1}{2017} = \frac{2016}{2017}$$

4. [6] A square can be divided into four congruent figures as shown:



For how many n with $1 \leq n \leq 100$ can a unit square be divided into n congruent figures?

Proposed by: Kevin Sun

Answer:

We can divide the square into congruent rectangles for all n , so the answer is .

5. [6] If $x + 2y - 3z = 7$ and $2x - y + 2z = 6$, determine $8x + y$.

Proposed by: Eshaan Nichani

Answer:

$$8x + y = 2(x + 2y - 3z) + 3(2x - y + 2z) = 2(7) + 3(6) = 32$$

6. [6] Let $ABCD$ be a rectangle, and let E and F be points on segment AB such that $AE = EF = FB$. If CE intersects the line AD at P , and PF intersects BC at Q , determine the ratio of BQ to CQ .

Proposed by: Eshaan Nichani

Answer:

Because $\triangle PAE \sim \triangle PDC$ and $AE : DC = 1 : 3$, we have that $PA : PD = 1 : 3 \implies PA : AB = PA : BC = 1 : 2$. Also, by similar triangles $\triangle PAF \sim \triangle QBF$, since $AF : BF = 2 : 1$, $PA : BQ = 2 : 1$.

Then $BQ = \frac{1}{2}PA = \frac{1}{2} \cdot \frac{1}{2}BC = \frac{1}{4}BC$. Then $BQ : CQ = \frac{1}{3}$.

7. [7] What is the minimum value of the product

$$\prod_{i=1}^6 \frac{a_i - a_{i+1}}{a_{i+2} - a_{i+3}}$$

given that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$? (note $a_7 = a_1, a_8 = a_2 \dots$)

Proposed by: Kevin Sun

Answer:

The product always evaluates to .

8. [7] Danielle picks a positive integer $1 \leq n \leq 2016$ uniformly at random. What is the probability that $\gcd(n, 2015) = 1$?

Proposed by: Evan Chen

Answer:

We split the interval $[1, 2016]$ into $[1, 2015]$ and 2016. The number of integers in $[1, 2015]$ that are relatively prime to 2015 is $\phi(2015) = \frac{4}{5} \cdot \frac{12}{13} \cdot \frac{30}{31} \cdot 2015 = 1440$. Also, 2016 is relatively prime to 2015, so there are a total of 1441 numbers in $[1, 2016]$ that are relatively prime to 2015. Then the probability

of picking a number relatively prime to 2015 is .

9. [7] How many 3-element subsets of the set $\{1, 2, 3, \dots, 19\}$ have sum of elements divisible by 4?

Proposed by: Sam Korsky

Answer:

Consider the elements of the sets mod 4. Then we would need to have sets of the form $\{0, 0, 0\}$, $\{0, 2, 2\}$, $\{0, 1, 3\}$, $\{1, 1, 2\}$, or $\{2, 3, 3\}$. In the set $\{1, 2, \dots, 19\}$ there four elements divisible by 4 and 5 elements congruent to each of 1, 2, 3 mod 4. Hence the desired number is given by

$$\binom{4}{3} + \binom{4}{1} \binom{5}{2} + \binom{4}{1} \binom{5}{1} \binom{5}{1} + \binom{5}{2} \binom{5}{1} + \binom{5}{1} \binom{5}{2} = \text{244}$$

10. [8] Michael is playing basketball. He makes 10% of his shots, and gets the ball back after 90% of his missed shots. If he does not get the ball back he stops playing. What is the probability that Michael eventually makes a shot?

Proposed by: Eshaan Nichani

Answer:

We find the probability Michael never makes a shot. We do casework on the number of shots Michael takes. He takes only one shot with probability $\frac{9}{10} \cdot \frac{1}{10}$ (he misses with probability $\frac{9}{10}$ and does not get the ball back with probability $\frac{1}{10}$). Similarly, he takes two shots with probability $\frac{9}{10} \cdot \frac{9}{10} \cdot \frac{9}{10} \cdot \frac{1}{10}$, three shots with probability $(\frac{9}{10})^5 \cdot \frac{1}{10}$, and so on. So we want to sum $\sum_{i=1}^{\infty} (\frac{9}{10})^{2i-1} \cdot \frac{1}{10} = \frac{\frac{1}{10} \cdot \frac{9}{10}}{1 - \frac{81}{100}} = \frac{9}{19}$. Then

the probability Michael makes a shot is $1 - \frac{9}{19} = \text{10/19}$.

11. [8] How many subsets S of the set $\{1, 2, \dots, 10\}$ satisfy the property that, for all $i \in [1, 9]$, either i or $i + 1$ (or both) is in S ?

Proposed by: Eshaan Nichani

Answer:

We do casework on the number of i 's not in S . Notice that these i 's that are not in S cannot be consecutive, otherwise there exists an index i such that both i and $i + 1$ are both not in S . Hence if

there are k i 's not in S , we want to arrange k black balls and $10 - k$ white balls such that no two black balls are consecutive. Take out $k - 1$ white balls to insert back between black balls later, then we want to arrange k black balls and $11 - 2k$ white balls arbitrarily, which can be done in $\binom{11-k}{k}$ ways. Hence we want to find the sum $\binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5}$, which is equal to $\boxed{144}$ ways.

12. [8] A positive integer \overline{ABC} , where A, B, C are digits, satisfies

$$\overline{ABC} = B^C - A$$

Find \overline{ABC} .

Proposed by: Henrik Boecken

The equation is equivalent to $100A + 10B + C = B^C - A$. Suppose $A = 0$, so that we get $10B + C = B^C$. Reducing mod B , we find that C must be divisible by B . $C \neq 0$, since otherwise $10B = 1$, contradiction, so $C \geq B$. Thus $10B + C \geq B^B$ for digits B, C . For $B \geq 4$, we have $100 > 10B + C \geq B^B > 100$, a contradiction, so $B = 1, 2, 3$. We can easily test that these do not yield solutions, so there are no solutions when $A = 0$.

Thus $A \geq 1$, and so $100 \leq 100A + 10B + C \leq 1000$, and thus $100 \leq B^C - A \leq 1000$. $1 \leq A \leq 10$, so we have $101 \leq B^C \leq 1010$. We can test that the only pairs (B, C) that satisfy this condition are $(2, 7), (2, 8), (2, 9), (3, 5), (3, 6), (4, 4), (5, 3), (6, 3), (7, 3), (8, 3), (9, 3)$. Of these pairs, only $(2, 7)$ yields a solution to the original equation, namely $A = 1, B = 2, C = 7$. Thus $\overline{ABC} = 127$.

13. [9] How many functions $f : \{0, 1\}^3 \rightarrow \{0, 1\}$ satisfy the property that, for all ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3) such that $a_i \geq b_i$ for all i , $f(a_1, a_2, a_3) \geq f(b_1, b_2, b_3)$?

Proposed by: Eshaan Nichani

Answer: $\boxed{20}$

Consider the unit cube with vertices $\{0, 1\}^3$. Let $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (0, 1, 0)$, $C = (0, 0, 1)$, $D = (0, 1, 1)$, $E = (1, 0, 1)$, $F = (1, 1, 0)$, and $P = (1, 1, 1)$. We want to find a function f on these vertices such that $f(1, y, z) \geq f(0, y, z)$ (and symmetric representations). For instance, if $f(A) = 1$, then $f(E) = f(F) = f(P) = 1$ as well, and if $f(D) = 1$, then $f(P) = 1$ as well.

We group the vertices into four levels: $L_0 = \{O\}$, $L_1 = \{A, B, C\}$, $L_2 = \{D, E, F\}$, and $L_3 = \{P\}$. We do casework on the lowest level of a 1 in a function.

- If the 1 is in L_0 , then f maps everything to 1, for a total of 1 way.
- If the 1 is in L_1 , then $f(O) = 0$. If there are 3 1's in L_1 , then everything but O must be mapped to 1, for 1 way. If there are 2 1's in L_1 , then $f(L_2) = f(L_3) = 1$, and there are 3 ways to choose the 2 1's in L_1 , for a total of 3 ways. If there is one 1, then WLOG $f(A) = 1$. Then $f(E) = f(F) = f(P) = 1$, and $f(D)$ equals either 0 or 1. There are $3 \cdot 2 = 6$ ways to do this. In total, there are $1 + 3 + 6 = 10$ ways for the lowest 1 to be in L_1 .
- If the lowest 1 is in L_2 , then $f(O) = f(L_1) = 0$. If there are 3 1's in L_2 , there is one way to make f . If there are 2 1's, then we can pick the 2 1's in 3 ways. Finally, if there is one 1, then we pick this 1 in 3 ways. There are $1 + 3 + 3 = 7$ ways.
- The lowest 1 is in L_3 . There is 1 way.
- There are no 1's. Then f sends everything to 0. There is 1 way.

In total, there are $1 + 10 + 7 + 1 + 1 = \boxed{20}$ total f 's.

14. [9] The very hungry caterpillar lives on the number line. For each non-zero integer i , a fruit sits on the point with coordinate i . The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?

Proposed by: Eshaan Nichani

Answer: $\boxed{9217}$

On the n th straight path, the caterpillar travels n units before hitting food and his weight is $n - 1$. Then his speed is 2^{1-n} . Then right before he turns around for the n th time, he has traveled a total time of $\sum_{i=1}^n \frac{i}{2^{1-i}} = \frac{1}{2} \sum_{i=1}^n i \cdot 2^i$. We want to know how many days the caterpillar moves before his weight is 10, so we want to take $n = 10$ so that his last straight path was taken at weight 9. Hence we want to evaluate $S = \frac{1}{2} \sum_{i=1}^{10} i \cdot 2^i$. Note that $2S = \frac{1}{2} \sum_{i=2}^{11} (i-1) \cdot 2^i$, so $S = 2S - S = \frac{1}{2} \left(11 \cdot 2^{11} - \sum_{i=1}^{10} 2^i \right) = \frac{1}{2} (10 \cdot 2^{11} - 2^{11} + 2) = \boxed{9217}$.

15. [9] Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 1$ and $CD = 2$ and height 1. Find the area of the region containing all points inside $ABCD$ whose projections onto the four sides of the trapezoid lie on the segments formed by AB, BC, CD and DA .

Proposed by: Allen Liu

Answer: $\boxed{\frac{5}{8}}$

Let E, F , be the projections of A, B on CD . A point whose projections lie on the sides must be contained in the square $ABFE$. Furthermore, the point must lie under the perpendicular to AD at A and the perpendicular to BC at B , which have slopes $\frac{1}{2}$ and $-\frac{1}{2}$. The area of the desired pentagon is

$$1 - \frac{1}{4} - \frac{1}{8} = \boxed{\frac{5}{8}}.$$

16. [10] Create a cube C_1 with edge length 1. Take the centers of the faces and connect them to form an octahedron O_1 . Take the centers of the octahedron's faces and connect them to form a new cube C_2 . Continue this process infinitely. Find the sum of all the surface areas of the cubes and octahedrons.

Proposed by: Shyam Narayanan

Answer: $\boxed{\frac{54+9\sqrt{3}}{8}}$

The lengths of the second cube are one-third of the lengths of the first cube, so the surface area decreases by a factor of one-ninth. Since the first cube has surface area 6 and the first octahedron has

surface area $\sqrt{3}$, the total area is $(6 + \sqrt{3}) \cdot \left(1 + \frac{1}{9} + \frac{1}{9^2} + \dots \right) = \boxed{\frac{54 + 9\sqrt{3}}{8}}$

17. [10] Let $p(x) = x^2 - x + 1$. Let α be a root of $p(p(p(x)))$. Find the value of

$$(p(\alpha) - 1)p(\alpha)p(p(\alpha))p(p(p(\alpha)))$$

Proposed by: Henrik Boecken

Answer: $\boxed{-1}$

Since $(x - 1)x = p(x) - 1$, we can set

$$\begin{aligned} (p(\alpha) - 1)p(\alpha)p(p(\alpha))p(p(p(\alpha))) &= (p(p(\alpha)) - 1)p(p(\alpha))p(p(p(\alpha))) \\ &= (p(p(p(\alpha))) - 1)p(p(p(\alpha))) \\ &= p(p(p(p(\alpha)))) - 1 \\ &= \boxed{-1} \end{aligned}$$

18. [10] An 8 by 8 grid of numbers obeys the following pattern:

1) The first row and first column consist of all 1s.

2) The entry in the i th row and j th column equals the sum of the numbers in the $(i - 1)$ by $(j - 1)$ sub-grid with row less than i and column less than j .

What is the number in the 8th row and 8th column?

Proposed by: Eshaan Nichani

Answer: $\boxed{2508}$

Let $x_{i,j}$ be the number in the i th row and the j th column. Then if $i, j \geq 2$, $x_{i+1,j+1} - x_{i+1,j} - x_{i,j+1} + x_{i,j}$ only counts the term $x_{i,j}$ since every other term is added and subtracted the same number of times. Thus $x_{i+1,j+1} = x_{i+1,j} + x_{i,j+1}$ when $i, j \geq 2$. Also, $x_{2,i} = x_{i,2} = i$ so $x_{i+1,j+1} = x_{i+1,j} + x_{i,j+1}$ holds for all $i, j \geq 1$ except for $(i, j) = (1, 1)$ where $x_{2,2}$ is one less than expected. This means that $x_{i,j}$ is the number ways of travelling from $(1, 1)$ to (i, j) , minus the number of ways of travelling from $(2, 2)$ to (i, j) , which is $\binom{14}{7} - \binom{12}{6} = \boxed{2508}$.

19. [11] Let S be the set of all positive integers whose prime factorizations only contain powers of the primes 2 and 2017 (1, powers of 2, and powers of 2017 are thus contained in S). Compute $\sum_{s \in S} \frac{1}{s}$.

Proposed by: Daniel Qu

Answer: $\boxed{\frac{2017}{1008}}$

Since every s can be written as $2^i \cdot 2017^j$ for non-negative integers i and j , the given sum can be written as $(\sum_{i=0}^{\infty} \frac{1}{2^i})(\sum_{j=0}^{\infty} \frac{1}{2017^j})$. We can easily find the sum of these geometric series since they both have common ratio of magnitude less than 1, giving us $(\frac{1}{1-\frac{1}{2}}) \cdot \frac{1}{1-\frac{1}{2017}} = \frac{2}{1} \cdot \frac{2017}{2016} = \frac{2017}{1008}$.

20. [11] Let \mathcal{V} be the volume enclosed by the graph

$$x^{2016} + y^{2016} + z^2 = 2016$$

Find \mathcal{V} rounded to the nearest multiple of ten.

Proposed by: Henrik Boecken

Answer: $\boxed{360}$

Let R be the region in question. Then we have

$$[-1, 1]^2 \times [-\sqrt{2014}, \sqrt{2014}] \subset R \subset [-\sqrt[2016]{2016}, \sqrt[2016]{2016}]^2 \times [\sqrt{2016}, \sqrt{2016}]$$

We find some bounds: we have

$$\sqrt{2016} < \sqrt{2025} = 45.$$

By concavity of $\sqrt{\cdot}$, we have the bound

$$\sqrt{2014} \leq \frac{11}{89}\sqrt{1936} + \frac{78}{89}\sqrt{2025} = 45 - \frac{11}{89}.$$

Finally, if we let $\sqrt[2016]{2016} = 1 + \epsilon$, then $(1 + \epsilon)^{2016} = 2016$, so

$$\binom{2016}{5} \epsilon^5 < 2016 \Rightarrow \epsilon < \sqrt[5]{\frac{120}{2015 \cdot 2014 \cdot 2013 \cdot 2012}} < \sqrt[5]{\frac{120}{2000^4}} = \sqrt[5]{\frac{7500}{10^{15}}} = \frac{\sqrt[5]{7500}}{1000} < \frac{6}{1000}.$$

Therefore, the volume of R is lower-bounded by

$$2^2 \cdot 2 \left(45 - \frac{11}{89}\right) = 360 - \frac{88}{89} > 355$$

and upper-bounded by

$$2^2 \left(1 + \frac{0.8}{100}\right)^2 \cdot 2(45) = 360 \left(1 + \frac{6}{1000}\right)^2 < 365.$$

Thus, R rounded to the nearest ten is 360.

21. [11] Zlatan has 2017 socks of various colours. He wants to proudly display one sock of each of the colours, and he counts that there are N ways to select socks from his collection for display. Given this information, what is the maximum value of N ?

Proposed by: Saranesh Prembabu

Answer: $\boxed{3^{671} \cdot 4}$

Say that there are k sock types labeled $1, 2, \dots, k$, and a_i socks of type i . The problem asks to maximize $\prod_{i=1}^k a_i$ subject to $\sum_{i=1}^k a_i = 2017$, over all k and all sequences of positive integers a_1, \dots, a_k .

The optimal (a_1, \dots, a_k) cannot have any $a_i = 1$ for any i , because if there exists $a_i = 1$ we can delete this a_i and add 1 to any a_j ($j \neq i$) to increase the product while keeping the sum constant.

There exists an optimal (a_1, \dots, a_k) without any $a_i \geq 4$ because if there exists $a_i \geq 4$ we can replace this a_i with $a_i - 2$ and 2, which nonstrictly increases the product while keeping the sum constant.

Therefore, there exists an optimal (a_1, \dots, a_k) whose terms are all 2 or 3. The optimal (a_1, \dots, a_k) cannot have more than two 2s, because we can replace three 2s with two 3s, which increases the sum by a factor of $\frac{3^2}{2^3} = \frac{9}{8}$ while keeping the sum constant.

It follows that we want to partition 2017 into 671 3s and two 2s, for a product of $3^{671} \cdot 4$.

22. [12] Let the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$ take only integer inputs and have integer outputs. For any integers x and y , f satisfies

$$f(x) + f(y) = f(x + 1) + f(y - 1)$$

If $f(2016) = 6102$ and $f(6102) = 2016$, what is $f(1)$?

Proposed by: Henrik Boecken

Answer: $\boxed{8117}$

We have

$$f(x + 1) = f(x) + f(y) - f(y - 1)$$

If y is fixed, we have

$$f(x + 1) = f(x) + \text{constant}$$

implying f is linear. Using our two points, then, we get $f(x) = 8118 - x$, so $f(1) = \boxed{8117}$

23. [12] Let d be a randomly chosen divisor of 2016. Find the expected value of

$$\frac{d^2}{d^2 + 2016}$$

Proposed by: Henrik Boecken

Answer: $\boxed{\frac{1}{2}}$

Let $ab = 2016$. Then

$$\frac{a^2}{a^2 + 2016} + \frac{b^2}{b^2 + 2016} = \frac{a^2}{a^2 + 2016} + \frac{\left(\frac{2016}{a}\right)^2}{\left(\frac{2016}{a}\right)^2 + 2016} = \frac{a^2}{a^2 + 2016} + \frac{2016}{a^2 + 2016} = 1$$

Thus, every divisor d pairs up with $\frac{2016}{d}$ to get 1, so our desired expected value is $\boxed{\frac{1}{2}}$.

24. [12] Consider an infinite grid of equilateral triangles. Each edge (that is, each side of a small triangle) is colored one of N colors. The coloring is done in such a way that any path between any two non-adjacent vertices consists of edges with at least two different colors. What is the smallest possible value of N ?

Proposed by: Henrik Boecken

Answer: $\boxed{6}$

Note that the condition is equivalent to having no edges of the same color sharing a vertex by just considering paths of length two. Consider a hexagon made out of six triangles. Six edges meet at the center, so $N \geq 6$. To prove $N = 6$, simply use two colors for each of the three possible directions of an edge, and color edges of the same orientation alternately with different colors.

25. [13] Chris and Paul each rent a different room of a hotel from rooms 1 – 60. However, the hotel manager mistakes them for one person and gives "Chris Paul" a room with Chris's and Paul's room concatenated. For example, if Chris had 15 and Paul had 9, "Chris Paul" has 159. If there are 360 rooms in the hotel, what is the probability that "Chris Paul" has a valid room?

Proposed by: Meghal Gupta

Answer: $\boxed{\frac{153}{1180}}$

There are $60 \cdot 59 = 3540$ total possible outcomes, and we need to count the number of these which concatenate into a number at most 60. Of these, $9 \cdot 8$ result from both Chris and Paul getting one-digit room numbers. If Chris gets a two-digit number, then he must get a number at most 35 and Paul should get a one-digit room number, giving $(35 - 9) \cdot 9$ possibilities. If Chris gets a one-digit number, it must be 1, 2, or 3. If Chris gets 1, 2 or 3, Paul can get any two-digit number from 10 to 60 to guarantee a valid room, giving $51 \cdot 3$ outcomes. The total number of correct outcomes is $72 + 51 \cdot 3 + 26 \cdot 9 = 459$, so the desired probability is $\boxed{\frac{153}{1180}}$.

26. [13] Find the number of ways to choose two nonempty subsets X and Y of $\{1, 2, \dots, 2001\}$, such that $|Y| = 1001$ and the smallest element of Y is equal to the largest element of X .

Proposed by: Allen Liu

Answer: $\boxed{2^{2000}}$

We claim that there is a bijection between pairs (X, Y) and sets S with at least 1001 elements. To get S from X and Y , take $S = X \cup Y$, which contains Y and thus has at least 1001 elements. To form (X, Y) from S , make Y the largest 1001 elements of S , and make X everything except the largest 1000 elements of S . Therefore we need to count the number of subsets of $\{1, 2, \dots, 2001\}$ with at least 1001 elements. For every subset of $\{1, 2, \dots, 2001\}$, either it or its complement has at least 1001 elements, so number of possible subsets is $\frac{1}{2} \cdot 2^{2001} = \boxed{2^{2000}}$.

27. [13] Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2 + r_3^2 + r_4^2} + \frac{r_2^2}{r_1^2 + r_3^2 + r_4^2} + \frac{r_3^2}{r_1^2 + r_2^2 + r_4^2} + \frac{r_4^2}{r_1^2 + r_2^2 + r_3^2}$$

Proposed by: Henrik Boecken

Answer: $\boxed{-4}$

Add 1 to each fraction to get

$$\frac{r_1^2 + r_2^2 + r_3^2 + r_4^2}{r_2^2 + r_3^2 + r_4^2} + \frac{r_1^2 + r_2^2 + r_3^2 + r_4^2}{r_1^2 + r_3^2 + r_4^2} + \frac{r_1^2 + r_2^2 + r_3^2 + r_4^2}{r_1^2 + r_2^2 + r_4^2} + \frac{r_1^2 + r_2^2 + r_3^2 + r_4^2}{r_1^2 + r_2^2 + r_3^2}$$

This seems like a difficult problem until one realizes that

$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (r_1 + r_2 + r_3 + r_4)^2 - 2(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) = 4^2 - 2 \cdot 8 = 0$$

Thus, our current expression is 0. Noting that we added 4, the original value had to be $\boxed{-4}$.

28. [15] The numbers 1 – 10 are written in a circle randomly. Find the expected number of numbers which are at least 2 larger than an adjacent number.

Proposed by: Shyam Narayanan

Answer: $\boxed{\frac{17}{3}}$

For $1 \leq i \leq 10$, let X_i be the random variable that is 1 if the i in the circle is at least 2 larger than one of its neighbors, and 0 otherwise. The random variable representing number of numbers that are at least 2 larger than one of their neighbors is then just $X_1 + X_2 + \cdots + X_{10}$. The expected value $\mathbb{E}[X_1 + X_2 + \cdots + X_{10}]$ is equal to $\mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{10}]$ by the linearity of expectation, so it suffices to compute $\mathbb{E}[X_i]$ for all $1 \leq i \leq 10$.

By the definition of expected value, $\mathbb{E}[X_i] = 1 \cdot P(\text{the } i \text{ is at least 2 larger than one of its neighbors}) + 0 \cdot P(\text{it is not at least 2 larger than either of its neighbors}) = P(\text{the } i \text{ is at least 2 larger than one of its neighbors}) = 1 - P(\text{the } i \text{ is at most 1 larger than both of its neighbors})$. For the last probability, i 's neighbors must be drawn from the set $\{\max(1, i-1), \max(1, i-1) + 1, \dots, 10\}$, excluding i itself. This set has $10 - \max(1, i-1)$ elements, so there are a total of $\binom{10 - \max(1, i-1)}{2}$ sets of two neighbors for i that satisfy the condition, out of a total of $\binom{9}{2}$ possible sets of two neighbors from all of the numbers that are not i . The last probability is then $\frac{\binom{10 - \max(1, i-1)}{2}}{\binom{9}{2}}$, so $\mathbb{E}[X_i] = 1 - \frac{\binom{10 - \max(1, i-1)}{2}}{\binom{9}{2}}$.

The final sum we wish to calculate then becomes $(1 - \frac{\binom{9}{2}}{\binom{9}{2}}) + (1 - \frac{\binom{8}{2}}{\binom{9}{2}}) + (1 - \frac{\binom{8}{2}}{\binom{9}{2}}) + (1 - \frac{\binom{7}{2}}{\binom{9}{2}}) + \cdots + (1 - \frac{\binom{1}{2}}{\binom{9}{2}}) = 0 + 0 + (1 - \frac{28}{36}) + (1 - \frac{21}{36}) + \cdots + (1 - 0) = \boxed{\frac{17}{3}}$.

29. [15] We want to design a new chess piece, the American, with the property that (i) the American can never attack itself, and (ii) if an American A_1 attacks another American A_2 , then A_2 also attacks A_1 . Let m be the number of squares that an American attacks when placed in the top left corner of an 8 by 8 chessboard. Let n be the maximal number of Americans that can be placed on the 8 by 8 chessboard such that no Americans attack each other, if one American must be in the top left corner. Find the largest possible value of mn .

Proposed by: Kevin Yang

Answer: $\boxed{1024}$

Since one of the Americans must be in the top left corner, that eliminates m squares from consideration for placing additional Americans. So $m + n$ is at most 64, which implies mn can be at most 1024. To achieve 1024, we can color a chessboard the normal way, and say that an American attacks all squares of the opposite color. Then the American in the top left corner attacks the 32 squares of the opposite color, and placing all Americans on the squares of the same color as the top-left corner guarantees no Americans attack each other.

30. [15] On the blackboard, Amy writes 2017 in base- a to get 133201_a . Betsy notices she can erase a digit from Amy's number and change the base to base- b such that the value of the the number remains the same. Catherine then notices she can erase a digit from Betsy's number and change the base to base- c such that the value still remains the same. Compute, in decimal, $a + b + c$.

Proposed by: Daniel Qu

Answer: $\boxed{22}$

$$2017 = 133201_4 = 13201_6 = 1201_{12}$$

31. [17] Define a number to be an anti-palindrome if, when written in base 3 as $a_n a_{n-1} \dots a_0$, then $a_i + a_{n-i} = 2$ for any $0 \leq i \leq n$. Find the number of anti-palindromes less than 3^{12} such that no two consecutive digits in base 3 are equal.

Proposed by: Shyam Narayanan

Answer: $\boxed{126}$

Note once the middle digit/pair of digits is determined, it suffices to choose the digits in the left half of the number and ensure no pair of consecutive digits are equal. For a number with an even number of digits, the middle pair is 02 or 20 while for a number with an odd number of digits, the middle digit is 1. We can now count recursively.

Let a_n be the number of ways to choose n digits no two of which are consecutive and equal such that the leading digit is nonzero and the ending digit is 1. Let b_n be the number ways to do the same such that the ending digit is 0 or 2.

Note $a_n = b_{n-1}$. Also $b_n = b_{n-1} + 2a_{n-1}$.

Solving for the terms of the sequence, they are $a_1 = 1, a_2 = b_1 = 1, a_3 = b_2 = 3, a_4 = b_3 = 5, a_5 = b_4 = 11, a_6 = b_5 = 21, b_6 = 43$. Therefore, there are 43 twelve-digit numbers satisfying the condition, 21 eleven-digit numbers, 21 ten-digit numbers.... and 1 one-digit number. The sum of these values gives us a final answer of 126

32. [17] Let $C_{k,n}$ denote the number of paths [17] on the Cartesian plane along which you can travel from $(0, 0)$ to (k, n) , given the following rules: 1) You can only travel directly upward or directly rightward 2) You can only change direction at lattice points 3) Each horizontal segment in the path must be at most 99 units long.

Find

$$\sum_{j=0}^{\infty} C_{100j+19,17}$$

Proposed by: Saranesh Prembabu

Answer: $\boxed{100^{17}}$

If we are traveling from $(0, 0)$ to $(n, 17)$, we first travel x_0 rightwards, then up one, then x_1 rightwards, then up one, ..., until we finally travel x_{17} rightwards. x_0, \dots, x_{17} are all at most 99 by our constraint, but can equal 0. Given that x_0, \dots, x_{16} are fixed, there is exactly one way to choose x_{17} so that $x_0 + \dots + x_{17}$ is congruent to 19 mod 100. Then, this means that the sum equals the total number of ways to choose x_0, \dots, x_{16} , which equals $100^{17} = 10^{34}$.

33. [17] Camille the snail lives on the surface of a regular dodecahedron. Right now he is on vertex P_1 of the face with vertices P_1, P_2, P_3, P_4, P_5 . This face has a perimeter of 5. Camille wants to get to the point on the dodecahedron farthest away from P_1 . To do so, he must travel along the surface a distance at least L . What is L^2 ?

Proposed by: Saranesh Prembabu

Answer: $\boxed{\frac{17+7\sqrt{5}}{2}}$

Consider the net of the dodecahedron. It suffices to look at three pentagons $ABCDE, EDFGH,$ and $GFIJK$, where $AJ = L$. This can be found by the law of cosines on triangle AEJ . We have $AE = 1, EJ = \tan 72^\circ$, and $\angle AEJ = 162^\circ$. Thus $L^2 = 1 + \tan^2 72^\circ + 2 \cdot \tan 72^\circ \cdot \cos 18^\circ = \frac{17+7\sqrt{5}}{2}$

34. [20] Find the sum of the ages of everyone who wrote a problem for this year's HMMT November contest. If your answer is X and the actual value is Y , your score will be $\max(0, 20 - |X - Y|)$

Proposed by: Allen Liu

Answer: $\boxed{258}$

There was one problem for which I could not determine author information, so I set the author as one of the problem czars at random. Then, I ran the following command on a folder containing TeX solutions files to all four contests:

```
evan@ArchMega ~/Downloads/November
$ grep --no-filename "Proposed by: " *.tex | sort | uniq -c
    15  {\em Proposed by: Allen Liu }
     1  {\em Proposed by: Brice Huang }
     2  {\em Proposed by: Christopher Shao }
     2  {\em Proposed by: Daniel Qu }
    21  {\em Proposed by: Eshaan Nichani }
```

```

3 {\em Proposed by: Evan Chen }
9 {\em Proposed by: Henrik Boecken }
2 {\em Proposed by: Kevin Sun }
2 {\em Proposed by: Kevin Yang }
1 {\em Proposed by: Meghal Gupta }
1 {\em Proposed by: Rachel Zhang }
1 {\em Proposed by: Sam Korsky }
3 {\em Proposed by: Saranesh Prembabu }
3 {\em Proposed by: Shyam Narayanan }

```

This gave the counts of problem proposals; there were 14 distinct authors of problems for November 2016. Summing their ages (concealed for privacy) gives 258.

35. [20] Find the total number of occurrences of the digits $0, 1, \dots, 9$ in the entire guts round. If your answer is X and the actual value is Y , your score will be $\max(0, 20 - \frac{|X-Y|}{2})$

Proposed by: Allen Liu

Answer:

To compute the answer, I extracted the flat text from the PDF file and ran word-count against the list of digit matches.

```

evan@ArchMega ~/Downloads/November
$ pdftotext HMMTNovember2016GutsTest.pdf guts-test-text.txt
evan@ArchMega ~/Downloads/November
$ cat guts-test-text.txt | egrep "[0-9]" --only-matching | wc -l
559

```

36. [20] Find the number of positive integers less than 1000000 which are less than or equal to the sum of their proper divisors. If your answer is X and the actual value is Y , your score will be $\max(0, 20 - 80|1 - \frac{X}{Y}|)$ rounded to the nearest integer.

Proposed by: Allen Liu

Answer:

```

N = 1000000
s = [0] * N
ans = 0
for i in range(1, N):
    if i <= s[i]:
        ans += 1
    for j in range(i + i, N, i):
        s[j] += i

print(ans)

```

HMMT November 2016

November 12, 2016

Team

1. [3] Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P . The common external tangent of the two circles intersects the line O_1O_2 at Q . What is the length of PQ ?

Proposed by: Eshaan Nichani

Answer: 12

Let the common external tangent intersect the circles centered at O_1, O_2 at X, Y respectively. Then $\frac{O_2Q}{O_1Q} = \frac{OY}{OX} = \frac{3}{2}$, so $\frac{O_1O_2}{O_1Q} = \frac{O_2Q - O_1Q}{O_1Q} = \frac{1}{2}$. Since $O_1O_2 = 2 + 3 = 5$, $O_1Q = 10$ and hence $PQ = O_1Q + O_1P = 12$.

2. [3] What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

Proposed by: Eshaan Nichani

Answer: 77

There exist a triangle with side lengths $4^2, 5^2, 6^2$, which has perimeter 77. If the sides have lengths a^2, b^2, c^2 with $0 < a < b < c$, then $a^2 + b^2 > c^2$ by the triangle inequality. Therefore $(b-1)^2 + b^2 \geq a^2 + b^2 > c^2 \geq (b+1)^2$. Solving this inequality gives $b > 4$. If $b \geq 6$, then $a^2 + b^2 + c^2 \geq 6^2 + 7^2 > 77$. If $b = 5$, then $c \geq 7$ is impossible, while $c = 6$ forces $a = 4$, which gives a perimeter of 77.

3. [3] Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

Proposed by: Henrik Boecken

Answer: 5

The value of $\omega^4 + \omega^3 + \omega^2 + \omega + 1 = \frac{\omega^5 - 1}{\omega - 1} = \frac{1}{\omega - 1}$. The sum of these values is therefore the sum of $\frac{1}{\omega - 1}$ over the five roots ω . Substituting $z = \omega - 1$, we have that $(z + 1)^5 = 2$, so $z^5 + 5z^4 + 10z^3 + 10z^2 + 5z - 1 = 0$. The sum of the reciprocals of the roots of this equation is $-\frac{5}{-1} = \span style="border: 1px solid black; padding: 2px;">5 by Vieta's.$

4. [5] Meghal is playing a game with 2016 rounds $1, 2, \dots, 2016$. In round n , two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

Proposed by: Rachel Zhang

Answer: 1019088

Let points O, A_1, A_2 lie in a plane such that $\angle A_1OA_2 = \frac{2\pi}{n+2}$. We represent the mirrors as line segments extending between O and A_1 , and O and A_2 . Also let points A_3, A_4, \dots, A_{n+2} lie in the plane such that A_{i+1} is the reflection of A_{i-1} over OA_i .

If Meghal shoots a laser along line l such that the first point of contact with a mirror is along OA_2 , the next point of contact, if it exists, is the point on OA_1 that is a reflection of the intersection of l with OA_3 . If we continue this logic, we find that the maximum score for round n is equal to the maximum number of intersection points between l and OA_i for some i . We do casework on whether n is even or odd. If n is even, there are at most $\frac{n+2}{2}$ spokes such that l can hit OA_i , and if n is odd, there are at most $\frac{n+3}{2}$ such spokes. Then we must sum $2 + 2 + 3 + 3 + \dots + 1009 + 1009 = 1009 \cdot 1010 - 1 - 1 = \span style="border: 1px solid black; padding: 2px;">1019088.$

5. [5] Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins?

Proposed by: Eshaan Nichani

Answer: $\boxed{5/12}$

Note that at any point in the game after the first roll, the probability that Allen wins depends only on the most recent roll, and not on any rolls before that one. So we may define p as the probability that Allen wins at any point in the game, given that the last roll was a 1, 2, or 3, and q as the probability that he wins given that the last roll was a 4, 5, or 6.

Suppose at some point, the last roll was $r_1 \in \{1, 2, 3\}$, and the next roll is $r_2 \in \{1, 2, 3, 4, 5, 6\}$. By the definition of p , Allen wins with probability p . Furthermore, if $r_2 = r_1$, which happens with probability $\frac{1}{6}$, Allen wins. If $r_2 \in \{1, 2, 3\}$ but $r_2 \neq r_1$, which happens with probability $\frac{2}{6}$, neither Allen nor Brian wins, so they continue playing the game, now where the last roll was r_2 . In this case, Allen wins with probability p . If $r_2 \in \{4, 5, 6\}$, which happens with probability $\frac{3}{6}$, neither Allen nor Brian wins, so they continue playing, now where the last roll was r_2 . In this case, Allen wins with probability q . Hence, the probability that Allen wins in this case can be expressed as $\frac{1}{6} + \frac{2}{6}p + \frac{3}{6}q$, and thus

$$p = \frac{1}{6} + \frac{2}{6}p + \frac{3}{6}q$$

By a similar analysis for q , we find that

$$q = \frac{1}{6} \cdot 0 + \frac{2}{6}p + \frac{3}{6}q$$

Solving, we get $p = \frac{1}{2}$ and $q = \frac{1}{3}$. Allen wins with probability $p = \frac{1}{2}$ if the first roll is 1, 2, or 3, and he wins with probability $q = \frac{1}{3}$ if the first roll is 4, 5, or 6. We conclude that the overall probability that

he wins the game is $\frac{1}{2}p + \frac{1}{2}q = \boxed{\frac{5}{12}}$.

6. [5] Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X . Find the length of EX .

Proposed by: Allen Liu

Answer: $\boxed{\frac{190}{49}}$

Since $\angle AFH = \angle AEH = 90^\circ$, E is on the circumcircle of AHF . So $\angle XEH = \angle HFD = \angle HBD$, which implies that $XE \parallel BD$. Hence $\frac{EX}{BD} = \frac{EY}{YB}$. Let DF and BE intersect at Y . Note that $\angle EDY = 180^\circ - \angle BDF - \angle CDE = 180^\circ - 2\angle A$, and $\angle BDY = \angle A$. Applying the sine rule to EYD and BYD , we get

$$\frac{EY}{YB} = \frac{ED}{BD} \cdot \frac{\sin \angle EDY}{\sin \angle BDY} = \frac{ED}{BD} \cdot \frac{\sin 2\angle A}{\sin \angle A} = \frac{ED}{BD} \cdot 2 \cos \angle A$$

Next, letting $x = CD$ and $y = AE$, by Pythagoras we have

$$AB^2 - (6 - x)^2 = AD^2 = AC^2 - x^2$$

$$BC^2 - (7 - y)^2 = BE^2 = BA^2 - y^2$$

Solving, we get $x = 5$, $y = \frac{19}{7}$. Drop the perpendicular from E to DC at Z . Then $ED \cos \angle A = ED \cos \angle EDZ = DZ$. But $AD \parallel EZ$, so $DZ = \frac{AE}{AC} \cdot DC = \frac{95}{49}$. Therefore

$$EX = \frac{EY}{YB} \cdot BD = 2ED \cos \angle A = 2DZ = \frac{190}{49}$$

7. [6] Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares. She can move the pieces in two ways:

- If a token has free square in front of it, then she can move this token one square to the right
- If the square immediately to the right of a token is occupied by the other token, then she can “leapfrog” the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.

If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

Proposed by: Christopher Shao

Answer:

We put a marker on (i, j) when a token is on i th and j th square and $i > j$. When the token in front/behind moves one step forward to a blank square, move the marker rightward/upward one unit correspondingly. When a “leapfrog” happens, the marker moves from $(x - 1, x)$ to $(x, x + 1)$. We can translate this movement into: 1. move the marker upward to (x, x) ; 2. move the marker rightward to $(x, x + 1)$. Thus, we set up a lattice path way from $(2, 1)$ to $(6, 5)$ staying under $y = x$. This is a bijection since every intersection of the path way and $y = x$ indicates a “leapfrog”. According to the definition of Catalan Number, the answer is the number of such lattice path ways, which is $C_5 = 42$.

8. [6] Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the i th row will toggle the state of each lightbulb in the i th row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the j th column will toggle the state of each bulb in the j th column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence? Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

Proposed by: Christopher Shao

Answer:

The switch flip operations are commutative, so for any given sequence of switch flips S , we get the same configuration regardless of the order we do them in. We can arrange the switch flips so that all of the flips of the same switch happen consecutively. Furthermore, two consecutive flips of the same switch leave the configuration unchanged, so we can remove them, resulting in a sequence of switch flips S' that involves flipping a switch for a row or column at most once that achieves the same configuration as S . The order of the flips in S' also doesn't matter, so we can treat S' as a set of switches that are flipped to produce the same configuration as S .

The desired number is then equal to the number of distinct configurations that can be obtained by flipping exactly the switches in some subset S' of the set of all of the switches. We claim that if S_1 and S_2 are distinct sets of switches that result in the same configuration of lights, then S_1 and S_2 are complements. Indeed, without loss of generality, suppose that the first row's switch is in S_1 and that it isn't in S_2 . In order to have the same configuration of lights in the first row, we must have that every column switch is in S_1 if and only if it isn't in S_2 . Applying the same argument to the first column yields that every row switch is in S_1 if and only if it isn't in S_2 , and the claim follows. Thus, for every set of switches, there is exactly one other set that attains the same configuration as it, namely its complement. There are 2^{m+n} sets of switches possible, and so the total number of possible configurations is $2^{m+n}/2 = 2^{m+n-1}$.

9. [7] A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x ?

Proposed by: Eshaan Nichani

Answer: $\boxed{\frac{15\sqrt{37}-75}{4}}$

Let O be the center of the large sphere, and let O_1, O_2, O_3 be the centers of the small spheres. Consider G , the center of equilateral $\triangle O_1O_2O_3$. Then if the radii of the small spheres are r , we have that $OG = 8+r$ and $O_1O_2 = O_2O_3 = O_3O_1 = 2r$, implying that $O_1G = \frac{2r}{\sqrt{3}}$. Then $OO_1 = \sqrt{OG^2 + OO_1^2} = \sqrt{(8+r)^2 + \frac{4}{3}r^2}$. Now draw the array OO_1 , and suppose it intersects the large sphere again at P . Then P is the point of tangency between the large sphere and the small sphere with center O_1 , so $OP = \sqrt{15^2 + 8^2} = 17 = OO_1 + O_1P = \sqrt{(8+r)^2 + \frac{4}{3}r^2} + r$. We rearrange this to be

$$\begin{aligned} 17 - r &= \sqrt{(8+r)^2 + \frac{4}{3}r^2} \\ \iff 289 - 34r + r^2 &= \frac{7}{3}r^2 + 16r + 64 \\ \iff \frac{4}{3}r^2 + 50r - 225 &= 0 \\ \implies r &= \frac{-50 \pm \sqrt{50^2 + 4 \cdot \frac{4}{3} \cdot 225}}{2 \cdot \frac{4}{3}} \\ &= \boxed{\frac{15\sqrt{37} - 75}{4}}. \end{aligned}$$

10. [7] Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x), p_2(x), p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in \mathbb{Z}$ such that $p_j(m) = i$.

Proposed by: Eshaan Nichani

Answer: $\boxed{9}$

The construction for $n = 9$ can be achieved with the polynomials $x^2 + x + 1$, $x^2 + x + 2$, and $x^2 + 5$.

First we consider what kinds of polynomials we can have. Let $p(x) = (x+h)^2 + k$. h is either an integer or half an integer. Let $k = 0$. If h is an integer then $p(x)$ hits the perfect squares 0, 1, 4, 9, etc. If h is half an integer, then let $k = 1/4$. Then $p(x)$ hits the product of two consecutive integers, i.e. 0, 2, 6, 12, etc.

Assume there is a construction for $n = 10$. In both of the cases above, the most a polynomial can hit out of 10 is 4, in the 0, 1, 4, 9 case. Thus p_1 must hit 1, 2, 5, 10, and p_2 and p_3 hit 3 integers each, out of 3, 4, 6, 7, 8, 9. The only ways we can hit 3 out of 7 consecutive integers is with the sequences 0, 2, 6 or 0, 1, 4. The only way a 0, 2, 6 works is if it hits 3, 5, and 9, which doesn't work since 5 was hit by p_2 . Otherwise, p_2 is 0, 1, 4, which doesn't work as p_2 hits 3, 4, and 7, and p_3 must hit 6, 8, and 9, which is impossible. Thus no construction for $n = 10$ exists.

HMMT November 2016

November 12, 2016

Theme Round

1. DeAndre Jordan shoots free throws that are worth 1 point each. He makes 40% of his shots. If he takes two shots find the probability that he scores at least 1 point.

Proposed by: Allen Liu

Answer: $\boxed{\frac{16}{25}}$

We want to find the probability of making at least one shot. The probability he makes no shots is $(\frac{3}{5})^2$, so the probability of making at least one is $1 - (\frac{3}{5})^2 = \boxed{\frac{16}{25}}$.

2. Point P_1 is located 600 miles West of point P_2 . At 7:00 AM a car departs from P_1 and drives East at a speed of 50 miles per hour. At 8:00 AM another car departs from P_2 and drives West at a constant speed of x miles per hour. If the cars meet each other exactly halfway between P_1 and P_2 , what is the value of x ?

Proposed by: Eshaan Nichani

Answer: $\boxed{60}$

Each car meets having traveled 300 miles. Therefore the first car traveled for $300/50 = 6$ hours, and so the second car traveled for 5 hours. The second car must have traveled $300/5 = \boxed{60}$ miles per hour.

3. The three points A, B, C form a triangle. $AB = 4, BC = 5, AC = 6$. Let the angle bisector of $\angle A$ intersect side BC at D . Let the foot of the perpendicular from B to the angle bisector of $\angle A$ be E . Let the line through E parallel to AC meet BC at F . Compute DF .

Proposed by: Allen Liu

Answer: $\boxed{1/2}$

Since AD bisects $\angle A$, by the angle bisector theorem $\frac{AB}{BD} = \frac{AC}{CD}$, so $BD = 2$ and $CD = 3$. Extend BE to hit AC at X . Since AE is the perpendicular bisector of BX , $AX = 4$. Since B, E, X are collinear, applying Menelaus' Theorem to the triangle ADC , we have

$$\frac{AE}{ED} \cdot \frac{DB}{BC} \cdot \frac{CX}{XA} = 1$$

This implies that $\frac{AE}{ED} = 5$, and since $EF \parallel AC$, $\frac{DF}{DC} = \frac{DE}{DA}$, so $DF = \frac{DC}{6} = \frac{1}{2}$.

4. A positive integer is written on each corner of a square such that numbers on opposite vertices are relatively prime while numbers on adjacent vertices are not relatively prime. What is the smallest possible value of the sum of these 4 numbers?

Proposed by: Eshaan Nichani

Answer: $\boxed{60}$

Two opposite vertices are relatively prime, but they both share a factor with their common neighbor. So that common neighbor must have two prime factors. So each of the 4 numbers has two prime factors, which are not shared with the opposite vertex. Moreover, it suffices to choose the vertices to be the numbers ab, bc, cd, da for some prime numbers a, b, c, d . It's clear that we should choose them to be the smallest primes 2, 3, 5, 7 in some order. The order that minimizes the sum of all of the numbers gives 14, 10, 15, 21 for a sum of 60.

5. Steph Curry is playing the following game and he wins if he has exactly 5 points at some time. Flip a fair coin. If heads, shoot a 3-point shot which is worth 3 points. If tails, shoot a free throw which is worth 1 point. He makes $\frac{1}{2}$ of his 3-point shots and all of his free throws. Find the probability he will win the game. (Note he keeps flipping the coin until he has exactly 5 or goes over 5 points)

Proposed by: Allen Liu

Answer: $\boxed{\frac{140}{243}}$

If he misses the shot, then the state of the game is the same as before he flipped the coin. Since the probability of making a free throw is $\frac{1}{2}$ and the probability of making a 3-point shot is $\frac{1}{4}$. Therefore, given that he earns some point, the probability it is a 3-point shot is $\frac{1}{3}$. The possible ways of earning points are 11111, 113, 131, and 311, which have probabilities $\frac{32}{243}$, $\frac{4}{27}$, $\frac{4}{27}$, and $\frac{4}{27}$, which sum to $\boxed{\frac{140}{243}}$.

6. Let P_1, P_2, \dots, P_6 be points in the complex plane, which are also roots of the equation $x^6 + 6x^3 - 216 = 0$. Given that $P_1P_2P_3P_4P_5P_6$ is a convex hexagon, determine the area of this hexagon.

Proposed by: Eshaan Nichani

Answer: $\boxed{9\sqrt{3}}$

Factor $x^6 + 6x^3 - 216 = (x^3 - 12)(x^3 + 18)$. This gives us 6 points equally spaced in terms of their angles from the origin, alternating in magnitude between $\sqrt[3]{12}$ and $\sqrt[3]{18}$. This means our hexagon is composed of 6 triangles, each with sides of length $\sqrt[3]{12}$ and $\sqrt[3]{18}$ and with a 60 degree angle in between them. This yields the area of each triangle as $\frac{3\sqrt{3}}{2}$, so the total area of the hexagon is $9\sqrt{3}$.

7. Seven lattice points form a convex heptagon with all sides having distinct lengths. Find the minimum possible value of the sum of the squares of the sides of the heptagon.

Proposed by: Allen Liu

Answer: $\boxed{42}$

Consider the vectors corresponding to the sides of the heptagon, and call them $[x_i, y_i]$ for i between 1 and 7. Then since $\sum x_i = \sum y_i = 0$, and a^2 has the same parity as a , we have that $\sum x_i^2 + y_i^2$ must be an even number. A side length of a lattice valued polygon must be expressible as $\sqrt{a^2 + b^2}$, so the smallest possible values are $\sqrt{1}, \sqrt{2}, \sqrt{4}, \sqrt{5}, \sqrt{8}, \sqrt{9}, \sqrt{10}$. However, using the seven smallest lengths violates the parity constraint. If we try $\sqrt{13}$, we indeed can get a heptagon with lengths $\sqrt{1}, \sqrt{2}, \sqrt{4}, \sqrt{5}, \sqrt{8}, \sqrt{9}, \sqrt{13}$. One example is the heptagon $(0, 0), (3, 0), (5, 1), (6, 2), (3, 4), (2, 4), (0, 2)$, and its sum of squares of side lengths is $1 + 2 + 4 + 5 + 8 + 9 + 13 = \boxed{42}$.

8. Let $P_1P_2 \dots P_8$ be a convex octagon. An integer i is chosen uniformly at random from 1 to 7, inclusive. For each vertex of the octagon, the line between that vertex and the vertex i vertices to the right is painted red. What is the expected number times two red lines intersect at a point that is not one of the vertices, given that no three diagonals are concurrent?

Proposed by: Eshaan Nichani

Answer: $\boxed{\frac{54}{7}}$

If $i = 1$ or $i = 7$, there are 0 intersections. If $i = 2$ or $i = 6$ there are 8. If $i = 3$ or $i = 5$ there are 16 intersections. When $i = 4$ there are 6 intersections (since the only lines drawn are the four long diagonals).

Thus the final answer is $\frac{8+16+6+16+8}{7} = \boxed{\frac{54}{7}}$

9. The vertices of a regular nonagon are colored such that 1) adjacent vertices are different colors and 2) if 3 vertices form an equilateral triangle, they are all different colors.

Let m be the minimum number of colors needed for a valid coloring, and n be the total number of colorings using m colors. Determine mn . (Assume each vertex is distinguishable.)

Proposed by: Eshaan Nichani

Answer: $\boxed{54}$

It's clear that m is more than 2 since it's impossible to alternate the color of the vertices without having two of the same color adjacent (since the graph is not bipartite). However, it's possible to use 3 colors. Number the vertices 1 through 9 in order and let the colors be A, B, C . Coloring the vertices in the order $BCBCACABA$ gives a configuration that works, so m is 3. To determine n , we can partition the nonagon into three equilateral triangles. Vertices 1, 4, 7 must be different colors, which we can choose in $3! = 6$ ways. Suppose WLOG that they're A, B, C respectively. Then we look at vertices 2, 5, 8. Vertex 2 can be colored B or C . If 2 is B , then vertex 8 must be A , and vertex 5 must be C . In this case there are two ways to color the remaining vertices 3, 6, 9. Otherwise, if vertex 2 is C , then vertex 5 must be A , and vertex 8 must be B . This gives us only 1 possible coloring for the remaining three vertices. So n is $6(2 + 1) = 18$. So our answer is $mn = 54$.

10. We have 10 points on a line $A_1, A_2 \cdots A_{10}$ in that order. Initially there are n chips on point A_1 . Now we are allowed to perform two types of moves. Take two chips on A_i , remove them and place one chip on A_{i+1} , or take two chips on A_{i+1} , remove them, and place a chip on A_{i+2} and A_i . Find the minimum possible value of n such that it is possible to get a chip on A_{10} through a sequence of moves.

Proposed by: Allen Liu

Answer: 46

We claim that $n = 46$ is the minimum possible value of n . As having extra chips cannot hurt, it is always better to perform the second operation than the first operation, except on point A_1 . Assign the value of a chip on point A_i to be i . Then the total value of the chips initially is n . Furthermore, both types of operations keep the total values of the chips the same, as $2 \cdot 1 = 2$ and $i + (i + 2) = 2 \cdot (i + 1)$.

When $n = 46$, we claim that any sequence of these moves will eventually lead to a chip reaching A_{10} . If, for the sake of contradiction, that there was a way to get stuck with no chip having reached A_{10} , then there could only be chips on A_1 through A_9 , and furthermore at most one chip on each. The total value of these chips is at most 45, which is less than the original value of chips 46.

However, if $n \leq 45$, we claim that it is impossible to get one chip to A_{10} . To get a chip to A_{10} , an operation must have been used on each of A_1 through A_9 at least once. Consider the last time the operation was used on A_k for $2 \leq k \leq 9$. After this operation, there must be a chip on A_{k-1} . Additionally, since no chip is ever moved past A_k again, there is no point to perform any operations on any chips left of A_k , which means that a chip will remain on A_{k-1} until the end. Therefore, if there is a way to get a chip to A_{10} , there must also be a way to get a chip to A_{10} and also A_1 through A_8 , which means that the original value of the chips must have been already $1 + 2 + \cdots + 8 + 10 = 46$.

HMMT Invitational Competition

April 2016

- [5] Theseus starts at the point $(0, 0)$ in the plane. If Theseus is standing at the point (x, y) in the plane, he can step one unit to the north to point $(x, y + 1)$, one unit to the west to point $(x - 1, y)$, one unit to the south to point $(x, y - 1)$, or one unit to the east to point $(x + 1, y)$. After a sequence of more than two such moves, starting with a step one unit to the south (to point $(0, -1)$), Theseus finds himself back at the point $(0, 0)$. He never visited any point other than $(0, 0)$ more than once, and never visited the point $(0, 0)$ except at the start and end of this sequence of moves.

Let X be the number of times that Theseus took a step one unit to the north, and then a step one unit to the west immediately afterward. Let Y be the number of times that Theseus took a step one unit to the west, and then a step one unit to the north immediately afterward. Prove that $|X - Y| = 1$.

Proposed by: Mitchell Lee

The path Theseus traces out is a closed, non-self-intersecting path in the plane. Since each move is along a line segment, the path forms the boundary of a polygon in the plane. WLOG this polygon is traversed counterclockwise (i.e. with the interior of the polygon always on Theseus's left); the argument in the clockwise case is analogous, with a sign flip.

Let $i = 0$ correspond to the direction east, $i = 1$ correspond to north, and so forth. For $0 \leq i \leq 3$ and $j = i \pm 1$ let X_{ij} be the number of times Theseus switches from going in direction i to going in direction j (where indices are taken mod 4). So, $X = X_{1,2}, Y = X_{2,1}$. We have the following relations among the X_{ij} :

The sum $n = \sum X_{i,j}$ is the number of vertices of the polygon. The sum of the interior angles of the polygon is thus $\pi(n-2)$. Each counterclockwise turn contributes $\frac{\pi}{2}$, and each clockwise turn contributes $\frac{3\pi}{2}$ to this sum. The number of the former is counted by $\sum X_{i,i+1}$, and the latter by $\sum X_{i+1,i}$. So, $\pi(n-2) = \frac{\pi}{2} \sum X_{i,i+1} + \frac{3\pi}{2} \sum X_{i+1,i} = \pi n + \frac{\pi}{2} \sum (X_{i+1,i} - X_{i,i+1})$, which gives $\sum (X_{i,i+1} - X_{i+1,i}) = 4$.

Finally, observe by starting from the middle of any edge that since we start and end in the same direction, the number of times we turn into direction i must equal the number of times we turn out of it. So $X_{i,i+1} + X_{i,i-1} = X_{i+1,i} + X_{i-1,i}$, which rearranges to give $X_{i,i+1} - X_{i+1,i} = X_{i-1,i} - X_{i-1,i}$. So $4 = \sum (X_{i,i+1} - X_{i+1,i}) = 4(X_{1,2} - X_{2,1}) = 4(X - Y)$.

So $X - Y = 1$ as desired.

- [7] Let ABC be an acute triangle with circumcenter O , orthocenter H , and circumcircle Ω . Let M be the midpoint of AH and N the midpoint of BH . Assume the points M, N, O, H are distinct and lie on a circle ω . Prove that the circles ω and Ω are internally tangent to each other.

Proposed by: Evan Chen

Let R be the circumradius of $\triangle ABC$. Recall that the circumcircle of $\triangle HBC$ has radius equal to R . By a homothety at H with factor 2 at H , it follows that $\triangle HMN$ has circumradius $R/2$. Thus ω has circumradius $R/2$, and since O lies on ω the conclusion follows.

Remark. This problem is a slight modification of a proposal by *Dhroova Aiylam*.

- [8] Denote by \mathbb{N} the positive integers. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that, for any $w, x, y, z \in \mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

Show that $f(n!) \geq n!$ for every positive integer n .

Proposed by: Pakawut Jiradilok

If $f(z_1) = f(z_2)$, then plugging in (w, x, y, z_1) and (w, x, y, z_2) yields $z_1 = z_2$. Thus, f is injective. Substitution of (z, w) with $(1, f(f(1)))$ in the main equation yields

$$f\left(f(f(1))xf(yf(1))\right) = f(xf(y))$$

Because f is injective, we have

$$f(yf(1)) \cdot f(f(1)) = f(y), \forall y \in \mathbb{N} \quad (*)$$

We can easily show by induction that

$$f(y) = f(yf(1)^n)f(f(1))^n, \forall n \in \mathbb{N}$$

Thus, $f(f(1))^n | f(y), \forall n \in \mathbb{N}$ which implies that $f(f(1)) = 1$. Using this equality in $(*)$ yields $f(1) = 1$. Substitution of (y, z) with $(1, 1)$ in the main equation yields $f(xy) = f(x)f(y)$.

So,

$$f(n!) = f\left(\prod_{i=1}^n i\right) = \prod_{i=1}^n f(i).$$

By injectivity, each $f(i)$ in this product is a distinct positive integer, so their product is at least $\prod_{i=1}^n i = n!$, as desired.

Remark 1. The equation condition of f in the problem is actually equivalent to the following two conditions combined:

$$\begin{aligned} \text{(C1)} \quad & f(xy) = f(x)f(y), \forall x, y \in \mathbb{N}, \text{ and} \\ \text{(C2)} \quad & f(f(f(x))) = x, \forall x \in \mathbb{N} \end{aligned}$$

4. [10] Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

Proposed by: Victor Wang

Proof. Let n be the (odd) degree of P . Suppose, for contradiction, that P has no integer roots, and let $Q(x) = xP(x) = a_n x^{n+1} + a_{n-1} x^n + \dots + a_0 x^1$. WLOG $a_n > 0$, so there exists $M > 0$ such that $Q(M) < Q(M+1) < \dots$ and $Q(-M) < Q(-M-1) < \dots$ (as $n+1$ is even).

There exist finitely many solutions (t, y) for any fixed t , so infinitely many solutions (x, y) with $|x|, |y| \geq M$. By definition of M , there WLOG exist infinitely many solutions with $|x| \geq M$ and $y \geq -x > 0$ (otherwise replace $P(t)$ with $P(-t)$).

Equivalently, there exist infinitely many (t, k) with $t \geq M$ and $k \geq 0$ such that

$$0 = Q(t+k) - Q(-t) = a_n[(t+k)^{n+1} - (-t)^{n+1}] + \dots + a_0[(t+k)^1 - (-t)^1].$$

If $k \neq 0$, then $Q(x+k) - Q(-x)$ has constant term $Q(k) - Q(0) = kP(k) \neq 0$, and if $k = 0$, then $Q(x+k) - Q(-x) = 2(a_0 x^1 + a_2 x^3 + \dots + a_{n-1} x^n)$ has x^1 coefficient nonzero (otherwise $P(0) = 0$), so no fixed k yields infinitely many solutions.

Hence for any $N \geq 0$, there exist $k = k_N \geq N$ and $t = t_N \geq M$ such that $0 = Q(t+k) - Q(-t)$. But then the triangle inequality yields

$$\underbrace{a_n[(t+k)^{n+1} - (-t)^{n+1}]}_{=\sum_{i=0}^{n-1} -a_i[(t+k)^{i+1} - (-t)^i]} \leq 2(t+k)^n(|a_{n-1}| + \dots + |a_0|). \quad (1)$$

But $(t+k)^{n+1} - t^{n+1} \geq (t+k)^{n+1} - t \cdot (t+k)^n = k \cdot (t+k)^n$,¹ so $a_n k_N \leq 2(|a_{n-1}| + \dots + |a_0|)$ for all N . Since $k_N \geq N$, this gives a contradiction for sufficiently large N . \square

¹However, the mean value theorem estimate $(t+k)^{n+1} - t^{n+1} \geq k \cdot (n+1)t^n$ will not suffice a priori, if k happens to be much larger than t . Of course, it is easy to prove that k cannot be too much larger than t (even before finishing the problem); the execution would just take slightly longer.

Remark 2. Another way to finish after the triangle inequality estimate (1) is to observe that $(t+k)^{n+1} - (-t)^{n+1} = (t+k)^{n+1} - t^{n+1} \geq t^n \cdot k + k^{n+1}$ (by binomial expansion), so $\frac{k_N(t_N^N + k_N^N)}{(t_N + k_N)^n}$ (which by Holder's inequality is at least $\frac{k_N}{2^{n-1}} \geq \frac{N}{2^{n-1}}$) is bounded above (independently of N), contradiction. (Here we use Holder's inequality to deduce $(1+1)^{n-1}(a^n + b^n) \geq (a+b)^n$ for positive reals a, b .)

Remark 3. The statement is vacuous for even-degree polynomials. The case of degree 3 polynomials was given as ISL 2002 A3.

5. [12] Let $S = \{a_1, \dots, a_n\}$ be a finite set of positive integers of size $n \geq 1$, and let T be the set of all positive integers that can be expressed as sums of perfect powers (including 1) of distinct numbers in S , meaning

$$T = \left\{ \sum_{i=1}^n a_i^{e_i} \mid e_1, e_2, \dots, e_n \geq 0 \right\}.$$

Show that there is a positive integer N (only depending on n) such that T contains no arithmetic progression of length N .

Proposed by: Yang Liu

Answer: N/A

In general we can assume that each $a_i > 1$, since replacing $a_i = 1$ by some large integer a creates a set T containing the original T as a subset (by setting $e_i = 0$).

We proceed by induction on n . For the base case $n = 1$, an arithmetic progression of length at least 3 would give $a_1^{e_1} + a_1^{e_3} = 2a_1^{e_2}$ where $e_3 > e_2 > e_1$. But $a_1^{e_3} \geq 2a_1^{e_2}$, so this is impossible. Thus our result holds for $n = 1$.

Assume the result is true for some $n - 1$, and let A_{n-1} be a number such that the longest progression when $|S| = n - 1$ has length less than A_{n-1} . Let M be a large integer that we will choose later. Take an arithmetic progression of length $2M + 1$, calling the terms $b_1, b_2, \dots, b_{2M+1}$. Note that $b_{2M+1} \leq 2b_{M+1}$, since b is a sequence of positive integers. For each term from b_{M+1} to b_{2M+1} assign to it the maximum power that is part of the sum. Call this value $c(b_i)$. More explicitly, if $b_i = \sum_{j=1}^n a_j^{e_j}$, then $c(b_i) = \max(a_1^{e_1}, a_2^{e_2}, \dots, a_n^{e_n})$.

Since $\frac{b_{M+1}}{n} \leq c(b_i) \leq b_{2M+1} \leq 2b_{M+1}$ for $M + 1 \leq i \leq 2M + 1$, there are at most $\sum_{j=1}^n (\log_{a_i} 2n + 1) \leq n(\log_2 2n + 1)$ different values of $c(b_i)$. By Van der Waerden's Theorem, there exists a value of M such that coloring an arithmetic progression of length M with $n(\log_2 2n + 1)$ colors yields a monochromatic arithmetic progression of length A_{n-1} . In particular, we can take $M = W(A_{n-1}, n(\log_2 2n + 1))$, where $W(n, k)$ denotes the Van der Waerden number. So, we color b_{M+1}, \dots, b_{2M+1} by their $c(b_i)$. Subtracting the common perfect power from each term of the monochromatic arithmetic progression obtained gives an arithmetic progression of A_{n-1} integers expressible as the sum of perfect powers of distinct numbers in $S \setminus a_j$. By the inductive hypothesis, this is a contradiction. So no arithmetic progression of length $2M + 1$ can be contained in T , and we can take $A_n = 2M + 1$.

By induction, we thus have such an A_n for all n .

HMMT February 2015
Saturday 21 February 2015
Algebra

1. Let Q be a polynomial

$$Q(x) = a_0 + a_1x + \cdots + a_nx^n,$$

where a_0, \dots, a_n are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

Answer: 254 Since each a_i is a nonnegative integer, $152 = Q(5) \equiv a_0 \pmod{5}$ and $Q(1) = 4 \implies a_i \leq 4$ for each i . Thus, $a_0 = 2$. Also, since $5^4 > 152 = Q(5)$, $a_4, a_5, \dots, a_n = 0$.

Now we simply need to solve the system of equations

$$\begin{aligned} 5a_1 + 5^2a_2^2 + 5^3a_3^3 &= 150 \\ a_1 + a_2 + a_3 &= 2 \end{aligned}$$

to get

$$a_2 + 6a_3 = 7.$$

Since a_2 and a_3 are nonnegative integers, $a_2 = 1$, $a_3 = 1$, and $a_1 = 0$. Therefore, $Q(6) = 6^3 + 6^2 + 2 = 254$.

2. The fraction $\frac{1}{2015}$ has a unique “(restricted) partial fraction decomposition” of the form

$$\frac{1}{2015} = \frac{a}{5} + \frac{b}{13} + \frac{c}{31},$$

where a, b, c are integers with $0 \leq a < 5$ and $0 \leq b < 13$. Find $a + b$.

Answer: 14 This is equivalent to $1 = 13 \cdot 31a + 5 \cdot 31b + 5 \cdot 13c$.¹ Taking modulo 5 gives $1 \equiv 3 \cdot 1a \pmod{5}$, so $a \equiv 2 \pmod{5}$. Taking modulo 13 gives $1 \equiv 5 \cdot 5b = 25b \equiv -b \pmod{13}$, so $b \equiv 12 \pmod{13}$. The size constraints on a, b give $a = 2$, $b = 12$, so $a + b = 14$.

Remark. This problem illustrates the analogy between polynomials and integers, with prime powers (here $5^1, 13^1, 31^1$) taking the role of powers of irreducible polynomials (such as $(x-1)^1$ or $(x^2+1)^3$, when working with polynomials over the real numbers).

Remark. The “partial fraction decomposition” needs to be restricted since it’s only unique “modulo 1”. Abstractly, the abelian group (or \mathbb{Z} -module) \mathbb{Q}/\mathbb{Z} has a “prime power direct sum decomposition” (more or less equivalent to Bezout’s identity, or the Chinese remainder theorem), but \mathbb{Q} itself (as an abelian group under addition) does not.

You may wonder whether there’s a similar “prime power decomposition” of \mathbb{Q} that accounts not just for addition, but also for multiplication (i.e. the full ring structure of the rationals). In some sense, the “adeles/ideles” serve this purpose, but it’s not as clean as the partial fraction decomposition (for additive structure alone)—in fact, the subtlety of adeles/ideles reflects much of the difficulty in number theory!

3. Let p be a real number and $c \neq 0$ an integer such that

$$c - 0.1 < x^p \left(\frac{1 - (1+x)^{10}}{1 + (1+x)^{10}} \right) < c + 0.1$$

for all (positive) real numbers x with $0 < x < 10^{-100}$. (The exact value 10^{-100} is not important. You could replace it with any “sufficiently small number”.)

Find the ordered pair (p, c) .

¹Note that this does actually have integer solutions by Bezout’s identity, as $\gcd(13 \cdot 31, 5 \cdot 31, 5 \cdot 13) = 1$.

Answer: $\boxed{(-1, -5)}$ This is essentially a problem about limits, but phrased concretely in terms of “small numbers” (like 0.1 and 10^{-100}).

We are essentially studying the rational function $f(x) := \frac{1-(1+x)^{10}}{1+(1+x)^{10}} = \frac{-10x+O(x^2)}{2+O(x)}$, where the “big-O” notation simply make precise the notion of “error terms”.²

Intuitively, $f(x) \approx \frac{-10x}{2} = -5x$ for “small nonzero x ”. (We could easily make this more precise if we wanted to, by specifying the error terms more carefully, but it’s not so important.) So $g(x) := x^p f(x) \approx -5x^{p+1}$ for “small nonzero x ”.

- If $p + 1 > 0$, g will approach 0 (“get very small”) as x approaches 0 (often denoted $x \rightarrow 0$), so there’s no way it can stay above the lower bound $c - 0.1$ for all small nonzero x .
- If $p + 1 < 0$, g will approach $-\infty$ (“get very large in the negative direction”) as $x \rightarrow 0$, so there’s no way it can stay below the upper bound $c + 0.1$ for all small nonzero x .
- If $p + 1 = 0$, $g \approx -5$ becomes approximately constant as $x \rightarrow 0$. Since c is an **integer**, we must have $c = -5$ (as -5 is the only integer within 0.1 of -5).

Remark. Why does $(p, c) = (-1, -5)$ actually satisfy the inequality? This is where the 10^{-100} kicks in: for such small values of x , the “error” $|g(x) - (-5)|$ of the approximation $g \approx -5$ does actually lie within the permitted threshold of ± 0.1 . (You can easily work out the details yourself, if you’re interested. It’s something you might want to work out once or twice in your life, but rational functions are “well-behaved” enough that we can usually rely on our intuition in these kinds of scenarios.)

4. Compute the number of sequences of integers (a_1, \dots, a_{200}) such that the following conditions hold.

- $0 \leq a_1 < a_2 < \dots < a_{200} \leq 202$.
- There exists a positive integer N with the following property: for every index $i \in \{1, \dots, 200\}$ there exists an index $j \in \{1, \dots, 200\}$ such that $a_i + a_j - N$ is divisible by 203.

Answer: $\boxed{20503}$ Let $m := 203$ be an integer not divisible by 3. We’ll show the answer for general such m is $m \lceil \frac{m-1}{2} \rceil$.

Let x, y, z be the three excluded residues. Then N works if and only if $\{x, y, z\} \equiv \{N-x, N-y, N-z\} \pmod{m}$. Since $x, y, z \pmod{m}$ has opposite orientation as $N-x, N-y, N-z \pmod{m}$, this is equivalent to x, y, z forming an arithmetic progression (in some order) modulo m centered at one of x, y, z (or algebraically, one of $N \equiv 2x \equiv y + z$, $N \equiv 2y \equiv z + x$, $N \equiv 2z \equiv x + y$ holds, respectively).

Since $3 \nmid m$, it’s impossible for more than one of these congruences to hold (or else x, y, z would have to be equally spaced modulo m , i.e. $x - y \equiv y - z \equiv z - x$). So the number of distinct 3-sets corresponding to arithmetic progressions is $m \lceil \frac{m-1}{2} \rceil$ (choose a center and a difference, noting that $\pm d$ give the same arithmetic progression). Since our specific $m = 203$ is odd this gives $m \frac{m-1}{2} = 203 \cdot 101 = 20503$.

Remark. This problem is a discrete analog of certain so-called Frieze patterns. (See also Chapter 6, Exercise 5.8 of Artin’s *Algebra* textbook.)

5. Let a, b, c be positive real numbers such that $a+b+c = 10$ and $ab+bc+ca = 25$. Let $m = \min\{ab, bc, ca\}$. Find the largest possible value of m .

Answer: $\boxed{\frac{25}{9}}$ Without loss of generality, we assume that $c \geq b \geq a$. We see that $3c \geq a+b+c = 10$. Therefore, $c \geq \frac{10}{3}$.

²For instance, the $O(x^2)$ refers to a function bounded by $C|x|^2$ for some positive constant C , whenever x is close enough to 0 (and as the 10^{-100} suggests, that’s all we care about).

Since

$$\begin{aligned}
 0 &\leq (a - b)^2 \\
 &= (a + b)^2 - 4ab \\
 &= (10 - c)^2 - 4(25 - c(a + b)) \\
 &= (10 - c)^2 - 4(25 - c(10 - c)) \\
 &= c(20 - 3c),
 \end{aligned}$$

we obtain $c \leq \frac{20}{3}$. Consider $m = \min\{ab, bc, ca\} = ab$, as $bc \geq ca \geq ab$. We compute $ab = 25 - c(a + b) = 25 - c(10 - c) = (c - 5)^2$. Since $\frac{10}{3} \leq c \leq \frac{20}{3}$, we get that $ab \leq \frac{25}{9}$. Therefore, $m \leq \frac{25}{9}$ in all cases and the equality can be obtained when $(a, b, c) = (\frac{5}{3}, \frac{5}{3}, \frac{20}{3})$.

6. Let a, b, c, d, e be nonnegative integers such that $625a + 250b + 100c + 40d + 16e = 15^3$. What is the maximum possible value of $a + b + c + d + e$?

Answer: 153 The intuition is that as much should be in e as possible. But divisibility obstructions like $16 \nmid 15^3$ are in our way. However, the way the coefficients $5^4 > 5^3 \cdot 2 > \dots$ are set up, we can at least easily avoid having a, b, c, d too large (specifically, ≥ 2). This is formalized below.

First, we observe that $(a_1, a_2, a_3, a_4, a_5) = (5, 1, 0, 0, 0)$ is a solution. Then given a solution, replacing (a_i, a_{i+1}) with $(a_i - 2, a_{i+1} + 5)$, where $1 \leq i \leq 4$, also yields a solution. Given a solution, it turns out all solutions can be achieved by some combination of these swaps (or inverses of these swaps).

Thus, to optimize the sum, we want $(a, b, c, d) \in \{0, 1\}^4$, since in this situation, there would be no way to make swaps to increase the sum. So the sequence of swaps looks like $(5, 1, 0, 0, 0) \rightarrow (1, 11, 0, 0, 0) \rightarrow (1, 1, 25, 0, 0) \rightarrow (1, 1, 1, 60, 0) \rightarrow (1, 1, 1, 0, 150)$, yielding a sum of $1 + 1 + 1 + 0 + 150 = 153$.

Why is this optimal? Suppose (a, b, c, d, e) maximizes $a + b + c + d + e$. Then $a, b, c, d \leq 1$, or else we could use a replacement $(a_i, a_{i+1}) \rightarrow (a_i - 2, a_{i+1} + 5)$ to strictly increase the sum. But modulo 2 forces a odd, so $a = 1$. Subtracting off and continuing in this manner³ shows that we must have $b = 1$, then $c = 1$, then $d = 0$, and finally $e = 150$.

Remark. The answer is coincidentally obtained by dropping the exponent of 15^3 into the one's place.

7. Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

- $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;
- there exist real numbers a, b, c such that

$$an^2 + bn + c = \cos(a_n)$$

for all $n \in \{1, 2, 3, 4\}$.

Compute the maximum possible value of

$$\cos(a_1) - \cos(a_4)$$

over all such sequences (a_1, a_2, a_3, a_4) .

Answer: $-9 + 3\sqrt{13}$ Let $f(n) = \cos a_n$ and $m = 1$. The second (“quadratic interpolation”) condition on $f(m), f(m + 1), f(m + 2), f(m + 3)$ is equivalent to having a vanishing third finite difference

$$f(m + 3) - 3f(m + 2) + 3f(m + 1) - f(m) = 0.$$

³This is analogous to the “number theoretic” proof of the uniqueness of the base 2 expansion of a nonnegative integer.

This is equivalent to

$$\begin{aligned} f(m+3) - f(m) &= 3[f(m+2) - f(m+1)] \\ \iff \cos(a_{m+3}) - \cos(a_m) &= 3(\cos(a_{m+2}) - \cos(a_{m+1})) \\ &= -6 \sin\left(\frac{a_{m+2} + a_{m+1}}{2}\right) \sin\left(\frac{a_{m+2} - a_{m+1}}{2}\right) \\ &= -6 \sin\left(\frac{a_{m+3}}{2}\right) \sin\left(\frac{a_m}{2}\right). \end{aligned}$$

Set $x = \sin\left(\frac{a_{m+3}}{2}\right)$ and $y = \sin\left(\frac{a_m}{2}\right)$. Then the above rearranges to

$$(1 - 2x^2) - (1 - 2y^2) = -6xy \iff x^2 - y^2 = 3xy.$$

Solving gives $y = x \frac{-3 \pm \sqrt{13}}{2}$. The expression we are trying to maximize is $2(x^2 - y^2) = 6xy$, so we want x, y to have the same sign; thus $y = x \frac{-3 + \sqrt{13}}{2}$.

Then $|y| \leq |x|$, so since $|x|, |y| \leq 1$, to maximize $6xy$ we can simply set $x = 1$, for a maximal value of $6 \cdot \frac{-3 + \sqrt{13}}{2} = -9 + 3\sqrt{13}$.

8. Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \dots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a “quadratic residue modulo $x^2 + 1$ and 35 ”, i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following **equivalent** conditions holds:

- there exist polynomials P, Q with integer coefficients such that $f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x)$;
- or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax + b)$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with (integer) coefficients all divisible by 35 .

Answer: 225 By the Chinese remainder theorem, we want the product of the answers modulo 5 and modulo 7 (i.e. when 35 is replaced by 5 and 7, respectively).

First we do the **modulo 7 case**. Since $x^2 + 1$ is irreducible modulo 7 (or more conceptually, in $\mathbb{F}_7[x]$), exactly half of the nonzero residues modulo $x^2 + 1$ and 7 (or just modulo $x^2 + \bar{1}$ if we’re working in $\mathbb{F}_7[x]$) are quadratic residues, i.e. our answer is $1 + \frac{7^2 - 1}{2} = 25$ (where we add back one for the zero polynomial).

Now we do the **modulo 5 case**. Since $x^2 + 1$ factors as $(x + 2)(x - 2)$ modulo 5 (or more conceptually, in $\mathbb{F}_5[x]$), by the **polynomial** Chinese remainder theorem modulo $x^2 + \bar{1}$ (working in $\mathbb{F}_5[x]$), we want the product of the number of **polynomial** quadratic residues modulo $x \pm \bar{2}$. By centering/evaluating polynomials at $\mp \bar{2}$ accordingly, the polynomial squares modulo these linear polynomials are just those reducing to **integer** squares modulo 5.⁴ So we have an answer of $(1 + \frac{5-1}{2})^2 = 9$ in this case.

Our final answer is thus $25 \cdot 9 = 225$.

Remark. This problem illustrates the analogy between integers and polynomials (specifically here, polynomials over the *finite field* of integers modulo 5 or 7), with $x^2 + 1 \pmod{7}$ or $x \pm 2 \pmod{5}$ taking the role of a prime number. Indeed, just as in the integer case, we expect exactly **half** of the (coprime) residues to be (coprime, esp. nonzero) quadratic residues.

9. Let $N = 30^{2015}$. Find the number of ordered 4-tuples of integers $(A, B, C, D) \in \{1, 2, \dots, N\}^4$ (not necessarily distinct) such that for every integer n , $An^3 + Bn^2 + 2Cn + D$ is divisible by N .

Answer: 24 Note that $n^0 = \binom{n}{0}$, $n^1 = \binom{n}{1}$, $n^2 = 2\binom{n}{2} + \binom{n}{1}$, $n^3 = 6\binom{n}{3} + 6\binom{n}{2} + \binom{n}{1}$ (generally see http://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind). Thus the polynomial rewrites as

$$6A \binom{n}{3} + (6A + 2B) \binom{n}{2} + (A + B + 2C) \binom{n}{1} + D \binom{n}{0},$$

⁴This is more explicit than necessary. By the same reasoning as in the previous paragraph, we can abstractly count $1 + \frac{5-1}{2}$ quadratic residues modulo $x \pm \bar{2}$ (irreducible polynomials in $\mathbb{F}_5[x]$) each (and then multiply/square to get the answer for $x^2 + 1$).

which by the classification of integer-valued polynomials is divisible by N always if and only if $6A, 6A + 2B, A + B + 2C, D$ are always divisible by N .

We can eliminate B and (trivially) D from the system: it's equivalent to the system $6A \equiv 0 \pmod{N}$, $4A - 4C \equiv 0 \pmod{N}$, $B \equiv -A - 2C \pmod{N}$, $D \equiv 0 \pmod{N}$. So we want 1^2 times the number of (A, C) with $A \equiv 0 \pmod{N/6}$, $C \equiv A \pmod{N/4}$. So there are $N/(N/6) = 6$ choices for A , and then given such a choice of A there are $N/(N/4) = 4$ choices for C . So we have $6 \cdot 4 \cdot 1^2 = 24$ solutions total.

10. Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

Answer: $\boxed{(5, -3, 2, 3)}$ We first give two systematic solutions using standard manipulations and divisibility conditions (with some casework), and then a third solution using quaternionic number theory (not very practical, so mostly for your cultural benefit).

Solution 1. Subtract the second equation from the third to get $a(c - b + 1) = 30$. Add the second and third to get $2a(b + c) = -4 - 2d$. Substitute into the fourth to get

$$2a(2ad - 31) = -4 - 2d \iff a(31 - 2ad) = 2 + d \iff d = \frac{31a - 2}{2a^2 + 1},$$

which in particular gives $a \not\equiv 1 \pmod{3}$. Then plugging in a factor of 30 for a gives us the system of equations $b + c = 2ad - 31$ and $c - b + 1 = 30/a$ in b, c . Here, observe that $b + c$ is odd, so $c - b + 1$ is even. Thus a must be odd (and from earlier $a \not\equiv 1 \pmod{3}$), so $a \in \{-1, \pm 3, 5, \pm 15\}$. Manually checking these, we see that the only possibilities we need to check are $(a, d) = (5, 3), (-1, -11), (-3, -5)$, corresponding to $(b, c) = (-3, 2), (11, -20), (5, -6)$. Then check the three candidates against first condition $a^2 - b^2 - c^2 - d^2 = c - b - 2$ to find our only solution $(a, b, c, d) = (5, -3, 2, 3)$.

Solution 2. Here's an alternative casework solution. From $2ad = b + c + 31$, we have that $b + c$ is odd. So, b and c has different parity. Thus, $b^2 + c^2 \equiv 1 \pmod{4}$. Plugging this into the first equation, we get that a and d also have the same parity.

So, $a^2 - b^2 - c^2 - d^2 \equiv -1 \pmod{4}$. Thus, $c - b - 2 \equiv -1 \pmod{4}$. So, $c \equiv b + 1 \pmod{4}$.

From taking modulo a in the second and third equation, we have $a \mid d + 32$ and $a \mid 28 - d$. So, $a \mid 60$.

Now, if a is even, let $a = 2k$ and $d = 2m$. Plugging this in the second and third equation, we get $2kc = 14 - k - m$ and $2kb = k - m - 16$. So, $k(c - b) = 15 - k$.

We can see that $k \neq 0$. Therefore, $c - b = \frac{15-k}{k} = \frac{15}{k} - 1$.

But $c - b \equiv 1 \pmod{4}$. So, $\frac{15}{k} - 1 \equiv 1 \pmod{4}$, or $\frac{15}{k} \equiv 2 \pmod{4}$ which leads to a contradiction.

So, a is odd. And we have $a \mid 60$. So, $a \mid 15$. This gives us 8 easy possibilities to check...

Solution 3. The left hand sides clue us in to the fact that this problem is secretly about quaternions. Indeed, we see that letting $z = a + bi + cj + dk$ gives

$$(z - i + j)z = -2 - 32i + 28j + 31k.$$

Taking norms gives $N(z - i + j)N(z) = 2^2 + 32^2 + 28^2 + 31^2 = 2773 = 47 \cdot 59$. By the triangle inequality, $N(z), N(z - i + j)$ aren't too far apart, so they must be 47, 59 (in some order).

Thus $z, z - i + j$ are Hurwitz primes.⁵ We rely on the following foundational lemma in quaternion number theory:

⁵For the purposes of quaternion number theory, it's simpler to work in the the Hurwitz quaternions $\mathbb{H} = \langle i, j, k, \frac{1+i+j+k}{2} \rangle_{\mathbb{Z}}$, which has a left- (or right-) division algorithm, left- (resp. right-) Euclidean algorithm, is a left- (resp. right-) principal ideal domain, etc. There's no corresponding division algorithms when we're working with the Lipschitz quaternions, i.e. those with integer coordinates.

Lemma. Let $p \in \mathbb{Z}$ be an integer prime, and A a Hurwitz quaternion. If $p \mid N(A)$, then the $\mathbb{H}A + \mathbb{H}p$ (a left ideal, hence principal) has all element norms divisible by p , hence is nontrivial. (So it's either $\mathbb{H}p$ or of the form $\mathbb{H}P$ for some Hurwitz prime P .)

In our case, it will suffice to apply the lemma for $A = -2 - 32i + 28j + 31k$ at primes $p = 47$ and $q = 59$ to get factorizations (unique up to suitable left/right unit multiplication) $A = QP$ and $A = P'Q'$ (respectively), with P, P' Hurwitz primes of norm p , and Q, Q' Hurwitz primes of norm q . Indeed, these factorizations come from $\mathbb{H}A + \mathbb{H}p = \mathbb{H}P$ and $\mathbb{H}A + \mathbb{H}q = \mathbb{H}Q'$.

We compute by the Euclidean algorithm:

$$\begin{aligned}
\mathbb{H}A + \mathbb{H}(47) &= \mathbb{H}(-2 - 32i + 28j + 31k) + \mathbb{H}(47) \\
&= \mathbb{H}(-2 + 15i - 19j - 16k) + \mathbb{H}(47) \\
&= [\mathbb{H}(47 \cdot 18) + \mathbb{H}(47)(-2 - 15i + 19j + 16k)] \frac{-2 + 15i - 19j - 16k}{47 \cdot 18} \\
&= [\mathbb{H}18 + \mathbb{H}(-2 + 3i + j - 2k)] \frac{-2 + 15i - 19j - 16k}{18} \\
&= \mathbb{H}(-2 + 3i + j - 2k) \frac{-2 + 15i - 19j - 16k}{18} \\
&= \mathbb{H} \frac{-54 - 90i + 54j - 36k}{18} \\
&= \mathbb{H}(-3 - 5i + 3j - 2k).
\end{aligned}$$

Thus⁶ there's a unit⁷ ϵ such that $P = \epsilon(-3 - 5i + 3j - 2k)$.

Similarly, to get P' , we compute

$$\begin{aligned}
A\mathbb{H} + 47\mathbb{H} &= (-2 - 32i + 28j + 31k)\mathbb{H} + 47\mathbb{H} \\
&= (-2 + 15i - 19j - 16k)\mathbb{H} + 47\mathbb{H} \\
&= \frac{-2 + 15i - 19j - 16k}{47 \cdot 18} [(47 \cdot 18)\mathbb{H} + 47(-2 - 15i + 19j + 16k)\mathbb{H}] \\
&= \frac{-2 + 15i - 19j - 16k}{18} [18\mathbb{H} + (-2 + 3i + j - 2k)\mathbb{H}] \\
&= \frac{-2 + 15i - 19j - 16k}{18} (-2 + 3i + j - 2k)\mathbb{H} \\
&= \frac{-54 + 18i + 18j + 108k}{18} \mathbb{H} \\
&= (-3 + i + j + 6k)\mathbb{H},
\end{aligned}$$

so there's a unit ϵ' with $P' = (-3 + i + j + 6k)\epsilon'$.

Finally, we have either $z = \epsilon(-3 - 5i + 3j - 2k)$ for some ϵ , or $z - i + j = (-3 + i + j + 6k)\epsilon'$ for some ϵ' . Checking the 24 + 24 cases (many of which don't have integer coefficients, and can be ruled out immediately) gives $z = iP = 5 - 3i + 2j + 3k$ as the only possibility.

Remark. We have presented the most conceptual proof possible. It's also possible to directly compute based on the norms only, and do some casework. For example, since $47 \equiv 3 \pmod{4}$, it's easy to check the only ways to write it as a sum of four squares are $(\pm 5)^2 + (\pm 3)^2 + (\pm 3)^2 + (\pm 2)^2$ and $(\pm 3)^2 + (\pm 1)^2 + (\pm 1)^2 + (\pm 6)^2$.

Remark. For a systematic treatment of quaternions (including the number theory used above), one good resource is *On Quaternions and Octonions: Their Geometry, Arithmetic, and Symmetry* by John H. Conway and Derek A. Smith. A more focused treatment is the expository paper *Factorization of Hurwitz Quaternions* by Boyd Coan and Cherng-tiao Perng.

For an example of interesting research in this rather exotic area, see the *Metacommutation of Hurwitz primes* paper by Henry Cohn and Abhinav Kumar.

⁶Hidden computations: we've used $47 \cdot 18 = 846 = 2^2 + 15^2 + 19^2 + 16^2$, and $18 = N(-2 + 3i + j - 2k)$.

⁷i.e. one of $\pm 1, \pm i, \pm j, \pm k, \frac{\pm 1 \pm i \pm j \pm k}{2}$

HMMT February 2015
Saturday 21 February 2015

Combinatorics

1. Evan's analog clock displays the time 12:13; the number of seconds is not shown. After 10 seconds elapse, it is still 12:13. What is the expected number of seconds until 12:14?

Answer: $\boxed{25}$ At first, the time is uniformly distributed between 12:13:00 and 12:13:50. After 10 seconds, the time is uniformly distributed between 12:13:10 and 12:14:00. Thus, it takes on average 25 seconds to reach 12:14(:00).

2. Victor has a drawer with 6 socks of 3 different types: 2 complex socks, 2 synthetic socks, and 2 trigonometric socks. He repeatedly draws 2 socks at a time from the drawer at random, and stops if the socks are of the same type. However, Victor is "synthetic-complex type-blind", so he also stops if he sees a synthetic and a complex sock.

What is the probability that Victor stops with 2 socks of the same type? Assume Victor returns both socks to the drawer after each step.

Answer: $\boxed{\frac{3}{7}}$ Let the socks be $C_1, C_2, S_1, S_2, T_1, T_2$, where C, S and T stand for complex, synthetic and trigonometric respectively. The possible stopping points consist of three pairs of socks of the same type plus four different complex-synthetic (C - S) pairs, for a total of 7. So the answer is $\frac{3}{7}$.

3. Starting with the number 0, Casey performs an infinite sequence of moves as follows: he chooses a number from $\{1, 2\}$ at random (each with probability $\frac{1}{2}$) and adds it to the current number. Let p_m be the probability that Casey ever reaches the number m . Find $p_{20} - p_{15}$.

Answer: $\boxed{\frac{11}{2^{20}}}$ We note that the only way n does not appear in the sequence is if $n-1$ and then $n+1$ appears. Hence, we have $p_0 = 1$, and $p_n = 1 - \frac{1}{2}p_{n-1}$ for $n > 0$. This gives $p_n - \frac{2}{3} = -\frac{1}{2}(p_{n-1} - \frac{2}{3})$, so that

$$p_n = \frac{2}{3} + \frac{1}{3} \cdot \left(-\frac{1}{2}\right)^n,$$

so $p_{20} - p_{15}$ is just

$$\frac{1 - (-2)^5}{3 \cdot 2^{20}} = \frac{11}{2^{20}}.$$

4. Alice Czarina is bored and is playing a game with a pile of rocks. The pile initially contains 2015 rocks. At each round, if the pile has N rocks, she removes k of them, where $1 \leq k \leq N$, with each possible k having equal probability. Alice Czarina continues until there are no more rocks in the pile. Let p be the probability that the number of rocks left in the pile after each round is a multiple of 5. If p is of the form $5^a \cdot 31^b \cdot \frac{c}{d}$, where a, b are integers and c, d are positive integers relatively prime to $5 \cdot 31$, find $a + b$.

Answer: $\boxed{-501}$ We claim that $p = \frac{1}{5} \frac{6}{10} \frac{11}{15} \frac{16}{20} \cdots \frac{2006}{2010} \frac{2011}{2015}$. Let p_n be the probability that, starting with n rocks, the number of rocks left after each round is a multiple of 5. Indeed, using recursions we have

$$p_{5k} = \frac{p_{5k-5} + p_{5k-10} + \cdots + p_5 + p_0}{5k}$$

for $k \geq 1$. For $k \geq 2$ we replace k with $k-1$, giving us

$$p_{5k-5} = \frac{p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0}{5k-5}$$

$$\implies (5k-5)p_{5k-5} = p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0$$

Substituting this back into the first equation, we have

$$5kp_{5k} = p_{5k-5} + (p_{5k-10} + p_{5k-15} + \cdots + p_5 + p_0) = p_{5k-5} + (5k-5)p_{5k-5},$$

which gives $p_{5k} = \frac{5k-4}{5k} p_{5k-5}$. Using this equation repeatedly along with the fact that $p_0 = 1$ proves the claim.

Now, the power of 5 in the denominator is $v_5(2015!) = 403 + 80 + 16 + 3 = 502$, and 5 does not divide any term in the numerator. Hence $a = -502$. (The sum counts multiples of 5 plus multiples of 5^2 plus multiples of 5^3 and so on; a multiple of 5^n but not 5^{n+1} is counted exactly n times, as desired.)

Noting that $2015 = 31 \cdot 65$, we found that the numbers divisible by 31 in the numerator are those of the form $31 + 155k$ where $0 \leq k \leq 12$, including $31^2 = 961$; in the denominator they are of the form $155k$ where $1 \leq k \leq 13$. Hence $b = (13 + 1) - 13 = 1$ where the extra 1 comes from 31^2 in the numerator. Thus $a + b = -501$.

5. For positive integers x , let $g(x)$ be the number of blocks of consecutive 1's in the binary expansion of x . For example, $g(19) = 2$ because $19 = 10011_2$ has a block of one 1 at the beginning and a block of two 1's at the end, and $g(7) = 1$ because $7 = 111_2$ only has a single block of three 1's. Compute $g(1) + g(2) + g(3) + \cdots + g(256)$.

Answer: 577 **Solution 1.** We prove that $g(1) + g(2) + \cdots + g(2^n) = 1 + 2^{n-2}(n+1)$ for all $n \geq 1$, giving an answer of $1 + 2^6 \cdot 9 = 577$. First note that $g(2^n) = 1$, and that we can view $0, 1, \dots, 2^n - 1$ as n -digit binary sequences by appending leading zeros as necessary. (Then $g(0) = 0$.)

Then for $0 \leq x \leq 2^n - 1$, x and $2^n - x$ are complementary n -digit binary sequences (of 0's and 1's), with x 's strings of 1's (0's) corresponding to $2^n - x$'s strings of 0's (resp. 1's). It follows that $g(x) + g(2^n - x)$ is simply 1 more than the number of digit changes in x (or $2^n - x$), i.e. the total number of 01 and 10 occurrences in x . Finally, because exactly half of all n -digit binary sequences have 0, 1 or 1, 0 at positions $k, k+1$ (for $1 \leq k \leq n-1$ fixed), we conclude that the average value of $g(x) + g(2^n - x)$ is $1 + \frac{n-1}{2} = \frac{n+1}{2}$, and thus that the total value of $g(x)$ is $\frac{1}{2} \cdot 2^n \cdot \frac{n+1}{2} = (n+1)2^{n-2}$, as desired.

Solution 2. We prove that $g(1) + g(2) + \cdots + g(2^n - 1) = 2^{n-2}(n+1)$. Identify each block of 1's with its *rightmost* 1. Then it suffices to count the number of these "rightmost 1's." For each $1 \leq k \leq n-1$, the k th digit from the left is a rightmost 1 if and only if the k and $k+1$ digits from the left are 1 and 0 respectively. Thus there are 2^{n-2} possible numbers. For the rightmost digit, it is a rightmost 1 if and only if it is a 1, so there are 2^{n-1} possible numbers. Sum up we have: $(n-1)2^{n-2} + 2^{n-1} = 2^{n-2}(n+1)$, as desired.

Remark. We can also solve this problem using recursion or generating functions.

6. Count the number of functions $f : \mathbb{Z} \rightarrow \{\text{'green'}, \text{'blue'}\}$ such that $f(x) = f(x+22)$ for all integers x and there does **not** exist an integer y with $f(y) = f(y+2) = \text{'green'}$.

Answer: 39601 It is clear that f is determined by $f(0), \dots, f(21)$. The colors of the 11 even integers are independent of those of the odd integers because evens and odds are never exactly 2 apart.

First, we count the number of ways to "color" the even integers. $f(0)$ can either be 'green' or 'blue'. If $f(0)$ is 'green', then $f(2) = f(20) = \text{'blue'}$. A valid coloring of the 8 other even integers corresponds bijectively to a string of 8 bits such that no two consecutive bits are 1. In general, the number of such length n strings is well known to be F_{n+2} (indexed according to $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$), which can be proven by recursion. Therefore, the number of colorings of even integers in this case is $F_{10} = 55$.

If $f(0)$ is 'blue', then a valid coloring of the 10 other even integers corresponds bijectively to a string as above, of 10 bits. The number of colorings for this case is $F_{12} = 144$. The total number of colorings of even integers is $55 + 144 = 199$.

Using the same reasoning for coloring the odd integers, we see that the number of colorings of all of the integers is $199^2 = 39601$.

7. 2015 people sit down at a restaurant. Each person orders a soup with probability $\frac{1}{2}$. Independently, each person orders a salad with probability $\frac{1}{2}$. What is the probability that the number of people who ordered a soup is exactly one more than the number of people who ordered a salad?

Answer: $\frac{\binom{4030}{2016}}{2^{4030}}$ OR $\frac{\binom{4030}{2014}}{2^{4030}}$ OR $\frac{\binom{4032}{2016} - 2\binom{4030}{2015}}{2^{4031}}$

Solution 1. Note that total soups = total salads + 1 is equivalent to total soups + total not-salads = 2016. So there are precisely $\binom{2015+2015}{2016}$ possibilities, each occurring with probability $(1/2)^{2015+2015}$. Thus our answer is $\frac{\binom{4030}{2016}}{2^{4030}}$.

Solution 2. To count the number of possibilities, we can directly evaluate the sum $\sum_{i=0}^{2014} \binom{2015}{i} \binom{2015}{i+1}$. One way is to note $\binom{2015}{i+1} = \binom{2015}{2014-i}$, and finish with Vandermonde's identity: $\sum_{i=0}^{2014} \binom{2015}{i} \binom{2015}{2014-i} = \binom{2015+2015}{2014} = \binom{4030}{2014}$ (which also equals $\binom{4030}{2016}$).

(We could have also used $\binom{2015}{i} = \binom{2015}{2015-i}$ to get $\sum_{i=0}^{2014} \binom{2015}{2015-i} \binom{2015}{i+1} = \binom{2015+2015}{2016}$ directly, which is closer in the spirit of the previous solution.)

Solution 3 (sketch). It's also possible to get a handle on $\sum_{i=0}^{2014} \binom{2015}{i} \binom{2015}{i+1}$ by squaring Pascal's identity $\binom{2015}{i} + \binom{2015}{i+1} = \binom{2016}{i+1}$ and summing over $0 \leq i \leq 2014$. This gives an answer of $\frac{\binom{4032}{2016} - 2\binom{4030}{2015}}{2^{4031}}$, which can be simplified by noting $\binom{4032}{2016} = \frac{4032}{2016} \binom{4031}{2015}$, and then applying Pascal's identity.

8. Let S be the set of all 3-digit numbers with all digits in the set $\{1, 2, 3, 4, 5, 6, 7\}$ (so in particular, **all three digits are nonzero**). For how many elements \overline{abc} of S is it true that at least one of the (not necessarily distinct) "digit cycles"

$$\overline{abc}, \overline{bca}, \overline{cab}$$

is divisible by 7? (Here, \overline{abc} denotes the number whose base 10 digits are a, b , and c in that order.)

Answer: $\boxed{127}$ Since the value of each digit is restricted to $\{1, 2, \dots, 7\}$, there is exactly one digit representative of each residue class modulo 7.

Note that $7 \mid \overline{abc}$ if and only if $100a + 10b + c \equiv 0 \pmod{7}$ or equivalently $2a + 3b + c \equiv 0$. So we want the number of triples of residues (a, b, c) such that at least one of $2a + 3b + c \equiv 0$, $2b + 3c + a \equiv 0$, $2c + 3a + b \equiv 0$ holds.

Let the solution sets of these three equations be S_1, S_2, S_3 respectively, so by PIE and cyclic symmetry we want to find $3|S_1| - 3|S_1 \cap S_2| + |S_1 \cap S_2 \cap S_3|$.

Clearly $|S_1| = 7^2$, since for each of a and b there is a unique c that satisfies the equation.

For $S_1 \cap S_2$, we may eliminate a to get the system $0 \equiv 2(2b + 3c) - (3b + c) = b + 5c$ (and $a \equiv -2b - 3c$), which has 7 solutions (one for each choice of c).

For $S_1 \cap S_2 \cap S_3 \subseteq S_1 \cap S_2$, we have from the previous paragraph that $b \equiv -5c$ and $a \equiv 10c - 3c \equiv 0$. By cyclic symmetry, $b, c \equiv 0$ as well, so there's exactly 1 solution in this case.

Thus the answer is $3 \cdot 7^2 - 3 \cdot 7 + 1 = 127$.

9. Calvin has a bag containing 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random (without replacement), he has pulled out 5 more red balls than blue balls, what is the probability that the next ball he pulls out is red?

Answer: $\boxed{\frac{9}{26}}$

Solution 1. The only information this gives us about the number of yellow balls left is that it is even. A bijection shows that the probability that there are k yellow balls left is equal to the probability that there are $30 - k$ yellow balls left (flip the colors of the red and blue balls, and then switch the 65 balls that have been picked with the 65 balls that have not been picked). So the expected number of yellow balls left is 15. Therefore the expected number of red balls left is 22.5. So the answer is $\frac{22.5}{65} = \frac{45}{130} = \frac{9}{26}$.

Solution 2. Let $w(b) = \binom{50}{b} \binom{50}{r=b+5} \binom{30}{60-2b}$ be the number of possibilities in which b blue balls have been drawn (precisely $15 \leq b \leq 30$ are possible). **For fixed** b , the probability of drawing red next is $\frac{50-r}{50+50+30-65} = \frac{45-b}{65}$. So we want to evaluate

$$\frac{\sum_{b=15}^{30} w(b) \frac{45-b}{65}}{\sum_{b=15}^{30} w(b)}$$

Note the symmetry of weights:

$$w(45 - b) = \binom{50}{45 - b} \binom{50}{50 - b} \binom{30}{2b - 30} = \binom{50}{b + 5} \binom{50}{b} \binom{30}{60 - 2b},$$

so the $\frac{45-b}{65}$ averages out with $\frac{45-(45-b)}{65}$ to give a final answer of $\frac{45/2}{65} = \frac{9}{26}$.

Remark. If one looks closely enough, the two approaches are not so different. The second solution may be more conceptually/symmetrically phrased in terms of the number of yellow balls.

10. A group of friends, numbered $1, 2, 3, \dots, 16$, take turns picking random numbers. Person 1 picks a number uniformly (at random) in $[0, 1]$, then person 2 picks a number uniformly (at random) in $[0, 2]$, and so on, with person k picking a number uniformly (at random) in $[0, k]$. What is the probability that the 16 numbers picked are strictly increasing?

Answer: $\frac{17^{15}}{16!^2}$ **Solution 1 (intuitive sketch).** If person i picks a_i , this is basically a continuous version of Catalan paths (always $y \leq x$) from $(0, 0)$ to $(17, 17)$, with “up-right corners” at the (i, a_i) .

A **cyclic shifts argument** shows that “ $\frac{1}{17}$ ” of the increasing sequences (x_1, \dots, x_{16}) in $[0, 17]^{16}$ work (i.e. have $x_i \in [0, i]$ for all i), so contribute volume¹ $\frac{1}{17} \frac{17^{16}}{16!}$. Explicitly, the cyclic shift we’re using is

$$T_C : (x_1, \dots, x_{16}) \mapsto (x_2 - x_1, \dots, x_{16} - x_1, C - x_1)$$

for $C = 17$ (though it’s the same for any $C > 0$), which sends increasing sequences in $[0, C]^{16}$ to increasing sequences in $[0, C]^{16}$. The “ $\frac{1}{17}$ ” essentially follows from the fact that T has period 17, and almost every² T -orbit (of 17 (increasing) sequences) contains exactly 1 working sequence.³ But to be more rigorous, we still need some more justification.⁴

The volume contribution of permitted sequences (i.e. $a_i \in [0, i]$ for all i ; those under consideration in the first place) $(a_1, \dots, a_{16}) \in [0, 17]^{16}$ is $16!$, so based on the previous paragraph, our final probability is $\frac{17^{15}}{16!^2}$.

Solution 2. Here we present a discrete version of the previous solution.

To do this, we consider several related events.

Let X be a 16-tuple chosen uniformly and randomly from $[0, 17]^{16}$ (used to define events A, B, C). Let Z be a 16-tuple chosen uniformly and randomly from $\{1, 2, \dots, 17\}^{16}$ (used to define event D).

- A is the event that X ’s coordinates are ordered ascending;
- B is the event that X lies in the “box” $[0, 1] \times \dots \times [0, 16]$;
- C is the event that when X ’s coordinates are sorted ascending to form Y (e.g. if $X = (1, 3.2, 3, 2, 5, 6, \dots, 16)$ then $Y = (1, 2, 3, 3.2, 5, 6, \dots, 16)$), Y lies in the box;
- D is the event that when Z ’s coordinates are sorted ascending to form W , W lies in the aforementioned box. When Z satisfies this condition, Z is known as a *parking function*.

¹If we wanted to be completely rigorous, we’d want to also show that the set of such sequences is *measurable*. However, our sets here are simple enough that this is not a major concern.

²i.e. “outside a set of measure 0”

³Try to visualize this yourself! Compare the “exactly 1 of 17” with the discrete solutions below.

⁴We’re using a symmetry/“bijection-of-integrals” argument here (to equate 17 integrals with sum $\frac{17^{16}}{16!}$), so we need to be a little careful with the bijections. In particular, we must make sure the cyclic shift

$$T : (x_1, \dots, x_{16}) \mapsto (x_2 - x_1, \dots, x_{16} - x_1, 17 - x_1)$$

is “measure-preserving” on \mathbb{R}^{16} . (An example of a non-measure-preserving bijection is $t \mapsto t^2$ on $[0, 1]$.) The standard way to check this is by taking a Jacobian (determinant); more conceptually, since $T^{17} = \text{Id}$, and T is affine (so the Jacobian is constant), we should be able to avoid direct computation by using the chain rule. Alternatively, more “combinatorially”, it should also be possible to do so by properly discretizing our space.

We want to find $P(A|B)$ because given that X is in B , X has a uniform distribution in the box, just as in the problem. Now note

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A \cap B)}{P(A)} \frac{P(A)}{P(B)} = P(B|A) \frac{P(A)}{P(B)}.$$

C is invariant with respect to permutations, so $\frac{1}{16!} = P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap B)}{P(C)}$. Since $P(A) = \frac{1}{16!}$, we have $P(B|A) = \frac{P(A \cap B)}{P(A)} = P(C)$.

Furthermore, $P(C) = P(D)$ because C only depends on the ceilings of the coordinates. So $P(A|B) = P(C) \frac{P(A)}{P(B)} = P(D) \frac{P(A)}{P(B)}$. (*)

Given a 16-tuple Z from $\{1, 2, \dots, 17\}^{16}$, let $Z + n$ (for integers n) be the 16-tuple formed by adding n to each coordinate and then reducing modulo 17 so that each coordinate lies in $[1, 17]$.

Key claim (discrete analog of cyclic shifts argument). Exactly one of $Z, Z + 1, \dots, Z + 16$ is a parking function.

First, assuming this claim, it easily follows that $P(D) = \frac{1}{17}$. Substituting $P(A) = \frac{1}{16!}$, $P(B) = \frac{16!}{17^{16}}$ into (*) gives $P(A|B) = \frac{17^{15}}{16!^2}$. It now remains to prove the claim.

Proof. Consider the following process.

Begin with 17 parking spots around a circle, labelled 1 to 17 clockwise and all unoccupied. There are 16 cars, 1 to 16, and they park one at a time, from 1 to 16. The i th car tries to park in the spot given by the i th coordinate of Z . If this spot is occupied, that car parks in the closest unoccupied spot in the clockwise direction. Because there are only 16 cars, each car will be able to park, and exactly one spot will be left.

Suppose that number 17 is left. For any integer n ($1 \leq n \leq 16$), the n cars that ended up parking in spots 1 through n must have corresponded to coordinates at most n . (If not, then the closest spot in the clockwise direction would have to be before spot 17 and greater than n , a contradiction.) It follows that the n th lowest coordinate is at most n and that when Z is sorted, it lies in the box.

Suppose now that D is true. For any integer n ($1 \leq n \leq 16$), the n th lowest coordinate is at most n , so there are (at least) n cars whose corresponding coordinates are at most n . At least one of these cars does not park in spots 1 through $n - 1$. Consider the first car to do so. It either parked in spot n , or skipped over it because spot n was occupied. Therefore spot n is occupied at the end. This is true for all n not equal to 17, so spot 17 is left.

It follows that Z is a parking function if and only if spot 17 is left. The same is true for $Z + 1$ (assuming that the process uses $Z + 1$ instead of Z), etc.

Observe that the process for $Z + 1$ is exactly that of Z , rotated by 1 spot clockwise. In particular, its empty spot is one more than that of Z , (where 1 is one more than 17.) It follows that exactly one of $Z, Z + 1, \dots, Z + 16$ leaves the spot 17, and that exactly one of these is a parking function. \square

Solution 3. Suppose that person i picks a number in the interval $[b_i - 1, b_i]$ where $b_i \leq i$. Then we have the condition: $b_1 \leq b_2 \leq \dots \leq b_{16}$. Let c_i be the number of b_j 's such that $b_j = i$. Then, for each admissible sequence b_1, b_2, \dots, b_{16} , there is the probability $\frac{1}{c_1!c_2!\dots c_{16}!}$ that the problem condition holds, since if c_i numbers are picked uniformly and randomly in the interval $[i - 1, i]$, then there is $\frac{1}{c_i!}$ chance of them being in an increasing order. Thus the answer we are looking for is

$$\frac{1}{16!} \sum_{\substack{b_i \leq i \\ b_1 \leq \dots \leq b_{16}}} \frac{1}{c_1!c_2!\dots c_{16}!} = \frac{1}{16!^2} \sum_{\substack{b_i \leq i \\ b_1 \leq \dots \leq b_{16}}} \binom{c_1 + \dots + c_{16}}{c_1, c_2, \dots, c_{16}}.$$

Thus it suffices to prove that

$$\sum_{\substack{b_i \leq i \\ b_1 \leq \dots \leq b_{16}}} \binom{c_1 + \dots + c_{16}}{c_1, c_2, \dots, c_{16}} = 17^{15}.$$

The left hand side counts the number of 16-tuple such that the n th smallest entry is less than or equal to n . In other words, this counts the number of parking functions of length 16.⁵ Since the number of parking functions of length n is $\frac{1}{n+1} \cdot (n+1)^n = (n+1)^{n-1}$ (as proven for $n = 16$ in the previous solution), we obtain the desired result.

⁵See <http://www-math.mit.edu/~rstan/transparencies/parking.pdf>.

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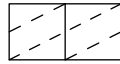
Geometry

1. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0)$, $(2, 0)$, $(2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown.



Pro selects a point P uniformly at random in the interior of R . Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

Answer: $\boxed{\frac{3}{4}}$



Precisely the middle two (of four) regions satisfy the problem conditions, and it's easy to compute the (fraction of) areas as $\frac{3}{4}$.

2. Let ABC be a triangle with orthocenter H ; suppose that $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC , and define G_B , G_C similarly. Determine the area of triangle $G_A G_B G_C$.

Answer: $\boxed{28/3}$ Let D, E, F be the midpoints of BC, CA , and AB , respectively. Then $G_A G_B G_C$ is the DEF about H with a ratio of $\frac{2}{3}$, and DEF is the dilation of ABC about H with a ratio of $-\frac{1}{2}$, so $G_A G_B G_C$ is the dilation of ABC about H with ratio $-\frac{1}{3}$. Thus $[G_A G_B G_C] = \frac{[ABC]}{9}$. By Heron's formula, the area of ABC is $\sqrt{21 \cdot 8 \cdot 7 \cdot 6} = 84$, so the area of $G_A G_B G_C$ is $[ABC]/9 = 84/9 = 28/3$.

3. Let $ABCD$ be a quadrilateral with $\angle BAD = \angle ABC = 90^\circ$, and suppose $AB = BC = 1$, $AD = 2$. The circumcircle of ABC meets \overline{AD} and \overline{BD} at points E and F , respectively. If lines AF and CD meet at K , compute EK .

Answer: $\boxed{\frac{\sqrt{2}}{2}}$ Assign coordinates such that B is the origin, A is $(0, 1)$, and C is $(1, 0)$. Clearly, E is the point $(1, 1)$. Since the circumcenter of ABC is $(\frac{1}{2}, \frac{1}{2})$, the equation of the circumcircle of ABC is $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$. Since line BD is given by $x = 2y$, we find that F is at $(\frac{6}{5}, \frac{3}{5})$. The intersection of AF with CD is therefore at $(\frac{3}{2}, \frac{1}{2})$, so K is the midpoint of CD . As a result, $EK = \frac{\sqrt{2}}{2}$.

This is in fact a special case of APMO 2013, Problem 5, when the quadrilateral is a square.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 2, DA = 4$. Let lines perpendicular to \overline{BC} from B and C meet \overline{AD} at B' and C' , respectively. Let lines perpendicular to \overline{AD} from A and D meet \overline{BC} at A' and D' , respectively. Compute the ratio $\frac{[BCC'B']}{[DAA'D']}$, where $[\varpi]$ denotes the area of figure ϖ .

Answer: $\boxed{\frac{37}{76}}$ To get a handle on the heights CB' , etc. perpendicular to BC and AD , let $X = BC \cap AD$, which lies on ray \overrightarrow{BC} and \overrightarrow{AD} since $\widehat{AB} > \widehat{CD}$ (as chords $AB > CD$).

By similar triangles we have equality of ratios $XC : XD : 2 = (XD + 4) : (XC + 2) : 3$, so we have a system of linear equations: $3XC = 2XD + 8$ and $3XD = 2XC + 4$, so $9XC = 6XD + 24 = 4XC + 32$ gives $XC = \frac{32}{5}$ and $XD = \frac{84/5}{3} = \frac{28}{5}$.

It's easy to compute the trapezoid area ratio $\frac{[BC'B'C]}{[AA'D'D]} = \frac{BC(CB'+BC')}{AD(AD'+DA')} = \frac{BC}{AD} \frac{XC+XB}{XA+XD}$ (where we have similar right triangles due to the common angle at X). This is just $\frac{BC}{AD} \frac{XC+BC/2}{XD+AD/2} = \frac{2}{4} \frac{32/5+1}{28/5+2} = \frac{37}{76}$.

5. Let I be the set of points (x, y) in the Cartesian plane such that

$$x > \left(\frac{y^4}{9} + 2015 \right)^{1/4}$$

Let $f(r)$ denote the area of the intersection of I and the disk $x^2 + y^2 \leq r^2$ of radius $r > 0$ centered at the origin $(0, 0)$. Determine the minimum possible real number L such that $f(r) < Lr^2$ for all $r > 0$.

Answer: $\boxed{\frac{\pi}{3}}$ Let $B(P, r)$ be the (closed) disc centered at P with radius r . Note that for all $(x, y) \in I$, $x > 0$, and $x > \left(\frac{y^4}{9} + 2015 \right)^{1/4} > \frac{|y|}{\sqrt{3}}$. Let $I' = \{(x, y) : x\sqrt{3} > |y|\}$. Then $I \subseteq I'$ and the intersection of I' with $B((0, 0), r)$ is $\frac{\pi}{3}r^2$, so the $f(r) =$ the area of $I \cap B((0, 0), r)$ is also less than $\frac{\pi}{3}r^2$. Thus $L = \frac{\pi}{3}$ works.

On the other hand, if $x > \frac{|y|}{\sqrt{3}} + 7$, then $x > \frac{|y|}{\sqrt{3}} + 7 > \left(\left(\frac{|y|}{9} \right)^4 + 7^4 \right)^{1/4} > \left(\frac{y^4}{9} + 2015 \right)^{1/4}$, which means that if $I'' = \{(x, y) : (x-7)\sqrt{3} > |y|\}$, then $I'' \subseteq I'$. However, for $r > 7$, the area of $I'' \cap B((7, 0), r-7)$ is $\frac{\pi}{3}(r-7)^2$, and $I'' \subseteq I$, $B((7, 0), r-7) \subseteq B((0, 0), r)$, which means that $f(r) > \frac{\pi}{3}(r-7)^2$ for all $r > 7$, from which it is not hard to see that $L = \frac{\pi}{3}$ is the minimum possible L .

Remark: The lines $y = \pm\sqrt{3}x$ are actually asymptotes for the graph of $9x^4 - y^4 = 2015$. The bulk of the problem generalizes to the curve $|\sqrt{3}x|^\alpha - |y|^\alpha = C$ (for a positive real $\alpha > 0$ and any real C); the case $\alpha = 0$ is the most familiar case of a hyperbola.

6. In triangle ABC , $AB = 2$, $AC = 1 + \sqrt{5}$, and $\angle CAB = 54^\circ$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD , determine the measure of $\angle ACM$, in degrees.

Answer: $\boxed{63}$ Let E be the midpoint of \overline{AD} . $EC = \sqrt{5} + 1 - \sqrt{5} = 1$, and $EM = 1$ by similar triangles ($ABD \sim EMD$). $\triangle ECM$ is isosceles, with $m\angle CEM = 54^\circ$. Thus $m\angle ACM = m\angle ECM = \frac{180-54}{2} = 63^\circ$.

7. Let $ABCDE$ be a square pyramid of height $\frac{1}{2}$ with square base $ABCD$ of side length $AB = 12$ (so E is the vertex of the pyramid, and the foot of the altitude from E to $ABCD$ is the center of square $ABCD$). The faces ADE and CDE meet at an acute angle of measure α (so that $0^\circ < \alpha < 90^\circ$). Find $\tan \alpha$.

Answer: $\boxed{\frac{17}{144}}$ Let X be the projection of A onto DE . Let $b = AB = 12$.

The key fact in this computation is that if Y is the projection of A onto face CDE , then the projection of Y onto line DE coincides with the projection of A onto line DE (i.e. X as defined above). We compute $AY = \frac{b}{\sqrt{b^2+1}}$ by looking at the angle formed by the faces and the square base (via $1/2-b/2-\sqrt{b^2+1}/2$ right triangle). Now we compute $AX = 2[AED]/ED = \frac{b\sqrt{b^2+1}/2}{\sqrt{2b^2+1}/2}$.

But $\alpha = \angle AXY$, so from $(b^2 + 1)^2 - (\sqrt{2b^2 + 1})^2 = (b^2)^2$, it easily follows that $\tan \alpha = \frac{\sqrt{2b^2+1}}{b^2} = \frac{17}{144}$.

8. Let S be the set of **discs** D contained completely in the set $\{(x, y) : y < 0\}$ (the region below the x -axis) and centered (at some point) on the curve $y = x^2 - \frac{3}{4}$. What is the area of the union of the elements of S ?

Answer: $\boxed{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}$ **Solution 1.** An arbitrary point (x_0, y_0) is contained in S if and only if there exists some (x, y) on the curve $(x, x^2 - \frac{3}{4})$ such that $(x - x_0)^2 + (y - y_0)^2 < y^2$, since the radius of the circle is at most the distance from (x, y) to the x -axis. Some manipulation yields $x^2 - 2y_0(x^2 - \frac{3}{4}) - 2xx_0 + x_0^2 + y_0^2 < 0$.

Observe that $(x_0, y_0) \in S$ if and only if the optimal choice for x that minimizes the expression satisfies the inequality. The minimum is achieved for $x = \frac{x_0}{1-2y_0}$. After substituting and simplifying, we obtain $y_0(\frac{-x_0^2}{1-2y_0} + x_0^2 + y_0^2 + \frac{3}{2}y_0) < 0$. Since $y_0 < 0$ and $1-2y_0 > 0$, we find that we need $-2x_0^2 - 2y_0^2 + \frac{3}{2} - 2y_0 > 0 \iff 1 > x_0 + (y_0 + \frac{1}{2})^2$.

S is therefore the intersection of the lower half-plane and a circle centered at $(0, -\frac{1}{2})$ of radius 1. This is a circle of sector angle $4\pi/3$ and an isosceles triangle with vertex angle $2\pi/3$. The sum of these areas is $\frac{2\pi}{3} + \frac{\sqrt{3}}{4}$.

Solution 2. Let $O = (0, -\frac{1}{2})$ and $\ell = \{y = -1\}$ be the focus and directrix of the given parabola. Let ℓ' denote the x -axis. Note that a point P' is in S iff there exists a point P on the parabola in the lower half-plane for which $d(P, P') < d(P, \ell')$. However, for all such P , $d(P, \ell') = 1 - d(P, \ell) = 1 - d(P, O)$, which means that P' is in S iff there exists a P on the parabola for which $d(P', P) + d(P, O) < 1$. It is not hard to see that this is precisely the intersection of the unit circle centered at O and the lower half-plane, so now we can proceed as in Solution 1.

9. Let $ABCD$ be a regular tetrahedron with side length 1. Let X be the point in triangle BCD such that $[XBC] = 2[XBD] = 4[XCD]$, where $[\varpi]$ denotes the area of figure ϖ . Let Y lie on segment AX such that $2AY = YX$. Let M be the midpoint of BD . Let Z be a point on segment AM such that the lines YZ and BC intersect at some point. Find $\frac{AZ}{ZM}$.

Answer: $\boxed{\frac{4}{7}}$ We apply three-dimensional barycentric coordinates with reference tetrahedron $ABCD$. The given conditions imply that

$$\begin{aligned} X &= (0 : 1 : 2 : 4) \\ Y &= (14 : 1 : 2 : 4) \\ M &= (0 : 1 : 0 : 1) \\ Z &= (t : 1 : 0 : 1) \end{aligned}$$

for some real number t . Normalizing, we obtain $Y = (\frac{14}{21}, \frac{1}{21}, \frac{2}{21}, \frac{4}{21})$ and $Z = (\frac{t}{t+2}, \frac{1}{t+2}, 0, \frac{1}{t+2})$. If YZ intersects line BC then there exist parameters $\alpha + \beta = 1$ such that $\alpha Y + \beta Z$ has zero A and D coordinates, meaning

$$\begin{aligned} \frac{14}{21}\alpha + \frac{t}{t+2}\beta &= 0 \\ \frac{4}{21}\alpha + \frac{1}{t+2}\beta &= 0 \\ \alpha + \beta &= 1. \end{aligned}$$

Adding twice the second equation to the first gives $\frac{22}{21}\alpha + \beta = 0$, so $\alpha = -22$, $\beta = 21$, and thus $t = \frac{7}{2}$. It follows that $Z = (7 : 2 : 0 : 2)$, and $\frac{AZ}{ZM} = \frac{2+2}{7} = \frac{4}{7}$.

10. Let \mathcal{G} be the set of all points (x, y) in the Cartesian plane such that $0 \leq y \leq 8$ and

$$(x - 3)^2 + 31 = (y - 4)^2 + 8\sqrt{y(8 - y)}.$$

There exists a unique line ℓ of **negative slope** tangent to \mathcal{G} and passing through the point $(0, 4)$. Suppose ℓ is tangent to \mathcal{G} at a **unique** point P . Find the coordinates (α, β) of P .

Answer: $\boxed{(\frac{12}{5}, \frac{8}{5})}$ Let G be \mathcal{G} restricted to the strip of plane $0 \leq y \leq 4$ (we only care about this region since ℓ has **negative slope** going down from $(0, 4)$). By completing the square, the original equation rearranges to $(x - 3)^2 + (\sqrt{y(8 - y)} - 4)^2 = 1$. One could finish the problem in a completely standard way via the single-variable parameterization $(x, \sqrt{y(8 - y)}) = (3 + \cos t, 4 + \sin t)$ on the appropriate interval of t —just take derivatives with respect to t to find slopes (the computations would probably not be too bad)—but we will present a slightly cleaner solution.

Consider the bijective plane transformation $\Phi : (x, y) \mapsto (x, \sqrt{y(8 - y)})$, with inverse $\Phi^{-1} : (x, y) \mapsto (x, 4 - \sqrt{16 - y^2})$. In general, Φ maps curves as follows: $\Phi(\{(x, y) : f(x, y) = c\}) = \{\Phi(x, y) : f(x, y) = c\} = \{(x', y') : f(\Phi^{-1}(x', y')) = c\}$.

Our line ℓ has the form $y - 4 = -mx$ for some $m > 0$. We have $\Phi(G) = \{(x - 3)^2 + (y - 4)^2 = 1 : 0 \leq y \leq 4\}$ and $\Phi(\{4 - y = mx : 0 \leq y \leq 4\}) = \{\sqrt{16 - y^2} = mx : 0 \leq y \leq 4\}$. Since ℓ is unique, m must

also be. But it's easy to see that $m = 1$ gives a tangency point, so if our original tangency point was (u, v) , then our new tangency point is $(u, \sqrt{v(8-v)}) = \frac{4}{5}(3, 4) = (\frac{12}{5}, \frac{16}{5})$, and so $(u, v) = (\frac{12}{5}, \frac{8}{5})$.

Remark. To see what \mathcal{G} looks like, see Wolfram Alpha using the plotting/graphing commands.

HMMT February 2015
Saturday 21 February 2015

Guts

1. [4] Let R be the rectangle in the Cartesian plane with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



The resulting figure has 7 segments of unit length, connecting neighboring lattice points (those lying on or inside R). Compute the number of paths from $(0,1)$ (the upper left corner) to $(2,0)$ (the lower right corner) along these 7 segments, where each segment can be used at most once.

Answer: [4] Just count them directly. If the first step is to the right, there are 2 paths. If the first step is downwards (so the next step must be to the right), there are again 2 paths. This gives a total of 4 paths.

2. [4] Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle ACD = \angle ADE = 90^\circ$ and $AB = BC = CD = DE = 1$. Compute AE .

Answer: [2] By Pythagoras,

$$AE^2 = AD^2 + 1 = AC^2 + 2 = AB^2 + 3 = 4,$$

so $AE = 2$.

3. [4] Find the number of pairs of union/intersection operations $(\square_1, \square_2) \in \{\cup, \cap\}^2$ satisfying the following condition: for any sets S, T , function $f : S \rightarrow T$, and subsets X, Y, Z of S , we have equality of sets

$$f(X)\square_1(f(Y)\square_2f(Z)) = f(X\square_1(Y\square_2Z)),$$

where $f(X)$ denotes the *image* of X : the set $\{f(x) : x \in X\}$, which is a subset of T . The images $f(Y)$ (of Y) and $f(Z)$ (of Z) are similarly defined.

Answer: [1] If and only if $\square_1 = \square_2 = \cup$. See <http://math.stackexchange.com/questions/359693/overview-of->

4. [4] Consider the function $z(x, y)$ describing the paraboloid

$$z = (2x - y)^2 - 2y^2 - 3y.$$

Archimedes and Brahmagupta are playing a game. Archimedes first chooses x . Afterwards, Brahmagupta chooses y . Archimedes wishes to minimize z while Brahmagupta wishes to maximize z . Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

Answer: [$-\frac{3}{8}$] Viewing x as a constant and completing the square, we find that

$$\begin{aligned} z &= 4x^2 - 4xy + y^2 - 2y^2 - 3y \\ &= -y^2 - (4x + 3)y + 4x^2 \\ &= -\left(y + \frac{4x + 3}{2}\right)^2 + \left(\frac{4x + 3}{2}\right)^2 + 4x^2. \end{aligned}$$

Brahmagupta wishes to maximize z , so regardless of the value of x , he will pick $y = -\frac{4x+3}{2}$. The expression for z then simplifies to

$$z = 8x^2 + 6x + \frac{9}{4}.$$

Archimedes knows this and will therefore pick x to minimize the above expression. By completing the square, we find that $x = -\frac{3}{8}$ minimizes z .

Alternatively, note that z is convex in x and concave in y , so we can use the minimax theorem to switch the order of moves. If Archimedes goes second, he will set $x = \frac{y}{2}$ to minimize z , so Brahmagupta will maximize $-2y^2 - 3y$ by setting $y = -\frac{3}{4}$. Thus Archimedes should pick $x = -\frac{3}{8}$, as above.

5. [5] Let \mathcal{H} be the unit hypercube of dimension 4 with a vertex at (x, y, z, w) for each choice of $x, y, z, w \in \{0, 1\}$. (Note that \mathcal{H} has $2^4 = 16$ vertices.) A bug starts at the vertex $(0, 0, 0, 0)$. In how many ways can the bug move to $(1, 1, 1, 1)$ (the opposite corner of \mathcal{H}) by taking exactly 4 steps along the edges of \mathcal{H} ?

Answer: [24] You may think of this as sequentially adding 1 to each coordinate of $(0, 0, 0, 0)$. There are 4 ways to choose the first coordinate, 3 ways to choose the second, and 2 ways to choose the third. The product is 24.

6. [5] Let D be a regular ten-sided polygon with edges of length 1. A triangle T is defined by choosing three vertices of D and connecting them with edges. How many different (non-congruent) triangles T can be formed?

Answer: [8] The problem is equivalent to finding the number of ways to partition 10 into a sum of three (unordered) positive integers. These can be computed by hand to be $(1, 1, 8)$, $(1, 2, 7)$, $(1, 3, 6)$, $(1, 4, 5)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, $(3, 3, 4)$.

7. [5] Let \mathcal{C} be a cube of side length 2. We color each of the faces of \mathcal{C} blue, then subdivide it into $2^3 = 8$ unit cubes. We then randomly rearrange these cubes (possibly with rotation) to form a new 3-dimensional cube.

What is the probability that its exterior is still completely blue?

Answer: $\frac{1}{2^{24}}$ or $\frac{1}{8^8}$ or $\frac{1}{16777216}$ Each vertex of the original cube must end up as a vertex of the new cube in order for all the old blue faces to show. There are 8 such vertices, each corresponding to one unit cube, and each has a probability $\frac{1}{8}$ of being oriented with the old outer vertex as a vertex of the new length-2 cube. Multiplying gives the answer.

8. [5] Evaluate

$$\sin(\arcsin(0.4) + \arcsin(0.5)) \cdot \sin(\arcsin(0.5) - \arcsin(0.4)),$$

where for $x \in [-1, 1]$, $\arcsin(x)$ denotes the unique real number $y \in [-\pi, \pi]$ such that $\sin(y) = x$.

Answer: [0.09 OR $\frac{9}{100}$] Use the difference of squares identity¹ $\sin(a-b)\sin(a+b) = \sin(a)^2 - \sin(b)^2$ to get $0.5^2 - 0.4^2 = 0.3^2 = 0.09 = \frac{9}{100}$.

9. [6] Let a, b, c be integers. Define $f(x) = ax^2 + bx + c$. Suppose there exist pairwise distinct integers u, v, w such that $f(u) = 0$, $f(v) = 0$, and $f(w) = 2$. Find the maximum possible value of the discriminant $b^2 - 4ac$ of f .

Answer: [16] By the factor theorem, $f(x) = a(x-u)(x-v)$, so the constraints essentially boil down to $2 = f(w) = a(w-u)(w-v)$. (It's not so important that $u \neq v$; we merely specified it for a shorter problem statement.)

We want to maximize the discriminant $b^2 - 4ac = a^2[(u+v)^2 - 4uv] = a^2(u-v)^2 = a^2[(w-v) - (w-u)]^2$. Clearly $a \mid 2$. If $a > 0$, then $(w-u)(w-v) = 2/a > 0$ means the difference $|u-v|$ is less than $2/a$, whereas if $a < 0$, since at least one of $|w-u|$ and $|w-v|$ equals 1, the difference $|u-v|$ of factors is greater than $2/|a|$.

So the optimal choice occurs either for $a = -1$ and $|u-v| = 3$, or $a = -2$ and $|u-v| = 2$. The latter wins, giving a discriminant of $(-2)^2 \cdot 2^2 = 16$.

¹proven most easily with complex numbers or the product-to-sum identity on $\sin(a-b)\sin(a+b)$ (followed by the double angle formula for cosine)

10. [6] Let $b(x) = x^2 + x + 1$. The polynomial $x^{2015} + x^{2014} + \cdots + x + 1$ has a unique “base $b(x)$ ” representation

$$x^{2015} + x^{2014} + \cdots + x + 1 = \sum_{k=0}^N a_k(x)b(x)^k,$$

where

- N is a nonnegative integer;
- each “digit” $a_k(x)$ (for $0 \leq k \leq N$) is either the zero polynomial (i.e. $a_k(x) = 0$) or a nonzero polynomial of degree less than $\deg b = 2$; and
- the “leading digit $a_N(x)$ ” is nonzero (i.e. not the zero polynomial).

Find $a_N(0)$ (the “leading digit evaluated at 0”).

Answer: $\boxed{-1006}$ Comparing degrees easily gives $N = 1007$. By ignoring terms of degree at most 2013, we see

$$a_N(x)(x^2 + x + 1)^{1007} \in x^{2015} + x^{2014} + O(x^{2013}).$$

Write $a_N(x) = ux + v$, so

$$\begin{aligned} a_N(x)(x^2 + x + 1)^{1007} &\in (ux + v)(x^{2014} + 1007x^{2013} + O(x^{2012})) \\ &\subseteq ux^{2015} + (v + 1007u)x^{2014} + O(x^{2013}). \end{aligned}$$

Finally, matching terms gives $u = 1$ and $v + 1007u = 1$, so $v = 1 - 1007 = -1006$.

Remark. This problem illustrates the analogy between integers and polynomials, with the nonconstant (degree ≥ 1) polynomial $b(x) = x^2 + x + 1$ taking the role of a positive integer base $b > 1$.

11. [6] Find

$$\sum_{k=0}^{\infty} \left\lfloor \frac{1 + \sqrt{\frac{2000000}{4^k}}}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .

Answer: $\boxed{1414}$ The k th floor (for $k \geq 0$) counts the number of **positive integer** solutions to $4^k(2x - 1)^2 \leq 2 \cdot 10^6$. So summing over all k , we want the number of integer solutions to $4^k(2x - 1)^2 \leq 2 \cdot 10^6$ with $k \geq 0$ and $x \geq 1$. But each positive integer can be uniquely represented as a power of 2 times an odd (positive) integer, so there are simply $\lfloor 10^3\sqrt{2} \rfloor = 1414$ solutions.

12. [6] For integers a, b, c, d , let $f(a, b, c, d)$ denote the number of ordered pairs of integers $(x, y) \in \{1, 2, 3, 4, 5\}^2$ such that $ax + by$ and $cx + dy$ are both divisible by 5. Find the sum of all possible values of $f(a, b, c, d)$.

Answer: $\boxed{31}$ Standard linear algebra over the field \mathbb{F}_5 (the integers modulo 5). The dimension of the solution set is at least 0 and at most 2, and any intermediate value can also be attained. So the answer is $1 + 5 + 5^2 = 31$.

This also can be easily reformulated in more concrete equation/congruence-solving terms, especially since there are few variables/equations.

13. [8] Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \geq 0$ for all $x \geq 0$, find the sum of all possible values of $P(-1)$.

Answer: $\boxed{9496}$ Since all the roots of $P(x)$ are integers, we can factor it as $P(x) = (x-r)(x-s)(x-t)$ for integers r, s, t . By Viète’s formula, the product of the roots is $rst = -2015$, so we need three integers to multiply to -2015 .

$P(x)$ cannot have two distinct positive roots u, v since otherwise, $P(x)$ would be negative at least in some infinitesimal region $x < u$ or $x > v$, or $P(x) < 0$ for $u < x < v$. Thus, in order to have two

positive roots, we must have a double root. Since $2015 = 5 \times 13 \times 31$, the only positive double root is a perfect square factor of 2015, which is at $x = 1$, giving us a possibility of $P(x) = (x - 1)^2(x + 2015)$.

Now we can consider when $P(x)$ only has negative roots. The possible unordered triplets are $(-1, -1, -2015)$, $(-1, -5, -403)$, $(-1, -13, -155)$, $(-1, -31, -65)$, $(-5, -13, -31)$, which yield the polynomials $(x + 1)^2(x + 2015)$, $(x + 1)(x + 5)(x + 403)$, $(x + 1)(x + 13)(x + 155)$, $(x + 1)(x + 31)(x + 65)$, $(x + 5)(x + 13)(x + 31)$, respectively.

Noticing that $P(-1) = 0$ for four of these polynomials, we see that the nonzero values are $P(-1) = (-1 - 1)^2(2014)$, $(5 - 1)(13 - 1)(31 - 1)$, which sum to $8056 + 1440 = 9496$.

14. [8] Find the smallest integer $n \geq 5$ for which there exists a set of n distinct pairs $(x_1, y_1), \dots, (x_n, y_n)$ of positive integers with $1 \leq x_i, y_i \leq 4$ for $i = 1, 2, \dots, n$, such that for any indices $r, s \in \{1, 2, \dots, n\}$ (not necessarily distinct), there exists an index $t \in \{1, 2, \dots, n\}$ such that 4 divides $x_r + x_s - x_t$ and $y_r + y_s - y_t$.

Answer: [8] In other words, we have a set S of n pairs in $(\mathbb{Z}/4\mathbb{Z})^2$ closed under addition. Since $1 + 1 + 1 + 1 \equiv 0 \pmod{4}$ and $1 + 1 + 1 \equiv -1 \pmod{4}$, $(0, 0) \in S$ and S is closed under (additive) inverses. Thus S forms a group under addition (a subgroup of $(\mathbb{Z}/4\mathbb{Z})^2$). By Lagrange's theorem (from basic group theory), $n \mid 4^2$, so $n \geq 8$. To achieve this bound, one possible construction is $\{1, 2, 3, 4\} \times \{2, 4\}$.

Remark. In fact, S is a finite abelian group. Such groups have a very clean classification; this is clarified by the fact that abelian groups are the same as modules over \mathbb{Z} , the ring of integers.

15. [8] Find the maximum possible value of $H \cdot M \cdot M \cdot T$ over all ordered triples (H, M, T) of integers such that $H \cdot M \cdot M \cdot T = H + M + M + T$.

Answer: [8] If any of H, M, T are zero, the product is 0. We can do better (examples below), so we may now restrict attention to the case when $H, M, T \neq 0$.

When $M \in \{-2, -1, 1, 2\}$, a little casework gives all the possible $(H, M, T) = (2, 1, 4), (4, 1, 2), (-1, -2, 1), (1, -2, -1)$.

- If $M = -2$, i.e. $H - 4 + T = 4HT$, then $-15 = (4H - 1)(4T - 1)$, so $4H - 1 \in \{\pm 1, \pm 3, \pm 5, \pm 15\}$ (only $-1, +3, -5, +15$ are possible) corresponding to $4T - 1 \in \{\mp 15, \mp 5, \mp 3, \mp 1\}$ (only $+15, -5, +3, -1$ are possible). But H, T are nonzero, we can only have $4H - 1 \in \{+3, -5\}$, yielding $(-1, -2, 1)$ and $(1, -2, -1)$.
- If $M = +2$, i.e. $H + 4 + T = 4HT$, then $17 = (4H - 1)(4T - 1)$, so $4H - 1 \in \{\pm 1, \pm 17\}$ (only $-1, -17$ are possible) corresponding to $4T - 1 \in \{\pm 17, \pm 1\}$ (only $-17, -1$ are possible). But H, T are nonzero, so there are no possibilities here.
- If $M = -1$, i.e. $H - 2 + T = HT$, then $-1 = (H - 1)(T - 1)$, so we have $H - 1 \in \{\pm 1\}$ and $T - 1 \in \{\mp 1\}$, neither of which is possible (as $H, T \neq 0$).
- If $M = +1$, i.e. $H + 2 + T = HT$, then $3 = (H - 1)(T - 1)$, so we have $H - 1 \in \{\pm 1, \pm 3\}$. Since $H, T \neq 0$, $H - 1 \in \{+1, +3\}$, yielding $(2, 1, 4)$ and $(4, 1, 2)$.

Now suppose there is such a triple (H, M, T) for $|M| \geq 3$. The equation in the problem gives $(M^2H - 1)(M^2T - 1) = 2M^3 + 1$. Note that since $H, T \neq 0$, $|2M^3 + 1| = |M^2H - 1| \cdot |M^2T - 1| \geq \min(M^2 - 1, M^2 + 1)^2 = M^4 - 2M^2 + 1 > 2|M|^3 + 1$ gives a contradiction.

16. [8] Determine the number of **unordered** triples of **distinct** points in the $4 \times 4 \times 4$ lattice grid $\{0, 1, 2, 3\}^3$ that are collinear in \mathbb{R}^3 (i.e. there exists a line passing through the three points).

Answer: [376] Define a main plane to be one of the xy, yz, zx planes. Define a space diagonal to be a set of collinear points not parallel to a main plane. We classify the lines as follows:

- (a) **Lines parallel to two axes (i.e. orthogonal to a main plane).** Notice that given a plane of the form $v = k$, where $v \in \{x, y, z\}$, $k \in \{0, 1, 2, 3\}$, there are 8 such lines, four in one direction and four in a perpendicular direction. There are $4 \times 3 = 12$ such planes. However, each line lies in two of these (v, k) planes, so there are $\frac{8 \times 4 \times 3}{2} = 48$ such lines. Each of these lines has 4 points, so there are 4 possible ways to choose 3 collinear points, giving $4 \times 48 = 192$ triplets.

- (b) **Diagonal lines containing four points parallel to some main plane.** Consider a plane of the form (v, k) , as defined above. These each have 2 diagonals that contain 4 collinear points. Each of these diagonals uniquely determines v, k so these diagonals are each counted once. There are 12 possible (v, k) pairs, yielding $12 \times 2 \times 4 = 96$ triplets.
- (c) **Diagonal lines containing three points parallel to some main plane.** Again, consider a plane (v, k) . By inspection, there are four such lines and one way to choose the triplet of points for each of these lines. This yields $4 \times 12 = 48$ triplets.
- (d) **Main diagonals.** There are four main diagonals, each with 4 collinear points, yielding $4 \times 4 = 16$ triplets.
- (e) **Space diagonals containing three points.** Choose one of the points in the set $\{1, 2\}^3$ to be the midpoint of the line. Since these 8 possibilities are symmetric, say we take the point $(1, 1, 1)$. There are four space diagonals passing through this point, but one is a main diagonal. So each of the 8 points has 3 such diagonals with 3 points each, yielding $8 \times 3 = 24$ ways.

Adding all these yields $192 + 96 + 48 + 16 + 24 = 376$.

17. [11] Find the least positive integer $N > 1$ satisfying the following two properties:

- There exists a positive integer a such that $N = a(2a - 1)$.
- The sum $1 + 2 + \dots + (N - 1)$ is divisible by k for every integer $1 \leq k \leq 10$.

Answer: [2016] The second condition implies that 16 divides $a(2a - 1)(2a^2 - a - 1)$, which shows that $a \equiv 0$ or 1 modulo 16. The case $a = 1$ would contradict the triviality-avoiding condition $N > 1$. a cannot be 16, because 7 does not divide $a(2a - 1)(2a^2 - a - 1)$. a cannot be 17, because 9 does not divide $a(2a - 1)(2a^2 - a - 1)$. It can be directly verified that $a = 32$ is the smallest positive integer for which $1 + 2 + \dots + (N - 1) = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31$ which is divisible by $1, 2, \dots, 10$. For this a , we compute $N = 32(2 \cdot 32 - 1) = 2016$.

18. [11] Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a function such that for any integers x, y , we have

$$f(x^2 - 3y^2) + f(x^2 + y^2) = 2(x + y)f(x - y).$$

Suppose that $f(n) > 0$ for all $n > 0$ and that $f(2015) \cdot f(2016)$ is a perfect square. Find the minimum possible value of $f(1) + f(2)$.

Answer: [246] Plugging in $-y$ in place of y in the equation and comparing the result with the original equation gives

$$(x - y)f(x + y) = (x + y)f(x - y)$$

This shows that whenever $a, b \in \mathbb{Z} - \{0\}$ with $a \equiv b \pmod{2}$, we have

$$\frac{f(a)}{a} = \frac{f(b)}{b}$$

which implies that there are constants $\alpha = f(1) \in \mathbb{Z}_{>0}, \beta = f(2) \in \mathbb{Z}_{>0}$ for which f satisfies the equation (*):

$$f(n) = \begin{cases} n \cdot \alpha & \text{when } 2 \nmid n \\ \frac{n}{2} \cdot \beta & \text{when } 2 \mid n \end{cases}$$

Therefore, $f(2015)f(2016) = 2015\alpha \cdot 1008\beta = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13 \cdot 31\alpha\beta$, so $\alpha\beta = 5 \cdot 7 \cdot 13 \cdot 31 \cdot t^2$ for some $t \in \mathbb{Z}_{>0}$. We claim that $(\alpha, \beta, t) = (5 \cdot 31, 7 \cdot 13, 1)$ is a triple which gives the minimum $\alpha + \beta$. In particular, we claim $\alpha + \beta \geq 246$.

Consider the case $t \geq 2$ first. We have, by AM-GM, $\alpha + \beta \geq 2 \cdot \sqrt{\alpha\beta} \geq 4 \cdot \sqrt{14105} > 246$. Suppose $t = 1$. We have $\alpha \cdot \beta = 5 \cdot 7 \cdot 13 \cdot 31$. Because $(\alpha + \beta)^2 - (\alpha - \beta)^2 = 4\alpha\beta$ is fixed, we want to have α as close as β as possible. This happens when one of α, β is $5 \cdot 31$ and the other is $7 \cdot 13$. In this case, $\alpha + \beta = 91 + 155 = 246$.

Finally, we note that the equality $f(1) + f(2) = 246$ can be attained. Consider $f : \mathbb{Z} \rightarrow \mathbb{Z}$ such that $f(n) = 91n$ for every odd $n \in \mathbb{Z}$ and $f(n) = \frac{155}{2}n$ for every even $n \in \mathbb{Z}$. It can be verified that f satisfies the condition in the problem and $f(1) + f(2) = 246$ as claimed.

19. [11] Find the smallest positive integer n such that the polynomial $(x+1)^n - 1$ is “divisible by $x^2 + 1$ modulo 3”, or more precisely, either of the following **equivalent** conditions holds:

- there exist polynomials P, Q with integer coefficients such that $(x+1)^n - 1 = (x^2 + 1)P(x) + 3Q(x)$;
- or more conceptually, the remainder when (the polynomial) $(x+1)^n - 1$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with (integer) coefficients all divisible by 3.

Answer: 8 **Solution 1.** We have $(x+1)^2 = x^2 + 2x + 1 \equiv 2x$, $(x+1)^4 \equiv (2x)^2 \equiv -4 \equiv -1$, and $(x+1)^8 \equiv (-1)^2 = 1$. So the order n divides 8, as $x+1$ and $x^2 + 1$ are relatively prime polynomials modulo 3 (or more conceptually, in $\mathbb{F}_3[x]$), but cannot be smaller by our computations of the 2nd and 4th powers.

Remark. This problem illustrates the analogy between integers and polynomials (specifically here, polynomials over the *finite field* of integers modulo 3), with $x^2 + 1 \pmod{3}$ taking the role of a prime number. Indeed, we can prove analogously to Fermat’s little theorem that $f(x)^{3^{\deg(x^2+1)}} \equiv f(x) \pmod{x^2 + 1, 3}$ for any polynomial $f(x)$: try to work out the proof yourself! (This kind of problem, with 3 replaced by any prime p and $x^2 + 1$ by any irreducible polynomial modulo p , is closely related to the theory of (*extensions of*) *finite fields*.)

Solution 2. Here’s a solution avoiding the terminology of groups and fields. Let $R(x) = (x+1)^n - 1 - P(x)(x^2 + 1) = 3Q(x)$ be the remainder (here P is the quotient), whose coefficients (upon expansion) must all be divisible by 3. Now consider $R(i)$. The coefficients of the even and odd powers of x are divisible by 3, so $R(i)$ must have real and imaginary parts divisible by 3. Notice that $(1+i)^2 = 2i$, $(1+i)^4 = -4$, $(1+i)^6 = -8i$, $(1+i)^8 = 16$, and that $(1+i)^8 - 1 = 15$, which has real and imaginary parts divisible by 3. Since any even power of $(1+i)$ (for $n \leq 6$) yields a purely real or purely imaginary number with a coefficient not divisible by 3, multiplying it by $1+i$ will yield an imaginary part not divisible by 3. To see that $n = 8$ works, notice that, when taken modulo 3,

$$\begin{aligned} (x+1)^8 - 1 &= x^8 + 8x^7 + 28x^6 + 56x^5 + 70x^4 + 56x^3 + 28x^2 + 8x \\ &\equiv x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x \\ &= (x^2 + 1)(x^6 - x^5 + x^2 - x) \pmod{3}, \end{aligned}$$

as desired.

20. [11] What is the largest real number θ less than π (i.e. $\theta < \pi$) such that

$$\prod_{k=0}^{10} \cos(2^k \theta) \neq 0$$

and

$$\prod_{k=0}^{10} \left(1 + \frac{1}{\cos(2^k \theta)} \right) = 1?$$

Answer: $\frac{2046\pi}{2047}$ For equality to hold, note that θ cannot be an integer multiple of π (or else $\sin = 0$ and $\cos = \pm 1$).

Let $z = e^{i\theta/2} \neq \pm 1$. Then in terms of complex numbers, we want

$$\prod_{k=0}^{10} \left(1 + \frac{2}{z^{2^{k+1}} + z^{-2^{k+1}}} \right) = \prod_{k=0}^{10} \frac{(z^{2^k} + z^{-2^k})^2}{z^{2^{k+1}} + z^{-2^{k+1}}},$$

which partially telescopes to

$$\frac{z + z^{-1}}{z^{2^{11}} + z^{-2^{11}}} \prod_{k=0}^{10} (z^{2^k} + z^{-2^k}).$$

Using a classical telescoping argument (or looking at binary representation; if you wish we may note that $z - z^{-1} \neq 0$, so the ultimate telescoping identity holds²), this simplifies to

$$\frac{z + z^{-1}}{z^{2^{11}} + z^{-2^{11}}} \frac{z^{2^{11}} - z^{-2^{11}}}{z - z^{-1}} = \frac{\tan(2^{10}\theta)}{\tan(\theta/2)}.$$

Since $\tan x$ is injective modulo π (i.e. π -periodic and injective on any given period), θ works if and only if $\frac{\theta}{2} + \ell\pi = 1024\theta$ for some integer ℓ , so $\theta = \frac{2\ell\pi}{2047}$. The largest value for ℓ such that $\theta < \pi$ is at $\ell = 1023$, which gives $\theta = \frac{2046\pi}{2047}$.³

Remark. It's also possible to do this without complex numbers, but it's less systematic. The steps are the same, though, first note that $1 + \sec 2^k\theta = \frac{1 + \cos 2^k\theta}{\cos 2^k\theta} = \frac{2 \cos^2 2^{k-1}\theta}{\cos 2^k\theta}$ using the identity $\cos 2x = 2 \cos^2 x - 1$ (what does this correspond to in complex numbers?). Then we telescope using the identity $2 \cos x = \frac{\sin 2x}{\sin x}$ (again, what does this correspond to in complex numbers?).

21. [14] Define a sequence $a_{i,j}$ of integers such that $a_{1,n} = n^n$ for $n \geq 1$ and $a_{i,j} = a_{i-1,j} + a_{i-1,j+1}$ for all $i, j \geq 1$. Find the last (decimal) digit of $a_{128,1}$.

Answer: 4 By applying the recursion multiple times, we find that $a_{1,1} = 1$, $a_{2,n} = n^n + (n+1)^{n+1}$, and $a_{3,n} = n^n + 2(n+1)^{n+1} + (n+2)^{n+2}$. At this point, we can conjecture and prove by induction that

$$a_{m,n} = \sum_{k=0}^{m-1} \binom{m-1}{k} (n+k)^{n+k} = \sum_{k \geq 0} \binom{m-1}{k} (n+k)^{n+k}.$$

(The second expression is convenient for dealing with boundary cases. The induction relies on $\binom{m}{0} = \binom{m-1}{0}$ on the $k=0$ boundary, as well as $\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}$ for $k \geq 1$.) We fix $m = 128$. Note that $\binom{127}{k} \equiv 1 \pmod{2}$ for all $1 \leq k \leq 127$ and $\binom{127}{k} \equiv 0 \pmod{5}$ for $3 \leq k \leq 124$, by Lucas' theorem on binomial coefficients. Therefore, we find that

$$a_{128,1} = \sum_{k=0}^{127} \binom{127}{k} (k+1)^{k+1} \equiv \sum_{k=0}^{127} (k+1)^{k+1} \equiv 0 \pmod{2}$$

and

$$a_{128,1} \equiv \sum_{k \in [0,2] \cup [125,127]} \binom{127}{k} (k+1)^{k+1} \equiv 4 \pmod{5}.$$

Therefore, $a_{128,1} \equiv 4 \pmod{10}$.

22. [14] Let $A_1, A_2, \dots, A_{2015}$ be distinct points on the unit circle with center O . For every two distinct integers i, j , let P_{ij} be the midpoint of A_i and A_j . Find the smallest possible value of

$$\sum_{1 \leq i < j \leq 2015} OP_{ij}^2.$$

Answer: $\frac{2015 \cdot 2013}{4}$ OR $\frac{4056195}{4}$ Use vectors. $\sum |a_i + a_j|^2/4 = \sum (2 + 2a_i \cdot a_j)/4 = \frac{1}{2} \binom{2015}{2} + \frac{1}{4} (|\sum a_i|^2 - \sum |a_i|^2) \geq 2015 \cdot \frac{2014}{4} - \frac{2015}{4} = \frac{2015 \cdot 2013}{4}$, with equality if and only if $\sum a_i = 0$, which occurs for instance for a regular 2015-gon.

²The identity still holds even if $z^{2^k} - z^{-2^k} = 0$ for some $k \geq 1$ used in the telescoping argument: why?

³This indeed works, since $\prod_{k=0}^{10} \cos(2^k\theta) \neq 0$: why?

23. [14] Let $S = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. A subset P of S is called *squarely* if it is nonempty and the sum of its elements is a perfect square. A squarely set Q is called *super squarely* if it is not a proper subset of any squarely set. Find the number of super squarely sets.

(A set A is said to be a *proper subset* of a set B if A is a subset of B and $A \neq B$.)

Answer: 5 Clearly we may biject squarely sets with binary representations of perfect squares between 1 and $2^0 + \dots + 2^8 = 2^9 - 1 = 511$, so there are 22 squarely sets, corresponding to n^2 for $n = 1, 2, \dots, 22$. For convenience, we say N is (super) squarely if and only if the set corresponding to N is (super) squarely.

The general strategy is to rule out lots of squares at time, by searching for squares with few missing digits (and ideally most 1's consecutive, for simplicity). We can restrict ourselves (for now) to odds; $(2k)^2$ is just k^2 with two additional zeros at the end. 1, 9, 25, 49, 81 are ineffective, but $121 = 2^7 - 7 = 2^6 + 2^5 + 2^4 + 2^3 + 2^0$ immediately rules out all odd squares up to 9^2 , as they must be 1 (mod 8).

Fortunately, $22^2 = 4 \cdot 11^2$ is in our range (i.e. less than 512), ruling out all even squares up to 20^2 as well.

This leaves us with $11^2, 13^2, 15^2, 17^2, 19^2, 21^2, 22^2$, with binary representations 001111001, 010101001, 011100001, 100100001, 101101001 (kills 17^2), 110111001 (kills 13^2), 111100100 (kills nothing by parity). Thus $11^2, 15^2, 19^2, 21^2, 22^2$ are the only super squarely numbers, for a total of 5.

24. [14] $ABCD$ is a cyclic quadrilateral with sides $AB = 10$, $BC = 8$, $CD = 25$, and $DA = 12$. A circle ω is tangent to segments DA , AB , and BC . Find the radius of ω .

Answer: $\sqrt{\frac{1209}{7}}$ OR $\frac{\sqrt{8463}}{7}$ Denote E an intersection point of AD and BC . Let $x = EA$ and $y = EB$. Because $ABCD$ is a cyclic quadrilateral, $\triangle EAB$ is similar to $\triangle ECD$. Therefore, $\frac{y+8}{x} = \frac{25}{10}$ and $\frac{x+12}{y} = \frac{25}{10}$. We get $x = \frac{128}{21}$ and $y = \frac{152}{21}$. Note that ω is the E -excircle of $\triangle EAB$, so we may finish by standard calculations.

Indeed, first we compute the semiperimeter $s = \frac{EA+AB+BE}{2} = \frac{x+y+10}{2} = \frac{35}{3}$. Now the radius of ω is (by Heron's formula for area)

$$r_E = \frac{[EAB]}{s - AB} = \sqrt{\frac{s(s-x)(s-y)}{s-10}} = \sqrt{\frac{1209}{7}} = \frac{\sqrt{8463}}{7}.$$

25. [17] Let r_1, \dots, r_n be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \dots + r_n^2$.

Answer: 8 Observe that

$$\begin{aligned} x^8 - 14x^4 - 8x^3 - x^2 + 1 &= (x^8 + 2x^4 + 1) - (16x^4 + 8x^3 + x^2) \\ &= (x^4 + 4x^2 + x + 1)(x^4 - 4x^2 - x + 1). \end{aligned}$$

The polynomial $x^4 + 4x^2 + x + 1 = x^4 + \frac{15}{4}x^2 + (\frac{x}{2} + 1)^2$ has no real roots. On the other hand, let $P(x) = x^4 - 4x^2 - x + 1$. Observe that $P(-\infty) = +\infty > 0$, $P(-1) = -1 < 0$, $P(0) = 1 > 0$, $P(1) = -3 < 0$, $P(+\infty) = +\infty > 0$, so by the intermediate value theorem, $P(x) = 0$ has four distinct real roots, which are precisely the real roots of the original degree 8 equation. By Vieta's formula on $P(x)$,

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 + r_4^2 &= (r_1 + r_2 + r_3 + r_4)^2 - 2 \cdot \left(\sum_{i < j} r_i r_j \right) \\ &= 0^2 - 2(-4) = 8. \end{aligned}$$

26. [17] Let $a = \sqrt{17}$ and $b = i\sqrt{19}$, where $i = \sqrt{-1}$. Find the maximum possible value of the ratio $|a - z|/|b - z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z| = 1$).

Answer: $\boxed{\frac{4}{3}}$ Let $|a - z|/|b - z| = k$. We wish to determine the minimum and maximum value of k . Squaring and expansion give:

$$\begin{aligned} |a - z|^2 &= |b - z|^2 \cdot k^2 \\ |a|^2 - 2a \cdot z + 1 &= (|b|^2 - 2b \cdot z + 1)k^2 \\ |a|^2 + 1 - (|b|^2 + 1)k^2 &= 2(a - bk^2) \cdot z, \end{aligned}$$

where \cdot is a dot product of complex numbers, i.e., the dot product of vectors corresponding to the complex numbers in the complex plane. Now, since z has modulus 1 but can assume any direction, the only constraint on the value of k is

$$||a|^2 + 1 - (|b|^2 + 1)k^2| \leq 2|a - bk^2|.$$

Squaring again and completing the square, the inequality reduces to:

$$\begin{aligned} (|a|^2 - 1)^2 + (|b|^2 - 1)^2 k^4 + 2(4a \cdot b - (|a|^2 + 1)(|b|^2 + 1))k^2 &\leq 0 \\ ((|a|^2 - 1) - (|b|^2 - 1)k^2)^2 - 4|a - b|^2 k^2 &\leq 0 \\ |(|a|^2 - 1) - (|b|^2 - 1)k^2| &\leq 2|a - b|k. \end{aligned}$$

At this stage all the relevant expressions are constant real numbers. Denote, for simplicity, $A = |a|^2 - 1$, $B = |b|^2 - 1$, and $C = |a - b|$. Then, we are looking for k such that $|A - Bk^2| \leq 2Ck$. If $B = 0$, then $k \geq |\frac{A}{2C}|$, so the minimum value is $|\frac{A}{2C}|$ and the maximum value is $+\infty$. Otherwise, consider

$$\begin{aligned} C^2 + AB &= (|a|^2 - 2a \cdot b + |b|^2) + (|a|^2 - 1)(|b|^2 - 1) \\ &= |ab|^2 - 2a \cdot b + 1 \\ &= |\bar{a}b|^2 - 2\Re(\bar{a}b) + 1 \\ &= |\bar{a}b - 1|^2 \end{aligned}$$

So let $D = |\bar{a}b - 1| = \sqrt{C^2 + AB}$. We may assume $B > 0$ (the another case is analogous: just substitute A, B with $-A, -B$). Then, k is determined by the following inequalities:

$$\begin{aligned} Bk^2 + 2Ck - A &\geq 0 \\ Bk^2 - 2Ck - A &\leq 0 \end{aligned}$$

The first inequality gives $k \leq \frac{-C-D}{B}$ or $k \geq \frac{-C+D}{B}$, and the second gives $\frac{C-D}{B} \leq k \leq \frac{C+D}{B}$. Combining, this gives $|\frac{C-D}{B}| \leq k \leq |\frac{C+D}{B}|$, as claimed.

To summarize the general answer, let $A = |a|^2 - 1$, $B = |b|^2 - 1$, $C = |a - b|$, $D = |\bar{a}b - 1|$. Then, if $|b| = 1$, min is $|\frac{A}{2C}|$ and max is $+\infty$; otherwise, min is $|\frac{C-D}{B}|$ and max is $|\frac{C+D}{B}|$.

In the special case $a = \sqrt{17}$ and $b = \sqrt{19}i$, we have $A = 16$, $B = 18$, $C = |\sqrt{17} - \sqrt{19}i| = \sqrt{36} = 6$, and $D = \sqrt{17 \cdot 19 + 1} = 18$. Thus the answer is $\frac{C+D}{B} = \frac{6+18}{18} = \frac{4}{3}$.

27. [17] Let a, b be integers chosen independently and uniformly at random from the set $\{0, 1, 2, \dots, 80\}$. Compute the expected value of the remainder when the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is divided by 3. (Here $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ whenever $a < b$.)

Answer: $\boxed{\frac{1816}{6561}}$ By Lucas' Theorem we're looking at

$$\prod_{i=1}^4 \binom{a_i}{b_i}$$

Guts

where the a_i and b_i are the digits of a and b in base 3. If any $a_i < b_i$, then the product is zero modulo 3.

Otherwise, the potential residues are $\binom{2}{0} = 1$, $\binom{2}{1} = 2$, $\binom{2}{2} = 1$, $\binom{1}{0} = 1$, $\binom{1}{1} = 1$, $\binom{0}{0} = 1$.

So each term in the product has a $\frac{1}{3}$ chance of being zero; given that everything is nonzero, each term has a $\frac{1}{6}$ chance of being 2 and a $\frac{5}{6}$ chance of being 1. The probability that an even number of terms are 1 given that none are zero is then given by the roots of unity filter

$$\frac{\left(\frac{5}{6} + \frac{1}{6} \cdot (1)\right)^4 + \left(\frac{5}{6} + \frac{1}{6} \cdot (-1)\right)^4}{2} = \frac{81 + 16}{162} = \frac{97}{162}.$$

Thus the expected value is

$$\left(\frac{2}{3}\right)^4 \left(2 - \frac{97}{162}\right) = \frac{1816}{6561}.$$

28. [17] Let w, x, y , and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z \\ 2\pi &= w + x + y + z \\ 3 \tan w &= k(1 + \sec w) \\ 4 \tan x &= k(1 + \sec x) \\ 5 \tan y &= k(1 + \sec y) \\ 6 \tan z &= k(1 + \sec z). \end{aligned}$$

(Here $\sec t$ denotes $\frac{1}{\cos t}$ when $\cos t \neq 0$.) Find k .

Answer: $\boxed{\sqrt{19}}$ From the identity $\tan \frac{u}{2} = \frac{\sin u}{1 + \cos u}$, the conditions work out to $3 \tan \frac{w}{2} = 4 \tan \frac{x}{2} = 5 \tan \frac{y}{2} = 6 \tan \frac{z}{2} = k$. Let $a = \tan \frac{w}{2}$, $b = \tan \frac{x}{2}$, $c = \tan \frac{y}{2}$, and $d = \tan \frac{z}{2}$. Using the identity $\tan(M + N) = \frac{\tan M + \tan N}{1 - \tan M \tan N}$, we obtain

$$\begin{aligned} \tan\left(\frac{w+x}{2} + \frac{y+z}{2}\right) &= \frac{\tan\left(\frac{w+x}{2}\right) + \tan\left(\frac{y+z}{2}\right)}{1 - \tan\left(\frac{w+x}{2}\right)\tan\left(\frac{y+z}{2}\right)} \\ &= \frac{\frac{a+b}{1-ab} + \frac{c+d}{1-cd}}{1 - \left(\frac{a+b}{1-ab}\right)\left(\frac{c+d}{1-cd}\right)} \\ &= \frac{a+b+c+d - abc - abd - bcd - acd}{1 + abcd - ab - ac - ad - bc - bd - cd}. \end{aligned}$$

Because $x + y + z + w = \pi$, we get that $\tan\left(\frac{x+y+z+w}{2}\right) = 0$ and thus $a+b+c+d = abc+abd+bcd+acd$. Substituting a, b, c, d corresponding to the variable k , we obtain that $k^3 - 19k = 0$. Therefore, k can be only $0, \sqrt{19}, -\sqrt{19}$. However, $k = 0$ is impossible as w, x, y, z will all be 0. Also, $k = -\sqrt{19}$ is impossible as w, x, y, z will exceed π . Therefore, $k = \sqrt{19}$.

29. [20] Let ABC be a triangle whose incircle has center I and is tangent to $\overline{BC}, \overline{CA}, \overline{AB}$, at D, E, F . Denote by X the midpoint of major arc \overline{BAC} of the circumcircle of ABC . Suppose P is a point on line XI such that $\overline{DP} \perp \overline{EF}$.

Given that $AB = 14$, $AC = 15$, and $BC = 13$, compute DP .

Answer: $\boxed{\frac{4\sqrt{5}}{5}}$ Let H be the orthocenter of triangle DEF . We claim that P is the midpoint of \overline{DH} . Indeed, consider an inversion at the incircle of ABC , denoting the inverse of a point with an asterik. It maps ABC to the nine-point circle of $\triangle DEF$. According to $\angle IAX = 90^\circ$, we have $\angle A^*X^*I = 90^\circ$. Hence line XI passes through the point diametrically opposite to A^* , which is the midpoint of \overline{DH} , as claimed.

The rest is a straightforward computation. The inradius of $\triangle ABC$ is $r = 4$. The length of EF is given by $EF = 2 \frac{AF \cdot r}{AI} = \frac{16}{\sqrt{5}}$. Then,

$$DP^2 = \left(\frac{1}{2} DH \right)^2 = \frac{1}{4} (4r^2 - EF^2) = 4^2 - \frac{64}{5} = \frac{16}{5}.$$

Hence $DP = \frac{4\sqrt{5}}{5}$.

Remark. This is also not too bad of a coordinate bash.

30. [20] Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

Answer: $\boxed{-\frac{7}{16}}$ For convenience denote the polynomial by $P(x)$. Notice $4 + 8 = 7 + 5 = 12$ and that the consecutive terms $12x^6 - 12x^5 + 12x^4$ are the leading terms of $12\Phi_{14}(x)$, which is suggestive. Indeed, consider ω a primitive 14-th root of unity; since $\omega^7 = -1$, we have $4\omega^{10} = -4\omega^3$, $-7\omega^9 = 7\omega^2$, and so on, so that

$$P(\omega) = 12(\omega^6 - \omega^5 + \dots + 1) = 12\Phi_{14}(\omega) = 0.$$

Dividing, we find

$$P(x) = \Phi_{14}(x)(4x^4 - 3x^3 - 2x^2 - 3x + 4).$$

This second polynomial is symmetric; since 0 is clearly not a root, we have

$$4x^4 - 3x^3 - 2x^2 - 3x + 4 = 0 \iff 4\left(x + \frac{1}{x}\right)^2 - 3\left(x + \frac{1}{x}\right) - 10 = 0.$$

Setting $y = x + 1/x$ and solving the quadratic gives $y = 2$ and $y = -5/4$ as solutions; replacing y with $x + 1/x$ and solving the two resulting quadratics give the double root $x = 1$ and the roots $(-5 \pm i\sqrt{39})/8$ respectively. Together with the primitive fourteenth roots of unity, these are all the roots of our polynomial.

Explicitly, the roots are

$$e^{\pi i/7}, e^{3\pi i/7}, e^{5\pi i/7}, e^{9\pi i/7}, e^{11\pi i/7}, e^{13\pi i/7}, 1, (-5 \pm i\sqrt{39})/8.$$

The sum of squares of the roots of unity (including 1) is just 0 by symmetry (or a number of other methods). The sum of the squares of the final conjugate pair is $\frac{2(5^2 - 39)}{8^2} = -\frac{14}{32} = -\frac{7}{16}$.

31. [20] Define a *power cycle* to be a set S consisting of the nonnegative integer powers of an integer a , i.e. $S = \{1, a, a^2, \dots\}$ for some integer a . What is the minimum number of power cycles required such that given any odd integer n , there exists some integer k in one of the power cycles such that $n \equiv k \pmod{1024}$?

Answer: $\boxed{10}$ **Solution 1.** Partition the odd residues mod 1024 into 10 classes:

- Class 1: $1 \pmod{4}$.
- Class n ($2 \leq n \leq 9$): $2^n - 1 \pmod{2^{n+1}}$.
- Class 10: $-1 \pmod{1024}$.

Let S_a be the power cycle generated by a . If a is in class 1, all of S_a is in class 1. If a is in class n ($2 \leq n \leq 9$), then S_a is in the union of class n and the residues $1 \pmod{2^{n+1}}$. If a is in class 10, then S_a is in the union of class n and the residues $1 \pmod{1024}$. Therefore, S_a cannot contain two of the following residues: $5, 2^2 - 1, 2^3 - 1, \dots, 2^{10} - 1$, and that at least 10 cycles are needed.

Note that $5^{128} - 1 = (5 - 1)(5 + 1)(5^2 + 1) \dots (5^{64} + 1)$ has exactly 9 factors of 2 in its prime factorization, while $5^{256} - 1 = (5^{128} - 1)(5^{128} + 1)$ is divisible by 1024 so the order of 5 modulo 1024, the smallest

positive power of 5 that is congruent to 1, is 256. Observe that among $5^0, 5^1, \dots, 5^{255}$, the ratio between any two is a positive power of 5 smaller than 5^{256} , so the ratio is not congruent to 1 and any two terms are not congruent mod 1024. In addition, all terms are in class 1, and class 1 has 256 members, so S_5 contains members congruent to each element of class 1.

Similarly, let $2 \leq n \leq 9$. Then the order of a , where $a = 2^n - 1$, is 2^{10-n} . The 2^{9-n} terms $a^1, a^3, \dots, a^{2^{10-n}-1}$ are pairwise not congruent and all in class n . Class n only has 2^{9-n} members, so S_a contains members congruent to each element of class n .

Finally, S_{-1} contains members congruent to the element of class 10.

The cycles S_5, S_{-1} , and 8 cycles S_a cover all the residues mod 1024, so the answer is 10.

Solution 2. Lemma. Given a positive integer $n \geq 3$, there exists an odd integer x such that the order of x modulo 2^n is 2^{n-2} .

Proof. We apply induction on n . The base cases of $n = 3, 4$ are clearly true with $x = 3$, so suppose we have the statement holds for $n - 1$ and we wish to show it for n where $n \geq 5$. Suppose no such integer x exists, so we have $x^{2^{n-3}} \equiv 1 \pmod{2^n}$ for all odd x . But then remark that $(x^{2^{n-4}} - 1)(x^{2^{n-4}} + 1) \equiv 0 \pmod{2^n}$. As $n - 4 \geq 1$ we have $x^{2^{n-4}} + 1 \equiv 2 \pmod{4}$ as all squares are 1 mod 4, so it follows for the above relation to be true we require 2^{n-1} divides $x^{2^{n-4}} - 1$ for all odd x . However, by taking x to have order 2^{n-3} modulo 2^{n-1} (which exists by the inductive hypothesis) we get a contradiction so we are done. \square

Now, let x have order 2^8 modulo 2^{10} . Remark that if for some integer k we had $x^k \equiv -1 \pmod{2^{10}}$, we would have $x^{2k} \equiv 1 \pmod{2^{10}}$ so $2^7 \mid k$. In that case k is even so as x is odd we have $x^k \equiv 1 \pmod{4}$ but $-1 + 1024m$ is never 1 mod 4 for any integer m so it follows -1 is not equal to x^k modulo 1024 for any integer k . Thus it follows that when we let S to be the set of powers of x , then no two elements in S and $-S$ are congruent modulo 1024. As the order of x is 256 and there are 512 possible odd remainders upon dividing by 1024, it immediately follows that every integer x is congruent modulo 1024 to $\pm x^k$ for some $1 \leq k \leq 256$ and some choice of sign.

Now, it is easy so see that $x, -x, -x^2, -x^4, \dots, -x^{2^7}, -x^{2^8}$ generating 10 power cycles works (x^a for any a is in the first, and then $-x^{2^k a}$ for odd a is in the power cycle generated by $-x^{2^k}$). To show that we cannot do any better suppose there exist 9 power cycles which include all odd integers modulo 1024. Then remark that $-x^{2^k}$ is congruent to a number in a power cycle modulo 1024 if and only if the power cycle is generated by an integer congruent to $x^{-2^k a} \pmod{1024}$ where a is an odd integer. It follows that 9 integers must be congruent to $-x^{a_1}, -x^{2a_2}, \dots, -x^{2^{56}a_9} \pmod{1024}$ for some odd integers a_1, a_2, \dots, a_9 . But then 3 is clearly not in any of the power cycles, contradiction so it follows we must have at least 10 power cycles so we are done.

32. [20] A wealthy king has his blacksmith fashion him a large cup, whose inside is a cone of height 9 inches and base diameter 6 inches (that is, the opening at the top of the cup is 6 inches in diameter). At one of his many feasts, he orders the mug to be filled to the brim with cranberry juice.

For each positive integer n , the king stirs his drink vigorously and takes a sip such that the height of fluid left in his cup after the sip goes down by $\frac{1}{n^2}$ inches. Shortly afterwards, while the king is distracted, the court jester adds pure Soy lent to the cup until it's once again full. The king takes sips precisely every minute, and his first sip is exactly one minute after the feast begins.

As time progresses, the amount of juice consumed by the king (in cubic inches) approaches a number r . Find r .

Answer: $\frac{216\pi^3 - 2187\sqrt{3}}{8\pi^2}$ First, we find the total amount of juice consumed. We can simply subtract the amount of juice remaining at infinity from the initial amount of juice in the cup, which of course is simply the volume of the cup; we'll denote this value by V .

Since volume in the cup varies as the cube of height, the amount of juice remaining in the cup after m minutes is

$$V \cdot \prod_{n=1}^m \left(\frac{9 - \frac{1}{n^2}}{9} \right)^3 = V \cdot \left(\prod_{n=1}^m \left(1 - \frac{1}{9n^2} \right) \right)^3.$$

We can now factor the term inside the product to find

$$V \left(\prod_{n=1}^m \frac{(3n+1)(3n-1)}{9n^2} \right)^3 = V \left(\frac{(3m+1)!}{3^{3m}(m!)^3} \right)^3.$$

If remains to evaluate the limit of this expression as m goes to infinity.

However, by Stirling's approximation, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \frac{(3m+1)!}{3^{3m}(m!)^3} &= \lim_{m \rightarrow \infty} \frac{\left(\frac{3n+1}{e}\right)^{3n+1} \cdot \sqrt{2\pi(3n+1)}}{\left(\frac{3n}{e}\right)^{3n} \sqrt{(2\pi n)^3}} \\ &= \lim_{m \rightarrow \infty} \frac{(3n+1)\sqrt{3}}{2\pi n e} \left(\frac{3n+1}{3n}\right)^{3n} \\ &= \frac{3\sqrt{3}}{2\pi}. \end{aligned}$$

Therefore the total amount of juice the king consumes is

$$V - V \left(\frac{3\sqrt{3}}{2\pi} \right)^3 = \left(\frac{3^2 \cdot \pi \cdot 9}{3} \right) \left(\frac{8\pi^3 - 81\sqrt{3}}{8\pi^3} \right) = \frac{216\pi^3 - 2187\sqrt{3}}{8\pi^2}.$$

Remark. We present another way to calculate the limit at $m \rightarrow \infty$ of $f(m) = \frac{(3m+1)!}{3^{3m}(m!)^3}$. We have

$$f(m+1) = \frac{(3m+4)!}{3^{3m+3}(m+1)!^3} = f(m) \frac{\left(m + \frac{2}{3}\right)\left(m + \frac{4}{3}\right)}{(m+1)^2},$$

whence we can write

$$f(m) = \frac{c\Gamma\left(m + \frac{2}{3}\right)\Gamma\left(m + \frac{4}{3}\right)}{\Gamma(m+1)^2}$$

for some constant c . We can find c by equating the expressions at $m = 0$; we have

$$1 = f(0) = \frac{c\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(1)^2},$$

so that $c = \Gamma(1)^2/\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)$.

Of course, $\Gamma(1) = 0! = 1$. We can evaluate the other product as follows:

$$\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right) = \frac{1}{3}\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{1}{3}\right) = \frac{1}{3} \cdot \frac{\pi}{\sin \pi/3} = \frac{2\pi}{3\sqrt{3}}.$$

Here the first step follows from $\Gamma(n+1) = n\Gamma(n)$, while the second follows from Euler's reflection formula. Thus $c = 3\sqrt{3}/2\pi$. We can now compute

$$\lim_{m \rightarrow \infty} f(m) = \lim_{m \rightarrow \infty} \frac{c\Gamma\left(m + \frac{2}{3}\right)\Gamma\left(m + \frac{4}{3}\right)}{\Gamma(m+1)^2} = \frac{3\sqrt{3}}{2\pi} \lim_{m \rightarrow \infty} \frac{\Gamma\left(m + \frac{2}{3}\right)\Gamma\left(m + \frac{4}{3}\right)}{\Gamma(m+1)^2}.$$

Since $\lim_{n \rightarrow \infty} \Gamma(n+\alpha)/[\Gamma(n)n^\alpha] = 1$, this final limit is 1 and $f(m) \rightarrow 3\sqrt{3}/2\pi$ as $m \rightarrow \infty$.

33. [25] Let N denote the sum of the decimal digits of $\binom{1000}{100}$. Estimate the value of N . If your answer is a positive integer A written fully in decimal notation (for example, 521495223), your score will be the greatest integer not exceeding $25 \cdot (0.99)^{|A-N|}$. Otherwise, your score will be zero.

Answer: 621 [http://www.wolframalpha.com/input/?i=sum+of+digits+of+nCr\(1000,100\)](http://www.wolframalpha.com/input/?i=sum+of+digits+of+nCr(1000,100))

To see this, one can estimate there are about 150 digits, and we expect the digits to be roughly random, for $150 \cdot 4.5 \approx 675$, which is already very close to the actual answer. The actual number of digits is 140, and here $140 \cdot 4.5 = 630$ is within 9 of the actual answer.

34. [25] For an integer n , let $f(n)$ denote the number of pairs (x, y) of integers such that $x^2 + xy + y^2 = n$. Compute the sum

$$\sum_{n=1}^{10^6} n f(n).$$

Write your answer in the form $a \cdot 10^b$, where b is an integer and $1 \leq a < 10$ is a decimal number.

If your answer is written in this form, your score will be $\max\{0, 25 - \lfloor 100|\log_{10}(A/N)| \rfloor\}$, where $N = a \cdot 10^b$ is your answer to this problem and A is the actual answer. Otherwise, your score will be zero.

Answer: $1.813759629294 \cdot 10^{12}$ Rewrite the sum as

$$\sum_{x^2+xy+y^2 \leq 10^6} (x^2 + xy + y^2),$$

where the sum is over all pairs (x, y) of integers with $x^2 + xy + y^2 \leq 10^6$. We can find a crude upper bound for this sum by noting that

$$x^2 + xy + y^2 = \frac{3}{4}x^2 + \left(\frac{x}{2} + y\right)^2 \geq \frac{3}{4}x^2,$$

so each term of this sum has $|x| \leq \frac{2}{\sqrt{3}}10^3$. Similarly, $|y| \leq \frac{2}{\sqrt{3}}10^3$. Therefore, the number of terms in the sum is at most

$$\left(\frac{4}{\sqrt{3}}10^3 + 1\right)^2 \approx 10^6.$$

(We are throwing away “small” factors like $\frac{16}{3}$ in the approximation.) Furthermore, each term in the sum is at most 10^6 , so the total sum is less than about 10^{12} . The answer $1 \cdot 10^{12}$ would unfortunately still get a score of 0.

For a better answer, we can approximate the sum by an integral:

$$\sum_{x^2+xy+y^2 \leq 10^6} (x^2 + xy + y^2) \approx \iint_{x^2+xy+y^2 \leq 10^6} (x^2 + xy + y^2) dy dx.$$

Performing the change of variables $(u, v) = \left(\frac{\sqrt{3}}{2}x, \frac{1}{2}x + y\right)$ and then switching to polar coordinates $(r, \theta) = (\sqrt{u^2 + v^2}, \tan^{-1}(v/u))$ yields

$$\begin{aligned} \iint_{x^2+xy+y^2 \leq 10^6} (x^2 + xy + y^2) dy dx &= \frac{2}{\sqrt{3}} \iint_{u^2+v^2 \leq 10^6} (u^2 + v^2) dv du \\ &= \frac{2}{\sqrt{3}} \int_0^{2\pi} \int_0^{10^3} r^3 dr d\theta \\ &= \frac{4\pi}{\sqrt{3}} \int_0^{10^3} r^3 dr \\ &= \frac{\pi}{\sqrt{3}} \cdot 10^{12}. \end{aligned}$$

This is approximately $1.8138 \cdot 10^{12}$, which is much closer to the actual answer. (An answer of $1.8 \cdot 10^{12}$ is good enough for full credit.)

The answer can also be computed exactly by the Common Lisp code:

```
(defconstant +MAX+ 1e6)
(defvar +lower+ -2000)
(defvar +upper+ 2000)
(princ
```

```

(loop for x from +lower+ to +upper+ sum
  (loop for y from +lower+ to +upper+
    sum
      (let ((S (+ (* x x) (* x y) (* y y))))
        (if (and (<= S +MAX+) (> S 0)) S 0))))))

```

35. [25] Let P denote the set of all subsets of $\{1, \dots, 23\}$. A subset $S \subseteq P$ is called *good* if whenever A, B are sets in S , the set $(A \setminus B) \cup (B \setminus A)$ is also in S . (Here, $A \setminus B$ denotes the set of all elements in A that are not in B , and $B \setminus A$ denotes the set of all elements in B that are not in A .) What **fraction** of the good subsets of P have between 2015 and 3015 elements, inclusive?

If your answer is a decimal number or a fraction (of the form m/n , where m and n are positive integers), then your score on this problem will be equal to $\max\{0, 25 - \lfloor 1000|A - N| \rfloor\}$, where N is your answer and A is the actual answer. Otherwise, your score will be zero.

Answer:

$\frac{18839183877670041942218307147122500601235}{47691684840486192422095701784512492731212} \approx 0.3950203047068107$
--

 Let $n = 23$, and $\ell = \lfloor n/2 \rfloor = 11$.

We use the well-known rephrasing of the symmetric difference $((A \setminus B) \cup (B \setminus A))$ in terms of addition modulo 2 of “indicators/characteristic vectors”. So we simply want the number $\binom{n}{\ell}_2$ of dimension ℓ subspaces of the $F := \mathbb{F}_2$ -vector space $V := \mathbb{F}_2^n$. Indeed, good subsets of 2^d elements simply correspond to dimension d subspaces (in particular, good subsets can only have sizes equal to powers of $|F| = 2$, and 2^ℓ is the only power between 2015 and 3015, inclusive).

To do this, it’s easier to first count the number of (ordered) tuples of ℓ linearly independent elements of V , and divide (to get the subspace count) by the number of (ordered) tuples of ℓ linearly independent elements of any ℓ -dimensional subspace of V (a well-defined number independent of the choice of subspace).

In general, if we want to count tuples of m linearly independent elements in an n -dimensional space (with $n \geq m$), just note that we are building on top of (0) (the zero-dimensional subspace), and once we’ve chosen $r \leq m$ elements (with $0 \leq r \leq m - 1$), there are $2^n - 2^r$ elements linearly independent to the previous r elements (which span a subspace of dimension r , hence of 2^r “bad” elements). Thus the number of m -dimensional subspaces of an n -dimensional space is

$$\binom{n}{m}_2 := \frac{(2^n - 2^0)(2^n - 2^1) \dots (2^n - 2^{m-1})}{(2^m - 2^0)(2^m - 2^1) \dots (2^m - 2^{m-1})},$$

a “Gaussian binomial coefficient.”

We want to estimate

$$\frac{\binom{n}{\ell}_2}{\sum_{m=0}^n \binom{n}{m}_2} = \frac{1}{2} \frac{\binom{23}{11}_2}{\sum_{m=0}^{11} \binom{23}{m}_2}.$$

To do this, note that $\binom{n}{m}_2 = \binom{n}{n-m}_2$, so we may restrict our attention to the lower half. Intuitively, the Gaussian binomial coefficients should decay exponentially (or similarly quickly) away from the center; indeed, if $m \leq n/2$, then

$$\binom{n}{m-1}_2 / \binom{n}{m}_2 = \frac{(2^m - 2^0) \cdot 2^{m-1}}{(2^n - 2^{m-1})} \approx 2^{2m-1-n}.$$

So in fact, the decay is super-exponential, starting (for $n = 23$ odd and $m \leq \ell = 11$) at an $\approx \frac{1}{4}$ rate. So most of the terms (past the first 2 to 4, say) are negligible in our estimation. If we use the first two terms, we get an approximation of $\frac{1}{2} \cdot \frac{1}{1+\frac{1}{4}} = \frac{2}{5} = 0.4$, which is enough for 20 points. (Including the next term gives an approximation of $\frac{32}{81} \approx 0.3950617$, which is good enough to get full credit.)

To compute the exact answer, we used the following python3 code:

```

from functools import lru_cache
from fractions import Fraction

@lru_cache(maxsize=None)
def gauss_binom(n, k, e):
    if k < 0 or k > n:
        return 0
    if k == 0 or k == n:
        return 1
    return e ** k * gauss_binom(n - 1, k, e) + \
        gauss_binom(n - 1, k - 1, e)

N = 23
K = 11
good = gauss_binom(N, K, 2)
total = sum(gauss_binom(N, i, 2) for i in range(N + 1))
print(Fraction(good, total))
print(float(good/total))

```

36. [25] A prime number p is *twin* if at least one of $p + 2$ or $p - 2$ is prime and *sexy* if at least one of $p + 6$ and $p - 6$ is prime.

How many sexy twin primes (i.e. primes that are both twin and sexy) are there less than 10^9 ? Express your answer as a positive integer N in decimal notation; for example, 521495223. If your answer is in this form, your score for this problem will be $\max\{0, 25 - \lfloor \frac{1}{10000} |A - N| \rfloor\}$, where A is the actual answer to this problem. Otherwise, your score will be zero.

Answer: 1462105 The Hardy-Littlewood conjecture states that given a set A of integers, the number of integers x such that $x + a$ is a prime for all $a \in A$ is

$$\frac{x}{(\ln x)^{|A|}} \prod_p \frac{1 - \frac{w(p;A)}{p}}{(1 - \frac{1}{p})^k} (1 + o(1))$$

where $w(p; A)$ is the number of distinct residues of A modulo p and the $o(1)$ term goes to 0 as x goes to infinity. Note that for the 4 tuples of the form $(0, \pm 2, \pm 6)$, $w(p; A) = 3$, and using the approximation $\frac{1-k/p}{(1-1/p)^k} \approx 1 - \binom{k}{2}/p^2 \approx (1 - \frac{1}{p^2})^{\binom{k}{2}}$, we have

$$\prod_{p>3} \frac{1 - \frac{k}{p}}{(1 - \frac{1}{p})^k} \approx \left(\frac{6}{\pi^2}\right)^{\binom{k}{2}} \cdot \left(\frac{4}{3}\right)^{\binom{k}{2}} \left(\frac{9}{8}\right)^{\binom{k}{2}} \approx \left(\frac{9}{10}\right)^{\binom{k}{2}}$$

Applying this for the four sets $A = (0, \pm 2, \pm 6)$, $x = 10^9$ (and approximating $\ln x = 20$ and just taking the $p = 2$ and $p = 3$ terms, we get the approximate answer

$$4 \cdot \frac{10^9}{20^3} \frac{1 - \frac{1}{2}}{(\frac{1}{2})^3} \frac{1 - \frac{2}{3}}{(\frac{1}{3})^3} \left(\frac{9}{10}\right)^3 = 1640250$$

One improvement we can make is to remove the double-counted tuples, in particular, integers x such that $x, x + 6, x - 6$, and one of $x \pm 2$ is prime. Again by the Hardy-Littlewood conjecture, the number of such x is approximately (using the same approximations)

$$2 \cdot \frac{10^9}{20^4} \frac{1 - \frac{1}{2}}{(\frac{1}{2})^4} \frac{1 - \frac{2}{3}}{(\frac{1}{3})^4} \left(\frac{9}{10}\right)^6 \approx 90000$$

Subtracting gives an estimate of about 1550000. Note that this is still an overestimate, as $\ln 10^9$ is actually about 20.7 and $\frac{1-k/p}{(1-1/p)^k} < (1 - \frac{1}{p^2})^{\binom{k}{2}}$.

Here is the C++ code that we used to generate the answer:


```

#include<iostream>
#include<cstring> // memset
using namespace std;

const int MAXN = 1e9;
bool is_prime[MAXN + 6];

int main(){
    // Sieve of Eratosthenes
    memset(is_prime, true, sizeof(is_prime));
    is_prime[0] = is_prime[1] = false;
    for (int i=2; i<MAXN + 6; i++){
        if (is_prime[i]){
            for (int j=2 * i; j < MAXN + 6; j += i){
                is_prime[j] = false;
            }
        }
    }

    // Count twin sexy primes.
    int ans = 1; // 5 is the only twin sexy prime < 6.
    for (int i=6; i<MAXN; i++){
        if (is_prime[i]
            && (is_prime[i-6] || is_prime[i+6])
            && (is_prime[i-2] || is_prime[i+2])) {
            ans++;
        }
    }

    cout << ans << endl;

    return 0;
}

```

HMMT February 2015
Saturday 21 February 2015

Team

1. [5] The complex numbers x, y, z satisfy

$$\begin{aligned}xyz &= -4 \\(x+1)(y+1)(z+1) &= 7 \\(x+2)(y+2)(z+2) &= -3.\end{aligned}$$

Find, with proof, the value of $(x+3)(y+3)(z+3)$.

Answer: $\boxed{-28}$ **Solution 1.** Consider the cubic polynomial $f(t) = (x+t)(y+t)(z+t)$. By the theory of finite differences, $f(3) - 3f(2) + 3f(1) - f(0) = 3! = 6$, since f is monic. Thus $f(3) = 6 + 3f(2) - 3f(1) + f(0) = 6 + 3(-3) - 3(7) + (-4) = -28$.

Solution 2. Alternatively, note that the system of equations is a (triangular) linear system in $w := xyz$, $v := xy + yz + zx$, and $u := x + y + z$. The unique solution (u, v, w) to this system is $(-\frac{27}{2}, \frac{47}{2}, -4)$. Plugging in yields

$$\begin{aligned}(x+3)(y+3)(z+3) &= w + 3v + 9u + 27 \\&= -4 + 3 \cdot \frac{47}{2} + 9 \cdot \left(-\frac{27}{2}\right) + 27 \\&= -28.\end{aligned}$$

Remark. Since $f(0) < 0$, $f(1) > 0$, $f(2) < 0$, $f(+\infty) > 0$, the intermediate value theorem tells us the roots $-x, -y, -z$ of f are real numbers in $(0, 1)$, $(1, 2)$, and $(2, +\infty)$, in some order. With a little more calculation, one finds that x, y, z are the three distinct zeroes of the polynomial $X^3 + \frac{27}{2}X^2 + \frac{47}{2}X + 4$. The three zeroes are approximately $-11.484, -1.825$, and -0.191 .

2. [10] Let P be a (non-self-intersecting) polygon in the plane. Let C_1, \dots, C_n be circles in the plane whose interiors cover the interior of P . For $1 \leq i \leq n$, let r_i be the radius of C_i . Prove that there is a single circle of radius $r_1 + \dots + r_n$ whose interior covers the interior of P .

Answer: $\boxed{\text{N/A}}$ If $n = 1$, we are done. Suppose $n > 1$. Since P is connected, there must be a point x on the plane which lies in the interiors of two circles, say C_i, C_j . Let O_i, O_j , respectively, be the centers of C_i, C_j . Since $O_i O_j < r_i + r_j$, we can choose O to be a point on segment $O_i O_j$ such that $O_i O \leq r_j$ and $O_j O \leq r_i$. Replace the two circles C_i and C_j with the circle C centered at O of radius $r_i + r_j$. Note that C covers both C_i and C_j . Induct to finish.

3. [15] Let $z = a + bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ (i.e. z is a Gaussian integer). If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p .

Answer: $\boxed{\text{N/A}}$ **Solution 1.** We directly compute/expand

$$\begin{aligned}\operatorname{Re}(z^p - z) &= \operatorname{Re}((a + bi)^p - (a + bi)) \\&= \left[a^p - \binom{p}{2} a^{p-2} b^2 + \binom{p}{4} a^{p-4} b^4 - \dots \right] - a.\end{aligned}$$

Since $\binom{p}{i}$ is divisible by p for all $i = 2, 4, 6, \dots$ (since $1 \leq i \leq p-1$), we have

$$\left[a^p - \binom{p}{2} a^{p-2} b^2 + \binom{p}{4} a^{p-4} b^4 - \dots \right] - a \equiv a^p - a \equiv 0 \pmod{p}$$

by Fermat's little theorem. Thus p divides the real part of $z^p - z$.

Remark. The fact that $\binom{p}{i}$ is divisible by p (for $1 \leq i \leq p-1$) is essentially equivalent to the polynomial congruence $(rX + sY)^p \equiv rX^p + sY^p \pmod{p}$ (here the coefficients are taken modulo p), a fundamental result often called that “Frobenius endomorphism”.

Solution 2. From the Frobenius endomorphism,

$$z^p = (a + bi)^p \equiv a^p + (bi)^p = a^p \pm b^p i \equiv a \pm bi \pmod{p \cdot \mathbb{Z}[i]},$$

where we’re using congruence of Gaussian integers (so that $u \equiv v \pmod{p}$ if and only if $\frac{u-v}{p}$ is a Gaussian integer). This is equivalent to the simultaneous congruence of the real and imaginary parts modulo p , so the real part of z^p is congruent to a , the real part of z . So indeed p divides the real part of $z^p - z$.

4. [15] (Convex) quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle Ω ; the diagonals of $ABCD$ meet at X . Suppose $AD < AB$, the circumcircle of triangle BCX intersects segment AB at a point $Y \neq B$, and ray \overrightarrow{CY} meets Ω again at a point $Z \neq C$. Prove that ray \overrightarrow{DY} bisects angle ZDB .

(We have only included the conditions $AD < AB$ and that Z lies on ray \overrightarrow{CY} for everyone’s convenience. With slight modifications, the problem holds in general. But we will only grade your team’s solution in this special case.)

Answer: N/A This is mostly just angle chasing. The conditions $AD < AB$ (or $\angle ABD < \angle ADB$) and the assumption $Z \in \overrightarrow{CY}$ are not crucial, as long as we’re careful with configurations (for example, DY may only be an external angle bisector of $\angle ZDB$ in some cases),¹ but it’s the easiest to visualize. In this case Y and Z lie between A and B , on the respective segment/arc. We’ll prove Y is the incenter of $\triangle ZDB$; it will follow that ray \overrightarrow{DY} indeed (internally) bisects $\angle ZDB$. It suffices to prove the following two facts:

- BY is the internal angle bisector of $\angle DBZ$.²
- ZY is the internal angle bisector of $\angle BZD$, since $CB = CD$. Indeed (more explicitly), arcs BC and CD are equal, so $\angle BZC = \angle CZD$, i.e. YZ bisects $\angle BZD$.

5. [20] For a convex quadrilateral P , let D denote the sum of the lengths of its diagonals and let S denote its perimeter. Determine, with proof, all possible values of $\frac{S}{D}$.

Answer: $1 < \frac{S}{D} < 2$ Suppose we have a convex quadrilateral $ABCD$ with diagonals AC and BD intersecting at E (convexity is equivalent to having E on the interiors of segments AC and BD).

To prove the lower bound, note that by the **triangle inequality**, $AB + BC > AC$ and $AD + DC > AC$, so $S = AB + BC + AD + DC > 2AC$. Similarly, $S > 2BD$, so $2S > 2AC + 2BD = 2D$ gives $S > D$.

To prove the upper bound, note that again by the **triangle inequality**, $AE + EB > AB$, $CE + BE > BC$, $AE + ED > AD$, $CE + ED > CD$. Adding these yields

$$2(AE + EC + BE + ED) > AB + BC + AD + CD = S.$$

Now since $ABCD$ is convex, E is inside the quadrilateral, so $AE + EC = AC$ and $BE + ED = BD$. Thus $2(AC + BD) = D > S$.

To achieve every real value in this range, first consider a square $ABCD$. This has $\frac{S}{D} = \sqrt{2}$. Suppose now that we have a rectangle with $AB = CD = 1$ and $BC = AD = x$, where $0 < x \leq 1$. As x

¹The cleanest way to see this is via directed angles, or more rigorously by checking that the problem is equivalent to a four-variable polynomial identity in complex numbers.

²This is true in general; it doesn’t require $CB = CD$. It’s part of the spiral similarity configuration centered at $B : YX \rightarrow ZA$ and $B : ZY \rightarrow AX$, due to $YZ \cap AX = C$ and $B = (CYX) \cap (CZA)$.

More explicitly, this follows from the angle chase

$$\angle DBA = \angle XBY = \angle XCY = \angle ACZ = \angle ABZ.$$

approaches 0 (i.e. **our rectangle gets thinner**), $\frac{S}{D}$ gets arbitrarily close to 1, so by intermediate value theorem, we hit every value $\frac{S}{D} \in (1, \sqrt{2}]$.

To achieve the other values, we let $AB = BC = CD = DA = 1$ and let $\theta = m\angle ABE$ vary from 45° down to 0° (i.e. **a rhombus that gets thinner**). This means $AC = 2 \sin \theta$ and $BD = 2 \cos \theta$. We have $S = 4$ and $D = 2(\sin \theta + \cos \theta)$. When $\theta = 45^\circ$, $\frac{S}{D} = \sqrt{2}$, and when $\theta = 0^\circ$, $\frac{S}{D} = 2$. Thus by the intermediate value theorem, we are able to choose θ to obtain any value in the range $[\sqrt{2}, 2)$.

Putting this construction together with the strict upper and lower bounds, we find that all possible values of $\frac{S}{D}$ are all real values in the open interval $(1, 2)$.

6. [30] \$indy has \$100 in pennies (worth \$0.01 each), nickels (worth \$0.05 each), dimes (worth \$0.10 each), and quarters (worth \$0.25 each). Prove that she can split her coins into two piles, each with total value exactly \$50.

Answer: N/A **Solution 1.** First, observe that if there are pennies in the mix, there must be a multiple of 5 pennies (since $5, 10, 25 \equiv 0 \pmod{5}$), so we can treat each group of 5 pennies as nickels and reduce the problem to the case of no pennies, treated below.

Indeed, suppose there are no pennies and that we are solving the problem with only quarters, dimes, and nickels. So we have $25q + 10d + 5n = 10000$ (or equivalently, $5q + 2d + n = 2000$), where q is the number of quarters, d is the number of dimes, and n is the number of nickels. Notice that if $q \geq 200$, $d \geq 500$, or $n \geq 1000$, then we can just take a subset of 200 quarters, 500 dimes, or 1000 nickels, respectively, and make that our first pile and the remaining coins our second pile.

Thus, we can assume that $q \leq 199$, $d \leq 499$, $n \leq 999$. Now if there are only quarters and dimes, we have $5q + 2d < 1000 + 1000 = 2000$, so there must be nickels in the mix. By similar reasoning, there must be dimes and quarters in the mix.

So to make our first pile, we **throw all the quarters in** that pile. The total money in the pile is at most \$49.75, so now,

- **if possible, we keep adding in dimes until we get an amount equal to or greater than \$50.** If it is \$50, then we are done. Otherwise, it must be \$50.05 since the total money value before the last dime is either \$49.90 or \$49.95. In that case, we take out the last dime and replace it with a nickel, giving us \$50.
- **if not possible (i.e. we run out of dimes before reaching \$50),** then all the quarters and dimes have been used up, so we only have nickels left. Of course, our money value now has a hundredth's place of 5 or 0 and is less than \$50, so we simply keep adding nickels until we are done.

Solution 2. First, observe that if there are pennies in the mix, there must be a multiple of 5 pennies (since $5, 10, 25 \equiv 0 \pmod{5}$), so we can treat each group of 5 pennies as nickels and reduce the problem to the case of no pennies, treated below. Similarly, we can treat each group of 2 nickels as dimes, and reduce the problem to the case of at most one nickel. We have two cases:

Case 1. There are no nickels. Then, either the dimes alone total to at least \$50, in which case \$indy can form a pile with \$50 worth of dimes and a pile with the rest of the money, or the quarters alone total to at least \$50, in which case \$indy can form a pile with \$50 worth of quarters and a pile with the rest of the money.

Case 2. There is exactly one nickel. By examining the value of the money modulo \$0.25, we find that there must be at least two dimes. Then, we treat the nickel and two dimes as a quarter, and reduce the problem to the previously solved case.

Remark. There are probably many other solutions/algorithms for this specific problem. For example, many teams did casework based on the parities of q, d, n , after “getting rid of pennies”.

Remark. It would certainly be interesting to investigate the natural generalizations of this problem (see e.g. the knapsack problem and partition problem). For instance, it's true that given a set of

positive integers ranging from 1 to n with sum no less than $2n!$, there exists a subset of them summing up to exactly $n!$. (See here and here for discussion on two AoPS blogs.)

Remark. The problem author (Carl Lian) says the problem comes from algebraic geometry!

7. [35] Let $f : [0, 1] \rightarrow \mathbb{C}$ be a *nonconstant* complex-valued function on the real interval $[0, 1]$. Prove that there exists $\epsilon > 0$ (possibly depending on f) such that for any polynomial P with complex coefficients, there exists a complex number z with $|z| \leq 1$ such that $|f(|z|) - P(z)| \geq \epsilon$.

Answer: N/A Here's a brief summary of how we'll proceed: Use a suitably-centered high-degree roots of unity filter, or alternatively integrate over suitably-centered circles ("continuous roots of unity filter").

We claim we can choose $\epsilon(f) = \max(|f(x_1) - f(x_2)|)/2$. Fix P and suppose for the sake of contradiction that for all z with $|z| \leq 1$ it is the case that $|f(z) - P(z)| < \epsilon$. We can write $z = re^{i\theta}$ so that

$$|f(r) - P(re^{i\theta})| < \epsilon.$$

Let p be a prime larger than $\deg(P)$ and set $\theta = 0, 2\pi/p, 4\pi/p, \dots$ in the above. Averaging, and using triangle inequality,

$$|f(r) - \frac{1}{p} \sum_{k=0}^{p-1} P(re^{2\pi ik/p})| < \epsilon.$$

The sum inside the absolute value is a roots of unity filter; since $p > \deg(P)$ it is simply the constant term of P - denote this value by p_0 . Thus for all r ,

$$|f(r) - p_0| < \epsilon.$$

Setting $r = x_1, x_2$ and applying triangle inequality gives the desired contradiction.

Remark. This is more or less a concrete instance of Runge's approximation theorem from complex analysis. However, do not be intimidated by the fancy names here: in fact, the key ingredient behind this more general theorem (namely, or similar³) uses the same core idea of the "continuous roots of unity filter".

8. [40] Let π be a permutation of $\{1, 2, \dots, 2015\}$. With proof, determine the maximum possible number of ordered pairs $(i, j) \in \{1, 2, \dots, 2015\}^2$ with $i < j$ such that $\pi(i) \cdot \pi(j) > i \cdot j$.

Answer: $\binom{2014}{2}$ Let $n = 2015$. The only information we will need about n is that $n > 5$.⁴ For the construction, take π to be the n -cycle defined by

$$\pi(k) = \begin{cases} k+1 & \text{if } 1 \leq k \leq n-1 \\ 1 & \text{if } k = n. \end{cases}$$

Then $\pi(i) > i$ for $1 \leq i \leq n-1$. So $\pi(i)\pi(j) > ij$ for at least $\binom{n-1}{2}$ pairs $i < j$.

For convenience let $z_i = \frac{\pi(i)}{i}$, so that we are trying to maximize the number of pairs $(i, j), i < j$ with $z_i z_j > 1$. Notice that over any cycle $c = (i_1 i_2 \dots i_k)$ in the cycle decomposition of π we have $\prod_{i \in c} z_i = 1$. In particular, multiplying over all such cycles gives $\prod_{i=1}^n z_i = 1$.

Construct a graph G on vertex set $V = [n]$ such that there is an edge between i and j whenever $\pi(i)\pi(j) > ij$. For any cycle $C = (v_1, v_2, \dots, v_k)$ in this graph we get $1 < \prod_{i=1}^k z_{v_i} z_{v_{i+1}} = \prod_{v \in C} z_v^2$. So, we get in particular that G is non-Hamiltonian.

By the contrapositive of Ore's theorem,⁵ there are two distinct indices $u, v \in [n]$ such that $d(u) + d(v) \leq n - 1$. The number of edges is then at most $\binom{n-2}{2} + (n-1) = \binom{n-1}{2} + 1$.

³which generally underlies a great deal of complex analysis

⁴If you want, you can try running a computer check for $n \leq 5$. One of the problem czars believes the answer is still $\binom{4}{2} = 6$ (compare with the Remark at the end for a looser general problem) based on computer program, but didn't check carefully.

⁵see also this StackExchange thread

If equality is to hold, we need $G \setminus \{u, v\}$ to be a complete graph. We also need $d(u) + d(v) = n - 1$, with uv not an edge. This implies $d(u), d(v) \geq 1$. Since $n > 5$, the pigeonhole principle gives that at least one of u, v has degree at least 3. WLOG $d(u) \geq 3$.

Let w be a neighbor of v and let a, b, c be neighbors of u ; WLOG $w \neq a, b$. Since $G \setminus \{u, v, w\}$ is a complete graph, we can pick a Hamiltonian path in $G \setminus \{u, v, w\}$ with endpoints a, b . Connecting u to the ends of this path forms an $(n - 2)$ -cycle C .

This gives us $\prod_{x \in C} z_x^2 > 1$. But we also have $z_v z_w > 1$, so $1 = \prod_{i=1}^n z_i > 1$, contradiction.

So, $\binom{n-1}{2} + 1$ cannot be attained, and $\binom{n-1}{2}$ is indeed the maximum number of pairs possible.

Remark. Let's think about the more general problem where we forget about the permutation structure (and only remember $\prod z_i = 1$ and $z_i > 0$). If $n = 4$ it's easy to show by hand that $\binom{n-1}{2}$ is still the best possible. If $n = 5$, then we can get $1 + \binom{n-1}{2} = 7$: consider exponentiating the zero-sum set $\{t, t, t^2 - t, t^2 - t, -2t^2\}$ for some sufficiently small t . A computer check would easily determine whether a construction exists in the original permutation setting. (If the following code is correct, the answer is 'no', i.e. 6 is the best: see <http://codepad.org/Efsv8Suy>.)

9. [40] Let $z = e^{\frac{2\pi i}{101}}$ and let $\omega = e^{\frac{2\pi i}{10}}$. Prove that

$$\prod_{a=0}^9 \prod_{b=0}^{100} \prod_{c=0}^{100} (\omega^a + z^b + z^c)$$

is an integer and find (with proof) its remainder upon division by 101.

Answer: 13 **Solution 1.** Let $p = 101$ and $r = 10$. Note that $p \nmid r$.

In the sequel, we will repeatedly use the polynomial identities $\prod_{k \pmod p} (x - z^k) = x^p - 1$, and $\prod_{j \pmod r} (x - \omega^j) = x^r - 1$.

The product is an integer by standard symmetric sum theory (concrete precursor to Galois theory, though one doesn't need the full language). More precisely, it expands into an integer-coefficient polynomial symmetric in the ω^j (for the r residues $j \pmod r$), and also symmetric in the z^k (for the p residues $k \pmod p$). So by the fundamental theorem of symmetric sums, it can be written as an integer-coefficient polynomial in the symmetric sums of the ω^j together with the symmetric sums of the z^k . But these symmetric sums are integers themselves, so the original expression is indeed an integer.

To actually compute the remainder modulo p , note⁶ by the Frobenius endomorphism $(u+v)^p \equiv u^p + v^p \pmod p$ that

$$\begin{aligned} \prod_{a,b,c} (\omega^a + z^b + z^c) &= \prod_{a,b} (-1)^p ((-\omega^a - z^b)^p - 1) \\ &\equiv \prod_{a,b} (\omega^{pa} + z^{pb} - (-1)^p) = \prod_{a,b} (\omega^{pa} + 2) \pmod p, \end{aligned}$$

where we use $p \equiv 1 \pmod 2$ in the last step. This simplifies further to

$$\begin{aligned} \prod_a (-1)^p (-2 - \omega^{pa})^p &= \prod_a (-1)^p (-2 - \omega^a)^p \\ &= (-1)^{pr} [(-2)^r - 1]^p \equiv (-1)^{pr} [(-2)^r - 1] \pmod p, \end{aligned}$$

where we've used the fact that $\{pa \pmod r\} = \{a \pmod r\}$ (since p is invertible mod r), as well as Fermat's little theorem on $(-2)^r - 1$.

Finally, we plug in the specific numbers:

$$(-1)^{101 \cdot 10} [(-2)^{10} - 1] = 1023 \equiv 13 \pmod{101}.$$

⁶Here we are using congruence of algebraic integers modulo p , so that $\alpha \equiv \beta \pmod p$ if and only if $\frac{\alpha - \beta}{p}$ is an algebraic integer. In particular, for integers u, v , $\frac{u-v}{p}$ is an algebraic integer if and only if $\frac{u-v}{p}$ is an integer.

Remark. By repeatedly getting rid of “ z ” terms and applying the Frobenius map, one can show that $\prod(\omega^a + z^{b_1} + \dots + z^{b_m})$ is an integer congruent to something like $m^r + 1$ modulo p (as long as p is odd, etc.).

Solution 2 (sketch). Here we sketch a better solution found by several teams, but missed by the author (primarily due to blindness from having first found the first solution). The proof of the first part (that the product is an integer) is the same, so we only sketch different proofs of the second part (computation of the remainder mod p).

For example, we can use the algebraic integer formulation from the previous solution. Indeed, note that $D := \prod_{a,b,c}(\omega^a + z^b + z^c) - \prod_{a,b,c}(\omega^a + 1^b + 1^c)$ is an algebraic integer divisible by $z - 1$. But D is also a difference of integers, hence an integer itself. The only way $\frac{D}{z-1}$ can be an algebraic integer is if $p \mid D$ (Why?), so it simply remains to compute the remainder when the integer $\prod_{a,b,c}(\omega^a + 1^b + 1^c)$ is divided by p , which is quite easy.

Alternatively (as some contestants/teams found), we can get rid of ω first (rather than z as in the previous solution), so (up to sign) we want to evaluate $\prod_{b,c}(1 + (z^b + z^c)^r) \pmod{p}$. But we have a polynomial identity $\prod_{b,c}(1 + (T^b + T^c)^r) \in \Phi_p(T) \cdot \mathbb{Z}[x] + M$ for some constant integer M , where $\Phi_p(T) = T^{p-1} + \dots + T + 1$ denotes the p th cyclotomic polynomial. But note that $\Phi_p(z) = 0$ is congruent to $\Phi_p(1) = p$ modulo p , so

$$\prod_{b,c}(1 + (z^b + z^c)^r) - \prod_{b,c}(1 + (1^b + 1^c)^r)$$

is an integer divisible by p . The rest is easy.

Remark. Without using the Frobenius map, this second solution gives another proof that $\prod(\omega^a + z^{b_1} + \dots + z^{b_m})$ is an integer congruent to something like $m^r + 1$ modulo p (as long as p is odd, etc.). Can you reconcile this difference (e.g. is the use of Frobenius in the first solution equivalent to a step of the second solution)?

Remark. The central idea behind the second solution is that if $f(z)$ is an integer for some polynomial $f \in \mathbb{Z}[x]$, then it’s congruent to (the integer) $f(1)$ modulo p . Can you generalize this, say when z is not necessarily a root of unity?

10. [40] The sequences of real numbers $\{a_i\}_{i=1}^\infty$ and $\{b_i\}_{i=1}^\infty$ satisfy $a_{n+1} = (a_{n-1} - 1)(b_n + 1)$ and $b_{n+1} = a_n b_{n-1} - 1$ for $n \geq 2$, with $a_1 = a_2 = 2015$ and $b_1 = b_2 = 2013$. Evaluate, with proof, the infinite sum

$$\sum_{n=1}^\infty b_n \left(\frac{1}{a_{n+1}} - \frac{1}{a_{n+3}} \right).$$

Answer: $1 + \frac{1}{2014 \cdot 2015}$ OR $\frac{4058211}{4058210}$ First note that a_n and b_n are weakly increasing and tend to infinity. In particular, $a_n, b_n \notin \{0, -1, 1\}$ for all n .

For $n \geq 1$, we have $a_{n+3} = (a_{n+1} - 1)(b_{n+2} + 1) = (a_{n+1} - 1)(a_{n+1} b_n)$, so

$$\frac{b_n}{a_{n+3}} = \frac{1}{a_{n+1}(a_{n+1} - 1)} = \frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1}}.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^\infty \frac{b_n}{a_{n+1}} - \frac{b_n}{a_{n+3}} &= \sum_{n=1}^\infty \frac{b_n}{a_{n+1}} - \left(\frac{1}{a_{n+1} - 1} - \frac{1}{a_{n+1}} \right) \\ &= \sum_{n=1}^\infty \frac{b_n + 1}{a_{n+1}} - \frac{1}{a_{n+1} - 1}. \end{aligned}$$

Furthermore, $b_n + 1 = \frac{a_{n+1}}{a_{n-1}-1}$ for $n \geq 2$. So the sum over $n \geq 2$ is

$$\begin{aligned}\sum_{n=2}^{\infty} \left(\frac{1}{a_{n-1}-1} - \frac{1}{a_{n+1}-1} \right) &= \lim_{N \rightarrow \infty} \sum_{n=2}^N \left(\frac{1}{a_{n-1}-1} - \frac{1}{a_{n+1}-1} \right) \\ &= \frac{1}{a_1-1} + \frac{1}{a_2-1} - \lim_{N \rightarrow \infty} \left(\frac{1}{a_N-1} + \frac{1}{a_{N+1}-1} \right) \\ &= \frac{1}{a_1-1} + \frac{1}{a_2-1}.\end{aligned}$$

Hence the final answer is

$$\left(\frac{b_1 + 1}{a_2} - \frac{1}{a_2 - 1} \right) + \left(\frac{1}{a_1 - 1} + \frac{1}{a_2 - 1} \right).$$

Cancelling the common terms and putting in our starting values, this equals

$$\frac{2014}{2015} + \frac{1}{2014} = 1 - \frac{1}{2015} + \frac{1}{2014} = 1 + \frac{1}{2014 \cdot 2015}.$$

HMMT November 2015

November 14, 2015

Individual

1. Find the number of triples (a, b, c) of positive integers such that $a + ab + abc = 11$.

Proposed by: Yang Liu

Answer: $\boxed{3}$

We can write $a + ab + abc = a(1 + b + bc)$. Since 11 is prime, $a = 11$ or $a = 1$. But since b, c are both positive integers, we cannot have $a = 11$, and so $a = 1$. Then $1 + b + bc = 11 \implies b + bc = 10 \implies b(c + 1) = 10$, and since c is a positive integer, only $b = 1, 2, 5$ are possible. This gives the $\boxed{3}$ triples $(a, b, c) = (1, 1, 9), (1, 2, 4), (1, 5, 1)$.

2. Let a and b be real numbers randomly (and independently) chosen from the range $[0, 1]$. Find the probability that a, b and 1 form the side lengths of an obtuse triangle.

Proposed by: Alexander Katz

Answer: $\boxed{\frac{\pi-2}{4}}$

We require $a + b > 1$ and $a^2 + b^2 < 1$. Geometrically, this is the area enclosed in the quarter-circle centered at the origin with radius 1, not including the area enclosed by $a + b < 1$ (an isosceles right triangle with side length 1). As a result, our desired probability is $\frac{\pi-2}{4}$.

3. Neo has an infinite supply of red pills and blue pills. When he takes a red pill, his weight will double, and when he takes a blue pill, he will lose one pound. If Neo originally weighs one pound, what is the minimum number of pills he must take to make his weight 2015 pounds?

Proposed by: Alexander Katz

Answer: $\boxed{13}$

Suppose instead Neo started at a weight of 2015 pounds, instead had green pills, which halve his weight, and purple pills, which increase his weight by a pound, and he wished to reduce his weight to one pound. It is clear that, if Neo were able to find such a sequence of pills in the case where he goes from 2015 pounds to 1 pound, he can perform the sequence in reverse (replacing green pills with red pills and purple pills with blue pills) to achieve the desired weight, so this problem is equivalent to the original.

Suppose at some point, Neo were to take two purple pills followed by a green pill; this changes his weight from $2k$ to $k + 1$. However, the same effect could be achieved using less pills by first taking a green pill and then taking a purple pill, so the optimal sequence will never contain consecutive purple pills. As a result, there is only one optimal sequence for Neo if he is trying to lose weight: take a purple pill when his weight is odd, and a green pill when his weight is even. His weight thus becomes

$$\begin{aligned} 2015 &\rightarrow 2016 \rightarrow 1008 \rightarrow 504 \rightarrow 252 \rightarrow 126 \rightarrow 63 \\ &\rightarrow 64 \rightarrow 32 \rightarrow 16 \rightarrow 8 \rightarrow 4 \rightarrow 2 \rightarrow 1 \end{aligned}$$

which requires a total of $\boxed{13}$ pills. Reversing this sequence solves the original problem directly.

4. Chords AB and CD of a circle are perpendicular and intersect at a point P . If $AP = 6$, $BP = 12$, and $CD = 22$, find the area of the circle.

Proposed by: Sam Korsky

Answer: $\boxed{130\pi}$

Let O be the center of the circle and let M be the midpoint of segment AB and let N be the midpoint of segment CD . Since quadrilateral OMP_N is a rectangle we have that $ON = MP = AM - AP = 3$ so

$$OC = \sqrt{ON^2 + NC^2} = \sqrt{9 + 121} = \sqrt{130}$$

Hence the desired area is $\boxed{130\pi}$.

5. Let S be a subset of the set $\{1, 2, 3, \dots, 2015\}$ such that for any two elements $a, b \in S$, the difference $a - b$ does not divide the sum $a + b$. Find the maximum possible size of S .

Proposed by: Sam Korsky

Answer: $\boxed{672}$

From each of the sets $\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \dots$ at most 1 element can be in S . This leads to an upper bound of $\lceil \frac{2015}{3} \rceil = \boxed{672}$ which we can obtain with the set $\{1, 4, 7, \dots, 2014\}$.

6. Consider all functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$f(f(x) + 2x + 20) = 15.$$

Call an integer n *good* if $f(n)$ can take any integer value. In other words, if we fix n , for any integer m , there exists a function f such that $f(n) = m$. Find the sum of all good integers x .

Proposed by: Yang Liu

Answer: $\boxed{-35}$

For almost all integers x , $f(x) \neq -x - 20$. If $f(x) = -x - 20$, then

$$f(-x - 20 + 2x + 20) = 15 \implies -x - 20 = 15 \implies x = -35.$$

Now it suffices to prove that the $f(-35)$ can take any value.

$f(-35) = 15$ in the function $f(x) \equiv 15$. Otherwise, set $f(-35) = c$, and $f(x) = 15$ for all other x . It is easy to check that these functions all work.

7. Let $\triangle ABC$ be a right triangle with right angle C . Let I be the incenter of ABC , and let M lie on AC and N on BC , respectively, such that M, I, N are collinear and \overline{MN} is parallel to AB . If $AB = 36$ and the perimeter of CMN is 48, find the area of ABC .

Proposed by: Alexander Katz

Answer: $\boxed{252}$

Note that $\angle MIA = \angle BAI = \angle CAI$, so $MI = MA$. Similarly, $NI = NB$. As a result, $CM + MN + NC = CM + MI + NI + NC = CM + MA + NB + NC = AC + BC = 48$. Furthermore, $AC^2 + BC^2 = 36^2$. As a result, we have $AC^2 + 2AC \cdot BC + BC^2 = 48^2$, so $2AC \cdot BC = 48^2 - 36^2 = 12 \cdot 84$, and so $\frac{AC \cdot BC}{2} = 3 \cdot 84 = \boxed{252}$.

8. Let $ABCD$ be a quadrilateral with an inscribed circle ω that has center I . If $IA = 5, IB = 7, IC = 4, ID = 9$, find the value of $\frac{AB}{CD}$.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{35}{36}}$

The I -altitudes of triangles AIB and CID are both equal to the radius of ω , hence have equal length. Therefore $\frac{[AIB]}{[CID]} = \frac{AB}{CD}$. Also note that $[AIB] = IA \cdot IB \cdot \sin AIB$ and $[CID] = IC \cdot ID \cdot \sin CID$, but since lines IA, IB, IC, ID bisect angles $\angle DAB, \angle ABC, \angle BCD, \angle CDA$ respectively we have that $\angle AIB + \angle CID = (180^\circ - \angle IAB - \angle IBA) + (180^\circ - \angle ICD - \angle IDC) = 180^\circ$. So, $\sin AIB = \sin CID$. Therefore $\frac{[AIB]}{[CID]} = \frac{IA \cdot IB}{IC \cdot ID}$. Hence

$$\frac{AB}{CD} = \frac{IA \cdot IB}{IC \cdot ID} = \boxed{\frac{35}{36}}.$$

9. Rosencrantz plays $n \leq 2015$ games of question, and ends up with a win rate (i.e. $\frac{\# \text{ of games won}}{\# \text{ of games played}}$) of k . Guildenstern has also played several games, and has a win rate less than k . He realizes that if, after playing some more games, his win rate becomes higher than k , then there must have been some point in time when Rosencrantz and Guildenstern had the exact same win-rate. Find the product of all possible values of k .

Proposed by: Alexander Katz

Answer: $\boxed{\frac{1}{2015}}$

Write $k = \frac{m}{n}$, for relatively prime integers m, n . For the property not to hold, there must exist integers a and b for which

$$\frac{a}{b} < \frac{m}{n} < \frac{a+1}{b+1}$$

(i.e. at some point, Guildenstern must "jump over" k with a single win)

$$\iff an + n - m > bm > an$$

hence there must exist a multiple of m strictly between an and $an + n - m$.

If $n - m = 1$, then the property holds as there is no integer between an and $an + n - m = an + 1$. We now show that if $n - m \neq 1$, then the property does not hold. By Bzout's Theorem, as n and m are relatively prime, there exist a and x such that $an = mx - 1$, where $0 < a < m$. Then $an + n - m \geq an + 2 = mx + 1$, so $b = x$ satisfies the conditions. As a result, the only possible k are those in the form $\frac{n}{n+1}$.

We know that Rosencrantz played at most 2015 games, so the largest non-perfect winrate he could possibly have is $\frac{2014}{2015}$. Therefore, $k \in \{\frac{1}{2}, \frac{2}{3}, \dots, \frac{2014}{2015}\}$, the product of which is $\boxed{\frac{1}{2015}}$.

10. Let N be the number of functions f from $\{1, 2, \dots, 101\} \rightarrow \{1, 2, \dots, 101\}$ such that $f^{101}(1) = 2$. Find the remainder when N is divided by 103.

Proposed by: Yang Liu

Answer: $\boxed{43}$

For convenience, let $n = 101$. Compute the number of functions such that $f^n(1) = 1$. Since n is a prime, there are 2 cases: the order of 1 is either 1 or n . The first case gives n^{n-1} functions, and the second case gives $(n-1)!$ functions. By symmetry, the number of ways for $f^n(1) = 2$ is

$$\frac{1}{n-1} \cdot (n^n - n^{n-1} - (n-1)!) = n^{n-1} - (n-2)!$$

Plugging in $n = 101$, we need to find

$$\begin{aligned} 101^{100} - 99! &\equiv (-2)^{-2} - \frac{101!}{6} \\ &= 1/4 - 1/6 = 1/12 = \boxed{43} \pmod{103}. \end{aligned}$$

HMMT November 2015

November 14, 2015

Guts round

1. [5] Farmer Yang has a 2015×2015 square grid of corn plants. One day, the plant in the very center of the grid becomes diseased. Every day, every plant adjacent to a diseased plant becomes diseased. After how many days will all of Yang's corn plants be diseased?

Proposed by: Alexander Katz

Answer:

After k minutes, the diseased plants are the ones with taxicab distance at most k from the center. The plants on the corner are the farthest from the center and have taxicab distance 2014 from the center, so all the plants will be diseased after 2014 minutes.

2. [5] The three sides of a right triangle form a geometric sequence. Determine the ratio of the length of the hypotenuse to the length of the shorter leg.

Proposed by: Alexander Katz

Answer:

Let the shorter leg have length ℓ , and the common ratio of the geometric sequence be $r > 1$. Then the length of the other leg is ℓr , and the length of the hypotenuse is ℓr^2 . Hence,

$$\begin{aligned}\ell^2 + (\ell r)^2 &= (\ell r^2)^2 \\ \implies \ell^2(r^2 + 1) &= \ell^2 r^4 \\ \implies r^2 + 1 &= r^4\end{aligned}$$

Hence, $r^4 - r^2 - 1 = 0$, and therefore $r^2 = \frac{1 \pm \sqrt{5}}{2}$. As $r > 1$, we have $r^2 = \frac{1 + \sqrt{5}}{2}$, completing the problem as the ratio of the hypotenuse to the shorter side is $\frac{\ell r^2}{\ell} = r^2$.

3. [5] A parallelogram has 2 sides of length 20 and 15. Given that its area is a positive integer, find the minimum possible area of the parallelogram.

Proposed by: Yang Liu

Answer:

The area of the parallelogram can be made arbitrarily small, so the smallest positive integer area is 1.

4. [6] Eric is taking a biology class. His problem sets are worth 100 points in total, his three midterms are worth 100 points each, and his final is worth 300 points. If he gets a perfect score on his problem sets and scores 60%, 70%, and 80% on his midterms respectively, what is the minimum possible percentage he can get on his final to ensure a passing grade? (Eric passes if and only if his overall percentage is at least 70%).

Proposed by: Alexander Katz

Answer:

We see there are a total of $100 + 3 \times 100 + 300 = 700$ points, and he needs $70\% \times 700 = 490$ of them. He has $100 + 60 + 70 + 80 = 310$ points before the final, so he needs 180 points out of 300 on the final, which is 60%.

5. [6] James writes down three integers. Alex picks some two of those integers, takes the average of them, and adds the result to the third integer. If the possible final results Alex could get are 42, 13, and 37, what are the three integers James originally chose?

Proposed by: Alexander Katz

Answer: $\boxed{-20, 28, 38}$

Let x, y, z be the integers. We have

$$\begin{aligned}\frac{x+y}{2} + z &= 42 \\ \frac{y+z}{2} + x &= 13 \\ \frac{x+z}{2} + y &= 37\end{aligned}$$

Adding these three equations yields $2(x+y+z) = 92$, so $\frac{x+y}{2} + z = 23 + \frac{z}{2} = 42$ so $z = 38$. Similarly, $x = -20$ and $y = 28$.

6. [6] Let AB be a segment of length 2 with midpoint M . Consider the circle with center O and radius r that is externally tangent to the circles with diameters AM and BM and internally tangent to the circle with diameter AB . Determine the value of r .

Proposed by: Sam Korsky

Answer: $\boxed{\frac{1}{3}}$

Let X be the midpoint of segment AM . Note that $OM \perp MX$ and that $MX = \frac{1}{2}$ and $OX = \frac{1}{2} + r$ and $OM = 1 - r$. Therefore by the Pythagorean theorem, we have

$$OM^2 + MX^2 = OX^2 \implies (1-r)^2 + \frac{1}{2^2} = \left(\frac{1}{2} + r\right)^2$$

which we can easily solve to find that $r = \boxed{\frac{1}{3}}$.

7. [7] Let n be the smallest positive integer with exactly 2015 positive factors. What is the sum of the (not necessarily distinct) prime factors of n ? For example, the sum of the prime factors of 72 is $2 + 2 + 2 + 3 + 3 = 14$.

Proposed by: Alexander Katz

Answer: $\boxed{116}$

Note that $2015 = 5 \times 13 \times 31$ and that $N = 2^{30} \cdot 3^{12} \cdot 5^4$ has exactly 2015 positive factors. We claim this is the smallest such integer. Note that $N < 2^{66}$.

If n has 3 distinct prime factors, it must be of the form $p^{30}q^{12}r^4$ for some primes p, q, r , so $n \geq 2^{30} \cdot 3^{12} \cdot 5^4$.

If n has 2 distinct prime factors, it must be of the form $p^e q^f > 2^{e+f}$ where $(e+1)(f+1) = 2015$. It is easy to see that this means $e+f > 66$ so $n > 2^{66} > N$.

If n has only 1 prime factor, we have $n \geq 2^{2014} > N$.

So N is the smallest such integer, and the sum of its prime factors is $2 \cdot 30 + 3 \cdot 12 + 5 \cdot 4 = 116$.

8. [7] For how many pairs of nonzero integers (c, d) with $-2015 \leq c, d \leq 2015$ do the equations $cx = d$ and $dx = c$ both have an integer solution?

Proposed by: Yang Liu

Answer: $\boxed{8060}$

We need both c/d and d/c to be integers, which is equivalent to $|c| = |d|$, or $d = \pm c$. So there are 4030 ways to pick c and 2 ways to pick d , for a total of 8060 pairs.

9. [7] Find the smallest positive integer n such that there exists a complex number z , with positive real and imaginary part, satisfying $z^n = (\bar{z})^n$.

Proposed by: Alexander Katz

Since $|z| = |\bar{z}|$ we may divide by $|z|$ and assume that $|z| = 1$. Then $\bar{z} = \frac{1}{z}$, so we are looking for the smallest positive integer n such that there is a $2n^{\text{th}}$ root of unity in the first quadrant. Clearly there is a sixth root of unity in the first quadrant but no fourth or second roots of unity, so $n = \boxed{3}$ is the smallest.

10. [8] Call a string of letters S an *almost palindrome* if S and the reverse of S differ in exactly two places. Find the number of ways to order the letters in $HMMTTHEMETEAM$ to get an almost palindrome.

Proposed by: Yang Liu

Answer: $\boxed{2160}$

Note that T, E, A are used an odd number of times. Therefore, one must go in the middle spot and the other pair must match up. There are $3 \cdot 2 \binom{6!}{2!} = 2160$ ways to fill in the first six spots with the letters T, H, E, M, M and a pair of different letters. The factor of 3 accounts for which letter goes in the middle.

11. [8] Find all integers n , not necessarily positive, for which there exist positive integers a, b, c satisfying $a^n + b^n = c^n$.

Proposed by: Rikhav Shah

Answer: $\boxed{\pm 1, \pm 2}$

By Fermat's Last Theorem, we know $n < 3$. Suppose $n \leq -3$. Then $a^n + b^n = c^n \implies (bc)^{-n} + (ac)^{-n} = (ab)^{-n}$, but since $-n \geq 3$, this is also impossible by Fermat's Last Theorem. As a result, $|n| < 3$.

Furthermore, $n \neq 0$, as $a^0 + b^0 = c^0 \implies 1 + 1 = 1$, which is false. We now just need to find constructions for $n = -2, -1, 1, 2$. When $n = 1$, $(a, b, c) = (1, 2, 3)$ suffices, and when $n = 2$, $(a, b, c) = (3, 4, 5)$ works nicely. When $n = -1$, $(a, b, c) = (6, 3, 2)$ works, and when $n = -2$, $(a, b, c) = (20, 15, 12)$ is one example. Therefore, the working values are $n = \boxed{\pm 1, \pm 2}$.

12. [8] Let a and b be positive real numbers. Determine the minimum possible value of

$$\sqrt{a^2 + b^2} + \sqrt{(a-1)^2 + b^2} + \sqrt{a^2 + (b-1)^2} + \sqrt{(a-1)^2 + (b-1)^2}$$

Proposed by: Alexander Katz

Answer: $\boxed{2\sqrt{2}}$

Let $ABCD$ be a square with $A = (0, 0), B = (1, 0), C = (1, 1), D = (0, 1)$, and P be a point in the same plane as $ABCD$. Then the desired expression is equivalent to $AP + BP + CP + DP$. By the triangle inequality, $AP + CP \geq AC$ and $BP + DP \geq BD$, so the minimum possible value is $AC + BD = 2\sqrt{2}$. This is achievable when $a = b = \frac{1}{2}$, so we are done.

13. [9] Consider a 4×4 grid of squares, each of which are originally colored red. Every minute, Piet can jump on one of the squares, changing the color of it and any adjacent squares (two squares are adjacent if they share a side) to blue. What is the minimum number of minutes it will take Piet to change the entire grid to blue?

Proposed by: Alexander Katz

Answer: $\boxed{4}$

Piet can change the colors of at most 5 squares per minute, so as there are 16 squares, it will take him at least four minutes to change the colors of every square. Some experimentation yields that it is indeed possible to make the entire grid blue after 4 minutes; one example is shown below:

	X		
			X
X			
		X	

Here, jumping on the squares marked with an X provides the desired all-blue grid.

14. [9] Let ABC be an acute triangle with orthocenter H . Let D, E be the feet of the A, B -altitudes respectively. Given that $AH = 20$ and $HD = 15$ and $BE = 56$, find the length of BH .

Proposed by: Sam Korsky

Answer: 50

Let x be the length of BH . Note that quadrilateral $ABDE$ is cyclic, so by Power of a Point, $x(56-x) = 20 \cdot 15 = 300$. Solving for x , we get $x = 50$ or 6 . We must have $BH > HD$ so $x = 50$ is the correct length.

15. [9] Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution".

Proposed by: Alexander Katz

Answer: 7

We have $1111_b = b^3 + b^2 + b + 1 = (b^2 + 1)(b + 1)$. Note that $\gcd(b^2 + 1, b + 1) = \gcd(b^2 + 1 - (b + 1)(b - 1), b + 1) = \gcd(2, b + 1)$, which is either 1 or 2. If the gcd is 1, then there is no solution as this implies $b^2 + 1$ is a perfect square, which is impossible for positive b . Hence the gcd is 2, and $b^2 + 1, b + 1$ are both twice perfect squares.

Let $b + 1 = 2a^2$. Then $b^2 + 1 = (2a^2 - 1)^2 + 1 = 4a^4 - 4a^2 + 2 = 2(2a^4 - 2a^2 + 1)$, so $2a^4 - 2a^2 + 1 = (a^2 - 1)^2 + (a^2)^2$ must be a perfect square. This first occurs when $a^2 - 1 = 3, a^2 = 4 \implies a = 2$, and thus $b = \boxed{7}$. Indeed, $1111_7 = 20^2$.

16. [10] For how many triples (x, y, z) of integers between -10 and 10 inclusive do there exist reals a, b, c that satisfy

$$ab = x$$

$$ac = y$$

$$bc = z?$$

Proposed by: Yang Liu

Answer: 4061

If none of x, y, z are zero, then there are $4 \cdot 10^3 = 4000$ ways, since xyz must be positive. Indeed, $(abc)^2 = xyz$. So an even number of them are negative, and the ways to choose an even number of 3 variables to be negative is 4 ways. If one of x, y, z is 0, then one of a, b, c is zero at least. So at least two of x, y, z must be 0. If all 3 are zero, this gives 1 more solution. If exactly 2 are negative, then this gives $3 \cdot 20$ more solutions. This comes from choosing one of x, y, z to be nonzero, and choosing its value in 20 ways.

Our final answer is $4000 + 60 + 1 = 4061$.

17. [10] Unit squares $ABCD$ and $EFGH$ have centers O_1 and O_2 respectively, and are originally situated such that B and E are at the same position and C and H are at the same position. The squares then rotate clockwise about their centers at the rate of one revolution per hour. After 5 minutes, what is the area of the intersection of the two squares?

Proposed by: Alexander Katz

Answer: $\frac{2-\sqrt{3}}{4}$

Note that $AE = BF = CG = DH = 1$ at all times. Suppose that the squares have rotated θ radians. Then $\angle O_1O_2H = \frac{\pi}{4} - \theta = \angle O_1DH$, so $\angle HDC = \frac{\pi}{4} - \angle O_1DH = \theta$. Let P be the intersection of AB and EH and Q be the intersection of BC and GH . Then $PH \parallel BQ$ and $HQ \parallel PB$, and $\angle PHG = \frac{\pi}{2}$, so $PBQH$ - our desired intersection - is a rectangle. We have $BQ = 1 - QC = 1 - \sin \theta$ and $HQ = 1 - \cos \theta$, so our desired area is $(1 - \cos \theta)(1 - \sin \theta)$. After 5 minutes, we have $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$,

so our answer is $\frac{2 - \sqrt{3}}{4}$.

18. [10] A function f satisfies, for all nonnegative integers x and y :

- $f(0, x) = f(x, 0) = x$
- If $x \geq y \geq 0$, $f(x, y) = f(x - y, y) + 1$
- If $y \geq x \geq 0$, $f(x, y) = f(x, y - x) + 1$

Find the maximum value of f over $0 \leq x, y \leq 100$.

Proposed by: Alexander Katz

Answer: 101

Firstly, $f(100, 100) = 101$.

To see this is maximal, note that $f(x, y) \leq \max\{x, y\} + 1$, say by induction on $x + y$.

19. [11] Each cell of a 2×5 grid of unit squares is to be colored white or black. Compute the number of such colorings for which no 2×2 square is a single color.

Proposed by: Alexander Katz

Answer: 634

Let a_n denote the number of ways to color a $2 \times n$ grid subject only to the given constraint, and b_n denote the number of ways to color a $2 \times n$ grid subject to the given constraint, but with the added restriction that the first column cannot be colored black-black.

Consider the first column of a $2 \times n$ grid that is not subject to the additional constraint. It can be colored black-white or white-black, in which case the leftmost 2×2 square is guaranteed not to be monochromatic, and so the remaining $2 \times (n - 1)$ subgrid can be colored in a_{n-1} ways. Otherwise, it is colored white-white or black-black; WLOG, assume that it's colored black-black. Then the remaining $2 \times (n - 1)$ subgrid is subject to both constraints, so there are b_{n-1} ways to color the remaining subgrid. Hence $a_n = 2a_{n-1} + 2b_{n-1}$.

Now consider the first column of a $2 \times n$ grid that is subject to the additional constraint. The first column cannot be colored black-black, and if it is colored white-black or black-white, there are a_{n-1} ways to color the remaining subgrid by similar logic to the previous case. If it is colored white-white, then there are b_{n-1} ways to color the remaining subgrid, again by similar logic to the previous case. Hence $b_n = 2a_{n-1} + b_{n-1}$.

Therefore, we have $b_n = 2a_{n-1} + \frac{1}{2}(a_n - 2a_{n-1})$, and so $a_n = 2a_{n-1} + 2b_{n-1} = 2a_{n-1} + 2(2a_{n-2} + \frac{1}{2}(a_{n-1} - 2a_{n-2})) = 3a_{n-1} + 2a_{n-2}$. Finally, we have $a_0 = 1$ (as the only possibility is to, well, do nothing) and $a_1 = 4$ (as any 2×1 coloring is admissible), so $a_2 = 14, a_3 = 50, a_4 = 178, a_5 = \boxed{634}$.

20. [11] Let n be a three-digit integer with nonzero digits, not all of which are the same. Define $f(n)$ to be the greatest common divisor of the six integers formed by any permutation of ns digits. For example, $f(123) = 3$, because $\gcd(123, 132, 213, 231, 312, 321) = 3$. Let the maximum possible value of $f(n)$ be k . Find the sum of all n for which $f(n) = k$.

Proposed by: Alexander Katz

Answer: 5994

Let $n = \overline{abc}$, and assume without loss of generality that $a \geq b \geq c$. We have $k \mid 100a + 10b + c$ and $k \mid 100a + 10c + b$, so $k \mid 9(b - c)$. Analogously, $k \mid 9(a - c)$ and $k \mid 9(a - b)$. Note that if $9 \mid n$, then 9 also divides any permutation of ns digits, so $9 \mid f(n)$ as well; ergo, $f(n) \geq 9$, implying that $k \geq 9$. If k is not a multiple of 3, then we have $k \mid c - a \implies k \leq c - a < 9$, contradiction, so $3 \mid k$.

Let $x = \min(a - b, b - c, a - c)$. If $x = 1$, then we have $k \mid 9$, implying $k = 9$ - irrelevant to our investigation. So we can assume $x \geq 2$. Note also that $x \leq 4$, as $2x \leq (a - b) + (b - c) = a - c \leq 9 - 1$, and if $x = 4$ we have $n = 951 \implies f(n) = 3$. If $x = 3$, then since $3 \mid k \mid 100a + 10b + c \implies 3 \mid a + b + c$, we have $a \equiv b \equiv c \pmod{3}$ (e.g. if $b - c = 3$, then $b \equiv c \pmod{3}$, so $a \equiv b \equiv c \pmod{3}$) - the other cases are analogous). This gives us the possibilities $n = 147, 258, 369$, which give $f(n) = 3, 3, 9$ respectively.

Hence we can conclude that $x = 2$; therefore $k \mid 18$. We know also that $k \geq 9$, so either $k = 9$ or $k = 18$. If $k = 18$, then all the digits of n must be even, and n must be a multiple of 9; it is clear that these are sufficient criteria. As n 's digits are all even, the sum of them is also even, and hence their sum is 18. Since $a \geq b \geq c$, we have $a + b + c = 18 \leq 3a \implies a \geq 6$, but if $a = 6$ then $a = b = c = 6$, contradicting the problem statement. Thus $a = 8$, and this gives us the solutions $n = 882, 864$ along with their permutations.

It remains to calculate the sum of the permutations of these solutions. In the $n = 882$ case, each digit is either 8, 8, or 2 (one time each), and in the $n = 864$ case, each digit is either 8, 6, or 4 (twice each). Hence the desired sum is $111(8 + 8 + 2) + 111(8 \cdot 2 + 6 \cdot 2 + 4 \cdot 2) = 111(54) = \boxed{5994}$.

21. [11] Consider a 2×2 grid of squares. Each of the squares will be colored with one of 10 colors, and two colorings are considered equivalent if one can be rotated to form the other. How many distinct colorings are there?

Proposed by: Sam Korsky

Answer: $\boxed{2530}$

This solution will be presented in the general case with n colors. Our problem asks for $n = 10$.

We isolate three cases:

Case 1: Every unit square has the same color

In this case there are clearly n ways to color the square.

Case 2: Two non-adjacent squares are the same color, and the other two squares are also the same color (but not all four squares are the same color).

In this case there are clearly $\binom{n}{2} = \frac{n(n-1)}{2}$ ways to color the square.

Case 3: Every other case

Since without the "rotation" condition there would be n^4 colorings, we have that in this case by complementary counting there are $\frac{n^4 - n(n-1) - n}{4}$ ways to color the square.

Therefore the answer is

$$n + \frac{n^2 - n}{2} + \frac{n^4 - n^2}{4} = \frac{n^4 + n^2 + 2n}{4} = \boxed{2530}$$

22. [12] Find all the roots of the polynomial $x^5 - 5x^4 + 11x^3 - 13x^2 + 9x - 3$.

Proposed by: Sam Korsky

Answer: $\boxed{1, \frac{3+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}}$

The $x^5 - 5x^4$ at the beginning of the polynomial motivates us to write it as $(x-1)^5 + x^3 - 3x^2 + 4x - 2$ and again the presence of the $x^3 - 3x^2$ motivates writing the polynomial in the form $(x-1)^5 + (x-1)^3 + (x-1)$. Let a and b be the roots of the polynomial $x^2 + x + 1$. It's clear that the roots of our polynomial are given by 1 and the roots of the polynomials $(x-1)^2 = a$ and $(x-1)^2 = b$. The quadratic formula shows that WLOG $a = \frac{-1+\sqrt{3}i}{2}$ and $b = \frac{-1-\sqrt{3}i}{2}$ so we find that either $x-1 = \pm\frac{1+\sqrt{3}i}{2}$ or $x-1 = \pm\frac{1-\sqrt{3}i}{2}$.

Hence our roots are $\boxed{1, \frac{3+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}, \frac{3-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}}$.

23. [12] Compute the smallest positive integer n for which

$$0 < \sqrt[n]{n} - \lfloor \sqrt[n]{n} \rfloor < \frac{1}{2015}.$$

Proposed by: Alexander Katz

Answer: $\boxed{4097}$

Let $n = a^4 + b$ where a, b are integers and $0 < b < 4a^3 + 6a^2 + 4a + 1$. Then

$$\begin{aligned} \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor &< \frac{1}{2015} \\ \sqrt[4]{a^4 + b} - a &< \frac{1}{2015} \\ \sqrt[4]{a^4 + b} &< a + \frac{1}{2015} \\ a^4 + b &< \left(a + \frac{1}{2015}\right)^4 \\ a^4 + b &< a^4 + \frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4} \end{aligned}$$

To minimize $n = a^4 + b$, we clearly should minimize b , which occurs at $b = 1$. Then

$$1 < \frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4}.$$

If $a = 7$, then $\frac{6a^2}{2015^2}, \frac{4a}{2015^3}, \frac{1}{2015^4} < \frac{1}{2015}$, so $\frac{4a^3}{2015} + \frac{6a^2}{2015^2} + \frac{4a}{2015^3} + \frac{1}{2015^4} < \frac{4 \cdot 7^3 + 3}{2015} < 1$, so $a \geq 8$. When $a = 8$, we have $\frac{4a^3}{2015} = \frac{2048}{2015} > 1$, so $a = 8$ is the minimum.

Hence, the minimum n is $8^4 + 1 = \boxed{4097}$.

24. [12] Three ants begin on three different vertices of a tetrahedron. Every second, they choose one of the three edges connecting to the vertex they are on with equal probability and travel to the other vertex on that edge. They all stop when any two ants reach the same vertex at the same time. What is the probability that all three ants are at the same vertex when they stop?

Proposed by: Anna Ellison

Answer: $\boxed{\frac{1}{16}}$

At every second, each ant can travel to any of the three vertices they are not currently on. Given that, at one second, the three ants are on different vertices, the probability of them all going to the same vertex is $\frac{1}{27}$ and the probability of them all going to different vertices is $\frac{11}{27}$, so the probability of the three ants all meeting for the first time on the n^{th} step is $(\frac{11}{27})^{n-1} \times \frac{1}{27}$. Then the probability the three ants all meet at the same time is $\sum_{i=0}^{\infty} (\frac{11}{27})^i \times \frac{1}{27} = \frac{\frac{1}{27}}{1 - \frac{11}{27}} = \frac{1}{16}$.

25. [13] Let ABC be a triangle that satisfies $AB = 13, BC = 14, AC = 15$. Given a point P in the plane, let P_A, P_B, P_C be the reflections of A, B, C across P . Call P *good* if the circumcircle of $P_A P_B P_C$ intersects the circumcircle of ABC at exactly 1 point. The locus of good points P encloses a region \mathcal{S} . Find the area of \mathcal{S} .

Proposed by: Yang Liu

Answer: $\boxed{\frac{4225}{64} \pi}$

By the properties of reflection, the circumradius of $P_A P_B P_C$ equals the circumradius of ABC . Therefore, the circumcircle of $P_A P_B P_C$ must be externally tangent to the circumcircle of ABC . Now it's easy to see that the midpoint of the 2 centers of ABC and $P_A P_B P_C$ lies on the circumcircle of ABC . So the locus of P is simply the circumcircle of ABC .

Since $[ABC] = \frac{abc}{4R}$, we find the circumradius is $R = \frac{13 \cdot 14 \cdot 15}{84 \cdot 4} = \frac{65}{8}$, so the enclosed region has area $\frac{4225}{64} \pi$.

26. [13] Let $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y . If $f(2) = 0$, compute $f(2015)$.

Proposed by: Alexander Katz

Answer: $\boxed{\log_2 2015 - 1}$

Let $g(x) = f(x) + 1$. Substituting g into the functional equation, we get that

$$g(xy) - 1 = g(x) - 1 + g(y) - 1 + 1$$

$$g(xy) = g(x) + g(y).$$

Also, $g(2) = 1$. Now substitute $x = e^{x'}$, $y = e^{y'}$, which is possible because $x, y \in \mathbb{R}^+$. Then set $h(x) = g(e^x)$. This gives us that

$$g(e^{x'+y'}) = g(e^{x'}) + g(e^{y'}) \implies h(x' + y') = h(x') + h(y')$$

for all $x', y' \in \mathbb{R}$. Also h is continuous. Therefore, by Cauchy's functional equation, $h(x) = cx$ for a real number c . Going all the way back to g , we can get that $g(x) = c \log x$. Since $g(2) = 1$, $c = \frac{1}{\log 2}$. Therefore, $g(2015) = c \log 2015 = \frac{\log 2015}{\log 2} = \log_2 2015$.

Finally, $f(2015) = g(2015) - 1 = \log_2 2015 - 1$.

27. [13] Let $ABCD$ be a quadrilateral with $A = (3, 4)$, $B = (9, -40)$, $C = (-5, -12)$, $D = (-7, 24)$. Let P be a point in the plane (not necessarily inside the quadrilateral). Find the minimum possible value of $AP + BP + CP + DP$.

Proposed by: Alexander Katz

Answer: $\boxed{16\sqrt{17} + 8\sqrt{5}}$

By the triangle inequality, $AP + CP \geq AC$ and $BP + DP \geq BD$. So P should be on AC and BD ; i.e. it should be the intersection of the two diagonals. Then $AP + BP + CP + DP = AC + BD$, which is easily computed to be $16\sqrt{17} + 8\sqrt{5}$ by the Pythagorean theorem.

Note that we require the intersection of the diagonals to actually *exist* for this proof to work, but $ABCD$ is convex and this is not an issue.

28. [15] Find the shortest distance between the lines $\frac{x+2}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{x-3}{-1} = \frac{y}{1} = \frac{z+1}{2}$

Proposed by: Sam Korsky

Answer: $\boxed{\frac{5\sqrt{3}}{3}}$

First we find the direction of a line perpendicular to both of these lines. By taking the cross product $(2, 3, 1) \times (-1, 1, 2) = (5, -5, 5)$ we find that the plane $x - y + z + 3 = 0$ contains the first line and is parallel to the second. Now we take a point on the second line, say the point $(3, 0, -1)$ and find the

distance between this point and the plane. This comes out to $\frac{|3-0+(-1)+3|}{\sqrt{1^2+1^2+1^2}} = \frac{5}{\sqrt{3}} = \boxed{\frac{5\sqrt{3}}{3}}$.

29. [15] Find the largest real number k such that there exists a sequence of positive reals $\{a_i\}$ for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^k}$ does not.

Proposed by: Alexander Katz

Answer: $\boxed{\frac{1}{2}}$

For $k > \frac{1}{2}$, I claim that the second sequence must converge. The proof is as follows: by the Cauchy-Schwarz inequality,

$$\left(\sum_{n \geq 1} \frac{\sqrt{a_n}}{n^k} \right)^2 \leq \left(\sum_{n \geq 1} a_n \right) \left(\sum_{n \geq 1} \frac{1}{n^{2k}} \right)$$

Since for $k > \frac{1}{2}$, $\sum_{n \geq 1} \frac{1}{n^{2k}}$ converges, the right hand side converges. Therefore, the left hand side must also converge.

For $k \leq \frac{1}{2}$, the following construction surprisingly works: $a_n = \frac{1}{n \log^2 n}$. It can be easily verified that $\sum_{n \geq 1} a_n$ converges, while

$$\sum_{n \geq 1} \frac{\sqrt{a_n}}{n^{\frac{1}{2}}} = \sum_{n \geq 1} \frac{1}{n \log n}$$

does not converge.

30. [15] Find the largest integer n such that the following holds: there exists a set of n points in the plane such that, for any choice of three of them, some two are unit distance apart.

Proposed by: Alexander Katz

Answer: $\boxed{7}$

We can obtain $n = 7$ in the following way: Consider a rhombus $ABCD$ made up of two equilateral triangles of side length 1, where $\angle DAB = 60^\circ$. Rotate the rhombus clockwise about A to obtain a new rhombus $AB'C'D'$ such that $DD' = 1$. Then one can verify that the seven points A, B, C, D, B', C', D' satisfy the problem condition.

To prove that $n = 8$ points is unobtainable, one interprets the problem in terms of graph theory. Consider a graph on 8 vertices, with an edge drawn between two vertices if and only if the vertices are at distance 1 apart. Assume for the sake of contradiction that this graph has no three points, no two of which are at distance 1 apart (in terms of graph theory, this means the graph has no independent set of size 3).

First, note that this graph cannot contain a complete graph of size 4 (it's clear that there can't exist four points in the plane with any two having the same pairwise distance).

I claim that every vertex has degree 4. It is easy to see that if a vertex has degree 5 or higher, then there exists an independent set of size 3 among its neighbors, contradiction (one can see this by drawing the 5 neighbors on a circle of radius 1 centered at our initial vertex and considering their pairwise distances). Moreover, if a vertex has degree 3 or lower then there are at least four vertices that are not at distance 1 from that vertex, and since not all four of these vertices can be at distance 1 from one another, there exists an independent set of size 3, contradiction.

Now, we consider the complement of our graph. Every vertex of this new graph has degree 3 and by our observations, contains no independent set of size 4. Moreover, by assumption this graph contains no triangle (a complete graph on three vertices). But we can check by hand that there are only six distinct graphs on eight vertices with each vertex having degree 3 (up to isomorphism), and five of these graphs contain a triangle, and the remaining graph contains an independent set of size 4, contradiction!

Hence the answer is $\boxed{n = 7}$

31. [17] Two random points are chosen on a segment and the segment is divided at each of these two points. Of the three segments obtained, find the probability that the largest segment is more than three times longer than the smallest segment.

Proposed by: Sam Korsky

Answer: $\boxed{\frac{27}{35}}$

We interpret the problem with geometric probability. Let the three segments have lengths $x, y, 1 - x - y$ and assume WLOG that $x \geq y \geq 1 - x - y$. The every possible (x, y) can be found in the triangle determined by the points $(\frac{1}{3}, \frac{1}{3}), (\frac{1}{2}, \frac{1}{2}), (1, 0)$ in \mathbb{R}^2 , which has area $\frac{1}{12}$. The line $x = 3(1 - x - y)$ intersects the lines $x = y$ and $y = 1 - x - y$ at the points $(\frac{3}{7}, \frac{3}{7})$ and $(\frac{3}{5}, \frac{1}{5})$. Hence $x \leq 3(1 - x - y)$ if (x, y) is in the triangle determined by points $(\frac{1}{3}, \frac{1}{3}), (\frac{3}{7}, \frac{3}{7}), (\frac{3}{5}, \frac{1}{5})$ which by shoelace has area $\frac{2}{105}$. Hence the desired probability is given by

$$\frac{\frac{1}{12} - \frac{2}{105}}{\frac{1}{12}} = \boxed{\frac{27}{35}}$$

32. [17] Find the sum of all positive integers $n \leq 2015$ that can be expressed in the form $\lceil \frac{x}{2} \rceil + y + xy$, where x and y are positive integers.

Proposed by: Calvin Deng

Answer:

Lemma: n is expressible as $\lceil \frac{x}{2} \rceil + y + xy$ iff $2n + 1$ is not a Fermat Prime.

Proof: Suppose n is expressible. If $x = 2k$, then $2n + 1 = (2k + 1)(2y + 1)$, and if $x = 2k - 1$, then $n = k(2y + 1)$. Thus, if $2n + 1$ isn't prime, we can factor $2n + 1$ as the product of two odd integers $2x + 1$, $2y + 1$ both greater than 1, resulting in positive integer values for x and y . Also, if n has an odd factor greater than 1, then we factor out its largest odd factor as $2y + 1$, giving a positive integer value for x and y . Thus n is expressible iff $2n + 1$ is not prime or n is not a power of 2. That leaves only the n such that $2n + 1$ is a prime one more than a power of two. These are well-known, and are called the Fermat primes.

It's a well-known fact that the only Fermat primes ≤ 2015 are 3, 5, 17, 257, which correspond to $n = 1, 2, 8, 128$. Thus the sum of all expressible numbers is $\frac{2015 \cdot 2016}{2} - (1 + 2 + 8 + 128) = 2029906$.

33. [17] How many ways are there to place four points in the plane such that the set of pairwise distances between the points consists of exactly 2 elements? (Two configurations are the same if one can be obtained from the other via rotation and scaling.)

Proposed by: Alexander Katz

Answer:

Let A, B, C, D be the four points. There are 6 pairwise distances, so at least three of them must be equal.

Case 1: There is no equilateral triangle. Then WLOG we have $AB = BC = CD = 1$.

- Subcase 1.1: $AD = 1$ as well. Then $AC = BD \neq 1$, so $ABCD$ is a square.
- Subcase 1.2: $AD \neq 1$. Then $AC = BD = AD$, so A, B, C, D are four points of a regular pentagon.

Case 2: There is an equilateral triangle, say ABC , of side length 1.

- Subcase 2.1: There are no more pairs of distance 1. Then D must be the center of the triangle.
- Subcase 2.2: There is one more pair of distance 1, say AD . Then D can be either of the two intersections of the unit circle centered at A with the perpendicular bisector of BC . This gives us 2 kites.
- Subcase 2.3: Both $AD = BD = 1$. Then $ABCD$ is a rhombus with a 60° angle.

This gives us 6 configurations total.

34. [20] Let n be the **second** smallest integer that can be written as the sum of two positive cubes in two different ways. Compute n . If your guess is a , you will receive $\max(25 - 5 \cdot \max(\frac{a}{n}, \frac{n}{a}), 0)$ points, rounded up.

Proposed by: Alexander Katz

Answer:

A computer search yields that the second smallest number is 4104. Indeed, $4104 = 9^3 + 15^3 = 2^3 + 16^3$

35. [20] Let n be the smallest positive integer such that any positive integer can be expressed as the sum of n integer 2015th powers. Find n . If your answer is a , your score will be $\max(20 - \frac{1}{5} |\log_{10} \frac{a}{n}|, 0)$, rounded up.

Proposed by: Alexander Katz

Answer:

In general, if $k \leq 471600000$, then any integer can be expressed as the sum of $2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2$ integer k th powers. This bound is optimal.

The problem asking for the minimum number of k -th powers needed to add to any positive integer is called Waring's problem.

36. [20] Consider the following seven false conjectures with absurdly high counterexamples. Pick any subset of them, and list their labels in order of their smallest counterexample (the smallest n for which the conjecture is false) from smallest to largest. For example, if you believe that the below list is already ordered by counterexample size, you should write "PECRSGA".

- P. (**Polya's conjecture**) For any integer n , at least half of the natural numbers below n have an odd number of prime factors.
- E. (**Euler's conjecture**) There is no perfect cube n that can be written as the sum of three positive cubes.
- C. (**Cyclotomic**) The polynomial with minimal degree whose roots are the primitive n th roots of unity has all coefficients equal to -1, 0, or 1.
- R. (**Prime race**) For any integer n , there are more primes below n equal to $2 \pmod{3}$ than there are equal to $1 \pmod{3}$.
- S. (**Seventeen conjecture**) For any integer n , $n^{17} + 9$ and $(n + 1)^{17} + 9$ are relatively prime.
- G. (**Goldbach's (other) conjecture**) Any odd composite integer n can be written as the sum of a prime and twice a square.
- A. (**Average square**) Let $a_1 = 1$ and $a_{k+1} = \frac{1+a_1^2+a_2^2+\dots+a_k^2}{k}$. Then a_n is an integer for any n .

If your answer is a list of $4 \leq n \leq 7$ labels in the correct order, your score will be $(n - 2)(n - 3)$. Otherwise, it will be 0.

Proposed by: Alexander Katz

Answer:

The smallest counterexamples are:

- Polya's conjecture: 906,150,257
- Euler's sum of powers: 31,858,749,840,007,945,920,321
- Cyclotomic polynomials: 105
- Prime race: 23,338,590,792
- Seventeen conjecture: 8,424,432,925,592,889,329,288,197,322,308,900,672,459,420,460,792,433
- Goldbach's other conjecture: 5777
- Average square: 44

HMMT November 2015

November 14, 2015

Team

1. [3] Triangle ABC is isosceles, and $\angle ABC = x^\circ$. If the sum of the possible measures of $\angle BAC$ is 240° , find x .

Proposed by: Alexander Katz

Answer: 20

There are three possible triangles: either $\angle ABC = \angle BCA$, in which case $\angle BAC = 180 - 2x$, $\angle ABC = \angle BAC$, in which case $\angle BAC = x$, or $\angle BAC = \angle BCA$, in which case $\angle BAC = \frac{180-x}{2}$. These sum to $\frac{540-3x}{2}$, so we have $\frac{540-3x}{2} = 240 \implies x = \boxed{20}$.

2. [3] Bassanio has three red coins, four yellow coins, and five blue coins. At any point, he may give Shylock any two coins of different colors in exchange for one coin of the other color; for example, he may give Shylock one red coin and one blue coin, and receive one yellow coin in return. Bassanio wishes to end with coins that are all the same color, and he wishes to do this while having as many coins as possible. How many coins will he end up with, and what color will they be?

Proposed by: Yang Liu

Answer: 7 yellow coins

Let r, y, b denote the numbers of red, yellow, and blue coins respectively. Note that each of the three possible exchanges do not change the parities of $y - r, b - y$, or $b - r$, and eventually one of these differences becomes zero. Since $b - r$ is the only one of these differences that is originally even, it must be the one that becomes zero, and so Bassanio will end with some number of yellow coins. Furthermore, Bassanio loses a coin in each exchange, and he requires at least five exchanges to rid himself of the blue coins, so he will have at most $12 - 5 = 7$ yellow coins at the end of his trading.

It remains to construct a sequence of trades that result in seven yellow coins. First, Bassanio will exchange one yellow and one blue coin for one red coin, leaving him with four red coins, three yellow coins, and four blue coins. He then converts the red and blue coins into yellow coins, resulting in 7 yellow coins, as desired.

3. [3] Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x , and let $\{x\}$ denote the fractional part of x . For example, $\lfloor \pi \rfloor = 3$, and $\{\pi\} = 0.14159\dots$, while $\lfloor 100 \rfloor = 100$ and $\{100\} = 0$. If n is the largest solution to the equation $\frac{\lfloor n \rfloor}{n} = \frac{2015}{2016}$, compute $\{n\}$.

Proposed by: Alexander Katz

Answer: $\frac{2014}{2015}$

Note that $n = \lfloor n \rfloor + \{n\}$, so

$$\begin{aligned}\frac{\lfloor n \rfloor}{n} &= \frac{\lfloor n \rfloor}{\lfloor n \rfloor + \{n\}} \\ &= \frac{2015}{2016} \\ \implies 2016\lfloor n \rfloor &= 2015\lfloor n \rfloor + 2015\{n\} \\ \implies \lfloor n \rfloor &= 2015\{n\}\end{aligned}$$

Hence, $n = \lfloor n \rfloor + \{n\} = \frac{2016}{2015}\lfloor n \rfloor$, and so n is maximized when $\lfloor n \rfloor$ is also maximized. As $\lfloor n \rfloor$ is an integer, and $\{n\} < 1$, the maximum possible value of $\lfloor n \rfloor$ is 2014. Therefore, $\{n\} = \frac{\lfloor n \rfloor}{2015} = \boxed{\frac{2014}{2015}}$.

4. [5] Call a set of positive integers *good* if there is a partition of it into two sets S and T , such that there do not exist three elements $a, b, c \in S$ such that $a^b = c$ and such that there do not exist three elements $a, b, c \in T$ such that $a^b = c$ (a and b need not be distinct). Find the smallest positive integer n such that the set $\{2, 3, 4, \dots, n\}$ is *not good*.

Proposed by: Sam Korsky

Answer: 65536

First, we claim that the set $\{2, 4, 8, 256, 65536\}$ is not good. Assume the contrary and say $2 \in S$. Then since $2^2 = 4$, we have $4 \in T$. And since $4^4 = 256$, we have $256 \in S$. Then since $256^2 = 65536$, we have $65536 \in T$. Now, note that we cannot place 8 in either S or T , contradiction.

Hence $n \leq 65536$. And the partition $S = \{2, 3\} \cup \{256, 257, \dots, 65535\}$ and $T = \{4, 5, \dots, 255\}$ shows that $n \geq 65536$. Therefore $n = \span style="border: 1px solid black; padding: 2px;">65536.$

5. [5] Kelvin the Frog is trying to hop across a river. The river has 10 lily pads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lily pad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side?

Proposed by: Alexander Katz

Answer: 176

Kelvin needs (at most) $i(10-i)$ hops to determine the i th lily pad he should jump to, then an additional 11 hops to actually get across the river. Thus he requires $\sum_{i=1}^{10} i(10-i) + 11 = \span style="border: 1px solid black; padding: 2px;">176 hops to guarantee success.$

6. [5] Marcus and four of his relatives are at a party. Each pair of the five people are either *friends* or *enemies*. For any two enemies, there is no person that they are both friends with. In how many ways is this possible?

Proposed by: Alexander Katz

Answer: 52

Denote friendship between two people a and b by $a \sim b$. Then, assuming everyone is friends with themselves, the following conditions are satisfied:

- $a \sim a$
- If $a \sim b$, then $b \sim a$
- If $a \sim b$ and $b \sim c$, then $a \sim c$

Thus we can separate the five people into a few groups (possibly one group), such that people are friends within each group, but two people are enemies when they are in different groups. Here comes the calculation. Since the number of group(s) can be 1, 2, 3, 4, or 5, we calculate for each of those cases. When there's only one group, then we only have 1 possibility that we have a group of 5, and the total number of friendship assignments in this case is $\binom{5}{5} = 1$; when there are two groups, we have $5 = 1 + 4 = 2 + 3$ are all possible numbers of the two groups, with a total of $\binom{5}{1} + \binom{5}{2} = 15$ choices; when there are three groups, then we have $5 = 1 + 1 + 3 = 1 + 2 + 2$, with $\binom{5}{3} + \frac{\binom{5}{1}\binom{5}{2}}{2} = 25$ possibilities; when there are four of them, then we have $5 = 1 + 1 + 1 + 2$ be its only possibility, with $\binom{5}{2} = 10$ separations; when there are 5 groups, obviously we have 1 possibility. Hence, we have a total of $1 + 15 + 25 + 10 + 1 = \span style="border: 1px solid black; padding: 2px;">52 possibilities.$

Alternatively, we can also solve the problem recursively. Let B_n be the number of friendship graphs with n people, and consider an arbitrary group. If this group has size k , then there are $\binom{n}{k}$ possible such groups, and B_{n-k} friendship graphs on the remaining $n - k$ people. Therefore, we have the recursion

$$B_n = \sum_{k=0}^n \binom{n}{k} B_{n-k}$$

with the initial condition $B_1 = 1$. Calculating routinely gives $B_5 = 52$ as before.

7. [6] Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at P . Let the area of triangle APB be 24 and let the area of triangle CPD be 25. What is the minimum possible area of quadrilateral $ABCD$?

Proposed by: Sam Korsky

Answer: $\boxed{49 + 20\sqrt{6}}$

Note that $\angle APB = 180^\circ - \angle BPC = \angle CPD = 180^\circ - \angle DPA$ so $4[BPC][DPA] = (PB \cdot PC \cdot \sin BPC)(PD \cdot PA \cdot \sin DPA) = (PA \cdot PB \cdot \sin APB)(PC \cdot PD \cdot \sin CPD) = 4[APB][CPD] = 2400 \implies [BPC][DPA] = 600$. Hence by AM-GM we have that

$$[BPC] + [DPA] \geq 2\sqrt{[BPC][DPA]} = 20\sqrt{6}$$

so the minimum area of quadrilateral $ABCD$ is $\boxed{49 + 20\sqrt{6}}$.

8. [6] Find **any** quadruple of positive integers (a, b, c, d) satisfying $a^3 + b^4 + c^5 = d^{11}$ and $abc < 10^5$.

Proposed by: Sam Korsky

Answer: $\boxed{(a, b, c, d) = (128, 32, 16, 4) \text{ or } (a, b, c, d) = (160, 16, 8, 4)}$

It's easy to guess that there are solutions such that a, b, c, d are in the form of n^x , where n is a rather small number. After a few attempts, we can see that we obtain simple equations when $n = 2$ or $n = 3$: for $n = 2$, the equation becomes in the form of $2^t + 2^t + 2^{t+1} = 2^{t+2}$ for some non-negative integer t ; for $n = 3$, the equation becomes in the form of $3^t + 3^t + 3^t = 3^{t+1}$ for some non-negative integer t . In the first case, we hope that t is a multiple of two of 3, 4, 5, that $t + 1$ is a multiple of the last one, and that $t + 2$ is a multiple of 11. Therefore, $t \equiv 15, 20, 24 \pmod{60}$ and $t \equiv 9 \pmod{11}$. It's easy to check that the only solution that satisfies the given inequality is the solution with $t = 20$, and $(a, b, c, d) = (128, 32, 16, 4)$. In the case where $n = 3$, we must have that t is a multiple of 60, which obviously doesn't satisfy the inequality restriction. Remark: By programming, we find that the only two solutions are $(a, b, c, d) = (128, 32, 16, 4)$ and $(a, b, c, d) = (160, 16, 8, 4)$, with the the former being the intended solution.

9. [7] A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is *good* if, starting from any vertex V connected to at least one other vertex, it is possible to draw a path starting and ending at V that traverses each edge exactly once. What is the probability that the graph is good?

Proposed by: Sam Korsky

Answer: $\boxed{\frac{507}{16384} \text{ or } \frac{2^{10}-10}{2^{15}} \text{ or } \frac{2^9-5}{2^{14}}}$

First, we find the probability that all vertices have even degree. Arbitrarily number the vertices 1, 2, 3, 4, 5, 6. Flip the coin for all the edges out of vertex 1; this vertex ends up with even degree with probability $\frac{1}{2}$. Next we flip for all the remaining edges out of vertex 2; regardless of previous edges, vertex 2 ends up with even degree with probability $\frac{1}{2}$, and so on through vertex 5. Finally, if vertices 1 through 5 all have even degree, vertex 6 must also have even degree. So all vertices have even degree with probability $\frac{1}{2^5} = \frac{1}{32}$. There are $\binom{6}{2} = 15$ edges total, so there are 2^{15} total possible graphs, of which 2^{10} have all vertices with even degree. Observe that exactly 10 of these latter graphs are not good, namely, the $\frac{1}{2}\binom{6}{3}$ graphs composed of two separate triangles. So $2^{10} - 10$ of our graphs are good, and the probability that a graph is good is $\frac{2^{10}-10}{2^{15}}$.

10. [7] A number n is *bad* if there exists some integer c for which $x^x \equiv c \pmod{n}$ has no integer solutions for x . Find the number of bad integers between 2 and 42 inclusive.

Proposed by: Sam Korsky

Answer: $\boxed{25}$

Call a number *good* if it is not *bad*. We claim all good numbers are products of distinct primes, none of which are equivalent to 1 modulo another.

We first show that all such numbers are *good*. Consider $n = p_1 p_2 \dots p_k$, and let x be a number satisfying $x \equiv c \pmod{p_1 p_2 \dots p_k}$ and $x \equiv 1 \pmod{(p_1 - 1)(p_2 - 1) \dots (p_k - 1)}$. Since, by assumption, $p_1 p_2 \dots p_k$ and $(p_1 - 1)(p_2 - 1) \dots (p_k - 1)$ are relatively prime, such an x must exist by CRT. Then $x^x \equiv c^1 = c \pmod{n}$, for any c , as desired.

We now show that all other numbers are *bad*. Suppose that there exist some $p_1, p_2 \mid n$ such that $\gcd(p_1, p_2 - 1) \neq 1$ (which must hold for some two primes by assumption), and hence $\gcd(p_1, p_2 - 1) = p_1$. Consider some c for which $p_1 c$ is not a p_1 th power modulo p_2 , which must exist as $p_1 c$ can take any value modulo p_2 (as p_1, p_2 are relatively prime). We then claim that $x^x \equiv p_1 c \pmod{n}$ is not solvable.

Since $p_1 p_2 \mid n$, we have $x^x \equiv p_1 c \pmod{p_1 p_2}$, hence $p_1 \mid x$. But then $x^x \equiv p_1 c$ is a p_1 th power modulo p_2 as $p_1 \mid x$, contradicting our choice of c . As a result, all such numbers are *bad*.

Finally, it is easy to see that n is *bad* if it is not squarefree. If p_1 divides n twice, then letting $c = p_1$ makes the given equivalence unsolvable.

Hence, there are 16 numbers (13 primes: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41; and 3 semiprimes: $3 \cdot 5 = 15$, $3 \cdot 11 = 33$, $5 \cdot 7 = 35$) that are *good*, which means that $41 - 16 = \boxed{25}$ numbers are *bad*.

HMMT November 2015

November 14, 2015

Theme round

1. Consider a 1×1 grid of squares. Let A, B, C, D be the vertices of this square, and let E be the midpoint of segment CD . Furthermore, let F be the point on segment BC satisfying $BF = 2CF$, and let P be the intersection of lines AF and BE . Find $\frac{AP}{PF}$.

Proposed by: Sam Korsky

Answer:

Let line BE hit line DA at Q . It's clear that triangles AQP and FBP are similar so

$$\frac{AP}{PF} = \frac{AQ}{BF} = \frac{2AD}{\frac{2}{3}BC} = \boxed{3}$$

2. Consider a 2×2 grid of squares. David writes a positive integer in each of the squares. Next to each row, he writes the product of the numbers in the row, and next to each column, he writes the product of the numbers in each column. If the sum of the eight numbers he writes down is 2015, what is the minimum possible sum of the four numbers he writes in the grid?

Proposed by: Alexander Katz

Answer:

Let the four numbers be a, b, c, d , so that the other four numbers are ab, ad, bc, bd . The sum of these eight numbers is $a + b + c + d + ab + ad + bc + bd = (a + c) + (b + d) + (a + c)(b + d) = 2015$, and so $(a + c + 1)(b + d + 1) = 2016$. Since we seek to minimize $a + b + c + d$, we need to find the two factors of 2016 that are closest to each other, which is easily calculated to be $42 \cdot 48 = 2016$; this makes $a + b + c + d = \boxed{88}$.

3. Consider a 3×3 grid of squares. A circle is inscribed in the lower left corner, the middle square of the top row, and the rightmost square of the middle row, and a circle O with radius r is drawn such that O is externally tangent to each of the three inscribed circles. If the side length of each square is 1, compute r .

Proposed by: Sam Korsky

Answer:

Let A be the center of the square in the lower left corner, let B be the center of the square in the middle of the top row, and let C be the center of the rightmost square in the middle row. It's clear that O is the circumcenter of triangle ABC - hence, the desired radius is merely the circumradius of triangle ABC minus $\frac{1}{2}$. Now note that by the Pythagorean theorem, $BC = \sqrt{2}$ and $AB = AC = \sqrt{5}$ so we easily find that the altitude from A in triangle ABC has length $\frac{3\sqrt{2}}{2}$. Therefore the area of triangle ABC is $\frac{3}{2}$. Hence the circumradius of triangle ABC is given by

$$\frac{BC \cdot CA \cdot AB}{4 \cdot \frac{3}{2}} = \frac{5\sqrt{2}}{6}$$

and so the answer is $\frac{5\sqrt{2}}{6} - \frac{1}{2} = \boxed{\frac{5\sqrt{2}-3}{6}}$.

4. Consider a 4×4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphales turn, he may color any uncolored square red, and on Crowleys turn, he may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphales score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize his score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphales score be?

Proposed by: Alexander Katz

Answer: 6

We claim that the answer is 6.

On Aziraphale's first two turns, it is always possible for him to take 2 adjacent squares from the central four; without loss of generality, suppose they are the squares at $(1, 1)$ and $(1, 2)$. If allowed, Aziraphale's next turn will be to take one of the remaining squares in the center, at which point there will be seven squares adjacent to a red square, and so Aziraphale can guarantee at least two more adjacent red squares. After that, since the number of blue squares is always at most the number of red squares, Aziraphale can guarantee another adjacent red square, making his score at least 6.

If, however, Crowley does not allow Aziraphale to attain another central red square – i.e. coloring the other two central squares blue – then Aziraphale will continue to take squares from the second row, WLOG $(1, 3)$. If Aziraphale is also allowed to take $(1, 0)$, he will clearly attain at least 6 adjacent red squares as each red square in this row has two adjacent squares to it, and otherwise (if Crowley takes $(1, 0)$), Aziraphale will take $(0, 1)$ and guarantee a score of at least $4 + \frac{4}{2} = 6$ as there are 4 uncolored squares adjacent to a red one.

Therefore, the end score will be at least 6. We now show that this is the best possible for Aziraphale; i.e. Crowley can always limit the score to 6. Crowley can play by the following strategy: if Aziraphale colors a square in the second row, Crowley will color the square below it, if Aziraphale colors a square in the third row, Crowley will color the square above it. Otherwise, if Aziraphale colors a square in the first or fourth rows, Crowley will color an arbitrary square in the same row. It is clear that the two "halves" of the board cannot be connected by red squares, and so the largest contiguous red region will occur entirely in one half of the grid, but then the maximum score is $4 + \frac{4}{2} = 6$.

The optimal score is thus both at least 6 and at most 6, so it must be 6 as desired.

5. Consider a 5×5 grid of squares. Vladimir colors some of these squares red, such that the centers of any four red squares do **not** form an axis-parallel rectangle (i.e. a rectangle whose sides are parallel to those of the squares). What is the maximum number of squares he could have colored red?

Proposed by: Sam Korsky

Answer: 12

We claim that the answer is 12. We first show that if 13 squares are colored red, then some four form an axis-parallel rectangle. Note that we can swap both columns and rows without affecting whether four squares form a rectangle, so we may assume without loss of generality that the top row has the most red squares colored; suppose it has k squares colored. We may further suppose that, without loss of generality, these k red squares are the first k squares in the top row from the left.

Consider the $k \times 5$ rectangle formed by the first k columns. In this rectangle, no more than 1 square per row can be red (excluding the top one), so there are a maximum of $k + 4$ squares colored red. In the remaining $(5 - k) \times 5$ rectangle, at most $4(5 - k)$ squares are colored red (as the top row of this rectangle has no red squares), so there are a maximum of $(k + 4) + 4(5 - k) = 24 - 3k$ squares colored red in the 5×5 grid. By assumption, at least 13 squares are colored red, so we have $13 \leq 24 - 3k \iff k \leq 3$.

Hence there are at most 3 red squares in any row. As there are at least 13 squares colored red, this implies that at least 3 rows have 3 red squares colored. Consider the 3×5 rectangle formed by these three rows. Suppose without loss of generality that the leftmost three squares in the top row are colored red, which forces the rightmost three squares in the second row to be colored red. But then, by the Pigeonhole Principle, some 2 of the 3 leftmost squares or some 2 of the 3 rightmost squares in the bottom row will be colored red, leading to an axis-parallel rectangle – a contradiction.

Hence there are at most 12 squares colored red. It remains to show that there exists some coloring where exactly 12 squares are colored red, one example of which is illustrated below:

	R	R	R	R
R	R			
R		R		
R			R	
R				R

The maximum number of red squares, therefore, is $\boxed{12}$.

6. Consider a 6×6 grid of squares. Edmond chooses four of these squares uniformly at random. What is the probability that the centers of these four squares form a square?

Proposed by: Alexander Katz

Answer: $\boxed{\frac{1}{561} \text{ OR } \frac{105}{\binom{36}{4}}}$

Firstly, there are $\binom{36}{4}$ possible combinations of points. Call a square *proper* if its sides are parallel to the coordinate axes and *improper* otherwise. Note that every *improper* square can be inscribed in a unique *proper* square. Hence, an $n \times n$ *proper* square represents a total of n squares: 1 proper and $n - 1$ improper.

There are thus a total of

$$\begin{aligned}
 \sum_{i=1}^6 i(6-i)^2 &= \sum_{i=1}^6 (i^3 - 12i^2 + 36i) \\
 &= \sum_{i=1}^6 i^3 - 12 \sum_{i=1}^6 i^2 + 36 \sum_{i=1}^6 i = 1^6 i \\
 &= 441 - 12(91) + 36(21) \\
 &= 441 - 1092 + 756 \\
 &= 105
 \end{aligned}$$

squares on the grid. Our desired probability is thus $\frac{105}{\binom{36}{4}} = \boxed{\frac{1}{561}}$.

7. Consider a 7×7 grid of squares. Let $f : \{1, 2, 3, 4, 5, 6, 7\} \rightarrow \{1, 2, 3, 4, 5, 6, 7\}$ be a function; in other words, $f(1), f(2), \dots, f(7)$ are each (not necessarily distinct) integers from 1 to 7. In the top row of the grid, the numbers from 1 to 7 are written in order; in every other square, $f(x)$ is written where x is the number above the square. How many functions have the property that the bottom row is identical to the top row, and no other row is identical to the top row?

Proposed by: Alexander Katz

Answer: $\boxed{1470}$

Consider the directed graph with $1, 2, 3, 4, 5, 6, 7$ as vertices, and there is an edge from i to j if and only if $f(i) = j$. Since the bottom row is equivalent to the top one, we have $f^6(x) = x$. Therefore, the graph must decompose into cycles of length 6, 3, 2, or 1. Furthermore, since no other row is equivalent to the top one, the least common multiple of the cycle lengths must be 6. The only partitions of 7 satisfying these constraints are $7 = 6 + 1$, $7 = 3 + 2 + 2$, and $7 = 3 + 2 + 1 + 1$.

If we have a cycle of length 6 and a cycle of length 1, there are 7 ways to choose which six vertices will be in the cycle of length 6, and there are $5! = 120$ ways to determine the values of f within this cycle (to see this, pick an arbitrary vertex in the cycle: the edge from it can connect to any of the remaining 5 vertices, which can connect to any of the remaining 4 vertices, etc.). Hence, there are $7 \cdot 120 = 840$ possible functions f in this case.

If we have a cycle of length 3 and two cycles of length 2, there are $\frac{\binom{7}{2}\binom{5}{2}}{2} = 105$ possible ways to assign which vertices will belong to which cycle (we divide by two to avoid double-counting the cycles of

length 2). As before, there are $2! \cdot 1! \cdot 1! = 2$ assignments of f within the cycles, so there are a total of 210 possible functions f in this case.

Finally, if we have a cycle of length 3, a cycle of length 2, and two cycles of length 1, there are $\binom{7}{3} \binom{4}{2} = 210$ possible ways to assign the cycles, and $2! \cdot 1! \cdot 0! \cdot 0! = 2$ ways to arrange the edges within the cycles, so there are a total of 420 possible functions f in this case.

Hence, there are a total of $840 + 210 + 420 = \boxed{1470}$ possible f .

8. Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

Proposed by: Alexander Katz

Answer: $\boxed{70}$

Let the expected number of minutes it will take the rook to reach the upper right corner from the top or right edges be E_e , and let the expected number of minutes it will take the rook to reach the upper right corner from any other square be E_c . Note that this is justified because the expected time from any square on the top or right edges is the same, as is the expected time from any other square (this is because swapping any two rows or columns doesn't affect the movement of the rook). This gives us two linear equations:

$$E_c = \frac{2}{14}(E_e + 1) + \frac{12}{14}(E_c + 1)$$

$$E_e = \frac{1}{14}(1) + \frac{6}{14}(E_e + 1) + \frac{7}{14}(E_c + 1)$$

which gives the solution $E_e = 63, E_c = \boxed{70}$.

9. Consider a 9×9 grid of squares. Haruki fills each square in this grid with an integer between 1 and 9, inclusive. The grid is called a *super-sudoku* if each of the following three conditions hold:

- Each column in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.
- Each row in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.
- Each 3×3 subsquare in the grid contains each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 exactly once.

How many possible super-sudoku grids are there?

Proposed by: Alexander Katz

Answer: $\boxed{0}$

Without loss of generality, suppose that the top left corner contains a 1, and examine the top left 3×4 :

1	x	x	x
x	x	x	*
x	x	x	*

There cannot be another 1 in any of the cells marked with an x, but the 3×3 on the right must contain a 1, so one of the cells marked with a * must be a 1. Similarly, looking at the top left 4×3 :

1	x	x
x	x	x
x	x	x
x	*	*

One of the cells marked with a * must also contain a 1. But then the 3×3 square diagonally below the top left one:

1	x	x	x
x	x	x	*
x	x	x	*
x	*	*	?

must contain multiple 1s, which is a contradiction. Hence no such supersudokus exist.

10. Consider a 10×10 grid of squares. One day, Daniel drops a burrito in the top left square, where a wingless pigeon happens to be looking for food. Every minute, if the pigeon and the burrito are in the same square, the pigeon will eat 10% of the burrito's original size and accidentally throw it into a random square (possibly the one it is already in). Otherwise, the pigeon will move to an adjacent square, decreasing the distance between it and the burrito. What is the expected number of minutes before the pigeon has eaten the entire burrito?

Proposed by: Sam Korsky

Answer: 71.8

Label the squares using coordinates, letting the top left corner be $(0, 0)$. The burrito will end up in 10 (not necessarily different) squares. Call them $p_1 = (x_1, y_1) = (0, 0), p_2 = (x_2, y_2), \dots, p_{10} = (x_{10}, y_{10})$. p_2 through p_{10} are uniformly distributed throughout the square. Let $d_i = |x_{i+1} - x_i| + |y_{i+1} - y_i|$, the taxicab distance between p_i and p_{i+1} .

After 1 minute, the pigeon will eat 10% of the burrito. Note that if, after eating the burrito, the pigeon throws it to a square taxicab distance d from the square it's currently in, it will take exactly d minutes for it to reach that square, regardless of the path it takes, and another minute for it to eat 10% of the burrito.

Hence, the expected number of minutes it takes for the pigeon to eat the whole burrito is

$$\begin{aligned}
 1 + E\left(\sum_{i=1}^9 (d_i + 1)\right) &= 1 + E\left(\sum_{i=1}^9 1 + |x_{i+1} - x_i| + |y_{i+1} - y_i|\right) \\
 &= 10 + 2 \cdot E\left(\sum_{i=1}^9 |x_{i+1} - x_i|\right) \\
 &= 10 + 2 \cdot \left(E(|x_2|) + E\left(\sum_{i=2}^9 |x_{i+1} - x_i|\right)\right) \\
 &= 10 + 2 \cdot (E(|x_2|) + 8 \cdot E(|x_{i+1} - x_i|)) \\
 &= 10 + 2 \cdot \left(4.5 + 8 \cdot \frac{1}{100} \cdot \sum_{k=1}^9 k(20 - 2k)\right) \\
 &= 10 + 2 \cdot (4.5 + 8 \cdot 3.3) \\
 &= 71.8
 \end{aligned}$$

HMMT February 2015
Saturday 21 February 2015

HMIC
Testing window: April 1–5

1. [20] Let S be the set of positive integers n such that the inequality

$$\phi(n) \cdot \tau(n) \geq \sqrt{\frac{n^3}{3}}$$

holds, where $\phi(n)$ is the number of positive integers $k \leq n$ that are relatively prime to n , and $\tau(n)$ is the number of positive divisors of n . Prove that S is finite.

Answer: N/A

Proof. Let S be the set of all positive integers n such that

$$\phi(n) \cdot \tau(n) \geq \sqrt{\frac{n^3}{3}}.$$

Define a function Φ on all positive integers n by

$$\Phi(n) = \frac{\phi(n)^2 \cdot \tau(n)^2}{n^3}.$$

An important observation is that Φ has the property that for every relatively prime positive integers m, n , we have $\Phi(mn) = \Phi(m)\Phi(n)$.

Define another function ψ on all ordered-pairs (a, p) of positive integer a and prime number p as follows:

$$\psi(a, p) := \frac{(a+1)^2(1-1/p)^2}{p^a}.$$

If we express n in its canonical form as $n = \prod_{i=1}^k p_i^{a_i}$, then we have

$$\Phi(n) = \prod_{i=1}^k \psi(a_i, p_i).$$

Therefore, S is actually the set of all $n = \prod_{i=1}^k p_i^{a_i}$ such that

$$\prod_{i=1}^k \psi(a_i, p_i) \geq \frac{1}{3}.$$

It is straightforward to establish the following: for every prime p and positive integer a ,

- if $p \geq 11$, then $\psi(a, p) < \frac{1}{3}$;
- if $p = 5$ and $a \geq 2$, then $\psi(a, p) < \frac{1}{3}$;
- if $p = 3$ and $a \geq 3$, then $\psi(a, p) < \frac{1}{3}$;
- if $p = 2$ and $a \geq 5$, then $\psi(a, p) < \frac{1}{3}$;
- $\psi(a, p)$ is always less than $\frac{1}{\sqrt{3}}$ **unless** $(a, p) = (1, 3)$ where $\psi(1, 3) = \frac{16}{27}$.

The data above shows that in the case $n = p^a$, in which there is only one prime dividing n , there are at most 8 possible n in S : $2^1, 2^2, 2^3, 2^4, 3^1, 3^2, 5^1, 7^1$. If n is divisible by at least two distinct primes,

then one of them must be 3^1 and 3^1 fully divides n (that is, $3^2 \nmid n$). In the latter case $n = 3 \cdot n_0$. Write $n_0 = \prod_{j=1}^l q_j^{b_j}$. In order for $\Phi(n) \geq \frac{1}{3}$, we require

$$\prod_{j=1}^l \psi(b_j, q_j) \geq \frac{9}{16}.$$

This is only possible when $(b, q) = (2, 2)$, where $n = 12$. Also, note that when $n = 1$, $\Phi(1) = 1$, so $1 \in S$. Hence, there are at most 10 (a finite number of) possible values of n in S . \square

Remark 0.1. This problem was proposed by Pakawut Jiradilok.

Remark 0.2. By explicit calculations, there are in fact exactly ten elements in S : 1, 2, 3, 4, 5, 7, 8, 9, 12, 16.

Remark 0.3. To only get a finite bound, one does not need to be nearly as careful as we are in the above proof.

2. [25] Let m, n be positive integers with $m \geq n$. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leq m$ and $a + b > m$.

For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ chosen with $v \geq 0$ minimal, and let $I(a, b)$ denote the (open) interval $(v/a, u/b)$.

Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

Answer: N/A

Proof. The fact that $I(a, b) \subseteq (0, 1)$ follows from the small-ness of $n \leq m < a + b$: the smallest solution (u, v) has $0 \leq v \leq a - 1$, so $u = \frac{n + bv}{a} < \frac{(a+b) + b(a-1)}{a} = b + 1$ forces $1 \leq u \leq b$.

For the main part of the problem, it suffices (actually, is equivalent) to show that

- (i) 0, 1 each appear exactly n times as endpoints of intervals $I(a, b)$; and
- (ii) each reduced rational $p/q \in (0, 1)$ appears an equal (possibly zero, if q is large) number of times as left and right endpoints of intervals $I(a, b)$.

We prove these separately as follows:

- (i) 0 is a (left) endpoint precisely when $v/a = 0$, or equivalently $(a, b) \in S$ has $a \mid n$ (look at $v \bmod a$). Since $n \leq m$, we get $\phi(d)$ good pairs $(d, b) \in S$ for fixed $d \mid n$. Indeed, for any coprime residue class modulo d , there's exactly one representative b with $b \leq m < b + d$. (Conceptually, it may be enlightening to think of S as part of the Euclidean algorithm tree generated by $(1, 1)$.)

Thus 0 occurs as an endpoint $\sum_{d \mid n} \phi(d) = n$ times. Similarly, 1 is an endpoint precisely when $u = b$, or equivalently $(a, b) \in S$ has $b \mid n$ (look at $u \bmod b$). By the same reasoning as before, we get n right endpoint occurrences of 1.

- (ii) Fix $p/q \in (0, 1)$ with p, q coprime, so $q \geq 2$. We want to show that $v_1/a_1 = p/q$ occurs for the same number of $(a_1, b_1) \in S$ as $u_2/b_2 = p/q$ occurs for $(a_2, b_2) \in S$. (In fact, we will show that the number of occurrences of the former in $(x, b_1) \in S$ equals the number of the latter in $(a_2, x) \in S$.)

The former occurs precisely when $1 \leq a_1 \leq m$, $q \mid a_1$, and (given the first two conditions) $v_1 \equiv p(a_1/q) \pmod{a_1}$. Given the first two conditions, the third is equivalent to $-nb_1^{-1} \equiv p(a_1/q) \pmod{a_1}$, or $b_1(a_1/q) \equiv -np^{-1} \pmod{a_1}$. This is equivalent to having $(a_1/q) \mid n$ and $b_1 \equiv (-n/(a_1/q))p^{-1} \pmod{q}$.

Similarly, the latter occurs precisely when $1 \leq b_2 \leq m$, $q \mid b_2$, $(b_2/q) \mid n$, and $a_2 \equiv (n/(b_2/q))p^{-1} \pmod{q}$.

As hinted at before, if we consider the occurrences for a fixed value $x = a_1 = b_2$, then the number of permitted residue classes $b_1 \pmod{a_1 = x}$ (in the former) is the same as the number of permitted residue classes $a_2 \pmod{b_2 = x}$ (in the latter): $y \pmod{x}$ is permitted in the former if and only if $-y \pmod{x}$ is permitted in the latter; note that $(y, x) = 1$ if and only if $(-y, x) = 1$.

□

Remark 0.4. This problem was proposed by Victor Wang.

Remark 0.5. This problem was inspired by ISL 2013 N7. Note that the $n = 1$ case gives the famous Farey fractions sequence (for fractions of denominator at most m).

3. [30] Let M be a 2014×2014 invertible matrix, and let $\mathcal{F}(M)$ denote the set of matrices whose rows are a permutation of the rows of M . Find the number of matrices $F \in \mathcal{F}(M)$ such that $\det(M + F) \neq 0$.

Answer: 2013!!²

Proof. Number the rows of M as $r_1, r_2, \dots, r_{2014}$, and remark that by invertibility, these rows are linearly independent. We can write $\det(M + F)$ as the sum of the determinants of the 2^{2014} matrices which can be obtained by replacing some set of the rows of M with the corresponding rows of F (this follows from distributivity and expansion by minors) - we'll call these the insertion matrices of M . Of course, the rows of F are themselves some permutation of the rows of M .

Now, all insertion matrices in which some row appears at least twice have determinant 0 (by linear dependence of rows), so the determinant of $M + F$ is the sum of the determinants of the insertion matrices whose rows are a permutation of the r_i . Let the i -th row of F be $r_{\sigma(i)}$ and partition σ into disjoint cycles; we claim $\det(M + F) = 0$ iff σ contains an even cycle.

Indeed, if we replace row r_i in M , in order for the rows of $M + F$ to remain a permutation of the r_i we must replace row $r_{\sigma(i)}$ as well, and so on. In other words, we either replace none of the rows in any particular cycle, or we replace all of them. Yet if any cycle is even, the determinants of any insertion matrix in which that cycle is replaced and the same insertion matrix, except not replacing that cycle, are the same in magnitude and opposite in sign (this follows from the fact that a row transposition flips the sign of the determinant, and we must perform an odd number of these). Thus $\det(M + F)$ vanishes if σ contains any even cycles. On the other hand, if σ consists of only odd cycles, every non-zero insertion matrix has determinant $\det(M) \neq 0$ by analogous reasoning (this time, the even number of row transpositions preserves sign). It remains only to count the number of elements of S_{2014} which can be decomposed into exclusively odd cycles.

We begin by writing a recursion for the general problem. Let x_n denote the number of permutations on $[n]$ which can be decomposed into exclusively odd cycles. By casework on the cycle containing 1, we find the recursion

$$x_n = \sum_i \binom{n-1}{2i-2} (2i-2)! x_{n-2i+1}$$

where $x_0 = x_1 = x_2 = 1$.

The intended solution from this point was combinatorial in nature, similar to the one from the fourth comment under Evan Chen's "Recursion" blog post (from June 2012). The following much nicer finish is due to Lawrence Sun:

Set $y_n = x_n/n!$ and rewrite the recursion as

$$ny_n = y_{n-1} + y_{n-3} + \dots$$

Shifting, we find in addition that

$$(n-2)y_{n-2} = y_{n-3} + \dots$$

Subtracting,

$$ny_n = y_{n-1} + (n-2)y_{n-2}$$

for all n . It now follows from a straightforward induction that $y_{2k} = [(2k-1)!!]^2/(2k)!$ and $y_{2k+1} = [(2k+1)!!(2k-1)!!]/(2k+1)!$. In particular,

$$x_{2014} = 2014!y_{2014} = (2013!!)^2$$

as desired.

Here is an alternative approach: define the power series $F(z)$ in the variable z by

$$F(z) = \sum_{n=0}^{\infty} x_n \frac{z^n}{n!}.$$

The recurrence relation is equivalent to the differential equation

$$F'(z) = \left(\sum_{n=0}^{\infty} z^{2n} \right) F(z) = \frac{1}{1-z^2} F(z).$$

Solving this equation using the initial condition $F(0) = 1$ gives

$$F(z) = \sqrt{\frac{1+z}{1-z}} = (1+z)(1-z^2)^{-1/2}.$$

By the binomial theorem, we have

$$F(z) = (1+z) \sum_{n=0}^{\infty} \binom{-1/2}{n} (-1)^n z^{2n},$$

so

$$x_{2014} = 2014! \binom{-1/2}{1007} (-1)^{1007} = (2013!!)^2$$

as desired. □

Remark 0.6. This problem was proposed by Dhroova Aiyalam.

Remark 0.7. We can write the matrices $F \in \mathcal{F}(M)$ as $F = PM$, where P ranges over all 2014×2014 permutation matrices. Then, we have $F + M = (P + 1_{2014 \times 2014})M$, which is invertible if and only if $P + 1_{2014 \times 2014}$ is. Using this, we can give an alternative proof that the answer to the question is equal to the number of permutations of $\{1, \dots, 2014\}$ with cycles of odd length.

4. [35] Prove that there exists a positive integer N such that for any positive integer $n \geq N$, there are at least 2015 non-empty subsets S of $\{n^2 + 1, n^2 + 2, \dots, n^2 + 3n\}$ with the property that the product of the elements of S is a perfect square.

Answer: N/A

Lemma 0.1. Let n, k be integers with $1 \leq k \leq n$. Then

$$\binom{n}{k} \leq \left(\frac{en}{k} \right)^k$$

where $e \approx 2.71828 \dots$ is Euler's number.

Proof. We have

$$\begin{aligned} \binom{n}{k} &= \frac{n(n-1) \cdots (n-k+1)}{k!} \\ &\leq \frac{n^k}{k!} \\ &= \left(\frac{n}{k} \right)^k \frac{k^k}{k!} \\ &\leq \left(\frac{n}{k} \right)^k \sum_{m=0}^{\infty} \frac{k^m}{m!} \\ &= \left(\frac{en}{k} \right)^k \end{aligned}$$

where the last step follows from the series expansion

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!}$$

applied to $x = k$. □

In the solution below, all logarithms are to the base e . If f and g are two functions of the variable n , then write $f(n) \sim g(n)$ to denote that

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1.$$

Proof of original problem. Let U be the set of elements of $\{n^2 + 1, \dots, n^2 + 3n\}$ with no prime factor greater than $3n$.¹ Let s be the number of primes $p \leq 3n$. By the pigeonhole principle, we can find distinct subsets $X_1, \dots, X_r \subset U$ with $r \geq \frac{2^{|U|}}{2^s}$ such that every prime $p \leq 3n$ either divides all the products

$$\prod_{x \in X_1} x, \dots, \prod_{x \in X_r} x$$

an even number of times or divides all of them an odd number of times. Define the subsets $Y_1, \dots, Y_r \subset U$ by

$$Y_i := X_1 \Delta X_i = (X_1 - X_i) \cup (X_i - X_1)$$

for $1 \leq i \leq r$. Then by construction, the sets Y_1, \dots, Y_r are distinct, and they have the property that every prime $p \leq 3n$ divides each of the products

$$\prod_{y \in Y_1} y, \dots, \prod_{y \in Y_r} y$$

an even number of times. Since $Y_1, \dots, Y_r \subset U$, the products

$$\prod_{y \in Y_1} y, \dots, \prod_{y \in Y_r} y$$

do not have any prime factor greater than $3n$, so they are perfect squares.

We have found r distinct subsets of $\{n^2 + 1, \dots, n^2 + 3n\}$ whose products are perfect squares. Therefore, it remains only to show that $r \geq 2016$ for n large enough. Since $r \geq \frac{2^{|U|}}{2^s}$, it suffices to show that $|U| \geq s + 11$ for n large enough.

Let T be the set of prime numbers $p > 3n$ that divide an element of $\{n^2 + 1, \dots, n^2 + 3n\}$. Every element of T evidently divides the binomial coefficient

$$\binom{n^2 + 3n}{3n} = \frac{(n^2 + 1) \cdots (n^2 + 3n)}{(3n)!},$$

so the product of the elements of T divides $\binom{n^2 + 3n}{3n}$ as well. We then have the chain of inequalities

$$\begin{aligned} (3n)^{|T|} &\leq \prod_{p \in T} p \\ &\leq \binom{n^2 + 3n}{3n} \\ &\leq \left(\frac{e(n+3)}{3} \right)^{3n} \end{aligned}$$

¹The reason for choosing the threshold $3n$ will become clear.

where the last inequality follows from the lemma. Therefore,

$$\begin{aligned} |T| &\leq \frac{\log\left(\left(\frac{\epsilon(n+3)}{3}\right)^{3n}\right)}{\log(3n)} \\ &= 3n \frac{1 + \log(n+3) - \log 3}{\log 3 + \log n}. \end{aligned}$$

It follows that

$$\begin{aligned} |U| &\geq 3n - |T| \\ &\geq 3n \left(1 - \frac{1 + \log(n+3) - \log 3}{\log 3 + \log n}\right) \\ &\sim (2 \log 3 - 1) \frac{3n}{\log(3n)}. \end{aligned}$$

On the other hand, by the prime number theorem we have

$$s \sim \frac{3n}{\log(3n)}.$$

Since $2 \log 3 - 1 > 1$, we conclude that $|U| \geq s + 11$ for n large enough, as desired. \square

Remark 0.8. This problem was proposed by Dhroova Aiyilam. Thanks to Mitchell Lee for writing the solution, and also working out Remark 0.10.

Remark 0.9. In fact, the above proof gives a lower bound for r of the form $2^{(2 \log 3 - 2 + o(1)) \frac{3n}{\log(3n)}}$ where $o(1)$ denotes a term that approaches 0 as $n \rightarrow \infty$.

Remark 0.10. In fact, for any real number $\epsilon > 0$ there is an integer N_ϵ such that the set

$$\left\{ n^2 + 1, \dots, n^2 + \left\lceil \left(\sqrt{\frac{3}{2}} e^{3/2} + \epsilon \right) \frac{n}{\log n} \right\rceil \right\}$$

contains a nonempty subset whose product is a square for any $n > N_\epsilon$. The proof of this is similar to the above solution with $3n$ replaced by

$$\left(\sqrt{\frac{3}{2}} e^{3/2} + \epsilon \right) \frac{n}{\log n}$$

and a more careful lower bound for

$$\prod_{p \in T} p.$$

Remark 0.11. Compare with the following problem (addressing higher powers): http://www.aops.com/community/c6h495623_perfect_powers_in_intervals.

5. [40] Let $\omega = e^{2\pi i/5}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

Answer: N/A

Proof. Let $\zeta = \omega = e^{2\pi i/5}$ be a primitive fifth root of unity. If $x = a + b\zeta + c\zeta^2 + d\zeta^3$ ($a, b, c, d \in \mathbb{Z}$), then from $\zeta + \zeta^{-1} = \frac{\sqrt{5}-1}{2}$ and $\zeta^2 + \zeta^{-2} = \zeta^3 + \zeta^{-3} = -\frac{\sqrt{5}+1}{2}$, we deduce $|x|^2 = (a^2 + b^2 + c^2 + d^2) + \frac{\sqrt{5}-1}{2}(ab + bc + cd) - \frac{\sqrt{5}+1}{2}(ac + bd + ad) \in \mathbb{Z}[\phi]$, where $\phi = \frac{1+\sqrt{5}}{2}$.

But $|x|^{2k} = |1 + \zeta|^2 = 2 + \phi^{-1} = \phi^2$, so $\phi^{2/k} = |x|^2 \in \mathbb{Z}[\phi]$. It follows that $|x|^2$ has norm $\phi^{2/k} \bar{\phi}^{2/k}$ in $\mathbb{Z}[\phi]$, and so must be a positive unit. Yet by the lemma below, the positive units are precisely the integer powers of ϕ , so $k > 1$ forces $k = 2$.

However, there must exist $f, g \in \mathbb{Z}[t]$ such that $f(t)^k = 1 + t + g(t)(t^4 + t^3 + t^2 + t + 1)$ (if $f(\zeta) = x$, then the minimal polynomial of ζ over \mathbb{Q} must divide $f(t)^k - 1 - t$), whence $f(1)^k \equiv 2 \pmod{5}$, so $2 \nmid k$, contradiction.

(Alternatively, $k = 2$ would force $a^2 + b^2 + c^2 + d^2 - ab - bc - cd = 0$, which can only occur if $a = b = c = d = 0$.) \square

Lemma 0.2. *The positive units of $\mathbb{Z}[\phi]$ are precisely the integer powers of ϕ . (In other words,³ ϕ is the fundamental unit of $\mathbb{Q}(\sqrt{5})$.)*

Proof. To prove this, one can either use Pell equation theory, or reason more directly as follows: a unit $m + n\phi > 1$ must be at least ϕ . (Sketch: unit is equivalent to norm ± 1 , and so its conjugate $m + n\bar{\phi} = \frac{\pm 1}{m+n\phi}$ lies in $(-1, 1)$, so $n\sqrt{5} = n(\phi - \bar{\phi}) > 0$ forces n positive, and then $-1 < m + n\bar{\phi}$ forces $m > -n\bar{\phi} - 1 = \frac{n}{\phi} - 1 > -1$. Hence $m + n\phi \geq 0 + 1\phi = \phi$, as desired.) If it's not a positive power of ϕ , then for some $\ell > 0$, $\frac{m+n\phi}{\phi^\ell}$ is a unit lying in $(1, \phi)$, contradiction. \square

Remark 0.12. This problem was proposed by Carl Lian.

Remark 0.13. The “morally correct” (purely algebraic) way to think about the above solution is via the tower of (number field) extensions

$$\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{5})/\mathbb{Q}.$$

Instead of thinking in terms of complex absolute value $|\cdot|$, it is better to reason in terms of the norm $N_{\mathbb{Q}(\zeta)/\mathbb{Q}(\sqrt{5})}(\cdot): \mathbb{Q}(\zeta) \rightarrow \mathbb{Q}(\sqrt{5})$ (which happens to be the same as looking at the square of complex absolute value, in our case), which maps algebraic integers to algebraic integers; the norm of $x^k = 1 + \zeta$ over $\mathbb{Q}(\sqrt{5})$ is ϕ^2 , and ϕ is the *fundamental unit* of $\mathbb{Q}(\sqrt{5})$. Then one proceeds as in the above solution.

²On a silly note, this inspired HMMT November 2013 General Problem 7.

³since $\mathbb{Z}[\phi]$ is the *ring of integers* of $\mathbb{Q}(\sqrt{5})$

HMMT 2014
Saturday 22 February 2014

Algebra

1. Given that x and y are nonzero real numbers such that $x + \frac{1}{y} = 10$ and $y + \frac{1}{x} = \frac{5}{12}$, find all possible values of x .

Answer: 4, 6 OR 6, 4 Let $z = \frac{1}{y}$. Then $x + z = 10$ and $\frac{1}{x} + \frac{1}{z} = \frac{5}{12}$. Since $\frac{1}{x} + \frac{1}{z} = \frac{x+z}{xz} = \frac{10}{xz}$, we have $xz = 24$. Thus, $x(10 - x) = 24$, so $x^2 - 10x + 24 = (x - 6)(x - 4) = 0$, whence $x = 6$ or $x = 4$.

Alternate solution: Clearing denominators gives $xy + 1 = 10y$ and $yx + 1 = \frac{5}{12}x$, so $x = 24y$. Thus we want to find all real (nonzero) x such that $\frac{x^2}{24} + 1 = \frac{5}{12}x$ (and for any such x , $y = x/24$ will satisfy the original system of equations). This factors as $(x - 4)(x - 6) = 0$, so precisely $x = 4, 6$ work.

2. Find the integer closest to

$$\frac{1}{\sqrt[4]{5^4 + 1} - \sqrt[4]{5^4 - 1}}.$$

Answer: 250 Let $x = (5^4 + 1)^{1/4}$ and $y = (5^4 - 1)^{1/4}$. Note that x and y are both approximately 5. We have

$$\begin{aligned} \frac{1}{x - y} &= \frac{(x + y)(x^2 + y^2)}{(x - y)(x + y)(x^2 + y^2)} = \frac{(x + y)(x^2 + y^2)}{x^4 - y^4} \\ &= \frac{(x + y)(x^2 + y^2)}{2} \approx \frac{(5 + 5)(5^2 + 5^2)}{2} = 250. \end{aligned}$$

Note: To justify the \approx , note that $1 = x^4 - 5^4$ implies

$$0 < x - 5 = \frac{1}{(x + 5)(x^2 + 5^2)} < \frac{1}{(5 + 5)(5^2 + 5^2)} = \frac{1}{500},$$

and similarly $1 = 5^4 - y^4$ implies

$$0 < 5 - y = \frac{1}{(5 + y)(5^2 + y^2)} < \frac{1}{(4 + 4)(4^2 + 4^2)} = \frac{1}{256}.$$

Similarly,

$$0 < x^2 - 5^2 = \frac{1}{x^2 + 5^2} < \frac{1}{2 \cdot 5^2} = \frac{1}{50}$$

and

$$0 < 5^2 - y^2 = \frac{1}{5^2 + y^2} < \frac{1}{5^2 + 4 \cdot 5^2} < \frac{1}{45}.$$

Now

$$|x + y - 10| = |(x - 5) - (5 - y)| < \max(|x - 5|, |5 - y|) < \frac{1}{256},$$

and similarly $|x^2 + y^2 - 2 \cdot 5^2| < \frac{1}{45}$. It's easy to check that $(10 - 1/256)(50 - 1/45) > 499.5$ and $(10 + 1/256)(50 + 1/45) < 500.5$, so we're done.

3. Let

$$A = \frac{1}{6} ((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3).$$

Compute 2^A .

Answer: 72 Let $a = \log_2(3)$, so $2^a = 3$ and $A = \frac{1}{6}[a^3 - (a + 1)^3 - (a + 2)^3 + (a + 3)^3]$. But $(x + 1)^3 - x^3 = 3x^2 + 3x + 1$, so $A = \frac{1}{6}[3(a + 2)^2 + 3(a + 2) - 3a^2 - 3a] = \frac{1}{2}[4a + 4 + 2] = 2a + 3$. Thus $2^A = (2^a)^2(2^3) = 9 \cdot 8 = 72$.

4. Let b and c be real numbers, and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

Answer: $\boxed{-\frac{3}{2} \text{ OR } -1.5 \text{ OR } -1\frac{1}{2}}$ Since $P(P(1)) = P(P(2)) = 0$, but $P(1) \neq P(2)$, it follows that $P(1) = 1 + b + c$ and $P(2) = 4 + 2b + c$ are the distinct roots of the polynomial $P(x)$. Thus, $P(x)$ factors:

$$\begin{aligned} P(x) &= x^2 + bx + c \\ &= (x - (1 + b + c))(x - (4 + 2b + c)) \\ &= x^2 - (5 + 3b + 2c)x + (1 + b + c)(4 + 2b + c). \end{aligned}$$

It follows that $-(5 + 3b + 2c) = b$, and that $c = (1 + b + c)(4 + 2b + c)$. From the first equation, we find $2b + c = -5/2$. Plugging in $c = -5/2 - 2b$ into the second equation yields

$$-5/2 - 2b = (1 + (-5/2) - b)(4 + (-5/2)).$$

Solving this equation yields $b = -\frac{1}{2}$, so $c = -5/2 - 2b = -\frac{3}{2}$.

5. Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

Answer: $\boxed{1}$ Rearrange the equation to $x^5 + (1 - x)^5 - 12 = 0$. It's easy to see this has two real roots, and that r is a root if and only if $1 - r$ is a root, so the answer must be 1.

Alternate solution: Note that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 5x(x - 1)(x^2 - x + 1) - 11 = 5u(u + 1) - 11$, where $u = x^2 - x$. Of course, $x^2 - x = u$ has real roots if and only if $u \geq -\frac{1}{4}$, and distinct real roots if and only if $u > -\frac{1}{4}$. But the roots of $5u(u + 1) - 11$ are $\frac{-5 \pm \sqrt{5^2 + 4(5)(11)}}{2 \cdot 5} = \frac{-5 \pm 7\sqrt{5}}{10}$, one of which is greater than $-\frac{1}{4}$ and the other less than $-\frac{1}{4}$. For the larger root, $x^2 - x = u$ has exactly two distinct real roots, which sum up to 1 by Vieta's.

6. Given that w and z are complex numbers such that $|w + z| = 1$ and $|w^2 + z^2| = 14$, find the smallest possible value of $|w^3 + z^3|$. Here, $|\cdot|$ denotes the absolute value of a complex number, given by $|a + bi| = \sqrt{a^2 + b^2}$ whenever a and b are real numbers.

Answer: $\boxed{\frac{41}{2} \text{ OR } 20.5 \text{ OR } 20\frac{1}{2}}$ We can rewrite $|w^3 + z^3| = |w + z||w^2 - wz + z^2| = |w^2 - wz + z^2| = |\frac{3}{2}(w^2 + z^2) - \frac{1}{2}(w + z)^2|$.

By the triangle inequality, $|\frac{3}{2}(w^2 + z^2) - \frac{1}{2}(w + z)^2| \leq |\frac{3}{2}(w^2 + z^2) - \frac{1}{2}(w + z)^2| + |\frac{1}{2}(w + z)^2|$. By rearranging and simplifying, we get $|w^3 + z^3| = |\frac{3}{2}(w^2 + z^2) - \frac{1}{2}(w + z)^2| \geq \frac{3}{2}|w^2 + z^2| - \frac{1}{2}|w + z|^2 = \frac{3}{2}(14) - \frac{1}{2} = \frac{41}{2}$.

To achieve $41/2$, it suffices to take w, z satisfying $w + z = 1$ and $w^2 + z^2 = 14$.

7. Find the largest real number c such that

$$\sum_{i=1}^{101} x_i^2 \geq cM^2$$

whenever x_1, \dots, x_{101} are real numbers such that $x_1 + \dots + x_{101} = 0$ and M is the median of x_1, \dots, x_{101} .

Answer: $\boxed{\frac{5151}{50} \text{ OR } 103.02 \text{ OR } 103\frac{1}{50}}$ Suppose without loss of generality that $x_1 \leq \dots \leq x_{101}$ and $M = x_{51} \geq 0$.

Note that $f(t) = t^2$ is a convex function over the reals, so we may "smooth" to the case $x_1 = \dots = x_{50} \leq x_{51} = \dots = x_{101}$ (the $x_{51} = \dots$ is why we needed to assume $x_{51} \geq 0$). Indeed, by Jensen's inequality, the map $x_1, x_2, \dots, x_{50} \rightarrow \frac{x_1 + \dots + x_{50}}{50}, \dots, \frac{x_1 + \dots + x_{50}}{50}$ will decrease or fix the LHS, while preserving the ordering condition and the zero-sum condition.

Similarly, we may without loss of generality replace x_{51}, \dots, x_{101} with their average (which will decrease or fix the LHS, but also either fix or increase the RHS). But this simplified problem has $x_1 = \dots =$

$x_{50} = -51r$ and $x_{51} = \dots = x_{101} = 50r$ for some $r \geq 0$, and by homogeneity, C works if and only if $C \leq \frac{50(51)^2 + 51(50)^2}{50^2} = \frac{51(101)}{50} = \frac{5151}{50}$.

Comment: One may also use the Cauchy-Schwarz inequality or the QM-AM inequality instead of Jensen's inequality.

Comment: For this particular problem, there is another solution using the identity $101 \sum x_i^2 - (\sum x_i)^2 = \sum (x_j - x_i)^2$. Indeed, we may set $u = x_{51} - (x_1 + \dots + x_{50})/50$ and $v = (x_{52} + \dots + x_{101})/50 - x_{51}$, and use the fact that $(u - v)^2 \leq u^2 + v^2$.

8. Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r .

Answer: $\boxed{\pm \frac{9}{4} \text{ (OR } \frac{9}{4}, -\frac{9}{4} \text{ OR } -\frac{9}{4}, \frac{9}{4}) \text{ OR } \pm 2\frac{1}{4} \text{ OR } \pm 2.25}$ Any real quartic has an even number of real roots *with multiplicity*, so there exists real r such that $x^4 + kx^3 + x^2 + 4kx + 16$ either takes the form $(x + r)^4$ (clearly impossible) or $(x + r)^2(x^2 + ax + b)$ for some real a, b with $a^2 < 4b$. Clearly $r \neq 0$, so $b = \frac{16}{r^2}$ and $4k = 4(k)$ yields $\frac{32}{r} + ar^2 = 4(2r + a) \implies a(r^2 - 4) = 8\frac{r^2 - 4}{r}$. Yet $a \neq \frac{8}{r}$ (or else $a^2 = 4b$), so $r^2 = 4$, and $1 = r^2 + 2ra + \frac{16}{r^2} \implies a = \frac{-7}{2r}$. Thus $k = 2r - \frac{7}{2r} = \pm \frac{9}{4}$ (since $r = \pm 2$).

It is easy to check that $k = \frac{9}{4}$ works, since $x^4 + (9/4)x^3 + x^2 + 4(9/4)x + 16 = \frac{1}{4}(x + 2)^2(4x^2 - 7x + 16)$. The polynomial given by $k = -\frac{9}{4}$ is just $\frac{1}{4}(-x + 2)^2(4x^2 + 7x + 16)$.

Alternate solution: $x^4 + kx^3 + x^2 + 4kx + 16 = (x^2 + \frac{k}{2}x + 4)^2 + (1 - 8 - \frac{k^2}{4})x^2$, so for some $\epsilon \in \{-1, 1\}$, $2x^2 + (k - \epsilon\sqrt{k^2 + 28})x + 8$ has a single real root and thus takes the form $2(x + r)^2$ (using the same notation as above). But then $(k - \epsilon\sqrt{k^2 + 28})^2 = 4(2)(8) = 8^2$, so we conclude that $(k \pm 8)^2 = (\epsilon\sqrt{k^2 + 28})^2$ and $k = \pm(4 - \frac{7}{4}) = \pm \frac{9}{4}$.

9. Given that a, b , and c are complex numbers satisfying

$$\begin{aligned} a^2 + ab + b^2 &= 1 + i \\ b^2 + bc + c^2 &= -2 \\ c^2 + ca + a^2 &= 1, \end{aligned}$$

compute $(ab + bc + ca)^2$. (Here, $i = \sqrt{-1}$.)

Answer: $\boxed{\frac{-11-4i}{3} \text{ OR } -\frac{11+4i}{3}}$ More generally, suppose $a^2 + ab + b^2 = z$, $b^2 + bc + c^2 = x$, $c^2 + ca + a^2 = y$ for some complex numbers a, b, c, x, y, z .

We show that

$$f(a, b, c, x, y, z) = \left(\frac{1}{2}(ab + bc + ca) \sin 120^\circ\right)^2 - \left(\frac{1}{4}\right)^2[(x + y + z)^2 - 2(x^2 + y^2 + z^2)]$$

holds in general. Plugging in $x = -2$, $y = 1$, $z = 1 + i$ will then yield the desired answer,

$$\begin{aligned} (ab + bc + ca)^2 &= \frac{16}{3} \frac{1}{16} [(x + y + z)^2 - 2(x^2 + y^2 + z^2)] \\ &= \frac{i^2 - 2(4 + 1 + (1 + i)^2)}{3} = \frac{-1 - 2(5 + 2i)}{3} = \frac{-11 - 4i}{3}. \end{aligned}$$

Solution 1: Plug in $x = b^2 + bc + c^2$, etc. to get a polynomial g in a, b, c (that agrees with f for every valid choice of a, b, c, x, y, z). It suffices to show that $g(a, b, c) = 0$ for all positive reals a, b, c , as then the polynomial g will be identically 0.

But this is easy: by the law of cosines, we get a geometrical configuration with a point P inside a triangle ABC with $PA = a$, $PB = b$, $PC = c$, $\angle PAB = \angle PBC = \angle PCA = 120^\circ$, $x = BC^2$,

$y = CA^2$, $z = AB^2$. By Heron's formula, we have

$$\begin{aligned} \left(\frac{1}{2}(ab + bc + ca) \sin 120^\circ\right)^2 &= [ABC]^2 \\ &= \frac{(\sqrt{x} + \sqrt{y} + \sqrt{z}) \prod_{\text{cyc}}(\sqrt{x} + \sqrt{y} - \sqrt{z})}{2^4} \\ &= \frac{1}{16}[(\sqrt{x} + \sqrt{y})^2 - z][(\sqrt{x} - \sqrt{y})^2 - z] \\ &= \frac{1}{16}[(x - y)^2 + z^2 - 2z(x + y)] \\ &= \left(\frac{1}{4}\right)^2[(x + y + z)^2 - 2(x^2 + y^2 + z^2)], \end{aligned}$$

as desired.

Solution 2: Let $s = a + b + c$. We have $x - y = b^2 + bc - ca - a^2 = (b - a)s$ and cyclic, so $x + as = y + bs = z + cs$ (they all equal $\sum a^2 + \sum bc$). Now add all equations to get

$$x + y + z = 2 \sum a^2 + \sum bc = s^2 + \frac{1}{2} \sum (b - c)^2;$$

multiplying both sides by $4s^2$ yields $4s^2(x + y + z) = 4s^4 + 2 \sum (z - y)^2$, so $[2s^2 - (x + y + z)]^2 = (x + y + z)^2 - 2 \sum (z - y)^2 = 6 \sum yz - 3 \sum x^2$. But $2s^2 - (x + y + z) = s^2 - \frac{1}{2} \sum (b - c)^2 = (a + b + c)^2 - \frac{1}{2} \sum (b - c)^2 = 3(ab + bc + ca)$, so

$$9(ab + bc + ca)^2 = 6 \sum yz - 3 \sum x^2 = 3[(x + y + z)^2 - 2(x^2 + y^2 + z^2)],$$

which easily rearranges to the desired.

Comment: Solution 2 can be done with less cleverness. Let $u = x + as = y + bs = z + cs$, so $a = \frac{u-x}{s}$, etc. Then $s = \sum \frac{u-x}{s}$, or $s^2 = 3u - s(x + y + z)$. But we get another equation in s, u by just plugging in directly to $a^2 + ab + b^2 = z$ (and after everything is in terms of s , we can finish without too much trouble).

10. For an integer n , let $f_9(n)$ denote the number of positive integers $d \leq 9$ dividing n . Suppose that m is a positive integer and b_1, b_2, \dots, b_m are real numbers such that $f_9(n) = \sum_{j=1}^m b_j f_9(n - j)$ for all $n > m$. Find the smallest possible value of m .

Answer: 28 Let $M = 9$. Consider the generating function

$$F(x) = \sum_{n \geq 1} f_M(n) x^n = \sum_{d=1}^M \sum_{k \geq 1} x^{dk} = \sum_{d=1}^M \frac{x^d}{1 - x^d}.$$

Observe that $f_M(n) = f_M(n + M!)$ for all $n \geq 1$ (in fact, all $n \leq 0$ as well). Thus $f_M(n)$ satisfies a degree m linear recurrence if and only if it *eventually* satisfies a degree m linear recurrence. But the latter occurs if and only if $P(x)F(x)$ is a polynomial for some degree m polynomial $P(x)$. (Why?)

Suppose $P(x)F(x) = Q(x)$ is a polynomial for some polynomial P of degree m . We show that $x^s - 1 \mid P(x)$ for $s = 1, 2, \dots, M$, or equivalently that $P(\omega) = 0$ for all primitive s th roots of unity $1 \leq s \leq M$. Fix a primitive s th root of unity ω , and define a function

$$F_\omega(z) = (1 - \omega^{-1}z) \sum_{s \nmid d \leq M} \frac{z^d}{1 - z^d} + \sum_{s \mid d \leq M} \frac{z^d}{1 + (\omega^{-1}z) + \dots + (\omega^{-1}z)^{d-1}}$$

for all z where all denominators are nonzero (in particular, this includes $z = \omega$).

Yet $F_\omega(z) - F(z)(1 - \omega^{-1}z) = 0$ for all complex z such that $z^1, z^2, \dots, z^M \neq 1$, so $P(z)F_\omega(z) - Q(z)(1 - \omega^{-1}z) = 0$ holds for all such z as well. In particular, the rational function $P(x)F_\omega(x) - Q(x)(1 - \omega^{-1}x)$

has infinitely many roots, so must be identically zero *once we clear denominators*. But no denominator vanishes at $x = \omega$, so we may plug in $x = \omega$ to the polynomial identity and then divide out by the original (nonzero) denominators to get $0 = P(\omega)F_\omega(\omega) - Q(\omega)(1 - \omega^{-1}\omega) = P(\omega)F_\omega(\omega)$. However,

$$F_\omega(\omega) = \sum_{s|d \leq M} \frac{\omega^d}{1 + (\omega^{-1}\omega) + \dots + (\omega^{-1}\omega)^{d-1}} = \sum_{s|d \leq M} \frac{1}{d}$$

is a positive integer multiple of $1/d$, and therefore nonzero. Thus $P(\omega) = 0$, as desired.

Conversely, if $x^s - 1 \mid P(x)$ for $s = 1, 2, \dots, M$, then $P(x)$ will clearly suffice. So we just want the degree of the least common multiple of the $x^s - 1$ for $s = 1, 2, \dots, M$, or just the number of roots of unity of order at most M , which is $\sum_{s=1}^M \phi(s) = 1 + 1 + 2 + 2 + 4 + 2 + 6 + 4 + 6 = 28$.

Comment: Only at the beginning do we treat $F(x)$ strictly as a formal power series; later once we get the rational function representation $\sum_{d=1}^6 \frac{x^d}{1-x^d}$, we can work with polynomial identities in general and don't have to worry about convergence issues for $|x| \geq 1$.

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Saturday 22 February 2014
Combinatorics

1. There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

Answer: $\boxed{19/165}$ There is a $\frac{60}{100} = \frac{3}{5}$ chance that Alex is in the class. If Alex is in the class, the probability that Zhu is in his section is $\frac{19}{99}$. So the answer is $\frac{3}{5} \cdot \frac{19}{99} = \frac{19}{165}$.

2. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

Answer: $\boxed{7560}$ There are $\binom{10}{5}$ ways to choose the 5-person committee. After choosing the committee, there are $2^5 - 2 = 30$ ways to choose the leaders. So the answer is $30 \cdot \binom{10}{5} = 7560$.

3. Bob writes a random string of 5 letters, where each letter is either A , B , C , or D . The letter in each position is independently chosen, and each of the letters A, B, C, D is chosen with equal probability. Given that there are at least two A 's in the string, find the probability that there are at least three A 's in the string.

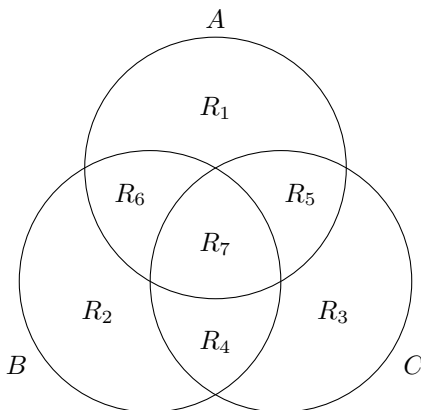
Answer: $\boxed{\frac{53}{188}}$ There are $\binom{5}{2}3^3 = 270$ strings with 2 A 's. There are $\binom{5}{3}3^2 = 90$ strings with 3 A 's. There are $\binom{5}{4}3^1 = 15$ strings with 4 A 's. There is $\binom{5}{5}3^0 = 1$ string with 5 A 's. The desired probability is $\frac{90+15+1}{270+90+15+1} = \frac{53}{188}$.

4. Find the number of triples of sets (A, B, C) such that:

- (a) $A, B, C \subseteq \{1, 2, 3, \dots, 8\}$.
(b) $|A \cap B| = |B \cap C| = |C \cap A| = 2$.
(c) $|A| = |B| = |C| = 4$.

Here, $|S|$ denotes the number of elements in the set S .

Answer: $\boxed{45360}$ We consider the sets drawn in a Venn diagram.



Note that each element that is in at least one of the subsets lies in these seven possible spaces. We split by casework, with the cases based on $N = |R_7| = |A \cap B \cap C|$.

Case 1: $N = 2$

Because we are given that $|R_4| + N = |R_5| + N = |R_6| + N = 2$, we must have $|R_4| = |R_5| = |R_6| = 0$. But we also know that $|R_1| + |R_5| + |R_6| + N = 4$, so $|R_1| = 2$. Similarly, $|R_2| = |R_3| = 2$. Since these regions are distinguishable, we multiply through and obtain $\binom{8}{2} \binom{6}{2} \binom{4}{2} \binom{2}{2} = 2520$ ways.

Case 2: $N = 1$

In this case, we can immediately deduce $|R_4| = |R_5| = |R_6| = 1$. From this, it follows that $|R_1| = 4 - 1 - 1 - 1 = 1$, and similarly, $|R_2| = |R_3| = 1$. All seven regions each contain one integer, so there are a total of $(8)(7) \dots (2) = 40320$ ways.

Case 3: $N = 0$

Because $|R_4| + N = |R_5| + N = |R_6| + N = 2$, we must have $|R_4| = |R_5| = |R_6| = 2$. Since $|R_1| + |R_5| + |R_6| + N = 4$, we immediately see that $|R_1| = 0$. Similarly, $|R_2| = |R_3| = 0$. The number of ways to fill R_4, R_5, R_6 is $\binom{8}{2} \binom{6}{2} \binom{4}{2} = 2520$.

This clearly exhausts all the possibilities, so adding gives us $40320 + 2520 + 2520 = 45360$ ways.

5. Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

- Every person must get a positive integer number of shares, and all 16 shares must be given out.
- No one person can have more shares than the other three people combined.

Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

Answer: 315 We are finding the number of integer solutions to $a + b + c + d = 16$ with $1 \leq a, b, c, d \leq 8$. We count the number of solutions to $a + b + c + d = 16$ over positive integers, and subtract the number of solutions in which at least one variable is larger than 8. If at least one variable is larger than 8, exactly one of the variables is larger than 8. We have 4 choices for this variable. The number of solutions to $a + b + c + d = 16$ over positive integers, where $a > 8$, is just the number of solutions to $a' + b + c + d = 8$ over positive integers, since we can substitute $a' = a - 8$. Thus, by the stars and bars formula (the number of positive integer solutions to $x_1 + \dots + x_m = n$ is $\binom{n-1}{m-1}$), the answer is $\binom{16-1}{4-1} - \binom{4}{1} \binom{(16-8)-1}{4-1} = 35 \cdot 13 - 4 \cdot 35 = 315$.

6. We have a calculator with two buttons that displays an integer x . Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y .)

Answer: 233 We consider the integers from this process written in binary. The first operation truncates the rightmost digit, while the second operation appends 01 to the right.

We cannot have a number with a substring 11. For simplicity, call a string *valid* if it has no consecutive 1's. Note that any number generated by this process is valid, as truncating the rightmost digit and appending 01 to the right of the digits clearly preserve validity.

Since we can effectively append a zero by applying the second operation and then the first operation, we see that we can achieve all valid strings.

Note that 2014 has eleven digits when written in binary, and any valid binary string with eleven digits is at most $10111111111 = 1535$. Therefore, our problem reduces to finding the number of eleven-digit valid strings. Let F_n denote the number of valid strings of length n . For any valid string of length n , we can create a valid string of length $n + 1$ by appending a 0, or we can create a valid string of length $n + 2$ by appending 01. This process is clearly reversible, so our recursion is given by $F_n = F_{n-1} + F_{n-2}$, with $F_1 = 2, F_2 = 3$. This yields a sequence of Fibonacci numbers starting from 2, and some computation shows that our answer is $F_{11} = 233$.

7. Six distinguishable players are participating in a tennis tournament. Each player plays one match of tennis against every other player. There are no ties in this tournament—each tennis match results in a win for one player and a loss for the other. Suppose that whenever A and B are players in the tournament such that A wins strictly more matches than B over the course of the tournament, it is also true that A wins the match against B in the tournament. In how many ways could the tournament have gone?

Answer: 2048 We first group the players by wins, so let G_1 be the set of all players with the most wins, G_2 be the set of all players with the second most wins, ..., G_n be the set of all players with the least wins. By the condition in the problem, everyone in group G_i must beat everyone in group G_j for all $i < j$. Now, consider the mini-tournament consisting of the matches among players inside a single group G_i . Each must have the same number of wins, say x_i . But the total number of games is $\binom{|G_i|}{2}$ and each game corresponds to exactly one win, so we must have $\binom{|G_i|}{2} = |G_i|x_i \implies |G_i| = 2x_i + 1$. Therefore, the number of players in each G_i is odd.

We now have $\sum |G_i| = 6$ and all $|G_i|$ are odd, so we can now do casework on the possibilities.

Case 1: G_i 's have sizes 5 and 1. In this case, there are 2 ways to permute the groups (i.e. either $|G_1| = 5, |G_2| = 1$ or $|G_1| = 1, |G_2| = 5$). There are 6 ways to distribute the players into the two groups. There are 24 possible mini-tournaments in the group of size 5; to prove this, we label the players p_1, \dots, p_5 and note that each player has 2 wins. Without loss of generality, let p_1 beat p_2 and p_3 , and also without loss of generality let p_2 beat p_3 . It's easy to verify that there are 2 possible mini-tournaments, depending on whether p_4 beats p_5 or p_5 beats p_4 . Since there are $\binom{4}{2} \cdot 2 = 12$ ways to pick the two players p_1 defeats and choose which one beats the other, there are indeed $12 \cdot 2 = 24$ tournaments. Then the total number of possible tournaments in this case is $2 \cdot 6 \cdot 24 = 288$.

Case 2: The sizes are 3, 3. In this case, there are $\binom{6}{3} = 20$ ways to distribute the players into the groups, and 2 possible mini-tournaments in either group, so the total here is $20 \cdot 2 \cdot 2 = 80$.

Case 3: The sizes are 3, 1, 1, 1. In this case, there are 4 ways to permute the groups, $\binom{6}{3} \cdot 6 = 120$ ways to distribute the players into groups, and 2 possible mini-tournaments in the group of size 3, for a total of $4 \cdot 120 \cdot 2 = 960$.

Case 4: The sizes are 1, 1, 1, 1, 1, 1. There are 720 ways to distribute the players into groups.

The final answer is $288 + 80 + 960 + 720 = 2048$.

8. The integers $1, 2, \dots, 64$ are written in the squares of a 8×8 chess board, such that for each $1 \leq i < 64$, the numbers i and $i + 1$ are in squares that share an edge. What is the largest possible sum that can appear along one of the diagonals?

Answer: 432 Our answer is $26 + 52 + 54 + 56 + 58 + 60 + 62 + 64$.

One possible configuration:

WLOG, we seek to maximize the sum of the numbers on the main diagonal (top left to bottom right). If we color the squares in a checker-board pattern and use the fact that a and $a + 1$ lie on different colored squares, we notice that all numbers appearing on the main diagonal must be of the same parity.

Consider the smallest value m on the main diagonal. All numbers from 1 to $m - 1$ must lie on one side of the diagonal since the main diagonal disconnects the board into two regions, and by assumption, all

26	25	24	23	18	17	8	7
27	52	53	22	19	16	9	6
28	51	54	21	20	15	10	5
29	50	55	56	57	14	11	4
30	49	44	43	58	13	12	3
31	48	45	42	59	60	61	2
32	47	46	41	40	39	62	1
33	34	35	36	37	38	63	64

numbers less than m cannot lie on the main diagonal. Therefore, $m \leq 29$ (one more than the seventh triangular number) But if $m = 29$, then the sum of the numbers on the main diagonal is at most $29 + 51 + 53 + 55 + 57 + 59 + 61 + 63 = 428$, as these numbers must be odd. Similarly, $m = 27$ is also not optimal.

This leaves $m = 28$ as a possibility. But if this were the case, the only way it beats our answer is if we have $28 + 52 + 54 + \dots + 64$, which would require $52, 54, \dots, 64$ to appear sequentially along the diagonal, forcing 28 to be in one of the corners.

Now label the squares (*row, column*) with $(1, 1)$ being the top left and $(8, 8)$ being the bottom right. Assume WLOG that 28 occupies $(1, 1)$. Since 62 and 64 are in $(7, 7)$ and $(8, 8)$, respectively, we must have 63 in $(7, 8)$ or $(8, 7)$, and WLOG, assume it's in $(8, 7)$. Since 61 is next to 60, it is not difficult to see that $(7, 8)$ must be occupied by 1 (all numbers a between 2 and 60 must have $a - 1$ and $a + 1$ as neighbors) . Since 1 is above the main diagonal, all numbers from 1 to 27 must also be above the main diagonal. Since there are 28 squares above the main diagonal, there is exactly one number above the main diagonal greater than 28.

Notice that 61 must occupy $(7, 6)$ or $(6, 7)$. If it occupies $(7, 6)$, then we are stuck at $(8, 6)$, since it must contain a number between 2 and 59, which is impossible. Therefore, 61 must occupy $(6, 7)$, and no more numbers greater than 28 can be above the main diagonal. This forces 59, 57, 55, and 53 to occupy $(6, 5)$, $(5, 4)$, $(4, 3)$, $(3, 2)$, respectively. But we see that 27 occupies $(1, 2)$ and 29 occupies $(2, 1)$, leaving nowhere for 51.

This is a contradiction, so our answer is therefore optimal.

Alternate solution: Another method of proving that $m \leq 26$ is to note that each side of the diagonal has 28 squares, 16 of which are one color and 12 of which are the other color. As the path has to alternate colors, one can make at most $13 + 12 = 25$ steps before moving on the diagonal.

9. There is a heads up coin on every integer of the number line. Lucky is initially standing on the zero point of the number line facing in the positive direction. Lucky performs the following procedure: he looks at the coin (or lack thereof) underneath him, and then,
- If the coin is heads up, Lucky flips it to tails up, turns around, and steps forward a distance of one unit.
 - If the coin is tails up, Lucky picks up the coin and steps forward a distance of one unit facing the same direction.
 - If there is no coin, Lucky places a coin heads up underneath him and steps forward a distance of one unit facing the same direction.

He repeats this procedure until there are 20 coins anywhere that are tails up. How many times has Lucky performed the procedure when the process stops?

Answer: 6098 We keep track of the following quantities: Let N be the sum of 2^k , where k ranges over all nonnegative integers such that position $-1 - k$ on the number line contains a tails-up coin. Let M be the sum of 2^k , where k ranges over all nonnegative integers such that position k contains a tails-up coin.

We also make the following definitions: A "right event" is the event that Lucky crosses from the negative integers on the number line to the non-negative integers. A "left event" is the event that Lucky crosses from the non-negative integers on the number line to the negative integers.

We now make the following claims:

- (a) Every time a right event or left event occurs, every point on the number line contains a coin.
- (b) Suppose that n is a positive integer. When the n th left event occurs, the value of M is equal to n . When the n th right event occurs, the value of N is equal to n .
- (c) For a nonzero integer n , denote by $\nu_2(n)$ the largest integer k such that 2^k divides n . The number of steps that elapse between the $(n - 1)$ st right event and the n th left event is equal to $2\nu_2(n) + 1$. The number of steps that elapse between the n th left event and the n th right event is also equal to $2\nu_2(n) + 1$. (If $n - 1 = 0$, then the " $(n - 1)$ st right event" refers to the beginning of the simulation.)
- (d) The man stops as soon as the 1023rd right event occurs. (Note that $1023 = 2^{10} - 1$.)

In other words, Lucky is keeping track of two numbers M and N , which are obtained by interpreting the coins on the number line as binary strings, and alternately incrementing each of them by one. We will prove claim 2; the other claims follow from very similar reasoning and their proofs will be omitted.

Clearly, left and right events alternate. That is, a left event occurs, then a right event, then a left event, and so on. So it's enough to prove that, between each right event and the following left event, the value of M is incremented by 1, and that between each left event and the following right event, the value of N is incremented by 1. We will show the first statement; the second follows from symmetry.

Suppose that a right event has just occurred. Then, by claim 1, every space on the number line contains a coin. So, there is some nonnegative integer ℓ for which positions $0, \dots, \ell - 1$ on the number line contain a tails up coin, and position ℓ contains a heads up coin. Since Lucky is standing at position 0 facing rightward, the following sequence of steps will occur:

- (a) Lucky will take ℓ steps to the right, eventually reaching position ℓ . During this process, he will pick up the coins at positions $0, \dots, \ell - 1$.
- (b) Then, Lucky turn the coin at position ℓ to a tails up coin and turn around.
- (c) Finally, Lucky will take $\ell + 1$ steps to the left, eventually reaching position -1 (at which point a left event occurs). During this process, he will place a heads up coin at positions $0, \dots, \ell - 1$.

During this sequence, the tails up coins at positions $0, \dots, \ell - 1$ have been changed to heads up coins, and the heads up coin at position ℓ has been changed to a tails up coin. So the value of M has been incremented by

$$2^\ell - \sum_{i=0}^{\ell-1} 2^i = 1$$

as desired.

Now, it remains to compute the answer to the question. By claims 3 and 4, the total number of steps taken by the simulation is

$$2 \sum_{n=1}^{1023} (2\nu_2(n) + 1).$$

This can be rewritten as

$$4 \sum_{n=1}^{1023} \nu_2(n) + 2 \cdot 1023 = 4\nu_2(1023!) + 2046.$$

We can compute $\nu_2(1023!) = 1013$ using Legendre's formula for the highest power of 2 dividing a factorial. This results in the final answer 6098.

10. An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \dots, (x_k, y_k)$ of points in \mathbb{R}^2 such that $(a, b) = (x_1, y_1)$, $(c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point.

Find the number of pairs (A, B) where A is an up-right path from $(0, 0)$ to $(4, 4)$, B is an up-right path from $(2, 0)$ to $(6, 4)$, and A and B do not intersect.

Answer: $\boxed{1750}$ The number of up-right paths from $(0, 0)$ to $(4, 4)$ is $\binom{8}{4}$ because any such up-right path is identical to a sequence of 4 U's and 4 R's, where U corresponds to a step upwards and R corresponds to a step rightwards. Therefore, the total number of pairs of (possibly intersecting) up-right paths from $(0, 0)$ to $(4, 4)$ and $(2, 0)$ to $(6, 4)$ is $\binom{8}{4}^2$.

We will now count the number of intersecting pairs of up-right paths and subtract it to get the answer. Consider an up-right path A from $(0, 0)$ to $(4, 4)$ and an up-right path B from $(2, 0)$ to $(6, 4)$. If they intersect, take the point (x, y) where they first meet each other, and switch the parts of the paths after (x, y) to make an up-right path A' from $(0, 0)$ to $(6, 4)$ and an up-right path B' from $(2, 0)$ to $(4, 4)$.

Conversely, given an up-right path A' from $(0, 0)$ to $(6, 4)$ and an up-right path B' from $(2, 0)$ to $(4, 4)$, they must intersect somewhere, so we can again take their first intersection point and switch the ends to get the original up-right path A from $(0, 0)$ to $(4, 4)$ and up-right path B from $(2, 0)$ to $(6, 4)$, where A and B intersect.

Consequently, the number of intersecting pairs of up-right paths is exactly equal to the number of pairs of up-right paths from $(0, 0)$ to $(6, 4)$ and $(2, 0)$ to $(4, 4)$, which is $\binom{10}{4}\binom{6}{4}$. The number of pairs that do not intersect is therefore $\binom{8}{4}^2 - \binom{10}{4}\binom{6}{4} = 4900 - 3150 = 1750$.

HMMT 2014
Saturday 22 February 2014
Geometry

1. Let O_1 and O_2 be concentric circles with radii 4 and 6, respectively. A chord AB is drawn in O_1 with length 2. Extend AB to intersect O_2 in points C and D . Find CD .

Answer: $\boxed{2\sqrt{21}}$ Let O be the common center of O_1 and O_2 , and let M be the midpoint of AB . Then $OM \perp AB$, so by the Pythagorean Theorem, $OM = \sqrt{4^2 - 1^2} = \sqrt{15}$. Thus $CD = 2CM = 2\sqrt{6^2 - 15} = 2\sqrt{21}$.

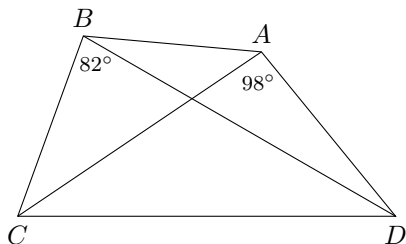
2. Point P and line ℓ are such that the distance from P to ℓ is 12. Given that T is a point on ℓ such that $PT = 13$, find the radius of the circle passing through P and tangent to ℓ at T .

Answer: $\boxed{169/24}$ Let O be the center of the given circle, Q be the foot of the altitude from P to ℓ , and M be the midpoint of PT . Then since $OM \perp PT$ and $\angle OTP = \angle TPQ$, $\triangle OMP \sim \triangle TPQ$. Thus the $OP = TP \cdot \frac{PM}{PQ} = 13 \cdot \frac{13/2}{12} = \frac{169}{24}$

3. ABC is a triangle such that $BC = 10$, $CA = 12$. Let M be the midpoint of side AC . Given that BM is parallel to the external bisector of $\angle A$, find area of triangle ABC . (Lines AB and AC form two angles, one of which is $\angle BAC$. The *external bisector* of $\angle A$ is the line that bisects the other angle.)

Answer: $\boxed{8\sqrt{14}}$ Since BM is parallel to the external bisector of $\angle A = \angle BAM$, it is perpendicular to the angle bisector of $\angle BAM$. Thus $BA = BM = \frac{1}{2}BC = 6$. By Heron's formula, the area of $\triangle ABC$ is therefore $\sqrt{(14)(8)(4)(2)} = 8\sqrt{14}$.

4. In quadrilateral $ABCD$, $\angle DAC = 98^\circ$, $\angle DBC = 82^\circ$, $\angle BCD = 70^\circ$, and $BC = AD$. Find $\angle ACD$.



Answer: $\boxed{28}$ Let B' be the reflection of B across CD . Note that $AD = BC$, and $\angle DAC + \angle CB'D = 180^\circ$, so $ACB'D$ is a cyclic trapezoid. Thus, $ACB'D$ is an isosceles trapezoid, so $\angle ACB' = 98^\circ$. Note that $\angle DCB' = \angle BCD = 70^\circ$, so $\angle ACD = \angle ACB' - \angle DCB' = 98^\circ - 70^\circ = 28^\circ$.

5. Let \mathcal{C} be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base \mathcal{C} and vertex P .

Answer: $\boxed{3 - \sqrt{5}}$ Consider the plane passing through P that is perpendicular to the plane of the circle. The intersection of the plane with the cone and sphere is a cross section consisting of a circle inscribed in a triangle with a vertex P . By symmetry, this circle is a great circle of the sphere, and hence has the same radius. The other two vertices of the triangle are the points of intersection between the plane and the unit circle, so the other two vertices are $(\frac{3}{5}, \frac{4}{5}, 0)$, $(-\frac{3}{5}, -\frac{4}{5}, 0)$.

Using the formula $A = rs$ and using the distance formula to find the side lengths, we find that $r = \frac{2A}{2s} = \frac{2 \cdot 8}{2 + 10 + 4\sqrt{5}} = 3 - \sqrt{5}$.

6. In quadrilateral $ABCD$, we have $AB = 5$, $BC = 6$, $CD = 5$, $DA = 4$, and $\angle ABC = 90^\circ$. Let AC and BD meet at E . Compute $\frac{BE}{ED}$.

Answer: $\boxed{\sqrt{3}}$ We find that $AC = \sqrt{61}$, and applying the law of cosines to triangle ACD tells us that $\angle ADC = 120$. Then $\frac{BE}{ED}$ is the ratio of the areas of triangles ABC and ADC , which is $\frac{(5)(6)}{(4)(5)\frac{\sqrt{3}}{2}} = \sqrt{3}$.

7. Triangle ABC has sides $AB = 14$, $BC = 13$, and $CA = 15$. It is inscribed in circle Γ , which has center O . Let M be the midpoint of AB , let B' be the point on Γ diametrically opposite B , and let X be the intersection of AO and MB' . Find the length of AX .

Answer: $\boxed{65/12}$ Since $B'B$ is a diameter, $\angle B'AB = 90^\circ$, so $B'A \parallel OM$, so $\frac{OM}{B'A} = \frac{BM}{BA} = \frac{1}{2}$. Thus $\frac{AX}{XO} = \frac{B'A}{OM} = 2$, so $AX = \frac{2}{3}R$, where $R = \frac{abc}{4A} = \frac{(13)(14)(15)}{4(84)} = \frac{65}{8}$ is the circumradius of ABC . Putting it all together gives $AX = \frac{65}{12}$.

8. Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC , respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN . Find the area of triangle NAY .

Answer: $\boxed{600/73}$ Let $G = AN \cap CM$ be the centroid of ABC . Then $GA = \frac{2}{3}GN = \frac{10}{3}$ and $GM = \frac{1}{3}CM = \frac{1}{3}\sqrt{8^2 + 3^2} = \frac{\sqrt{73}}{3}$. By power of a point, $(GM)(GY) = GA^2$, so $GY = \frac{GA^2}{GM} = \frac{(10/3)^2}{\frac{\sqrt{73}}{3}} = \frac{100}{3\sqrt{73}}$. Thus

$$\begin{aligned} [NAY] &= [GAM] \cdot \frac{[GAY]}{[GAM]} \cdot \frac{[NAY]}{[GAY]} \\ &= \frac{1}{6}[ABC] \cdot \frac{GY}{GM} \cdot \frac{NA}{GA} \\ &= 4 \cdot \frac{100}{73} \cdot \frac{3}{2} = \frac{600}{73} \end{aligned}$$

9. Two circles are said to be *orthogonal* if they intersect in two points, and their tangents at either point of intersection are perpendicular. Two circles ω_1 and ω_2 with radii 10 and 13, respectively, are externally tangent at point P . Another circle ω_3 with radius $2\sqrt{2}$ passes through P and is orthogonal to both ω_1 and ω_2 . A fourth circle ω_4 , orthogonal to ω_3 , is externally tangent to ω_1 and ω_2 . Compute the radius of ω_4 .

Answer: $\boxed{\frac{92}{61}}$ Let ω_i have center O_i and radius r_i . Since ω_3 is orthogonal to $\omega_1, \omega_2, \omega_4$, it has equal power r_3^2 to each of them. Thus O_3 is the radical center of $\omega_1, \omega_2, \omega_4$, which is equidistant to the three sides of $\triangle O_1O_2O_4$ and therefore its incenter.

For distinct $i, j \in \{1, 2, 4\}$, $\omega_i \cap \omega_j$ lies on the circles with diameters O_3O_i and O_3O_j , and hence ω_3 itself. It follows that ω_3 is the incircle of $\triangle O_1O_2O_4$, so $8 = r_3^2 = \frac{r_1r_2r_4}{r_1+r_2+r_4} = \frac{130r_4}{23+r_4} \implies r_4 = \frac{92}{61}$.

Comment: The condition $P \in \omega_3$ is unnecessary.

10. Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let Γ be the circumcircle of ABC , let O be its circumcenter, and let M be the midpoint of minor arc \widehat{BC} . Circle ω_1 is internally tangent to Γ at A , and circle ω_2 , centered at M , is externally tangent to ω_1 at a point T . Ray AT meets segment BC at point S , such that $BS - CS = 4/15$. Find the radius of ω_2 .

Answer: $\boxed{1235/108}$ Let N be the midpoint of BC . Notice that $BS - CS = \frac{4}{15}$ means that $NS = \frac{2}{15}$. Let lines MN and AS meet at P , and let D be the foot of the altitude from A to BC . Then $BD = 5$ and $AD = 12$, so $DN = 2$ and $DS = \frac{32}{15}$. Thus $NP = AD \frac{SN}{SD} = 12 \frac{2/15}{32/15} = \frac{3}{4}$. Now $OB = R = \frac{abc}{4A} = \frac{(13)(14)(15)}{4(84)} = \frac{65}{8}$, so $ON = \sqrt{OB^2 - BN^2} = \sqrt{\left(\frac{65}{8}\right)^2 - 7^2} = \frac{33}{8}$. Thus $OP = \frac{27}{8}$ and $PM = OM - OP = \frac{19}{4}$. By Monge's theorem, the exsimilicenter of ω_1 and Γ (which is A), the insimilicenter of ω_1 and ω_2 (which is T), and the insimilicenter of ω_2 and Γ (call this P') are collinear. But notice that this means $P' = OM \cap AT = P$. From this we get

$$\frac{\text{radius of } \omega_2}{R} = \frac{MP}{OP} = \frac{38}{27}.$$

Thus the radius of ω_2 is $\frac{65}{8} \cdot \frac{38}{27} = \frac{1235}{108}$.

HMMT 2014
Saturday 22 February 2014

Guts

1. [4] Compute the prime factorization of 159999.

Answer: $\boxed{3 \cdot 7 \cdot 19 \cdot 401}$ We have $159999 = 160000 - 1 = 20^4 - 1 = (20 - 1)(20 + 1)(20^2 + 1) = 3 \cdot 7 \cdot 19 \cdot 401$.

2. [4] Let x_1, \dots, x_{100} be defined so that for each i , x_i is a (uniformly) random integer between 1 and 6 inclusive. Find the expected number of integers in the set $\{x_1, x_1 + x_2, \dots, x_1 + x_2 + \dots + x_{100}\}$ that are multiples of 6.

Answer: $\boxed{\frac{50}{3}}$ Note that for any i , the probability that $x_1 + x_2 + \dots + x_i$ is a multiple of 6 is $\frac{1}{6}$ because exactly 1 value out of 6 possible values of x_i works. Because these 100 events are independent, the expected value is $100 \cdot \frac{1}{6} = \frac{50}{3}$.

3. [4] Let $ABCDEF$ be a regular hexagon. Let P be the circle inscribed in $\triangle BDF$. Find the ratio of the area of circle P to the area of rectangle $ABDE$.

Answer: $\boxed{\frac{\pi\sqrt{3}}{12}}$ Let the side length of the hexagon be s . The length of BD is $s\sqrt{3}$, so the area of rectangle $ABDE$ is $s^2\sqrt{3}$. Equilateral triangle BDF has side length $s\sqrt{3}$. The inradius of an equilateral triangle is $\frac{\sqrt{3}}{6}$ times the length of its side, and so has length $\frac{s}{2}$. Thus, the area of circle P is $\frac{\pi s^2}{4}$, so the ratio is $\frac{\pi s^2/4}{s^2\sqrt{3}} = \frac{\pi\sqrt{3}}{12}$.

4. [4] Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z ?

Answer: $\boxed{\frac{217}{900}}$ As $100 = 2^2 \cdot 5^2$, there are $3 \cdot 3 = 9$ divisors of 100, so there are 900 possible pairs of d and z that can be chosen.

If d is chosen, then there are $\frac{100}{d}$ possible values of z such that d divides z , so the total number of valid pairs of d and z is $\sum_{d|100} \frac{100}{d} = \sum_{d|100} d = (1 + 2 + 2^2)(1 + 5 + 5^2) = 7 \cdot 31 = 217$. The answer is therefore $\frac{217}{900}$.

5. [5] If four fair six-sided dice are rolled, what is the probability that the lowest number appearing on any die is exactly 3?

Answer: $\boxed{175/1296}$ The probability that all the die rolls are at least 3 is $\frac{4^4}{6^4}$. The probability they are all at least 4 is $\frac{3^4}{6^4}$. The probability of being in the former category but not the latter is thus $\frac{4^4}{6^4} - \frac{3^4}{6^4} = \frac{256-81}{1296} = \frac{175}{1296}$.

6. [5] Find all integers n for which $\frac{n^3 + 8}{n^2 - 4}$ is an integer.

Answer: $\boxed{0, 1, 3, 4, 6}$ We have $\frac{n^3+8}{n^2-4} = \frac{(n+2)(n^2-2n+4)}{(n+2)(n-2)} = \frac{n^2-2n+4}{n-2}$ for all $n \neq -2$. Then $\frac{n^2-2n+4}{n-2} = n + \frac{4}{n-2}$, which is an integer if and only if $\frac{4}{n-2}$ is an integer. This happens when $n-2 = -4, -2, -1, 1, 2, 4$, corresponding to $n = -2, 0, 1, 3, 4, 6$, but we have $n \neq -2$ so the answers are 0, 1, 3, 4, 6.

7. [5] The Evil League of Evil is plotting to poison the city's water supply. They plan to set out from their headquarters at (5, 1) and put poison in two pipes, one along the line $y = x$ and one along the line $x = 7$. However, they need to get the job done quickly before Captain Hammer catches them. What's the shortest distance they can travel to visit both pipes and then return to their headquarters?

Answer: $\boxed{4\sqrt{5}}$ After they go to $y = x$, we reflect the remainder of their path in $y = x$, along with the second pipe and their headquarters. Now, they must go from (5, 1) to $y = 7$ crossing $y = x$,

and then go to $(1, 5)$. When they reach $y = 7$, we reflect the remainder of their path again, so now their reflected headquarters is at $(1, 9)$. Thus, they just go from $(5, 1)$ to $(1, 9)$ in some path that inevitably crosses $y = x$ and $y = 7$. The shortest path they can take is a straight line with length $\sqrt{4^2 + 8^2} = 4\sqrt{5}$.

Comment. These ideas can be used to prove that the orthic triangle of an acute triangle has the smallest possible perimeter of all inscribed triangles.

Also, see if you can find an alternative solution using Minkowski's inequality!

8. [5] The numbers $2^0, 2^1, \dots, 2^{15}, 2^{16} = 65536$ are written on a blackboard. You repeatedly take two numbers on the blackboard, subtract one from the other, erase them both, and write the result of the subtraction on the blackboard. What is the largest possible number that can remain on the blackboard when there is only one number left?

Answer: $\boxed{131069}$ If we reverse the order of the numbers in the final subtraction we perform, then the final number will be negated. Thus, the possible final numbers come in pairs with opposite signs. Therefore, the largest possible number is the negative of the smallest possible number. To get the smallest possible number, clearly we can take the smallest number originally on the board and subtract all of the other numbers from it (you can make this rigorous pretty easily if needed), so the smallest possible number is $1 - \sum_{k=1}^{16} 2^k = 1 - 131070 = -131069$, and thus the largest possible number is 131069.

9. [6] Compute the side length of the largest cube contained in the region

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 25 \text{ and } x \geq 0\}$$

of three-dimensional space.

Answer: $\boxed{\frac{5\sqrt{6}}{3}}$ The given region is a hemisphere, so the largest cube that can fit inside it has one face centered at the origin and the four vertices of the opposite face on the spherical surface. Let the side length of this cube be s . Then, the radius of the circle is the hypotenuse of a triangle with side lengths s and $\frac{\sqrt{2}}{2}s$. So, by the Pythagorean Theorem, the radius equals $\frac{\sqrt{6}}{2}s$. Since the radius of the hemisphere is 5, the side length of the cube is $\frac{5\sqrt{6}}{3}$.

10. [6] Find the number of nonempty sets \mathcal{F} of subsets of the set $\{1, \dots, 2014\}$ such that:

- (a) For any subsets $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$.
 (b) If $S \in \mathcal{F}$, $T \subseteq \{1, \dots, 2014\}$, and $S \subseteq T$, then $T \in \mathcal{F}$.

Answer: $\boxed{2^{2014}}$ For a subset S of $\{1, \dots, 2014\}$, let \mathcal{F}_S be the set of all sets T such that $S \subseteq T \subseteq \{1, \dots, 2014\}$. It can be checked that the sets \mathcal{F}_S satisfy the conditions 1 and 2. We claim that the \mathcal{F}_S are the only sets of subsets of $\{1, \dots, 2014\}$ satisfying the conditions 1 and 2. (Thus, the answer is the number of subsets S of $\{1, \dots, 2014\}$, which is 2^{2014} .)

Suppose that \mathcal{F} satisfies the conditions 1 and 2, and let S be the intersection of all the sets of \mathcal{F} . We claim that $\mathcal{F} = \mathcal{F}_S$. First, by definition of S , all elements $T \in \mathcal{F}$ are supersets of S , so $\mathcal{F} \subseteq \mathcal{F}_S$. On the other hand, by iterating condition 1, it follows that S is an element of \mathcal{F} , so by condition 2 any set T with $S \subseteq T \subseteq \{1, \dots, 2014\}$ is an element of \mathcal{F} . So $\mathcal{F} \supseteq \mathcal{F}_S$. Thus $\mathcal{F} = \mathcal{F}_S$.

11. [6] Two fair octahedral dice, each with the numbers 1 through 8 on their faces, are rolled. Let N be the remainder when the product of the numbers showing on the two dice is divided by 8. Find the expected value of N .

Answer: $\boxed{\frac{11}{4}}$ If the first die is odd, which has $\frac{1}{2}$ probability, then N can be any of 0, 1, 2, 3, 4, 5, 6, 7 with equal probability, because multiplying each element of $\{0, \dots, 7\}$ with an odd number and taking modulo 8 results in the same numbers, as all odd numbers are relatively prime to 8. The expected value in this case is 3.5.

If the first die is even but not a multiple of 4, which has $\frac{1}{4}$ probability, then using similar reasoning, N can be any of 0, 2, 4, 6 with equal probability, so the expected value is 3.

If the first die is 4, which has $\frac{1}{8}$ probability, then N can be any of 0, 4 with equal probability, so the expected value is 2.

Finally, if the first die is 8, which has $\frac{1}{8}$ probability, then $N = 0$. The total expected value is $\frac{1}{2}(3.5) + \frac{1}{4}(3) + \frac{1}{8}(2) + \frac{1}{8}(0) = \frac{11}{4}$.

12. [6] Find a nonzero monic polynomial $P(x)$ with integer coefficients and minimal degree such that $P(1 - \sqrt[3]{2} + \sqrt[3]{4}) = 0$. (A polynomial is called *monic* if its leading coefficient is 1.)

Answer: $x^3 - 3x^2 + 9x - 9$ Note that $(1 - \sqrt[3]{2} + \sqrt[3]{4})(1 + \sqrt[3]{2}) = 3$, so $1 - \sqrt[3]{2} + \sqrt[3]{4} = \frac{3}{1 + \sqrt[3]{2}}$.

Now, if $f(x) = x^3 - 2$, we have $f(\sqrt[3]{2}) = 0$, so if we let $g(x) = f(x - 1) = (x - 1)^3 - 2 = x^3 - 3x^2 + 3x - 3$, then $g(1 + \sqrt[3]{2}) = f(\sqrt[3]{2}) = 0$. Finally, we let $h(x) = g(\frac{x}{3}) = \frac{27}{x^3} - \frac{27}{x^2} + \frac{9}{x} - 3$ so $h(\frac{3}{1 + \sqrt[3]{2}}) = g(1 + \sqrt[3]{2}) = 0$.

To make this a monic polynomial, we multiply $h(x)$ by $-\frac{x^3}{3}$ to get $x^3 - 3x^2 + 9x - 9$.

13. [8] An auditorium has two rows of seats, with 50 seats in each row. 100 indistinguishable people sit in the seats one at a time, subject to the condition that each person, except for the first person to sit in each row, must sit to the left or right of an occupied seat, and no two people can sit in the same seat. In how many ways can this process occur?

Answer: $\binom{100}{50} 2^{98}$ First, note that there are 2^{49} ways a single row can be filled, because each of the 49 people after the first in a row must sit to the left or to the right of the current group of people in the row, so there are 2 possibilities for each of these 49 people.

Now, there are $\binom{100}{50}$ ways to choose the order in which people are added to the rows, and 2^{49} ways to fill up each row separately, for a total of $\binom{100}{50} 2^{98}$ ways to fill up the auditorium.

14. [8] Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle D = 90^\circ$. Suppose that there is a point E on CD such that $AE = BE$ and that triangles AED and CEB are similar, but not congruent. Given that $\frac{CD}{AB} = 2014$, find $\frac{BC}{AD}$.

Answer: $\sqrt{4027}$ Let M be the midpoint of AB . Let $AM = MB = ED = a$, $ME = AD = b$, and $AE = BE = c$. Since $\triangle BEC \sim \triangle DEA$, but $\triangle BEC$ is not congruent to $\triangle DAE$, we must have $\triangle BEC \sim \triangle DEA$. Thus, $BC/BE = AD/DE = b/a$, so $BC = bc/a$, and $CE/EB = AE/ED = c/a$, so $EC = c^2/a$. We are given that $CD/AB = \frac{\frac{c^2}{a} + a}{2a} = \frac{c^2}{2a^2} + \frac{1}{2} = 2014 \Rightarrow \frac{c^2}{a^2} = 4027$. Thus, $BC/AD = \frac{bc/a}{b} = c/a = \sqrt{4027}$.

15. [8] Given a regular pentagon of area 1, a *pivot line* is a line not passing through any of the pentagon's vertices such that there are 3 vertices of the pentagon on one side of the line and 2 on the other. A *pivot point* is a point inside the pentagon with only finitely many non-pivot lines passing through it. Find the area of the region of pivot points.

Answer: $\frac{1}{2}(7 - 3\sqrt{5})$ Let the pentagon be labeled $ABCDE$. First, no pivot point can be on the same side of AC as vertex B . Any such point P has the infinite set of non-pivot lines within the hourglass shape formed by the acute angles between lines PA and PC . Similar logic can be applied to points on the same side of BD as C , and so on. The set of pivot points is thus a small pentagon with sides on AC, BD, CE, DA, EB . The side ratio of this small pentagon to the large pentagon is

$$(2 \cos(72^\circ))^2 = \frac{3 - \sqrt{5}}{2},$$

so the area of the small pentagon is

$$\left(\frac{3 - \sqrt{5}}{2}\right)^2 = \frac{1}{2}(7 - 3\sqrt{5}).$$

16. [8] Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

Answer: $\boxed{\frac{72+32\sqrt{2}}{7}}$ Let $u = x^2 + 2y^2$. By AM-GM, $u \geq \sqrt{8}xy$, so $xy \leq \frac{u}{\sqrt{8}}$. If we let $xy = ku$ where $k \leq \frac{1}{\sqrt{8}}$, then we have

$$u(1 - k) = 8$$

$$u(1 + k) = x^2 + xy + 2y^2$$

that is, $u(1 + k) = 8 \cdot \frac{1 + k}{1 - k}$. It is not hard to see that the maximum value of this expression occurs at

$$k = \frac{1}{\sqrt{8}}, \text{ so the maximum value is } 8 \cdot \frac{1 + \frac{1}{\sqrt{8}}}{1 - \frac{1}{\sqrt{8}}} = \frac{72 + 32\sqrt{2}}{7}.$$

17. [11] Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function satisfying the following conditions:

- (a) $f(1) = 1$.
- (b) $f(a) \leq f(b)$ whenever a and b are positive integers with $a \leq b$.
- (c) $f(2a) = f(a) + 1$ for all positive integers a .

How many possible values can the 2014-tuple $(f(1), f(2), \dots, f(2014))$ take?

Answer: $\boxed{1007}$ Note that $f(2014) = f(1007) + 1$, so there must be exactly one index $1008 \leq i \leq 2014$ such that $f(i) = f(i - 1) + 1$, and for all $1008 \leq j \leq 2014, j \neq i$ we must have $f(j) = f(j - 1)$. We first claim that each value of i corresponds to exactly one 2014-tuple $(f(1), \dots, f(2014))$. To prove this, note that $f(1024) = 11$, so each i uniquely determines the values of $f(1007), \dots, f(2014)$. Then all of $f(1), \dots, f(1006)$ can be uniquely determined from these values because for any $1 \leq k \leq 1006$, there exists a unique n such that $1007 \leq k \cdot 2^n \leq 2014$. It's also clear that these values satisfy the condition that f is nondecreasing, so we have a correspondence from each $1008 \leq i \leq 2014$ to a unique 2014-tuple.

Also, given any valid 2014-tuple $(f(1), \dots, f(2014))$, we know that $f(1), \dots, f(1006)$ can be uniquely determined by $f(1007), \dots, f(2014)$, which yields some $1008 \leq i \leq 2014$ where $f(i) = f(i - 1) + 1$, so we actually have a bijection between possible values of i and 2014-tuples. Therefore, the total number of possible 2014-tuples is 1007.

18. [11] Find the number of ordered quadruples of positive integers (a, b, c, d) such that a, b, c , and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

Answer: $\boxed{1940}$ Since $abcd > 900 \iff \frac{30}{a} \frac{30}{b} \frac{30}{c} \frac{30}{d} < 900$, and there are $\binom{4}{2}^3$ solutions to $abcd = 2^2 3^2 5^2$, the answer is $\frac{1}{2}(8^4 - \binom{4}{2}^3) = 1940$ by symmetry.

19. [11] Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E , the bisectors of $\angle ABC$ and $\angle BCD$ meet at F , the bisectors of $\angle BCD$ and $\angle CDA$ meet at G , and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H . Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG .

Answer: $\boxed{\frac{256}{7}}$ Let M, N be the midpoints of AD, BC respectively. Since AE and DE are bisectors of supplementary angles, the triangle AED is right with right angle E . Then EM is the median of a right triangle from the right angle, so triangles EMA and EMD are isosceles with vertex M . But then $\angle MEA = \angle EAM = \angle EAB$, so $EM \parallel AB$. Similarly, $FN \parallel BA$. Thus, both E and F are on the midline of this trapezoid. Let the length of EF be x . Triangle EFH has area 1 and is similar to triangle ABH , which has area 25, so $AB = 5x$. Then, letting the heights of trapezoids $EABF$ and $EDCF$ be h (they are equal since EF is on the midline), the area of trapezoid $EABF$ is $\frac{6xh}{2} = 24$. So the area of trapezoid $EDCF$ is $36 = \frac{9xh}{2}$. Thus $DC = 8x$. Then, triangle GEF is similar to and has $\frac{1}{64}$ times the area of triangle CDG . So the area of triangle CDG is $\frac{64}{63}$ times the area of quadrilateral $EDCF$, or $\frac{256}{7}$.

20. [11] A deck of 8056 cards has 2014 ranks numbered 1–2014. Each rank has four suits—hearts, diamonds, clubs, and spades. Each card has a rank and a suit, and no two cards have the same rank and the same suit. How many subsets of the set of cards in this deck have cards from an odd number of distinct ranks?

Answer: $\boxed{\frac{1}{2}(16^{2014} - 14^{2014})}$ There are $\binom{2014}{k}$ ways to pick k ranks, and 15 ways to pick the suits in each rank (because there are 16 subsets of suits, and we must exclude the empty one). We therefore want to evaluate the sum $\binom{2014}{1}15^1 + \binom{2014}{3}15^3 + \cdots + \binom{2014}{2013}15^{2013}$.

Note that $(1 + 15)^{2014} = 1 + \binom{2014}{1}15^1 + \binom{2014}{2}15^2 + \cdots + \binom{2014}{2013}15^{2013} + 15^{2014}$ and $(1 - 15)^{2014} = 1 - \binom{2014}{1}15^1 + \binom{2014}{2}15^2 - \cdots - \binom{2014}{2013}15^{2013} + 15^{2014}$, so our sum is simply $\frac{(1+15)^{2014} - (1-15)^{2014}}{2} = \frac{1}{2}(16^{2014} - 14^{2014})$.

21. [14] Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \leq a_1, a_2, a_3, a_4, a_5 \leq 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

Answer: $\boxed{6528}$ Let $f(n)$ denote the number of n -tuples (a_1, \dots, a_n) such that $0 \leq a_1, \dots, a_n \leq 7$ and $5 \mid 2^{a_1} + \dots + 2^{a_n}$. To compute $f(n+1)$ from $f(n)$, we note that given any n -tuple (a_1, \dots, a_n) such that $0 \leq a_1, \dots, a_n \leq 7$ and $5 \nmid 2^{a_1} + \dots + 2^{a_n}$, there are exactly two possible values for a_{n+1} such that $0 \leq a_{n+1} \leq 7$ and $5 \mid 2^{a_1} + \dots + 2^{a_{n+1}}$, because $2^n \equiv 1, 2, 4, 3, 1, 2, 4, 3 \pmod{5}$ for $n = 0, 1, 2, 3, 4, 5, 6, 7$ respectively.

Also, given any valid $(n+1)$ -tuple (a_1, \dots, a_{n+1}) , we can remove a_{n+1} to get an n -tuple (a_1, \dots, a_n) such that $0 \leq a_1, \dots, a_n \leq 7$ and $5 \nmid 2^{a_1} + \dots + 2^{a_n}$, so these are in bijection. There are a total of 8^n n -tuples, $f(n)$ of which satisfy $5 \mid 2^{a_1} + \dots + 2^{a_n}$, so there are $8^n - f(n)$ for which $5 \nmid 2^{a_1} + \dots + 2^{a_n}$. Therefore, $f(n+1) = 2(8^n - f(n))$.

We now have $f(1) = 0$, $f(2) = 2(8 - 0) = 16$, $f(3) = 2(64 - 16) = 96$, $f(4) = 2(512 - 96) = 832$, $f(5) = 2(4096 - 832) = 6528$.

22. [14] Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω . Suppose that BD and AC intersect at a point E . The tangent to ω at B meets line AC at a point F , so that C lies between E and F . Given that $AE = 6$, $EC = 4$, $BE = 2$, and $BF = 12$, find DA .

Answer: $\boxed{2\sqrt{42}}$ By power of a point, we have $ED \cdot EB = EA \cdot EC$, whence $ED = 12$. Additionally, by power of a point, we have $144 = FB^2 = FC \cdot FA = FC(FC + 10)$, so $FC = 8$. Note that $\angle FBC = \angle FAB$ and $\angle CFB = \angle AFB$, so $\triangle FBC \sim \triangle FAB$. Thus, $AB/BC = FA/FB = 18/12 = 3/2$, so $AB = 3k$ and $BC = 2k$ for some k . Since $\triangle BEC \sim \triangle AED$, we have $AD/BC = AE/BE = 3$, so $AD = 3BC = 6k$. By Stewart's theorem on $\triangle EBF$, we have

$$(4)(8)(12) + (2k)^2(12) = (2)^2(8) + (12)^2(4) \implies 8 + k^2 = 8/12 + 12,$$

whence $k^2 = 14/3$. Thus,

$$DA = 6k = 6\sqrt{14/3} = 6 \frac{\sqrt{42}}{3} = 2\sqrt{42}.$$

23. [14] Let $S = \{-100, -99, -98, \dots, 99, 100\}$. Choose a 50-element subset T of S at random. Find the expected number of elements of the set $\{|x| : x \in T\}$.

Answer: $\boxed{\frac{8825}{201}}$ Let us solve a more generalized version of the problem: Let S be a set with $2n + 1$ elements, and partition S into sets A_0, A_1, \dots, A_n such that $|A_0| = 1$ and $|A_1| = |A_2| = \cdots = |A_n| = 2$. (In this problem, we have $A_0 = \{0\}$ and $A_k = \{k, -k\}$ for $k = 1, 2, \dots, 100$.) Let T be a randomly chosen m -element subset of S . What is the expected number of A_k 's that have a representative in T ? For $k = 0, 1, \dots, n$, let $w_k = 1$ if $T \cap A_k \neq \emptyset$ and 0 otherwise, so that the number of A_k 's that have a representative in T is equal to $\sum_{k=0}^n w_k$. It follows that the expected number of A_k 's that have a representative in T is equal to

$$E[w_0 + w_1 + \cdots + w_n] = E[w_0] + E[w_1] + \cdots + E[w_n] = E[w_0] + nE[w_1],$$

since $E[w_1] = E[w_2] = \dots = E[w_n]$ by symmetry.

Now $E[w_0]$ is equal to the probability that $T \cap A_0 \neq \emptyset$, that is, the probability that the single element of A_0 is in T , which is $|T|/|S| = m/(2n+1)$. Similarly, $E[w_1]$ is the probability that $T \cap A_1 \neq \emptyset$, that is, the probability that at least one of the two elements of A_1 is in T . Since there are $\binom{2n-1}{m}$ m -element subsets of S that exclude both elements of A_1 , and there are $\binom{2n+1}{m}$ m -element subsets of S in total, we have that

$$E[w_1] = 1 - \frac{\binom{2n-1}{m}}{\binom{2n+1}{m}} = 1 - \frac{(2n-m)(2n-m+1)}{2n(2n+1)}.$$

Putting this together, we find that the expected number of A_k 's that have a representative in T is

$$\frac{m}{2n+1} + n - \frac{(2n-m+1)(2n-m)}{2(2n+1)}.$$

In this particular problem, we have $n = 100$ and $m = 50$, so substituting these values gives our answer

of $\boxed{\frac{8825}{201}}$.

24. [14] Let $A = \{a_1, a_2, \dots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A .

Answer: $\boxed{1267}$ For $2 \leq i \leq 6$, we claim that $a_1 \equiv \dots \equiv a_7 \pmod{i}$. This is because if we consider any $i-1$ of the 7 numbers, the other $7-(i-1) = 8-i$ of them must all be equal modulo i , because we want the sum of all subsets of size i to be a multiple of i . However, $8-i \geq 2$, and this argument applies to any $8-i$ of the 7 integers, so in fact all of them must be equal modulo i .

We now have that all of the integers are equivalent modulo all of $2, \dots, 6$, so they are equivalent modulo 60, their least common multiple. Therefore, if the smallest integer is k , then the other 6 integers must be at least $k+60, k+60 \cdot 2, \dots, k+60 \cdot 6$. This means the sum is $7k + 60 \cdot 21 \geq 7 + 60 \cdot 21 = 1267$. 1267 is achievable with $\{1, 1+60, \dots, 1+60 \cdot 6\}$, so it is the answer.

25. [17] Let ABC be an equilateral triangle of side length 6 inscribed in a circle ω . Let A_1, A_2 be the points (distinct from A) where the lines through A passing through the two trisection points of BC meet ω . Define B_1, B_2, C_1, C_2 similarly. Given that $A_1, A_2, B_1, B_2, C_1, C_2$ appear on ω in that order, find the area of hexagon $A_1A_2B_1B_2C_1C_2$.

Answer: $\boxed{\frac{846\sqrt{3}}{49}}$ Let A' be the point on BC such that $2BA' = A'C$. By law of cosines on triangle $AA'B$, we find that $AA' = 2\sqrt{7}$. By power of a point, $A'A_1 = \frac{2 \cdot 4}{2\sqrt{7}} = \frac{4}{\sqrt{7}}$. Using side length ratios, $A_1A_2 = 2 \frac{AA_1}{AA'} = 2 \frac{2\sqrt{7} + \frac{4}{\sqrt{7}}}{2\sqrt{7}} = \frac{18}{7}$.

Now our hexagon can be broken down into equilateral triangle $A_1B_1C_1$ and three copies of triangle $A_1C_1C_2$. Since our hexagon has rotational symmetry, $\angle C_2 = 120$, and using law of cosines on this triangle with side lengths $\frac{18}{7}$ and 6, a little algebra yields $A_1C_2 = \frac{30}{7}$ (this is a 3-5-7 triangle with an angle 120).

The area of the hexagon is therefore $\frac{6^2\sqrt{3}}{4} + 3 * \frac{1}{2} \frac{18}{7} \frac{30}{7} \frac{\sqrt{3}}{2} = \frac{846\sqrt{3}}{49}$

26. [17] For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})$$

where the product is over all $i \in \{1, \dots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_{2014}}.$$

Answer: $\boxed{\frac{1}{2014!^{2014}}}$ We perform Lagrange interpolation on the polynomial $P(x) = 1$ through the points $1^{2014}, 2^{2014}, \dots, 2014^{2014}$. We have

$$1 = P(x) = \sum_{j=1}^{2014} \frac{\prod_{i=1, i \neq j}^{2014} (x - i^{2014})}{\prod_{i=1, i \neq j}^{2014} (j^{2014} - i^{2014})}.$$

Thus,

$$1 = P(0) = \sum_{j=1}^{2014} \frac{((-1)^{2013}) \frac{2014!^{2014}}{j^{2014}}}{(-1)^{2013} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})},$$

which equals

$$2014!^{2014} \sum_{j=1}^{2014} \frac{1}{j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})} = 2014!^{2014} \left(\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_{2014}} \right),$$

so the desired sum is $\frac{1}{2014!^{2014}}$.

27. [17] Suppose that (a_1, \dots, a_{20}) and (b_1, \dots, b_{20}) are two sequences of integers such that the sequence $(a_1, \dots, a_{20}, b_1, \dots, b_{20})$ contains each of the numbers $1, \dots, 40$ exactly once. What is the maximum possible value of the sum

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j)?$$

Answer: $\boxed{5530}$ Let x_k , for $1 \leq k \leq 40$, be the number of integers i with $1 \leq i \leq 20$ such that $a_i \geq k$. Let y_k , for $1 \leq k \leq 40$, be the number of integers j with $1 \leq j \leq 20$ such that $b_j \geq k$. It follows from the problem statement that $x_k + y_k$ is the number of elements of the set $\{1, \dots, 40\}$ which are greater than or equal to k , which is just $41 - k$.

Note that if $1 \leq i, j \leq 20$, and $1 \leq k \leq 40$, then $\min(a_i, b_j) \geq k$ if and only if $a_i \geq k$ and $b_j \geq k$. So for a fixed k with $1 \leq k \leq 40$, the number of pairs (i, j) with $1 \leq i, j \leq 20$ such that $\min(a_i, b_j) \geq k$ is equal to $x_k y_k$. So we can rewrite

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j) = \sum_{k=1}^{40} x_k y_k.$$

Since $x_k + y_k = 41 - k$ for $1 \leq k \leq 40$, we have

$$x_k y_k \leq \left\lfloor \frac{41 - k}{2} \right\rfloor \left\lceil \frac{41 - k}{2} \right\rceil$$

by a convexity argument. So

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j) \leq \sum_{k=1}^{40} \left\lfloor \frac{41 - k}{2} \right\rfloor \left\lceil \frac{41 - k}{2} \right\rceil = 5530.$$

Equality holds when $(a_1, \dots, a_{20}) = (2, 4, \dots, 38, 40)$ and $(b_1, \dots, b_{20}) = (1, 3, \dots, 37, 39)$.

28. [17] Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \dots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

Answer: $\boxed{\frac{3^{2014}}{2^{2014} \cdot 2014!}}$ Define the polynomial functions h_1 and h_2 by $h_1(x) = f(2x) + g(2x)$ and $h_2(x) = f(2x-1) - g(2x-1)$. Then, the problem conditions tell us that $h_1(x) = 2^{2x}$ and $h_2(x) = 2^{2x-1}$ for $x = 1, 2, \dots, 2015$.

By the Lagrange interpolation formula, the polynomial h_1 is given by

$$h_1(x) = \sum_{i=1}^{2015} 2^{2i} \prod_{\substack{j=1 \\ i \neq j}}^{2015} \frac{x-j}{i-j}.$$

So the coefficient of x^{2014} in $h_1(x)$ is

$$\sum_{i=1}^{2015} 2^{2i} \prod_{\substack{j=1 \\ i \neq j}}^{2015} \frac{1}{i-j} = \frac{1}{2014!} \sum_{i=1}^{2015} 2^{2i} (-1)^{2015-i} \binom{2014}{i-1} = \frac{4 \cdot 3^{2014}}{2014!}$$

where the last equality follows from the binomial theorem. By a similar argument, the coefficient of x^{2014} in $h_2(x)$ is $\frac{2 \cdot 3^{2014}}{2014!}$.

We can write $g(x) = \frac{1}{2}(h_1(x/2) - h_2((x+1)/2))$. So, the coefficient of x^{2014} in $g(x)$ is

$$\frac{1}{2} \left(\frac{4 \cdot 3^{2014}}{2^{2014} \cdot 2014!} - \frac{2 \cdot 3^{2014}}{2^{2014} \cdot 2014!} \right) = \frac{3^{2014}}{2^{2014} \cdot 2014!}.$$

29. [20] Natalie has a copy of the unit interval $[0, 1]$ that is colored white. She also has a black marker, and she colors the interval in the following manner: at each step, she selects a value $x \in [0, 1]$ uniformly at random, and

- If $x \leq \frac{1}{2}$ she colors the interval $[x, x + \frac{1}{2}]$ with her marker.
- If $x > \frac{1}{2}$ she colors the intervals $[x, 1]$ and $[0, x - \frac{1}{2}]$ with her marker.

What is the expected value of the number of steps Natalie will need to color the entire interval black?

Answer: 5 The first choice always wipes out half the interval. So we calculate the expected value of the amount of time needed to wipe out the other half.

Solution 1 (non-calculus):

We assume the interval has $2n$ points and we start with the last n colored black. We let $f(k)$ be the expected value of the number of turns we need if there are k white points left. So we must calculate $f(n)$.

We observe that

$$f(k) = 1 + \frac{(n-k+1) \cdot 0 + (n-k+1) \cdot f(k) + 2 \sum_{i=1}^{k-1} f(i)}{2n}$$

$$f(k) \frac{n+k-1}{2n} = 1 + \frac{\sum_{i=1}^{k-1} f(i)}{n}$$

$$f(k+1) \frac{n+k}{2n} = 1 + \frac{\sum_{i=1}^k f(i)}{n}$$

$$f(k+1) = f(k) \frac{n+k+1}{n+k}$$

$$f(k) = f(1) \frac{n+k}{n+1}$$

And note that $f(1) = 2$ so $f(n) = \frac{4n}{n+1}$ and $\lim_{n \rightarrow \infty} f(n) = 4$.

Therefore adding the first turn, the expected value is 5.

Solution 2 (calculus):

We let $f(x)$ be the expected value with length x uncolored. Like above, $\lim_{x \rightarrow 0} f(x) = 2$.

Similarly we have the recursion

$$f(x) = 1 + \left(\frac{1}{2} - x\right)f(x) + 2 \int_0^x f(y)dy$$

$$f'(x) = 0 + \frac{1}{2}f'(x) - f(x) - xf'(x) + 2f(x)$$

$$\frac{f'(x)}{f(x)} = \frac{1}{x + \frac{1}{2}}$$

And solving yields $f(x) = c(\frac{1}{2} + x)$ and since $\lim_{x \rightarrow 0} f(x) = 2$, $c = 4$. So $f(x) = 2 + 4x$ and $f(\frac{1}{2}) = 4$. Therefore adding the first turn, our expected value is 5.

30. [20] Let ABC be a triangle with circumcenter O , incenter I , $\angle B = 45^\circ$, and $OI \parallel BC$. Find $\cos \angle C$.

Answer: $\boxed{1 - \frac{\sqrt{2}}{2}}$ Let M be the midpoint of BC , and D the foot of the perpendicular of I with BC . Because $OI \parallel BC$, we have $OM = ID$. Since $\angle BOC = 2\angle A$, the length of OM is $OA \cos \angle BOM = OA \cos A = R \cos A$, and the length of ID is r , where R and r are the circumradius and inradius of $\triangle ABC$, respectively.

Thus, $r = R \cos A$, so $1 + \cos A = (R + r)/R$. By Carnot's theorem, $(R + r)/R = \cos A + \cos B + \cos C$, so we have $\cos B + \cos C = 1$. Since $\cos B = \frac{\sqrt{2}}{2}$, we have $\cos C = 1 - \frac{\sqrt{2}}{2}$.

31. [20] Compute

$$\sum_{k=1}^{1007} \left(\cos \left(\frac{\pi k}{1007} \right) \right)^{2014}.$$

Answer: $\boxed{\frac{2014(1 + \binom{2013}{1007})}{2^{2014}}}$ Let $\omega = e^{\frac{2\pi i}{2014}}$. We have $\omega^{2014} = 1$. Note that $\cos(\frac{\pi k}{1007}) = \frac{1}{2}(\omega^k + \omega^{-k})$. Our desired expression is

$$\frac{1}{2^{2014}} \sum_{k=1}^{1007} (\omega^k + \omega^{-k})^{2014}$$

Using binomial expansion and switching the order of the resulting summation, this is equal to

$$\frac{1}{2^{2014}} \sum_{j=0}^{2014} \binom{2014}{j} \sum_{k=1}^{1007} (\omega^{2014-2j})^k$$

Note that unless $\omega^{2014-2j} = 1$, the summand

$$\sum_{k=1}^{1007} (\omega^{2014-2j})^k$$

is the sum of roots of unity spaced evenly around the unit circle in the complex plane (in particular the 1007th, 19th, and 53rd roots of unity), so it is zero. Thus, we must only sum over those j for which $\omega^{2014-2j} = 1$, which holds for $j = 0, 1007, 2014$. This yields the answer

$$\frac{1}{2^{2014}} \left(1007 + 1007 \binom{2014}{1007} + 1007 \right) = \frac{2014 \left(1 + \binom{2013}{1007} \right)}{2^{2014}}.$$

32. [20] Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$, $a + \frac{10b}{a^2 + b^2} = 5$, and $b + \frac{10a}{a^2 + b^2} = 4$.

Answer: $\boxed{(1, 2), (4, 2), (\frac{5}{2}, 2 \pm \frac{3}{2}i)}$ **Solution 1.** First, it is easy to see that $ab \neq 0$. Thus, we can write

$$\frac{5 - a}{b} = \frac{4 - b}{a} = \frac{10}{a^2 + b^2}.$$

Then, we have

$$\frac{10}{a^2 + b^2} = \frac{4a - ab}{a^2} = \frac{5b - ab}{b^2} = \frac{4a + 5b - 2ab}{a^2 + b^2}.$$

Therefore, $4a + 5b - 2ab = 10$, so $(2a - 5)(b - 2) = 0$. Now we just plug back in and get the four solutions: $(1, 2), (4, 2), (\frac{5}{2}, 2 \pm \frac{3}{2}i)$. It's not hard to check that they all work.

Solution 2. The first equation plus i times the second yields $5 + 4i = a + bi + \frac{10(b+ai)}{a^2+b^2} = a + bi - \frac{10i}{a+bi}$, which is equivalent to $a + bi = \frac{(5\pm 3)+4i}{2}$ by the quadratic formula.

Similarly, the second equation plus i times the first yields $4 + 5i = b + ai - \frac{10i}{b+ai}$, which is equivalent to $b + ai = \frac{4+(5\pm 3)i}{2}$.

Letting $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ be the signs in $a+bi$ and $b+ai$, we get $(a, b) = \frac{1}{2}(a+bi, b+ai) - \frac{1}{2}i(b+ai, a+bi) = (\frac{10+(\epsilon_1+\epsilon_2)3}{4}, \frac{8+(\epsilon_2-\epsilon_1)3i}{4})$.

Comment. Many alternative approaches are possible. For instance, $\frac{5-a}{b} = \frac{4-b}{a} \implies b - 2 = \epsilon\sqrt{(a-1)(a-4)}$ for some $\epsilon \in \{-1, 1\}$, and substituting in and expanding gives $0 = (-2a^2+5a)\epsilon\sqrt{(a-1)(a-4)}$.

More symmetrically, we may write $a = \lambda(4-b), b = \lambda(5-a)$ to get $(a, b) = \frac{\lambda}{1-\lambda^2}(4-5\lambda, 5-4\lambda)$, and then plug into $a^2 + b^2 = 10\lambda$ to get $0 = 10(\lambda^4 + 1) - 41(\lambda^3 + \lambda) + 60\lambda^2 = (\lambda - 2)(2\lambda - 1)(5\lambda^2 - 8\lambda + 5)$.

33. [25] An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \dots, (x_k, y_k)$ of points in \mathbb{R}^2 such that $(a, b) = (x_1, y_1), (c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$.

Let S be the set of all up-right paths from $(-400, -400)$ to $(400, 400)$. What fraction of the paths in S do not contain any point (x, y) such that $|x|, |y| \leq 10$? Express your answer as a decimal number between 0 and 1.

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - 10|C - A|), 0\} \rceil$.

Answer: 0.2937156494680644... Note that any up-right path must pass through exactly one point of the form $(n, -n)$ (i.e. a point on the upper-left to lower-right diagonal), and the number of such paths is $\binom{800}{400-n}^2$ because there are $\binom{800}{400-n}$ up-right paths from $(-400, -400)$ to $(n, -n)$ and another $\binom{800}{400-n}$ from $(n, -n)$ to $(400, 400)$. An up-right path contains a point (x, y) with $|x|, |y| \leq 10$ if and only if $-10 \leq n \leq 10$, so the probability that this happens is

$$\frac{\sum_{n=-10}^{10} \binom{800}{400-n}^2}{\sum_{n=-400}^{400} \binom{800}{400-n}^2} = \frac{\sum_{n=-10}^{10} \binom{800}{400-n}^2}{\binom{1600}{800}}$$

To estimate this, recall that if we normalize $\binom{800}{n}$ to be a probability density function, then it will be approximately normal with mean 400 and variance $800 \cdot \frac{1}{4} = 200$. If this is squared, then it is proportional to a normal distribution with half the variance and the same mean, because the probability density function of a normal distribution is proportional to $e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, where μ is the mean and σ^2 is the variance.

Therefore, the $\binom{800}{n}^2$ probability density function is roughly proportional to a normal distribution with mean 400 and variance 100, or standard deviation 10. So $\sum_{n=-10}^{10} \binom{800}{400-n}^2$ represents roughly one standard deviation. Recall that approximately 68 percent of a normal distribution lies within one standard deviation of the mean (look up the 68-95-99.7 rule to read more), so a good guess would be around .32. This guess can be improved by noting that we're actually summing 21 values instead of 20, so you'd have approximately $.68 \cdot \frac{21}{20} \approx .71$ of the normal distribution, giving an answer of .29.

34. [25] Consider a number line, with a lily pad placed at each integer point. A frog is standing at the lily pad at the point 0 on the number line, and wants to reach the lily pad at the point 2014 on the number line. If the frog stands at the point n on the number line, it can jump directly to either point

$n + 2$ or point $n + 3$ on the number line. Each of the lily pads at the points $1, \dots, 2013$ on the number line has, independently and with probability $1/2$, a snake. Let p be the probability that the frog can make some sequence of jumps to reach the lily pad at the point 2014 on the number line, without ever landing on a lily pad containing a snake. What is $p^{1/2014}$? Express your answer as a decimal number.

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - 20|C - A|), 0\} \rceil$.

Answer: 0.9102805441016536

First, we establish a rough upper bound for the probability p . Let q be the probability that the frog can reach the lily pad at the point 2014 on the number line if it is allowed to jump from a point n on the number line to the point $n + 1$, in addition to the points $n + 2$ and $n + 3$. Clearly, $p \leq q$. Furthermore, p is approximated by q ; it should be easy to convince one's self that jumps from a point n to the point $n + 1$ are only useful for reaching the lily pad at point 2014 in very few situations.

Now we compute q . We note that, if the frog can jump from points n to points $n + 1$, $n + 2$, and $n + 3$, then it can reach the lily pad at the point 2014 on the number line if and only if each snake-free lily pad is at most 3 units away from the closest snake-free lily pad on the left.

Define the sequence $\{a_m\}_{m=1}^{\infty}$ by $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, and $a_{m+3} = a_{m+2} + a_{m+1} + a_m$ for $m \geq 0$. Then, it can be shown by induction that a_m is the number of possible arrangements of snakes on lily pads at points $1, \dots, m - 1$ so that the frog can make some sequence of jumps (of size 1, 2, or 3) from the lily pad at point 0 to the lily pad at point m without landing on a lily pad containing a snake. It follows that $q = a_{2014}/2^{2013}$. So

$$p^{1/2014} \approx q^{1/2014} = (a_{2014})^{1/2014}/2^{2013/2014} \approx (a_{2014})^{1/2014}/2.$$

Analyzing the recurrence relation $a_{m+3} = a_{m+2} + a_{m+1} + a_m$ yields that $(a_{2014})^{1/2014}$ is approximately equal to the largest real root r of the characteristic polynomial equation $r^3 - r^2 - r - 1 = 0$. So to roughly approximate p , it suffices to find the largest real root of this equation.

For this, we apply Newton's method, or one of many other methods for computing the roots of a polynomial. With an initial guess of 2, one iteration of Newton's method yields $r \approx 13/7$, so $p \approx r/2 \approx 13/14 \approx 0.928571$. A second iteration yields $r \approx 1777/966$, so $p \approx r/2 \approx 1777/1932 \approx 0.919772$. (It turns out that the value of r is $1.839286\dots$, yielding $p \approx r/2 = 0.919643\dots$)

Using tools from probability theory, we can get an even better estimate for p . We model the problem using a discrete-time Markov chain. The state of the Markov chain at time n , for $n = 0, 1, \dots, 2013$, indicates which of the lily pads at positions $n - 2, n - 1, n$ are reachable by the frog. It is clear that the state of the Markov chain at time n only depends (randomly) on its state at time $n - 1$. There are $2^3 = 8$ possible states for this Markov chain, because each of the lily pads at positions $n - 2, n - 1, n$ can be either reachable or unreachable by the frog. Number each state using the number $1 + d_2 + 2d_1 + 4d_0$, where d_i is 1 if the lily pad at point $n - i$ is reachable, and 0 otherwise. So, for example, at time $n = 0$, the lily pad at point n is reachable ($d_0 = 1$) whereas the lily pads at points $n - 1$ and $n - 2$ are unreachable ($d_1 = d_2 = 0$), so the Markov chain is in state number $1 + d_2 + 2d_1 + 4d_0 = 5$.

The transition matrix M for the Markov chain can now be computed directly from the conditions of the problem. It is equal to

$$M := \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}.$$

(The verification of this transition matrix is left as an exercise for the reader.) So the state vector v for the Markov chain at time 2013 is

$$v := M^{2014}[0, 1, 0, 0, 0, 0, 0, 0]^t.$$

Now, the lily pad at point 2014 is reachable by the frog if and only if the Markov chain is in state 3, 4, 5, 6, 7, or 8 at time 2013. This happens with probability

$$p = [0, 0, 1, 1, 1, 1, 1, 1] v.$$

By expanding $[0, 1, 0, 0, 0, 0, 0, 0]^t$ in an eigenbasis for M , we find that $p^{1/2014}$ is approximately equal to the second-largest real eigenvalue of the matrix M . The characteristic polynomial of M is

$$\det(\lambda I - M) = -\frac{\lambda^3}{8} + \frac{3\lambda^4}{8} + \frac{\lambda^6}{4} - \frac{3\lambda^7}{2} + \lambda^8,$$

so its eigenvalues are the roots of this polynomial. The largest real root of this characteristic polynomial is $\lambda = 1$, and the second-largest real root is $0.9105247383471604\dots$ (which can be found, again, using Newton's method, after factoring out $(\lambda - 1)\lambda^3$ from the polynomial), which is a good approximation for p .

35. [25] How many times does the letter “e” occur in all problem statements in this year’s HMMT February competition?

If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1 - |\log_2(C/A)|), 0\} \rceil$.

Answer: 1661 It is possible to arrive at a good estimate using Fermi estimation. See http://en.wikipedia.org/wiki/Fermi_problem for more details.

For example, there are 76 problems on the HMMT this year. You might guess that the average number of words in a problem is approximately 40, and the average number of letters in a word is about 5. The frequency of the letter “e” in the English language is about 10%, resulting in an estimate of

$$76 \cdot 40 \cdot 5 \cdot 0.1 = 1520.$$

This is remarkably close to the actual answer.

36. [25] We have two concentric circles C_1 and C_2 with radii 1 and 2, respectively. A random chord of C_2 is chosen. What is the probability that it intersects C_1 ?

Your answer to this problem must be expressed in the form $\frac{m}{n}$, where m and n are positive integers. If your answer is in this form, your score for this problem will be $\lfloor \frac{25 \cdot X}{Y} \rfloor$, where X is the total number of teams who submit the answer $\frac{m}{n}$ (including your own team), and Y is the total number of teams who submit a valid answer. Otherwise, your score is 0. (Your answer is *not* graded based on correctness, whether your fraction is in lowest terms, whether it is at most 1, etc.)

Answer: N/A The question given at the beginning of the problem statement is a famous problem in probability theory widely known as Bertrand’s paradox. Depending on the interpretation of the phrase “random chord,” there are at least three different possible answers to this question:

- If the random chord is chosen by choosing two (uniform) random endpoints on circle C_2 and taking the chord joining them, the answer to the question is $1/3$.
- If the random chord is chosen by choosing a (uniformly) random point P the interior of C_2 (other than the center) and taking the chord with midpoint P , the answer to the question becomes $1/4$.
- If the random chord is chosen by choosing a (uniformly) random diameter d of C , choosing a point P on d , and taking the chord passing through P and perpendicular to d , the answer to the question becomes $1/2$. (This is also the answer resulting from taking a uniformly random *horizontal* chord of C_2 .)

You can read more about Bertrand’s paradox online at [http://en.wikipedia.org/wiki/Bertrand_paradox_\(probability\)](http://en.wikipedia.org/wiki/Bertrand_paradox_(probability)). We expect that many of the valid submissions to this problem will be equal to $1/2, 1/3$, or $1/4$.

However, your score on this problem is not based on correctness, but is rather proportional to the number of teams who wrote the same answer as you! Thus, this becomes a problem of finding what

is known in game theory as the “focal point,” or “Schelling point.” You can read more about focal points at [http://en.wikipedia.org/wiki/Focal_point_\(game_theory\)](http://en.wikipedia.org/wiki/Focal_point_(game_theory)) or in economist Thomas Schelling’s book *The Strategy Of Conflict*.

HMMT 2014
Saturday 22 February 2014

Team

1. [10] Let ω be a circle, and let A and B be two points in its interior. Prove that there exists a circle passing through A and B that is contained in the interior of ω .

Answer: N/A WLOG, suppose $OA \geq OB$. Let ω' be the circle of radius OA centered at O . We have that B lies inside ω' . Thus, it is possible to scale ω' down about the point A to get a circle ω'' passing through both A and B . Since ω'' lies inside ω' and ω' lies inside ω , ω'' lies inside ω .

Alternative solution 1: WLOG, suppose $OA \geq OB$. Since $OA \geq OB$, the perpendicular bisector of AB intersects segment OA at some point C . We claim that the circle ω' passing through A and B and centered at C lies entirely in ω . Let $x = OA$ and $y = AC = BC$. Note that y is the length of the radius of ω' . By definition, any point P contained in ω' is of distance at most y from C . Applying the triangle inequality to OC , we see that $OP \leq OC + CP \leq (x - y) + y = x$, so P lies in ω . Since P was arbitrary, it follows that ω' lies entirely in ω .

Alternative solution 2: Draw line AB , and let it intersect ω at A' and B' , where A and A' are on the same side of B . Choose X inside the segment AB so that $A'X/AX = B'X/BX$; such a point exists by the intermediate value theorem. Notice that X is the center of a dilation taking $A'B'$ to AB - the same dilation carries ω to ω' which goes through A and B . Since ω' is ω dilated with respect to a point in its interior, it's clear that ω' must be contained entirely within ω , and so we are done.

2. [15] Let a_1, a_2, \dots be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is the maximal number of distinct terms in the sequence b_1, b_2, \dots ?

Answer: 127 It is clear that the sequence $\{a_i\}$ will be a concatenation of sequences of the form $\{v_i\}_{i=1}^{N_0}, \{w_i \cdot p_1\}_{i=1}^{N_1}, \{x_i \cdot p_1 p_2\}_{i=1}^{N_2}, \{y_i \cdot p_1 p_2 p_3\}_{i=1}^{N_3}$, and $\{z_i \cdot p_1 p_2 p_3 p_4\}_{i=1}^{N_4}$, for some permutation (p_1, p_2, p_3, p_4) of $(2, 3, 5, 7)$ and some sequences of integers $\{v_i\}, \{w_i\}, \{x_i\}, \{y_i\}, \{z_i\}$, each coprime with 210.

In $\{v_i\}_{i=1}^{N_0}$, there are a maximum of $\phi(210)$ distinct terms mod 210. In $\{w_i \cdot p_1\}_{i=1}^{N_1}$, there are a maximum of $\phi(\frac{210}{p_1})$ distinct terms mod 210. In $\{x_i \cdot p_1 p_2\}_{i=1}^{N_2}$, there are a maximum of $\phi(\frac{210}{p_1 p_2})$ distinct terms mod 210. In $\{y_i \cdot p_1 p_2 p_3\}_{i=1}^{N_3}$, there are a maximum of $\phi(\frac{210}{p_1 p_2 p_3})$ distinct terms mod 210. In $\{z_i \cdot p_1 p_2 p_3 p_4\}_{i=1}^{N_4}$, there can only be one distinct term mod 210.

Therefore we wish to maximize $\phi(210) + \phi(\frac{210}{p_1}) + \phi(\frac{210}{p_1 p_2}) + \phi(\frac{210}{p_1 p_2 p_3}) + 1$ over all permutations (p_1, p_2, p_3, p_4) of $(2, 3, 5, 7)$. It's easy to see that the maximum occurs when we take $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7$ for an answer of $\phi(210) + \phi(105) + \phi(35) + \phi(7) + 1 = 127$. This upper bound is clearly attainable by having the v_i 's cycle through the $\phi(210)$ integers less than 210 coprime to 210, the w_i 's cycle through the $\phi(\frac{210}{p_1})$ integers less than $\frac{210}{p_1}$ coprime to $\frac{210}{p_1}$, etc.

3. [15] There are n girls G_1, \dots, G_n and n boys B_1, \dots, B_n . A pair (G_i, B_j) is called *suitable* if and only if girl G_i is willing to marry boy B_j . Given that there is exactly one way to pair each girl with a distinct boy that she is willing to marry, what is the maximal possible number of suitable pairs?

Answer: $\frac{n(n+1)}{2}$ We represent the problem as a graph with vertices $G_1, \dots, G_n, B_1, \dots, B_n$ such that there is an edge between vertices G_i and B_j if and only if (G_i, B_j) is suitable, so we want to maximize the number of edges while having a unique matching.

We claim the answer is $\frac{n(n+1)}{2}$. First, note that this can be achieved by having an edge between G_i and B_j for all pairs $j \leq i$, because the only possible matching in this case is pairing G_i with B_i for all i . To prove that this is maximal, we first assume without loss of generality that our unique matching consists of pairing G_i with B_i for all i , which takes n edges. Now, note that for any i, j , at most one of the two edges $G_i B_j$ and $G_j B_i$ can be added, because if both were added, we could pair G_i with B_j

and G_j with B_i instead to get another valid matching. Therefore, we may add at most $\binom{n}{2} \cdot 1 = \frac{n(n-1)}{2}$ edges, so the maximal number of edges is $n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$ as desired.

4. [20] Compute

$$\sum_{k=0}^{100} \left\lfloor \frac{2^{100}}{2^{50} + 2^k} \right\rfloor.$$

(Here, if x is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to x .)

Answer: $\boxed{101 \cdot 2^{49} - 50}$ Let $a_k = \frac{2^{100}}{2^{50} + 2^k}$. Notice that, for $k = 0, 1, \dots, 49$,

$$a_k + a_{100-k} = \frac{2^{100}}{2^{50} + 2^k} + \frac{2^{100}}{2^{50} + 2^{100-k}} = \frac{2^{100}}{2^{50} + 2^k} + \frac{2^{50+k}}{2^k + 2^{50}} = 2^{50}$$

It is clear that for $k = 0, 1, \dots, 49$, $a_k, a_{100-k} \notin \mathbb{Z}$, so $\lfloor a_k \rfloor + \lfloor a_{100-k} \rfloor = 2^{50} - 1$ (since the sum of floors is an integer less than $a_k + a_{100-k}$ but greater than $a_k - 1 + a_{100-k} - 1$). Thus,

$$\sum_{k=0}^{100} \lfloor a_k \rfloor = 50 \cdot (2^{50} - 1) + 2^{49} = 101 \cdot 2^{49} - 50$$

5. [25] Prove that there exists a nonzero complex number c and a real number d such that

$$\left| \frac{1}{1+z+z^2} \right| - \left| \frac{1}{1+z+z^2} - c \right| = d$$

for all z with $|z| = 1$ and $1+z+z^2 \neq 0$. (Here, $|z|$ denotes the absolute value of the complex number z , so that $|a+bi| = \sqrt{a^2+b^2}$ for real numbers a, b .)

Answer: $\boxed{\frac{4}{3}}$ Let $f(z) = \frac{1}{1+z+z^2}$. Parametrize $z = e^{it} = \cos t + i \sin t$ and let $g(t) = f(e^{it})$, $0 \leq t < 2\pi$. Writing out $\frac{1}{1+z+z^2}$ in terms of t and simplifying, we find that $g(t) = \frac{\cos t - i \sin t}{1+2 \cos t}$. Letting $x(t) = \Re(g(t))$ and $y(t) = \Im(g(t))$ (the real and imaginary parts of $g(t)$, respectively), what we wish to prove is equivalent to showing that $\{(x(t), y(t)) \mid 0 \leq t < 2\pi\}$ is a hyperbola with one focus at $(0, 0)$. However

$$9 \left(x(t) - \frac{2}{3} \right)^2 - 3y(t)^2 = 1$$

holds for all t , so from this equation we find that the locus of points $(x(t), y(t))$ is a hyperbola, with center $(\frac{2}{3}, 0)$ and focal length $\frac{2}{3}$, so the foci are at $(0, 0)$ and $(\frac{4}{3}, 0)$. Hence $c = \frac{4}{3}$.

6. [25] Let n be a positive integer. A sequence (a_0, \dots, a_n) of integers is *acceptable* if it satisfies the following conditions:

- $0 = |a_0| < |a_1| < \dots < |a_{n-1}| < |a_n|$.
- The sets $\{|a_1 - a_0|, |a_2 - a_1|, \dots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{1, 3, 9, \dots, 3^{n-1}\}$ are equal.

Prove that the number of acceptable sequences of integers is $(n+1)!$.

Answer: $\boxed{N/A}$ We actually prove a more general result via strong induction on n .

First, we state the more general result we wish to prove.

For $n > 0$, define a *great sequence* to be a sequence of integers (a_0, \dots, a_n) such that

- $0 = |a_0| < |a_1| < \dots < |a_{n-1}| < |a_n|$
- Let (b_1, b_2, \dots, b_n) be a set of positive integers such that $3b_i \leq b_{i+1}$ for all i . The sets $\{|a_1 - a_0|, |a_2 - a_1|, \dots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{b_1, b_2, \dots, b_n\}$ are equal.

Then, the number of great sequences is $(n+1)!$.

If we prove this statement, then we can just consider the specific case of $b_1 = 1$, $3b_i = b_{i+1}$ to solve our problem.

Before we proceed, we will prove a lemma.

Lemma: Let (b_1, b_2, \dots, b_n) be a set of positive integers such that $3b_i \leq b_{i+1}$ for all i . Then $b_i > 2 \sum_{k=1}^{i-1} b_k$.

Proof: $\frac{b_i}{2} = \frac{b_i}{3} + \frac{b_i}{9} + \dots \geq \sum_{k=1}^{i-1} b_k$. Equality only occurs when the sequence b_i is infinite, which is not the case, so the inequality holds.

We can now proceed with the induction. The base case is obvious. Assume it is true up to $n = j - 1$. Next, consider some permutation of the set $B = \{b_1, b_2, b_3, \dots, b_j\}$. Denote it as $C = \{c_1, c_2, \dots, c_j\}$. Find the element $c_k = b_j$. For the first $k - 1$ elements of C , we can put them in order and apply the inductive hypothesis. The number of great sequences such that the set of differences $\{|a_1 - a_0|, \dots, |a_{k-1} - a_{k-2}|\}$ is equal to $\{c_1, c_2, \dots, c_{k-1}\}$ is $k!$.

Then, a_k can be either $a_{k-1} + c_k$ or $a_{k-1} - c_k$. This is because, by the lemma, $c_k > 2 \sum_{m=1}^{k-1} c_m = 2 \sum_{m=1}^{k-1} |a_m - a_{m-1}| \geq 2|a_{k-1} - a_{k-2} + a_{k-2} - a_{k-3} + \dots + a_1 - a_0| = 2|a_{k-1}|$. It is easy to check that for either possible value of a_k , $|a_k| > |a_{k-1}|$. After that, there is only one possible value for a_{k+1}, \dots, a_j because only one of $a_i \pm c_{i+1}$ will satisfy $|a_i| > |a_{i-1}|$.

There are $\binom{j-1}{k-1}$ possible ways to choose c_1, c_2, \dots, c_{k-1} from B . Given those elements of C , there are $k!$ ways to make a great sequence $(a_0, a_1, \dots, a_{k-1})$. Then, there are 2 possible values for a_k . After that, there are $(j - k)!$ ways to order the remaining elements of C , and for each such ordering, there is exactly 1 possible great sequence (a_0, a_1, \dots, a_j) .

Now, counting up all the possible ways to do this over all values of k , we get that the number of great sequences is equal to $\sum_{k=1}^j \binom{j-1}{k-1} k! 2(j-k)! = \sum_{k=1}^j 2(j-1)!(k) = 2(j-1)! \frac{j(j+1)}{2} = (j+1)!$. The induction is complete, and this finishes the proof.

Alternate solution: Another method of performing the induction is noting that any acceptable sequence (a_0, \dots, a_n) can be matched with $n + 2$ acceptable sequences of length $n + 2$ because we can take $(3a_0, \dots, 3a_n)$ and add an element with a difference of 1 in any of $n + 2$ positions.

7. [30] Find the maximum possible number of diagonals of equal length in a convex hexagon.

Answer: 7 First, we will prove that 7 is possible. Consider the following hexagon $ABCDEF$ whose vertices are located at $A(0, 0)$, $B(\frac{1}{2}, 1 - \frac{\sqrt{3}}{2})$, $C(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $D(0, 1)$, $E(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $F(-\frac{1}{2}, 1 - \frac{\sqrt{3}}{2})$. One can easily verify that all diagonals but BE and CF have length 1.

Now suppose that there are at least 8 diagonals in a certain convex hexagon $ABCDEF$ whose lengths are equal. There must be a diagonal such that, with this diagonal taken out, the other 8 have equal length. There are two cases.

Case I: The diagonal is one of AC, BD, CE, DF, EA, FB . WLOG, assume it is AC . We have $EC = EB = FB = FC$. Thus, B and C are both on the perpendicular bisector of EF . Since $ABCDEF$ is convex, both B and C must be on the same side of line EF , but this is impossible as one of B or C , must be contained in triangle CEF . Contradiction.

Case II: The diagonal is one of AD, BE, CF . WLOG, assume it is AD . Again, we have $EC = EB = FB = FC$. By the above reasoning, this is a contradiction.

Thus, 7 is the maximum number of possible diagonals.

8. [35] Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, and $BC = 6$. Let D be the foot of the altitude from A to BC , and E be the intersection of AO with BC . Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX .

Answer: $\boxed{96/41}$ Let AX intersect the circumcircle of $\triangle ABC$ again at K . Let OY intersect AK and BC at T and L , respectively. We have $\angle LOA = \angle OYX = \angle TDX = \angle LAK$, so AL is tangent to the circumcircle. Furthermore, $OL \perp AK$, so $\triangle ALK$ is isosceles with $AL = AK$, so AK is also tangent to the circumcircle. Since BC and the tangents to the circumcircle at A and K all intersect at the same point L , CL is a symmedian of $\triangle ACK$. Then AK is a symmedian of $\triangle ABC$. Then we can use $\frac{BX}{XC} = \frac{(AB)^2}{(AC)^2}$ to compute $BX = \frac{96}{41}$.

9. [35] For integers $m, n \geq 1$, let $A(n, m)$ be the number of sequences (a_1, \dots, a_{nm}) of integers satisfying the following two properties:

- (a) Each integer k with $1 \leq k \leq n$ occurs exactly m times in the sequence (a_1, \dots, a_{nm}) .
- (b) If i, j , and k are integers such that $1 \leq i \leq nm$ and $1 \leq j \leq k \leq n$, then j occurs in the sequence (a_1, \dots, a_i) at least as many times as k does.

For example, if $n = 2$ and $m = 5$, a possible sequence is $(a_1, \dots, a_{10}) = (1, 1, 2, 1, 2, 2, 1, 2, 1, 2)$. On the other hand, the sequence $(a_1, \dots, a_{10}) = (1, 2, 1, 2, 2, 1, 1, 1, 2, 2)$ does not satisfy property (2) for $i = 5$, $j = 1$, and $k = 2$.

Prove that $A(n, m) = A(m, n)$.

Answer: $\boxed{N/A}$ Solution 1: We show that $A(n, m)$ is equal to the the number of standard Young tableaux with n rows and m columns (i.e. fillings of an $n \times m$ matrix with the numbers $1, 2, \dots, nm$ so that numbers are increasing in each row and column). Consider the procedure where every time a k appears in the sequences, you add a number to the leftmost empty spot of the k -th row. Doing this procedure will result in a valid standard Young tableau. The entries are increasing along every row because new elements are added from left to right. The elements are also increasing along every column. This is because the condition about the sequences implies that there will always be at least as many elements in row i as there are in row j for $i < j$. At the end of this procedure, the Young Tableaux has been filled because each of the n numbers have been added m times.

Now, consider a $n \times m$ standard Young tableau. If the number p is in row k , you add a k to the sequence. This will produce a valid sequence. To see this, suppose that p appears in the entry (x, y) . Then all of the entries (q, y) where $q < x$ have already been added to the sequence because they must contain entries less than p . Thus, the numbers 1 through $x - 1$ have all already been added at least y times. Then, when we process p , we are adding the y -th x , which is valid. At the end of this procedure, n numbers have been added m times to the sequence.

The two procedures given above are inverses of each other. Thus, $A(n, m)$ is equal to the number of $n \times m$ standard Young tableaux. For every $n \times m$ Tableaux, we can transpose it to form an $m \times n$ tableau. The number of such tableaux is $A(m, n)$. Thus, $A(n, m) = A(m, n)$.

Solution 2: We can also form a direct bijection to show $A(n, m) = A(m, n)$, as follows. Suppose that $a = (a_1, \dots, a_{nm})$ is a sequence satisfying properties 1 and 2. We will define a sequence $f(a) = (b_1, \dots, b_{nm})$ satisfying the same properties 1 and 2, but with m and n switched.

The bijection f is simple: just define b_i , for $1 \leq i \leq nm$, to be equal to the number of j with $1 \leq j \leq i$ such that $a_j = i$. In other words, to obtain $f(a)$ from a , replace the k th occurrence of each number with the number k . For example, if $n = 2$ and $m = 5$, then $f(1, 1, 2, 1, 2, 2, 1, 2, 1, 2) = (1, 2, 1, 3, 2, 3, 4, 4, 5, 5)$.

First, it is clear that $f(a)$ satisfies property 1 with m and n switched. Indeed, it follows directly from the definition of f that for each pair (k_1, k_2) with $1 \leq k_1 \leq n$ and $1 \leq k_2 \leq m$, there is exactly one index i for which $(a_i, b_i) = (k_1, k_2)$. This implies the desired result.

Second, we show that $f(a)$ satisfies property 2 with m and n switched. For this, suppose that i, j, k are integers with $1 \leq i \leq nm$ and $1 \leq j \leq k \leq m$. Then, the number of times that k appears in the sequence (b_1, \dots, b_i) is exactly equal to the number of integers ℓ with $1 \leq \ell \leq n$ such that ℓ appears at least k times in the sequence (a_1, \dots, a_i) . Similarly, the number of times that j appears in the sequence (b_1, \dots, b_i) is exactly equal to the number of integers ℓ with $1 \leq j \leq n$ such that ℓ appears at least j

times in the sequence (a_1, \dots, a_i) . In particular, the number j appears in the sequence (b_1, \dots, b_i) at least as many times as k does.

Now, we show that f is a bijection. We claim that f is actually an involution; that is, $f(f(a)) = a$. (Note in particular that this implies that f is a bijection, and its inverse is itself.)

Fix an index i with $1 \leq i \leq mn$. Let $\ell = a_i$ and $k = b_i$; then, k is the number of times that the term ℓ appears in the sequence (a_1, \dots, a_i) . It is enough to show that ℓ is the number of times that the term k appears in the sequence (b_1, \dots, b_i) . First, note that by definition of f , the number of times k appears in the sequence (b_1, \dots, b_i) is equal to the number of integers j such that the sequence (a_1, \dots, a_i) contains the number j at least k times.

Now, ℓ occurs exactly k times in the sequence (a_1, \dots, a_i) . So by property 2, each j with $j \leq \ell$ appears at least k times in the sequence (a_1, \dots, a_i) . Furthermore, each j with $j > \ell$ appears at most $k - 1$ times in the sequence (a_1, \dots, a_{i-1}) and thus appears at most $k - 1$ times in the sequence (a_1, \dots, a_i) as well (because $a_i = \ell$). Therefore, the number of integers j such that the sequence (a_1, \dots, a_i) contains the number j at least k times is exactly equal to ℓ . This shows the desired result.

Note: The bijections given in solutions 1 and 2 can actually be shown to be the same.

10. [40] Fix a positive real number $c > 1$ and positive integer n . Initially, a blackboard contains the numbers $1, c, \dots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2b$. Prove that after $n - 1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L} - 1}{c^{1/L} - 1} \right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

Answer: N/A By a simple reverse induction, we can show that at any instant, any number on the board takes the form $\sum_{\alpha \in A} c^{r_\alpha} c^\alpha$ for a certain $A \subseteq \{0, \dots, n-1\}$ for non-negative integer weights r_α satisfying $\sum_{\alpha \in A} \phi^{-r_\alpha} = 1$, where these subsets A partition $\{0, \dots, n-1\}$, and after each minute the number of parts decreases by 1. (Note that $1 = \phi^{-1} + \phi^{-2}$, and when we “merge” two of these sets A, B corresponding to $a, b \rightarrow ca + c^2b$, we add 1 to all the weights of A , and 2 to all the weights of B). Therefore, the final number takes the form $\sum_{i=0}^{n-1} c^{t_i} c^i$ for positive integer weights t_0, \dots, t_{n-1} satisfying $\sum_{i=0}^{n-1} \phi^{-t_i} = 1$.

Let $M = \log_\phi(c) > 0$ (so $L = 1 + M$). Then $c = \phi^M$, so by Holder’s inequality,

$$\left(\sum_{i=0}^{n-1} c^{t_i} c^i \right)^1 \left(\sum_{i=0}^{n-1} \phi^{-t_i} \right)^M \geq \left(\sum_{i=0}^{n-1} [c^i (c\phi^{-M})^{t_i}]^{1/L} \right)^L = \left(\frac{c^{n/L} - 1}{c^{1/L} - 1} \right)^L,$$

as desired.

Comment: Note that the same method applies for any choice of initial numbers x_1, x_2, \dots, x_n (the choice c^0, c^1, \dots, c^{n-1} simply gives a relatively clean closed form). The invariant $\sum \phi^{-t_i} = 1$ was inspired by the corresponding invariant $\sum 2^{-t_i} = 1$ for ISL 2007 A5, where the relevant operation is $\{a, b\} \mapsto \{c^1a + c^2b\}$ (specifically, for $c = 2$, but this is not important) rather than $\{a, b\} \mapsto \{c^1a + c^2b\}$.

Alternate solution: We have $cx^L + c^2y^L \geq (x + y)^L$ for any nonnegative reals x, y , since $(cx^L + c^2y^L)(c^{-1/(L-1)} + c^{-2/(L-1)})^{L-1} \geq (x + y)^L$ by Holder’s inequality, $c^{1/(L-1)} = c^{1/\log_\phi(c)} = \phi$, and of course $\phi^{-1} + \phi^{-2} = 1$.

In particular, $(ca + c^2b)^{1/L} \geq a^{1/L} + b^{1/L}$. Thus $\sum_{i=0}^{n-1} (c^i)^{1/L} \leq N^{1/L}$, where N is the final (nonnegative) number on the board. Hence $N^{1/L} \geq \frac{c^{n/L} - 1}{c^{1/L} - 1}$.

HMMT November 2014
Saturday 15 November 2014
General Test

1. Two circles ω and γ have radii 3 and 4 respectively, and their centers are 10 units apart. Let x be the shortest possible distance between a point on ω and a point on γ , and let y be the longest possible distance between a point on ω and a point on γ . Find the product xy .

Answer: 51 Let ℓ be the line connecting the centers of ω and γ . Let A and B be the intersections of ℓ with ω , and let C and D be the intersections of ℓ with γ , so that $A, B, C,$ and D are collinear, in that order. The shortest distance between a point on ω and a point on γ is $BC = 3$. The longest distance is $AD = 3 + 10 + 4 = 17$. The product is 51.

2. Let ABC be a triangle with $\angle B = 90^\circ$. Given that there exists a point D on AC such that $AD = DC$ and $BD = BC$, compute the value of the ratio $\frac{AB}{BC}$.

Answer: $\sqrt{3}$ D is the circumcenter of ABC because it is the midpoint of the hypotenuse. Therefore, $DB = DA = DC$ because they are all radii of the circumcircle, so DBC is an equilateral triangle, and $\angle C = 60^\circ$. This means that ABC is a $30^\circ - 60^\circ - 90^\circ$ triangle, with $\frac{AB}{BC} = \sqrt{3}$.

3. Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

Answer: 15 Let $d = \gcd(a, b)$ for some $a, b \in \mathbb{Z}^+$. Then, we can write $d = ax - by$, where $x, y \in \mathbb{Z}^+$, and

$$2^a - 1 \mid 2^{ax} - 1 \tag{1}$$

$$2^b - 1 \mid 2^{by} - 1 \tag{2}$$

Multiplying the right-hand side of (2) by 2^d , we get,

$$2^b - 1 \mid 2^{ax} - 2^d$$

Thus, $\gcd(2^a - 1, 2^b - 1) = 2^d - 1 = 2^{\gcd(a,b)} - 1$.

Using $a = 16$ and $b = 36$, we get

$$\gcd(2^{16} - 1, 2^{36} - 1) = 2^{\gcd(16,36)} - 1 = 2^4 - 1 = 15$$

4. In rectangle $ABCD$ with area 1, point M is selected on \overline{AB} and points X, Y are selected on \overline{CD} such that $AX < AY$. Suppose that $AM = BM$. Given that the area of triangle MXY is $\frac{1}{2014}$, compute the area of trapezoid $AXYB$.

Answer: $\frac{1}{2} + \frac{1}{2014}$ OR $\frac{504}{1007}$ Notice that $[AMX] + [BYM] = \frac{1}{2}[ABCD] = \frac{1}{2}$. Thus,

$$[AXYB] = [AMX] + [BYM] + [MXY] = \frac{1}{2} + \frac{1}{2014} = \frac{504}{1007}$$

5. Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2^{100} wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move?

(By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

Answer: 33 We will work in the binary system in this solution.

Let multiplying the stored value by 2 and adding 1 be *Move A* and multiplying the stored value by 4 and adding 3 be *Move B*. Let the stored value be S . Then, *Move A* affixes one 1 to S , while *Move B* affixes two 1s. The goal is to have greater than or equal to 101 1s. If any player makes the number of 1s in S congruent to $2 \pmod{3}$, then no matter what the other player does, he will lose, since the number of 1s in S reaches 101 or 102 only from $99 \equiv 0 \pmod{3}$ or $100 \equiv 1 \pmod{3}$.

Mark's winning strategy: Do *Move A*. In the succeeding moves, if William does *Move B*, then Mark does *Move A*, and vice versa, which in total, affixes three 1s to S . This ensures that William always takes his turn while the number of 1s in S is congruent to $2 \pmod{3}$. Note that Mark has to follow this strategy because once he does not, then William can follow the same strategy and make Mark lose, a contradiction to the required optimal play. Since S starts out with one 1, this process gives William a maximum of 33 moves.

6. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The angle bisector of C intersects side AB at X . Points M and N are drawn on sides BC and AC , respectively, such that $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$. Compute the length MN .

Answer: $\boxed{\frac{3\sqrt{14}}{5}}$ By Stewart's Theorem on the angle bisector,

$$CX^2 = AC \cdot BC \left(1 - \frac{AB}{AC + BC}\right)^2$$

Thus,

$$CX^2 = 4 \cdot 6 \left(1 - \frac{5}{10}\right)^2 = 18$$

Since $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$, we produce equal angles. So, by similar triangles, $XM = XN = \frac{4 \cdot 6}{10} = \frac{12}{5}$.

Moreover, triangles MCX and NCX are congruent isosceles triangles with vertices M and N , respectively. Since CX is an angle bisector, then CX and MN are perpendicular bisectors of each other.

Therefore,

$$MN^2 = 4(XN^2 - (CX/2)^2) = 4 \cdot \left(\frac{12}{5}\right)^2 - 18 = \frac{126}{25}$$

and

$$MN = \frac{3\sqrt{14}}{5}$$

7. Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *perfect* if for any distinct indices i, j, k , the three numbers a_i, a_j, a_k do *not* form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

Answer: $\boxed{780}$ There are two situations.

1. The multiset is $aabbc$; the only condition here is $c \neq \frac{1}{2}(a + b)$, for $\binom{5}{3} - |S| \cdot \binom{3}{1} = 18$ such triples, where S is the set of unordered triples (a, b, c) which do not satisfy the condition, and $S = \{(1, 2, 3), (2, 3, 4), (3, 4, 5), (1, 3, 5)\}$. Each one gives $\frac{5!}{2!2!} = 30$ orderings, so $18 \cdot 30 = 540$ in this case.

2. There are four distinct elements in the tuple. Then, the elements must be $\{1, 2, 4, 5\}$. All of them work, for an additional $4 \cdot 60 = 240$.

Therefore, there are $540 + 240 = 780$ such tuples.

8. Let a, b, c, x be reals with $(a + b)(b + c)(c + a) \neq 0$ that satisfy

$$\frac{a^2}{a + b} = \frac{a^2}{a + c} + 20, \quad \frac{b^2}{b + c} = \frac{b^2}{b + a} + 14, \quad \text{and} \quad \frac{c^2}{c + a} = \frac{c^2}{c + b} + x.$$

Compute x .

Answer: $\boxed{-34}$ Note that

$$\begin{aligned} \frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+a} - \frac{a^2}{c+a} - \frac{b^2}{a+b} - \frac{c^2}{b+c} &= \frac{a^2-b^2}{a+b} + \frac{b^2-c^2}{b+c} + \frac{c^2-a^2}{c+a} \\ &= (a-b) + (b-c) + (c-a) \\ &= 0. \end{aligned}$$

Thus, when we sum up all the given equations, we get that $20 + 14 + x = 0$. Therefore, $x = -34$.

9. For any positive integers a and b , define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$. (The operation \oplus is called the *exclusive or*.) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} \left(k \oplus \left\lfloor \frac{k}{2} \right\rfloor \right).$$

Here $\lfloor x \rfloor$ is the greatest integer not exceeding x .

Answer: $\boxed{2^{2013}(2^{2014}-1) \text{ OR } 2^{4027}-2^{2013}}$ Let $k = a_{2013}a_{2012}\dots a_0$ in base 2. Then $\lfloor \frac{k}{2} \rfloor = 0a_{2013}\dots a_1$ in base 2. So the leftmost digit of $k \oplus \lfloor \frac{k}{2} \rfloor$ is 1 if and only if $a_{2013} = 1$, and the n th digit from the right is 1 if and only if $a_n \neq a_{n-1}$ ($1 \leq n \leq 2013$).

In either case, the probability of each digit being 1 is $\frac{1}{2}$. Therefore, the sum of all such numbers is

$$\frac{1}{2} \cdot 2^{2014} \cdot \underbrace{11\dots 11}_2 = 2^{2013}(2^{2014}-1).$$

10. Suppose that m and n are integers with $1 \leq m \leq 49$ and $n \geq 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m ?

Answer: $\boxed{29}$ If n is even, $n+1 \mid n^{n+1} + 1$, so we can cover all odd m .

If m is even and $m \mid n^{n+1} + 1$, then n must be odd, so $n+1$ is even, and m cannot be divisible by 4 or any prime congruent to 3 (mod 4). Conversely, if $m/2$ has all factors 1 (mod 4), then by CRT there exists $N \equiv 1 \pmod{4}$ such that $m \mid N^2 + 1 \mid N^{N+1} + 1$ (note $(N+1)/2$ is odd).

So the only bad numbers take the form $2k$, where $1 \leq k \leq 24$ is divisible by at least one of 2, 3, 7, 11, 19, 23, 31, ... We count $k = 2, 4, \dots, 24$ (there are 12 numbers here), $k = 3, 9, 15, 21$ (another four), $k = 7, 11, 19, 23$ (another four), giving a final answer of $49 - 12 - 4 - 4 = 29$.

HMMT November 2014
Saturday 15 November 2014
Guts Round

1. [5] Solve for x in the equation $20 \cdot 14 + x = 20 + 14 \cdot x$.

Answer: 20 By inspection, $20 + 14 \cdot 20 = 20 \cdot 14 + 20$. Alternatively, one can simply compute $x = \frac{20 \cdot 14 - 20}{14 - 1} = 20$.

2. [5] Find the area of a triangle with side lengths 14, 48, and 50.

Answer: 336 Note that this is a multiple of the 7-24-25 right triangle. The area is therefore

$$\frac{14(48)}{2} = 336.$$

3. [5] Victoria wants to order at least 550 donuts from Dunkin' Donuts for the HMMT 2014 November contest. However, donuts only come in multiples of twelve. Assuming every twelve donuts cost \$7.49, what is the minimum amount Victoria needs to pay, in dollars? (Because HMMT is affiliated with MIT, the purchase is tax exempt. Moreover, because of the size of the order, there is no delivery fee.)

Answer: 344.54 The smallest multiple of 12 larger than 550 is $552 = 12 \cdot 46$. So the answer is $46 \cdot \$7.49$. To make the multiplication easier, we can write this as $46 \cdot (\$7.5 - \$0.01) = \$345 - \$0.46 = \$344.54$.

Note: this is the actual cost of donuts at the 2014 HMMT November contest.

4. [6] How many two-digit prime numbers have the property that both digits are also primes?

Answer: 4 When considering the 16 two-digit numbers with 2, 3, 5, and 7 as digits, we find that only 23, 37, 53, and 73 have this property.

5. [6] Suppose that x, y, z are real numbers such that

$$x = y + z + 2, \quad y = z + x + 1, \quad \text{and} \quad z = x + y + 4.$$

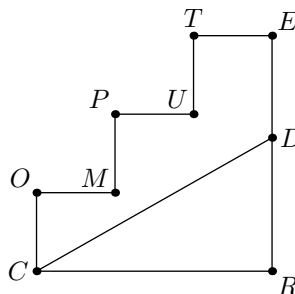
Compute $x + y + z$.

Answer: -7 Adding all three equations gives

$$x + y + z = 2(x + y + z) + 7,$$

from which we find that $x + y + z = -7$.

6. [6] In the octagon *COMPUTER* exhibited below, all interior angles are either 90° or 270° and we have $CO = OM = MP = PU = UT = TE = 1$.



Point D (not to scale in the diagram) is selected on segment RE so that polygons *COMPUTED* and *CDR* have the same area. Find DR .

Answer: 2 The area of the octagon *COMPUTER* is equal to 6. So, the area of *CDR* must be 3. So, we have the equation $\frac{1}{2} * CD * DR = [CDR] = 3$. And from $CD = 3$, we have $DR = 2$.

7. [7] Let $ABCD$ be a quadrilateral inscribed in a circle with diameter \overline{AD} . If $AB = 5$, $AC = 6$, and $BD = 7$, find CD .

Answer: $\boxed{\sqrt{38}}$ We have $AD^2 = AB^2 + BD^2 = AC^2 + CD^2$, so $CD = \sqrt{AB^2 + BD^2 - AC^2} = \sqrt{38}$.

8. [7] Find the number of digits in the decimal representation of 2^{41} .

Answer: $\boxed{13}$ Noticing that $2^{10} = 1024 \approx 1000$ allows for a good estimate. Alternatively, the number of decimal digits of n is given by $\lfloor \log_{10}(n) \rfloor + 1$. Using $\log_{10}(2) \approx 0.31$ also gives the correct answer. The exact value of 2^{41} is 219902325552.

9. [7] Let f be a function from the nonnegative integers to the positive reals such that $f(x+y) = f(x) \cdot f(y)$ holds for all nonnegative integers x and y . If $f(19) = 524288k$, find $f(4)$ in terms of k .

Answer: $\boxed{16k^{4/19}}$ The given condition implies $f(mn) = f(m)^n$, so

$$f(4)^{19} = f(4 \cdot 19) = f(19 \cdot 4) = f(19)^4$$

and it follows that $f(4) = 16k^{4/19}$.

10. [8] Let ABC be a triangle with $CA = CB = 5$ and $AB = 8$. A circle ω is drawn such that the interior of triangle ABC is completely contained in the interior of ω . Find the smallest possible area of ω .

Answer: $\boxed{16\pi}$ We need to contain the interior of \overline{AB} , so the diameter is at least 8. This bound is sharp because the circle with diameter \overline{AB} contains all of ABC . Hence the minimal area is 16π .

11. [8] How many integers n in the set $\{4, 9, 14, 19, \dots, 2014\}$ have the property that the sum of the decimal digits of n is even?

Answer: $\boxed{201}$

We know that 2014 does not qualify the property. So, we'll consider $\{4, 9, 14, \dots, 2009\}$ instead. Now, we partition this set into 2 sets: $\{4, 14, 24, \dots, 2004\}$ and $\{9, 19, 29, \dots, 2009\}$.

For each so the first and second set are basically $x4$ and $x9$, where $x = 0, 1, 2, \dots, 200$, respectively. And we know that for each value of x , x must be either even or odd, which makes *exactly* one of $\{x4, x9\}$ has even sum of decimal digits. Therefore, there are in total of 201 such numbers.

12. [8] Sindy writes down the positive integers less than 200 in increasing order, but skips the multiples of 10. She then alternately places + and - signs before each of the integers, yielding an expression $+1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 11 + 12 - \dots - 199$. What is the value of the resulting expression?

Answer: $\boxed{-100}$ Group the numbers into $(1 - 2 + 3 - 4 + \dots + 18 - 19) + (21 - 22 + \dots + 38 - 39) + \dots + (181 - 182 + \dots + 198 - 199)$. We can easily show that each group is equal to -10 , and so the answer is -100 .

13. [9] Let ABC be a triangle with $AB = AC = \frac{25}{14}BC$. Let M denote the midpoint of \overline{BC} and let X and Y denote the projections of M onto \overline{AB} and \overline{AC} , respectively. If the areas of triangle ABC and quadrilateral $AXMY$ are both positive integers, find the minimum possible sum of these areas.

Answer: $\boxed{1201}$ By similar triangles, one can show that $[AXMY] = 2 \cdot [AMX] = \left(\frac{24}{25}\right)^2 \cdot 2[ABM] = \left(\frac{24}{25}\right)^2 \cdot [ABC]$. Thus the answer is $25^2 + 24^2 = 1201$.

14. [9] How many ways can the eight vertices of a three-dimensional cube be colored red and blue such that no two points connected by an edge are both red? Rotations and reflections of a given coloring are considered distinct.

Answer: $\boxed{35}$ We do casework on R , the number of red vertices. Let the cube be called $ABCDEFGH$, with opposite faces $ABCD$ and $EFGH$, such that A is directly above E .

- $R = 0$: There is one such coloring, which has only blue vertices.

- $R = 1$: There are 8 ways to choose the red vertex, and all other vertices must be blue. There are 8 colorings in this case.
- $R = 2$: Any pair not an edge works, so the answer is $\binom{8}{2} - 12 = 16$.
- $R = 3$: Each face $ABCD$ and $EFGH$ has at most two red spots. Assume WLOG $ABCD$ has exactly two and $EFGH$ has exactly one (multiply by 2 at the end). There are two ways to pick those in $ABCD$ (two opposite corners), and two ways after that to pick $EFGH$. Hence the grand total for this subcase is $2 \cdot 2 \cdot 2 = 8$.
- $R = 4$: There are only two ways to do this.

Hence, the sum is 35.

15. [9] Carl is on a vertex of a regular pentagon. Every minute, he randomly selects an adjacent vertex (each with probability $\frac{1}{2}$) and walks along the edge to it. What is the probability that after 10 minutes, he ends up where he had started?

Answer: $\boxed{\frac{127}{512}}$ Let A denote a clockwise move and B denote a counterclockwise move. We want to have some combination of 10 A's and B's, with the number of A's and the number of B's differing by a multiple of 5. We have $\binom{10}{0} + \binom{10}{5} + \binom{10}{10} = 254$. Hence the answer is $\frac{254}{2^{10}} = \frac{127}{512}$.

16. [10] A particular coin has a $\frac{1}{3}$ chance of landing on heads (H), $\frac{1}{3}$ chance of landing on tails (T), and $\frac{1}{3}$ chance of landing vertically in the middle (M). When continuously flipping this coin, what is the probability of observing the continuous sequence HMMT before HMT?

Answer: $\boxed{\frac{1}{4}}$ For a string of coin flips S , let P_S denote the probability of flipping HMMT before HMT if S is the starting sequence of flips. We know that the desired probability, p , is $\frac{1}{3}P_H + \frac{1}{3}P_M + \frac{1}{3}P_T$. Now, using conditional probability, we find that

$$\begin{aligned} P_H &= \frac{1}{3}P_{HH} + \frac{1}{3}P_{HM} + \frac{1}{3}P_{HT} \\ &= \frac{1}{3}P_H + \frac{1}{3}P_{HM} + \frac{1}{3}P_T. \end{aligned}$$

We similarly find that

$$\begin{aligned} P_M &= P_T = \frac{1}{3}P_H + \frac{1}{3}P_M + \frac{1}{3}P_T \\ P_{HM} &= \frac{1}{3}P_{HMM} + \frac{1}{3}P_H \\ P_{HMM} &= \frac{1}{3} + \frac{1}{3}P_M + \frac{1}{3}P_H. \end{aligned}$$

Solving gives $P_H = P_M = P_T = \frac{1}{4}$. Thus, $p = \frac{1}{4}$.

17. [10] Let ABC be a triangle with $AB = AC = 5$ and $BC = 6$. Denote by ω the circumcircle of ABC . We draw a circle Ω which is externally tangent to ω as well as to the lines AB and AC (such a circle is called an A -mixtilinear excircle). Find the radius of Ω .

Answer: $\boxed{\frac{75}{8}}$ Let M be the midpoint of BC . Let D be the point diametrically opposite A on the circumcircle, and let the A -mixtilinear excircle be tangent to lines AB and AC at X and Y . Let O be the center of the A -mixtilinear excircle.

Notice that $\triangle AOX \sim \triangle ABM$. If we let x be the desired radius, we have

$$\frac{x + AD}{x} = \frac{5}{3}.$$

We can compute $\frac{AD}{5} = \frac{5}{4}$ since $\triangle ADB \sim \triangle ABM$, we derive $AD = \frac{25}{4}$. From here it follows that $x = \frac{75}{8}$.

18. [10] For any positive integer x , define $\text{Accident}(x)$ to be the set of ordered pairs (s, t) with $s \in \{0, 2, 4, 5, 7, 9, 11\}$ and $t \in \{1, 3, 6, 8, 10\}$ such that $x + s - t$ is divisible by 12. For any nonnegative integer i , let a_i denote the number of $x \in \{0, 1, \dots, 11\}$ for which $|\text{Accident}(x)| = i$. Find

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2.$$

Answer: 26 Modulo twelve, the first set turns out to be $\{-1 \cdot 7, 0 \cdot 7, \dots, 5 \cdot 7\}$ and the second set turns out to be $\{6 \cdot 7, \dots, 10 \cdot 7\}$. We can eliminate the factor of 7 and shift to reduce the problem to $s \in \{0, 1, \dots, 6\}$ and $t \in \{7, \dots, 11\}$. With this we can easily compute $(a_0, a_1, a_2, a_3, a_4, a_5) = (1, 2, 2, 2, 2, 3)$. Therefore, the answer is 26.

It turns out that the choice of the sets correspond to the twelve keys of a piano in an octave. The first set gives the locations of the white keys, while the second set gives the location of the black keys. Steps of seven then correspond to perfect fifths. See, for example:

- http://en.wikipedia.org/wiki/Music_and_mathematics
- http://en.wikipedia.org/wiki/Guerino_Mazzola

19. [11] Let a sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_0 = \sqrt{2}$, $a_1 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 1$. The sequence of remainders when a_0, a_1, a_2, \dots are divided by 2014 is eventually periodic with some minimal period p (meaning that $a_m = a_{m+p}$ for all sufficiently large integers m , and p is the smallest such positive integer). Find p .

Answer: 12 Let $a_n = 2^{b_n}$, so notice $b_1 = 1, b_2 = 2$, and $b_{n+1} = b_n + 2b_{n-1}$ for $n \geq 1$, so by inspection $b_n = 2^{n-1}$ for all n ; thus $a_n = 2^{2^{n-1}}$. $2014 = 2 \cdot 19 \cdot 53$ so we just want to find the lcm of the eventual periods of $2^n \pmod{\text{ord}_{19}(2)}$ and $\text{ord}_{53}(2)$. These orders divide 18 and 52 respectively, and we can manually check $\text{ord}_9(2) = 6$ and $\text{ord}_{13}(2) = 12$. The lcm of these is 12, so to show the answer is 12, it suffices to show that $13 \mid \text{ord}_{53}(2)$. This is true since $2^4 \not\equiv 1 \pmod{53}$. So, the answer is 12.

20. [11] Determine the number of sequences of sets S_1, S_2, \dots, S_{999} such that

$$S_1 \subseteq S_2 \subseteq \dots \subseteq S_{999} \subseteq \{1, 2, \dots, 999\}.$$

Here $A \subseteq B$ means that all elements of A are also elements of B .

Answer: 10²⁹⁹⁷ OR 1000⁹⁹⁹ The idea is to look at each element individually, rather than each subset. For each $k \in \{1, 2, \dots, 999\}$, there are 1000 choices for the first subset in the chain that contains k . This count includes the possibility that k doesn't appear in any of the subsets. If S_i is the first subset containing k , for some $i \in \{1, 2, \dots, 999\}$, then k is also in S_j , for all $i < j \leq 999$. As a result, picking the first subset that contains k uniquely determines the appearance of k in all the subsets. It follows that there are 1000^{999} such subset chains.

21. [11] If you flip a fair coin 1000 times, what is the expected value of the product of the number of heads and the number of tails?

Answer: 249750 We solve the problem for n coins. We want to find

$$E(n) = \sum_{k=0}^n \frac{1}{2^n} \binom{n}{k} k(n-k).$$

We present three methods for evaluating this sum.

Method 1: Discard the terms $k = 0, k = n$. Since $\binom{n}{k} k(n-k) = n(n-1) \binom{n-2}{k-1}$ by the factorial definition, we may rewrite the sum as

$$E(n) = \frac{n(n-1)}{2^n} \cdot \sum_{k=1}^{n-1} \binom{n-2}{k-1}.$$

But clearly $\sum_{k=1}^{n-1} \binom{n-2}{k-1} = 2^{n-2}$, so the answer is $\frac{n(n-1)}{4}$.

Method 2: Let $\mathbb{E}[\cdot]$ denote expected value. Using linearity of expectation, we want to find the expected value of

$$\mathbb{E}[X(n - X)] = n\mathbb{E}[X] - \mathbb{E}[X^2]$$

where X is the number of heads. Moreover, we have

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

The variance of each individual coin flip is $\frac{1}{4}$, so $\text{Var}(X) = \frac{n}{4}$. Hence $\mathbb{E}[X^2] = \frac{1}{4}n^2 + \frac{n}{4}$. Consequently

$$\mathbb{E}[X(n - X)] = n \cdot \frac{n}{2} - \left(\frac{1}{4}n^2 + \frac{n}{4}\right) = \frac{n(n-1)}{4}.$$

Method 3: Differentiating the binomial theorem, we obtain

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (x + y)^n = \sum_{k=0}^n \frac{\partial}{\partial x} \frac{\partial}{\partial y} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^n \binom{n}{k} k(n-k) x^{k-1} y^{n-k-1}.$$

We also know that

$$\frac{\partial}{\partial x} \frac{\partial}{\partial y} (x + y)^n = n(n-1)(x + y)^{n-2}.$$

Plugging in $x = y = 1$, we find that $E(n) = \frac{n(n-1)}{4}$.

22. [12] Evaluate the infinite sum

$$\sum_{n=2}^{\infty} \log_2 \left(\frac{1 - \frac{1}{n}}{1 - \frac{1}{n+1}} \right).$$

Answer: $\boxed{-1}$ Using the identity $\log_2 \left(\frac{a}{b} \right) = \log_2 a - \log_2 b$, the sum becomes

$$\sum_{n=2}^{\infty} \log_2 \left(\frac{n-1}{n} \right) - \sum_{n=2}^{\infty} \log_2 \left(\frac{n}{n+1} \right).$$

Most of the terms cancel out, except the $\log_2 \left(\frac{1}{2} \right)$ term from the first sum. Therefore, the answer is -1 .

23. [12] Seven little children sit in a circle. The teacher distributes pieces of candy to the children in such a way that the following conditions hold.

- Every little child gets at least one piece of candy.
- No two little children have the same number of pieces of candy.
- The numbers of candy pieces given to any two adjacent little children have a common factor other than 1.
- There is no prime dividing every little child's number of candy pieces.

What is the smallest number of pieces of candy that the teacher must have ready for the little children?

Answer: $\boxed{44}$ An optimal arrangement is 2-6-3-9-12-4-8.

Note that at least two prime factors must appear. In addition, any prime factor that appears must appear in at least two non-prime powers unless it is not used as a common factor between any two adjacent little children. Thus with the distinctness condition we easily see that, if we are to beat 44, 5 and 7 cannot be included. More comparison shows that 12 or something higher cannot be avoided, so this is optimal.

24. [12] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. We construct isosceles right triangle ACD with $\angle ADC = 90^\circ$, where D, B are on the same side of line AC , and let lines AD and CB meet at F . Similarly, we construct isosceles right triangle BCE with $\angle BEC = 90^\circ$, where E, A are on the same side of line BC , and let lines BE and CA meet at G . Find $\cos \angle AGF$.

Answer: $\boxed{-\frac{5}{13}}$ We see that $\angle GAF = \angle GBF = 45^\circ$, hence quadrilateral $GFBA$ is cyclic. Consequently $\angle AGF + \angle FBA = 180^\circ$. So $\cos \angle AGF = -\cos \angle FBA$. One can check directly that $\cos \angle CBA = \frac{5}{13}$ (say, by the Law of Cosines).

25. [13] What is the smallest positive integer n which cannot be written in any of the following forms?

- $n = 1 + 2 + \dots + k$ for a positive integer k .
- $n = p^k$ for a prime number p and integer k .
- $n = p + 1$ for a prime number p .
- $n = pq$ for some distinct prime numbers p and q

Answer: $\boxed{40}$ The first numbers which are neither of the form p^k nor pq are 12, 18, 20, 24, 28, 30, 36, 40, \dots . Of these 12, 18, 20, 24, 30 are of the form $p + 1$ and 28, 36 are triangular. Hence the answer is 40.

26. [13] Consider a permutation $(a_1, a_2, a_3, a_4, a_5)$ of $\{1, 2, 3, 4, 5\}$. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *flawless* if for all $1 \leq i < j < k \leq 5$, the sequence (a_i, a_j, a_k) is *not* an arithmetic progression (in that order). Find the number of flawless 5-tuples.

Answer: $\boxed{20}$ We do casework on the position of 3.

- If $a_1 = 3$, then the condition is that 4 must appear after 5 and 2 must appear after 1. It is easy to check there are six ways to do this.
- If $a_2 = 3$, then there are no solutions; since there must be an index $i \geq 3$ with $a_i = 6 - a_1$.
- If $a_3 = 3$, then we must have $\{\{a_1, a_2\}, \{a_4, a_5\}\} = \{\{1, 5\}, \{2, 4\}\}$. It's easy to see there are $2^3 = 8$ such assignments.
- The case $a_4 = 3$ is the same as $a_2 = 3$, for zero solutions.
- The case $a_5 = 3$ is the same as $a_1 = 3$, for six solutions.

Hence, the total is $6 + 8 + 6 = 20$.

27. [13] In triangle ABC , let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2 , respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2 , respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2 , respectively.

If triangle ABC has sides of length 5, 12, and 13, find the area of the triangle determined by lines A_1C_2 , B_1A_2 and C_1B_2 .

Answer: $\boxed{\frac{6728}{3375}}$ By the definition of a parabola, we get $AA_1 = A_1B \sin B$ and similarly for the other points. So $\frac{AB_2}{AB} = \frac{AC_1}{AC}$, giving $B_2C_1 \parallel BC$, and similarly for the other sides. So DEF (WLOG, in that order) is similar to ABC . It suffices to scale after finding the length of EF , which is

$$B_2C_1 - B_2F - EC_1$$

The parallel lines also give us $B_2A_1F \sim BAC$ and so forth, so expanding out the ratios from these similarities in terms of sines eventually gives

$$\frac{EF}{BC} = \frac{2 \prod_{cyc} \sin A + \sum_{cyc} \sin A \sin B - 1}{\prod_{cyc} (1 + \sin A)}$$

Plugging in, squaring the result, and multiplying by $K_{ABC} = 30$ gives the answer.

28. [15] Let x be a complex number such that $x + x^{-1}$ is a root of the polynomial $p(t) = t^3 + t^2 - 2t - 1$. Find all possible values of $x^7 + x^{-7}$.

Answer: 2 Since $x + x^{-1}$ is a root,

$$\begin{aligned} 0 &= (x + x^{-1})^3 + (x + x^{-1})^2 - 2(x + x^{-1}) - 1 \\ &= x^3 + x^{-3} + 3x + 3x^{-1} + x^2 + 2 + x^{-2} - 2x - 2x^{-1} - 1 \\ &= x^3 + x^{-3} + x^2 + x^{-2} + x + x^{-1} + 1 \\ &= x^{-3} (1 + x + x^2 + \cdots + x^6). \end{aligned}$$

Since $x \neq 0$, the above equality holds only if x is a primitive seventh root of unity, i.e. $x^7 = 1$ and $x \neq 1$. Therefore, the only possible value of $x^7 + x^{-7}$ is $1 + 1 = 2$.

29. [15] Let ω be a fixed circle with radius 1, and let BC be a fixed chord of ω such that $BC = 1$. The locus of the incenter of ABC as A varies along the circumference of ω bounds a region \mathcal{R} in the plane. Find the area of \mathcal{R} .

Answer: $\pi \left(\frac{3-\sqrt{3}}{3} \right) - 1$ We will make use of the following lemmas.

Lemma 1: If ABC is a triangle with incenter I , then $\angle BIC = 90 + \frac{A}{2}$.

Proof: Consider triangle BIC . Since I is the intersection of the angle bisectors, $\angle IBC = \frac{B}{2}$ and $\angle ICB = \frac{C}{2}$. It follows that $\angle BIC = 180 - \frac{B}{2} - \frac{C}{2} = 90 + \frac{A}{2}$.

Lemma 2: If A is on major arc BC , then the circumcenter of $\triangle BIC$ is the midpoint of minor arc BC , and vice-versa.

Proof: Let M be the midpoint of minor arc BC . It suffices to show that $\angle BMC + 2\angle BIC = 360^\circ$, since $BM = MC$. This follows from Lemma 1 and the fact that $\angle BMC = 180 - \angle A$. The other case is similar.

Let O be the center of ω . Since BC has the same length as a radius, $\triangle OBC$ is equilateral. We now break the problem into cases depending on the location of A .

Case 1: If A is on major arc BC , then $\angle A = 30^\circ$ by inscribed angles. If M is the midpoint of minor arc BC , then $\angle BMC = 150^\circ$. Therefore, if I is the incenter of $\triangle ABC$, then I traces out a circular segment bounded by BC with central angle 150° , on the same side of BC as A .

Case 2: A similar analysis shows that I traces out a circular segment bounded by BC with central angle 30° , on the other side of BC .

The area of a circular segment of angle θ (in radians) is given by $\frac{1}{2}\theta R^2 - \frac{1}{2}R^2 \sin \theta$, where R is the radius of the circular segment. By the Law of Cosines, since $BC = 1$, we also have that $2R^2 - 2R^2 \cos \theta = 1$. Computation now gives the desired answer.

30. [15] Suppose we keep rolling a fair 2014-sided die (whose faces are labelled $1, 2, \dots, 2014$) until we obtain a value less than or equal to the previous roll. Let E be the expected number of times we roll the die. Find the nearest integer to $100E$.

Answer: 272 Let $n = 2014$. Let p_k denote the probability the sequence has length at least k . We observe that

$$p_k = \frac{\binom{n}{k}}{n^k}$$

since every sequence of k rolls can be sorted in exactly one way. Now the answer is

$$\sum_{k \geq 0} p_k = \left(1 + \frac{1}{n} \right)^n.$$

As $n \rightarrow \infty$, this approaches e . Indeed, one can check from here that the answer is 272.

31. [17] Flat Albert and his buddy Mike are watching the game on Sunday afternoon. Albert is drinking lemonade from a two-dimensional cup which is an isosceles triangle whose height and base measure 9cm and 6cm; the opening of the cup corresponds to the base, which points upwards. Every minute after the game begins, the following takes place: if n minutes have elapsed, Albert stirs his drink vigorously and takes a sip of height $\frac{1}{n^2}$ cm. Shortly afterwards, while Albert is busy watching the game, Mike adds cranberry juice to the cup until it's once again full in an attempt to create Mike's cranberry lemonade. Albert takes sips precisely every minute, and his first sip is exactly one minute after the game begins. After an infinite amount of time, let A denote the amount of cranberry juice that has been poured (in square centimeters). Find the integer nearest $\frac{27}{\pi^2}A$.

Answer: 26 Let $A_0 = \frac{1}{2}(6)(9) = 27$ denote the area of Albert's cup; since area varies as the square of length, at time n Mike adds

$$A \left(1 - \left(1 - \frac{1}{9n^2} \right)^2 \right)$$

whence in all, he adds

$$A_0 \sum_{n=1}^{\infty} \left(\frac{2}{9n^2} - \frac{1}{81n^4} \right) = \frac{2A_0\zeta(2)}{9} - \frac{A_0\zeta(4)}{81} = 6\zeta(2) - \frac{1}{3}\zeta(4)$$

where ζ is the Riemann zeta function. Since $\zeta(2) = \frac{\pi^2}{6}$ and $\zeta(4) = \frac{\pi^4}{90}$, we find that $A = \pi^2 - \frac{\pi^4}{270}$, so $\frac{27A}{\pi^2} = 27 - \frac{\pi^2}{10}$, which gives an answer 26.

Note that while the value of $\zeta(2)$ is needed to reasonable precision, we only need the fact that $0.5 < \frac{9}{\pi^2}\zeta(4) < 1.5$ in order to obtain a sufficiently accurate approximation. This is not hard to obtain because the terms of the expansion $\zeta(4)$ decrease rapidly.

32. [17] Let $f(x) = x^2 - 2$, and let f^n denote the function f applied n times. Compute the remainder when $f^{24}(18)$ is divided by 89.

Answer: 47 Let L_n denote the Lucas numbers given by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$. Note that $L_n^2 - 2 = L_{2n}$ when n is even (one can show this by induction, or explicitly using $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$). So, $f^{24}(L_6) = L_{3 \cdot 2^{25}}$.

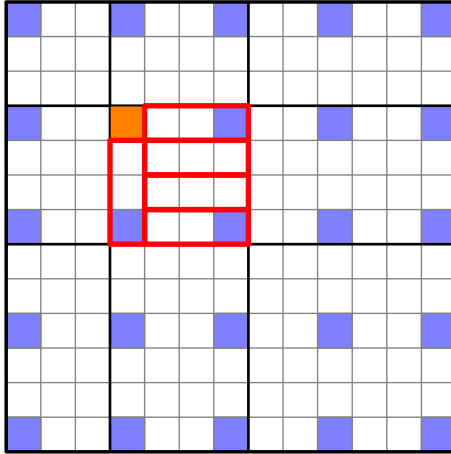
Now note that since $89 \equiv 4 \pmod{5}$, we have $5^{\frac{p-1}{2}} \equiv 1 \pmod{89}$ so $L_{89} = \left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p \equiv L_1 \pmod{89}$ and similarly $L_{90} \equiv L_2$, so the sequence $L_n \pmod{89}$ is periodic with period 88. (Alternatively, reason by analog of Fermat's little theorem, since we can substitute an integer residue for $\sqrt{5}$.)

We have $3 \cdot 2^{25} \equiv 3 \cdot 2^5 \equiv 8 \pmod{11}$ and $\equiv 0 \pmod{8}$, so $L_{3 \cdot 2^{25}} \equiv L_8 \pmod{89}$. Computing $L_8 = 47$ gives the answer.

33. [17] How many ways can you remove one tile from a 2014×2014 grid such that the resulting figure can be tiled by 1×3 and 3×1 rectangles?

Answer: 451584 Number the rows and columns of the grid from 0 to 2013, thereby assigning an ordered pair to each tile. We claim that a tile (i, j) may be selected if and only if $i \equiv j \equiv 0 \pmod{3}$; call such a square good.

First, let us show that this condition is sufficient. Observe that any such square s is the corner of a canonical 4×4 square S whose vertices are all good. Then the sides of S partition the board into nine distinct regions. It's easy to see that all of them can be suitably tiled.



Now we show that only good squares can be removed. Let ω be a non-real cube root of unity. In the tile with coordinates (i, j) , place the complex number ω^{i+j} . Note that any 1×3 or 3×1 rectangle placed on the grid must cover three squares with sum $1 + \omega + \omega^2 = 0$. Now, note that the sum of the numbers on the whole 2014×2014 grid, including the removed tile, is

$$\sum_{k=0}^{2013} \sum_{l=0}^{2013} \omega^{k+l} = \left(\sum_{k=0}^{2013} \omega^k \right)^2$$

which can be simplified to 1 using the identity $1 + \omega + \omega^2 = 0$. Therefore, it is necessary that $i + j \equiv 0 \pmod{3}$. By placing the complex number ω^{i-j} instead of ω^{i+j} , the same calculations show that $i - j \equiv 0 \pmod{3}$ is necessary. This can only occur if $i \equiv j \equiv 0 \pmod{3}$.

Hence the answer is exactly the set of good squares, of which there are $672^2 = 451584$.

34. [20] Let M denote the number of positive integers which divide $2014!$, and let N be the integer closest to $\ln(M)$. Estimate the value of N . If your answer is a positive integer A , your score on this problem will be the larger of 0 and $\lfloor 20 - \frac{1}{8}|A - N| \rfloor$. Otherwise, your score will be zero.

Answer: 439 Combining Legendre's Formula and the standard prime approximations, the answer is

$$\prod_p \left(1 + \frac{2014 - s_p(2014)}{p - 1} \right)$$

where $s_p(n)$ denotes the sum of the base p -digits of n .

Estimate $\ln 1000 \approx 8$, and $\ln 2014 \approx 9$. Using the Prime Number Theorem or otherwise, one might estimate about 150 primes less than 1007 and 100 primes between 1008 and 2014. Each prime between 1008 and 2014 contributes exactly $\ln 2$. For the other 150 primes we estimate $\ln 2014/p$ as their contribution, which gives $\sum_{p < 1000} (\ln 2014 - \ln p)$. Estimating the average $\ln p$ for $p < 1000$ to be $\ln 1000 - 1 \approx 7$ (i.e. an average prime less than 1000 might be around $1000/e$), this becomes $150 \cdot 2 = 300$. So these wildly vague estimates give $300 + 150 \ln 2 \approx 400$, which is not far from the actual answer.

The following program in Common Lisp then gives the precise answer of 438.50943.

```
;;; First, generate a list of all the primes
(defconstant +MAXP+ 2500)
(defun is-prime (p)
  (loop for k from 2 to (isqrt p) never (zerop (mod p k))))
(defparameter *primes* (loop for p from 2 to +MAXP+
  if (is-prime p) collect p))

;;; Define NT functions
```

```

(defconstant +MAXDIGITS+ 15)
(defun base-p-digit (p i n)
  (mod (truncate n (expt p i)) p))
(defun sum-base-p-digit (p n)
  (loop for i from 0 to +MAXDIGITS+ sum (base-p-digit p i n)))
(defun vp-n-factorial (p n)
  (/ (- n (sum-base-p-digit p n)) (1- p)))

;;; Compute product
(princ (loop for p in *primes*
  sum (log (1+ (vp-n-factorial p 2014))))))

```

35. [20] Ten points are equally spaced on a circle. A *graph* is a set of segments (possibly empty) drawn between pairs of points, so that every two points are joined by either zero or one segments. Two graphs are considered the same if we can obtain one from the other by rearranging the points.

Let N denote the number of graphs with the property that for any two points, there exists a path from one to the other among the segments of the graph. Estimate the value of N . If your answer is a positive integer A , your score on this problem will be the larger of 0 and $\lfloor 20 - 5 |\ln(A/N)| \rfloor$. Otherwise, your score will be zero.

Answer: 11716571 The question asks for the number of isomorphism classes of connected graphs on 10 vertices. This is enumerated in <http://oeis.org/A001349>; the answer is 11716571.

In fact, of the $2^{45} \approx 3.51 \cdot 10^{13} \approx 3 \cdot 10^{13}$ graphs on 10 labelled vertices, virtually all (about $3.45 \cdot 10^{13}$) are connected. You might guess this by noticing that an “average” graph has 22.5 edges, which is fairly dense (and virtually all graphs with many edges are connected). Moreover, a “typical” isomorphism class contains $10! \approx 3 \cdot 10^6$ elements, one for each permutation of the vertices. So estimating the quotient $\frac{3 \cdot 10^{13}}{3 \cdot 10^6} = 10^7$ gives a very close estimate.

36. [20] Pick a subset of at least four of the following geometric theorems, order them from earliest to latest by publication date, and write down their **labels** (a single capital letter) in that order. If a theorem was discovered multiple times, use the publication date corresponding to the geometer for which the theorem is named.

C. (**Ceva**) Three cevians AD , BE , CF of a triangle ABC are concurrent if and only if $\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = 1$.

E. (**Euler**) In a triangle ABC with incenter I and circumcenter O , we have $IO^2 = R(R - 2r)$, where r is the inradius and R is the circumradius of ABC .

H. (**Heron**) The area of a triangle ABC is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

M. (**Menelaus**) If D , E , F lie on lines BC , CA , AB , then they are collinear if and only if $\frac{BD}{DC} \frac{CE}{EA} \frac{AF}{FB} = -1$, where the ratios are directed.

P. (**Pascal**) Intersections of opposite sides of cyclic hexagons are collinear.

S. (**Stewart**) Let ABC be a triangle and D a point on BC . Set $m = BD$, $n = CD$, $d = AD$. Then $man + dad = bmb + cnc$.

V. (**Varignon**) The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

If your answer is a list of $4 \leq N \leq 7$ labels in a correct order, your score will be $(N-2)(N-3)$. Otherwise, your score will be zero.

Answer: HMPCVSE The publication dates were as follows.

- Heron: 60 AD, in his book *Metrica*.
- Menelaus: We could not find the exact date the theorem was published in his book *Spherics*, but because Menelaus lived from 70AD to around 130AD, this is the correct placement.
- Pascal: 1640 AD, when he was just 17 years old. He wrote of the theorem in a note one year before that.

- Ceva: 1678 AD, in his work *De lineis rectis*. But it was already known at least as early as the 11th century.
- Varignon: 1731 AD.
- Stewart: 1746 AD.
- Euler: 1764 AD, despite already being published in 1746.

HMMT November 2014
Saturday 15 November 2014
Team Round

1. [3] What is the smallest positive integer n which cannot be written in any of the following forms?

- $n = 1 + 2 + \cdots + k$ for a positive integer k .
- $n = p^k$ for a prime number p and integer k .
- $n = p + 1$ for a prime number p .

Answer: 22 Consider $1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19$ are in the form p^k . So we are left with $6, 10, 12, 14, 15, 18, 20, 21, 22, \dots$

Next, $6, 12, 14, 18, 20$ are in the form $p + 1$, so we are left with $10, 15, 21, 22, \dots$

Finally, $10, 15, 21$ are in the form $n = 1 + 2 + \cdots + k$, so we are left with $22, \dots$

Since $22 = 2 \cdot 11$ is not a prime power, $22 - 1 = 21$ is not prime, and $1 + 2 + \cdots + 6 = 21 < 22 < 28 = 1 + 2 + \cdots + 7$, 22 is the smallest number not in the three forms, as desired.

2. [5] Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x .

Answer: 23 Consider that $f(x) = x^2 + 6x + 7 = (x + 3)^2 - 2$. So $f(x) \geq -2$ for real numbers x . Also, f is increasing on the interval $[-3, \infty)$. Therefore

$$f(f(x)) \geq f(-2) = -1,$$

$$f(f(f(x))) \geq f(-1) = 2,$$

and

$$f(f(f(f(x)))) \geq f(2) = 23.$$

Thus, the minimum value of $f(f(f(f(x))))$ is 23 and equality is obtained when $x = -3$.

3. [5] The side lengths of a triangle are distinct positive integers. One of the side lengths is a multiple of 42 , and another is a multiple of 72 . What is the minimum possible length of the third side?

Answer: 7 Suppose that two of the side lengths are $42a$ and $72b$, for some positive integers a and b . Let c be the third side length. We know that $42a$ is not equal to $72b$, since the side lengths are distinct. Also, $6 \mid 42a - 72b$. Therefore, by the triangle inequality, we get $c > |42a - 72b| \geq 6$ and thus $c \geq 7$. Hence, the minimum length of the third side is 7 and equality is obtained when $a = 7$ and $b = 4$.

4. [3] How many ways are there to color the vertices of a triangle red, green, blue, or yellow such that no two vertices have the same color? Rotations and reflections are considered distinct.

Answer: 24 There are 4 ways to color the first vertex, then 3 ways to color the second vertex to be distinct from the first, and finally 2 ways to color the third vertex to be distinct from the earlier two vertices. Multiplying gives 24 ways.

5. [5] Let A, B, C, D, E be five points on a circle; some segments are drawn between the points so that each of the $\binom{5}{2} = 10$ pairs of points is connected by either zero or one segments. Determine the number of sets of segments that can be drawn such that:

- It is possible to travel from any of the five points to any other of the five points along drawn segments.
- It is possible to divide the five points into two nonempty sets S and T such that each segment has one endpoint in S and the other endpoint in T .

Answer: 195 First we show that we can divide the five points into sets S and T according to the second condition in only one way. Assume that we can divide the five points into $S \cup T$ and $S' \cup T'$. Then, let $A = S' \cap S, B = S' \cap T, C = T' \cap S$, and $D = T' \cap T$. Since S, T and S', T' partition the set of five points, A, B, C, D also partition the set of five points.

Now, according to the second condition, there can only be segments between S and T and between S' and T' . Therefore, the only possible segments are between points in A and D , or between points in B and C . Since, according to the first condition, the points are all connected via segments, it must be that $A = D = \emptyset$ or $B = C = \emptyset$. If $A = D = \emptyset$, then it follows that $S' = T$ and $T' = S$. Otherwise, if $B = C = \emptyset$, then $S' = S$ and $T' = T$. In either case, S, T and S', T' are the same partition of the five points, as desired.

We now determine the possible sets of segments with regard to the sets S and T .

Case 1: the two sets contain 4 points and 1 point. Then, there are $\binom{5}{1} = 5$ ways to partition the points in this manner. Moreover, the 1 point (in its own set) must be connected to each of the other 4 points, and these are the only possible segments. Therefore, there is only 1 possible set of segments, which, combining with the 5 ways of choosing the sets, gives 5 possible sets of segments.

Case 2: the two sets contain 3 points and 2 points. Then, there are $\binom{5}{2} = 10$ ways to partition the points in this manner. Let S be the set containing 3 points and T the set containing 2 points. We consider the possible degrees of the points in T .

- If both points have degree 3, then each point must connect to all points in S , and the five points are connected via segments. So the number of possible sets of segments is 1.
- If the points have degree 3 and 2. Then, we can swap the points in 2 ways, and, for the point with degree 2, we can choose the elements of S it connects to in $\binom{3}{2} = 3$ ways. In each case, the five points are guaranteed to be connected via segments. Hence 6 ways.
- If the points have degree 3 and 1. Similarly, we can swap the points in 2 ways and connect the point with degree 1 to the elements of S in $\binom{3}{1} = 3$ ways. Since all five points are connected in all cases, we have 6 ways.
- If both points have degree 2. Then, in order for the five points to be connected, the two points must connect to a common element of S . Call this common element A . Then, for the other two elements of S , each one must be connected to exactly one element of T . We can choose A in 3 ways, and swap the correspondence between the other two elements of S with the elements of T in 2 ways. Hence 6 ways.
- If the points have degree 2 and 1. Then, in order to cover S , the point with degree 2 must connect to 2 points in S , and the point with degree 1 to the remaining point in S . But then, the five points will not be connected via segments, an impossibility.
- If both points have degree 1. Then, similar to the previous case, it is impossible to cover all the 3 points in S with only 2 segments, a contradiction.

Combining the subcases, we have $1 + 6 + 6 + 6 = 19$ possible sets of segments with regard to a partition. With 10 possible partitions, we have a total of $19 \cdot 10 = 190$ possible sets of segments.

Finally, combining this number with the 5 possibilities from case 1, we have a total of $5 + 190 = 195$ possibilities, as desired.

6. [6] Find the number of strictly increasing sequences of nonnegative integers with the following properties:
- The first term is 0 and the last term is 12. In particular, the sequence has at least two terms.
 - Among any two consecutive terms, exactly one of them is even.

Answer: 144 For a natural number n , let A_n be a set containing all sequences which satisfy the problem conditions but which 12 is replaced by n . Also, let a_n be the size of A_n .

We first consider a_1 and a_2 . We get $a_1 = 1$, as the only sequence satisfying the problem conditions is $0, 1$. We also get $a_2 = 1$, as the only possible sequence is $0, 1, 2$.

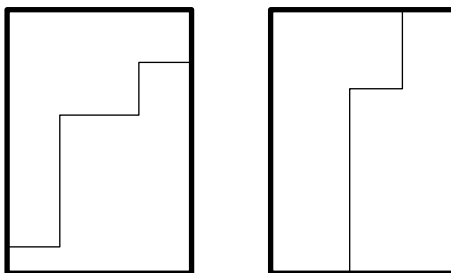
Next, we show that $a_{n+2} = a_{n+1} + a_n$ for all natural number n . We consider the second-to-last terms of each sequence in A_{n+2} .

Case 1. The second-to-last term is $n + 1$. When we leave out the last term, the remaining sequence will still satisfy the problem conditions, and hence is in A_{n+1} . Conversely, for a sequence in A_{n+1} , we could add $n + 2$ at the end of that sequence, and since $n + 1$ and $n + 2$ have different parities, the resulting sequence will be in A_{n+2} . Therefore, there is a one-to-one correspondence between the sequences in this case and the sequences in A_{n+1} . So the number of sequences in this case is a_{n+1} .

Case 2. The second-to-last term is less than or equal n . But n and $n + 2$ have the same parity, so the second-to-last term cannot exceed $n - 1$. When we substitute the last term ($n + 2$) with n , the resulting sequence will satisfy the problem conditions and will be in A_n . Conversely, for a sequence in A_n , we could substitute its last term n , with $n + 2$. As n and $n + 2$ have the same parity, the resulting sequence will be in A_n . Hence, in this case, the number of sequences is a_n .

Now, since $a_{n+2} = a_{n+1} + a_n$ for all natural numbers n , we can recursively compute that the number of all possible sequences having their last terms as 12 is $a_{12} = 144$. Note that the resulting sequence (a_n) is none other than the Fibonacci numbers.

7. [7] Sammy has a wooden board, shaped as a rectangle with length 2^{2014} and height 3^{2014} . The board is divided into a grid of unit squares. A termite starts at either the left or bottom edge of the rectangle, and walks along the gridlines by moving either to the right or upwards, until it reaches an edge opposite the one from which the termite started. Depicted below are two possible paths of the termite.



The termite's path dissects the board into two parts. Sammy is surprised to find that he can still arrange the pieces to form a new rectangle not congruent to the original rectangle. This rectangle has perimeter P . How many possible values of P are there?

Answer: [4] Let R be the original rectangle and R' the new rectangle which is different from R . We see that the perimeter of R' depends on the possibilities for the side lengths of R' .

We will prove that the dividing line must have the following characterization: starting from the lower left corner of R , walk to the right by distance a , then walk up by distance b , for some positive number a and b , and repeat the two steps until one reaches the upper right corner of R , with the condition that the last step is a walk to the right. (The directions stated here depends on the orientation of R , but we can always orient R so as to fit the description.) Let there be $n + 1$ walks to the right and n walks to the top, then we have that this division would rearrange a rectangle of dimension $(n + 1)a \times nb$ into a rectangle of dimension $na \times (n + 1)b$.

Let us first assume the above. Now, according to the problem, it suffices to find n, a, b such that $(n + 1)a = 2^{2014}, nb = 3^{2014}$ or $(n + 1)a = 3^{2014}, nb = 2^{2014}$. This means that $n + 1$ and n are a power of 3 and a power of 2, whose exponents do not exceed 2014. This corresponds to finding nonnegative integers $k, l \leq 2014$ such that $|2^k - 3^l| = 1$. The only possible pairs of $(2^k, 3^l)$ are $(2, 1), (2, 3), (3, 4)$ and $(8, 9)$. So there are 4 possible configurations of R' .

Now, we prove our claim. For completeness, we will actually prove the claim more generally for any cut, not just ones that move right and up (hence the length of the solution which follows, but only the above two paragraphs are relevant for the purposes of finding the answer).

First we show that the dividing boundary between the two pieces must meet the boundary of R at two points, each being on opposite sides of R as the other. To see why, consider that otherwise, there would be two consecutive sides of R which belong to the same piece. Then, the smallest rectangle containing such a configuration must have each side being as large as each of the two sides, and thus it is R . Since this piece is also part of R' , R' must contain R , but their areas are equal, so $R' = R$, a contradiction.

Now, let the dividing boundary go from the top side to the bottom side of R , and call the right piece "piece 1" and the left piece "piece 2." We orient R' in such a way that piece 1 is fixed and piece 2 is moved from the original position in some way to create R' . We will show that piece 2 must be moved by translation by some vector v , t_v . Otherwise, piece 2 is affected by t_v as well as a rotation by 90° or 180° . We show that these cases are impossible.

First, consider the case where there is a 90° rotation. Let the distance from the top side to the bottom side of R be x . Then, the two pieces are contained between a pair of horizontal lines which are of distance x apart from one another. If piece 2 is rotated by 90° , then these horizontal lines become a pair of *vertical* lines which are of distance x apart from one another. So R' is contained within a union of regions between a pair of horizontal lines and a pair of vertical lines.

Now, we show that R' must be contained within only *one* of these regions. Consider if there exists points (x_1, y_1) and (x_2, y_2) in R' such that (x_1, y_1) is not in the horizontal region (so that y_1 is out of range) and (x_2, y_2) is not in the vertical region (so that x_2 is out of range). Then, it follows that (x_2, y_1) is also in the rectangle R' . But (x_2, y_1) cannot be contained in either region, since both of its x and y coordinates are out of range, a contradiction.

So let us assume, without loss of generality, that R' is contained in the vertical region (the one which contains piece 2). Then, the horizontal side of R' cannot have length greater than x , the width of the region. However, piece 2 is contained in the region, and its width is *exactly* x . Therefore, the width of R' must be exactly x , rendering it to be the same shape as R , a contradiction.

Next, we show that the case with a 180° rotation is also impossible. We modify our considerations from the previous case by considering a *half*-region of the region between a pair of horizontal lines (which are still of distance x apart), which we define as a part of the region on the right or on the left of a certain vertical line. Then, piece 2 is contained within a certain half-region going to the *right* and piece 1 is contained within a certain half-region going to the *left*. Now, in R' , since piece 2 is rotated by 180° , we would have both half-regions going to the left, and R' is contained within a union of them.

Now, consider the "end" of each half-region (the part of the boundary that is vertical). The ends of both half-regions must be contained in R' , since they are part of piece 1 and piece 2. Consider a vector that maps the end of one half-region to the other. If the vector is horizontal, then the union of the regions have vertical distance x . Similarly to the previous case, we deduce that the vertical side of R' must be of length no more than x , and so must be exactly x , but then $R' = R$, a contradiction.

Now, if the vector has both nonzero horizontal and vertical components, then the parallelogram generated by the locus of the end of a half-region being translated by the vector to the end of the other half-region must be contained within R' (since R' is convex). However, the parallelogram is not contained within the union of the two half-regions, a contradiction.

Finally, if the vector is vertical, then the two half-regions must be on top of one another, and so will have no region in common. Then, since R' is a rectangle, the intersection of R' with each half-region will also be a rectangle. So pieces 1 and 2 must be rectangles. But then a rotation of 180° would map piece 2 to itself. So this reduces to a case of pure translation.

We now consider the translation t_v by a vector v on piece 2. Since R' must contain the ends of the half-regions (which retain their original orientations), the vertical side of R' must be at least of length x . But $R' \neq R$, so the vertical side of R' has length strictly greater than x . This implies that the horizontal side of R' must be strictly shorter than that of R , since they have equal area. However, the horizontal side of R' is at least as long as the horizontal distance between the ends of the two half-regions, so the ends of the two half-regions must have moved closer to one another horizontally.

This implies that the vector v has a positive x component. Also, v cannot be entirely horizontal, because there is no more space for piece 2 to move into. So v has a nonzero y component. Without loss of generality, let us assume that v has a positive y component.

Before we continue further, let us label the vertices of R as A, B, C, D , going in counter-clockwise direction, with the left side of R being AB and the right side of R being CD . So AB is in piece 2 and CD is in piece 1. Call the translated piece 2 that is part of the rectangle R' piece 2', with the corresponding points A' and B' .

Now, consider the half-regions of piece 1 and piece 2. They are half-regions with the end AB going to the right and the one with the end CD going to the left. So, in R' , the half-regions are with end $A'B'$ going to the right and with end CD going to the left, and R' is contained within the union of these two. Now, there cannot be a point in R' that is to the left of $A'B'$, since the smallest rectangle containing that point and A' would not be contained in the union of the two half-regions. Similarly, there cannot be a point in R' that is to the right of CD for the same reason. These restrictions imply that R' must be contained within the union of the two half-regions that lie horizontally between $A'B'$ and CD . However, since the smallest rectangle containing A' and C is precisely this region, R' must be this region. Let $A'B'$ intersect BC at L and let CD intersect the line passing through A' which is parallel to AD at M . We have $R' = A'LCM$.

Now, consider the segments BL and LB' . We know that BL is a boundary of piece 2. Also, since LB' is a boundary of R' and it is below piece 2', it cannot be a boundary of piece 2'. Therefore, it must be a boundary of piece 1. Since LB' is not a boundary of R but is a boundary of piece 1, it must be part of the dividing boundary between piece 1 and 2, and so must also be a boundary of piece 2.

We now prove that: from a sequence of B, B', B'', \dots , each being translated from the preceding one by v , one of them must eventually lie on AD . Also, if L, L', L'', \dots is also a sequence of L being successively translated by v , then, using B_i and L_i to designate the i th term of each sequence: B_iL_i and L_iB_{i+1} must be part of the boundary of piece 2 for all $i \leq N-1$, where B_N is the last point, the one that lies on AD .

We have already proved the assertion for $i = 1$. We now set out to prove, by induction, that B_iL_i and L_iB_{i+1} must be part of the boundary of piece 2 for all i such that B_{i+1} is still within R or on the edge of R .

Consider, by induction hypothesis, that $B_{i-1}L_{i-1}$ and $L_{i-1}B_i$ are parts of the boundary of piece 2. Then, by mapping, B_iL_i and L_iB_{i+1} must be parts of the boundary of piece 2'. Since the dividing boundary of R go from top to bottom, L_iB_{i+1} cannot be on the rightmost edge of R , which is also the rightmost edge of R' . This means that L_iB_{i+1} is not a boundary of R' . So we have that B_iL_i and L_iB_{i+1} are not boundaries of R' but are boundaries of piece 2', so they must be boundaries of piece 1. Now, since they are not boundaries of R either but are boundaries of piece 1, they must be boundaries of piece 2, as desired.

Now we are left to show that one of B_i must lie exactly on AD . Let B_i be the last term of the sequence that is contained within R or its boundary. Then, by the previous result, $B_{i-1}L_{i-1}$ and $L_{i-1}B_i$ are boundaries of piece 2. Then, by mapping, B_iL_i is a boundary of piece 2'. Since B_iL_i is on or below AD , it cannot be on the boundary of R' , but since it is a boundary of piece 2', it must also be a boundary of piece 1. Now, if B_i is not on AD , then B_iL_i will not be a boundary of R , but since it is a boundary of piece 1, it must also be a boundary of piece 2. By mapping, this implies that $B_{i+1}L_{i+1}$ must be a boundary of piece 2'. However, $B_{i+1}L_{i+1}$ is not contained within R or its boundary, and so cannot be on piece 1's boundary. Therefore, since $B_{i+1}L_{i+1}$ is a boundary of piece 2' but not of piece 1, it must be a boundary of R' . This means that $B_{i+1}L_{i+1}$ is on the upper edge of R' . Mapping back, we get that B_iL_i must be on the upper edge of R , a contradiction to the assumption that B_i is not on AD (the upper edge of R). So B_i is on AD , as desired.

Let B_N be the term of the sequence that is on AD . We have now shown that

$$L_1B_2, B_2L_2, \dots, B_{N-1}L_{N-1}, L_{N-1}B_N$$

completely defines the dividing line between piece 1 and piece 2. Moreover, $B_1 = B$, defining the starting point of the dividing line. We now add the final description: $L_N = D$. To see why, note that

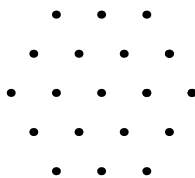
L_N must be on the upper edge of R , as it is in the same horizontal level as B_N . However, since L_{N-1} is one of furthest points to the right of piece 2, by mapping, L_N must be one of the furthest points to the right of piece 2', and so must be on the rightmost edge of piece 2', which is the rightmost edge of R' and of R . Therefore, L_N is on the upper edge and the rightmost edge of R , and so it must be D , as desired.

8. [3] Let \mathcal{H} be a regular hexagon with side length one. Peter picks a point P uniformly and at random within \mathcal{H} , then draws the largest circle with center P that is contained in \mathcal{H} . What is this probability that the radius of this circle is less than $\frac{1}{2}$?

Answer: $\boxed{\frac{2\sqrt{3}-1}{3}}$ We first cut the regular hexagon \mathcal{H} by segments connecting its center to each vertex into six different equilateral triangles with side lengths 1. Therefore, each point inside \mathcal{H} is contained in some equilateral triangle. We first see that for each point inside an equilateral triangle, the radius of the largest circle with center P which is contained in \mathcal{H} equals the shortest distance from P to the nearest side of the hexagon, which is also a side of the triangle in which it is contained.

Consider that the height of each triangle is $\frac{\sqrt{3}}{2}$. Therefore, the region inside the triangle containing all points with distance more than $\frac{1}{2}$ to the side of the hexagon is an equilateral triangle with a height of $\frac{\sqrt{3}-1}{2}$. Consequently, the area inside the triangle containing all points with distance less than $\frac{1}{2}$ to the side of the hexagon has area $\frac{\sqrt{3}}{4} \left(1 - \left(\frac{\sqrt{3}-1}{2}\right)^2\right) = \frac{\sqrt{3}}{4} \cdot \left(\frac{2\sqrt{3}-1}{3}\right)$. This is of the ratio $\frac{2\sqrt{3}-1}{3}$ to the area of the triangle, which is $\frac{\sqrt{3}}{4}$. Since all triangles are identical and the point P is picked uniformly within \mathcal{H} , the probability that the radius of the largest circle with center P which is contained in \mathcal{H} is less than $\frac{1}{2}$ is $\frac{2\sqrt{3}-1}{3}$, as desired.

9. [5] How many lines pass through exactly two points in the following hexagonal grid?



Answer: $\boxed{60}$ *First solution.* From a total of 19 points, there are $\binom{19}{2} = 171$ ways to choose two points. We consider lines that pass through more than 2 points.

- There are $6 + 6 + 3 = 15$ lines that pass through exactly three points. These are: the six sides of the largest hexagon, three lines through the center (perpendicular to the sides of the largest hexagon), and the other six lines perpendiculars to the sides of the largest hexagon.
- There are 6 lines that pass through exactly four points. (They are parallel to the sides of the largest hexagon.)
- There are 3 lines that pass through exactly five points. (They all pass through the center.)

For each $n = 3, 4, 5$, a line that passes through n points will be counted $\binom{n}{2}$ times, and so the corresponding amount will have to be subtracted. Hence the answer is

$$171 - \binom{3}{2} \cdot 15 - \binom{4}{2} \cdot 6 - \binom{5}{2} \cdot 3 = 171 - 45 - 36 - 30 = 60.$$

Second solution. We divide the points into 4 groups as follows.

- Group 1 consists of the center point.
- Group 2 consists of the 6 points surrounding the center.
- Group 3 consists of the 6 vertices of the largest hexagon.

- Group 4 consists of the 6 midpoints of the sides of the largest hexagon.

We wish to count the number of lines that pass through exactly 2 points. Consider: all lines connecting points in group 1 and 2, 1 and 3, and 1 and 4 pass through more than 2 points. So it is sufficient to restrict our attention to group 2, 3 and 4.

- For lines connecting group 2 and 2, the only possibilities are those that the two endpoints are 120 degrees apart with respect to the center, so 6 possibilities.
- For lines connecting group 3 and 3, it is impossible.
- For lines connecting group 4 and 4, the two endpoints must be 60 degrees apart with respect to the center, so 6 possibilities.
- For lines connecting group 3 and 2. For each point in group 3, the only possible points in group 2 are those that are 120 degrees apart from the point in group 3. So $2 \cdot 6 = 12$ possibilities.
- For lines connecting group 4 and 2, the endpoints must be 150 degrees apart with respect to the center, so $2 \cdot 6 = 12$ possibilities.
- For lines connecting group 4 and 3. For each point in group 4, any point in group 3 works except those that are on the side on the largest hexagon of which the point in group 4 is the midpoint. Hence $4 \cdot 6 = 24$ possibilities.

Therefore, the number of lines passing through 2 points is $6 + 6 + 12 + 12 + 24 = 60$, as desired.

10. [8] Let $ABCDEF$ be a convex hexagon with the following properties.

- \overline{AC} and \overline{AE} trisect $\angle BAF$.
- $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$.
- $AB = 2AC = 4AE = 8AF$.

Suppose that quadrilaterals $ACDE$ and $ADEF$ have area 2014 and 1400, respectively. Find the area of quadrilateral $ABCD$.

Answer: 7295 From conditions (a) and (c), we know that triangles AFE , AEC and ACB are similar to one another, each being twice as large as the preceding one in each dimension. Let $\overline{AE} \cap \overline{FC} = P$ and $\overline{AC} \cap \overline{EB} = Q$. Then, since the quadrilaterals $AFEC$ and $AECB$ are similar to one another, we have $AP : PE = AQ : QC$. Therefore, $\overline{PQ} \parallel \overline{EC}$.

Let $\overline{PC} \cap \overline{QE} = T$. We know by condition (b) that $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$. Therefore, triangles PQT and ECD have their three sides parallel to one another, and so must be similar. From this we deduce that the three lines joining the corresponding vertices of the two triangles must meet at a point, i.e., that PE, TD, QC are concurrent. Since PE and QC intersect at A , the points A, T, D are collinear. Now, because $TCDE$ is a parallelogram, \overline{TD} bisects \overline{EC} . Therefore, since A, T, D are collinear, \overline{AD} also bisects \overline{EC} . So the triangles ADE and ACD have equal area.

Now, since the area of quadrilateral $ACDE$ is 2014, the area of triangle ADE is $2014/2 = 1007$. And since the area of quadrilateral $ADEF$ is 1400, the area of triangle AFE is $1400 - 1007 = 393$. Therefore, the area of quadrilateral $ABCD$ is $16 \cdot 393 + 1007 = 7295$, as desired.

HMMT November 2014
Saturday 15 November 2014
Theme Round

1. Find the probability that the townspeople win if there are initially two townspeople and one goon.

Answer: $\boxed{\frac{1}{3}}$ The goon is chosen on the first turn with probability $\frac{1}{3}$, and this is necessary and sufficient for the townspeople to win.

2. Find the smallest positive integer n such that, if there are initially $2n$ townspeople and 1 goon, then the probability the townspeople win is greater than 50%.

Answer: $\boxed{3}$ We instead consider the probability the goon wins. The game clearly must last n days. The probability the goon is not sent to jail on any of these n days is then

$$\frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3}.$$

If $n = 2$ then the probability the goon wins is $\frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15} > \frac{1}{2}$, but when $n = 3$ we have $\frac{6}{7} \cdot \frac{8}{15} = \frac{16}{35} < \frac{1}{2}$, so the answer is $n = 3$.

Alternatively, let p_n be the probability that $2n$ townspeople triumph against 1 goon. There is a $\frac{1}{2n+1}$ chance that the goon is jailed during the first morning and the townspeople win. Otherwise, the goon eliminates one townsperson during the night. We thus have $2n - 2$ townspeople and 1 goon left, so the probability that the town wins is p_{n-1} . We obtain the recursion

$$p_n = \frac{1}{2n+1} + \frac{2n}{2n+1}p_{n-1}.$$

By the previous question, we have the initial condition $p_1 = \frac{1}{3}$. We find that $p_2 = \frac{7}{15} < \frac{1}{2}$ and $p_3 = \frac{19}{35} > \frac{1}{2}$, yielding $n = 3$ as above.

3. Find the smallest positive integer n such that, if there are initially $n + 1$ townspeople and n goons, then the probability the townspeople win is less than 1%.

Answer: $\boxed{6}$ By a similar inductive argument, the probability for a given n is

$$p_n = \frac{n!}{(2n+1)!!}.$$

Clearly this is decreasing in n . It is easy to see that

$$p_5 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 5 \cdot 7 \cdot 9 \cdot 11} = \frac{8}{693} > 0.01$$

and

$$p_6 = \frac{6}{13}p_5 = \frac{48}{693 \cdot 13} < 0.01.$$

Hence the answer is $n = 6$. Heuristically, $p_{n+1} \approx \frac{1}{2}p_n$ for each n , so arriving at these estimates for the correct answer of n is not difficult.

4. Suppose there are initially 1001 townspeople and two goons. What is the probability that, when the game ends, there are exactly 1000 people in jail?

Answer: $\boxed{\frac{3}{1003}}$ By considering the parity of the number of people in jail, we see that this situation arises if and only if the goons win after the 500th night. That means that at this point we must have exactly one townsperson and two goons remaining. In other words, this situation arises if and only if no goon is ever sent to jail. The probability that this occurs is

$$\frac{1001}{1003} \cdot \frac{999}{1001} \cdot \frac{997}{999} \cdots \frac{3}{5} = \frac{3}{1003}.$$

5. Suppose that there are initially eight townspeople and one goon. One of the eight townspeople is named *Jester*. If *Jester* is sent to jail during some morning, then the game ends immediately in his sole victory. (However, the *Jester* does not win if he is sent to jail during some night.)

Find the probability that only the *Jester* wins.

Answer: $\boxed{\frac{1}{3}}$ Let a_n denote the answer when there are $2n - 1$ regular townies, one *Jester*, and one goon. It is not hard to see that $a_1 = \frac{1}{3}$. Moreover, we have a recursion

$$a_n = \frac{1}{2n+1} \cdot 1 + \frac{1}{2n+1} \cdot 0 + \frac{2n-1}{2n+1} \left(\frac{1}{2n-1} \cdot 0 + \frac{2n-2}{2n-1} \cdot a_{n-1} \right).$$

The recursion follows from the following consideration: during the day, there is a $\frac{1}{2n+1}$ chance the *Jester* is sent to jail and a $\frac{1}{2n+1}$ chance the goon is sent to jail, at which point the game ends. Otherwise, there is a $\frac{1}{2n-1}$ chance that the *Jester* is selected to be jailed from among the townies during the evening. If none of these events occur, then we arrive at the situation of a_{n-1} .

Since $a_1 = \frac{1}{3}$, we find that $a_n = \frac{1}{3}$ for all values of n . This gives the answer.

6. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be pairwise distinct parabolas in the plane. Find the maximum possible number of intersections between two or more of the \mathcal{P}_i . In other words, find the maximum number of points that can lie on two or more of the parabolas $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$.

Answer: $\boxed{12}$ Note that two distinct parabolas intersect in at most 4 points, which is not difficult to see by drawing examples.¹ Given three parabolas, each pair intersects in at most 4 points, for at most $4 \cdot 3 = 12$ points of intersection in total. It is easy to draw an example achieving this maximum, for example, by slanting the parabolas at different angles.

7. Let \mathcal{P} be a parabola with focus F and directrix ℓ . A line through F intersects \mathcal{P} at two points A and B . Let D and C be the feet of the altitudes from A and B onto ℓ , respectively. Given that $AB = 20$ and $CD = 14$, compute the area of $ABCD$.

Answer: $\boxed{140}$ Observe that $AD + BC = AF + FB = 20$, and that $ABCD$ is a trapezoid with height $BC = 14$. Hence the answer is $\frac{1}{2}(AD + BC)(14) = 140$.

8. Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$(y + x) = (y - x)^2 + 3(y - x) + 3.$$

Find the minimum possible value of y .

Answer: $\boxed{-\frac{1}{2}}$ Let $w = y - x$. Adding w to both sides and dividing by two gives

$$y = \frac{w^2 + 4w + 3}{2} = \frac{(w + 2)^2 - 1}{2},$$

which is minimized when $w = -2$. This yields $y = -\frac{1}{2}$.

9. In equilateral triangle ABC with side length 2, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2 , respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2 , respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2 , respectively.

Find the perimeter of the triangle formed by lines A_1A_2, B_1B_2, C_1C_2 .

Answer: $\boxed{66 - 36\sqrt{3}}$ Since everything is equilateral it's easy to find the side length of the wanted triangle. By symmetry, it's just $AA_1 + 2A_1B_2 = 3AA_1 - AB$. Using the definition of a parabola, $AA_1 = \frac{\sqrt{3}}{2}A_1B$ so some calculation gives a side length of $2(11 - 6\sqrt{3})$, thus the perimeter claimed.

¹This can be seen rigorously by showing that five points in the plane determine at most one quadratic equation $ax^2 + by^2 + cxy + dx + ey + f = 0$, up to scaling.

10. Let z be a complex number and k a positive integer such that z^k is a positive real number other than 1. Let $f(n)$ denote the real part of the complex number z^n . Assume the parabola $p(n) = an^2 + bn + c$ intersects $f(n)$ four times, at $n = 0, 1, 2, 3$. Assuming the smallest possible value of k , find the largest possible value of a .

Answer: $\boxed{\frac{1}{3}}$ Let $r = |z|, \theta = \arg z$, and $C = \frac{\Re z}{|z|} = \cos \theta = \cos \frac{2\pi j}{k}$ for some j with $\gcd(j, k) = 1$. The condition of the four consecutive points lying on a parabola is equivalent to having the finite difference

$$f(3) - 3f(2) + 3f(1) - f(0) = 0.$$

This implies

$$\begin{aligned} f(3) - f(0) &= 3[f(2) - f(1)] \\ \iff r^3 \cos(3\theta) - 1 &= 3(r^2 \cos(2\theta) - r \cos(\theta)) \\ \iff r^3(4C^3 - 3C) - 1 &= 3(r^2(2C^2 - 1) - rC). \end{aligned}$$

Now we simply test the first few possible values of k .

$k = 1$ implies $C = 1$, which gives $r^3 - 1 = 3(r^2 - r) \implies (r - 1)^3 = 0 \implies r = 1$. This is not allowed since $r = 1$ implies a periodic function.

$k = 2$ implies $C = -1$, which gives $-r^3 - 1 = 3r^2 + r \implies (r + 1)^3 = 0$, again not allowed since $r > 0$.

$k = 3$ implies $C = -\frac{1}{2}$. This gives $r^3 - 1 = \frac{-3}{2}(r^2 - r) \implies (r - 1)(r + \frac{1}{2})(r + 2) = 0$. These roots are either negative or 1, again not allowed.

$k = 4$ implies $C = 0$. This gives $-1 = -3r^2 \implies r = \pm \frac{1}{\sqrt{3}}$. $r = \frac{1}{\sqrt{3}}$ is allowed, so this will generate our answer.

Again by finite differences (or by any other method of interpolating with a quadratic), we get $2a = f(0) + f(2) - 2f(1) = \frac{2}{3}$, so $a = \frac{1}{3}$.

HMMT 2014
Saturday 22 February 2014
HMIC

1. [20] Consider a regular n -gon with $n > 3$, and call a line *acceptable* if it passes through the interior of this n -gon. Draw m different acceptable lines, so that the n -gon is divided into several smaller polygons.
- (a) Prove that there exists an m , depending only on n , such that any collection of m acceptable lines results in one of the smaller polygons having 3 or 4 sides.
- (b) Find the smallest possible m which guarantees that at least one of the smaller polygons will have 3 or 4 sides.

Answer: N/A We will prove that if $m \geq n - 4$, then there is guaranteed to be a smaller polygon with 3 or 4 sides, while if $m \leq n - 5$, there might not be a polygon with 3 or 4 sides. This will solve both parts of the problem.

Given a configuration of lines, let P_1, \dots, P_k be all of the resulting smaller polygons. Let $E(P_i)$ be the number of edges in polygon P_i , and let $E = E(P_1) + \dots + E(P_k)$. First, note that whenever a new polygon is formed, it must have been because a larger polygon was split into two smaller polygons by a line passing through it. When this happens, k increases by 1 and E increases by at most 4 (it might be less than 4 if the line passes through vertices of the larger polygon). Therefore, if adding an acceptable line increases the number of polygons by a , then E increases by at most $4a$.

Now, assume $m \geq n - 4$. At the beginning, we have $k = 1$ and $E = n$. If the number of polygons at the end is $p + 1$, then $E \leq n + 4p$, so the average number of edges per polygon is less than or equal to $\frac{n+4p}{1+p}$. Now, note that each acceptable line introduces at least one new polygon, so $p \geq m \geq n - 4$. Also, note that as p increases, $\frac{n+4p}{1+p}$ strictly decreases, so it is maximized at $p = n - 4$, where $\frac{n+4p}{1+p} = \frac{5n-16}{n-3} = 5 - \frac{1}{n-3} < 5$. Therefore, the average number of edges per polygon is less than 5, so there must exist a polygon with either 3 or 4 edges, as desired.

We will now show that if $m \leq n - 5$, then we can draw m acceptable lines in a regular n -gon $A_1A_2 \dots A_n$ such that there are no polygons with 3 or 4 sides. Let M_1 be the midpoint of A_3A_4 , M_2 be the midpoint of A_4A_5 , ..., M_{n-5} be the midpoint of $A_{n-3}A_{n-4}$. Let the m acceptable lines be $A_1M_1, A_1M_2, \dots, A_1M_m$. We can see that all resulting polygons have 5 sides or more, so we are done.

2. [25] 2014 triangles have non-overlapping interiors contained in a circle of radius 1. What is the largest possible value of the sum of their areas?

Answer: N/A This problem turned out to be much trickier than we expected. We have yet to see a complete solution, but let us know if you find one!

Comment. We apologize for the oversight on our part. Our test-solvers essentially all misread the problem to contain the additional assumption that all triangle vertices lie on the circumference (which is not the case).

3. [30] Fix positive integers m and n . Suppose that a_1, a_2, \dots, a_m are reals, and that pairwise distinct vectors $v_1, \dots, v_m \in \mathbb{R}^n$ satisfy

$$\sum_{j \neq i} a_j \frac{v_j - v_i}{\|v_j - v_i\|^3} = 0$$

for $i = 1, 2, \dots, m$.

Prove that

$$\sum_{1 \leq i < j \leq m} \frac{a_i a_j}{\|v_j - v_i\|} = 0.$$

Answer: N/A Since $v_i \cdot (v_j - v_i) + v_j \cdot (v_i - v_j) = -\|v_j - v_i\|^2$ for any $1 \leq i < j \leq m$, we have

$$0 = \sum_{i=1}^m a_i v_i \cdot 0 = \sum_{i=1}^m a_i v_i \sum_{j \neq i} a_j \frac{v_j - v_i}{\|v_j - v_i\|^3} = - \sum_{1 \leq i < j \leq m} \frac{a_i a_j}{\|v_j - v_i\|},$$

as desired.

Alternative Solution. Fix a_1, \dots, a_m , and define $f : (\mathbb{R}^n)^m \rightarrow \mathbb{R}$ by

$$f(x_1, \dots, x_m) = \sum_{1 \leq i < j \leq m} \frac{a_i a_j}{\|x_j - x_i\|}.$$

Now if we view $f(x_1, \dots, x_m)$ individually as a function in x_k (k fixed), then the problem condition for $i = k$ says precisely that the gradient of f with respect to x_k (but of course, multiplied by a_k) evaluated at (v_1, \dots, v_m) is 0.

Finally, the multivariate chain rule shows that $F(t) := f(tv_1, \dots, tv_m) = \frac{1}{t} f(v_1, \dots, v_m)$ must satisfy $F'(1) = 0$. Yet $F'(1) = -\frac{1}{1^2} f(v_1, \dots, v_m)$, so we have $f(v_1, \dots, v_m) = 0$, as desired.

Comment. This is (sort of) an energy minimization problem from physics (more precisely, minimizing electrostatic or gravitational potential).

Comment. It would be interesting to determine all differentiable $A : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that the problem still holds when the terms are replaced by $\frac{v_j - v_i}{\|v_j - v_i\|} A'(\|v_j - v_i\|)$ and $A(\|v_j - v_i\|)$. (Here $A(t) = 1/t$, which has a nice physical interpretation, but that's not important.) The obvious examples are $A(t) = ct^s$ for constant c and real s (where both solutions easily carry through). But there may be others where instead of defining $F(t)$ through the scaling transformation $(v_1, \dots, v_m) \rightarrow (tv_1, \dots, tv_m)$, there's a subtler translation.

4. [35] Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

Answer: N/A Suppose ω is a primitive n th root, so that $\Phi_n(x)$ is the minimal polynomial of ω (over \mathbb{Q}). Thus $f(\omega)f(\omega^{-1}) = 1$ implies $\Phi_n(x) \mid f(x)f(x^{-1}) - 1$ (divisibility as Laurent polynomials). Hence $|f(\omega^k)| = 1$ whenever $\gcd(k, n) = 1$, and $\prod_{\gcd(k, n)=1} (t - f(\omega^k))$ is a monic integer-coefficient polynomial with all roots within the unit disk, so its roots must all be roots of unity (since 0 is clearly not a root), and we're done.

Comment. The result used at the end (a monic integer-coefficient polynomial with all roots within the unit disk has every root either 0 or a root of unity) is classical. See for instance the discussion here or here. This result is sometimes called Kronecker's theorem, and appears, for instance, in *Problems from the Book*.

Comment. Note that the proof works as long as $f \in \mathbb{Q}[x]$ and $f(\omega)$ is an algebraic integer. *Once is it known that $f(\omega)$ is a root of unity*, more can be said about the order m of $f(\omega)$. If ω is a primitive n th root of unity, then the conjugates of $f(\omega)$ lie among the $\leq \phi(n)$ numbers in $\{f(\omega^k) : \gcd(k, n) = 1\}$. In particular, $\phi(m) \leq \phi(n)$, which (e.g. through a naive estimate using $p - 1 \geq \sqrt{p}$ for $p \geq 3$ to get $\phi(m) \geq \sqrt{m/2}$) shows that $m \leq 2\phi(n)^2$ (for instance), so there exists $g \in \mathbb{Q}[x]$ with $g(\omega)$ a root of unity of maximal order N , which then must generate the unit group of $\mathbb{Q}(\omega)$ (or else we could find a larger order). In particular, since ω lies in this group, $n \mid N$. But $\phi(N) \leq \phi(n)$ by our earlier argument, so either $N = n$ or $N = 2n$ if n is odd. Either way, it's clear that $g(\omega) = \pm\omega^k$ for some k , so our original $f(\omega)$ must also be $\pm\omega^j$ for some j .

Alternative Solution. We follow the paper here, but fill in some of the details to make the ideas as accessible as possible. We will use the fact that if A is a finitely generated abelian group, then any subgroup B is also finitely generated, and furthermore, $\text{rank}(B) \leq \text{rank}(A)$. (For instance, see Theorem 14.6.5 in Artin's *Algebra, 3rd ed.* for a proof of finite generation, which for our particular case of modules over a PID (namely, \mathbb{Z}) can be done in parallel to the $\text{rank}(B) \leq \text{rank}(A)$ proof—the proof is very similar to Gaussian elimination.)

Let $K = \mathbb{Q}(\omega)$, V the set of algebraic integers in K of modulus 1 (which must be units of K), $U \supseteq V$ the set of units in K , $R = U \cap \mathbb{R}$ the set of real units in K (which, importantly, is also the set of units in $K \cap \mathbb{R}$), and $L = K \cap \mathbb{R}$ denote the maximal real subfield of K .

The key is that $\text{rank}(V) = \text{rank}(U) - \text{rank}(R)$. To prove this, we note that for $u \in U$, $\bar{u} \in U$ as well (since $\bar{K} = K$), so $u\bar{u} = |u|^2 \in R$ and $u/\bar{u} \in V$, which gives $u^2 = (u\bar{u})(u/\bar{u}) \in RV$. Thus $S := \{u^2 : u \in U\} \subseteq RV \subseteq U$. But U is a finitely generated abelian group by Dirichlet's unit theorem, so S, RV are as well, and furthermore $\text{rank}(S) \leq \text{rank}(RV) \leq \text{rank}(U)$. Now take multiplicatively independent $u_1, \dots, u_{\text{rank}(U)} \in U$; then clearly $u_1^2, \dots, u_{\text{rank}(U)}^2 \in S$ are multiplicatively independent as well, so $\text{rank}(U) \leq \text{rank}(S)$, whence $\text{rank}(RV) = \text{rank}(U)$. Of course, $R \cap V = \{\pm 1\}$, so $\text{rank}(RV) = \text{rank}(R) + \text{rank}(V) - \text{rank}(R \cap V) = \text{rank}(R) + \text{rank}(V)$ (the proof here for \mathbb{Z} -modules is essentially the same as that for \mathbb{Q} -vector spaces), finishing the proof.

V contains only roots of unity if and only if $\text{rank}(V) = 0$ (no free elements), so we must prove $\text{rank}(U) = \text{rank}(R)$. Let r, s denote the number of real and complex embeddings of a number field, so $[K : \mathbb{Q}] = r(K) + 2s(K)$, $[L : \mathbb{Q}] = r(L) + 2s(L)$, and thus $r(K) + 2s(K) = [K : L](r(L) + 2s(L))$. By Dirichlet's unit theorem, $\text{rank}(U) = r(K) + s(K) - 1$ and $\text{rank}(R) = r(L) + s(L) - 1$.

If $K = L$ (equivalently, K is totally real), we're trivially done. Otherwise, if $[K : L] > 1$, it's easy to check that we have $\text{rank}(U) = \text{rank}(R)$ if and only if $[K : L] = 2$ and $r(K) = s(L) = 0$ (K is totally imaginary and L is totally real)—then $r(L) = s(K)$ automatically holds. This is precisely the condition that K is a CM-field.

(CM-field K just means K is totally imaginary (no real embeddings), and K is a quadratic extension of $K \cap \mathbb{R}$.)

In particular, the cyclotomic field $K = \mathbb{Q}(\omega) = \mathbb{Q}[\omega]$ is a totally imaginary quadratic extension of the totally real field $\mathbb{Q}(\omega + \omega^{-1}) = \mathbb{Q}[\omega + \omega^{-1}]$. Indeed, it is easy to check that the latter is simply $K \cap \mathbb{R}$, and $K = \mathbb{Q}(\omega + \omega^{-1}, \omega - \omega^{-1} = \pm\sqrt{(\omega + \omega^{-1})^2 - 4})$. So we're done.

5. [40] Let n be a positive integer, and let A and B be $n \times n$ matrices with complex entries such that $A^2 = B^2$. Show that there exists an $n \times n$ invertible matrix S with complex entries that satisfies $S(AB - BA) = (BA - AB)S$.

Answer: $\boxed{N/A}$ Let $X = A + B$ and $Y = A - B$, so $XY = BA - AB$ and $YX = AB - BA$. Note that $XY = -YX$.

It suffices (actually is equivalent) to show that $AB - BA$ and $BA - AB$ have the same Jordan forms. In other words, we need to show that for any complex number λ , the Jordan λ -block decompositions of $AB - BA$ and $BA - AB$ are the same.

However, the λ -decomposition of a matrix M is uniquely determined by the infinite (eventually constant) sequence $(\dim \ker(M - \lambda I)^1, \dim \ker(M - \lambda I)^2, \dots)$, so we only have to show that $\dim \ker(XY - \lambda I)^k = \dim \ker(YX - \lambda I)^k$ for all positive integers k and complex numbers λ .

For $\lambda = 0$, this follows trivially from $XY = -YX$: we have $(XY)^k v = 0$ if and only if $(YX)^k v = 0$.

Now suppose $\lambda \neq 0$, and define the (matrix) polynomial $p(T) = (T - \lambda I)^k$. Let $V_1 = \{v \in \mathbb{C}^n : p(XY)v = 0\}$, and $V_2 = \{v \in \mathbb{C}^n : p(YX)v = 0\}$. Note that if $v \in V_1$, then $0 = [Yp(XY)]v = [p(YX)Y]v = p(YX)[Yv]$, so $Yv \in V_2$. Furthermore, if $v \neq 0$, then $Yv \neq 0$, or else $p(XY)v = p(0)v = (-\lambda I)^k v \neq 0$ (since $(XY)^j v = 0$ for all $j \geq 1$), contradiction. Viewing Y (more precisely, $v \rightarrow Yv$) as a linear map from V_1 to V_2 , we have $\dim V_1 = \dim YV_1 + \dim \ker Y = \dim YV_1 \leq \dim V_2$ (alternatively, if v_1, \dots, v_r is a basis of V_1 , then Yv_1, \dots, Yv_r are linearly independent in V_2). By symmetry, $\dim V_1 = \dim V_2$, and we're done.

Comment. If A, B are additionally real, then we can also choose S to be *real* (not just complex). Indeed, note that if $S = P + Qi$ for real P, Q , then since A, B are real, it suffices to find $c \in \mathbb{R}$ such that $\det(P + cQ) \neq 0$. However, $\det(P + xQ)$ is a polynomial in x that is not identically zero (it is nonzero for $x = i$), so it has finitely many (complex, and thus real) roots. Thus the desired c clearly exists.

Comment. Note that $A(AB - BA) = (BA - AB)A$, since $A^2B = B^3 = BA^2$. (The same holds for $A \rightarrow B$.) Thus the problem is easy unless all linear combinations of A, B are noninvertible. However, it seems difficult to finish without analyzing generalized eigenvectors as above.

Comment. See this paper by Ross Lippert and Gilbert Strang for a general analysis of the Jordan forms of MN and NM .

HMMT 2013
Saturday 16 February 2013
Algebra Test

1. Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x .

Answer: $\boxed{3 + \sqrt{7}}$ We have $(xy)^2 + (x + y)^2 = 40$ and $xy + (x + y) = 8$. Squaring the second equation and subtracting the first gives $xy(x + y) = 12$ so $xy, x + y$ are the roots of the quadratic $a^2 - 8a + 12 = 0$. It follows that $\{xy, x + y\} = \{2, 6\}$. If $x + y = 2$ and $xy = 6$, then x, y are the roots of the quadratic $b^2 - 2b + 6 = 0$, which are non-real, so in fact $x + y = 6$ and $xy = 2$, and x, y are the roots of the quadratic $b^2 - 6b + 2 = 0$. Because $x > y$, we take the larger root, which is $\frac{6 + \sqrt{28}}{2} = 3 + \sqrt{7}$.

2. Let $\{a_n\}_{n \geq 1}$ be an arithmetic sequence and $\{g_n\}_{n \geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

Answer: $\boxed{-54}$ Let the terms of the geometric sequence be a, ra, r^2a, r^3a . Then, the terms of the arithmetic sequence are $-a, -ra, -r^2a + 1, -r^3a$. However, if the first two terms of this sequence are $-a, -ra$, the next two terms must also be $(-2r + 1)a, (-3r + 2)a$. It is clear that $a \neq 0$ because $a_3 + g_3 \neq 0$, so $-r^3 = -3r + 2 \Rightarrow r = 1$ or -2 . However, we see from the arithmetic sequence that $r = 1$ is impossible, so $r = -2$. Finally, by considering a_3 , we see that $-4a + 1 = 5a$, so $a = 1/9$. We also see that $a_n = (3n - 4)a$ and $g_n = (-2)^{n-1}a$, so our answer is $a_{10} + g_{10} = (26 - 512)a = -486a = -54$.

3. Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S .

Answer: $\boxed{577}$ S is the set of positive integers with exactly three ones in its binary representation. The number of such integers with at most d total bits is $\binom{d}{3}$, and noting that $\binom{9}{3} = 84$ and $\binom{10}{3} = 120$, we want the 16th smallest integer of the form $2^9 + 2^x + 2^y$, where $y < x < 9$. Ignoring the 2^9 term, there are $\binom{d'}{2}$ positive integers of the form $2^x + 2^y$ with at most d' total bits. Because $\binom{6}{2} = 15$, our answer is $2^9 + 2^6 + 2^0 = 577$. (By a *bit*, we mean a digit in base 2.)

4. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$\begin{aligned}x_1(x_1 + 1) &= A \\x_2(x_2 + 1) &= A \\x_1^4 + 3x_1^3 + 5x_1 &= x_2^4 + 3x_2^3 + 5x_2.\end{aligned}$$

Answer: $\boxed{-7}$ Applying polynomial division,

$$\begin{aligned}x_1^4 + 3x_1^3 + 5x_1 &= (x_1^2 + x_1 - A)(x_1^2 + 2x_1 + (A - 2)) + (A + 7)x_1 + A(A - 2) \\&= (A + 7)x_1 + A(A - 2).\end{aligned}$$

Thus, in order for the last equation to hold, we need $(A + 7)x_1 = (A + 7)x_2$, from which it follows that $A = -7$. These steps are reversible, so $A = -7$ indeed satisfies the needed condition.

5. Let a and b be real numbers, and let r, s , and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2 , and t^2 . If $g(-1) = -5$, find the maximum possible value of b .

Answer: $\boxed{1 + \sqrt{5}}$ By Vieta's Formulae, $m = -(r^2 + s^2 + t^2) = -a^2 + 2b$, $n = r^2s^2 + s^2t^2 + t^2r^2 = b^2 + 2a$, and $p = -1$. Therefore, $g(-1) = -1 - a^2 + 2b - b^2 - 2a - 1 = -5 \Leftrightarrow (a + 1)^2 + (b - 1)^2 = 5$. This is an equation of a circle, so b reaches its maximum when $a + 1 = 0 \Rightarrow a = -1$. When $a = -1$, $b = 1 \pm \sqrt{5}$, so the maximum is $1 + \sqrt{5}$.

6. Find the number of integers n such that

$$1 + \left\lfloor \frac{100n}{101} \right\rfloor = \left\lceil \frac{99n}{100} \right\rceil.$$

Answer: $\boxed{10100}$ Consider $f(n) = \lceil \frac{99n}{100} \rceil - \lfloor \frac{100n}{101} \rfloor$. Note that $f(n + 10100) = \lceil \frac{99n}{100} + 99 \cdot 101 \rceil - \lfloor \frac{100n}{101} + 100 \rfloor = f(n) + 99 \cdot 101 - 100 = f(n) - 1$. Thus, for each residue class r modulo 10100, there is exactly one value of n for which $f(n) = 1$ and $n \equiv r \pmod{10100}$. It follows immediately that the answer is 10100.

7. Compute

$$\sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1 + a_2 + \cdots + a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

Answer: $\boxed{15309/256}$ Note that, since this is symmetric in a_1 through a_7 ,

$$\begin{aligned} \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1 + a_2 + \cdots + a_7}{3^{a_1+a_2+\cdots+a_7}} &= 7 \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1}{3^{a_1+a_2+\cdots+a_7}} \\ &= 7 \left(\sum_{a_1=0}^{\infty} \frac{a_1}{3^{a_1}} \right) \left(\sum_{a=0}^{\infty} \frac{1}{3^a} \right)^6. \end{aligned}$$

If $S = \sum \frac{a}{3^a}$, then $3S - S = \sum \frac{1}{3^a} = 3/2$, so $S = 3/4$. It follows that the answer equals $7 \cdot \frac{3}{4} \cdot \left(\frac{3}{2}\right)^6 = \frac{15309}{256}$. Alternatively, let $f(z) = \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} z^{a_1+a_2+\cdots+a_7}$. Note that we can rewrite $f(z) = (\sum_{a=0}^{\infty} z^a)^7 = \frac{1}{(1-z)^7}$. Furthermore, note that $zf'(z) = \sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} (a_1 + a_2 + \cdots + a_7) z^{a_1+a_2+\cdots+a_7}$, so the sum in question is simply $\frac{f'(1/3)}{3}$. Since $f'(x) = \frac{7}{(1-x)^8}$, it follows that the sum is equal to $\frac{7 \cdot 3^7}{2^8} = \frac{15309}{256}$.

8. Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $\frac{x^6+y^6}{x^5+y^5}$.

Answer: $\boxed{10 \pm 2\sqrt{17}}$ Let $A = \frac{1}{x} + \frac{1}{y}$ and let $B = \frac{x}{y} + \frac{y}{x}$. Then

$$\frac{B}{A} = \frac{x^2 + y^2}{x + y} = 4,$$

so $B = 4A$. Next, note that

$$B^2 - 2 = \frac{x^4 + y^4}{x^2 y^2} \quad \text{and} \quad AB - A = \frac{x^3 + y^3}{x^2 y^2},$$

so

$$\frac{B^2 - 2}{AB - A} = 2.$$

Substituting $B = 4A$ and simplifying, we find that $4A^2 + A - 1 = 0$, so $A = \frac{-1 \pm \sqrt{17}}{8}$. Finally, note that

$$64A^3 - 12A = B^3 - 3B = \frac{x^6 + y^6}{x^3 y^3} \quad \text{and} \quad 16A^3 - 4A^2 - A = A(B^2 - 2) - (AB - A) = \frac{x^5 + y^5}{x^3 y^3},$$

so

$$\frac{x^6 + y^6}{x^5 + y^5} = \frac{64A^2 - 12}{16A^2 - 4A - 1} = \frac{4 - 16A}{3 - 8A},$$

where the last inequality follows from the fact that $4A^2 = 1 - A$. If $A = \frac{-1 + \sqrt{17}}{8}$, then this value equals $10 + 2\sqrt{17}$. Similarly, if $A = \frac{-1 - \sqrt{17}}{8}$, then this value equals $10 - 2\sqrt{17}$.

(It is not hard to see that these values are achievable by noting that with the values of A and B we can solve for $x + y$ and xy , and thus for x and y .)

9. Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

Answer: 46/3 *First solution:* Note that

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}} = \frac{1}{3} + \sum_{k=1}^{22} \frac{1 - z^k}{1 - z^{3k}} = \frac{1}{3} + \sum_{k=1}^{22} \frac{1 - (z^{24})^k}{1 - z^{3k}} = \frac{1}{3} + \sum_{k=1}^{22} \sum_{\ell=0}^7 z^{3k\ell}.$$

3 and 23 are prime, so every non-zero residue modulo 23 appears in an exponent in the last sum exactly 7 times, and the summand 1 appears 22 times. Because the sum of the 23rd roots of unity is zero, our answer is $\frac{1}{3} + (22 - 7) = \frac{46}{3}$.

Second solution: For an alternate approach, we first prove the following identity for an arbitrary complex number a :

$$\sum_{k=0}^{22} \frac{1}{a - z^k} = \frac{23a^{22}}{a^{23} - 1}.$$

To see this, let $f(x) = x^{23} - 1 = (x - 1)(x - z)(x - z^2) \dots (x - z^{22})$. Note that the sum in question is merely $\frac{f'(a)}{f(a)}$, from which the identity follows.

Now, returning to our original sum, let $\omega \neq 1$ satisfy $\omega^3 = 1$. Then

$$\begin{aligned} \sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}} &= \frac{1}{\omega^2 - \omega} \sum_{k=0}^{22} \frac{1}{\omega - z^k} - \frac{1}{\omega^2 - z^k} \\ &= \frac{1}{\omega^2 - \omega} \left(\sum_{k=0}^{22} \frac{1}{\omega - z^k} - \sum_{k=0}^{22} \frac{1}{\omega^2 - z^k} \right) \\ &= \frac{1}{\omega^2 - \omega} \left(\frac{23\omega^{22}}{\omega^{23} - 1} - \frac{23\omega^{44}}{\omega^{46} - 1} \right) \\ &= \frac{23}{\omega^2 - \omega} \left(\frac{\omega}{\omega^2 - 1} - \frac{\omega^2}{\omega - 1} \right) \\ &= \frac{23}{\omega^2 - \omega} \frac{(\omega^2 - \omega) - (\omega - \omega^2)}{2 - \omega - \omega^2} \\ &= \frac{46}{3}. \end{aligned}$$

10. Let N be a positive integer whose decimal representation contains 11235 as a contiguous substring, and let k be a positive integer such that $10^k > N$. Find the minimum possible value of

$$\frac{10^k - 1}{\gcd(N, 10^k - 1)}.$$

Answer: [89] Set $m = \frac{10^k - 1}{\gcd(N, 10^k - 1)}$. Then, in lowest terms, $\frac{N}{10^k - 1} = \frac{a}{m}$ for some integer a . On the other hand, the decimal expansion of $\frac{N}{10^k - 1}$ simply consists of the decimal expansion of N , possibly with some padded zeros, repeating. Since N contains 11235 as a contiguous substring, the decimal representation of $\frac{a}{m}$ must as well.

Conversely, if m is relatively prime to 10 and if there exists an a such that the decimal representation of $\frac{a}{m}$ contains the substring 11235, we claim that m is an attainable value for $\frac{10^k - 1}{\gcd(N, 10^k - 1)}$. To see this, note that since m is relatively prime to 10, there exists a value of k such that m divides $10^k - 1$ (for example, $k = \phi(m)$). Letting $ms = 10^k - 1$ and $N = as$, it follows that $\frac{a}{m} = \frac{as}{ms} = \frac{N}{10^k - 1}$. Since the decimal expansion of this fraction contains the substring 11235, it follows that N must also, and therefore m is an attainable value.

We are therefore looking for a fraction $\frac{a}{m}$ which contains the substring 11235 in its decimal expansion. Since 1, 1, 2, 3, and 5 are the first five Fibonacci numbers, it makes sense to look at the value of the infinite series

$$\sum_{i=1}^{\infty} \frac{F_i}{10^i}.$$

A simple generating function argument shows that $\sum_{i=1}^{\infty} F_i x^i = \frac{x}{1-x-x^2}$, so substituting $x = 1/10$ leads us to the fraction $10/89$ (which indeed begins 0.11235...).

How do we know no smaller values of m are possible? Well, if a'/m' contains the substring 11235 somewhere in its infinitely repeating decimal expansion, then note that there is an i such that the decimal expansion of the fractional part of $10^i(a'/m')$ begins with 0.11235.... We can therefore, without loss of generality, assume that the decimal representation of a'/m' begins 0.11235.... But since the decimal representation of $10/89$ begins 0.11235..., it follows that

$$\left| \frac{10}{89} - \frac{a'}{m'} \right| \leq 10^{-5}.$$

On the other hand, this absolute difference, if non-zero, is at least $\frac{1}{89m'}$. If $m' < 89$, this is at least $\frac{1}{89^2} > 10^{-5}$, and therefore no smaller values of m' are possible.

HMMT 2013
Saturday 16 February 2013
Combinatorics Test

1. A standard 52-card deck contains cards of 4 suits and 13 numbers, with exactly one card for each pairing of suit and number. If Maya draws two cards with replacement from this deck, what is the probability that the two cards have the same suit or have the same number, but not both?

Answer: $\boxed{\frac{15}{52}}$ After drawing the first card, there are 12 other cards from the same suit and 3 other cards with the same number, so the probability is $\frac{12+3}{52}$.

2. If Alex does not sing on Saturday, then she has a 70% chance of singing on Sunday; however, to rest her voice, she never sings on both days. If Alex has a 50% chance of singing on Sunday, find the probability that she sings on Saturday.

Answer: $\boxed{\frac{2}{7}}$ Let p be the probability that Alex sings on Saturday. Then, the probability that she sings on Sunday is $.7(1 - p)$; setting this equal to $.5$ gives $p = \frac{2}{7}$.

3. On a game show, Merble will be presented with a series of 2013 marbles, each of which is either red or blue on the outside. Each time he sees a marble, he can either keep it or pass, but cannot return to a previous marble; he receives 3 points for keeping a red marble, loses 2 points for keeping a blue marble, and gains 0 points for passing. All distributions of colors are equally likely and Merble can only see the color of his current marble. If his goal is to end with exactly one point and he plays optimally, what is the probability that he fails?

Answer: $\boxed{\frac{1}{2^{2012}}}$ First, we note that if all the marbles are red or all are blue, then it is impossible for Merble to win; we claim that he can guarantee himself a win in every other case. In particular, his strategy should be to keep the first red and first blue marble that he encounters, and to ignore all the others. Consequently, the probability that he cannot win is $\frac{2}{2^{2013}} = \frac{1}{2^{2012}}$.

4. How many orderings (a_1, \dots, a_8) of $(1, 2, \dots, 8)$ exist such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 = 0$?

Answer: $\boxed{4608}$ We can divide the numbers up based on whether they have a + or - before them. Both the numbers following +'s and -'s must add up to 18. Without loss of generality, we can assume the +'s contain the number 1 (and add a factor of 2 at the end to account for this). The possible 4-element sets containing a 1 which add to 18 are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, $\{1, 4, 6, 7\}$. Additionally, there are $4!$ ways to order the numbers following a + and $4!$ ways to order the numbers following a -. Thus the total number of possibilities is $4 \times 2 \times 4! \times 4! = 4608$.

5. At a certain chocolate company, each bar is 1 unit long. To make the bars more interesting, the company has decided to combine dark and white chocolate pieces. The process starts with two bars, one completely dark and one completely white. At each step of the process, a new number p is chosen uniformly at random between 0 and 1. Each of the two bars is cut p units from the left, and the pieces on the left are switched: each is grafted onto the opposite bar where the other piece of length p was previously attached. For example, the bars might look like this after the first step:



Each step after the first operates on the bars resulting from the previous step. After a total of 100 steps, what is the probability that on each bar, the chocolate $1/3$ units from the left is the same type of chocolate as that $2/3$ units from the left?

Answer: $\frac{1}{2} \left[\left(\frac{1}{3} \right)^{100} + 1 \right]$ If the values of p chosen are p_1, \dots, p_{100} , then note that the color of a bar changes at each value of p_i . Consequently, we want to find the probability that exactly an even number of p_i are in $(\frac{1}{3}, \frac{2}{3})$. Summing, this is equal to

$$\binom{100}{0} \left(\frac{1}{3} \right)^0 \left(\frac{2}{3} \right)^{100} + \binom{100}{2} \left(\frac{1}{3} \right)^2 \left(\frac{2}{3} \right)^{98} + \dots + \binom{100}{100} \left(\frac{1}{3} \right)^{100} \left(\frac{2}{3} \right)^0.$$

To compute, we note that this is equal to

$$\frac{1}{2} \left[\left(\frac{2}{3} + \frac{1}{3} \right)^{100} + \left(\frac{2}{3} - \frac{1}{3} \right)^{100} \right]$$

after expanding using the binomial theorem, since any terms with odd exponents are cancelled out between the two terms.

6. Values a_1, \dots, a_{2013} are chosen independently and at random from the set $\{1, \dots, 2013\}$. What is expected number of distinct values in the set $\{a_1, \dots, a_{2013}\}$?

Answer: $\frac{2013^{2013} - 2012^{2013}}{2013^{2012}}$ For each $n \in \{1, 2, \dots, 2013\}$, let $X_n = 1$ if n appears in $\{a_1, a_2, \dots, a_{2013}\}$ and 0 otherwise. Defined this way, $E[X_n]$ is the probability that n appears in $\{a_1, a_2, \dots, a_{2013}\}$. Since each a_i ($1 \leq i \leq 2013$) is not n with probability $2012/2013$, the probability that n is none of the a_i 's is $(\frac{2012}{2013})^{2013}$, so $E[X_n]$, the probability that n is one of the a_i 's, is $1 - (\frac{2012}{2013})^{2013}$. The expected number of distinct values in $\{a_1, \dots, a_{2013}\}$ is the expected number of $n \in \{1, 2, \dots, 2013\}$ such that $X_n = 1$, that is, the expected value of $X_1 + X_2 + \dots + X_{2013}$. By linearity of expectation, $E[X_1 + X_2 + \dots + X_{2013}] = E[X_1] + E[X_2] + \dots + E[X_n] = 2013 \left(1 - (\frac{2012}{2013})^{2013} \right) = \frac{2013^{2013} - 2012^{2013}}{2013^{2012}}$.

7. A single-elimination ping-pong tournament has 2^{2013} players, seeded in order of ability. If the player with seed x plays the player with seed y , then it is possible for x to win if and only if $x \leq y + 3$. For how many players P it is possible for P to win? (In each round of a single elimination tournament, the remaining players are randomly paired up; each player plays against the other player in his pair, with the winner from each pair progressing to the next round and the loser eliminated. This is repeated until there is only one player remaining.)

Answer: 6038 We calculate the highest seed n that can win. Below, we say that a player x *vicariously* defeats a player y if x defeats y directly or indirectly through some chain (i.e. x defeats x_1 , who defeated x_2 , ..., who defeated x_n , who defeated y for some players x_1, \dots, x_n).

We first consider the highest seeds that are capable of making the semifinals. The eventual winner must be able to beat two of these players and thus must be able to beat the second best player in the semifinals. The seed of the player who vicariously beats the 1-seed is maximized if 1 loses to 4 in the first round, 4 to 7 in the second round, etc. Therefore $3 \cdot 2011 + 1 = 6034$ is the maximum value of the highest seed in the semifinals. If 1, and 2 are in different quarters of the draw, then by a similar argument 6035 is the largest possible value of the second best player in the semis, and thus 6038 is the highest that can win. If 1 and 2 are in the same quarter, then in one round the highest remaining seed will not be able to go up by 3, when the player who has vicariously beaten 1 plays the player who vicariously beat 2, so $3 \cdot 2011 - 1 = 6032$ is the highest player the semifinalist from that quarter could be. But then the eventual winner still must be seeded at most 6 above this player, and thus 6038 is still the upper bound.

Therefore 6038 is the worst seed that could possibly win, and can do so if 6034, 6035, 6036, 6038 all make the semis, which is possible (it is not difficult to construct such a tournament). Then, note that any player x with a lower seed can also win for some tournament – in particular, it suffices to take

the tournament where it is possible for player 6038 to win and switch the positions of 6038 and x . Consequently, there are 6038 players for whom it is possible to win under some tournament.

8. It is known that exactly one of the three (distinguishable) musketeers stole the truffles. Each musketeer makes one statement, in which he either claims that one of the three is guilty, or claims that one of the three is innocent. It is possible for two or more of the musketeers to make the same statement. After hearing their claims, and knowing that exactly one musketeer lied, the inspector is able to deduce who stole the truffles. How many ordered triplets of statements could have been made?

Answer: 99 We divide into cases, based on the number of distinct people that statements are made about.

- The statements are made about 3 distinct people. Then, since exactly one person is guilty, and because exactly one of the three lied, there are either zero statements of guilt or two statements of guilt possible; in either case, it is impossible by symmetry to determine who is guilty or innocent.
- The statements are made about 2 distinct people or 1 distinct person. Then, either at least two of the statements are the same, or all are different.
 - If two statements are the same, then those two statements must be true because only one musketeer lied. Consequently, the lone statement must be false. If all the statements are about the same person, there there must be 2 guilty claims and 1 innocence claim (otherwise we would not know which of the other two people was guilty). Then, there are 3 choices for who the statement is about and 3 choices for who makes the innocence claim, for a $3 \cdot 3 = 9$ triplets of statements. Meanwhile, if the statements are about two different people, then this is doable unless both of the distinct statements imply guilt for the person concerned (i.e. where there are two guilty accusations against one person and one claim of innocence against another). Consequently, there are 3 sets of statements that can be made, $3 \cdot 2 = 6$ ways to determine who they are made about, and 3 ways to determine who makes which statement, for a total of $3 \cdot 6 \cdot 3 = 54$ triplets in this case.
 - If all the statements are different, then they must be about two different people. Here, there must be one person, who we will call A, who has both a claim of innocence and an accusation of guilt against him. The last statement must concern another person, B. If the statement accuses B of being guilty, then we can deduce that he is the guilty one. On the other hand, if the statement claims that B is innocent, either of the other two musketeers could be guilty. Consequently, there are $3 \cdot 2 = 6$ ways to choose A and B, and $3! = 6$ ways to choose who makes which statement, for a total of $6 \cdot 6 = 36$ triplets of statements.

In total, we have $9 + 54 + 36 = 99$ possible triplets of statements.

9. Given a permutation σ of $\{1, 2, \dots, 2013\}$, let $f(\sigma)$ to be the number of fixed points of σ – that is, the number of $k \in \{1, 2, \dots, 2013\}$ such that $\sigma(k) = k$. If S is the set of all possible permutations σ , compute

$$\sum_{\sigma \in S} f(\sigma)^4.$$

(Here, a *permutation* σ is a bijective mapping from $\{1, 2, \dots, 2013\}$ to $\{1, 2, \dots, 2013\}$.)

Answer: 15(2013!) First, note that

$$\sum_{\sigma \in S} f(\sigma)^4 = \sum_{\sigma \in S} \sum_{1 \leq a_1, a_2, a_3, a_4 \leq 2013} g(\sigma, a_1, a_2, a_3, a_4),$$

where $g(\sigma, a_1, a_2, a_3, a_4) = 1$ if all a_i are fixed points of σ and 0 otherwise. (The a_i 's need not be distinct.) Switching the order of summation, we find that the desired sum is

$$\sum_{1 \leq a_1, a_2, a_3, a_4 \leq 2013} \sum_{\sigma \in S} g(\sigma, a_1, a_2, a_3, a_4).$$

Note that the inner sum is equal to the number of permutations on $\{1, 2, \dots, 2013\}$ that fix a_1, a_2, a_3 , and a_4 . This depends on the number of distinct values the a_i s take. If they take on exactly k distinct values, then the inner sum will evaluate to $(2013 - k)!$, because σ can be any permutation of the remaining $2013 - k$ elements. (For example, if $a_1 = a_2$ but a_1, a_3 , and a_4 are distinct, the inner sum is $2010!$ because σ can be any permutation that fixes a_1, a_3 , and a_4 .)

Now, suppose we are given which of the a_i are equal (for example, we could be given $a_1 = a_2$ but a_1, a_3, a_4 mutually distinct, as per the above example). Assuming there are k distinct values among the a_i , there are $2013(2013 - 1) \cdots (2013 - k + 1)$ ways to choose the a_i . At this point, there are $(2013 - k)!$ ways to choose σ on the remaining $(2013 - k)$ values such that it fixes the a_i , for a total of $2013!$ choices for $(\sigma, a_1, a_2, a_3, a_4)$ such that $g(\sigma, a_1, a_2, a_3, a_4) = 1$ and the a_i satisfy the correct equality relations.

Thus the answer is $2013!$ times the number of ways to choose equivalence classes on the a_i , so the problem reduces to finding the number of ways to partition 4 elements into nonempty sets. This process can be accelerated by doing casework based on the number of sets:

- One set must contain all four elements, only one possibility. (i.e. all the a_i s are equal)
- Either one set contains 3 elements and the other contains the fourth (4 possibilities) or one set contains 2 elements and the other contains the other two (3 possibilities). (i.e. there are two distinct values of a_i)
- One set contains two elements, the other two each contain one. There are $\binom{4}{2} = 6$ ways to choose the two elements in the set with two elements, and this uniquely determines the partition. (i.e. there are three distinct values of a_i)
- All sets contain one element, in which case there is only one possibility. (i.e. all the a_i are distinct)

Thus the number of ways to construct such a partition is $1 + 4 + 3 + 6 + 1 = 15$, and our answer is $15 \cdot 2013!$.

10. Rosencrantz and Guildenstern each start with \$2013 and are flipping a fair coin. When the coin comes up heads Rosencrantz pays Guildenstern \$1 and when the coin comes up tails Guildenstern pays Rosencrantz \$1. Let $f(n)$ be the number of dollars Rosencrantz is ahead of his starting amount after n flips. Compute the expected value of $\max\{f(0), f(1), f(2), \dots, f(2013)\}$.

Answer: $\frac{-1}{2} + \frac{(1007) \binom{2013}{1006}}{2^{2012}}$ We want to calculate $\Gamma = \sum_{i=0}^{\infty} i \cdot P(\text{max profit} = i)$, where we consider the maximum profit Rosencrantz has at any point over the first 2013 coin flips. By summation by parts this is equal to $\sum_{a=1}^{2013} P(\text{max profit} \geq a)$.

Let p_a be the probability that Rosencrantz' max profit is at least a and let $E_{n,a}$ be the set of sequences of flips such that Rosencrantz first reaches a profit of a on exactly the n th flip. Let Q_a^+, Q_a^-, Q_a be the sets such that after all 2013 flips Rosencrantz' final profit is (respectively) greater than, less than, or equal to a .

Then,

$$\begin{aligned} \Gamma &= \sum_{a=1}^{2013} P(\text{max profit} \geq a) \\ &= \sum_{a=1}^{2013} \sum_{n=a}^{2013} P(E_{n,a}) \\ &= \sum_{a=1}^{2013} \sum_{n=a}^{2013} P(E_{n,a} \cap Q_a^+) + P(E_{n,a} \cap Q_a^-) + P(E_{n,a} \cap Q_a). \end{aligned}$$

By symmetry, $P(E_{n,a} \cap Q_a^+) = P(E_{n,a} \cap Q_a^-)$ because, for any sequence in $E_{n,a} \cap Q_a^+$, we can reverse all the flips after the n th flip to get a sequence in $E_{n,a} \cap Q_a^-$, and vice-versa.

Furthermore, $\sum_{n=a}^{2013} P(E_{n,a} \cap Q_a^+) = P(Q_a^+)$ and $\sum_{n=a}^{2013} P(E_{n,a} \cap Q_a) = P(Q_a)$.

So we have

$$\sum_{a=1}^{2013} P(\max \text{ profit} \geq a) = \sum_{a=1}^{2013} (P(Q_a) + 2P(Q_a^+)).$$

Since by symmetry $P(Q_a) = P(Q_{-a})$ and we have an odd number of flips, we have $\sum_{a=1}^{2013} P(Q_a) = \frac{1}{2}$.

$$\text{Also } P(Q_a^+) = \frac{1}{2^{2013}} \sum_{k=\lceil \frac{2014+a}{2} \rceil}^{2013} \binom{2013}{k}.$$

So the rest is just computation. We have:

$$\begin{aligned} \Gamma &= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{a=1}^{2013} \sum_{k=\lceil \frac{2014+a}{2} \rceil}^{2013} \binom{2013}{k} \\ &= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} \sum_{a=1}^{2k-2014} \binom{2013}{k} \\ &= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} \binom{2013}{k} (k + k - 2013 - 1) \\ &= \frac{1}{2} + \frac{1}{2^{2012}} \sum_{k=1008}^{2013} 2013 \binom{2012}{k-1} - 2013 \binom{2012}{k} - \binom{2013}{k} \\ &= \frac{1}{2} + \frac{2013 \binom{2012}{1007} - 2^{2012} + \binom{2013}{1007}}{2^{2012}} \\ &= \frac{-1}{2} + \frac{(1007) \binom{2013}{1006}}{2^{2012}}. \end{aligned}$$

So the answer is $\frac{-1}{2} + \frac{(1007) \binom{2013}{1006}}{2^{2012}}$ (for reference, approximately 35.3).

HMMT 2013
Saturday 16 February 2013
Geometry Test

1. Jarris the triangle is playing in the (x, y) plane. Let his maximum y coordinate be k . Given that he has side lengths 6, 8, and 10 and that no part of him is below the x -axis, find the minimum possible value of k .

Answer: $\boxed{\frac{24}{5}}$ By playing around, we find that Jarris should have his hypotenuse flat on the x -axis. The desired minimum value of k is then the length of the altitude to the hypotenuse. Thus, by computing the area of the triangle in two ways, $\frac{1}{2} \cdot 10 \cdot k = \frac{1}{2} \cdot 6 \cdot 8$ and so $k = \frac{24}{5}$.

2. Let $ABCD$ be an isosceles trapezoid such that $AD = BC$, $AB = 3$, and $CD = 8$. Let E be a point in the plane such that $BC = EC$ and $AE \perp EC$. Compute AE .

Answer: $\boxed{2\sqrt{6}}$ Let $r = BC = EC = AD$. $\triangle ACE$ has right angle at E , so by the Pythagorean Theorem,

$$AE^2 = AC^2 - CE^2 = AC^2 - r^2$$

Let the height of $\triangle ACD$ at A intersect DC at F . Once again, by the Pythagorean Theorem,

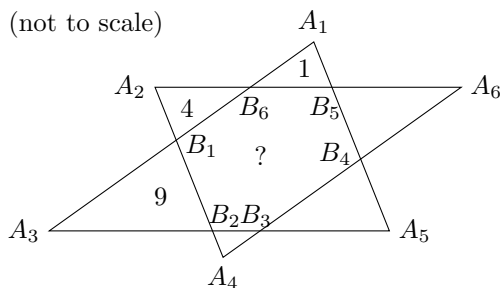
$$AC^2 = FC^2 + AF^2 = \left(\frac{8-3}{2} + 3\right)^2 + AD^2 - DF^2 = \left(\frac{11}{2}\right)^2 + r^2 - \left(\frac{5}{2}\right)^2$$

Plugging into the first equation,

$$AE^2 = \left(\frac{11}{2}\right)^2 + r^2 - \left(\frac{5}{2}\right)^2 - r^2,$$

so $AE = 2\sqrt{6}$.

3. Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon such that $A_iA_{i+2} \parallel A_{i+3}A_{i+5}$ for $i = 1, 2, 3$ (we take $A_{i+6} = A_i$ for each i). Segment A_iA_{i+2} intersects segment $A_{i+1}A_{i+3}$ at B_i , for $1 \leq i \leq 6$, as shown. Furthermore, suppose that $\triangle A_1A_3A_5 \cong \triangle A_4A_6A_2$. Given that $[A_1B_5B_6] = 1$, $[A_2B_6B_1] = 4$, and $[A_3B_1B_2] = 9$ (by $[XYZ]$ we mean the area of $\triangle XYZ$), determine the area of hexagon $B_1B_2B_3B_4B_5B_6$.



Answer: $\boxed{22}$ Because $B_6A_3B_3A_6$ and $B_1A_4B_4A_1$ are parallelograms, $B_6A_3 = A_6B_3$ and $A_1B_1 = A_4B_4$. By the congruence of the large triangles $A_1A_3A_5$ and $A_2A_4A_6$, $A_1A_3 = A_4A_6$. Thus, $B_6A_3 + A_1B_1 - A_1A_3 = A_6B_3 + A_4B_4 - A_4A_6$, so $B_6B_1 = B_3B_4$. Similarly, opposite sides of hexagon $B_1B_2B_3B_4B_5B_6$ are equal, and implying that the triangles opposite each other on the outside of this hexagon are congruent.

Furthermore, by definition $B_5B_6 \parallel A_3A_5$, $B_3B_4 \parallel A_1A_3$, $B_6B_1 \parallel A_4A_6$ and $B_1B_2 \parallel A_1A_5$. Let the area of triangle $A_1A_3A_5$ and triangle $A_2A_4A_6$ be k^2 . Then, by similar triangles,

$$\begin{aligned}\sqrt{\frac{1}{k^2}} &= \frac{A_1B_6}{A_1A_3} \\ \sqrt{\frac{4}{k^2}} &= \frac{B_6B_1}{A_4A_6} = \frac{B_1B_6}{A_1A_3} \\ \sqrt{\frac{9}{k^2}} &= \frac{A_3B_1}{A_1A_3}\end{aligned}$$

Summing yields $6/k = 1$, so $k^2 = 36$. To finish, the area of $B_1B_2B_3B_4B_5B_6$ is equivalent to the area of the triangle $A_1A_3A_5$ minus the areas of the smaller triangles provided in the hypothesis. Thus, our answer is $36 - 1 - 4 - 9 = 22$.

4. Let ω_1 and ω_2 be circles with centers O_1 and O_2 , respectively, and radii r_1 and r_2 , respectively. Suppose that O_2 is on ω_1 . Let A be one of the intersections of ω_1 and ω_2 , and B be one of the two intersections of line O_1O_2 with ω_2 . If $AB = O_1A$, find all possible values of $\frac{r_1}{r_2}$.

Answer: $\left[\frac{-1+\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2} \right]$ There are two configurations to this problem, namely, B in between the segment O_1O_2 and B on the ray O_1O_2 passing through the side of O_2 . Case 1: Let us only consider the triangle ABO_2 . $AB = AO_1 = O_1O_2 = r_1$ because of the hypothesis and AO_1 and O_1O_2 are radii of ω_1 . $O_2B = O_2A = r_2$ because they are both radii of ω_2 .

Then by the isosceles triangles, $\angle AO_1B = \angle ABO_1 = \angle ABO_2 = \angle O_2AB$. Thus can establish that $\triangle ABO_1 \sim \triangle O_2AB$.

Thus,

$$\begin{aligned}\frac{r_2}{r_1} &= \frac{r_1}{r_2 - r_1} \\ r_1^2 - r_2^2 + r_1r_2 &= 0\end{aligned}$$

By straightforward quadratic equation computation and discarding the negative solution,

$$\frac{r_1}{r_2} = \frac{-1 + \sqrt{5}}{2}$$

Case 2: Similar to case 1, let us only consider the triangle ABO_1 . $AB = AO_1 = O_1O_2 = r_1$ because of the hypothesis and AO_1 and O_1O_2 are radii of ω_1 . $O_2B = O_2A = r_2$ because they are both radii of ω_2 .

Then by the isosceles triangles, $\angle AO_1B = \angle ABO_1 = \angle ABO_2 = \angle O_2AB$. Thus can establish that $\triangle ABO_1 \sim \triangle O_2AB$.

Now,

$$\begin{aligned}\frac{r_2}{r_1} &= \frac{r_1}{r_2 + r_1} \\ r_1^2 - r_2^2 - r_1r_2 &= 0\end{aligned}$$

By straightforward quadratic equation computation and discarding the negative solution,

$$\frac{r_1}{r_2} = \frac{1 + \sqrt{5}}{2}$$

5. In triangle ABC , $\angle A = 45^\circ$ and M is the midpoint of \overline{BC} . \overline{AM} intersects the circumcircle of ABC for the second time at D , and $AM = 2MD$. Find $\cos \angle AOD$, where O is the circumcenter of ABC .

Answer: $\left[-\frac{1}{8} \right]$ $\angle BAC = 45^\circ$, so $\angle BOC = 90^\circ$. If the radius of the circumcircle is r , $BC = \sqrt{2}r$, and $BM = CM = \frac{\sqrt{2}}{2}r$. By power of a point, $BM \cdot CM = AM \cdot DM$, so $AM = r$ and $DM = \frac{1}{2}r$, and $AD = \frac{3}{2}r$. Using the law of cosines on triangle AOD gives $\cos \angle AOD = -\frac{1}{8}$.

6. Let $ABCD$ be a quadrilateral such that $\angle ABC = \angle CDA = 90^\circ$, and $BC = 7$. Let E and F be on BD such that AE and CF are perpendicular to BD . Suppose that $BE = 3$. Determine the product of the smallest and largest possible lengths of DF .

Answer: $\boxed{9}$ By inscribed angles, $\angle CDB = \angle CAB$, and $\angle ABD = \angle ACD$. By definition, $\angle AEB = \angle CDA = \angle ABC = \angle CFA$. Thus, $\triangle ABE \sim \triangle ADC$ and $\triangle CDF \sim \triangle CAB$. This shows that

$$\frac{BE}{AB} = \frac{CD}{CA} \text{ and } \frac{DF}{CD} = \frac{AB}{BD}$$

Based on the previous two equations, it is sufficient to conclude that $3 = EB = FD$. Thus, FD must equal to 3, and the product of its largest and smallest length is 9.

7. Let ABC be an obtuse triangle with circumcenter O such that $\angle ABC = 15^\circ$ and $\angle BAC > 90^\circ$. Suppose that AO meets BC at D , and that $OD^2 + OC \cdot DC = OC^2$. Find $\angle C$.

Answer: $\boxed{35}$ Let the radius of the circumcircle of $\triangle ABC$ be r .

$$\begin{aligned} OD^2 + OC \cdot CD &= OC^2 \\ OC \cdot CD &= OC^2 - OD^2 \\ OC \cdot CD &= (OC + OD)(OC - OD) \\ OC \cdot CD &= (r + OD)(r - OD) \end{aligned}$$

By the power of the point at D ,

$$\begin{aligned} OC \cdot CD &= BD \cdot DC \\ r &= BD \end{aligned}$$

Then, $\triangle OBD$ and $\triangle OAB$ and $\triangle AOC$ are isosceles triangles. Let $\angle DOB = \alpha$. $\angle BAO = 90 - \frac{\alpha}{2}$. In $\triangle ABD$, $15 + 90 - \frac{\alpha}{2} = \alpha$. This means that $\alpha = 70$. Furthermore, $\angle ACB$ intercepts minor arc AB , thus $\angle ACB = \frac{\angle AOB}{2} = \frac{70}{2} = 35$

8. Let $ABCD$ be a convex quadrilateral. Extend line CD past D to meet line AB at P and extend line CB past B to meet line AD at Q . Suppose that line AC bisects $\angle BAD$. If $AD = \frac{7}{4}$, $AP = \frac{21}{2}$, and $AB = \frac{14}{11}$, compute AQ .

Answer: $\boxed{\frac{42}{13}}$ We prove the more general statement $\frac{1}{AB} + \frac{1}{AP} = \frac{1}{AD} + \frac{1}{AQ}$, from which the answer easily follows.

Denote $\angle BAC = \angle CAD = \gamma$, $\angle BCA = \alpha$, $\angle ACD = \beta$. Then we have that by the law of sines, $\frac{AC}{AB} + \frac{AC}{AP} = \frac{\sin(\gamma+\alpha)}{\sin(\alpha)} + \frac{\sin(\gamma-\beta)}{\sin(\beta)} = \frac{\sin(\gamma-\alpha)}{\sin(\alpha)} + \frac{\sin(\gamma+\beta)}{\sin(\beta)} = \frac{AC}{AD} + \frac{AC}{AQ}$ where we have simply used the sine addition formula for the middle step.

Dividing the whole equation by AC gives the desired formula, from which we compute $AQ = (\frac{11}{14} + \frac{2}{21} - \frac{4}{7})^{-1} = \frac{42}{13}$.

9. Pentagon $ABCDE$ is given with the following conditions:

- $\angle CBD + \angle DAE = \angle BAD = 45^\circ$, $\angle BCD + \angle DEA = 300^\circ$
- $\frac{BA}{DA} = \frac{2\sqrt{2}}{3}$, $CD = \frac{7\sqrt{5}}{3}$, and $DE = \frac{15\sqrt{2}}{4}$
- $AD^2 \cdot BC = AB \cdot AE \cdot BD$

Compute BD .

Answer: $\boxed{\sqrt{39}}$ As a preliminary, we may compute that by the law of cosines, the ratio $\frac{AD}{BD} = \frac{3}{\sqrt{5}}$.

Now, construct the point P in triangle ABD such that $\triangle APB \sim \triangle AED$. Observe that $\frac{AP}{AD} = \frac{AE \cdot AB}{AD \cdot AD} = \frac{BC}{BD}$ (where we have used first the similarity and then condition 3). Furthermore, $\angle CBD = \angle DAB - \angle DAE = \angle DAB - \angle PAB = \angle PAD$ so by SAS, we have that $\triangle CBD \sim \triangle PAD$.

Therefore, by the similar triangles, we may compute $PB = DE \cdot \frac{AB}{AD} = 5$ and $PD = CD \cdot \frac{AD}{BD} = 7$. Furthermore, $\angle BPD = 360 - \angle BPA - \angle DPA = 360 - \angle BCD - \angle DEA = 60$ and therefore, by the law of cosines, we have that $BD = \sqrt{39}$.

10. Triangle ABC is inscribed in a circle ω . Let the bisector of angle A meet ω at D and BC at E . Let the reflections of A across D and C be D' and C' , respectively. Suppose that $\angle A = 60^\circ$, $AB = 3$, and $AE = 4$. If the tangent to ω at A meets line BC at P , and the circumcircle of APD' meets line BC at F (other than P), compute FC' .

Answer: $\boxed{2\sqrt{13} - 6\sqrt{3}}$ First observe that by angle chasing, $\angle PAE = 180 - \frac{1}{2}\angle BAC - \angle ABC = \angle AEP$, so by the cyclic quadrilateral $APD'F$, $\angle EFD' = \angle PAE = \angle PEA = \angle D'EF$. Thus, $ED'F$ is isosceles.

Define B' to be the reflection of A about B , and observe that $B'C' \parallel EF$ and $B'D'C'$ is isosceles. It follows that $B'EFC'$ is an isosceles trapezoid, so $FC' = B'E$, which by the law of cosines, is equal to $\sqrt{AB'^2 + AE^2 - 2AB \cdot AE \cos 30} = 2\sqrt{13} - 6\sqrt{3}$.

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Guts Round

1. [4] Arpon chooses a positive real number k . For each positive integer n , he places a marker at the point (n, nk) in the (x, y) plane. Suppose that two markers whose x coordinates differ by 4 have distance 31. What is the distance between the markers at $(7, 7k)$ and $(19, 19k)$?

Answer: $\boxed{93}$ The difference of the x -coordinates of the markers is $12 = 3 \cdot 4$. Thus, by similar triangles (where we draw right triangles whose legs are parallel to the axes and whose hypotenuses lie along the line $y = kx$), the distance between the markers is $3 \cdot 31 = 93$.

2. [4] The real numbers x, y, z satisfy $0 \leq x \leq y \leq z \leq 4$. If their squares form an arithmetic progression with common difference 2, determine the minimum possible value of $|x - y| + |y - z|$.

Answer: $\boxed{4 - 2\sqrt{3}}$ Clearly $|x - y| + |y - z| = z - x = \frac{z^2 - x^2}{z + x} = \frac{4}{z + x}$, which is minimized when $z = 4$ and $x = \sqrt{12}$. Thus, our answer is $4 - \sqrt{12} = 4 - 2\sqrt{3}$.

3. [4] Find the rightmost non-zero digit of the expansion of $(20)(13!)$.

Answer: $\boxed{6}$ We can rewrite this as $(10 \cdot 2)(13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = (10^3)(2 \cdot 13 \cdot 12 \cdot 11 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 4 \cdot 3)$; multiplying together the units digits for the terms not equal to 10 reveals that the rightmost non-zero digit is 6.

4. [4] Spencer is making burritos, each of which consists of one wrap and one filling. He has enough filling for up to four beef burritos and three chicken burritos. However, he only has five wraps for the burritos; in how many orders can he make exactly five burritos?

Answer: $\boxed{25}$ Spencer's burrito-making can include either 3, 2, or 1 chicken burrito; consequently, he has $\binom{5}{3} + \binom{5}{2} + \binom{5}{1} = 25$ orders in which he can make burritos.

5. [5] Rahul has ten cards face-down, which consist of five distinct pairs of matching cards. During each move of his game, Rahul chooses one card to turn face-up, looks at it, and then chooses another to turn face-up and looks at it. If the two face-up cards match, the game ends. If not, Rahul flips both cards face-down and keeps repeating this process. Initially, Rahul doesn't know which cards are which. Assuming that he has perfect memory, find the smallest number of moves after which he can guarantee that the game has ended.

Answer: $\boxed{4}$ Label the 10 cards $a_1, a_2, \dots, a_5, b_1, b_2, \dots, b_5$ such that a_i and b_i match for $1 \leq i \leq 5$.

First, we'll show that Rahul cannot always end the game in less than 4 moves, in particular, when he turns up his fifth card (during the third move), it is possible that the card he flips over is not one which he has yet encountered; consequently, he will not guarantee being able to match it, so he cannot guarantee that the game can end in three moves.

However, Rahul can always end the game in 4 moves. To do this, he can turn over 6 distinct cards in his first 3 moves. If we consider the 5 sets of cards $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}, \{a_5, b_5\}$, then by the pigeonhole principle, at least 2 of the 6 revealed cards must be from the same set. Rahul can then turn over those 2 cards on the fourth move, ending the game.

6. [5] Let R be the region in the Cartesian plane of points (x, y) satisfying $x \geq 0, y \geq 0$, and $x + y + \lfloor x \rfloor + \lfloor y \rfloor \leq 5$. Determine the area of R .

Answer: $\boxed{\frac{9}{2}}$ We claim that a point in the first quadrant satisfies the desired property if the point is below the line $x + y = 3$ and does not satisfy the desired property if it is above the line.

To see this, for a point inside the region, $x + y < 3$ and $\lfloor x \rfloor + \lfloor y \rfloor \leq x + y < 3$. However, $\lfloor x \rfloor + \lfloor y \rfloor$ must equal to an integer. Thus, $\lfloor x \rfloor + \lfloor y \rfloor \leq 2$. Adding these two equations, $x + y + \lfloor x \rfloor + \lfloor y \rfloor < 5$, which satisfies the desired property. Conversely, for a point outside the region, $\lfloor x \rfloor + \lfloor y \rfloor + \{x\} + \{y\} = x + y > 3$. However, $\{x\} + \{y\} < 2$. Thus, $\lfloor x \rfloor + \lfloor y \rfloor > 1$, so $\lfloor x \rfloor + \lfloor y \rfloor \geq 2$, implying that $x + y + \lfloor x \rfloor + \lfloor y \rfloor > 5$.

To finish, R is the region bounded by the x-axis, the y-axis, and the line $x + y = 3$ is a right triangle whose legs have length 3. Consequently, R has area $\frac{9}{2}$.

7. [5] Find the number of positive divisors d of $15! = 15 \cdot 14 \cdot \dots \cdot 2 \cdot 1$ such that $\gcd(d, 60) = 5$.

Answer: $\boxed{36}$ Since $\gcd(d, 60) = 5$, we know that $d = 5^i d'$ for some integer $i > 0$ and some integer d' which is relatively prime to 60. Consequently, d' is a divisor of $(15!)/5$; eliminating common factors with 60 gives that d' is a factor of $(7^2)(11)(13)$, which has $(2 + 1)(1 + 1)(1 + 1) = 12$ factors. Finally, i can be 1, 2, or 3, so there are a total of $3 \cdot 12 = 36$ possibilities.

8. [5] In a game, there are three indistinguishable boxes; one box contains two red balls, one contains two blue balls, and the last contains one ball of each color. To play, Raj first predicts whether he will draw two balls of the same color or two of different colors. Then, he picks a box, draws a ball at random, looks at the color, and replaces the ball in the same box. Finally, he repeats this; however, the boxes are not shuffled between draws, so he can determine whether he wants to draw again from the same box. Raj wins if he predicts correctly; if he plays optimally, what is the probability that he will win?

Answer: $\boxed{\frac{5}{6}}$ Call the box with two red balls box 1, the box with one of each color box 2, and the box with two blue balls box 3. Without loss of generality, assume that the first ball that Bob draws is red. If Bob picked box 1, then he would have picked a red ball with probability 1, and if Bob picked box 2, then he would have picked a red ball with probability $\frac{1}{2}$. Therefore, the probability that he picked box 1 is $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$, and the probability that he picked box 2 is $\frac{1}{3}$. We will now consider both possible predictions and find which one gives a better probability of winning, assuming optimal play.

If Bob predicts that he will draw two balls of the same color, then there are two possible plays: he draws from the same box, or he draws from a different box. If he draws from the same box, then in the $\frac{2}{3}$ chance that he originally picked box 1, he will always win, and in the $\frac{1}{3}$ chance that he picked box 2, he will win with probability $\frac{1}{2}$, for a total probability of $\frac{2}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{6}$. If he draws from a different box, then if he originally picked box 1, he will win with probability $\frac{1}{4}$ and if he originally picked box 2, he will win with probability $\frac{1}{2}$, for a total probability of $\frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{3}$.

If Bob predicts that he will draw two balls of different colors, then we can consider the same two possible plays. Using similar calculations, if he draws from the same box, then he will win with probability $\frac{1}{6}$, and if he draws from a different box, then he will win with probability $\frac{2}{3}$. Looking at all cases, Bob's best play is to predict that he will draw two balls of the same color and then draw the second ball from the same box, with a winning probability of $\frac{5}{6}$.

9. [6] I have 8 unit cubes of different colors, which I want to glue together into a $2 \times 2 \times 2$ cube. How many distinct $2 \times 2 \times 2$ cubes can I make? Rotations of the same cube are not considered distinct, but reflections are.

Answer: $\boxed{1680}$ Our goal is to first pin down the cube, so it can't rotate. Without loss of generality, suppose one of the unit cubes is purple, and let the purple cube be in the top left front position. Now, look at the three positions that share a face with the purple cube. There are $\binom{7}{3}$ ways to pick the three cubes that fill those positions and two ways to position them that are rotationally distinct. Now, we've taken care of any possible rotations, so there are simply $4!$ ways to position the final four cubes. Thus, our answer is $\binom{7}{3} \cdot 2 \cdot 4! = 1680$ ways.

10. [6] Wesyu is a farmer, and she's building a cao (a relative of the cow) pasture. She starts with a triangle $A_0A_1A_2$ where angle A_0 is 90° , angle A_1 is 60° , and A_0A_1 is 1. She then extends the pasture. First, she extends A_2A_0 to A_3 such that $A_3A_0 = \frac{1}{2}A_2A_0$ and the new pasture is triangle $A_1A_2A_3$. Next, she extends A_3A_1 to A_4 such that $A_4A_1 = \frac{1}{6}A_3A_1$. She continues, each time extending A_nA_{n-2} to A_{n+1} such that $A_{n+1}A_{n-2} = \frac{1}{2^{n-2}}A_nA_{n-2}$. What is the smallest K such that her pasture never exceeds an area of K ?

Answer: $\boxed{\sqrt{3}}$ First, note that for any i , after performing the operation on triangle $A_iA_{i+1}A_{i+2}$, the resulting pasture is triangle $A_{i+1}A_{i+2}A_{i+3}$. Let K_i be the area of triangle $A_iA_{i+1}A_{i+2}$. From

$A_{n+1}A_{n-2} = \frac{1}{2^{n-2}}A_nA_{n-2}$ and $A_nA_{n+1} = A_nA_{n-2} + A_{n-2}A_{n+1}$, we have $A_nA_{n+1} = (1 + \frac{1}{2^{n-2}})A_nA_{n-2}$. We also know that the area of a triangle is half the product of its base and height, so if we let the base of triangle $A_{n-2}A_{n-1}A_n$ be A_nA_{n-2} , its area is $K_{n-2} = \frac{1}{2}hA_nA_{n-2}$. The area of triangle $A_{n-1}A_nA_{n+1}$ is $K_{n-1} = \frac{1}{2}hA_nA_{n+1}$. The h 's are equal because the distance from A_{n-1} to the base does not change.

We now have $\frac{K_{n-1}}{K_{n-2}} = \frac{A_nA_{n+1}}{A_nA_{n-2}} = 1 + \frac{1}{2^{n-2}} = \frac{2^n-1}{2^{n-2}}$. Therefore, $\frac{K_1}{K_0} = \frac{3}{2}$, $\frac{K_2}{K_0} = \frac{K_2}{K_1} \frac{K_1}{K_0} = \frac{7}{6} \cdot \frac{3}{2} = \frac{7}{4}$, $\frac{K_3}{K_0} = \frac{K_3}{K_2} \frac{K_2}{K_0} = \frac{15}{14} \cdot \frac{7}{4} = \frac{15}{8}$. We see the pattern $\frac{K_n}{K_0} = \frac{2^{n+1}-1}{2^n}$, which can be easily proven by induction. As n approaches infinity, $\frac{K_n}{K_0}$ grows arbitrarily close to 2, so the smallest K such that the pasture never exceeds an area of K is $2K_0 = \sqrt{3}$.

11. [6] Compute the prime factorization of 1007021035035021007001. (You should write your answer in the form $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where p_1, \dots, p_k are distinct prime numbers and e_1, \dots, e_k are positive integers.)

Answer: $\boxed{7^7 \cdot 11^7 \cdot 13^7}$ The number in question is

$$\sum_{i=0}^7 \binom{7}{i} 1000^i = (1000 + 1)^7 = 1001^7 = 7^7 \cdot 11^7 \cdot 13^7.$$

12. [6] For how many integers $1 \leq k \leq 2013$ does the decimal representation of k^k end with a 1?

Answer: $\boxed{202}$ We claim that this is only possible if k has a units digit of 1. Clearly, it is true in these cases. Additionally, k^k cannot have a units digit of 1 when k has a units digit of 2, 4, 5, 6, or 8. If k has a units digit of 3 or 7, then k^k has a units digit of 1 if and only if $4|k$, a contradiction. Similarly, if k has a units digit of 9, then k^k has a units digit of 1 if and only if $2|k$, also a contradiction. Since there are 202 integers between 1 and 2013, inclusive, with a units digit of 1, there are 202 such k which fulfill our criterion.

13. [8] Find the smallest positive integer n such that $\frac{5^{n+1} + 2^{n+1}}{5^n + 2^n} > 4.99$.

Answer: $\boxed{7}$ Writing $5^{n+1} = 5 \cdot 5^n$ and $2^{n+1} = 2 \cdot 2^n$ and cross-multiplying yields $0.01 \cdot 5^n > 2.99 \cdot 2^n$, and re-arranging yields $(2.5)^n > 299$. A straightforward calculation shows that the smallest n for which this is true is $n = 7$.

14. [8] Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D , and the angle bisector from C intersects \overline{AB} at E . If $\frac{DE}{DC} = \frac{1}{3}$, compute $\frac{AB}{AC}$.

Answer: $\boxed{\frac{7}{9}}$ Let $AE = x$ and $BE = y$. Using angle-bisector theorem on $\triangle ACE$ we have $x : DE = AC : DC$, so $AC = 3x$. Using some angle chasing, it is simple to see that $\angle ADE = \angle AED$, so $AD = AE = x$. Then, note that $\triangle CDA \sim \triangle CEB$, so $y : (DC + DE) = x : DC$, so $y : x = 1 + \frac{1}{3} = \frac{4}{3}$, so $AB = x + \frac{4}{3}x = \frac{7}{3}x$. Thus the desired answer is $AB : AC = \frac{7}{3}x : 3x = \frac{7}{9}$.

15. [8] Tim and Allen are playing a match of *tenus*. In a match of *tenus*, the two players play a series of games, each of which is won by one of the two players. The match ends when one player has won exactly two more games than the other player, at which point the player who has won more games wins the match. In odd-numbered games, Tim wins with probability $3/4$, and in the even-numbered games, Allen wins with probability $3/4$. What is the expected number of games in a match?

Answer: $\boxed{\frac{16}{3}}$ Let the answer be E . If Tim wins the first game and Allen wins the second game or vice versa, which occurs with probability $(3/4)^2 + (1/4)^2 = 5/8$, the expected number of additional games is just E , so the expected total number of games is $E + 2$. If, on the other hand, one of Tim and Allen wins both of the first two games, with probability $1 - (5/8) = 3/8$, there are exactly 2 games in the match. It follows that

$$E = \frac{3}{8} \cdot 2 + \frac{5}{8} \cdot (E + 2),$$

and solving gives $E = \frac{16}{3}$.

16. [8] The walls of a room are in the shape of a triangle ABC with $\angle ABC = 90^\circ$, $\angle BAC = 60^\circ$, and $AB = 6$. Chong stands at the midpoint of BC and rolls a ball toward AB . Suppose that the ball bounces off AB , then AC , then returns exactly to Chong. Find the length of the path of the ball.

Answer: $\boxed{3\sqrt{21}}$ Let C' be the reflection of C across AB and B' be the reflection of B across AC' ; note that $\overline{B'A}, C$ are collinear by angle chasing. The image of the path under these reflections is just the line segment MM' , where M is the midpoint of BC and M' is the midpoint of $B'C'$, so our answer is just the length of MM' . Applying the Law of Cosines to triangle $M'C'M$, we have $MM'^2 = 27 + 243 - 2 \cdot 3\sqrt{3} \cdot 9\sqrt{3} \cdot \frac{1}{2} = 189$, so $MM' = 3\sqrt{21}$.

17. [11] The lines $y = x$, $y = 2x$, and $y = 3x$ are the three medians of a triangle with perimeter 1. Find the length of the longest side of the triangle.

Answer: $\boxed{\frac{\sqrt{58}}{2+\sqrt{34}+\sqrt{58}}}$ The three medians of a triangle contain its vertices, so the three vertices of the triangle are (a, a) , $(b, 2b)$ and $(c, 3c)$ for some a, b , and c . Then, the midpoint of (a, a) and $(b, 2b)$, which is $(\frac{a+b}{2}, \frac{a+2b}{2})$, must lie along the line $y = 3x$. Therefore,

$$\begin{aligned}\frac{a+2b}{2} &= 3 \cdot \frac{a+b}{2}, \\ a+2b &= 3a+3b, \\ -2a &= b.\end{aligned}$$

Similarly, the midpoint of $(b, 2b)$ and $(c, 3c)$, which is $(\frac{b+c}{2}, \frac{2b+3c}{2})$, must lie along the line $y = x$. Therefore,

$$\begin{aligned}\frac{2b+3c}{2} &= \frac{b+c}{2}, \\ 2b+3c &= b+c, \\ b &= -2c, \\ c &= -\frac{1}{2}b = a.\end{aligned}$$

From this, three points can be represented as (a, a) , $(-2a, -4a)$, and $(a, 3a)$. Using the distance formula, the three side lengths of the triangle are $2|a|$, $\sqrt{34}|a|$, and $\sqrt{58}|a|$. Since the perimeter of the triangle is 1, we find that $|a| = \frac{1}{2+\sqrt{34}+\sqrt{58}}$ and therefore the longest side length is $\frac{\sqrt{58}}{2+\sqrt{34}+\sqrt{58}}$.

18. [11] Define the sequence of positive integers $\{a_n\}$ as follows. Let $a_1 = 1, a_2 = 3$, and for each $n > 2$, let a_n be the result of expressing a_{n-1} in base $n-1$, then reading the resulting numeral in base n , then adding 2 (in base n). For example, $a_2 = 3_{10} = 11_2$, so $a_3 = 11_3 + 2_3 = 6_{10}$. Express a_{2013} in base ten.

Answer: $\boxed{23097}$ We claim that for nonnegative integers m and for $0 \leq n < 3 \cdot 2^m$, $a_{3 \cdot 2^m + n} = (3 \cdot 2^m + n)(m+2) + 2n$. We will prove this by induction; the base case for $a_3 = 6$ (when $m = 0, n = 0$) is given in the problem statement. Now, suppose that this is true for some pair m and n . We will divide this into two cases:

- Case 1: $n < 3 \cdot 2^m - 1$. Then, we want to prove that this is true for m and $n+1$. In particular, writing $a_{3 \cdot 2^m + n}$ in base $3 \cdot 2^m + n$ results in the digits $m+2$ and $2n$. Consequently, reading it in base $3 \cdot 2^m + n + 1$ gives $a_{3 \cdot 2^m + n + 1} = 2 + (3 \cdot 2^m + n + 1)(m+2) + 2n = (2 \cdot 2^m + n + 1)(m+2) + 2(n+1)$, as desired.
- Case 2: $n = 3 \cdot 2^m - 1$. Then, we want to prove that this is true for $m+1$ and 0. Similarly to the previous case, we get that $a_{3 \cdot 2^m + n + 1} = a_{3 \cdot 2^{m+1}} = 2 + (3 \cdot 2^m + n + 1)(m+2) + 2n = 2 + (3 \cdot 2^{m+1})(m+2) + 2(3 \cdot 2^m - 1) = (3 \cdot 2^{m+1} + 0)((m+1) + 2) + 2(0)$, as desired.

In both cases, we have proved our claim.

19. [11] An isosceles trapezoid $ABCD$ with bases AB and CD has $AB = 13$, $CD = 17$, and height 3. Let E be the intersection of AC and BD . Circles Ω and ω are circumscribed about triangles ABE and CDE . Compute the sum of the radii of Ω and ω .

Answer: 39 Let Ω have center O and radius R and let ω have center P and radius M . Let Q be the intersection of AB and OE . Note that OE is the perpendicular bisector of AB because the trapezoid is isosceles. Also, we see OE is the circumradius of Ω . On the other hand, we know by similarity of $\triangle AEB$ and $\triangle CED$ that $QE = \frac{13}{13+17} \cdot 3 = \frac{13}{30} \cdot 3$. And, because $BQ = 13/2$ and is perpendicular to OQ , we can apply the Pythagorean theorem to $\triangle OQB$ to see $OQ = \sqrt{R^2 - \left(\frac{13}{2}\right)^2}$. Since $OE = OQ + QE$, $R = \frac{13}{30} \cdot 3 + \sqrt{R^2 - \left(\frac{13}{2}\right)^2}$. Solving this equation for R yields $R = \frac{13}{30} \cdot 39$. Since by similarity $M = \frac{17}{13}R$, we know $R + M = \frac{30}{13}R$, so $R + M = 39$.

20. [11] The polynomial $f(x) = x^3 - 3x^2 - 4x + 4$ has three real roots r_1 , r_2 , and r_3 . Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1 , s_2 , and s_3 , where $s_1 = r_1 + r_2z + r_3z^2$, $s_2 = r_1z + r_2z^2 + r_3$, $s_3 = r_1z^2 + r_2 + r_3z$, and $z = \frac{-1+i\sqrt{3}}{2}$. Find the real part of the sum of the coefficients of $g(x)$.

Answer: -26 Note that $z = e^{\frac{2\pi}{3}i} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$, so that $z^3 = 1$ and $z^2 + z + 1 = 0$. Also, $s_2 = s_1z$ and $s_3 = s_1z^2$.

Then, the sum of the coefficients of $g(x)$ is $g(1) = (1-s_1)(1-s_2)(1-s_3) = (1-s_1)(1-s_1z)(1-s_1z^2) = 1 - (1+z+z^2)s_1 + (z+z^2+z^3)s_1^2 - z^3s_1^3 = 1 - s_1^3$.

Meanwhile, $s_1^3 = (r_1 + r_2z + r_3z^2)^3 = r_1^3 + r_2^3 + r_3^3 + 3r_1^2r_2z + 3r_1^2r_3z^2 + 3r_2^2r_3z + 3r_2^2r_1z^2 + 3r_3^2r_1z + 3r_3^2r_2z^2 + 6r_1r_2r_3$.

Since the real parts of both z and z^2 are $-\frac{1}{2}$, and since all of r_1 , r_2 , and r_3 are real, the real part of s_1^3 is $r_1^3 + r_2^3 + r_3^3 - \frac{3}{2}(r_1^2r_2 + \dots + r_3^2r_2) + 6r_1r_2r_3 = (r_1 + r_2 + r_3)^3 - \frac{9}{2}(r_1 + r_2 + r_3)(r_1r_2 + r_2r_3 + r_3r_1) + \frac{27}{2}r_1r_2r_3 = 3^3 - \frac{9}{2} \cdot 3 \cdot -4 + \frac{27}{2} \cdot -4 = 27$.

Therefore, the answer is $1 - 27 = -26$.

21. [14] Find the number of positive integers $j \leq 3^{2013}$ such that

$$j = \sum_{k=0}^m \left((-1)^k \cdot 3^{a_k} \right)$$

for some strictly increasing sequence of nonnegative integers $\{a_k\}$. For example, we may write $3 = 3^1$ and $55 = 3^0 - 3^3 + 3^4$, but 4 cannot be written in this form.

Answer: 2²⁰¹³ Clearly m must be even, or the sum would be negative. Furthermore, if $a_m \leq 2013$, the sum cannot exceed 3^{2013} since $j = 3^{a_m} + \sum_{k=0}^{m-1} \left((-1)^k \cdot 3^{a_k} \right) \leq 3^{a_m}$. Likewise, if $a_m > 2013$, then the sum necessarily exceeds 3^{2013} , which is not hard to see by applying the Triangle Inequality and summing a geometric series. Hence, the elements of $\{a_k\}$ can be any subset of $\{0, 1, \dots, 2013\}$ with an odd number of elements. Since the number of even-sized subsets is equal to the number of odd-sized elements, there are $\frac{2^{2014}}{2} = 2^{2013}$ such subsets.

Now, it suffices to show that given such an $\{a_k\}$, the value of j can only be obtained in this way. Suppose for the the sake of contradiction that there exist two such sequences $\{a_k\}_{0 \leq k \leq m_a}$ and $\{b_k\}_{0 \leq k \leq m_b}$ which produce the same value of j for j positive or negative, where we choose $\{a_k\}, \{b_k\}$ such that $\min(m_a, m_b)$ is as small as possible. Then, we note that since $3^{a_0} + 3^{a_1} + \dots + 3^{(a_{m_a}-1)} \leq 3^0 + 3^1 + \dots + 3^{(a_{m_a}-1)} < 2(3^{(a_{m_a}-1)})$, we have that $\sum_{k=0}^{m_a} \left((-1)^k \cdot 3^{a_k} \right) > 3^{(a_{m_a}-1)}$. Similarly, we get that $3^{(a_{m_b}-1)} \geq \sum_{k=0}^{m_b} \left((-1)^k \cdot 3^{a_k} \right) > 3^{(m_b-1)}$; for the two to be equal, we must have $m_a = m_b$. However, this means that the sequences obtained by removing a_{m_a} and a_{m_b} from $\{a_k\}, \{b_k\}$ have smaller maximum value but still produce the same alternating sum, contradicting our original assumption.

22. [14] Sherry and Val are playing a game. Sherry has a deck containing 2011 red cards and 2012 black cards, shuffled randomly. Sherry flips these cards over one at a time, and before she flips each card over, Val guesses whether it is red or black. If Val guesses correctly, she wins 1 dollar; otherwise, she loses 1 dollar. In addition, Val must guess red exactly 2011 times. If Val plays optimally, what is her expected profit from this game?

Answer: $\boxed{\frac{1}{4023}}$ We will prove by induction on $r + b$ that the expected profit for guessing if there are r red cards, b black cards, and where g guesses must be red, is equal to $(b - r) + \frac{2(r-b)}{(r+b)}g$. It is not difficult to check that this holds in the cases $(r, b, g) = (1, 0, 0), (0, 1, 0), (1, 0, 1), (0, 1, 1)$. Then, suppose that this is true as long as the number of cards is strictly less than $r + b$; we will prove that it also holds true when there are r red and b blue cards.

Let $f(r, b, g)$ be her expected profit under these conditions. If she guesses red, her expected profit is

$$\frac{r}{r+b}(1 + f(r-1, b, g-1)) + \frac{b}{r+b}(-1 + f(r, b-1, g-1)) = (b-r) + \frac{2(r-b)}{(r+b)}g.$$

Similarly, if she guesses black, her expected profit is

$$\frac{r}{r+b}(-1 + f(r-1, b, g)) + \frac{b}{r+b}(1 + f(r, b-1, g)) = (b-r) + \frac{2(r-b)}{(r+b)}g.$$

Plugging in the our starting values gives an expected profit of $\frac{1}{4023}$.

23. [14] Let $ABCD$ be a parallelogram with $AB = 8$, $AD = 11$, and $\angle BAD = 60^\circ$. Let X be on segment CD with $CX/XD = 1/3$ and Y be on segment AD with $AY/YD = 1/2$. Let Z be on segment AB such that AX , BY , and DZ are concurrent. Determine the area of triangle XYZ .

Answer: $\boxed{\frac{19\sqrt{3}}{2}}$ Let AX and BD meet at P . We have $DP/PB = DX/AB = 3/4$. Now, applying Ceva's Theorem in triangle ABD , we see that

$$\frac{AZ}{ZB} = \frac{DP}{PB} \cdot \frac{AY}{YD} = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}.$$

Now,

$$\frac{[AYZ]}{[ABCD]} = \frac{[AYZ]}{2[ABD]} = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{11} = \frac{1}{22},$$

and similarly

$$\frac{[DYZ]}{[ABCD]} = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}.$$

Also,

$$\frac{[XCBZ]}{[ABCD]} = \frac{1}{2} \left(\frac{1}{4} + \frac{8}{11} \right) = \frac{43}{88}.$$

The area of XYZ is the rest of the fraction of the area of $ABCD$ not covered by the three above polygons, which by a straightforward calculation $19/88$ the area of $ABCD$, so our answer is

$$8 \cdot 11 \cdot \sin 60^\circ \cdot \frac{19}{88} = \frac{19\sqrt{3}}{2}.$$

24. [14] Given a point p and a line segment l , let $d(p, l)$ be the distance between them. Let A , B , and C be points in the plane such that $AB = 6$, $BC = 8$, $AC = 10$. What is the area of the region in the (x, y) -plane formed by the ordered pairs (x, y) such that there exists a point P inside triangle ABC with $d(P, AB) + x = d(P, BC) + y = d(P, AC)$?

Answer: $\boxed{\frac{288}{5}}$ Place ABC in the coordinate plane so that $A = (0, 6), B = (0, 0), C = (8, 0)$. Consider a point $P = (a, b)$ inside triangle ABC . Clearly, $d(P, AB) = a, d(P, BC) = b$. Now, we see

that the area of triangle ABC is $\frac{6 \cdot 8}{2} = 24$, but may also be computed by summing the areas of triangles PAB, PBC, PCA . The area of triangle PAB is $\frac{6 \cdot a}{2} = 3a$, and similarly the area of triangle PBC is $4b$. Thus, it follows easily that $d(P, CA) = \frac{24 - 3a - 4b}{5}$. Now, we have

$$(x, y) = \left(\frac{24}{5} - \frac{8}{5}a - \frac{4}{5}b, \frac{24}{5} - \frac{3}{5}a - \frac{9}{5}b \right).$$

The desired region is the set of (x, y) obtained by those (a, b) subject to the constraints $a \geq 0, b \geq 0, 6a + 8b \leq 48$.

Consequently, our region is the triangle whose vertices are obtained by evaluating (x, y) at the vertices (a, b) of the triangle. To see this, let $f(a, b)$ output the corresponding (x, y) according to the above. Then, we can write every point P in ABC as $P = m(0, 0) + n(0, 6) + p(8, 0)$ for some $m + n + p = 1$. Then, $f(P) = mf(0, 0) + nf(0, 6) + pf(8, 0) = m(\frac{24}{5}, \frac{24}{5}) + n(-8, 0) + p(0, -6)$, so $f(P)$ ranges over the triangle with those three vertices.

Therefore, we need the area of the triangle with vertices $(\frac{24}{5}, \frac{24}{5}), (0, -6), (-8, 0)$, which is easily computed (for example, using determinants) to be $\frac{288}{5}$.

25. [17] The sequence (z_n) of complex numbers satisfies the following properties:

- z_1 and z_2 are not real.
- $z_{n+2} = z_{n+1}^2 z_n$ for all integers $n \geq 1$.
- $\frac{z_{n+3}}{z_n^2}$ is real for all integers $n \geq 1$.
- $\left| \frac{z_3}{z_4} \right| = \left| \frac{z_4}{z_5} \right| = 2$.

Find the product of all possible values of z_1 .

Answer: 65536 All complex numbers can be expressed as $r(\cos \theta + i \sin \theta) = re^{i\theta}$. Let z_n be $r_n e^{i\theta_n}$.

$$\frac{z_{n+3}}{z_n^2} = \frac{z_{n+2} z_{n+1}}{z_n^2} = \frac{z_{n+1}^2 z_n^2}{z_n^2} = z_{n+1}^5 \text{ is real for all } n \geq 1, \text{ so } \theta_n = \frac{\pi k_n}{5} \text{ for all } n \geq 2, \text{ where } k_n \text{ is an}$$

integer. $\theta_1 + 2\theta_2 = \theta_3$, so we may write $\theta_1 = \frac{\pi k_1}{5}$ with k_1 an integer.

$$\frac{r_3}{r_4} = \frac{r_4}{r_5} \Rightarrow r_5 = \frac{r_4^2}{r_3} = r_4^2 r_3, \text{ so } r_3 = 1. \quad \frac{r_3}{r_4} = 2 \Rightarrow r_4 = \frac{1}{2}, r_4 = r_3^2 r_2 \Rightarrow r_2 = \frac{1}{2}, \text{ and } r_3 = r_2^2 r_1 \Rightarrow r_1 = 4.$$

Therefore, the possible values of z_1 are the nonreal roots of the equation $x^{10} - 4^{10} = 0$, and the product of the eight possible values is $\frac{4^{10}}{4^2} = 4^8 = 65536$. For these values of z_1 , it is not difficult to construct a sequence which works, by choosing z_2 nonreal so that $|z_2| = \frac{1}{2}$.

26. [17] Triangle ABC has perimeter 1. Its three altitudes form the side lengths of a triangle. Find the set of all possible values of $\min(AB, BC, CA)$.

Answer: $(\frac{3-\sqrt{5}}{4}, \frac{1}{3}]$ Let a, b, c denote the side lengths BC, CA , and AB , respectively. Without loss of generality, assume $a \leq b \leq c$; we are looking for the possible range of a .

First, note that the maximum possible value of a is $\frac{1}{3}$, which occurs when ABC is equilateral. It remains to find a lower bound for a .

Now rewrite $c = xa$ and $b = ya$, where we have $x \geq y \geq 1$. Note that for a non-equilateral triangle, $x > 1$. The triangle inequality gives us $a + b > c$, or equivalently, $y > x - 1$. If we let K be the area, the condition for the altitudes gives us $\frac{2K}{c} + \frac{2K}{b} > \frac{2K}{a}$, or equivalently, $\frac{1}{b} > \frac{1}{a} - \frac{1}{c}$, which after some manipulation yields $y < \frac{x}{x-1}$. Putting these conditions together yields $x - 1 < \frac{x}{x-1}$, and after rearranging and solving a quadratic, we get $x < \frac{3+\sqrt{5}}{2}$.

We now use the condition $a(1+x+y) = 1$, and to find a lower bound for a , we need an upper bound for $1+x+y$. We know that $1+x+y < 1+x+\frac{x}{x-1} = x-1+\frac{1}{x-1}+3$.

Now let $f(x) = x-1+\frac{1}{x-1}+3$. If $1 < x < 2$, then $1+x+y \leq 1+2x < 5$. But for $x \geq 2$, we see that $f(x)$ attains a minimum of 5 at $x=2$ and continues to strictly increase after that point. Since $x < \frac{3+\sqrt{5}}{2}$, we have $f(x) < f\left(\frac{3+\sqrt{5}}{2}\right) = 3+\sqrt{5} > 5$, so this is a better upper bound than the case for which $1 < x < 2$. Therefore, $a > \left(\frac{1}{3+\sqrt{5}}\right) = \frac{3-\sqrt{5}}{4}$.

For any a such that $\sqrt{5}-2 \geq a > \frac{3-\sqrt{5}}{4}$, we can let $b = \frac{1+\sqrt{5}}{2}a$ and $c = 1-a-b$. For any other possible a , we can let $b=c = \frac{1-a}{2}$. The triangle inequality and the altitude condition can both be verified algebraically.

We now conclude that the set of all possible a is $\frac{3-\sqrt{5}}{4} < a \leq \frac{1}{3}$.

27. [17] Let W be the hypercube $\{(x_1, x_2, x_3, x_4) \mid 0 \leq x_1, x_2, x_3, x_4 \leq 1\}$. The intersection of W and a hyperplane parallel to $x_1+x_2+x_3+x_4=0$ is a non-degenerate 3-dimensional polyhedron. What is the maximum number of faces of this polyhedron?

Answer: 8 The number of faces in the polyhedron is equal to the number of distinct cells (3-dimensional faces) of the hypercube whose interior the hyperplane intersects. However, it is possible to arrange the hyperplane such that it intersects all 8 cells. Namely, $x_1+x_2+x_3+x_4 = \frac{3}{2}$ intersects all 8 cells because it passes through $(0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ (which is on the cell $x_1=0$), $(1, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ (which is on the cell $x_1=1$), and the points of intersection with the other 6 cells can be found by permuting these quadruples.

28. [17] Let $z_0+z_1+z_2+\dots$ be an infinite complex geometric series such that $z_0=1$ and $z_{2013} = \frac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

Answer: $\frac{2013^{2014}}{2013^{2013}-1}$ Clearly, the possible common ratios are the 2013 roots $r_1, r_2, \dots, r_{2013}$ of the equation $r^{2013} = \frac{1}{2013^{2013}}$. We want the sum of the values of $x_n = \frac{1}{1-r_n}$, so we consider the polynomial whose roots are $x_1, x_2, \dots, x_{2013}$. It is easy to see that $(1-\frac{1}{x_n})^{2013} = \frac{1}{2013^{2013}}$, so it follows that the x_n are the roots of the polynomial equation $\frac{1}{2013^{2013}}x^{2013} - (x-1)^{2013} = 0$. The leading coefficient of this polynomial is $\frac{1}{2013^{2013}} - 1$, and it follows easily from the Binomial Theorem that the next coefficient is 2013, so our answer is, by Vieta's Formulae,

$$-\frac{2013}{\frac{1}{2013^{2013}} - 1} = \frac{2013^{2014}}{2013^{2013} - 1}.$$

29. [20] Let A_1, A_2, \dots, A_m be finite sets of size 2012 and let B_1, B_2, \dots, B_m be finite sets of size 2013 such that $A_i \cap B_j = \emptyset$ if and only if $i=j$. Find the maximum value of m .

Answer: $\binom{4025}{2012}$ In general, we will show that if each of the sets A_i contain a elements and if each of the sets B_j contain b elements, then the maximum value for m is $\binom{a+b}{a}$.

Let U denote the union of all the sets A_i and B_j and let $|U|=n$. Consider the $n!$ orderings of the elements of U . Note that for any specific ordering, there is at most one value of i such that all the elements in A_i come before all the elements in B_i in this ordering; this follows since A_j shares at least one element with B_i and B_j shares at least one element with A_i for any other $j \neq i$.

On the other hand, the number of ways to permute the $(a+b)$ elements in $A_i \cup B_i$ so that all the elements in A_i come first is equal to $a!b!$. Therefore, the number of permutations of U where all the elements in A_i come before all the elements in B_i is equal to:

$$n! \cdot \frac{a!b!}{(a+b)!} = \frac{n!}{\binom{a+b}{a}}$$

Summing over all m values of i , the total number of orderings where, for some i , the elements in A_i come before B_i is equal to

$$\frac{n!m}{\binom{a+b}{a}}$$

But there are at most $u!$ such orderings, since there are $u!$ total orderings, so it follows that $m \leq \binom{a+b}{a}$. Equality is attained by taking U to be a set containing $(a+b)$ elements, letting A_i range over all a -element subsets of U , and letting $B_i = U \setminus A_i$ for each i .

30. [20] How many positive integers k are there such that

$$\frac{k}{2013}(a+b) = \text{lcm}(a,b)$$

has a solution in positive integers (a,b) ?

Answer: 1006 First, we can let $h = \gcd(a,b)$ so that $(a,b) = (hA,hB)$ where $\gcd(A,B) = 1$. Making these substitutions yields $\frac{k}{2013}(hA+hB) = hAB$, so $k = \frac{2013AB}{A+B}$. Because A and B are relatively prime, $A+B$ shares no common factors with neither A nor B , so in order to have k be an integer, $A+B$ must divide 2013, and since A and B are positive, $A+B > 1$.

We first show that for different possible values of $A+B$, the values of k generated are distinct. In particular, we need to show that $\frac{2013AB}{A+B} \neq \frac{2013A'B'}{A'+B'}$ whenever $A+B \neq A'+B'$. Assume that such an equality exists, and cross-multiplying yields $AB(A'+B') = A'B'(A+B)$. Since AB is relatively prime to $A+B$, we must have $A+B$ divide $A'+B'$. With a similar argument, we can show that $A'+B'$ must divide $A+B$, so $A+B = A'+B'$.

Now, we need to show that for the same denominator $A+B$, the values of k generated are also distinct for some relatively prime non-ordered pair (A,B) . Let $n = A+B = C+D$. Assume that $\frac{2013AB}{n} = \frac{2013CD}{n}$, or equivalently, $A(n-A) = C(n-C)$. After some rearrangement, we have $(C+A)(C-A) = n(C-A)$. This implies that either $C = A$ or $C = n - A = B$. But in either case, (C,D) is some permutation of (A,B) .

Our answer can therefore be obtained by summing up the totients of the factors of 2013 (excluding 1) and dividing by 2 since (A,B) and (B,A) correspond to the same k value, so our answer is $\frac{2013-1}{2} = 1006$.

Remark: It can be proven that the sum of the totients of all the factors of any positive integer N equals N , but in this case, the sum of the totients can be computed by hand.

31. [20] Let $ABCD$ be a quadrilateral inscribed in a unit circle with center O . Suppose that $\angle AOB = \angle COD = 135^\circ$, $BC = 1$. Let B' and C' be the reflections of A across BO and CO respectively. Let H_1 and H_2 be the orthocenters of $AB'C'$ and BCD , respectively. If M is the midpoint of OH_1 , and O' is the reflection of O about the midpoint of MH_2 , compute OO' .

Answer: $\frac{1}{4}(8 - \sqrt{6} - 3\sqrt{2})$ Put the diagram on the complex plane with O at the origin and A at 1. Let B have coordinate b and C have coordinate c . We obtain easily that B' is b^2 , C' is c^2 , and D is bc . Therefore, H_1 is $1 + b^2 + c^2$ and H_2 is $b + c + bc$ (we have used the fact that for triangles on the unit circle, their orthocenter is the sum of the vertices). Finally, we have that M is $\frac{1}{2}(1 + b^2 + c^2)$, so the reflection of O about the midpoint of MH_2 is $\frac{1}{2}(1 + b^2 + c^2 + 2b + 2c + 2bc) = \frac{1}{2}(b + c + 1)^2$, so we just seek $\frac{1}{2}|b + c + 1|^2$. But we know that $b = \text{cis}135^\circ$ and $c = \text{cis}195^\circ$, so we obtain that this value is $\frac{1}{4}(8 - \sqrt{6} - 3\sqrt{2})$.

32. [20] For an even integer positive integer n Kevin has a tape of length $4n$ with marks at $-2n, -2n + 1, \dots, 2n - 1, 2n$. He then randomly picks n points in the set $-n, -n + 1, -n + 2, \dots, n - 1, n$, and places a stone on each of these points. We call a stone 'stuck' if it is on $2n$ or $-2n$, or either all the points to the right, or all the points to the left, all contain stones. Then, every minute, Kevin shifts the unstuck stones in the following manner:

- He picks an unstuck stone uniformly at random and then flips a fair coin.

- If the coin came up heads, he then moves that stone and every stone in the largest contiguous set containing that stone one point to the left. If the coin came up tails, he moves every stone in that set one point right instead.

- He repeats until all the stones are stuck.

Let p_k be the probability that at the end of the process there are exactly k stones in the right half. Evaluate

$$\frac{p_{n-1} - p_{n-2} + p_{n-3} - \dots + p_3 - p_2 + p_1}{p_{n-1} + p_{n-2} + p_{n-3} + \dots + p_3 + p_2 + p_1}$$

in terms of n .

Answer: $\boxed{\frac{1}{n-1}}$ After we have selected the positions of the initial n stones, we number their positions: $a_1 < a_2 < \dots < a_n$. The conditions on how we move the stones imply that the expected value of $(a_i - a_j)$ after t minutes is still equal to $a_i - a_j$. In addition, if b_i is the final position of the i th stone, $E(b_{i+1} - b_i) = E(a_{i+1} - a_i)$. But this quantity is also equal to $(3n + 2) \cdot p_i + 1 \cdot (1 - p_i)$.

Now, let's calculate the expected value of $a_{i+1} - a_i$. This is the sum over $g = a_{i+1} - a_i$, and j , the number of spaces before a_i of $g \cdot \binom{j}{i-1} \binom{2n-j-g}{n-i+1}$, so we get

$$\frac{1}{\binom{2n+1}{n}} \sum_g g \cdot \sum_j \binom{j}{i-1} \binom{2n-j-g}{n-i-1}$$

But $\sum_j \binom{j}{i-1} \binom{2n-j-g}{n-i-1}$ is just $\binom{2n-g+1}{n-1}$. Therefore the expected value of $a_{i+1} - a_i$ is independent of i , so p_i is constant for all $i \neq 0, n$. It follows that the answer is $\frac{1}{n-1}$.

33. [25] Compute the value of $1^{25} + 2^{24} + 3^{23} + \dots + 24^2 + 25^1$. If your answer is A and the correct answer is C , then your score on this problem will be $\lfloor 25 \min\left(\left(\frac{A}{C}\right)^2, \left(\frac{C}{A}\right)^2\right) \rfloor$.

Answer: $\boxed{66071772829247409}$ The sum is extremely unimodal, so we want to approximate it using its largest term. Taking logs of each term, we see that the max occurs when $(26 - n) \log n$ peaks, and taking derivatives gives

$$x + x \log x = 26$$

From here it's easy to see that the answer is around 10, and slightly less (it's actually about 8.3, but in any case it's hard to find powers of anything except 10). Thus the largest term will be something like 10^{16} , which is already an order of magnitude within the desired answer 6.6×10^{16} .

To do better we'd really need to understand the behavior of the function x^{26-x} , but what approximately happens is that only the four or five largest terms in the sum are of any substantial size; thus it is reasonable here to pick some constant from 4 to 20 to multiply our guess 10^{16} ; any guess between 4.0×10^{16} and 2.0×10^{17} is reasonable.

34. [25] For how many unordered sets $\{a, b, c, d\}$ of positive integers, none of which exceed 168, do there exist integers w, x, y, z such that $(-1)^w a + (-1)^x b + (-1)^y c + (-1)^z d = 168$? If your answer is A and the correct answer is C , then your score on this problem will be $\lfloor 25e^{-3\frac{|C-A|}{C}} \rfloor$.

Answer: $\boxed{761474}$ As an approximation, we assume a, b, c, d are ordered to begin with (so we have to divide by 24 later) and add to 168 with a unique choice of signs; then, it suffices to count $e + f + g + h = 168$ with each e, f, g, h in $[-168, 168]$ and then divide by 24 (we drop the condition that none of them can be zero because it shouldn't affect the answer that much).

One way to do this is generating functions. We want the coefficient of t^{168} in the generating function $(t^{-168} + t^{-167} + \dots + t^{167} + t^{168})^4 = (t^{169} - t^{-168})^4 / (t - 1)^4$

Clearing the negative powers, it suffices to find the coefficient of t^{840} in $(t^{337} - 1)^4 / (t - 1)^4 = (1 - 4t^{337} + 6t^{674} - \dots) \frac{1}{(t-1)^4}$.

To do this we expand the bottom as a power series in t :

$$\frac{1}{(t-1)^4} = \sum_{n \geq 0} \binom{n+3}{3} t^n$$

It remains to calculate $\binom{840+3}{3} - 4 \cdot \binom{840-337+3}{3} + 6 \cdot \binom{840-674+3}{3}$. This is almost exactly equal to $\frac{1}{6}(843^3 - 4 \cdot 506^3 + 6 \cdot 169^3) \approx 1.83 \times 10^7$.

Dividing by 24, we arrive at an estimation 762500. Even if we use a bad approximation $\frac{1}{6 \cdot 24}(850^3 - 4 \cdot 500^3 + 6 \cdot 150^3)$ we get approximately 933000, which is fairly close to the answer.

35. [25] Let P be the number to partition 2013 into an ordered tuple of prime numbers? What is $\log_2(P)$? If your answer is A and the correct answer is C , then your score on this problem will be $\lfloor \frac{125}{2} (\min(\frac{C}{A}, \frac{A}{C}) - \frac{3}{5}) \rfloor$ or zero, whichever is larger.

Answer: 614.519... We use the following facts and heuristics.

(1) The ordered partitions of n into any positive integers (not just primes) is 2^{n-1} . This can be guessed by checking small cases and finding a pattern, and is not difficult to prove.

(2) The partitions of $\frac{2013}{n}$ into any positive integers equals the partitions of 2013 into integers from the set $\{n, 2n, 3n, \dots\}$.

(3) The small numbers matter more when considering partitions.

(4) The set of primes $\{2, 3, 5, 7, \dots\}$ is close in size (near the small numbers) to $\{3, 6, 9, \dots\}$ or $\{2, 4, 6, \dots\}$.

(5) The prime numbers get very sparse compared to the above two sets in the larger numbers.

Thus, using these heuristics, the number of partitions of 2013 into primes is approximately $2^{\frac{2013}{3}-1}$ or $2^{\frac{2013}{2}-1}$, which, taking logarithms, give 670 and 1005.5, respectively. By (5), we should estimate something that is slightly less than these numbers.

36. [24] (Mathematicians A to Z) Below are the names of 26 mathematicians, one for each letter of the alphabet. Your answer to this question should be a subset of $\{A, B, \dots, Z\}$, where each letter represents the corresponding mathematician. If two mathematicians in your subset have birthdates that are within 20 years of each other, then your score is 0. Otherwise, your score is $\max(3(k-3), 0)$ where k is the number of elements in your subset.

Niels Abel	Isaac Newton
Étienne Bézout	Nicole Oresme
Augustin-Louis Cauchy	Blaise Pascal
René Descartes	Daniel Quillen
Leonhard Euler	Bernhard Riemann
Pierre Fatou	Jean-Pierre Serre
Alexander Grothendieck	Alan Turing
David Hilbert	Stanislaw Ulam
Kenkichi Iwasawa	John Venn
Carl Jacobi	Andrew Wiles
Andrey Kolmogorov	Leonardo Ximenes
Joseph-Louis Lagrange	Shing-Tung Yau
John Milnor	Ernst Zermelo

Answer: {O, D, P, E, B, C, R, H, K, S, Y} A knowledgeable math historian might come up with this 11-element subset, earning 24 points: $\{O, D, P, E, B, C, R, H, K, S, Y\}$.

HMMT 2013
Saturday 16 February 2013
Team Round

1. [10] Let a and b be real numbers such that $\frac{ab}{a^2 + b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2 - b^2|}{a^2 + b^2}$.

Answer: $\boxed{\frac{\sqrt{3}}{2}}$ The hypothesis statement is equivalent to

$$a^2 + b^2 = 4ab$$

$$1 : (a + b)^2 = 6ab$$

$$2 : (a - b)^2 = 2ab$$

Multiplying equations 1 and 2,

$$(a^2 - b^2)^2 = 12(ab)^2$$

$$|a^2 - b^2| = \pm\sqrt{12}ab$$

Since the left hand side and ab are both positive,

$$|a^2 - b^2| = \sqrt{12}ab$$

$$\frac{|a^2 - b^2|}{a^2 + b^2} = \frac{\sqrt{12}ab}{a^2 + b^2} = \frac{\sqrt{12}}{4} = \frac{\sqrt{3}}{2}$$

(It is clear that such a and b exist: for example, we can take $a = 1$ and solve for b by way of the quadratic formula.)

2. [15] A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$?

Answer: $\boxed{16}$ We first show that $M + N \geq 16$. Consider the point right before the last table is occupied. We have two cases: first suppose there exists at least one open counter seat. Then, every table must contribute at least 3 to the value of $M + N$, because no groups of 1 will have taken a table with one of the counter seats open. By the end, the counter must contribute at least $5 + 2 = 7$ to $M + N$, as there must be at least two groups sitting at the counter. It follows that $M + N \geq 16$. For the second case, assume the counter is full right before the last table is taken. Then, everybody sitting at the counter must have entered as a singleton, since they entered when a table was still available. Consequently, the counter must contribute 10 to $M + N$, and each table contributes at least 2, so once again $M + N \geq 16$.

Now, $M + N = 16$ is achievable with eight groups of one, who first fill the counter seats, then the three tables. Thus, our answer is 16.

3. [15] Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F . Let FO intersect BC at E . Compute BE .

Answer: $\boxed{EB = \frac{7}{2}}$ O is the circumcenter of $\triangle ABC \implies AO = CO \implies \angle OCA = \angle OAC$. Because AC is an inscribed arc of circumcircle $\triangle AOC$, $\angle OCA = \angle OFA$. Furthermore BC is tangent to circumcircle $\triangle AOC$, so $\angle OAC = \angle OCB$. However, again using the fact that O is the circumcenter of $\triangle ABC$, $\angle OCB = \angle OBC$.

We now have that CO bisects $\angle ACB$, so it follows that triangle $CA = CB$. Also, by AA similarity we have $EOB \sim EBF$. Thus, $EB^2 = EO \cdot EF = EC^2$ by the similarity and power of a point, so $EB = BC/2 = AC/2 = 7/2$.

4. [20] Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine with proof the smallest possible value of

$$\sum_{1 \leq i < j \leq 5} \lfloor a_i + a_j \rfloor.$$

Answer: 72 We claim that the minimum is 72. This can be achieved by taking $a_1 = a_2 = a_3 = a_4 = 0.4$ and $a_5 = 18.4$. To prove that this is optimal, note that

$$\sum_{1 \leq i < j \leq 5} \lfloor a_i + a_j \rfloor = \sum_{1 \leq i < j \leq 5} (a_i + a_j) - \{a_i + a_j\} = 80 - \sum_{1 \leq i < j \leq 5} \{a_i + a_j\},$$

so it suffices to maximize

$$\sum_{1 \leq i < j \leq 5} \{a_i + a_j\} = \sum_{i=1}^5 \{a_i + a_{i+2}\} + \sum_{i=1}^5 \{a_i + a_{i+1}\},$$

where $a_6 = a_1$ and $a_7 = a_2$. Taking each sum modulo 1, it is clear that both are integers. Thus, the above sum is at most $2 \cdot 4 = 8$, and our original expression is at least $80 - 8 = 72$, completing the proof.

5. [25] Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

1. Choose a row, and subtract 1 from each entry.
2. Chooses a column, and add 1 to each entry.

He would like to get an array where all integers are divisible by 2013. On how many arrays is this possible?

Answer: 2013^{4025} We claim that the set of grids on which it is possible to obtain an array of all zeroes (mod 2013) is indexed by ordered 4025-tuples of residues (mod 2013), corresponding to the starting entries in the first row and first column of the grid, giving the answer of 2013^{4025} . To do this, we show that given after fixing all of the entries in the first row and column, there is a unique starting grid which can become an array of all zeroes after applying the appropriate operations.

Let $a_{i,j}$ be the entry in the i -th row and the j -th column. Suppose there is a sequence of operations giving all zeroes in the array; let r_i be the number of times we operate on row i , and let c_j be the number of times we operate on column j . It is enough to take all of these values to be residues modulo 2013. Clearly, $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j . In particular, $r_1 + c_1 \equiv a_{1,1}$. Now, for each i, j , we have

$$\begin{aligned} a_{i,j} &\equiv -r_i - c_j \\ &\equiv (a_{i,1} + c_1) + (a_{1,j} + r_1) \\ &\equiv a_{i,1} + a_{1,j} - a_{1,1}, \end{aligned}$$

which is fixed. Thus, the rest of the entries in the grid are forced.

Conversely, if we set $a_{i,j}$ to be the appropriate representative of the residue class of $a_{i,1} + a_{1,j} - a_{1,1}$ modulo 2013, we may take $r_i \equiv -a_{i,1} \pmod{2013}$, and $c_j \equiv a_{1,1} - a_{1,j} \pmod{2013}$ for each i, j . It is clear that $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j , so we're done.

6. [25] Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω . Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD .

Answer: N/A Since AD is an angle bisector, D is the midpoint of arc BC opposite A on ω . It is well-known that B, I , and C lie on a circle centered at D . Thus $BD = DC = DI$. Applying Ptolemy's theorem to cyclic quadrilateral $ABDC$, we get

$$AB \cdot DC + AC \cdot BD = AD \cdot BC = AD \cdot (AB + AC)/2$$

Using $BD = DC$ we have immediately that $AD = 2BD = 2DI$ so I is the midpoint of AD as desired.

Solution 2: Let P and Q be the midpoints of AB and AC , and take the point E on segment BC such that $BE = BP$. Note that $CE = AB - BE = AB - BP = \frac{AB+AC}{2} - \frac{AB}{2} = \frac{AC}{2} = CQ$, so triangles BPE and CQE are isosceles. In addition, $\frac{BE}{EC} = \frac{AB/2}{AC/2} = \frac{AB}{AC}$, so by the angle bisector theorem, AE bisects $\angle CAB$, whence E must lie on the bisector of $\angle A$.

Since triangles BPE and CQE are isosceles, the bisectors of angles B and C are the perpendicular bisectors of segments PE and EQ , respectively. Thus, the circumcenter of $\triangle PQE$ is I , so the perpendicular bisector of PQ meets the bisector of $\angle A$ at I .

Furthermore, since $\angle DAB = \angle CAD$, arcs \widehat{BD} and \widehat{DC} have the same measure, so $BD = DC$, whence the perpendicular bisector of BC meets the bisector of $\angle A$ at D . A homothety centered at A with factor $1/2$ maps BC to PQ , and so maps D to I . Thus, D is the midpoint of AI .

Solution 3: Let $a = BC$, $b = CA$, and $c = AB$, let r and R denote the lengths of the inradius and circumradius of $\triangle ABC$, respectively, let E be the intersection of segments AD and BC , and let O be the circumcenter of $\triangle ABC$. I is the midpoint of chord AD if and only if $OI \perp AD$, which is true if and only if $OA^2 = OE^2 + AE^2$. By Euler's distance formula, $OE^2 = R(R - 2r)$, and by Stewart's theorem and the angle bisector theorem we can find $AE^2 = \left(\frac{b+c}{a+b+c}\right)^2 bc \left(1 - \left(\frac{a}{b+c}\right)^2\right) = \frac{bc}{3}$. Thus, it remains to show that $6Rr = bc$.

Now, we use the combine the well-known formulas $\frac{abc}{4K} = R$ and $K = rs$ to get $abc = 4Rrs$, where K is the area of $\triangle ABC$ and s is its semiperimeter. We have $bc = Rr \frac{4s}{a} = Rr \frac{2a+2(b+c)}{a} = Rr \frac{2a+2(2a)}{a} = 6Rr$, as desired.

7. [30] There are n children and n toys such that each child has a strict preference ordering on the toys. We want to distribute the toys: say a distribution A dominates a distribution $B \neq A$ if in A , each child receives at least as preferable of a toy as in B . Prove that if some distribution is not dominated by any other, then at least one child gets his/her favorite toy in that distribution.

Answer: N/A Suppose we have a distribution A assigning each child C_i , $i = 1, 2, \dots, n$, toy T_i , such that no child C_i gets their top preference $T'_i \neq T_i$. Then, pick an arbitrary child C_1 and construct the sequence of children $C_{i_1}, C_{i_2}, C_{i_3}, \dots$ where $i_1 = 1$ and $C_{i_{k+1}}$ was assigned the favorite toy T'_{i_k} of the last child C_{i_k} . Eventually, some $C_{i_k} = C_{i_1}$; at this point, just pass the toys around this cycle so that each of these children gets their favorite toy. Clearly the resulting distribution dominates the original, so we're done.

8. [35] Let points A and B be on circle ω centered at O . Suppose that ω_A and ω_B are circles not containing O which are internally tangent to ω at A and B , respectively. Let ω_A and ω_B intersect at C and D such that D is inside triangle ABC . Suppose that line BC meets ω again at E and let line EA intersect ω_A at F . If $FC \perp CD$, prove that O , C , and D are collinear.

Answer: N/A Let $H = CA \cap \omega$, and $G = BH \cap \omega_B$. There are homotheties centered at A and B taking $\omega_A \rightarrow \omega$ and $\omega_B \rightarrow \omega$ that take $A : F \mapsto E$, $A : C \mapsto H$, $B : C \mapsto E$ and $B : G \mapsto H$. In particular $CF \parallel EH \parallel CG$, so C, F, G are collinear, lying on a line perpendicular to DC .

Because of the right angles at C , DF and DG are diameters of ω_A, ω_B , respectively. Also, we have that the ratio of the sizes of ω_A and ω_B , under the two homotheties above, is $CF/EH \cdot EH/CG = CF/CG$. Therefore, $DF/DG = CF/CG$, but then $\triangle DCF$ and $\triangle DCG$ are both right triangles which share one side and have hypotenuse and other side in proportion; it is obvious now that the two circles ω_A and ω_B are congruent.

Therefore, O has the same distance to A and B , and so the same distances to the centers of the two circles as well. As a result, O lies on the radical axis CD as desired.

9. [35] Let m be an odd positive integer greater than 1. Let S_m be the set of all non-negative integers less than m which are of the form $x + y$, where $xy - 1$ is divisible by m . Let $f(m)$ be the number of elements of S_m .

- (a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.
(b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and p is an odd prime.

Answer: N/A For a positive integer n , let $\mathbb{Z}/n\mathbb{Z}$ denote the set of residues modulo n and $(\mathbb{Z}/n\mathbb{Z})^*$ denote the set of residues modulo n that are relatively prime to n . Then, rephrased, S_m is the set of residues modulo m of the form $x + x^{-1}$, where $x \in (\mathbb{Z}/m\mathbb{Z})^*$.

For part (a), suppose $a = x + x^{-1} \in S_m$ and $b = y + y^{-1} \in S_n$. By the Chinese Remainder Theorem, there exists a residue $z \in (\mathbb{Z}/mn\mathbb{Z})^*$ such that $m|(x - z)$ and $n|(y - z)$, and thus $z + z^{-1} \equiv x + x^{-1} \pmod{m}$ and $z + z^{-1} \equiv y + y^{-1} \pmod{n}$. Therefore, all $f(m)f(n)$ residues modulo mn which result from applying the Chinese Remainder Theorem to an element each of S_m and S_n are in S_{mn} . Conversely, given $z + z^{-1} \in (\mathbb{Z}/mn\mathbb{Z})^*$, taking $z + z^{-1}$ modulo m and n gives elements of S_m and S_n , so indeed $f(mn) = f(m)f(n)$.

We now proceed to part (b). For each $x \in (\mathbb{Z}/p^k\mathbb{Z})^*$, denote $q(x)$ to be the largest non-negative integer $i \leq k$ such that p^i divides $x^2 - 1$ (this is clearly well-defined). For a given x , let $g(x)$ be the number of $y \in (\mathbb{Z}/p^k\mathbb{Z})^*$ such that $x + x^{-1} \equiv y + y^{-1} \pmod{p^k}$. Note that this condition is equivalent to $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$.

First, consider the case in which $q(x) \geq k/2$, in which case we have $p^{\lceil k/2 \rceil} | (x - 1)(x + 1)$, and because p is odd, $x \equiv \pm 1 \pmod{p^{\lceil k/2 \rceil}}$. Thus, either $(x - 1)^2 \equiv 0 \pmod{p^k}$ or $(x + 1)^2 \equiv 0 \pmod{p^k}$, and it follows that $x + x^{-1} \equiv \pm 2 \pmod{p^k}$ (clearly 2 and -2 are distinct). Conversely, $x + x^{-1} \equiv \pm 2$ implies $(x \pm 1) \equiv 0 \pmod{p^{\lceil k/2 \rceil}}$, which in turn implies $q(x) \geq k/2$. It is now clear that there are exactly two elements of S_m corresponding to residues of the form $x + x^{-1}$ with $q(x) \geq k/2$, and all other elements of S_m come from x with $q(x) < k/2$.

Fix x with $q(x) < k/2$; we will compute $g(x)$. Suppose $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$, and say $x - y, xy - 1$ have j, j' factors of p , respectively. If $j \leq j'$, note that $x^2 - 1 \equiv xy - 1 \equiv 0 \pmod{p^j}$, so $j \leq q(x)$. Similarly, if $j' \leq j$, $x \equiv y \pmod{p^{j'}}$, so $x^2 - 1 \equiv xy - 1 \equiv 0 \pmod{p^{j'}}$, and so $j' \leq q(x)$. It follows that $\min(j, j') \leq q(x)$, and thus $\max(j, j') \geq k - q(x)$.

Suppose $p^{k-q(x)} | (x - y)$. Then, $xy - 1 \equiv x^2 - 1 \equiv 0 \pmod{p^{q(x)}}$ because $q(x) < k/2$, so any y with $p^{k-q(x)} | (x - y)$ satisfies $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$. Now, suppose $p^{k-q(x)} | (xy - 1)$, that is, $y \equiv x^{-1} \pmod{p^{k-q(x)}}$. Then, $xy \equiv 1 \pmod{p^{q(x)}}$, and since $x^2 \equiv 1 \pmod{p^{q(x)}}$, we have $x \equiv y \pmod{p^{q(x)}}$, so again we have $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$. It follows that the set of y satisfying $(x - y)(xy - 1) \equiv 0 \pmod{p^k}$ is exactly the set of y with $y \equiv x \pmod{p^{k-q(x)}}$ or $y \equiv x^{-1} \pmod{p^{k-q(x)}}$. x, x^{-1} are distinct residues modulo $p^{k-q(x)}$, because $x^2 - 1$ has fewer than $k/2$ factors of p , so it follows that $g(x) = 2p^{q(x)}$. In particular, note that these values are distinct for different values of $q(x) < k/2$, so $x + x^{-1} \equiv y + y^{-1}$ implies $q(x) = q(y)$ or $q(x), q(y) \geq k/2$.

For each integer i with $0 \leq i < k/2$, we need to compute the number of $x \in (\mathbb{Z}/p^k\mathbb{Z})$ with $q(x) = i$. Clearly, this is the number of x with $p^i | x^2 - 1$ minus the number of x with $p^{i+1} | x^2 - 1$. When $i = 0$, the number of x with $p^i | x^2 - 1$ is clearly $p^{k-1}(p - 1)$, and when $i > 0$, this number is $2p^{k-i}$, as we have $x \equiv \pm 1 \pmod{p^i}$.

We can now count the number of elements of S_{p^k} using casework on the value of $q(x)$ where we take x to be such that $x + x^{-1}$ is a particular element of S_m . Applying the results from the previous two paragraphs, our answer is

$$2 + \frac{p^{k-1}(p-3)}{2} + \sum_{i=1}^{\lceil k/2 \rceil - 1} \frac{2p^{k-i} - 2p^{k-i-1}}{2p^i},$$

where the summand 2 comes from $\pm 2 \in S_{p^k}$, corresponding to all x with $q(x) > k/2$, the next summand comes from those x with $q(x) = 0$, and each additional summand comes from those x with $q(x) = i$ in the relevant range. We may evaluate the last sum as a geometric series, to obtain the final closed form answer of

$$2 + \frac{p^{k-1}(p-3)}{2} + \frac{p^{k-1} - p^{\frac{1+(-1)^k}{2}}}{p+1}.$$

10. [40] Chim Tu has a large rectangular table. On it, there are finitely many pieces of paper with non-overlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in finitely many steps?

Answer: N/A Let the pieces of paper be P_1, P_2, \dots, P_n in the Cartesian plane. It suffices to show that for any constant distance D , they can be slid so that each pairwise distance is at least D . Then, we can apply this using D equal to the diameter of the rectangle, sliding all but at most one of the pieces of paper off the table, and then slide this last one off arbitrarily.

We show in particular that there is always a polygon P_i which can be slid arbitrarily far to the right (i.e. in the positive x -direction). For each P_i let B_i be a bottommost (i.e. lowest y -coordinate) point on the boundary of P_i . Define a point Q to be *exposed* if the ray starting at Q in the positive x direction meets the interior of no piece of paper.

Consider the set of all exposed B_i ; this set is nonempty because certainly the bottommost of all the B_i is exposed. Of this set, let B_k be the exposed B_i with maximal y -coordinate, and if there are more than one such, choose the one with maximal x -coordinate. We claim that the corresponding P_k can be slid arbitrarily far to the right.

Suppose for the sake of contradiction that there is some polygon blocking this path. To be precise, if A_k is the highest point of P_k , then the region R formed by the right-side boundary of P_k and the rays pointing in the positive x direction from A_k and B_k , must contain interior point(s) of some set of polygon(s) P_j in its interior. All of their bottommost points B_j must lie in R , since none of them can have boundary intersecting the ray from B_k , by the construction of B_k .

Because B_k was chosen to be rightmost out of all the exposed B_i with that y -coordinate, it must be that all of the B_j corresponding to the blocking P_j have larger y -coordinate. Now, choose of these the one with smallest y -coordinate - it must be exposed, and it has strictly higher y -coordinate than B_k , contradiction. It follows that the interior of R intersects no pieces of paper.

Now, for a fixed D such that D is at least the largest distance between any two points of two polygons, we can shift this exposed piece of paper nD to the right, the next one of the remaining pieces $(n-1)D$, and so on, so that the pairwise distances between pieces of paper, even when projected onto the x -axis, are at least D each. We're done.

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General Test

1. [2] What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

Answer: $\boxed{24}$ The smallest two integers that are the product of four primes are $2^4 = 16$ and $2^3 \cdot 3 = 24$. Since 16 is a perfect square and 24 is not, the answer is 24.

2. [3] Plot points A, B, C at coordinates $(0, 0)$, $(0, 1)$, and $(1, 1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC . Let X_1 be the area swept out when Bobby rotates S counterclockwise 45 degrees about point A . Let X_2 be the area swept out when Calvin rotates S clockwise 45 degrees about point A . Find $\frac{X_1 + X_2}{2}$.

Answer: $\boxed{\frac{\pi}{4}}$ It's easy to see $X_1 = X_2$. Simple cutting and pasting shows that X_1 equals the area of $\frac{1}{8}$ of a circle with radius $AC = \sqrt{2}$, so $\frac{X_1 + X_2}{2} = X_1 = \frac{1}{8}\pi(\sqrt{2})^2 = \frac{\pi}{4}$.

3. [4] A 24-hour digital clock shows times $h : m : s$, where h, m , and s are integers with $0 \leq h \leq 23$, $0 \leq m \leq 59$, and $0 \leq s \leq 59$. How many times $h : m : s$ satisfy $h + m = s$?

Answer: $\boxed{1164}$ We are solving $h + m = s$ in $0 \leq s \leq 59$, $0 \leq m \leq 59$, and $0 \leq h \leq 23$. If $s \geq 24$, each h corresponds to exactly 1 solution, so we get $24(59 - 23) = 24(36)$ in this case. If $s \leq 23$, we want the number of nonnegative integer solutions to $h + m \leq 23$, which by lattice point counting (or balls and urns) is $\binom{23 + 2}{2} = (23 + 2)(23 + 1)/2 = 25 \cdot 12$. Thus our total is $12(72 + 25) = 12(100 - 3) = 1164$.

4. [4] A 50-card deck consists of 4 cards labeled “ i ” for $i = 1, 2, \dots, 12$ and 2 cards labeled “13”. If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

Answer: $\boxed{\frac{73}{1225}}$ All pairs of distinct cards (where we distinguish cards even with the same label) are equally likely. There are $\binom{2}{2} + 12\binom{4}{2} = 73$ pairs of cards with the same label and $\binom{50}{2} = 100 \cdot \frac{49}{4} = 1225$ pairs of cards overall, so the desired probability is $\frac{73}{1225}$.

5. [5] Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC , respectively. Suppose that there exists a point F on ray \overrightarrow{DE} outside of ABC such that triangle BFA is similar to triangle ABC . Compute $\frac{AB}{BC}$.

Answer: $\boxed{\sqrt{2}}$ Let $\alpha = \angle ABC = \angle ACB$, $AB = 2x$, and $BC = 2y$, so $AD = DB = AE = EC = x$ and $DE = y$. Since $\triangle BFA \sim \triangle ABC$ and $BA = AC$, we in fact have $\triangle BFA \cong \triangle ABC$, so $BF = BA = 2x$, $FA = 2y$, and $\angle DAF = \alpha$. But $DE \parallel BC$ yields $\angle ADF = \angle ABC = \alpha$ as well, whence $\triangle FAD \sim \triangle ABC$ gives $\frac{2y}{x} = \frac{FA}{AD} = \frac{AB}{BC} = \frac{2x}{2y} \implies \frac{AB}{BC} = \frac{x}{y} = \sqrt{2}$.

6. [5] Find the number of positive integer divisors of $12!$ that leave a remainder of 1 when divided by 3.

Answer: $\boxed{66}$ First we factor $12! = 2^{10}3^55^27^111^1$, and note that $2, 5, 11 \equiv -1 \pmod{3}$ while $7 \equiv 1 \pmod{3}$. The desired divisors are precisely $2^a5^b7^c11^d$ with $0 \leq a \leq 10$, $0 \leq b \leq 2$, $0 \leq c \leq 1$, $0 \leq d \leq 1$, and $a + b + d$ even. But then for any choice of a, b , exactly one $d \in \{0, 1\}$ makes $a + b + d$ even, so we have exactly one 1 (mod 3)-divisor for every triple (a, b, c) satisfying the inequality constraints. This gives a total of $(10 + 1)(2 + 1)(1 + 1) = 66$.

7. [6] Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d .

Answer: $\boxed{\frac{3}{2}}$ Let $f(a, b, c, d) = (a^2 + b^2 + c^2 + d^2) - (ab + \lambda bc + cd)$. For fixed (b, c, d) , f is minimized at $a = \frac{b}{2}$, and for fixed (a, b, c) , f is minimized at $d = \frac{c}{2}$, so simply we want the largest λ such that $f(\frac{b}{2}, b, c, \frac{c}{2}) = \frac{3}{4}(b^2 + c^2) - \lambda bc$ is always nonnegative. By AM-GM, this holds if and only if $\lambda \leq 2 \cdot \frac{3}{4} = \frac{3}{2}$.

8. [6] How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, \dots$?

Answer: $\boxed{105}$ We want to find which integers have only 0's and 1's in their base 3 representation. Note that $1000_{10} = 1101001_3$. We can construct a bijection from all such numbers to the binary strings, by mapping $x_3 \leftrightarrow x_2$. Since $1101001_2 = 105_{10}$, we conclude that the answer is 105.

9. [7] Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}$, $AC = \sqrt{3}$, $\angle BAD = 30^\circ$, and $\angle CAD = 45^\circ$. Find AD .

Answer: $\boxed{\frac{\sqrt{6}}{2} \text{ OR } \frac{\sqrt{3}}{\sqrt{2}}}$ Note that $[BAD] + [CAD] = [ABC]$. If $\alpha_1 = \angle BAD$, $\alpha_2 = \angle CAD$, then we deduce $\frac{\sin(\alpha_1 + \alpha_2)}{AD} = \frac{\sin \alpha_1}{AC} + \frac{\sin \alpha_2}{AB}$ upon division by $AB \cdot AC \cdot AD$. Now

$$AD = \frac{\sin(30^\circ + 45^\circ)}{\frac{\sin 30^\circ}{\sqrt{3}} + \frac{\sin 45^\circ}{\sqrt{2}}}.$$

But $\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \sin 45^\circ \cos 30^\circ = \sin 30^\circ \frac{1}{\sqrt{2}} + \sin 45^\circ \frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2} \left(\frac{\sin 30^\circ}{\sqrt{3}} + \frac{\sin 45^\circ}{\sqrt{2}} \right)$, so our answer is $\frac{\sqrt{6}}{2}$.

10. [8] How many functions $f : \{1, 2, \dots, 2013\} \rightarrow \{1, 2, \dots, 2013\}$ satisfy $f(j) < f(i) + j - i$ for all integers i, j such that $1 \leq i < j \leq 2013$?

Answer: $\boxed{\binom{4025}{2013}}$ Note that the given condition is equivalent to $f(j) - j < f(i) - i$ for all $1 \leq i < j \leq 2013$. Let $g(i) = f(i) - i$, so that the condition becomes $g(j) < g(i)$ for $i < j$ and $1 - i \leq g(i) \leq 2013 - i$. However, since g is decreasing, we see by induction that $g(i + 1)$ is in the desired range so long as $g(i)$ is in the desired range. Hence, it suffices to choose 2013 values for $g(1), \dots, g(2013)$ in decreasing order from $[-2012, 2012]$, for a total of $\binom{4025}{2013}$ possible functions.

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Guts Round

1. [5] Evaluate $2 + 5 + 8 + \cdots + 101$.

Answer: $\boxed{1751}$ There are $\frac{102}{3} = 34$ terms with average $\frac{2+101}{2}$, so their sum is $17 \cdot 103 = 1751$.

2. [5] Two fair six-sided dice are rolled. What is the probability that their sum is at least 10?

Answer: $\boxed{\frac{1}{6}}$ There are 3, 2, 1 outcomes with sum 10, 11, 12, so the probability is $\frac{3+2+1}{6^2} = \frac{1}{6}$.

3. [5] A square is inscribed in a circle of radius 1. Find the perimeter of the square.

Answer: $\boxed{4\sqrt{2} \text{ OR } \frac{8}{\sqrt{2}}}$ The square has diagonal length 2, so side length $\sqrt{2}$ and perimeter $4\sqrt{2}$.

4. [6] Find the minimum possible value of $(x^2 + 6x + 2)^2$ over all real numbers x .

Answer: $\boxed{0}$ This is $((x + 3)^2 - 7)^2 \geq 0$, with equality at $x + 3 = \pm\sqrt{7}$.

5. [6] How many positive integers less than 100 are relatively prime to 200? (Two numbers are *relatively prime* if their greatest common factor is 1.)

Answer: $\boxed{40}$ $1 \leq n < 100$ is relatively prime to 200 if and only if it's relatively prime to 100 (200, 100 have the same prime factors). Thus our answer is $\phi(100) = 100 \cdot \frac{4}{5} = 40$.

6. [6] A right triangle has area 5 and a hypotenuse of length 5. Find its perimeter.

Answer: $\boxed{5 + 3\sqrt{5}}$ If x, y denote the legs, then $xy = 10$ and $x^2 + y^2 = 25$, so $x + y + \sqrt{x^2 + y^2} = \sqrt{(x^2 + y^2) + 2xy} + 5 = \sqrt{45} + 5 = 5 + 3\sqrt{5}$.

7. [7] Marty and three other people took a math test. Everyone got a non-negative integer score. The average score was 20. Marty was told the average score and concluded that everyone else scored below average. What was the minimum possible score Marty could have gotten in order to definitively reach this conclusion?

Answer: $\boxed{61}$ Suppose for the sake of contradiction Marty obtained a score of 60 or lower. Since the mean is 20, the total score of the 4 test takers must be 80. Then there exists the possibility of 2 students getting 0, and the last student getting a score of 20 or higher. If so, Marty could not have concluded with certainty that everyone else scored below average.

With a score of 61, any of the other three students must have scored points lower or equal to 19 points. Thus Marty is able to conclude that everyone else scored below average.

8. [7] Evaluate the expression

$$\frac{1}{2 - \frac{1}{2 - \frac{1}{2 - \cdots \frac{1}{2 - \frac{1}{2}}}}},$$

where the digit 2 appears 2013 times.

Answer: $\boxed{\frac{2013}{2014}}$ Let $f(n)$ denote the corresponding expression with the digit 2 appearing exactly n times. Then $f(1) = \frac{1}{2}$ and for $n > 1$, $f(n) = \frac{1}{2 - f(n-1)}$. By induction using the identity $\frac{1}{2 - \frac{1}{N}} = \frac{N}{N+1}$, $f(n) = \frac{n}{n+1}$ for all $n \geq 1$, so $f(2013) = \frac{2013}{2014}$.

9. [7] Find the remainder when $1^2 + 3^2 + 5^2 + \dots + 99^2$ is divided by 1000.

Answer: 650 We have $S = \sum_{i=0}^{49} (2i+1)^2 = \sum_{i=0}^{49} 4i^2 + 4i + 1 = 4 \cdot \frac{49 \cdot 50 \cdot 99}{6} + 4 \cdot \frac{49 \cdot 50}{2} + 50 \equiv 700 + 900 + 50 \pmod{1000} \equiv 650 \pmod{1000}$.

10. [8] How many pairs of real numbers (x, y) satisfy the equation

$$y^4 - y^2 = xy^3 - xy = x^3y - xy = x^4 - x^2 = 0?$$

Answer: 9 We can see that if they solve the first and fourth equations, they are automatically solutions to the second and third equations. Hence, the solutions are just the $3^2 = 9$ points where x, y can be any of $-1, 0, 1$.

11. [8] David has a unit triangular array of 10 points, 4 on each side. A *looping path* is a sequence A_1, A_2, \dots, A_{10} containing each of the 10 points exactly once, such that A_i and A_{i+1} are adjacent (exactly 1 unit apart) for $i = 1, 2, \dots, 10$. (Here $A_{11} = A_1$.) Find the number of looping paths in this array.

Answer: 60 There are $10 \cdot 2$ times as many loop sequences as loops. To count the number of loops, first focus on the three corners of the array: their edges are uniquely determined. It's now easy to see there are 3 loops (they form "V-shapes"), so the answer is $10 \cdot 2 \cdot 3 = 60$.

12. [8] Given that $62^2 + 122^2 = 18728$, find positive integers (n, m) such that $n^2 + m^2 = 9364$.

Answer: (30, 92) OR (92, 30) If $a^2 + b^2 = 2c$, then $(\frac{a+b}{2})^2 + (\frac{a-b}{2})^2 = \frac{2a^2+2b^2}{4} = \frac{a^2+b^2}{2} = c$. Thus, $n = \frac{62+122}{2} = 92$ and $m = \frac{122-62}{2} = 30$ works.

13. [9] Let $S = \{1, 2, \dots, 2013\}$. Find the number of ordered triples (A, B, C) of subsets of S such that $A \subseteq B$ and $A \cup B \cup C = S$.

Answer: 5^{2013} OR 125^{671} Let $n = 2013$. Each of the n elements can be independently placed in 5 spots: there are $2^3 - 1$ choices with element x in at least one set, and we subtract the 2^1 choices with element x in set A but not B . Specifying where the elements go uniquely determines A, B, C , so there are $5^n = 5^{2013}$ ordered triples.

14. [9] Find all triples of positive integers (x, y, z) such that $x^2 + y - z = 100$ and $x + y^2 - z = 124$.

Answer: (12, 13, 57) Cancel z to get $24 = (y-x)(y+x-1)$. Since x, y are positive, we have $y+x-1 \geq 1+1-1 > 0$, so $0 < y-x < y+x-1$. But $y-x$ and $y+x-1$ have opposite parity, so $(y-x, y+x-1) \in \{(1, 24), (3, 8)\}$ yields $(y, x) \in \{(13, 12), (6, 3)\}$.

Finally, $0 < z = x^2 + y - 100$ forces $(x, y, z) = (12, 13, 57)$.

15. [9] Find all real numbers x between 0 and 360 such that $\sqrt{3} \cos 10^\circ = \cos 40^\circ + \sin x^\circ$.

Answer: 70, 110 (need both, but order doesn't matter) Note that $\sqrt{3} = 2 \cos 30^\circ$, so $\sin x^\circ = \cos 20^\circ \implies x \in \{70, 110\}$.

16. [10] A bug is on one exterior vertex of solid S , a $3 \times 3 \times 3$ cube that has its center $1 \times 1 \times 1$ cube removed, and wishes to travel to the opposite exterior vertex. Let O denote the *outer* surface of S (formed by the surface of the $3 \times 3 \times 3$ cube). Let $L(S)$ denote the length of the shortest path through S . (Note that such a path cannot pass through the missing center cube, which is empty space.) Let $L(O)$ denote the length of the shortest path through O . What is the ratio $\frac{L(S)}{L(O)}$?

Answer: $\frac{\sqrt{29}}{3\sqrt{5}}$ OR $\frac{\sqrt{145}}{15}$ By (*), the shortest route in O has length $2\sqrt{1.5^2 + 3^2} = 3\sqrt{5}$. By (**), the shortest route overall (in S) has length $2\sqrt{1.5^2 + 1^2 + 2^2} = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29}$. Therefore the desired ratio is $\frac{\sqrt{29}}{3\sqrt{5}} = \frac{\sqrt{145}}{15}$.

(*) Suppose we're trying to get from $(0, 0, 0)$ to $(3, 3, 3)$ through O . Then one minimal-length path through O is $(0, 0, 0) \rightarrow (1.5, 0, 3) \rightarrow (3, 3, 3)$.

(**) Suppose we're trying to get from $(0, 0, 0)$ to $(3, 3, 3)$ through S . Then the inner hole is $[1, 2] \times [1, 2] \times [1, 2]$, and one minimal-length path is $(0, 0, 0) \rightarrow (1.5, 1, 2) \rightarrow (3, 3, 3)$.

To justify (*), note that we must at some point hit one of the three faces incident to $(3, 3, 3)$, and therefore one of the edges of those faces. Without loss of generality, the *first* of these edges (which must lie on a face incident to $(0, 0, 0)$) is $\{(t, 0, 3) : 0 \leq t \leq 3\}$. Then the shortest path goes directly from the origin to the edge, and then directly to $(3, 3, 3)$; $t = 1.5$ minimizes the resulting distance. (One may either appeal to the classic geometric “unfolding” argument, or just direct algebraic minimization.)

To justify (**), consider the portion of S “visible” from $(0, 0, 0)$. It sees 3 mutually adjacent faces of the center cube “hole” (when looking inside the solid) and its 6 edges. $(3, 3, 3)$ can also see these 6 edges. The shortest path through S must be a straight line from the start vertex to some point P on the surface of the center cube (+), and it's easy to see, using the triangle inequality, that this point P must be on one of the 6 edges. Without loss of generality, it's on the edge $\{(t, 1, 2) : 1 \leq t \leq 2\}$. The remaining path is a straight line to $(3, 3, 3)$. Then once again, $t = 1.5$ minimizes the distance. (And once again, one may either appeal to the classic geometric “unfolding” argument—though this time it's a little trickier—or just direct algebraic minimization.)

Comment. (+) can be proven in many ways. The rough physical intuition is that “a fully stretched rubber band” with ends at $(0, 0, 0)$ and $(3, 3, 3)$ must not have any “wiggle room” (and so touches the inner cube). Perhaps more rigorously, we can try shortening a path that does not hit the center cube, for instance by “projecting the path down” towards the closest edge of the inner cube.

17. [10] Find the sum of $\frac{1}{n}$ over all positive integers n with the property that the decimal representation of $\frac{1}{n}$ terminates.

Answer: $\boxed{\frac{5}{2}}$ The decimal representation of $\frac{1}{n}$ terminates if and only if $n = 2^i 5^j$ for some nonnegative integers i, j , so our desired sum is

$$\sum_{i \geq 0} \sum_{j \geq 0} 2^{-i} 5^{-j} = \sum_{i \geq 0} 2^{-i} \sum_{j \geq 0} 5^{-j} = (1 - 2^{-1})^{-1} (1 - 5^{-1})^{-1} = \frac{2}{1} \frac{5}{4} = \frac{5}{2}.$$

18. [10] The rightmost nonzero digit in the decimal expansion of $101!$ is the same as the rightmost nonzero digit of $n!$, where n is an integer greater than 101 . Find the smallest possible value of n .

Answer: $\boxed{103}$ $101!$ has more factors of 2 than 5, so its rightmost nonzero digit is one of 2, 4, 6, 8. Notice that if the rightmost nonzero digit of $101!$ is $2k$ ($1 \leq k \leq 4$), then $102!$ has rightmost nonzero digit $102(2k) \equiv 4k \pmod{10}$, and $103!$ has rightmost nonzero digit $103(4k) \equiv 2k \pmod{10}$. Hence $n = 103$.

19. [11] Let p, q, r, s be distinct primes such that $pq - rs$ is divisible by 30. Find the minimum possible value of $p + q + r + s$.

Answer: $\boxed{54}$ The key is to realize none of the primes can be 2, 3, or 5, or else we would have to use one of them twice. Hence p, q, r, s must lie among 7, 11, 13, 17, 19, 23, 29, ... These options give remainders of 1 (mod 2) (obviously), 1, -1 , 1 , -1 , -1 , ... modulo 3, and 2, 1, 3, 2, 4, 3, 4, ... modulo 5. We automatically have $2 \mid pq - rs$, and we have $3 \mid pq - rs$ if and only if $pqr s \equiv (pq)^2 \equiv 1 \pmod{3}$, i.e. there are an even number of $-1 \pmod{3}$'s among p, q, r, s .

If $\{p, q, r, s\} = \{7, 11, 13, 17\}$, then we cannot have $5 \mid pq - rs$, or else $12 \equiv pqr s \equiv (pq)^2 \pmod{5}$ is a quadratic residue. Our next smallest choice (in terms of $p + q + r + s$) is $\{7, 11, 17, 19\}$, which works: $7 \cdot 17 - 11 \cdot 19 \equiv 2^2 - 4 \equiv 0 \pmod{5}$. This gives an answer of $7 + 17 + 11 + 19 = 54$.

20. [11] There exist unique nonnegative integers A, B between 0 and 9, inclusive, such that

$$(1001 \cdot A + 110 \cdot B)^2 = 57, 108, 249.$$

Find $10 \cdot A + B$.

Answer: $\boxed{75}$ We only need to bound for $AB00$; in other words, $AB^2 \leq 5710$ but $(AB+1)^2 \geq 5710$. A quick check gives $AB = 75$. (Lots of ways to get this...)

21. [11] Suppose A, B, C , and D are four circles of radius $r > 0$ centered about the points $(0, r)$, $(r, 0)$, $(0, -r)$, and $(-r, 0)$ in the plane. Let O be a circle centered at $(0, 0)$ with radius $2r$. In terms of r , what is the area of the union of circles A, B, C , and D subtracted by the area of circle O that is not contained in the union of A, B, C , and D ?

(The *union* of two or more regions in the plane is the set of points lying in at least one of the regions.)

Answer: $\boxed{8r^2}$ **Solution 1.** Let U denote the union of the four circles, so we seek

$$U - ([O] - U) = 2U - [O] = 2[(2r)^2 + 4 \cdot \frac{1}{2}\pi r^2] - \pi(2r)^2 = 8r^2.$$

(Here we decompose U into the square S with vertices at $(\pm r, \pm r)$ and the four semicircular regions of radius r bordering the four sides of U .)

Solution 2. There are three different kinds of regions: let x be an area of a small circle that does not contain the two intersections with the other two small circles, y be an area of intersection of two small circles, and z be one of those four areas that is inside the big circle but outside all of the small circles.

Then the key observation is $y = z$. Indeed, adopting the union U notation from the previous solution, we have $4z = [O] - U = \pi(2r)^2 - U$, and by the inclusion-exclusion principle, $4y = [A] + [B] + [C] + [D] - U = 4\pi r^2 - U$, so $y = z$. Now $U = (4x + 4y) - (4z) = 4x$. But the area of each x is simply $2r^2$ by moving the curved outward parts to fit into the curved inward parts to get a $r \times 2r$ rectangle. So the answer is $8r^2$.

22. [12] Let S be a subset of $\{1, 2, 3, \dots, 12\}$ such that it is impossible to partition S into k disjoint subsets, each of whose elements sum to the same value, for any integer $k \geq 2$. Find the maximum possible sum of the elements of S .

Answer: $\boxed{77}$ We note that the maximum possible sum is 78 (the entire set). However, this could be partitioned into 2 subsets with sum 39: $\{1, 2, 3, 10, 11, 12\}$ and $\{4, 5, 6, 7, 8, 9\}$. The next largest possible sum is 77 (the entire set except 1). If $k \geq 2$ subsets each had equal sum, then they would have to be 7 subsets with sum 11 each or 11 subsets with sum 7 each. However, the subset containing 12 will have sum greater than 11; hence there is no way to partition the subset $\{2, \dots, 12\}$ into equal subsets.

23. [12] The number $989 \cdot 1001 \cdot 1007 + 320$ can be written as the product of three distinct primes p, q, r with $p < q < r$. Find (p, q, r) .

Answer: $\boxed{(991, 997, 1009)}$ Let $f(x) = x(x-12)(x+6) + 320 = x^3 - 6x^2 - 72x + 320$, so that $f(1001) = 989 \cdot 1001 \cdot 1007 + 320$. But $f(4) = 4(-8)(10) + 320 = 0$, so $f(x) = (x-4)(x^2 - 2x - 80) = (x-4)(x-10)(x+8)$.

Thus $f(1001) = 991 \cdot 997 \cdot 1009$, as desired.

24. [12] Find the number of subsets S of $\{1, 2, \dots, 6\}$ satisfying the following conditions:

- S is non-empty.
- No subset of S has the property that the sum of its elements is 10.

Answer: $\boxed{34}$ We do casework based on the largest element of S . Call a set n -free if none of its subsets have elements summing to n .

Case 1: The largest element of S is 6. Then $4 \notin S$. If $5 \notin S$, then we wish to find all 4-free subsets of $\{1, 2, 3\}$ (note that $1 + 2 + 3 = 6 < 10$). We just cannot include both 1, 3, so we have $2(2^2 - 1) = 6$ choices here.

If $5 \in S$, then we want 4, 5-free subsets of $\{1, 2, 3\}$. The only 4-but-not-5-free subset is $\{2, 3\}$, so we have $6 - 1$ choices here, for a case total of $6 + 5 = 11$.

Case 2: The largest element of S is 5. We seek 5, 10-free subsets of $\{1, 2, 3, 4\}$. We just cannot have both 1, 4 or both 2, 3 (note that getting 10 requires the whole set), so we have $(2^2 - 1)^2 = 9$ subsets in this case.

Case 3: The largest element of S is at most 4. (So we want a 4-free subset of $\{1, 2, 3, 4\}$.) The only way to sum to 10 with 1, 2, 3, 4 is by using all the terms, so we simply discount the empty set and $\{1, 2, 3, 4\}$, for a total of $2^4 - 2 = 14$ subsets.

In conclusion, the total number of subsets is $11 + 9 + 14 = 34$.

25. [13] Let a, b be positive reals with $a > b > \frac{1}{2}a$. Place two squares of side lengths a, b next to each other, such that the larger square has lower left corner at $(0, 0)$ and the smaller square has lower left corner at $(a, 0)$. Draw the line passing through $(0, a)$ and $(a + b, 0)$. The region in the two squares lying above the line has area 2013. If (a, b) is the unique pair maximizing $a + b$, compute $\frac{a}{b}$.

Answer: $\boxed{\frac{5}{3}}$ Let $t = \frac{a}{b} \in (1, 2)$; we will rewrite the sum $a + b$ as a function of t . The area condition easily translates to $\frac{a^2 - ab + 2b^2}{2} = 2013$, or $b^2(t^2 - t + 2) = 4026 \iff b = \sqrt{\frac{4026}{t^2 - t + 2}}$. Thus $a + b$ is a function $f(t) = (1 + t)\sqrt{\frac{4026}{t^2 - t + 2}}$ of t , and our answer is simply the value of t maximizing f , or equivalently $g(t) = \frac{f^2}{4026} = \frac{(1+t)^2}{t^2 - t + 2}$, over the interval $(1, 2)$. (In general, such maximizers/maximums need not exist, but we shall prove there's a unique maximum here.)

We claim that $\lambda = \frac{16}{7}$ is the maximum of $\frac{(1+t)^2}{t^2 - t + 2}$. Indeed,

$$\begin{aligned} \lambda - g(t) &= \frac{(\lambda - 1)t^2 - (\lambda + 2)t + (2\lambda - 1)}{t^2 - t + 2} \\ &= \frac{1}{7} \frac{9t^2 - 30t + 25}{t^2 - t + 2} = \frac{1}{7} \frac{(3t - 5)^2}{(t - \frac{1}{2})^2 + \frac{7}{4}} \geq 0 \end{aligned}$$

for all reals t (not just $t \in (1, 2)$), with equality at $t = \frac{5}{3} \in (1, 2)$.

Comment. To motivate the choice of λ , suppose λ were the maximum of f , attained at $t_0 \in (1, 2)$; then $h(t) = \lambda(t^2 - t + 2) - (t + 1)^2$ is quadratic and nonnegative on $(1, 2)$, but zero at $t = t_0$. If g is a nontrivial quadratic (nonzero leading coefficient), then t_0 must be a double root, so g has determinant 0. Of course, g could also be constant or linear over $(1, 2)$, but we can easily rule out both of these possibilities.

Alternatively, we can simply take a derivative of f to find critical points.

26. [13] Trapezoid $ABCD$ is inscribed in the parabola $y = x^2$ such that $A = (a, a^2)$, $B = (b, b^2)$, $C = (-b, b^2)$, and $D = (-a, a^2)$ for some positive reals a, b with $a > b$. If $AD + BC = AB + CD$, and $AB = \frac{3}{4}$, what is a ?

Answer: $\boxed{\frac{27}{40}}$ Let $t = \frac{3}{4}$, so $2a + 2b = 2\sqrt{(a - b)^2 + (a^2 - b^2)^2} = 2t$ gives $a = b = t$ and $t^2 = (a - b)^2[1 + (a + b)^2] = (a - b)^2[1 + t^2]$. Thus $a = \frac{t + \frac{t}{\sqrt{1+t^2}}}{2} = \frac{\frac{3}{4} + \frac{3}{4}}{2} = \frac{27}{40}$.

27. [13] Find all triples of real numbers (a, b, c) such that $a^2 + 2b^2 - 2bc = 16$ and $2ab - c^2 = 16$.

Answer: $\boxed{(4, 4, 4), (-4, -4, -4)}$ (need both, but order doesn't matter) $a^2 + 2b^2 - 2bc$ and $2ab - c^2$ are both homogeneous degree 2 polynomials in a, b, c , so we focus on the *homogeneous* equation $a^2 + 2b^2 - 2bc = 2ab - c^2$, or $(a - b)^2 + (b - c)^2 = 0$. So $a = b = c$, and $a^2 = 2ab - c^2 = 16$ gives the solutions $(4, 4, 4)$ and $(-4, -4, -4)$.

28. [15] Triangle ABC has $AB = 4$, $BC = 3$, and a right angle at B . Circles ω_1 and ω_2 of equal radii are drawn such that ω_1 is tangent to AB and AC , ω_2 is tangent to BC and AC , and ω_1 is tangent to ω_2 . Find the radius of ω_1 .

Answer: $\boxed{\frac{5}{7}}$ **Solution 1.** Denote by r the common radius of ω_1, ω_2 , and let O_1, O_2 be the centers of ω_1 and ω_2 respectively. Suppose ω_i hits AC at B_i for $i = 1, 2$, so that $O_1O_2 = B_1B_2 = 2r$.

Extend angle bisector AO_1 to hit BC at P . By the angle bisector theorem and triangle similarity $\triangle AB_1O_1 \sim \triangle ABP$, we deduce $\frac{r}{AB_1} = \frac{BP}{AB} = \frac{3}{4+5}$. Similarly, $\frac{r}{CB_2} = \frac{4}{3+5}$, so

$$5 = AC = AB_1 + B_1B_2 + B_2C = 3r + 2r + 2r = 7r,$$

or $r = \frac{5}{7}$.

Solution 2. Use the same notation as in the previous solution, and let $\alpha = \frac{1}{2}\angle A$. By constructing right triangles with hypotenuses AO_1 , O_1O_2 , and O_2C and legs parallel to AB and BC , we obtain

$$4 = AB = r \cot \alpha + 2r \cos \angle A + r.$$

But $\cot \alpha = \frac{1+\cos 2\alpha}{\sin 2\alpha} = \frac{1+\frac{4}{5}}{\frac{3}{5}} = 3$ and $\cos \angle A = \frac{4}{5}$, so the above equation simplifies to

$$4 = r\left(3 + \frac{8}{5} + 1\right) = \frac{28r}{5},$$

or $r = \frac{5}{7}$.

29. [15] Let $\triangle XYZ$ be a right triangle with $\angle XYZ = 90^\circ$. Suppose there exists an infinite sequence of equilateral triangles $X_0Y_0T_0, X_1Y_1T_1, \dots$ such that $X_0 = X, Y_0 = Y, X_i$ lies on the segment XZ for all $i \geq 0, Y_i$ lies on the segment YZ for all $i \geq 0, X_iY_i$ is perpendicular to YZ for all $i \geq 0, T_i$ and Y are separated by line XZ for all $i \geq 0$, and X_i lies on segment $Y_{i-1}T_{i-1}$ for $i \geq 1$.

Let \mathcal{P} denote the union of the equilateral triangles. If the area of \mathcal{P} is equal to the area of XYZ , find $\frac{XY}{YZ}$.

Answer: 1 For any region R , let $[R]$ denote its area.

Let $a = XY, b = YZ, ra = X_1Y_1$. Then $[\mathcal{P}] = [XYT_0](1 + r^2 + r^4 + \dots), [XYZ] = [XY_1X_1](1 + r^2 + r^4 + \dots), YY_1 = ra\sqrt{3}$, and $b = ra\sqrt{3}(1 + r + r^2 + \dots)$ (although we can also get this by similar triangles).

Hence $\frac{a^2\sqrt{3}}{4} = \frac{1}{2}(ra + a)(ra\sqrt{3})$, or $2r(r + 1) = 1 \implies r = \frac{\sqrt{3}-1}{2}$. Thus $\frac{XY}{YZ} = \frac{a}{b} = \frac{1-r}{r\sqrt{3}} = 1$.

30. [15] Find the number of ordered triples of integers (a, b, c) with $1 \leq a, b, c \leq 100$ and $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2$.

Answer: 29800 This factors as $(a-b)(b-c)(c-a) = 0$. By the inclusion-exclusion principle, we get $3 \cdot 100^2 - 3 \cdot 100 + 100 = 29800$.

31. [17] Chords \overline{AB} and \overline{CD} of circle ω intersect at E such that $AE = 8, BE = 2, CD = 10$, and $\angle AEC = 90^\circ$. Let R be a rectangle inside ω with sides parallel to \overline{AB} and \overline{CD} , such that no point in the interior of R lies on $\overline{AB}, \overline{CD}$, or the boundary of ω . What is the maximum possible area of R ?

Answer: 26 + 6\sqrt{17} By power of a point, $(CE)(ED) = (AE)(EB) = 16$, and $CE + ED = CD = 10$. Thus CE, ED are 2, 8. Without loss of generality, assume $CE = 8$ and $DE = 2$.

Assume our circle is centered at the origin, with points $A = (-3, 5), B = (-3, -5), C = (5, -3), D = (-5, -3)$, and the equation of the circle is $x^2 + y^2 = 34$. Clearly the largest possible rectangle must lie in the first quadrant, and if we let (x, y) be the upper-right corner of the rectangle, then the area of the rectangle is $(x+3)(y+3) = 9 + 6(x+y) + xy \leq 9 + 12\sqrt{\frac{x^2+y^2}{2} + \frac{x^2+y^2}{2}} = 26 + 6\sqrt{17}$, where equality holds if and only if $x = y = \sqrt{17}$.

32. [17] Suppose that x and y are chosen randomly and uniformly from $(0, 1)$. What is the probability that $\lfloor \sqrt{\frac{x}{y}} \rfloor$ is even? *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Answer: 1 - \frac{\pi^2}{24} \text{ OR } \frac{24-\pi^2}{24} Note that for every positive integer n , the probability that $\lfloor \sqrt{\frac{x}{y}} \rfloor = n$ is just the area of the triangle formed between $(0, 0), (1, \frac{1}{n^2}), (1, \frac{1}{(n+1)^2})$, which is just $\frac{1}{2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} \right)$.

Thus the probability that $\lfloor \sqrt{\frac{x}{y}} \rfloor$ is odd is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{(2k-1)^2} - \frac{1}{(2k)^2} \right) &= \frac{1}{2} \sum_{k=1}^{\infty} \left(\frac{1}{(2k-1)^2} + \frac{1}{(2k)^2} \right) - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} \\ &= \frac{\pi^2}{12} - \frac{\pi^2}{24} = \frac{\pi^2}{24}. \end{aligned}$$

Thus our answer is just $1 - \frac{\pi^2}{24}$.

33. [17] On each side of a 6 by 8 rectangle, construct an equilateral triangle with that side as one edge such that the interior of the triangle intersects the interior of the rectangle. What is the total area of all regions that are contained in exactly 3 of the 4 equilateral triangles?

Answer: $\frac{96\sqrt{3}-154}{\sqrt{3}}$ OR $\frac{288-154\sqrt{3}}{3}$ OR $96 - \frac{154}{\sqrt{3}}$ OR $96 - \frac{154\sqrt{3}}{3}$ Let the rectangle be $ABCD$ with $AB = 8$ and $BC = 6$. Let the four equilateral triangles be ABP_1 , BCP_2 , CDP_3 , and DAP_4 (for convenience, call them the P_1 -, P_2 -, P_3 -, P_4 - triangles). Let $W = AP_1 \cap DP_3$, $X = AP_1 \cap DP_4$, and $Y = DP_4 \cap CP_2$. Reflect X, Y over the line P_2P_4 (the line halfway between AB and DC) to points X', Y' .

First we analyze the basic configuration of the diagram. Since $AB = 8 < 2 \cdot 6 \frac{\sqrt{3}}{2}$, the P_2 -, P_4 - triangles intersect. Furthermore, $AP_1 \perp BP_2$, so if $T = BP_2 \cap AP_1$, then $BP_2 = 6 < 4\sqrt{3} = BT$. Therefore P_2 lies inside triangle P_1BA , and by symmetry, also triangle P_3DC .

It follows that the area we wish to compute is the union of two (congruent) concave hexagons, one of which is $WXY P_2 Y' X'$. (The other is its reflection over YY' , the mid-line of AD and BC .) So we seek

$$2[WXY P_2 Y' X'] = 2([WXP_4X'] - [P_2YP_4Y']).$$

It's easy to see that $[WXP_4X'] = \frac{1}{3}[ADP_4] = \frac{1}{3} \frac{6^2\sqrt{3}}{4} = 3\sqrt{3}$, since WXP_4X' and its reflections over lines DWX' and AWX partition $\triangle ADP_4$.

It remains to consider P_2YP_4Y' , a rhombus with (perpendicular) diagonals P_2P_4 and YY' . If O denotes the intersection of these two diagonals (also the center of $ABCD$), then OP_2 is $P_2B \frac{\sqrt{3}}{2} - \frac{1}{2}AB = 3\sqrt{3} - 4$, the difference between the lengths of the P_2 -altitude in $\triangle CBP_2$ and the distance between the parallel lines YY', CB . Easy angle chasing gives $OY = \frac{OP_2}{\sqrt{3}}$, so

$$[P_2YP_4Y'] = 4 \cdot \frac{OP_2 \cdot OY}{2} = \frac{2}{\sqrt{3}} OP_2^2 = \frac{2}{\sqrt{3}} (3\sqrt{3} - 4)^2 = \frac{86 - 48\sqrt{3}}{\sqrt{3}},$$

and our desired area is

$$2[WXP_4X'] - 2[P_2YP_4Y'] = 6\sqrt{3} - \frac{172 - 96\sqrt{3}}{\sqrt{3}} = \frac{96\sqrt{3} - 154}{\sqrt{3}},$$

or $\frac{288-154\sqrt{3}}{3}$.

34. [20] Find the number of positive integers less than 1000000 that are divisible by some perfect cube greater than 1. Your score will be $\max\{0, \lfloor 20 - 200|1 - \frac{k}{S}| \rfloor\}$, where k is your answer and S is the actual answer.

Answer: $\boxed{168089}$ Using the following code, we get the answer (denoted by the variable ans):

```
ans = 0
for n in xrange(1,1000000):
```

```

divisible_by_cube = True
for i in xrange(2,101):
    if n%(i*i*i)==0:
        divisible_by_cube = False
        break
    if divisible_by_cube: ans = ans + 1
print ans

```

This gives the output

168089

Alternatively, let $N = 1000000$ and denote by P the set of primes. Then by PIE, the number of $n \in (0, N)$ divisible by a nontrivial cube, or equivalently, by p^3 for some $p \in P$, is

$$\sum_{p \in P} \left\lfloor \frac{N-1}{p^3} \right\rfloor - \sum_{p < q \in P} \left\lfloor \frac{N-1}{p^3 q^3} \right\rfloor \pm \dots,$$

which deviates from

$$\sum_{p \in P} \frac{N-1}{p^3} - \sum_{p < q \in P} \frac{N-1}{p^3 q^3} \pm \dots = (N-1) \left(1 - \prod_{p \in P} (1 - p^{-3})\right)$$

by at most the sum of

- $N^{1/3} \sup_{t \in \mathbb{R}} |t - \lfloor t \rfloor| = N^{1/3}$, for terms $\lfloor \frac{N-1}{p_1^3 \dots p_r^3} \rfloor$ with $p_1 \dots p_r \leq (N-1)^{1/3}$, and
- $(N-1) \sum_{k > (N-1)^{1/3}} k^{-3} < (N-1) [(N-1)^{-1} + \int_{(N-1)^{1/3}}^{\infty} x^{-3} dx] = 1 + (N-1) \frac{(N-1)^{-2/3}}{2} = O(N^{1/3})$, for the remaining terms.

So we are really interested in $10^6 - 10^6 \prod_{p \in P} (1 - p^{-3})$ (which, for completeness, is 168092.627...). There are a few simple ways to approximate this:

- We can use a partial product of $\prod_{p \in P} (1 - p^{-3})$. Using just $1 - 2^{-3} = 0.875$ gives an answer of 125000 (this is also just the number of $x \leq N$ divisible by $2^3 = 8$), $(1 - 2^{-3})(1 - 3^{-3}) \approx 0.843$ gives 157000 (around the number of x divisible by 2^3 or 3^3), etc. This will give a lower bound, of course, so we can guess a bit higher. For instance, while 157000 gives a score of around 7, rounding up to 160000 gives ≈ 10 .
- We can note that $\prod_{p \in P} (1 - p^{-3}) = \zeta(3)^{-1}$ is the inverse of $1 + 2^{-3} + 3^{-3} + \dots$. This is a bit less efficient, but successive partial sums (starting with $1 + 2^{-3}$) give around 111000, 139000, 150000, 157000, etc. Again, this gives a lower bound, so we can guess a little higher.
- We can optimize the previous approach with integral approximation after the r th term: $\zeta(3)$ is the sum of $1 + 2^{-3} + \dots + r^{-3}$ plus something between $\int_{r+1}^{\infty} x^{-3} dx = \frac{1}{2}(r+1)^{-2}$ and $\int_r^{\infty} x^{-3} dx = \frac{1}{2}r^{-2}$. Then starting with $r = 1$, we get intervals of around (111000, 334000), (152000, 200000), (161000, 179000), (165000, 173000), etc. Then we can take something like the average of the two endpoints as our guess; such a strategy gets a score of around 10 for $r = 2$ already, and ≈ 17 for $r = 3$.

35. [20] Consider the following 4 by 4 grid with one corner square removed:

You may start at any square in this grid and at each move, you may either stop or travel to an adjacent square (sharing a side, not just a corner) that you have not already visited (the square you start at is automatically marked as visited). Determine the distinct number of paths you can take. Your score will be $\max\{0, \lfloor 20 - 200|1 - \frac{k}{S}| \rfloor\}$, where k is your answer and S is the actual answer.

Answer: 14007

36. [20] Pick a subset of at least four of the following seven numbers, order them from least to greatest, and write down their **labels** (corresponding letters from A through G) in that order: (A) π ; (B) $\sqrt{2} + \sqrt{3}$; (C) $\sqrt{10}$; (D) $\frac{355}{113}$; (E) $16 \tan^{-1} \frac{1}{5} - 4 \tan^{-1} \frac{1}{240}$; (F) $\ln(23)$; and (G) $2^{\sqrt{e}}$. If the ordering of the numbers you picked is correct and you picked at least 4 numbers, then your score for this problem will be $(N - 2)(N - 3)$, where N is the size of your subset; otherwise, your score is 0.

Answer: F, G, A, D, E, B, C OR $F < G < A < D < E < B < C$ OR $C > B > E > D > A > G > F$

We have $\ln(23) < 2^{\sqrt{e}} < \pi < \frac{355}{113} < 16 \tan^{-1} \frac{1}{5} - 4 \tan^{-1} \frac{1}{240} < \sqrt{2} + \sqrt{3} < \sqrt{10}$.

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Team Round

1. [3] Tim the Beaver can make three different types of geometrical figures: squares, regular hexagons, and regular octagons. Tim makes a random sequence $F_0, F_1, F_2, F_3, \dots$ of figures as follows:

- F_0 is a square.
- For every positive integer i , F_i is randomly chosen to be one of the 2 figures *distinct from* F_{i-1} (each chosen with equal probability $\frac{1}{2}$).
- Tim takes 4 seconds to make squares, 6 to make hexagons, and 8 to make octagons. He makes one figure after another, with no breaks in between.

Suppose that exactly 17 seconds after he starts making F_0 , Tim is making a figure with n sides. What is the expected value of n ?

Answer: $\boxed{7}$ We write $F_i = n$ as shorthand for “the i th figure is an n -sided polygon.”

If $F_1 = 8$, the $F_2 = 6$ or $F_2 = 4$. If $F_2 = 6$, Tim is making a 6-gon at time 13 (probability contribution $1/4$). If $F_2 = 4$, $F_3 = 6$ or $F_3 = 8$ will take the time 13 mark ($1/8$ contribution each).

If $F_1 = 6$, $F_2 = 8$ or $F_2 = 4$. If $F_2 = 8$, it takes the 13 mark ($1/4$ contribution). If $F_2 = 4$, $F_3 = 6$ or $F_3 = 8$ will take the 13 mark ($1/8$ contribution each).

Thus, the expected value of the number of sides at time 13 is $0(4) + (\frac{1}{4} + \frac{1}{8} + \frac{1}{8})(6) + (\frac{1}{8} + \frac{1}{4} + \frac{1}{8})(8) = 7$.

2. [4] Gary plays the following game with a fair n -sided die whose faces are labeled with the positive integers between 1 and n , inclusive: if $n = 1$, he stops; otherwise he rolls the die, and starts over with a k -sided die, where k is the number his n -sided die lands on. (In particular, if he gets $k = 1$, he will stop rolling the die.) If he starts out with a 6-sided die, what is the expected number of rolls he makes?

Answer: $\boxed{\frac{197}{60}}$ If we let a_n be the expected number of rolls starting with an n -sided die, we see immediately that $a_1 = 0$, and $a_n = 1 + \frac{1}{n} \sum_{i=1}^n a_i$ for $n > 1$. Thus $a_2 = 2$, and for $n \geq 3$, $a_n = 1 + \frac{1}{n} a_n + \frac{n-1}{n} (a_{n-1} - 1)$, or $a_n = a_{n-1} + \frac{1}{n-1}$. Thus $a_n = 1 + \sum_{i=1}^{n-1} \frac{1}{i}$ for $n \geq 2$, so $a_6 = 1 + \frac{60+30+20+15+12}{60} = \frac{197}{60}$.

3. [6] The digits 1, 2, 3, 4, 5, 6 are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

Answer: $\boxed{143745}$ By linearity of expectation and symmetry,

$$\mathbb{E}[MN] = \mathbb{E}[(100A + 10B + C)(100D + 10E + F)] = 111^2 \cdot \mathbb{E}[AD].$$

Since

$$\mathbb{E}[AD] = \frac{(1 + 2 + 3 + 4 + 5 + 6)^2 - (1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2)}{6 \cdot 5} = \frac{350}{30},$$

our answer is $111 \cdot 35 \cdot 37 = 111 \cdot 1295 = 143745$.

4. [4] Consider triangle ABC with side lengths $AB = 4$, $BC = 7$, and $AC = 8$. Let M be the midpoint of segment AB , and let N be the point on the interior of segment AC that also lies on the circumcircle of triangle MBC . Compute BN .

Answer: $\boxed{\frac{\sqrt{210}}{4} \text{ OR } \frac{\sqrt{105}}{2\sqrt{2}}}$ Let $\angle BAC = \theta$. Then, $\cos \theta = \frac{4^2 + 8^2 - 7^2}{2 \cdot 4 \cdot 8}$. Since $AM = \frac{4}{2} = 2$, and power of a point gives $AM \cdot AB = AN \cdot AC$, we have $AN = \frac{2 \cdot 4}{8} = 1$, so $NC = 8 - 1 = 7$. Law of cosines on triangle BAN gives

$$BN^2 = 4^2 + 1^2 - 2 \cdot 4 \cdot 1 \cdot \frac{4^2 + 8^2 - 7^2}{2 \cdot 4 \cdot 8} = 17 - \frac{16 + 15}{8} = 15 - \frac{15}{8} = \frac{105}{8},$$

so $BN = \frac{\sqrt{210}}{4}$.

5. [4] In triangle ABC , $\angle BAC = 60^\circ$. Let ω be a circle tangent to segment AB at point D and segment AC at point E . Suppose ω intersects segment BC at points F and G such that F lies in between B and G . Given that $AD = FG = 4$ and $BF = \frac{1}{2}$, find the length of CG .

Answer: $\boxed{\frac{16}{5}}$ Let $x = CG$. First, by power of a point, $BD = \sqrt{BF(BF + FG)} = \frac{3}{2}$, and $CE = \sqrt{x(x+4)}$. By the law of cosines, we have

$$\left(x + \frac{9}{2}\right)^2 = \left(\frac{11}{2}\right)^2 + (4 + \sqrt{x(x+4)})^2 - \frac{11}{2}(4 + \sqrt{x(x+4)}),$$

which rearranges to $2(5x - 4) = 5\sqrt{x(x+4)}$. Squaring and noting $x > \frac{4}{5}$ gives $(5x - 16)(15x - 4) = 0 \implies x = \frac{16}{5}$.

6. [6] Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D . If $B'D = 4$ and $C'D = 6$, compute BC .

Answer: $\boxed{\frac{24}{5}}$ Let $x = AB$ and $y = AC$, and define $t > 0$ such that $BB' = tx$ and $CC' = ty$. Then $10 = B'C' = (1+t)\sqrt{x^2+y^2}$, $4^2 = t(1+t)x^2$, and $6^2 = t(1+t)y^2$ (by power of a point), so $52 = 4^2 + 6^2 = t(1+t)(x^2+y^2)$ gives $\frac{13}{25} = \frac{52}{10^2} = \frac{t(1+t)}{(1+t)^2} = \frac{t}{1+t} \implies t = \frac{13}{12}$. Hence $BC = \sqrt{x^2+y^2} = \frac{10}{1+t} = \frac{10}{25/12} = \frac{24}{5}$.

7. [7] In equilateral triangle ABC , a circle ω is drawn such that it is tangent to all three sides of the triangle. A line is drawn from A to point D on segment BC such that AD intersects ω at points E and F . If $EF = 4$ and $AB = 8$, determine $|AE - FD|$.

Answer: $\boxed{\frac{4}{\sqrt{5}} \text{ OR } \frac{4\sqrt{5}}{5}}$ Without loss of generality, A, E, F, D lie in that order. Let $x = AE$, $y = DF$. By power of a point, $x(x+4) = 4^2 \implies x = 2\sqrt{5} - 2$, and $y(y+4) = (x+4+y)^2 - (4\sqrt{3})^2 \implies y = \frac{48-(x+4)^2}{2(x+2)} = \frac{12-(1+\sqrt{5})^2}{\sqrt{5}}$. It readily follows that $x - y = \frac{4}{\sqrt{5}} = \frac{4\sqrt{5}}{5}$.

8. [2] Define the sequence $\{x_i\}_{i \geq 0}$ by $x_0 = x_1 = x_2 = 1$ and $x_k = \frac{x_{k-1} + x_{k-2} + 1}{x_{k-3}}$ for $k > 2$. Find x_{2013} .

Answer: $\boxed{9}$ We have $x_3 = \frac{1+1+1}{1} = 3$, $x_4 = \frac{3+1+1}{1} = 5$, $x_5 = \frac{5+3+1}{1} = 9$, $x_6 = \frac{9+5+1}{3} = 5$. By the symmetry of our recurrence (or just further computation—it doesn't matter much), $x_7 = 3$ and $x_8 = x_9 = x_{10} = 1$, so our sequence has period 8. Thus $x_{2013} = x_{13} = x_5 = 9$.

9. [7] For an integer $n \geq 0$, let $f(n)$ be the smallest possible value of $|x + y|$, where x and y are integers such that $3x - 2y = n$. Evaluate $f(0) + f(1) + f(2) + \dots + f(2013)$.

Answer: $\boxed{2416}$ First, we can use $3x - 2y = n$ to get $x = \frac{n+2y}{3}$. Thus $|x + y| = \left|\frac{n+5y}{3}\right|$. Given a certain n , the only restriction on y is that $3 \mid n + 2y \iff 3 \mid n + 5y$. Hence the set of possible $x + y$ equals the set of integers of the form $\frac{n+5y}{3}$, which in turn equals the set of integers congruent to $3^{-1}n \equiv 2n \pmod{5}$. (Prove this!)

Thus $f(n) = |x + y|$ is minimized when $x + y$ equals the *least absolute remainder* $(2n)_5$ when $2n$ is divided by 5, i.e. the number between -2 and 2 (inclusive) congruent to $2n$ modulo 5. We immediately find $f(n) = f(n + 5m)$ for all integers m , and the following initial values of f : $f(0) = |(0)_5| = 0$; $f(1) = |(2)_5| = 2$; $f(2) = |(4)_5| = 1$; $f(3) = |(6)_5| = 1$; and $f(4) = |(8)_5| = 2$.

Since $2013 = 403 \cdot 5 - 2$, it follows that $f(0) + f(1) + \dots + f(2013) = 403[f(0) + f(1) + \dots + f(4)] - f(2014) = 403 \cdot 6 - 2 = 2416$.

10. [7] Let $\omega = \cos \frac{2\pi}{727} + i \sin \frac{2\pi}{727}$. The imaginary part of the complex number

$$\prod_{k=8}^{13} \left(1 + \omega^{3^{k-1}} + \omega^{2 \cdot 3^{k-1}}\right)$$

is equal to $\sin \alpha$ for some angle α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Find α .

Answer: $\boxed{\frac{12\pi}{727}}$ Note that $727 = 3^6 - 2$. Our product telescopes to $\frac{1-\omega^{3^{13}}}{1-\omega^{3^7}} = \frac{1-\omega^{12}}{1-\omega^6} = 1 + \omega^6$, which has imaginary part $\sin \frac{12\pi}{727}$, giving $\alpha = \frac{12\pi}{727}$.

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Theme Round

1. [2] Two cars are driving directly towards each other such that one is twice as fast as the other. The distance between their starting points is 4 miles. When the two cars meet, how many miles is the faster car from its starting point?

Answer: $\boxed{\frac{8}{3}}$ Note that the faster car traveled twice the distance of the slower car, and together, the two cars traveled the total distance between the starting points, which is 4 miles. Let the distance that the faster car traveled be x . Then, $x + \frac{x}{2} = 4 \implies x = \frac{8}{3}$. Thus, the faster car traveled $\frac{8}{3}$ miles from the starting point.

2. [4] You are standing at a pole and a snail is moving directly away from the pole at 1 cm/s. When the snail is 1 meter away, you start "Round 1". In Round n ($n \geq 1$), you move directly toward the snail at $n + 1$ cm/s. When you reach the snail, you immediately turn around and move back to the starting pole at $n + 1$ cm/s. When you reach the pole, you immediately turn around and Round $n + 1$ begins. At the start of Round 100, how many **meters** away is the snail?

Answer: $\boxed{5050}$ Suppose the snail is x_n meters away at the start of round n , so $x_1 = 1$, and the runner takes $\frac{100x_n}{(n+1)-1} = \frac{100x_n}{n}$ seconds to catch up to the snail. But the runner takes the same amount of time to run back to the start, so during round n , the snail moves a distance of $x_{n+1} - x_n = \frac{200x_n}{n} \cdot \frac{1}{100} = \frac{2x_n}{n}$. Finally, we have $x_{100} = \frac{101}{99}x_{99} = \frac{101}{99} \cdot \frac{100}{98}x_{98} = \dots = \frac{101! \cdot 2!}{99!}x_1 = 5050$.

3. [5] Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. Initially, there is an ant at each vertex. The ants start walking at a rate of 1 unit per second, in the direction $A \rightarrow B \rightarrow C \rightarrow A$ (so the ant starting at A moves along ray \overrightarrow{AB} , etc.). For a positive real number t less than 3, let $A(t)$ be the area of the triangle whose vertices are the positions of the ants after t seconds have elapsed. For what positive real number t less than 3 is $A(t)$ minimized?

Answer: $\boxed{\frac{47}{24}}$ We instead maximize the area of the remaining triangles. This area (using $\frac{1}{2}xy \sin \theta$) is $\frac{1}{2}(t)(5-t)\frac{3}{5} + \frac{1}{2}(t)(3-t)\frac{4}{5} + \frac{1}{2}(t)(4-t)1 = \frac{1}{10}(-12t^2 + 47t)$, which has a maximum at $t = \frac{47}{24} \in (0, 3)$.

4. [7] There are 2 runners on the perimeter of a regular hexagon, initially located at adjacent vertices. Every second, each of the runners independently moves either one vertex to the left, with probability $\frac{1}{2}$, or one vertex to the right, also with probability $\frac{1}{2}$. Find the probability that after a 2013 second run (in which the runners switch vertices 2013 times each), the runners end up at adjacent vertices once again.

Answer: $\boxed{\frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^{2013} \text{ OR } \frac{2^{4027}+1}{3 \cdot 2^{4026}} \text{ OR } \frac{2}{3} + \frac{1}{3}\left(\frac{1}{2}\right)^{4026} \text{ OR } \frac{2}{3} + \frac{1}{3}\left(\frac{1}{64}\right)^{671}}$ Label the runners A and B and arbitrarily fix an orientation of the hexagon. Let $p_t(i)$ be the probability that A is $i \pmod{6}$ vertices to the right of B at time t , so without loss of generality $p_0(1) = 1$ and $p_0(2) = \dots = p_0(6) = 0$. Then for $t > 0$, $p_t(i) = \frac{1}{4}p_{t-1}(i-2) + \frac{1}{2}p_{t-1}(i) + \frac{1}{4}p_{t-1}(i+2)$.

In particular, $p_t(2) = p_t(4) = p_t(6) = 0$ for all t , so we may restrict our attention to $p_t(1), p_t(3), p_t(5)$. Thus $p_t(1) + p_t(3) + p_t(5) = 1$ for all $t \geq 0$, and we deduce $p_t(i) = \frac{1}{4} + \frac{1}{4}p_{t-1}(i)$ for $i = 1, 3, 5$.

Finally, let $f(t) = p_t(1) + p_t(5)$ denote the probability that A, B are 1 vertex apart at time t , so $f(t) = \frac{1}{2} + \frac{1}{4}f(t-1) \implies f(t) - \frac{2}{3} = \frac{1}{4}[f(t-1) - \frac{2}{3}]$, and we conclude that $f(2013) = \frac{2}{3} + \frac{1}{3}\left(\frac{1}{4}\right)^{2013}$.

5. [7] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Company XYZ wants to locate their base at the point P in the plane minimizing the total distance to their workers, who are located at vertices A , B , and C . There are 1, 5, and 4 workers at A , B , and C , respectively. Find the minimum possible total distance Company XYZ's workers have to travel to get to P .

Answer: $\boxed{69}$ We want to minimize $1 \cdot PA + 5 \cdot PB + 4 \cdot PC$. By the triangle inequality, $(PA + PB) + 4(PB + PC) \geq AB + 4BC = 13 + 56 = 69$, with equality precisely when $P = [AB] \cap [BC] = B$.

6. [2] Evaluate 1201201_{-4} .

Answer: 2017 The answer is $1 + 2(-4)^2 + (-4)^3 + 2(-4)^5 + (-4)^6 = 1 - 2 \cdot 4^2 + 2 \cdot 4^5 = 2049 - 32 = 2017$.

7. [3] Express -2013 in base -4 .

Answer: 200203 $-2013 \equiv 3 \pmod{4}$, so the last digit is 3; now $\frac{-2013-3}{-4} = 504 \equiv 0$, so the next digit (to the left) is 0; then $\frac{504-0}{-4} = -126 \equiv 2$; $\frac{-126-2}{-4} = 32 \equiv 0$; $\frac{32-0}{-4} = -8 \equiv 0$; $\frac{-8-0}{-4} = 2$.

Thus $-2013_{10} = 200203_{-4}$.

8. [5] Let $b(n)$ be the number of digits in the base -4 representation of n . Evaluate $\sum_{i=1}^{2013} b(i)$.

Answer: 12345 We have the following:

- $b(n) = 1$ for n between 1 and 3.
- $b(n) = 3$ for n between $4^2 - 3 \cdot 4 = 4$ and $3 \cdot 4^2 + 3 = 51$. (Since $a \cdot 4^2 - b \cdot 4 + c$ takes on $3 \cdot 4 \cdot 4$ distinct values over $1 \leq a \leq 3, 0 \leq b \leq 3, 0 \leq c \leq 3$, with minimum 4 and maximum 51.)
- $b(n) = 5$ for n between $4^4 - 3 \cdot 4^3 - 3 \cdot 4 = 52$ and $3 \cdot 4^4 + 3 \cdot 4^2 + 3 = 819$.
- $b(n) = 7$ for n between $4^6 - 3 \cdot 4^5 - 3 \cdot 4^3 - 3 \cdot 4^1 = 820$ and $3 \cdot 4^6 + 3 \cdot 4^4 + 3 \cdot 4^2 + 3 > 2013$.

Thus

$$\sum_{i=1}^{2013} b(i) = 7(2013) - 2(819 + 51 + 3) = 14091 - 2(873) = 14091 - 1746 = 12345.$$

9. [7] Let N be the largest positive integer that can be expressed as a 2013-digit base -4 number. What is the remainder when N is divided by 210?

Answer: 51 The largest is $\sum_{i=0}^{1006} 3 \cdot 4^{2i} = 3 \frac{16^{1007} - 1}{16 - 1} = \frac{16^{1007} - 1}{5}$.

This is $1 \pmod{2}$, $0 \pmod{3}$, $3 \cdot 1007 \equiv 21 \equiv 1 \pmod{5}$, and $3(2^{1007} - 1) \equiv 3(2^8 - 1) \equiv 3(2^2 - 1) \equiv 2 \pmod{7}$, so we need $1 \pmod{10}$ and $9 \pmod{21}$, which is $9 + 2 \cdot 21 = 51 \pmod{210}$.

10. [8] Find the sum of all positive integers n such that there exists an integer b with $|b| \neq 4$ such that the base -4 representation of n is the same as the base b representation of n .

Answer: 1026 All 1 digit numbers, 0, 1, 2, 3, are solutions when, say, $b = 5$. (Of course, $d \in \{0, 1, 2, 3\}$ works for any base b of absolute value greater than d but not equal to 4.)

Consider now positive integers $n = (a_d \dots a_1 a_0)_4$ with more than one digit, so $d \geq 1$, $a_d \neq 0$, and $0 \leq a_k \leq 3$ for $k = 0, 1, \dots, d$. Then n has the same representation in base b if and only if $|b| > \max a_k$ and $\sum_{k=0}^d a_k (-4)^k = \sum_{k=0}^d a_k b^k$, or equivalently, $\sum_{k=0}^d a_k (b^k - (-4)^k) = 0$.

First we prove that $b \leq 3$. Indeed, if $b \geq 4$, then $b \neq 4 \implies b \geq 5$, so $b^k - (-4)^k$ is positive for all $k \geq 1$ (and zero for $k = 0$). But then $\sum_{k=0}^d a_k (b^k - (-4)^k) \geq a_d (b^d - (-4)^d)$ must be positive, and cannot vanish.

Next, we show $b \geq 2$. Assume otherwise for the sake of contradiction; b cannot be $0, \pm 1$ (these bases don't make sense in general) or -4 , so we may label two distinct negative integers $-r, -s$ with $r - 1 \geq s \geq 2$ such that $\{r, s\} = \{4, -b\}$, $s > \max a_k$, and $\sum_{k=0}^d a_k ((-r)^k - (-s)^k) = 0$, which, combined with the fact that $r^k - s^k \geq 0$ (equality only at $k = 0$), yields

$$\begin{aligned} r^d - s^d &\leq a_d (r^d - s^d) = \sum_{k=0}^{d-1} (-1)^{d-1-k} a_k (r^k - s^k) \\ &\leq \sum_{k=0}^{d-1} (s-1)(r^k - s^k) = (s-1) \frac{r^d - 1}{r-1} - (s^d - 1). \end{aligned}$$

Hence $r^d - 1 \leq (s-1) \frac{r^d - 1}{r-1} < (r-1) \frac{r^d - 1}{r-1} = r^d - 1$, which is absurd.

Thus $b \geq 2$, and since $b \leq 3$ we must either have $b = 2$ or $b = 3$. In particular, all a_k must be at most $b - 1$. We now rewrite our condition as

$$a_d(4^d - (-b)^d) = \sum_{k=0}^{d-1} (-1)^{d-1-k} a_k (4^k - (-b)^k).$$

Since $4^k - (-b)^k \geq 0$ for $k \geq 0$, with equality only at $k = 0$, we deduce

$$a_d(4^d - (-b)^d) \leq \sum_{k \equiv d-1 \pmod{2}} (b-1)(4^k - (-b)^k).$$

If $d - 1$ is even (d is odd), this gives

$$a_d(4^d + b^d) \leq (b-1) \frac{4^{d+1} - 4^0}{4^2 - 1} - (b-1) \frac{b^{d+1} - b^0}{b^2 - 1},$$

so $4^d < (b-1) \frac{4^{d+1}}{15} \implies b > 1 + \frac{15}{4}$, which is impossible.

Thus $d - 1$ is odd (d is even), and we get

$$a_d(4^d - b^d) \leq (b-1) \frac{4^{d+1} - 4^1}{4^2 - 1} + (b-1) \frac{b^{d+1} - b^1}{b^2 - 1} \iff \frac{b^d - 1}{4^d - 1} \geq \frac{a_d - \frac{4}{15}(b-1)}{a_d + \frac{b}{b+1}}.$$

If $b = 2$, then $a_d = 1$, so $\frac{1}{2^{d+1}} = \frac{2^d - 1}{4^d - 1} \geq \frac{11}{25}$, which is clearly impossible ($d \geq 2$).

If $b = 3$ and $a_d = 2$, then $\frac{9^{d/2} - 1}{16^{d/2} - 1} \leq \frac{8}{15}$. Since d is even, it's easy to check this holds only for $d/2 = 1$, with equality, so $a_k = b - 1$ if $k \equiv d - 1 \pmod{2}$. Thus $(a_d, \dots, a_0) = (2, 2, a_0)$, yielding solutions $(22x)_3$ (which do work; note that the last digit doesn't matter).

Otherwise, if $b = 3$ and $a_d = 14$, then $\frac{9^{d/2} - 1}{16^{d/2} - 1} \leq \frac{4}{15}$. It's easy to check $d/2 \in \{1, 2\}$.

If $d/2 = 1$, we're solving $16a_2 - 4a_1 + a_0 = 9a_2 + 3a_1 + a_0 \iff a_2 = a_1$. We thus obtain the working solution $(11x)_3$. (Note that $110 = \frac{1}{2}220$ in bases $-4, 3$.)

If $d/2 = 2$, we want $256a_4 - 64a_3 + 16a_2 - 4a_1 + a_0 = 81a_4 + 27a_3 + 9a_2 + 3a_1 + a_0$, or $175 = 91a_3 - 7a_2 + 7a_1$, which simplifies to $25 = 13a_3 - a_2 + a_1$. This gives the working solutions $(1210x)_3, (1221x)_3$. (Note that $12100 = 110^2$ and $12210 = 110^2 + 110$ in bases $-4, 3$.)

The list of all nontrivial (≥ 2 -digit) solutions (in base -4 and b) is then $11x, 22x, 1210x, 1221x$, where $b = 3$ and $x \in \{0, 1, 2\}$. In base 10, they are $12 + x, 2 \cdot 12 + x, 12^2 + x, 12^2 + 12 + x$, with sum $3(2 \cdot 12^2 + 4 \cdot 12) + 4(0 + 1 + 2) = 1020$.

Finally, we need to include the trivial solutions $n = 1, 2, 3$, for a total sum of 1026.

HMMT 2013
Saturday 16 February 2013
HMIC

1. [5] Let \mathcal{S} be a set of size n , and k be a positive integer. For each $1 \leq i \leq kn$, there is a subset $S_i \subset \mathcal{S}$ such that $|S_i| = 2$. Furthermore, for each $e \in \mathcal{S}$, there are exactly $2k$ values of i such that $e \in S_i$. Show that it is possible to choose one element from S_i for each $1 \leq i \leq kn$ such that every element of \mathcal{S} is chosen exactly k times.

Answer: N/A Consider the undirected graph $G = (\mathcal{S}, E)$ where the elements of \mathcal{S} are the vertices, and for each $1 \leq i \leq kn$, there is an edge between the two elements of S_i . (Note that there might be multiedges if two subsets are the same, but there are no self-loops.) Consider any connected component C of G , which must be a $2k$ -regular graph, and because $2k$ is even, C has an Eulerian circuit. Pick an orientation of the circuit, and hence a direction for each edge in C . Then, for each i such that the edge corresponding to S_i is in C , pick the element that is pointed to by that edge. Since the circuit goes into each vertex of C k times, each element in C is picked exactly k times as desired. Repeating for each connected component finishes the problem.

2. [7] Find all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that, for all real numbers x, y ,

$$(x - y)(f(x) - f(y)) = f(x - f(y))f(f(x) - y).$$

Answer: $f(x) = 0, f(x) = x$ First, suppose that $f(0) \neq 0$. Then, by letting $x = y = 0$, we get that either $f(f(0))$ or $f(-f(0))$ is zero. The former gives a contradiction by plugging in $x = f(0), y = 0$ into the original equation. Thus, $f(-f(0)) = 0$.

Now plugging in $x = 0, y = -f(0)$ gives that $f(2f(0)) = f(0)$. Finally, we note that by symmetry of the left hand side of the functional equation, $f(x - f(y))f(f(x) - y) = f(y - f(x))f(f(y) - x)$, and letting $y = 0$ and $x = 2f(0)$ we get $f(f(0))f(f(0)) = f(-f(0))f(-f(0)) = 0$, giving and so $f(f(0)) = 0$, again a contradiction. Therefore, $f(0) = 0$.

Setting $y = 0$ yields $xf(x) = f(x)f(f(x))$, so if $f(x) \neq 0$, then $f(f(x)) = x$. On the other hand, setting $y = f(x)$ yields $(x - f(x))(f(x) - f(f(x))) = f(x - f(f(x)))f(0)$, so in fact $f(f(x)) = f(x) \rightarrow f(x) = x$. Thus, for all x , $f(x) = 0$ or $f(x) = x$.

Clearly, $f \equiv 0$ satisfies the original equation, so suppose there exists a non-zero x with $f(x) = x$. Then, for all $y \neq x$, if $f(y) \neq 0$, the original equation yields $x(x - y) = xf(x - y)$. Thus, $f(x - y) = x - y$. Similarly, if $f(y) = y$, we find $(x - y)^2 = f(x - y)^2$, so $f(x - y) = x - y$. It follows that f must be the identity, which indeed is a solution to the original equation. The proof is complete.

3. [8] Triangle ABC is inscribed in a circle ω such that $\angle A = 60^\circ$ and $\angle B = 75^\circ$. Let the bisector of angle A meet BC and ω at E and D , respectively. Let the reflections of A across D and C be D' and C' , respectively. If the tangent to ω at A meets line BC at P , and the circumcircle of APD' meets line AC at $F \neq A$, prove that the circumcircle of $C'FE$ is tangent to BC at E .

Answer: N/A We will show that $CE^2 = (CF)(CC')$. By a simple computation using the given angles, one may find that this is equivalent to $CF = AC - AB$, or $AF = 2AC - AB$.

We compute AF by trigonometry. Assume for simplicity that $AC = \frac{1}{2}$, so $AD' = 2AD = 2AC = 1$ because $\triangle ACD$ is isosceles by angle chasing. We first compute PD' by the law of cosines on triangle APD' , which yields $PD'^2 = AP^2 + AD'^2 - 2(AP)(AD') \cos 75 = AP^2 + 1 - 2AP \cos 75$. We may easily compute AP by the law of sines in triangle BAP to be $\frac{1}{\sqrt{2}}$. Thus, $PD'^2 = \frac{1}{2} + 1 - \sqrt{2} \frac{\sqrt{6} - \sqrt{2}}{4} = \frac{4 - \sqrt{3}}{2}$.

By the law of sines within the circumcircle of $APD'F$, we have $\frac{FD'}{\sin 30} = \frac{PD'}{\sin 75}$. From this, we find that $FD' = \frac{PD' \sin 30}{\sin 75} = \frac{4 - \sqrt{3}}{\sqrt{2} + \sqrt{6}}$. Thus, we have by the law of cosines on triangle $D'AF$ that $AF^2 + AD'^2 - 2(AF)(AD') \cos 30 = AF^2 + 1 - \sqrt{3}AF = FD'^2 = \frac{4 - \sqrt{3}}{4 + 2\sqrt{3}}$. Finally, solving for AF yields $AF = \frac{\sqrt{3} \pm (2\sqrt{3} - 3)}{2}$ from which we indeed get $AF = \frac{3 - \sqrt{3}}{2}$ which equals $2AC - AB$.

4. [10] A subset $U \subset \mathbb{R}$ is *open* if for any $x \in U$, there exist real numbers a, b such that $x \in (a, b) \subset U$. Suppose $S \subset \mathbb{R}$ has the property that any open set intersecting $(0, 1)$ also intersects S . Let T be a countable collection of open sets containing S . Prove that the intersection of all of the sets of T is not a countable subset of \mathbb{R} .

(A set Γ is *countable* if there exists a bijective function $f : \Gamma \rightarrow \mathbb{Z}$.)

Answer: N/A If S is uncountable then we're done, so assume S is countable. We may also assume that the supersets are a chain $V_1 \supset V_2 \supset V_3 \supset \dots$ by taking intersections.

We will use the following fact from point set topology:

If $K_1 \supset K_2 \supset \dots$ is a sequence of nonempty compact sets, then their intersection is nonempty.

Now, we construct an uncountable family of such sequences $K_1 \supset K_2 \supset \dots$ such that K_i is a nontrivial (i.e. $[a, b]$ with $b > a$) closed interval contained in V_i , and any two such sequences have disjoint intersection. The construction proceeds as follows. At each step, we choose out of countably many options one closed interval $K_{i+1} \subset K_i$.

To choose K_1 , we claim that there exist countably many disjoint closed intervals in V_1 . To do this, choose an open interval in V_1 of length at most $1/2$, and take some closed interval inside it. In the remainder of $(0, 1)$, which has measure at least $1/2$ and where V_1 is still dense, choose another open interval of length at most $1/4$, and a closed interval inside it. This process can be repeated indefinitely to find a countably infinite family of nontrivial disjoint closed intervals in V_1 . Choose one of them to be K_1 .

Now, inductively apply this construction on K_i to find countably many choices for K_{i+1} , all disjoint. As a result, we get a total of $\mathbb{N}^{\mathbb{N}}$, i.e. uncountably many, choices of sequences K_i satisfying the given properties. Any pair of sequences eventually have disjoint K_i after some point, so no two intersections will intersect. Thus we can choose a point out of the intersection of each K_i sequence to find uncountably many points in the intersection of all the V_i , as desired.

5. [12]

(a) Given a finite set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X . Prove that if m, n are integers with $m \geq n > 2$, then $f_X(m) + f_X(n) \geq f_X(m+1) + f_X(n-1)$.

(b) Let P_0 be a 1-by-2 rectangle (including its interior), and inductively define the polygon P_i to be the result of folding P_{i-1} over some line that cuts P_{i-1} into two **connected** parts. The *diameter* of a polygon P_i is the maximum distance between two points of P_i . Determine the smallest possible diameter of P_{2013} .

(In other words, given a polygon P_{i-1} , a fold of P_{i-1} consists of a line l dividing P_{i-1} into two connected parts A and B , and the folded polygon $P_i = A \cup B_l$, where B_l is the reflection of B over the line l .)

Answer: N/A

Note from <http://artofproblemsolving.com/community/c129h529051p3039219>. Minor slip: In Section 1.3, Case 2 (for #5), how are B' and A' defined exactly? Right now it's not clear to me that they have $m+1$ and $n-1$ vertices, respectively, since it's not always possible to find $B_i, B_{i'}$ with $f(i') - f(i) = i' - i$, is it? (The simplest counterexample is when $m = 3, n = 5$, and the A_j, B_k are pairwise distinct.)

[b]Edit:[/b] I think it should be $f(i') - f(i) = i' - i + 1$ (which is not hard to find using a discrete continuity argument) and then $B' = B_1 \dots B_i A_{f(i)+1} \dots A_{f(i')} B_{i'+1} \dots B_m$ (A' the complement of B'), in which case everything seems to work out. (We are assuming at most two points lie between B_k, B_{k+1} for all k ; otherwise we can just use Case 1.) Thanks to Tim for clarifying most of this! I probably should have tried the $m = 3, n = 5$ case a little harder before posting this, since it generalizes fairly easily.

1 Convexity of the largest area of a polygon function

(a)

Let V be a finite set of points in the plane. Let f_V be a function that takes integers ≥ 3 as input, and outputs a polygon of largest area with vertices in V .

Lemma 1.1. $[f_V(a+1)] + [f_V(a-1)] \leq 2[f_V(a)]$, where $[P]$ represents the area of polygon P .

Proof. Let the vertices of $f_V(a+1)$ be $1_{a+1}, 2_{a+1}, \dots, (a+1)_{a+1}$, and let the vertices of $f_V(a-1)$ be $1_{a-1}, 2_{a-1}, \dots, (a-1)_{a-1}$.

Lemma 1.2. *The collection of vertices*

$$W = (1_{a-1}, 2_{a-1}, \dots, (a-1)_{a-1}, 1_{a+1}, \dots, (a+1)_{a+1})$$

. *It's clear that $W \subset V$. are in convex configuration.*

Proof. Suppose that the vertex $s_x \in W$ (for some integers $x \in \{a-1, a+1\}$, s) is inside the convex hull of W . Then there exists a vertex k on the convex hull of W such that replacing vertex s_x in $f_V(x)$ with k increases the area of the polygon $f_V(x)$. However, $f_V(x)$ is defined to be the polygon with x sides with maximum area. Contradiction. This proves Lemma 2. \square

Therefore the vertices in W are in convex configuration. Label them with $1, 2, \dots, 2a$ in clockwise order, starting from an arbitrary vertex. Let $(2a)$ denote the vertex labeled $(2a)$. (The vertex (1) will be shorthanded as 1 when there is no ambiguity).

Lemma 1.3. *We have the inequality*

$$[f_V(a+1)] + [f_V(a-1)] \leq [\mathcal{P}1357 \dots (2a-1)] + [\mathcal{P}2468 \dots (2a)].$$

(Throughout, the symbol \mathcal{P} denotes an arbitrary polygon.)

Proof. It suffices to show that

$$\begin{aligned} & 2[\mathcal{P}1234 \dots (2a)] - [f_V(a+1)] - [f_V(a-1)] \\ & \geq 2[\mathcal{P}1234 \dots (2a)] - [\mathcal{P}1357 \dots (2a-1)] - [\mathcal{P}2468 \dots (2a)]. \end{aligned}$$

We will do this using the next few lemmas.

Lemma 1.4. *Let x and y be integers $\leq 2a$. If ray $\overrightarrow{(x)(x+1)}$ intersects (or is parallel to) ray $\overrightarrow{(y)(y-1)}$. Then $[\mathcal{P}(x)(x+1)(x+2) \dots (y-1)(y)] \geq [\Delta(x)(x+1)(x+2)] + [\Delta(x+1)(x+2)(x+3)] + \dots + [\Delta((y-2)(y-1)(y))]$ (where $(2a+k)$ is equivalent to (k) for all k).*

Proof. $[\mathcal{P}(x)(x+1)(x+2) \dots (y-1)(y)] = [\Delta(x)(x+1)(x+2)] + [\Delta(x)(x+2)(x+3)] + \dots + [\Delta(x)(y-1)(y)]$. Suppose ray $\overrightarrow{(x)(x+1)}$ intersects ray $\overrightarrow{(y)(y-1)}$ at point D , the convexity of W tells us that all vertices in between x and y exclusive going clockwise from x are contained inside triangle $(x)D(y)$. It follows that for any x' between x and y exclusive going clockwise from x , that (x) is farther from line $\overrightarrow{(x')(x'+1)}$ than $(x'-1)$. Therefore

$$[\Delta(x)(x')(x'+1)] \leq [\Delta(x'-1)(x')(x'+1)].$$

Summing these relations over all x in between x and y exclusive going clockwise from x gives the statement of Lemma 4, as desired.

An identical argument works if ray $\overrightarrow{(x)(x+1)}$ and ray $\overrightarrow{(y)(y-1)}$ are parallel, except instead of D , all the points between x and y are bounded by the two rays and the line $\overrightarrow{(x)(y)}$.

This proves Lemma 4. \square

Let z and s be integers such that $z \in \{a-1, a+1\}$ and both vertices s_z and $(s+1)_z$ are adjacent vertices of $f_V(z)$. u and v are positive integers defined such that s_z has label (u) and s_{z+1} has label (v) .

Lemma 1.5. Rays $\overrightarrow{(u)(u+1)}$ and $\overrightarrow{(v)(v-1)}$ intersect or are parallel to each other.

Proof. Since $(u), (u+1), (v-1), (v)$ are in convex formation, then either $\overrightarrow{(u)(u+1)}$ and $\overrightarrow{(v)(v-1)}$ intersect or are parallel to each other, or else $\overrightarrow{(u+1)(u)}$ and $\overrightarrow{(v-1)(v)}$ intersect. If we suppose that the lemma statement does not hold for the sake of contradiction, then it follows that rays $\overrightarrow{(u+1)(u)}$ and $\overrightarrow{(v-1)(v)}$ must intersect. Let Q be the collection of vertices between v and u in W , going clockwise, inclusive. Since Q is in convex formation (it's a subset of W), and $\overrightarrow{(u+1)(u)}$ and $\overrightarrow{(v-1)(v)}$ intersect, it follows that $u+1$ is farther than u from any line joining two vertices of Q . Therefore removing s_z (which is the same vertex as (u)) from $f_V(z)$ and replacing it with vertex $(u+1)$ increases the area of $f_V(z)$, contradicting the assertion that $f_V(z)$ was the polygon of z vertices with maximal area on the set V . Therefore Lemma 5 is proven. \square

Lemma 5 allows us to apply Lemma 4 to the polygons formed by taking $(\mathcal{P}1234 \dots (2a))$ and subtracting out $f_V(a+1]$, as well as the polygons formed by taking $(\mathcal{P}1234 \dots (2a))$ and subtracting out $f_V(a-1)$. Adding the results tells us that

$$\begin{aligned} & 2[\mathcal{P}1234 \dots (2a)] - [f_V(a+1)] - [f_V(a-1)] \\ & \geq 2[\mathcal{P}1234 \dots (2a)] - [\mathcal{P}1357 \dots (2a-1)] - [\mathcal{P}2468 \dots (2a)], \end{aligned}$$

which is sufficient to prove Lemma 3. \square

Since $(\mathcal{P}135 \dots (2a-1)), (\mathcal{P}246 \dots (2a)) \leq f_V(a)$ since $(1), (2), \dots, (2a) \in V$. Using this fact with Lemma 3, we have proven Lemma 1 as desired. \square

Lemma 1.6. Any function g from integers to real numbers that satisfies $g(x+1) + g(x-1) \leq 2g(x)$ satisfies the inequality $g(x) + g(y) \leq g(x+1) + g(y-1)$ for $y-1 \geq x+1$, and is thus a convex function.

Proof. We proceed by induction on $y-x$. When $y-x=2$, the problem is trivial.

For $y-x > 2$, note that $g(x)+g(x+2)+g(y)+g(y-2) \leq 2g(x+1)+2g(y-1)$, but the inductive statement shows that $g(x+2) + g(y-2) \geq g(x+1) + g(y-1)$. It follows that $g(x) + g(y) \leq g(x+1) + g(y-1)$, which completes our induction as desired. \square

It follows from the preceding lemma that $f_V(a)$ is a convex function, as desired.

2 Folding a paper

Let the smallest diameter be M . We claim that M is $\frac{\sqrt{2}}{2^{1006}}$. Let the P_0 be a 1 by 2 rectangle. For $0 \leq i \leq 2012$, polygon P_{i+1} is polygon P_i folded across line l_i . Line l_i splits P_i into P'_i and P''_i , and P''_i is reflected to P'''_i , and $P_{i+1} = P'_i \cup P'''_i$.

Claim 1. For $1 \leq i \leq 2013$, P_i can be expressed as the union of no more than 2^i convex polygons, and the average number of sides of these polygons is not more than 4.

Proof. We proceed by induction. P_0 clearly satisfies the inductive hypothesis. Suppose that P_i is the union of convex polygons A_1, A_2, \dots, A_n . Folding A_1, A_2, \dots, A_n across l_i and taking the union of the resulting polygons yields P_{i+1} .

Let $S(P)$ be the number of sides of polygon P .

Lemma. A convex polygon Z is folded once. Z' and Z''' are convex, and $S(Z') + S(Z''') \leq S(Z) + 4$.

Proof. Any line l through Z splits Z into two convex polygons, and four new vertices are formed by l . The number of vertices in a convex polygon equals the number of sides, proving our lemma. \square

Thus P_{i+1} can be expressed as the union of no more than 2^{i+1} convex polygons, and the average number of sides of these convex polygons does not exceed 4. Claim 1 is proven. \square

Claim 2. For a quadrilateral of area A , the diameter has length less than $\sqrt{2A}$.

Proof. Without loss of generality, let the fixed area be 1. The diagonal is not longer than the diameter of the quadrilateral. Half of the product of the diagonals is not less than the area, and Claim 2 follows. \square

Claim 1 tells us that P_{2013} is the union of no more than 2^{2013} convex polygons, whose average number of sides is no more than 4. Part (a) of the problem tells us that there is some quadrilateral with area $\geq \frac{1}{2^{2012}}$, with vertices at the vertices of the final folded figure. Claim 2 tells us that the diameter of this quadrilateral, and thus of P_{2013} , is $\geq \frac{\sqrt{2}}{2^{1006}}$.

15th Annual Harvard-MIT Mathematics Tournament
Saturday 11 February 2012

Algebra Test

1. Let f be the function such that

$$f(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $f(\underbrace{f(\dots f(x)\dots)}_{2012f's})$ from $x = 0$ to $x = 1$?

Answer: $\boxed{\sqrt{4^{2012} + 1}}$ When there are n copies of f , the graph consists of 2^n segments, each of which goes $1/2^n$ units to the right, and alternately 1 unit up or down. So, the length is

$$2^n \sqrt{1 + \frac{1}{2^{2n}}} = \sqrt{4^n + 1}$$

Taking $n = 2012$, the answer is

$$\sqrt{4^{2012} + 1}$$

2. You are given an unlimited supply of red, blue, and yellow cards to form a hand. Each card has a point value and your score is the sum of the point values of those cards. The point values are as follows: the value of each red card is 1, the value of each blue card is equal to twice the number of red cards, and the value of each yellow card is equal to three times the number of blue cards. What is the maximum score you can get with fifteen cards?

Answer: $\boxed{168}$ If there are B blue cards, then each red card contributes $1 + 2B$ points (one for itself and two for each blue card) and each yellow card contributes $3B$ points. Thus, if $B > 1$, it is optimal to change all red cards to yellow cards. When $B = 0$, the maximum number of points is 15. When $B = 1$, the number of points is always 42. When $B > 1$, the number of points is $3BY$, where Y is the number of yellow cards. Since $B + Y = 15$, the desired maximum occurs when $B = 7$ and $Y = 8$, which gives 168 points.

3. Given points a and b in the plane, let $a \oplus b$ be the unique point c such that abc is an equilateral triangle with a, b, c in the clockwise orientation.

Solve $(x \oplus (0, 0)) \oplus (1, 1) = (1, -1)$ for x .

Answer: $\boxed{\left(\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)}$ It is clear from the definition of \oplus that $b \oplus (a \oplus b) = a$ and if $a \oplus b = c$ then $b \oplus c = a$ and $c \oplus a = b$. Therefore $x \oplus (0, 0) = (1, 1) \oplus (1, -1) = (1 - \sqrt{3}, 0)$. Now this means $x = (0, 0) \oplus (1 - \sqrt{3}, 0) = \left(\frac{1-\sqrt{3}}{2}, \frac{3-\sqrt{3}}{2}\right)$.

4. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at $z^3, z^5, z^7, \dots, z^{2013}$, in that order; on Sunday, he begins at 1 and delivers milk to houses located at $z^2, z^4, z^6, \dots, z^{2012}$, in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2 .

Answer: $\boxed{\frac{1005}{1006}}$ Note that the distance between two points in the complex plane, m and n , is $|m - n|$. We have that

$$\sum_{k=1}^{1006} |z^{2k+1} - z^{2k-1}| = \sum_{k=1}^{1006} |z^{2k} - z^{2k-2}| = \sqrt{2012}.$$

However, noting that

$$|z| \cdot \sum_{k=1}^{1006} |z^{2k} - z^{2k-2}| = \sum_{k=1}^{1006} |z^{2k+1} - z^{2k-1}|,$$

we must have $|z| = 1$. Then, since Eli travels a distance of $\sqrt{2012}$ on each day, we have

$$\begin{aligned} \sum_{k=1}^{1006} |z^{2k} - z^{2k-2}| &= |z^2 - 1| \cdot \sum_{k=1}^{1006} |z^{2k-2}| = |z^2 - 1| \cdot \sum_{k=1}^{1006} |z|^{2k-2} \\ &= 1006 |z^2 - 1| = \sqrt{2012}, \end{aligned}$$

so $|z^2 - 1| = \frac{\sqrt{2012}}{1006}$. Since $|z| = 1$, we can write $z = \cos(\theta) + i \sin(\theta)$ and then $z^2 = \cos(2\theta) + i \sin(2\theta)$. Hence,

$$|z^2 - 1| = \sqrt{(\cos(2\theta) - 1)^2 + \sin^2(2\theta)} = \sqrt{2 - 2\cos(2\theta)} = \frac{\sqrt{2012}}{1006},$$

so $2 - 2\cos(2\theta) = \frac{2}{1006}$. The real part of z^2 , $\cos(2\theta)$, is thus $\frac{1005}{1006}$.

5. Find all ordered triples (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3$, $a \lfloor b \rfloor c = 4$, and $ab \lfloor c \rfloor = 5$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .

Answer: $\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5} \right), \left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5} \right)$ Write $p = abc$, $q = \lfloor a \rfloor \lfloor b \rfloor \lfloor c \rfloor$. Note that q is an integer. Multiplying the three equations gives:

$$p = \sqrt{\frac{60}{q}}$$

Substitution into the first equation,

$$p = 3 \frac{a}{\lfloor a \rfloor} < 3 \frac{\lfloor a \rfloor + 1}{\lfloor a \rfloor} \leq 6$$

Looking at the last equation:

$$p = 5 \frac{c}{\lfloor c \rfloor} \geq 5 \frac{\lfloor c \rfloor}{\lfloor c \rfloor} \geq 5$$

Here we've used $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$, and also the apparent fact that $\lfloor a \rfloor \geq 1$. Now:

$$5 \leq \sqrt{\frac{60}{q}} \leq 6$$

$$\frac{12}{5} \geq q \geq \frac{5}{3}$$

Since q is an integer, we must have $q = 2$. Since q is a product of 3 positive integers, we must have those be 1,1, and 2 in some order, so there are three cases:

Case 1: $\lfloor a \rfloor = 2$. By the equations, we'd need $a = \frac{2}{3}\sqrt{30} = \sqrt{120/9} > 3$, a contradiction, so there are no solutions in this case.

Case 2: $\lfloor b \rfloor = 2$. We have the solution

$$\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{2}, \frac{\sqrt{30}}{5} \right)$$

Case 3: $\lfloor c \rfloor = 2$. We have the solution

$$\left(\frac{\sqrt{30}}{3}, \frac{\sqrt{30}}{4}, \frac{2\sqrt{30}}{5} \right)$$

6. Let $a_0 = -2$, $b_0 = 1$, and for $n \geq 0$, let

$$\begin{aligned} a_{n+1} &= a_n + b_n + \sqrt{a_n^2 + b_n^2}, \\ b_{n+1} &= a_n + b_n - \sqrt{a_n^2 + b_n^2}. \end{aligned}$$

Find a_{2012} .

Answer: $\boxed{2^{1006}\sqrt{2^{2010} + 2} - 2^{2011}}$ We have

$$\begin{aligned} a_{n+1} + b_{n+1} &= 2(a_n + b_n) \\ a_{n+1}b_{n+1} &= (a_n + b_n)^2 - a_n^2 - b_n^2 = 2a_nb_n \end{aligned}$$

Thus,

$$\begin{aligned} a_n + b_n &= -2^n \\ a_nb_n &= -2^{n+1} \end{aligned}$$

Using Viète's formula, a_{2012} and b_{2012} are the roots of the following quadratic, and, since the square root is positive, a_{2012} is the bigger root:

$$x^2 + 2^{2012}x - 2^{2013}$$

Thus,

$$a_{2012} = 2^{1006}\sqrt{2^{2010} + 2} - 2^{2011}$$

7. Let \otimes be a binary operation that takes two positive real numbers and returns a positive real number. Suppose further that \otimes is continuous, commutative ($a \otimes b = b \otimes a$), distributive across multiplication ($a \otimes (bc) = (a \otimes b)(a \otimes c)$), and that $2 \otimes 2 = 4$. Solve the equation $x \otimes y = x$ for y in terms of x for $x > 1$.

Answer: $\boxed{y = \sqrt{2}}$ We note that $(a \otimes b)^k = (a \otimes b)^k$ for all positive integers k . Then for all rational numbers $\frac{p}{q}$ we have $a \otimes b^{\frac{p}{q}} = (a \otimes b^{\frac{1}{q}})^p = (a \otimes b)^{\frac{p}{q}}$. So by continuity, for all real numbers a, b , it follows that $2^a \otimes 2^b = (2 \otimes 2)^{ab} = 4^{ab}$. Therefore given positive reals x, y , we have $x \otimes y = 2^{\log_2(x)} \otimes 2^{\log_2(y)} = 4^{\log_2(x)\log_2(y)}$.

If $x = 4^{\log_2(x)\log_2(y)} = 2^{2\log_2(x)\log_2(y)}$ then $\log_2(x) = 2\log_2(x)\log_2(y)$ and $1 = 2\log_2(y) = \log_2(y^2)$. Thus $y = \sqrt{2}$ regardless of x .

8. Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let $x_n = x_{n-1}y_{n-2} + x_{n-2}y_{n-1}$ and $y_n = y_{n-1}y_{n-2} - x_{n-1}x_{n-2}$. What are the last two digits of $|x_{2012}|$?

Answer: $\boxed{84}$ Let $z_n = y_n + x_n i$. Then the recursion implies that:

$$\begin{aligned} z_1 &= z_2 = 1 + i, \\ z_n &= z_{n-1}z_{n-2}. \end{aligned}$$

This implies that

$$z_n = (z_1)^{F_n},$$

where F_n is the n^{th} Fibonacci number ($F_1 = F_2 = 1$). So, $z_{2012} = (1 + i)^{F_{2012}}$. Notice that

$$(1 + i)^2 = 2i.$$

Also notice that every third Fibonacci number is even, and the rest are odd. So:

$$z_{2012} = (2i)^{\frac{F_{2012}-1}{2}}(1 + i)$$

Let $m = \frac{F_{2012}-1}{2}$. Since both real and imaginary parts of $1+i$ are 1, it follows that the last two digits of $|x_{2012}|$ are simply the last two digits of $2^m = 2^{\frac{F_{2012}-1}{2}}$.

By the Chinese Remainder Theorem, it suffices to evaluate 2^m modulo 4 and 25. Clearly, 2^m is divisible by 4. To evaluate it modulo 25, it suffices by Euler's Totient theorem to evaluate m modulo 20.

To determine $(F_{2012}-1)/2$ modulo 4 it suffices to determine F_{2012} modulo 8. The Fibonacci sequence has period 12 modulo 8, and we find

$$\begin{aligned} F_{2012} &\equiv 5 \pmod{8}, \\ m &\equiv 2 \pmod{4}. \end{aligned}$$

$2 * 3 \equiv 1 \pmod{5}$, so

$$m \equiv 3F_{2012} - 3 \pmod{5}.$$

The Fibonacci sequence has period 20 modulo 5, and we find

$$m \equiv 4 \pmod{5}.$$

Combining,

$$\begin{aligned} m &\equiv 14 \pmod{20} \\ 2^m &\equiv 2^{14} = 4096 \equiv 21 \pmod{25} \\ |x_{2012}| &\equiv 4 \cdot 21 = 84 \pmod{100}. \end{aligned}$$

9. How many real triples (a, b, c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real y ?

Answer: 18 Let p have roots r, r, s, t . Using Vieta's on the coefficient of the cubic and linear terms, we see that $2r + s + t = r^2s + r^2t + 2rst$. Rearranging gives $2r(1-st) = (r^2-1)(s+t)$.

If $r^2 - 1 = 0$, then since $r \neq 0$, we require that $1 - st = 0$ for the equation to hold. Conversely, if $1 - st = 0$, then since $st = 1$, $s + t = 0$ cannot hold for real s, t , we require that $r^2 - 1 = 0$ for the equation to hold. So one valid case is where both these values are zero, so $r^2 = st = 1$. If $r = \tan y$ (here we stipulate that $0 \leq y < \pi$), then either $y = \frac{\pi}{4}$ or $y = \frac{3\pi}{4}$. In either case, the value of $\tan 2y$ is undefined. If $r = \tan 2y$, then we have the possible values $y = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$. In each of these cases, we must check if $\tan y \tan 3y = 1$. But this is true if $y + 3y = 4y$ is an odd integer multiple of $\frac{\pi}{2}$, which is the case for all such values. If $r = \tan 3y$, then we must have $\tan y \tan 2y = 1$, so that $3y$ is an odd integer multiple of $\frac{\pi}{2}$. But then $\tan 3y$ would be undefined, so none of these values can work.

Now, we may assume that $r^2 - 1$ and $1 - st$ are both nonzero. Dividing both sides by $(r^2 - 1)(1 - st)$ and rearranging yields $0 = \frac{2r}{1-r^2} + \frac{s+t}{1-st}$, the tangent addition formula along with the tangent double angle formula. By setting r to be one of $\tan y$, $\tan 2y$, or $\tan 3y$, we have one of the following:

- (a) $0 = \tan 2y + \tan 5y$
- (b) $0 = \tan 4y + \tan 4y$
- (c) $0 = \tan 6y + \tan 3y$.

We will find the number of solutions y in the interval $[0, \pi)$. Case 1 yields six multiples of $\frac{\pi}{7}$. Case 2 yields $\tan 4y = 0$, which we can readily check has no solutions. Case 3 yields eight multiples of $\frac{\pi}{9}$. In total, we have $4 + 6 + 8 = 18$ possible values of y .

10. Suppose that there are 16 variables $\{a_{i,j}\}_{0 \leq i,j \leq 3}$, each of which may be 0 or 1. For how many settings of the variables $a_{i,j}$ do there exist positive reals $c_{i,j}$ such that the polynomial

$$f(x, y) = \sum_{0 \leq i,j \leq 3} a_{i,j} c_{i,j} x^i y^j$$

$(x, y \in \mathbb{R})$ is bounded below?

Answer: 126 For some choices of the $a_{i,j}$, let $S = \{(i, j) | a_{i,j} = 1\}$, and let $S' = S \cup \{(0, 0)\}$. Let $C(S')$ denote the convex hull of S' . We claim that there exist the problem conditions are satisfied (there exist positive coefficients for the terms so that the polynomial is bounded below) if and only if the vertices of $C(S')$ all have both coordinates even.

For one direction, suppose that $C(S')$ has a vertex $v = (i', j')$ with at least one odd coordinate; WLOG, suppose it is i' . Since v is a vertex, it maximizes some objective function $ai + bj$ over $C(S')$ uniquely, and thus also over S' . Since $(0, 0) \in S'$, we must have $ai' + bj' > 0$. Now consider plugging in $(x, y) = (-t^a, t^b)$ ($t > 0$) into f . This gives the value

$$f(-t^a, t^b) = \sum_{(i,j) \in S} (-1)^i c_{i,j} t^{ai+bj}.$$

But no matter what positive $c_{i,j}$ we choose, this expression is not bounded below as t grows infinitely large, as there is a $-c_{i',j'} t^{ai'+bj'}$ term, with $ai' + bj' > 0$, and all other terms have smaller powers of t . So the polynomial cannot be bounded below.

For the other direction, suppose the vertices of $C(S')$ all have both coordinates even. If all points in S' are vertices of $C(S')$, then the polynomial is a sum of squares, so it is bounded below. Otherwise, we assume that some points in S' are not vertices of $C(S')$. It suffices to consider the case where there is exactly one such point. Call this point $w = (i', j')$. Let $V(S')$ denote the set of the vertices of $C(S')$, and let $n = |V(S')|$. Enumerate the points of $V(S')$ as v_1, v_2, \dots, v_n . Let i_k, j_k denote the i and j coordinates of v_k , respectively.

Since $w \in C(S')$, there exist nonnegative constants $\lambda_1, \lambda_2, \dots, \lambda_n$ such that $\sum_{k=1}^n \lambda_k = 1$ and $\sum_{k=1}^n \lambda_k v_k = w$. (Here, we are treating the ordered pairs as vectors.) Then, by weighted AM-GM, we have

$$\sum_{k=1}^n \lambda_k |x|^{i_k} |y|^{j_k} \geq |x|^{i'} |y|^{j'}.$$

Let c be the λ -value associated with $(0, 0)$. Then by picking $c_{i_k, j_k} = \lambda_k$ and $c_{i', j'} = 1$, we find that $p(x, y) \geq -c$ for all x, y , as desired.

We now find all possible convex hulls $C(S')$ (with vertices chosen from $(0, 0), (0, 2), (2, 0),$ and $(2, 2)$), and for each convex hull, determine how many possible settings of $a_{i,j}$ give that convex hull. There are 8 such possible convex hulls: the point $(0, 0)$ only, 3 lines, 3 triangles, and the square. The point has 2 possible choices, each line has 4 possible choices, each triangle has 16 possible choices, and the square has 64 possible choices, giving $2 + 3 \cdot 4 + 3 \cdot 16 + 64 = 126$ total choices.

15th Annual Harvard-MIT Mathematics Tournament
Saturday 11 February 2012
Combinatorics Test

1. In the game of Minesweeper, a number on a square denotes the number of mines that share at least one vertex with that square. A square with a number may not have a mine, and the blank squares are undetermined. How many ways can the mines be placed in this configuration?

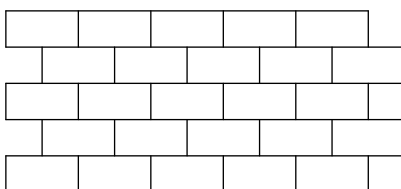
	2		1		2

Answer: 95 Let A be the number of mines in the first two columns. Let B, C, D, E be the number of mines in the third, fourth, fifth, and sixth columns, respectively. We need to have $A + B = 2$, $B + C + D = 1$, and $D + E = 2$. This can happen in three ways, which are $(A, B, C, D, E) = (2, 0, 1, 0, 2), (2, 0, 0, 1, 1), (1, 1, 0, 0, 2)$. This gives $(10)(2)(1) + (10)(3)(2) + (5)(3)(1) = 95$ possible configurations.

2. Brian has a 20-sided die with faces numbered from 1 to 20, and George has three 6-sided dice with faces numbered from 1 to 6. Brian and George simultaneously roll all their dice. What is the probability that the number on Brian's die is larger than the sum of the numbers on George's dice?

Answer: $\frac{19}{40}$ Let Brian's roll be d and let George's rolls be x, y, z . By pairing the situation d, x, y, z with $21 - d, 7 - x, 7 - y, 7 - z$, we see that the probability that Brian rolls higher is the same as the probability that George rolls higher. Given any of George's rolls x, y, z , there is exactly one number Brian can roll which will make them tie, so the probability that they tie is $\frac{1}{20}$. So the probability that Brian wins is $\frac{1 - \frac{1}{20}}{2} = \frac{19}{40}$.

3. In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



Answer: 61 The number of valid selections is equal to the number of paths which start at a top brick and end at a bottom brick. We compute these by writing 1 in each of the top bricks and letting lower bricks be the sum of the one or two bricks above them. Thus, the number inside each brick is the number of paths from that brick to the top. The bottom row is 6, 14, 16, 15, 10, which sums to 61.

1	1	1	1	1	
	2	2	2	2	1
2	4	4	4	3	
	6	8	8	7	3
6	14	16	15	10	

4. A frog is at the point $(0, 0)$. Every second, he can jump one unit either up or right. He can only move to points (x, y) where x and y are not both odd. How many ways can he get to the point $(8, 14)$?

Answer: 330 When the frog is at a point (x, y) where x and y are both even, then if that frog chooses to move right, his next move will also have to be a step right; similarly, if he moves up, his next move will have to be up.

If we “collapse” each double step into one step, the problem simply becomes how many ways are there to move to the point $(4, 7)$ using only right and up steps, with no other restrictions. That is 11 steps total, so the answer is $\binom{11}{4} = 330$.

5. Dizzy Daisy is standing on the point $(0, 0)$ on the xy -plane and is trying to get to the point $(6, 6)$. She starts facing rightward and takes a step 1 unit forward. On each subsequent second, she either takes a step 1 unit forward or turns 90 degrees counterclockwise then takes a step 1 unit forward. She may never go on a point outside the square defined by $|x| \leq 6, |y| \leq 6$, nor may she ever go on the same point twice. How many different paths may Daisy take?

Answer: 131922 Because Daisy can only turn in one direction and never goes to the same square twice, we see that she must travel in an increasing spiral about the origin. Clearly, she must arrive at $(6, 6)$ coming from below. To count her paths, it therefore suffices to consider the horizontal and vertical lines along which she travels (out of 5 choices to move upward, 6 choices leftward, 6 choices downward, and 6 choices rightward). Breaking up the cases by the number of complete rotations she performs, the total is $\binom{5}{0}\binom{6}{0}^3 + \binom{5}{1}\binom{6}{1}^3 + \binom{5}{2}\binom{6}{2}^3 + \binom{5}{3}\binom{6}{3}^3 + \binom{5}{4}\binom{6}{4}^3 + \binom{5}{5}\binom{6}{5}^3 = 131922$.

6. For a permutation σ of $1, 2, \dots, 7$, a *transposition* is a swapping of two elements. (For instance, we could apply a transposition to the permutation $3, 7, 1, 4, 5, 6, 2$ and get $3, 7, 6, 4, 5, 1, 2$ by swapping the 1 and the 6.)

Let $f(\sigma)$ be the minimum number of transpositions necessary to turn σ into the permutation $1, 2, 3, 4, 5, 6, 7$. Find the sum of $f(\sigma)$ over all permutations σ of $1, 2, \dots, 7$.

Answer: 22212 To solve this problem, we use the idea of a *cycle* in a permutation. If σ is a permutation, we say that $(a_1 a_2 \dots a_k)$ is a cycle if $\sigma(a_i) = a_{i+1}$ for $1 \leq i \leq k-1$ and $\sigma(a_k) = a_1$. Any permutation can be decomposed into disjoint cycles; for instance, the permutation $3, 7, 6, 4, 5, 1, 2$, can be written as $(1\ 3\ 6)(2\ 7)(4)(5)$. For a permutation σ , let $g(\sigma)$ be the number of cycles in its cycle decomposition. (This includes single-element cycles.)

Claim. For any permutation σ on n elements, $f(\sigma) = n - g(\sigma)$.

Proof. Given a cycle $(a_1 a_2 \dots a_k)$ (with $k \geq 2$) of a permutation σ , we can turn this cycle into the identity permutation with $k - 1$ transpositions; first we swap a_1 and a_2 ; that is, we replace σ with a permutation σ' such that instead of $\sigma(a_k) = a_1$ and $\sigma(a_1) = a_2$, we have $\sigma'(a_k) = a_2$ and $\sigma'(a_1) = a_1$. Now, σ' takes a_1 to itself, so we are left with the cycle $(a_2 \dots a_n)$. We continue until the entire cycle is replaced by the identity, which takes $k - 1$ transpositions. Now, for any σ , we resolve each cycle in this way, making a total of $n - g(\sigma)$ transpositions, to turn σ into the identity permutation.

This shows that $n - g(\sigma)$ transpositions; now let us that we cannot do it in less. We show that whenever we make a transposition, the value of $n - g(\sigma)$ can never decrease by more than 1. Whenever we swap two elements, if they are in different cycles, then those two cycles merge into one; thus $n - g(\sigma)$ actually increased. If the two elements are in one cycle, then the one cycle splits into two cycles, so $n - g(\sigma)$ decreased by only one, and this proves the claim.

Thus, we want to find

$$\sum_{\sigma \in S_7} (7 - g(\sigma)) = 7 \cdot 7! - \sum_{\sigma \in S_7} g(\sigma)$$

to evaluate the sum, we instead sum over every cycle the number of permutations it appears in. For any $1 \leq k \leq 7$, the number of cycles of size k is $\frac{n!}{(n-k)!k}$, and the number of permutations each such cycle can appear in is $(n - k)!$. Thus we get that the answer is

$$7 \cdot 7! - \sum_{k=1}^7 \frac{7!}{k} = 22212.$$

7. You are repeatedly flipping a fair coin. What is the expected number of flips until the first time that your previous 2012 flips are 'HTHT...HT'?

Answer: $\boxed{(2^{2014} - 4)/3}$ Let S be our string, and let $f(n)$ be the number of binary strings of length n which do not contain S . Let $g(n)$ be the number of strings of length n which contain S but whose prefix of length $n - 1$ does not contain S (so it contains S for the "first" time at time n).

Consider any string of length n which does not contain S and append S to it. Now, this new string contains S , and in fact it must contain S for the first time at either time $n + 2$, $n + 4$, ..., or $n + 2012$. It's then easy to deduce the relation

$$f(n) = g(n + 2) + g(n + 4) + \cdots + g(n + 2012)$$

Now, let's translate this into a statement about probabilities. Let t be the first time our sequence of coin flips contains the string S . Dividing both sides by 2^n , our equality becomes

$$P(t > n) = 4P(t = n + 2) + 16P(t = n + 4) + \cdots + 2^{2012}P(t = n + 2012)$$

Summing this over all n from 0 to ∞ , we get

$$\sum P(t > n) = 4 + 16 + \cdots + 2^{2012} = (2^{2014} - 4)/3$$

But it is also easy to show that since t is integer-valued, $\sum P(t > n) = E(t)$, and we are done.

8. How many ways can one color the squares of a 6x6 grid red and blue such that the number of red squares in each row and column is exactly 2?

Answer: $\boxed{67950}$ Assume the grid is $n \times n$. Let $f(n)$ denote the number of ways to color exactly two squares in each row and column red. So $f(1) = 0$ and $f(2) = 1$. We note that coloring two squares red in each row and column partitions the set $1, 2, \dots, n$ into cycles such that i is in the same cycle as, and adjacent to, j iff column i and column j have a red square in the same row. Each i is adjacent to two other, (or the same one twice in a 2-cycle).

Now consider the cycle containing 1, and let it have size k . There are $\binom{n}{2}$ ways to color two squares red in the first column. Now we let the column that is red in the same row as the top ball in the first column, be the next number in the cycle. There are $n - 1$ ways to pick this column, and $n - 2$ ways to pick the second red square in this column (unless $k = 2$). Then there are $(n - 2)(n - 3)$ ways to pick the red squares in the third column. and $(n - j)(n - j + 1)$ ways to pick the j th ones for $j \leq k - 1$. Then when we pick the k th column, the last one in the cycle, it has to be red in the same row as the second red square in column 1, so there are just $n - k + 1$ choices. Therefore if the cycle has length k there are

$\frac{n(n-1)}{2} \times (n - 1)(n - 2) \times \cdots \times (n - k + 1)(n - k + 2) \times (n - k + 1)$ ways, which equals: $\frac{n!(n-1)!}{2(n-k)!(n-k)!}$.

Summing over the size of the cycle containing the first column, we get

$$f(n) = \sum_{k=2}^n \frac{1}{2} f(n - k) \frac{(n)!(n - 1)!}{(n - k)!(n - k)!}$$

$$\frac{2nf(n)}{n!n!} = \sum_{k=2}^n \frac{f(n - k)}{(n - k)!(n - k)!}$$

$$\frac{2nf(n)}{n!n!} - \frac{2(n - 1)f(n - 1)}{(n - 1)!(n - 1)!} = \frac{f(n - 2)}{(n - 2)!(n - 2)!}$$

We thus obtain the recursion:

$$f(n) = n(n - 1)f(n - 1) + \frac{n(n - 1)^2}{2}f(n - 2)$$

Then we get:

$$f(1) = 0$$

$$f(2) = 1$$

$$f(3) = 6$$

$$f(4) = 12 \times 6 + 18 = 90$$

$$f(5) = 20 \times 90 + 40 \times 6 = 2040$$

$$f(6) = 30 \times 2040 + 75 \times 90 = 67950.$$

9. A parking lot consists of 2012 parking spots equally spaced in a line, numbered 1 through 2012. One by one, 2012 cars park in these spots under the following procedure: the first car picks from the 2012 spots uniformly randomly, and each following car picks uniformly randomly among all possible choices which maximize the minimal distance from an already parked car. What is the probability that the last car to park must choose spot 1?

Answer: $\frac{1}{2062300}$ We see that for 1 to be the last spot, 2 must be picked first (with probability $\frac{1}{n}$), after which spot n is picked. Then, cars from 3 to $n - 1$ will be picked until there are only gaps of 1 or 2 remaining. At this point, each of the remaining spots (including spot 1) is picked uniformly at random, so the probability that spot 1 is chosen last here will be the reciprocal of the number of remaining slots.

Let $f(n)$ denote the number of empty spots that will be left if cars park in $n + 2$ consecutive spots whose ends are occupied, under the same conditions, except that the process stops when a car is forced to park immediately next to a car. We want to find the value of $f(2009)$. Given the gap of n cars, after placing a car, there are gaps of $f(\lfloor \frac{n-1}{2} \rfloor)$ and $f(\lceil \frac{n-1}{2} \rceil)$ remaining. Thus, $f(n) = f(\lfloor \frac{n-1}{2} \rfloor) + f(\lceil \frac{n-1}{2} \rceil)$. With the base cases $f(1) = 1, f(2) = 2$, we can determine with induction that

$$f(x) = \begin{cases} x - 2^{n-1} + 1 & \text{if } 2^n \leq x \leq \frac{3}{2} \cdot 2^n - 2, \\ 2^n & \text{if } \frac{3}{2} \cdot 2^n - 1 \leq x \leq 2 \cdot 2^n - 1. \end{cases}$$

Thus, $f(2009) = 1024$, so the total probability is $\frac{1}{2012} \cdot \frac{1}{1024+1} = \frac{1}{2062300}$.

10. Jacob starts with some complex number x_0 other than 0 or 1. He repeatedly flips a fair coin. If the n^{th} flip lands heads, he lets $x_n = 1 - x_{n-1}$, and if it lands tails he lets $x_n = \frac{1}{x_{n-1}}$. Over all possible choices of x_0 , what are all possible values of the probability that $x_{2012} = x_0$?

Answer: $\left\{1, \frac{2^{2011}+1}{3 \cdot 2^{2011}}\right\}$ Let $f(x) = 1 - x, g(x) = \frac{1}{x}$. Then for any $x, f(f(x)) = x$ and $g(g(x)) = x$. Furthermore, $f(g(x)) = 1 - \frac{1}{x}, g(f(g(x))) = \frac{x}{x-1}, f(g(f(g(x)))) = \frac{1}{1-x}, g(f(g(f(g(x)))) = 1 - x = f(x)$, so for all n, x_n is one of $x, \frac{1}{x}, 1 - \frac{1}{x}, \frac{x}{x-1}, \frac{1}{1-x}, 1 - x$, and we can understand the coin flipping procedure as moving either left or right with equal probability along this cycle of values.

For most x , all six of these values are distinct. In this case, suppose that we move right R times and left $2012 - R$ times between x_0 and x_{2012} . For $x_{2012} = x_0$, we need to have that $R - 2012 + R \equiv 0 \pmod{6}$, or $R \equiv 1 \pmod{3}$. The number of possible ways to return to x_0 is then $a = \binom{2012}{1} + \binom{2012}{4} + \dots + \binom{2012}{2011}$. Let $b = \binom{2012}{0} + \binom{2012}{3} + \dots + \binom{2012}{2010} = \binom{2012}{2} + \binom{2012}{5} + \dots + \binom{2012}{2012}$. Then we have $a + 2b = 2^{2012}$ and that $b = \frac{(1+1)^{2012} + (1+\omega)^{2012} + (1+\omega^2)^{2012}}{3}$, where ω is a primitive third root of unity. It can be seen that $1 + \omega$ is a primitive sixth root of unity and $1 + \omega^2$ is its inverse, so $(1 + \omega)^{2012} = (1 + \omega)^2 = \omega$, and similarly $(1 + \omega^2)^{2012} = \omega^2$. Therefore, $b = \frac{2^{2012}-1}{3}$, so $a = 2^{2012} - 2b = \frac{2^{2012}+2}{3}$, and our desired probability is then $\frac{a}{2^{2012}} = \frac{2^{2012}+2}{3 \cdot 2^{2012}} = \frac{2^{2011}+1}{3 \cdot 2^{2011}}$.

For some x_0 , however, the cycle of values can become degenerate. It could be the case that two adjacent values are equal. Let y be a value that is equal to an adjacent value. Then $y = \frac{1}{y}$ or $y = 1 - y$, which gives $y \in \{-1, \frac{1}{2}\}$. Therefore, this only occurs in the cycle of values $-1, 2, \frac{1}{2}, \frac{1}{2}, 2, -1$. In this case, note that after 2012 steps we will always end up an even number of steps away from our starting point, and each of the numbers occupies two spaces of opposite parity, so we would need to return to our

original location, just as if all six numbers were distinct. Therefore in this case we again have that the probability that $x_{2012} = x_0$ is $\frac{2^{2011}+1}{3 \cdot 2^{2011}}$.

It is also possible that two numbers two apart on the cycle are equal. For this to be the case, let y be the value such that $f(g(y)) = y$. Then $1 - \frac{1}{x} = x$, or $x - 1 = x^2$, so $x = \frac{1 \pm i\sqrt{3}}{2}$. Let $\zeta = \frac{1+i\sqrt{3}}{2}$. Then we get that the cycle of values is $\zeta, \bar{\zeta}, \zeta, \bar{\zeta}, \zeta, \bar{\zeta}$, and since at the end we are always an even number of spaces away from our starting location, the probability that $x_{2012} = x_0$ is 1.

Finally, we need to consider the possibility that two opposite numbers are equal. In this case we have a y such that $f(g(f(y))) = y$, or $\frac{x}{x-1} = x$, so $x = 2$. In this case we obtain the same cycle of numbers in the case where two adjacent numbers are equal, and so we again obtain the probability $\frac{2^{2011}+1}{3 \cdot 2^{2011}}$. Therefore, the only possibilities are 1, $\frac{2^{2011}+1}{3 \cdot 2^{2011}}$.

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Geometry Test

1. ABC is an isosceles triangle with $AB = 2$ and $\angle ABC = 90^\circ$. D is the midpoint of BC and E is on AC such that the area of $AEDB$ is twice the area of ECD . Find the length of DE .

Answer: $\boxed{\frac{\sqrt{17}}{3}}$ Let F be the foot of the perpendicular from E to BC . We have $[AEDB] + [EDC] = [ABC] = 2 \Rightarrow [EDC] = \frac{2}{3}$. Since we also have $[EDC] = \frac{1}{2}(EF)(DC)$, we get $EF = FC = \frac{4}{3}$. So $FD = \frac{1}{3}$, and $ED = \frac{\sqrt{17}}{3}$ by the Pythagorean Theorem.

2. Let ABC be a triangle with $\angle A = 90^\circ$, $AB = 1$, and $AC = 2$. Let ℓ be a line through A perpendicular to BC , and let the perpendicular bisectors of AB and AC meet ℓ at E and F , respectively. Find the length of segment EF .

Answer: $\boxed{\frac{3\sqrt{5}}{4}}$ Let M, N be the midpoints of AB and AC , respectively. Then we have $\angle EAB = \angle ACB$ and $\angle EAC = \angle ABC$, so $AEM \sim CBA \Rightarrow AE = \frac{\sqrt{5}}{4}$ and $FAN \sim CBA \Rightarrow AF = \sqrt{5}$. Consequently, $EF = AF - AE = \frac{3\sqrt{5}}{4}$.

3. Let ABC be a triangle with incenter I . Let the circle centered at B and passing through I intersect side AB at D and let the circle centered at C passing through I intersect side AC at E . Suppose DE is the perpendicular bisector of AI . What are all possible measures of angle BAC in degrees?

Answer: $\boxed{\frac{540}{7}}$ Let $\alpha = \angle BAC$. DE is the perpendicular bisector of AI , so $DA = DI$, and $\angle DIA = \angle DAI = \alpha/2$. Thus, $\angle IDB = \angle DIB = \alpha$, since $BD = BI$. This gives $\angle DBI = 180^\circ - 2\alpha$, so that $\angle ABC = 360^\circ - 4\alpha$. Similarly, $\angle ACB = 360^\circ - 4\alpha$. Now, summing the angles in ABC , we find $720^\circ - 7\alpha = 180^\circ$, so that $\alpha = \frac{540}{7}^\circ$.

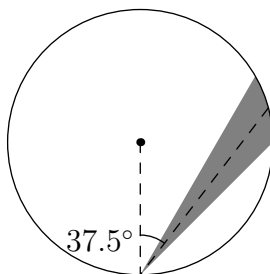
4. There are circles ω_1 and ω_2 . They intersect in two points, one of which is the point A . B lies on ω_1 such that AB is tangent to ω_2 . The tangent to ω_1 at B intersects ω_2 at C and D , where D is the closer to B . AD intersects ω_1 again at E . If $BD = 3$ and $CD = 13$, find EB/ED .

Answer: $\boxed{4\sqrt{3}/3}$

[diagram]

By power of a point, $BA = \sqrt{BD \cdot BC} = 4\sqrt{3}$. Also, $DEB \sim DBA$, so $EB/ED = BA/BD = 4\sqrt{3}/3$.

5. A mouse lives in a circular cage with completely reflective walls. At the edge of this cage, a small flashlight with vertex on the circle whose beam forms an angle of 15° is centered at an angle of 37.5° away from the center. The mouse will die in the dark. What fraction of the total area of the cage can keep the mouse alive?



Answer: $\boxed{\frac{3}{4}}$ We claim that the lit region is the entire cage except for a circle of half the radius of the cage in the center, along with some isolated points on the boundary of the circle and possibly minus a set of area 0. Note that the region is the same except for a set of area 0 if we disallow the light paths at the very edge of the beam. In that case, we can see that the lit region is an open subset of the disk, as clearly the region after k bounces is open for each k and the union of open sets is again open. We will then show that a dense subset of the claimed region of the cage is lit.

First, let us show that no part of the inner circle is lit. For any given light path, each chord of the circle is the same length, and in particular the minimum distance from the center of the circle is the same on each chord of the path. Since none of the initial chords can come closer than half the cage's radius to the center, the circle with half the cage's radius is indeed dark.

Now we need to show that for each open subset of the outer region, there is a light path passing through it, which will imply that the unlit region outside the small circle contains no open set, and thus has area 0. To do this, simply consider a light path whose angle away from the center is irrational such that the distance d from the center of the cage to the midpoint of the first chord is sufficiently close to the distance from the center of the cage to a point in the open set we're considering. Each successive chord of the light path can be seen as a rotation of the original one, and since at each step it is translated by an irrational angle, we obtain a dense subset of all the possible chords. This means that we obtain a dense subset of the circumference of the circle of radius d centered at the center of the cage, and in particular a point inside the open set under consideration, as desired.

Therefore, the lit region of the cage is the area outside the concentric circle of half the radius plus or minus some regions of area 0, which tells us that $\frac{3}{4}$ of the cage is lit.

6. Triangle ABC is an equilateral triangle with side length 1. Let X_0, X_1, \dots be an infinite sequence of points such that the following conditions hold:

- X_0 is the center of ABC
- For all $i \geq 0$, X_{2i+1} lies on segment AB and X_{2i+2} lies on segment AC .
- For all $i \geq 0$, $\angle X_i X_{i+1} X_{i+2} = 90^\circ$.
- For all $i \geq 1$, X_{i+2} lies in triangle $AX_i X_{i+1}$.

Find the maximum possible value of $\sum_{i=0}^{\infty} |X_i X_{i+1}|$, where $|PQ|$ is the length of line segment PQ .

Answer: $\boxed{\frac{\sqrt{6}}{3}}$ Let Y be the foot of the perpendicular from A to $X_0 X_1$: note that the sum we wish to minimize is simply $X_0 Y + Y A$. However, it is not difficult to check (for example, by AM-GM) that $AY + Y X_0 \geq \sqrt{2} A X_0 = \sqrt{6}/3$. This may be achieved by making $\angle Y X_0 A = 45^\circ$, so that $\angle A X_1 X_0 = 105^\circ$.

7. Let S be the set of the points $(x_1, x_2, \dots, x_{2012})$ in 2012-dimensional space such that $|x_1| + |x_2| + \dots + |x_{2012}| \leq 1$. Let T be the set of points in 2012-dimensional space such that $\max_{i=1}^{2012} |x_i| = 2$. Let p be a randomly chosen point on T . What is the probability that the closest point in S to p is a vertex of S ?

Answer: $\boxed{\frac{1}{2^{2011}}}$ Note that T is a hypercube in 2012-dimensional space, containing the rotated hyperoctahedron S . Let v be a particular vertex of S , and we will consider the set of points x on T such that v is the closest point to x in S . Let w be another point of S and let ℓ be the line between v and w . Then in order for v to be the closest point to x in S , it must also be so on the region of ℓ contained in S . This condition is then equivalent to the projection of x lying past v on the line ℓ , or alternatively that v lies in the opposite halfspace of w defined by the hyperplane perpendicular to ℓ and passing through v . This can be written algebraically as $(x - v) \cdot (w - v) \leq 0$. Therefore, v is the closest point to x if and only if $(x - v) \cdot (w - v) \leq 0$ for all w in S .

Note that these conditions do not depend on where w is on the line, so for each line intersecting S nontrivially, let us choose w such that w lies on the hyperplane H containing all the vertices of S except for v and $-v$. We can further see that the conditions are linear in w , so them holding for all w

in H is equivalent to them holding on the vertices of the region $S \cap H$, which are simply the vertices of S except for v and $-v$. Let us now compute what these conditions look like.

Without loss of generality, let $v = (1, 0, \dots, 0)$ and $w = (0, 1, 0, \dots, 0)$. Then the equation is of the form $(x-v) \cdot (-1, 1, 0, \dots, 0) \leq 0$, which we can rewrite as $(x_1-1, x_2, x_3, \dots, x_{2012}) \cdot (-1, 1, 0, \dots, 0) = 1 - x_1 + x_2 \leq 0$. For the other choices of w , we get the similar conditions that $1 - x_1 + x_i \leq 0$ and also $1 - x_1 - x_i \leq 0$ for each $i \in \{2, \dots, 2012\}$. Note that if any $x_i \in \{2, -2\}$ for $i \neq 1$, then one of these conditions trivially fails, as it would require $3 - x_1 \leq 0$. Therefore, the only face of T where x can lie is the face defined by $x_1 = 2$, which gives us the conditions that $-1 + x_i \leq 0$ and $-1 - x_i \leq 0$, so $x_i \in [-1, 1]$ for all $i \in \{2, \dots, 2012\}$. This defines a 2011-dimensional hypercube of side length 2 on the face of T defined by $x_1 = 2$, and we obtain similar regions on each of the other faces corresponding to the other vertices of S . Therefore, the volume of the set of x for which x is closest to a vertex is $2 \cdot 2012 \cdot 2^{2011}$ and the volume of all the choices of x is $2 \cdot 2012 \cdot 4^{2011}$, so the desired probability is $\frac{1}{2^{2011}}$.

8. Hexagon $ABCDEF$ has a circumscribed circle and an inscribed circle. If $AB = 9$, $BC = 6$, $CD = 2$, and $EF = 4$. Find $\{DE, FA\}$.

Answer: $\left\{ \frac{9+\sqrt{33}}{2}, \frac{9-\sqrt{33}}{2} \right\}$ By Brianchon's Theorem, AD, BE, CF concur at some point P . Also, it follows from the fact that tangents from a point to a circle have equal lengths that $AB + CD + EF = BC + DE + FA$. Let $DE = x$, so that $FA = 9 - x$.

Note that $APB \sim EPD$, $BPC \sim FPE$, and $CPD \sim APF$. The second similarity gives $BP/FP = 3/2$, so that $BP = 3y, FP = 2y$ for some y . From here, the first similarity gives $DP = xy/3$. Now, the third similarity gives $4y = \frac{(9-x)xy}{3}$, so that $x^2 - 9x + 12 = 0$. It follows that $x = \frac{9 \pm \sqrt{33}}{2}$, giving our answer.

9. Let O, O_1, O_2, O_3, O_4 be points such that O_1, O, O_3 and O_2, O, O_4 are collinear in that order, $OO_1 = 1, OO_2 = 2, OO_3 = \sqrt{2}, OO_4 = 2$, and $\angle O_1OO_2 = 45^\circ$. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the circles with respective centers O_1, O_2, O_3, O_4 that go through O . Let A be the intersection of ω_1 and ω_2 , B be the intersection of ω_2 and ω_3 , C be the intersection of ω_3 and ω_4 , and D be the intersection of ω_4 and ω_1 , with A, B, C, D all distinct from O . What is the largest possible area of a convex quadrilateral $P_1P_2P_3P_4$ such that P_i lies on O_i and that A, B, C, D all lie on its perimeter?

Answer: $8 + 4\sqrt{2}$ We first maximize the area of triangle P_1OP_2 , noting that the sum of the area of P_1OP_2 and the three other analogous triangles is the area of $P_1P_2P_3P_4$. Note that if $A \neq P_1, P_2$, without loss of generality say $\angle OAP_1 < 90^\circ$. Then, $\angle OO_1P_1 = 2\angle OAP_1$, and since $\angle OAP_2 = 180^\circ - \angle OAP_1 > 90^\circ$, we see that $\angle OO_2P_2 = 2\angle OAP_1$ as well, and it follows that $OO_1P_1 \sim OO_2P_2$. This is a spiral similarity, so $OO_1O_2 \sim OP_1P_2$, and in particular $\angle P_1OP_2 = \angle O_1OO_2$, which is fixed. By the sine area formula, to maximize $OP_1 \cdot OP_2$, which is bounded above by the diameters $2(OO_1), 2(OO_2)$. In a similar way, we want P_3, P_4 to be diametrically opposite O_3, O_4 in their respective circles.

When we take these P_i , we indeed have $A \in P_1P_2$ and similarly for B, C, D , since $\angle OAP_1 = \angle OAP_2 = 90^\circ$. To finish, the area of the quadrilateral is the sum of the areas of the four triangles, which is

$$\frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot 2^2 \cdot (1 \cdot 2 + 2 \cdot \sqrt{2} + \sqrt{2} \cdot 2 + 2 \cdot 1) = 8 + 4\sqrt{2}.$$

10. Let C denote the set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1$. A sequence $A_i = (x_i, y_i) | i \geq 0$ of points in \mathbb{R}^2 is 'centric' if it satisfies the following properties:

- $A_0 = (x_0, y_0) = (0, 0), A_1 = (x_1, y_1) = (1, 0)$.
- For all $n \geq 0$, the circumcenter of triangle $A_nA_{n+1}A_{n+2}$ lies in C .

Let K be the maximum value of $x_{2012}^2 + y_{2012}^2$ over all centric sequences. Find all points (x, y) such that $x^2 + y^2 = K$ and there exists a centric sequence such that $A_{2012} = (x, y)$.

Answer: $\boxed{(-1006, 1006\sqrt{3}), (-1006, -1006\sqrt{3})}$ Consider any triple of points $\triangle A_n A_{n+1} A_{n+2}$ with circumcenter P_n . By the Triangle Inequality we have $A_n P_n \leq A_n A_0 + A_0 P_n \leq A_n A_0 + 1$. Since P_n is the circumcenter, we have $P_n A_n = P_n A_{n+2}$. Finally we have $A_{n+2} a_0 \leq P_n A_{n+2} + 1 = A_n P_n + 1 \leq A_n A_0 + 2$. Therefore $\sqrt{x_{n+2}^2 + y_{n+2}^2} \leq \sqrt{x_n^2 + y_n^2} + 2$.

It is also clear that equality occurs if and only if A_n, A_0, P_n, A_{n+2} are all collinear and P_n lies on the unit circle.

From the above inequality it is clear that $\sqrt{x_{2012}^2 + y_{2012}^2} \leq 2012$. So the maximum value of K is 2012^2 .

Now we must find all points A_2 that conforms to the conditions of the equality case. P_0 must lie on the unit circle, so it lies on the intersection of the unit circle with the perpendicular bisector of $A_0 A_1$, which is the line $x = \frac{1}{2}$. Thus P_0 must be one of $(\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$. From now on we assume that we take the positive root, as the negative root just reflects all successive points about the x -axis.

If $P_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ then A_0, P_0, A_2 must be collinear, so $A_2 = (1, \sqrt{3})$.

Then since we must have $A_0, P_{2n}, A_{2n}, A_{2n+2}$ collinear and P_{2n} on the unit circle, it follows that $P_{2n} = (-1)^n (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then by induction we have $A_{2n} = (-1)^{n+1} (n, n\sqrt{3})$. To fill out the rest of the sequence, we may take $A_{2n+1} = (-1)^n (n+1, -n\sqrt{3})$ and $P_{2n+1} = (-1)^{n+1} (\frac{1}{2}, -\frac{\sqrt{3}}{2})$ so that we get the needed properties.

Therefore the answer is

$$A_{2012} \in \{(-1006, 1006\sqrt{3}), (-1006, -1006\sqrt{3})\}$$

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Guts

1. [2] Square $ABCD$ has side length 2, and X is a point outside the square such that $AX = XB = \sqrt{2}$. What is the length of the longest diagonal of pentagon $AXBCD$?

Answer: $\boxed{\sqrt{10}}$ Since $AX = XB = \sqrt{2}$ and $AB = 2$, we have $\angle AXB = 90^\circ$. Hence, the distance from X to \overline{AB} is 1 and the distance from X to CD is 3. By inspection, the largest diagonals are thus $BX = CX = \sqrt{3^2 + 1^2}$.

2. [2] Let a_0, a_1, a_2, \dots denote the sequence of real numbers such that $a_0 = 2$ and $a_{n+1} = \frac{a_n}{1+a_n}$ for $n \geq 0$. Compute a_{2012} .

Answer: $\boxed{\frac{2}{4025}}$ Calculating out the first few terms, note that they follow the pattern $a_n = \frac{2}{2n+1}$. Plugging this back into the recursion shows that it indeed works.

3. [2] Suppose x and y are real numbers such that $-1 < x < y < 1$. Let G be the sum of the geometric series whose first term is x and whose ratio is y , and let G' be the sum of the geometric series whose first term is y and ratio is x . If $G = G'$, find $x + y$.

Answer: $\boxed{1}$ We note that $G = x/(1-y)$ and $G' = y/(1-x)$. Setting them equal gives $x/(1-y) = y/(1-x) \Rightarrow x^2 - x = y^2 - x \Rightarrow (x+y-1)(x-y) = 0$, so we get that $x+y-1 = 0 \Rightarrow x+y = 1$.

4. [2] Luna has an infinite supply of red, blue, orange, and green socks. She wants to arrange 2012 socks in a line such that no red sock is adjacent to a blue sock and no orange sock is adjacent to a green sock. How many ways can she do this?

Answer: $\boxed{4 \cdot 3^{2011}}$ Luna has 4 choices for the first sock. After that, she has 3 choices for each of 2011 remaining socks for a total of $4 \cdot 3^{2011}$.

5. [3] Mr. Canada chooses a positive real a uniformly at random from $(0, 1]$, chooses a positive real b uniformly at random from $(0, 1]$, and then sets $c = a/(a+b)$. What is the probability that c lies between $1/4$ and $3/4$?

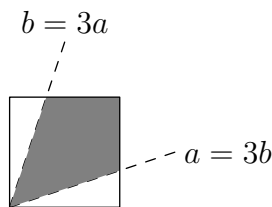
Answer: $\boxed{2/3}$ From $c \geq 1/4$ we get

$$\frac{a}{a+b} \geq \frac{1}{4} \iff b \leq 3a$$

and similarly $c \leq 3/4$ gives

$$\frac{a}{a+b} \leq \frac{3}{4} \iff a \leq 3b.$$

Choosing a and b randomly from $[0, 1]$ is equivalent to choosing a single point uniformly and randomly from the unit square, with a on the horizontal axis and b on the vertical axis:



To find the probability that $b \leq 3a$ and $a \leq 3b$, we need to find the area of the shaded region of the square. The area of each of the triangles on the side is $(1/2)(1)(1/3) = 1/6$, and so the area of the shaded region is $1 - 2(1/6) = 2/3$.

6. [3] Let rectangle $ABCD$ have lengths $AB = 20$ and $BC = 12$. Extend ray BC to Z such that $CZ = 18$. Let E be the point in the interior of $ABCD$ such that the perpendicular distance from E to \overline{AB} is 6 and the perpendicular distance from E to \overline{AD} is 6. Let line EZ intersect AB at X and CD at Y . Find the area of quadrilateral $AXYD$.

Answer: 72 Draw the line parallel to \overline{AD} through E , intersecting \overline{AB} at F and \overline{CD} at G . It is clear that $\triangle XFE$ and $\triangle YGE$ are congruent, so the area of $AXYD$ is equal to that of $AFGD$. But $AFGD$ is simply a 12 by 6 rectangle, so the answer must be 72. (Note: It is also possible to directly compute the values of AX and DY , then use the formula for the area of a trapezoid.)

7. [3] M is an 8×8 matrix. For $1 \leq i \leq 8$, all entries in row i are at least i , and all entries on column i are at least i . What is the minimum possible sum of the entries of M ?

Answer: 372 Let s_n be the minimum possible sum for an n by n matrix. Then, we note that increasing it by adding row $n + 1$ and column $n + 1$ gives $2n + 1$ additional entries, each of which has minimal size at least $n + 1$. Consequently, we obtain $s_{n+1} = s_n + (2n + 1)(n + 1) = s_n + 2n^2 + 3n + 1$. Since $s_0 = 0$, we get that $s_8 = 2(7^2 + \dots + 0^2) + 3(7 + \dots + 0) + 8 = 372$.

8. [3] Amy and Ben need to eat 1000 total carrots and 1000 total muffins. The muffins can not be eaten until all the carrots are eaten. Furthermore, Amy can not eat a muffin within 5 minutes of eating a carrot and neither can Ben. If Amy eats 40 carrots per minute and 70 muffins per minute and Ben eats 60 carrots per minute and 30 muffins per minute, what is the minimum number of minutes it will take them to finish the food?

Answer: 23.5 or 47/2 Amy and Ben will continuously eat carrots, then stop (not necessarily at the same time), and continuously eat muffins until no food is left. Suppose that Amy and Ben finish eating the carrots in T_1 minutes and the muffins T_2 minutes later; we wish to find the minimum value of $T_1 + T_2$. Furthermore, suppose Amy finishes eating the carrots at time a_1 , and Ben does so at time b_1 , so that $T_1 = \max(a_1, b_1)$.

First, suppose that $a_1 \leq b_1$, and let $b_1 - a_1 = c$. We have $40(T_1 - c) + 60T_1 = 1000$, so T_1 is minimized when $c = 0$. Also, $30(T_2 - 5) + 70(T_2 - \max(5 - c, 0)) = 1000$. We see that $T_1 + T_2$ is minimized when $c = 5$, and $T_1 + T_2 = 23.5$. In a similar way, we see that when $b_1 \leq a_1$, $T_1 + T_2 > 23.5$, so our answer is 23.5.

9. [5] Given $\triangle ABC$ with $AB < AC$, the altitude AD , angle bisector AE , and median AF are drawn from A , with D, E, F all lying on \overline{BC} . If $\angle BAD = 2\angle DAE = 2\angle EAF = \angle FAC$, what are all possible values of $\angle ACB$?

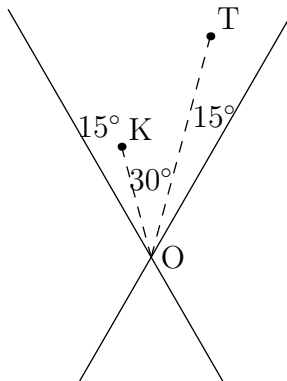
Answer: 30° or $\pi/6$ radians Let H and O be the orthocenter and circumcenter of ABC , respectively: it is well-known (and not difficult to check) that $\angle BAH = \angle CAO$. However, note that $\angle BAH = \angle BAD = \angle CAF$, so $\angle CAF = \angle CAO$, that is, O lies on median AF , and since $AB < AC$, it follows that $F = O$. Therefore, $\angle BAC = 90^\circ$.

Now, we compute $\angle ACB = \angle BAD = \frac{2}{6}\angle BAC = 30^\circ$.

10. [5] Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Compute $P(3)$.

Answer: 5 Plugging in $x = -1, 1, 2$ results in the trio of equations $1 = P(-1) = P(0) - P(1) + P(2)$, $P(1) = P(0) + P(1) + P(2) \Rightarrow P(1) + P(2) = 0$, and $P(2) = P(0) + 2P(1) + 4P(2)$. Solving these as a system of equations in $P(0), P(1), P(2)$ gives $P(0) = -1, P(1) = -1, P(2) = 1$. Consequently, $P(x) = x^2 - x - 1 \Rightarrow P(3) = 5$.

11. [5] Knot is on an epic quest to save the land of Hyruler from the evil Gammadorf. To do this, he must collect the two pieces of the Lineforce, then go to the Temple of Lime. As shown on the figure, Knot starts on point K , and must travel to point T , where $OK = 2$ and $OT = 4$. However, he must first reach both solid lines in the figure below to collect the pieces of the Lineforce. What is the minimal distance Knot must travel to do so?



Answer: $2\sqrt{5}$ Let l_1 and l_2 be the lines as labeled in the above diagram. First, suppose Knot visits l_1 first, at point P_1 , then l_2 , at point P_2 . Let K' be the reflection of K over l_1 , and let T' be the reflection of T over l_2 . The length of Knot's path is at least

$$KP_1 + P_1P_2 + P_2T = K'P_1 + P_1P_2 + P_2T' \geq K'T'$$

by the Triangle Inequality (This bound can be achieved by taking P_1, P_2 to be the intersections of $K'T'$ with l_1, l_2 , respectively.) Also, note that $\angle K'OT' = 90^\circ$, so that $K'T' = 2\sqrt{5}$.

Now, suppose Knot instead visits l_2 first, at point Q_2 , then l_1 , at point Q_1 . Letting K'' be the reflection of K over l_2 and T'' be the reflection of T over l_1 , by similar logic to before the length of his path is at least the length of $K''T''$. However, by inspection $K''T'' > K'T'$, so our answer is $2\sqrt{5}$.

12. [5] Knot is ready to face Gammadorf in a card game. In this game, there is a deck with twenty cards numbered from 1 to 20. Each player starts with a five card hand drawn from this deck. In each round, Gammadorf plays a card in his hand, then Knot plays a card in his hand. Whoever played a card with greater value gets a point. At the end of five rounds, the player with the most points wins. If Gammadorf starts with a hand of 1, 5, 10, 15, 20, how many five-card hands of the fifteen remaining cards can Knot draw which always let Knot win (assuming he plays optimally)?

Answer: 2982 Knot can only lose if all of his cards are lower than 10; if not he can win by playing the lowest card that beats Gammadorf's card, or if this is not possible, his lowest card, each turn. There are $\binom{7}{5} = 21$ losing hands, so he has $\binom{15}{5} - \binom{7}{5}$ possible winning hands.

13. [7] Niffy's favorite number is a positive integer, and Stebbysaurus is trying to guess what it is. Niffy tells her that when expressed in decimal without any leading zeros, her favorite number satisfies the following:

- Adding 1 to the number results in an integer divisible by 210.
- The sum of the digits of the number is twice its number of digits.
- The number has no more than 12 digits.
- The number alternates in even and odd digits.

Given this information, what are all possible values of Niffy's favorite number?

Answer: 1010309 Note that Niffy's favorite number must end in 9, since adding 1 makes it divisible by 10. Also, the sum of the digits of Niffy's favorite number must be even (because it is equal to twice the number of digits) and congruent to 2 modulo 3 (because adding 1 gives a multiple of 3). Furthermore, the sum of digits can be at most 24, because there at most 12 digits in Niffy's favorite number, and must be at least 9, because the last digit is 9. This gives the possible sums of digits 14 and 20. However, if the sum of the digits of the integer is 20, there are 10 digits, exactly 5 of which are odd, giving an odd sum of digits, which is impossible. Thus, Niffy's favorite number is a 7 digit number with sum of digits 14.

The integers which we seek must be of the form $\overline{ABCDEF9}$, where A, C, E are odd, B, D, F are even, and $A+B+C+D+E+F = 5$. Now, note that $\{A, C, E\} = \{1, 1, 1\}$ or $\{1, 1, 3\}$, and these correspond to $\{B, D, F\} = \{0, 0, 2\}$ and $\{0, 0, 0\}$, respectively. It suffices to determine which of these six integers are congruent to $-1 \pmod{7}$, and we see that Niffy's favorite number must be 1010309.

14. [7] Let triangle ABC have $AB = 5$, $BC = 6$, and $AC = 7$, with circumcenter O . Extend ray AB to point D such that $BD = 5$, and extend ray BC to point E such that $OD = OE$. Find CE .

Answer: $\boxed{\sqrt{59} - 3}$ Because $OD = OE$, D and E have equal power with respect to the circle, so $(EC)(EB) = (DB)(DA) = 50$. Letting $EC = x$, we have $x(x+6) = 50$, and taking the positive root gives $x = \sqrt{59} - 3$.

15. [7] Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct real polynomials such that the x -coordinate of the vertex of f is a root of g , the x -coordinate of the vertex of g is a root of f and both f and g have the same minimum value. If the graphs of the two polynomials intersect at the point $(2012, -2012)$, what is the value of $a + c$?

Answer: $\boxed{-8048}$ It is clear, by symmetry, that 2012 is the equidistant from the vertices of the two quadratics. Then it is clear that reflecting f about the line $x = 2012$ yields g and vice versa. Thus the average of each pair of roots is 2012. Thus the sum of the four roots of f and g is 8048, so $a + c = -8048$.

16. [7] Let A, B, C , and D be points randomly selected independently and uniformly within the unit square. What is the probability that the six lines \overline{AB} , \overline{AC} , \overline{AD} , \overline{BC} , \overline{BD} , and \overline{CD} all have positive slope?

Answer: $\boxed{\frac{1}{24}}$ Consider the sets of x -coordinates and y -coordinates of the points. In order to make 6 lines of positive slope, we must have smallest x -coordinate must be paired with the smallest y -coordinate, the second smallest together, and so forth. If we fix the order of the x -coordinates, the probability that the corresponding y -coordinates are in the same order is $1/24$.

17. [11] Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at $\frac{1}{1000}$ meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

Answer: $\boxed{2^{17}/125}$ If the two orbs leave their respective walls at the same time, then they will return to their walls at the same time (because colliding affects both their speeds). After returning to the wall n times, Mark's orb will travel at $\frac{4^n}{1000}$ meter/s and William's will travel at 2^n meter/s. Mark wins when his orb is traveling faster at $n = 10$. $\frac{4^{10}}{1000} = \frac{2^{17}}{125}$

18. [11] Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$.

Answer: $\boxed{\frac{\sqrt{6}}{2}}$ Solution 1: Let $x = \cos(\theta)$ and $y = \sin(\theta)$. Then, by the triple angle formulae, we have that $3x - 4x^3 = -\cos(3\theta)$ and $3y - 4y^3 = \sin(3\theta)$, so $-\sin(3\theta)\cos(3\theta) = -\frac{1}{2}$. We can write this as $2\sin(3\theta)\cos(3\theta) = \sin(6\theta) = 1$, so $\theta = \frac{1}{6}\sin^{-1}(1) = \frac{\pi}{12}$. Thus, $x + y = \cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{12}\right) = \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{\sqrt{6}}{2}$.

Solution 2: Expanding gives $9xy + 16x^3y^3 - 12xy^3 - 12x^3y = 9(xy) + 16(xy)^3 - 12(xy)(x^2 + y^2) = -\frac{1}{2}$, and since $x^2 + y^2 = 1$, this is $-3(xy) + 16(xy)^3 = -\frac{1}{2}$, giving $xy = -\frac{1}{2}, \frac{1}{4}$. However, since x and y are positive reals, we must have $xy = \frac{1}{4}$. Then, $x + y = \sqrt{x^2 + y^2 + 2xy} = \sqrt{1 + 2 \cdot \frac{1}{4}} = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}$.

19. [11] Given that P is a real polynomial of degree at most 2012 such that $P(n) = 2^n$ for $n = 1, 2, \dots, 2012$, what choice(s) of $P(0)$ produce the minimal possible value of $P(0)^2 + P(2013)^2$?

Answer: $\boxed{1 - 2^{2012}}$ Define $\Delta^1(n) = P(n+1) - P(n)$ and $\Delta^i(n) = \Delta^{i-1}(n+1) - \Delta^{i-1}(n)$ for $i > 1$. Since $P(n)$ has degree at most 2012, we know that $\Delta^{2012}(n)$ is constant. Computing, we obtain $\Delta^1(0) = 2 - P(0)$ and $\Delta^i(0) = 2^{i-1}$ for $1 < i \leq 2012$. We see that continuing on gives $\Delta^{2012}(0) = \Delta^{2012}(1) = P(0)$ and $\Delta^i(2012-i) = 2^{2013-i}$ for $1 \leq i \leq 2011$. Then, $P(2013) = P(2012) + \Delta^1(2012) = \dots = P(2012) + \Delta^1(2011) + \dots + \Delta^{2012}(0) = P(0) + 2^{2013} - 2$. Now, we want to minimize the value of $P(0)^2 + P(2013)^2 = 2P(0)^2 + 2P(0)(2^{2013} - 2) + (2^{2013} - 2)^2$, but this occurs simply when $P(0) = -\frac{1}{2}(2^{2013} - 2) = 1 - 2^{2012}$.

20. [11] Let n be the maximum number of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Let k be the number of ways to put n bishops on an 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$. (Two bishops are considered to be attacking each other if they lie on the same diagonal. Equivalently, if we label the squares with coordinates (x, y) , with $1 \leq x, y \leq 6$, then the bishops on (a, b) and (c, d) are attacking each other if and only if $|a - c| = |b - d|$.)

Answer: $\boxed{74}$ Color the square with coordinates (i, j) black if $i + j$ is odd and white otherwise, for all $1 \leq i, j \leq 6$. Looking at the black squares only, we note that there are six distinct diagonals which run upward and to the right, but that two of them consist only of a corner square; we cannot simultaneously place bishops on both of these corner squares. Consequently, we can place at most five bishops on black squares. (This can be achieved by placing bishops on $(1, 2), (1, 4), (6, 1), (6, 3), (6, 5)$.) If there are five bishops on black squares, there must be exactly one bishop on one of the two black corner squares, $(6, 1)$ and $(1, 6)$: suppose without loss of generality that we place a bishop on $(1, 6)$. Then, exactly one of $(3, 6)$ and $(1, 4)$ must also contain a bishop, and there are 2 ways to place two bishops on the four remaining black squares that are not yet under attack. Thus, we have a total of $2 \cdot 2 \cdot 2$ possible placements on black squares.

Similarly, there are at most 5 bishops which can be placed on white squares and 2^3 ways to place them, so that $n = 10$ and $k = 2^6$. Finally, $n + k = 10 + 2^6 = 74$.

21. [13] Let N be a three-digit integer such that the difference between any two positive integer factors of N is divisible by 3. Let $d(N)$ denote the number of positive integers which divide N . Find the maximum possible value of $N \cdot d(N)$.

Answer: $\boxed{5586}$ We first note that all the prime factors of n must be 1 modulo 3 (and thus 1 modulo 6). The smallest primes with this property are 7, 13, 19, \dots . Since $7^4 = 2401 > 1000$, the number can have at most 3 prime factors (including repeats). Since $7 \cdot 13 \cdot 19 = 1729 > 1000$, the most factors N can have is 6. Consider the number $7^2 \cdot 19 = 931$, which has 6 factors. For this choice of N , $N \cdot d(N) = 5586$. For another N to do better, it must have at least 6 factors, for otherwise, $N \cdot d(N) < 1000 \cdot 5 = 5000$. It is easy to verify that $7^2 \cdot 19$ is the greatest number with 6 prime factors satisfying our conditions, so the answer must be 5586.

22. [13] For each positive integer n , there is a circle around the origin with radius n . Rainbow Dash starts off somewhere on the plane, but not on a circle. She takes off in some direction in a straight path. She moves $\frac{\sqrt{5}}{5}$ units before crossing a circle, then $\sqrt{5}$ units, then $\frac{3\sqrt{5}}{5}$ units. What distance will she travel before she crosses another circle?

Answer: $\boxed{\frac{2\sqrt{170} - 9\sqrt{5}}{5}}$ Note that the distance from Rainbow Dash's starting point to the first place in which she hits a circle is irrelevant, except in checking that this distance is small enough that she does not hit another circle beforehand. It will be clear at the end that our configuration does not allow this (by the Triangle Inequality). Let O be the origin, and let Rainbow Dash's first three meeting points be A, B, C so that $AB = \sqrt{5}$ and $BC = \frac{3\sqrt{5}}{5}$.

Consider the lengths of OA, OB, OC . First, note that if $OA = OC = n$ (i.e. A and C lie on the same circle), then we need $OB = n - 1$, but since she only crosses the circle containing B once, it follows that the circle passing through B is tangent to AC , which is impossible since $AB \neq AC$. If $OA = OB = n$,

note that $OC = n + 1$. Dropping a perpendicular from O to AB , we see that by the Pythagorean Theorem,

$$n^2 - \frac{5}{4} = (n + 1)^2 - \frac{121}{20},$$

from which we get that n is not an integer. Similarly, when $OB = OC = n$, we have $OA = n + 1$, and n is not an integer.

Therefore, either $OA = n + 2, OB = n + 1, OC = n$ or $OA = n, OB = n + 1, OC = n + 2$. In the first case, by Stewart's Theorem,

$$\frac{24\sqrt{5}}{5} + (n + 1)^2 \cdot \frac{8\sqrt{5}}{5} = n^2 \cdot \sqrt{5} + (n + 2)^2 \cdot \frac{3\sqrt{5}}{5}.$$

This gives a negative value of n , so the configuration is impossible. In the final case, we have, again by Stewart's Theorem,

$$\frac{24\sqrt{5}}{5} + (n + 1)^2 \cdot \frac{8\sqrt{5}}{5} = (n + 2)^2 \cdot \sqrt{5} + n^2 \cdot \frac{3\sqrt{5}}{5}.$$

Solving gives $n = 3$, so $OA = 3, OB = 4, OC = 5$.

Next, we compute, by the Law of Cosines, $\cos \angle OAB = -\frac{1}{3\sqrt{5}}$, so that $\sin \angle OAB = \frac{2\sqrt{11}}{3\sqrt{5}}$. Let the projection from O to line AC be P ; we get that $OP = \frac{2\sqrt{11}}{\sqrt{5}}$. Rainbow Dash will next hit the circle of radius 6 at D . Our answer is now $CD = PD - PC = \frac{2\sqrt{170}}{5} - \frac{9\sqrt{5}}{5}$ by the Pythagorean Theorem.

23. [13] Points X and Y are inside a unit square. The score of a vertex of the square is the minimum distance from that vertex to X or Y . What is the minimum possible sum of the scores of the vertices of the square?

Answer: $\frac{\sqrt{6} + \sqrt{2}}{2}$ Let the square be $ABCD$. First, suppose that all four vertices are closer to X than Y . Then, by the triangle inequality, the sum of the scores is $AX + BX + CX + DX \geq AB + CD = 2$. Similarly, suppose exactly two vertices are closer to X than Y . Here, we have two distinct cases: the vertices closer to X are either adjacent or opposite. Again, by the Triangle Inequality, it follows that the sum of the scores of the vertices is at least 2.

On the other hand, suppose that A is closer to X and B, C, D are closer to Y . We wish to compute the minimum value of $AX + BY + CY + DY$, but note that we can make $X = A$ to simply minimize $BY + CY + DY$. We now want Y to be the Fermat point of triangle BCD , so that $\angle BYC = \angle CYD = \angle DYB = 120^\circ$. Note that by symmetry, we must have $\angle BCY = \angle DCY = 45^\circ$, so $\angle CBY = \angle CDY = 15^\circ$.

And now we use the law of sines: $BY = DY = \frac{\sin 45^\circ}{\sin 120^\circ}$ and $CY = \frac{\sin 15^\circ}{\sin 120^\circ}$. Now, we have $BY + CY + DY = \frac{\sqrt{2} + \sqrt{6}}{2}$, which is less than 2, so this is our answer.

24. [13] Franklin has four bags, numbered 1 through 4. Initially, the first bag contains fifteen balls, numbered 1 through 15, and the other bags are empty. Franklin randomly pulls a pair of balls out of the first bag, throws away the ball with the lower number, and moves the ball with the higher number into the second bag. He does this until there is only one ball left in the first bag. He then repeats this process in the second and third bag until there is exactly one ball in each bag. What is the probability that ball 14 is in one of the bags at the end?

Answer: $\frac{2}{3}$ Pretend there is a 16th ball numbered 16. This process is equivalent to randomly drawing a tournament bracket for the 16 balls, and playing a tournament where the higher ranked ball always wins. The probability that a ball is left in a bag at the end is the probability that it loses to ball 16. Of the three balls 14, 15, 16, there is a $\frac{1}{3}$ chance 14 plays 15 first, a $\frac{1}{3}$ chance 14 plays 16 first, and a $\frac{1}{3}$ chance 15 plays 16 first. In the first case, 14 does not lose to 16, and instead loses to 15; otherwise 14 loses to 16, and ends up in a bag. So the answer is $\frac{2}{3}$.

25. [17] FemtoPravis is walking on an 8×8 chessboard that wraps around at its edges (so squares on the left edge of the chessboard are adjacent to squares on the right edge, and similarly for the top and bottom edges). Each femtosecond, FemtoPravis moves in one of the four diagonal directions uniformly at random. After 2012 femtoseconds, what is the probability that FemtoPravis is at his original location?

Answer: $\left(\frac{1+2^{1005}}{2^{1007}}\right)^2$ We note the probability that he ends up in the same row is equal to the probability that he ends up in the same column by symmetry. Clearly these are independent, so we calculate the probability that he ends up in the same row.

Now we number the rows $0 - 7$ where 0 and 7 are adjacent. Suppose he starts at row 0. After two more turns, the probability he is in row 2 (or row 6) is $\frac{1}{4}$, and the probability he is in row 0 again is $\frac{1}{2}$.

Let a_n, b_n, c_n and d_n denote the probability he is in row $0, 2, 4, 6$ respectively after $2n$ moves.

We have $a_0 = 1$, and for $n \geq 0$ we have the following equations:

$$a_{n+1} = \frac{1}{2}a_n + \frac{1}{4}b_n + \frac{1}{4}d_n$$

$$b_{n+1} = \frac{1}{2}b_n + \frac{1}{4}a_n + \frac{1}{4}c_n$$

$$c_{n+1} = \frac{1}{2}c_n + \frac{1}{4}b_n + \frac{1}{4}d_n$$

$$d_{n+1} = \frac{1}{2}d_n + \frac{1}{4}a_n + \frac{1}{4}c_n$$

From which we get the following equations:

$$a_n + c_n = \frac{1}{2}$$

$$x_n = a_n - c_n = \frac{1}{2}(a_{n-1} - c_{n-1}) = \frac{x_{n-1}}{2}$$

So

$$a_{1006} + c_{1006} = \frac{1}{2}$$

$$x_0 = 1, x_{1006} = \frac{1}{2^{1006}}$$

$$a_{1006} = \frac{1 + 2^{1005}}{2^{1007}}$$

And thus the answer is $\left(\frac{1+2^{1005}}{2^{1007}}\right)^2$.

26. [17] Suppose ABC is a triangle with circumcenter O and orthocenter H such that A, B, C, O , and H are all on distinct points with integer coordinates. What is the second smallest possible value of the circumradius of ABC ?

Answer: $\sqrt{10}$ Assume without loss of generality that the circumcenter is at the origin. By well known properties of the Euler line, the centroid G is such that O, G , and H are collinear, with G in between O and H , such that $GH = 2GO$. Thus, since $G = \frac{1}{3}(A + B + C)$, and we are assuming O is the origin, we have $H = A + B + C$. This means that as long as A, B , and C are integer points, H will be as well.

However, since H needs to be distinct from A, B , and C , we must have $\triangle ABC$ not be a right triangle, since in right triangles, the orthocenter is the vertex where the right angle is.

Now, if a circle centered at the origin has any integer points, it will have at least four integer points. (If it has a point of the form $(a, 0)$, then it will also have $(-a, 0), (0, a)$, and $(0, -a)$. If it has a point of the form (a, b) , with $a, b \neq 0$, it will have each point of the form $(\pm a, \pm b)$.) But in any of these cases where there are only four points, any triangle which can be made from those points is a right triangle.

Thus we need the circumcircle to contain at least eight lattice points. The smallest radius this occurs at is $\sqrt{1^2 + 2^2} = \sqrt{5}$, which contains the eight points $(\pm 1, \pm 2)$ and $(\pm 2, \pm 1)$. We get at least one valid triangle with this circumradius:

$$A = (-1, 2), B = (1, 2), C = (2, 1).$$

The next valid circumradius is $\sqrt{1^2 + 3^2} = \sqrt{10}$ which has the valid triangle

$$A = (-1, 3), B = (1, 3), C = (3, 1).$$

27. [17] Let S be the set $\{1, 2, \dots, 2012\}$. A perfectutation is a bijective function h from S to itself such that there exists an $a \in S$ such that $h(a) \neq a$, and that for any pair of integers $a \in S$ and $b \in S$ such that $h(a) \neq a$, $h(b) \neq b$, there exists a positive integer k such that $h^k(a) = b$. Let n be the number of ordered pairs of perfectutations (f, g) such that $f(g(i)) = g(f(i))$ for all $i \in S$, but $f \neq g$. Find the remainder when n is divided by 2011.

Answer: $\boxed{2}$ Note that both f and g , when written in cycle notation, must contain exactly one cycle that contains more than 1 element. Assume f has k fixed points, and that the other $2012 - k$ elements form a cycle, (of which there are $(2011 - k)!$ ways).

Then note that if f fixes a then $f(g(a)) = g(f(a)) = g(a)$ implies f fixes $g(a)$. So g must send fixed points of f to fixed points of f . It must, therefore, send non-fixed points to non-fixed points. This partitions S into two sets, at least one of which must be fixed by g , since g is a perfectutation.

If g fixes all of the non-fixed points of f , then, since any function commutes with the identity, g fixes some m of the fixed points and cycles the rest in $(k - m - 1)!$ ways. So there are $\sum_{m=0}^{k-2} \binom{k}{m} (k - m - 1)!$ choices, which is $\sum_{m=0}^{k-2} \frac{k!}{(k-m)m!}$.

If g fixes all of the fixed points of f , then order the non-fixed points of f $a_1, a_2, \dots, a_{2012-k}$ such that $f(a_i) = a_{i+1}$. If $g(a_i) = a_j$ then $f(g(a_i)) = a_{j+1}$ thus $g(a_{i+1}) = a_{j+1}$. Therefore the choice of $g(a_1)$ uniquely determines $g(a_i)$ for the rest of the i , and $g(a_m) = a_{m+j-i}$. But g has to be a perfectutation, so g cycles through all the non-fixed points of f , which happens if and only if $j - i$ is relatively prime to $2012 - k$. So there are $\phi(2012 - k)$ choices.

Therefore for any f there are $\sum_{m=0}^{k-2} \frac{k!}{(k-m)m!} + \phi(2012 - k)$ choices of g , but one of them will be $g = f$, which we cannot have by the problem statement. So there are $-1 + \sum_{m=0}^{k-2} \frac{k!}{(k-m)m!} + \phi(2012 - k)$ options.

Now note that a permutation can not fix all but one element. So $n = \sum_{k=0}^{2010} \binom{2012}{k} (2011 - k)! (-1 + \sum_{m=0}^{k-2} \frac{k!}{(k-m)m!} + \phi(2012 - k))$

Modulo 2011 (which is prime), note that all terms in the summand except the one where $k = 1$ vanish. Thus, $n \equiv (2010)!(-1 + (-1)) \equiv 2 \pmod{2011}$ by Wilson's Theorem.

28. [17] Alice is sitting in a teacup ride with infinitely many layers of spinning disks. The largest disk has radius 5. Each succeeding disk has its center attached to a point on the circumference of the previous disk and has a radius equal to $2/3$ of the previous disk. Each disk spins around its center (relative to the disk it is attached to) at a rate of $\pi/6$ radians per second. Initially, at $t = 0$, the centers of the disks are aligned on a single line, going outward. Alice is sitting at the limit point of all these disks. After 12 seconds, what is the length of the trajectory that Alice has traced out?

Answer: $\boxed{18\pi}$ Suppose the center of the largest teacup is at the origin in the complex plane, and let $z = \frac{2}{3}e^{\pi it/6}$. The center of the second disk is at $5e^{\pi it/6}$ at time t ; that is, $\frac{15}{2}z$. Then the center of the third disk relative to the center of the second disk is at $\frac{15}{2}z^2$, and so on. Summing up a geometric series, we get that Alice's position is

$$\begin{aligned} \frac{15}{2}(z + z^2 + z^3 + \dots) &= \frac{15}{2}(1 + z^2 + z^3 + \dots) - \frac{15}{2} \\ &= \frac{15}{2} \left(\frac{1}{1 - z} \right) - \frac{15}{2}. \end{aligned}$$

Now, after 12 seconds, z has made a full circle in the complex plane centered at 0 and of radius $2/3$. Thus $1 - z$ is a circle centered at 1 of radius $2/3$.

So $1 - z$ traces a circle, and now we need to find the path that $\frac{1}{1-z}$ traces. In the complex plane, taking the reciprocal corresponds to a reflection about the real axis followed by a geometric inversion about the unit circle centered at 0. It is well known that geometric inversion maps circles not passing through the center of the inversion to circles.

Now, the circle traced by $1 - z$ contains the points $1 - 2/3 = 1/3$, and $1 + 2/3 = 5/3$. Therefore the circle $\frac{1}{1-z}$ contains the points 3 and $3/5$, with the center lying halfway between. So the radius of the circle is

$$\frac{1}{2} \left(3 - \frac{3}{5} \right) = \frac{6}{5}$$

and so the perimeter is $2\pi(6/5) = 12\pi/5$. Scaling by $15/2$ gives an answer of

$$\frac{15}{2} \left(\frac{12\pi}{5} \right) = 18\pi.$$

29. [19] Consider the cube whose vertices are the eight points (x, y, z) for which each of x , y , and z is either 0 or 1. How many ways are there to color its vertices black or white such that, for any vertex, if all of its neighbors are the same color then it is also that color? Two vertices are neighbors if they are the two endpoints of some edge of the cube.

Answer: 118 Divide the 8 vertices of the cube into two sets A and B such that each set contains 4 vertices, any two of which are diagonally adjacent across a face of the cube. We do casework based on the number of vertices of each color in set A .

- Case 1: 4 black. Then all the vertices in B must be black, for 1 possible coloring.
- Case 2: 3 black, 1 white. Then there are 4 ways to assign the white vertex. The vertex in B surrounded by the black vertices must also be black. Meanwhile, the three remaining vertices in B may be any configuration except all black, for a total of $4(2^3 - 1) = 28$ possible colorings.
- Case 3: 2 black, 2 white. Then, there are 6 ways to assign the 2 white vertices. The 4 vertices of B cannot all be the same color. Additionally, we cannot have 3 black vertices of B surround a white vertex of A with the other vertex of B white, and vice-versa, so we have a total of $6(2^4 - 2 - 4) = 60$ possible colorings.
- Case 4: 1 black, 3 white. As in case 2, there are 28 possible colorings.
- Case 5: 5 white. As in case 1, there is 1 possible coloring.

So there is a total of $1 + 28 + 60 + 28 + 1 = 118$ possible colorings.

30. [19] You have a twig of length 1. You repeatedly do the following: select two points on the twig independently and uniformly at random, make cuts on these two points, and keep only the largest piece. After 2012 repetitions, what is the expected length of the remaining piece?

Answer: (11/18)²⁰¹² First let $p(x)$ be the probability density of x being the longest length.

Let a_n be the expected length after n cuts. $a_n = \int_0^1 p(x) \cdot (xa_{n-1})dx = a_{n-1} \int_0^1 xp(x)dx = a_1 a_{n-1}$.

It follows that $a_n = a_1^n$, so our answer is $(a_1)^{2012}$. We now calculate a_1 .

Let $P(z)$ be the probability that the longest section is $\leq z$. Clearly $P(z) = 0$ for $z \leq \frac{1}{3}$.

To simulate making two cuts we pick two random numbers x, y from $[0, 1]$, and assume without loss of generality that $x \leq y$. Then picking two such points is equivalent to picking a point in the top left triangle half of the unit square. This figure has area $\frac{1}{2}$ so our $P(z)$ will be double the area where $x \leq z$, $y \geq 1 - z$ and $y - x \leq z$. For $\frac{1}{3} \leq z \leq \frac{1}{2}$ the probability is double the area bounded by $x = z, 1 - z = y, y - x = z$. This is $2(\frac{1}{2}(3z - 1)^2) = (3z - 1)^2$.

For $\frac{1}{2} \leq z \leq 1$ this value is double the hexagon bounded by $x = 0$, $y = 1 - z$, $y = x$, $x = z$, $y = 1$, $y = x + z$. The complement of this set, however, is three triangles of area $\frac{(1-z)^2}{2}$, so $P(z) = 1 - 3(1-z)^2$ for $\frac{1}{2} \leq z \leq 1$.

Now note that $P'(z) = p(z)$. Therefore by integration by parts $a_1 = \int_0^1 zp(z)dz = \int_0^1 zP'(z)dz = \int_0^1 zP(z) - \int_0^1 P(z)dz$. This equals

$$\begin{aligned} & 1 - \int_{\frac{1}{3}}^{\frac{1}{2}} (3z-1)^2 dz - \int_{\frac{1}{2}}^1 1 - 3(1-z)^2 dz \\ &= 1 - \left[\frac{1}{3} \frac{(3z-1)^3}{9} - \frac{1}{2} + \left[\frac{1}{2} (z-1)^3 \right] \right] \\ &= 1 - \frac{1}{72} - \frac{1}{2} + \frac{1}{8} = \frac{1}{2} + \frac{1}{9} \\ &= \frac{11}{18} \end{aligned}$$

So the answer is $(\frac{11}{18})^{2012}$.

31. [19] Let S_7 denote all the permutations of $1, 2, \dots, 7$. For any $\pi \in S_7$, let $f(\pi)$ be the smallest positive integer i such that $\pi(1), \pi(2), \dots, \pi(i)$ is a permutation of $1, 2, \dots, i$. Compute $\sum_{\pi \in S_7} f(\pi)$.

Answer: [29093] Extend the definition of f to apply for any permutation of $1, 2, \dots, n$, for any positive integer n . For positive integer n , let $g(n)$ denote the number of permutations π of $1, 2, \dots, n$ such that $f(\pi) = n$. We have $g(1) = 1$. For fixed n, k (with $k \leq n$), the number of permutations π of $1, 2, \dots, n$ such that $f(\pi) = k$ is $g(k)(n-k)!$. This gives us the recursive formula $g(n) = n! - \sum_{k=1}^{n-1} g(k)(n-k)!$. Using this formula, we find that the first 7 values of g are 1, 1, 3, 13, 71, 461, 3447. Our sum is then equal to $\sum_{k=1}^7 k \cdot g(k)(7-k)!$. Using our computed values of g , we get that the sum evaluates to 29093.

32. [19] Let S be a set of size 3. How many collections T of subsets of S have the property that for any two subsets $U \in T$ and $V \in T$, both $U \cap V$ and $U \cup V$ are in T ?

Answer: [74] Let us consider the collections T grouped based on the size of the set $X = \bigcup_{U \in T} U$, which we can see also must be in T as long as T contains at least one set. This leads us to count the number of collections on a set of size at most 3 satisfying the desired property with the additional property that the entire set must be in the collection. Let C_n denote that number of such collections on a set of size n . Our answer will then be $1 + \binom{3}{0}C_0 + \binom{3}{1}C_1 + \binom{3}{2}C_2 + \binom{3}{3}C_3$, with the additional 1 coming from the empty collection.

Now for such a collection T on a set of n elements, consider the set $I = \bigcap_{U \in T} U$. Suppose this set has size k . Then removing all these elements from consideration gives us another such collection on a set of size $n - k$, but now containing the empty set. We can see that for each particular choice of I , this gives a bijection to the collections on the set S to the collections on the set $S - I$. This leads us to consider the further restricted collections that must contain both the entire set and the empty set.

It turns out that such restricted collections are a well-studied class of objects called *topological spaces*. Let T_n be the number of topological spaces on n elements. Our argument before shows that $C_n = \sum_{k=0}^n \binom{n}{k} T_k$. It is relatively straightforward to see that $T_0 = 1, T_1 = 1$, and $T_2 = 4$. For a set of size 3, there are the following spaces. The number of symmetric versions is shown in parentheses.

- $\emptyset, \{a, b, c\}$ (1)
- $\emptyset, \{a, b\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{a, b\}, \{a, b, c\}$ (6)
- $\emptyset, \{a\}, \{b, c\}, \{a, b, c\}$ (3)

- $\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}$ (3)
- $\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, b, c\}$ (6)
- $\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$ (1)

which gives $T_3 = 29$. Tracing back our reductions, we have that $C_0 = \binom{0}{0}T_0 = 1, C_1 = \binom{1}{0}T_0 + \binom{1}{1}T_1 = 2, C_2 = \binom{2}{0}T_0 + \binom{2}{1}T_1 + \binom{2}{2}T_2 = 7, C_3 = \binom{3}{0}T_0 + \binom{3}{1}T_1 + \binom{3}{2}T_2 + \binom{3}{3}T_3 = 45$, and then our answer is $1 + \binom{3}{0}C_0 + \binom{3}{1}C_1 + \binom{3}{2}C_2 + \binom{3}{3}C_3 = 1 + 1 + 6 + 21 + 45 = 74$.

33. [23] Compute the decimal expansion of $\sqrt{\pi}$. Your score will be $\min(23, k)$, where k is the number of consecutive correct digits immediately following the decimal point in your answer.

Answer: 1.77245385090551602729816... For this problem, it is useful to know the following square root algorithm that allows for digit-by-digit extraction of \sqrt{x} and gives one decimal place of \sqrt{x} for each two decimal places of x . We will illustrate how to extract the second digit after the decimal point of $\sqrt{\pi}$, knowing that $\pi = 3.1415 \dots$ and $\sqrt{\pi} = 1.7 \dots$.

Let d be the next decimal digit. Then d should be the largest digit such that $(1.7 + 0.01d)^2 < \pi$, which in this case we will treat as $(1.7 + 0.01d)^2 < 3.1415$. Expanding this, we get $2.89 + 0.034d + 0.0001d^2 < 3.1415$, from which we get the value of d to be approximately $\lfloor \frac{3.1415 - 2.89}{0.034} \rfloor = \lfloor \frac{0.2515}{0.034} \rfloor = 7$, since the $0.0001d^2$ term is negligible. Indeed, 7 is the largest such digit, and so $d = 7$ is the second digit of $\sqrt{\pi}$. Because we are constantly subtracting the square of our extracted answer so far, we can record the difference in a manner similar to long division, which yields a quick method of extracting square roots by hand.

34. [23] Let Q be the product of the sizes of all the non-empty subsets of $\{1, 2, \dots, 2012\}$, and let $M = \log_2(\log_2(Q))$. Give lower and upper bounds L and U for M . If $0 < L \leq M \leq U$, then your score will be $\min(23, \lfloor \frac{23}{3^{(U-L)}} \rfloor)$. Otherwise, your score will be 0.

Answer: 2015.318180... In this solution, all logarithms will be taken in base 2. It is clear that $\log(Q) = \sum_{k=1}^{2012} \binom{2012}{k} \log(k)$. By pairing k with $2012 - k$, we get $\sum_{k=1}^{2011} 0.5 * \log(k(2012 - k)) \binom{2012}{k} + \log(2012)$, which is between $0.5 * \log(2012) \sum_{k=0}^{2012} \binom{2012}{k}$ and $\log(2012) \sum_{k=0}^{2012} \binom{2012}{k}$; i.e., the answer is between $\log(2012)2^{2011}$ and $\log(2012)2^{2012}$. Thus $\log(\log(Q))$ is between $2011 + \log(\log(2012))$ and $2012 + \log(\log(2012))$. Also $3 < \log(\log(2012)) < 4$. So we get $2014 < M < 2016$.

35. [23] Let N be the number of distinct roots of $\prod_{k=1}^{2012} (x^k - 1)$. Give lower and upper bounds L and U on N . If $0 < L \leq N \leq U$, then your score will be $\lfloor \frac{23}{(U/L)^{1.7}} \rfloor$. Otherwise, your score will be 0.

Answer: 1231288 For x to be such a number is equivalent to x being a k^{th} root of unity for some k up to 2012. For each k , there are $\varphi(k)$ primitive k^{th} roots of unity, so the total number of roots is $\sum_{k=1}^{2012} \varphi(k)$.

We will give a good approximation of this number using well known facts about the Möbius function, defined by $\mu(n) = \begin{cases} 0 & \text{if } n \text{ is not squarefree} \\ (-1)^r & \text{if } n \text{ has } r \text{ distinct prime factors} \end{cases}$. It turns out that if $f(n) = \sum_{d|n} g(d)$, then $g(n) = \sum_{d|n} \mu(d) f(\frac{n}{d})$. Using this fact, since $n = \sum_{d|n} \varphi(d)$, we have that $\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$.

Now we have reduced the problem to estimating $\sum_{k=1}^{2012} \sum_{d|k} \mu(d) \frac{k}{d}$. Let $a = \frac{k}{d}$, so we obtain $\sum_{k=1}^{2012} \sum_{d|k} a \mu(d)$.

We can interchange the order of summation by writing

$$\begin{aligned} \sum_{d=1}^{2012} \sum_{a=1}^{\lfloor \frac{2012}{d} \rfloor} a\mu(d) &\approx \sum_{d=1}^{2012} \mu(d) \frac{1}{2} \left(\lfloor \frac{2012}{d} \rfloor \right)^2 \\ &\approx \sum_{d=1}^{2012} \mu(d) \frac{2012^2}{2d^2} \\ &= \frac{2012^2}{2} \sum_{d=1}^{2012} \frac{\mu(d)}{d^2} \\ &\approx \frac{2012^2}{2} \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \end{aligned}$$

The Möbius function also satisfies the property that $\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$, which can be seen

as a special case of the theorem above (letting $f(n) = 1, g(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$). We can then see that

$\left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} \right) \left(\sum_{c=1}^{\infty} \frac{1}{c^2} \right) = \frac{1}{1^2} = 1$, so $\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2}$. Therefore, we have $\sum_{k=1}^{2012} \varphi(k) \approx \frac{3}{\pi^2} \cdot 2012^2 = 1230488.266\dots$ 2012 is large enough that all of our approximations are pretty accurate and we should be comfortable perturbing this estimate by a small factor to give bounding values.

36. [23] Maria is hopping up a flight of stairs with 100 steps. At every hop, she advances some integer number of steps. Each hop she makes has fewer steps. However, the positive difference between the length of consecutive hops decreases. Let P be the number of distinct ways she can hop up the stairs. Find lower and upper bounds L and U for P . If $0 < L \leq P \leq U$, your score will be $\left\lfloor \frac{23}{\sqrt{U/L}} \right\rfloor$. Otherwise, your score will be 0.

Answer: 6922 Consider the sequence of hops backwards. It is an increasing sequence where the first finite differences are increasing, so all the second finite differences are all positive integers. Furthermore, given positive integers a, e_0 (representing the initial value and initial first finite difference), and a sequence of positive integers e_1, e_2, \dots, e_k , representing the second finite differences, we obtain a sequence of $k+2$ integers satisfying our constraints where the i -th term is equal to $a + \sum_{j=0}^{i-2} (i-j-1)e_j$. The sum of these values is equal to $(k+2)a + \sum_{j=0}^k \frac{(j+1)(j+2)}{2} e_{k-j}$. The number of sequences of length $k+2$ with sum 100 is then the number of positive integer solutions to $(k+2)a + \sum_{j=0}^k \frac{(j+1)(j+2)}{2} e_{k-j} = 100$ in a, e_0, \dots, e_k .

We can now approximate a count of the total number of solutions by caseworking over the possible values of k . First, we must consider all sequences of length 1 or 2; it easy to see that there are 1 and 49 of these, respectively. Otherwise, we want to consider the following equations:

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 100, \\ x_1 + 4x_2 + 3x_3 + 6x_4 &= 100, \\ x_1 + 5x_2 + 3x_3 + 6x_4 + 10x_5 &= 100, \\ x_1 + 6x_2 + 3x_3 + 6x_4 + 10x_5 + 15x_6 &= 100, \\ x_1 + 7x_2 + 3x_3 + 6x_4 + 10x_5 + 15x_6 + 21x_7 &= 100, \\ x_1 + 8x_2 + 3x_3 + 6x_4 + 10x_5 + 15x_6 + 21x_7 + 28x_8 &= 100. \end{aligned}$$

(Any larger values of k will yield equations with no solutions, as the sum of the coefficients will be greater than 100 in all of these.) We can compute the number of solutions to the last equation easily: there are 8. For the remaining 5 equations, we can use the following observation to estimate the number of solutions. Suppose we wanted to count the number of positive integer solutions to

$\sum_{i=1}^k c_i x_i = n$, where $c_i = 1$. This is equivalent to finding the number of nonnegative integer solutions to $\sum_{i=1}^k c_i x_i = n - \sum_{i=1}^k c_i$, which is also equivalent to finding the number of nonnegative integer solutions to $\sum_{i=2}^k c_i x_i \leq n - \sum_{i=1}^k c_i$. Let $n' = n - \sum_{i=1}^k c_i$. Each x_i can take values from 0 to $\frac{n'}{c_i}$, giving about $\frac{(n')^{k-1}}{\prod_{i=2}^k c_i}$ choices. Of course, not all of these choices work; we try to estimate the probability that one does. We can think of a random selection of x_i as approximated by choosing X_i uniformly from $[0, 1]$, then setting $x_i = X_i \cdot \frac{n'}{c_i}$. The condition would then be $\sum_{i=2}^k X_k \leq 1$. The probability of this occurring is $\frac{1}{(k-1)!}$, so an approximate number of solutions would be $\frac{(n')^{k-1}}{(k-1)! \prod_{i=2}^k c_i}$. We can use these values to get a lower bound of around 4000, and we can also use n instead of n' for a reasonable upper bound of 16000.

15th Annual Harvard-MIT Mathematics Tournament
Saturday 11 February 2012

Team A

1. [20] Let ABC be a triangle with $AB < AC$. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D . Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC , respectively. Prove that $BE = CF$.

Answer: see below Note that DE, DF are the distances from D to AB, AC , respectively, and because AD is the angle bisector of $\angle BAC$, we have $DE = DF$. Also, $DB = DC$ because D is on the perpendicular bisector of BC . Finally, $\angle DEB = \angle DFC = 90^\circ$, so it follows that $DEB \cong DFC$, and $BE = CF$.

2. [20] For what positive integers n do there exist functions $f, g : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that for each $1 \leq i \leq n$, either $f(g(i)) = i$ or $g(f(i)) = i$, but not both?

Answer: n even We claim that this is possible for all even n . First, a construction: set $f(2m-1) = f(2m) = 2m-1$ and $g(2m-1) = g(2m) = 2m$ for $m = 1, \dots, \frac{n}{2}$. It is easy to verify that this solution works.

Now, we show that this is impossible for odd n . Without loss of generality, suppose that $f(g(1)) = 1$ and that $g(1) = a \Rightarrow f(a) = 1$. Then, we have $g(f(a)) = g(a) = 1$. Consequently, $a \neq 1$. In this case, call 1 and a a *pair* (we likewise regard i and j as a pair when $g(f(i)) = i$ and $f(i) = j$). Now, to show that n is even it suffices to show that all pairs are disjoint. Suppose for the sake of contradiction that some integer $b \neq a$ is also in a pair with 1 (note that 1 is arbitrary). Then, we have $f(g(b)) = b, g(b) = 1$ or $g(f(b)) = b, f(b) = 1$. But we already know that $g(1) = a$, so we must have $f(g(b)) = b, g(b) = 1$. But that would mean that both $f(g(1)) = 1$ and $g(f(1)) = 1$, a contradiction.

3. [20] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

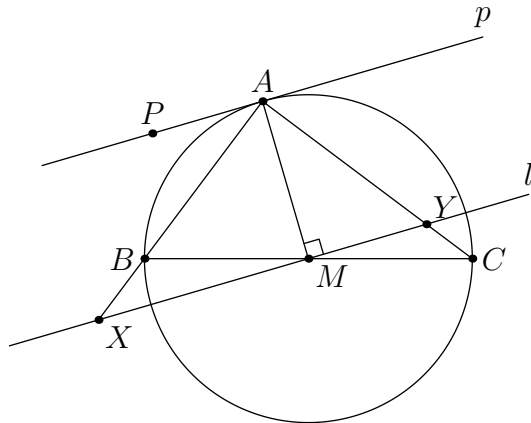
- (a) Suppose that a token can legally move to any horizontally or vertically adjacent square. Show that Bob has a winning strategy for this game.
- (b) Suppose instead that a token can legally move to any horizontally, vertically, or diagonally adjacent square. Show that Alice has a winning strategy for this game.

Answer: see below For part (a), color the checkerboard in the standard way so that half of the squares are black and the other half are white. Bob's winning strategy is to place the two coins on the same color, so that Alice must always move her coin on to a square with the opposite color as the square containing Bob's coin.

For part (b), consider any starting configuration. By considering only the column that the tokens are in, it is easy to see that Alice can get to the same column as Bob (immediately after her move) in 7 rounds. (This is just a game on a 1×8 chessboard.) Following this, Alice can stay on the same column as Bob each turn, while getting to the same row as him. This too also takes at most 7 rounds. Thus, Alice can catch Bob in $14 < 2012$ rounds from any starting position.

4. [20] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC . Line l is drawn through M so that it is perpendicular to AM , and intersects line AB at point X and line AC at point Y . Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral XYC is cyclic.

Answer: see below



First, note that $XBYC$ cyclic is equivalent to $\angle BXM = \angle ACB$. However, note that $\angle BXM = 90^\circ - \angle BAM$, so $XBYC$ cyclic is in turn equivalent to $\angle BAM + \angle ACB = 90^\circ$.

Let the line tangent to the circumcircle of $\triangle ABC$ at A be p , and let P be an arbitrary point on p on the same side of AM as B . Note that $\angle PAB = \angle ACB$. If $\angle ACB = 90^\circ - \angle BAM$ we have $l \perp AM$ and thus the circumcenter O of $\triangle ABC$ lies on AM . Since $AB < AC$, we must have $O = M$, and $\angle BAC = 90^\circ$. Conversely, if $\angle BAC = 90^\circ$, $\angle PAM = 90^\circ$, and it follows that $\angle ACB = 90^\circ - \angle BAM$.

5. [20] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes five pizzas, each with some subset of the five toppings. For each pizza, Arno states (with either a “yes” or a “no”) if the pizza has any toppings that he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she has enough information to make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

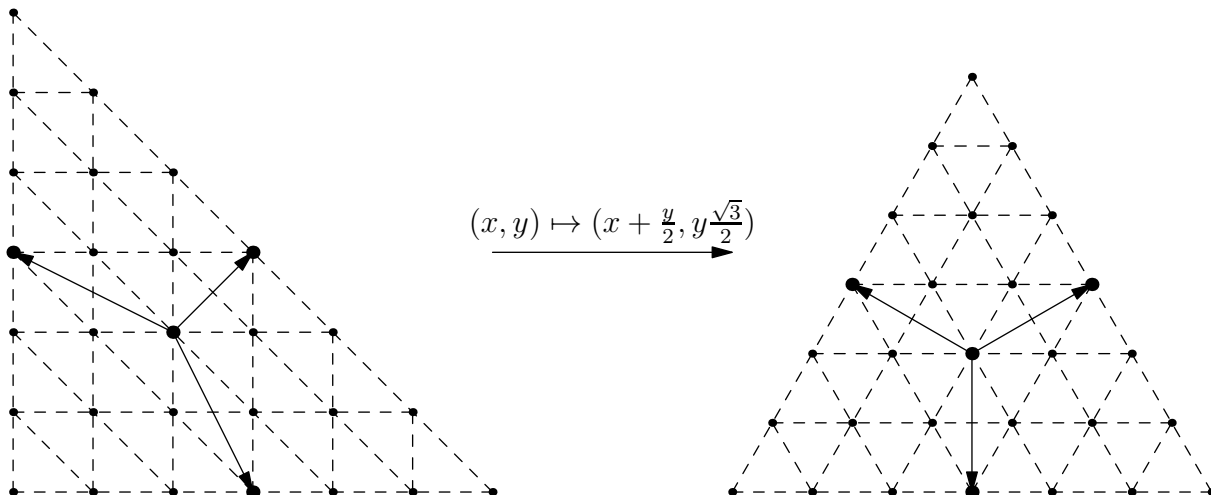
Answer: see below We claim the only way for Purineqa to deduce Arno’s preferences is for each pizza to contain exactly one topping, with no topping be repeated. It is obvious that he can deduce the toppings in this case.

We now claim that this is not possible with any other combination. Suppose that Arno tells Purineqa that he does not like any of the five pizzas. Then, Purineqa should be able to rule out at least one of the possibilities that Arno likes none of the toppings and that Arno likes exactly one of the toppings T . It is clear that this is possible if and only if there is a pizza with only T on it. This is true for all five toppings T , so we’re done.

6. [30] It has recently been discovered that the right triangle with vertices $(0,0)$, $(0,2012)$, and $(2012,0)$ is a giant pond that is home to many frogs. Frogs have the special ability that, if they are at a lattice point (x, y) , they can hop to any of the three lattice points $(x+1, y+1)$, $(x-2, y+1)$, and $(x+1, y-2)$, assuming the given lattice point lies in or on the boundary of the triangle.

Frog Jeff starts at the corner $(0, 0)$, while Frog Kenny starts at the corner $(0, 2012)$. Show that the set of points that Jeff can reach is equal in size to the set of points that Kenny can reach.

Answer: see below



We transform the triangle as follows: map each lattice point (x, y) to the point

$$x(1, 0) + y(1/2, \sqrt{3}/2) = (x + y/2, y\sqrt{3}/2).$$

This transforms the right triangle into an equilateral triangle as shown above.

Now, the three allowed movements

$$\begin{aligned} (x, y) &\mapsto (x + 1, y + 1), \\ (x, y) &\mapsto (x - 2, y + 1), \\ (x, y) &\mapsto (x + 1, y - 2) \end{aligned}$$

become the movements

$$\begin{aligned} (x, y) &\mapsto (x + 3/2, y + \sqrt{3}/2), \\ (x, y) &\mapsto (x - 3/2, y + \sqrt{3}/2), \\ (x, y) &\mapsto (x, y - \sqrt{3}) \end{aligned}$$

That is, each step is a movement of $\sqrt{3}$ in any of these three directions, which are separated by 120° angles. The pond is now completely symmetrical with respect to 120° rotations, so it does not matter which vertex you start at. The lower left vertex corresponds to the original point $(0, 0)$, and the top vertex corresponds to the original point $(0, 2012)$.

7. [30] Five points are chosen on a sphere of radius 1. What is the maximum possible volume of their convex hull?

Answer: $\frac{\sqrt{3}}{2}$ Let the points be A, B, C, X, Y so that X and Y are on opposite sides of the plane defined by triangle ABC . The volume is $1/3$ the product of the area of ABC and sum of the distances from X and Y to the plane defined by ABC . The area of ABC is maximized when the plane containing it intersects the sphere in the largest possible cross section, and ABC is equilateral: this gives an area of $3\sqrt{3}/4$. Then, the sum of the distances from X and Y to the plane of ABC is at most 2. This is clearly obtainable when $A, B,$ and C form an equilateral triangle circumscribed by the equator of the sphere, and X and Y are at the poles, and we get a volume of $\sqrt{3}/2$.

8. [30] For integer $n, m \geq 1$, let $A(n, m)$ denote the number of functions $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, m\}$ such that $f(j) - f(i) \leq j - i$ for all $1 \leq i < j \leq n$, and let $B(n, m)$ denote the number of functions $g : \{0, 1, \dots, 2n + m\} \rightarrow \{0, 1, \dots, m\}$ such that $g(0) = 0$, $g(2n + m) = m$, and $|g(i) - g(i - 1)| = 1$ for all $1 \leq i \leq 2n + m$. Prove that $A(n, m) = B(n, m)$.

Answer: see below We first note that the condition for f is equivalent to $i - f(i) \leq j - f(j)$ for all $1 \leq i < j \leq n$. Letting $f'(x) = x - f(x)$, we see this is equivalent to saying that f' is decreasing. Thus, we only need that $f'(x) \leq f'(x+1)$; in other words, we only require the statement to be true for $j = i+1$.

Fix m, n . For any function g satisfying the conditions for B , we construct a function f satisfying the conditions for A as follows. For a given function g and $1 \leq i \leq 2n+m$, say that g has a *up step* at i if $g(i) - g(i-1) = 1$, and say it has a *down step* at i otherwise. We see that g must be composed of $m+n$ up steps and n down steps. Let i_1, i_2, \dots, i_n be the indices for which down steps occur, in ascending order. Let f be the function such that $f(j) = (m+1) - g(i_j)$ for $1 \leq j \leq n$. By our argument in the first paragraph, it suffices to show that $f(k) - f(k-1) \leq 1$ for $1 < k \leq n$, or that $g(i_{k-1}) - 1 \leq g(i_k)$. If this were not the case for some k , then there would be at least 1 down step in between i_{k-1} and i_k , a contradiction, so the condition indeed holds.

We now claim that this construction is a bijection. For injectivity, note that for any two distinct g, g' , there exists a k for which the values of $g(i_k), g'(i'_k)$ are distinct, in which case the functions f, f' must be distinct. For surjectivity, consider any suitable f . Let f' be the function such that $f'(k) = (m+1) - f(k)$ for all $1 \leq k \leq n$. (The range of this function is still $\{1, 2, \dots, m\}$.) Then, we can find a g as follows: for each $1 \leq k \leq n$ in sequence, have g make up steps until it reaches the value $f'(k)$, then take one down step. This is always possible, as $f'(k+1) - f'(k) \leq 1$. Thus, our claim is true, and our proof is complete.

9. [40] For any positive integer n , let $N = \varphi(1) + \varphi(2) + \dots + \varphi(n)$. Show that there exists a sequence

$$a_1, a_2, \dots, a_N$$

containing exactly $\varphi(k)$ instances of k for all positive integers $k \leq n$ such that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_N a_1} = 1.$$

Answer: see below We write all fractions of the form b/a , where a and b are relatively prime, and $0 \leq b \leq a \leq n$, in ascending order. For instance, for $n = 5$, this is the sequence

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

This sequence is known as the *Farey sequence*.

Now, if we look at the the sequence of the denominators of the fractions, we see that k appears $\varphi(k)$ times when $k > 1$, although 1 appears twice. Thus, there are $N+1$ elements in the Farey sequence. Let the Farey sequence be

$$\frac{b_1}{a_1}, \frac{b_2}{a_2}, \dots, \frac{b_{N+1}}{a_{N+1}}$$

Now, $a_{N+1} = 1$, so the sequence a_1, a_2, \dots, a_N contains $\varphi(k)$ instances of k for every $1 \leq k \leq n$. We claim that this sequence also satisfies

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_N a_1} = 1.$$

Since $a_1 = a_{N+1} = 1$, we have

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_N a_1} = \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_N a_{N+1}}.$$

Now, it will suffice to show that $\frac{1}{a_i a_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$. Once we have shown this, the above sum will telescope to $\frac{b_{N+1}}{a_{N+1}} - \frac{b_1}{a_1} = 1 - 0 = 1$.

To see why $\frac{1}{a_i a_{i+1}} = \frac{b_{i+1}}{a_{i+1}} - \frac{b_i}{a_i}$ holds, we note that this is equivalent to $1 = b_{i+1} a_i - b_i a_{i+1}$. We can prove this fact geometrically: consider the triangle in the plane with vertices $(0, 0)$, (a_i, b_i) , and

(a_{i+1}, b_{i+1}) . This triangle contains these three boundary points, but it contains no other boundary or interior points since a_i and a_{i+1} are relatively prime to b_i and b_{i+1} , respectively, and since no other fraction with denominator at most n lies between $\frac{b_i}{a_i}$ and $\frac{b_{i+1}}{a_{i+1}}$. Thus, by Pick's theorem, this triangle has area $1/2$. But the area of the triangle can also be computed as the cross product $\frac{1}{2}(b_{i+1}a_i - b_ia_{i+1})$; hence $b_{i+1}a_i - b_ia_{i+1} = 1$ and we are done.

10. [40] For positive odd integer n , let $f(n)$ denote the number of matrices A satisfying the following conditions:

- A is $n \times n$.
- Each row and column contains each of $1, 2, \dots, n$ exactly once in some order.
- $A^T = A$. (That is, the element in row i and column j is equal to the one in row j and column i , for all $1 \leq i, j \leq n$.)

Prove that $f(n) \geq \frac{n!(n-1)!}{\varphi(n)}$.

Answer: see below We first note that main diagonal (the squares with row number equal to column number) is a permutation of $1, 2, \dots, n$. This is because each number i ($1 \leq i \leq n$) appears an even number of times off the main diagonal, so must appear an odd number of times on the main diagonal. Thus, we may assume that the main diagonal's values are $1, 2, \dots, n$ in that order. Call any matrix satisfying this condition and the problem conditions *good*. Let $g(n)$ denote the number of good matrices. It now remains to show that $g(n) \geq \frac{(n-1)!}{\varphi(n)}$.

Now, consider a round-robin tournament with n teams, labeled from 1 through n , with the matches spread over n days such that on day i , all teams except team i play exactly one match (so there are $\frac{n-1}{2}$ pairings), and at the end of n days, each pair of teams has played exactly once. We consider two such tournaments distinct if there is some pairing of teams i, j which occurs on different days in the tournaments. We claim that the tournaments are in bijection with the good matrices.

Proof of Claim: Given any good matrix A , we construct a tournament by making day k have matches between team i and j for each i, j such that $A_{i,j} = k$, besides $(i, j) = (k, k)$. Every pair will play some day, and since each column and row contains exactly one value of each number, no team will play more than once a day. Furthermore, given two distinct good matrices, there exists a value (off the main diagonal) on which they differ; this value corresponds to the same pair playing on different dates, so the corresponding tournaments must be distinct. For the other direction, take any tournament. Make a matrix A with the main diagonal as $1, 2, \dots, n$, and for each k , set $A_{i,j} = k$ for each i, j such that teams i, j play each other on day k . This gives a good matrix. Similarly, given any two distinct tournaments, there exists a team pair i, j which play each other on different days; this corresponds to a differing value on the corresponding good matrices.

It now suffices to exhibit $\frac{(n-1)!}{\varphi(n)}$ distinct tournaments. (It may be helpful here to think of the days in the tournament as an unordered collection of sets of pairings, with the order implicitly imposed by the team not present in the set of pairings.) For our construction, consider a regular n -gon with center O . Label the points as A_1, A_2, \dots, A_n as an arbitrary permutation (so there are $n!$ possible labelings). The team k will be represented by A_k . For each k , consider the line A_kO . The remaining $n-1$ vertices can be paired into $\frac{n-1}{2}$ groups which are perpendicular to this line; use these pairings for day k . Of course, this doesn't generate $n!$ distinct tournaments— but how many does it make?

Consider any permutation of labels. Starting from an arbitrary point, let the points of the polygon be $A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(n)}$ in clockwise order. Letting $\pi(0) = \pi(n)$ and $\pi(n+1) = \pi(1)$, we note that $\pi(i-1)$ and $\pi(i+1)$ play each other on day $\pi(i)$. We then see that any other permutation of labels representing the same tournament must have $A_{\pi(i-1)}A_{\pi(i)} = A_{\pi(i)}A_{\pi(i+1)}$ for all i . Thus, if $A_{\pi(1)}$ is k vertices clockwise of $A_{\pi(0)}$, then $A_{\pi(2)}$ is k vertices clockwise of $A_{\pi(1)}$, and so on all the way up to $A_{\pi(n-1)}$ being k vertices clockwise of $A_{\pi(n)}$. This is only possible if k is relatively prime to n , so there are $\varphi(n)$ choices of k . There are n choices of the place to put $A_{\pi(1)}$, giving $n\varphi(n)$ choices of permutations meeting this condition. It is clear that each permutation meeting this condition provides the same tournament, so the $n!$ permutations can be partitioned into equivalence classes of size $n\varphi(n)$ each. Thus, there are $\frac{n!}{n\varphi(n)}$ distinct equivalence classes, and we are done.

15th Annual Harvard-MIT Mathematics Tournament
Saturday 11 February 2012

Team B

1. [10] Triangle ABC has $AB = 5$, $BC = 3\sqrt{2}$, and $AC = 1$. If the altitude from B to AC and the angle bisector of angle A intersect at D , what is BD ?

Answer: $\boxed{\frac{5}{3}}$ Let E be the foot of the perpendicular from B to line AC . By the Law of Cosines, $\cos \angle BAC = \frac{4}{5}$, and it follows that $BE = 3$ and $AE = 4$. Now, by the Angle Bisector Theorem, $\frac{BD}{DE} = \frac{AB}{AE} = \frac{5}{4}$, so $BD = \frac{5}{3}$.

2. [10] You are given two line segments of length 2^n for each integer $0 \leq n \leq 10$. How many distinct nondegenerate triangles can you form with three of the segments? Two triangles are considered distinct if they are not congruent.

Answer: $\boxed{55}$ First, observe that if we have three sticks of distinct lengths $2^a < 2^b < 2^c$, then $2^a + 2^b < 2^{b+1} \leq 2^c$, so we cannot form a triangle. Thus, we must have (exactly) two of our sticks the same length, so that our triangle has side lengths $2^a, 2^a, 2^b$. This triangle is non-degenerate if and only if $2^{a+1} > 2^b$, and since $a \neq b$, this happens if and only if $a > b$. Clearly, there are $\binom{11}{2} = 55$ ways to choose such a, b .

3. [10] Mac is trying to fill 2012 barrels with apple cider. He starts with 0 energy. Every minute, he may rest, gaining 1 energy, or if he has n energy, he may expend k energy ($0 \leq k \leq n$) to fill up to $n(k+1)$ barrels with cider. What is the minimal number of minutes he needs to fill all the barrels?

Answer: $\boxed{46}$ First, suppose that Mac fills barrels during two consecutive minutes. Let his energy immediately before doing so be n , and the energy spent in the next two minutes be k_1, k_2 , respectively. It is not difficult to check that he can fill at least as many barrels by spending $k_1 + k_2 + 1$ energy and resting for an additional minute before doing so, so that his starting energy is $n + 1$: this does not change the total amount of time. Furthermore, this does not affect the amount of energy Mac has remaining afterward. We may thus assume that Mac first rests for a (not necessarily fixed) period of time, then spends one minute filling barrels, and repeats this process until all of the barrels are filled.

Next, we check that he only needs to fill barrels once. Suppose that Mac first rests for n_1 minutes, then spends k_1 energy, and next rests for n_2 minutes and spends k_2 energy. It is again not difficult to check that Mac can instead rest for $n_1 + n_2 + 1$ minutes and spend $k_1 + k_2 + 1$ energy, to increase the number of barrels filled, while not changing the amount of time nor the energy remaining. Iterating this operation, we can reduce our problem to the case in which Mac first rests for n minutes, then spends n energy filling all of the barrels.

We need $n(n+1) \geq 2012$, so $n \geq 45$, and Mac needs a minimum of 46 minutes.

4. [10] A restaurant has some number of seats, arranged in a line. Its customers are in parties arranged in a queue. To seat its customers, the restaurant takes the next party in the queue and attempts to seat all of the party's member(s) in a contiguous block of unoccupied seats. If one or more such blocks exist, then the restaurant places the party in an arbitrarily selected block; otherwise, the party leaves.

Suppose the queue has parties of sizes 6, 4, 2, 5, 3, 1 from front to back, and all seats are initially empty. What is the minimal number of seats the restaurant needs to guarantee that it will seat all of these customers?

Answer: $\boxed{29}$ First, note that if there are only 28 seats, it is possible for the seating not to be possible, in the following way. The party of six could be seated in such a way that the remaining contiguous regions have sizes 10 and 12. Then, the party of 4 is seated in the middle of the region of size 12, and the party of 2 is seated in the middle of the region of size 10, so that the remaining regions all of size 4. Now, it is impossible to seat the party of 5. It is clear that fewer than 28 seats also make it impossible to guarantee that everyone can be seated.

Now, given 29 seats, clearly, the first party can be seated. Afterward, 23 seats remain in at most 2 contiguous regions, one of which has to have size at least 4. Next, 19 seats remain in at most 3

contiguous regions, one of which has size at least 2. 17 seats in at most 4 contiguous regions remain for the party of 5, and one region must have size at least 5. Finally, 12 seats in at most 5 contiguous regions are available for the party of 3, and the party of 1 can take any remaining seat. Our answer is therefore 29.

Remark: We can arrive at the answer of 29 by doing the above argument in reverse.

5. [10] Steph and Jeff each start with the number 4, and Travis is flipping a coin. Every time he flips a heads, Steph replaces her number x with $2x - 1$, and Jeff replaces his number y with $y + 8$. Every time he flips a tails, Steph replaces her number x with $\frac{x+1}{2}$, and Jeff replaces his number y with $y - 3$. After some (positive) number of coin flips, Steph and Jeff miraculously end up with the same number below 2012. How many times was the coin flipped?

Answer: 137 Suppose that a heads and b tails are flipped. Jeff's number at the end is $4 + 8a - 3b$. Note that the operations which Steph applies are inverses of each other, and as a result it is not difficult to check by induction that her final number is simply $1 + 3 \cdot 2^{a-b}$.

We now have $3 + 3(a - b) + 5a = 3 \cdot 2^{a-b}$. Letting $n = a - b$, we see that $2^n - n - 1$ must be divisible by 5, so that a is an integer. In particular, n is a positive integer. Furthermore, we have $1 + 3 \cdot 2^n < 2012$, so that $n \leq 9$. We see that the only possibility is for $n = 7 = a - b$, and thus $4 + 8a - 3b = 385$. Solving, we get $a = 72, b = 65$, so our answer is $72 + 65 = 137$.

6. [20] Let ABC be a triangle with $AB < AC$. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D . Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC , respectively. Prove that $BE = CF$.

Answer: see below Note that DE, DF are the distances from D to AB, AC , respectively, and because AD is the angle bisector of $\angle BAC$, we have $DE = DF$. Also, $DB = DC$ because D is on the perpendicular bisector of BC . Finally, $\angle DEB = \angle DFC = 90^\circ$, so it follows that $DEB \cong DFC$, and $BE = CF$.

7. [20] For what positive integers n do there exist functions $f, g : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that for each $1 \leq i \leq n$, either $f(g(i)) = i$ or $g(f(i)) = i$, but not both?

Answer: n even We claim that this is possible for all even n . First, a construction: set $f(2m - 1) = f(2m) = 2m - 1$ and $g(2m - 1) = g(2m) = 2m$ for $m = 1, \dots, \frac{n}{2}$. It is easy to verify that this solution works.

Now, we show that this is impossible for odd n . Without loss of generality, suppose that $f(g(1)) = 1$ and that $g(1) = a \Rightarrow f(a) = 1$. Then, we have $g(f(a)) = g(a) = 1$. Consequently, $a \neq 1$. In this case, call 1 and a a *pair* (we likewise regard i and j as a pair when $g(f(i)) = i$ and $f(i) = j$). Now, to show that n is even it suffices to show that all pairs are disjoint. Suppose for the sake of contradiction that some integer $b \neq a$ is also in a pair with 1 (note that 1 is arbitrary). Then, we have $f(g(b)) = b, g(b) = 1$ or $g(f(b)) = b, f(b) = 1$. But we already know that $g(1) = a$, so we must have $f(g(b)) = b, g(b) = 1$. But that would mean that both $f(g(1)) = 1$ and $g(f(1)) = 1$, a contradiction.

8. [20] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

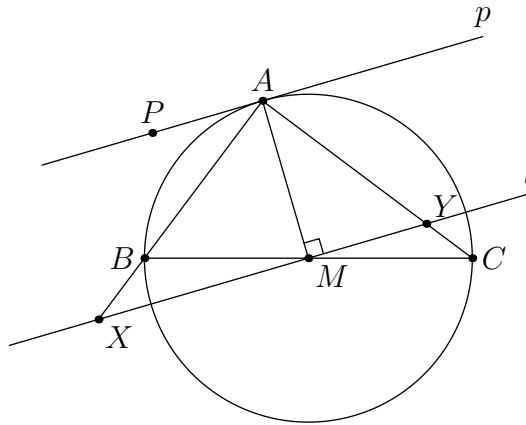
- (a) Suppose that a token can legally move to any horizontally or vertically adjacent square. Show that Bob has a winning strategy for this game.
- (b) Suppose instead that a token can legally move to any horizontally, vertically, or diagonally adjacent square. Show that Alice has a winning strategy for this game.

Answer: see below For part (a), color the checkerboard in the standard way so that half of the squares are black and the other half are white. Bob's winning strategy is to place the two coins on the same color, so that Alice must always move her coin on to a square with the opposite color as the square containing Bob's coin.

For part (b), consider any starting configuration. By considering only the column that the tokens are in, it is easy to see that Alice can get to the same column as Bob (immediately after her move) in 7 rounds. (This is just a game on a 1×8 chessboard.) Following this, Alice can stay on the same column as Bob each turn, while getting to the same row as him. This too also takes at most 7 rounds. Thus, Alice can catch Bob in $14 < 2012$ rounds from any starting position.

9. [20] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC . Line l is drawn through M so that it is perpendicular to AM , and intersects line AB at point X and line AC at point Y . Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

Answer: see below



First, note that $XBYC$ cyclic is equivalent to $\angle BXM = \angle ACB$. However, note that $\angle BXM = 90^\circ - \angle BAM$, so $XBYC$ cyclic is in turn equivalent to $\angle BAM + \angle ACB = 90^\circ$.

Let the line tangent to the circumcircle of $\triangle ABC$ at A be p , and let P be an arbitrary point on p on the same side of AM as B . Note that $\angle PAB = \angle ACB$. If $\angle ACB = 90^\circ - \angle BAM$ we have $l \perp AM$ and thus the circumcenter O of $\triangle ABC$ lies on AM . Since $AB < AC$, we must have $O = M$, and $\angle BAC = 90^\circ$. Conversely, if $\angle BAC = 90^\circ$, $\angle PAM = 90^\circ$, and it follows that $\angle ACB = 90^\circ - \angle BAM$.

10. [20] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes five pizzas, each with some subset of the five toppings. For each pizza, Arno states (with either a “yes” or a “no”) if the pizza has any toppings that he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she has enough information to make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

Answer: see below We claim the only way for Purineqa to deduce Arno’s preferences is for each pizza to contain exactly one topping, with no topping be repeated. It is obvious that he can deduce the toppings in this case.

We now claim that this is not possible with any other combination. Suppose that Arno tells Purineqa that he does not like any of the five pizzas. Then, Purineqa should be able to rule out at least one of the possibilities that Arno likes none of the toppings and that Arno likes exactly one of the toppings T . It is clear that this is possible if and only if there is a pizza with only T on it. This is true for all five toppings T , so we’re done.

HMMT November 2012
Saturday 10 November 2012
General Test

1. [3] What is the sum of all of the distinct prime factors of $25^3 - 27^2$?

Answer: $\boxed{28}$ We note that $25^3 - 27^2 = 5^6 - 3^6 = (5^3 - 3^3)(5^3 + 3^3) = (5 - 3)(5^2 + 5 \cdot 3 + 3^2)(5 + 3)(5^2 - 5 \cdot 3 + 3^2) = 2 \cdot 7^2 \cdot 2^3 \cdot 19$, so the sum of the distinct prime factors is $2 + 7 + 19 = 28$.

2. [3] Let $Q(x) = x^2 + 2x + 3$, and suppose that $P(x)$ is a polynomial such that

$$P(Q(x)) = x^6 + 6x^5 + 18x^4 + 32x^3 + 35x^2 + 22x + 8.$$

Compute $P(2)$.

Answer: $\boxed{2}$ Note that $Q(-1) = 2$. Therefore, $P(2) = P(Q(-1)) = 1 - 6 + 18 - 32 + 35 - 22 + 8 = 2$.

3. [3] $ABCD$ is a rectangle with $AB = 20$ and $BC = 3$. A circle with radius 5, centered at the midpoint of DC , meets the rectangle at four points: W, X, Y , and Z . Find the area of quadrilateral $WXYZ$.

Answer: $\boxed{27}$ Suppose that X and Y are located on AB with X closer to A than B . Let O be the center of the circle, and let P be the midpoint of AB . We have $OP \perp AB$ so OPX and OPY are right triangles with right angles at P . Because $OX = OY = 5$ and $OP = 3$, we have $XP = PY = 4$ by the Pythagorean theorem. Now, $WXYZ$ is a trapezoid with $WZ = WO + OZ = 5 + 5 = 10$, $XY = XP + PY = 8$, and height 3, so its area is $(\frac{10+8}{2}) \times 3 = 27$.

4. [4] If you roll four fair 6-sided dice, what is the probability that at least three of them will show the same value?

Answer: $\boxed{\frac{7}{72}}$ We have two cases: either three of the dice show one value and the last shows a different value, or all four dice show the same value. In the first case, there are six choices for the value of the dice which are the same and $\binom{4}{3}$ choice for which dice show that value. Then there are 5 choices for the last die. In total, there are $6\binom{4}{3}5 = 120$ possibilities. For the second case, there are 6 values that the last die can show. Consequently, the overall probability is, $\frac{120+6}{6^4} = \frac{126}{6^4} = \frac{7}{72}$.

5. [4] How many ways are there to arrange three indistinguishable rooks on a 6×6 board such that no two rooks are attacking each other? (Two rooks are attacking each other if and only if they are in the same row or the same column.)

Answer: $\boxed{2400}$ There are $6 \times 6 = 36$ possible places to place the first rook. Since it cannot be in the same row or column as the first, the second rook has $5 \times 5 = 25$ possible places, and similarly, the third rook has $4 \times 4 = 16$ possible places. However, the rooks are indistinguishable, so there are $3! = 6$ ways to reorder them. Therefore, the number of arrangements is $\frac{36 \times 25 \times 16}{6} = 2400$.

6. [5] $ABCD$ is a parallelogram satisfying $AB = 7$, $BC = 2$, and $\angle DAB = 120^\circ$. Parallelogram $ECFA$ is contained in $ABCD$ and is similar to it. Find the ratio of the area of $ECFA$ to the area of $ABCD$.

Answer: $\boxed{\frac{39}{67}}$ First, note that BD is the long diagonal of $ABCD$, and AC is the long diagonal of $ECFA$. Because the ratio of the areas of similar figures is equal to the square of the ratio of their side lengths, we know that the ratio of the area of $ECFA$ to the area of $ABCD$ is equal to the ratio $\frac{AC^2}{BD^2}$. Using law of cosines on triangle ABD , we have $BD^2 = AD^2 + AB^2 - 2(AD)(AB)\cos(120^\circ) = 2^2 + 7^2 - 2(2)(7)(-\frac{1}{2}) = 67$.

Using law of cosines on triangle ABC , we have $AC^2 = AB^2 + BC^2 - 2(AB)(BC)\cos(60^\circ) = 7^2 + 2^2 - 2(7)(2)(\frac{1}{2}) = 39$.

Finally, $\frac{AC^2}{BD^2} = \frac{39}{67}$.

7. [6] Find the number of ordered 2012-tuples of integers $(x_1, x_2, \dots, x_{2012})$, with each integer between 0 and 2011 inclusive, such that the sum $x_1 + 2x_2 + 3x_3 + \dots + 2012x_{2012}$ is divisible by 2012.

Answer: $\boxed{2012^{2011}}$ We claim that for any choice of $x_2, x_3, \dots, x_{2012}$, there is exactly one possible value of x_1 satisfying the condition. We have $x_1 + 2x_2 + \dots + 2012x_{2012} \equiv 0 \pmod{2012}$ or $x_1 \equiv -(2x_2 + \dots + 2012x_{2012}) \pmod{2012}$. Indeed, we see that the right hand side is always an integer between 0 and 2011, so x_1 must equal this number.

Now, there are 2012 choices for each of the 2011 variables x_2, \dots, x_{2012} , and each of the 2012^{2011} possible combinations gives exactly one valid solution, so the total number of 2012-tuples is 2012^{2011} .

8. [7] Let n be the 200th smallest positive real solution to the equation $x - \frac{\pi}{2} = \tan x$. Find the greatest integer that does not exceed $\frac{n}{2}$.

Answer: $\boxed{314}$ Drawing the graphs of the functions $y = x - \frac{\pi}{2}$ and $y = \tan x$, we may observe that the graphs intersect exactly once in each of the intervals $(\frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2})$ for each $k = 1, 2, \dots$. Hence, the 200th intersection has x in the range $(\frac{399\pi}{2}, \frac{401\pi}{2})$. At this intersection, $y = x - \frac{\pi}{2}$ is large, and thus, the intersection will be slightly less than $\frac{401\pi}{2}$. We have that $\lfloor \frac{401\pi}{4} \rfloor = \lfloor 100\pi + \frac{\pi}{4} \rfloor = \lfloor 314.16 + \frac{\pi}{4} \rfloor = 314$.

9. [7] Consider triangle ABC where $BC = 7$, $CA = 8$, and $AB = 9$. D and E are the midpoints of BC and CA , respectively, and AD and BE meet at G . The reflection of G across D is G' , and $G'E$ meets CG at P . Find the length PG .

Answer: $\boxed{\frac{\sqrt{145}}{9}}$ Observe that since G' is a reflection and $GD = \frac{1}{2}AG$, we have $AG = GG'$ and therefore, P is the centroid of triangle ACG' . Thus, extending CG to hit AB at F , $PG = \frac{1}{3}CG = \frac{2}{9}CF = \frac{2}{9}\sqrt{\frac{2(8^2+7^2)-9^2}{4}} = \frac{\sqrt{145}}{9}$ by the formula for the length of a median.

10. [8] Let α and β be reals. Find the least possible value of

$$(2 \cos \alpha + 5 \sin \beta - 8)^2 + (2 \sin \alpha + 5 \cos \beta - 15)^2.$$

Answer: $\boxed{100}$ Let the vector $\vec{v} = (2 \cos \alpha, 2 \sin \alpha)$ and $\vec{w} = (5 \sin \beta, 5 \cos \beta)$. The locus of ends of vectors expressible in the form $\vec{v} + \vec{w}$ are the points which are five units away from a point on the circle of radius two about the origin. The expression that we desire to minimize is the square of the distance from this point to $X = (8, 15)$. Thus, the closest distance from such a point to X is when the point is 7 units away from the origin along the segment from the origin to X . Thus, since X is 17 units away from the origin, the minimum is $10^2 = 100$.

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1. [5]

Answer: 10 The prime numbers under 30 are: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29. There are 10 in total.

2. [5]

Answer: 180 Albert's choice of burgers, sides, and drinks are independent events. Therefore, we can use the Multiplication Principle of Counting Independent Events to find the total number of different meals that Albert can get is:

$$5 \times 3 \times 12 = 180$$

3. [5]

Answer: 199 π The area of a circle of radius 100 is $100^2\pi$, and the area of a circle of radius 99 is $99^2\pi$. Therefore, the area of the region between them is $(100^2 - 99^2)\pi = (100 + 99)(100 - 99)\pi = 199\pi$.

4. [6]

Answer: 10 $\angle CBD = \angle CDB$ because $BC = CD$. Notice that $\angle BCD = 80 + 50 + 30 = 160$, so $\angle CBD = \angle CDB = 10$.

5. [6]

Answer: $\frac{20}{3}$

$$\begin{aligned} 20 &= 4a^2 + 9b^2 \\ 20 + 12ab &= 4a^2 + 12ab + 9b^2 \\ 20 + 12ab &= (2a + 3b)^2 \\ 20 + 12ab &= 100 \\ 12ab &= 80 \\ ab &= \frac{20}{3} \end{aligned}$$

6. [6]

Answer: 25502400

$$\begin{aligned} &(1^3 + 3 \cdot 1^2 + 3 \cdot 1) + (2^3 + 3 \cdot 2^2 + 3 \cdot 2) + \cdots + (99^3 + 3 \cdot 99^2 + 3 \cdot 99) \\ &= (1^3 + 3 \cdot 1^2 + 3 \cdot 1 + 1) + (2^3 + 3 \cdot 2^2 + 3 \cdot 2 + 1) + \cdots + (99^3 + 3 \cdot 99^2 + 3 \cdot 99 + 1) - 99 \\ &= (1 + 1)^3 + (2 + 1)^3 + \cdots + (99 + 1)^3 - 99 \\ &= 1^3 + (1 + 1)^3 + (2 + 1)^3 + \cdots + (99 + 1)^3 - 100 \\ &= \left(\frac{100(100 + 1)}{2} \right)^2 - 100 \\ &= 25502400 \end{aligned}$$

7. [7]

Answer: $\boxed{335}$

$$\begin{aligned}a_n &= 1 + 3 + 3^2 + \dots + 3^n \\3a_n &= 3 + 3^2 + 3^3 + \dots + 3^{n+1} \\3a_n &= 3^{n+1} - 1 + S_n \\a_n &= \frac{3^{n+1} - 1}{2}\end{aligned}$$

Because $\gcd(2, 7) = 1$, to demonstrate that $7|a_n = \frac{3^{n+1}-1}{2}$ is equivalent to showing that $3^{n+1} - 1 \equiv 0 \pmod{7}$

However $3^{n+1} - 1 \equiv 0 \pmod{7} \implies n + 1 \equiv 0 \pmod{6}$, hence the values of $n \leq 2012$ are 5, 11, 17, ..., 2009, which is in total

$$\frac{2009 + 1}{6} = 335$$

possible terms.

8. [7]

Answer: $\boxed{\frac{169}{425}}$ There are 13 cards of each suit. The probability that the second card has a different suit than the first is $\frac{3 \cdot 13}{52-1}$. The probability that the third card has a different suit than the first and second card is $\frac{2 \cdot 13}{52-2}$

Since the two events need to both occur, we use the multiplicative rule:

$$\begin{aligned}\frac{3 \cdot 13}{52-1} \cdot \frac{2 \cdot 13}{52-2} \\= \frac{169}{425}\end{aligned}$$

9. [7]

Answer: $\boxed{4}$ Because 23 is odd, we must have an odd number of odd numbers in our set. Since the smallest odd composite number is 9, we cannot have more than 2 odd numbers, as otherwise the sum would be at least 27. Therefore, the set has exactly one odd number.

The only odd composite numbers less than 23 are 9, 15, and 21. If we include 21, then the rest of the set must include composite numbers that add up to 2, which is impossible. If we include 15, then the rest of the set must include distinct even composite numbers that add up to 8. The only possibility is the set $\{8\}$. If we include 9, the rest of the set must contain distinct even composite numbers that add to 14. The only possibilities are $\{14\}$, $\{4, 10\}$, and $\{6, 8\}$. We have exhausted all cases, so there are a total of 4 sets.

10. [8]

Answer: $\boxed{1006}$ Taking both sides modulo 2012, we see that $a_n \equiv a_{n-1} + n \pmod{2012}$. Therefore, $a_{2012} \equiv a_{2011} + 2012 \equiv a_{2010} + 2011 + 2012 \equiv \dots \equiv 1 + 2 + \dots + 2012 \equiv \frac{(2012)(2013)}{2} \equiv (1006)(2013) \equiv (1006)(1) \equiv 1006 \pmod{2012}$.

11. [8]

Answer: $\boxed{45}$ The number of zeroes that $n!$ ends with is the largest power of 10 dividing $n!$. The exponent of 5 dividing $n!$ exceeds the exponent of 2 dividing $n!$, so we simply seek the exponent of 5 dividing $n!$.

For a number less than 125, this exponent is just the number of multiples of 5, but not 25, less than n plus twice the number of multiples of 25 less than n .

Counting up, we see that $24!$ ends with 4 zeroes while $25!$ ends with 6 zeroes, so $n!$ cannot end with 5 zeroes. Continuing to count up, we see that the smallest n such that $n!$ ends with 10 zeroes is 45.

12. [8]

Answer: $\boxed{\frac{2\pi-\sqrt{3}}{2}}$ Let A_Δ be the equilateral triangle. Let A_1 be the area of the region outside of the equilateral triangle but inside the second and third circles. Define A_2, A_3 analogously. We have $A_1 = A_2 = A_3 = A_k =$

$$\left(\frac{1^2 \cdot \pi}{3} - \frac{1^2 \cdot \sin 120}{2} \right) = \frac{4\pi - 3\sqrt{3}}{12},$$

and

$$A_\Delta = \frac{1^2 \cdot \sin 60}{2} = \frac{\sqrt{3}}{4}.$$

Thus, the total area is

$$A_1 + A_2 + A_3 + A_\Delta = 3 \cdot \frac{4\pi - 3\sqrt{3}}{12} + \frac{\sqrt{3}}{4} = \frac{2\pi - \sqrt{3}}{2}.$$

13. [9]

Answer: $\boxed{6\sqrt{3}}$ Let AB be x . Then, $AC = 18 - 7 - x = 11 - x$. Using the law of cosines gives:

$$x^2 + (11 - x)^2 - 2x(11 - x) \cos 60^\circ = 7^2.$$

$$3x^2 - 33x + 72 = 0.$$

$$x = 3 \text{ or } 8.$$

Therefore, $AB = 8$ and $AC = 3$ or $AB = 3$ and $AC = 8$. In both cases, the area of the triangle is:

$$\frac{1}{2} \cdot 8 \cdot 3 \sin 60^\circ = \boxed{6\sqrt{3}}.$$

14. [9]

Answer: $\boxed{32}$ First, we can fix the position of the 1. Then, by the condition that the numbers are increasing along each arc from 1, we know that the 2 must be adjacent to the 1; so we have two options for its placement. Similarly, we have two options for placing each of 3,4,5,6 in that order. Finally, the 7 must go in the remaining space, for a total of $2^5 = 32$ orderings.

15. [9]

Answer: $\boxed{49\pi}$ To find the region in question, we want to find (a, b) such that the discriminant of the quadratic is not positive. In other words, we want

$$4(a + b - 7)^2 - 4(a)(2b) \leq 0 \Leftrightarrow a^2 + b^2 - 7a - 7b + 49 \leq 0 \Leftrightarrow (a - 7)^2 + (b - 7)^2 \leq 49,$$

which is a circle of radius 7 centered at $(7, 7)$ and hence has area 49π .

16. [10]

Answer: $\boxed{11}$ We divide this problem into cases based on the relative position of the two red beads:

- They are adjacent. Then, there are 4 possible placements of the green and blue beads: GGBB, GBBG, GBGB, BGGB.
- They are 1 bead apart. Then, there are two choices for the bead between them and 2 choices for the other bead of that color, for a total of 4.
- They are opposite. Then, there are three choices for the placement of the green beads.

This gives a total of 11 arrangements.

17. [10]

Answer: $\left(\frac{5}{4}, \frac{1}{2}, \frac{1}{4}\right)$ We can rewrite the equation in terms of $\ln 2, \ln 3, \ln 5$, to get

$$3 \ln 2 + 3 \ln 3 + 2 \ln 5 = \ln 5400 = px + qy + rz = (p + 3q + r) \ln 2 + (p + 2q + 3r) \ln 3 + (p + q + r) \ln 5.$$

Consequently, since p, q, r are rational we want to solve the system of equations $p + 3q + r = 3, p + 2q + 3r = 3, p + q + r = 2$, which results in the ordered triple $\left(\frac{5}{4}, \frac{1}{2}, \frac{1}{4}\right)$.

18. [10]

Answer: $1 + \frac{\sqrt{3}}{3}$ DBC is a right triangle with hypotenuse DC . Since $DE = EC, E$ is the midpoint of this right triangle's hypotenuse, and it follows that E is the circumcenter of the triangle. It follows that $BE = DE = CE$, as these are all radii of the same circle. A similar argument shows that $BD = DE = AD$.

Thus, $BD = DE = CE$, and triangle BDE is equilateral. So, $\angle DBE = \angle BED = \angle EDB = 60^\circ$. We have $\angle BEC = 180^\circ - \angle BED = 120^\circ$. Because $BE = CE$, triangle BEC is isosceles and $\angle ECB = 30^\circ$. Therefore, DBC is a right triangle with $\angle DBC = 90^\circ, \angle BCD = 30^\circ$, and $\angle CDB = 60^\circ$. This means that $CD = \frac{2}{\sqrt{3}}BC$. Combined with $CD = \frac{2}{3}$, we have $BC = \frac{\sqrt{3}}{3}$. Similarly, $AB = \frac{\sqrt{3}}{3}$, so $AB + AC = 1 + \frac{\sqrt{3}}{3}$.

19. [11]

Answer: $\boxed{21}$ First, note that $525 = 3 \times 7 \times 5 \times 5$. Then, taking the equation modulo 7 gives that $7 \mid x$; let $x = 7x'$ for some nonnegative integer x' . Similarly, we can write $y = 5y'$ and $z = 3z'$ for some nonnegative integers y', z' . Then, after substitution and division of both sides by 105, the given equation is equivalent to $x' + y' + z' = 5$. This is the same as the problem of placing 2 dividers among 5 balls, so is $\binom{7}{2} = 21$.

20. [11]

Answer: $\frac{81^{10}}{82^{10}}$ The teacher will leave if the students from each pair are either both present or both not present; the probability that both are present is $\frac{81}{100}$ and the probability that neither are present is $\frac{1}{100}$. If the teacher leaves, then the probability that both students in any given pair did not get lost is $\frac{\frac{81}{100}}{\frac{81}{100} + \frac{1}{100}} = \frac{81}{82}$. Since there are ten pairs, the overall probability is $\left(\frac{81}{82}\right)^{10} = \frac{81^{10}}{82^{10}}$.

21. [11]

Answer: $\frac{80\sqrt{3}}{9}$ By shared bases, we know that

$$[EFC] = \left(\frac{5}{6}\right) [AEC] = \left(\frac{5}{6}\right) \left(\frac{4}{5}\right) [ADC] = \left(\frac{5}{6}\right) \left(\frac{4}{5}\right) \left(\frac{2}{3}\right) [ABC].$$

By Heron's formula, we find that $[ABC] = \sqrt{(15)(8)(2)(5)} = 20\sqrt{3}$, so $[AEC] = \frac{80\sqrt{3}}{9}$.

22. [12]

Answer: $\boxed{\frac{1}{2}}$ Let X_n be the n th number rolled. The number formed, $0.\overline{X_1X_2} \dots$, is simply $\sum_{n=1}^{\infty} \frac{X_n}{10^n}$. By linearity of expectation, the expected value is $\sum_{n=1}^{\infty} \mathbb{E}\left(\frac{X_n}{10^n}\right) = \sum_{n=1}^{\infty} \frac{\mathbb{E}(X_n)}{10^n}$. However, the rolls are independent: for all n , $\mathbb{E}(X_n) = \frac{1}{6}(1+2+3+5+7+9) = \frac{9}{2}$. So, our answer is $\frac{9}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{10^n} = \frac{9}{2} \cdot \frac{1}{9} = \frac{1}{2}$.

23. [12]

Answer: $\boxed{\frac{32}{11}}$ Since AP is a tangent to Ω , we know that $\angle PAB = \angle PCA$, so $\triangle PAB \sim \triangle PCA$, so we get that

$$\frac{PB}{PA} = \frac{4}{7} = \frac{PA}{PB+6}.$$

Solving, we get that $7PB = 4PA$, so

$$4(PB+6) = 7PA = \frac{49}{4}PB \Rightarrow \frac{33}{4}PB = 24 \Rightarrow PB = \frac{32}{11}.$$

24. [12]

Answer: $\boxed{105}$ Since 4 pieces of candy are distributed, there must be exactly 8 children who do not receive any candy; since no two consecutive children do receive candy, the 8 who do not must consist of 4 groups of consecutive children. We divide into cases based on the sizes of these groups:

- $\{5, 1, 1, 1\}$: there are 12 places to begin the group of 5 children who do not receive any candy
- $\{4, 2, 1, 1\}$: there are 12 places to begin the group of 4 children who do not receive candy and then 3 choices for the group of 2 children which does not receive candy, for a total of 36 choices
- $\{3, 3, 1, 1\}$: these 8 children can either be bunched in the order 3,3,1,1, or in the order 3,1,3,1; the first has 12 positions in which to begin the first group of 3 non-candy receiving children and the second has 6 possibilities (due to symmetry), for a total of 18
- $\{3, 2, 2, 1\}$: there are 12 places to begin the group of 3 children who do not receive candy and then 3 choices for the group of 1 child which does not receive candy, for a total of 36 choices
- $\{2, 2, 2, 2\}$: there are $12/4 = 3$ ways in which this can occur

This gives a total of $12 + 36 + 18 + 36 + 3 = 105$

25. [13]

Answer: $\boxed{\frac{83}{80}}$ Using the Law of Sines, we have

$$\frac{\sin^2 B + \sin^2 C - \sin^2 A}{\sin B \sin C} = \frac{\sin B}{\sin C} + \frac{\sin C}{\sin B} - \frac{\sin A \sin A}{\sin B \sin C} = \frac{AC}{AB} + \frac{AB}{AC} - \frac{BC}{AC} \frac{BC}{AB} = \frac{83}{80}.$$

26. [13]

Answer: $\boxed{2^{2013} - 6036}$ Let $y_n = x_n - 2^{n+1}$. Note that

$$x_{n+1} = 2x_n - x_{n-1} + 2^n \Leftrightarrow y_{n+1} = 2y_n - y_{n-1} \Leftrightarrow y_{n+1} - y_n = y_n - y_{n-1}.$$

Using the values for x_1, x_2 , we get that $y_1 = -3$ and $y_2 = -6$, so $y_{n+1} - y_n = (-6) - (-3) = -3$. By induction, $y_n = -3n$. Then, we get that $x_n = 2^{n+1} - 3n$, so $x_{2012} = 2^{2013} - 6036$.

27. [13]

Answer: $\boxed{1755}$ We write this as $(a-b)(a+b) + (a-b)(c) = (a-b)(a+b+c) = 2012$. Since a, b, c are positive integers, $a-b < a+b+c$. So, we have three possibilities: $a-b = 1$ and $a+b+c = 2012$, $a-b = 2$ and $a+b+c = 1006$, and $a-b = 4$ and $a+b+c = 503$.

The first solution gives $a = b + 1$ and $c = 2011 - 2b$, so b can range from 1 through 1005, which determines a and c completely.

Similarly, the second solution gives $a = b + 2$ and $c = 1004 - 2b$, so b can range from 1 through 501.

Finally, the third solution gives $a = b + 4$ and $c = 499 - 2b$, so b can range from 1 through 249.

Hence, the total number of solutions is $1005 + 501 + 249 = 1755$.

28. [15]

Answer: $\boxed{(0, \frac{1+\sqrt{2}}{2}]}$ Suppose that $k = \frac{ab+b^2}{a^2+b^2}$ for some positive real a, b . We claim that k lies in $(0, \frac{1+\sqrt{2}}{2}]$. Let $x = \frac{a}{b}$. We have that $\frac{ab+b^2}{a^2+b^2} = \frac{\frac{a}{b}+1}{(\frac{a}{b})^2+1} = \frac{x+1}{x^2+1}$. Thus, $x+1 = k(x^2+1)$, so the quadratic $kt^2 - t + k - 1 = 0$ has a positive real root. Thus, its discriminant must be nonnegative, so $1^2 \geq 4(k-1)(k) \implies (2k-1)^2 \leq 2$, which implies $\frac{1-\sqrt{2}}{2} \leq k \leq \frac{1+\sqrt{2}}{2}$. Since $x > 0$, we also have $k > 0$, so we know that k must lie in $(0, \frac{1+\sqrt{2}}{2}]$.

Now, take any k in the interval $(0, \frac{1+\sqrt{2}}{2}]$. We thus know that $1^2 \geq 4k(k-1)$, so the quadratic $kt^2 - t + k - 1 = 0$ has a positive solution, $\frac{1+\sqrt{1-4k(k-1)}}{2k}$. Call this solution x . Then $k(x^2+1) = x+1$, so $\frac{x+1}{x^2+1} = k$. If we set $a = x$ and $b = 1$, we get that $\frac{ab+b^2}{a^2+b^2} = k$. Thus, the set of all attainable values of $\frac{ab+b^2}{a^2+b^2}$ is the interval $(0, \frac{1+\sqrt{2}}{2}]$.

29. [15]

Answer: $\boxed{-\frac{25+5\sqrt{17}}{8}}$ First, note that $(x+1)(2x+1)(3x+1)(4x+1) = ((x+1)(4x+1))((2x+1)(3x+1)) = (4x^2+5x+1)(6x^2+5x+1) = (5x^2+5x+1-x^2)(5x^2+5x+1+x^2) = (5x^2+5x+1)^2 - x^4$. Therefore, the equation is equivalent to $(5x^2+5x+1)^2 = 17x^4$, or $5x^2+5x+1 = \pm\sqrt{17}x^2$.

If $5x^2+5x+1 = \sqrt{17}x^2$, then $(5-\sqrt{17})x^2+5x+1 = 0$. The discriminant of this is $25-4(5-\sqrt{17}) = 5+4\sqrt{17}$, so in this case, there are two real roots and they sum to $-\frac{5}{5-\sqrt{17}} = -\frac{25+5\sqrt{17}}{8}$.

If $5x^2+5x+1 = -\sqrt{17}x^2$, then $(5+\sqrt{17})x^2+5x+1 = 0$. The discriminant of this is $25-4(5+\sqrt{17}) = 5-4\sqrt{17}$. This is less than zero, so there are no real solutions in this case. Therefore, the sum of all real solutions to the equation is $-\frac{25+5\sqrt{17}}{8}$.

30. [15]

Answer: $\boxed{\frac{3}{7}}$ It suffices to assume that the monkey starts all over as soon as he has typed a string that ends in no prefix of either abc or aaa . For instance, if the monkey gets to abb we can throw these out because there's no way to finish one of those strings from this without starting all over.

Now, we draw the tree of all possible intermediate stages under this assumption; there are not many possibilities. The paths from the root "a" are:

- a- aa- aaa
- a- aa- aab- aabc
- a- ab- abc

The first and last possibilities have probability $1/27$ each and the middle one has probability $1/81$, so in total the probability of getting the first before the second or the third is $\frac{1/27}{1/27+1/81+1/27} = \frac{3}{7}$.

31. [17]

Answer: $\frac{3\sqrt{3}}{20}$ We place the points in the coordinate plane. We let $A = (0, 0, \frac{\sqrt{6}}{3})$, $B = (0, \frac{\sqrt{3}}{3}, 0)$, $C = (-\frac{1}{2}, -\frac{\sqrt{3}}{6}, 0)$, and $D = (\frac{1}{2}, \frac{\sqrt{3}}{6}, 0)$. The point P is the origin, while M is $(0, 0, \frac{\sqrt{6}}{6})$. The line through B and M is the line $x = 0$, $y = \frac{\sqrt{3}}{3} - z\sqrt{2}$. The plane through A , C , and D has equation $z = 2\sqrt{2}y + \sqrt{\frac{2}{3}}$. The coordinates of Q are the coordinates of the intersection of this line and this plane. Equating the equations and solving for y and z , we see that $y = -\frac{1}{5\sqrt{3}}$ and $z = \frac{\sqrt{6}}{5}$, so the coordinates of Q are $(0, -\frac{1}{5\sqrt{3}}, \frac{\sqrt{6}}{5})$.

Let N be the midpoint of CD , which has coordinates $(0, -\frac{\sqrt{3}}{6}, 0)$. By the distance formula, $QN = \frac{3\sqrt{3}}{10}$. Thus, the area of QCD is $\frac{QN \cdot CD}{2} = \frac{3\sqrt{3}}{20}$.

32. [17]

Answer: 2530 Note that $\phi(23) = 22$ and $\phi(22) = 10$, so if $\text{lcm}(23, 22, 10) = 2530|k$ then $f(n+k) \equiv f(n) \pmod{23}$ is always true. We show that this is necessary as well.

Choosing $n \equiv 0 \pmod{23}$, we see that $k \equiv 0 \pmod{23}$. Thus $n+k \equiv n \pmod{23}$ always, and we can move to the exponent by choosing n to be a generator modulo 23:

$$(n+k)^{n+k} \equiv n^n \pmod{22}$$

The choice of n here is independent of the choice $\pmod{23}$ since 22 and 23 are coprime. Thus we must have again that $22|k$, by choosing $n \equiv 0 \pmod{22}$. But then $n+k \equiv n \pmod{11}$ always, and we can go to the exponent modulo $\phi(11) = 10$ by choosing n a generator modulo 11:

$$n+k \equiv n \pmod{10}$$

From here it follows that $10|k$ as well. Thus $2530|k$ and 2530 is the minimum positive integer desired.

33. [17]

Answer: $\tan^{-1}(\frac{1009}{1005})$ As per usual with reflection problems instead of bouncing off the sides of a 1×1 square we imagine the ball to travel in a straight line from origin in an infinite grid of 1×1 squares, "bouncing" every time it meets a line $x = m$ or $y = n$. Let the lattice point it first meets after leaving the origin be (a, b) , so that $b > a$. Note that a and b are coprime, otherwise the ball will reach a vertex before the 2012th bounce. We wish to minimize the slope of the line to this point from origin, which is b/a .

Now, the number of bounces up to this point is $a - 1 + b - 1 = a + b - 2$, so the given statement is just $a + b = 2014$. To minimize b/a with a and b relatively prime, we must have $a = 1005$, $b = 1009$, so that the angle is $\tan^{-1}(\frac{1009}{1005})$.

34. [20]

Answer: 15612 Note that the number of integers between 1 and 2012 that have n as a divisor is $\lfloor \frac{2012}{n} \rfloor$. Therefore, if we sum over the possible divisors, we see that the sum is equivalent to $\sum_{d=1}^{2012} \lfloor \frac{2012}{d} \rfloor$. This can be approximated by $\sum_{d=1}^{2012} \frac{2012}{d} = 2012 \sum_{d=1}^{2012} \frac{1}{d} \approx 2012 \ln(2012)$. As it turns out, $2012 \ln(2012) \approx 15300$, which is worth 18 points. Using the very rough approximation $\ln(2012) \approx 7$ still gives 14 points.

35. [20]

Answer: 3.1415875473 The answer is $1006 \sin \frac{\pi}{1006}$. Approximating directly by $\pi = 3.1415\dots$ is worth only 3 points.

Using the third-degree Taylor polynomial for \sin we can approximate $\sin x \approx x - \frac{x^3}{6}$. This gives an answer of 3.1415875473 worth full points. If during the calculation we use the approximation $\pi^3 \approx 30$, this gives an answer worth 9 points.

36. [20]

Answer: Submit solution

HMMT November 2012
Saturday 10 November 2012
Team Round

1. [3] Find the number of integers between 1 and 200 inclusive whose distinct prime divisors sum to 16. (For example, the sum of the distinct prime divisors of 12 is $2 + 3 = 5$.)

Answer: 6 The primes less than 16 are 2, 3, 5, 7, 11, and 13. We can write 16 as the sum of such primes in three different ways and find the integers less than 200 with those prime factors:

- $13 + 3: 3 \cdot 13 = 39$ and $3^2 \cdot 13 = 117$.
- $11 + 5: 5 \cdot 11 = 55$
- $11 + 3 + 2: 2 \cdot 3 \cdot 11 = 66, 2^2 \cdot 3 \cdot 11 = 132,$ and $2 \cdot 3^2 \cdot 11 = 198$.

There are therefore 6 numbers less than 200 whose prime divisors sum to 16.

2. [5] Find the number of ordered triples of divisors (d_1, d_2, d_3) of 360 such that $d_1 d_2 d_3$ is also a divisor of 360.

Answer: 800 Since $360 = 2^3 \cdot 3^2 \cdot 5$, the only possible prime divisors of d_i are 2, 3, and 5, so we can write $d_i = 2^{a_i} \cdot 3^{b_i} \cdot 5^{c_i}$, for nonnegative integers $a_i, b_i,$ and c_i . Then, $d_1 d_2 d_3 | 360$ if and only if the following three inequalities hold.

$$\begin{aligned}a_1 + a_2 + a_3 &\leq 3 \\b_1 + b_2 + b_3 &\leq 2 \\c_1 + c_2 + c_3 &\leq 1\end{aligned}$$

Now, one can count that there are 20 assignments of a_i that satisfy the first inequality, 10 assignments of b_i that satisfy the second inequality, and 4 assignments of c_i that satisfy the third inequality, for a total of 800 ordered triples (d_1, d_2, d_3) .

(Alternatively, instead of counting, it is possible to show that the number of nonnegative-integer triples (a_1, a_2, a_3) satisfying $a_1 + a_2 + a_3 \leq n$ equals $\binom{n+3}{3}$, since this is equal to the number of nonnegative-integer quadruplets (a_1, a_2, a_3, a_4) satisfying $a_1 + a_2 + a_3 + a_4 = n$.)

3. [6] Find the largest integer less than 2012 all of whose divisors have at most two 1's in their binary representations.

Answer: 1536 Call a number *good* if all of its positive divisors have at most two 1's in their binary representations. Then, if p is an odd prime divisor of a good number, p must be of the form $2^k + 1$. The only such primes less than 2012 are 3, 5, 17, and 257, so the only possible prime divisors of n are 2, 3, 5, 17, and 257.

Next, note that since $(2^i + 1)(2^j + 1) = 2^{i+j} + 2^i + 2^j + 1$, if either i or j is greater than 1, then there will be at least 3 1's in the binary representation of $(2^i + 1)(2^j + 1)$, so $(2^i + 1)(2^j + 1)$ cannot divide a good number. On the other hand, if $i = j = 1$, then $(2^1 + 1)(2^1 + 1) = 9 = 2^3 + 1$, so 9 is a good number and can divide a good number. Finally, note that since multiplication by 2 in binary just appends additional 0s, so if n is a good number, then $2n$ is also a good number.

It therefore follows that any good number less than 2012 must be of the form $c \cdot 2^k$, where c belongs to $\{1, 3, 5, 9, 17, 257\}$ (and moreover, all such numbers are good). It is then straightforward to check that the largest such number is $1536 = 3 \cdot 2^9$.

4. [3] Let π be a permutation of the numbers from 2 through 2012. Find the largest possible value of $\log_2 \pi(2) \cdot \log_3 \pi(3) \cdots \log_{2012} \pi(2012)$.

Answer: $\boxed{1}$ Note that

$$\begin{aligned}\prod_{i=2}^{2012} \log_i \pi(i) &= \prod_{i=2}^{2012} \frac{\log \pi(i)}{\log i} \\ &= \frac{\prod_{i=2}^{2012} \log \pi(i)}{\prod_{i=2}^{2012} \log i} \\ &= 1,\end{aligned}$$

where the last equality holds since π is a permutation of the numbers 2 through 2012.

5. [4] Let π be a randomly chosen permutation of the numbers from 1 through 2012. Find the probability that $\pi(\pi(2012)) = 2012$.

Answer: $\boxed{\frac{1}{1006}}$ There are two possibilities: either $\pi(2012) = 2012$ or $\pi(2012) = i$ and $\pi(i) = 2012$ for $i \neq 2012$. The first case occurs with probability $2011!/2012! = 1/2012$, since any permutation on the remaining 2011 elements is possible. Similarly, for any fixed i , the second case occurs with probability $2010!/2012! = 1/(2011 \cdot 2012)$, since any permutation on the remaining 2010 elements is possible. Since there are 2011 possible values for i , and since our two possibilities are disjoint, the overall probability that $\pi(\pi(2012)) = 2012$ equals

$$\frac{1}{2012} + (2011) \frac{1}{2011 \cdot 2012} = \frac{1}{1006}.$$

6. [6] Let π be a permutation of the numbers from 1 through 2012. What is the maximum possible number of integers n with $1 \leq n \leq 2011$ such that $\pi(n)$ divides $\pi(n+1)$?

Answer: $\boxed{1006}$ Since any proper divisor of n must be less than or equal to $n/2$, none of the numbers greater than 1006 can divide any other number less than or equal to 2012. Since there are at most 1006 values of n for which $\pi(n) \leq 1006$, this means that there can be at most 1006 values of n for which $\pi(n)$ divides $\pi(n+1)$.

On the other hand, there exists a permutation for which $\pi(n)$ divides $\pi(n+1)$ for exactly 1006 values of n , namely the permutation:

$$(1, 2, 2^2, 2^3, \dots, 2^{10}, 3, 2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, \dots, 2^9 \cdot 3, 5, \dots)$$

Formally, for each odd number $\ell < 2012$, we construct the sequence $\ell, 2\ell, 4\ell, \dots, 2^k\ell$, where k is the largest integer such that $2^k\ell \leq 2012$. We then concatenate all of these sequences to form a permutation of the numbers 1 through ℓ (note that no number occurs in more than one sequence). It follows that if $\pi(n) \leq 1006$, then $\pi(n+1)$ will equal $2\pi(n)$, and therefore $\pi(n)$ will divide $\pi(n+1)$ for all 1006 values of n satisfying $1 \leq \pi(n) \leq 1006$.

7. [8] Let $A_1 A_2 \dots A_{100}$ be the vertices of a regular 100-gon. Let π be a randomly chosen permutation of the numbers from 1 through 100. The segments $A_{\pi(1)} A_{\pi(2)}, A_{\pi(2)} A_{\pi(3)}, \dots, A_{\pi(99)} A_{\pi(100)}, A_{\pi(100)} A_{\pi(1)}$ are drawn. Find the expected number of pairs of line segments that intersect at a point in the interior of the 100-gon.

Answer: $\boxed{\frac{4850}{3}}$ By linearity of expectation, the expected number of total intersections is equal to the sum of the probabilities that any given intersection will occur.

Let us compute the probability $p_{i,j}$ that $A_{\pi(i)} A_{\pi(i+1)}$ intersects $A_{\pi(j)} A_{\pi(j+1)}$ (where $1 \leq i, j \leq 100$, $i \neq j$, and indices are taken modulo 100). Note first that if $j = i + 1$, then these two segments share vertex $\pi(i + 1)$ and therefore will not intersect in the interior of the 100-gon; similarly, if $i = j + 1$, these two segments will also not intersect. On the other hand, if $\pi(i), \pi(i + 1), \pi(j)$, and $\pi(j + 1)$ are all distinct, then there is a $1/3$ chance that $A_{\pi(i)} A_{\pi(i+1)}$ intersects $A_{\pi(j)} A_{\pi(j+1)}$; in any set of four

points that form a convex quadrilateral, exactly one of the three ways of pairing the points into two pairs (two pairs of opposite sides and the two diagonals) forms two segments that intersect inside the quadrilateral (namely, the two diagonals).

Now, there are 100 ways to choose a value for i , and 97 ways to choose a value for j which is not i , $i + 1$, or $i - 1$, there are 9700 ordered pairs (i, j) where $p_{i,j} = 1/3$. Since each pair is counted twice (once as (i, j) and once as (j, i)), there are $9700/2 = 4850$ distinct possible intersections, each of which occurs with probability $1/3$, so the expected number of intersections is equal to $4850/3$.

8. [4] ABC is a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. The altitude from A to BC is extended to meet the circumcircle of ABC at D . Find AD .

Answer: $\boxed{\frac{63}{4}}$ Let the altitude from A to BC meet BC at E . The altitude AE has length 12; one way to see this is that it splits the triangle ABC into a $9 - 12 - 15$ right triangle and a $5 - 12 - 13$ right triangle; from this, we also know that $BE = 9$ and $CE = 5$.

Now, by Power of a Point, $AE \cdot DE = BE \cdot CE$, so $DE = (BE \cdot CE)/AE = (9 \cdot 5)/(12) = 15/4$. It then follows that $AD = AE + DE = 63/4$.

9. [5] Triangle ABC satisfies $\angle B > \angle C$. Let M be the midpoint of BC , and let the perpendicular bisector of BC meet the circumcircle of $\triangle ABC$ at a point D such that points A, D, C , and B appear on the circle in that order. Given that $\angle ADM = 68^\circ$ and $\angle DAC = 64^\circ$, find $\angle B$.

Answer: $\boxed{86^\circ}$ Extend DM to hit the circumcircle at E . Then, note that since $ADEB$ is a cyclic quadrilateral, $\angle ABE = 180^\circ - \angle ADE = 180^\circ - \angle ADM = 180^\circ - 68^\circ = 112^\circ$.

We also have that $\angle MEC = \angle DEC = \angle DAC = 64^\circ$. But now, since M is the midpoint of BC and since $EM \perp BC$, triangle BEC is isosceles. This implies that $\angle BEM = \angle MEC = 64^\circ$, and $\angle MBE = 90^\circ - \angle MEB = 26^\circ$. It follows that $\angle B = \angle ABE - \angle MBE = 112^\circ - 26^\circ = 86^\circ$.

10. [6] Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. Points A', B', C' are such that $B'C'$ is tangent to the circumcircle of $\triangle ABC$ at A , $C'A'$ is tangent to the circumcircle at B , and $A'B'$ is tangent to the circumcircle at C . Find the length $B'C'$.

Answer: $\boxed{\frac{80}{3}}$ Note that by equal tangents, $B'A = B'C$, $C'A = C'B$, and $A'B = A'C$. Moreover, since the line segments $A'B'$, $B'C'$, and $C'A'$ are tangent to the circumcircle of ABC at C , A , and B respectively, we have that $\angle A'BC = \angle A'CB = \angle A$, $\angle B'AC = \angle B'CA = \angle B$, and $\angle C'BA = \angle C'AB = \angle C$. By drawing the altitudes of the isosceles triangles $BC'A$ and $AC'B$, we therefore have that $C'A = 2/\cos C$ and $B'A = 3/\cos B$.

Now, by the Law of Cosines, we have that

$$\begin{aligned}\cos B &= \frac{a^2 + c^2 - b^2}{2ac} = \frac{25 + 16 - 36}{2(5)(4)} = \frac{1}{8} \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab} = \frac{25 + 36 - 16}{2(5)(6)} = \frac{3}{4}.\end{aligned}$$

Therefore,

$$B'C' = C'A + B'A = 2\left(\frac{4}{3}\right) + 3(8) = \frac{80}{3}.$$

HMMT November 2012
Saturday 10 November 2012
Theme Round

1. [3] If $4^{4^4} = \sqrt[128]{2^{2^{2^n}}}$, find n .

Answer: 4 We rewrite the left hand side as

$$(2^2)^{4^4} = 2^{2 \cdot 4^4} = 2^{2^9},$$

and the right hand side as

$$(2^{2^{2^n}})^{\frac{1}{128}} = 2^{2^{(2^n-7)}}.$$

Equating, we find $2^n - 7 = 9$, yielding $n = 4$.

2. [4] If $x^x = 2012^{2012^{2013}}$, find x .

Answer: 2012^{2012} We have

$$2012^{2012^{2013}} = 2012^{2012 \cdot 2012^{2012}} = (2012^{2012})^{2012^{2012}}.$$

Thus, $x = 2012^{2012}$.

3. [4] Find the smallest positive integer n such that $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_n > 3^{3^{3^3}}$. (The notation $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_n$ is used to denote a power tower with n 2's. For example, $\underbrace{2^{2^{\cdot^{\cdot^2}}}}_4$ would equal $2^{2^{2^2}}$.)

Answer: 6 Clearly, $n \geq 5$. When we take $n = 5$, we have

$$2^{2^{2^{2^2}}} = 2^{2^{16}} < 3^{3^{27}} = 3^{3^{3^3}}.$$

On the other hand, when $n = 6$, we have

$$2^{2^{2^{2^{2^2}}}} = 2^{2^{65536}} = 4^{2^{65535}} > 4^{4^{27}} > 3^{3^{27}} = 3^{3^{3^3}}.$$

Our answer is thus $n = 6$.

4. [7] Find the sum of all real solutions for x to the equation $(x^2 + 2x + 3)^{(x^2 + 2x + 3)^{(x^2 + 2x + 3)}} = 2012$.

Answer: -2 When $y = x^2 + 2x + 3$, note that there is a unique real number y such that $y^{y^y} = 2012$ because y^{y^y} is increasing in y . The sum of the real distinct solutions of the equation $x^2 + 2x + 3 = y$ is -2 by Vieta's Formulae as long as $2^2 + 4(y - 3) > 0$, which is equivalent to $y > 2$. This is easily seen to be the case; therefore, our answer is -2 .

5. [7] Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^4 3^{2^1}}}$ repeatedly until a single base 12 digit remains. Find this digit.

Answer: 4 For a positive integer n , let $s(n)$ be the sum of digits when n is expressed in base 12. We claim that $s(n) \equiv n \pmod{11}$ for all positive integers n . Indeed, if $n = d_k 12^k + d_{k-1} 12^{k-1} + \dots + d_0$ with each d_i an integer between 0 and 11, inclusive, because $12 \equiv 1 \pmod{11}$, reducing modulo 11 gives exactly $s(n)$. Thus, our answer is congruent to $N = 7^{6^{5^4 3^{2^1}}}$ modulo 11, and furthermore must be a one-digit integer in base 12; these two conditions uniquely determine the answer.

By Fermat's Little Theorem, $7^{10} \equiv 1 \pmod{11}$, and also observe that $6^{5^4 3^{2^1}} \equiv 6 \pmod{11}$ because $6 \equiv 0 \pmod{2}$ and $6 \equiv 1 \pmod{5}$. Thus, $N \equiv 7^6 \equiv 343^2 \equiv 2^2 \equiv 4 \pmod{11}$, which is our answer. (Additionally, we note that this process of writing the number in base twelve and summing the digits must eventually terminate because the value decreases after each step.)

6. [3] A rectangular piece of paper with vertices $ABCD$ is being cut by a pair of scissors. The pair of scissors starts at vertex A , and then cuts along the angle bisector of DAB until it reaches another edge of the paper. One of the two resulting pieces of paper has 4 times the area of the other piece. What is the ratio of the longer side of the original paper to the shorter side?

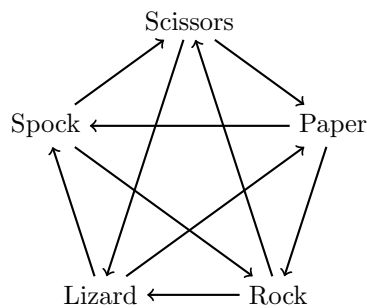
Answer: $\boxed{\frac{5}{2}}$ Without loss of generality, let $AB > AD$, and let $x = AD$, $y = AB$. Let the cut along the angle bisector of $\angle DAB$ meet CD at E . Note that ADE is a 45-45-90 triangle, so $DE = AD = x$, and $EC = y - x$. Now, $[ADE] = \frac{x^2}{2}$, and $[AECB] = x(y - \frac{x}{2}) = 4[ADE]$. Equating and dividing both sides by x , we find that $2x = y - \frac{x}{2}$, so $y/x = \frac{5}{2}$.

7. [4] The game of rock-scissors is played just like rock-paper-scissors, except that neither player is allowed to play paper. You play against a poorly-designed computer program that plays rock with 50% probability and scissors with 50% probability. If you play optimally against the computer, find the probability that after 8 games you have won at least 4.

Answer: $\boxed{\frac{163}{256}}$ Since rock will always win against scissors, the optimum strategy is for you to always play rock; then, you win a game if and only if the computer plays scissors. Let p_n be the probability that the computer plays scissors n times; we want $p_0 + p_1 + p_2 + p_3 + p_4$. Note that by symmetry, $p_n = p_{8-n}$ for $n = 0, 1, \dots, 8$, and because $p_0 + p_1 + \dots + p_8 = 1$, $p_0 + \dots + p_3 = p_5 + \dots + p_8 = (1 - p_4)/2$. Our answer will thus be $(1 + p_4)/2$.

If the computer is to play scissors exactly 4 times, there are $\binom{8}{4}$ ways in which it can do so, compared to 2^8 possible combinations of eight plays. Thus, $p_4 = \binom{8}{4}/2^8 = 35/128$. Our answer is thus $\frac{1 + \frac{35}{128}}{2} = \frac{163}{256}$.

8. [4] In the game of rock-paper-scissors-lizard-Spock, rock defeats scissors and lizard, paper defeats rock and Spock, scissors defeats paper and lizard, lizard defeats paper and Spock, and Spock defeats rock and scissors, as shown in the below diagram. As before, if two players choose the same move, then there is a draw. If three people each play a game of rock-paper-scissors-lizard-Spock at the same time by choosing one of the five moves at random, what is the probability that one player beats the other two?



Answer: $\boxed{\frac{12}{25}}$ Let the three players be A, B, C . Our answer will simply be the sum of the probability that A beats both B and C , the probability that B beats both C and A , and the probability that C beats A and B , because these events are all mutually exclusive. By symmetry, these three probabilities are the same, so we only need to compute the probability that A beats both B and C . Given A 's play, the probability that B 's play loses to that of A is $2/5$, and similarly for C . Thus, our answer is $3 \cdot (\frac{2}{5}) \cdot (\frac{2}{5}) = \frac{12}{25}$.

9. [6] 64 people are in a single elimination rock-paper-scissors tournament, which consists of a 6-round knockout bracket. Each person has a different rock-paper-scissors skill level, and in any game, the person with the higher skill level will always win. For how many players P is it possible that P wins the first four rounds that he plays?

(A 6-round knockout bracket is a tournament which works as follows:

- (a) In the first round, all 64 competitors are paired into 32 groups, and the two people in each group play each other. The winners advance to the second round, and the losers are eliminated.
- (b) In the second round, the remaining 32 players are paired into 16 groups. Again, the winner of each group proceeds to the next round, while the loser is eliminated.
- (c) Each round proceeds in a similar way, eliminating half of the remaining players. After the sixth round, only one player will not have been eliminated. That player is declared the champion.)

Answer: $\boxed{49}$ Note that a sub-bracket, that is, a subset of games of the tournament that themselves constitute a bracket, is always won by the person with the highest skill level. Therefore, a person wins her first four rounds if and only if she has the highest skill level among the people in her 16-person sub-bracket. This is possible for all but the people with the $16 - 1 = 15$ lowest skill levels, so our answer is $64 - 15 = 49$.

10. [8] In a game of rock-paper-scissors with n people, the following rules are used to determine a champion:

- (a) In a round, each person who has not been eliminated randomly chooses one of rock, paper, or scissors to play.
- (b) If at least one person plays rock, at least one person plays paper, and at least one person plays scissors, then the round is declared a tie and no one is eliminated. If everyone makes the same move, then the round is also declared a tie.
- (c) If exactly two moves are represented, then everyone who made the losing move is eliminated from playing in all further rounds (for example, in a game with 8 people, if 5 people play rock and 3 people play scissors, then the 3 who played scissors are eliminated).
- (d) The rounds continue until only one person has not been eliminated. That person is declared the champion and the game ends.

If a game begins with 4 people, what is the expected value of the number of rounds required for a champion to be determined?

Answer: $\boxed{\frac{45}{14}}$ For each positive integer n , let E_n denote the expected number of rounds required to determine a winner among n people. Clearly, $E_1 = 0$. When $n = 2$, on the first move, there is a $\frac{1}{3}$ probability that there is a tie, and a $\frac{2}{3}$ probability that a winner is determined. In the first case, the expected number of additional rounds needed is exactly E_2 ; in the second, it is E_1 . Therefore, we get the relation

$$E_2 = \frac{1}{3}(E_2 + 1) + \frac{2}{3}(E_1 + 1),$$

from which it follows that $E_2 = \frac{3}{2}$.

Next, if $n = 3$, with probability $\frac{1}{9}$ there is only one distinct play among the three players, and with probability $\frac{6}{27} = \frac{2}{9}$ all three players make different plays. In both of these cases, no players are eliminated. In all remaining situations, which occur with total probability $\frac{2}{3}$, two players make one play and the third makes a distinct play; with probability $\frac{1}{3}$ two players are eliminated and with probability $\frac{1}{3}$ one player is eliminated. This gives the relation

$$E_3 = \frac{1}{3}(E_3 + 1) + \frac{1}{3}(E_2 + 1) + \frac{1}{3}(E_1 + 1),$$

from which we find that $E_3 = \frac{9}{4}$.

Finally, suppose $n = 4$. With probability $\frac{1}{27}$, all four players make the same play, and with probability $\frac{3 \cdot 6 \cdot 2}{81} = \frac{4}{9}$, two players make one play, and the other two players make the other two plays; in both cases no players are eliminated, with total probability $\frac{1}{27} + \frac{4}{9} = \frac{13}{27}$ over the two cases. With probability $\frac{6 \cdot 4}{81} = \frac{8}{27}$, three players make one play and the fourth makes another; thus, there is a probability of $\frac{4}{27}$ for exactly one player being eliminated and a probability of $\frac{4}{27}$ of three players being eliminated.

Then, there is a remaining probability of $\frac{6 \cdot 3}{81} = \frac{2}{9}$, two players make one play and the other two players make another. Similar analysis from before yields

$$E_4 = \frac{13}{27}(E_4 + 1) + \frac{4}{27}(E_3 + 1) + \frac{2}{9}(E_2 + 1) + \frac{4}{27}(E_1 + 1),$$

so it follows that $E_4 = \frac{45}{14}$.

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Algebra & Calculus Individual Test

1. Let a , b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

2. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

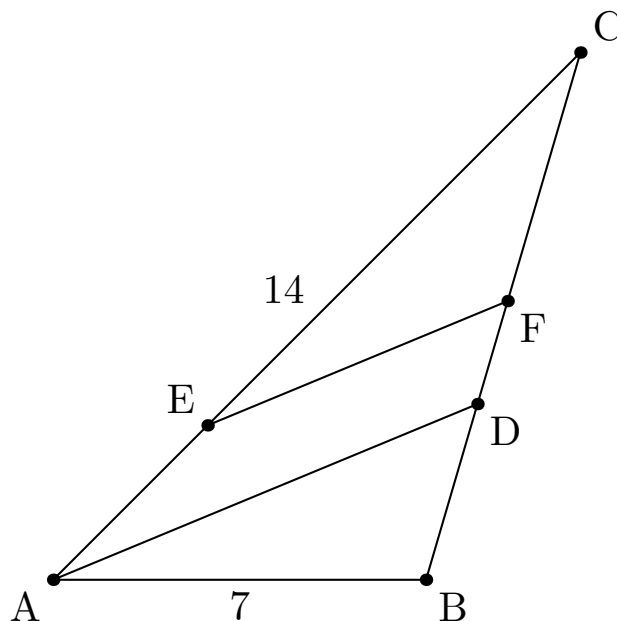
Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \leq 2$ for all real numbers x . If a and b are real numbers such that the set of possible values of $\int_0^1 f(x) dx$ is the open interval (a, b) , determine $b - a$.

Answer: $\boxed{\frac{3}{4}}$ Draw lines of slope ± 2 passing through $(0, 0)$ and $(1, 1)$. These form a parallelogram with vertices $(0, 0)$, $(.75, 1.5)$, $(1, 1)$, $(.25, -.5)$. By the mean value theorem, no point of $(x, f(x))$ lies outside this parallelogram, but we can construct functions arbitrarily close to the top or the bottom of the parallelogram while satisfying the condition of the problem. So $(b - a)$ is the area of this parallelogram, which is $\frac{3}{4}$.

4. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \quad (1)$$

($[\]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC .

5. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

6. Let $a \star b = ab + a + b$ for all integers a and b . Evaluate $1 \star (2 \star (3 \star (4 \star \dots (99 \star 100) \dots)))$.

Answer: $\boxed{101! - 1}$ We will first show that \star is both commutative and associative.

- Commutativity: $a \star b = ab + a + b = b \star a$
- Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

So we need only calculate $((\dots (1 \star 2) \star 3) \star 4) \dots \star 100$. We will prove by induction that

$$((\dots (1 \star 2) \star 3) \star 4) \dots \star n = (n + 1)! - 1.$$

- Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
- Inductive step:

Suppose that

$$(((\dots (1 \star 2) \star 3) \star 4) \dots \star n) = (n + 1)! - 1.$$

Then,

$$\begin{aligned} (((\dots (1 \star 2) \star 3) \star 4) \dots \star n) \star (n + 1) &= ((n + 1)! - 1) \star (n + 1) \\ &= (n + 1)!(n + 1) - (n + 1) + (n + 1)! - 1 + (n + 1) \\ &= (n + 2)! - 1 \end{aligned}$$

Hence, $((\dots(1 \star 2) \star 3) \star 4) \dots \star n) = (n + 1)! - 1$ for all n . For $n = 100$, this results to $101! - 1$.

7. Let $f : [0, 1) \rightarrow \mathbb{R}$ be a function that satisfies the following condition: if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1 a_2 a_3 \dots$$

is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \geq k$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$

Determine $f'(\frac{1}{3})$.

Answer: $\boxed{0}$ Note that $\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n}$.

Clearly f is an increasing function. Also for any integer $n \geq 1$, we see from decimal expansions that $f(\frac{1}{3} \pm \frac{1}{10^n}) - f(\frac{1}{3}) = \pm \frac{1}{10^{2n}}$.

Consider h such that $10^{-n-1} \leq |h| < 10^{-n}$. The two properties of f outlined above show that $|f(\frac{1}{3} + h) - f(\frac{1}{3})| < \frac{1}{10^{2n}}$. And from $|\frac{1}{h}| \leq 10^{n+1}$, we get $|\frac{f(\frac{1}{3}+h)-f(\frac{1}{3})}{h}| < \frac{1}{10^{n-1}}$. Taking $n \rightarrow \infty$ gives $h \rightarrow 0$ and $f'(\frac{1}{3}) = \lim_{n \rightarrow \infty} \frac{1}{10^{n-1}} = 0$.

8. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

Answer: $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x - 3)(x + 2)$. Suppose a positive integer a divides both $2x - 3$ and $x + 2$. Then a must also divide $2(x + 2) - (2x - 3) = 7$. Hence, a can either be 1 or 7. As a result, $2x - 3 = 7^n$ or -7^n for some positive integer n , or either $x + 2$ or $2x - 3$ is ± 1 . We consider the following cases:

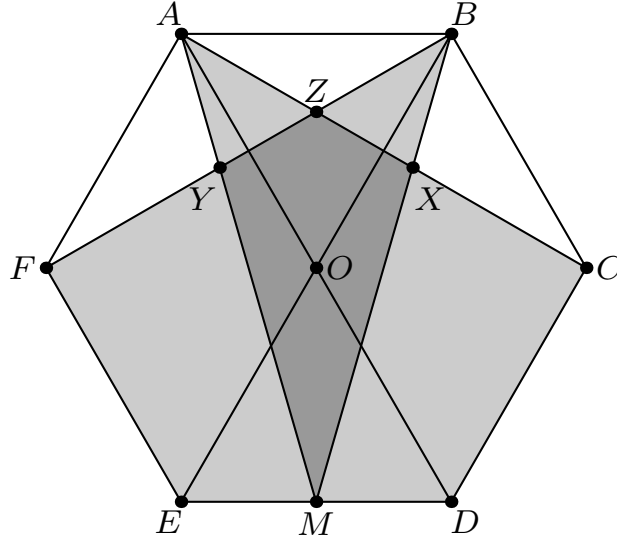
- $(2x - 3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
- $(2x - 3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
- $(x + 2) = 1$. Then $x = -1$, which yields $f(x) = -5$ not a prime power.
- $(x + 2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
- $(2x - 3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
- $(2x - 3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
- $(2x - 3) = \pm 7^n$, for $n \geq 2$. Then, since $x + 2 = \frac{(2x - 3) + 7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

So x can be either -3, 2 or 5.

(Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer -3, 2, 5 was marked as correct.)

9. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

10. For all real numbers x , let

$$f(x) = \frac{1}{\sqrt[2011]{1-x^{2011}}}.$$

Evaluate $(f(f(\dots(f(2011))\dots)))^{2011}$, where f is applied 2010 times.

Answer: $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x)) = \frac{\sqrt[2011]{1-x^{2011}}}{-x}$ and $f(f(f(x))) = x$. Hence $(f(f(\dots(f(x))\dots))) = x$, where f is applied 2010 times. So $(f(f(\dots(f(2011))\dots)))^{2011} = 2011^{2011}$.

11. Evaluate $\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} dx$.

Answer: $\boxed{\frac{2011!}{2010^{2012}}}$ By the chain rule, $\frac{d}{dx}(\ln x)^n = \frac{n \ln^{n-1} x}{x}$.

We calculate the definite integral using integration by parts:

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \left[\frac{(\ln x)^n}{-2010x^{2010}} \right]_{x=1}^{x=\infty} - \int_{x=1}^\infty \frac{n(\ln x)^{n-1}}{-2010x^{2011}} dx$$

But $\ln(1) = 0$, and $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^{2010}} = 0$ for all $n > 0$. So

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \int_{x=1}^\infty \frac{n(\ln x)^{n-1}}{2010x^{2011}} dx$$

It follows that

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \frac{n!}{2010^n} \int_{x=1}^\infty \frac{1}{x^{2011}} dx = \frac{n!}{2010^{n+1}}$$

So the answer is $\frac{2011!}{2010^{2012}}$.

12. Let $f(x) = x^2 + 6x + c$ for all real numbers x , where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

Answer: $\boxed{\frac{11-\sqrt{13}}{2}}$ Suppose f has only one distinct root r_1 . Then, if x_1 is a root of $f(f(x))$, it must be the case that $f(x_1) = r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence f has two distinct roots. Let them be $r_1 \neq r_2$.

Since $f(f(x))$ has just three distinct roots, either $f(x) = r_1$ or $f(x) = r_2$ has one distinct root. Assume without loss of generality that r_1 has one distinct root. Then $f(x) = x^2 + 6x + c = r_1$ has one root, so that $x^2 + 6x + c - r_1$ is a square polynomial. Therefore, $c - r_1 = 9$, so that $r_1 = c - 9$. So $c - 9$ is a root of f . So $(c - 9)^2 + 6(c - 9) + c = 0$, yielding $c^2 - 11c + 27 = 0$, or $(c - \frac{11}{2})^2 = \frac{13}{4}$. This results to $c = \frac{11 \pm \sqrt{13}}{2}$.

If $c = \frac{11 - \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 - \sqrt{13}}{2} = (x + \frac{7 + \sqrt{13}}{2})(x + \frac{5 - \sqrt{13}}{2})$. We know $f(x) = \frac{-7 - \sqrt{13}}{2}$ has a double root, -3. Now $\frac{-5 + \sqrt{13}}{2} > \frac{-7 - \sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

If $c = \frac{11 + \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 + \sqrt{13}}{2} = (x + \frac{7 - \sqrt{13}}{2})(x + \frac{5 + \sqrt{13}}{2})$. We know $f(x) = \frac{-7 + \sqrt{13}}{2}$ has a double root, -3, and this is the value of f at the vertex of the parabola, so it is its minimum value. Since $\frac{-5 - \sqrt{13}}{2} < \frac{-7 + \sqrt{13}}{2}$, $f(x) = \frac{-5 - \sqrt{13}}{2}$ has no solutions. So in this case, f has only one real root.

So the answer is $c = \frac{11 - \sqrt{13}}{2}$.

Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

13. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the n th turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1} . The player then draws the largest circle that is centered at p_n and contained in O_{n-1} , and calls this circle O_n . The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

Answer: $\boxed{\frac{6\pi}{7}}$ Let $f(r)$ be the average area colored red on a dartboard of radius r if Sarah plays first. Then $f(r)$ is proportional to r^2 . Let $f(r) = (\pi x)r^2$ for some constant x . We want to find $f(1) = \pi x$.

In the first throw, if Sarah's dart hits a point with distance r from the center of O_0 , the radius of O_1 will be $1 - r$. The expected value of the area colored red will be $(\pi - \pi(1 - r)^2) + (\pi(1 - r)^2 - f(1 - r)) = \pi - f(1 - r)$. The value of $f(1)$ is the average value of $\pi - f(1 - r)$ over all points in O_0 . Using polar coordinates, we get

$$f(1) = \frac{\int_0^{2\pi} \int_0^1 (\pi - f(1 - r)) r dr d\theta}{\int_0^{2\pi} \int_0^1 r dr d\theta}$$

$$\pi x = \frac{\int_0^1 (\pi - \pi x(1 - r)^2) r dr}{\int_0^1 r dr}$$

$$\frac{\pi x}{2} = \int_0^1 \pi r - \pi x r(1 - r)^2 dr$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \pi x \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \frac{\pi x}{12}$$

$$\pi x = \frac{6\pi}{7}$$

14. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

Answer: 86400000 For each nonnegative integer i , let $x^i = x(x-1)\cdots(x-i+1)$. (Define $x^0 = 1$.)

Lemma: Each polynomial with integer coefficients f can be uniquely written in the form

$$f(x) = a_n x^n + \dots + a_1 x^1 + a_0 x^0, a_n \neq 0.$$

Proof: Induct on the degree. The base case (degree 0) is clear. If f has degree m with leading coefficient c , then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^n$ can be uniquely written as $a_{n-1}x^{n-1}(x) + \dots + a_1x^1 + a_0x^0$.

There are 120 possible choices for a_0 , namely any integer in $[0, 120)$. Once a_0, \dots, a_{i-1} have been chosen so $0 \leq P(0), \dots, P(i-1) < 120$, for some $0 \leq i \leq 5$, then we have

$$P(i) = a_i i! + a_{i-1} i^{i-1} + \dots + a_0$$

so by choosing a_i we can make $P(i)$ any number congruent to $a_{i-1}i^{i-1} + \dots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for a_i . Note the choice of a_i does not affect the value of $P(0), \dots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

$$\prod_{i=0}^5 \frac{120}{i!} = 86400000.$$

Note: There is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the i th finite difference at the point 0.

15. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\int_0^1 f(x) dx$.

Answer: $[\frac{3}{4}, 1]$ Since the maximum value of f is 1, $\int_0^1 f(x) dx \leq 1$.

By our condition $f(f(x)) = 1$, f is 1 at any point within the range of f . Clearly, 1 is in the range of f , so $f(1) = 1$. Now $f(x)$ is continuous on a closed interval so it attains a minimum value c . Since c is in the range of f , $f(c) = 1$.

If $c = 1$, $f(x) = 1$ for all x and $\int_0^1 f(x) dx = 1$.

Now assume $c < 1$. By the intermediate value theorem, since f is continuous it attains all values between c and 1. So for all $x \geq c$, $f(x) = 1$. Therefore,

$$\int_0^1 f(x) dx = \int_0^c f(x) dx + (1 - c).$$

Since $f(x) \geq c$, $\int_0^c f(x) dx > c^2$, and the equality is strict because f is continuous and thus cannot be c for all $x < c$ and 1 at c . So

$$\int_0^1 f(x) dx > c^2 + (1 - c) = (c - \frac{1}{2})^2 + \frac{3}{4} \geq \frac{3}{4}.$$

Therefore $\frac{3}{4} < \int_0^1 f(x) dx \leq 1$, and it is easy to show that every value in this interval can be reached.

16. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}, 0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c . The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a-1 \\ \frac{1-(b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x + 2 + \varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following.

- (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4 + 2\varepsilon} > -\sqrt{4 + 2\varepsilon + \varepsilon^2} = -2 - \varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N - 1$ by backwards induction on i , so the desired inequalities follow.
- (ii) G_i is increasing for $i \geq 2N + 1$. Indeed, if $x \geq 2 + \varepsilon$, then $x^2 - x = x(x - 1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \dots$.
- (iii) G_i is unbounded. This follows since $h(x) - x = x(x - 2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so G_i increases faster and faster for $i \geq 2N + 1$.

Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to

take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\cdots g(2)\cdots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin \left(\frac{\theta}{2} \right) = 2 \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \cdots + (-1)^n \left(\theta - \frac{\pi}{2^n} \right) \right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \cdots \right) = -2 \cos \left(\frac{\pi}{3} \right) = -1$, as needed.

17. Let $f : (0, 1) \rightarrow (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

Answer: $\{-1, 1\}$ The key step is to notice that for such a function f , $f'(x) \neq 0$ for any x .

Assume, for sake of contradiction that there exists $0 < y < 1$ such that $f'(y) = 0$. Since f' is a continuous function, there is some small interval (c, d) containing y such that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$. Now there exists some n, a such that $\frac{a}{2^n}, \frac{a+1}{2^n}$ are both in the interval (c, d) . From the definition, $\frac{f\left(\frac{a+1}{2^n}\right) - f\left(\frac{a}{2^n}\right)}{\frac{a+1}{2^n} - \frac{a}{2^n}} = 2^n \left(\frac{b'}{2^n} - \frac{b}{2^n} \right) = b' - b$ where b, b' are integers; one is odd, and one is even. So $b' - b$ is an odd integer. Since f is differentiable, by the mean value theorem there exists a point where $f' = b' - b$. But this point is in the interval (c, d) , and $|b' - b| > \frac{1}{2}$. This contradicts the assumption that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$.

Since $f'(x) \neq 0$, and f' is a continuous function, f' is either always positive or always negative. So f is either increasing or decreasing. $f\left(\frac{1}{2}\right) = \frac{1}{2}$ always. If f is increasing, it follows that $f\left(\frac{1}{4}\right) = \frac{1}{4}$, $f\left(\frac{3}{4}\right) = \frac{3}{4}$, and we can show by induction that indeed $f\left(\frac{a}{2^n}\right) = \frac{a}{2^n}$ for all integers a, n . Since numbers of this form are dense in the interval $(0, 1)$, and f is a continuous function, $f(x) = x$ for all x .

It can be similarly shown that if f is decreasing $f(x) = 1 - x$ for all x . So the only possible values of $f'\left(\frac{1}{2}\right)$ are $-1, 1$.

Query: if the condition that the derivative is continuous were omitted, would the same result still hold?

18. Let $z = \cos \frac{2\pi}{2011} + i \sin \frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \cdots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x . Evaluate $P(z)P(z^2)P(z^3)\cdots P(z^{2010})$.

Answer: $2011^{2009} \cdot (1005^{2011} - 1004^{2011})$ Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x - 1) = x^{2009} + 2x^{2008} + \cdots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x - 1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \cdots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$\begin{aligned} (x - 1)(P(x)(x - 1) + \frac{2010 \cdot 2011}{2}) &= x^{2010} + x^{2009} + \cdots + x - 2010, \\ &= (x^{2010} + x^{2009} + \cdots + x + 1) - 2011. \end{aligned}$$

Hence,

$$P(x) = \frac{(x^{2010} + x^{2009} + \dots + x + 1) - 2011}{x - 1} - 2011 \cdot 1005$$

Note that $x^{2010} + x^{2009} + \dots + x + 1$ has z, z^2, \dots, z^{2010} as roots, so they vanish at those points. Plugging those 2010 powers of z into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011) \cdot 1005 \cdot (x - \frac{1004}{1005})}{(x - 1)^2}.$$

Note that $(x - z)(x - z^2) \dots (x - z^{2010}) = x^{2010} + x^{2009} + \dots + 1$. Using this, the product turns out to be $2011^{2009} \cdot (1005^{2011} - 1004^{2011})$.

19. Let

$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$

and note that F may be expanded as a power series so that $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of

positive real numbers (c, d) such that $\lim_{n \rightarrow \infty} \frac{a_n}{n^d} = c$.

Answer: $(\frac{1}{6^{2011} 2010!}, 2010)$ First notice that all the roots of $2 - x - x^5$ that are not 1 lie strictly outside the unit circle. As such, we may write $2 - x - x^5$ as $2(1 - x)(1 - r_1x)(1 - r_2x)(1 - r_3x)(1 - r_4x)$ where $|r_i| < 1$, and let $\frac{1}{(2 - x - x^5)} = \frac{b_0}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)}$. We calculate b_0 as $\lim_{x \rightarrow 1} \frac{(1 - x)}{(2 - x - x^5)} = \lim_{x \rightarrow 1} \frac{(-1)}{(-1 - 5x^4)} = \frac{1}{6}$.

Now raise the equation above to the 2011th power.

$$\frac{1}{(2 - x - x^5)^{2011}} = \left(\frac{1/6}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)} \right)^{2011}$$

Expand the right hand side using multinomial expansion and then apply partial fractions. The result will be a sum of the terms $(1 - x)^{-k}$ and $(1 - r_i x)^{-k}$, where $k \leq 2011$.

Since $|r_i| < 1$, the power series of $(1 - r_i x)^{-k}$ will have exponentially decaying coefficients, so we only need to consider the $(1 - x)^{-k}$ terms. The coefficient of x^n in the power series of $(1 - x)^{-k}$ is $\binom{n+k-1}{k-1}$, which is a $(k - 1)$ th degree polynomial in variable n . So when we sum up all coefficients, only the power series of $(1 - x)^{-2011}$ will have impact on the leading term n^{2010} .

The coefficient of the $(1 - x)^{-2011}$ term in the multinomial expansion is $(\frac{1}{6})^{2011}$. The coefficient of the x^n term in the power series of $(1 - x)^{-2011}$ is $\binom{n+2010}{2010} = \frac{1}{2010!} n^{2010} + \dots$. Therefore, $(c, d) = (\frac{1}{6^{2011} 2010!}, 2010)$.

20. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2; b_0 = 2$, and $a_{n+1} = a_n \sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n \sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4 .

Answer: $\boxed{1000001100111222 \text{ and } 2211100110000012}$

Note first that $\sqrt{1 + a_n^2 + b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note that $1 + a_{n+1}^2 + b_{n+1}^2 = 1 + a_n^2(1 + a_n^2 + b_n^2) + b_n^2 - 2a_n b_n \sqrt{1 + a_n^2 + b_n^2} + b_n^2(1 + a_n^2 + b_n^2) + a_n^2 + 2a_n b_n \sqrt{1 + a_n^2 + b_n^2} = 1 + (a_n^2 + b_n^2)(1 + a_n^2 + b_n^2) + a_n^2 + b_n^2 = (1 + a_n^2 + b_n^2)^2$.

Invoking the inductive hypothesis, we see that $\sqrt{1 + a_{n+1}^2 + b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.

The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_n i$ for all nonnegative integers n . It should be clear that $z_0 = 2 + 2i$ and $z_{n+1} = z_n(3^{2^n} + i)$. Therefore, $z_4 = (2 + 2i)(3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$. This product is difficult to evaluate in

the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*¹, in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2 . Let $x + yi = (3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$, and consider the balanced ternary representation of x and y . For all $0 \leq j \leq 15$, let x_j denote the digit in the 3^j place of x , let y_j denote the digit in the 3^j place of y , and let $b(j)$ denote the number of ones in the binary representation of j . It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod{4}$, $x_j = 0$ if $b(j) \equiv 1 \pmod{2}$, and $x_j = 1$ if $b(j) \equiv 0 \pmod{4}$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod{4}$, $y_j = 0$ if $b(j) \equiv 0 \pmod{2}$, and $y_j = 1$ if $b(j) \equiv 3 \pmod{4}$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

¹http://en.wikipedia.org/wiki/Balanced_ternary

14th Annual Harvard-MIT Mathematics Tournament
 Saturday 12 February 2011

Algebra & Combinatorics Individual Test

1. Let a , b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

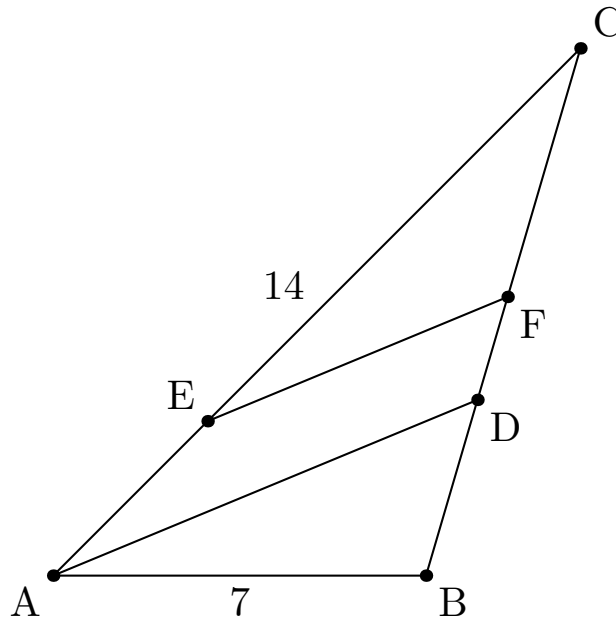
2. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

Answer: $\boxed{2 \cdot 15!^2}$ If we color the desks of the class in a checkerboard pattern, we notice that all of one gender must go in the squares colored black, and the other gender must go in the squares colored white. There are 2 ways to pick which gender goes in which color, $15!$ ways to put the boys into desks and $15!$ ways to put the girls into desks. So the number of ways is $2 \cdot 15!^2$.

(There is a little ambiguity in the problem statement as to whether the 15 boys and the 15 girls are distinguishable or not. If they are not distinguishable, the answer is clearly 2. Given the number of contestants who submitted the answer 2, the graders judged that there was enough ambiguity to justify accepting 2 as a correct answer. So both 2 and $2 \cdot 15!^2$ were accepted as correct answers.)

3. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

($[\]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned}d + 7 &> 14 \\7 + 14 &> d\end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC .

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

5. Let $a \star b = ab + a + b$ for all integers a and b . Evaluate $1 \star (2 \star (3 \star (4 \star \dots (99 \star 100) \dots)))$.

Answer: $\boxed{101! - 1}$ We will first show that \star is both commutative and associative.

- Commutativity: $a \star b = ab + a + b = b \star a$
- Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

So we need only calculate $((\dots (1 \star 2) \star 3) \star 4) \dots \star 100$. We will prove by induction that

$$((\dots (1 \star 2) \star 3) \star 4) \dots \star n = (n + 1)! - 1.$$

- Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
- Inductive step:

Suppose that

$$(((\dots (1 \star 2) \star 3) \star 4) \dots \star n) = (n + 1)! - 1.$$

Then,

$$\begin{aligned}(((\dots (1 \star 2) \star 3) \star 4) \dots \star n) \star (n + 1) &= ((n + 1)! - 1) \star (n + 1) \\ &= (n + 1)!(n + 1) - (n + 1) + (n + 1)! - 1 + (n + 1) \\ &= (n + 2)! - 1\end{aligned}$$

Hence, $((\dots (1 \star 2) \star 3) \star 4) \dots \star n = (n + 1)! - 1$ for all n . For $n = 100$, this results to $101! - 1$.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins

immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

Answer: $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x - 3)(x + 2)$. Suppose a positive integer a divides both $2x - 3$ and $x + 2$. Then a must also divide $2(x + 2) - (2x - 3) = 7$. Hence, a can either be 1 or 7. As a result, $2x - 3 = 7^n$ or -7^n for some positive integer n , or either $x + 2$ or $2x - 3$ is ± 1 . We consider the following cases:

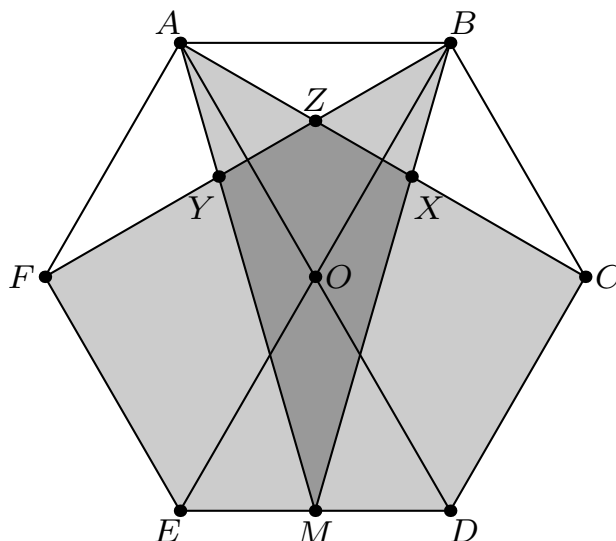
- $(2x - 3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
- $(2x - 3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
- $(x + 2) = 1$. Then $x = -1$, which yields $f(x) = -5$ not a prime power.
- $(x + 2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
- $(2x - 3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
- $(2x - 3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
- $(2x - 3) = \pm 7^n$, for $n \geq 2$. Then, since $x + 2 = \frac{(2x - 3) + 7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

So x can be either -3, 2 or 5.

(Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer -3, 2, 5 was marked as correct.)

8. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

9. For all real numbers x , let

$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$

Evaluate $(f(f(\dots(f(2011))\dots)))^{2011}$, where f is applied 2010 times.

Answer: $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x)) = \frac{\sqrt[2011]{1 - x^{2011}}}{-x}$ and $f(f(f(x))) = x$. Hence $(f(f(\dots(f(x))\dots))) = x$, where f is applied 2010 times. So $(f(f(\dots(f(2011))\dots)))^{2011} = 2011^{2011}$.

10. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

Answer: $\boxed{4022}$ If n is odd and greater than 1, then Goliath makes the last move. No matter what two numbers are on the board, Goliath can combine them to make an even number. Hence Goliath has a winning strategy in this case.

Now suppose n is even. We can replace all numbers on the board by their residues modulo 2. Initially the board reads $1, 0, 1, 0, \dots, 1, 0$. David combines the rightmost 1 and 0 by addition to make 1, so now the board reads $1, 0, 1, 0, \dots, 0, 1$. We call a board of this form a “good” board. When it is Goliath’s turn, and there is a good board, no matter where he moves, David can make a move to restore a good board. Indeed, Goliath must combine a neighboring 0 and 1; David can then combine that number with a neighboring 1 to make 1 and create a good board with two fewer numbers.

David can ensure a good board after his last turn. But a good board with one number is simply 1, so David wins. So David has a winning strategy if n is even. Therefore, the 2011th smallest positive integer greater than 1 for which David can guarantee victory is the 2011th even positive integer, which is 4022.

11. Let $f(x) = x^2 + 6x + c$ for all real numbers x , where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

Answer: $\boxed{\frac{11 - \sqrt{13}}{2}}$ Suppose f has only one distinct root r_1 . Then, if x_1 is a root of $f(f(x))$, it must be the case that $f(x_1) = r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence f has two distinct roots. Let them be $r_1 \neq r_2$.

Since $f(f(x))$ has just three distinct roots, either $f(x) = r_1$ or $f(x) = r_2$ has one distinct root. Assume without loss of generality that r_1 has one distinct root. Then $f(x) = x^2 + 6x + c = r_1$ has one root, so that $x^2 + 6x + c - r_1$ is a square polynomial. Therefore, $c - r_1 = 9$, so that $r_1 = c - 9$. So $c - 9$ is a root of f . So $(c - 9)^2 + 6(c - 9) + c = 0$, yielding $c^2 - 11c + 27 = 0$, or $(c - \frac{11}{2})^2 = \frac{13}{2}$. This results to $c = \frac{11 \pm \sqrt{13}}{2}$.

If $c = \frac{11 - \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 - \sqrt{13}}{2} = (x + \frac{7 + \sqrt{13}}{2})(x + \frac{5 - \sqrt{13}}{2})$. We know $f(x) = \frac{-7 - \sqrt{13}}{2}$ has a double root, -3. Now $\frac{-5 + \sqrt{13}}{2} > \frac{-7 - \sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

If $c = \frac{11 + \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 + \sqrt{13}}{2} = (x + \frac{7 - \sqrt{13}}{2})(x + \frac{5 + \sqrt{13}}{2})$. We know $f(x) = \frac{-7 + \sqrt{13}}{2}$ has a double root, -3, and this is the value of f at the vertex of the parabola, so it is its minimum value. Since $\frac{-5 - \sqrt{13}}{2} < \frac{-7 + \sqrt{13}}{2}$, $f(x) = \frac{-5 - \sqrt{13}}{2}$ has no solutions. So in this case, f has only one real root.

So the answer is $c = \frac{11-\sqrt{13}}{2}$.

Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

12. Mike and Harry play a game on an 8×8 board. For some positive integer k , Mike chooses k squares and writes an M in each of them. Harry then chooses $k + 1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming “ HMM ” or “ MMH ,” when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

Answer: 16 Suppose Mike writes k M 's. Let a be the number of squares which, if Harry writes an H in, will yield either HMM or MMH horizontally, and let b be the number of squares which, if Harry writes an H in, will yield either HMM or MMH vertically. We will show that $a \leq k$ and $b \leq k$. Then, it will follow that there are at most $a + b \leq 2k$ squares which Harry cannot write an H in. There will be at least $64 - k - 2k = 64 - 3k$ squares which Harry can write in. If $64 - 3k \geq k + 1$, or $k \leq 15$, then Harry wins.

We will show that $a \leq k$ (that $b \leq k$ will follow by symmetry). Suppose there are a_i M 's in row i . In each group of 2 or more consecutive M 's, Harry cannot write H to the left or right of that group, giving at most 2 forbidden squares. Hence a_i is at most the number of M 's in row i . Summing over the rows gives the desired result.

Mike can win by writing 16 M 's according to the following diagram:

```

. . . . .
. M M . . M M .
. M M . . M M .
. . . . .
. . . . .
. M M . . M M .
. M M . . M M .
. . . . .

```

13. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

Answer: 86400000 For each nonnegative integer i , let $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$. (Define $x^{\underline{0}} = 1$.)

Lemma: Each polynomial with integer coefficients f can be uniquely written in the form

$$f(x) = a_n x^{\underline{n}} + \dots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}, a_n \neq 0.$$

Proof: Induct on the degree. The base case (degree 0) is clear. If f has degree m with leading coefficient c , then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^{\underline{n}}$ can be uniquely written as $a_{n-1}x^{\underline{n-1}} + \dots + a_1x^{\underline{1}} + a_0x^{\underline{0}}$.

There are 120 possible choices for a_0 , namely any integer in $[0, 120)$. Once a_0, \dots, a_{i-1} have been chosen so $0 \leq P(0), \dots, P(i-1) < 120$, for some $0 \leq i \leq 5$, then we have

$$P(i) = a_i i! + a_{i-1} i^{\underline{i-1}} + \dots + a_0$$

so by choosing a_i we can make $P(i)$ any number congruent to $a_{i-1}i^{\underline{i-1}} + \dots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for a_i . Note the choice of a_i does not affect the value of $P(0), \dots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

$$\prod_{i=0}^5 \frac{120}{i!} = 86400000.$$

Note: There is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the i th finite difference at the point 0.

14. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \dots, (1008, 1005), (1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j) , with (x_i, y_i) left of (x_j, y_j) , erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

Answer: $\boxed{504510}$ First, note that none of the numbers will ever be 0. Let \star denote the replacement operation. For each pair on the board (x_i, y_i) define its *primary form* to be (x_i, y_i) and its *secondary form* to be $\left[x_i y_i, \frac{x_i}{y_i}\right]$. Note that the primary form determines the secondary form uniquely and vice versa. In secondary form,

$$[a_1, b_1] \star [a_2, b_2] = \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}}\right) \star \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}}\right) = \left(a_1 b_2, \frac{a_1}{b_2}\right) = [a_1^2, b_2^2].$$

Thus we may replace all pairs on the board by their secondary form and use the above rule for \star instead. From the above rule, we see that if the leftmost number on the board is x , then after one minute it will be x or x^2 depending on whether it was erased in the intervening step, and similarly for the rightmost number. Let k be the number of times the leftmost pair is erased and n be the number of times the rightmost pair is erased. Then the final pair is

$$\left[4022^{2^k}, \left(\frac{1007}{1006}\right)^{2^n}\right]. \quad (2)$$

Any step except the last cannot involve both the leftmost and rightmost pair, so $k + n \leq 1005$. Since every pair must be erased at least once, $k, n \geq 1$. Every pair of integers satisfying the above can occur, for example, by making $1005 - k - n$ moves involving only the pairs in the middle, then making $k - 1$ moves involving the leftmost pair, and finally n moves involving the rightmost pair.

In light of (2), the answer is the number of possible pairs (k, n) , which is $\sum_{k=1}^{1004} \sum_{n=1}^{1005-k} 1 = \sum_{k=1}^{1004} 1005 - k =$

$$\sum_{k=1}^{1004} k = \frac{1004 \cdot 1005}{2} = 504510.$$

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}, 0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c .

The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a-1 \\ \frac{1-(b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b-a-1 < -2$ or $b-a-1 > 2$, but $b > a$, so the latter must hold and $(b-a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b-a)^2$ (the square of the difference of the two roots) and $(r_2+1)^2 - 4r_3$ (from the coefficients), so $(r_2+1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x+2+\varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following.

- (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4+2\varepsilon} > -\sqrt{4+2\varepsilon+\varepsilon^2} = -2 - \varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N-1$ by backwards induction on i , so the desired inequalities follow.
- (ii) G_i is increasing for $i \geq 2N+1$. Indeed, if $x \geq 2 + \varepsilon$, then $x^2 - x = x(x-1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \dots$.
- (iii) G_i is unbounded. This follows since $h(x) - x = x(x-2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so G_i increases faster and faster for $i \geq 2N+1$.

Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\dots g(2)\dots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin\left(\frac{\theta}{2}\right) = 2 \cos\left(\frac{\pi}{2} - \frac{\theta}{2}\right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos\left(\frac{\pi}{2} - \frac{\pi}{4} + \dots + (-1)^n\left(\theta - \frac{\pi}{2^n}\right)\right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos\left(\frac{\pi}{2} - \frac{\pi}{4} + \dots\right) = -2 \cos\left(\frac{\pi}{3}\right) = -1$, as needed.

16. Let $A = \{1, 2, \dots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

Answer: $\boxed{2^{2011} - 2012}$ Let n be the element of A not in the range of f . Let m be the element of A that is hit twice.

We now sum the total number of functions over n, m . Clearly $f(1) = 1$, and by induction, for $x \leq m, f(x) = x$. Also unless $n = 2011, f(2011) = 2011$ because f can take no other number to 2011. It follows from backwards induction that for $x > n, f(x) = x$. Therefore $n > m$, and there are only $n - m$ values of f that are not fixed.

Now $f(m+1) = m$ or $f(m+1) = m+1$. For $m < k < n$, given the selection of $f(1), f(2), \dots, f(k-1), k-1$ of the $k+1$ possible values of $f(k+1)$ ($1, 2, 3, \dots, k$, and counting m twice) have been taken, so there are two distinct values that $f(k+1)$ can take (one of them is $k+1$, and the other is not, so they are distinct). For $f(n)$, when the other 2010 values of f have been assigned, there is only one missing, so $f(n)$ is determined.

For each integer in $[m, n)$, there are two possible values of f , so there are 2^{n-m-1} different functions f for a given m, n . So our answer is

$$\begin{aligned} \sum_{m=1}^{2010} \sum_{n=m+1}^{2011} 2^{n-m-1} &= \sum_{m=1}^{2010} 2^{-m-1} \sum_{n=m+1}^{2011} 2^n \\ &= \sum_{m=1}^{2010} 2^{-m-1} (2^{2012} - 2^{m+1}) \\ &= \sum_{m=1}^{2010} 2^{2011-m} - 1 \\ &= \left(\sum_{m=1}^{2010} 2^m \right) - 2010 \\ &= 2^{2011} - 2012 \end{aligned}$$

17. Let $z = \cos \frac{2\pi}{2011} + i \sin \frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \dots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x . Evaluate $P(z)P(z^2)P(z^3) \dots P(z^{2010})$.

Answer: $\boxed{2011^{2009} \cdot (1005^{2011} - 1004^{2011})}$ Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x - 1) = x^{2009} + 2x^{2008} + \dots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x - 1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \dots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$\begin{aligned} (x - 1)(P(x)(x - 1) + \frac{2010 \cdot 2011}{2}) &= x^{2010} + x^{2009} + \dots + x - 2010, \\ &= (x^{2010} + x^{2009} + \dots + x + 1) - 2011. \end{aligned}$$

Hence,

$$P(x) = \frac{(x^{2010} + x^{2009} + \dots + x + 1) - 2011}{x - 1} - 2011 \cdot 1005$$

Note that $x^{2010} + x^{2009} + \dots + x + 1$ has z, z^2, \dots, z^{2010} as roots, so they vanish at those points. Plugging those 2010 powers of z into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011) \cdot 1005 \cdot \left(x - \frac{1004}{1005}\right)}{(x-1)^2}.$$

Note that $(x-z)(x-z^2)\dots(x-z^{2010}) = x^{2010} + x^{2009} + \dots + 1$. Using this, the product turns out to be $2011^{2009} \cdot (1005^{2011} - 1004^{2011})$.

18. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n - 1$ members vote randomly.

Answer: $\frac{2^n - 1}{n2^{n-1}}$ Let x be the probability Hermia is elected if Lysander votes for her, and let y be the probability that she wins if Lysander does not vote for her. We are trying to find x , and do so by first finding y . If Lysander votes for Hermia with probability $\frac{1}{2}$ then the probability that Hermia is elected chairman is $\frac{x}{2} + \frac{y}{2}$, but it is also $\frac{1}{n}$ by symmetry. If Lysander does not vote for Hermia, Hermia can get at most 1 vote, and then can only be elected if everyone gets one vote and she wins the tiebreaker. The probability she wins the tiebreaker is $\frac{1}{n}$, and chasing around the circle, the probability that every person gets 1 vote is $\frac{1}{2^{n-1}}$. (Everyone votes for the person to the left, or everyone votes for the person to the right.) Hence

$$y = \frac{1}{n2^{n-1}}.$$

Then $\frac{x}{2} + \frac{1}{n2^n} = \frac{1}{n}$, so solving for x gives

$$x = \frac{2^n - 1}{n2^{n-1}}.$$

19. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2; b_0 = 2$, and $a_{n+1} = a_n\sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n\sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4 .

Answer: $\boxed{1000001100111222 \text{ and } 2211100110000012}$

Note first that $\sqrt{1 + a_n^2 + b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note that $1 + a_{n+1}^2 + b_{n+1}^2 = 1 + a_n^2(1 + a_n^2 + b_n^2) + b_n^2 - 2a_nb_n\sqrt{1 + a_n^2 + b_n^2} + b_n^2(1 + a_n^2 + b_n^2) + a_n^2 + 2a_nb_n\sqrt{1 + a_n^2 + b_n^2} = 1 + (a_n^2 + b_n^2)(1 + a_n^2 + b_n^2) + a_n^2 + b_n^2 = (1 + a_n^2 + b_n^2)^2$. Invoking the inductive hypothesis, we see that $\sqrt{1 + a_{n+1}^2 + b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.

The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_n i$ for all nonnegative integers n . It should be clear that $z_0 = 2 + 2i$ and $z_{n+1} = z_n(3^{2^n} + i)$. Therefore, $z_4 = (2 + 2i)(3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$. This product is difficult to evaluate in the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*¹, in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2 . Let $x + yi = (3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$, and consider the balanced ternary representation of x and y . For all $0 \leq j \leq 15$, let x_j denote the digit in the 3^j place of x , let y_j denote the digit in the 3^j place of y , and let $b(j)$ denote the number of ones in the binary representation of j . It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod{4}$, $x_j = 0$ if $b(j) \equiv 1 \pmod{2}$, and $x_j = 1$ if $b(j) \equiv 0 \pmod{4}$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod{4}$, $y_j = 0$ if $b(j) \equiv 0 \pmod{2}$, and $y_j = 1$ if $b(j) \equiv 3 \pmod{4}$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It

¹http://en.wikipedia.org/wiki/Balanced_ternary

remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

20. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \dots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

Answer: 1

Consider the grid whose entries in the j th row are, in order, $2011j - 2010, 2011j - 2009, \dots, 2011j$. Call this grid A_0 . For $k = 1, 2, \dots, 2010$, let grid A_k be the grid obtained from A_0 by swapping the rightmost entry of the k th row with the leftmost entry of the $k+1$ st row. We claim that if $A \in \{A_0, A_1, \dots, A_{2010}\}$, then given any legally numbered grid B such that A and B differ in at least one entry, there exist two integers in the same column of B that occur in the same row of A .

We first consider A_0 . Assume for the sake of contradiction B is a legally numbered grid distinct from A_0 , such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Since the numbers $1, 2, \dots, 2011$ occur in the same row of A_0 , they must all occur in different columns of B . Clearly 1 is the leftmost entry in B 's first row. Let m be the smallest number that does not occur in the first row of B . Since each row is in order, m must be the first entry in its row. But then 1 and m are in the same column of B , a contradiction. It follows that the numbers $1, 2, \dots, 2011$ all occur in the first row of B . Proceeding by induction, $2011j - 2010, 2011j - 2009, \dots, 2011j$ must all occur in the j th row of B for all $1 \leq j \leq 2011$. Since A_0 is the only legally numbered grid satisfying this condition, we have reached the desired contradiction.

Now note that if $A \in \{A_1, \dots, A_{2010}\}$, there exist two integers in the same column of A_0 that occur in the same row of A . In particular, if $A = A_k$ and $1 \leq k \leq 2010$, then the integers $2011k - 2010$ and $2011k + 1$ occur in the same column of A_0 and in the same row of A_k . Therefore, it suffices to show that for all $1 \leq k \leq 2010$, there is no legally numbered grid B distinct from A_k and A_0 such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Assume for the sake of contradiction that there does exist such a grid B . By the same logic as above, applied to the first $k - 1$ rows and applied backwards to the last $2010 - k - 1$ rows, we see that B may only differ from A_k in the k th and $k + 1$ st rows. However, there are only two legally numbered grids that are identical to A_k outside of rows k and $k + 1$, namely A_0 and A_k . This proves the claim.

It remains only to note that, by the pigeonhole principle, if one of Alice's grids is A_0 , then there exists a positive integer k , $1 \leq k \leq 2010$, such that A_k is not one of the Alice's grids. Therefore, if Bob sets his initial grid to be A_0 , he will require only one swap to switch his grid to A_k after examining Alice's grids. If A_0 is not among Alice's grids, then if Bob sets his initial grid to be A_0 , he will not in fact require any swaps at all.

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

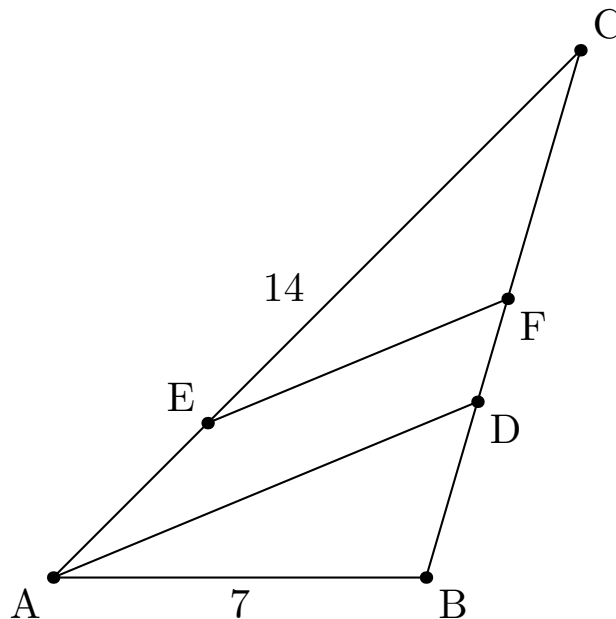
Algebra & Geometry Individual Test

1. Let a , b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

Answer: 4 If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: 13



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

($[]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

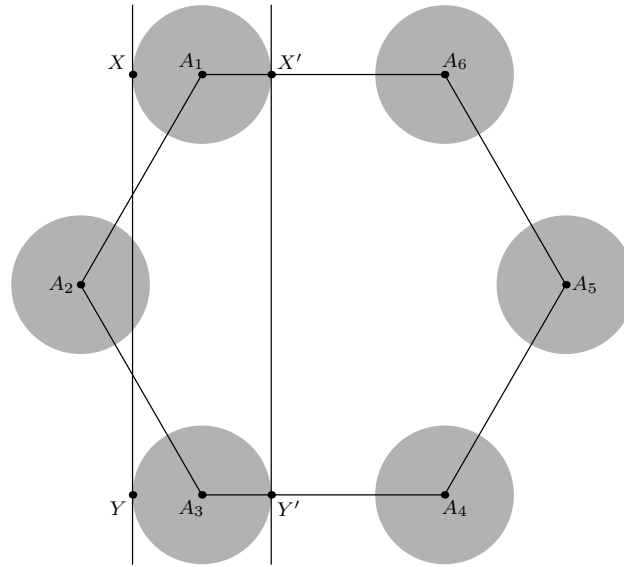
Hence $7 < d < 21$ and there are 13 possible integral values for BC .

3. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

Answer: 2^{n-1} Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

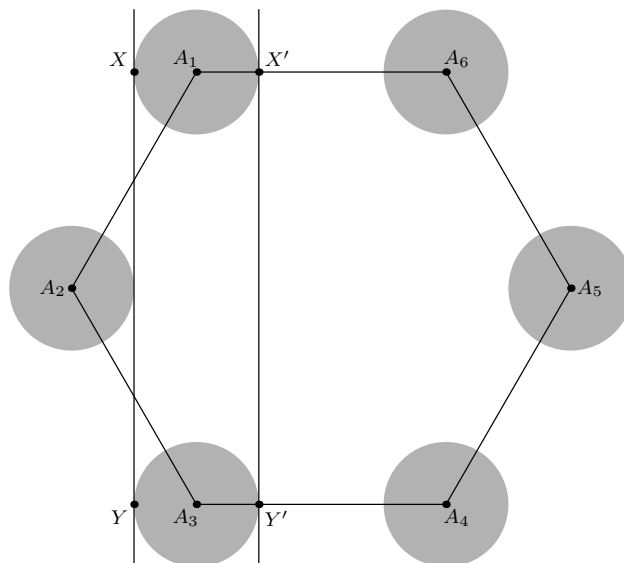
4. Let H be a regular hexagon of side length x . Call a hexagon in the same plane a “distortion” of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

Answer: $\boxed{4}$



Let $H = A_1A_2A_3A_4A_5A_6$ be the hexagon, and for all $1 \leq i \leq 6$, let points A'_i be considered such that $A_iA'_i < 1$. Let $H' = A'_1A'_2A'_3A'_4A'_5A'_6$, and consider all indices modulo 6. For any point P in the plane, let $D(P)$ denote the unit disk $\{Q \mid PQ < 1\}$ centered at P ; it follows that $A'_i \in D(A_i)$.

Let X and X' be points on line A_1A_6 , and let Y and Y' be points on line A_3A_4 such that $A_1X = A_1X' = A_3Y = A_3Y' = 1$ and X and X' lie on opposite sides of A_1 and Y and Y' lie on opposite sides of A_3 . If X' and Y' lie on segments A_1A_6 and A_3A_4 , respectively, then segment $A'_1A'_3$ lies between the lines XY and $X'Y'$. Note that $\frac{x}{2}$ is the distance from A_2 to A_1A_3 .



If $\frac{x}{2} \geq 2$, then $C(A_2)$ cannot intersect line XY , since the distance from XY to A_1A_3 is 1 and the distance from XY to A_2 is at least 1. Therefore, A_1A_3 separates A_2 from the other 3 vertices of the hexagon. By analogous reasoning applied to the other vertices, we may conclude that H' is convex.

If $\frac{x}{2} < 2$, then $C(A_2)$ intersects XY , so by choosing $A'_1 = X$ and $A'_3 = Y$, we see that we may choose A'_2 on the opposite side of XY , in which case H' will be concave. Hence the answer is 4, as desired.

5. Let $a \star b = ab + a + b$ for all integers a and b . Evaluate $1 \star (2 \star (3 \star (4 \star \dots (99 \star 100) \dots)))$.

Answer: $101! - 1$

We will first show that \star is both commutative and associative.

- Commutativity: $a \star b = ab + a + b = b \star a$
- Associativity: $a \star (b \star c) = a(bc + b + c) + a + bc + b + c = abc + ab + ac + bc + a + b + c$ and $(a \star b) \star c = (ab + a + b)c + ab + a + b + c = abc + ab + ac + bc + a + b + c$. So $a \star (b \star c) = (a \star b) \star c$.

So we need only calculate $((\dots (1 \star 2) \star 3) \star 4) \dots \star 100$. We will prove by induction that

$$((\dots (1 \star 2) \star 3) \star 4) \dots \star n = (n + 1)! - 1.$$

- Base case ($n = 2$): $(1 \star 2) = 2 + 1 + 2 = 5 = 3! - 1$
- Inductive step:

Suppose that

$$(((\dots (1 \star 2) \star 3) \star 4) \dots \star n) = (n + 1)! - 1.$$

Then,

$$\begin{aligned} (((\dots (1 \star 2) \star 3) \star 4) \dots \star n) \star (n + 1) &= ((n + 1)! - 1) \star (n + 1) \\ &= (n + 1)!(n + 1) - (n + 1) + (n + 1)! - 1 + (n + 1) \\ &= (n + 2)! - 1 \end{aligned}$$

Hence, $((\dots (1 \star 2) \star 3) \star 4) \dots \star n = (n + 1)! - 1$ for all n . For $n = 100$, this results to $101! - 1$.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

Answer: $\boxed{-3, 2, 5}$ Let $f(x) = 2x^2 + x - 6 = (2x - 3)(x + 2)$. Suppose a positive integer a divides both $2x - 3$ and $x + 2$. Then a must also divide $2(x + 2) - (2x - 3) = 7$. Hence, a can either be 1 or 7. As a result, $2x - 3 = 7^n$ or -7^n for some positive integer n , or either $x + 2$ or $2x - 3$ is ± 1 . We consider the following cases:

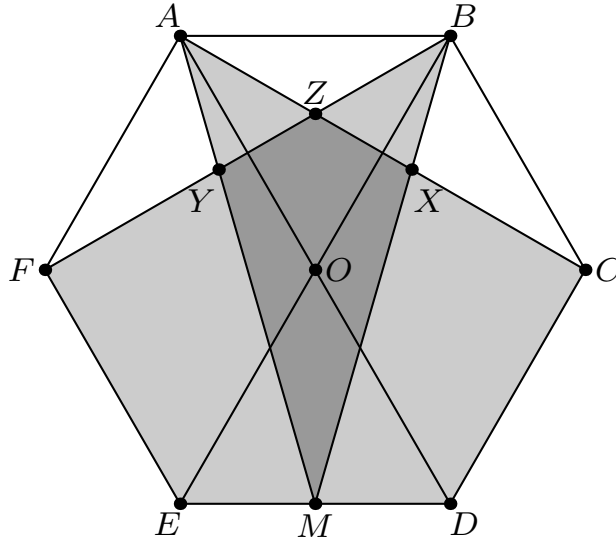
- $(2x - 3) = 1$. Then $x = 2$, which yields $f(x) = 4$, a prime power.
- $(2x - 3) = -1$. Then $x = 1$, which yields $f(x) = -3$, not a prime power.
- $(x + 2) = 1$. Then $x = -1$, which yields $f(x) = -5$ not a prime power.
- $(x + 2) = -1$. Then $x = -3$, which yields $f(x) = 9$, a prime power.
- $(2x - 3) = 7$. Then $x = 5$, which yields $f(x) = 49$, a prime power.
- $(2x - 3) = -7$. Then $x = -2$, which yields $f(x) = 0$, not a prime power.
- $(2x - 3) = \pm 7^n$, for $n \geq 2$. Then, since $x + 2 = \frac{(2x - 3) + 7}{2}$, we have that $x + 2$ is divisible by 7 but not by 49. Hence $x + 2 = \pm 7$, yielding $x = 5, -9$. The former has already been considered, while the latter yields $f(x) = 147$.

So x can be either -3, 2 or 5.

(Note: In the official solutions packet we did not list the answer -3. This oversight was quickly noticed on the day of the test, and only the answer $-3, 2, 5$ was marked as correct.

8. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

9. For all real numbers x , let

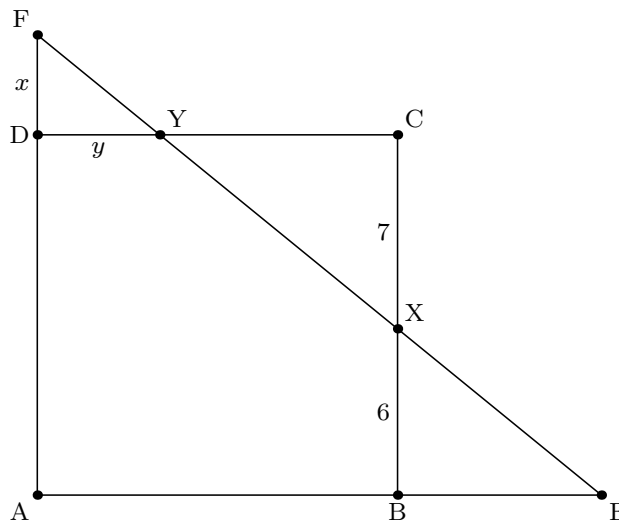
$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$

Evaluate $(f(f(\dots(f(2011))\dots)))^{2011}$, where f is applied 2010 times.

Answer: $\boxed{2011^{2011}}$ Direct calculation shows that $f(f(x)) = \frac{\sqrt[2011]{1 - x^{2011}}}{-x}$ and $f(f(f(x))) = x$. Hence $(f(f(\dots(f(x))\dots))) = x$, where f is applied 2010 times. So $(f(f(\dots(f(2011))\dots)))^{2011} = 2011^{2011}$.

10. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD , respectively, so that the area of square $ABCD$ equals the area of triangle AEF . If EF intersects BC at X and $BX = 6$, determine DF .

Answer: $\boxed{\sqrt{13}}$



First Solution

Let Y be the point of intersection of lines EF and CD . Note that $[ABCD] = [AEF]$ implies that $[BEX] + [DYF] = [CYX]$. Since $\triangle BEX \sim \triangle CYX \sim \triangle DYF$, there exists some constant r such that $[BEX] = r \cdot BX^2$, $[DYF] = r \cdot CX^2$, and $[CYX] = r \cdot DF^2$. Hence $BX^2 + DF^2 = CX^2$, so $DF = \sqrt{CX^2 - BX^2} = \sqrt{49 - 36} = \sqrt{13}$.

Second Solution

Let $x = DF$ and $y = YD$. Since $\triangle BXE \sim \triangle CXY \sim \triangle DFY$, we have

$$\frac{BE}{BX} = \frac{CY}{CX} = \frac{DY}{DF} = \frac{y}{x}.$$

Using $BX = 6$, $XC = 7$ and $CY = 13 - y$ we get $BE = \frac{6y}{x}$ and $\frac{13-y}{7} = \frac{y}{x}$. Solving this last equation for y gives $y = \frac{13x}{x+7}$. Now $[ABCD] = [AEF]$ gives

$$\begin{aligned} 169 &= \frac{1}{2}AE \cdot AF = \frac{1}{2} \left(13 + \frac{6y}{x} \right) (13 + x). \\ 169 &= 6y + 13x + \frac{78y}{x} \\ 13 &= \frac{6x}{x+7} + x + \frac{78}{x+7} \\ 0 &= x^2 - 13. \end{aligned}$$

Thus $x = \sqrt{13}$.

11. Let $f(x) = x^2 + 6x + c$ for all real numbers x , where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

Answer: $\boxed{\frac{11-\sqrt{13}}{2}}$ Suppose f has only one distinct root r_1 . Then, if x_1 is a root of $f(f(x))$, it must be the case that $f(x_1) = r_1$. As a result, $f(f(x))$ would have at most two roots, thus not satisfying the problem condition. Hence f has two distinct roots. Let them be $r_1 \neq r_2$.

Since $f(f(x))$ has just three distinct roots, either $f(x) = r_1$ or $f(x) = r_2$ has one distinct root. Assume without loss of generality that r_1 has one distinct root. Then $f(x) = x^2 + 6x + c = r_1$ has one root, so that $x^2 + 6x + c - r_1$ is a square polynomial. Therefore, $c - r_1 = 9$, so that $r_1 = c - 9$. So $c - 9$ is a root of f . So $(c - 9)^2 + 6(c - 9) + c = 0$, yielding $c^2 - 11c + 27 = 0$, or $(c - \frac{11}{2})^2 = \frac{13}{4}$. This results to $c = \frac{11 \pm \sqrt{13}}{2}$.

If $c = \frac{11 - \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 - \sqrt{13}}{2} = (x + \frac{7 + \sqrt{13}}{2})(x + \frac{5 - \sqrt{13}}{2})$. We know $f(x) = \frac{-7 - \sqrt{13}}{2}$ has a double root, -3 . Now $\frac{-5 + \sqrt{13}}{2} > \frac{-7 - \sqrt{13}}{2}$ so the second root is above the vertex of the parabola, and is hit twice.

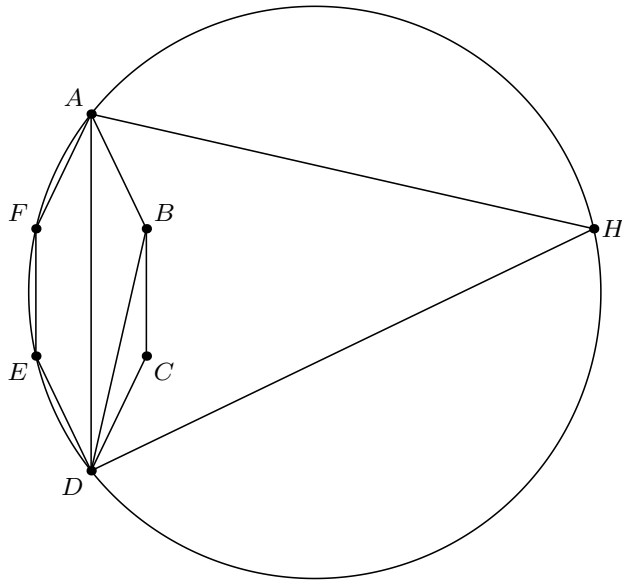
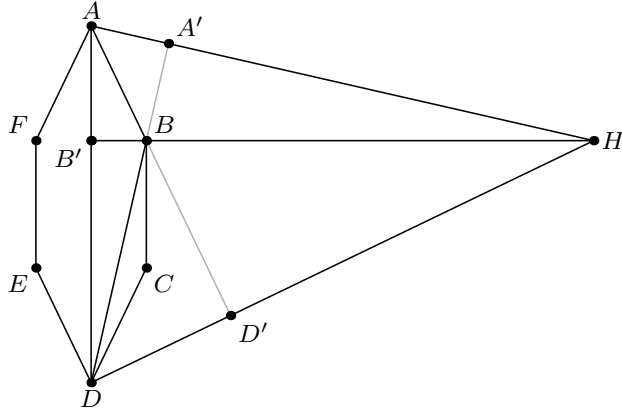
If $c = \frac{11 + \sqrt{13}}{2}$, $f(x) = x^2 + 6x + \frac{11 + \sqrt{13}}{2} = (x + \frac{7 - \sqrt{13}}{2})(x + \frac{5 + \sqrt{13}}{2})$. We know $f(x) = \frac{-7 + \sqrt{13}}{2}$ has a double root, -3 , and this is the value of f at the vertex of the parabola, so it is its minimum value. Since $\frac{-5 - \sqrt{13}}{2} < \frac{-7 + \sqrt{13}}{2}$, $f(x) = \frac{-5 - \sqrt{13}}{2}$ has no solutions. So in this case, f has only one real root.

So the answer is $c = \frac{11 - \sqrt{13}}{2}$.

Note: In the solutions packet we had both roots listed as the correct answer. We noticed this oversight on the day of the test and awarded points only for the correct answer.

12. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC , AD , and EF are parallel. Let H be the orthocenter of triangle ABD . If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

Answer: $\boxed{3}$



Note that $ABCD$ and $DEFA$ are isosceles trapezoids, so $\angle BAD = \angle CDA$ and $\angle FAD = \angle EDA$. In order for the hexagon to be convex, the angles at $B, C, E,$ and F have to be obtuse, so $\angle A = \angle D = 4^\circ$. Letting s be a side length of the hexagon, $AD = AB \cos \angle BAD + BC + CD \cos \angle CDA = s(1 + 2 \cos \angle BAD)$, so $\angle BAD$ is uniquely determined by AD . Since the same equation holds for trapezoid $DEFA$, it follows that $\angle BAD = \angle FAD = \angle CDA = \angle EDA = 2^\circ$. Then $\angle BCD = 180^\circ - 2^\circ = 178^\circ$. Since $\triangle BCD$ is isosceles, $\angle CDB = 1^\circ$ and $\angle BDA = 1^\circ$. (One may also note that $\angle BDA = 1^\circ$ by observing that equal lengths AB and BC must intercept equal arcs on the circumcircle of isosceles trapezoid $ABCD$).

Let $A', B',$ and D' be the feet of the perpendiculars from $A, B,$ and D to $BD, DA,$ and AB , respectively. Angle chasing yields

$$\begin{aligned} \angle AHD &= \angle AHB' + \angle DHB' = (90^\circ - \angle A'AB') + (90^\circ - \angle D'DB') \\ &= \angle BDA + \angle BAD = 1^\circ + 2^\circ = 3^\circ \\ \angle HAD &= 90^\circ - \angle AHB' = 89^\circ \\ \angle HDA &= 90^\circ - \angle DHB' = 88^\circ \end{aligned}$$

Hence the smallest angle in $\triangle HAD$ is 3° .

It is faster, however, to draw the circumcircle of $DEFA$, and to note that since H is the orthocenter of triangle ABD , B is the orthocenter of triangle HAD . Then since F is the reflection of B across AD , quadrilateral $HAFD$ is cyclic, so $\angle AHD = \angle ADF + \angle DAF = 1^\circ + 2^\circ = 3^\circ$, as desired.

13. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

Answer: 86400000

For each nonnegative integer i , let $x^{\underline{i}} = x(x-1)\cdots(x-i+1)$. (Define $x^{\underline{0}} = 1$.)

Lemma: Each polynomial with integer coefficients f can be uniquely written in the form

$$f(x) = a_n x^{\underline{n}} + \dots + a_1 x^{\underline{1}} + a_0 x^{\underline{0}}, a_n \neq 0.$$

Proof: Induct on the degree. The base case (degree 0) is clear. If f has degree m with leading coefficient c , then by matching leading coefficients we must have $m = n$ and $a_n = c$. By the induction hypothesis, $f(x) - cx^{\underline{n}}$ can be uniquely written as $a_{n-1}x^{\underline{n-1}} + \dots + a_1x^{\underline{1}} + a_0x^{\underline{0}}$.

There are 120 possible choices for a_0 , namely any integer in $[0, 120)$. Once a_0, \dots, a_{i-1} have been chosen so $0 \leq P(0), \dots, P(i-1) < 120$, for some $0 \leq i \leq 5$, then we have

$$P(i) = a_i i! + a_{i-1} i^{\underline{i-1}} + \dots + a_0$$

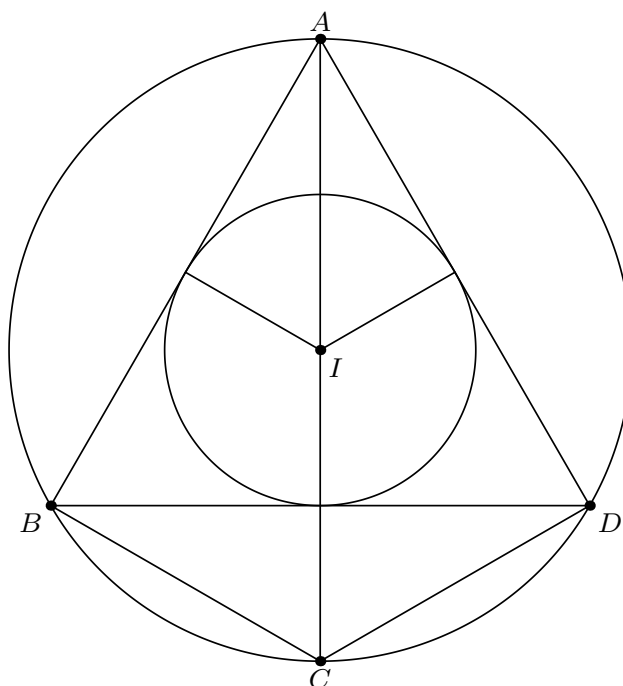
so by choosing a_i we can make $P(i)$ any number congruent to $a_{i-1} i^{\underline{i-1}} + \dots + a_0$ modulo $i!$. Thus there are $\frac{120}{i!}$ choices for a_i . Note the choice of a_i does not affect the value of $P(0), \dots, P(i-1)$. Thus all polynomials we obtain in this way are valid. The answer is

$$\prod_{i=0}^5 \frac{120}{i!} = 86400000.$$

Note: There is also a solution involving finite differences that is basically equivalent to this solution. One proves that for $i = 0, 1, 2, 3, 4, 5$ there are $\frac{5!}{i!}$ ways to pick the i th finite difference at the point 0.

14. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD . If $AI = 2$ as well, find the minimum value of the length of diagonal BD .

Answer: $2\sqrt{3}$



Let T be the point where the incircle intersects AD , and let r be the inradius and R be the circumradius of $\triangle ABD$. Since $BC = CD = 2$, C is on the midpoint of arc BD on the opposite side of BD as A , and hence on the angle bisector of A . Thus A , I , and C are collinear. We have the following formulas:

$$AI = \frac{IM}{\sin \angle IAM} = \frac{r}{\sin \frac{A}{2}}$$

$$BC = 2R \sin \frac{A}{2}$$

$$BD = 2R \sin A$$

The last two equations follow from the extended law of sines on $\triangle ABC$ and $\triangle ABD$, respectively.

Using $AI = 2 = BC$ gives $\sin^2 \frac{A}{2} = \frac{r}{2R}$. However, it is well-known that $R \geq 2r$ with equality for an equilateral triangle (one way to see this is the identity $1 + \frac{r}{R} = \cos A + \cos B + \cos D$). Hence $\sin^2 \frac{A}{2} \leq \frac{1}{4}$ and $\frac{A}{2} \leq 30^\circ$. Then

$$BD = 2R \left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right) = BC \cdot 2 \cos \frac{A}{2} \geq 2 \left(2 \cdot \frac{\sqrt{3}}{2} \right) = 2\sqrt{3}$$

with equality when $\triangle ABD$ is equilateral.

Remark: Similar but perhaps simpler computations can be made by noting that if AC intersects BD at X , then $AB/BX = AD/DX = 2$, which follows from the exterior angle bisector theorem; if I_A is the A -excenter of triangle ABC , then $AI_A/XI_A = 2$ since it is well-known that C is the circumcenter of cyclic quadrilateral $BIDI_A$.

15. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: $\boxed{2}$

Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}$, $0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c . The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$f\left(\frac{b+a-1}{2}\right) < c$$

$$\left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} < a-1$$

$$\frac{1 - (b-a)^2}{4} + \frac{b-a+1}{2} < 0$$

$$\frac{3}{4} < \frac{(b-a)^2}{4} - \frac{b-a}{2}$$

$$4 < (b-a-1)^2.$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x + 2 + \varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following.

- (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4 + 2\varepsilon} > -\sqrt{4 + 2\varepsilon + \varepsilon^2} = -2 - \varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N - 1$ by backwards induction on i , so the desired inequalities follow.
- (ii) G_i is increasing for $i \geq 2N + 1$. Indeed, if $x \geq 2 + \varepsilon$, then $x^2 - x = x(x - 1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \dots$.
- (iii) G_i is unbounded. This follows since $h(x) - x = x(x - 2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so G_i increases faster and faster for $i \geq 2N + 1$.

Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\dots g(2)\dots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

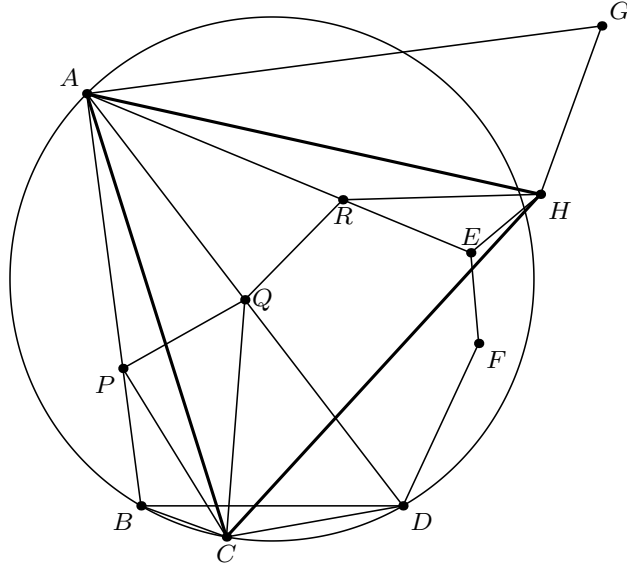
But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin \left(\frac{\theta}{2} \right) = 2 \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots + (-1)^n \left(\theta - \frac{\pi}{2^n} \right) \right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots \right) = -2 \cos \left(\frac{\pi}{3} \right) = -1$, as needed.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD , respectively. Determine the maximum value of m .

Answer: 2



For a fixed quadrilateral $ABCD$ as described, we first show that m , the minimum possible length of $CP + PQ + QC$, equals the length of AC . Reflect B , C , and P across line AD to points E , F , and R , respectively, and then reflect D and F across AE to points G and H , respectively. These two reflections combine to give a 60° rotation around A , so triangle ACH is equilateral. It also follows that RH is a 60° rotation of PC around A , so, in particular, these segments have the same length. Because $QR = QP$ by reflection,

$$CP + PQ + QC = CQ + QR + RH.$$

The latter is the length of a broken path $CQRH$ from C to H , and by the “shortest path is a straight line” principle, this total length is at least as long as $CH = CA$. (More directly, this follows from the triangle inequality: $(CQ + QR) + RH \geq CR + RH \geq CH$). Therefore, the lower bound $m \geq AC$ indeed holds. To see that this is actually an equality, note that choosing Q as the intersection of segment CH with ray AD , and choosing P so that its reflection R is the intersection of CH with ray AE , aligns path $CQRH$ with segment CH , thus obtaining the desired minimum $m = AC$.

We may conclude that the largest possible value of m is the largest possible length of AC , namely 2: the length of a diameter of the circle.

17. Let $z = \cos \frac{2\pi}{2011} + i \sin \frac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \dots + \frac{2008 \cdot 2009}{2}x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x . Evaluate $P(z)P(z^2)P(z^3) \dots P(z^{2010})$.

Answer: $\boxed{2011^{2009} \cdot (1005^{2011} - 1004^{2011})}$

Multiply $P(x)$ by $x - 1$ to get

$$P(x)(x - 1) = x^{2009} + 2x^{2008} + \dots + 2009x - \frac{2009 \cdot 2010}{2},$$

or,

$$P(x)(x - 1) + 2010 \cdot 1005 = x^{2009} + 2x^{2008} + \dots + 2009x + 2010.$$

Multiplying by $x - 1$ once again:

$$\begin{aligned}(x-1)(P(x)(x-1) + \frac{2010 \cdot 2011}{2}) &= x^{2010} + x^{2009} + \dots + x - 2010, \\ &= (x^{2010} + x^{2009} + \dots + x + 1) - 2011.\end{aligned}$$

Hence,

$$P(x) = \frac{\frac{(x^{2010} + x^{2009} + \dots + x + 1) - 2011}{x-1} - 2011 \cdot 1005}{x-1}$$

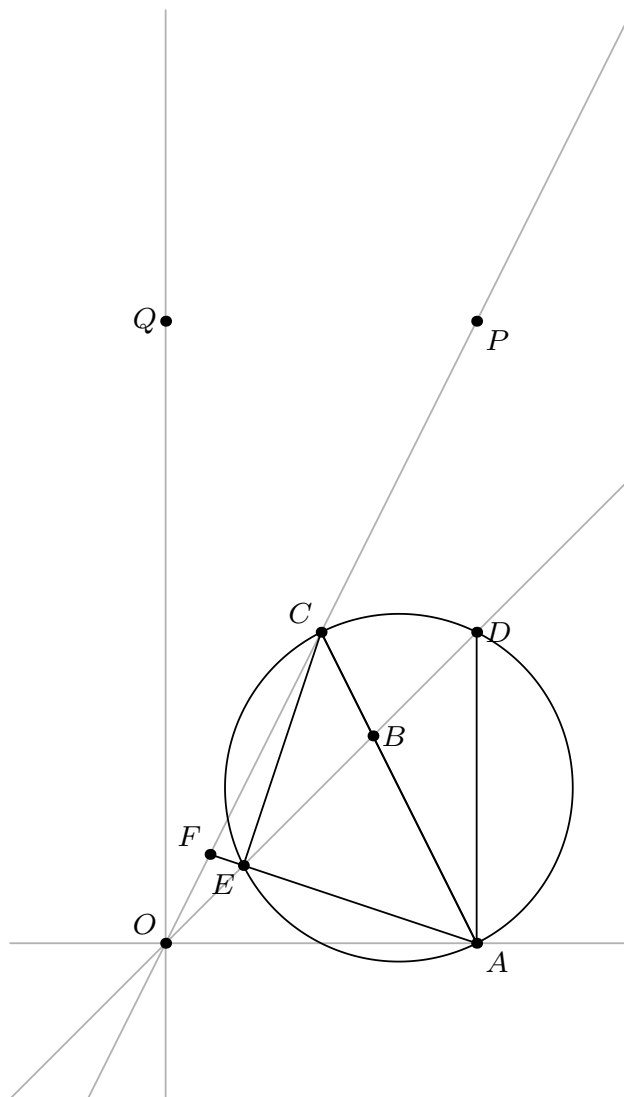
Note that $x^{2010} + x^{2009} + \dots + x + 1$ has z, z^2, \dots, z^{2010} as roots, so they vanish at those points. Plugging those 2010 powers of z into the last equation, and multiplying them together, we obtain

$$\prod_{i=1}^{2010} P(z^i) = \frac{(-2011) \cdot 1005 \cdot (x - \frac{1004}{1005})}{(x-1)^2}.$$

Note that $(x-z)(x-z^2)\dots(x-z^{2010}) = x^{2010} + x^{2009} + \dots + 1$. Using this, the product turns out to be $2011^{2009} \cdot (1005^{2011} - 1004^{2011})$.

18. Collinear points A , B , and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x -axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E , and ray AE intersects $y = 2x$ at F , evaluate AE/EF .

Answer: $\boxed{7}$



Let points O , P , and Q be located at $(0, 0)$, $(a, 2a)$, and $(0, 2a)$, respectively. Note that $BC/AB = 1/2$ implies $[OCD]/[OAD] = 1/2$, so since $[OPD] = [OAD]$, $[OCD]/[OPD] = 1/2$. It follows that $[OCD] = [OPD]$. Hence $OC = CP$. We may conclude that triangles OCQ and PCA are congruent, so $C = (a/2, a)$.

It follows that $\angle ADC$ is right, so the circumcircle of triangle ADC is the midpoint of AC , which is located at $(3a/4, a/2)$. Let $(3a/4, a/2) = H$, and let $E = (b, b)$. Then the power of the point O with respect to the circumcircle of ADC is $OD \cdot OE = 2ab$, but it may also be computed as $OH^2 - HA^2 = 13a/16 - 5a/16 = a/2$. It follows that $b = a/4$, so $E = (a/4, a/4)$.

We may conclude that line AE is $x + 3y = a$, which intersects $y = 2x$ at an x -coordinate of $a/7$. Therefore, $AE/EF = (a - a/4)/(a/4 - a/7) = (3a/4)/(3a/28) = 7$.

Remark: The problem may be solved more quickly if one notes from the beginning that lines OA , OD , OP , and OQ form a harmonic pencil because D is the midpoint of AP and lines OQ and AP are parallel.

19. Let $\{a_n\}$ and $\{b_n\}$ be sequences defined recursively by $a_0 = 2$; $b_0 = 2$, and $a_{n+1} = a_n \sqrt{1 + a_n^2 + b_n^2} - b_n$; $b_{n+1} = b_n \sqrt{1 + a_n^2 + b_n^2} + a_n$. Find the ternary (base 3) representation of a_4 and b_4 .

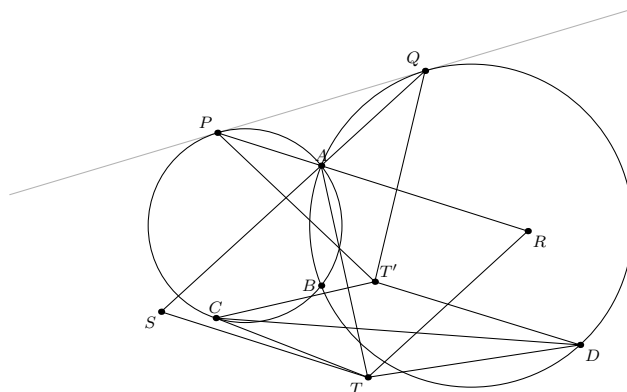
Answer: 1000001100111222 and 2211100110000012 Note first that $\sqrt{1 + a_n^2 + b_n^2} = 3^{2^n}$. The proof is by induction; the base case follows trivially from what is given. For the inductive step, note

that $1 + a_{n+1}^2 + b_{n+1}^2 = 1 + a_n^2(1 + a_n^2 + b_n^2) + b_n^2 - 2a_nb_n\sqrt{1 + a_n^2 + b_n^2} + b_n^2(1 + a_n^2 + b_n^2) + a_n^2 + 2a_nb_n\sqrt{1 + a_n^2 + b_n^2} = 1 + (a_n^2 + b_n^2)(1 + a_n^2 + b_n^2) + a_n^2 + b_n^2 = (1 + a_n^2 + b_n^2)^2$. Invoking the inductive hypothesis, we see that $\sqrt{1 + a_{n+1}^2 + b_{n+1}^2} = (3^{2^n})^2 = 3^{2^{n+1}}$, as desired.

The quickest way to finish from here is to consider a sequence of complex numbers $\{z_n\}$ defined by $z_n = a_n + b_n i$ for all nonnegative integers n . It should be clear that $z_0 = 2 + 2i$ and $z_{n+1} = z_n(3^{2^n} + i)$. Therefore, $z_4 = (2 + 2i)(3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$. This product is difficult to evaluate in the decimal number system, but in ternary the calculation is a cinch! To speed things up, we will use *balanced ternary*¹, in which the three digits allowed are $-1, 0$, and 1 rather than $0, 1$, and 2 . Let $x + yi = (3^{2^0} + i)(3^{2^1} + i)(3^{2^2} + i)(3^{2^3} + i)$, and consider the balanced ternary representation of x and y . For all $0 \leq j \leq 15$, let x_j denote the digit in the 3^j place of x , let y_j denote the digit in the 3^j place of y , and let $b(j)$ denote the number of ones in the binary representation of j . It should be clear that $x_j = -1$ if $b(j) \equiv 2 \pmod{4}$, $x_j = 0$ if $b(j) \equiv 1 \pmod{2}$, and $x_j = 1$ if $b(j) \equiv 0 \pmod{4}$. Similarly, $y_j = -1$ if $b(j) \equiv 1 \pmod{4}$, $y_j = 0$ if $b(j) \equiv 0 \pmod{2}$, and $y_j = 1$ if $b(j) \equiv 3 \pmod{4}$. Converting to ordinary ternary representation, we see that $x = 221211221122001_3$ and $y = 110022202212120_3$. It remains to note that $a_4 = 2x - 2y$ and $b_4 = 2x + 2y$ and perform the requisite arithmetic to arrive at the answer above.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B . Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B . Let points R and S lie along rays PA and QA , respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q . Let O be the circumcenter of triangle ASR , and let C and D be the midpoints of major arcs AP and AQ , respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

Answer: 142.5



We use directed angles throughout the solution.

Let T denote the point such that $\angle TCD = 1/2\angle APQ$ and $\angle TDC = 1/2\angle AQP$. We claim that T is the circumcenter of triangle SAR .

Since $CP = CA$, $QP = RA$, and $\angle CPQ = \angle CPA + \angle APQ = \angle CPA + \angle ACP = \angle CAR$, we have $\triangle CPQ \cong \triangle CAR$. By spiral similarity, we have $\triangle CPA \sim \triangle CQR$.

Let T' denote the reflection of T across CD . Since $\angle TCT' = \angle APQ = \angle ACP$, we have $\triangle TCT' \sim \triangle ACP \sim \triangle RCQ$. Again, by spiral similarity centered at C , we have $\triangle CTR \sim \triangle CT'Q$. But $CT = CT'$, so $\triangle CTR \cong \triangle CT'Q$ and $TR = T'Q$. Similarly, $\triangle DTT' \sim \triangle DAQ$, and spiral similarity centered at D shows that $\triangle DTA \cong \triangle DT'Q$. Thus $TA = T'Q = TR$.

We similarly have $TA = T'P = TS$, so T is indeed the circumcenter. Therefore, we have $\angle COD = \angle CTD = 180^\circ - \frac{45^\circ}{2} - \frac{30^\circ}{2} = 142.5^\circ$.

¹http://en.wikipedia.org/wiki/Balanced_ternary

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Calculus & Combinatorics Individual Test

1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

Answer: $\boxed{2 \cdot 15!^2}$ If we color the desks of the class in a checkerboard pattern, we notice that all of one gender must go in the squares colored black, and the other gender must go in the squares colored white. There are 2 ways to pick which gender goes in which color, $15!$ ways to put the boys into desks and $15!$ ways to put the girls into desks. So the number of ways is $2 \cdot 15!^2$.

(There is a little ambiguity in the problem statement as to whether the 15 boys and the 15 girls are distinguishable or not. If they are not distinguishable, the answer is clearly 2. Given the number of contestants who submitted the answer 2, the graders judged that there was enough ambiguity to justify accepting 2 as a correct answer. So both 2 and $2 \cdot 15!^2$ were accepted as correct answers.)

2. Let a , b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \leq 2$ for all real numbers x . If a and b are real numbers such that the set of possible values of $\int_0^1 f(x) dx$ is the open interval (a, b) , determine $b - a$.

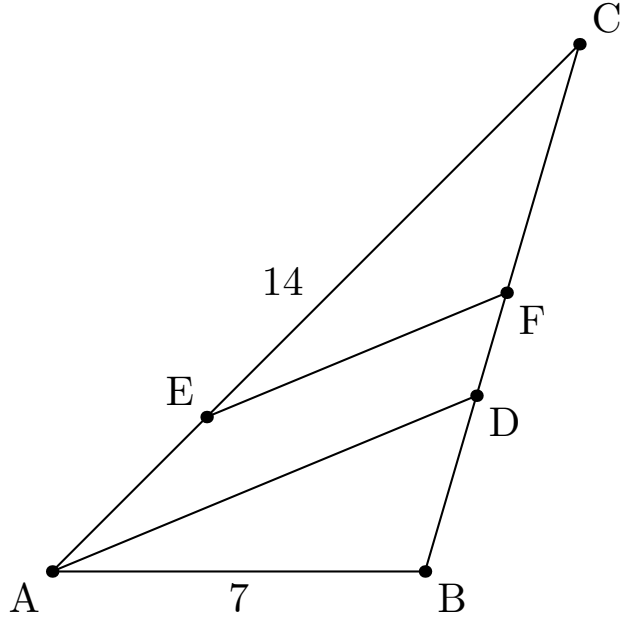
Answer: $\boxed{\frac{3}{4}}$ Draw lines of slope ± 2 passing through $(0, 0)$ and $(1, 1)$. These form a parallelogram with vertices $(0, 0), (.75, 1.5), (1, 1), (.25, -.5)$. By the mean value theorem, no point of $(x, f(x))$ lies outside this parallelogram, but we can construct functions arbitrarily close to the top or the bottom of the parallelogram while satisfying the condition of the problem. So $(b - a)$ is the area of this parallelogram, which is $\frac{3}{4}$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

5. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \quad (1)$$

($[\]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC .

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

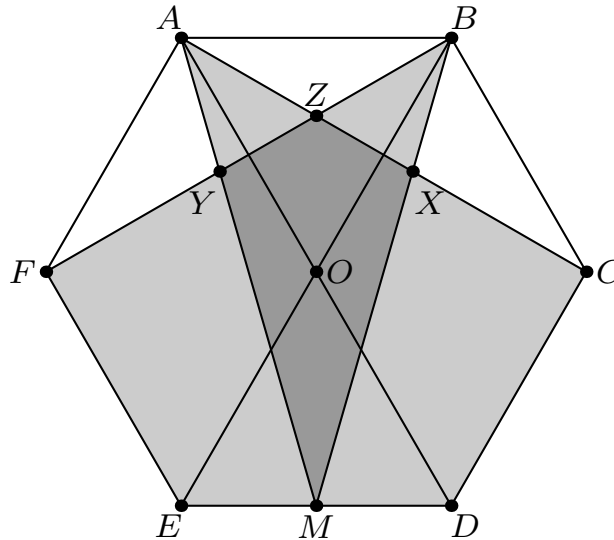
$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

8. Let $f : [0, 1) \rightarrow \mathbb{R}$ be a function that satisfies the following condition: if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3\dots$$

is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \geq k$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$

Determine $f'(\frac{1}{3})$.

Answer: $\boxed{0}$ Note that $\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n}$.

Clearly f is an increasing function. Also for any integer $n \geq 1$, we see from decimal expansions that $f(\frac{1}{3} \pm \frac{1}{10^n}) - f(\frac{1}{3}) = \pm \frac{1}{10^{2n}}$.

Consider h such that $10^{-n-1} \leq |h| < 10^{-n}$. The two properties of f outlined above show that $|f(\frac{1}{3} + h) - f(\frac{1}{3})| < \frac{1}{10^{2n}}$. And from $|\frac{1}{h}| \leq 10^{n+1}$, we get $|\frac{f(\frac{1}{3}+h)-f(\frac{1}{3})}{h}| < \frac{1}{10^{n-1}}$. Taking $n \rightarrow \infty$ gives $h \rightarrow 0$ and $f'(\frac{1}{3}) = \lim_{n \rightarrow \infty} \frac{1}{10^{n-1}} = 0$.

9. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

Answer: 4022 If n is odd and greater than 1, then Goliath makes the last move. No matter what two numbers are on the board, Goliath can combine them to make an even number. Hence Goliath has a winning strategy in this case.

Now suppose n is even. We can replace all numbers on the board by their residues modulo 2. Initially the board reads $1, 0, 1, 0, \dots, 1, 0$. David combines the rightmost 1 and 0 by addition to make 1, so now the board reads $1, 0, 1, 0, \dots, 0, 1$. We call a board of this form a “good” board. When it is Goliath’s turn, and there is a good board, no matter where he moves, David can make a move to restore a good board. Indeed, Goliath must combine a neighboring 0 and 1; David can then combine that number with a neighboring 1 to make 1 and create a good board with two fewer numbers.

David can ensure a good board after his last turn. But a good board with one number is simply 1, so David wins. So David has a winning strategy if n is even. Therefore, the 2011th smallest positive integer greater than 1 for which David can guarantee victory is the 2011th even positive integer, which is 4022.

10. Evaluate $\int_1^\infty \left(\frac{\ln x}{x}\right)^{2011} dx$.

Answer: $\frac{2011!}{2010^{2012}}$ By the chain rule, $\frac{d}{dx}(\ln x)^n = \frac{n \ln^{n-1} x}{x}$.

We calculate the definite integral using integration by parts:

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \left[\frac{(\ln x)^n}{-2010x^{2010}} \right]_{x=1}^{x=\infty} - \int_{x=1}^\infty \frac{n(\ln x)^{n-1}}{-2010x^{2011}} dx$$

But $\ln(1) = 0$, and $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^{2010}} = 0$ for all $n > 0$. So

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \int_{x=1}^\infty \frac{n(\ln x)^{n-1}}{2010x^{2011}} dx$$

It follows that

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \frac{n!}{2010^n} \int_{x=1}^\infty \frac{1}{x^{2011}} dx = \frac{n!}{2010^{n+1}}$$

So the answer is $\frac{2011!}{2010^{2012}}$.

11. Mike and Harry play a game on an 8×8 board. For some positive integer k , Mike chooses k squares and writes an M in each of them. Harry then chooses $k + 1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming “ HMM ” or “ MMH ,” when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

Answer: 16

Suppose Mike writes k M ’s. Let a be the number of squares which, if Harry writes an H in, will yield either HMM or MMH horizontally, and let b be the number of squares which, if Harry writes an H in, will yield either HMM or MMH vertically. We will show that $a \leq k$ and $b \leq k$. Then, it will follow that there are at most $a + b \leq 2k$ squares which Harry cannot write an H in. There will be at least $64 - k - 2k = 64 - 3k$ squares which Harry can write in. If $64 - 3k \geq k + 1$, or $k \leq 15$, then Harry wins.

We will show that $a \leq k$ (that $b \leq k$ will follow by symmetry). Suppose there are a_i M ’s in row i . In each group of 2 or more consecutive M ’s, Harry cannot write H to the left or right of that group, giving at most 2 forbidden squares. Hence a_i is at most the number of M ’s in row i . Summing over the rows gives the desired result.

Mike can win by writing 16 M ’s according to the following diagram:

$\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$
 $\cdot M M \cdot \cdot M M \cdot$
 $\cdot M M \cdot \cdot M M \cdot$
 $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$
 $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$
 $\cdot M M \cdot \cdot M M \cdot$
 $\cdot M M \cdot \cdot M M \cdot$
 $\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot$

12. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the n th turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1} . The player then draws the largest circle that is centered at p_n and contained in O_{n-1} , and calls this circle O_n . The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

Answer: $\boxed{\frac{6\pi}{7}}$ Let $f(r)$ be the average area colored red on a dartboard of radius r if Sarah plays first. Then $f(r)$ is proportional to r^2 . Let $f(r) = (\pi x)r^2$ for some constant x . We want to find $f(1) = \pi x$.

In the first throw, if Sarah's dart hits a point with distance r from the center of O_0 , the radius of O_1 will be $1 - r$. The expected value of the area colored red will be $(\pi - \pi(1 - r)^2) + (\pi(1 - r)^2 - f(1 - r)) = \pi - f(1 - r)$. The value of $f(1)$ is the average value of $\pi - f(1 - r)$ over all points in O_0 . Using polar coordinates, we get

$$f(1) = \frac{\int_0^{2\pi} \int_0^1 (\pi - f(1 - r)) r dr d\theta}{\int_0^{2\pi} \int_0^1 r dr d\theta}$$

$$\pi x = \frac{\int_0^1 (\pi - \pi x(1 - r)^2) r dr}{\int_0^1 r dr}$$

$$\frac{\pi x}{2} = \int_0^1 \pi r - \pi x r(1 - r)^2 dr$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \pi x \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \frac{\pi x}{12}$$

$$\pi x = \frac{6\pi}{7}$$

13. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \dots, (1008, 1005), (1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j) , with (x_i, y_i) left of (x_j, y_j) , erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j} \right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

Answer: $\boxed{504510}$ First, note that none of the numbers will ever be 0. Let \star denote the replacement operation. For each pair on the board (x_i, y_i) define its *primary form* to be (x_i, y_i) and its *secondary*

form to be $[x_i y_i, \frac{x_i}{y_i}]$. Note that the primary form determines the secondary form uniquely and vice versa. In secondary form,

$$[a_1, b_1] \star [a_2, b_2] = \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}} \right) \star \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}} \right) = \left(a_1 b_2, \frac{a_1}{b_2} \right) = [a_1^2, b_2^2].$$

Thus we may replace all pairs on the board by their secondary form and use the above rule for \star instead. From the above rule, we see that if the leftmost number on the board is x , then after one minute it will be x or x^2 depending on whether it was erased in the intervening step, and similarly for the rightmost number. Let k be the number of times the leftmost pair is erased and n be the number of times the rightmost pair is erased. Then the final pair is

$$\left[4022^{2^k}, \left(\frac{1007}{1006} \right)^{2^n} \right]. \quad (2)$$

Any step except the last cannot involve both the leftmost and rightmost pair, so $k + n \leq 1005$. Since every pair must be erased at least once, $k, n \geq 1$. Every pair of integers satisfying the above can occur, for example, by making $1005 - k - n$ moves involving only the pairs in the middle, then making $k - 1$ moves involving the leftmost pair, and finally n moves involving the rightmost pair.

In light of (2), the answer is the number of possible pairs (k, n) , which is

$$\sum_{k=1}^{1004} \sum_{n=1}^{1005-k} 1 = \sum_{k=1}^{1004} 1005 - k = \sum_{k=1}^{1004} k = \frac{1004 \cdot 1005}{2} = 504510.$$

14. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\int_0^1 f(x) dx$.

Answer: $\left[\frac{3}{4}, 1 \right]$ Since the maximum value of f is 1, $\int_0^1 f(x) dx \leq 1$.

By our condition $f(f(x)) = 1$, f is 1 at any point within the range of f . Clearly, 1 is in the range of f , so $f(1) = 1$. Now $f(x)$ is continuous on a closed interval so it attains a minimum value c . Since c is in the range of f , $f(c) = 1$.

If $c = 1$, $f(x) = 1$ for all x and $\int_0^1 f(x) dx = 1$.

Now assume $c < 1$. By the intermediate value theorem, since f is continuous it attains all values between c and 1. So for all $x \geq c$, $f(x) = 1$. Therefore,

$$\int_0^1 f(x) dx = \int_0^c f(x) dx + (1 - c).$$

Since $f(x) \geq c$, $\int_0^c f(x) dx > c^2$, and the equality is strict because f is continuous and thus cannot be c for all $x < c$ and 1 at c . So

$$\int_0^1 f(x) dx > c^2 + (1 - c) = \left(c - \frac{1}{2} \right)^2 + \frac{3}{4} \geq \frac{3}{4}.$$

Therefore $\frac{3}{4} < \int_0^1 f(x) dx \leq 1$, and it is easy to show that every value in this interval can be reached.

15. Let $A = \{1, 2, \dots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

Answer: $2^{2011} - 2012$ Let n be the element of A not in the range of f . Let m be the element of A that is hit twice.

We now sum the total number of functions over n, m . Clearly $f(1) = 1$, and by induction, for $x \leq m$, $f(x) = x$. Also unless $n = 2011$, $f(2011) = 2011$ because f can take no other number to 2011. It follows from backwards induction that for $x > n$, $f(x) = x$. Therefore $n > m$, and there are only $n - m$ values of f that are not fixed.

Now $f(m+1) = m$ or $f(m+1) = m+1$. For $m < k < n$, given the selection of $f(1), f(2), \dots, f(k-1)$, $k-1$ of the $k+1$ possible values of $f(k+1)$ ($1, 2, 3, \dots, k$, and counting m twice) have been taken, so there are two distinct values that $f(k+1)$ can take (one of them is $k+1$, and the other is not, so they are distinct). For $f(n)$, when the other 2010 values of f have been assigned, there is only one missing, so $f(n)$ is determined.

For each integer in $[m, n)$, there are two possible values of f , so there are 2^{n-m-1} different functions f for a given m, n . So our answer is

$$\begin{aligned} \sum_{m=1}^{2010} \sum_{n=m+1}^{2011} 2^{n-m-1} &= \sum_{m=1}^{2010} 2^{-m-1} \sum_{n=m+1}^{2011} 2^n \\ &= \sum_{m=1}^{2010} 2^{-m-1} (2^{2012} - 2^{m+1}) \\ &= \sum_{m=1}^{2010} 2^{2011-m} - 1 \\ &= \left(\sum_{m=1}^{2010} 2^m \right) - 2010 \\ &= 2^{2011} - 2012 \end{aligned}$$

16. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: 2

Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}$, $0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c . The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right) \left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a-1 \\ \frac{1 - (b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x + 2 + \varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following.

- (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4 + 2\varepsilon} > -\sqrt{4 + 2\varepsilon + \varepsilon^2} = -2 - \varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N - 1$ by backwards induction on i , so the desired inequalities follow.
- (ii) G_i is increasing for $i \geq 2N + 1$. Indeed, if $x \geq 2 + \varepsilon$, then $x^2 - x = x(x - 1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \dots$.
- (iii) G_i is unbounded. This follows since $h(x) - x = x(x - 2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so G_i increases faster and faster for $i \geq 2N + 1$.

Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\dots g(2)\dots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin \left(\frac{\theta}{2} \right) = 2 \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots + (-1)^n \left(\theta - \frac{\pi}{2n} \right) \right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots \right) = -2 \cos \left(\frac{\pi}{3} \right) = -1$, as needed.

17. Let $f : (0, 1) \rightarrow (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

Answer: $\boxed{\{-1, 1\}}$ The key step is to notice that for such a function $f, f'(x) \neq 0$ for any x .

Assume, for sake of contradiction that there exists $0 < y < 1$ such that $f'(y) = 0$. Since f' is a continuous function, there is some small interval (c, d) containing y such that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$. Now there exists some n, a such that $\frac{a}{2^n}, \frac{a+1}{2^n}$ are both in the interval (c, d) . From the definition, $\frac{f\left(\frac{a+1}{2^n}\right) - f\left(\frac{a}{2^n}\right)}{\frac{a+1}{2^n} - \frac{a}{2^n}} = 2^n \left(\frac{b'}{2^n} - \frac{b}{2^n} \right) = b' - b$ where b, b' are integers; one is odd, and one is even. So $b' - b$ is an odd integer. Since f is differentiable, by the mean value theorem there exists a point where $f' = b' - b$. But this point is in the interval (c, d) , and $|b' - b| > \frac{1}{2}$. This contradicts the assumption that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$.

Since $f'(x) \neq 0$, and f' is a continuous function, f' is either always positive or always negative. So f is either increasing or decreasing. $f\left(\frac{1}{2}\right) = \frac{1}{2}$ always. If f is increasing, it follows that $f\left(\frac{1}{4}\right) = \frac{1}{4}, f\left(\frac{3}{4}\right) = \frac{3}{4}$, and we can show by induction that indeed $f\left(\frac{a}{2^n}\right) = \frac{a}{2^n}$ for all integers a, n . Since numbers of this form are dense in the interval $(0, 1)$, and f is a continuous function, $f(x) = x$ for all x .

It can be similarly shown that if f is decreasing $f(x) = 1 - x$ for all x . So the only possible values of $f'(\frac{1}{2})$ are $-1, 1$.

Query: if the condition that the derivative is continuous were omitted, would the same result still hold?

18. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n - 1$ members vote randomly.

Answer: $\left[\frac{2^n - 1}{n2^{n-1}} \right]$ Let x be the probability Hermia is elected if Lysander votes for her, and let y be the probability that she wins if Lysander does not vote for her. We are trying to find x , and do so by first finding y . If Lysander votes for Hermia with probability $\frac{1}{2}$ then the probability that Hermia is elected chairman is $\frac{x}{2} + \frac{y}{2}$, but it is also $\frac{1}{n}$ by symmetry. If Lysander does not vote for Hermia, Hermia can get at most 1 vote, and then can only be elected if everyone gets one vote and she wins the tiebreaker. The probability she wins the tiebreaker is $\frac{1}{n}$, and chasing around the circle, the probability that every person gets 1 vote is $\frac{1}{2^{n-1}}$. (Everyone votes for the person to the left, or everyone votes for the person to the right.) Hence

$$y = \frac{1}{n2^{n-1}}.$$

Then $\frac{x}{2} + \frac{1}{n2^n} = \frac{1}{n}$, so solving for x gives

$$x = \frac{2^n - 1}{n2^{n-1}}.$$

19. Let

$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$

and note that F may be expanded as a power series so that $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers (c, d) such that $\lim_{n \rightarrow \infty} \frac{a_n}{n^d} = c$.

Answer: $\left(\frac{1}{6^{2011} 2010!}, 2010 \right)$

First notice that all the roots of $2 - x - x^5$ that are not 1 lie strictly outside the unit circle. As such, we may write $2 - x - x^5$ as $2(1 - x)(1 - r_1x)(1 - r_2x)(1 - r_3x)(1 - r_4x)$ where $|r_i| < 1$, and let $\frac{1}{(2 - x - x^5)} = \frac{b_0}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)}$. We calculate b_0 as $\lim_{x \rightarrow 1} \frac{(1 - x)}{(2 - x - x^5)} = \lim_{x \rightarrow 1} \frac{(-1)}{(-1 - 5x^4)} = \frac{1}{6}$. Now raise the equation above to the 2011th power.

$$\frac{1}{(2 - x - x^5)^{2011}} = \left(\frac{1/6}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)} \right)^{2011}$$

Expand the right hand side using multinomial expansion and then apply partial fractions. The result will be a sum of the terms $(1 - x)^{-k}$ and $(1 - r_i x)^{-k}$, where $k \leq 2011$.

Since $|r_i| < 1$, the power series of $(1 - r_i x)^{-k}$ will have exponentially decaying coefficients, so we only need to consider the $(1 - x)^{-k}$ terms. The coefficient of x^n in the power series of $(1 - x)^{-k}$ is $\binom{n+k-1}{k-1}$, which is a $(k - 1)$ th degree polynomial in variable n . So when we sum up all coefficients, only the power series of $(1 - x)^{-2011}$ will have impact on the leading term n^{2010} .

The coefficient of the $(1-x)^{-2011}$ term in the multinomial expansion is $(\frac{1}{6})^{2011}$. The coefficient of the x^n term in the power series of $(1-x)^{-2011}$ is $\binom{n+2010}{2010} = \frac{1}{2010!}n^{2010} + \dots$. Therefore, $(c, d) = (\frac{1}{6^{2011}2010!}, 2010)$.

20. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \dots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

Answer: 1

Consider the grid whose entries in the j th row are, in order, $2011j - 2010, 2011j - 2009, \dots, 2011j$. Call this grid A_0 . For $k = 1, 2, \dots, 2010$, let grid A_k be the grid obtained from A_0 by swapping the rightmost entry of the k th row with the leftmost entry of the $k+1$ st row. We claim that if $A \in \{A_0, A_1, \dots, A_{2010}\}$, then given any legally numbered grid B such that A and B differ in at least one entry, there exist two integers in the same column of B that occur in the same row of A .

We first consider A_0 . Assume for the sake of contradiction B is a legally numbered grid distinct from A_0 , such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Since the numbers $1, 2, \dots, 2011$ occur in the same row of A_0 , they must all occur in different columns of B . Clearly 1 is the leftmost entry in B 's first row. Let m be the smallest number that does not occur in the first row of B . Since each row is in order, m must be the first entry in its row. But then 1 and m are in the same column of B , a contradiction. It follows that the numbers $1, 2, \dots, 2011$ all occur in the first row of B . Proceeding by induction, $2011j - 2010, 2011j - 2009, \dots, 2011j$ must all occur in the j th row of B for all $1 \leq j \leq 2011$. Since A_0 is the only legally numbered grid satisfying this condition, we have reached the desired contradiction.

Now note that if $A \in \{A_1, \dots, A_{2010}\}$, there exist two integers in the same column of A_0 that occur in the same row of A . In particular, if $A = A_k$ and $1 \leq k \leq 2010$, then the integers $2011k - 2010$ and $2011k + 1$ occur in the same column of A_0 and in the same row of A_k . Therefore, it suffices to show that for all $1 \leq k \leq 2010$, there is no legally numbered grid B distinct from A_k and A_0 such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Assume for the sake of contradiction that there does exist such a grid B . By the same logic as above, applied to the first $k - 1$ rows and applied backwards to the last $2010 - k - 1$ rows, we see that B may only differ from A_k in the k th and $k + 1$ st rows. However, there are only two legally numbered grids that are identical to A_k outside of rows k and $k + 1$, namely A_0 and A_k . This proves the claim.

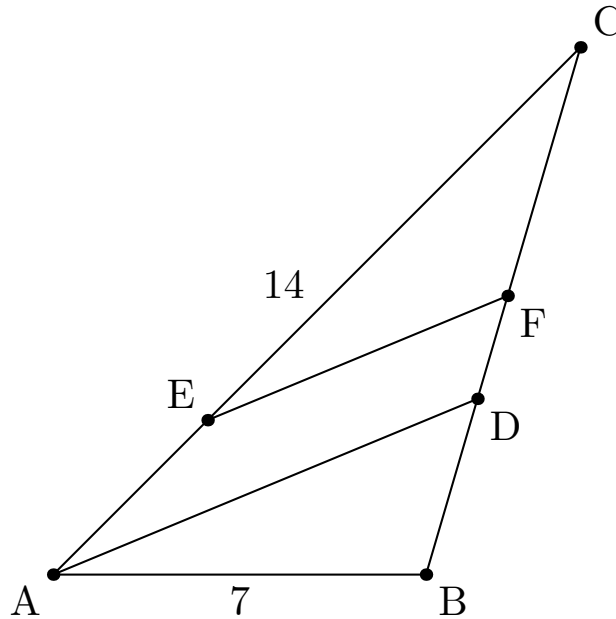
It remains only to note that, by the pigeonhole principle, if one of Alice's grids is A_0 , then there exists a positive integer k , $1 \leq k \leq 2010$, such that A_k is not one of the Alice's grids. Therefore, if Bob sets his initial grid to be A_0 , he will require only one swap to switch his grid to A_k after examining Alice's grids. If A_0 is not among Alice's grids, then if Bob sets his initial grid to be A_0 , he will not in fact require any swaps at all.

14th Annual Harvard-MIT Mathematics Tournament
 Saturday 12 February 2011

Calculus & Geometry Individual Test

1. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: 13



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

($[\]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC .

2. Let a, b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

Answer: 4 If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \leq 2$ for all real numbers x . If a and b are real numbers such that the set of possible values of $\int_0^1 f(x) dx$ is the open interval (a, b) , determine $b - a$.

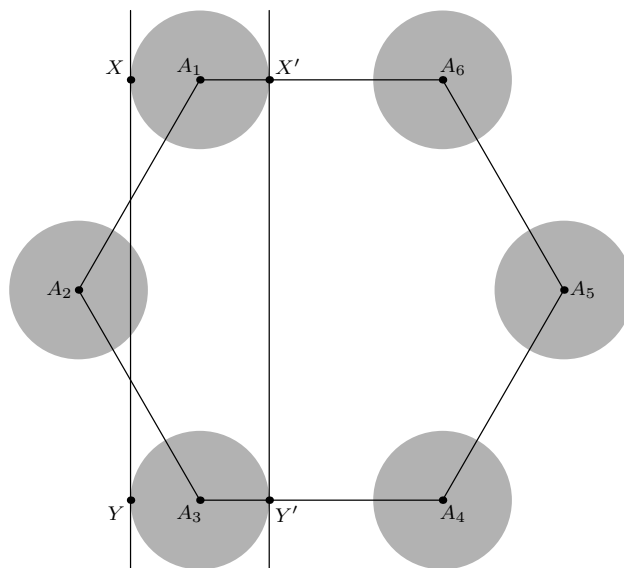
Answer: $\boxed{\frac{3}{4}}$ Draw lines of slope ± 2 passing through $(0, 0)$ and $(1, 1)$. These form a parallelogram with vertices $(0, 0), (.75, 1.5), (1, 1), (.25, -.5)$. By the mean value theorem, no point of $(x, f(x))$ lies outside this parallelogram, but we can construct functions arbitrarily close to the top or the bottom of the parallelogram while satisfying the condition of the problem. So $(b - a)$ is the area of this parallelogram, which is $\frac{3}{4}$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

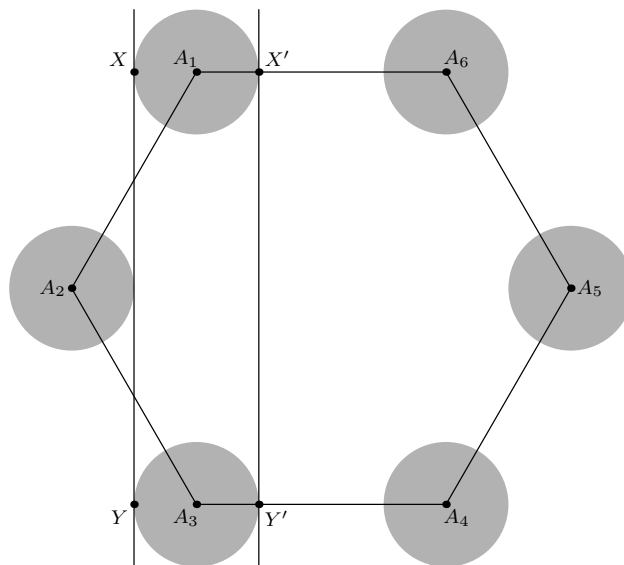
5. Let H be a regular hexagon of side length x . Call a hexagon in the same plane a “distortion” of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

Answer: $\boxed{4}$



Let $H = A_1A_2A_3A_4A_5A_6$ be the hexagon, and for all $1 \leq i \leq 6$, let points A'_i be considered such that $A_iA'_i < 1$. Let $H' = A'_1A'_2A'_3A'_4A'_5A'_6$, and consider all indices modulo 6. For any point P in the plane, let $D(P)$ denote the unit disk $\{Q \mid PQ < 1\}$ centered at P ; it follows that $A'_i \in D(A_i)$.

Let X and X' be points on line A_1A_6 , and let Y and Y' be points on line A_3A_4 such that $A_1X = A_1X' = A_3Y = A_3Y' = 1$ and X and X' lie on opposite sides of A_1 and Y and Y' lie on opposite sides of A_3 . If X' and Y' lie on segments A_1A_6 and A_3A_4 , respectively, then segment $A'_1A'_3$ lies between the lines XY and $X'Y'$. Note that $\frac{x}{2}$ is the distance from A_2 to A_1A_3 .



If $\frac{x}{2} \geq 2$, then $C(A_2)$ cannot intersect line XY , since the distance from XY to A_1A_3 is 1 and the distance from XY to A_2 is at least 1. Therefore, A_1A_3 separates A_2 from the other 3 vertices of the hexagon. By analogous reasoning applied to the other vertices, we may conclude that H' is convex.

If $\frac{x}{2} < 2$, then $C(A_2)$ intersects XY , so by choosing $A'_1 = X$ and $A'_3 = Y$, we see that we may choose A'_2 on the opposite side of XY , in which case H' will be concave. Hence the answer is 4, as desired.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

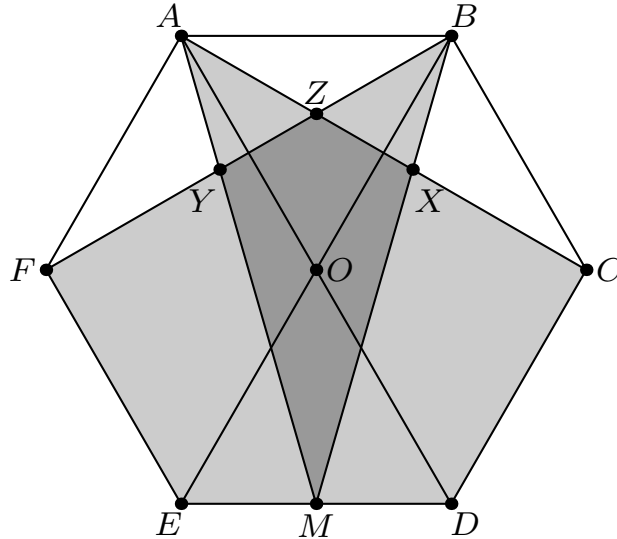
$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

Answer: $\boxed{0}$



Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

8. Let $f : [0, 1) \rightarrow \mathbb{R}$ be a function that satisfies the following condition: if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3 \dots$$

is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \geq k$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$

Determine $f'(\frac{1}{3})$.

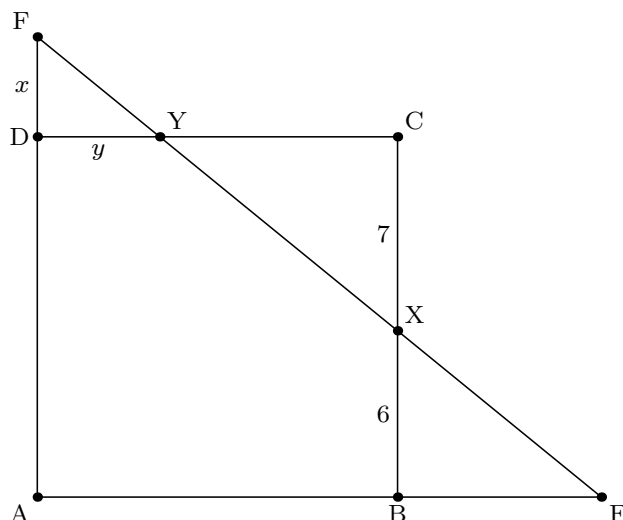
Answer: $\boxed{0}$ Note that $\frac{1}{3} = \sum_{n=1}^{\infty} \frac{3}{10^n}$.

Clearly f is an increasing function. Also for any integer $n \geq 1$, we see from decimal expansions that $f(\frac{1}{3} \pm \frac{1}{10^n}) - f(\frac{1}{3}) = \pm \frac{1}{10^{2n}}$.

Consider h such that $10^{-n-1} \leq |h| < 10^{-n}$. The two properties of f outlined above show that $|f(\frac{1}{3} + h) - f(\frac{1}{3})| < \frac{1}{10^{2n}}$. And from $|\frac{1}{h}| \leq 10^{n+1}$, we get $|\frac{f(\frac{1}{3}+h)-f(\frac{1}{3})}{h}| < \frac{1}{10^{n-1}}$. Taking $n \rightarrow \infty$ gives $h \rightarrow 0$ and $f'(\frac{1}{3}) = \lim_{n \rightarrow \infty} \frac{1}{10^{n-1}} = 0$.

9. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD , respectively, so that the area of square $ABCD$ equals the area of triangle AEF . If EF intersects BC at X and $BX = 6$, determine DF .

Answer: $\boxed{\sqrt{13}}$



First Solution

Let Y be the point of intersection of lines EF and CD . Note that $[ABCD] = [AEF]$ implies that $[BEX] + [DYF] = [CYX]$. Since $\triangle BEX \sim \triangle CYX \sim \triangle DYF$, there exists some constant r such that $[BEX] = r \cdot BX^2$, $[YDF] = r \cdot CX^2$, and $[CYX] = r \cdot DF^2$. Hence $BX^2 + DF^2 = CX^2$, so $DF = \sqrt{CX^2 - BX^2} = \sqrt{49 - 36} = \sqrt{13}$.

Second Solution

Let $x = DF$ and $y = YD$. Since $\triangle BXE \sim \triangle CXY \sim \triangle DFY$, we have

$$\frac{BE}{BX} = \frac{CY}{CX} = \frac{DY}{DF} = \frac{y}{x}.$$

Using $BX = 6$, $XC = 7$ and $CY = 13 - y$ we get $BE = \frac{6y}{x}$ and $\frac{13-y}{7} = \frac{y}{x}$. Solving this last equation for y gives $y = \frac{13x}{x+7}$. Now $[ABCD] = [AEF]$ gives

$$\begin{aligned} 169 &= \frac{1}{2}AE \cdot AF = \frac{1}{2} \left(13 + \frac{6y}{x} \right) (13 + x). \\ 169 &= 6y + 13x + \frac{78y}{x} \\ 13 &= \frac{6x}{x+7} + x + \frac{78}{x+7} \\ 0 &= x^2 - 13. \end{aligned}$$

Thus $x = \sqrt{13}$.

10. Evaluate $\int_1^\infty \left(\frac{\ln x}{x} \right)^{2011} dx$.

Answer: $\frac{2011!}{2010^{2012}}$ By the chain rule, $\frac{d}{dx} (\ln x)^n = \frac{n \ln^{n-1} x}{x}$.

We calculate the definite integral using integration by parts:

$$\int_{x=1}^\infty \frac{(\ln x)^n}{x^{2011}} dx = \left[\frac{(\ln x)^n}{-2010x^{2010}} \right]_{x=1}^{x=\infty} - \int_{x=1}^\infty \frac{n(\ln x)^{n-1}}{-2010x^{2011}} dx$$

But $\ln(1) = 0$, and $\lim_{x \rightarrow \infty} \frac{(\ln x)^n}{x^{2010}} = 0$ for all $n > 0$. So

$$\int_{x=1}^{\infty} \frac{(\ln x)^n}{x^{2011}} dx = \int_{x=1}^{\infty} \frac{n(\ln x)^{n-1}}{2010x^{2011}} dx$$

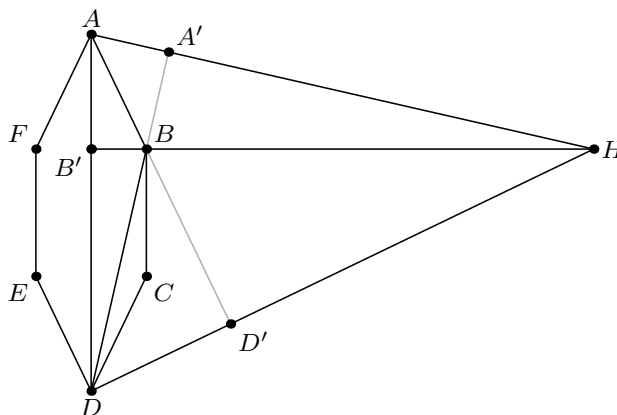
It follows that

$$\int_{x=1}^{\infty} \frac{(\ln x)^n}{x^{2011}} dx = \frac{n!}{2010^n} \int_{x=1}^{\infty} \frac{1}{x^{2011}} dx = \frac{n!}{2010^{n+1}}$$

So the answer is $\frac{2011!}{2010^{2012}}$.

11. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC , AD , and EF are parallel. Let H be the orthocenter of triangle ABD . If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

Answer: 3

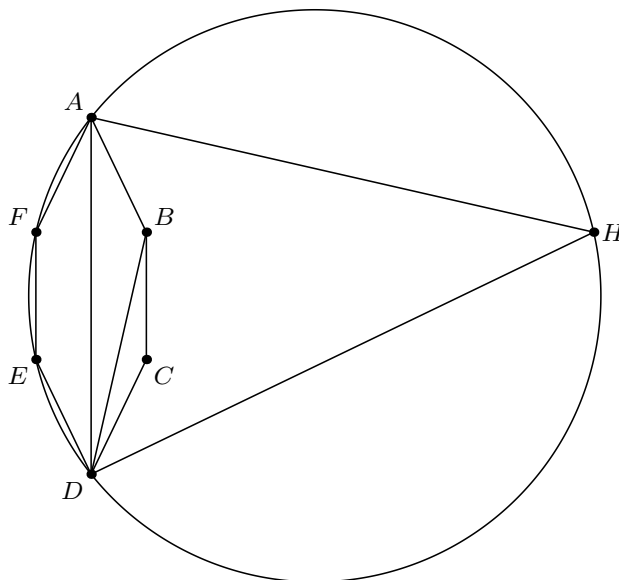


Note that $ABCD$ and $DEFA$ are isosceles trapezoids, so $\angle BAD = \angle CDA$ and $\angle FAD = \angle EDA$. In order for the hexagon to be convex, the angles at B , C , E , and F have to be obtuse, so $\angle A = \angle D = 4^\circ$. Letting s be a side length of the hexagon, $AD = AB \cos \angle BAD + BC + CD \cos \angle CDA = s(1 + 2 \cos \angle BAD)$, so $\angle BAD$ is uniquely determined by AD . Since the same equation holds for trapezoid $DEFA$, it follows that $\angle BAD = \angle FAD = \angle CDA = \angle EDA = 2^\circ$. Then $\angle BCD = 180^\circ - 2^\circ = 178^\circ$. Since $\triangle BCD$ is isosceles, $\angle CDB = 1^\circ$ and $\angle BDA = 1^\circ$. (One may also note that $\angle BDA = 1^\circ$ by observing that equal lengths AB and BC must intercept equal arcs on the circumcircle of isosceles trapezoid $ABCD$).

Let A' , B' , and D' be the feet of the perpendiculars from A , B , and D to BD , DA , and AB , respectively. Angle chasing yields

$$\begin{aligned} \angle AHD &= \angle AHB' + \angle DHB' = (90^\circ - \angle A'AB') + (90^\circ - \angle D'DB') \\ &= \angle BDA + \angle BAD = 1^\circ + 2^\circ = 3^\circ \\ \angle HAD &= 90^\circ - \angle AHB' = 89^\circ \\ \angle HDA &= 90^\circ - \angle DHB' = 88^\circ \end{aligned}$$

Hence the smallest angle in $\triangle HAD$ is 3° .



It is faster, however, to draw the circumcircle of $DEFA$, and to note that since H is the orthocenter of triangle ABD , B is the orthocenter of triangle HAD . Then since F is the reflection of B across AD , quadrilateral $HAFD$ is cyclic, so $\angle AHD = \angle ADF + \angle DAF = 1^\circ + 2^\circ = 3^\circ$, as desired.

12. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the n th turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1} . The player then draws the largest circle that is centered at p_n and contained in O_{n-1} , and calls this circle O_n . The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

Answer: $\boxed{\frac{6\pi}{7}}$ Let $f(r)$ be the average area colored red on a dartboard of radius r if Sarah plays first. Then $f(r)$ is proportional to r^2 . Let $f(r) = (\pi x)r^2$ for some constant x . We want to find $f(1) = \pi x$.

In the first throw, if Sarah's dart hits a point with distance r from the center of O_0 , the radius of O_1 will be $1 - r$. The expected value of the area colored red will be $(\pi - \pi(1 - r)^2) + (\pi(1 - r)^2 - f(1 - r)) = \pi - f(1 - r)$. The value of $f(1)$ is the average value of $\pi - f(1 - r)$ over all points in O_0 . Using polar coordinates, we get

$$f(1) = \frac{\int_0^{2\pi} \int_0^1 (\pi - f(1 - r)) r dr d\theta}{\int_0^{2\pi} \int_0^1 r dr d\theta}$$

$$\pi x = \frac{\int_0^1 (\pi - \pi x(1 - r)^2) r dr}{\int_0^1 r dr}$$

$$\frac{\pi x}{2} = \int_0^1 \pi r - \pi x r(1 - r)^2 dr$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \pi x \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right)$$

$$\frac{\pi x}{2} = \frac{\pi}{2} - \frac{\pi x}{12}$$

$$\pi x = \frac{6\pi}{7}$$

13. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD . If $AI = 2$ as well, find the minimum value of the length of diagonal BD .

Answer: $\boxed{2\sqrt{3}}$ Let T be the point where the incircle intersects AD , and let r be the inradius and R be the circumradius of $\triangle ABD$. Since $BC = CD = 2$, C is on the midpoint of arc BD on the opposite side of BD as A , and hence on the angle bisector of A . Thus A , I , and C are collinear. We have the following formulas:

$$AI = \frac{IM}{\sin \angle IAM} = \frac{r}{\sin \frac{A}{2}}$$

$$BC = 2R \sin \frac{A}{2}$$

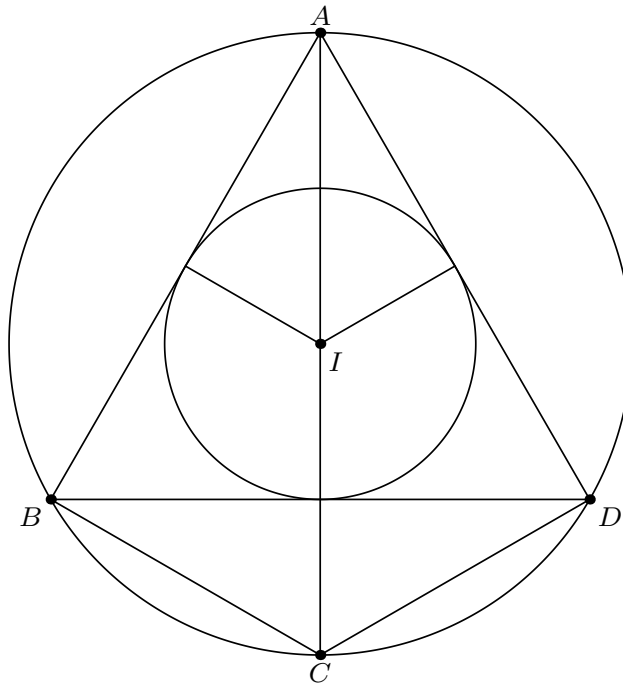
$$BD = 2R \sin A$$

The last two equations follow from the extended law of sines on $\triangle ABC$ and $\triangle ABD$, respectively.

Using $AI = 2 = BC$ gives $\sin^2 \frac{A}{2} = \frac{r}{2R}$. However, it is well-known that $R \geq 2r$ with equality for an equilateral triangle (one way to see this is the identity $1 + \frac{r}{R} = \cos A + \cos B + \cos D$). Hence $\sin^2 \frac{A}{2} \leq \frac{1}{4}$ and $\frac{A}{2} \leq 30^\circ$. Then

$$BD = 2R \left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right) = BC \cdot 2 \cos \frac{A}{2} \geq 2 \left(2 \cdot \frac{\sqrt{3}}{2} \right) = 2\sqrt{3}$$

with equality when $\triangle ABD$ is equilateral.



Remark: Similar but perhaps simpler computations can be made by noting that if AC intersects BD at X , then $AB/BX = AD/DX = 2$, which follows from the exterior angle bisector theorem; if I_A is the A -excenter of triangle ABC , then $AI_A/XI_A = 2$ since it is well-known that C is the circumcenter of cyclic quadrilateral $BIDI_A$.

14. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\int_0^1 f(x) dx$.

Answer: $\left[\frac{3}{4}, 1\right]$ Since the maximum value of f is 1, $\int_0^1 f(x) dx \leq 1$.

By our condition $f(f(x)) = 1$, f is 1 at any point within the range of f . Clearly, 1 is in the range of f , so $f(1) = 1$. Now $f(x)$ is continuous on a closed interval so it attains a minimum value c . Since c is in the range of f , $f(c) = 1$.

If $c = 1$, $f(x) = 1$ for all x and $\int_0^1 f(x) dx = 1$.

Now assume $c < 1$. By the intermediate value theorem, since f is continuous it attains all values between c and 1. So for all $x \geq c$, $f(x) = 1$. Therefore,

$$\int_0^1 f(x) dx = \int_0^c f(x) dx + (1 - c).$$

Since $f(x) \geq c$, $\int_0^c f(x) dx > c^2$, and the equality is strict because f is continuous and thus cannot be c for all $x < c$ and 1 at c . So

$$\int_0^1 f(x) dx > c^2 + (1 - c) = \left(c - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}.$$

Therefore $\frac{3}{4} < \int_0^1 f(x) dx \leq 1$, and it is easy to show that every value in this interval can be reached.

15. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}$, $0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c . The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right)\left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a-1 \\ \frac{1 - (b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x+2+\varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following. (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N-1$. First note $G_{2N} = -\sqrt{4+2\varepsilon} > -\sqrt{4+2\varepsilon+\varepsilon^2} = -2-\varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2-\varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2-\varepsilon$, and it sends $(-2-\varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2-\varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N-1$ by backwards induction on i , so the desired inequalities follow. (ii) G_i is increasing for $i \geq 2N+1$. Indeed, if $x \geq 2+\varepsilon$, then $x^2 - x = x(x-1) > 2+\varepsilon$ so $h(x) > x$. Hence $2+\varepsilon = G_{2N+1} < G_{2N+2} < \dots$. (iii) G_i is unbounded. This follows since $h(x) - x = x(x-2) - 2 - \varepsilon$ is increasing for $x > 2+\varepsilon$, so G_i increases faster and faster for $i \geq 2N+1$. Now define $f(x) = h(x+G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2+\varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\dots g(2)\dots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

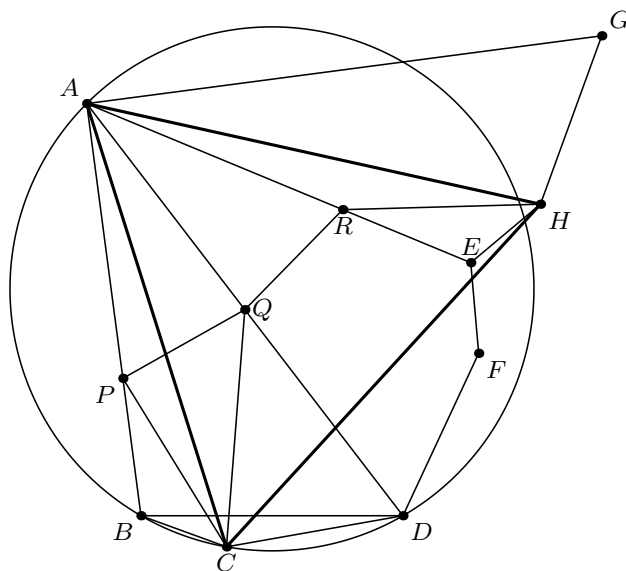
But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin \left(\frac{\theta}{2} \right) = 2 \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots + (-1)^n \left(\theta - \frac{\pi}{2^n} \right) \right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots \right) = -2 \cos \left(\frac{\pi}{3} \right) = -1$, as needed.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD , respectively. Determine the maximum value of m .

Answer: 2



For a fixed quadrilateral $ABCD$ as described, we first show that m , the minimum possible length of $CP + PQ + QC$, equals the length of AC . Reflect B , C , and P across line AD to points E , F , and R , respectively, and then reflect D and F across AE to points G and H , respectively. These two reflections combine to give a 60° rotation around A , so triangle ACH is equilateral. It also follows that RH is a 60° rotation of PC around A , so, in particular, these segments have the same length. Because $QR = QP$ by reflection,

$$CP + PQ + QC = CQ + QR + RH.$$

The latter is the length of a broken path $CQRH$ from C to H , and by the “shortest path is a straight line” principle, this total length is at least as long as $CH = CA$. (More directly, this follows from the triangle inequality: $(CQ + QR) + RH \geq CR + RH \geq CH$). Therefore, the lower bound $m \geq AC$ indeed holds. To see that this is actually an equality, note that choosing Q as the intersection of segment CH with ray AD , and choosing P so that its reflection R is the intersection of CH with ray AE , aligns path $CQRH$ with segment CH , thus obtaining the desired minimum $m = AC$.

We may conclude that the largest possible value of m is the largest possible length of AC , namely 2: the length of a diameter of the circle.

17. Let $f : (0, 1) \rightarrow (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f(\frac{a}{2^n}) = \frac{b}{2^n}$. Determine the set of possible values of $f'(\frac{1}{2})$.

Answer: $\{-1, 1\}$ The key step is to notice that for such a function f , $f'(x) \neq 0$ for any x .

Assume, for sake of contradiction that there exists $0 < y < 1$ such that $f'(y) = 0$. Since f' is a continuous function, there is some small interval (c, d) containing y such that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$. Now there exists some n, a such that $\frac{a}{2^n}, \frac{a+1}{2^n}$ are both in the interval (c, d) . From the definition, $\frac{f(\frac{a+1}{2^n}) - f(\frac{a}{2^n})}{\frac{a+1}{2^n} - \frac{a}{2^n}} = 2^n(\frac{b'}{2^n} - \frac{b}{2^n}) = b' - b$ where b, b' are integers; one is odd, and one is even. So $b' - b$ is an odd integer. Since f is differentiable, by the mean value theorem there exists a point where $f' = b' - b$. But this point is in the interval (c, d) , and $|b' - b| > \frac{1}{2}$. This contradicts the assumption that $|f'(x)| \leq \frac{1}{2}$ for all $x \in (c, d)$.

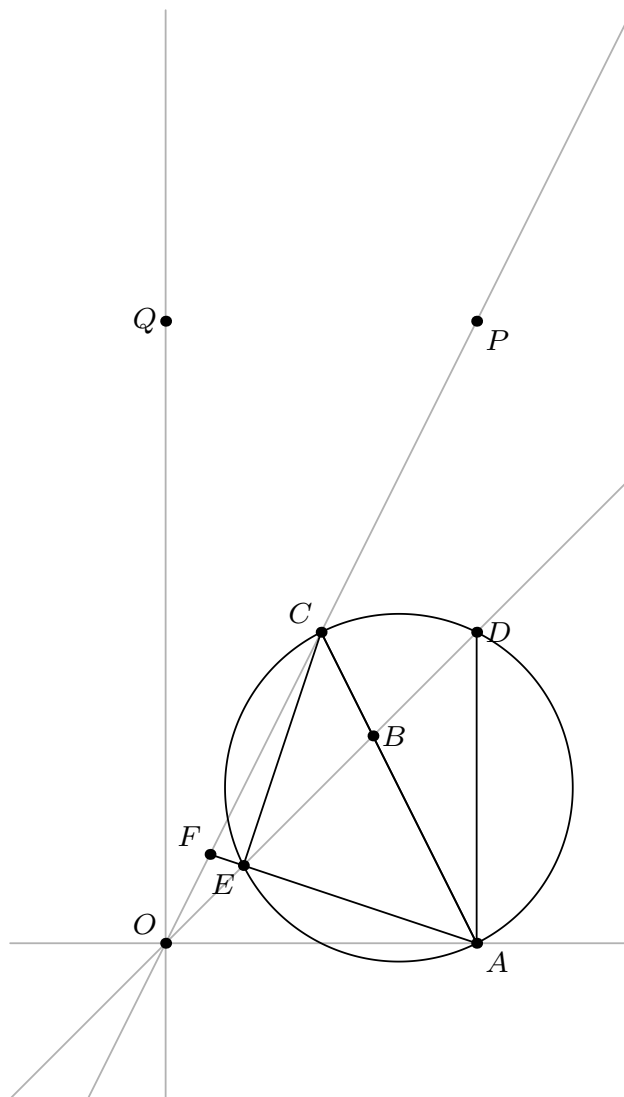
Since $f'(x) \neq 0$, and f' is a continuous function, f' is either always positive or always negative. So f is either increasing or decreasing. $f(\frac{1}{2}) = \frac{1}{2}$ always. If f is increasing, it follows that $f(\frac{1}{4}) = \frac{1}{4}$, $f(\frac{3}{4}) = \frac{3}{4}$, and we can show by induction that indeed $f(\frac{a}{2^n}) = \frac{a}{2^n}$ for all integers a, n . Since numbers of this form are dense in the interval $(0, 1)$, and f is a continuous function, $f(x) = x$ for all x .

It can be similarly shown that if f is decreasing $f(x) = 1 - x$ for all x . So the only possible values of $f'(\frac{1}{2})$ are $-1, 1$.

Query: if the condition that the derivative is continuous were omitted, would the same result still hold?

18. Collinear points A , B , and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x -axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E , and ray AE intersects $y = 2x$ at F , evaluate AE/EF .

Answer: $\boxed{7}$



Let points O , P , and Q be located at $(0, 0)$, $(a, 2a)$, and $(0, 2a)$, respectively. Note that $BC/AB = 1/2$ implies $[OCD]/[OAD] = 1/2$, so since $[OPD] = [OAD]$, $[OCD]/[OPD] = 1/2$. It follows that $[OCD] = [OPD]$. Hence $OC = CP$. We may conclude that triangles OCQ and PCA are congruent, so $C = (a/2, a)$.

It follows that $\angle ADC$ is right, so the circumcircle of triangle ADC is the midpoint of AC , which is located at $(3a/4, a/2)$. Let $(3a/4, a/2) = H$, and let $E = (b, b)$. Then the power of the point O with respect to the circumcircle of ADC is $OD \cdot OE = 2ab$, but it may also be computed as $OH^2 - HA^2 = 13a/16 - 5a/16 = a/2$. It follows that $b = a/4$, so $E = (a/4, a/4)$.

We may conclude that line AE is $x + 3y = a$, which intersects $y = 2x$ at an x -coordinate of $a/7$. Therefore, $AE/EF = (a - a/4)/(a/4 - a/7) = (3a/4)/(3a/28) = 7$.

Remark: The problem may be solved more quickly if one notes from the beginning that lines OA , OD , OP , and OQ form a harmonic pencil because D is the midpoint of AP and lines OQ and AP are parallel.

19. Let

$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$

and note that F may be expanded as a power series so that $F(x) = \sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers (c, d) such that $\lim_{n \rightarrow \infty} \frac{a_n}{n^d} = c$.

Answer: $\left(\frac{1}{6^{2011} 2010!}, 2010\right)$ First notice that all the roots of $2 - x - x^5$ that are not 1 lie strictly outside the unit circle. As such, we may write $2 - x - x^5$ as $2(1 - x)(1 - r_1x)(1 - r_2x)(1 - r_3x)(1 - r_4x)$ where $|r_i| < 1$, and let $\frac{1}{(2 - x - x^5)} = \frac{b_0}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)}$. We calculate b_0 as $\lim_{x \rightarrow 1} \frac{(1 - x)}{(2 - x - x^5)} = \lim_{x \rightarrow 1} \frac{(-1)}{(-1 - 5x^4)} = \frac{1}{6}$.

Now raise the equation above to the 2011th power.

$$\frac{1}{(2 - x - x^5)^{2011}} = \left(\frac{1/6}{(1 - x)} + \frac{b_1}{(1 - r_1x)} + \dots + \frac{b_4}{(1 - r_4x)} \right)^{2011}$$

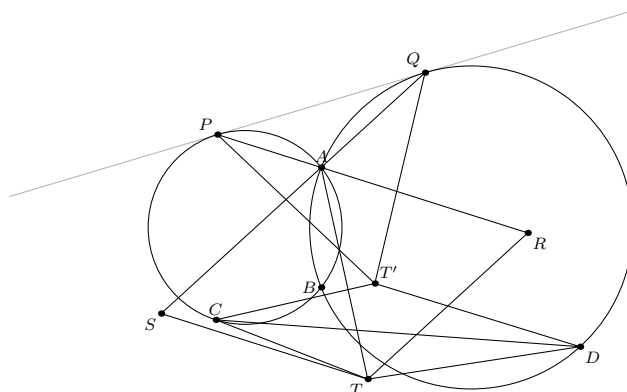
Expand the right hand side using multinomial expansion and then apply partial fractions. The result will be a sum of the terms $(1 - x)^{-k}$ and $(1 - r_i x)^{-k}$, where $k \leq 2011$.

Since $|r_i| < 1$, the power series of $(1 - r_i x)^{-k}$ will have exponentially decaying coefficients, so we only need to consider the $(1 - x)^{-k}$ terms. The coefficient of x^n in the power series of $(1 - x)^{-k}$ is $\binom{n+k-1}{k-1}$, which is a $(k - 1)$ th degree polynomial in variable n . So when we sum up all coefficients, only the power series of $(1 - x)^{-2011}$ will have impact on the leading term n^{2010} .

The coefficient of the $(1 - x)^{-2011}$ term in the multinomial expansion is $\left(\frac{1}{6}\right)^{2011}$. The coefficient of the x^n term in the power series of $(1 - x)^{-2011}$ is $\binom{n+2010}{2010} = \frac{1}{2010!} n^{2010} + \dots$. Therefore, $(c, d) = \left(\frac{1}{6^{2011} 2010!}, 2010\right)$.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B . Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B . Let points R and S lie along rays PA and QA , respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q . Let O be the circumcenter of triangle ASR , and let C and D be the midpoints of major arcs AP and AQ , respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

Answer: $\boxed{142.5}$



We use directed angles throughout the solution.

Let T denote the point such that $\angle TCD = 1/2\angle APQ$ and $\angle TDC = 1/2\angle AQP$. We claim that T is the circumcenter of triangle SAR .

Since $CP = CA$, $QP = RA$, and $\angle CPQ = \angle CPA + \angle APQ = \angle CPA + \angle ACP = \angle CAR$, we have $\triangle CPQ \cong \triangle CAR$. By spiral similarity, we have $\triangle CPA \sim \triangle CQR$.

Let T' denote the reflection of T across CD . Since $\angle TCT' = \angle APQ = \angle ACP$, we have $\triangle TCT' \sim \triangle ACP \sim \triangle RCQ$. Again, by spiral similarity centered at C , we have $\triangle CTR \sim \triangle CT'Q$. But $CT = CT'$, so $\triangle CTR \cong \triangle CT'Q$ and $TR = T'Q$. Similarly, $\triangle DTT' \sim \triangle DAQ$, and spiral similarity centered at D shows that $\triangle DTA \cong \triangle DT'Q$. Thus $TA = T'Q = TR$.

We similarly have $TA = T'P = TS$, so T is indeed the circumcenter. Therefore, we have $\angle COD = \angle CTD = 180^\circ - \frac{45^\circ}{2} - \frac{30^\circ}{2} = 142.5^\circ$.

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Combinatorics & Geometry Individual Test

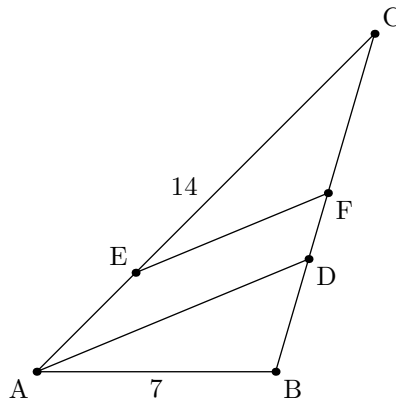
1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

Answer: $\boxed{2 \cdot 15!^2}$ If we color the desks of the class in a checkerboard pattern, we notice that all of one gender must go in the squares colored black, and the other gender must go in the squares colored white. There are 2 ways to pick which gender goes in which color, $15!$ ways to put the boys into desks and $15!$ ways to put the girls into desks. So the number of ways is $2 \cdot 15!^2$.

(There is a little ambiguity in the problem statement as to whether the 15 boys and the 15 girls are distinguishable or not. If they are not distinguishable, the answer is clearly 2. Given the number of contestants who submitted the answer 2, the graders judged that there was enough ambiguity to justify accepting 2 as a correct answer. So both 2 and $2 \cdot 15!^2$ were accepted as correct answers.)

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D . If there exist points E and F on sides AC and BC , respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC .

Answer: $\boxed{13}$



Note that such E, F exist if and only if

$$\frac{[ADC]}{[ADB]} = 2. \tag{1}$$

($[\]$ denotes area.) Since AD is the angle bisector, and the ratio of areas of triangles with equal height is the ratio of their bases,

$$\frac{AC}{AB} = \frac{DC}{DB} = \frac{[ADC]}{[ADB]}.$$

Hence (1) is equivalent to $AC = 2AB = 14$. Then BC can be any length d such that the triangle inequalities are satisfied:

$$\begin{aligned} d + 7 &> 14 \\ 7 + 14 &> d \end{aligned}$$

Hence $7 < d < 21$ and there are 13 possible integral values for BC .

3. Let a , b , and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

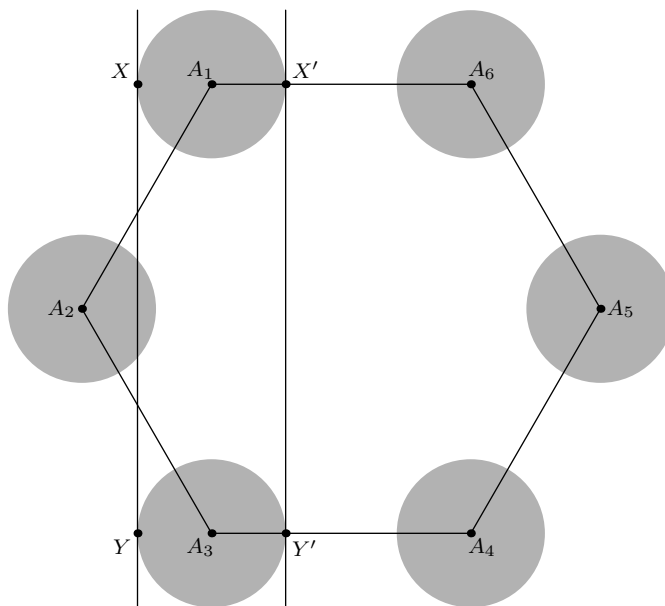
Answer: $\boxed{4}$ If all the polynomials had real roots, their discriminants would all be nonnegative: $a^2 \geq 4bc$, $b^2 \geq 4ca$, and $c^2 \geq 4ab$. Multiplying these inequalities gives $(abc)^2 \geq 64(abc)^2$, a contradiction. Hence one of the quadratics has no real roots. The maximum of 4 real roots is attainable: for example, the values $(a, b, c) = (1, 5, 6)$ give $-2, -3$ as roots to $x^2 + 5x + 6$ and $-1, -\frac{1}{5}$ as roots to $5x^2 + 6x + 1$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

Answer: $\boxed{2^{n-1}}$ Note that Josh must pass through the center square of each row. There are 2 ways to get from the center square of row k to the center square of row $k + 1$. So there are 2^{n-1} ways to get to the center square of row n .

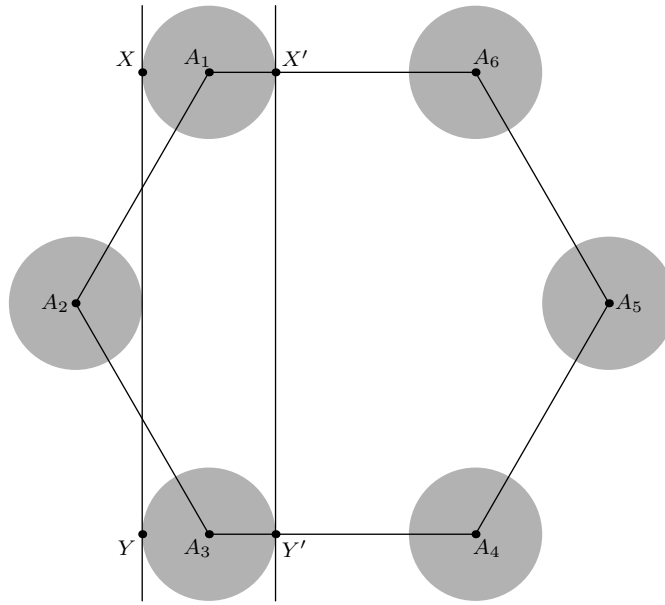
5. Let H be a regular hexagon of side length x . Call a hexagon in the same plane a “distortion” of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

Answer: $\boxed{4}$



Let $H = A_1A_2A_3A_4A_5A_6$ be the hexagon, and for all $1 \leq i \leq 6$, let points A'_i be considered such that $A_iA'_i < 1$. Let $H' = A'_1A'_2A'_3A'_4A'_5A'_6$, and consider all indices modulo 6. For any point P in the plane, let $D(P)$ denote the unit disk $\{Q \mid PQ < 1\}$ centered at P ; it follows that $A'_i \in D(A_i)$.

Let X and X' be points on line A_1A_6 , and let Y and Y' be points on line A_3A_4 such that $A_1X = A_1X' = A_3Y = A_3Y' = 1$ and X and X' lie on opposite sides of A_1 and Y and Y' lie on opposite sides of A_3 . If X' and Y' lie on segments A_1A_6 and A_3A_4 , respectively, then segment $A'_1A'_3$ lies between the lines XY and $X'Y'$. Note that $\frac{x}{2}$ is the distance from A_2 to A_1A_3 .



If $\frac{x}{2} \geq 2$, then $C(A_2)$ cannot intersect line XY , since the distance from XY to A_1A_3 is 1 and the distance from XY to A_2 is at least 1. Therefore, A_1A_3 separates A_2 from the other 3 vertices of the hexagon. By analogous reasoning applied to the other vertices, we may conclude that H' is convex.

If $\frac{x}{2} < 2$, then $C(A_2)$ intersects XY , so by choosing $A'_1 = X$ and $A'_3 = Y$, we see that we may choose A'_2 on the opposite side of XY , in which case H' will be concave. Hence the answer is 4, as desired.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

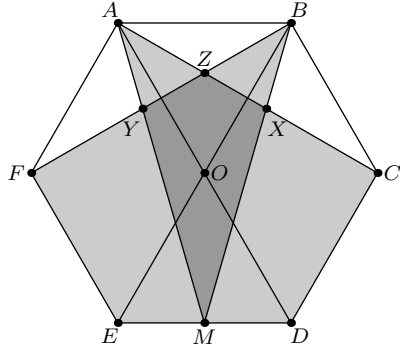
Answer: $\boxed{\frac{5}{11}}$ For $1 \leq k \leq 6$, let x_k be the probability that the current player, say A , will win when the number on the tally at the beginning of his turn is k modulo 7. The probability that the total is l modulo 7 after his roll is $\frac{1}{6}$ for each $l \not\equiv k \pmod{7}$; in particular, there is a $\frac{1}{6}$ chance he wins immediately. The chance that A will win if he leaves l on the board after his turn is $1 - x_l$. Hence for $1 \leq k \leq 6$,

$$x_k = \frac{1}{6} \sum_{1 \leq l \leq 6, l \neq k} (1 - x_l) + \frac{1}{6}.$$

Letting $s = \sum_{l=1}^6 x_l$, this becomes $x_k = \frac{x_k - s}{6} + 1$ or $\frac{5x_k}{6} = -\frac{s}{6} + 1$. Hence $x_1 = \dots = x_6$, and $6x_k = s$ for every k . Plugging this in gives $\frac{11x_k}{6} = 1$, or $x_k = \frac{6}{11}$.

Since Nathaniel cannot win on his first turn, he leaves Obediah with a number not divisible by 7. Hence Obediah's chance of winning is $\frac{6}{11}$ and Nathaniel's chance of winning is $\frac{5}{11}$.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE . Let X be the intersection of AC and BM , let Y be the intersection of BF and AM , and let Z be the intersection of AC and BF . If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.



Answer:

Let O be the center of the hexagon. The desired area is $[ABCDEF] - [ACDM] - [BFEM]$. Note that $[ADM] = [ADE]/2 = [ODE] = [ABC]$, where the last equation holds because $\sin 60^\circ = \sin 120^\circ$. Thus, $[ACDM] = [ACD] + [ADM] = [ACD] + [ABC] = [ABCD]$, but the area of $ABCD$ is half the area of the hexagon. Similarly, the area of $[BFEM]$ is half the area of the hexagon, so the answer is zero.

8. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

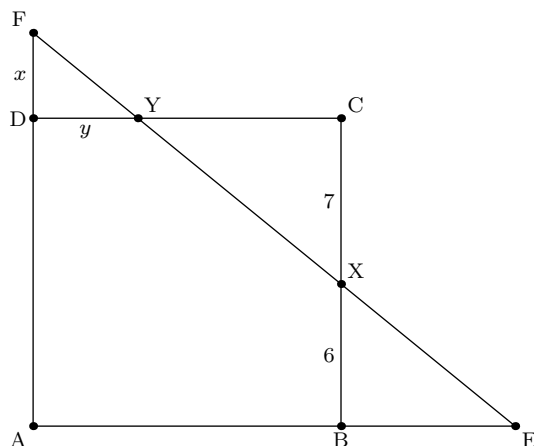
Answer: If n is odd and greater than 1, then Goliath makes the last move. No matter what two numbers are on the board, Goliath can combine them to make an even number. Hence Goliath has a winning strategy in this case.

Now suppose n is even. We can replace all numbers on the board by their residues modulo 2. Initially the board reads $1, 0, 1, 0, \dots, 1, 0$. David combines the rightmost 1 and 0 by addition to make 1, so now the board reads $1, 0, 1, 0, \dots, 0, 1$. We call a board of this form a “good” board. When it is Goliath’s turn, and there is a good board, no matter where he moves, David can make a move to restore a good board. Indeed, Goliath must combine a neighboring 0 and 1; David can then combine that number with a neighboring 1 to make 1 and create a good board with two fewer numbers.

David can ensure a good board after his last turn. But a good board with one number is simply 1, so David wins. So David has a winning strategy if n is even. Therefore, the 2011th smallest positive integer greater than 1 for which David can guarantee victory is the 2011th even positive integer, which is 4022.

9. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD , respectively, so that the area of square $ABCD$ equals the area of triangle AEF . If EF intersects BC at X and $BX = 6$, determine DF .

Answer:



First Solution

Let Y be the point of intersection of lines EF and CD . Note that $[ABCD] = [AEF]$ implies that $[BEX] + [DYF] = [CYX]$. Since $\triangle BEX \sim \triangle CYX \sim \triangle DYF$, there exists some constant r such that $[BEX] = r \cdot BX^2$, $[YDF] = r \cdot CX^2$, and $[CYX] = r \cdot DF^2$. Hence $BX^2 + DF^2 = CX^2$, so $DF = \sqrt{CX^2 - BX^2} = \sqrt{49 - 36} = \sqrt{13}$.

Second Solution

Let $x = DF$ and $y = YD$. Since $\triangle BXE \sim \triangle CXY \sim \triangle DFY$, we have

$$\frac{BE}{BX} = \frac{CY}{CX} = \frac{DY}{DF} = \frac{y}{x}.$$

Using $BX = 6$, $XC = 7$ and $CY = 13 - y$ we get $BE = \frac{6y}{x}$ and $\frac{13-y}{7} = \frac{y}{x}$. Solving this last equation for y gives $y = \frac{13x}{x+7}$. Now $[ABCD] = [AEF]$ gives

$$\begin{aligned} 169 &= \frac{1}{2}AE \cdot AF = \frac{1}{2} \left(13 + \frac{6y}{x} \right) (13 + x). \\ 169 &= 6y + 13x + \frac{78y}{x} \\ 13 &= \frac{6x}{x+7} + x + \frac{78}{x+7} \\ 0 &= x^2 - 13. \end{aligned}$$

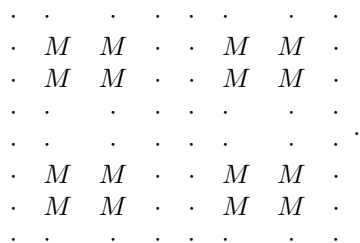
Thus $x = \sqrt{13}$.

10. Mike and Harry play a game on an 8×8 board. For some positive integer k , Mike chooses k squares and writes an M in each of them. Harry then chooses $k + 1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming “ HMM ” or “ MMH ,” when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

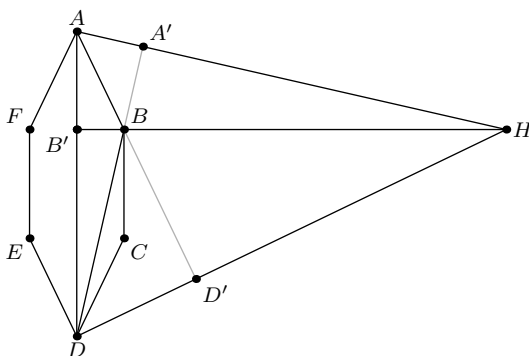
Answer: 16 Suppose Mike writes k M 's. Let a be the number of squares which, if Harry writes an H in, will yield either HMM or MMH horizontally, and let b be the number of squares which, if Harry writes an H in, will yield either HMM or MMH vertically. We will show that $a \leq k$ and $b \leq k$. Then, it will follow that there are at most $a + b \leq 2k$ squares which Harry cannot write an H in. There will be at least $64 - k - 2k = 64 - 3k$ squares which Harry can write in. If $64 - 3k \geq k + 1$, or $k \leq 15$, then Harry wins.

We will show that $a \leq k$ (that $b \leq k$ will follow by symmetry). Suppose there are a_i M 's in row i . In each group of 2 or more consecutive M 's, Harry cannot write H to the left or right of that group, giving at most 2 forbidden squares. Hence a_i is at most the number of M 's in row i . Summing over the rows gives the desired result.

Mike can win by writing 16 M 's according to the following diagram:



11. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC , AD , and EF are parallel. Let H be the orthocenter of triangle ABD . If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.



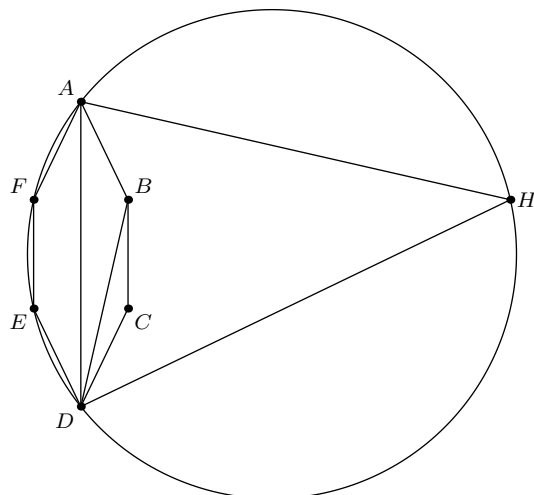
Answer: 3

Note that $ABCD$ and $DEFA$ are isosceles trapezoids, so $\angle BAD = \angle CDA$ and $\angle FAD = \angle EDA$. In order for the hexagon to be convex, the angles at B , C , E , and F have to be obtuse, so $\angle A = \angle D = 4^\circ$. Letting s be a side length of the hexagon, $AD = AB \cos \angle BAD + BC + CD \cos \angle CDA = s(1 + 2 \cos \angle BAD)$, so $\angle BAD$ is uniquely determined by AD . Since the same equation holds for trapezoid $DEFA$, it follows that $\angle BAD = \angle FAD = \angle CDA = \angle EDA = 2^\circ$. Then $\angle BCD = 180^\circ - 2^\circ = 178^\circ$. Since $\triangle BCD$ is isosceles, $\angle CDB = 1^\circ$ and $\angle BDA = 1^\circ$. (One may also note that $\angle BDA = 1^\circ$ by observing that equal lengths AB and BC must intercept equal arcs on the circumcircle of isosceles trapezoid $ABCD$).

Let A' , B' , and D' be the feet of the perpendiculars from A , B , and D to BD , DA , and AB , respectively. Angle chasing yields

$$\begin{aligned} \angle AHD &= \angle AHB' + \angle DHB' = (90^\circ - \angle A'AB') + (90^\circ - \angle D'DB') \\ &= \angle BDA + \angle BAD = 1^\circ + 2^\circ = 3^\circ \\ \angle HAD &= 90^\circ - \angle AHB' = 89^\circ \\ \angle HDA &= 90^\circ - \angle DHB' = 88^\circ \end{aligned}$$

Hence the smallest angle in $\triangle HAD$ is 3° .



It is faster, however, to draw the circumcircle of $DEFA$, and to note that since H is the orthocenter of triangle ABD , B is the orthocenter of triangle HAD . Then since F is the reflection of B across AD , quadrilateral $HAFD$ is cyclic, so $\angle AHD = \angle ADF + \angle DAF = 1^\circ + 2^\circ = 3^\circ$, as desired.

12. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \dots, (1008, 1005), (1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j) , with (x_i, y_i) left of (x_j, y_j) , erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

Answer: 504510 First, note that none of the numbers will ever be 0. Let \star denote the replacement operation. For each pair on the board (x_i, y_i) define its *primary form* to be (x_i, y_i) and its *secondary form* to be $[x_i y_i, \frac{x_i}{y_i}]$. Note that the primary form determines the secondary form uniquely and vice versa. In secondary form,

$$[a_1, b_1] \star [a_2, b_2] = \left(\sqrt{a_1 b_1}, \sqrt{\frac{a_1}{b_1}}\right) \star \left(\sqrt{a_2 b_2}, \sqrt{\frac{a_2}{b_2}}\right) = \left(a_1 b_2, \frac{a_1}{b_2}\right) = [a_1^2, b_2^2].$$

Thus we may replace all pairs on the board by their secondary form and use the above rule for \star instead. From the above rule, we see that if the leftmost number on the board is x , then after one minute it will be x or x^2 depending on whether it was erased in the intervening step, and similarly for the rightmost number. Let k be the number of times the leftmost pair is erased and n be the number of times the rightmost pair is erased. Then the final pair is

$$\left[4022^{2^k}, \left(\frac{1007}{1006}\right)^{2^n}\right]. \tag{2}$$

Any step except the last cannot involve both the leftmost and rightmost pair, so $k + n \leq 1005$. Since every pair must be erased at least once, $k, n \geq 1$. Every pair of integers satisfying the above can occur, for example, by making $1005 - k - n$ moves involving only the pairs in the middle, then making $k - 1$ moves involving the leftmost pair, and finally n moves involving the rightmost pair.

In light of (2), the answer is the number of possible pairs (k, n) , which is

$$\sum_{k=1}^{1004} \sum_{n=1}^{1005-k} 1 = \sum_{k=1}^{1004} 1005 - k = \sum_{k=1}^{1004} k = \frac{1004 \cdot 1005}{2} = 504510.$$

13. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD . If $AI = 2$ as well, find the minimum value of the length of diagonal BD .

Answer: $2\sqrt{3}$ Let T be the point where the incircle intersects AD , and let r be the inradius and R be the circumradius of $\triangle ABD$. Since $BC = CD = 2$, C is on the midpoint of arc BD on the opposite side of BD as A , and hence on the angle bisector of A . Thus A , I , and C are collinear. We have the following formulas:

$$AI = \frac{IM}{\sin \angle IAM} = \frac{r}{\sin \frac{A}{2}}$$

$$BC = 2R \sin \frac{A}{2}$$

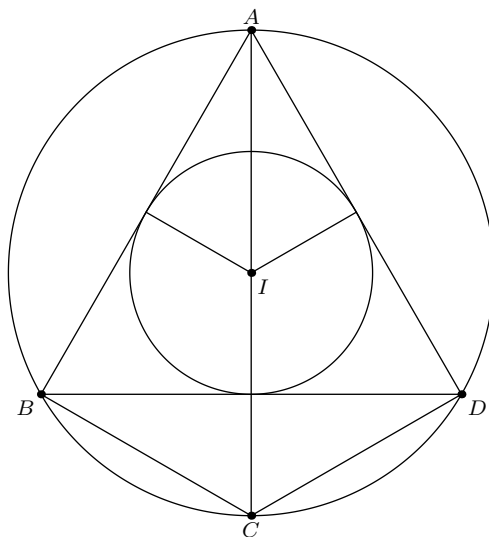
$$BD = 2R \sin A$$

The last two equations follow from the extended law of sines on $\triangle ABC$ and $\triangle ABD$, respectively.

Using $AI = 2 = BC$ gives $\sin^2 \frac{A}{2} = \frac{r}{2R}$. However, it is well-known that $R \geq 2r$ with equality for an equilateral triangle (one way to see this is the identity $1 + \frac{r}{R} = \cos A + \cos B + \cos D$). Hence $\sin^2 \frac{A}{2} \leq \frac{1}{4}$ and $\frac{A}{2} \leq 30^\circ$. Then

$$BD = 2R \left(2 \sin \frac{A}{2} \cos \frac{A}{2} \right) = BC \cdot 2 \cos \frac{A}{2} \geq 2 \left(2 \cdot \frac{\sqrt{3}}{2} \right) = 2\sqrt{3}$$

with equality when $\triangle ABD$ is equilateral.



Remark: Similar but perhaps simpler computations can be made by noting that if AC intersects BD at X , then $AB/BX = AD/DX = 2$, which follows from the exterior angle bisector theorem; if I_A is the A -excenter of triangle ABC , then $AI_A/XI_A = 2$ since it is well-known that C is the circumcenter of cyclic quadrilateral $BIDI_A$.

14. Let $A = \{1, 2, \dots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

Answer: $2^{2011} - 2012$ Let n be the element of A not in the range of f . Let m be the element of A that is hit twice.

We now sum the total number of functions over n, m . Clearly $f(1) = 1$, and by induction, for $x \leq m$, $f(x) = x$. Also unless $n = 2011$, $f(2011) = 2011$ because f can take no other number to 2011. It

follows from backwards induction that for $x > n$, $f(x) = x$. Therefore $n > m$, and there are only $n - m$ values of f that are not fixed.

Now $f(m + 1) = m$ or $f(m + 1) = m + 1$. For $m < k < n$, given the selection of $f(1), f(2), \dots, f(k - 1)$, $k - 1$ of the $k + 1$ possible values of $f(k + 1)$ ($1, 2, 3, \dots, k$, and counting m twice) have been taken, so there are two distinct values that $f(k + 1)$ can take (one of them is $k + 1$, and the other is not, so they are distinct). For $f(n)$, when the other 2010 values of f have been assigned, there is only one missing, so $f(n)$ is determined.

For each integer in $[m, n)$, there are two possible values of f , so there are 2^{n-m-1} different functions f for a given m, n . So our answer is

$$\begin{aligned} \sum_{m=1}^{2010} \sum_{n=m+1}^{2011} 2^{n-m-1} &= \sum_{m=1}^{2010} 2^{-m-1} \sum_{n=m+1}^{2011} 2^n \\ &= \sum_{m=1}^{2010} 2^{-m-1} (2^{2012} - 2^{m+1}) \\ &= \sum_{m=1}^{2010} 2^{2011-m} - 1 \\ &= \left(\sum_{m=1}^{2010} 2^m \right) - 2010 \\ &= 2^{2011} - 2012 \end{aligned}$$

15. Let $f(x) = x^2 - r_2x + r_3$ for all real numbers x , where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A .

Answer: $\boxed{2}$ Consider the function $f(x) - x$. By the constraints of the problem, $f(x) - x$ must be negative for some x , namely, for $x = g_{2i+1}$, $0 \leq i \leq 2011$. Since $f(x) - x$ is positive for x of large absolute value, the graph of $f(x) - x$ crosses the x -axis twice and $f(x) - x$ has two real roots, say $a < b$. Factoring gives $f(x) - x = (x - a)(x - b)$, or $f(x) = (x - a)(x - b) + x$.

Now, for $x < a$, $f(x) > x > a$, while for $x > b$, $f(x) > x > b$. Let $c \neq b$ be the number such that $f(c) = f(b) = b$. Note that b is not the vertex as $f(a) = a < b$, so by the symmetry of quadratics, c exists and $\frac{b+c}{2} = \frac{r_2}{2}$ as the vertex of the parabola. By the same token, $\frac{b+a}{2} = \frac{r_2+1}{2}$ is the vertex of $f(x) - x$. Hence $c = a - 1$. If $f(x) > b$ then $x < c$ or $x > b$. Consider the smallest j such that $g_j > b$. Then by the above observation, $g_{j-1} < c$. (If $g_i \geq b$ then $f(g_i) \geq g_i \geq b$ so by induction, $g_{i+1} \geq g_i$ for all $i \geq j$. Hence $j > 1$; in fact $j \geq 4025$.) Since $g_{j-1} = f(g_{j-2})$, the minimum value of f is less than c . The minimum value is the value of f evaluated at its vertex, $\frac{b+a-1}{2}$, so

$$\begin{aligned} f\left(\frac{b+a-1}{2}\right) &< c \\ \left(\frac{b+a-1}{2} - a\right) \left(\frac{b+a-1}{2} - b\right) + \frac{b+a-1}{2} &< a-1 \\ \frac{1 - (b-a)^2}{4} + \frac{b-a+1}{2} &< 0 \\ \frac{3}{4} &< \frac{(b-a)^2}{4} - \frac{b-a}{2} \\ 4 &< (b-a-1)^2. \end{aligned}$$

Then either $b - a - 1 < -2$ or $b - a - 1 > 2$, but $b > a$, so the latter must hold and $(b - a)^2 > 9$. Now, the discriminant of $f(x) - x$ equals $(b - a)^2$ (the square of the difference of the two roots) and $(r_2 + 1)^2 - 4r_3$ (from the coefficients), so $(r_2 + 1)^2 > 9 + 4r_3$. But $r_3 = g_1 > g_0 = 0$ so $|r_2| > 2$.

We claim that we can make $|r_2|$ arbitrarily close to 2, so that the answer is 2. First define $G_i, i \geq 0$ as follows. Let $N \geq 2012$ be an integer. For $\varepsilon > 0$ let $h(x) = x^2 - 2 - \varepsilon, g_\varepsilon(x) = -\sqrt{x + 2 + \varepsilon}$ and $G_{2N+1} = 2 + \varepsilon$, and define G_i recursively by $G_i = g_\varepsilon(G_{i+1}), G_{i+1} = h(G_i)$. (These two equations are consistent.) Note the following. (i) $G_{2i} < G_{2i+1}$ and $G_{2i+1} > G_{2i+2}$ for $0 \leq i \leq N - 1$. First note $G_{2N} = -\sqrt{4 + 2\varepsilon} > -\sqrt{4 + 2\varepsilon + \varepsilon^2} = -2 - \varepsilon$. Let l be the negative solution to $h(x) = x$. Note that $-2 - \varepsilon < G_{2N} < l < 0$ since $h(G_{2N}) > 0 > G_{2N}$. Now $g_\varepsilon(x)$ is defined as long as $x \geq -2 - \varepsilon$, and it sends $(-2 - \varepsilon, l)$ into $(l, 0)$ and $(l, 0)$ into $(-2 - \varepsilon, l)$. It follows that the $G_i, 0 \leq i \leq 2N$ are well-defined; moreover, $G_{2i} < l$ and $G_{2i+1} > l$ for $0 \leq i \leq N - 1$ by backwards induction on i , so the desired inequalities follow. (ii) G_i is increasing for $i \geq 2N + 1$. Indeed, if $x \geq 2 + \varepsilon$, then $x^2 - x = x(x - 1) > 2 + \varepsilon$ so $h(x) > x$. Hence $2 + \varepsilon = G_{2N+1} < G_{2N+2} < \dots$. (iii) G_i is unbounded. This follows since $h(x) - x = x(x - 2) - 2 - \varepsilon$ is increasing for $x > 2 + \varepsilon$, so G_i increases faster and faster for $i \geq 2N + 1$. Now define $f(x) = h(x + G_0) - G_0 = x^2 + 2G_0x + G_0^2 - G_0 - 2 - \varepsilon$. Note $G_{i+1} = h(G_i)$ while $g_{i+1} = f(g_i) = h(g_i + G_0) - G_0$, so by induction $g_i = G_i - G_0$. Since $\{G_i\}_{i=0}^\infty$ satisfies (i), (ii), and (iii), so does g_i .

We claim that we can make G_0 arbitrarily close to -1 by choosing N large enough and ε small enough; this will make $r_2 = -2G_0$ arbitrarily close to 2. Choosing N large corresponds to taking G_0 to be a larger iterate of $2 + \varepsilon$ under $g_\varepsilon(x)$. By continuity of this function with respect to x and ε , it suffices to take $\varepsilon = 0$ and show that (letting $g = g_0$)

$$g^{(n)}(2) = \underbrace{g(\dots g(2)\dots)}_n \rightarrow -1 \text{ as } n \rightarrow \infty.$$

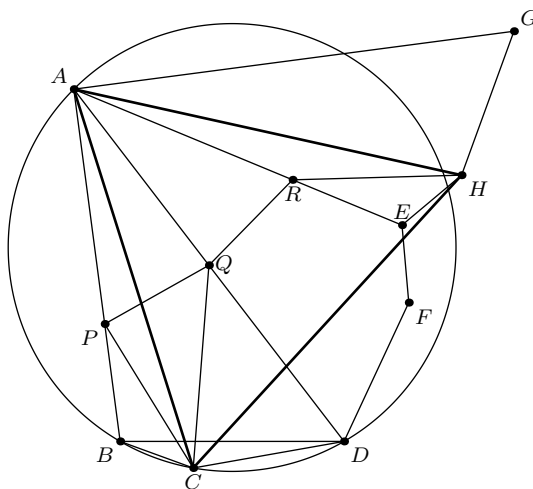
But note that for $0 \leq \theta \leq \frac{\pi}{2}$,

$$g(-2 \cos \theta) = -\sqrt{2 - 2 \cos \theta} = -2 \sin \left(\frac{\theta}{2} \right) = 2 \cos \left(\frac{\pi}{2} - \frac{\theta}{2} \right).$$

Hence by induction, $g^{(n)}(-2 \cos \theta) = -2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots + (-1)^n \left(\theta - \frac{\pi}{2^n} \right) \right)$. Hence $g^{(n)}(2) = g^{(n-1)}(-2 \cos 0)$ converges to $-2 \cos \left(\frac{\pi}{2} - \frac{\pi}{4} + \dots \right) = -2 \cos \left(\frac{\pi}{3} \right) = -1$, as needed.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD , respectively. Determine the maximum value of m .

Answer: 2



For a fixed quadrilateral $ABCD$ as described, we first show that m , the minimum possible length of $CP + PQ + CQ$, equals the length of AC . Reflect B, C , and P across line AD to points E, F , and R , respectively, and then reflect D and F across AE to points G and H , respectively. These two

reflections combine to give a 60° rotation around A , so triangle ACH is equilateral. It also follows that RH is a 60° rotation of PC around A , so, in particular, these segments have the same length. Because $QR = QP$ by reflection,

$$CP + PQ + QC = CQ + QR + RH.$$

The latter is the length of a broken path $CQRH$ from C to H , and by the “shortest path is a straight line” principle, this total length is at least as long as $CH = CA$. (More directly, this follows from the triangle inequality: $(CQ + QR) + RH \geq CR + RH \geq CH$). Therefore, the lower bound $m \geq AC$ indeed holds. To see that this is actually an equality, note that choosing Q as the intersection of segment CH with ray AD , and choosing P so that its reflection R is the intersection of CH with ray AE , aligns path $CQRH$ with segment CH , thus obtaining the desired minimum $m = AC$.

We may conclude that the largest possible value of m is the largest possible length of AC , namely 2: the length of a diameter of the circle.

17. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander’s left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n - 1$ members vote randomly.

Answer: $\boxed{\frac{2^n - 1}{n2^{n-1}}}$ Let x be the probability Hermia is elected if Lysander votes for her, and let y be the probability that she wins if Lysander does not vote for her. We are trying to find x , and do so by first finding y . If Lysander votes for Hermia with probability $\frac{1}{2}$ then the probability that Hermia is elected chairman is $\frac{x}{2} + \frac{y}{2}$, but it is also $\frac{1}{n}$ by symmetry. If Lysander does not vote for Hermia, Hermia can get at most 1 vote, and then can only be elected if everyone gets one vote and she wins the tiebreaker. The probability she wins the tiebreaker is $\frac{1}{n}$, and chasing around the circle, the probability that every person gets 1 vote is $\frac{1}{2^{n-1}}$. (Everyone votes for the person to the left, or everyone votes for the person to the right.) Hence

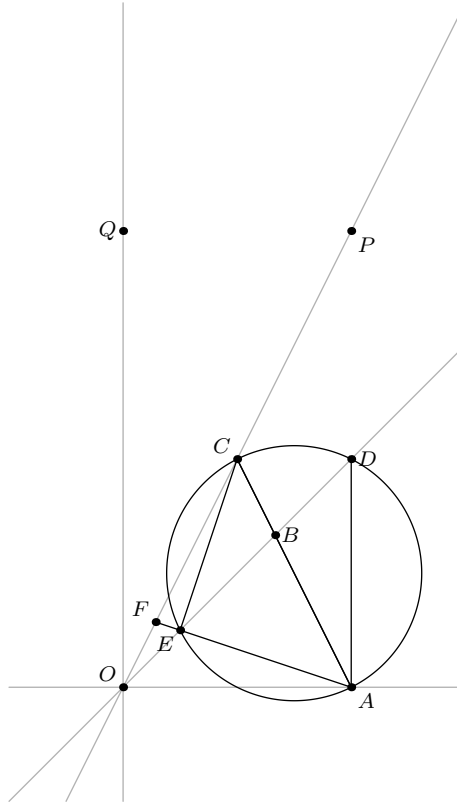
$$y = \frac{1}{n2^{n-1}}.$$

Then $\frac{x}{2} + \frac{1}{n2^n} = \frac{1}{n}$, so solving for x gives

$$x = \frac{2^n - 1}{n2^{n-1}}.$$

18. Collinear points A , B , and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x -axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E , and ray AE intersects $y = 2x$ at F , evaluate AE/EF .

Answer: $\boxed{7}$



Let points O , P , and Q be located at $(0, 0)$, $(a, 2a)$, and $(0, 2a)$, respectively. Note that $BC/AB = 1/2$ implies $[OCD]/[OAD] = 1/2$, so since $[OPD] = [OAD]$, $[OCD]/[OPD] = 1/2$. It follows that $[OCD] = [OPD]$. Hence $OC = CP$. We may conclude that triangles OCQ and PCA are congruent, so $C = (a/2, a)$.

It follows that $\angle ADC$ is right, so the circumcircle of triangle ADC is the midpoint of AC , which is located at $(3a/4, a/2)$. Let $(3a/4, a/2) = H$, and let $E = (b, b)$. Then the power of the point O with respect to the circumcircle of ADC is $OD \cdot OE = 2ab$, but it may also be computed as $OH^2 - HA^2 = 13a/16 - 5a/16 = a/2$. It follows that $b = a/4$, so $E = (a/4, a/4)$.

We may conclude that line AE is $x + 3y = a$, which intersects $y = 2x$ at an x -coordinate of $a/7$. Therefore, $AE/EF = (a - a/4)/(a/4 - a/7) = (3a/4)/(3a/28) = 7$.

Remark: The problem may be solved more quickly if one notes from the beginning that lines OA , OD , OP , and OQ form a harmonic pencil because D is the midpoint of AP and lines OQ and AP are parallel.

19. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \dots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

Answer: 1 Consider the grid whose entries in the j th row are, in order, $2011j - 2010, 2011j - 2009, \dots, 2011j$. Call this grid A_0 . For $k = 1, 2, \dots, 2010$, let grid A_k be the grid obtained from A_0 by swapping the rightmost entry of the k th row with the leftmost entry of the $k + 1$ st row. We claim

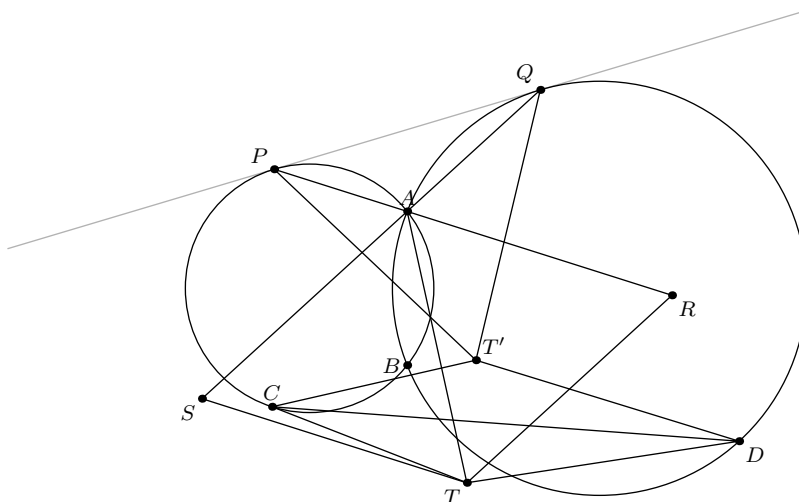
that if $A \in \{A_0, A_1, \dots, A_{2010}\}$, then given any legally numbered grid B such that A and B differ in at least one entry, there exist two integers in the same column of B that occur in the same row of A . We first consider A_0 . Assume for the sake of contradiction B is a legally numbered grid distinct from A_0 , such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Since the numbers $1, 2, \dots, 2011$ occur in the same row of A_0 , they must all occur in different columns of B . Clearly 1 is the leftmost entry in B 's first row. Let m be the smallest number that does not occur in the first row of B . Since each row is in order, m must be the first entry in its row. But then 1 and m are in the same column of B , a contradiction. It follows that the numbers $1, 2, \dots, 2011$ all occur in the first row of B . Proceeding by induction, $2011j - 2010, 2011j - 2009, \dots, 2011j$ must all occur in the j th row of B for all $1 \leq j \leq 2011$. Since A_0 is the only legally numbered grid satisfying this condition, we have reached the desired contradiction.

Now note that if $A \in \{A_1, \dots, A_{2010}\}$, there exist two integers in the same column of A_0 that occur in the same row of A . In particular, if $A = A_k$ and $1 \leq k \leq 2010$, then the integers $2011k - 2010$ and $2011k + 1$ occur in the same column of A_0 and in the same row of A_k . Therefore, it suffices to show that for all $1 \leq k \leq 2010$, there is no legally numbered grid B distinct from A_k and A_0 such that there do not exist two integers in the same column of B that occur in the same row of A_0 . Assume for the sake of contradiction that there does exist such a grid B . By the same logic as above, applied to the first $k - 1$ rows and applied backwards to the last $2010 - k - 1$ rows, we see that B may only differ from A_k in the k th and $k + 1$ st rows. However, there are only two legally numbered grids that are identical to A_k outside of rows k and $k + 1$, namely A_0 and A_k . This proves the claim.

It remains only to note that, by the pigeonhole principle, if one of Alice's grids is A_0 , then there exists a positive integer k , $1 \leq k \leq 2010$, such that A_k is not one of the Alice's grids. Therefore, if Bob sets his initial grid to be A_0 , he will require only one swap to switch his grid to A_k after examining Alice's grids. If A_0 is not among Alice's grids, then if Bob sets his initial grid to be A_0 , he will not in fact require any swaps at all.

20. Let ω_1 and ω_2 be two circles that intersect at points A and B . Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B . Let points R and S lie along rays PA and QA , respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q . Let O be the circumcenter of triangle ASR , and let C and D be the midpoints of major arcs AP and AQ , respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

Answer: 142.5



We use directed angles throughout the solution.

Let T denote the point such that $\angle TCD = 1/2\angle APQ$ and $\angle TDC = 1/2\angle AQP$. We claim that T is the circumcenter of triangle SAR .

Since $CP = CA$, $QP = RA$, and $\angle CPQ = \angle CPA + \angle APQ = \angle CPA + \angle ACP = \angle CAR$, we have $\triangle CPQ \cong \triangle CAR$. By spiral similarity, we have $\triangle CPA \sim \triangle CQR$.

Let T' denote the reflection of T across CD . Since $\angle TCT' = \angle APQ = \angle ACP$, we have $\triangle TCT' \sim \triangle ACP \sim \triangle RCQ$. Again, by spiral similarity centered at C , we have $\triangle CTR \sim \triangle CT'Q$. But $CT = CT'$, so $\triangle CTR \cong \triangle CT'Q$ and $TR = T'Q$. Similarly, $\triangle DTT' \sim \triangle DAQ$, and spiral similarity centered at D shows that $\triangle DTA \cong \triangle DT'Q$. Thus $TA = T'Q = TR$.

We similarly have $TA = T'P = TS$, so T is indeed the circumcenter. Therefore, we have $\angle COD = \angle CTD = 180^\circ - \frac{45^\circ}{2} - \frac{30^\circ}{2} = 142.5^\circ$.

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Guts Round

1. [4] Let ABC be a triangle with area 1. Let points D and E lie on AB and AC , respectively, such that DE is parallel to BC and $DE/BC = 1/3$. If F is the reflection of A across DE , find the area of triangle FBC .

Answer: $\boxed{\frac{1}{3}}$

Let AF intersect BC at H . Since $DE/BC = 1/3$ and F and A are equidistant from DE , we have $AF = \frac{2}{3}AH$ and $FH = AH - AF = \frac{1}{3}AH$. Furthermore, since AF is perpendicular to DE , we have AH and FH are the altitudes of triangles ABC and FBC respectively. Therefore the area of triangle FBC is $\frac{1}{2} \cdot FH \cdot BC = \frac{1}{2} \cdot \frac{1}{3} \cdot AH \cdot BC = \frac{1}{3}$.

2. [4] Let $a \star b = \sin a \cos b$ for all real numbers a and b . If x and y are real numbers such that $x \star y - y \star x = 1$, what is the maximum value of $x \star y + y \star x$?

Answer: $\boxed{1}$

We have $x \star y + y \star x = \sin x \cos y + \cos x \sin y = \sin(x + y) \leq 1$. Equality is achieved when $x = \frac{\pi}{2}$ and $y = 0$. Indeed, for these values of x and y , we have $x \star y - y \star x = \sin x \cos y - \cos x \sin y = \sin(x - y) = \sin \frac{\pi}{2} = 1$.

3. [4] Evaluate $2011 \times 20122012 \times 201320132013 - 2013 \times 20112011 \times 201220122012$.

Answer: $\boxed{0}$

Both terms are equal to $2011 \times 2012 \times 2013 \times 1 \times 10001 \times 100010001$.

4. [4] Let p be the answer to this question. If a point is chosen uniformly at random from the square bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, what is the probability that at least one of its coordinates is greater than p ?

Answer: $\boxed{\frac{\sqrt{5}-1}{2}}$

The probability that a randomly chosen point has both coordinates less than p is p^2 , so the probability that at least one of its coordinates is greater than p is $1 - p^2$. Since p is the answer to this question, we have $1 - p^2 = p$, and the only solution of p in the interval $[0, 1]$ is $\frac{\sqrt{5}-1}{2}$.

5. [5] Rachele picks a positive integer a and writes it next to itself to obtain a new positive integer b . For instance, if $a = 17$, then $b = 1717$. To her surprise, she finds that b is a multiple of a^2 . Find the product of all the possible values of $\frac{b}{a^2}$.

Answer: $\boxed{77}$

Suppose a has k digits. Then $b = a(10^k + 1)$. Thus a divides $10^k + 1$. Since $a \geq 10^{k-1}$, we have $\frac{10^k+1}{a} \leq 11$. But since none of 2, 3, or 5 divide $10^k + 1$, the only possibilities are 7 and 11. These values are obtained when $a = 143$ and $a = 1$, respectively.

6. [5] Square $ABCD$ is inscribed in circle ω with radius 10. Four additional squares are drawn inside ω but outside $ABCD$ such that the lengths of their diagonals are as large as possible. A sixth square is drawn by connecting the centers of the four aforementioned small squares. Find the area of the sixth square.

Answer: $\boxed{144}$

Let $DEGF$ denote the small square that shares a side with AB , where D and E lie on AB . Let O denote the center of ω , K denote the midpoint of FG , and H denote the center of $DEGF$. The area of the sixth square is $2 \cdot OH^2$.

Let $KF = x$. Since $KF^2 + OK^2 = OF^2$, we have $x^2 + (2x + 5\sqrt{2})^2 = 10^2$. Solving for x , we get $x = \sqrt{2}$. Thus, we have $OH = 6\sqrt{2}$ and $2 \cdot OH^2 = 144$.

7. [6] For any positive real numbers a and b , define $a \circ b = a + b + 2\sqrt{ab}$. Find all positive real numbers x such that $x^2 \circ 9x = 121$.

Answer: $\boxed{\frac{31-3\sqrt{53}}{2}}$

Since $a \circ b = (\sqrt{a} + \sqrt{b})^2$, we have $x^2 \circ 9x = (x + 3\sqrt{x})^2$. Moreover, since x is positive, we have $x + 3\sqrt{x} = 11$, and the only possible solution is that $\sqrt{x} = \frac{-3+\sqrt{53}}{2}$, so $x = \frac{31-3\sqrt{53}}{2}$.

8. [6] Find the smallest k such that for any arrangement of 3000 checkers in a 2011×2011 checkerboard, with at most one checker in each square, there exist k rows and k columns for which every checker is contained in at least one of these rows or columns.

Answer: $\boxed{1006}$

If there is a chip in every square along a main diagonal, then we need at least 1006 rows and columns to contain all these chips. We are left to show that 1006 is sufficient.

Take the 1006 rows with greatest number of chips. Assume without loss of generality they are the first 1006 rows. If the remaining 1005 rows contain at most 1005 chips, then we can certainly choose 1006 columns that contain these chips. Otherwise, there exists a row that contains at least 2 chips, so every row in the first 1006 rows must contain at least 2 chips. But this means that there are at least $2 \times 1006 + 1006 = 3018$ chips in total. Contradiction.

9. [6] Segments AA' , BB' , and CC' , each of length 2, all intersect at a point O . If $\angle AOC' = \angle BOA' = \angle COB' = 60^\circ$, find the maximum possible value of the sum of the areas of triangles AOC' , BOA' , and COB' .

Answer: $\boxed{\sqrt{3}}$

Extend OA to D and OC' to E such that $AD = OA'$ and $C'E = OC$. Since $OD = OE = 2$ and $\angle DOE = 60^\circ$, we have ODE is an equilateral triangle. Let F be the point on DE such that $DF = OB$ and $EF = OB'$. Clearly we have $\triangle DFA \cong \triangle OBA'$ and $\triangle EFC' \cong \triangle OB'C$. Thus the sum of the areas of triangles AOC' , BOA' , and COB' is the same as the sum of the areas of triangle DFA , FEC' , and OAC' , which is at most the area of triangle ODE . Since ODE is an equilateral triangle with side length 2, its area is $\sqrt{3}$. Equality is achieved when $OC = OA' = 0$.

10. [6] In how many ways can one fill a 4×4 grid with a 0 or 1 in each square such that the sum of the entries in each row, column, and long diagonal is even?

Answer: $\boxed{256}$

First we name the elements of the square as follows:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}$$

We claim that for any given values of $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{32}$, and a_{33} (the + signs in the diagram below), there is a unique way to assign values to the rest of the entries such that all necessary sums are even.

$$\begin{array}{cccc} + & + & + & a_{14} \\ + & + & + & a_{24} \\ a_{31} & + & + & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}$$

Taking additions mod 2, we have

$$\begin{aligned} a_{14} &= a_{11} + a_{12} + a_{13} \\ a_{24} &= a_{21} + a_{22} + a_{23} \\ a_{44} &= a_{11} + a_{22} + a_{33} \\ a_{42} &= a_{12} + a_{22} + a_{32} \\ a_{43} &= a_{13} + a_{23} + a_{33} \end{aligned}$$

Since the 4th column, the 4th row, and the 1st column must have entries that sum to 0, we have

$$\begin{aligned} a_{34} &= a_{14} + a_{24} + a_{44} = a_{12} + a_{13} + a_{21} + a_{23} + a_{33} \\ a_{41} &= a_{42} + a_{43} + a_{44} = a_{11} + a_{12} + a_{13} + a_{23} + a_{32} \\ a_{31} &= a_{11} + a_{21} + a_{41} = a_{12} + a_{13} + a_{21} + a_{23} + a_{32} \end{aligned}$$

It is easy to check that the sum of entries in every row, column, and the main diagonal is even. Since there are $2^8 = 256$ ways to assign the values to the initial 8 entries, there are exactly 256 ways to fill the board.

11. [8] Rosencrantz and Guildenstern play a game in which they repeatedly flip a fair coin. Let $a_1 = 4$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. On the n th flip, if the coin is heads, Rosencrantz pays Guildenstern a_n dollars, and, if the coin is tails, Guildenstern pays Rosencrantz a_n dollars. If play continues for 2010 turns, what is the probability that Rosencrantz ends up with more money than he started with?

Answer: $\boxed{\frac{1}{2} - \frac{1}{2^{1341}}}$

Since Rosencrantz and Guildenstern have an equal chance of winning each toss, both have the same probability of ending up with a positive amount of money. Let x denote the probability that they both end up with zero dollars. We wish to find $\frac{1-x}{2}$.

We have x is equal to the probability that

$$s_{2010} := i_1 a_1 + i_2 a_2 + \cdots + i_{2010} a_{2010} = 0,$$

where i_n has an equal probability of being either 1 or -1 .

We claim that $s_{2010} = 0$ if and only if $i_{3n} = -i_{3n-1} = -i_{3n-2}$ for all $n \leq 670$. We start with the following lemma.

Lemma. We have $a_n > \sum_{k=1}^{n-3} a_k$ for all $n \geq 4$.

Proof: For the case $n = 4$, $a_4 = a_3 + a_2 = 2a_2 + a_1 > a_1$. In case $n > 4$, we have

$$a_n = a_{n-2} + a_{n-1} > a_{n-2} + \sum_{k=1}^{n-4} a_k = a_{n-4} + \sum_{k=1}^{n-3} a_k > \sum_{k=1}^{n-3} a_k.$$

It suffices to show that $s_{3n} = 0$ only if $i_{3k} = -i_{3k-1} = -i_{3k-2}$ for all $k \leq n$. The triangle inequality implies the following:

$$\begin{aligned} 0 &\leq \left| i_{3n-2} a_{3n-2} + s_{3n-3} \right| - \left| i_{3n-1} a_{3n-1} + i_{3n} a_{3n} \right| \leq |s_{3n}| = 0 \\ 0 &\leq \left| i_{3n-1} a_{3n-1} + s_{3n-3} \right| - \left| i_{3n-2} a_{3n-2} + i_{3n} a_{3n} \right| \leq |s_{3n}| = 0 \end{aligned}$$

By the lemma, we have

$$\begin{aligned}
 a_{3n-2} + \sum_{k=1}^{3n-3} a_k &< a_{3n-2} + a_{3n} &< a_{3n-1} + a_{3n} \\
 a_{3n-1} + \sum_{k=1}^{3n-3} a_k &< a_{3n-1} + a_{3n-3} + a_{3n-4} + a_{3n-2} &= a_{3n-2} + a_{3n} \\
 i_{3n} = i_{3n-1} \text{ implies } \left| a_{3n-2} + \sum_{k=1}^{3n-3} a_k \right| &< |i_{3n-1}a_{3n-1} + i_{3n}a_{3n}| \text{ and } i_{3n} = i_{3n-2} \text{ implies } \left| a_{3n-1} + \right. \\
 \left. \sum_{k=1}^{3n-3} a_k \right| &< |i_{3n-2}a_{3n-2} + i_{3n}a_{3n}|, \text{ which are both contradictions; therefore, we must have } i_{3n} = -i_{3n-1} \\
 \text{and } i_{3n} &= -i_{3n-2}.
 \end{aligned}$$

The probability that $i_{3n} = -i_{3n-1} = -i_{3n-2}$ is $\frac{1}{4}$, so $x = \left(\frac{1}{4}\right)^{670} = \frac{1}{2^{1340}}$, and $\frac{1-x}{2} = \frac{1}{2} - \frac{1}{2^{1341}}$.

12. [8] A sequence of integers $\{a_i\}$ is defined as follows: $a_i = i$ for all $1 \leq i \leq 5$, and $a_i = a_1 a_2 \cdots a_{i-1} - 1$ for all $i > 5$. Evaluate $a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2$.

Answer: -1941

For all $i \geq 6$, we have $a_i = a_1 a_2 \cdots a_{i-1} - 1$. So

$$\begin{aligned}
 a_{i+1} &= a_1 a_2 \cdots a_i - 1 \\
 &= (a_1 a_2 \cdots a_{i-1}) a_i - 1 \\
 &= (a_i + 1) a_i - 1 \\
 &= a_i^2 + a_i - 1.
 \end{aligned}$$

Therefore, for all $i \geq 6$, we have $a_i^2 = a_{i+1} - a_i + 1$, and we obtain that

$$\begin{aligned}
 &a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2 \\
 &= a_{2012} + 1 - \sum_{i=1}^5 a_i^2 - \sum_{i=6}^{2011} a_i^2 \\
 &= a_{2012} + 1 - \sum_{i=1}^5 i^2 - \sum_{i=6}^{2011} (a_{i+1} - a_i + 1) \\
 &= a_{2012} + 1 - 55 - (a_{2012} - a_6 + 2006) \\
 &= a_6 - 2060 \\
 &= -1941
 \end{aligned}$$

13. [8] Let a, b , and c be the side lengths of a triangle, and assume that $a \leq b$ and $a \leq c$. Let $x = \frac{b+c-a}{2}$. If r and R denote the inradius and circumradius, respectively, find the minimum value of $\frac{ax}{rR}$.

Answer: 3

It is well-known that both $\frac{abc}{4R}$ and $\frac{r(a+b+c)}{2}$ are equal to the area of triangle ABC . Thus $\frac{abc}{4R} = \frac{r(a+b+c)}{2}$, and

$$Rr = \frac{abc}{2(a+b+c)}.$$

Since $a \leq b$ and $a \leq c$, we have $\frac{a^2}{bc} \leq 1$. We thus obtain that

$$\begin{aligned} \frac{ax}{rR} &= \frac{a(b+c-a)/2}{\frac{abc}{2(a+b+c)}} \\ &= \frac{(a+b+c)(b+c-a)}{bc} \\ &= \frac{(b+c)^2 - a^2}{bc} \\ &= \frac{(b+c)^2}{bc} - \frac{a^2}{bc} \\ &= \frac{b}{c} + \frac{c}{b} + 2 - \frac{a^2}{bc} \\ &\geq \frac{b}{c} + \frac{c}{b} + 2 - 1 \\ &\geq 2 + 2 - 1 \\ &= 3 \end{aligned}$$

Equality is achieved when $a = b = c$.

14. [8] Danny has a set of 15 pool balls, numbered $1, 2, \dots, 15$. In how many ways can he put the balls in 8 indistinguishable bins such that the sum of the numbers of the balls in each bin is 14, 15, or 16?

Answer: 122

Clearly, the balls numbered $15, 14, \dots, 9, 8$ must be placed in separate bins, so we number the bins $15, 14, \dots, 9, 8$. Note that bins 15 and 14 may contain only one ball while all other bins must contain at least two balls. We have two cases to examine.

Case 1: Only one bin contains exactly one ball. Let a_i denote the number of ways to place the balls numbered $1, 2, \dots, i-1$ into the bins numbered $15, 14, \dots, 15-i+1$. We can place either $i-1$ or $i-2$ into the bin numbered $15-i+1$. If we place $i-1$ in there, then there are a_{i-1} ways to finish packing the rest. If we place $i-2$ in this bin, then $i-1$ must be placed in the bin numbered $15-i+2$, so there are a_{i-2} ways to place the rest of the balls. Therefore $a_i = a_{i-1} + a_{i-2}$. Since $a_1 = 2$ and $a_2 = 3$, the sequence $\{a_i\}$ is the Fibonacci sequence, and $a_7 = 34$.

Case 2: Both bins 14 and 15 contain only one ball. A pair of balls from 1-7 must be put together to one of the bins numbered 8 through 13. This pair has sum at most 8, so we can count for all the cases.

Balls	Number of packings
1, 2	16
1, 3	10
1, 4	12
1, 5	12
1, 6	10
1, 7	8
2, 3	6
2, 4	4
2, 5	4
2, 6	3
3, 4	2
3, 5	1

Therefore, there are 88 possibilities in this case, and the total number of possibilities is 122.

15. [10] Find all irrational numbers x such that $x^3 - 17x$ and $x^2 + 4x$ are both rational numbers.

Answer: $-2 \pm \sqrt{5}$

From $x^2 + 4x \in \mathbb{Q}$, we deduce that $(x + 2)^2 = x^2 + 4x + 4$ is also rational, and hence $x = -2 \pm \sqrt{y}$, where y is rational. Then $x^3 - 17x = (26 - 6y) \pm (y - 5)\sqrt{y}$, which forces y to be 5. Hence $x = -2 \pm \sqrt{5}$. It is easy to check that both values satisfy the problem conditions.

16. [10] Let R be a semicircle with diameter XY . A trapezoid $ABCD$ in which AB is parallel to CD is circumscribed about R such that AB contains XY . If $AD = 4$, $CD = 5$, and $BC = 6$, determine AB .

Answer: 10

We claim that $AB = AD + BC$. Let O denote the center of R . Since DA and DC are both tangent to R , we have $\angle ADO = \angle ODC$. Since CD is parallel to AB , we also have $\angle ODC = \angle DOA$. Thus $\angle ADO = \angle DOA$, and it follows that $AD = AO$. Similarly, we have $BC = BO$. Therefore, we

$$AB = AO + BO = AD + BC = 10.$$

17. [10] Given positive real numbers x , y , and z that satisfy the following system of equations:

$$\begin{aligned} x^2 + y^2 + xy &= 1, \\ y^2 + z^2 + yz &= 4, \\ z^2 + x^2 + zx &= 5, \end{aligned}$$

find $x + y + z$.

Answer: $\sqrt{5 + 2\sqrt{3}}$

Let O denote the origin. Construct vectors OA , OB , and OC as follows: The lengths of OA , OB , and OC are x , y , and z , respectively, and the angle between any two vectors is 120° . By the Law of Cosines, we have $AB = 1$, $BC = 2$, and $AC = \sqrt{5}$. Thus ABC is a right triangle, and O is its Fermat point. The area of triangle ABC is equal to $\frac{1}{2} \cdot 1 \cdot 2 = 1$. But the area is also equal to the sum of the areas of triangles AOB , BOC , and COA , which is equal to $\frac{1}{2} \cdot xy \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot yz \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot zx \cdot \frac{\sqrt{3}}{2}$. We thus obtain $xy + yz + zx = \frac{4}{\sqrt{3}}$. Adding the three equations given in the problem and subtracting both sides by $xy + yz + zx$, we obtain $x^2 + y^2 + z^2 = 5 - \frac{2}{\sqrt{3}}$. Therefore $(x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + yz + zx) = 5 + 2\sqrt{3}$.

18. [10] In how many ways can each square of a 4×2011 grid be colored red, blue, or yellow such that no two squares that are diagonally adjacent are the same color?

Answer: $64 \cdot 3^{4020}$

If we first color the board in a checkerboard pattern, it is clear that the white squares are independent of the black squares in diagonal coloring, so we calculate the number of ways to color the white squares of a $4 \times n$ board and then square it.

Let a_n be the number of ways to color the white squares of a $4 \times n$ board in this manner such that the two squares in the last column are the same color, and b_n the number of ways to color it such that they are different. We want to find their sum x_n . We have $a_1 = 3$, $b_1 = 6$. Given any filled $4 \times n - 1$ grid with the two white squares in the last column different, there is only 1 choice for the middle square in the n th row, and two choices for the outside square, 1 choice makes them the same color, 1 makes them different. If the two white squares are the same, there are 2 choices for the middle square and the outer square, so 4 choices. Of these, in 2 choices, the two new squares are the same color, and in the other 2, the two squares are different. It follows that $a_n = 2a_{n-1} + b_{n-1}$ and $b_n = 2a_{n-1} + b_{n-1}$, so $a_n = b_n$ for $n \geq 2$. We have $x_n = 8 \cdot 3^{n-1}$ and $x_{2011} = 8 \cdot 3^{2010}$. So the answer is $64 \cdot 3^{4020}$.

19. [12] Find the least positive integer N with the following property: If all lattice points in $[1, 3] \times [1, 7] \times [1, N]$ are colored either black or white, then there exists a rectangular prism, whose faces are parallel to the xy , xz , and yz planes, and whose eight vertices are all colored in the same color.

Answer: 127

First we claim that if the lattice points in $[1, 3] \times [1, 7]$ are colored either black or white, then there exists a rectangle whose faces are parallel to the x and y axes, whose vertices are all the same color (a.k.a. monochromatic). Indeed, in every row $y = i$, $1 \leq i \leq 7$, there are two lattice points with the same color. Note there are 3 combinations of 2 columns to choose from (for the two similarly-colored lattice points to be in), and 2 colors to choose from. By the Pigeonhole Principle, in the $2 \cdot 3 + 1 = 7$ rows two rows must have a pair of similarly-colored lattice points in the same columns, i.e. there is a monochromatic rectangle.

This shows that in each cross section $z = i$, $1 \leq i \leq N$ there is a monochromatic rectangle. Next, note there are $\binom{3}{2}\binom{7}{2}$ possibilities for this rectangle ($\binom{3}{2}$ ways to choose the 2 x -coordinates and $\binom{7}{2}$ ways to choose the 2 y -coordinates), and 2 possible colors. Thus if $N = 2\binom{3}{2}\binom{7}{2} + 1 = 127$, then by the Pigeonhole Principle there are two values of i such that the same-colored rectangle has the same x and y coordinates in the plane $z = i$, i.e. there is a monochromatic rectangular prism.

For $N = 126$ the assertion is not true. In each cross section $z = i$, we can color so that there is exactly 1 monochromatic rectangle, and in the 126 cross sections, have all 126 possible monochromatic rectangles represented. To do this, in each cross section we color so that each row has exactly 2 lattice points of the same color, and such that 6 of the rows give all possible combinations of 2 points having the same color. This way, there will be exactly 1 monochromatic rectangle in each cross section; we can obviously vary it for the different cross sections.

20. [12] Let $ABCD$ be a quadrilateral circumscribed about a circle with center O . Let O_1, O_2, O_3 , and O_4 denote the circumcenters of $\triangle AOB, \triangle BOC, \triangle COD$, and $\triangle DOA$. If $\angle A = 120^\circ, \angle B = 80^\circ$, and $\angle C = 45^\circ$, what is the acute angle formed by the two lines passing through O_1O_3 and O_2O_4 ?

Answer: 82.5

Lemma: Given a triangle $\triangle ABC$, let I be the incenter, I_A be the excenter opposite A , and \check{S} be the second intersection of AI with the circumcircle. Then \check{S} is the center of the circle through B, I, C , and I_A .

Proof. First, note

$$\angle IBI_A = \angle IBC + \angle CBI_A = \frac{\angle ABC}{2} + \frac{180^\circ - \angle ABC}{2} = 90^\circ.$$

Similarly $\angle ICI_A = 90^\circ$. Therefore $BICI_A$ is cyclic. Now note that A, I, \check{S} , and I_A are collinear because they are all on the angle bisector of $\angle BAC$. Hence

$$\angle CI\check{S} = 180^\circ - \angle CIA = \angle CAI + \angle ACI = \angle BC\check{S} + \angle ICB = \angle IC\check{S}.$$

(Note $\angle CAI = \angle BA\check{S} = \angle BC\check{S}$ since A, B, \check{S} , and C are concyclic.) Hence $\check{S}C = \check{S}I$. Similarly $\check{S}B = \check{S}I$. Thus \check{S} is the center of the circle passing through B, I , and C , and therefore I_A as well. \square

Let BA and CD intersect at E and DA and CB intersect at F . We first show that F, O_1, O , and O_3 are collinear.

Let O'_1 and O'_3 denote the intersections of FO with the circumcircles of triangles FAB and FDC . Since O is the excenter of triangle FAB , by the lemma O'_1 is the circumcenter of $\triangle ABO$; since O is incenter of triangle FDC , by the lemma O'_3 is the circumcenter of $\triangle DOC$. Hence $O'_1 = O_1$ and $O'_3 = O_3$. Thus, points F, O_1, O , and O_3 are collinear, and similarly, we have E, O_2, O , and O_3 are collinear. Now $\angle BEC = 55^\circ$ and $\angle DFC = 20^\circ$ so considering quadrilateral $EOFC$, the angle between O_1O_3 and O_2O_4 is

$$\begin{aligned} \angle EOF &= \angle OEC + \angle OFC + \angle FCE \\ &= \frac{\angle BEC}{2} + \frac{\angle DFC}{2} + \angle FCE \\ &= 27.5^\circ + 10^\circ + 45^\circ = 82.5^\circ. \end{aligned}$$

21. [12] Let $ABCD$ be a quadrilateral inscribed in a circle with center O . Let P denote the intersection of AC and BD . Let M and N denote the midpoints of AD and BC . If $AP = 1$, $BP = 3$, $DP = \sqrt{3}$, and AC is perpendicular to BD , find the area of triangle MON .

Answer: $\boxed{\frac{3}{4}}$

We first prove that $ONPM$ is a parallelogram. Note that APD and BPC are both $30^\circ - 60^\circ - 90^\circ$ triangles. Let M' denote the intersection of MP and BC . Since $\angle BPM' = \angle MPD = 30^\circ$, we have $MP \perp BC$. Since ON is the perpendicular bisector of BC , we have $MP \parallel NO$. Similarly, we have $MO \parallel NP$. Thus $ONPM$ is a parallelogram. It follows that the area of triangle MON is equal to the area of triangle MPN , which is equal to $\frac{1}{2} \cdot 1 \cdot 3 \cdot \sin \angle MPN = \frac{1}{2} \cdot 1 \cdot 3 \cdot \sin 150^\circ = \frac{3}{4}$.

22. [12] Find the number of ordered triples (a, b, c) of pairwise distinct integers such that $-31 \leq a, b, c \leq 31$ and $a + b + c > 0$.

Answer: $\boxed{117690}$

We will find the number of such triples with $a < b < c$. The answer to the original problem will then be six times what we will get. By symmetry, the number of triples (a, b, c) with $a + b + c > 0$ is equal to the number of those with $a + b + c < 0$. Our main step is thus to find the number of triples with sum 0.

If $b = 0$, then $a = -c$, and there are 31 such triples. We will count the number of such triples with $b > 0$ since the number of those with $b < 0$ will be equal by symmetry.

For all positive n such that $1 \leq n \leq 15$, if $a = -2n$, there are $n - 1$ pairs (b, c) such that $a + b + c = 0$ and $b > 0$, and for all positive n such that $1 \leq n \leq 16$, if $a = -2n + 1$, there are also $n - 1$ such pairs (b, c) . In total, we have $1 + 1 + 2 + 2 + 3 + 3 + \dots + 14 + 14 + 15 = 225$ triples in the case $b > 0$ (and hence likewise for $b < 0$.)

In total, there are $31 + 225 + 225 = 481$ triples such that $a < b < c$ and $a + b + c = 0$. Since there are $\binom{63}{3} = 39711$ triples (a, b, c) such that $-31 \leq a < b < c \leq 31$, the number of triples with the additional restriction that $a + b + c > 0$ is $\frac{39711 - 481}{2} = 19615$. So the answer to the original problem is $19615 \times 6 = 117690$.

23. [14] Let S be the set of points (x, y, z) in \mathbb{R}^3 such that x, y , and z are positive integers less than or equal to 100. Let f be a bijective map between S and the $\{1, 2, \dots, 1000000\}$ that satisfies the following property: if $x_1 \leq x_2$, $y_1 \leq y_2$, and $z_1 \leq z_2$, then $f(x_1, y_1, z_1) \leq f(x_2, y_2, z_2)$. Define

$$A_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(i, j, k),$$

$$B_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, i, k),$$

and $C_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, k, i)$.

Determine the minimum value of $A_{i+1} - A_i + B_{j+1} - B_j + C_{k+1} - C_k$.

Answer: $\boxed{30604}$

We examine the 6 planes, their intersections and the lines between 2 points in one of the three pairs of parallel planes. The expression is equivalent to summing differences in values along all these lines. We examine the planes intersections. There is one cube, $3 \cdot 98$ squares and $3 \cdot 98 \cdot 98$ lines. The minimum value of the difference along a line is 1. For a square, to minimize the differences we take four consecutive numbers, and the minimum value is 6. To find the minimum value along a cube, we take 8 consecutive numbers. Since we are taking differences, we can add or subtract any constant to the numbers, so we assume the numbers are 1-8. Examining the cube, we see there's 1 spot where the number is multiplied by -3, 3 spots where the number is multiplied by -1, 1 spot where

the number is multiplied by 3, and 3 spots where the number is multiplied by 1. 1 and 8 must go in the corners, and 2,3,4,5 must go in spots multiplied by -1, -1, 1,1, respectively. To minimize the differences we put 5 in the final spot multiplied by -1, and 4 in the spot multiplied by 1 opposite 5. Then the sum of all the differences is 28, so the minimum for a cube is 28. So the answer is $28 + 18(100 - 2) + 3(100 - 2)^2 = 30000 + 600 + 12 - 36 + 28 = 30604$. It is clear that this value can be obtained.

24. [14] In how many ways may thirteen beads be placed on a circular necklace if each bead is either blue or yellow and no two yellow beads may be placed in adjacent positions? (Beads of the same color are considered to be identical, and two arrangements are considered to be the same if and only if each can be obtained from the other by rotation).

Answer: $\boxed{41}$

Let t_n be the number of arrangements of n beads in a row such that bead i and $i + 1$ are not both yellow for $1 \leq i < n$. Let a_n and b_n be the number of arrangements satisfying the additional condition that beads n and 1 are not both yellow, and that beads n and 1 are both yellow, respectively. Clearly $t_n = a_n + b_n$.

First consider t_n . If bead n is blue, there are t_{n-1} ways to choose the remaining colors. If bead n is yellow, then bead $n - 1$ must be blue, and there are t_{n-2} ways to choose the remaining colors. Hence

$$t_n = t_{n-1} + t_{n-2}.$$

Next consider b_n . In this case beads 1 and n are both yellow, so beads 2 and $n - 1$ must be blue; the remaining colors can be chosen in t_{n-4} ways. Hence

$$b_n = t_{n-4}.$$

From $t_0 = 1$ and $t_1 = 2$ we see that $t_n = F_{n+2}$ and $b_n = F_{n-2}$ and so $a_n = F_{n+2} - F_{n-2}$, where F_n denotes the n th Fibonacci number.

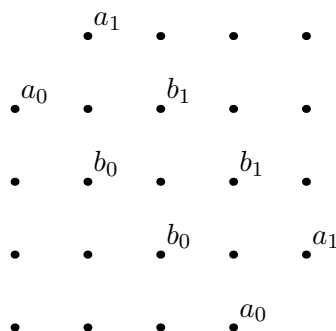
Next, since 13 is prime, a circular arrangement corresponds to exactly 13 straight-line arrangements, except for when all beads are the same color. The all-blue's arrangement satisfies our condition, while the all-yellow does not. Hence our answer is

$$\frac{a_{13} - 1}{13} + 1 = \frac{F_{15} - F_{11} + 12}{13} = \frac{610 - 89 + 12}{13} = 41.$$

25. [14] Let n be an integer greater than 3. Let R be the set of lattice points (x, y) such that $0 \leq x, y \leq n$ and $|x - y| \leq 3$. Let A_n be the number of paths from $(0, 0)$ to (n, n) that consist only of steps of the form $(x, y) \rightarrow (x, y + 1)$ and $(x, y) \rightarrow (x + 1, y)$ and are contained entirely within R . Find the smallest positive real number that is greater than $\frac{A_{n+1}}{A_n}$ for all n .

Answer: $\boxed{2 + \sqrt{2}}$

We first find A_n in terms of n . Let a_n be the number of ways to get to the point $(n, n + 3)$, and let b_n be the number of ways to get to the point $(n + 1, n + 2)$. By symmetry, a_n is also the number of ways to get to $(n + 3, n)$ and b_n is also the number of ways to get to the point $(n + 2, n + 1)$.



We can easily see that $a_0 = 1$ and $b_0 = 3$. This also means that $A_n = a_{n-3} + 3b_{n-3} + 3b_{n-3} + a_{n-3} = 2a_{n-3} + 6b_{n-3}$.

We also get the recurrence:

$$\begin{aligned} a_{i+1} &= a_i + b_i \\ b_{i+1} &= a_i + 3b_i \end{aligned}$$

We have both $3a_{i+1} = 3a_i + 3b_i$ and $a_{i+2} = a_{i+1} + b_{i+1}$. Subtracting these gives

$$\begin{aligned} a_{i+2} - 3a_{i+1} &= a_{i+1} - 3a_i + b_{i+1} - 3b_i \\ a_{i+2} - 3a_{i+1} &= a_{i+1} - 3a_i + a_i \\ a_{i+2} &= 4a_{i+1} - 2a_i \end{aligned}$$

Now we can solve this recurrence using its characteristic polynomial $x^2 - 4x + 2$, which has roots of $2 + \sqrt{2}$ and $2 - \sqrt{2}$. We can then write $a_i = A(2 + \sqrt{2})^i + B(2 - \sqrt{2})^i$ for some constants A and B . Now, $a_0 = a$ and $a_1 = a_0 + b_0 = 4$. Using this, we solve for A and B to get

$$a_i = \left(\frac{1 + \sqrt{2}}{2} \right) (2 + \sqrt{2})^i + \left(\frac{1 - \sqrt{2}}{2} \right) (2 - \sqrt{2})^i$$

Then,

$$\begin{aligned} b_i &= a_{i+1} - a_i \\ &= \left(\frac{1 + \sqrt{2}}{2} \right) ((2 + \sqrt{2})^{i+1} - (2 + \sqrt{2})^i) + \left(\frac{1 - \sqrt{2}}{2} \right) ((2 - \sqrt{2})^{i+1} - (2 - \sqrt{2})^i) \\ &= \left(\frac{1 + \sqrt{2}}{2} \right) (1 + \sqrt{2})(2 + \sqrt{2})^i + \left(\frac{1 - \sqrt{2}}{2} \right) (1 - \sqrt{2})(2 - \sqrt{2})^i \\ &= \left(\frac{3 + 2\sqrt{2}}{2} \right) (2 + \sqrt{2})^i + \left(\frac{3 - 2\sqrt{2}}{2} \right) (2 - \sqrt{2})^i \end{aligned}$$

Therefore,

$$A_n = 2a_{n-3} + 6b_{n-3} = (10 + 7\sqrt{2})(2 + \sqrt{2})^{n-3} + (10 - 7\sqrt{2})(2 - \sqrt{2})^{n-3}.$$

We can then easily see that $A_n < (2 + \sqrt{2})A_{n-1}$. Also, since $2 - \sqrt{2} < 1$, as n approaches infinity, the ratio $\frac{A_n}{A_{n-1}}$ approaches $2 + \sqrt{2}$. Hence the least upper bound of $\frac{A_n}{A_{n-1}}$ is $2 + \sqrt{2}$.

26. [14] In how many ways can 13 bishops be placed on an 8×8 chessboard such that (i) a bishop is placed on the second square in the second row, (ii) at most one bishop is placed on each square, (iii) no bishop is placed on the same diagonal as another bishop, and (iv) every diagonal contains a bishop? (For the purposes of this problem, consider all diagonals of the chessboard to be diagonals, not just the main diagonals).

Answer: 1152

We color the squares of the chessboard white and black such that B2 (the second square in the second row) is black. Note that at most 7 bishops can go on the white squares, and if there is a bishop on b2, at most 5 more can be on the white squares. So of the other 12 bishops, 7 go on white squares and 5 go on black squares.

Consider the long diagonal on the white squares, and the 6 white diagonals parallel to it. Of the 7 bishops placed on the white squares, exactly one must go on each of these diagonals (this also proves

that at most 7 can go on the white squares). Of these diagonals there is 1 of length 8, and 2 of length 2, 4, and 6. There are 2 ways to place 2 bishops on the diagonals of length 2, then 2 ways to place 2 bishops on the diagonals of length 4, then 2 ways to place 2 bishops on the diagonals of length 2, then the long diagonal bishop can go on either corner. So there are 16 ways to place 7 bishops on the white squares.

Now we can divide the black squares of the board into the 6 diagonals parallel to the long white diagonal, and the long black diagonal. The bishop on b2 accounts for two of these diagonals. We are left with a diagonal of length 3, and two diagonals of length 5, 7. There are 3 ways to pick the bishop on the diagonal of length 3, 6 ways to pick two bishop for the diagonals of length 5, and 6 ways to pick the bishop on the diagonals of length 7. So there are 72 ways to pick 5 other bishops for the black squares. So the answer is $72 \cdot 16 = 1152$.

27. [16] Find the number of polynomials $p(x)$ with integer coefficients satisfying $p(x) \geq \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ and $p(x) \leq \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ for all $x \in \mathbb{R}$.

Answer: 4

We first find the intersection points of $f(x) = 2x^4 - 6x^2 + 1$ and $g(x) = 4 - 5x^2$. If $2x^4 - 6x^2 + 1 = 4 - 5x^2$, then $2x^4 - x^2 - 3 = 0$, so $(2x^2 - 3)(x^2 + 1) = 0$, and $x = \pm\sqrt{\frac{3}{2}}$. Note that this also demonstrates that $g(x) \geq f(x)$ if and only if $|x| \leq \sqrt{\frac{3}{2}}$ and that $p(x)$ must satisfy $p(x) \leq g(x)$ iff $|x| \leq \sqrt{\frac{3}{2}}$. We must have $p(\sqrt{\frac{3}{2}}) = -\frac{7}{2}$, so $p(x) = -\frac{7}{2} + (2x^2 - 3)q(x)$ for a polynomial q of degree 0, 1 or 2. We now examine cases.

Case 1: q is constant. We have $q = \frac{n}{2}$ for an integer n since $1 \leq p(0) \leq 4$, $n = -\frac{3}{2}$ or $-\frac{5}{2}$. Clearly $q = -\frac{5}{2}$ is an appropriate choice because then $p = g$. Let $p_1(x) = -\frac{7}{2} + -\frac{3}{2}(2x^2 - 3) = 1 - 3x^2$. We have $p_1(x)$ is $\leq g(x)$ and $\leq f(x)$ in the right places, so this function works. Thus, there are 2 solutions.

Case 2: q is linear. We have $p(x)$ is a cubic, so $p(x) - g(x)$ is also a cubic, which means it can't be positive for both arbitrarily large positive values of x and arbitrarily large negative values of x . Thus, there are no solutions.

Case 3: q is quadratic. We have $q(x) = ax^2 + bx + c$. Apply the same argument for the case when q is a constant, we have $c = -\frac{3}{2}$ or $-\frac{5}{2}$. Since p must have integer coefficients, we have b must be an integer. Since $f(x) \geq p(x) \geq g(x)$ for large x , the leading coefficient of f must be greater than or equal to the leading coefficient of p , which must be greater than 0. Thus $a = 1$ or $a = \frac{1}{2}$. However, if $a = \frac{1}{2}$, then the quadratic term of p is not an integer, so $a = 1$.

Now if $c = -\frac{5}{2}$, then $p(0) = 4 = g(0)$. But this is the maximum value of $g(x)$, so it must be a local maximum of p . Thus p must not have a linear term odd (otherwise the function behaves like $-3bx + 4$ around $x = 0$). So $p(x)$ must be $2x^4 - 8x^2 + 4$. This is indeed bounded between $f(x)$ and $g(x)$ at all points.

Now suppose $c = -\frac{3}{2}$. We have a must equal to 1. If $b \neq 0$, then p will have a cubic term, which means $f(x) - p(x)$ can't be positive for both arbitrarily large positive x and arbitrarily large negative x , so $b = 0$. Therefore $p(x) = -\frac{7}{2} + (2x^2 - 3)(x^2 - \frac{3}{2}) = f(x)$. It is easy to check that this choice of p is indeed bounded between $f(x)$ and $g(x)$.

Therefore, there are 4 solutions in total.

28. [16] Let ABC be a triangle, and let points P and Q lie on BC such that P is closer to B than Q is. Suppose that the radii of the incircles of triangles ABP , APQ , and AQC are all equal to 1, and that the radii of the corresponding excircles opposite A are 3, 6, and 5, respectively. If the radius of the incircle of triangle ABC is $\frac{3}{2}$, find the radius of the excircle of triangle ABC opposite A .

Answer: 135

Let t denote the radius of the excircle of triangle $\triangle ABC$.

Lemma: Let $\triangle ABC$ be a triangle, and let r and r_A be the inradius and exradius opposite A . Then

$$\frac{r}{r_A} = \tan \frac{B}{2} \tan \frac{C}{2}.$$

Proof. Let I and J denote the incenter and the excenter with respect to A . Let D and E be the foot of the perpendicular from I and J to BC , respectively. Then

$$\begin{aligned} r &= ID = BI \sin \frac{B}{2} \\ r_A &= JE = BJ \sin \frac{180^\circ - B}{2} = BJ \cos \frac{B}{2} \\ BI &= BJ \tan \angle AJB = BJ \tan \frac{C}{2}. \end{aligned}$$

The last equation followed from

$$\angle AJB = 180^\circ - \angle ABJ - \angle JAB = \frac{180^\circ - B}{2} - \frac{A}{2} = \frac{C}{2}.$$

Hence

$$\frac{r}{r_A} = \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \cdot \frac{BI}{BJ} = \tan \frac{B}{2} \cdot \tan \frac{C}{2}.$$

□

Noting $\tan \frac{\angle APB}{2} \tan \frac{\angle APQ}{2} = \tan \frac{\angle AQP}{2} \tan \frac{\angle AQC}{2} = 1$ and applying the lemma to $\triangle ABC$, $\triangle ABP$, $\triangle APQ$, and $\triangle AQC$ give

$$\begin{aligned} \frac{3/2}{t} &= \tan \frac{\angle ABC}{2} \cdot \tan \frac{\angle ACB}{2} \\ &= \left(\tan \frac{\angle ABC}{2} \cdot \tan \frac{\angle APB}{2} \right) \cdot \left(\tan \frac{\angle APQ}{2} \cdot \tan \frac{\angle AQP}{2} \right) \cdot \left(\tan \frac{\angle AQC}{2} \cdot \tan \frac{\angle ACB}{2} \right) \\ &= \frac{1}{3} \cdot \frac{1}{6} \cdot \frac{1}{5} \end{aligned}$$

Therefore, $t = 135$.

29. [16] Let ABC be a triangle such that $AB = AC = 182$ and $BC = 140$. Let X_1 lie on AC such that $CX_1 = 130$. Let the line through X_1 perpendicular to BX_1 at X_1 meet AB at X_2 . Define X_2, X_3, \dots , as follows: for n odd and $n \geq 1$, let X_{n+1} be the intersection of AB with the perpendicular to $X_{n-1}X_n$ through X_n ; for n even and $n \geq 2$, let X_{n+1} be the intersection of AC with the perpendicular to $X_{n-1}X_n$ through X_n . Find $BX_1 + X_1X_2 + X_2X_3 + \dots$

Answer: $\boxed{\frac{1106}{5}}$

Let M and N denote the perpendiculars from X_1 and A to BC , respectively. Since triangle ABC is isosceles, we have M is the midpoint of BC . Moreover, since AM is parallel to X_1N , we have $\frac{NC}{X_1C} = \frac{MC}{AC} \Leftrightarrow \frac{X_1N}{130} = \frac{70}{182} = \frac{5}{13}$, so $NC = 50$. Moreover, since $X_1N \perp BC$, we find $X_1C = 120$ by the Pythagorean Theorem. Also, $BN = BC - NC = 140 - 50 = 90$, so by the Pythagorean Theorem, $X_1B = 150$.

We want to compute $X_2X_1 = X_1B \tan(\angle ABX_1)$. We have

$$\begin{aligned} \tan(\angle ABX_1) &= \tan(\angle ABC - \angle X_1BC) = \frac{1 + \tan(\angle ABC) \tan(\angle X_1BC)}{\tan(\angle ABC) - \tan(\angle X_1BC)} = \frac{\left(\frac{12}{5}\right) - \left(\frac{4}{3}\right)}{1 + \left(\frac{12}{5}\right)\left(\frac{4}{3}\right)} \\ &= \frac{\frac{16}{15}}{\frac{63}{15}} = \frac{16}{63}. \end{aligned}$$

Hence $X_2X_1 = 150 \cdot \frac{16}{63}$, and by the Pythagorean Theorem again, $X_2B = 150 \cdot \frac{65}{63}$.

Next, notice that $\frac{AX_n}{AX_{n+2}}$ is constant for every nonnegative integer n (where we let $B = X_0$). Indeed, since X_nX_{n+1} is parallel to $X_{n+2}X_{n+3}$ for each n , the dilation taking X_n to X_{n+2} for some n also takes X_k to X_{k+2} for all k .

Since $\triangle AX_{n+2}X_{n+3} \sim \triangle AX_nX_{n+1}$ with ratio $\frac{AX_n}{AX_{n+2}}$ for each even n , we can compute that $\frac{X_{n+2}X_{n+3}}{X_nX_{n+1}} = \frac{AX_{n+2}}{AX_n} = 1 - \frac{150 \cdot \frac{65}{182}}{182}$ for every nonnegative integer n . Notice we use all three sides of the above similar triangles.

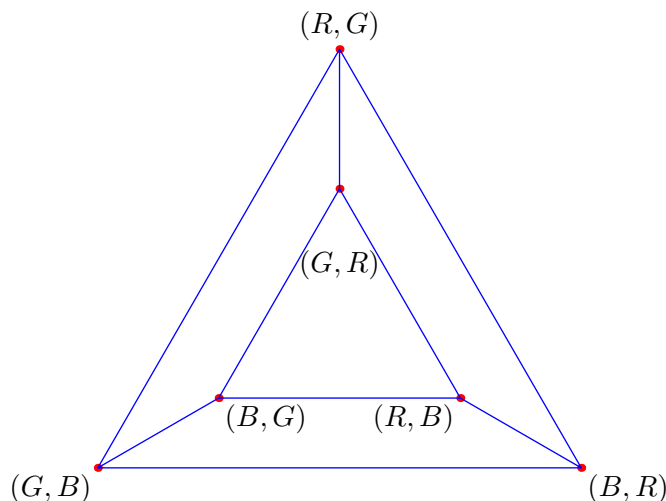
We now split our desired sum into two geometric series, one with the even terms and one with the odd terms, to obtain

$$\begin{aligned} BX_1 + X_1X_2 + \dots &= (BX_1 + X_2X_3 + \dots) + (X_1X_2 + X_3X_4 + \dots) = \frac{150}{\frac{150 \cdot \frac{65}{182}}{182}} + \frac{150 \cdot \frac{16}{63}}{\frac{150 \cdot \frac{65}{182}}{182}} \\ &= \frac{\frac{79}{63} \cdot 150}{\frac{150 \cdot \frac{65}{182}}{182}} = \frac{1106}{5}. \end{aligned}$$

30. [16] How many ways are there to color the vertices of a $2n$ -gon with three colors such that no vertex has the same color as its either of its two neighbors or the vertex directly across from it?

Answer: $\boxed{3^n + (-2)^{n+1} - 1}$

Let the $2n$ -gon have vertices A_1, A_2, \dots, A_{2n} , in that order. Consider the diagonals $d_1 = (A_1, A_{n+1})$, $d_2 = (A_2, A_{n+2})$, \dots , $d_n = (A_n, A_{2n})$. Suppose the three colors are red (R), green (G), and blue (B). Each diagonal can either be colored (R, G) , (G, R) , (G, B) , (B, G) , (B, R) , or (R, B) . We first choose one of the six colorings for d_1 , which then constrains the possible colorings for d_2 , which constrains the possible colorings for d_3 , and so on. This graph shows the possible configurations; two pairs of colors are connected by an edge if they can be the colors for d_i and d_{i+1} for any $1 \leq i \leq n - 1$.

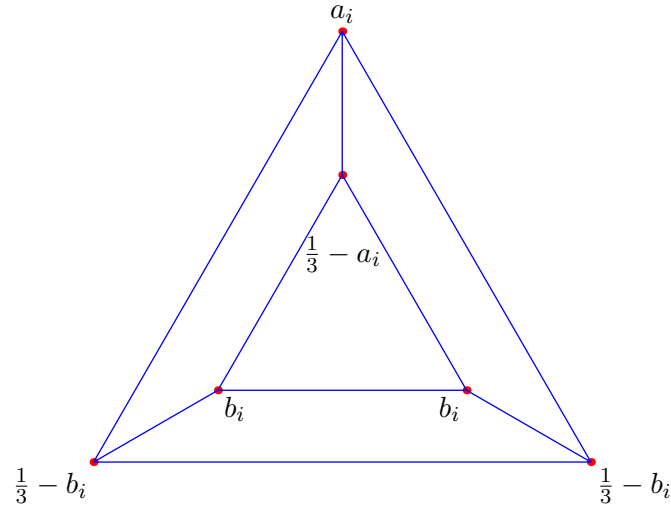


Suppose without loss of generality that d_1 is colored (R, G) . (At the end, we multiply our answer by 6.) Then d_n must be either (R, G) , (B, G) , or (R, B) . Now, we simply need to count the number of paths of length $n - 1$ within this graph from (R, G) to one of these three points.

Suppose we are making a random walk of $n - 1$ steps, where at each move we pick one of the three possible edges with probability $\frac{1}{3}$. We will calculate the probability that the walk ends at one of (R, G) , (B, G) , or (R, B) .

Let a_i and b_i be the probability that, after i steps, we are at (R, G) and (B, G) , respectively. By symmetry, b_i is also the probability that we are at (R, B) after i steps.

Observe that after each move, the probability of arriving at either (R, G) or (G, R) will always be $\frac{1}{3}$. Therefore, the probability of being at (G, R) after i steps is $\frac{1}{3} - a_i$. Similarly, the probability of being at (G, B) is $\frac{1}{3} - b_i$ and the probability of being at (B, R) is $\frac{1}{3} - b_i$.



Now, for $i \geq 1$ we have the recurrences

$$\begin{aligned} a_{i+1} &= \frac{1}{3} \left(\left(\frac{1}{3} - a_i \right) + \left(\frac{1}{3} - b_i \right) + \left(\frac{1}{3} - b_i \right) \right) \\ &= \frac{1}{3} - \frac{1}{3} a_i - \frac{2}{3} b_i \end{aligned}$$

$$\begin{aligned} b_{i+1} &= \frac{1}{3} \left(\left(\frac{1}{3} - a_i \right) + \left(\frac{1}{3} - b_i \right) + b_i \right) \\ &= \frac{2}{9} - \frac{1}{3} a_i \end{aligned}$$

So then

$$\begin{aligned} a_{i+2} &= \frac{1}{3} - \frac{1}{3} a_{i+1} - \frac{2}{3} b_{i+1} \\ a_{i+2} &= \frac{1}{3} - \frac{1}{3} a_{i+1} - \frac{2}{3} \left(\frac{2}{9} - \frac{1}{3} a_i \right) \\ a_{i+2} &= \frac{5}{27} - \frac{1}{3} a_{i+1} + \frac{2}{9} a_i \\ \left(a_{i+2} - \frac{1}{6} \right) &= -\frac{1}{3} \left(a_{i+1} - \frac{1}{6} \right) + \frac{2}{9} \left(a_i - \frac{1}{6} \right) \end{aligned}$$

This recurrence has a characteristic polynomial $x^2 + \frac{1}{3}x - \frac{2}{9}$, which has roots $\frac{1}{3}$ and $-\frac{2}{3}$. We can write $a_i = \frac{1}{6} + A\left(\frac{1}{3}\right)^i + B\left(-\frac{2}{3}\right)^i$ for some constants A and B for $i \geq 1$. Since $a_1 = 0$ and $a_2 = \frac{1}{3}$, we can solve for A and B and get

$$a_i = \frac{1}{6} + \frac{1}{6} \left(\frac{1}{3} \right)^i + \frac{1}{3} \left(-\frac{2}{3} \right)^i$$

The answer to the problem is then

$$\begin{aligned}
 6 \cdot 3^{n-1}(a_{n-1} + 2b_{n-1}) &= 6 \cdot 3^{n-1}(a_{n-1} + 1 - a_{n-1} - 3a_n) \\
 &= 6 \cdot 3^{n-1}(1 - 3a_n) \\
 &= 6 \cdot 3^{n-1} \left(1 - 3 \left(\frac{1}{6} + \frac{1}{6} \left(\frac{1}{3} \right)^n + \frac{1}{3} \left(-\frac{2}{3} \right)^n \right) \right) \\
 &= 6 \cdot 3^{n-1} \left(\frac{1}{2} - \frac{1}{2} \left(\frac{1}{3} \right)^n - \left(-\frac{2}{3} \right)^n \right) \\
 &= 3^n + (-2)^{n+1} - 1.
 \end{aligned}$$

31. [18] Let $A = \{1, 2, 3, \dots, 9\}$. Find the number of bijective functions $f : A \rightarrow A$ for which there exists at least one $i \in A$ such that

$$|f(i) - f^{-1}(i)| > 1.$$

Answer: 359108

We count the complement — the number of functions f such that for all $i \in A$, $|f(i) - f^{-1}(i)| \leq 1$.

The condition is equivalent to $|f(f(i)) - i| \leq 1$ for all $i \in A$. If $f(j) = j$, the inequality is automatically satisfied for $i = j$. Otherwise, if $f(f(j)) = j$ but $f(j) = k \neq j$, then we will have $f(f(k)) = k$, allowing the inequality to be satisfied for $i = j, k$. Else, if $f(f(i)) \neq i$, say $f(f(i)) = i + 1$ and $f(i) = k$, then $f(f(k)) = f(i + 1) = k + 1$ or $k - 1$. Thus the function f allows us to partition the elements of A into three groups:

- (a) those such that $f(i) = i$,
- (b) those that form pairs $\{i, j\}$ such that $f(i) = j$ and $f(j) = i$, and
- (c) those that form quartets $\{i, i + 1, j, j + 1\}$ such that f permutes them as $(i \ j \ i + 1 \ j + 1)$ or $(i \ j + 1 \ i + 1 \ j)$, in cycle notation.

Let a be the number of elements of the second type. Note that a is even.

Case 1: There are no elements of the third type. If $a = 8$, there are $9 \cdot 7 \cdot 5 \cdot 3 = 945$ possibilities. If $a = 6$, there are $\binom{9}{3} \cdot 5 \cdot 3 = 1260$ possibilities. If $a = 4$, there are $\binom{9}{5} \cdot 3 = 378$ possibilities. If $a = 2$, there are $\binom{9}{7} = 36$ possibilities. If $a = 0$, there is 1 possibility. In total, case 1 offers $945 + 1260 + 378 + 36 + 1 = 2620$ possibilities.

Case 2: There are 4 elements of the third type. There are 21 ways to choose the quartet $\{i, i + 1, j, j + 1\}$. For each way, there are two ways to assign the values of the function to each element (as described above). For the remaining 5 elements, we divide into cases according to the value of a . If $a = 4$, there are $5 \times 3 = 15$ possibilities. If $a = 2$, there are $\binom{5}{3} = 10$ possibilities. If $a = 0$, there is one possibility. In total, case 2 offers $21 \times 2 \times (15 + 10 + 1) = 1092$ possibilities.

Case 3: There are 8 elements of the third type. There are 5 ways to choose the unique element not of the third type. Of the remaining eight, there are 3 ways to divide them into two quartets, and for each quartet, there are 2 ways to assign values of f . In total, case 3 offers $5 \times 3 \times 2^2 = 60$ possibilities.

Therefore, the number of functions $f : A \rightarrow A$ such that for at least one $i \in A$, $|f(i) - f^{-1}(i)| > 1$ is $9! - 2620 - 1092 - 60 = 359108$.

32. [18] Let p be a prime positive integer. Define a mod- p recurrence of degree n to be a sequence $\{a_k\}_{k \geq 0}$ of numbers modulo p satisfying a relation of the form $a_{i+n} = c_{n-1}a_{i+n-1} + \dots + c_1a_{i+1} + c_0a_i$ for all $i \geq 0$, where c_0, c_1, \dots, c_{n-1} are integers and $c_0 \not\equiv 0 \pmod{p}$. Compute the number of distinct linear recurrences of degree at most n in terms of p and n .

Answer: $1 - n \frac{p-1}{p+1} + \frac{p^2(p^{2n}-1)}{(p+1)^2}$

In the solution all polynomials are taken modulo p . Call a polynomial *nice* if it is monic with nonzero constant coefficient. We can associate each recurrence relation with a polynomial: associate

$$c_n a_{i+n} + c_{n-1} a_{i+n-1} + \dots + c_1 a_{i+1} + c_0 a_i = 0$$

with

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0.$$

Let D_i be the set of mod- p recurrences $\{a_k\}_{k \geq 0}$ where i is the least integer so that $\{a_k\}_{k \geq 0}$ has degree i , and let $d_i = |D_i|$.

Let S_n be the set of pairs $(\{a_k\}_{k \geq 0}, P)$ where $\{a_k\}_{k \geq 0}$ is a mod- p recurrence, and P is a nice polynomial associated to a recurrence relation of degree at most n satisfied by $\{a_k\}_{k \geq 0}$. To find d_n generally, we count the number of elements in S_n in two ways.

One the one hand, for each sequence $\{a_k\}_{k \geq 0}$ in D_i , there exist p^{n-i} polynomials P such that $(\{a_k\}_{k \geq 0}, P) \in S$. Indeed, $\{a_k\}_{k \geq 0}$ satisfies any recurrence relation associated with a polynomial multiple of P . When $j = i$ there is just one nice degree j polynomial that is a multiple of P , P itself. For $j > i$, there are $(p-1)p^{j-i-1}$ nice polynomials of degree j that are multiples of P , namely QP where Q is a nice polynomial of degree $j-i$. (There are p choices for the coefficients of x, \dots, x^{j-i-1} and $p-1$ choices for the constant term.) So the number of nice polynomial multiples of degree at most n is

$$1 + \sum_{j=i+1}^n (p-1)p^{j-i-1} = 1 + (p-1) \left(\frac{p^{n-i} - 1}{p-1} \right) = p^{n-i}.$$

Hence

$$|S_n| = \sum_{i=0}^n d_i p^{n-i}. \tag{1}$$

On the other hand, given a monic polynomial P of degree i , there are p^i recurrences $\{a_k\}_{k \geq 0}$ such that $(\{a_k\}_{k \geq 0}, P) \in S$, since a_0, \dots, a_{i-1} can be chosen arbitrarily and the rest of the terms are determined. Since there are $(p-1)p^{i-1}$ nice polynomials of degree $i \neq 0$ (and 1 nice polynomial for $i = 0$), summing over i gives

$$|S_n| = 1 + \sum_{i=1}^n (p-1)p^{2i-1} \tag{2}$$

Now clearly $d_0 = 1$. Setting (1) and (2) equal for n and $n+1$ give

$$\sum_{i=0}^{n+1} d_i p^{n+1-i} = 1 + (p-1) \sum_{i=1}^{n+1} p^{2i-1} \tag{3}$$

$$\begin{aligned} \sum_{i=0}^n d_i p^{n-i} &= 1 + (p-1) \sum_{i=1}^n p^{2i-1} \\ \implies \sum_{i=0}^n d_i p^{n+1-i} &= p + (p-1) \sum_{i=1}^n p^{2i}. \end{aligned} \tag{4}$$

Subtracting (4) from (3) yields:

$$\begin{aligned}
 d_{n+1} &= 1 - p + (p-1) \sum_{i=1}^{2n+1} (-1)^{i+1} p^i \\
 &= (p-1) \sum_{i=0}^{2n+1} (-1)^{i+1} p^i \\
 &= (p-1)^2 \sum_{i=0}^n p^{2m} \\
 &= (p-1)^2 \left(\frac{p^{2n+2} - 1}{p^2 - 1} \right) \\
 &= \frac{(p-1)(p^{2n+2} - 1)}{p+1}
 \end{aligned}$$

Thus the answer is

$$\begin{aligned}
 \sum_{i=0}^n d_i &= 1 + \frac{p-1}{p+1} \sum_{i=1}^n (p^{2i} - 1) \\
 &= 1 + \frac{p-1}{p+1} \left(-n + p^2 \cdot \frac{p^{2n} - 1}{p^2 - 1} \right) \\
 &= 1 - n \frac{p-1}{p+1} + \frac{p^2(p^{2n} - 1)}{(p+1)^2}
 \end{aligned}$$

33. [25] Find the number of sequences consisting of 100 R 's and 2011 S 's that satisfy the property that among the first k letters, the number of S 's is strictly more than 20 times the number of R 's for all $1 \leq k \leq 2111$.

Answer: $\frac{11}{2111} \binom{2111}{100}$

Given positive integers r and s such that $s \geq 20r$, let $N(s, r)$ denote the number of sequences of s copies of S and r copies of R such that for all $1 \leq k \leq r + s - 1$, among the first k letters, the number of S 's is strictly more than 20 times the number of R 's. We claim that

$$N(s, r) = \frac{s - 20r}{s + r} \binom{s + r}{s}.$$

We induct on $s + r$. For the base case, note that $N(20r, r) = 0$ for all r and $N(s, 0) = 1$ for all s .

Assume the formula holds up to $r + s - 1$. We now construct a path in the Cartesian plane as follows: Let R represent moving one unit to the left and S represent moving one unit up. This induces a bijection between sequences of S 's and R s and lattice paths from the origin to the point (r, s) that are above the line $y = 20x$. In order to reach the point (r, s) , we must go through either $(r - 1, s)$ or $(r, s - 1)$. Therefore, we have $N(r, s) = N(s, r - 1) + N(s - 1, r) = \frac{s - 20(r - 1)}{s + r - 1} \binom{s + r - 1}{s} + \frac{s - 1 - 20r}{s + r - 1} \binom{s + r - 1}{s} = \frac{s - 20r}{s + r} \binom{s + r}{s}$.

Remark This is a special case of the ballot problem, first studied in 1887 by Joseph Bertrand, generalizing the Catalan numbers. A good expository article on this problem is "Four Proofs of the Ballot Problem" by Marc Renault available on his website at <http://webpace.ship.edu/msrenault/ballotproblem/>.

In 2009, Yufei Zhao studied a variant problem called bidirectional ballot sequences, which he used to construct More-Sums-Then-Differences sets in additive combinatorics. His paper is available on his website at http://web.mit.edu/yufeiz/www/mstd_construction.pdf.

34. [25] Let $w = w_1, w_2, \dots, w_6$ be a permutation of the integers $\{1, 2, \dots, 6\}$. If there do not exist indices $i < j < k$ such that $w_i < w_j < w_k$ or indices $i < j < k < l$ such that $w_i > w_j > w_k > w_l$, then w is said to be *exquisite*. Find the number of exquisite permutations.

Answer: 25

Given a permutation $w = w_1, \dots, w_n$ for some n , call a sequence $w_{i_1}, w_{i_2}, \dots, w_{i_m}$ an increasing subsequence if $i_1 < \dots < i_m$ and $w_{i_1} < \dots < w_{i_m}$. Define decreasing subsequences similarly. Let $is(w)$ denote the length of the longest increasing sequence and $ds(w)$ denote the length of the longest decreasing sequence. We wish to find the number of permutations for $n = 6$ such that $is(w) \leq 2$ and $ds(w) \leq 3$. We note here that $6 = 2 \times 3$ is not a coincidence.

Erdos and Szekeres first studied problems on the longest increasing and decreasing subsequences. In 1935, they showed that for any permutation w of $\{1, 2, \dots, pq + 1\}$, either $is(w) > p$ or $ds(w) > q$, which later appeared on the Russian Math Olympiad.

In 1961, Schensted proved that the bound $pq + 1$ is sharp, and he enumerated the number of permutations for $n = pq$ such that $is(w) \leq p$ and $ds(w) \leq q$ (exquisite permutations for simplicity), with an elegant combinatorial proof based on the RSK-algorithm relating Young Tableaux and permutations.

The main idea of his proof is as follows. Consider a $p \times q$ rectangle. A Young Tableau is an assignment of $1, 2, \dots, pq$, one to each unit square of the rectangle, such that every row and column is in increasing order. There is a bijection between set of exquisite permutations and pairs of Young Tableaux. Since the number of ways to write $1, 2, \dots, 6$ on a 2×3 rectangle with every row and column in increasing order is 5, there are exactly 25 exquisite permutations.

For a thorough exposition of increasing and decreasing subsequences and a collection of interesting open questions, see Richard Stanley's note for his undergraduate research students at MIT, "Increasing and decreasing subsequences and their variants" available at <http://www.math.mit.edu/~rstan/papers/ids.pdf>.

35. [25] An *independent set* of a graph G is a set of vertices of G such that no two vertices among these are connected by an edge. If G has 2000 vertices, and each vertex has degree 10, find the maximum possible number of independent sets that G can have.

Answer: 2047¹⁰⁰

The upper bound is obtained when G is a disjoint union of bipartite graphs, each of which has 20 vertices with 10 in each group such that every pair of vertices not in the same group are connected.

In 1991, during his study of the Cameron—Erdos conjecture on the number of sum-free sets, Noga Alon came across this problem on independent sets and conjectured that our construction gives the best bound.

This problem received considerable attention due to its application to combinatorial group theory and statistical mechanics, but no solution was found until 2009, when Yufei Zhao resolved Alon's conjecture with a beautiful and elementary approach.

For the reader to enjoy the full insight of Yufei's argument, we omit the proof here and refer to his paper "The Number of Independent Sets in a Regular Graph" available on his website at http://web.mit.edu/yufeiz/www/indep_reg.pdf.

36. [25] An *ordering* of a set of n elements is a bijective map between the set and $\{1, 2, \dots, n\}$. Call an ordering p of the 10 unordered pairs of distinct integers from the set $\{1, 2, 3, 4, 5\}$ admissible if, for any $1 \leq a < b < c \leq 5$, either $p(\{a, b\}) < p(\{a, c\}) < p(\{b, c\})$ or $p(\{b, c\}) < p(\{a, c\}) < p(\{a, b\})$. Find the total number of admissible orderings.

Answer: 768

This problem is a special case of the *higher Bruhat order*, a class of combinatorial object widely studied for its connection to an assortment of mathematical areas such as algebraic geometry, algebraic combinatorics, and computational geometry.

An admissible order in our problem—the higher Bruhat order $B(5, 2)$ —are best viewed as the set of *reduced decompositions* of the permutation 4321. Loosely speaking, a reduced decomposition is a sequence of adjacent transpositions that changes the permutation $n, n - 1, \dots, 1$ to $1, 2, \dots, n$. For example, for $n = 4$, the sequence $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$ induces the following reduced decomposition:

$$4321 \rightarrow 4\mathbf{3}12 \rightarrow 41\mathbf{3}2 \rightarrow 14\mathbf{3}2 \rightarrow 1\mathbf{4}23 \rightarrow 12\mathbf{4}3 \rightarrow 1234.$$

For each permutation above, switching the two bolded numbers yields the next permutation in the chain. For example, switching 3 and 1 in $4\mathbf{3}12$ yields $41\mathbf{3}2$.

Readers interested in the connection between the higher Bruhat order and reduced decompositions are referred to Delong Meng’s paper “Reduced decompositions and permutation patterns generalized to the higher Bruhat order” for background as well as recent development of this subject. The paper is available at

<http://web.mit.edu/delong13/www/papers.html>.

The number of reduced decompositions of $n, n - 1, \dots, 1$ is equal to the number of $(n - 1)$ st *standard Young Tableaux* of staircase shape, given by the formula

$$\frac{\binom{n}{2}!}{1^{n-1} \cdot 3^{n-2} \dots (2n - 3)}.$$

When $n = 5$, the formula gives 768.

Young Tableaux are one of the most important tools in algebraic combinatorics, especially for problems involving permutations, integer partitions, and posets. For a concise and accessible introduction to the Young Tableaux and reduced decompositions (and much more cool combinatorial stuff!), see Section 7 of Richard Stanley’s note “A Combinatorial Miscellany” at <http://math.mit.edu/~rstan/papers/comb.pdf>.

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14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Team Round A

Warm-Up [50]

1. [20] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. Alice goes first, and the players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y , subject to the constraint that $0 < y - x < 1$.
 - (a) [10] If the first player to write a number greater than or equal to 2010 wins, determine, with proof, who has the winning strategy.
 - (b) [10] If the first player to write a number greater than or equal to 2010 on her 2011th turn or later wins (if a player writes a number greater than or equal to 2010 on her 2010th turn or earlier, she loses immediately), determine, with proof, who has the winning strategy.

Solution: Each turn of the game is equivalent to adding a number between 0 and 1 to the number on the board.

- (a) Barbara has the winning strategy: whenever Alice adds z to the number on the board, Barbara adds $1 - z$. After Barbara's i th turn, the number on the board will be i . Therefore, after Barbara's 2009th turn, Alice will be forced to write a number between 2009 and 2010, after which Barbara can write 2010 and win the game.
 - (b) Alice has the winning strategy: she writes any number a on her first turn, and, after that, whenever Barbara adds z to the number on the board, Alice adds $1 - z$. After Alice's i th turn, the number on the board will be $(i - 1) + a$, so after Alice's 2010th turn, the number will be $2009 + a$. Since Barbara cannot write a number greater than or equal to 2010 on her 2010th turn, she will be forced to write a number between $2009 + a$ and 2010, after which Alice can write 2010 and win the game.
2. [15] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left, the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?

Solution: Both operations, run and fight, move the black checker exactly one square to the right, so the game will end after exactly 2008 moves regardless of Brian's choices. Furthermore, it is easy to see that if we view running and fighting as operations, they commute. So the order of the moves does not matter, all that matters is how many times Rachel runs and how many times Rachel fights. Each fight adds one white checker to the grid, so two games with different numbers of fights will end up in different final configurations. There are 2009 possible values for the number of fights, so there are 2009 possible final configurations.

3. [15] Let n be a positive integer, and let a_1, a_2, \dots, a_n be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n - 1$, where, for all positive integers k , $\varphi(k)$ denotes the number of positive integers less than or equal to k that are relatively prime to k . Prove that $a_n \geq 2^{n-1}$.

Solution: We first note that $\varphi(s) < s$ for all positive integers $s \geq 2$, so $a_m > 2$ for all $m > 1$.

For integers $s > 2$, let A_s be the set of all positive integers $x \leq s$ such that $\gcd(s, x) = 1$. Since $\gcd(s, x) = \gcd(s, s - x)$ for all x , if a is a positive integer in A_s , so is $s - a$. Moreover, if a is in A_s , a and $s - a$ are different since $\gcd(s, \frac{s}{2}) = \frac{s}{2} > 1$, meaning $\frac{s}{2}$ is not in A_s . Hence we may evenly pair

up the elements of A_s that sum to s , so $\varphi(s)$, the number of elements of A_s , must be even. It follows that a_m is even for all $m \leq n - 1$.

If $t > 2$ is even, A_t will not contain any even number, so $\varphi(t) \leq \frac{t}{2}$. We may conclude that $a_m \geq 2a_{m-1}$ for all $m \leq n - 1$, so $a_n \geq a_{n-1} \geq 2^{n-2}a_1 = 2^{n-1}$, as desired.

Finally, note that such a set exists for all n by letting $a_i = 2^i$ for all i .

Complex Numbers [35]

4. [15] Let a , b , and c be complex numbers such that $|a| = |b| = |c| = |a + b + c| = 1$. If $|a - b| = |a - c|$ and $b \neq c$, prove that $|a + b||a + c| = 2$.

Solution:

First Solution. Since $|a| = 1$, a cannot be 0. Let $u = \frac{b}{a}$ and $v = \frac{c}{a}$. Dividing the given equations by $|a| = 1$ gives $|u| = |v| = |1 + u + v| = 1$ and $|1 - u| = |1 - v|$. The goal is to prove that $|1 + u||1 + v| = 2$. By squaring $|1 - u| = |1 - v|$, we get $(1 - u)\overline{(1 - u)} = (1 - v)\overline{(1 - v)}$, and thus $1 - u - \bar{u} + |u|^2 = 1 - v - \bar{v} + |v|^2$, or $u + \bar{u} = v + \bar{v}$. This implies $\operatorname{Re}(u) = \operatorname{Re}(v)$. Since u and v are on the unit circle in the complex plane, u is equal to either v or \bar{v} . However, $b \neq c$ implies $u \neq v$, so $u = \bar{v}$.

Therefore, $1 = |1 + u + \bar{u}| = |1 + 2\operatorname{Re}(u)|$. Since $\operatorname{Re}(u)$ is real, we either have $\operatorname{Re}(u) = 0$ or $\operatorname{Re}(u) = -1$. The first case gives $u = \pm i$ and $|1 + u||1 + v| = |1 + i||1 - i| = 2$, as desired. It remains only to note that $\operatorname{Re}(u) = -1$ is in fact impossible because u is of norm 1 and $u = -1$ would imply $u = \bar{u} = v$.

Remark: by the rotational symmetry of the circle, it is acceptable to skip the first step of this solution and assume $a = 1$ without loss of generality.

Second Solution. Let a , b , and c , be the vertices of a triangle inscribed in the unit circle in the complex plane. Since the complex coordinate of the circumcenter is 0 and the complex coordinate of the centroid is $\frac{a+b+c}{3}$, it follows from well-known facts about the Euler line that the complex coordinate of the orthocenter is $a + b + c$. Hence the orthocenter lies on the unit circle as well. Is it not possible for the orthocenter not to be among the three vertices of the triangle, for, if it were, two opposite angles of the convex cyclic quadrilateral formed by the three vertices and the orthocenter would each measure greater than 90 degrees. It follows that the triangle is right. However, since $|a - b| = |a - c|$, the right angle cannot occur at b or c , so it must occur at a , and the desired conclusion follows immediately.

5. [20] Let a and b be positive real numbers. Define two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ for all positive integers n by $(a + bi)^n = a_n + b_n i$. Prove that

$$\frac{|a_{n+1}| + |b_{n+1}|}{|a_n| + |b_n|} \geq \frac{a^2 + b^2}{a + b}$$

for all positive integers n .

Solution: Let $z = a + bi$. It is easy to see that what we are asked to show is equivalent to

$$\frac{|z^{n+1} + \bar{z}^{n+1}| + |z^{n+1} - \bar{z}^{n+1}|}{|z^n + \bar{z}^n| + |z^n - \bar{z}^n|} \geq \frac{2z\bar{z}}{|z + \bar{z}| + |z - \bar{z}|}$$

Cross-multiplying, we see that it suffices to show

$$\begin{aligned} & |z^{n+2} + z^{n+1}\bar{z} + z\bar{z}^{n+1} + \bar{z}^{n+2}| + |z^{n+2} - z^{n+1}\bar{z} + z\bar{z}^{n+1} - \bar{z}^{n+2}| \\ & + |z^{n+2} + z^{n+1}\bar{z} - z\bar{z}^{n+1} - \bar{z}^{n+2}| + |z^{n+2} - z^{n+1}\bar{z} - z\bar{z}^{n+1} + \bar{z}^{n+2}| \\ & \geq 2|z^{n+1}\bar{z} + z\bar{z}^{n+1}| + 2|z^{n+1}\bar{z} - z\bar{z}^{n+1}| \end{aligned}$$

However, by the triangle inequality,

$$|z^{n+2} + z^{n+1}\bar{z} + z\bar{z}^{n+1} + \bar{z}^{n+2}| + |z^{n+2} - z^{n+1}\bar{z} - z\bar{z}^{n+1} + \bar{z}^{n+2}| \geq 2|z^{n+1}\bar{z} + z\bar{z}^{n+1}|$$

and

$$|z^{n+2} - z^{n+1}\bar{z} + z\bar{z}^{n+1} - \bar{z}^{n+2}| + |z^{n+2} + z^{n+1}\bar{z} - z\bar{z}^{n+1} - \bar{z}^{n+2}| \geq 2|z^{n+1}\bar{z} - z\bar{z}^{n+1}|$$

This completes the proof.

Remark: more computationally intensive trigonometric solutions are also possible by reducing the problem to maximizing and minimizing the values of the sine and cosine functions.

Coin Flipping [75]

In a one-player game, the player begins with $4m$ fair coins. On each of m turns, the player takes 4 unused coins, flips 3 of them randomly to heads or tails, and then selects whether the 4th one is heads or tails (these four coins are then considered used). After m turns, when the sides of all $4m$ coins have been determined, if half the coins are heads and half are tails, the player wins; otherwise, the player loses.

6. [10] Prove that whenever the player must choose the side of a coin, the optimal strategy is to choose heads if more coins have been determined tails than heads and to choose tails if more coins have been determined heads than tails.

Solution: Let $q_n(k)$ be the probability that, with n turns left and $|H - T| = k$, the player wins playing the optimal strategy. Note that k is always even. We prove by induction on n that with n turns left in the game, this is the optimal strategy, and that $q_n(k) \geq q_n(k + 2)$.

Base Case: $n = 1$, if there is one more turn in the game, then clearly if we flip 3 coins and get $|H - T| \geq 3$ then our play does not matter. If we get $|H - T| = 1$ then the optimal strategy is to pick the side so that $|H - T| = 0$. So this is the optimal strategy.

Now $q_1(0) = \frac{3}{4}$, $q_1(2) = \frac{1}{2}$, $q_1(4) = \frac{1}{8}$ and $q_1(2k) = 0$ for $k \geq 3$. So $q_1(k) \geq q_1(k + 2)$ for all k .

Induction Step: Assume the induction hypothesis for $n - 1$ turns left in the game. With n turns left in the game, since $q_{n-1}(k)$ decreases when $k = |H - T|$ increases, a larger value of $|H - T|$ is never more desirable. So picking the side of the coin that minimizes $|H - T|$ is the optimal strategy.

Now for $k \geq 2$, $q_n(2k) = \frac{1}{8}q_{n-1}(2k - 4) + \frac{3}{8}q_{n-1}(2k - 2) + \frac{3}{8}q_{n-1}(2k) + \frac{1}{8}q_{n-1}(2k + 2)$, and it is clear, by the induction hypothesis that $q_n(2k) \geq q_n(2k + 2)$ for all $k \geq 2$. Similar computations make it clear that $q_n(0) \geq q_n(2) \geq q_n(4)$. This completes the induction step.

7. [15] Let T denote the number of coins determined tails and H denote the number of coins determined heads at the end of the game. Let $p_m(k)$ be the probability that $|T - H| = k$ after a game with $4m$ coins, assuming that the player follows the optimal strategy outlined in problem 6. Clearly $p_m(k) = 0$ if k is an odd integer, so we need only consider the case when k is an even integer. (By definition, $p_0(0) = 1$ and $p_0(k) = 0$ for $k \geq 1$). Prove that $p_m(0) \geq p_{m+1}(0)$ for all nonnegative integers m .

Solution:

First solution. Using the sequences $q_n(k)$ defined above, it is clear that $q_{n+1}(0) = \frac{3}{4}q_n(0) + \frac{1}{4}q_n(2) \leq q_n(0)$. So $q_{n+1}(0) \leq q_n(0)$. Now note that for $k = 0$, $p_n(k) = q_n(k)$, because they are both equal to the probability that after n turns of the optimal strategy $|H - T| = 0$ (this is only true for $k = 0$). It follows that $p_{n+1}(0) \leq p_n(0)$.

Second solution. When we play the game with $n + 1$ turns, there is a $\frac{3}{4}$ chance that after 1 turn $|H - T| = 0$ and a $\frac{1}{4}$ chance that $|H - T| = 2$.

Now suppose we play the game, and get 2 tails and 1 heads on the first turn. Consider the following two strategies for the rest of the game.

Strategy A: We pick the fourth coin to be heads, and play the optimal strategy for the other n turns. Our probability of winning is $p_n(0)$.

Strategy B: We pick the fourth coin to be heads with probability $\frac{3}{4}$ and tails with probability $\frac{1}{4}$, then proceed with the optimal strategy for the rest of the game. This is equivalent to throwing the first 3 coins over again and applying the optimal strategy. Our probability of winning is $p_{n+1}(0)$.

By the theorem in the previous problem, Strategy A is the optimal strategy, and thus our probability of winning employing Strategy B does not exceed our probability of winning employing Strategy A. So $p_n(0) \geq p_{n+1}(0)$.

8. (a) [5] Find a_1 , a_2 , and a_3 , so that the following equation holds for all $m \geq 1$:

$$p_m(0) = a_1 p_{m-1}(0) + a_2 p_{m-1}(2) + a_3 p_{m-1}(4)$$

- (b) [5] Find b_1 , b_2 , b_3 , and b_4 , so that the following equation holds for all $m \geq 1$:

$$p_m(2) = b_1 p_{m-1}(0) + b_2 p_{m-1}(2) + b_3 p_{m-1}(4) + b_4 p_{m-1}(6)$$

- (c) [5] Find c_1 , c_2 , c_3 , and c_4 , so that the following equation holds for all $m \geq 1$ and $j \geq 2$:

$$p_m(2j) = c_1 p_{m-1}(2j-2) + c_2 p_{m-1}(2j) + c_3 p_{m-1}(2j+2) + c_4 p_{m-1}(2j+4)$$

Solution: We will show how to find a_1 in the first equation; other coefficients can be evaluated in the same way. a_1 is the probability that, beginning with $|T - H| = 0$, the value of $|T - H|$ remains 0 after using the optimal strategy for one round. This only happens when the first three coins are not all heads or all tails. Therefore, $a_1 = \frac{3}{4}$. It follows that $b_1 = \frac{1}{4}$. The rest of the terms are the binomial coefficients of the expansion of $(\frac{1+1}{2})^3$.

- (a) $a_1 = \frac{3}{4}$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{8}$.
 (b) $b_1 = \frac{1}{4}$, $b_2 = \frac{3}{8}$, $b_3 = \frac{3}{8}$, $b_4 = \frac{1}{8}$.
 (c) $c_1 = \frac{1}{8}$, $c_2 = \frac{3}{8}$, $c_3 = \frac{3}{8}$, $c_4 = \frac{1}{8}$.

9. [15] We would now like to examine the behavior of $p_m(k)$ as m becomes arbitrarily large; specifically, we would like to discern whether $\lim_{m \rightarrow \infty} p_m(0)$ exists and, if it does, to determine its value. Let $\lim_{m \rightarrow \infty} p_m(k) = A_k$.

- (a) [5] Prove that $\frac{2}{3}p_m(k) \geq p_m(k+2)$ for all m and k .
 (b) [10] Prove that A_0 exists and that $A_0 > 0$. Feel free to assume the result of analysis that a non-increasing sequence of real numbers that is bounded below by a constant c converges to a limit that is greater than or equal to c .

Solution:

- (a) We proceed by induction.

Base Case: When $n = 1$, we get $\frac{2}{3}p_1(0) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2} > \frac{1}{4} = p_1(2)$. And since $p_1(2k) = 0$ for $k \geq 2$, we get $\frac{2}{3}p_1(k) \geq p_1(k+2)$ for all k .

Induction Step: Assume that $\frac{2}{3}p_i(k) \geq p_i(k+2)$ for all $i \leq n$. This clearly implies $\frac{2}{3}p_{n+1}(k) \geq p_{n+1}(k+2)$ for all $k \geq 4$ by the formula (c) from the previous problem.

Now, for $k = 2$,

$$\begin{aligned} \frac{2}{3}p_{n+1}(2) &= \frac{2}{3} \left(\frac{1}{4}p_n(0) + \frac{3}{8}p_n(2) + \frac{3}{8}p_n(4) + \frac{1}{8}p_n(6) \right) \\ &\geq \frac{1}{12}p_n(0) + p_{n+1}(4) \\ &\geq p_{n+1}(4) \end{aligned}$$

by the formula (b) from the previous problem and our induction hypothesis.

For $k = 0$, we write out formula (a) from the previous problem.

$$\begin{aligned} \frac{2}{3}p_{n+1}(0) &= \frac{2}{3} \left(\frac{3}{4}p_n(0) + \frac{1}{2}p_n(2) + \frac{1}{8}p_n(4) \right) \\ &\geq \frac{1}{4}p_n(0) + \frac{3}{8}p_n(2) + \frac{1}{2}p_n(4) + \frac{1}{8}p_n(6) \end{aligned}$$

by our induction hypothesis. But $p_{n+1}(2) = \frac{1}{4}p_n(0) + \frac{3}{8}p_n(2) + \frac{3}{8}p_n(4) + \frac{1}{8}p_n(6)$, which is clearly less than the last line above. Thus $\frac{2}{3}p_{n+1}(0) \geq p_{n+1}(2)$. This completes the induction step.

Remark: This is nowhere near the strongest bound. We intentionally asked for a bound strong enough to be helpful in part b, but not strong enough to help with other problems. The strongest bound is $p_n(0) \geq \frac{2+\sqrt{5}}{2}p_n(2)$ and $p_n(2k) \geq (2+\sqrt{5})p_n(2k+2)$ for $k \geq 1$. This is intuitive because $A_0 = \frac{2+\sqrt{5}}{2}A_2$ and for $k \geq 1$, $A_{2k} = (2+\sqrt{5})A_{2k+2}$.

(b) For any n and k , $p_n(2k) \leq (\frac{2}{3})^k p_n(0)$. Now

$$1 = \sum_{k=0}^{\infty} p_n(2k) \leq p_n(0) \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = 3p_n(0).$$

So $p_n(0) \geq \frac{1}{3}$ for all n . This means $p_n(0)$ is a non-increasing sequence that is bounded below by $\frac{1}{3}$. So A_0 exists and $A_0 \geq \frac{1}{3} > 0$.

10. [20] Once it has been demonstrated that $\lim_{n \rightarrow \infty} p_n(0)$ exists and is greater than 0, it follows that

$\lim_{n \rightarrow \infty} p_n(k)$ exists and is greater than 0 for all even positive integers k and that $\sum_{k=0}^{\infty} A_{2k} = 1$. It also follows that $A_0 = a_1A_0 + a_2A_2 + a_3A_4$, $A_2 = b_1A_0 + b_2A_2 + b_3A_4 + b_4A_6$, and $A_{2j} = c_1A_{2j-2} + c_2A_{2j} + c_3A_{2j+2} + c_4A_{2j+4}$ for all positive integers $j \geq 2$, where $a_1, a_2, a_3, b_1, b_2, b_3, b_4$, and c_1, c_2, c_3, c_4 are the constants you found in problem 8. Assuming these results, determine, with proof, the value of A_0 .

Solution: By our recurrence relations,

$$\frac{1}{4}A_0 = \frac{1}{2}A_2 + \frac{1}{8}A_4$$

$$\frac{5}{8}A_2 = \frac{1}{4}A_0 + \frac{3}{8}A_4 + \frac{1}{8}A_6$$

and

$$\frac{5}{8}A_{2k} = \frac{1}{8}A_{2k-2} + \frac{3}{8}A_{2k+2} + \frac{1}{8}A_{2k+4}$$

for all $k \geq 2$. So the terms A_{2k} ($k \geq 1$) satisfy a linear recurrence with characteristic equation $x^3 + 3x^2 + 1 = 5x$. The roots of the characteristic equation are 1 and $-2 \pm \sqrt{5}$, so for $k \geq 2$, $A_{2k} = c_1 + c_2(-2 + \sqrt{5})^k + c_3(-2 - \sqrt{5})^k$. But clearly, $\lim_{k \rightarrow \infty} A_{2k} = 0$, so $c_1 = c_3 = 0$. Thus for $k \geq 2$, we are left with $A_{2k} = c_2(-2 + \sqrt{5})^k = A_4(-2 + \sqrt{5})^{k-2}$.

Now, since $\sum_{k=0}^{\infty} A_{2k} = 1$, we get $1 = A_0 + A_2 + A_4 \sum_{j=0}^{\infty} (\sqrt{5} - 2)^j = A_0 + A_2 + \frac{A_4}{1 - (\sqrt{5} - 2)} = A_0 + A_2 + \frac{3 + \sqrt{5}}{4}A_4$.

We now have the following equations.

$$1 = A_0 + A_2 + \frac{3 + \sqrt{5}}{4}A_4$$

$$0 = 2A_0 - 4A_2 - A_4$$

$$0 = 2A_0 - 5A_2 + 3A_4 + A_6 = 2A_0 - 5A_2 + (1 + \sqrt{5})A_4$$

We have three equations with three unknowns. Solving this linear system gives $A_0 = \frac{\sqrt{5}-1}{2}$.

Remark: The fact that $A_0 = \frac{\sqrt{5}-1}{2} > \frac{1}{2}$ means that for arbitrarily large n , the player wins more often than they lose, so the player is at an advantage. In addition the final answer is the reciprocal of the golden ratio, which is simply beautiful.

The probabilistic experiment explored in this problem is a type of random walk. A random walk is an infinite set of moves in some set (usually the R^n lattice but sometimes weird shapes or even fractals)

where each move is determined by a random variable. The most basic random walk is a walk on the real line, where the walker starts at 0, and at every move has probability $\frac{1}{2}$ of moving one unit to the left, and probability $\frac{1}{2}$ of moving one unit to the right.

The random walk explored in this problem is what is known as a 'biased' random walk, where the probabilities of movement are asymmetric, and weighted towards a certain direction or outcome. In this example the choice of the side of 1 out of every 4 coins biases the walk towards the origin.

An interesting inquiry for the reader to ponder is, what is the value of A_0 if the game is played with 6 coins flipped each turn, or 8, or $2m$. If the limits A_{2k} exist, they will satisfy a linear recurrence with characteristic equation $\left(\frac{x+1}{2}\right)^{2m-1} = x^{m-1}$, as is evident from this problem, where we solved the $m = 2$ case.

We now give proofs that A_k exists for all even k , and that $\sum_{k=0}^{\infty} A_{2k} = 1$

First note that we can use the inductive argument in problem 9a to prove the stronger bound $p_n(0) \geq \frac{2+\sqrt{5}}{2}p_n(2)$ and $p_n(2k) \geq (2 + \sqrt{5})p_n(2k+2)$ for all $k \geq 1$.

Now,

$$\begin{aligned} p_{n+2}(0) + \frac{1}{4}p_{n+1}(2) &= \frac{3}{4}p_{n+1}(0) + \frac{3}{4}p_{n+1}(2) + \frac{1}{8}p_{n+1}(4) \\ &= \frac{3}{4}p_n(0) + \frac{43}{64}p_n(2) + \frac{27}{64}p_n(4) + \frac{9}{64}p_n(6) + \frac{1}{64}p_n(8) \\ &= p_{n+1}(0) + \frac{11}{64}p_n(2) + \frac{19}{64}p_n(4) + \frac{9}{64}p_n(6) + \frac{1}{64}p_n(8) \\ &= p_{n+1}(0) + \frac{1}{4}p_n(2) - \frac{5}{64}p_n(2) + \frac{19}{64}p_n(4) + \frac{9}{64}p_n(6) + \frac{1}{64}p_n(8) \\ &\leq p_{n+1}(0) + \frac{1}{4}p_n(2) \end{aligned}$$

So $p_{n+1}(0) + p_n(2)$ is a non-increasing sequence that is bounded below, so it converges to a limit, call it $A_0 + A_2$. Now $A_2 = -\lim_{n \rightarrow \infty} p_{n+1}(0) + \lim_{n \rightarrow \infty} (p_{n+1}(0) + p_n(2)) = \lim_{n \rightarrow \infty} p_n(2)$.

So $\lim_{n \rightarrow \infty} p_n(2)$ exists. It follows from our recursions that A_{2k} exists for all k . Since $p_n(2k) \geq 0$ for all k , it follows that all the A_i are non-negative.

Now we must show that $\sum_{k=0}^{\infty} A_{2k} = 1$. This is not immediately clear because, while a finite sum of limits of sequences equals the limit of the finite sum of the sequences, the sum of the limits does not necessarily equal the limit of the sum if it is an infinite sum.

Now it follows from part 8a, that for any $\epsilon > 0$ there exists N such that $\sum_{k=0}^N p_n(2k) > 1 - \frac{\epsilon}{2}$ for all n .

So $\sum_{k=0}^N A_{2k} \geq 1 - \frac{\epsilon}{2} > 1 - \epsilon$. Since $A_{2k} \geq 0$ for all k , $\sum_{k=0}^{\infty} A_{2k} \geq 1$.

Now if $\sum_{k=0}^{\infty} A_{2k}$ is greater than 1, or diverges, then there exists a pair j and $\epsilon > 0$ such that $\sum_{k=0}^j A_{2k} =$

$1 + \epsilon$. But, $\sum_{k=0}^j p_n(2k) \leq 1$ for all n , so $\sum_{k=0}^j A_{2k} \leq 1$, contradiction.

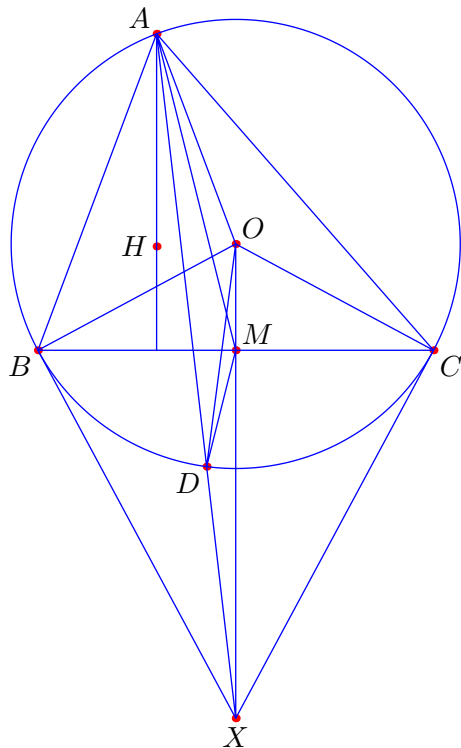
So $\sum_{k=0}^{\infty} A_{2k} = 1$.

Geometry [90]

11. [20] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC . Let the circumcircle of triangle AOM intersect ω again at D . If H is the orthocenter of triangle ABC , prove that $\angle DAH = \angle MAO$.

Solution:

First solution.



Let X be the intersection of the line tangent to ω at B with the line tangent to ω at C . Note that $\triangle OMC \sim \triangle OCX$ since $\angle OMC = \angle OCX = \frac{\pi}{2}$. Hence $\frac{OM}{OC} = \frac{OC}{OX}$, or, equivalently, $\frac{OM}{OA} = \frac{OC}{OX}$. By SAS similarity, it follows that $\triangle OAM \sim \triangle OXA$. Therefore, $\angle OAM = \angle OXA$.

We claim now that $\angle OAD = \angle OAX$. By the similarity $\triangle OAM \sim \triangle OXA$, we have that $\angle OAX = \angle OMA$. Since $AOMD$ is a cyclic quadrilateral, we have that $\angle OMA = \angle ODA$. Since $OA = OD$, we have that $\angle ODA = \angle OAD$. Combining these equations tells us that $\angle OAX = \angle OAD$, so A , D , and X are collinear.

Finally, since both AH and OX are perpendicular to BC , it follows that $AH \parallel OX$, so $\angle DAH = \angle DXO = \angle AXO = \angle MAO$, as desired.

Second solution.

Let Y be the intersection of BC with the line tangent to ω at A . Then the circumcircle of triangle AOM has diameter OY , so AD is perpendicular to OY because the radical axis of two circles is perpendicular to the line between their centers. Since Y is on the polar of A , it follows that A is on the polar of Y , so $AD \perp OX$ implies that AD is the polar of Y , i.e. AD is the symmedian from A in triangle ABC . Hence AD and AM are isogonal. Since AH and AO are also isogonal, the desired conclusion follows immediately.

Remark: with sufficient perserverance, angle-chasing solutions involving only the points given in the diagram may also be devised.

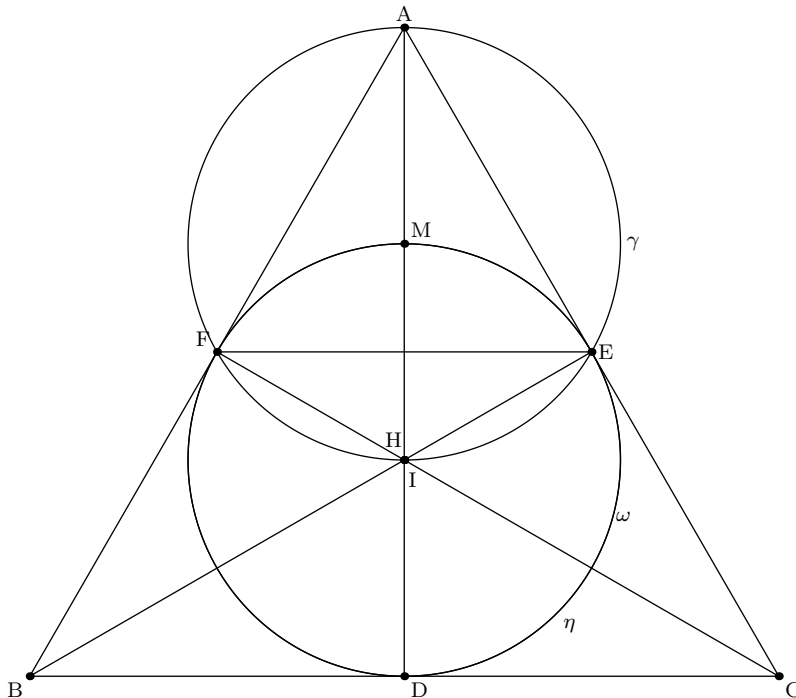
12. [70] Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C , respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF .

Solution:

This problem is arguably the most difficult among all those appearing in the 2011 Harvard-MIT Mathematics Tournament. Do not feel badly if your team wasted time in a vain attempt to find a solution. It was intended by the author as a serious test for serious geometers.

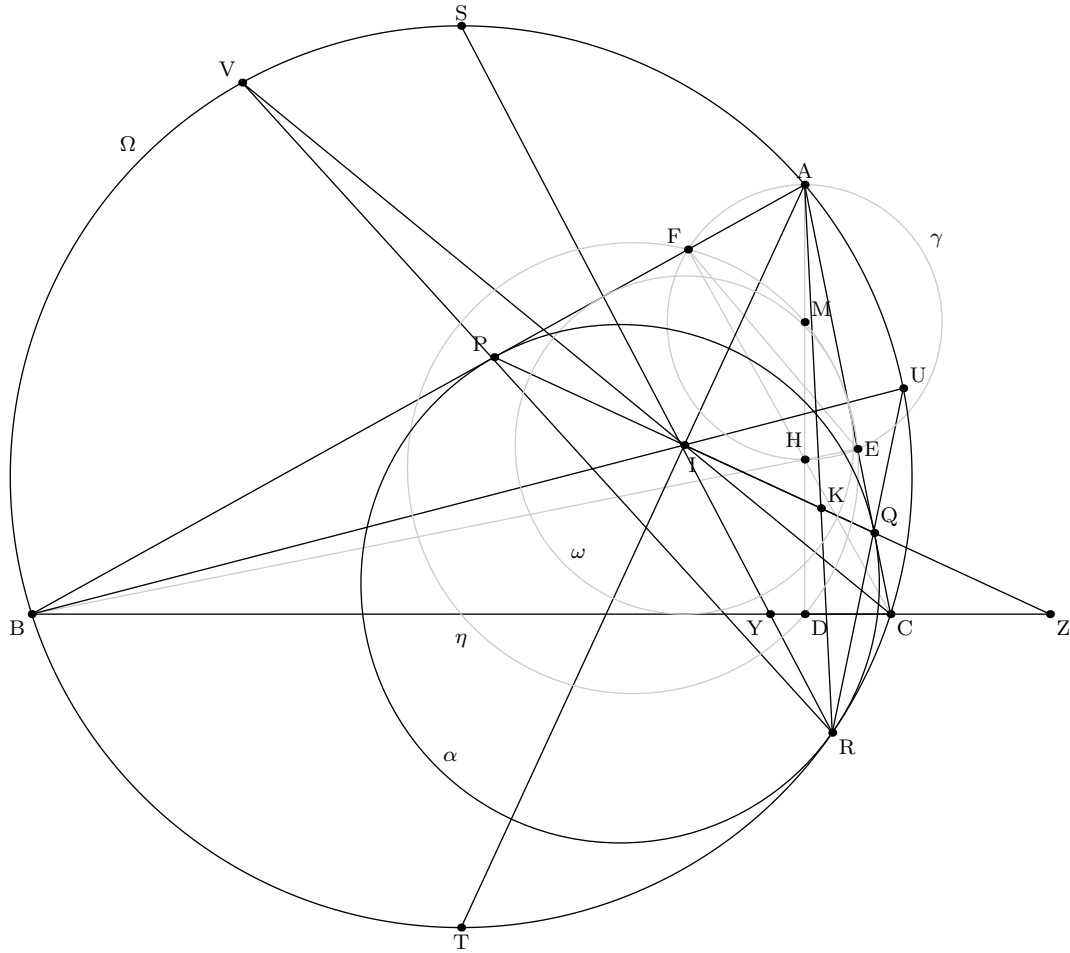
Let D be the foot of the altitude from A . Let H be the orthocenter of triangle ABC . Let M be the midpoint of AH . Let I be the incenter of triangle ABC . Let ω be the incircle of triangle ABC . Let γ be the circumcircle of AEF . Let η be the nine-point circle of triangle ABC .

We first dispense with the case in which $AB = AC$. Since M is the circumcenter of triangle AEF , if the circumcenter of triangle AEF lies on the incircle of triangle ABC , then ω and η intersect in two points: M and D . Since ω and η are tangent by Feuerbach's Theorem, it follows that ω and η are coincident, in which case triangle ABC is equilateral.

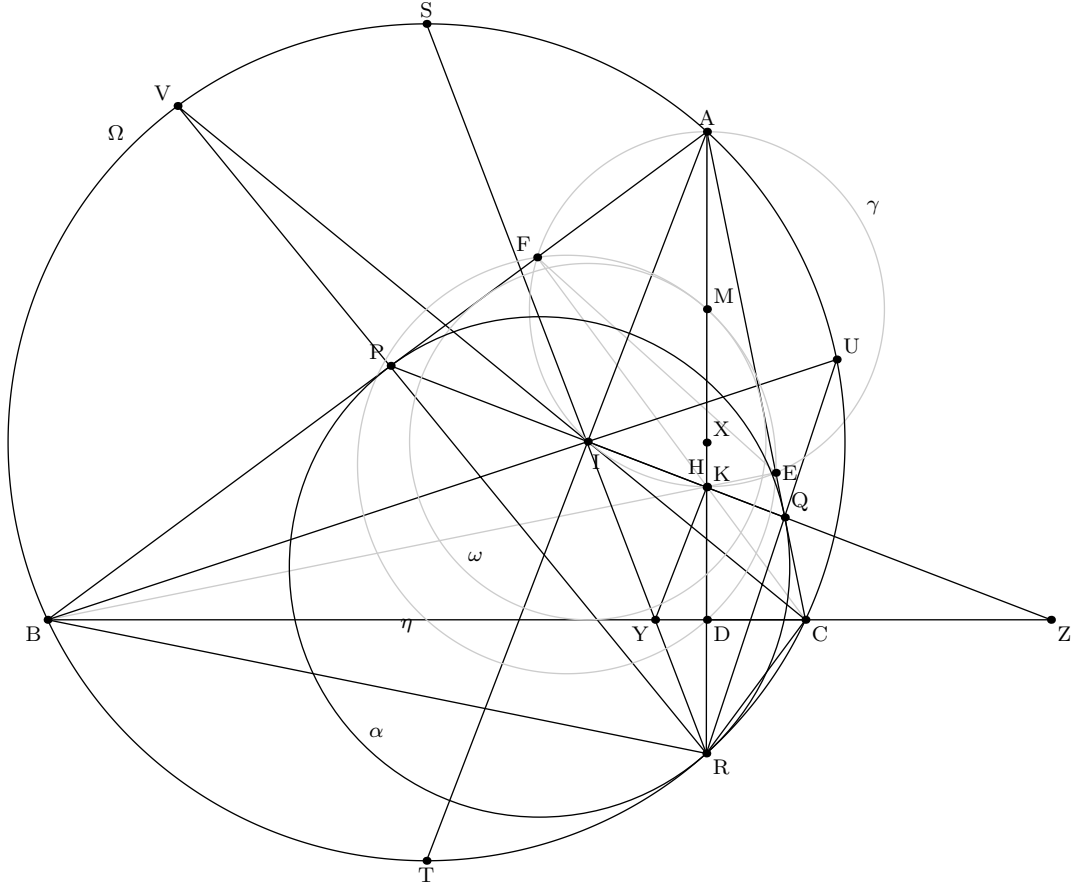


If the incenter of triangle ABC lies on the circumcircle of triangle AEF , then A , I , and H lie on γ . Since no circle may intersect a line in three distinct points and A is distinct from I and H , it follows that I and H are coincident, in which case triangle ABC is equilateral. Since it is clear that if ABC is equilateral, M lies on ω and I lies on γ , this completes the proof in the case that $AB = AC$.

Throughout the remainder of this solution, we assume that $AB \neq AC$. Let Ω be the circumcircle of triangle ABC , and let α be the A -mixtilinear incircle. Let α be tangent to AB at P , to AC at Q , and to Ω at R . Since $AB \neq AC$, lines PQ and BC are not parallel. Let their intersection be Z , and let the bisectors of $\angle A$, $\angle B$, and $\angle C$ intersect Ω at T , U , and V , respectively. Let the point diametrically opposite T on Ω be S . Let lines RS and BC intersect at Y . We introduce a lemma here, namely that I lies on PQ and that Y and Z are harmonic conjugates with respect to B and C .



Let lines AR and PQ intersect at K . Since PA and QA are tangent to α at P and Q , respectively, RK is the symmedian from R in triangle PQR . By Pascal's Theorem applied to cyclic hexagon $ACVRUB$, points Q , I , and P are collinear. Since $AP = AQ$ and AI bisects $\angle PAQ$, I is the midpoint of PQ . Hence RI is the median from R in triangle PQR . It follows that $\angle ICQ = \angle PRA = \angle IRQ$ and $\angle IBP = \angle QRA = \angle IRP$. Hence quadrilaterals $RBPI$ and $RCQI$ are cyclic, so RI is the bisector of $\angle BRC$. It follows also that $\angle BIP = \angle BCI$, so line PQ is tangent to the circumcircle of triangle BIC at I . Since T is the circumcenter of triangle BIC and S is diametrically opposite T on Ω , BS and CS are tangent to the circumcircle of triangle BIC at B and C , respectively. It follows that IS is the symmedian from I in triangle BIC ; since RI is the bisector of $\angle BRC$, R , I , and S are collinear. Hence RS is the polar of Z with respect to the circumcircle of triangle BIC , so Y and Z are harmonic conjugates with respect to B and C , as desired.



We now take up the main problem. We first prove that I lies on γ if M lies on ω . If M lies on ω , then M is the Feuerbach point. Let X be the center of the homothety of positive magnitude mapping ω to Ω . Since the center of the homothety of positive magnitude ω to η is M and the center of the homothety of positive magnitude mapping η to Ω is H , it follows by Monge's Circle Theorem that X lies on line MH . Note that R is the center of the homothety of positive magnitude mapping α to Ω . Since the center of homothety of positive magnitude mapping α to ω is A , it follows by the same theorem that R lies on line AX , so points A , M , H , and R are collinear.

Note that triangle BRC is the reflection of triangle BHC about line BC . Since line RY bisects angle BRC , it follows that line HY bisects $\angle BHC$. Let the line through H parallel to PQ intersect BC at Z' . Since $\angle HBA = \angle HCA$, the internal bisectors of $\angle BAC$ and $\angle BHC$ are parallel. Since PQ is perpendicular to AI and BZ' is parallel to PQ , it follows that BZ' is the external bisector of $\angle BHC$. Hence Y and Z' are harmonic conjugates with respect to B and C . It follows that Z and Z' are coincident, and, therefore, that H and K are coincident. We may conclude that $\angle AIH$ is right, which implies the desired result because γ is the circle on diameter AH . (Do think carefully about why this argument fails if $AB = AC$).

Conversely, if I lies on γ , then $\angle AIH$ is right, so points P , H , I , and Q are collinear. Since PQ is perpendicular to AI and the internal bisectors of $\angle BAC$ and $\angle BHC$ are parallel, it follows that PQ is the external bisector of $\angle BHC$. Let the internal bisector of $\angle BHC$ intersect BC at Y' . Then Y' and Z are harmonic conjugates with respect to B and C , so Y and Y' are coincident. Let R' be the reflection of H about line BC . Clearly $R'Y$ is the bisector of $\angle BR'C$, so R' , Y , and S are collinear. Since R , Y , and S are collinear and R' also lies on Ω , it follows that R and R' are coincident. Since HR' is perpendicular to BC , it follows that points A , H , and R are collinear.

Let X' be the center of the homothety of positive magnitude mapping ω to Ω . Since the center of the homothety of positive magnitude mapping ω to α is A , and the center of the homothety of positive magnitude mapping α to Ω is R , it follows by Monge's Circle Theorem that X' lies on line AR . Note

that H is the center of the homothety of positive magnitude mapping Ω to η . Since the center of the homothety of positive magnitude mapping ω to η is the Feuerbach point, it follows by the same theorem that the Feuerbach point lies on line AR as well. Since AR intersects η at M and D , the Feuerbach point may be either M or D . However, D may not be the Feuerbach point, else the altitude from A and the bisector of $\angle A$ in triangle ABC would be coincident, contradicting the assumption that $AB \neq AC$. We may conclude that M is the Feuerbach point, so, in particular, M lies on ω , as desired. This completes the proof.

Remark: more computationally intensive solutions are also possible; in particular, it may be observed that the condition $IJ = AM$ is equivalent to the condition $MD/DA' = IJ/JA'$, where J denotes the point where ω is tangent to BC and A' denotes the midpoint of BC . It is left to the reader to determine how to proceed. Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C , respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF .

Last Writes [110]

13. [30] Given positive integers a and b such that $a > b$, define a sequence of ordered pairs (a_l, b_l) for nonnegative integers l by $a_0 = a$, $b_0 = b$, and $(a_{l+1}, b_{l+1}) = (b_l, a_l \bmod b_l)$, where, for all positive integers x and y , $x \bmod y$ is defined to be the remainder left by x upon division by y . Define $f(a, b)$ to be the smallest positive integer j such that $b_j = 0$. Given a positive integer n , define $g(n)$ to be $\max_{1 \leq k \leq n-1} f(n, k)$.
- (a) [15] Given a positive integer m , what is the smallest positive integer n_m such that $g(n_m) = m$?
- (b) [15] What is the second smallest?

Solution:

- (a) The answer is F_{m+1} , where $F_1 = 1, F_2 = 2$, and $F_{i+1} = F_i + F_{i-1}$ for all $i \geq 2$.

We consider a reverse sequence as follows: starting at $p_0 = (k_0, 0)$ for some positive integer k_0 , at each step we can take a pair $p_i = (r_i, s_i)$ to any pair $p_{i+1} = (s_i + k_{i+1}r_i, r_i) = (r_{i+1}, s_{i+1})$ for some positive integer k_{i+1} . It is clear that any such sequence is the reverse of a legal sequence. Thus, n_m is equal to the smallest possible value of the first integer in a possible p_m of a reverse sequence. The pair p_m is uniquely determined by the choice of k_0, k_1, \dots, k_m , and lowering any k_i lowers the first number of p_m . Thus, the minimum possible value occurs when all k_i are equal to 1. In this case, $p_0 = (1, 0), p_1 = (2, 1)$, and $p_{i+1} = (r_i + s_i, r_i)$. It is apparent that $p_i = (F_{i+1}, F_i)$, so the minimum is F_{m+1} .

- (b) The answer is L_{m+1} , where $L_1 = 1, L_2 = 3$, and $L_{i+1} = L_i + L_{i-1}$.

The only integer n satisfying $g(n) = 1$ is 2, as it is the only positive integer $n > 1$ such that all integers $1 \leq i < n$ divide n . Thus, for $m = 1$, there is no such second smallest integer.

For $m \geq 2$, we once again consider all possible reverse sequences. For any sequence, characterized by k_0, \dots, k_m , if there are at least two k_i not equal to 1, then we can find a smaller possible value (of r_{m+1}) not equal to the minimum possible by setting any one of these k_i to 1. Similarly, if any k_i is at least 3, we can find a smaller possible value not equal to the minimum by setting this k_i to be 2. Thus, the second smallest obtainable value of r_{m+1} must occur when $k_i = 1$ for all $0 \leq i \leq m$ except for some k_j which is equal to 2.

We now claim that the value of r_m is minimized with respect to the above conditions by letting $k_1 = 2$, and $k_i = 1$ for all other $0 \leq i \leq m$. Doing so yields $r_m = L_{m+1}$, with $\{L_i\}$ defined as in the answer. Before we begin our proof, we first note that letting $k_0 = 2$ instead yields $r_m = 2F_{m+1}$, which is strictly larger than L_{m+1} (since it is larger for $m = 1, 2$, and both sequences satisfy the same recurrence). We therefore assume that $k_0 = 1$.

Proof of claim. Our recurrences for pairs give us that $r_{i+1} = k_{i+1}r_i + s_i, s_{i+1} = r_i$. Thus, we have $r_{i+1} = k_{i+1}r_i + r_{i-1}$. Now suppose we have $k_j = 2$ for a particular j , and for all $i \neq j, 0 \leq i \leq m$,

we have $k_i = 1$. Then we have $r_0 = 1, r_1 = 1 + k_1$, and for $2 \leq i \leq m, i \neq j$, we have $r_i = r_{i-1} + r_{i-2}$. We also have $r_j = 2r_{j-1} + r_{j-2}$.

By our reasoning in part (a), $r_i = F_{i+1}$ for $i < j$. Thus, $r_j = 2F_j + F_{j-1} = F_{j+1} + F_j = F_{j+2}$ (where we define $F_0 = 1$). Expressing r_{j-1} as $F_{j+1} - F_{j-1}$, we find $r_{j+1} = F_{j+3} - F_{j-1}, r_{j+2} = F_{j+4} - F_{j-1}, r_{j+3} = F_{j+5} - 2F_{j-1}$, and so on. It is easy to show by induction that for $i > j$, we have $r_i = F_{i+2} - F_{i-j-1}F_{j-1}$. Thus, to minimize $r_m = F_{m+2} - F_{m-j-1}F_{j-1}$, we must maximize $F_{m-j-1}F_{j-1}$.

We claim that $F_{m-j-1}F_{j-1} \leq F_{m-2}$ for $1 \leq j \leq m$.

Proof of claim. It is easy to see (and in fact well-known) that for all positive integers n , F_n counts the number of distinct ways to tile a $1 \times n$ board with 1×1 squares and 1×2 dominoes. Thus $F_{m-j-1}F_{j-1}$ counts the number of ways to tile a $1 \times m-2$ board with 1×1 squares and 1×2 dominoes such that the $j-1$ th and j th square are not both covered by the same domino, which is at most the number of ways to tile a $1 \times m-2$ board, as desired.

Since this minimum is reached by letting $j = 1$, we may conclude that the answer is indeed L_{m+1} .

14. [25] Given a positive integer n , a sequence of integers a_1, a_2, \dots, a_r , where $0 \leq a_i \leq k$ for all $1 \leq i \leq r$, is said to be a “ k -representation” of n if there exists an integer c such that

$$\sum_{i=1}^r a_i = \sum_{i=1}^r a_i k^{c-i} = n.$$

Prove that every positive integer n has a k -representation, and that the k -representation is unique if and only if 0 does not appear in the base- k representation of $n-1$.

Solution: Given a positive integer n , a sequence of integers a_1, a_2, \dots, a_r , where $0 \leq a_i \leq k$ for all $1 \leq i \leq r$, is said to be a “ k -representation” of n if there exists an integer c such that

$$\sum_{i=1}^r a_i = \sum_{i=1}^r a_i k^{c-i} = n.$$

Prove that every positive integer n has a k -representation, and that the k -representation is unique if and only if 0 does not appear in the base- k representation of $n-1$.

Equivalently, a k -representation is given by a sequence a_p, \dots, a_q , for some $p < q$, such that

$$\sum_{i=p}^q a_i = \sum_{i=p}^q a_i k^i.$$

We first show existence. Let the representation of $n-1$ in base k be given by $\sum_{i=0}^r a_i k^i = n-1$. Let $l = \sum_{i=1}^r a_i (k^{i-1} + k^{i-2} + \dots + 1)$. Now we extend the sequence $\{a_i\}$ by letting $a_i = k-1$ if $-l+1 \leq i \leq -1$ and $a_{-l} = k$. We claim that $a_{-l}, \dots, a_{r-1}, a_r$ is a k -representation of n . Indeed,

$$\begin{aligned} \sum_{i=-l}^r a_i k^i &= (n-1) + \left(\sum_{i=-l}^{-1} (k-1)n^i \right) + k^{-l} \\ &= (n-1) + (k-1)k^{-1} \frac{1-k^{-l}}{1-k^{-1}} + k^{-l} = n \\ \sum_{i=-l}^r a_i &= \left(\sum_{i=0}^r a_i \right) + l(k-1) + 1 \\ &= \left(\sum_{i=0}^r a_i \right) + \left(\sum_{i=1}^r a_i (k^i - 1) \right) + 1 \\ &= \left(\sum_{i=0}^r a_i k^i \right) + 1 = n \end{aligned}$$

Now suppose there is another k -representation $\{b_i\}$ of n ; then $\sum b_i = \sum b_i k^i = n$. This implies that $\sum c_i = \sum c_i k^i = 0$ where $c_i = a_i - b_i$. The following claim specifies all possibilities of $\{c_i\}$.

Claim: Suppose that $\sum_{i=p}^q c_i k^i = 0$, where $c_i, p, q \in \mathbb{Z}$, $p < q$, and $c_i \in [-k, k]$. Then the sequence $\{c_i\}$ must be the concatenation of subsequences of the form

$$\pm(1, 1 - k, 1 - k, \dots, 1 - k, -k)$$

, possibly with 0's in between.

Proof. If c_i is not the zero sequence, then without loss of generality, we may assume $c_q > 0$.

Since $\sum_{i=p}^q c_i k^i = 0$, we have

$$|c_q k^q| = |c_{q-1} k^{q-1} + \dots + c_p k^p| \leq k^q + k^{q-1} + \dots + k^{p+1} < \frac{k}{k-1} k^q \leq 2k^q$$

so $c_q = 1$. Now we have

$$k^q + c_{q-1} k^{q-1} + \dots + c_p k^p = 0. \tag{1}$$

This means $(k + c_{q-1})k^{q-1} + \dots + c_p k^p = 0$. Hence, as above,

$$|(k + c_{q-1})k^{q-1}| = |c_{q-2} k^{q-2} + \dots + c_p k^p| < 2k^{q-1}.$$

Therefore, $|(k + c_{q-1})| < 2$, which means $c_{q-1} = -k$ or $-k + 1$.

If $c_{q-1} = -k$, then $c_q k^q + c_{q-1} k^{q-1} = 0$, and we get a subsequence $(1, -k)$.

If $c_{q-1} = -k + 1$, then $c_q k^q + c_{q-1} k^{q-1} = k^{q-1}$. Thus

$$k^{q-1} + c_{q-2} k^{q-2} + \dots + c_p k^p = 0,$$

which has the exact same form as (1). We can then repeat this procedure to obtain a subsequence $(1, 1 - k, 1 - k, \dots, 1 - k, -k)$.

Once we have such a subsequence, the terms in the sum $\sum_{i=p}^q c_i k^i = 0$ corresponding to that subsequence sum to 0, so we may remove them and apply the same argument. \square

We now consider the cases.

If the base k representation of $n - 1$ contains no 0's, then $a_i \neq 0$ so it is impossible to have $c_i = a_i - b_i = -k$. On the other hand, we know that c_i is composed of subsequences of the form $\pm(1, 1 - k, 1 - k, \dots, 1 - k, -k)$. Therefore, if $\{c_i\}$ is not the zero sequence, then the fact that $\sum c_i = 0$ implies that we must have both a subsequence $(1, 1 - k, 1 - k, \dots, 1 - k, -k)$ and a subsequence $-(1, 1 - k, 1 - k, \dots, 1 - k, -k)$, meaning that there exists i for which $c_i = -k$, contradiction.

If the base k representation of $n - 1$ contains a 0, then picking the largest i such that $a_i = 0$, we can change a_i to k and a_{i+1} to $a_{i+1} - 1$, and append a $(1, -k)$ to the end of the sequence. This yields another k -representation of n , so a k -representation of n is unique if and only if the base k representation of $n - 1$ contains no 0's, as desired.

15. [55] Denote $\{1, 2, \dots, n\}$ by $[n]$, and let S be the set of all permutations of $[n]$. Call a subset T of S *good* if every permutation σ in S may be written as $t_1 t_2$ for elements t_1 and t_2 of T , where the product of two permutations is defined to be their composition. Call a subset U of S *extremely good* if every permutation σ in S may be written as $s^{-1}us$ for elements s of S and u of U . Let τ be the smallest value of $|T|/|S|$ for all good subsets T , and let v be the smallest value of $|U|/|S|$ for all extremely good subsets U . Prove that $\sqrt{v} \geq \tau$.

Solution: Denote $\{1, 2, \dots, n\}$ by $[n]$, and let S be the set of all permutations of $[n]$. Call a subset T of S *good* if every permutation σ in S may be written as $t_1 t_2$ for elements t_1 and t_2 of T , where the product of two permutations is defined to be their composition. Call a subset U of S *extremely good* if every permutation σ in S may be written as $s^{-1}us$ for elements s of S and u of U . Let τ be

the smallest value of $|T|/|S|$ for all good subsets T , and let v be the smallest value of $|U|/|S|$ for all extremely good subsets U . Prove that $\sqrt{v} \geq \tau$.

Call an element $t \in S$ an *involution* if and only if t^2 is the identity permutation. We claim that the set of all involutions in S constitutes a good subset of S . The proof is simple. Let s be an arbitrary permutation in S . Note that s may be decomposed into a product of disjoint cycles of length at least 2, and suppose that there are m such cycles in its decomposition. For $1 \leq i \leq m$, let l_i denote the length of the i th cycle, so that s may be written as

$$(a_{1,1}a_{1,2} \dots a_{1,l_1})(a_{2,1}a_{2,2} \dots a_{2,l_2}) \dots (a_{m,1}a_{m,2} \dots a_{m,l_m})$$

for some pairwise distinct elements

$$a_{1,1}, a_{1,2}, \dots, a_{1,l_1}, a_{2,1}, a_{2,2}, \dots, a_{2,l_2}, \dots, a_{m,1}, a_{m,2}, \dots, a_{m,l_m}$$

of $[n]$. Consider the permutations q and r defined by $q(a_{i,j}) = a_{i,l_i+1-j}$ for all $1 \leq j \leq l_i$ and $r(a_{i,1}) = a_{i,1}$ and $r(a_{i,k}) = a_{i,l_i+2-k}$ for all $2 \leq k \leq l_i$, for all $1 \leq i \leq m$, and by $q(x) = r(x) = x$ for all $x \in [n]$ otherwise. Since $q, r \in S$, $q^2 = r^2 = 1$, and $rq = s$, it follows that the set of all involutions in S is indeed good, as desired.

For all integer partitions λ of n , let f^λ denote the number of standard Young tableaux of shape λ . By the Robinson-Schensted-Knuth correspondence, the permutations of $[n]$ are in bijection with pairs of standard Young tableaux of the same shape in such a way that the involutions of $[n]$ are in bijection with pairs of identical standard Young tableaux. In other words, the number of permutations of $[n]$ is equal to the number of pairs of identically shaped standard Young tableaux whose shape is a partition of n , and the number of involutions of $[n]$ is equal to the number of standard Young tableaux whose shape is a partition of n . Hence $n! = \sum_\lambda (f^\lambda)^2$ and $\tau \leq \sum_\lambda \frac{f^\lambda}{n!}$, where both sums range over all partitions λ of n .

For all elements $u \in S$, define the conjugacy class of u to be the set of elements that may be written in the form $s^{-1}us$ for some $s \in S$. It is easy to see that for all $u, u' \in S$, the conjugacy classes of u and u' are either identical or disjoint. It follows that S may be partitioned into disjoint conjugacy classes and that any extremely good subset of S must contain at least one element from each distinct conjugacy class. We claim that the number of distinct conjugacy classes of S is at least the number of integer partitions of n . It turns out that the two numbers are in fact equal, but such a result is not necessary for the purposes of this problem. Let u be an arbitrary permutation in S . Recall that u may be written as

$$(a_{1,1}a_{1,2} \dots a_{1,l_1})(a_{2,1}a_{2,2} \dots a_{2,l_2}) \dots (a_{m,1}a_{m,2} \dots a_{m,l_m})$$

for some pairwise distinct elements

$$a_{1,1}, a_{1,2}, \dots, a_{1,l_1}, a_{2,1}, a_{2,2}, \dots, a_{2,l_2}, \dots, a_{m,1}, a_{m,2}, \dots, a_{m,l_m}$$

of $[n]$. Associate to the conjugacy class of u the partition $n = l_{w_1} + l_{w_2} + \dots + l_{w_m} + 1 + \dots + 1$, where w_1, w_2, \dots, w_m is a partition of $1, 2, \dots, m$ such that $w_1 \geq w_2 \geq \dots \geq w_m$ and the 1's represent all the fixed points of s . Now note that if s is any permutation in S , $s^{-1}us$ may be written in the form

$$s^{-1}(a_{1,1}a_{1,2} \dots a_{1,l_1})ss^{-1}(a_{2,1}a_{2,2} \dots a_{2,l_2})s \dots s^{-1}(a_{m,1}a_{m,2} \dots a_{m,l_m})s,$$

which may be written as

$$(b_{1,1}b_{1,2} \dots b_{1,l_1})(b_{2,1}b_{2,2} \dots b_{2,l_2}) \dots (b_{m,1}b_{m,2} \dots b_{m,l_m})$$

for some pairwise distinct elements

$$b_{1,1}, b_{1,2}, \dots, b_{1,l_1}, b_{2,1}, b_{2,2}, \dots, b_{2,l_2}, \dots, b_{m,1}, b_{m,2}, \dots, b_{m,l_m}$$

of $[n]$ because multiplying by s^{-1} on the left and by s on the right is equivalent to re-indexing the letters $1, 2, \dots, n$. Hence if partitions of n are associated to all conjugacy classes of S analogously, the

same partition that is associated to u is associated to $s^{-1}us$ for all $s \in S$. Since exactly one partition is associated to each conjugacy class, it follows that the number of conjugacy classes cannot exceed the number of partitions, as desired.

To conclude, we need only observe that $v \geq \sum_{\lambda} \frac{1}{n!}$ by our above claim, for then

$$n!\sqrt{v} = \sqrt{n! \sum_{\lambda} 1} = \sqrt{\left(\sum_{\lambda} (f^{\lambda})^2\right) \left(\sum_{\lambda} 1\right)} \geq \sum_{\lambda} f^{\lambda} \geq n!\tau$$

by the Cauchy-Schwarz inequality, and this completes the proof.

Remark: more computationally intensive solutions that do not use Young tableaux but instead calculate the number of involutions explicitly and make use of the generating function for the number of integer partitions are also possible.

14th Annual Harvard-MIT Mathematics Tournament
Saturday 12 February 2011

Team Round B

Think Carefully! [55]

The problems in this section require only short answers.

1. [55] Tom, Dick, and Harry play a game in which they each pick an integer between 1 and 2011. Tom picks a number first and informs Dick and Harry of his choice. Then Dick picks a different number and informs Harry of his choice. Finally, Harry picks a number different from both Tom's and Dick's. After all the picks are complete, an integer is randomly selected between 1 and 2011. The player whose number is closest wins 2 dollars, unless there is a tie, in which case each of the tied players wins 1 dollar. If Tom knows that Dick and Harry will each play optimally and select randomly among equally optimal choices, there are two numbers Tom can pick to maximize his expected profit; what are they?

Answer: 503, 1509 Let x denote the number Tom chooses. By the symmetry of the problem, picking x and picking $2012 - x$ yield the same expected profit. If Tom picks 1006, Dick sees that if he picks 1007, Harry's best play is to pick 1005, and Dick will win with probability $\frac{1005}{2011}$, and clearly this is the best outcome he can achieve. So Dick will pick 1007 (or 1005) and Harry will pick 1005 (or 1007), and Tom will win with probability $\frac{1}{2011}$. Picking the number 2 will make the probability of winning at least $\frac{2}{2011}$ since Dick and Harry would both be foolish to pick 1, so picking 1006 is suboptimal. It is now clear that the answer to the problem is some pair $(x, 2012 - x)$.

By the symmetry of the problem we can assume without loss of generality that $1 \leq x \leq 1005$. We will show that of these choices, $x = 503$ maximizes Tom's expected profit. The trick is to examine the relationship between Tom's choice and Dick's choice. We claim that (a) if $x > 503$, Dick's choice is $2013 - x$ and (b) if $x < 503$, Dick's choice is $1341 + \lfloor \frac{x+1}{3} \rfloor$. Let y denote the number Dick chooses.

Proof (a). Note first that if $x > 503$, $2013 - x$ is the smallest value of y for which Harry always chooses a number less than x . This is obvious, for if $y < 2011 - x$, a choice of $2012 - x$ will give Harry a greater expected profit than a choice of any number less than x , and if $y = 2012 - x$, Harry never chooses a number between x and y since $x - 1 > \frac{2012 - x - x}{2} = 1006 - x$ holds for all integers x greater than 503, so, by symmetry, Harry chooses a number less than x exactly half the time. The desired result actually follows immediately because the fact that Harry always chooses a number less than x when $y = 2013 - x$ implies that there would be no point in Dick choosing a number greater than $2013 - x$, so it suffices to compare Dick's expected profit when $y = 2013 - x$ with that of all smaller values of y , which is trivial.

Proof (b). This case is somewhat more difficult. First, it should be obvious that if $x < 503$, Harry never chooses a number less than x . The proof is by contradiction. If $y \leq 2011 - x$, Harry can obtain greater expected profit by choosing $2012 - x$ than by choosing any number less than x . If $y \geq 1002 + x$, Harry can obtain greater expected profit by choosing any number between x and y than by choosing a number less than x . Hence if Harry chooses a number less than x , x and y must satisfy $y > 2011 - x$ and $y < 1002 + x$, which implies $2x > 2009$, contradiction. We next claim that if $y \geq 1342 + \lfloor \frac{x+1}{3} \rfloor$, then Harry chooses a number between x and y . The proof follows from the inequality $\frac{1342 + \lfloor \frac{x+1}{3} \rfloor - x}{2} > 2011 - (1342 + \lfloor \frac{x+1}{3} \rfloor)$, which is equivalent to $3\lfloor \frac{x+1}{3} \rfloor > x - 4$, which is obviously true. We also claim that if $y \leq 1340 + \lfloor \frac{x+1}{3} \rfloor$ then Harry chooses a number greater than y .

The proof follows from the inequality $\frac{1340 + \lfloor \frac{x+1}{3} \rfloor - x}{2} < 2011 - (1340 + \lfloor \frac{x+1}{3} \rfloor)$, which is equivalent to $3\lfloor \frac{x+1}{3} \rfloor < x + 2$, which is also obviously true. Now if $x \not\equiv 1 \pmod{3}$, then we have $3\lfloor \frac{x+1}{3} \rfloor > x - 1$, so, in fact, if $y = 1341 + \lfloor \frac{x+1}{3} \rfloor$, Harry still chooses a number between x and y . In this case it is clear that such a choice of y maximizes Dick's expected profit. It turns out that the same holds true even if $x \equiv 1 \pmod{3}$; the computations are only slightly more involved. We omit them here because they bear tangential relation to the main proof.

We may conclude that if $x > 503$, the optimal choice for x is 504, and if $x < 503$, the optimal choice for x is 502. In the first case, $y = 1509$, and, in the second case, $y = 1508$. Since a choice of $x = 503$ and $y = 1509$ clearly outperforms both of these combinations when evaluated based on Tom's expected

profit, it suffices now to show that if Tom chooses 503, Dick chooses 1509. Since the computations are routine and almost identical to those shown above, the proof is left as an exercise to the reader.

Remark: For the more determined, intuition alone is sufficient to see the answer. Assuming Tom's number x is less than 1006, Dick will never pick a number y that makes Harry's best move $y + 1$, so Harry's best move will either be $x - 1$ or anything in between x and y . Clearly Tom does not want Harry to pick $x - 1$, and the biggest number he can pick so that Harry will have to pick in between him and Dick is $\frac{1}{4}$ th of the way along the set of numbers, which in this case is 503. In a rigorous solution, however, the calculations must be handled with grave care.

An interesting inquiry is, what is the optimal strategy with n players?

Complex Numbers [90]

The problems in this section require only short answers.

The *norm* of a complex number $z = a + bi$, denoted by $|z|$, is defined to be $\sqrt{a^2 + b^2}$. In the following problems, it may be helpful to note that the norm is multiplicative and that it obeys the triangle inequality. In other words, please observe that for all complex numbers x and y , $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$. (You may verify these facts for yourself if you like).

2. [20] Let a , b , and c be complex numbers such that $|a| = |b| = |c| = |a + b + c| = 1$. If $|a - b| = |a - c|$ and $b \neq c$, evaluate $|a + b||a + c|$.

Answer: $\boxed{2}$ *First solution.*

Since $|a| = 1$, a cannot be 0. Let $u = \frac{b}{a}$ and $v = \frac{c}{a}$. Dividing the given equations by $|a| = 1$ gives $|u| = |v| = |1 + u + v| = 1$ and $|1 - u| = |1 - v|$. The goal is to prove that $|1 + u||1 + v| = 2$.

By squaring $|1 - u| = |1 - v|$, we get $(1 - u)(\overline{1 - u}) = (1 - v)(\overline{1 - v})$, and thus $1 - u - \bar{u} + |u|^2 = 1 - v - \bar{v} + |v|^2$, or $u + \bar{u} = v + \bar{v}$. This implies $\operatorname{Re}(u) = \operatorname{Re}(v)$. Since u and v are on the unit circle in the complex plane, u is equal to either v or \bar{v} . However, $b \neq c$ implies $u \neq v$, so $u = \bar{v}$.

Therefore, $1 = |1 + u + \bar{u}| = |1 + 2\operatorname{Re}(u)|$. Since $\operatorname{Re}(u)$ is real, we either have $\operatorname{Re}(u) = 0$ or $\operatorname{Re}(u) = -1$. The first case gives $u = \pm i$ and $|1 + u||1 + v| = |1 + i||1 - i| = 2$, as desired. It remains only to note that $\operatorname{Re}(u) = -1$ is in fact impossible because u is of norm 1 and $u = -1$ would imply $u = \bar{u} = v$.

Remark: by the rotational symmetry of the circle, it is acceptable to skip the first step of this solution and assume $a = 1$ without loss of generality.

Second solution.

Let a , b , and c , be the vertices of a triangle inscribed in the unit circle in the complex plane. Since the complex coordinate of the circumcenter is 0 and the complex coordinate of the centroid is $\frac{a+b+c}{3}$, it follows from well-known facts about the Euler line that the complex coordinate of the orthocenter is $a + b + c$. Hence the orthocenter lies on the unit circle as well. Is it not possible for the orthocenter not to be among the three vertices of the triangle, for, if it were, two opposite angles of the convex cyclic quadrilateral formed by the three vertices and the orthocenter would each measure greater than 90 degrees. It follows that the triangle is right. However, since $|a - b| = |a - c|$, the right angle cannot occur at b or c , so it must occur at a , and the desired conclusion follows immediately.

3. [30] Let x and y be complex numbers such that $|x| = |y| = 1$.
- (a) [15] Determine the maximum value of $|1 + x| + |1 + y| - |1 + xy|$.
- (b) [15] Determine the maximum value of $|1 + x| + |1 + xy| + |1 + xy^2| + \dots + |1 + xy^{2011}| - 1006|1 + y|$.

Solution:

(a) **Answer:** $\boxed{2\sqrt{2}}$

(b) **Answer:** $\boxed{2012\sqrt{2}}$ We divide the terms into 1006 sums of the form

$$|1 + xy^{2k}| + |1 + xy^{2k+1}| - |1 + y|.$$

For each of these, we obtain, as in part a,

$$\begin{aligned} |1 + xy^{2k}| + |1 + xy^{2k+1}| - |1 + y| &\leq |1 + xy^{2k}| + |xy^{2k+1} - y| \\ &= |1 + xy^{2k}| + |y| |1 - xy^{2k}| \\ &= |1 + xy^{2k}| + |1 - xy^{2k}| \\ &\leq 2\sqrt{|1 - x^2y^{4k}|} \end{aligned}$$

Again, this is maximized when $x^2y^4 = -1$, which leaves a total sum of at most $2012\sqrt{2}$. If we let $x = i$ and $y = -1$, every sum of three terms will be $2\sqrt{2}$, for a total of $\boxed{2012\sqrt{2}}$, as desired.

4. [40] Let a , b , and c be complex numbers such that $|a| = |b| = |c| = 1$. If

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 1$$

as well, determine the product of all possible values of $|a + b + c|$.

Answer: $\boxed{2}$ Let $s = a + b + c$. Then

$$\begin{aligned} s^3 &= a^3 + b^3 + c^3 + 3(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2) + 6abc \\ &= abc\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} + 3\left(\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c}\right) + 6\right) \\ &= abc\left(1 + 3(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 9\right) + 6 \\ &= abc\left(3s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 2\right) \\ &= abc(3s\bar{s} - 2) \quad (\text{because } \bar{s} = \bar{a} + \bar{b} + \bar{c} = 1/a + 1/b + 1/c) \\ &= abc(3|s|^2 - 2) \end{aligned}$$

Taking absolute values, we find $|s|^3 = |3|s|^2 - 2|$. It follows that $|s|$ must be a positive real root of $x^3 - 3x^2 + 2 = 0$ or $x^3 + 3x^2 - 2 = 0$. However, since the negative real roots of $x^3 - 3x^2 + 2 = 0$ are exactly the additive inverses of the positive real roots of $x^3 - 3x^2 + 2 = 0$, and all three roots of $x^3 - 3x^2 + 2 = 0$ are real ($x^3 - 3x^2 + 2 = 0$ may be factored as $(x - 1)(x^2 - 2x - 2) = 0$, and the discriminant of $x^2 - 2x - 2$ is positive), the product of all possible values of $|s|$ is $(-2) \cdot (-1)^n$, where n denotes the number of negative real roots of $x^3 - 3x^2 + 2 = 0$. By Descartes's Rule of Signs, we see that n is odd, so the answer is 2, as desired.

Warm Up Your Proof Skills! [40]

The problems in this section require complete proofs.

5. (a) [20] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y , subject to the constraint that $0 < y - x < 1$. The first player to write a number greater than or equal to 2010 wins. If Alice goes first, determine, with proof, who has the winning strategy.
- (b) [20] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y , subject to the constraint that $y - x \in (0, 1)$. The first player to write a number greater than or equal to 2010 on her 2011th turn or later wins. If a player writes a number greater than or equal to 2010 on her 2010th turn or before, she loses immediately. If Alice goes first, determine, with proof, who has the winning strategy.

Solution: Each turn of the game is equivalent to adding a number between 0 and 1 to the number on the board.

- (a) Barbara has the winning strategy: whenever Alice adds z to the number on the board, Barbara adds $1 - z$. After Barbara's i th turn, the number on the board will be i . Therefore, after Barbara's 2009th turn, Alice will be forced to write a number between 2009 and 2010, after which Barbara can write 2010 and win the game.
- (b) Alice has the winning strategy: she writes any number a on her first turn, and, after that, whenever Barbara adds z to the number on the board, Alice adds $1 - z$. After Alice's i th turn, the number on the board will be $(i - 1) + a$, so after Alice's 2010th turn, the number will be $2009 + a$. Since Barbara cannot write a number greater than or equal to 2010 on her 2010th turn, she will be forced to write a number between $2009 + a$ and 2010, after which Alice can write 2010 and win the game.

The Euler Totient Function [40]

The problems in this section require complete proofs.

Euler's *totient* function, denoted by φ , is a function whose domain is the set of positive integers. It is especially important in number theory, so it is often discussed on the radio or on national TV. (Just kidding). But what is it, exactly? For all positive integers k , $\varphi(k)$ is defined to be the number of positive integers less than or equal to k that are relatively prime to k . It turns out that φ is what you would call a *multiplicative function*, which means that if a and b are relatively prime positive integers, $\varphi(ab) = \varphi(a)\varphi(b)$. Unfortunately, the proof of this result is highly nontrivial. However, there is much more than that to φ , as you are about to discover!

6. [10] Let n be a positive integer such that $n > 2$. Prove that $\varphi(n)$ is even.

Solution: Let A_n be the set of all positive integers $x \leq n$ such that $\gcd(n, x) = 1$. Since $\gcd(n, x) = \gcd(n, n - x)$ for all x , if a is a positive integer in A_n , so is $n - a$. Moreover, if a is in A_n , a and $n - a$ are different since $\gcd(n, \frac{n}{2}) = \frac{n}{2} > 1$, meaning $\frac{n}{2}$ is not in A_n . Hence we may evenly pair up the elements of A_n that sum to n , so $\varphi(n)$, the number of elements of A_n , must be even, as desired.

7. [10] Let n be an even positive integer. Prove that $\varphi(n) \leq \frac{n}{2}$.

Solution: Again, let A_n be the set of all positive integers $x \leq n$ such that $\gcd(n, x) = 1$. Since n is even, no element of A_n may be even, and, by definition, every element of A_n must be at most n . It follows that $\varphi(n)$, the number of elements of A_n , must be at most $\frac{n}{2}$, as desired.

8. [20] Let n be a positive integer, and let a_1, a_2, \dots, a_n be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n - 1$. Prove that $a_n \geq 2^{n-1}$.

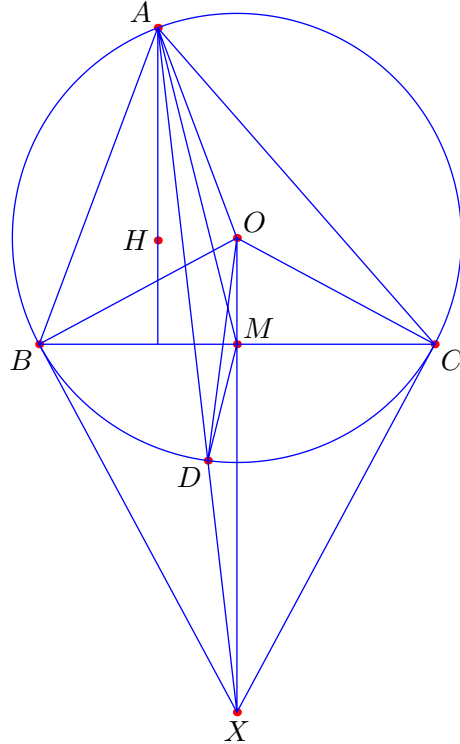
Solution: From problem six, it follows that a_m is even for all $m \leq n - 1$. From problem seven, it follows that $a_m \geq 2a_{m-1}$ for all $m \leq n - 1$. We may conclude that $a_n \geq a_{n-1} \geq 2^{n-2}a_1 = 2^{n-1}$, as desired.

Introduction to the Symmedian [70]

The problems in this section require complete proofs.

If A , B , and C are three points in the plane that do not all lie on the same line, the symmedian from A in triangle ABC is defined to be the reflection of the median from A in triangle ABC about the bisector of angle A . Like the φ function, it turns out that the symmedian satisfies some interesting properties, too. For instance, just like how the medians from A , B , and C all intersect at the centroid of triangle ABC , the symmedians from A , B , and C all intersect at what is called (no surprises here) the symmedian point of triangle ABC . The proof of this fact is not easy, but it is unremarkable. In this section, you will investigate some surprising alternative constructions of the symmedian.

9. [25] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC . Let the tangents to ω at B and C intersect at X . Prove that $\angle OAM = \angle OXA$. (Hint: use SAS similarity).



Solution:

Note that $\triangle OMC \sim \triangle OCX$ since $\angle OMC = \angle OCX = \frac{\pi}{2}$. Hence $\frac{OM}{OC} = \frac{OC}{OX}$, or, equivalently, $\frac{OM}{OA} = \frac{OA}{OX}$. By SAS similarity, it follows that $\triangle OAM \sim \triangle OXA$. Therefore, $\angle OAM = \angle OXA$.

10. [15] Let the circumcircle of triangle AOM intersect ω again at D . Prove that points A , D , and X are collinear.

Solution: By the similarity $\triangle OAM \sim OXA$, we have that $\angle OAX = \angle OMA$. Since $AOMD$ is a cyclic quadrilateral, we have that $\angle OMA = \angle ODA$. Since $OA = OD$, we have that $\angle ODA = \angle OAD$. Combining these equations tells us that $\angle OAX = \angle OAD$, so A , D , and X are collinear, as desired.

11. [10] Let H be the intersection of the three altitudes of triangle ABC . (This point is usually called the orthocenter). Prove that $\angle DAH = \angle MAO$.

Solution: Note that since both AH and OX are perpendicular to BC , it follows that $AH \parallel OX$, so $\angle DAH = \angle DXO = \angle AXO = \angle MAO$, as desired.

12. [10] Prove that line AD is the symmedian from A in triangle ABC by showing that $\angle DAB = \angle MAC$.

Solution: First, we have that

$$\angle OAC = \frac{1}{2}(\pi - \angle AOC) = \frac{\pi}{2} - \angle ABC = \angle HAB$$

so, by the previous problem, we obtain

$$\angle DAB = \angle DAH + \angle HAB = \angle MAO + \angle OAC = \angle MAC.$$

It follows that the lines AD and AM are reflections of each other across the bisector of $\angle BAC$, so AD is the symmedian from A in triangle ABC , as desired.

13. [10] Prove that line AD is also the symmedian from D in triangle DBC .

Solution: In the previous problems, we showed that the symmedian from A in triangle ABC is the line AX , where X is the intersection of the tangents to the circumcircle at B and C . Since the choice of triangle was arbitrary, we may conclude that DX is the symmedian from D in triangle DBC . Since A , D , and X , are collinear, this completes the proof.

Last Writes [65]

The problems in this section require complete proofs.

14. [25] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left; the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?

Answer: [2009] Both operations, run and fight, move the black checker exactly one square to the right, so the game will end after exactly 2008 moves regardless of Brian's choices. Furthermore, it is easy to see that the order of the operations does not matter, so two games with the same number of fights will end up in the same final configuration. Finally, note that each fight adds one white checker to the grid, so two games with different numbers of fights will end up in different final configurations. There are 2009 possible values for the number of fights, so there are 2009 possible final configurations.

15. [40] On Facebook, there is a group of people that satisfies the following two properties: (i) there exists a positive integers k such that any subset of $2k - 1$ people in the group contains a subset of k people in the group who are all friends with each other, and (ii) every member of the group has 2011 friends or fewer.

- (a) [15] If $k = 2$, determine, with proof, the maximum number of people the group may contain.
(b) [25] If $k = 776$, determine, with proof, the maximum number of people the group may contain.

Solution:

- (a) **Answer:** [4024] If $k = 2$, then among any three people at least two of them are friends. Clearly if we have 4024 people divided into two sets of 2012 such that everyone is friends with everyone in their set but no one in the other set, then any triple of three people will contain two people from the same set, who will automatically be friends. Therefore, the group may contain 4024 people.

We now prove that 4024 is the maximum. Given any set with at least 4025 people, consider any two people who are not friends. Between them they have at most 4022 friends, so by the pigeonhole principle there exists someone who is not friends with either one. Therefore, there exists a triple of three people who are all not friends with each other, so the group may not contain 4025 people or more, so 4024 is the maximum, as we claimed.

- (b) **Answer:** [4024] The answer remains the same. Considering the construction identical to that in the above solution, we see that the group may still contain 4024 people and satisfy the desired criterion.

We now prove that 4024 is the maximum. Consider a group with at least 4025 people. From the previous part, we know there exist three people who are not friends. Pick such a threesome and call them X, Y , and Z . We now consider the rest of the people and successively pick pairs of persons A_i, B_i for $1 \leq i \leq 774$ as follows: Once we have picked A_i, B_i , of the remaining $4019 - 2i$ people, we find two who are not friends, which is always possible since $4019 - 2i > 2013$, and we name one of those people A_{i+1} and the other B_{i+1} . Once we have done this, consider the set containing X, Y , and Z as well as A_i and B_i for all $1 \leq i \leq 774$. This is a set of $2k - 1 = 1551$ people. Any subset where everyone is friends with each other can contain at most 1 of A_i, B_i for all i , and at most 1 of X, Y , and Z , meaning that any such subset may contain at most 775 people. Hence there exists a subset containing 1551 people that does not have 776 people who are all friends. Thus, the group may not contain 4025 people or more, so the answer is still 4024, as desired.

4th Annual Harvard-MIT November Tournament
Saturday 12 November 2011
General Test

1. [3] Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

Answer: $\boxed{(1, 3), (3, \frac{1}{3})}$ Multiplying the second equation by x gives

$$x^2 + x^2y = 4x,$$

and substituting our known value of x^2y gives the quadratic

$$x^2 - 4x + 3 = 0,$$

so $x = 1$ or $x = 3$. Hence, we obtain the solutions $(x, y) = (1, 3), (3, 1/3)$.

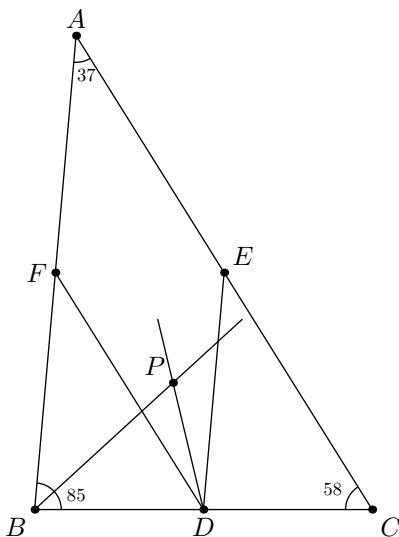
2. [3] Let ABC be a triangle, and let $D, E,$ and F be the midpoints of sides $BC, CA,$ and $AB,$ respectively. Let the angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P . Given that $\angle BAC = 37^\circ$ and $\angle CBA = 85^\circ$, determine the degree measure of $\angle BPD$.

Answer: $\boxed{61^\circ}$ Because D, E, F are midpoints, we have $ABC \sim DEF$. Furthermore, we know that $FD \parallel AC$ and $DE \parallel AB$, so we have

$$\angle BDF = \angle BCA = 180 - 37 - 85 = 58^\circ.$$

Also, $\angle FDE = \angle BAC = 37^\circ$. Hence, we have

$$\angle BPD = 180^\circ - \angle PBD - \angle PDB = 180^\circ - \frac{85^\circ}{2} - \left(\frac{37^\circ}{2} + 58^\circ\right) = 61^\circ.$$



3. [4] Alberto, Bernardo, and Carlos are collectively listening to three different songs. Each is simultaneously listening to exactly two songs, and each song is being listened to by exactly two people. In how many ways can this occur?

Answer: $\boxed{6}$ We have $\binom{3}{2} = 3$ choices for the songs that Alberto is listening to. Then, Bernardo and Carlos must both be listening to the third song. Thus, there are 2 choices for the song that Bernardo shares with Alberto. From here, we see that the songs that everyone is listening to are forced. Thus, there are a total of $3 \times 2 = 6$ ways for the three to be listening to songs.

4. [4] Determine the remainder when

$$2^{\frac{1 \cdot 2}{2}} + 2^{\frac{2 \cdot 3}{2}} + \dots + 2^{\frac{2011 \cdot 2012}{2}}$$

is divided by 7.

Answer: $\boxed{1}$ We have that $2^3 \equiv 1 \pmod{7}$. Hence, it suffices to consider the exponents modulo 3. We note that the exponents are the triangular number and upon division by 3 give the pattern of remainders 1, 0, 0, 1, 0, 0, \dots , so what we want is

$$\begin{aligned} 2^{\frac{1 \cdot 2}{2}} + \dots + 2^{\frac{2011 \cdot 2012}{2}} &\equiv 2^1 + 2^0 + 2^0 + 2^1 + \dots + 2^0 + 2^1 \pmod{7} \\ &\equiv \frac{2010}{3}(2^1 + 2^0 + 2^0) + 2^1 \\ &\equiv (670)(4) + 2 \\ &\equiv 1. \end{aligned}$$

5. [5] Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}.$$

Answer: $\boxed{\frac{257}{16}}$ We note that

$$\begin{aligned} \frac{1}{4} &= \frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} \\ &= \frac{\sqrt{x} - \sqrt{x-2}}{(\sqrt{x} + \sqrt{x-2})(\sqrt{x} - \sqrt{x-2})} + \frac{\sqrt{x+2} - \sqrt{x}}{(\sqrt{x+2} + \sqrt{x})(\sqrt{x+2} - \sqrt{x})} \\ &= \frac{\sqrt{x} - \sqrt{x-2}}{2} + \frac{\sqrt{x+2} - \sqrt{x}}{2} \\ &= \frac{1}{2}(\sqrt{x+2} - \sqrt{x-2}), \end{aligned}$$

so that

$$2\sqrt{x+2} - 2\sqrt{x-2} = 1.$$

Squaring, we get that

$$8x - 8\sqrt{(x+2)(x-2)} = 1 \Rightarrow 8x - 1 = 8\sqrt{(x+2)(x-2)}.$$

Squaring again gives

$$64x^2 - 16x + 1 = 64x^2 - 256,$$

so we get that $x = \frac{257}{16}$.

6. [5] Five people of heights 65, 66, 67, 68, and 69 inches stand facing forwards in a line. How many orders are there for them to line up, if no person can stand immediately before or after someone who is exactly 1 inch taller or exactly 1 inch shorter than himself?

Answer: $\boxed{14}$ Let the people be A, B, C, D, E so that their heights are in that order, with A the tallest and E the shortest. We will do casework based on the position of C .

- *Case 1:* C is in the middle. Then, B must be on one of the two ends, for two choices. This leaves only one choice for D —the other end. Then, we know the positions of A and E since A cannot neighbor B and E cannot neighbor D . So we have 2 options for this case.

- *Case 2:* C is in the second or fourth spot. Then, we have two choices for the position of C . Without loss of generality, let C be in the second spot. Then, the first and third spots must be A and E , giving us two options. This fixes the positions of B and D , so we have a total of $2 \times 2 = 4$ options for this case.
- *Case 3:* C is in the first or last spot. Then, we have two choices for the position of C . Without loss of generality, let it be in the first spot. Either A or E is in the second spot, giving us two choices. Without loss of generality, let it be A . Then, if D is in the third spot, the positions of B and E are fixed. If E is in third spot, the positions of B and D are fixed, so we have a total of $2 \times 2 \times (1 + 1) = 8$ options for this case.

Hence, we have a total of $2 + 4 + 8 = 14$ possibilities.

7. [5] Determine the number of angles θ between 0 and 2π , other than integer multiples of $\pi/2$, such that the quantities $\sin \theta$, $\cos \theta$, and $\tan \theta$ form a geometric sequence in some order.

Answer: 4 If $\sin \theta$, $\cos \theta$, and $\tan \theta$ are in a geometric progression, then the product of two must equal the square of the third. Using this criterion, we have 3 cases.

- *Case 1:* $\sin \theta \cdot \tan \theta = \cos^2 \theta$. This implies that $(\sin^2 \theta) = (\cos^3 \theta)$. Writing $\sin^2 \theta$ as $1 - \cos^2 \theta$ and letting $\cos \theta = x$, we have that $x^3 + x^2 - 1 = 0$. We wish to find the number of solutions of this where $|x| \leq 1$. Clearly -1 is not a root. If $-1 < x \leq 0$, we have that $x^2 + x^3 \leq x^2 < 1$ so $x^3 + x^2 - 1 < 0$ and there are no roots. If $0 < x \leq 1$, then $x^3 + x^2 - 1$ is a strictly increasing function. Since it has value -1 at $x = 0$ and value 1 at $x = 1$, there is exactly one root between 0 and 1, non-inclusive. There are 2 values of θ such that $\cos \theta$ equals this root, and thus, two solutions in this case.
- *Case 2:* $\sin \theta \cdot \cos \theta = \tan^2 \theta$. This implies that $\cos^3 \theta = \sin \theta$. To find the number of solutions in this case, we can analyze the graphs of the functions in different ranges. Note that from $\theta = 0$ to $\theta = \frac{\pi}{2}$, $\cos^3 \theta$ decreases strictly from 1 to 0 while $\sin \theta$ increases strictly from 0 to 1. Hence, there is one solution in this range. By a similar argument, a solution exists between $\theta = \pi$ and $\theta = \frac{3\pi}{2}$. In the intervals $[\frac{\pi}{2}, \pi]$ and $[\frac{3\pi}{2}, 2\pi]$, we have that one function is negative and the other is positive, so there are no solutions. Thus, there are two solutions in this case.
- *Case 3:* $\cos \theta \cdot \tan \theta = \sin^2 \theta$. This implies that $\sin \theta = \sin^2 \theta$, so $\sin \theta = 0, 1$. Clearly the only solutions of these have θ as an integer multiple of $\frac{\pi}{2}$. Thus, there are no pertinent solutions in this case.

We can see that the solutions for the first two cases are mutually exclusive. Hence, there are 4 solutions in total.

8. [6] Find the number of integers x such that the following three conditions all hold:

- x is a multiple of 5
- $121 < x < 1331$
- When x is written as an integer in base 11 with no leading 0s (i.e. no 0s at the very left), its rightmost digit is strictly greater than its leftmost digit.

Answer: 99 We will work in base 11, so let $x = \overline{def}_{11}$ such that $d > 0$. Then, based on the first two conditions, we aim to find multiples of 5 between 100_{11} and 1000_{11} . We note that

$$\overline{def}_{11} \equiv 11^2 \cdot d + 11 \cdot e + f \equiv d + e + f \pmod{5}.$$

Hence, x a multiple of 5 if and only if the sum of its digits is a multiple of 5. Thus, we wish to find triples (d, e, f) with elements in $0, 1, 2, \dots, 9, 10$ such that $d + e + f \equiv 0 \pmod{5}$ and $0 < d < f$.

Note that if we choose d and f such that $d < f$, there is exactly one value of e modulo 5 that would make $d + e + f \equiv 0 \pmod{5}$. Once the this value of e is fixed, then there are two possibilities for e unless $e \equiv 0 \pmod{5}$, in which case there are three possibilities. Thus, our answer is twice the number of ways to choose d and f such that $0 < d < f$ plus the number of ways to choose d and f such that

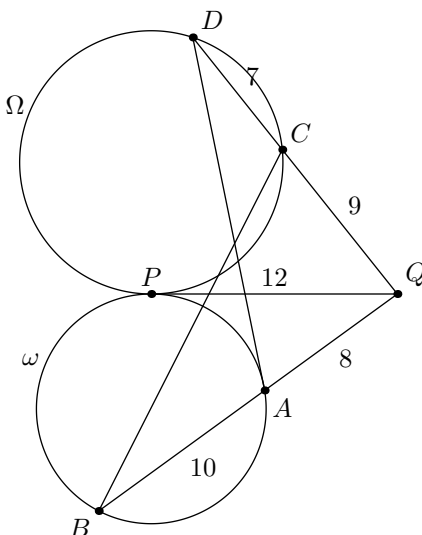
$d + f \equiv 0 \pmod{5}$ and $0 < d < f$ (to account for the extra choice for the value of e). Note that the number of ways to choose $0 < d < f$ is just $\binom{10}{2}$ since any any choice of two digits yields exactly one way to order them. The number of ways to choose $d + f \equiv 0 \pmod{5}$ and $0 < d < f$ can be found by listing: $(d, f) = (1, 4), (1, 9), (2, 3), (2, 8), (3, 7), (4, 6), (5, 10), (6, 9), (7, 8)$, for 9 such pairings.

Hence, the total is $2\binom{10}{2} + 9 = 99$ possibilities for x .

9. [7] Let P and Q be points on line l with $PQ = 12$. Two circles, ω and Ω , are both tangent to l at P and are externally tangent to each other. A line through Q intersects ω at A and B , with A closer to Q than B , such that $AB = 10$. Similarly, another line through Q intersects Ω at C and D , with C closer to Q than D , such that $CD = 7$. Find the ratio AD/BC .

Answer: $\boxed{\frac{8}{9}}$ We first apply the Power of a Point theorem repeatedly. Note that $QA \cdot QB = QP^2 = QC \cdot QD$. Substituting in our known values, we obtain $QA(QA + 10) = 12^2 = QC(QC + 7)$. Solving these quadratics, we get that $QA = 8$ and $QC = 9$.

We can see that $\frac{AQ}{DQ} = \frac{CQ}{BQ}$ and that $\angle AQD = \angle CQB$, so $QAD \sim QCB$. (Alternatively, going back to the equality $QA \cdot QB = QC \cdot QD$, we realize that this is just a Power of a Point theorem on the quadrilateral $ABDC$, and so this quadrilateral is cyclic. This implies that $\angle ADQ = \angle ADC = \angle ABC = \angle QBC$.) Thus, $\frac{AD}{BC} = \frac{AQ}{QC} = \frac{8}{9}$.



10. [8] Let r_1, r_2, \dots, r_7 be the distinct complex roots of the polynomial $P(x) = x^7 - 7$. Let

$$K = \prod_{1 \leq i < j \leq 7} (r_i + r_j),$$

that is, the product of all numbers of the form $r_i + r_j$, where i and j are integers for which $1 \leq i < j \leq 7$. Determine the value of K^2 .

Answer: $\boxed{117649}$ We first note that $x^7 - 7 = (x - r_1)(x - r_2) \cdots (x - r_7)$, which implies, replacing x by $-x$ and taking the negative of the equation, that $(x + r_1)(x + r_2) \cdots (x + r_7) = x^7 + 7$. Also note that the product of the r_i is just the constant term, so $r_1 r_2 \cdots r_7 = 7$.

Now, we have that

$$\begin{aligned}
2^7 \cdot 7 \cdot K^2 &= \left(\prod_{i=1}^7 2r_i \right) K^2 \\
&= \prod_{i=1}^7 2r_i \prod_{1 \leq i < j \leq 7} (r_i + r_j)^2 \\
&= \prod_{1 \leq i=j \leq 7} (r_i + r_j) \prod_{1 \leq i < j \leq 7} (r_i + r_j) \prod_{1 \leq j < i \leq 7} (r_i + r_j) \\
&= \prod_{1 \leq i, j \leq 7} (r_i + r_j) \\
&= \prod_{i=1}^7 \prod_{j=1}^7 (r_i + r_j).
\end{aligned}$$

However, note that for any fixed i , $\prod_{j=1}^7 (r_i + r_j)$ is just the result of substituting $x = r_i$ into $(x + r_1)(x + r_2) \cdots (x + r_7)$. Hence,

$$\prod_{j=1}^7 (r_i + r_j) = r_i^7 + 7 = (r_i^7 - 7) + 14 = 14.$$

Therefore, taking the product over all i gives 14^7 , which yields $K^2 = 7^6 = 117649$.

4th Annual Harvard-MIT November Tournament
Saturday 12 November 2011
Guts Round

1. [5] Determine the remainder when $1 + 2 + \dots + 2014$ is divided by 2012.

Answer: 1009 We wish to find the value of $1 + 2 + \dots + 2014$ modulo 2012. We have

$$1 + 2 + \dots + 2014 = \frac{1}{2}(2014)(2015) = 1007 \cdot 2015 \equiv 1007 \cdot 3 = 3021 \equiv 1009 \pmod{2012}.$$

Remark: Note that, since 2 is not relatively prime to 2012, that this is not the same as

$$\frac{1}{2}(2)(3) \equiv 3 \pmod{2012}.$$

2. [5] Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 4$. Let E be the point on BC with $BE = 3$, and let F be the point on segment AE such that F lies halfway between the segments AB and CD . If G is the point of intersection of DF and BC , find BG .

Answer: 1 Note that since F is a point halfway between AB and CD , the diagram must be symmetric about the line through F parallel to AB . Hence, G must be the reflection of E across the midpoint of BC . Therefore, $BG = EC = 1$.

3. [5] Let x be a real number such that $2^x = 3$. Determine the value of 4^{3x+2} .

Answer: 11664 We have

$$4^{3x+2} = 4^{3x} \cdot 4^2 = (2^2)^{3x} \cdot 16 = 2^{6x} \cdot 16 = (2^x)^6 \cdot 16 = 3^6 \cdot 16 = 11664.$$

4. [6] Determine which of the following numbers is smallest in value: $54\sqrt{3}$, 144 , $108\sqrt{6} - 108\sqrt{2}$.

Answer: $54\sqrt{3}$ We can first compare $54\sqrt{3}$ and 144 . Note that $\sqrt{3} < 2$ and $\frac{144}{54} = \frac{8}{3} > 2$. Hence, $54\sqrt{3}$ is less. Now, we wish to compare this to $108\sqrt{6} - 108\sqrt{2}$. This is equivalent to comparing $\sqrt{3}$ to $2(\sqrt{6} - \sqrt{2})$. We claim that $\sqrt{3} < 2(\sqrt{6} - \sqrt{2})$. To prove this, square both sides to get $3 < 4(8 - 4\sqrt{3})$ or $\sqrt{3} < \frac{29}{16}$ which is true because $\frac{29^2}{16^2} = \frac{841}{256} > 3$. We can reverse this sequence of squarings because, at each step, we make sure that both our values are positive after taking the square root. Hence, $54\sqrt{3}$ is the smallest.

5. [6] Charlie folds an $\frac{17}{2}$ -inch by 11-inch piece of paper in half twice, each time along a straight line parallel to one of the paper's edges. What is the smallest possible perimeter of the piece after two such folds?

Answer: $\frac{39}{2}$ Note that a piece of paper is folded in half, one pair of opposite sides is preserved and the other pair is halved. Hence, the net effect on the perimeter is to decrease it by one of the side lengths. Hence, the original perimeter is $2\left(\frac{17}{2}\right) + 2 \cdot 11 = 39$ and by considering the cases of folding twice along one edge or folding once along each edge, one can see that this perimeter can be decreased by at most $11 + \frac{17}{2} = \frac{39}{2}$. Hence, the minimal perimeter is $\frac{39}{2}$.

6. [6] To survive the coming Cambridge winter, Chim Tu doesn't wear one T-shirt, but instead wears up to FOUR T-shirts, all in different colors. An *outfit* consists of three or more T-shirts, put on one on top of the other in some order, such that two outfits are distinct if the sets of T-shirts used are different or the sets of T-shirts used are the same but the order in which they are worn is different. Given that Chim Tu changes his outfit every three days, and otherwise never wears the same outfit twice, how many days of winter can Chim Tu survive? (Needless to say, he only has four t-shirts.)

Answer: $\boxed{144}$ We note that there are 4 choices for Chim Tu's innermost T-shirt, 3 choices for the next, and 2 choices for the next. At this point, he has exactly 1 T-shirt left, and 2 choices: either he puts that one on as well or he discards it. Thus, he has a total of $4 \times 3 \times 2 \times 2 = 48$ outfits, and can survive for $48 \times 3 = 144$ days.

7. [7] How many ordered triples of positive integers (a, b, c) are there for which $a^4 b^2 c = 54000$?

Answer: $\boxed{16}$ We note that $54000 = 2^4 \times 3^3 \times 5^3$. Hence, we must have $a = 2^{a_1} 3^{a_2} 5^{a_3}$, $b = 2^{b_1} 3^{b_2} 5^{b_3}$, $c = 2^{c_1} 3^{c_2} 5^{c_3}$. We look at each prime factor individually:

- $4a_1 + 2b_1 + c_1 = 4$ gives 4 solutions: $(1, 0, 0), (0, 2, 0), (0, 1, 2), (0, 0, 4)$
- $4a_2 + 2b_2 + c_2 = 3$ and $4a_3 + 2b_3 + c_3 = 3$ each give 2 solutions: $(0, 1, 1), (0, 1, 3)$.

Hence, we have a total of $4 \times 2 \times 2 = 16$ solutions.

8. [7] Let a, b, c be not necessarily distinct integers between 1 and 2011, inclusive. Find the smallest possible value of $\frac{ab+c}{a+b+c}$.

Answer: $\boxed{\frac{2}{3}}$ We have

$$\frac{ab+c}{a+b+c} = \frac{ab-a-b}{a+b+c} + 1.$$

We note that $\frac{ab-a-b}{a+b+c} < 0 \Leftrightarrow (a-1)(b-1) < 1$, which only occurs when either $a = 1$ or $b = 1$. Without loss of generality, let $a = 1$. Then, we have a value of

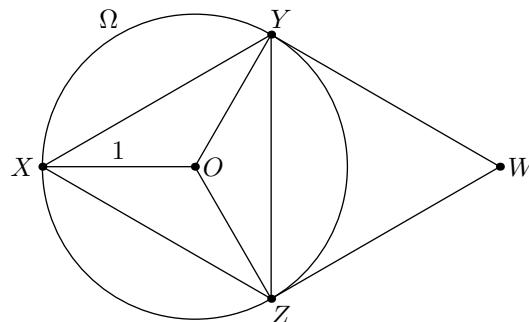
$$\frac{-1}{b+c+a} + 1.$$

We see that this is minimized when b and c are also minimized (so $b = c = 1$), for a value of $\frac{2}{3}$.

9. [7] Unit circle Ω has points X, Y, Z on its circumference so that XYZ is an equilateral triangle. Let W be a point other than X in the plane such that triangle WYZ is also equilateral. Determine the area of the region inside triangle WYZ that lies outside circle Ω .

Answer: $\boxed{\frac{3\sqrt{3}-\pi}{3}}$ Let O be the center of the circle. Then, we note that since $\angle WYZ = 60^\circ = \angle YXZ$, that YW is tangent to Ω . Similarly, WZ is tangent to Ω . Now, we note that the circular segment corresponding to YZ is equal to $\frac{1}{3}$ the area of Ω less the area of triangle OYZ . Hence, our total area is

$$[WYZ] - \frac{1}{3}[\Omega] + [YOZ] = \frac{3\sqrt{3}}{4} - \frac{1}{3}\pi + \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}-\pi}{3}.$$



10. [8] Determine the number of integers D such that whenever a and b are both real numbers with $-1/4 < a, b < 1/4$, then $|a^2 - Db^2| < 1$.

Answer: $\boxed{32}$ We have

$$-1 < a^2 - Db^2 < 1 \Rightarrow \frac{a^2 - 1}{b^2} < D < \frac{a^2 + 1}{b^2}.$$

We have $\frac{a^2 - 1}{b^2}$ is maximal at $-15 = \frac{.25^2 - 1}{.25^2}$ and $\frac{a^2 + 1}{b^2}$ is minimal at $\frac{0^2 + 1}{.25^2} = 16$. However, since we cannot have $a, b = \pm .25$, checking border cases of -15 and 16 shows that both of these values are possible for D . Hence, $-15 \leq D \leq 16$, so there are 32 possible values of D .

11. [8] For positive integers m, n , let $\gcd(m, n)$ denote the largest positive integer that is a factor of both m and n . Compute

$$\sum_{n=1}^{91} \gcd(n, 91).$$

Answer: $\boxed{325}$ Since $91 = 7 \times 13$, we see that the possible values of $\gcd(n, 91)$ are 1, 7, 13, 91. For $1 \leq n \leq 91$, there is only one value of n such that $\gcd(n, 91) = 91$. Then, we see that there are 12 values of n for which $\gcd(n, 91) = 7$ (namely, multiples of 7 other than 91), 6 values of n for which $\gcd(n, 91) = 13$ (the multiples of 13 other than 91), and $91 - 1 - 6 - 12 = 72$ values of n for which $\gcd(n, 91) = 1$. Hence, our answer is $1 \times 91 + 12 \times 7 + 6 \times 13 + 72 \times 1 = 325$.

12. [8] Joe has written 5 questions of different difficulties for a test with problems numbered 1 through 5. He wants to make sure that problem i is harder than problem j whenever $i - j \geq 3$. In how many ways can he order the problems for his test?

Answer: $\boxed{25}$ We will write $p_i > p_j$ for integers i, j when the i th problem is harder than the j th problem. For the problem conditions to be true, we must have $p_4 > p_1$, $p_5 > p_2$, and $p_5 > p_1$.

Then, out of $5! = 120$ total orderings, we see that in half of them satisfy $p_4 > p_1$ and half satisfy $p_5 > p_2$, and that these two events occur independently. Hence, there are $\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)(120) = 30$ orderings which satisfy the first two conditions. Then, we see that there are $\frac{4!}{2!2!} = 6$ orderings of p_1, p_2, p_4, p_5 which work; of these, only $p_4 > p_1 > p_5 > p_2$ violates the condition $p_5 > p_1$. Consequently, we have $\frac{5}{6}(30) = 25$ good problem orderings.

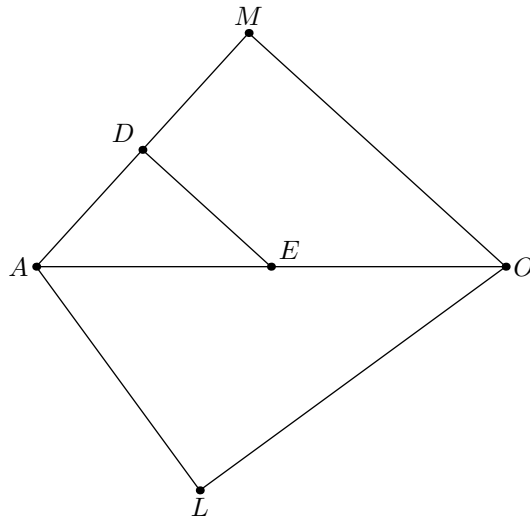
13. [8] Tac is dressing his cat to go outside. He has four indistinguishable socks, four indistinguishable shoes, and 4 indistinguishable show-shoes. In a hurry, Tac randomly pulls pieces of clothing out of a door and tries to put them on a random one of his cat's legs; however, Tac never tries to put more than one of each type of clothing on each leg of his cat. What is the probability that, after Tac is done, the snow-shoe on each of his cat's legs is on top of the shoe, which is on top of the sock?

Answer: $\boxed{\frac{1}{1296}}$ On each leg, Tac's cat will get a shoe, a sock, and a snow-shoe in a random order. Thus, the probability that they will be put on in order for any given leg is $\frac{1}{3!} = \frac{1}{6}$. Thus, the probability that this will occur for all 4 legs is $\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$.

14. [8] Let $AMOL$ be a quadrilateral with $AM = 10$, $MO = 11$, and $OL = 12$. Given that the perpendicular bisectors of sides AM and OL intersect at the midpoint of segment AO , find the length of side LA .

Answer: $\boxed{\sqrt{77}}$ Let D be the midpoint of AM and E be the midpoint of AO . Then, we note that $ADE \sim AMO$, so M is a right angle. Similarly, L is a right angle. Consequently, we get that

$$AO^2 = OM^2 + AM^2 \Rightarrow AL = \sqrt{AO^2 - OL^2} = \sqrt{11^2 + 10^2 - 12^2} = \sqrt{77}.$$



15. [8] For positive integers n , let $L(n)$ be the largest factor of n other than n itself. Determine the number of ordered pairs of composite positive integers (m, n) for which $L(m)L(n) = 80$.

Answer: [12] Let x be an integer, and let p_x be the smallest prime factor of x . Then, if $L(a) = x$, we note that we must have $a = px$ for some prime $p \leq p_x$. (Otherwise, if $p > p_x$, then $\frac{px}{p_x} > x$. If p is composite, then $kx > x$ for some factor k of x .)

So we have:

- $L(a) = 2, 4, 8, 10, 16, 20, 40 \Rightarrow 1$ value for a
- $L(a) = 5 \Rightarrow 3$ values for a

Hence, we note that, since m and n are composite, we cannot have $L(m) = 1$ or $L(n) = 1$, so the possible pairs $(L(m), L(n))$ are $(2, 40), (4, 20), (5, 16), (8, 10)$ and vice-versa.

We add the number of choices for each pair, and double since m and n are interchangeable, to get $2(1 \times 1 + 1 \times 1 + 3 \times 1 + 1 \times 1) = 12$ possible ordered pairs (m, n) .

16. [10] A small fish is holding 17 cards, labeled 1 through 17, which he shuffles into a random order. Then, he notices that although the cards are not currently sorted in ascending order, he can sort them into ascending order by removing one card and putting it back in a *different* position (at the beginning, between some two cards, or at the end). In how many possible orders could his cards currently be?

Answer: [256] Instead of looking at moves which put the cards in order, we start with the cards in order and consider possible starting positions by backtracking one move: each of 17 cards can be moved to 16 new places. But moving card k between card $k + 1$ and card $k + 2$ is equivalent to moving card $k + 1$ between card $k - 1$ and card k . We note that these are the only possible pairs of moves which produce the same result, so we have double counted 16 moves. Thus, we have a total of $17 \times 16 - 16 = 256$ possible initial positions.

17. [10] For a positive integer n , let $p(n)$ denote the product of the positive integer factors of n . Determine the number of factors n of 2310 for which $p(n)$ is a perfect square.

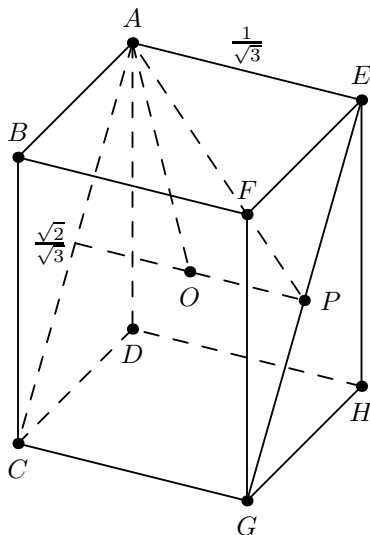
Answer: [27] Note that $2310 = 2 \times 3 \times 5 \times 7 \times 11$. In general, we see that if n has $d(n)$ positive integer factors, then $p(n) = n^{\frac{d(n)}{2}}$ since we can pair factors $(d, \frac{n}{d})$ which multiply to n . As a result, $p(n)$ is a square if and only if n is a square or d is a multiple of 4.

Thus, because 2310 is not divisible by the square of any prime, we claim that for integers n dividing 2310, $p(n)$ is even if and only if n is not prime. Clearly, $p(n)$ is simply equal to n when n is prime, and $p(1) = 1$, so it suffices to check the case when n is composite. Suppose that $n = p_1 p_2 \cdots p_k$, where $k > 1$ and $\{p_1, \dots, p_k\}$ is some subset of $\{2, 3, 5, 7, 11\}$. Then, we see that n has 2^k factors, and that $4 \mid 2^k$, so $p(n)$ is a square.

Since 2310 has $2^5 = 32$ factors, five of which are prime, 27 of them have $p(n)$ even.

18. [10] Consider a cube $ABCDEFGH$, where $ABCD$ and $EFGH$ are faces, and segments AE, BF, CG, DH are edges of the cube. Let P be the center of face $EFGH$, and let O be the center of the cube. Given that $AG = 1$, determine the area of triangle AOP .

Answer: $\frac{\sqrt{2}}{24}$ From $AG = 1$, we get that $AE = \frac{1}{\sqrt{3}}$ and $AC = \frac{\sqrt{2}}{\sqrt{3}}$. We note that triangle AOP is located in the plane of rectangle $ACGE$. Since $OP \parallel CG$ and O is halfway between AC and EG , we get that $[AOP] = \frac{1}{8}[ACGE]$. Hence, $[AOP] = \frac{1}{8}(\frac{1}{\sqrt{3}})(\frac{\sqrt{2}}{\sqrt{3}}) = \frac{\sqrt{2}}{24}$.



19. [10] Let $ABCD$ be a rectangle with $AB = 3$ and $BC = 7$. Let W be a point on segment AB such that $AW = 1$. Let X, Y, Z be points on segments BC, CD, DA , respectively, so that quadrilateral $WXYZ$ is a rectangle, and $BX < XC$. Determine the length of segment BX .

Answer: $\frac{7-\sqrt{41}}{2}$ We note that

$$\angle YXC = 90 - \angle WXB = \angle XWB = 90 - \angle AWZ = \angle AZW$$

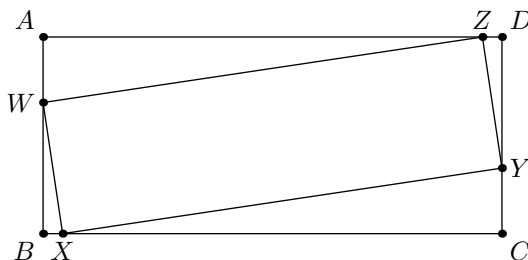
gives us that $XYC \cong ZWA$ and $XYZ \sim WXB$. Consequently, we get that $YC = AW = 1$. From $XYZ \sim WXB$, we get that

$$\frac{BX}{BW} = \frac{CY}{CX} \Rightarrow \frac{BX}{2} = \frac{1}{7 - BX}$$

from which we get

$$BX^2 - 7BX + 2 = 0 \Rightarrow BX = \frac{7 - \sqrt{41}}{2}$$

(since we have $BX < CX$).



20. [10] The UEFA Champions League playoffs is a 16-team soccer tournament in which Spanish teams always win against non-Spanish teams. In each of 4 rounds, each remaining team is *randomly* paired against one other team; the winner advances to the next round, and the loser is permanently knocked out of the tournament. If 3 of the 16 teams are Spanish, what is the probability that there are 2 Spanish teams in the final round?

Answer: $\boxed{\frac{4}{5}}$ We note that the probability there are not two Spanish teams in the final two is the probability that the 3 of them have already competed against each other in previous rounds. Note that the random pairings in each round is equivalent, by the final round, to dividing the 16 into two groups of 8 and taking a winner from each. Now, letting the Spanish teams be A , B , and C , once we fix the group in which A is contained, the probability that B is contained in this group as well is $7/15$. Likewise, the probability that C will be in the same group as A and B is now $6/14$. Our answer is thus

$$1 - \left(\frac{7}{15}\right) \left(\frac{6}{14}\right) = \frac{4}{5}.$$

21. [10] Let $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ be a polynomial with roots r_1, r_2, r_3, r_4 . Let Q be the quartic polynomial with roots $r_1^2, r_2^2, r_3^2, r_4^2$, such that the coefficient of the x^4 term of Q is 1. Simplify the quotient $Q(x^2)/P(x)$, leaving your answer in terms of x . (You may assume that x is not equal to any of r_1, r_2, r_3, r_4).

Answer: $\boxed{x^4 - 2x^3 - 13x^2 + 14x + 24}$ We note that we must have

$$Q(x) = (x - r_1^2)(x - r_2^2)(x - r_3^2)(x - r_4^2) \Rightarrow Q(x^2) = (x^2 - r_1^2)(x^2 - r_2^2)(x^2 - r_3^2)(x^2 - r_4^2)$$

. Since $P(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)$, we get that

$$Q(x^2)/P(x) = (x + r_1)(x + r_2)(x + r_3)(x + r_4).$$

Thus, $Q(x^2)/P(x) = (-1)^4 P(-x) = P(-x)$, so it follows that

$$Q(x^2)/P(x) = x^4 - 2x^3 - 13x^2 + 14x + 24.$$

22. [12] Let ABC be a triangle with $AB = 23$, $BC = 24$, and $CA = 27$. Let D be the point on segment AC such that the incircles of triangles BAD and BCD are tangent. Determine the ratio CD/DA .

Answer: $\boxed{\frac{14}{13}}$ Let X, Z, E be the points of tangency of the incircle of ABD to AB, BD, DA respectively. Let Y, Z, F be the points of tangency of the incircle of BCD to CB, BD, DC respectively. We note that

$$CB + BD + DC = CY + YB + BZ + ZD + DF + FC = 2(CY) + 2(BY) + 2(DF)2(24) + 2(DF)$$

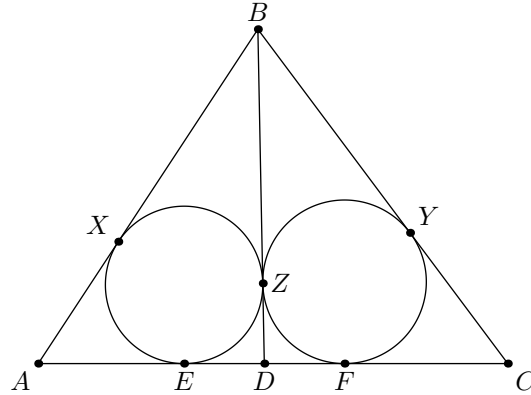
by equal tangents, and that similarly

$$AB + BD + DA = 2(23) + 2(DE).$$

Since $DE = DZ = DF$ by equal tangents, we can subtract the equations above to get that

$$CB + CD - AB - AD = 2(24) - 2(23) \Rightarrow CD - DA = 1.$$

Since we know that $CD + DA = 27$, we get that $CD = 14$, $DA = 13$, so the desired ratio is $\frac{14}{13}$.



23. [12] Let $N = \overline{5AB37C2}$, where A, B, C are digits between 0 and 9, inclusive, and N is a 7-digit positive integer. If N is divisible by 792, determine all possible ordered triples (A, B, C) .

Answer: $\boxed{(0, 5, 5), (4, 5, 1), (6, 4, 9)}$ First, note that $792 = 2^3 \times 3^2 \times 11$. So we get that

$$8 \mid N \Rightarrow 8 \mid \overline{7C2} \Rightarrow 8 \mid 10C + 6 \Rightarrow C = 1, 5, 9$$

$$9 \mid N \Rightarrow 9 \mid 5 + A + B + 3 + 7 + C + 2 \Rightarrow A + B + C = 1, 10, 19$$

$$11 \mid N \Rightarrow 11 \mid 5 - A + B - 3 + 7 - C + 2 \Rightarrow -A + B - C = -11, 0$$

Adding the last two equations, and noting that they sum to $2B$, which must be even, we get that $B = 4, 5$.

Checking values of C we get possible triplets of $(0, 5, 5)$, $(4, 5, 1)$, and $(6, 4, 9)$.

24. [12] Three not necessarily distinct positive integers between 1 and 99, inclusive, are written in a row on a blackboard. Then, the numbers, without including any leading zeros, are concatenated to form a new integer N . For example, if the integers written, in order, are 25, 6, and 12, then $N = 25612$ (and not $N = 250612$). Determine the number of possible values of N .

Answer: $\boxed{825957}$ We will divide this into cases based on the number of digits of N .

- *Case 1:* 6 digits. Then each of the three numbers must have two digits, so we have 90 choices for each. So we have a total of $90^3 = 729000$ possibilities.
- *Case 2:* 5 digits. Then, exactly one of the three numbers is between 1 and 9, inclusive. We consider cases on the presence of 0s in N .
 - No 0s. Then, we have 9 choices for each digit, for a total of $9^5 = 59049$ choices.
 - One 0. Then, the 0 can be the second, third, fourth, or fifth digit, and 9 choices for each of the other 4 digits. Then, we have a total of $4 \times 9^4 = 26244$ choices.
 - Two 0s. Then, there must be at least one digit between them and they cannot be in the first digit, giving us 3 choices for the positioning of the 0s. Then, we have a total of $3 \times 9^3 = 2187$ choices.

So we have a total of $59049 + 26244 + 2187 = 87480$ choices in this case.

- *Case 3:* 4 digits. Again, we casework on the presence of 0s.
 - No 0s. Then, we have $9^4 = 6561$ choices.
 - One 0. Then, the 0 can go in the second, third, or fourth digit, so we have $3 \times 9^3 = 2187$ choices.

So we have a total of $6561 + 2187 = 8748$ choices in this case.

- *Case 4:* 3 digits. Then, we cannot have any 0s, so we have a total of $9^3 = 729$ choices.

Hence, we have a total of $729000 + 87480 + 8748 + 729 = 825957$ choices for N .

25. [12] Let XYZ be an equilateral triangle, and let K, L, M be points on sides XY, YZ, ZX , respectively, such that $XK/KY = B$, $YL/LZ = 1/C$, and $ZM/MX = 1$. Determine the ratio of the area of triangle KLM to the area of triangle XYZ .

Answer: $\boxed{\frac{1}{5}}$ First, we note that

$$[KLM] = [XYZ] - [XKM] - [YLK] - [ZML].$$

Then, note that

$$[XKM] = \frac{XK}{XY} \cdot \frac{XM}{XZ} \cdot [XYZ] = \frac{B}{B+1} \cdot \frac{1}{2} \cdot [XYZ]$$

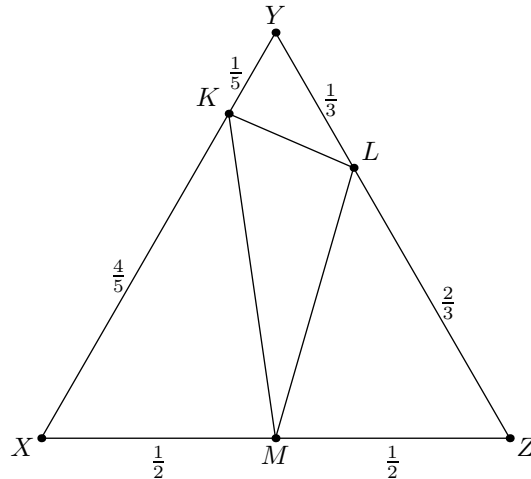
$$[YLK] = \frac{YL}{YZ} \cdot \frac{YK}{YX} \cdot [XYZ] = \frac{1}{C+1} \cdot \frac{1}{B+1} \cdot [XYZ]$$

$$[ZML] = \frac{ZM}{ZX} \cdot \frac{ZL}{ZY} \cdot [XYZ] = \frac{1}{2} \cdot \frac{1}{C+1} \cdot [XYZ]$$

Consequently,

$$\begin{aligned} A &= \frac{[KLM]}{[XYZ]} \\ &= 1 - \frac{B}{B+1} \cdot \frac{1}{2} - \frac{1}{C+1} \cdot \frac{1}{B+1} - \frac{C}{C+1} \cdot \frac{1}{2} \\ &= \frac{B+C}{(B+1)(C+1)(2)} \end{aligned}$$

If we solve our system of equations for A, B, C , we get that $A = \frac{1}{5}$.



26. [12] Determine the positive real value of x for which

$$\sqrt{2+AC+2Cx} + \sqrt{AC-2+2Ax} = \sqrt{2(A+C)x+2AC}.$$

Answer: $\boxed{4}$ Note that if we have $\sqrt{a} + \sqrt{b} = \sqrt{a+b}$ for non-negative reals a, b , then squaring gives us that $2\sqrt{ab} = 0$, so that either $a = 0$ or $b = 0$.

Now, note that

$$(2+AC+2Cx) + (AC-2+2Ax) = (2(A+C)x+2AC).$$

Consequently, either $(2 + AC + 2Cx)$ or $(AC - 2 + 2Ax)$ must be equal to 0. However, we observe from the problems that both A , C , and x must be non-negative, so $(2 + AC + 2Cx) > 0$. As a result, we know that $AC - 2 + 2Ax = 0$, or that

$$B = x = \frac{2 - AC}{2A}.$$

If we solve our system of equations for A, B, C , we get that $B = 4$.

27. [12] In-Young generates a string of B zeroes and ones using the following method:

- First, she flips a fair coin. If it lands heads, her first digit will be a 0, and if it lands tails, her first digit will be a 1.
- For each subsequent bit, she flips an unfair coin, which lands heads with probability A . If the coin lands heads, she writes down the number (zero or one) different from previous digit, while if the coin lands tails, she writes down the previous digit again.

What is the expected value of the number of zeroes in her string?

Answer: [2] Since each digit is dependent on the previous, and the first digit is random, we note that the probability that In Young obtains a particular string is the same probability as that she obtains the inverse string (i.e. that where the positions of the 0s and 1s are swapped). Consequently, we would expect that half of her digits are 0s, so that

$$C = \frac{B}{2}.$$

If we solve our system of equations for A, B, C , we get that $C = 2$.

Solution of the system of equations for Problems 25, 26, 27:

Thus, we have the three equations

$$A = \frac{B + C}{(B + 1)(C + 1)(2)}, \quad B = \frac{2 - AC}{2A}, \quad C = \frac{B}{2}$$

Plugging the last equation into the first two results in

$$A = \frac{3B}{(B + 1)(B + 2)(2)} \Rightarrow B = \frac{4 - AB}{4A}$$

Rearranging the second equation gives

$$4AB = 4 - AB \Rightarrow AB = \frac{4}{5} \Rightarrow A = \frac{4}{5B}$$

Then, plugging this into the first equation gives

$$\begin{aligned} \frac{4}{5B} &= \frac{3B}{(B + 1)(B + 2)(2)} \\ \Rightarrow 15B^2 &= 8B^2 + 24B + 16 \\ \Rightarrow 7B^2 - 24B - 16 &= 0 \\ \Rightarrow (7B + 4)(B - 4) &= 0 \end{aligned}$$

Since we know that $B > 0$, we get that $B = 4$. Plugging this back in gives $A = \frac{1}{5}$ and $C = 2$.

28. [14] Determine the value of

$$\sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!}.$$

Answer: $\boxed{\frac{2009(2^{2010})+1}{2011!}}$ We note that

$$\begin{aligned} (2011!) \sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!} &= \sum_{k=1}^{2011} \frac{(2011!)(k-1)}{k!(2011-k)!} \\ &= \sum_{k=1}^{2011} \frac{k(2011)!}{k!(2011-k)!} - \sum_{k=1}^{2011} \frac{2011!}{k!(2011-k)!} \\ &= \sum_{k=1}^{2011} k \binom{2011}{k} - \sum_{k=1}^{2011} \binom{2011}{k} \\ &= (2011)(2^{2010}) - (2^{2011} - 1) \end{aligned}$$

Thus, we get an answer of $(2009(2^{2010}) + 1) / (2011!)$.

Note: To compute the last two sums, observe that

$$\sum_{k=0}^{2011} \binom{2011}{k} = (1+1)^{2011} = 2^{2011}$$

by the Binomial Theorem, and that

$$\sum_{k=0}^{2011} k \binom{2011}{k} = \frac{1}{2} \left(\sum_{k=0}^{2011} k \binom{2011}{k} + \sum_{k=0}^{2011} (2011-k) \binom{2011}{2011-k} \right) = 2011(2^{2010}).$$

29. [14] Let ABC be a triangle with $AB = 4$, $BC = 8$, and $CA = 5$. Let M be the midpoint of BC , and let D be the point on the circumcircle of ABC so that segment AD intersects the interior of ABC , and $\angle BAD = \angle CAM$. Let AD intersect side BC at X . Compute the ratio AX/AD .

Answer: $\boxed{\frac{9}{41}}$ Let E be the intersection of AM with the circumcircle of ABC . We note that, by equal angles $ADC \sim ABM$, so that

$$AD = AC \left(\frac{AB}{AM} \right) = \frac{20}{AM}.$$

Using the law of cosines on ABC , we get that

$$\cos B = \frac{4^2 + 8^2 - 5^2}{2(4)(8)} = \frac{55}{64}.$$

Then, using the law of cosines on ABM , we get that

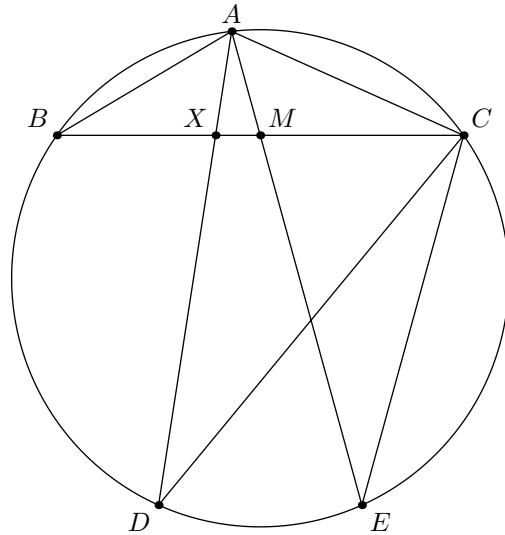
$$AM = \sqrt{4^2 + 4^2 - 2(4)(4) \cos B} = \frac{3}{\sqrt{2}} \Rightarrow AD = \frac{20\sqrt{2}}{3}.$$

Applying Power of a Point on M ,

$$(AM)(ME) = (BM)(MC) \Rightarrow ME = \frac{16\sqrt{2}}{3} \Rightarrow AE = \frac{41\sqrt{2}}{6}.$$

Then, we note that $AXB \sim ACE$, so that

$$AX = AB \left(\frac{AC}{AE} \right) = \frac{60\sqrt{2}}{41} \Rightarrow \frac{AX}{AD} = \frac{9}{41}$$



30. [14] Let S be a set of consecutive positive integers such that for any integer n in S , the sum of the digits of n is not a multiple of 11. Determine the largest possible number of elements of S .

Answer: 38 We claim that the answer is 38. This can be achieved by taking the smallest integer in the set to be 999981. Then, our sums of digits of the integers in the set are

$$45, \dots, 53, 45, \dots, 54, 1, \dots, 10, 2, \dots, 10,$$

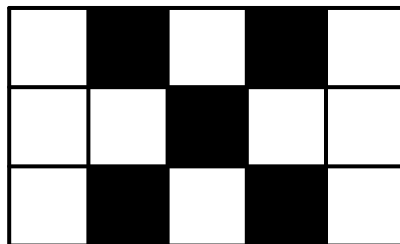
none of which are divisible by 11.

Suppose now that we can find a larger set S : then we can then take a 39-element subset of S which has the same property. Note that this implies that there are consecutive integers $a - 1, a, a + 1$ for which $10b, \dots, 10b + 9$ are all in S for $b = a - 1, a, a + 1$. Now, let $10a$ have sum of digits N . Then, the sums of digits of $10a + 1, 10a + 2, \dots, 10a + 9$ are $N + 1, N + 2, \dots, N + 9$, respectively, and it follows that $n \equiv 1 \pmod{11}$.

If the tens digit of $10a$ is not 9, note that $10(a + 1) + 9$ has sum of digits $N + 10$, which is divisible by 11, a contradiction. On the other hand, if the tens digit of $10a$ is 9, the sum of digits of $10(a - 1)$ is $N - 1$, which is also divisible by 11. Thus, S has at most 38 elements.

Motivation: We want to focus on subsets of S of the form $\{10a, \dots, 10a + 9\}$, since the sum of digits goes up by 1 most of the time. If the tens digit of $10a$ is anything other than 0 or 9, we see that S can at most contain the integers between $10a - 8$ and $10a + 18$, inclusive. However, we can attempt to make $10(a - 1) + 9$ have sum of digits congruent to $N + 9$ modulo 11, as to be able to add as many integers to the beginning as possible, which can be achieved by making $10(a - 1) + 9$ end in the appropriate number of nines. We see that we want to take $10(a - 1) + 9 = 999999$ so that the sum of digits upon adding 1 goes down by $53 \equiv 9 \pmod{11}$, giving the example we constructed previously.

31. [17] Each square in a 3×10 grid is colored black or white. Let N be the number of ways this can be done in such a way that no five squares in an 'X' configuration (as shown by the black squares below) are all white or all black. Determine \sqrt{N} .



Answer: 25636 Note that we may label half of the cells in our board the number 0 and the other half 1, in such a way that squares labeled 0 are adjacent only to squares labeled 1 and vice versa. In other words, we make this labeling in a 'checkerboard' pattern. Since cells in an 'X' formation are all labeled with the same number, the number of ways to color the cells labeled 0 is \sqrt{N} , and the same is true of coloring the cells labeled 1.

Let a_{2n} be the number of ways to color the squares labeled 0 in a 3 by $2n$ grid without a monochromatic 'X' formation; we want to find a_{10} . Without loss of generality, let the rightmost column of our grid have two cells labeled 0. Let b_{2n} be the number of such colorings on a 3 by $2n$ grid which do not have two black squares in the rightmost column and do not contain a monochromatic 'X', which we note is also the number of such colorings which do not have two white squares in the rightmost column.

Now, we will establish a recursion on a_{2n} and b_{2n} . We have two cases:

- *Case 1:* All three squares in the last two columns are the same color. For a_{2n} , there are 2 ways to color these last three squares, and for b_{2n} there is 1 way to color them. Then, we see that there are b_{2n-2} ways to color the remaining $2n - 2$ columns.
- *Case 2:* The last three squares are not all the same color. For a_{2n} , there are 6 ways to color the last three squares, and for b_{2n} there are 5 ways to color them. Then, there are a_{2n-2} ways to color the remaining $2n - 2$ columns.

Consequently, we get the recursions $a_{2n} = 6a_{2n-2} + 2b_{2n-2}$ and $b_{2n} = 5a_{2n-2} + b_{2n-2}$. From the first equation, we get that $b_{2n} = \frac{1}{2}a_{2n+2} - 3a_{2n}$. Plugging this in to the second equations results in the recursion

$$\frac{1}{2}a_{2n+2} - 3a_{2n} = 5a_{2n-2} + \frac{1}{2}a_{2n} - 3a_{2n-2} \Rightarrow a_{2n+2} = 7a_{2n} + 4a_{2n-2}.$$

Now, we can easily see that $a_0 = 1$ and $a_2 = 2^3 = 8$, so we compute $a_{10} = 25636$.

32. [17] Find all real numbers x satisfying

$$x^9 + \frac{9}{8}x^6 + \frac{27}{64}x^3 - x + \frac{219}{512} = 0.$$

Answer: $\frac{1}{2}, \frac{-1 \pm \sqrt{13}}{4}$ Note that we can re-write the given equation as

$$\sqrt[3]{x - \frac{3}{8}} = x^3 + \frac{3}{8}.$$

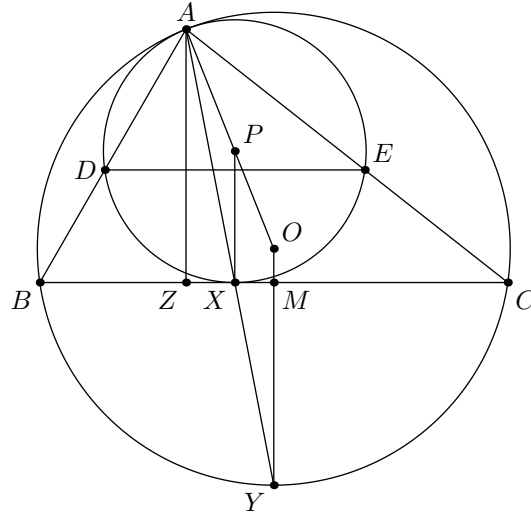
Furthermore, the functions of x on either side, we see, are inverses of each other and increasing. Let $f(x) = \sqrt[3]{x - \frac{3}{8}}$. Suppose that $f(x) = y = f^{-1}(x)$. Then, $f(y) = x$. However, if $x < y$, we have $f(x) > f(y)$, contradicting the fact that f is increasing, and similarly, if $y < x$, we have $f(x) < f(y)$, again a contradiction. Therefore, if $f(x) = f^{-1}(x)$ and both are increasing functions in x , we require $f(x) = x$. This gives the cubic

$$x^3 - x + \frac{3}{8} = 0 \rightarrow \left(x - \frac{1}{2}\right) \left(x^2 + \frac{1}{2}x - \frac{3}{4}\right) = 0,$$

giving $x = \frac{1}{2}, \frac{-1 \pm \sqrt{13}}{4}$.

33. [17] Let ABC be a triangle with $AB = 5$, $BC = 8$, and $CA = 7$. Let Γ be a circle internally tangent to the circumcircle of ABC at A which is also tangent to segment BC . Γ intersects AB and AC at points D and E , respectively. Determine the length of segment DE .

Answer: $\frac{40}{9}$



First, note that a homothety h centered at A takes Γ to the circumcircle of ABC , D to B and E to C , since the two circles are tangent. As a result, we have $DE \parallel BC$. Now, let P be the center of Γ and O be the circumcenter of ABC : by the homothety h , we have $DE/BC = AP/AO$.

Let Γ be tangent to BC at X , and let ray \overrightarrow{AX} meet the circumcircle of ABC at Y . Note that Y is the image of X under h . Furthermore, h takes BC to the tangent line l to the circumcircle of ABC at Y , and since $BC \parallel l$, we must have that Y is the midpoint of arc \widehat{BC} . Therefore, AX bisects $\angle BAC$.

Now, let Z be the foot of the altitude from A to BC , and let M be the midpoint of BC , so that $OM \perp BC$. Note that $AP/AO = ZX/ZM$. Now, letting $BC = a = 8$, $CA = b = 7$, and $AB = c = 5$, we compute

$$BZ = c \cos B = \frac{c^2 + a^2 - b^2}{2a} = \frac{5}{2}$$

by the Law of Cosines,

$$BX = \frac{ac}{b+c} = \frac{10}{3}$$

by the Angle Bisector Theorem, and

$$BM = 4.$$

To finish,

$$DE = \frac{(AP)(BC)}{AO} = \frac{(ZX)(BC)}{ZM} = \frac{(5/6)(8)}{(3/2)} = \frac{40}{9}$$

34. [20] The integer 843301 is prime. The *primorial* of a prime number p , denoted $p\#$, is defined to be the product of all prime numbers less than or equal to p . Determine the number of digits in $843301\#$. Your score will be

$$\max \left\{ \left\lfloor 60 \left(\frac{1}{3} - \left| \ln \left(\frac{A}{d} \right) \right| \right) \right\rfloor, 0 \right\},$$

where A is your answer and d is the actual answer.

Answer: 365851 **Remark:** $843301\# - 1$ is the largest known prime number of the form $p\# - 1$, where p is prime.

35. [20] Let G be the number of Google hits of “guts round” at 10:31PM on October 31, 2011. Let B be the number of Bing hits of “guts round” at the same time. Determine B/G . Your score will be

$$\max \left(0, \left\lfloor 20 \left(1 - \frac{20|a-k|}{k} \right) \right\rfloor \right),$$

where k is the actual answer and a is your answer.

Answer: The number of Google hits was 7350. The number of Bing hits was 6080. The answer is thus $6080/7350 = .82721$.

36. [20] Order any subset of the following twentieth century mathematical achievements chronologically, from earliest to most recent. If you correctly place at least six of the events in order, your score will be $2(n - 5)$, where n is the number of events in your sequence; otherwise, your score will be zero. Note: if you order any number of events with one error, your score will be zero.
- A). Axioms for Set Theory published by Zermelo
 - B). Category Theory introduced by Mac Lane and Eilenberg
 - C). Collatz Conjecture proposed
 - D). Erdos number defined by Goffman
 - E). First United States delegation sent to International Mathematical Olympiad
 - F). Four Color Theorem proven with computer assistance by Appel and Haken
 - G). Harvard-MIT Math Tournament founded
 - H). Hierarchy of grammars described by Chomsky
 - I). Hilbert Problems stated
 - J). Incompleteness Theorems published by Godel
 - K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute
 - L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis
 - M). Nash Equilibrium introduced in doctoral dissertation
 - N). Proof of Fermat's Last Theorem completed by Wiles
 - O). Quicksort algorithm invented by Hoare

Write your answer as a list of letters, without any commas or parentheses.

Answer: The dates are as follows:

- A). Axioms for Set Theory published by Zermelo 1908
- B). Category Theory introduced by Mac Lane and Eilenberg 1942-1945
- C). Collatz Conjecture proposed 1937
- D). Erdos number defined by Goffman 1969
- E). First United States delegation sent to International Mathematical Olympiad 1974
- F). Four Color Theorem proven with computer assistance by Appel and Haken 1976
- G). Harvard-MIT Math Tournament founded 1998
- H). Hierarchy of grammars described by Chomsky 1956
- I). Hilbert Problems stated 1900
- J). Incompleteness Theorems published by Godel 1931
- K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute 2000
- L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis 1992
- M). Nash Equilibrium introduced in doctoral dissertation 1950
- N). Proof of Fermat's Last Theorem completed by Wiles 1994
- O). Quicksort algorithm invented by Hoare 1960

so the answer is *IAJCBMHODEFLNGK*.

4th Annual Harvard-MIT November Tournament
Saturday 12 November 2011
Team Round

1. [2] Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

Answer: 0 We claim that our integer is divisible by 3 for all positive integers x . Indeed, we have

$$\begin{aligned} 3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x &\equiv (0)^x + (-1)^x + (1)^x + (-1)^x + (1)^x + (-1)^x + (1)^x \\ &\equiv 3[(1)^x + (-1)^x] \\ &\equiv 0 \pmod{3}. \end{aligned}$$

It is clear that for all $x \geq 1$, our integer is strictly greater than 3, so it will always be composite, making our answer 0.

2. [4] Determine the set of all real numbers p for which the polynomial $Q(x) = x^3 + px^2 - px - 1$ has three distinct real roots.

Answer: $p > 1$ and $p < -3$ First, we note that

$$x^3 + px^2 - px - 1 = (x - 1)(x^2 + (p + 1)x + 1).$$

Hence, $x^2 + (p + 1)x + 1$ has two distinct roots. Consequently, the discriminant of this equation must be positive, so $(p + 1)^2 - 4 > 0$, so either $p > 1$ or $p < -3$. However, the problem specifies that the quadratic must have *distinct* roots (since the original cubic has distinct roots), so to finish, we need to check that 1 is not a double root—we will do this by checking that 1 is not a root of $x^2 + (p + 1)x + 1$ for any value p in our range. But this is clear, since $1 + (p + 1) + 1 = 0 \Rightarrow p = -3$, which is not in the aforementioned range. Thus, our answer is all p satisfying $p > 1$ or $p < -3$.

3. [6] Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

Answer: -3193 Define $Q(x) = P(x) - x - 6 = x^4 - 29x^3 + ax^2 + (b - 1)x + (c - 6)$ and notice that $Q(5) = Q(11) = Q(17) = 0$. $Q(x)$ has degree 4 and by Vieta's Formulas the sum of its roots is 29, so its last root is $29 - 17 - 11 - 5 = -4$, giving us $Q(x) = (x - 5)(x - 11)(x - 17)(x + 4)$. This means that $P(1) = Q(1) + 7 = (-4)(-10)(-16)(5) + 7 = -3200 + 7 = -3193$.

4. [7] Determine the number of quadratic polynomials $P(x) = p_1x^2 + p_2x - p_3$, where p_1, p_2, p_3 are not necessarily distinct (positive) prime numbers less than 50, whose roots are distinct rational numbers.

Answer: 31 The existence of distinct rational roots means that the given quadratic splits into linear factors. Then, since p_1, p_3 are both prime, we get that the following are the only possible factorizations:

- $(p_1x - p_3)(x + 1) \Rightarrow p_2 = p_1 - p_3$
- $(p_1x + p_3)(x - 1) \Rightarrow p_2 = -p_1 + p_3$
- $(p_1x - 1)(x + p_3) \Rightarrow p_2 = p_1p_3 - 1$
- $(p_1x + 1)(x - p_3) \Rightarrow p_2 = -p_1p_3 + 1$

In the first case, observe that since $p_2 + p_3 = p_1$, we have $p_1 > 2$, so p_1 is odd and exactly one of p_2, p_3 is equal to 2. Thus, we get a solutions for every pair of twin primes below 50, which we enumerate to be (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), giving 12 solutions in total. Similarly, the second case gives $p_1 + p_2 = p_3$, for another 12 solutions.

In the third case, if p_1, p_3 are both odd, then p_2 is even and thus equal to 2. However, this gives $p_1 p_3 = 3$, which is impossible. Therefore, at least one of p_1, p_3 is equal to 2. If $p_1 = 2$, we get $p_2 = 2p_3 - 1$, which we find has 4 solutions: $(p_2, p_3) = (3, 2), (5, 3), (13, 7), (37, 19)$. Similarly, there are four solutions with $p_3 = 2$. However, we count the solution $(p_1, p_2, p_3) = (2, 3, 2)$ twice, so we have a total of 7 solutions in this case. Finally, in the last case

$$p_2 = -p_1 p_3 + 1 < -(2)(2) + 1 < 0,$$

so there are no solutions. Hence, we have a total of $12 + 12 + 7 = 31$ solutions.

5. [3] Sixteen wooden Cs are placed in a 4-by-4 grid, all with the same orientation, and each is to be colored either red or blue. A *quadrant operation* on the grid consists of choosing one of the four two-by-two subgrids of Cs found at the corners of the grid and moving each C in the subgrid to the adjacent square in the subgrid that is 90 degrees away in the clockwise direction, without changing the orientation of the C. Given that two colorings are considered the same if and only if one can be obtained from the other by a series of quadrant operations, determine the number of distinct colorings of the Cs.

C	C	C	C
C	C	C	C
C	C	C	C
C	C	C	C

Answer: 1296 For each quadrant, we have three distinct cases based on the number of Cs in each color:

- *Case 1:* all four the same color: 2 configurations (all red or all blue)
- *Case 2:* 3 of one color, 1 of the other: 2 configurations (three red or three blue)
- *Case 3:* 2 of each color: 2 configurations (red squares adjacent or opposite)

Thus, since there are 4 quadrants, there are a total of $(2 + 2 + 2)^4 = 1296$ possible grids.

6. [5] Ten Cs are written in a row. Some Cs are upper-case and some are lower-case, and each is written in one of two colors, green and yellow. It is given that there is at least one lower-case C, at least one green C, and at least one C that is both upper-case and yellow. Furthermore, no lower-case C can be followed by an upper-case C, and no yellow C can be followed by a green C. In how many ways can the Cs be written?

Answer: 36 By the conditions of the problem, we must pick some point in the line where the green Cs transition to yellow, and some point where the upper-case Cs transition to lower-case. We see that the first transition must occur before the second, and that they cannot occur on the same C. Hence, the answer is $\binom{9}{2} = 36$.

7. [7] Julia is learning how to write the letter C. She has 6 differently-colored crayons, and wants to write Cc Cc Cc Cc Cc. In how many ways can she write the ten Cs, in such a way that each upper case C is a different color, each lower case C is a different color, and in each pair the upper case C and lower case C are different colors?

Answer: 222480 Suppose Julia writes Cc a sixth time, coloring the upper-case C with the unique color different from that of the first five upper-case Cs, and doing the same with the lower-case C (note: we allow the sixth upper-case C and lower-case c to be the same color). Note that because the colors on the last Cc are forced, and any forced coloring of them is admissible, our problem is equivalent to coloring these six pairs.

There are $6!$ ways for Julia to color the upper-case Cs. We have two cases for coloring the lower-case Cs:

- *Case 1:* the last pair of Cs use two different colors. In this case, all six lower-case Cs have a different color to their associated upper-case C, and in addition the six lower-case Cs all use each color exactly once. In other words, we have a derangement* of the six colors, based on the colors of the upper-case Cs. We calculate $D_6 = 265$ ways to color the lower-case Cs here.
- *Case 2:* the last pair of Cs have both Cs the same color. Then, the color of the last lower-case C is forced, and with the other five Cs we, in a similar way to before, have a derangement of the remaining five colors based on the colors of the first five lower-case Cs, so we have $D_5 = 44$ ways to finish the coloring.

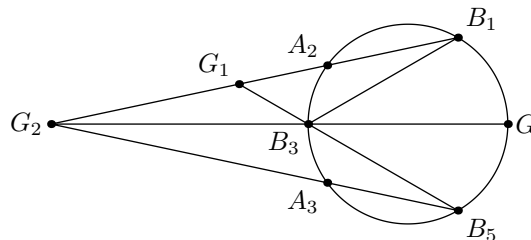
Our answer is thus $720(265 + 44) = 222480$.

*A **derangement** is a permutation π of the set $\{1, 2, \dots, n\}$ such that $\pi(k) \neq k$ for all k , i.e. there are no fixed points of the permutation. To calculate D_n , the number of derangements of an n -element set, we can use an inclusion-exclusion argument. There are $n!$ ways to permute the elements of the set. Now, we subtract the number of permutations with at least one fixed point, which is $\binom{n}{1}(n-1)! = \frac{n!}{1!}$, since we choose a fixed point, then permute the other $n-1$ elements. Correcting for overcounting, we add back the number of permutations with at least two fixed points, which is $\binom{n}{2}(n-2)! = \frac{n!}{2!}$. Continuing in this fashion by use of the principle of inclusion-exclusion, we get

$$D_n = n! \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{(-1)^n}{n!} \right).$$

8. [4] Let $G, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, B_5$ be ten points on a circle such that $GA_1A_2A_3A_4$ is a regular pentagon and $GB_1B_2B_3B_4B_5$ is a regular hexagon, and B_1 lies on minor arc GA_1 . Let B_5B_3 intersect B_1A_2 at G_1 , and let B_5A_3 intersect GB_3 at G_2 . Determine the degree measure of $\angle GG_2G_1$.

Answer: 12°

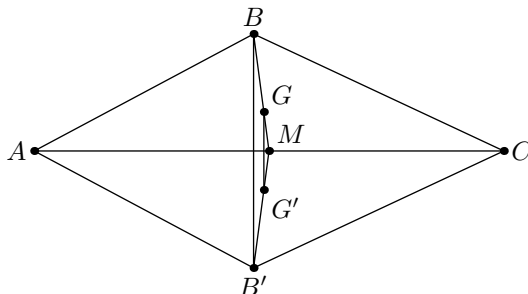


Note that GB_3 is a diameter of the circle. As a result, A_2, A_3 are symmetric with respect to GB_3 , as are B_1, B_5 . Therefore, B_1A_2 and B_5A_3 intersect along line GB_3 , so in fact, B_1, A_2, G_1, G_2 are collinear. We now have

$$\angle GG_2G_1 = \angle GG_2B_1 = \frac{\widehat{GB_1} - \widehat{B_3A_2}}{2} = \frac{60^\circ - 36^\circ}{2} = 12^\circ.$$

9. [4] Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B over the line CA . Let G be the centroid of triangle ABC , and let G' be the centroid of triangle $AB'C$. Determine the length of segment GG' .

Answer: $\boxed{\frac{48}{17}}$



Let M be the midpoint of AC . For any triangle, we know that the centroid is located $2/3$ of the way from the vertex, so we have $MG/MB = MG'/MB' = 1/3$, and it follows that $MGG' \sim MBB'$. Thus, $GG' = BB'/3$. However, note that BB' is twice the altitude to AC in triangle ABC . To finish, we calculate the area of ABC in two different ways. By Heron's Formula, we have

$$[ABC] = \sqrt{18(18-9)(18-10)(18-17)} = 36,$$

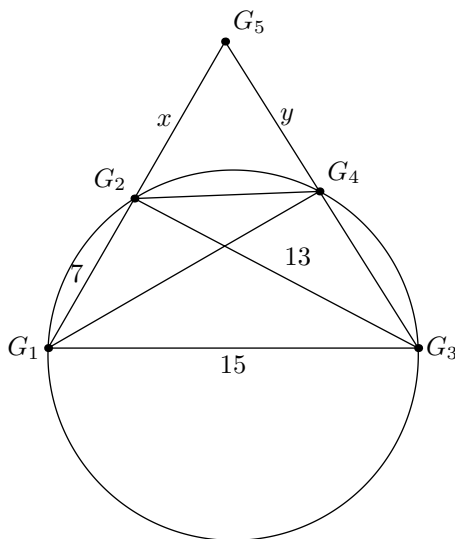
and we also have

$$[ABC] = \frac{1}{4}BB' \cdot AC = \frac{17}{4}(BB'),$$

from which it follows that $GG' = BB'/3 = 48/17$.

10. [8] Let $G_1G_2G_3$ be a triangle with $G_1G_2 = 7$, $G_2G_3 = 13$, and $G_3G_1 = 15$. Let G_4 be a point outside triangle $G_1G_2G_3$ so that ray $\overrightarrow{G_1G_4}$ cuts through the interior of the triangle, $G_3G_4 = G_4G_2$, and $\angle G_3G_1G_4 = 30^\circ$. Let G_3G_4 and G_1G_2 meet at G_5 . Determine the length of segment G_2G_5 .

Answer: $\boxed{\frac{169}{23}}$



We first show that quadrilateral $G_1G_2G_4G_3$ is cyclic. Note that by the law of cosines,

$$\cos \angle G_2 G_1 G_3 = \frac{7^2 + 15^2 - 13^2}{2 \cdot 7 \cdot 15} = \frac{1}{2},$$

so $\angle G_2 G_1 G_3 = 60^\circ$. However, we know that $\angle G_3 G_1 G_4 = 30^\circ$, so $G_1 G_4$ is an angle bisector. Now, let $G_1 G_4$ intersect the circumcircle of triangle $G_1 G_2 G_3$ at X . Then, the minor arcs $\widehat{G_2 X}$ and $\widehat{G_3 X}$ are subtended by the equal angles $\angle G_2 G_1 X$ and $\angle G_3 G_1 X$, implying that $G_2 X = G_3 X$, i.e. X is on the perpendicular bisector of $G_2 G_3$, l . Similarly, since $G_4 G_2 = G_4 G_3$, G_4 lies on l . However, since l and $G_1 G_4$ are distinct (in particular, G_1 lies on $G_1 G_4$ but not l), we in fact have $X = G_4$, so $G_1 G_2 G_4 G_3$ is cyclic.

We now have $G_5 G_2 G_4 \sim G_5 G_3 G_1$ since $G_1 G_2 G_4 G_3$ is cyclic. Now, we have $\angle G_4 G_3 G_2 = \angle G_4 G_1 G_2 = 30^\circ$, and we may now compute $G_2 G_4 = G_4 G_3 = 13/\sqrt{3}$. Let $G_5 G_2 = x$ and $G_5 G_4 = y$. Now, from $G_5 G_4 G_2 \sim G_5 G_1 G_3$, we have:

$$\frac{x}{y + 13/\sqrt{3}} = \frac{13/\sqrt{3}}{15} = \frac{y}{x + 7}.$$

Equating the first and second expressions and cross-multiplying, we get

$$y + \frac{13\sqrt{3}}{3} = \frac{15\sqrt{3}x}{13}.$$

Now, equating the first and third expressions and substituting gives

$$\left(\frac{15\sqrt{3}x}{13} - \frac{13\sqrt{3}}{3} \right) \left(\frac{15\sqrt{3}x}{13} \right) = x(x + 7).$$

Upon dividing both sides by x , we obtain a linear equation from which we can solve to get $x = 169/23$.

4th Annual Harvard-MIT November Tournament
Saturday 12 November 2011
Theme Round

1. [3] Five of James' friends are sitting around a circular table to play a game of Fish. James chooses a place between two of his friends to pull up a chair and sit. Then, the six friends divide themselves into two disjoint teams, with each team consisting of three consecutive players at the table. If the order in which the three members of a team sit does not matter, how many possible (unordered) pairs of teams are possible?

Answer: $\boxed{5}$ Note that the team not containing James must consist of three consecutive players who are already seated. We have 5 choices for the player sitting furthest clockwise on the team of which James is not a part. The choice of this player uniquely determines the teams, so we have a total of 5 possible pairs.

2. [3] In a game of Fish, R2 and R3 are each holding a positive number of cards so that they are collectively holding a total of 24 cards. Each player gives an integer estimate for the number of cards he is holding, such that each estimate is an integer between 80% of his actual number of cards and 120% of his actual number of cards, inclusive. Find the smallest possible sum of the two estimates.

Answer: $\boxed{20}$ To minimize the sum, we want each player to say an estimate as small as possible—i.e. an estimate as close to 80% of his actual number of cards as possible. We claim that the minimum possible sum is 20.

First, this is achievable when R2 has 10 cards and estimates 8, and when R3 has 14 cards and estimates 12.

Then, suppose that R2 has x cards and R3 has $24 - x$. Then, the sum of their estimates is

$$\left\lceil \frac{4}{5}(x) \right\rceil + \left\lceil \frac{4}{5}(24 - x) \right\rceil \geq \left\lceil \frac{4}{5}(x) + \frac{4}{5}(24 - x) \right\rceil \geq \left\lceil \frac{4}{5}(24) \right\rceil \geq 20.$$

Note: We use the fact that for all real numbers a, b , $\lceil a \rceil + \lceil b \rceil \geq \lceil a + b \rceil$.

3. [5] In preparation for a game of Fish, Carl must deal 48 cards to 6 players. For each card that he deals, he runs through the entirety of the following process:
1. He gives a card to a random player.
 2. A player Z is randomly chosen from the set of players who have at least as many cards as every other player (i.e. Z has the most cards or is tied for having the most cards).
 3. A player D is randomly chosen from the set of players other than Z who have at most as many cards as every other player (i.e. D has the fewest cards or is tied for having the fewest cards).
 4. Z gives one card to D .

He repeats steps 1-4 for each card dealt, including the last card. After all the cards have been dealt, what is the probability that each player has exactly 8 cards?

Answer: $\boxed{\frac{5}{6}}$ After any number of cards are dealt, we see that the difference between the number of cards that any two players hold is at most one. Thus, after the first 47 cards have been dealt, there is only one possible distribution: there must be 5 players with 8 cards and 1 player with 7 cards. We have two cases:

- Carl gives the last card to the player with 7 cards. Then, this player must give a card to another, leading to a uneven distribution of cards.
- Carl gives the last card to a player already with 8 cards. Then, that player must give a card to another; however, our criteria specify that he can only give it to the player with 7 cards, leading to an even distribution.

The probability of the second case happening, as Carl deals at random, is $\frac{5}{6}$.

4. [6] Toward the end of a game of Fish, the 2 through 7 of spades, inclusive, remain in the hands of three distinguishable players: DBR, RB, and DB, such that each player has at least one card. If it is known that DBR either has more than one card or has an even-numbered spade, or both, in how many ways can the players' hands be distributed?

Answer: 450 First, we count the number of distributions where each player has at least 1 card. The possible distributions are:

- *Case 1:* 4/1/1: There are 3 choices for who gets 4 cards, 6 choices for the card that one of the single-card players holds, and 5 choices for the card the other single-card player holds, or $3 \times 6 \times 5 = 90$ choices.
- *Case 2:* 3/2/1: There are 6 choices for the single card, $\binom{5}{2} = 10$ choices for the pair of cards, and $3! = 6$ choices for which player gets how many cards, for a total of $6 \times 10 \times 6 = 360$ choices.
- *Case 3:* 2/2/2: There are $\binom{6}{2} = 15$ choices for the cards DBR gets, $\binom{4}{2} = 6$ for the cards that RB gets, and DB gets the remaining two cards. This gives a total of $15 \times 6 = 90$ choices.

Thus, we have a total of $90 + 360 + 90 = 540$ ways for the cards to be distributed so that each person holds at least one.

Next, we look at the number of ways that the condition cannot be satisfied if each player has at least one card. Then, DBR must have no more than one card, and cannot have an even spade. We only care about cases where he has a non-zero number of cards, so he must have exactly 1 odd spade. Then, we see that there are $2^5 - 2 = 30$ ways to distribute the other 5 cards among RB and DB so that neither has 0 cards. Since there are 3 odd spades, this gives 3×30 bad cases, so we have $540 - 90 = 450$ ones where all the problem conditions hold.

5. [8] For any finite sequence of positive integers π , let $S(\pi)$ be the number of strictly increasing sub-sequences in π with length 2 or more. For example, in the sequence $\pi = \{3, 1, 2, 4\}$, there are five increasing sub-sequences: $\{3, 4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, and $\{1, 2, 4\}$, so $S(\pi) = 5$. In an eight-player game of Fish, Joy is dealt six cards of distinct values, which she puts in a random order π from left to right in her hand. Determine

$$\sum_{\pi} S(\pi),$$

where the sum is taken over all possible orders π of the card values.

Answer: 8287 For each subset of Joy's set of cards, we compute the number of orders of cards in which the cards in the subset are arranged in increasing order. When we sum over all subsets of Joy's cards, we will obtain the desired sum.

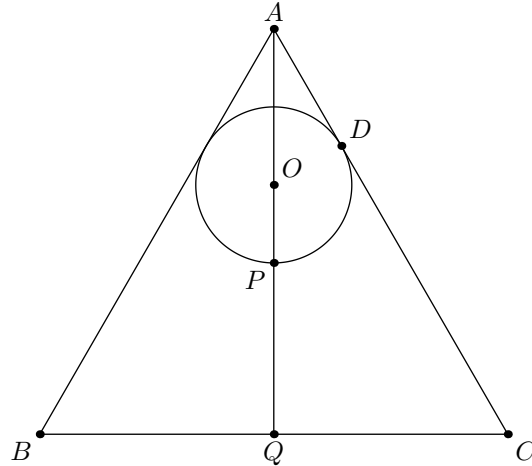
Consider any subset of k cards. The probability that they are arranged in increasing order is precisely $1/k!$ (since we can form a $k!$ -to-1 correspondence between all possible orders and orders in which the cards in our subset are in increasing order), and there are $6! = 720$ total arrangements so exactly $720/k!$ of them give an increasing subsequence in the specified cards. Now for any for $k = 2, 3, 4, 5, 6$, we have $\binom{6}{k}$ subsets of k cards, so we sum to get

$$\sum_{\pi} S(\pi) = \sum_{k=2}^6 \binom{6}{k} \cdot \frac{6!}{k!} = 8287.$$

6. [3] Let ABC be an equilateral triangle with $AB = 3$. Circle ω with diameter 1 is drawn inside the triangle such that it is tangent to sides AB and AC . Let P be a point on ω and Q be a point on segment BC . Find the minimum possible length of the segment PQ .

Answer: $\boxed{\frac{3\sqrt{3}-3}{2}}$ Let P, Q , be the points which minimize the distance. We see that we want both to lie on the altitude from A to BC . Hence, Q is the foot of the altitude from A to BC and $AQ = \frac{3\sqrt{3}}{2}$. Let O , which must also lie on this line, be the center of ω , and let D be the point of tangency between ω and AC . Then, since $OD = \frac{1}{2}$, we have $AO = 2OD = 1$ because $\angle OAD = 30^\circ$, and $OP = \frac{1}{2}$. Consequently,

$$PQ = AQ - AO - OP = \frac{3\sqrt{3}-3}{2}.$$



7. [4] Let XYZ be a triangle with $\angle XYZ = 40^\circ$ and $\angle YZX = 60^\circ$. A circle Γ , centered at the point I , lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of Γ with YZ , and let ray \overrightarrow{XI} intersect side YZ at B . Determine the measure of $\angle AIB$.

Answer: $\boxed{10^\circ}$ Let D be the foot of the perpendicular from X to YZ . Since I is the incenter and A the point of tangency, $IA \perp YZ$, so

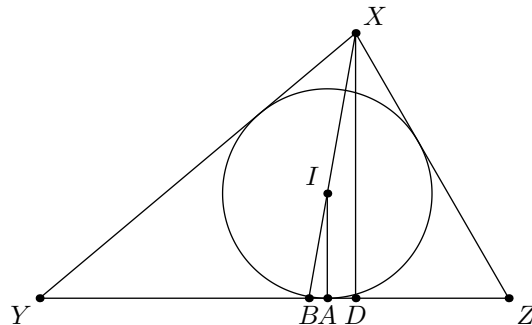
$$AI \parallel XD \Rightarrow \angle AIB = \angle DXB.$$

Since I is the incenter,

$$\angle BXZ = \frac{1}{2}\angle YXZ = \frac{1}{2}(180^\circ - 40^\circ - 60^\circ) = 40^\circ.$$

Consequently, we get that

$$\angle AIB = \angle DXB = \angle ZXB - \angle ZXD = 40^\circ - (90^\circ - 60^\circ) = 10^\circ.$$

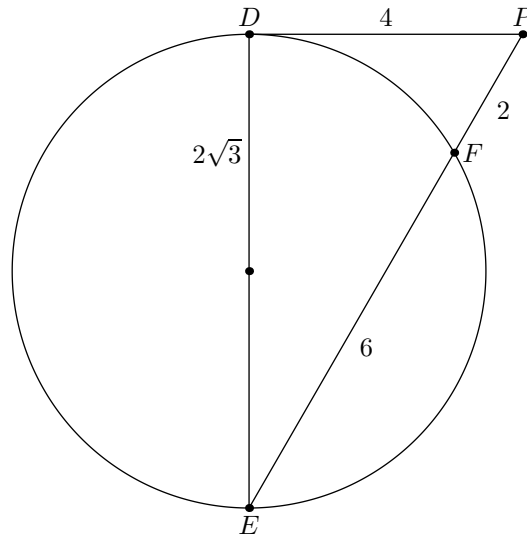


8. [5] Points D, E, F lie on circle O such that the line tangent to O at D intersects ray \overrightarrow{EF} at P . Given that $PD = 4$, $PF = 2$, and $\angle FPD = 60^\circ$, determine the area of circle O .

Answer: $\boxed{12\pi}$ By the power of a point on P , we get that

$$16 = PD^2 = (PF)(PE) = 2(PE) \Rightarrow PE = 8.$$

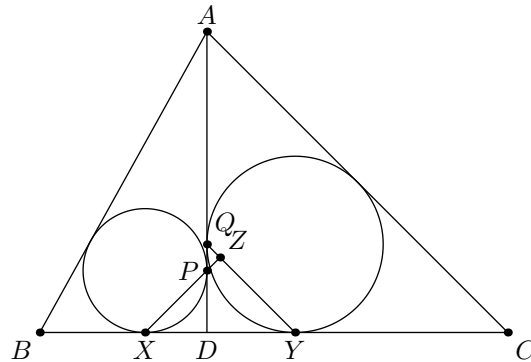
However, since $PE = 2PD$ and $\angle FPD = 60^\circ$, we notice that PDE is a $30 - 60 - 90$ triangle, so $DE = 4\sqrt{3}$ and we have $ED \perp DP$. It follows that DE is a diameter of the circle, since tangents the tangent at D must be perpendicular to the radius containing D . Hence, the area of the circle is $(\frac{1}{2}DE)^2\pi = 12\pi$.



9. [6] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC . The inscribed circles of triangles ABD and ACD are tangent to AD at P and Q , respectively, and are tangent to BC at X and Y , respectively. Let PX and QY meet at Z . Determine the area of triangle XYZ .

Answer: $\boxed{\frac{25}{4}}$ First, note that $AD = 12$, $BD = 5$, $CD = 9$.

By equal tangents, we get that $PD = DX$, so PDX is isosceles. Because D is a right angle, we get that $\angle PXD = 45^\circ$. Similarly, $\angle QYZ = 45^\circ$, so XYZ is an isosceles right triangle with hypotenuse XY . However, by tangents to the incircle, we get that $XD = \frac{1}{2}(12+5-13) = 2$ and $YD = \frac{1}{2}(12+9-15) = 3$. Hence, the area of the XYZ is $\frac{1}{4}(XY)^2 = \frac{1}{4}(2+3)^2 = \frac{25}{4}$.

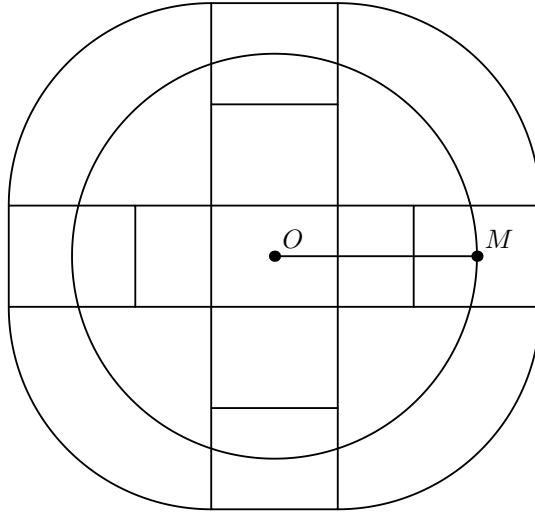


10. [7] Let Ω be a circle of radius 8 centered at point O , and let M be a point on Ω . Let S be the set of points P such that P is contained within Ω , or such that there exists some rectangle $ABCD$ containing P whose center is on Ω with $AB = 4$, $BC = 5$, and $BC \parallel OM$. Find the area of S .

Answer: $164 + 64\pi$ We wish to consider the union of all rectangles $ABCD$ with $AB = 4$, $BC = 5$, and $BC \parallel OM$, with center X on Ω . Consider translating rectangle $ABCD$ along the radius XO to a rectangle $A'B'C'D'$ now centered at O . It is now clear that every point inside $ABCD$ is a translate of a point in $A'B'C'D'$, and furthermore, any rectangle $ABCD$ translates along the appropriate radius to the same rectangle $A'B'C'D'$.

We see that the boundary of this region can be constructed by constructing a quarter-circle at each vertex, then connecting these quarter-circles with tangents to form four rectangular regions. Now, splitting our region in to four quarter circles and five rectangles, we compute the desired area to be

$$4 \cdot \frac{1}{4}(8)^2\pi + 2(4 \cdot 8) + 2(5 \cdot 8) + (4 \cdot 5) = 164 + 64\pi.$$



13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
Algebra Subject Test

1. [3] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

Find x .

Answer: $\boxed{\frac{3+\sqrt{17}}{2}}$ Squaring both sides of $x - y^2 = 3$ gives $x^2 + y^4 - 2xy^2 = 9$. Subtract this equation from twice the second given to get $x^2 + 2xy^2 + y^4 = 17 \implies x + y^2 = \pm 17$. Combining this equation with the first given, we see that $x = \frac{3 \pm \sqrt{17}}{2}$. Since x is a positive real, x must be $\frac{3+\sqrt{17}}{2}$.

2. [3] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \dots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \dots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3) .

Answer: $\boxed{(5, 21, 421)}$ Suppose that A and B were rational numbers of rank 3 less than $\frac{1}{4}$, and let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive integers so that $A = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$ and $B = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}$ are the expressions for A and B as stated in the problem. If $b_1 < a_1$ then $A < \frac{1}{a_1-1} \leq \frac{1}{b_1} < B$. In other words, of all the rationals less than $\frac{1}{4}$ with rank 3, those that have $a_1 = 5$ are greater than those that have $a_1 = 6, 7, 8, \dots$. Therefore we can “build” q greedily, adding the largest unit fraction that keeps q less than $\frac{1}{4}$:

$\frac{1}{5}$ is the largest unit fraction less than $\frac{1}{4}$, hence $a_1 = 5$;

$\frac{1}{21}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5}$, hence $a_2 = 21$;

$\frac{1}{421}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5} - \frac{1}{21}$, hence $a_3 = 421$.

3. [4] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \dots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

Answer: $\boxed{1092}$ Suppose that $S_N = 0$ for some $N \geq 0$. Then $a_{N+1} = 1$ because $N + 1 \geq S_N$. The following table lists the values of a_k and S_k for a few $k \geq N$:

k	a_k	S_k
N		0
$N + 1$	1	$N + 1$
$N + 2$	1	$2N + 3$
$N + 3$	-1	N
$N + 4$	1	$2N + 4$
$N + 5$	-1	$N - 1$
$N + 6$	1	$2N + 5$
$N + 7$	-1	$N - 2$

We see inductively that, for every $i \geq 1$,

$$S_{N+2i} = 2N + 2 + i$$

and

$$S_{N+1+2i} = N + 1 - i$$

thus $S_{3N+3} = 0$ is the next k for which $S_k = 0$. The values of k for which $S_k = 0$ satisfy the recurrence relation $p_{n+1} = 3p_n + 3$, and we compute that the first terms of the sequence are 0, 3, 12, 39, 120, 363, 1092; hence 1092 is our answer.

4. [4] Suppose that there exist nonzero complex numbers $a, b, c,$ and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

Answer: $\boxed{1, -1, i, -i}$ Let k be a root of both polynomials. Multiplying the first polynomial by k and subtracting the second, we have $ak^4 - a = 0$, which means that k is either 1, $-1, i,$ or $-i$. If $a = b = c = d = 1$, then $-1, i,$ and $-i$ are roots of both polynomials. If $a = b = c = 1$ and $d = -3$, then 1 is a root of both polynomials. So k can be 1, $-1, i,$ and $-i$.

5. [5] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

Answer: $\boxed{-90}$ We have $x^2 + y^2 + 2xy = 1$. Define $a = 2xy$ and $b = x^2 + y^2$ for convenience. Then $a + b = 1$ and $b - a = x^2 + y^2 - 2xy = (x - y)^2 = 2b - 1$ so that $x, y = \frac{\sqrt{2b-1} \pm 1}{2}$. Then

$$\begin{aligned} x^{20} + y^{20} &= \left(\frac{\sqrt{2b-1} + 1}{2} \right)^{20} + \left(\frac{\sqrt{2b-1} - 1}{2} \right)^{20} \\ &= \frac{1}{2^{20}} [(\sqrt{2b-1} + 1)^{20} + (\sqrt{2b-1} - 1)^{20}] \\ &= \frac{2}{2^{20}} \left[(\sqrt{2b-1})^{20} + \binom{20}{2} (\sqrt{2b-1})^{18} + \binom{20}{4} (\sqrt{2b-1})^{16} + \dots \right] \\ &= \frac{2}{2^{20}} \left[(2b-1)^{10} + \binom{20}{2} (2b-1)^9 + \binom{20}{4} (2b-1)^8 + \dots \right] \\ &= 20 \end{aligned}$$

We want to find the sum of distinct roots of the above polynomial in b ; we first prove that the original polynomial is square-free. The conditions $x + y = 1$ and $x^{20} + y^{20} = 20$ imply that $x^{20} + (1-x)^{20} - 20 = 0$; let $p(x) = x^{20} + (1-x)^{20} - 20$. p is square-free if and only if $GCD(p, p') = c$ for some constant c :

$$\begin{aligned} GCD(p, p') &= GCD(x^{20} + (1-x)^{20} - 20, 20(x^{19} - (1-x)^{19})) \\ &= GCD(x^{20} - x(1-x)^{19} + (1-x)^{19} - 20, 20(x^{19} - (1-x)^{19})) \\ &= GCD((1-x)^{19} - 20, x^{19} - (1-x)^{19}) \\ &= GCD((1-x)^{19} - 20, x^{19} - 20) \end{aligned}$$

The roots of $x^{19} - 20$ are $\sqrt[19]{20^k} \exp\left(\frac{2\pi ik}{19}\right)$ for some $k = 0, 1, \dots, 18$; the roots of $(1-x)^{19} - 20$ are $1 - \sqrt[19]{20^k} \exp\left(\frac{2\pi ik}{19}\right)$ for some $k = 0, 1, \dots, 18$. If $x^{19} - 20$ and $(1-x)^{19} - 20$ share a common root, then there exist integers m, n such that $\sqrt[19]{20^m} \exp\left(\frac{2\pi im}{19}\right) = 1 - \sqrt[19]{20^n} \exp\left(\frac{2\pi in}{19}\right)$; since the imaginary parts of both sides must be the same, we have $m = n$ and $\sqrt[19]{20^m} \exp\left(\frac{2\pi im}{19}\right) = \frac{1}{2} \implies 20^m = \frac{1}{2^{19}}$, a contradiction. Thus we have proved that the polynomial in x has no double roots. Since for each b there exists a unique pair (x, y) (up to permutations) that satisfies $x^2 + y^2 = b$ and $(x + y)^2 = 1$, the polynomial in b has no double roots.

Let the coefficient of b^n in the above equation be $[b^n]$. By Vieta's Formulas, the sum of all possible values of $b = x^2 + y^2$ is equal to $-\frac{[b^9]}{[b^{10}]}$. $[b^{10}] = \frac{2}{2^{20}} (2^{10})$ and $[b^9] = \frac{2}{2^{20}} \left(-\binom{10}{1} 2^9 + \binom{20}{2} 2^9\right)$; thus $-\frac{[b^9]}{[b^{10}]} = -\frac{\binom{10}{1} 2^9 - \binom{20}{2} 2^9}{2^{10}} = -90$.

6. [5] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \dots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p ?

Answer: $\boxed{1005}$ Let $p(x)$ be a polynomial with the maximum number of minus signs. $p(x)$ cannot have more than 1005 minus signs, otherwise $p(1) < 0$ and $p(2) \geq 2^{2010} - 2^{2009} - \dots - 2 - 1 = 1$, which implies, by the Intermediate Value Theorem, that p must have a root greater than 1.

Let $p(x) = \frac{x^{2011} + 1}{x + 1} = x^{2010} - x^{2009} + x^{2008} - \dots - x + 1$. -1 is the only real root of $x^{2011} + 1 = 0$ but $p(-1) = 2011$; therefore p has no real roots. Since p has 1005 minus signs, it is the desired polynomial.

7. [5] Let $a, b, c, x, y,$ and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz .

Answer: -5892 Manipulate the equations to get a common denominator: $a = \frac{b+c}{x-2} \implies x-2 = \frac{b+c}{a} \implies x-1 = \frac{a+b+c}{a} \implies \frac{1}{x-1} = \frac{a}{a+b+c}$; similarly, $\frac{1}{y-1} = \frac{b}{a+b+c}$ and $\frac{1}{z-1} = \frac{c}{a+b+c}$. Thus

$$\begin{aligned} \frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} &= 1 \\ (y-1)(z-1) + (x-1)(z-1) + (x-1)(y-1) &= (x-1)(y-1)(z-1) \\ xy + yz + zx - 2(x+y+z) + 3 &= xyz - (xy + yz + zx) + (x+y+z) - 1 \\ xyz - 2(xy + yz + zx) + 3(x+y+z) - 4 &= 0 \\ xyz - 2(67) + 3(2010) - 4 &= 0 \\ xyz &= -5892 \end{aligned}$$

8. [6] How many polynomials of degree exactly 5 with real coefficients send the set $\{1, 2, 3, 4, 5, 6\}$ to a permutation of itself?

Answer: 714 For every permutation σ of $\{1, 2, 3, 4, 5, 6\}$, Lagrange Interpolation¹ gives a polynomial of degree at most 5 with $p(x) = \sigma(x)$ for every $x = 1, 2, 3, 4, 5, 6$. Additionally, this polynomial is unique: assume that there exist two polynomials p, q of degree ≤ 5 such that they map $\{1, 2, 3, 4, 5, 6\}$ to the same permutation. Then $p - q$ is a nonzero polynomial of degree ≤ 5 with 6 distinct roots, a contradiction. Thus an upper bound for the answer is $6! = 720$ polynomials.

However, not every polynomial obtained by Lagrange interpolation is of degree 5 (for example, $p(x) = x$). We can count the number of invalid polynomials using finite differences.² A polynomial has degree less than 5 if and only if the sequence of 5th finite differences is 0. The 5th finite difference of $p(1), p(2), p(3), p(4), p(5), p(6)$ is $p(1) - 5p(2) + 10p(3) - 10p(4) + 5p(5) - p(6)$; thus we want to solve $p(1) - 5p(2) + 10p(3) - 10p(4) + 5p(5) - p(6) = 0$ with $\{p(1), p(2), p(3), p(4), p(5), p(6)\} = \{1, 2, 3, 4, 5, 6\}$.

Taking the above equation modulo 5, we get $p(1) = p(6) \pmod{5} \implies \{p(1), p(6)\} = \{1, 6\}$. Note that $1 - 5p(2) + 10p(3) - 10p(4) + 5p(5) - 6 = 0$ if and only if $6 - 5p(5) + 10p(4) - 10p(3) + 5p(2) - 1 = 0$, so we may assume that $p(1) = 1$ and double our result later. Then we have $\{p(2), p(3), p(4), p(5)\} = \{2, 3, 4, 5\}$ and

$$-p(2) + 2p(3) - 2p(4) + p(5) = 1.$$

The above equation taken modulo 2 implies that $p(2), p(5)$ are of opposite parity, so $p(3), p(4)$ are of opposite parity. We do casework on $\{p(2), p(5)\}$:

- (a) $p(2) = 2, p(5) = 3; 2p(3) - 2p(4) = 0$ is a contradiction
- (b) $p(2) = 2, p(5) = 5; 2p(3) - 2p(4) = -2 \implies p(3) - p(4) = -1 \implies p(3) = 3, p(4) = 4$
- (c) $p(2) = 3, p(5) = 2; 2p(3) - 2p(4) = -2 \implies p(3) - p(4) = -1 \implies p(3) = 4, p(4) = 5$
- (d) $p(2) = 3, p(5) = 4; 2p(3) - 2p(4) = 0$ is a contradiction
- (e) $p(2) = 4, p(5) = 3; 2p(3) - 2p(4) = 2 \implies p(3) - p(4) = 1$ but $\{p(3), p(4)\} = \{2, 5\}$, contradiction
- (f) $p(2) = 4, p(5) = 5; 2p(3) - 2p(4) = 0$ is a contradiction
- (g) $p(2) = 5, p(5) = 2; 2p(3) - 2p(4) = 4 \implies p(3) - p(4) = 2$, contradiction
- (h) $p(2) = 5, p(5) = 4; 2p(3) - 2p(4) = 2 \implies p(3) - p(4) = 1 \implies p(3) = 3, p(4) = 2$

Hence there are a total of $720 - 2(3) = 714$ polynomials.

¹See http://en.wikipedia.org/wiki/Lagrange_interpolation.

²See http://www.artofproblemsolving.com/Forum/weblog_entry.php?p=1263378.

9. [7] Let $f(x) = cx(x-1)$, where c is a positive real number. We use $f^n(x)$ to denote the polynomial obtained by composing f with itself n times. For every positive integer n , all the roots of $f^n(x)$ are real. What is the smallest possible value of c ?

Answer: 2 We first prove that all roots of $f^n(x)$ are greater than or equal to $-\frac{c}{4}$ and less than or equal to $1 + \frac{c}{4}$. Suppose that r is a root of $f^n(x)$. If $r = -\frac{c}{4}$, $f^{-1}(r) = \{\frac{1}{2}\}$ and $-\frac{c}{4} < \frac{1}{2} < 1 + \frac{c}{4}$ since c is positive. Suppose $r \neq -\frac{c}{4}$; by the quadratic formula, there exist two complex numbers r_1, r_2 such that $r_1 + r_2 = 1$ and $f(r_1) = f(r_2) = r$. Thus all the roots of $f^n(x)$ (except $\frac{1}{2}$) come in pairs that sum to 1. No root r of $f^n(x)$ can be less than $-\frac{c}{4}$, otherwise $f^{n+1}(x)$ has an imaginary root, $f^{-1}(r)$. Also, no root r of $f^n(x)$ can be greater than $1 + \frac{c}{4}$, otherwise its "conjugate" root will be less than $-\frac{c}{4}$.

Define $g(x) = \frac{1}{2} \left(1 + \sqrt{1 + \frac{4x}{c}} \right)$, the larger inverse of $f(x)$. Note that $g^n(x)$ is the largest element of $f^{-n}(x)$ (which is a set). $g^n(0)$ should be less than or equal to $1 + \frac{c}{4}$ for all n . Let x_0 be the nonzero real number such that $g(x_0) = x_0$; then $cx_0(x_0 - 1) = x_0 \implies x_0 = 1 + \frac{1}{c}$. $x_0 < g(x) < x$ if $x > x_0$ and $x < g(x) < x_0$ if $x < x_0$; it can be proved that g^n converges to x_0 . Hence we have the requirement that $x_0 = 1 + \frac{1}{c} \leq 1 + \frac{c}{4} \implies c \geq 2$.

We verify that $c = 2$ is possible. All the roots of $f^{-n}(x)$ will be real if $g(0) \leq 1 + \frac{c}{4} = \frac{3}{2}$. We know that $0 < \frac{3}{2} \implies g(0) < \frac{3}{2}$, so $g^2(0) < \frac{3}{2}$ and $g^n(0) < g^{n+1}(0) < \frac{3}{2}$ for all n . Therefore all the roots of $f^n(x)$ are real.

10. [8] Let $p(x)$ and $q(x)$ be two cubic polynomials such that $p(0) = -24$, $q(0) = 30$, and

$$p(q(x)) = q(p(x))$$

for all real numbers x . Find the ordered pair $(p(3), q(6))$.

Answer: (3, -24) Note that the polynomials $f(x) = ax^3$ and $g(x) = -ax^3$ commute under composition. Let $h(x) = x + b$ be a linear polynomial, and note that its inverse $h^{-1}(x) = x - b$ is also a linear polynomial. The composite polynomials $h^{-1}fh$ and $h^{-1}gh$ commute, since function composition is associative, and these polynomials are also cubic.

We solve for the a and b such that $(h^{-1}fh)(0) = -24$ and $(h^{-1}gh)(0) = 30$. We must have:

$$ab^3 - b = -24, \quad -ab^3 - b = 30 \implies a = 1, b = -3$$

These values of a and b yield the polynomials $p(x) = (x-3)^3 + 3$ and $q(x) = -(x-3)^3 + 3$. The polynomials take on the values $p(3) = 3$ and $q(6) = -24$.

Remark: The pair of polynomials found in the solution is not unique. There is, in fact, an entire family of commuting cubic polynomials with $p(0) = -24$ and $q(0) = 30$. They are of the form

$$p(x) = tx(x-3)(x-6) - 24, \quad q(x) = -tx(x-3)(x-6) + 30$$

where t is any real number. However, the values of $p(3)$ and $q(6)$ are the same for all polynomials in this family. In fact, if we give the initial conditions $p(0) = k_1$ and $q(0) = k_2$, then we get a general solution of

$$p(x) = t \left(x^3 - \frac{3}{2}(k_1 + k_2)x^2 + \frac{1}{2}(k_1 + k_2)^2 x \right) + \frac{k_2 - k_1}{k_2 + k_1} x + k_1$$

$$q(x) = -t \left(x^3 - \frac{3}{2}(k_1 + k_2)x^2 + \frac{1}{2}(k_1 + k_2)^2 x \right) - \frac{k_2 - k_1}{k_2 + k_1} x + k_2.$$

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
Calculus Subject Test

1. [3] Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

Answer: 50 Observe that $p(x)$ must be quadratic. Let $p(x) = ax^2 + bx + c$. Comparing coefficients gives $a = 1$, $b - 2a = 2$, and $c - b = 1$. So $b = 4$, $c = 5$, $p(x) = x^2 + 4x + 5$ and $p(5) = 25 + 20 + 5 = 50$.

2. [3] Let f be a function such that $f(0) = 1$, $f'(0) = 2$, and

$$f''(t) = 4f'(t) - 3f(t) + 1$$

for all t . Compute the 4th derivative of f , evaluated at 0.

Answer: 54 Putting $t = 0$ gives $f''(0) = 6$. By differentiating both sides, we get $f^{(3)}(t) = 4f''(t) - 3f'(t)$ and $f^{(3)}(0) = 4 \cdot 6 - 3 \cdot 2 = 18$. Similarly, $f^{(4)}(t) = 4f^{(3)}(t) - 3f''(t)$ and $f^{(4)}(0) = 4 \cdot 18 - 3 \cdot 6 = 54$.

3. [4] Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeros of $p'(x)$ are also zeros of $p(x)$. Find p . Note: monic means that the leading coefficient is 1.

Answer: $(x + 1)^3$ A root of a polynomial p will be a double root if and only if it is also a root of p' . Let a and b be the roots of p' . Since a and b are also roots of p , they are double roots of p . But p can have only three roots, so $a = b$ and a becomes a double root of p' . This makes $p'(x) = 3c(x - a)^2$ for some constant $3c$, and thus $p(x) = c(x - a)^3 + d$. Because a is a root of p and p is monic, $d = 0$ and $c = 1$. From $p(0) = 1$ we get $p(x) = (x + 1)^3$.

4. [4] Compute $\lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n |\cos(k)|}{n}$.

Answer: $\frac{2}{\pi}$ The main idea lies on the fact that positive integers are uniformly distributed modulo π . (In the other words, if each integer n is written as $q\pi + r$ where q is an integer and $0 \leq r < \pi$, the value of r will distribute uniformly in the interval $[0, \pi]$.) Using this fact, the summation is equivalent to the average value (using the Riemann summation) of the function $|\cos(k)|$ over the interval $[0, \pi]$. Therefore, the answer is $\frac{1}{\pi} \int_0^\pi |\cos(k)| = \frac{2}{\pi}$.

5. [4] Let the functions $f(\alpha, x)$ and $g(\alpha)$ be defined as

$$f(\alpha, x) = \frac{\left(\frac{x}{2}\right)^\alpha}{x-1} \qquad g(\alpha) = \left. \frac{d^4 f}{dx^4} \right|_{x=2}$$

Then $g(\alpha)$ is a polynomial in α . Find the leading coefficient of $g(\alpha)$.

Answer: $\frac{1}{16}$ Write the first equation as $(x - 1)f = \left(\frac{x}{2}\right)^\alpha$. For now, treat α as a constant. From this equation, repeatedly applying derivative with respect to x gives

$$\begin{aligned} (x-1)f' + f &= \left(\frac{\alpha}{2}\right) \left(\frac{x}{2}\right)^{\alpha-1} \\ (x-1)f'' + 2f' &= \left(\frac{\alpha}{2}\right) \left(\frac{\alpha-1}{2}\right) \left(\frac{x}{2}\right)^{\alpha-2} \\ (x-1)f^{(3)} + 3f'' &= \left(\frac{\alpha}{2}\right) \left(\frac{\alpha-1}{2}\right) \left(\frac{\alpha-2}{2}\right) \left(\frac{x}{2}\right)^{\alpha-3} \\ (x-1)f^{(4)} + 4f^{(3)} &= \left(\frac{\alpha}{2}\right) \left(\frac{\alpha-1}{2}\right) \left(\frac{\alpha-2}{2}\right) \left(\frac{\alpha-3}{2}\right) \left(\frac{x}{2}\right)^{\alpha-4} \end{aligned}$$

Substituting $x = 2$ to all equations gives $g(\alpha) = f^{(4)}(\alpha, 2) = \left(\frac{\alpha}{2}\right) \left(\frac{\alpha-1}{2}\right) \left(\frac{\alpha-2}{2}\right) \left(\frac{\alpha-3}{2}\right) - 4f^{(3)}(\alpha, 2)$. Because $f^{(3)}(\alpha, 2)$ is a cubic polynomial in α , the leading coefficient of $g(\alpha)$ is $\frac{1}{16}$.

6. [5] Let $f(x) = x^3 - x^2$. For a given value of c , the graph of $f(x)$, together with the graph of the line $c + x$, split the plane up into regions. Suppose that c is such that exactly two of these regions have finite area. Find the value of c that minimizes the sum of the areas of these two regions.

Answer: $\boxed{-\frac{11}{27}}$ Observe that $f(x)$ can be written as $(x - \frac{1}{3})^3 - \frac{1}{3}(x - \frac{1}{3}) - \frac{2}{27}$, which has 180° symmetry around the point $(\frac{1}{3}, -\frac{2}{27})$. Suppose the graph of f cuts the line $y = c + x$ into two segments of lengths a and b . When we move the line toward point P with a small distance Δx (measured along the line perpendicular to $y = x + c$), the sum of the enclosed areas will increase by $|a - b|(\Delta x)$. As long as the line $x + c$ does not pass through P , we can find a new line $x + c^*$ that increases the sum of the enclosed areas. Therefore, the sum of the areas reaches its maximum when the line passes through P . For that line, we can find that $c = y - x = -\frac{2}{27} - \frac{1}{3} = -\frac{11}{27}$.

7. [6] Let a_1, a_2 , and a_3 be nonzero complex numbers with non-negative real and imaginary parts. Find the minimum possible value of

$$\frac{|a_1 + a_2 + a_3|}{\sqrt[3]{|a_1 a_2 a_3|}}.$$

Answer: $\boxed{\sqrt{3}\sqrt[3]{2}}$ Write a_1 in its polar form $re^{i\theta}$ where $0 \leq \theta \leq \frac{\pi}{2}$. Suppose a_2, a_3 and r are fixed so that the denominator is constant. Write $a_2 + a_3$ as $se^{i\phi}$. Since a_2 and a_3 have non-negative real and imaginary parts, the angle ϕ lies between 0 and $\frac{\pi}{2}$. Consider the function

$$f(\theta) = |a_1 + a_2 + a_3|^2 = |re^{i\theta} + se^{i\phi}|^2 = r^2 + 2rs \cos(\theta - \phi) + s^2.$$

Its second derivative is $f''(\theta) = -2rs(\cos(\theta - \phi))$. Since $-\frac{\pi}{2} \leq (\theta - \phi) \leq \frac{\pi}{2}$, we know that $f''(\theta) < 0$ and f is concave. Therefore, to minimize f , the angle θ must be either 0 or $\frac{\pi}{2}$. Similarly, each of a_1, a_2 and a_3 must be either purely real or purely imaginary to minimize f and the original fraction.

By the AM-GM inequality, if a_1, a_2 and a_3 are all real or all imaginary, then the minimum value of the fraction is 3. Now suppose only two of the a_i 's, say, a_1 and a_2 are real. Since the fraction is homogenous, we may fix $a_1 + a_2$ - let the sum be 2. The term $a_1 a_2$ in the denominator achieves its maximum only when a_1 and a_2 are equal, i.e. when $a_1 = a_2 = 1$. Then, if $a_3 = ki$ for some real number k , then the expression equals

$$\frac{\sqrt{k^2 + 4}}{\sqrt[3]{k}}.$$

Squaring and taking the derivative, we find that the minimum value of the fraction is $\sqrt{3}\sqrt[3]{2}$, attained when $k = \sqrt{2}$. With similar reasoning, the case where only one of the a_i 's is real yields the same minimum value.

8. [6] Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Calculate $\sum_{n=2}^{\infty} f(n)$.

Answer: $\boxed{3 - e}$

$$\begin{aligned} \sum_{n=2}^{\infty} f(n) &= \sum_{k=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{k^n \cdot k!} \\ &= \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{n=2}^{\infty} \frac{1}{k^n} \\ &= \sum_{k=2}^{\infty} \frac{1}{k!} \cdot \frac{1}{k(k-1)} \\ &= \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \cdot \frac{1}{k^2(k-1)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=2}^{\infty} \frac{1}{(k-1)!} \left(\frac{1}{k-1} - \frac{1}{k^2} - \frac{1}{k} \right) \\
&= \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)(k-1)!} - \frac{1}{k \cdot k!} - \frac{1}{k!} \right) \\
&= \sum_{k=2}^{\infty} \left(\frac{1}{(k-1)(k-1)!} - \frac{1}{k \cdot k!} \right) - \sum_{k=2}^{\infty} \frac{1}{k!} \\
&= \frac{1}{1 \cdot 1!} - \left(e - \frac{1}{0!} - \frac{1}{1!} \right) \\
&= 3 - e
\end{aligned}$$

9. [7] Let $x(t)$ be a solution to the differential equation

$$(x + x')^2 + x \cdot x'' = \cos t$$

with $x(0) = x'(0) = \sqrt{\frac{2}{5}}$. Compute $x\left(\frac{\pi}{4}\right)$.

Answer: $\sqrt[4]{\frac{450}{5}}$ Rewrite the equation as $x^2 + 2xx' + (xx')' = \cos t$. Let $y = x^2$, so $y' = 2xx'$ and the equation becomes $y + y' + \frac{1}{2}y'' = \cos t$. The term $\cos t$ suggests that the particular solution should be in the form $A \sin t + B \cos t$. By substitution and coefficient comparison, we get $A = \frac{4}{5}$ and $B = \frac{2}{5}$. Since the function $y(t) = \frac{4}{5} \sin t + \frac{2}{5} \cos t$ already satisfies the initial conditions $y(0) = x(0)^2 = \frac{2}{5}$ and $y'(0) = 2x(0)x'(0) = \frac{4}{5}$, the function y also solves the initial value problem. Note that since x is positive at $t = 0$ and $y = x^2$ never reaches zero before t reaches $\frac{\pi}{4}$, the value of $x\left(\frac{\pi}{4}\right)$ must be positive. Therefore, $x\left(\frac{\pi}{4}\right) = +\sqrt{y\left(\frac{\pi}{4}\right)} = \sqrt{\frac{6}{5} \cdot \frac{\sqrt{2}}{2}} = \sqrt[4]{\frac{450}{5}}$.

10. [8] Let $f(n) = \sum_{k=1}^n \frac{1}{k}$. Then there exists constants γ , c , and d such that

$$f(n) = \ln(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + O\left(\frac{1}{n^3}\right),$$

where the $O\left(\frac{1}{n^3}\right)$ means terms of order $\frac{1}{n^3}$ or lower. Compute the ordered pair (c, d) .

Answer: $\left(\frac{1}{2}, -\frac{1}{12}\right)$ From the given formula, we pull out the term $\frac{k}{n^3}$ from $O\left(\frac{1}{n^4}\right)$, making $f(n) = \log(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + \frac{k}{n^3} + O\left(\frac{1}{n^4}\right)$. Therefore,

$$f(n+1) - f(n) = \log\left(\frac{n+1}{n}\right) - c\left(\frac{1}{n} - \frac{1}{n+1}\right) - d\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) - k\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right) + O\left(\frac{1}{n^4}\right).$$

For the left hand side, $f(n+1) - f(n) = \frac{1}{n+1}$. By substituting $x = \frac{1}{n}$, the formula above becomes

$$\frac{x}{x+1} = \log(1+x) - cx^2 \cdot \frac{1}{x+1} - dx^3 \cdot \frac{x+2}{(x+1)^2} - kx^4 \cdot \frac{x^2+3x+3}{(x+1)^3} + O(x^4).$$

Because x is on the order of $\frac{1}{n}$, $\frac{1}{(x+1)^3}$ is on the order of a constant. Therefore, all the terms in the expansion of $kx^4 \cdot \frac{x^2+3x+3}{(x+1)^3}$ are of order x^4 or higher, so we can collapse it into $O(x^4)$. Using the Taylor expansions, we get

$$x(1-x+x^2) + O(x^4) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) - cx^2(1-x) - dx^3(2) + O(x^4).$$

Coefficient comparison gives $c = \frac{1}{2}$ and $d = -\frac{1}{12}$.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010

Combinatorics Subject Test

1. [2] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x , if x is in T and $2x$ is in S then $2x$ is also in T ?

Answer: 180 We partition the elements of S into the following subsets: $\{1, 2, 4, 8\}$, $\{3, 6\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$. Consider the first subset, $\{1, 2, 4, 8\}$. Say 2 is an element of T . Because $2 \cdot 2 = 4$ is in S , 4 must also be in T . Furthermore, since $4 \cdot 2 = 8$ is in S , 8 must also be in T . So if T contains 2, it must also contain 4 and 8. Similarly, if T contains 1, it must also contain 2, 4, and 8. So T can contain the following subsets of the subset $\{1, 2, 4, 8\}$: the empty set, $\{8\}$, $\{4, 8\}$, $\{2, 4, 8\}$, or $\{1, 2, 4, 8\}$. This gives 5 possibilities for the first subset. In general, we see that if T contains an element q of one of these subsets, it must also contain the elements in that subset that are larger than q , because we created the subsets for this to be true. So there are 3 possibilities for $\{3, 6\}$, 3 for $\{5, 10\}$, 2 for $\{7\}$, and 2 for $\{9\}$. This gives a total of $5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 180$ possible subsets T .

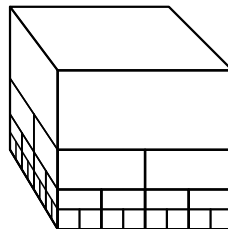
2. [3] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider $0!$ and $1!$ to be distinct.

Answer: 39 Note that $1 = 0!$, $2 = 0! + 1!$, $3 = 0! + 2!$, and $4 = 0! + 1! + 2!$. These are the only numbers less than 6 that can be written as the sum of factorials. The only other factorials less than 240 are $3! = 6$, $4! = 24$, and $5! = 120$. So a positive integer less than or equal to 240 can only contain $3!$, $4!$, $5!$, and/or one of 1, 2, 3, or 4 in its sum. If it contains any factorial larger than $5!$, it will be larger than 240. So a sum less than or equal to 240 will either include $3!$ or not (2 ways), $4!$ or not (2 ways), $5!$ or not (2 ways), and add an additional 0, 1, 2, 3 or 4 (5 ways). This gives $2 \cdot 2 \cdot 2 \cdot 5 = 40$ integers less than 240. However, we want only positive integers, so we must not count 0. So there are 39 such positive integers.

3. [4] How many ways are there to choose 2010 functions f_1, \dots, f_{2010} from $\{0, 1\}$ to $\{0, 1\}$ such that $f_{2010} \circ f_{2009} \circ \dots \circ f_1$ is constant? Note: a function g is constant if $g(a) = g(b)$ for all a, b in the domain of g .

Answer: $4^{2010} - 2^{2010}$ If all 2010 functions are bijective¹, then the composition $f_{2010} \circ f_{2009} \circ \dots \circ f_1$ will be bijective also, and therefore not constant. If, however, one of f_1, \dots, f_{2010} is not bijective, say f_k , then $f_k(0) = f_k(1) = q$, so $f_{2010} \circ f_{2009} \circ \dots \circ f_{k+1} \circ f_k \circ \dots \circ f_1(0) = f_{2010} \circ f_{2009} \circ \dots \circ f_{k+1}(q) = f_{2010} \circ f_{2009} \circ \dots \circ f_{k+1} \circ f_k \circ \dots \circ f_1(1)$. So the composition will be constant unless all f_i are bijective. Since there are 4 possible functions² from $\{0, 1\}$ to $\{0, 1\}$ and 2 of them are bijective, we subtract the cases where all the functions are bijective from the total to get $4^{2010} - 2^{2010}$.

4. [4] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the k th layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



¹See <http://en.wikipedia.org/wiki/Bijection>.

²The four functions are $a, b, c,$ and d , defined as follows: $a(0) = 0, a(1) = 0; b(0) = 0, b(1) = 1; c(0) = 1, c(1) = 0; d(0) = 1, d(1) = 1$.

Answer: $\boxed{3384}$ Each time Laura removes a block, 4 additional blocks are exposed, increasing the total number of exposed blocks by 3. She removes 5 blocks, for a total of $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13$ ways. However, the stack originally only has 4 layers, so we must subtract the cases where removing a block on the bottom layer does not expose any new blocks. There are $1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 256$ of these (the last factor of 4 is from the 4 blocks that we counted as being exposed, but were not actually). So our final answer is $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 - 1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 3384$.

5. [5] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

Answer: $\boxed{20503}$

Let the number of 2, 5, and 10 dollar notes John can use be x , y , and z respectively. We wish to find the number of nonnegative integer solutions to $2x + 5y + 10z = 2010$. Consider this equation mod 2. Because $2x$, $10z$, and 2010 are even, $5y$ must also be even, so y must be even. Now consider the equation mod 5. Because $5y$, $10z$, and 2010 are divisible by 5, $2x$ must also be divisible by 5, so x must be divisible by 5. So both $2x$ and $5y$ are divisible by 10. So the equation is equivalent to $10x' + 10y' + 10z = 2010$, or $x' + y' + z = 201$, with x' , y' , and z nonnegative integers. There is a well-known bijection between solutions of this equation and picking 2 of 203 balls in a row on the table (explained in further detail below), so there are $\binom{203}{2} = 20503$ ways.

The bijection between solutions of $x' + y' + z = 201$ and arrangements of 203 balls in a row is as follows. Given a solution of the equation, we put x' white balls in a row, then a black ball, then y' white balls, then a black ball, then z white balls. This is like having 203 balls in a row on a table and picking two of them to be black. To go from an arrangement of balls to a solution of the equation, we just read off x' , y' , and z from the number of white balls in a row. There are $\binom{203}{2}$ ways to choose 2 of 203 balls to be black, so there are $\binom{203}{2}$ solutions to $x' + y' + z = 201$.

6. [5] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y) , it can move to $(x - 1, y - 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$, or $(x + 1, y + 1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

Answer: $\boxed{\binom{4020}{1005}^2}$ Note that each of the coordinates either increases or decreases the x and y -coordinates by 1. In order to reach 2010 after 4020 steps, each of the coordinates must be increased 3015 times and decreased 1005 times. A permutation of 3015 plusses and 1005 minuses for each of x and y uniquely corresponds to a path the ant could take to $(2010, 2010)$, because we can take ordered pairs from the two lists and match them up to a valid step the ant can take. So the number of ways the ant can end up at $(2010, 2010)$ after 4020 seconds is equal to the number of ways to arrange plusses and minuses for both x and y , or $\left(\binom{4020}{1005}\right)^2$.

7. [6] For each integer x with $1 \leq x \leq 10$, a point is randomly placed at either $(x, 1)$ or $(x, -1)$ with equal probability. What is the expected area of the convex hull of these points? Note: the convex hull of a finite set is the smallest convex polygon containing it.

Answer: $\boxed{\frac{1793}{128}}$ Let $n = 10$. Given a random variable X , let $\mathbb{E}(X)$ denote its expected value. If all points are collinear, then the convex hull has area zero. This happens with probability $\frac{2}{2^n}$ (either all points are at $y = 1$ or all points are at $y = -1$). Otherwise, the points form a trapezoid with height 2 (the trapezoid is possibly degenerate, but this won't matter for our calculation). Let $x_{1,l}$ be the x -coordinate of the left-most point at $y = 1$ and $x_{1,r}$ be the x -coordinate of the right-most point at $y = 1$. Define $x_{-1,l}$ and $x_{-1,r}$ similarly for $y = -1$. Then the area of the trapezoid is

$$2 \cdot \frac{(x_{1,r} - x_{1,l}) + (x_{-1,r} - x_{-1,l})}{2} = x_{1,r} + x_{-1,r} - x_{1,l} - x_{-1,l}.$$

The expected area of the convex hull (assuming the points are not all collinear) is then, by linearity of expectation,

$$\mathbb{E}(x_{1,r} + x_{-1,r} - x_{1,l} - x_{-1,l}) = \mathbb{E}(x_{1,r}) + \mathbb{E}(x_{-1,r}) - \mathbb{E}(x_{1,l}) - \mathbb{E}(x_{-1,l}).$$

We need only compute the expected values given in the above equation. Note that $x_{1,r}$ is equal to k with probability $\frac{2^{k-1}}{2^n-2}$, except that it is equal to n with probability $\frac{2^{n-1}-1}{2^n-2}$ (the denominator is 2^n-2 instead of 2^n because we need to exclude the case where all points are collinear). Therefore, the expected value of $x_{1,r}$ is equal to

$$\begin{aligned} & \frac{1}{2^n-2} \left(\left(\sum_{k=1}^n k \cdot 2^{k-1} \right) - n \cdot 1 \right) \\ &= \frac{1}{2^n-2} \left((1+2+\dots+2^{n-1}) + (2+4+\dots+2^{n-1}) + \dots + 2^{n-1} - n \right) \\ &= \frac{1}{2^n-2} \left((2^n-1) + (2^n-2) + \dots + (2^n-2^{n-1}) - n \right) \\ &= \frac{1}{2^n-2} (n \cdot 2^n - (2^n-1) - n) \\ &= (n-1) \frac{2^n-1}{2^n-2} \end{aligned}$$

Similarly, the expected value of $x_{-1,r}$ is also $(n-1)\frac{2^n-1}{2^n-2}$. By symmetry, the expected value of both $x_{1,l}$ and $x_{-1,l}$ is $n+1 - (n-1)\frac{2^n-1}{2^n-2}$. This says that if the points are not all collinear then the expected area is $2 \cdot \left((n-1)\frac{2^n-1}{2^{n-1}-1} - (n+1) \right)$. So, the expected area is

$$\begin{aligned} & \frac{2}{2^n} \cdot 0 + \left(1 - \frac{2}{2^n} \right) \cdot 2 \cdot \left((n-1) \frac{2^n-1}{2^{n-1}-1} - (n+1) \right) \\ &= 2 \cdot \frac{2^{n-1}-1}{2^{n-1}} \cdot \left((n-1) \frac{2^n-1}{2^{n-1}-1} - (n+1) \right) \\ &= 2 \cdot \frac{(n-1)(2^n-1) - (n+1)(2^{n-1}-1)}{2^{n-1}} \\ &= 2 \cdot \frac{((2n-2) - (n+1))2^{n-1} + 2}{2^{n-1}} \\ &= 2n - 6 + \frac{1}{2^{n-3}} \end{aligned}$$

Plugging in $n = 10$, we get $14 + \frac{1}{128} = \frac{1793}{128}$.

8. [6] How many functions f from $\{-1005, \dots, 1005\}$ to $\{-2010, \dots, 2010\}$ are there such that the following two conditions are satisfied?

- If $a < b$ then $f(a) < f(b)$.
- There is no n in $\{-1005, \dots, 1005\}$ such that $|f(n)| = |n|$.

Answer: **Note:** the intended answer was $\binom{4019}{2011}$, but the original answer was incorrect. The correct answer is:

1173346782666677300072441773814388000553179587006710786401225043842699552460942166630860
5302966355504513409792805200762540756742811158611534813828022157596601875355477425764387
2333935841666957750009216404095352456877594554817419353494267665830087436353494075828446
0070506487793628698617665091500712606599653369601270652785265395252421526230453391663029
1476263072382369363170971857101590310272130771639046414860423440232291348986940615141526
0247281998288175423628757177754777309519630334406956881890655029018130367627043067425502
2334151384481231298380228052789795136259575164777156839054346649261636296328387580363485
2904329986459861362633348204891967272842242778625137520975558407856496002297523759366027

1506637984075036473724713869804364399766664507880042495122618597629613572449327653716600
6715747717529280910646607622693561789482959920478796128008380531607300324374576791477561
5881495035032334387221203759898494171708240222856256961757026746724252966598328065735933
6668742613422094179386207330487537984173936781232801614775355365060827617078032786368164
8860839124954588222610166915992867657815394480973063139752195206598739798365623873142903
28539769699667459275254643229234106717245366005816917271187760792

This obviously cannot be computed by hand, but there is a polynomial-time dynamic programming algorithm that will compute it.

9. [7] Rosencrantz and Guildenstern are playing a game where they repeatedly flip coins. Rosencrantz wins if 1 heads followed by 2009 tails appears. Guildenstern wins if 2010 heads come in a row. They will flip coins until someone wins. What is the probability that Rosencrantz wins?

Answer: $\frac{2^{2009}-1}{3 \cdot 2^{2008}-1}$ We can assume the first throw is heads (because neither player can win starting from a string of only tails). Let x be the probability that Rosencrantz wins. Let y be the probability that Rosencrantz wins after HT.

Whenever there is a string of less than 2009 tails followed by a heads, the heads basically means the two are starting from the beginning, where Rosencrantz has probability x of winning.

We also know that $x = y(1 - \frac{1}{2^{2009}})$. This is because from the initial heads there is a $(1 - \frac{1}{2^{2009}})$ chance Rosencrantz doesn't lose, and in this case the last two flips are HT, in which case Rosencrantz has probability y of winning.

If the first two throws are HT, there is a $\frac{1}{2^{2008}}$ chance Rosencrantz wins; otherwise, there is eventually a heads, and so we are back in the case of starting from a heads, which corresponds to x . Therefore, $y = \frac{1}{2^{2008}} + x(1 - \frac{1}{2^{2008}})$. Putting this together with the previous equation, we get:

$$\begin{aligned} x &= \left(\frac{1}{2^{2008}} + x\left(1 - \frac{1}{2^{2008}}\right)\right)\left(1 - \frac{1}{2^{2009}}\right) \\ \implies x &= \left(\frac{1+2^{2008}x-x}{2^{2008}}\right)\left(\frac{2^{2009}-1}{2^{2009}}\right) \\ \implies 2^{4017}x &= x\left(2^{4017} - 2^{2009} - 2^{2008} + 1\right) + 2^{2009} - 1 \\ \implies x &= \frac{2^{2009}-1}{2^{2009}+2^{2008}-1}, \end{aligned}$$

so the answer is $\frac{2^{2009}-1}{2^{2009}+2^{2008}-1} = \frac{2^{2009}-1}{3 \cdot 2^{2008}-1}$.

10. [8] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?

Answer: [49] First, we show that 50 is too big. Assume for sake of contradiction that a labeling with at least 50 distinct integers exists. By the Pigeonhole Principle, there must be at least one row, say the first row, with at least 4 distinct integers in it; in this case, that is exactly 4, since that is the maximum number of distinct integers in one row. Then, in the remaining 15 rows there must be at least 46 distinct integers (these 46 will also be distinct from the 4 in the first row). Using Pigeonhole again, there will be another row, say the second row, with 4 distinct integers in it. Call the set of integers in the first and second rows S . Because the 4 distinct integers in the second row are distinct from the 4 in the first row, there are 8 distinct values in the first two rows, so $|S| = 8$. Now consider the subcolumns containing the cells in rows 3 to 16. In each subcolumn, there are at most 2 values not in S , because there are already two distinct values in that column from the cells in the first two rows. So, the maximum number of distinct values in the table is $16 \cdot 2 + 8 = 40$, a contradiction. So a valid labeling must have fewer than 50 distinct integers. Below, we show by example that 49 is attainable.

1	17	33	-	-	-	-	-	-	-	-	-	-	-	-	-
-	2	18	34	-	-	-	-	-	-	-	-	-	-	-	-
-	-	3	19	35	-	-	-	-	-	-	-	-	-	-	-
-	-	-	4	20	36	-	-	-	-	-	-	-	-	-	-
-	-	-	-	5	21	37	-	-	-	-	-	-	-	-	-
-	-	-	-	-	6	22	38	-	-	-	-	-	-	-	-
-	-	-	-	-	-	7	23	39	-	-	-	-	-	-	-
-	-	-	-	-	-	-	8	24	40	-	-	-	-	-	-
-	-	-	-	-	-	-	-	9	25	41	-	-	-	-	-
-	-	-	-	-	-	-	-	-	10	26	42	-	-	-	-
-	-	-	-	-	-	-	-	-	-	11	27	43	-	-	-
-	-	-	-	-	-	-	-	-	-	-	12	28	44	-	-
-	-	-	-	-	-	-	-	-	-	-	-	13	29	45	-
-	-	-	-	-	-	-	-	-	-	-	-	-	14	30	46
47	-	-	-	-	-	-	-	-	-	-	-	-	-	15	31
32	48	-	-	-	-	-	-	-	-	-	-	-	-	-	16

Cells that do not contain a number are colored with color 49.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
General Test, Part 1

1. [3] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

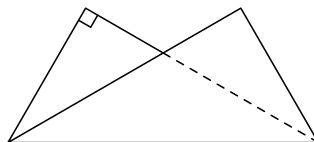
Find x .

Answer: $\boxed{\frac{3+\sqrt{17}}{2}}$ Squaring both sides of $x - y^2 = 3$ gives $x^2 + y^4 - 2xy^2 = 9$. Subtract this equation from twice the second given to get $x^2 + 2xy^2 + y^4 = 17 \implies x + y^2 = \pm 17$. Combining this equation with the first given, we see that $x = \frac{3 \pm \sqrt{17}}{2}$. Since x is a positive real, x must be $\frac{3+\sqrt{17}}{2}$.

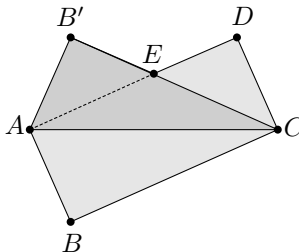
2. [3] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x , if x is in T and $2x$ is in S then $2x$ is also in T ?

Answer: $\boxed{180}$ We partition the elements of S into the following subsets: $\{1, 2, 4, 8\}$, $\{3, 6\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$. Consider the first subset, $\{1, 2, 4, 8\}$. Say 2 is an element of T . Because $2 \cdot 2 = 4$ is in S , 4 must also be in T . Furthermore, since $4 \cdot 2 = 8$ is in S , 8 must also be in T . So if T contains 2, it must also contain 4 and 8. Similarly, if T contains 1, it must also contain 2, 4, and 8. So T can contain the following subsets of the subset $\{1, 2, 4, 8\}$: the empty set, $\{8\}$, $\{4, 8\}$, $\{2, 4, 8\}$, or $\{1, 2, 4, 8\}$. This gives 5 possibilities for the first subset. In general, we see that if T contains an element q of one of these subsets, it must also contain the elements in that subset that are larger than q , because we created the subsets for this to be true. So there are 3 possibilities for $\{3, 6\}$, 3 for $\{5, 10\}$, 2 for $\{7\}$, and 2 for $\{9\}$. This gives a total of $5 \cdot 3 \cdot 3 \cdot 2 \cdot 2 = 180$ possible subsets T .

3. [4] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



Answer: $\boxed{\sqrt{5}}$



Given a polygon $P_1P_2 \cdots P_k$, let $[P_1P_2 \cdots P_k]$ denote its area. Let $ABCD$ be the rectangle. Suppose we fold B across \overline{AC} , and let E be the intersection of \overline{AD} and $\overline{B'C}$. Then we end up with the pentagon $ACDEB'$, depicted above. Let's suppose, without loss of generality, that $ABCD$ has area 1. Then $\triangle AEC$ must have area $\frac{3}{10}$, since

$$\begin{aligned}
[ABCD] &= [ABC] + [ACD] \\
&= [AB'C] + [ACD] \\
&= [AB'E] + 2[AEC] + [EDC] \\
&= [ACDEB'] + [AEC] \\
&= \frac{7}{10}[ABCD] + [AEC],
\end{aligned}$$

That is, $[AEC] = \frac{3}{10}[ABCD] = \frac{3}{10}$.

Since $\triangle ECD$ is congruent to $\triangle EAB'$, both triangles have area $\frac{1}{5}$. Note that $\triangle AB'C$, $\triangle ABC$, and $\triangle CDA$ are all congruent, and all have area $\frac{1}{2}$. Since $\triangle AEC$ and $\triangle EDC$ share altitude \overline{DC} , $\frac{DE}{EA} = \frac{[DEC]}{[AEC]} = \frac{2}{3}$. Because $\triangle CAE$ is isosceles, $CE = EA$. Let $AE = 3x$. The $CE = 3x$, $DE = 2x$, and $CD = x\sqrt{9-4} = x\sqrt{5}$. Then $\frac{AD}{DC} = \frac{AE+ED}{DC} = \frac{3+2}{\sqrt{5}} = \sqrt{5}$.

4. [4] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \dots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

Answer: 1092 Suppose that $S_N = 0$ for some $N \geq 0$. Then $a_{N+1} = 1$ because $N + 1 \geq S_N$. The following table lists the values of a_k and S_k for a few $k \geq N$:

k	a_k	S_k
N		0
$N + 1$	1	$N + 1$
$N + 2$	1	$2N + 3$
$N + 3$	-1	N
$N + 4$	1	$2N + 4$
$N + 5$	-1	$N - 1$
$N + 6$	1	$2N + 5$
$N + 7$	-1	$N - 2$

We see inductively that, for every $i \geq 1$,

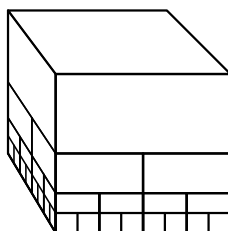
$$S_{N+2i} = 2N + 2 + i$$

and

$$S_{N+1+2i} = N + 1 - i$$

thus $S_{3N+3} = 0$ is the next k for which $S_k = 0$. The values of k for which $S_k = 0$ satisfy the recurrence relation $p_{n+1} = 3p_n + 3$, and we compute that the first terms of the sequence are 0, 3, 12, 39, 120, 363, 1092; hence 1092 is our answer.

5. [4] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the k th layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



Answer: 3384 Each time Laura removes a block, 4 additional blocks are exposed, increasing the total number of exposed blocks by 3. She removes 5 blocks, for a total of $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13$ ways. However, the stack originally only has 4 layers, so we must subtract the cases where removing a block on the bottom layer does not expose any new blocks. There are $1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 256$ of these (the last factor of 4 is from the 4 blocks that we counted as being exposed, but were not actually). So our final answer is $1 \cdot 4 \cdot 7 \cdot 10 \cdot 13 - 1 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 3384$.

6. [5] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

Answer: 20503

Let the number of 2, 5, and 10 dollar notes John can use be x , y , and z respectively. We wish to find the number of nonnegative integer solutions to $2x + 5y + 10z = 2010$. Consider this equation mod 2. Because $2x$, $10z$, and 2010 are even, $5y$ must also be even, so y must be even. Now consider the equation mod 5. Because $5y$, $10z$, and 2010 are divisible by 5, $2x$ must also be divisible by 5, so x must be divisible by 5. So both $2x$ and $5y$ are divisible by 10. So the equation is equivalent to $10x' + 10y' + 10z = 2010$, or $x' + y' + z = 201$, with x' , y' , and z nonnegative integers. There is a well-known bijection between solutions of this equation and picking 2 of 203 balls in a row on the table (explained in further detail below), so there are $\binom{203}{2} = 20503$ ways.

The bijection between solutions of $x' + y' + z = 201$ and arrangements of 203 balls in a row is as follows. Given a solution of the equation, we put x' white balls in a row, then a black ball, then y' white balls, then a black ball, then z white balls. This is like having 203 balls in a row on a table and picking two of them to be black. To go from an arrangement of balls to a solution of the equation, we just read off x' , y' , and z from the number of white balls in a row. There are $\binom{203}{2}$ ways to choose 2 of 203 balls to be black, so there are $\binom{203}{2}$ solutions to $x' + y' + z = 201$.

7. [6] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \dots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p ?

Answer: 1005 Let $p(x)$ be a polynomial with the maximum number of minus signs. $p(x)$ cannot have more than 1005 minus signs, otherwise $p(1) < 0$ and $p(2) \geq 2^{2010} - 2^{2009} - \dots - 2 - 1 = 1$, which implies, by the Intermediate Value Theorem, that p must have a root greater than 1.

Let $p(x) = \frac{x^{2011} + 1}{x + 1} = x^{2010} - x^{2009} + x^{2008} - \dots - x + 1$. -1 is the only real root of $x^{2011} + 1 = 0$ but $p(-1) = 2011$; therefore p has no real roots. Since p has 1005 minus signs, it is the desired polynomial.

8. [6] A sphere is the set of points at a fixed positive distance r from its center. Let \mathcal{S} be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of \mathcal{S} is a finite number n . Find the maximum possible value of n .

Answer: 2 The answer is 2 for any number of dimensions. We prove this by induction on the dimension.

Note that 1-dimensional spheres are pairs of points, and 2-dimensional spheres are circles.

Base case, $d = 2$: The intersection of two circles is either a circle (if the original circles are identical, and in the same place), a pair of points, a single point (if the circles are tangent), or the empty set. Thus, in dimension 2, the largest finite number of intersection points is 2, because the number of pairwise intersection points is 0, 1, or 2 for distinct circles.

We now prove that the intersection of two k -dimensional spheres is either the empty set, a $(k - 1)$ -dimensional sphere, a k -dimensional sphere (which only occurs if the original spheres are identical and coincident). Consider two spheres in k -dimensional space, and impose a coordinate system such that the centers of the two spheres lie on one coordinate axis. Then the equations for the two spheres become identical in all but one coordinate:

$$\begin{aligned}(x_1 - a_1)^2 + x_2^2 + \dots + x_k^2 &= r_1^2 \\(x_1 - a_2)^2 + x_2^2 + \dots + x_k^2 &= r_2^2\end{aligned}$$

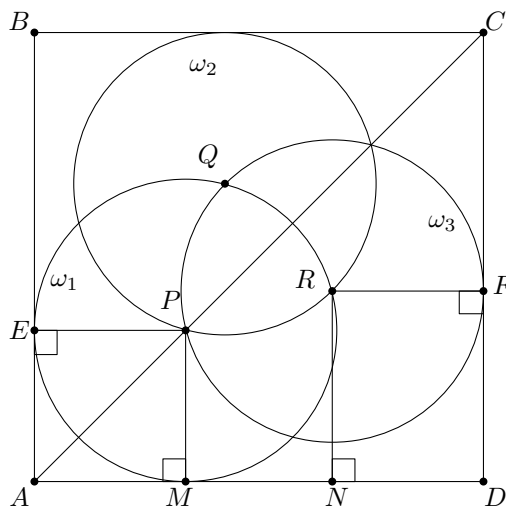
If $a_1 = a_2$, the spheres are concentric, and so they are either nonintersecting or coincident, intersecting in a k -dimensional sphere. If $a_1 \neq a_2$, then subtracting the equations and solving for x_1 yields $x_1 = \frac{r_1^2 - a_1^2 - r_2^2 + a_2^2}{2(a_2 - a_1)}$. Plugging this in to either equation above yields a single equation that describes a $(k - 1)$ -dimensional sphere.

Assume we are in dimension d , and suppose for induction that for all k less than d , any two distinct k -dimensional spheres intersecting in a finite number of points intersect in at most two points. Suppose we have a collection of d -dimensional spheres s_1, s_2, \dots, s_m . Without loss of generality, suppose the s_i are distinct. Let t_i be the intersection of s_i and s_{i+1} for $1 \leq i < m$. If any t_i are the empty set, then the intersection of the t_i is empty. None of the t_i is a d -dimensional sphere because the s_i are distinct. Thus each of t_1, t_2, \dots, t_{m-1} is a $(d - 1)$ -dimensional sphere, and the intersection of all of them is the same as the intersection of the d -dimensional spheres. We can then apply the inductive hypothesis to find that t_1, \dots, t_{m-1} intersect in at most two points. Thus, by induction, a set of spheres in any dimension which intersect at only finitely many points intersect at at most two points.

We now exhibit a set of 2^{2009} 2010-dimensional spheres, and prove that their intersection contains exactly two points. Take the spheres with radii $\sqrt{2013}$ and centers $(0, \pm 1, \pm 1, \dots, \pm 1)$, where the sign of each coordinate is independent from the sign of every other coordinate. Because of our choice of radius, all these spheres pass through the points $(\pm 2, 0, 0, \dots, 0)$. Then the intersection is the set of points $(x_1, x_2, \dots, x_{2010})$ which satisfy the equations $x_1^2 + (x_2 \pm 1)^2 + \dots + (x_{2010} \pm 1)^2 = 2013$. The only solutions to these equations are the points $(\pm 2, 0, 0, \dots, 0)$ (since $(x_i + 1)^2$ must be the same as $(x_i - 1)^2$ for all $i > 1$, because we may hold all but one of the \pm choices constant, and change the remaining one). \square

9. [7] Three unit circles ω_1, ω_2 , and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3 . Find the side length of the square S .

Answer: $\boxed{\frac{\sqrt{6} + \sqrt{2} + 8}{4}}$



By the Pigeonhole Principle, two of the sides must be tangent to the same circle, say ω_1 . Since S surrounds the circles, these two sides must be adjacent, so we can let A denote the common vertex of the two sides tangent to ω_1 . Let B, C , and D be the other vertices of S in clockwise order, and let P, Q , and R be the centers of ω_1, ω_2 , and ω_3 respectively, and suppose WLOG that they are also in clockwise order. Then AC passes through the center of ω_1 , and by symmetry (since $AB = AD$) it

must also pass through the other intersection point of ω_2 and ω_3 . That is, AC is the radical axis of ω_2 and ω_3 .

Now, let M and N be the feet of the perpendiculars from P and R , respectively, to side AD . Let E and F be the feet of the perpendiculars from P to AB and from R to DC , respectively. Then $PEAM$ and $NRFD$ are rectangles, and PE and RF are radii of ω_1 and ω_2 respectively. Thus $AM = EP = 1$ and $ND = RF = 1$. Finally, we have

$$\begin{aligned} MN &= PR \cdot \cos(180^\circ - \angle EPR) \\ &= \cos(180^\circ - \angle EPQ - \angle RPQ) \\ &= -\cos((270^\circ - 60^\circ)/2 + 60^\circ) \\ &= -\cos(165^\circ) \\ &= \cos(15^\circ) \\ &= \frac{\sqrt{6} + \sqrt{2}}{4}. \end{aligned}$$

Thus $AD = AM + MN + ND = 1 + \frac{\sqrt{6} + \sqrt{2}}{4} + 1 = \frac{\sqrt{6} + \sqrt{2} + 8}{4}$ as claimed. \square

10. [8] Let $a, b, c, x, y,$ and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

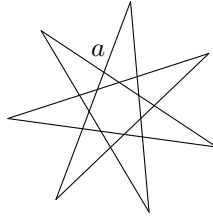
If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz .

Answer: $\boxed{-5892}$ Manipulate the equations to get a common denominator: $a = \frac{b+c}{x-2} \implies x-2 = \frac{b+c}{a} \implies x-1 = \frac{a+b+c}{a} \implies \frac{1}{x-1} = \frac{a}{a+b+c}$; similarly, $\frac{1}{y-1} = \frac{b}{a+b+c}$ and $\frac{1}{z-1} = \frac{c}{a+b+c}$. Thus

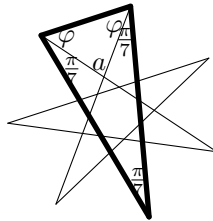
$$\begin{aligned} \frac{1}{x-1} + \frac{1}{y-1} + \frac{1}{z-1} &= 1 \\ (y-1)(z-1) + (x-1)(z-1) + (x-1)(y-1) &= (x-1)(y-1)(z-1) \\ xy + yz + zx - 2(x+y+z) + 3 &= xyz - (xy + yz + zx) + (x+y+z) - 1 \\ xyz - 2(xy + yz + zx) + 3(x+y+z) - 4 &= 0 \\ xyz - 2(67) + 3(2010) - 4 &= 0 \\ xyz &= -5892 \end{aligned}$$

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
General Test, Part 2

1. [3] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



Answer: $\boxed{\frac{3\pi}{7}}$ The measure of the interior angle of a point of the star is $\frac{\pi}{7}$ because it is an inscribed angle on the circumcircle which intercepts a seventh of the circle.¹



Consider the triangle shown above in bold. Because the sum of the angles in any triangle is π ,

$$2\varphi + 3\left(\frac{\pi}{7}\right) = \pi = 2\varphi + a$$

Canceling the 2φ on the right-hand side and on the left-hand side, we obtain

$$a = \frac{3\pi}{7}.$$

□

2. [3] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \dots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \dots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3) .

Answer: $\boxed{(5, 21, 421)}$ Suppose that A and B were rational numbers of rank 3 less than $\frac{1}{4}$, and let $a_1, a_2, a_3, b_1, b_2, b_3$ be positive integers so that $A = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$ and $B = \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3}$ are the expressions for A and B as stated in the problem. If $b_1 < a_1$ then $A < \frac{1}{a_1-1} \leq \frac{1}{b_1} < B$. In other words, of all the rationals less than $\frac{1}{4}$ with rank 3, those that have $a_1 = 5$ are greater than those that have $a_1 = 6, 7, 8, \dots$. Therefore we can “build” q greedily, adding the largest unit fraction that keeps q less than $\frac{1}{4}$:

$\frac{1}{5}$ is the largest unit fraction less than $\frac{1}{4}$, hence $a_1 = 5$;

$\frac{1}{21}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5}$, hence $a_2 = 21$;

$\frac{1}{421}$ is the largest unit fraction less than $\frac{1}{4} - \frac{1}{5} - \frac{1}{21}$, hence $a_3 = 421$.

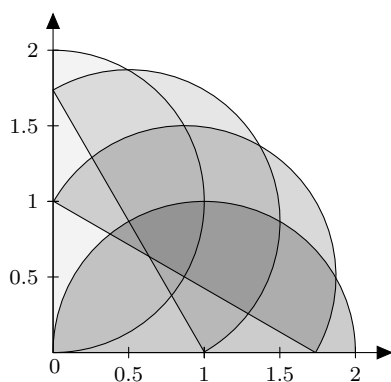
¹http://en.wikipedia.org/wiki/Inscribed_angle_theorem

3. [4] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider $0!$ and $1!$ to be distinct.

Answer: $\boxed{39}$ Note that $1 = 0!$, $2 = 0! + 1!$, $3 = 0! + 2!$, and $4 = 0! + 1! + 2!$. These are the only numbers less than 6 that can be written as the sum of factorials. The only other factorials less than 240 are $3! = 6$, $4! = 24$, and $5! = 120$. So a positive integer less than or equal to 240 can only contain $3!$, $4!$, $5!$, and/or one of 1, 2, 3, or 4 in its sum. If it contains any factorial larger than $5!$, it will be larger than 240. So a sum less than or equal to 240 will either include $3!$ or not (2 ways), $4!$ or not (2 ways), $5!$ or not (2 ways), and add an additional 0, 1, 2, 3 or 4 (5 ways). This gives $2 \cdot 2 \cdot 2 \cdot 5 = 40$ integers less than 240. However, we want only positive integers, so we must not count 0. So there are 39 such positive integers.

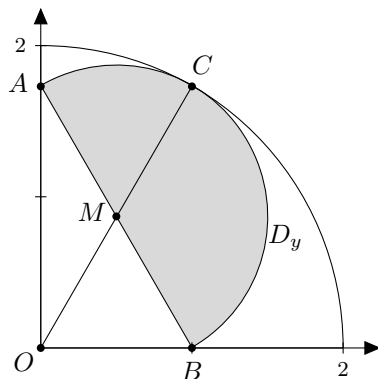
4. [4] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x -axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

Answer: $\boxed{\pi}$



From the picture above, we see that the union of the half-disks will be a quarter-circle with radius 2, and therefore area π . To prove that this is the case, we first prove that the boundary of every half-disk intersects the quarter-circle with radius 2, and then that the half-disk is internally tangent to the quarter-circle at that point. This is sufficient because it is clear from the diagram that we need not worry about covering the interior of the quarter-circle.

Let O be the origin. For a given half-disk D_y , label the vertex on the y -axis A and the vertex on the x -axis B . Let M be the midpoint of line segment \overline{AB} . Draw segment OM , and extend it until it intersects the curved boundary of D_y . Label the intersection point C . This construction is shown in the diagram below.



We first prove that C lies on the quarter-circle, centered at the origin, with radius 2. Since M is the midpoint of \overline{AB} , and A is on the y -axis, M is horizontally halfway between B and the y -axis. Since

O and B are on the x -axis (which is perpendicular to the y -axis), segments \overline{OM} and \overline{MB} have the same length. Since M is the midpoint of \overline{AB} , and $AB = 2$, $OM = 1$. Since D_y is a half-disk with radius 1, all points on its curved boundary are 1 away from its center, M . Then C is 2 away from the origin, and the quarter-circle consists of all points which are 2 away from the origin. Thus, C is an intersection of the half-disk D_y with the positive quarter-circle of radius 2.

It remains to show that the half-disk D_y is internally tangent to the quarter-circle. Since \overline{OC} is a radius of the quarter-circle, it is perpendicular to the tangent of the quarter-circle at C . Since \overline{MC} is a radius of the half-disk, it is perpendicular to the tangent of the half-disk at C . Then the tangents lines of the half-disk and the quarter-circle coincide, and the half-disk is tangent to the quarter-circle. It is obvious from the diagram that the half-disk lies at least partially inside of the quarter-circle, the half-disk D_y is internally tangent to the quarter-circle.

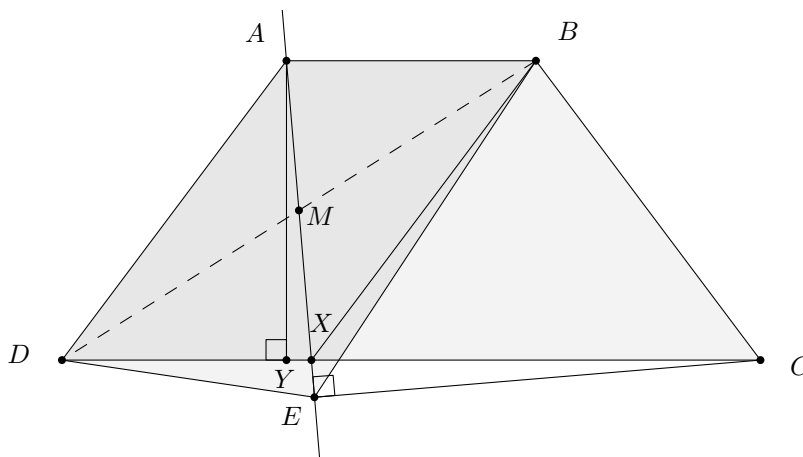
Then the union of the half-disks is be a quarter-circle with radius 2, and has area π . □

5. [5] Suppose that there exist nonzero complex numbers a, b, c , and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

Answer: $\boxed{1, -1, i, -i}$ Let k be a root of both polynomials. Multiplying the first polynomial by k and subtracting the second, we have $ak^4 - a = 0$, which means that k is either 1, -1 , i , or $-i$. If $a = b = c = d = 1$, then $-1, i$, and $-i$ are roots of both polynomials. If $a = b = c = 1$ and $d = -3$, then 1 is a root of both polynomials. So k can be 1, $-1, i$, and $-i$.

6. [5] Let $ABCD$ be an isosceles trapezoid such that $AB = 10, BC = 15, CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC .

Answer: $\boxed{\frac{216}{\sqrt{145}}}$



The locus of points E such that $[AED] = [AEB]$ forms a line, since area is a linear function of the coordinates of E ; setting the areas equal gives a linear equation in the coordinates E^2 . Note that A and M , the midpoint of \overline{DB} , are on this line; A because both areas are 0, and M because the triangles share an altitude, and bases \overline{DM} and \overline{MB} are equal in length. Then \overline{AM} is the set of points satisfying the area condition. The point E , then, is such that $\triangle AEC$ is a right angle (to make the distance minimal) and E lies on \overline{AM} .

Let X be the point of intersection of \overline{AM} and \overline{CD} . Then $\triangle AMB \sim \triangle XMD$, and since $MD = BM$, they are in fact congruent. Thus $DX = AB = 10$, and $XC = 18$. Similarly, $BX = 15$, so $ABXD$ is a parallelogram. Let Y be the foot of the perpendicular from A to \overline{DC} , so that $DY = \frac{DC-AB}{2} = 9$. Then

²The notation $[A_1A_2 \dots A_n]$ means the area of polygon $A_1A_2 \dots A_n$.

$AY = \sqrt{AD^2 - DY^2} = \sqrt{225 - 81} = 12$. Then $YX = DX - DY = 1$ and $AX = \sqrt{AY^2 + YX^2} = \sqrt{144 + 1} = \sqrt{145}$. Since both $\triangle AXY$ and $\triangle CXE$ have a right angle, and $\angle EXC$ and $\angle YXA$ are congruent because they are vertical angles, $\triangle AXY \sim \triangle CXE$. Then $\frac{CE}{AY} = \frac{CX}{AX}$, so $CE = 12 \cdot \frac{18}{\sqrt{145}} = \frac{216}{\sqrt{145}}$.

7. [5] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

Answer: $\boxed{-90}$ We have $x^2 + y^2 + 2xy = 1$. Define $a = 2xy$ and $b = x^2 + y^2$ for convenience. Then $a + b = 1$ and $b - a = x^2 + y^2 - 2xy = (x - y)^2 = 2b - 1$ so that $x, y = \frac{\sqrt{2b-1} \pm 1}{2}$. Then

$$\begin{aligned} x^{20} + y^{20} &= \left(\frac{\sqrt{2b-1} + 1}{2} \right)^{20} + \left(\frac{\sqrt{2b-1} - 1}{2} \right)^{20} \\ &= \frac{1}{2^{20}} [(\sqrt{2b-1} + 1)^{20} + (\sqrt{2b-1} - 1)^{20}] \\ &= \frac{2}{2^{20}} \left[(\sqrt{2b-1})^{20} + \binom{20}{2} (\sqrt{2b-1})^{18} + \binom{20}{4} (\sqrt{2b-1})^{16} + \dots \right] \\ &= \frac{2}{2^{20}} \left[(2b-1)^{10} + \binom{20}{2} (2b-1)^9 + \binom{20}{4} (2b-1)^8 + \dots \right] \\ &= 20 \end{aligned}$$

We want to find the sum of distinct roots of the above polynomial in b ; we first prove that the original polynomial is square-free. The conditions $x + y = 1$ and $x^{20} + y^{20} = 20$ imply that $x^{20} + (1-x)^{20} - 20 = 0$; let $p(x) = x^{20} + (1-x)^{20} - 20$. p is square-free if and only if $GCD(p, p') = c$ for some constant c :

$$\begin{aligned} GCD(p, p') &= GCD(x^{20} + (1-x)^{20} - 20, 20(x^{19} - (1-x)^{19})) \\ &= GCD(x^{20} - x(1-x)^{19} + (1-x)^{19} - 20, 20(x^{19} - (1-x)^{19})) \\ &= GCD((1-x)^{19} - 20, x^{19} - (1-x)^{19}) \\ &= GCD((1-x)^{19} - 20, x^{19} - 20) \end{aligned}$$

The roots of $x^{19} - 20$ are $\sqrt[19]{20^k} \exp(\frac{2\pi ik}{19})$ for some $k = 0, 1, \dots, 18$; the roots of $(1-x)^{19} - 20$ are $1 - \sqrt[19]{20^k} \exp(\frac{2\pi ik}{19})$ for some $k = 0, 1, \dots, 18$. If $x^{19} - 20$ and $(1-x)^{19} - 20$ share a common root, then there exist integers m, n such that $\sqrt[19]{20^m} \exp(\frac{2\pi im}{19}) = 1 - \sqrt[19]{20^n} \exp(\frac{2\pi in}{19})$; since the imaginary parts of both sides must be the same, we have $m = n$ and $\sqrt[19]{20^m} \exp(\frac{2\pi im}{19}) = \frac{1}{2} \implies 20^m = \frac{1}{2^{19}}$, a contradiction. Thus we have proved that the polynomial in x has no double roots. Since for each b there exists a unique pair (x, y) (up to permutations) that satisfies $x^2 + y^2 = b$ and $(x + y)^2 = 1$, the polynomial in b has no double roots.

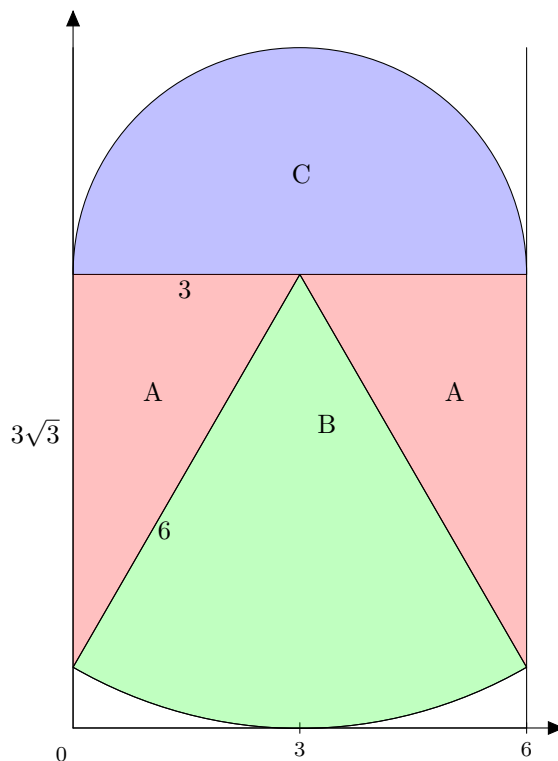
Let the coefficient of b^n in the above equation be $[b^n]$. By Vieta's Formulas, the sum of all possible values of $b = x^2 + y^2$ is equal to $-\frac{[b^9]}{[b^{10}]}$. $[b^{10}] = \frac{2}{2^{20}} (2^{10})$ and $[b^9] = \frac{2}{2^{20}} (-\binom{10}{1} 2^9 + \binom{20}{2} 2^9)$; thus $-\frac{[b^9]}{[b^{10}]} = -\frac{\binom{10}{1} 2^9 - \binom{20}{2} 2^9}{2^{10}} = -90$.

8. [6] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y) , it can move to $(x - 1, y - 1)$, $(x - 1, y + 1)$, $(x + 1, y - 1)$, or $(x + 1, y + 1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

Answer: $\boxed{\binom{4020}{1005}^2}$ Note that each of the coordinates either increases or decreases the x and y -coordinates by 1. In order to reach 2010 after 4020 steps, each of the coordinates must be increased 3015 times and decreased 1005 times. A permutation of 3015 plusses and 1005 minuses for each of x and y uniquely corresponds to a path the ant could take to $(2010, 2010)$, because we can take ordered pairs from the two lists and match them up to a valid step the ant can take. So the number of ways the ant can end up at $(2010, 2010)$ after 4020 seconds is equal to the number of ways to arrange plusses and minuses for both x and y , or $(\binom{4020}{1005})^2$.

9. [7] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

Answer: $9\sqrt{3} + \frac{21\pi}{2}$



If you draw concentric circles around the destination point, the condition is equivalent to the restriction that you must always go inwards towards the destination. In the diagram above, the regions through which you might pass are shaded.

We find the areas of regions A, B, and C separately, and add them up (doubling the area of region A, because there are two of them).

The hypotenuse of triangle A is of length 6, and the base is of length 3, so it is a $\frac{\pi}{6}-\frac{\pi}{3}-\frac{\pi}{2}$ triangle (30-60-90 triangle) with area $\frac{9\sqrt{3}}{2}$. Then the total area of the regions labeled A is $9\sqrt{3}$.

Since the angle of triangle A nearest the center of the circle (the destination point) is $\frac{\pi}{3}$, sector B has central angle $\frac{\pi}{3}$. Then the area of sector B is $\frac{1}{2}r^2\theta = \frac{1}{2} \cdot 36 \cdot \frac{\pi}{3} = 6\pi$.

Region C is a half-disc of radius 3, so its area is $\frac{9\pi}{2}$.

Thus, the total area is $9\sqrt{3} + \frac{21\pi}{2}$.

10. [8] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?

Answer: [49] First, we show that 50 is too big. Assume for sake of contradiction that a labeling with at least 50 distinct integers exists. By the Pigeonhole Principle, there must be at least one row, say the first row, with at least 4 distinct integers in it; in this case, that is exactly 4, since that is the maximum number of distinct integers in one row. Then, in the remaining 15 rows there must be at

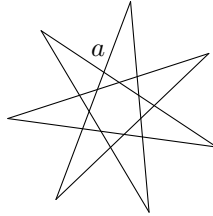
least 46 distinct integers (these 46 will also be distinct from the 4 in the first row). Using Pigeonhole again, there will be another row, say the second row, with 4 distinct integers in it. Call the set of integers in the first and second rows S . Because the 4 distinct integers in the second row are distinct from the 4 in the first row, there are 8 distinct values in the first two rows, so $|S| = 8$. Now consider the subcolumns containing the cells in rows 3 to 16. In each subcolumn, there are at most 2 values not in S , because there are already two distinct values in that column from the cells in the first two rows. So, the maximum number of distinct values in the table is $16 \cdot 2 + 8 = 40$, a contradiction. So a valid labeling must have fewer than 50 distinct integers. Below, we show by example that 49 is attainable.

1	17	33	-	-	-	-	-	-	-	-	-	-	-	-	-
-	2	18	34	-	-	-	-	-	-	-	-	-	-	-	-
-	-	3	19	35	-	-	-	-	-	-	-	-	-	-	-
-	-	-	4	20	36	-	-	-	-	-	-	-	-	-	-
-	-	-	-	5	21	37	-	-	-	-	-	-	-	-	-
-	-	-	-	-	6	22	38	-	-	-	-	-	-	-	-
-	-	-	-	-	-	7	23	39	-	-	-	-	-	-	-
-	-	-	-	-	-	-	8	24	40	-	-	-	-	-	-
-	-	-	-	-	-	-	-	9	25	41	-	-	-	-	-
-	-	-	-	-	-	-	-	-	10	26	42	-	-	-	-
-	-	-	-	-	-	-	-	-	-	11	27	43	-	-	-
-	-	-	-	-	-	-	-	-	-	-	12	28	44	-	-
-	-	-	-	-	-	-	-	-	-	-	-	13	29	45	-
-	-	-	-	-	-	-	-	-	-	-	-	-	14	30	46
47	-	-	-	-	-	-	-	-	-	-	-	-	-	15	31
32	48	-	-	-	-	-	-	-	-	-	-	-	-	-	16

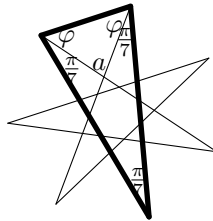
Cells that do not contain a number are colored with color 49.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
Geometry Subject Test

1. [3] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



Answer: $\boxed{\frac{3\pi}{7}}$ The measure of the interior angle of a point of the star is $\frac{\pi}{7}$ because it is an inscribed angle on the circumcircle which intercepts a seventh of the circle.¹



Consider the triangle shown above in bold. Because the sum of the angles in any triangle is π ,

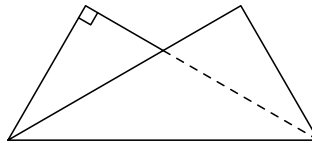
$$2\varphi + 3\left(\frac{\pi}{7}\right) = \pi = 2\varphi + a$$

Canceling the 2φ on the right-hand side and on the left-hand side, we obtain

$$a = \frac{3\pi}{7}.$$

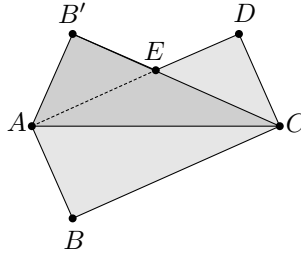
□

2. [3] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



Answer: $\boxed{\sqrt{5}}$

¹http://en.wikipedia.org/wiki/Inscribed_angle_theorem



Given a polygon $P_1P_2 \cdots P_k$, let $[P_1P_2 \cdots P_k]$ denote its area. Let $ABCD$ be the rectangle. Suppose we fold B across \overline{AC} , and let E be the intersection of \overline{AD} and $\overline{B'C}$. Then we end up with the pentagon $ACDEB'$, depicted above. Let's suppose, without loss of generality, that $ABCD$ has area 1. Then $\triangle AEC$ must have area $\frac{3}{10}$, since

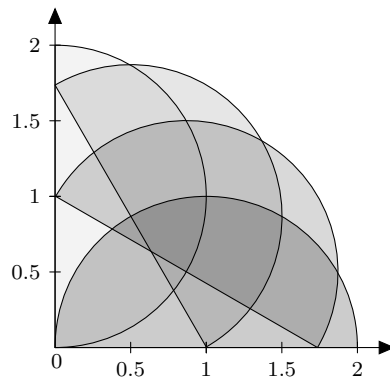
$$\begin{aligned}
 [ABCD] &= [ABC] + [ACD] \\
 &= [AB'C] + [ACD] \\
 &= [AB'E] + 2[AEC] + [EDC] \\
 &= [ACDEB'] + [AEC] \\
 &= \frac{7}{10}[ABCD] + [AEC],
 \end{aligned}$$

That is, $[AEC] = \frac{3}{10}[ABCD] = \frac{3}{10}$.

Since $\triangle ECD$ is congruent to $\triangle EAB'$, both triangles have area $\frac{1}{5}$. Note that $\triangle AB'C$, $\triangle ABC$, and $\triangle CDA$ are all congruent, and all have area $\frac{1}{2}$. Since $\triangle AEC$ and $\triangle EDC$ share altitude \overline{DC} , $\frac{DE}{EA} = \frac{[DEC]}{[AEC]} = \frac{2}{3}$. Because $\triangle CAE$ is isosceles, $CE = EA$. Let $AE = 3x$. The $CE = 3x$, $DE = 2x$, and $CD = x\sqrt{9-4} = x\sqrt{5}$. Then $\frac{AD}{DC} = \frac{AE+ED}{DC} = \frac{3+2}{\sqrt{5}} = \sqrt{5}$.

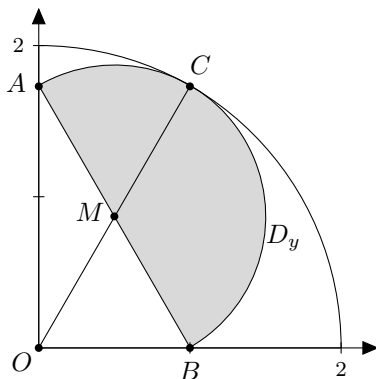
3. [4] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x -axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

Answer: $\boxed{\pi}$



From the picture above, we see that the union of the half-disks will be a quarter-circle with radius 2, and therefore area π . To prove that this is the case, we first prove that the boundary of every half-disk intersects the quarter-circle with radius 2, and then that the half-disk is internally tangent to the quarter-circle at that point. This is sufficient because it is clear from the diagram that we need not worry about covering the interior of the quarter-circle.

Let O be the origin. For a given half-disk D_y , label the vertex on the y -axis A and the vertex on the x -axis B . Let M be the midpoint of line segment \overline{AB} . Draw segment OM , and extend it until it intersects the curved boundary of D_y . Label the intersection point C . This construction is shown in the diagram below.



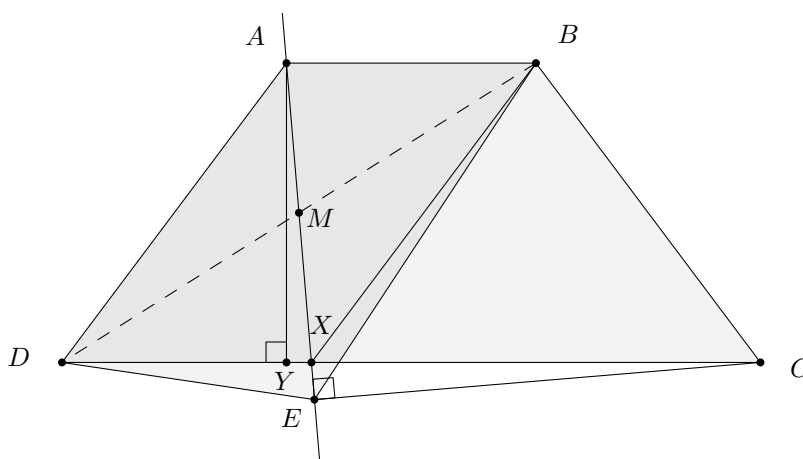
We first prove that C lies on the quarter-circle, centered at the origin, with radius 2. Since M is the midpoint of \overline{AB} , and A is on the y -axis, M is horizontally halfway between B and the y -axis. Since O and B are on the x -axis (which is perpendicular to the y -axis), segments \overline{OM} and \overline{MB} have the same length. Since M is the midpoint of \overline{AB} , and $AB = 2$, $OM = 1$. Since D_y is a half-disk with radius 1, all points on its curved boundary are 1 away from its center, M . Then C is 2 away from the origin, and the quarter-circle consists of all points which are 2 away from the origin. Thus, C is an intersection of the half-disk D_y with the positive quarter-circle of radius 2.

It remains to show that the half-disk D_y is internally tangent to the quarter-circle. Since \overline{OC} is a radius of the quarter-circle, it is perpendicular to the tangent of the quarter-circle at C . Since \overline{MC} is a radius of the half-disk, it is perpendicular to the tangent of the half-disk at C . Then the tangents lines of the half-disk and the quarter-circle coincide, and the half-disk is tangent to the quarter-circle. It is obvious from the diagram that the half-disk lies at least partially inside of the quarter-circle, the half-disk D_y is internally tangent to the quarter-circle.

Then the union of the half-disks is be a quarter-circle with radius 2, and has area π . □

4. [4] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC .

Answer: $\frac{216}{\sqrt{145}}$



The locus of points E such that $[AED] = [AEB]$ forms a line, since area is a linear function of the coordinates of E ; setting the areas equal gives a linear equation in the coordinates E^2 . Note that A and M , the midpoint of \overline{DB} , are on this line; A because both areas are 0, and M because the triangles share an altitude, and bases \overline{DM} and \overline{MB} are equal in length. Then \overline{AM} is the set of points satisfying the area condition. The point E , then, is such that $\triangle AEC$ is a right angle (to make the distance minimal) and E lies on \overline{AM} .

Let X be the point of intersection of \overline{AM} and \overline{CD} . Then $\triangle AMB \sim \triangle XMD$, and since $MD = BM$, they are in fact congruent. Thus $DX = AB = 10$, and $XC = 18$. Similarly, $BX = 15$, so $ABXD$ is a parallelogram. Let Y be the foot of the perpendicular from A to \overline{DC} , so that $DY = \frac{DC-AB}{2} = 9$. Then $AY = \sqrt{AD^2 - DY^2} = \sqrt{225 - 81} = 12$. Then $YX = DX - DY = 1$ and $AX = \sqrt{AY^2 + YX^2} = \sqrt{144 + 1} = \sqrt{145}$. Since both $\triangle AXD$ and $\triangle CXE$ have a right angle, and $\angle EXC$ and $\angle YXA$ are congruent because they are vertical angles, $\triangle AXD \sim \triangle CXE$. Then $\frac{CE}{AX} = \frac{CX}{AY}$, so $CE = 12 \cdot \frac{18}{\sqrt{145}} = \frac{216}{\sqrt{145}}$.

5. [4] A sphere is the set of points at a fixed positive distance r from its center. Let \mathcal{S} be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of \mathcal{S} is a finite number n . Find the maximum possible value of n .

Answer: [2] The answer is 2 for any number of dimensions. We prove this by induction on the dimension.

Note that 1-dimensional spheres are pairs of points, and 2-dimensional spheres are circles.

Base case, $d = 2$: The intersection of two circles is either a circle (if the original circles are identical, and in the same place), a pair of points, a single point (if the circles are tangent), or the empty set. Thus, in dimension 2, the largest finite number of intersection points is 2, because the number of pairwise intersection points is 0, 1, or 2 for distinct circles.

We now prove that the intersection of two k -dimensional spheres is either the empty set, a $(k - 1)$ -dimensional sphere, a k -dimensional sphere (which only occurs if the original spheres are identical and coincident). Consider two spheres in k -dimensional space, and impose a coordinate system such that the centers of the two spheres lie on one coordinate axis. Then the equations for the two spheres become identical in all but one coordinate:

$$\begin{aligned}(x_1 - a_1)^2 + x_2^2 + \dots + x_k^2 &= r_1^2 \\(x_1 - a_2)^2 + x_2^2 + \dots + x_k^2 &= r_2^2\end{aligned}$$

If $a_1 = a_2$, the spheres are concentric, and so they are either nonintersecting or coincident, intersecting in a k -dimensional sphere. If $a_1 \neq a_2$, then subtracting the equations and solving for x_1 yields $x_1 = \frac{r_1^2 - a_1^2 - r_2^2 + a_2^2}{2(a_2 - a_1)}$. Plugging this in to either equation above yields a single equation that describes a $(k - 1)$ -dimensional sphere.

Assume we are in dimension d , and suppose for induction that for all k less than d , any two distinct k -dimensional spheres intersecting in a finite number of points intersect in at most two points. Suppose we have a collection of d -dimensional spheres s_1, s_2, \dots, s_m . Without loss of generality, suppose the s_i are distinct. Let t_i be the intersection of s_i and s_{i+1} for $1 \leq i < m$. If any t_i are the empty set, then the intersection of the t_i is empty. None of the t_i is a d -dimensional sphere because the s_i are distinct. Thus each of t_1, t_2, \dots, t_{m-1} is a $(d - 1)$ -dimensional sphere, and the intersection of all of them is the same as the intersection of the d -dimensional spheres. We can then apply the inductive hypothesis to find that t_1, \dots, t_{m-1} intersect in at most two points. Thus, by induction, a set of spheres in any dimension which intersect at only finitely many points intersect at at most two points.

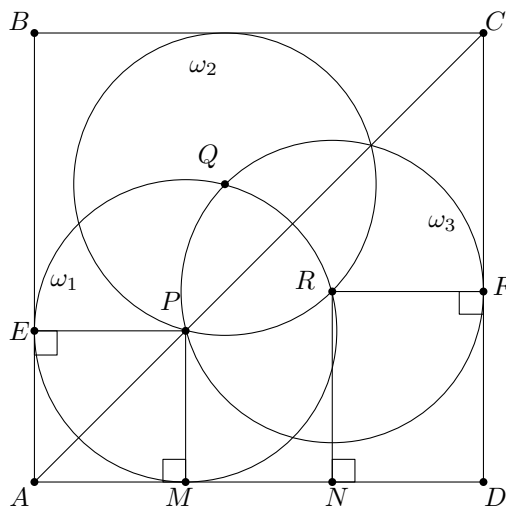
We now exhibit a set of 2^{2009} 2010-dimensional spheres, and prove that their intersection contains exactly two points. Take the spheres with radii $\sqrt{2013}$ and centers $(0, \pm 1, \pm 1, \dots, \pm 1)$, where the sign of each coordinate is independent from the sign of every other coordinate. Because of our choice of radius, all these spheres pass through the points $(\pm 2, 0, 0, \dots, 0)$. Then the intersection is the set of

²The notation $[A_1A_2 \dots A_n]$ means the area of polygon $A_1A_2 \dots A_n$.

points $(x_1, x_2, \dots, x_{2010})$ which satisfy the equations $x_1^2 + (x_2 \pm 1)^2 + \dots + (x_{2010} \pm 1)^2 = 2013$. The only solutions to these equations are the points $(\pm 2, 0, 0, \dots, 0)$ (since $(x_i + 1)^2$ must be the same as $(x_i - 1)^2$ for all $i > 1$, because we may hold all but one of the \pm choices constant, and change the remaining one). \square

6. [5] Three unit circles ω_1, ω_2 , and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3 . Find the side length of the square S .

Answer: $\boxed{\frac{\sqrt{6} + \sqrt{2} + 8}{4}}$



By the Pigeonhole Principle, two of the sides must be tangent to the same circle, say ω_1 . Since S surrounds the circles, these two sides must be adjacent, so we can let A denote the common vertex of the two sides tangent to ω_1 . Let B, C , and D be the other vertices of S in clockwise order, and let P, Q , and R be the centers of ω_1, ω_2 , and ω_3 respectively, and suppose WLOG that they are also in clockwise order. Then AC passes through the center of ω_1 , and by symmetry (since $AB = AD$) it must also pass through the other intersection point of ω_2 and ω_3 . That is, AC is the radical axis of ω_2 and ω_3 .

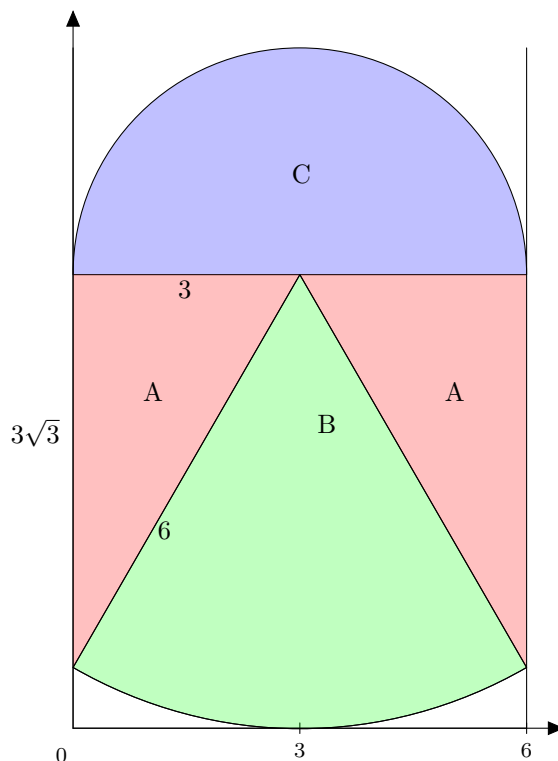
Now, let M and N be the feet of the perpendiculars from P and R , respectively, to side AD . Let E and F be the feet of the perpendiculars from P to AB and from R to DC , respectively. Then $PEAM$ and $NRFD$ are rectangles, and PE and RF are radii of ω_1 and ω_2 respectively. Thus $AM = EP = 1$ and $ND = RF = 1$. Finally, we have

$$\begin{aligned}
 MN &= PR \cdot \cos(180^\circ - \angle EPR) \\
 &= \cos(180^\circ - \angle EPQ - \angle RPQ) \\
 &= -\cos((270^\circ - 60^\circ)/2 + 60^\circ) \\
 &= -\cos(165^\circ) \\
 &= \cos(15^\circ) \\
 &= \frac{\sqrt{6} + \sqrt{2}}{4}.
 \end{aligned}$$

Thus $AD = AM + MN + ND = 1 + \frac{\sqrt{6} + \sqrt{2}}{4} + 1 = \frac{\sqrt{6} + \sqrt{2} + 8}{4}$ as claimed. \square

7. [6] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

Answer: $9\sqrt{3} + \frac{21\pi}{2}$



If you draw concentric circles around the destination point, the condition is equivalent to the restriction that you must always go inwards towards the destination. In the diagram above, the regions through which you might pass are shaded.

We find the areas of regions A, B, and C separately, and add them up (doubling the area of region A, because there are two of them).

The hypotenuse of triangle A is of length 6, and the base is of length 3, so it is a $\frac{\pi}{6}$ - $\frac{\pi}{3}$ - $\frac{\pi}{2}$ triangle (30-60-90 triangle) with area $\frac{9\sqrt{3}}{2}$. Then the total area of the regions labeled A is $9\sqrt{3}$.

Since the angle of triangle A nearest the center of the circle (the destination point) is $\frac{\pi}{3}$, sector B has central angle $\frac{\pi}{3}$. Then the area of sector B is $\frac{1}{2}r^2\theta = \frac{1}{2} \cdot 36 \cdot \frac{\pi}{3} = 6\pi$.

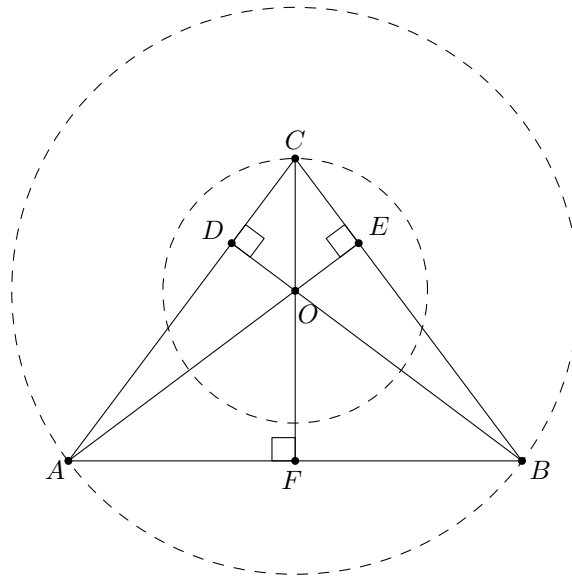
Region C is a half-disc of radius 3, so its area is $\frac{9\pi}{2}$.

Thus, the total area is $9\sqrt{3} + \frac{21\pi}{2}$.

8. [6] Let O be the point $(0, 0)$. Let A, B, C be three points in the plane such that $AO = 15$, $BO = 15$, and $CO = 7$, and such that the area of triangle ABC is maximal. What is the length of the shortest side of ABC ?

Answer: $\boxed{20}$ We claim that O should be the orthocenter of the triangle ABC . If O is not on an altitude of $\triangle ABC$, suppose (without loss of generality) that \overline{AO} is not perpendicular to \overline{BC} . We can

rotate A around O , leaving B and C fixed, to make \overline{AO} perpendicular to \overline{BC} , which strictly increases the area. Therefore, if $[ABC]$ is maximal then $\triangle ABC$ is an isosceles triangle with orthocenter O and base \overline{AB} .

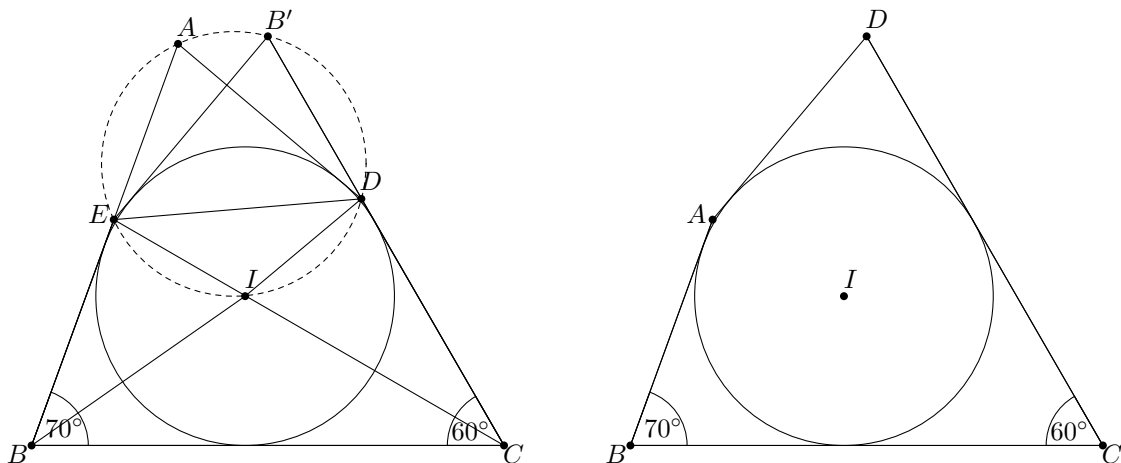


Let F be the foot of the perpendicular from C to \overline{AB} . Since $\angle FOA$ and $\angle COE$ are vertical, $\angle FAO = \angle OCE$. Then $\triangle FAO$ is similar to $\triangle FCB$, so we have $\frac{AF}{OF} = \frac{CF}{BF} = \frac{OF+I}{AF}$, so $AF^2 = OF^2 + 7 \cdot OF$. Since $AF^2 = 225 - OF^2$, $2 \cdot OF^2 + 7 \cdot OF - 225 = 0$, so $OF = 9$. Then $AF = 12$, so $AB = 24$ and $BC = 20$. Thus, the length of the shortest side of $\triangle ABC$ is 20.

9. [7] Let $ABCD$ be a quadrilateral with an inscribed circle centered at I . Let CI intersect AB at E . If $\angle IDE = 35^\circ$, $\angle ABC = 70^\circ$, and $\angle BCD = 60^\circ$, then what are all possible measures of $\angle CDA$?

Answer: 70° and 160°

Arbitrarily defining B and C determines I and E up to reflections across BC . D lies on both the circle determined by $\angle EDI = 35^\circ$ and the line through C tangent to the circle (and on the opposite side of B); since the intersection of a line and a circle has at most two points, there are only two cases for $ABCD$. The diagram below on the left shows the construction made in this solution, containing both cases. The diagram below on the right shows only the degenerate case.



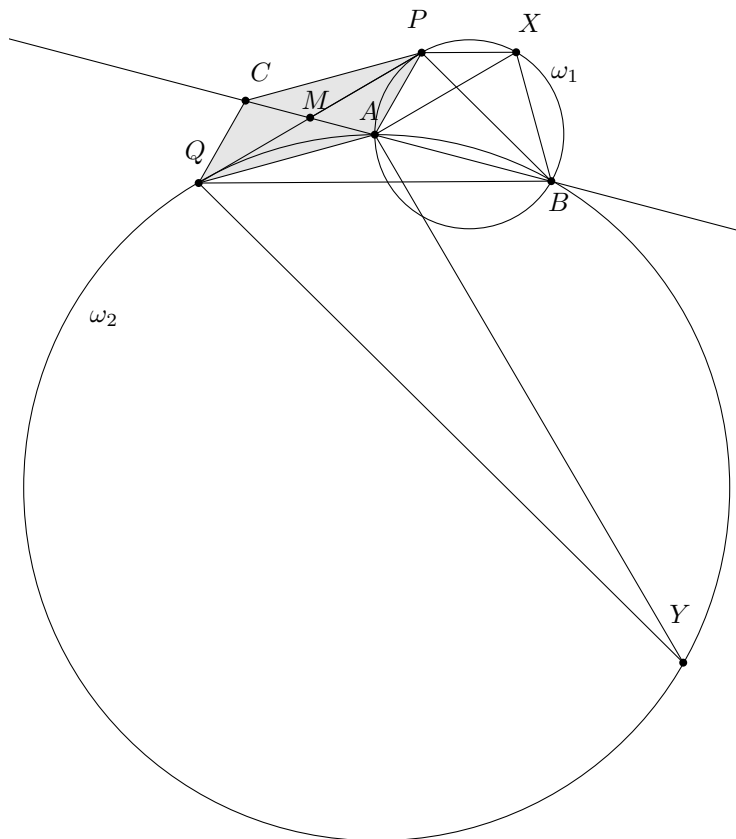
Reflect B across EC to B' ; then $CB = CB'$. Since BA and BC are tangent to the circle centered at I , IB is the angle bisector of $\angle ABC$. Then $\angle IBE = \angle IB'E = 35^\circ$. If $B' = D$, then $\angle ADC = \angle EB'C = 70^\circ$. Otherwise, since $\angle IB'E = 35^\circ = \angle IDE$ (given), $EB'DI$ is a cyclic quadrilateral. Then $\angle IED = \angle IB'D = 35^\circ$ and $\angle BCI = \angle ECD = 30^\circ$, so $\triangle CED \sim \triangle CBI$.

Since $\angle CID$ is exterior to $\triangle DIE$, $\angle CID = \angle IDE + \angle DEI = 70^\circ$. Then $\triangle CDI \sim \triangle CEB$. Because $EB'DI$ is cyclic, $\angle IDC = \angle IEB' = \angle IEB = 180^\circ - 70^\circ - 30^\circ = 80^\circ$. Then $\angle ADC = 2\angle IDC = 160^\circ$.

Thus, the two possible measures are 70° and 160° .

10. [8] Circles ω_1 and ω_2 intersect at points A and B . Segment PQ is tangent to ω_1 at P and to ω_2 at Q , and A is closer to PQ than B . Point X is on ω_1 such that $PX \parallel QB$, and point Y is on ω_2 such that $QY \parallel PB$. Given that $\angle APQ = 30^\circ$ and $\angle PQA = 15^\circ$, find the ratio AX/AY .

Answer: $2 - \sqrt{3}$



Let C be the fourth vertex of parallelogram $APCQ$. The midpoint M of \overline{PQ} is the intersection of the diagonals of this parallelogram. Because M has equal power³ with respect to the two circles ω_1 and ω_2 , it lies on \overleftrightarrow{AB} , the circles' radical axis⁴. Therefore, C lies on \overleftrightarrow{AB} as well.

Using a series of parallel lines and inscribed arcs, we have:

$$\angle APC = \angle APQ + \angle CPQ = \angle APQ + \angle PQA = \angle ABP + \angle QBA = \angle PBQ = \angle XPB,$$

where the last equality follows from the fact that $PX \parallel QB$.

³http://en.wikipedia.org/wiki/Power_of_a_point

⁴http://en.wikipedia.org/wiki/Radical_axis

We also know that $\angle BXP = 180^\circ - \angle PAB = \angle CAP$, so triangles BXP and CAP are similar. By the spiral similarity theorem⁵, triangles BPC and XPA are similar, too.

By analogous reasoning, triangles BQC and YQA are similar. Then we have:

$$\frac{AX}{AY} = \frac{AX/BC}{AY/BC} = \frac{AP/CP}{AQ/CQ} = \frac{AP^2}{AQ^2}$$

where the last inequality holds because $APCQ$ is a parallelogram. Using the Law of Sines, the last expression equals $\frac{\sin^2 15^\circ}{\sin^2 30^\circ} = 2 - \sqrt{3}$.

⁵This theorem states that if $\triangle PAB$ and $\triangle PXY$ are similar and oriented the same way, then $\triangle PAX$ and $\triangle PBY$ are similar too. It is true because the first similarity implies that $AP/BP = XP/YP$ and $\angle APB = \angle XPY$, which proves the second similarity.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
Guts Round

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 13th ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 20 FEBRUARY 2010 — GUTS ROUND

1. [4] If $A = 10^9 - 987654321$ and $B = \frac{123456789+1}{10}$, what is the value of \sqrt{AB} ?

Answer: 12345679 Both A and B equal 12345679, so $\sqrt{AB} = 12345679$ as well.

2. [4] Suppose that $x, y,$ and z are non-negative real numbers such that $x + y + z = 1$. What is the maximum possible value of $x + y^2 + z^3$?

Answer: 1 Since $0 \leq y, z \leq 1$, we have $y^2 \leq y$ and $z^3 \leq z$. Therefore $x + y^2 + z^3 \leq x + y + z = 1$. We can get $x + y^2 + z^3 = 1$ by setting $(x, y, z) = (1, 0, 0)$.

3. [4] In a group of people, there are 13 who like apples, 9 who like blueberries, 15 who like cantaloupe, and 6 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes blueberries also likes exactly one of apples and cantaloupe. Each person who likes cantaloupe also likes exactly one of blueberries and dates. Find the minimum possible number of people in the group.

Answer: 22 Everyone who likes cantaloupe likes exactly one of blueberries and dates. However, there are 15 people who like cantaloupe, 9 who like blueberries, and 6 who like dates. Thus, everyone who likes blueberries or dates must also like cantaloupes (because if any of them didn't, we would end up with less than 15 people who like cantaloupe).

Since everyone who likes blueberries likes cantaloupes, none of them can like apples. However, the 6 people who like both cantaloupe and dates can also like apples. So, we could have a group where 7 people like apples alone, 9 like blueberries and cantaloupe, and 6 like apples, cantaloupe, and dates. This gives 22 people in the group, which is optimal.

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 13th ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 20 FEBRUARY 2010 — GUTS ROUND

4. [5] To set up for a Fourth of July party, David is making a string of red, white, and blue balloons. He places them according to the following rules:

- No red balloon is adjacent to another red balloon.
- White balloons appear in groups of exactly two, and groups of white balloons are separated by at least two non-white balloons.
- Blue balloons appear in groups of exactly three, and groups of blue balloons are separated by at least three non-blue balloons.

If David uses over 600 balloons, determine the smallest number of red balloons that he can use.

Answer: 99 It is possible to achieve 99 red balloons with the arrangement

$$\text{WWBBBW} \underbrace{\text{RBBWWRBBBW} \dots \text{RBBBW}}_{99 \text{ RBBBW's}}$$

which contains $99 \cdot 6 + 7 = 601$ balloons.

Now assume that one can construct a chain with 98 or fewer red balloons. Then there can be 99 blocks of non-red balloons, which in total must contain more than 502 balloons. The only valid combinations of white and blue balloons are WWBBB, BBBWW, and WWBBBW (Any others contain the subsequence BBBWWBBB, which is invalid). The sequence $\dots \text{WWR}$ must be followed by BBBWW; otherwise two groups of white balloons would be too close. Similarly, the sequence

RWW ... must be preceded by WWBBB. It follows that WWBBBWW can be used at most once in a valid sequence, meaning that there can be at most $98 \cdot 5 + 7 = 497$ non-red balloons. Contradiction. Therefore the minimum is $\boxed{99}$ red balloons. (Better if the party's outdoors; then we'd have 99 red balloons floating in the summer sky. :-p)

5. [5] You have a length of string and 7 beads in the 7 colors of the rainbow. You place the beads on the string as follows – you randomly pick a bead that you haven't used yet, then randomly add it to either the left end or the right end of the string. What is the probability that, at the end, the colors of the beads are the colors of the rainbow in order? (The string cannot be flipped, so the red bead must appear on the left side and the violet bead on the right side.)

Answer: $\boxed{\frac{1}{5040}}$ The threading method does not depend on the colors of the beads, so at the end all configurations are equally likely. Since there are $7! = 5040$ configurations in total, the probability of any particular configuration is $\frac{1}{5040}$.

6. [5] How many different numbers are obtainable from five 5s by first concatenating some of the 5s, then multiplying them together? For example, we could do $5 \cdot 55 \cdot 55$, $555 \cdot 55$, or 55555 , but not $5 \cdot 5$ or 2525 .

Answer: $\boxed{7}$ If we do 55555 , then we're done.

Note that 5, 55, 555, and 5555 all have completely distinguishable prime factorizations. This means that if we are given a product of them, we can obtain the individual terms. The number of 5555's is the exponent of 101, the number of 555's is the exponent of 37, the number of 55's is the exponent of 11 minus the exponent of 101, and the number of 5's is just whatever we need to get the proper exponent of 5. Then the answer is the number of ways we can split the five 5's into groups of at least one. This is the number of unordered partitions of 5, which is 7.

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7. [6] What are the last 8 digits of

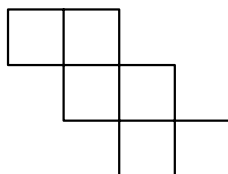
$$11 \times 101 \times 1001 \times 10001 \times 100001 \times 1000001 \times 111?$$

Answer: $\boxed{19754321}$ Multiply terms in a clever order.

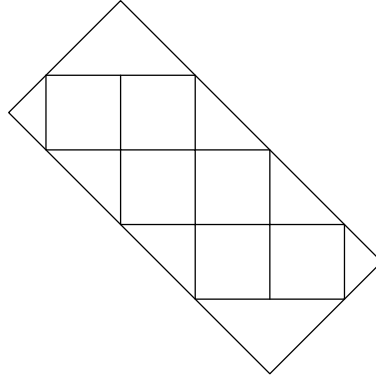
$$\begin{aligned} 11 \cdot 101 \cdot 10001 &= 11,111,111, \\ 111 \cdot 1001 \cdot 1000001 &= 111,111,111,111. \end{aligned}$$

The last eight digits of 11, 111, 111 · 111, 111, 111, 111 are 87654321. We then just need to compute the last 8 digits of $87654321 \cdot 100001 = 87654321 + \dots 32100000$, which are 19754321.

8. [6] Each square in the following hexomino has side length 1. Find the minimum area of any rectangle that contains the entire hexomino.



Answer: $\boxed{\frac{21}{2}}$ If a rectangle contains the entire hexomino, it must also contain its convex hull, which is an origin-symmetric hexagon. It is fairly clear that the smallest rectangle that contains such a hexagon must share one set of parallel sides with the hexagon. There are three such rectangles, and checking them all, we find that the one shown below is the smallest. It has area $\frac{21}{2}$.



9. [6] Indecisive Andy starts out at the midpoint of the 1-unit-long segment \overline{HT} . He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H , and if the coin is tails, he moves halfway towards endpoint T . After his 2010 moves, what is the expected distance between Andy and the midpoint of \overline{HT} ?

Answer: $\boxed{\frac{1}{4}}$ Let Andy's position be x units from the H end after 2009 flips. If Andy moves towards the H end, he ends up at $\frac{x}{2}$, a distance of $\frac{1-x}{2}$ from the midpoint. If Andy moves towards the T end, he ends up at $\frac{1+x}{2}$, a distance of $\frac{x}{2}$ from the midpoint. His expected distance from the midpoint is then

$$\frac{\frac{1-x}{2} + \frac{x}{2}}{2} = \frac{1}{4}.$$

Since this does not depend on x , $\frac{1}{4}$ is the answer.

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10. [7] Let ABC be a triangle with $AB = 8$, $BC = 15$, and $AC = 17$. Point X is chosen at random on line segment AB . Point Y is chosen at random on line segment BC . Point Z is chosen at random on line segment CA . What is the expected area of triangle XYZ ?

Answer: $\boxed{15}$ Let $\mathbb{E}(X)$ denote the expected value of X , and let $[S]$ denote the area of S . Then

$$\begin{aligned} \mathbb{E}([\triangle XYZ]) &= \mathbb{E}([\triangle ABC] - [\triangle XYB] - [\triangle ZYC] - [\triangle XBZ]) \\ &= [\triangle ABC] - \mathbb{E}([\triangle XYB]) - \mathbb{E}([\triangle ZYC]) - [\triangle XBZ], \end{aligned}$$

where the last step follows from linearity of expectation¹. But $[\triangle XYB] = \frac{1}{2} \cdot BX \cdot BY \cdot \sin(B)$. The $\frac{1}{2} \sin(B)$ term is constant, and BX and BY are both independent with expected values $\frac{AB}{2}$ and $\frac{BC}{2}$, respectively. Thus $\mathbb{E}([\triangle XYB]) = \frac{1}{8} AB \cdot BC \cdot \sin(B) = \frac{1}{4} [\triangle ABC]$. Similarly, $\mathbb{E}([\triangle ZYC]) = \mathbb{E}([\triangle ZBX]) = \frac{1}{4} [\triangle ABC]$.

Then we have $\mathbb{E}([\triangle XYZ]) = (1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{4}) [\triangle ABC] = \frac{1}{4} [\triangle ABC] = 15$.

Note: We can also solve this problem (and the more general case of polygons) by noting that the area of XYZ is linear in the coordinates of X , Y , and Z , so the expected area of XYZ is the same as the area of $X'Y'Z'$, where X' is the expected location of X , Y' is the expected location of Y , and Z' is the expected location of Z . In our case, this corresponds to the midpoints of the three sides AB , BC , and CA .

¹See <http://www-math.mit.edu/~spielman/AdvComplexity/handout.ps> for an introduction to linearity of expectation and other important tools in probability

11. [7] From the point (x, y) , a *legal move* is a move to $(\frac{x}{3} + u, \frac{y}{3} + v)$, where u and v are real numbers such that $u^2 + v^2 \leq 1$. What is the area of the set of points that can be reached from $(0, 0)$ in a finite number of legal moves?

Answer: $\boxed{\frac{9\pi}{4}}$ We claim that the set of points is the disc with radius $\frac{3}{2}$ centered at the origin, which clearly has area $\frac{9\pi}{4}$.

First, we show that the set is contained in this disc. This is because if we are currently at a distance of r from the origin, then we can't end up at a distance of greater than $\frac{r}{3} + 1$ from the origin after a single move. Since $\frac{r}{3} + 1 < \frac{3}{2}$ if $r < \frac{3}{2}$, we will always end up in the disc of radius $\frac{3}{2}$ if we start in it. Since the origin is inside this disc, any finite number of moves will leave us inside this disc.

Next, we show that all points in this disc can be reached in a finite number of moves. Indeed, after one move we can get all points within a distance of 1. After two moves, we can get all points within a distance of $\frac{4}{3}$. After three moves, we can get all points within a distance of $\frac{13}{9}$. In general, after n moves we can get all points within a distance of $\frac{3}{2} - \frac{1}{2 \cdot 3^{n-1}}$. This means that for any distance $d < \frac{3}{2}$, we will eventually get all points within a distance of d , so all points in the disc of radius $\frac{3}{2}$ can be reached after some number of moves.

12. [7] How many different collections of 9 letters are there? A letter can appear multiple times in a collection. Two collections are equal if each letter appears the same number of times in both collections.

Answer: $\boxed{\binom{34}{9}}$ We put these collections in bijections with binary strings of length 34 containing 9 zeroes and 25 ones. Take any such string - the 9 zeroes will correspond to the 9 letters in the collection. If there are n ones before a zero, then that zero corresponds to the $(n + 1)$ st letter of the alphabet. This scheme is an injective map from the binary strings to the collections, and it has an inverse, so the number of collections is $\binom{34}{9}$.

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13. [8] A triangle in the xy -plane is such that when projected onto the x -axis, y -axis, and the line $y = x$, the results are line segments whose endpoints are $(1, 0)$ and $(5, 0)$, $(0, 8)$ and $(0, 13)$, and $(5, 5)$ and $(7.5, 7.5)$, respectively. What is the triangle's area?

Answer: $\boxed{\frac{17}{2}}$ Sketch the lines $x = 1$, $x = 5$, $y = 8$, $y = 13$, $y = 10 - x$, and $y = 15 - x$. The triangle has to be contained in the hexagonal region contained in all these lines. If all the projections are correct, every other vertex of the hexagon must be a vertex of the triangle, which gives us two possibilities for the triangle. One of these triangles has vertices at $(2, 8)$, $(1, 13)$, and $(5, 10)$, and has an area of $\frac{17}{2}$. It is easy to check that the other triangle has the same area, so the answer is unique.

14. [8] In how many ways can you fill a 3×3 table with the numbers 1 through 9 (each used once) such that all pairs of adjacent numbers (sharing one side) are relatively prime?

Answer: $\boxed{2016}$ The numbers can be separated into four sets. Numbers in the set $A = \{1, 5, 7\}$ can be placed next to anything. The next two sets are $B = \{2, 4, 8\}$ and $C = \{3, 9\}$. The number 6, which forms the final set D , can only be placed next to elements of A . The elements of each group can be interchanged without violating the condition, so without loss of generality, we can pretend we have three 1's, three 2's, two 3's, and one 6, as long as we multiply our answer by $3!3!2!$ at the end. The available arrangements are, grouped by the position of the 6, are:

When 6 is in contact with three numbers:

1	2	3
6	1	2
1	2	3

When 6 is in contact with two numbers:

6	1	2
1	2	3
2	3	1

6	1	2
1	1	3
2	3	2

The next two can be flipped diagonally to create different arrangements:

6	1	2
1	2	3
1	3	2

6	1	2
1	2	3
3	1	2

Those seven arrangements can be rotated 90, 180, and 270 degrees about the center to generate a total of 28 arrangements. $28 \cdot 3!3!2! = 2016$.

15. [8] Pick a random integer between 0 and 4095, inclusive. Write it in base 2 (without any leading zeroes). What is the expected number of consecutive digits that are not the same (that is, the expected number of occurrences of either 01 or 10 in the base 2 representation)?

Answer: $\frac{20481}{4096}$ Note that every number in the range can be written as a 12-digit binary string. For $i = 1, 2, \dots, 11$, let R_i be a random variable which is 1 if the i th and $(i + 1)$ st digits differ in a randomly chosen number in the range. By linearity of expectation, $E(\sum_i R_i) = \sum E(R_i)$.

Since we choose every binary string of length 12 with equal probability, the sum of the expectations is $\frac{11}{2}$. However, this is not the expected number of 01s and 10s - we need to subtract the occasions where the leading digit is zero. There is a $\frac{1}{2}$ chance that the number starts with a 0, in which case we must ignore the first digit change - unless the number was 0, in which case there are no digit changes. Therefore, our answer is $\frac{11}{2} - \frac{1}{2} + \frac{1}{4096} = \frac{20481}{4096}$.

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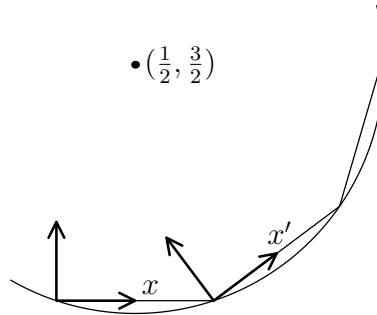
16. [9] Jessica has three marbles colored red, green, and blue. She randomly selects a non-empty subset of them (such that each subset is equally likely) and puts them in a bag. You then draw three marbles from the bag with replacement. The colors you see are red, blue, red. What is the probability that the only marbles in the bag are red and blue?

Answer: $\frac{27}{35}$ There are two possible sets of marbles in the bag, {red,blue} and {red,blue,green}. Initially, both these sets are equally likely to be in the bag. However, the probability of red, blue, red being drawn from a set S of marbles is proportional to $|S|^{-3}$, as long as red and blue are both in S . By Bayes's Rule, we must weight the probability of these two sets by $|S|^{-3}$. The answer is $\frac{(1/2)^3}{(1/2)^3 + (1/3)^3}$.

17. [9] An ant starts at the origin, facing in the positive x -direction. Each second, it moves 1 unit forward, then turns counterclockwise by $\sin^{-1}(\frac{3}{5})$ degrees. What is the least upper bound on the distance between the ant and the origin? (The least upper bound is the smallest real number r that is at least as big as every distance that the ant ever is from the origin.)

Answer: $\sqrt{\frac{10}{2}}$ We claim that the points the ant visits lie on a circle of radius $\frac{\sqrt{10}}{2}$. We show this by saying that the ant stays a constant distance $\frac{\sqrt{10}}{2}$ from the point $(\frac{1}{2}, \frac{3}{2})$.

Suppose the ant moves on a plane P . Consider a transformation of the plane P' such that after the first move, the ant is at the origin of P' and facing in the direction of the x' axis (on P'). The transformation to get from P to P' can be gotten by rotating P about the origin counterclockwise through an angle $\sin^{-1}(\frac{3}{5})$ and then translating it 1 unit to the right. Observe that the point $(\frac{1}{2}, \frac{3}{2})$ is fixed under this transformation, which can be shown through the expression $(\frac{1}{2} + \frac{3}{2}i)(\frac{4}{5} + \frac{3}{5}i) + 1 = \frac{1}{2} + \frac{3}{2}i$. It follows that at every point the ant stops, it will always be the same distance from $(\frac{1}{2}, \frac{3}{2})$. Since it starts at $(0, 0)$, this fixed distance is $\frac{\sqrt{10}}{2}$.



Since $\sin^{-1}(\frac{3}{5})$ is not a rational multiple of π , the points the ant stops at form a dense subset of the circle in question. As a result, the least upper bound on the distance between the ant and the origin is the diameter of the circle, which is $\sqrt{10}$.

18. [9] Find two lines of symmetry of the graph of the function $y = x + \frac{1}{x}$. Express your answer as two equations of the form $y = ax + b$.

Answer: $y = (1 + \sqrt{2})x$ and $y = (1 - \sqrt{2})x$ The graph of the function $y = x + \frac{1}{x}$ is a hyperbola. We can see this more clearly by writing it out in the standard form $x^2 - xy + 1 = 0$ or $(\frac{y}{2})^2 - (x - \frac{1}{2}y)^2 = 1$. The hyperbola has asymptotes given by $x = 0$ and $y = x$, so the lines of symmetry will be the (interior and exterior) angle bisectors of these two lines. This means that they will be $y = \tan(67.5^\circ)x$ and $y = -\cot(67.5^\circ)x$, which, using the tangent half-angle formula $\tan(\frac{x}{2}) = \sqrt{\frac{1+\cos(x)}{1-\cos(x)}}$, gives the two lines $y = (1 + \sqrt{2})x$ and $y = (1 - \sqrt{2})x$.

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19. [10] A 5-dimensional ant starts at one vertex of a 5-dimensional hypercube of side length 1. A *move* is when the ant travels from one vertex to another vertex at a distance of $\sqrt{2}$ away. How many ways can the ant make 5 moves and end up on the same vertex it started at?

Answer: $\boxed{6240}$ We let the cube lie in \mathbb{R}^5 with each corner with coordinates 1 or 0. Assume the ant starts at $(0, 0, 0, 0, 0)$. Every move the ant adds or subtracts 1 to two of the places. Note that this means the ant can only land on a vertex with the sum of its coordinates an even number. Every move the ant has $\binom{5}{2} = 10$ choices.

From any vertex there are 10 two-move sequences that put the ant at the same vertex it started at.

There are 6 two-move sequences to move from one vertex to a different, chosen vertex. If your chosen vertex differs from your current vertex by 2 of the 5 coordinates, your first move corrects for one of these two. There are 2 ways to choose which coordinate to correct for on the first move, and there are 3 ways to choose the second coordinate you change, yielding 6 sequences. If your chosen vertex differs from your current vertex by 4 of the 5 coordinates, each move corrects for two of these four. This yields $\binom{4}{2} = 6$ sequences.

Finally, there are 60 three-move sequences that put the ant at the same vertex it started at. There are 10 ways to choose the first move, and there are 6 ways to make two moves to return to your original position.

The motion of the ant can be split into two cases.

Case 1: After the 3rd move the ant is on the vertex it started at. There are $(60)(10) = 600$ different possible paths.

Case 2: After the third move the ant is on a vertex different from the one it started on. There are $(10^3 - 60)(6) = (940)(6) = 5640$ different possible paths.

So there are 6240 total possible paths.

20. [10] Find the volume of the set of points (x, y, z) satisfying

$$\begin{aligned} x, y, z &\geq 0 \\ x + y &\leq 1 \\ y + z &\leq 1 \\ z + x &\leq 1 \end{aligned}$$

Answer: $\boxed{\frac{1}{4}}$ Without loss of generality, assume that $x \geq y$ - half the volume of the solid is on this side of the plane $x = y$. For each value of c from 0 to $\frac{1}{2}$, the region of the intersection of this half of the solid with the plane $y = c$ is a trapezoid. The trapezoid has height $1 - 2c$ and average base $\frac{1}{2}$, so it has an area of $\frac{1}{2} - c$.

The total volume of this region is $\frac{1}{2}$ times the average area of the trapezoids, which is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. Double that to get the total volume, which is $\frac{1}{4}$.

21. [10] Let $\triangle ABC$ be a scalene triangle. Let h_a be the locus of points P such that $|PB - PC| = |AB - AC|$. Let h_b be the locus of points P such that $|PC - PA| = |BC - BA|$. Let h_c be the locus of points P such that $|PA - PB| = |CA - CB|$. In how many points do all of $h_a, h_b,$ and h_c concur?

Answer: $\boxed{2}$ The idea is similar to the proof that the angle bisectors concur or that the perpendicular bisectors concur. Assume WLOG that $BC > AB > CA$. Note that h_a and h_b are both hyperbolas. Therefore, h_a and h_b intersect in four points (each branch of h_a intersects exactly once with each branch of h_b). Note that the branches of h_a correspond to the cases when $PB > PC$ and when $PB < PC$. Similarly, the branches of h_b correspond to the cases when $PC > PA$ and $PC < PA$.

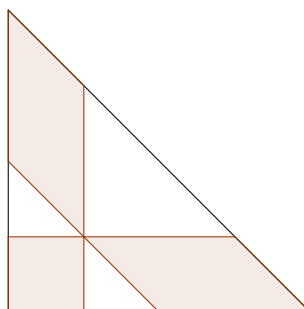
If either $PA < PB < PC$ or $PC < PB < PA$ (which each happens for exactly one point of intersection of h_a and h_b), then $|PC - PA| = |PC - PB| + |PB - PA| = |AB - AC| + |BC - BA| = |BC - AC|$, and so P also lies on h_c . So, exactly two of the four points of intersection of h_a and h_b lie on h_c , meaning that $h_a, h_b,$ and h_c concur in four points.

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22. [12] You are the general of an army. You and the opposing general both have an equal number of troops to distribute among three battlefields. Whoever has more troops on a battlefield always wins (you win ties). An *order* is an ordered triple of non-negative real numbers (x, y, z) such that $x + y + z = 1$, and corresponds to sending a fraction x of the troops to the first field, y to the second, and z to the third. Suppose that you give the order $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and that the other general issues an order chosen uniformly at random from all possible orders. What is the probability that you win two out of the three battles?

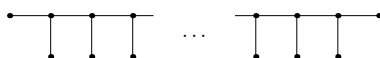
Answer: $\boxed{\frac{5}{8}}$ Let x be the portion of soldiers the opposing general sends to the first battlefield, and y the portion he sends to the second. Then $1 - x - y$ is the portion he sends to the third. Then $x \geq 0$, $y \geq 0$, and $x + y \leq 1$. Furthermore, you win if one of the three conditions is satisfied: $x \leq \frac{1}{4}$ and $y \leq \frac{1}{4}$, $x \leq \frac{1}{4}$ and $1 - x - y \leq \frac{1}{2}$, or $y \leq \frac{1}{4}$ and $1 - x - y \leq \frac{1}{2}$. This is illustrated in the picture below.



This triangle is a linear projection of the region of feasible orders, so it preserves area and probability ratios. The probability that you win, then is given by the portion of the triangle that satisfies one of the three above constraints — in other words, the area of the shaded region divided by the area of the entire triangle. We can easily calculate this to be $\frac{\frac{5}{16}}{\frac{1}{2}} = \frac{5}{8}$.

23. [12] In the country of Francisca, there are 2010 cities, some of which are connected by roads. Between any two cities, there is a unique path which runs along the roads and which does not pass through any city twice. What is the maximum possible number of cities in Francisca which have at least 3 roads running out of them?

Answer: 1004 The restrictions on how roads connect cities directly imply that the graph of the cities of Francisca with the roads as edges is a tree. Therefore the sum of the degrees of all the vertices is $2009 \cdot 2 = 4018$. Suppose that b vertices have degree ≥ 3 . The other $2010 - b$ vertices must have a degree of at least 1, so $3b + (2010 - b) \leq 4018$ or $2b \leq 2008$. So b is at most 1004. We can achieve $b = 1004$ with the following graph:



24. [12] Define a sequence of polynomials as follows: let $a_1 = 3x^2 - x$, let $a_2 = 3x^2 - 7x + 3$, and for $n \geq 1$, let $a_{n+2} = \frac{5}{2}a_{n+1} - a_n$. As n tends to infinity, what is the limit of the sum of the roots of a_n ?

Answer: $\frac{13}{3}$ By using standard methods for solving linear recurrences², we see that this recurrence has a characteristic polynomial of $x^2 - \frac{5}{2}x + 1 = (x - \frac{1}{2})(x - 2)$, hence $a_n(x) = c(x) \cdot 2^n + d(x) \cdot 2^{-n}$ for some polynomials c and d . Plugging in $n = 1$ and $n = 2$ gives

$$2c(x) + \frac{1}{2}d(x) = 3x^2 - x$$

and

$$4c(x) + \frac{1}{4}d(x) = 3x^2 - 7x + 3.$$

Subtracting the first equation from two times the second equation gives $6c(x) = 3x^2 - 13x + 6$, so $c(x) = \frac{3x^2 - 13x + 6}{6}$. As n grows large, the $c(x)2^n$ term dominates compared to the $d(x)2^{-n}$ term, so the roots of $a_n(x)$ converge to the roots of $c(x)$. Thus the roots of $a_n(x)$ converge to the roots of $3x^2 - 13x + 6$, which by Vieta's formula³ have a sum of $\frac{13}{3}$.

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25. [15] How many functions $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ have the property that $f(\{1, 2, 3\})$ and $f(f(\{1, 2, 3\}))$ are disjoint?

Answer: 94 Let $f(\{1, 2, 3\})$ be A . Then $A \cap f(A) = \emptyset$, so A must be a subset of $\{4, 5\}$. If $B = \{4, 5\}$, there are $2^3 - 2$ ways to assign each element in $\{1, 2, 3\}$ to a value in $\{4, 5\}$, and 9 ways to assign each element of $\{4, 5\}$ to a value in $\{1, 2, 3\}$, for a total of 54 choices of f . If $A = \{4\}$, there is 1 possible value for each element of $\{1, 2, 3\}$, 4 ways to assign $\{4\}$ with a value from $\{1, 2, 3, 5\}$, and 5 ways to assign a value to $\{5\}$. Similarly, if $A = \{5\}$, there are $4 \cdot 5 = 20$ choices for f . In total, there are $54 + 20 \cdot 2 = 94$ possible functions.

²See the sections "Linear homogeneous recurrence relations with constant coefficients" and "Theorem" at http://en.wikipedia.org/wiki/Recurrence_relation.

³See <http://mathworld.wolfram.com/VietasFormulas.html>.

26. [15] Express the following in closed form, as a function of x :

$$\sin^2(x) + \sin^2(2x) \cos^2(x) + \sin^2(4x) \cos^2(2x) \cos^2(x) + \cdots + \sin^2(2^{2010}x) \cos^2(2^{2009}x) \cdots \cos^2(2x) \cos^2(x).$$

Answer: $\boxed{1 - \frac{\sin^2(2^{2011}x)}{4^{2011} \sin^2(x)}}$ Note that

$$\begin{aligned} & \sin^2(x) + \sin^2(2x) \cos^2(x) + \cdots + \sin^2(2^{2010}x) \cos^2(2^{2009}x) \cdots \cos^2(2x) \cos^2(x) \\ &= (1 - \cos^2(x)) + (1 - \cos^2(2x)) \cos^2(x) + \cdots + (1 - \cos^2(2^{2010}x)) \cos^2(2^{2009}x) \cdots \cos^2(x), \end{aligned}$$

which telescopes to $1 - \cos^2(x) \cos^2(2x) \cos^2(4x) \cdots \cos^2(2^{2010}x)$. To evaluate $\cos^2(x) \cos^2(2x) \cdots \cos^2(2^{2010}x)$, multiply and divide by $\sin^2(x)$. We then get

$$1 - \frac{\sin^2(x) \cos^2(x) \cos^2(2x) \cdots \cos^2(2^{2010}x)}{\sin^2(x)}.$$

Using the double-angle formula for sin, we get that $\sin^2(y) \cos^2(y) = \frac{\sin^2(2y)}{4}$. Applying this 2011 times makes the above expression

$$1 - \frac{\sin^2(2^{2011}x)}{4^{2011} \sin^2(x)},$$

which is in closed form.

27. [15] Suppose that there are real numbers $a, b, c \geq 1$ and that there are positive reals x, y, z such that

$$\begin{aligned} a^x + b^y + c^z &= 4 \\ xa^x + yb^y + zc^z &= 6 \\ x^2a^x + y^2b^y + z^2c^z &= 9. \end{aligned}$$

What is the maximum possible value of c ?

Answer: $\boxed{\sqrt[3]{4}}$ The Cauchy-Schwarz inequality states that given 2 sequences of n real numbers x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n , then $(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) \geq (x_1y_1 + x_2y_2 + \cdots + x_ny_n)^2$ with equality holding if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \cdots = \frac{x_n}{y_n}$. Applying this to $\{a^{x/2}, b^{y/2}, c^{z/2}\}$ and $\{xa^{x/2}, yb^{y/2}, zc^{z/2}\}$ yields $(a^x + b^y + c^z)(x^2a^x + y^2b^y + z^2c^z) \geq (xa^x + yb^y + zc^z)^2$ with equality holding if and only if $x = y = z$.

However, equality does hold (both sides evaluate to 36), so $x = y = z$. The second equation then becomes $x(a^x + b^x + c^x) = 6$, which implies $x = \frac{3}{2}$. Then we have $a^{3/2} + b^{3/2} + c^{3/2} = 4$. To maximize c , we minimize a and b by setting $a = b = 1$. Then $c^{3/2} = 2$ or $c = \sqrt[3]{4}$.

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28. [18] Danielle Bellatrix Robinson is organizing a poker tournament with 9 people. The tournament will have 4 rounds, and in each round the 9 players are split into 3 groups of 3. During the tournament, each player plays every other player exactly once. How many different ways can Danielle divide the 9 people into three groups in each round to satisfy these requirements?

Answer: $\boxed{20160}$ We first split the 9 people up arbitrarily into groups of 3. There are $\frac{\binom{9}{3}\binom{6}{3}\binom{3}{3}}{3!} = 280$ ways of doing this. Without loss of generality, label the people 1 through 9 so that the first round groups are $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$. We will use this numbering to count the number of ways DBR can divide the 9 players in rounds 2, 3, and 4.

In round 2, because players 1, 2, and 3 are together in the first round, they must be in separate groups, and likewise for $\{4, 5, 6\}$ and $\{7, 8, 9\}$. Disregarding ordering of the three groups in a single round, we will first place 1, 2, and 3 into their groups, then count the number of ways to place $\{4, 5, 6\}$ and $\{7, 8, 9\}$ in the groups with them. We do this by placing one member from each of $\{4, 5, 6\}$ and $\{7, 8, 9\}$ into each group. There are $(3!)^2$ ways to do this. Now, because of symmetry, we can use the round 2 grouping $\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}$ to list out the remaining possibilities for round 3 and 4 groupings.

Casework shows that there are 2 ways to group the players in the remaining two rounds. We multiply $280 \cdot (3!)^2 \cdot 2$ to get 20160.

29. [18] Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

Answer: [29] The main idea is the following lemma:

Lemma. For any non-negative integer n and prime p , $\sum_{k=n+1}^{n+p^2-p} k^k \equiv 1 \pmod{p}$.

Proof. Note that a^b depends only on the value of $a \pmod{p}$ and the value of $b \pmod{p-1}$. Since p and $p-1$ are relatively prime, the Chinese Remainder Theorem⁴ implies that any p^2-p consecutive integers will take on each possible pair of a residue mod p and a residue mod $p-1$. In other words, if we let $(a, b) = (k \pmod{p}, k \pmod{p-1})$, then as k ranges through p^2-p consecutive integers, (a, b) will range through all p^2-p ordered pairs of residues mod p and residues mod $p-1$. This implies that

$$\sum_{k=n+1}^{n+p^2-p} k^k \equiv \sum_{b=1}^{p-1} \sum_{a=1}^p a^b.$$

It is well-known that $\sum_{a=1}^p a^b = \begin{cases} -1 & p-1 \mid b \\ 0 & p-1 \nmid b \end{cases}$. We will sketch a proof here. When $p-1 \mid b$, the result follows from Fermat's Little Theorem⁵. When $p-1 \nmid b$, it suffices to consider the case when $b \mid p-1$, since the b th powers mod p are the same as the $\gcd(b, p-1)$ th powers mod p , and there are an equal number of every non-zero b th power. But in this case, the b th powers are just the solutions to $x^{\frac{p-1}{b}} - 1$, which add up to zero by Vieta's formulas.

Now, using the formula for $\sum a^b$, we get that

$$\sum_{b=1}^{p-1} \sum_{a=1}^p a^b \equiv -1 \pmod{p},$$

which completes the lemma. □

We now apply the lemma with $p = 101$ and $n = 3, 10103$, and 20103 to get that $\sum_{k=1}^{30303} k^k \equiv \left(\sum_{k=1}^3 k^k\right) - 3$. But $\sum_{k=1}^3 k^k = 1^1 + 2^2 + 3^3 = 1 + 4 + 27 = 32$, so the answer is $32 - 3 = 29$.

30. [18] A monomial term $x_{i_1} x_{i_2} \dots x_{i_k}$ in the variables x_1, x_2, \dots, x_8 is *square-free* if i_1, i_2, \dots, i_k are distinct. (A constant term such as 1 is considered square-free.) What is the sum of the coefficients of the square-free terms in the following product?

$$\prod_{1 \leq i < j \leq 8} (1 + x_i x_j)$$

⁴See http://en.wikipedia.org/wiki/Chinese_remainder_theorem.

⁵See http://en.wikipedia.org/wiki/Fermat's_little_theorem.

Answer: 764 Let a_n be the sum of the coefficients of the square-terms in the product $\prod_{1 \leq i < j \leq n} (1 + x_i x_j)$. Square-free terms in this product come in two types: either they include x_n , or they do not. The sum of the coefficients of the terms that include x_n is $(n - 1)a_{n-2}$, since we can choose any of the $n - 1$ other variables to be paired with x_n , and then choose any square-free term from the product taken over the other $n - 2$ variables. The sum of the coefficients of the terms that do not include x_n are a_{n-1} , because they all come from the product over the other $n - 1$ variables. Therefore, $a_n = a_{n-1} + (n - 1)a_{n-2}$.

We use this recursion to find a_8 . As base cases, a_0 and a_1 are both 1. Then $a_2 = 2$, $a_3 = 4$, $a_4 = 10$, $a_5 = 26$, $a_6 = 76$, $a_7 = 232$, and finally, $a_8 = 764$.

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31. [21] In the Democratic Republic of Irun, 5 people are voting in an election among 5 candidates. If each person votes for a single candidate at random, what is the expected number of candidates that will be voted for?

Answer: $\frac{2101}{625}$ The probability that a chosen candidate will receive no votes at all is $(\frac{4}{5})^5$, or the probability that every person will vote for someone other than that one candidate. Then the probability that a chosen candidate will receive at least one vote is $1 - (\frac{4}{5})^5 = \frac{2101}{3125}$. By linearity of expectations, the final answer is this probability times the number of candidates, or $5 \cdot \frac{2101}{3125} = \frac{2101}{625}$.

32. [21] There are 101 people participating in a Secret Santa gift exchange. As usual each person is randomly assigned another person for whom (s)he has to get a gift, such that each person gives and receives exactly one gift and no one gives a gift to themselves. What is the probability that the first person neither gives gifts to or receives gifts from the second or third person? Express your answer as a decimal rounded to five decimal places.

Answer: 0.96039 Let D_k denote the number of derangements of $\{1, 2, \dots, k\}$. (A derangement is a permutation in which no element appears in its original position.)

Call the first three people A , B , and C . Let $X \rightarrow Y$ denote that X gives a gift to Y and let $X \not\rightarrow Y$ denote that X gives a gift to anyone other than Y . We are fine unless we have $A \rightarrow B$, $B \rightarrow A$, $A \rightarrow C$, or $C \rightarrow A$. We will compute the number of ways for various things to occur, then combine it into what we want.

There are D_{n-1} ways to have $A \rightarrow B \not\rightarrow A$. This is because if $A \rightarrow B$, we can treat A and B as a single vertex, and since $B \not\rightarrow A$, we have a derangement. Similarly, there are D_{n-1} ways to have $B \rightarrow A \not\rightarrow B$. Thirdly, there are D_{n-2} ways to have $A \rightarrow B \rightarrow A$, since D_{n-2} says how many ways the other $n - 2$ people can exchange their gifts. So there are $2D_{n-1} + D_{n-2}$ ways to have a conflict between A and B .

Similarly, there are $2D_{n-1} + D_{n-2}$ ways to have a conflict between A and C .

Using similar arguments, we can show that there are D_{n-2} ways for $B \rightarrow A \rightarrow C \not\rightarrow B$ to occur and D_{n-3} ways for $B \rightarrow A \rightarrow C \rightarrow B$ to occur. We get the same results when B and C are reversed. This gives $2D_{n-2} + 2D_{n-3}$ ways for a conflict to occur within all three people.

By the Principle of Inclusion-Exclusion, the total number of ways to have a conflict is

$$(\# \text{ conflicts between } A \text{ and } B) + (\# \text{ conflicts between } A \text{ and } C) - (\# \text{ conflicts between } A, B, \text{ and } C),$$

which evaluates to $4D_{n-1} - 2D_{n-3}$.

Approximating D_n as $\frac{n!}{e}$ (the actual formula is this quantity rounded to the nearest integer, so this is a great approximation), we find that the probability of no conflicts is

$$1 - \frac{4D_{n-1} - 2D_{n-3}}{D_n} \approx 1 - 4 \left(\frac{(n-1)!/e}{n!/e} \right) + 2 \left(\frac{(n-3)!/e}{n!/e} \right) = \frac{n(n-1)(n-2) - 4(n-1)(n-2) + 2}{n(n-1)(n-2)}.$$

Substitute $m = n - 1$ (this makes $m = 100$, so the expression is easier to evaluate) to get a probability of

$$\frac{m^3 - m - 4m^2 + 4m - 2}{m^3 - m} = \frac{m^3 - 4m^2 + 3m - 2}{m^3 - m} = \frac{1,000,000 - 40,000 + 300 - 2}{100 \cdot 9999} = \frac{960298}{100 \cdot 9999}$$

$$= 0.960208 \cdot (1.000100010001 \dots) = 0.960208 + 0.0000960208 + \dots = 0.9603940 \dots$$

To five decimal places, the desired probability is 0.96039.

33. [21] Let $a_1 = 3$, and for $n > 1$, let a_n be the largest real number such that

$$4(a_{n-1}^2 + a_n^2) = 10a_{n-1}a_n - 9.$$

What is the largest positive integer less than a_8 ?

Answer: [335] Let t_n be the larger real such that $a_n = t_n + \frac{1}{t_n}$. Then $t_1 = \frac{3+\sqrt{5}}{2}$. We claim that $t_n = 2t_{n-1}$. Writing the recurrence as a quadratic polynomial in a_n , we have:

$$4a_n^2 - 10a_{n-1}a_n + 4a_{n-1}^2 + 9 = 0.$$

Using the quadratic formula, we see that $a_n = \frac{5}{4}a_{n-1} + \frac{3}{4}\sqrt{a_{n-1}^2 - 4}$. (We ignore the negative square root, since a_n is the largest real number satisfying the polynomial.) Substituting $t_{n-1} + \frac{1}{t_{n-1}}$ for a_{n-1} , we see that $\sqrt{a_{n-1}^2 - 4} = \sqrt{t_{n-1}^2 - 2 + \frac{1}{t_{n-1}^2}}$, so we have:

$$a_n = \frac{5}{4} \left(t_{n-1} + \frac{1}{t_{n-1}} \right) + \frac{3}{4} \sqrt{\left(t_{n-1} - \frac{1}{t_{n-1}} \right)^2} = 2t_{n-1} + \frac{1}{2t_{n-1}}$$

so $t_n = 2t_{n-1}$, as claimed. Then $a_8 = \frac{128(3+\sqrt{5})}{2} + \frac{2}{128(3+\sqrt{5})}$. The second term is vanishingly small, so $\lfloor a_8 \rfloor = \lfloor 64(3 + \sqrt{5}) \rfloor$. We approximate $\sqrt{5}$ to two decimal places as 2.24, making this expression $\lfloor 335.36 \rfloor = 335$. Since our value of $\sqrt{5}$ is correct to within 0.005, the decimal is correct to within 0.32, which means the final answer is exact.

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34. [25] 3000 people each go into one of three rooms randomly. What is the most likely value for the maximum number of people in any of the rooms? Your score for this problem will be 0 if you write down a number less than or equal to 1000. Otherwise, it will be $25 - 27 \frac{|A-C|}{\min(A,C)-1000}$.

Answer: [1019] To get a rough approximation, we can use the fact that a sum of identical random variables converges to a Gaussian distribution,⁶ in this case with a mean of 1000 and a variance of $3000 \cdot \frac{2}{9} = 667$. Since $\sqrt{667} \approx 26$, 1026 is a good guess, as Gaussians tend to differ from their mean by approximately their variance.

The actual answer was computed with the following python program:

```
facts = [0]*3001
facts[0]=1
for a in range(1,3001):
    facts[a]=a*facts[a-1]

def binom(n,k):
    return facts[n]/(facts[k]*facts[n-k])
```

⁶See http://en.wikipedia.org/wiki/Central_limit_theorem.

```

maxes = [0]*3001

M = 1075

for a in range(0,3001):
    for b in range(0,3001-a):
        c = 3000-a-b
        m = max(a,max(b,c))
        if m < M:
            maxes[m] += facts[3000]/(facts[a]*facts[b]*facts[c])
            print [a,b]

best = 1000
for a in range(1000,1050):
    print maxes[a],a
    if maxes[best] <= maxes[a]:
        best = a

print maxes[best]
print best

```

We can use arguments involving the Chernoff bound⁷ to show that the answer is necessarily less than 1075. Alternately, if we wanted to be really careful, we could just set $M = 3001$, but then we'd have to wait a while for the script to finish.

35. [25] Call a positive integer *almost-square* if it can be written as $a \cdot b$, where a and b are integers and $a \leq b \leq \frac{4}{3}a$. How many almost-square positive integers are less than or equal to 1000000? Your score will be equal to $25 - 65 \frac{|A-C|}{\min(A,C)}$.

Answer: 130348 To get a good estimate for the number of almost-square integers, note that any number of the form $a \cdot b$, with $b \leq \frac{4}{3}a$, will be by definition almost-square. Let's assume that it's relatively unlikely that a number is almost-square in more than one way. Then the number of almost-square numbers less than n will be approximately

$$\sum_{a=1}^{\sqrt{n}} \sum_{b=a}^{\frac{4}{3}a} 1 = \frac{1}{3} \sum_{a=1}^{\sqrt{n}} a = \frac{1}{6} \sqrt{n} (\sqrt{n} + 1),$$

which is about $\frac{n}{6}$. So, $\frac{n}{6}$ will be a fairly good estimate for the number of almost-square numbers less than n , making 160000 a reasonable guess.

We can do better, though. For example, we summed $\frac{a}{3}$ all the way up to \sqrt{n} , but we are really overcounting here because when a is close to \sqrt{n} , $a \cdot b$ will be less than n only when $b \leq \frac{n}{a}$, as opposed to $b \leq \frac{4a}{3}$. So we should really be taking the sum

⁷See http://en.wikipedia.org/wiki/Chernoff_bound.

$$\begin{aligned}
& \sum_{a=1}^{\sqrt{\frac{3n}{4}}} \sum_{b=a}^{\frac{4a}{3}} 1 + \sum_{a=\sqrt{\frac{3n}{4}}}^{\sqrt{n}} \sum_{b=a}^{\frac{n}{a}} 1 \\
&= \sum_{a=1}^{\sqrt{\frac{3n}{4}}} \frac{a}{3} + \sum_{a=\sqrt{\frac{3n}{4}}}^{\sqrt{n}} \left(\frac{n}{a} - a \right) \\
&\approx \frac{1}{6} \frac{3n}{4} + n \left(\log(\sqrt{n}) - \log\left(\sqrt{\frac{3n}{4}}\right) \right) - \left(\frac{n}{2} - \frac{3n}{8} \right) \\
&= \frac{n}{8} + n \frac{\log(4) - \log(3)}{2} - \frac{n}{8} \\
&= n \frac{\log(4) - \log(3)}{2}.
\end{aligned}$$

In the process of taking the sum, we saw that we had something between $\frac{n}{8}$ and $\frac{n}{6}$, so we could also guess something between 166000 and 125000, which would give us about 145000, an even better answer. If we actually calculate $\frac{\log(4) - \log(3)}{2}$, we see that it's about 0.14384, so 143840 would be the best guess if we were to use this strategy. In reality, we would want to round down a bit in both cases, since we are overcounting (because numbers could be square-free in multiple ways), so we should probably answer something like 140000.

A final refinement to our calculation (and perhaps easier than the previous one), is to assume that the products $a \cdot b$ that we consider are randomly distributed between 1 and n , and to compute the expected number of distinct numbers we end up with. This is the same type of problem as number 31 on this contest, and we compute that if we randomly distribute k numbers between 1 and n then we expect to end up with $n \left(1 - \left(1 - \frac{1}{n} \right)^k \right)$ distinct numbers. When $k = n \frac{\log(4) - \log(3)}{2}$, we get that this equals

$$\begin{aligned}
n \left(1 - \left(\left(1 - \frac{1}{n} \right)^n \right)^{\frac{\log(4) - \log(3)}{2}} \right) &= n \left(1 - \sqrt{e^{\log(3) - \log(4)}} \right) \\
&= n \left(1 - \sqrt{\frac{3}{4}} \right) \\
&= n \left(1 - \frac{\sqrt{3}}{2} \right) \\
&\approx 0.134n,
\end{aligned}$$

Giving us an answer of 134000, which is very close to the correct answer.

The actual answer was found by computer, using the following C++ program:

```

#include <stdio.h>

using namespace std;

bool isAlmostSquare(int n){
    for(int k=1;k*k<=n;k++)
        if(n%k==0 && 3*(n/k) <= 4*k) return true;
    return false;
}

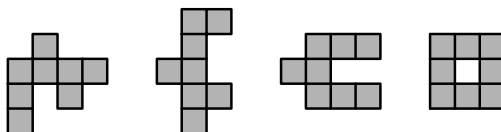
```

```

int main(){
    int c = 0;
    for(int n=1;n<=1000000;n++)
        if(isAlmostSquare(n)) c++;
    printf("%d\n",c);
    return 0;
}

```

36. [25] Consider an infinite grid of unit squares. An n -omino is a subset of n squares that is connected. Below are depicted examples of 8-ominoes. Two n -ominoes are considered equivalent if one can be obtained from the other by translations and rotations. What is the number of distinct 15-ominoes? Your score will be equal to $25 - 13|\ln(A) - \ln(C)|$.



Answer: 3426576 We claim that there are approximately $\frac{3^{n-1}}{4}$ n -ominoes. First, we define an order on the squares in an n -omino, as follows: we order the squares from left to right, and within a column, we order the squares from top to bottom.

We construct an n -omino by starting with a single square and attaching squares, one by one, as close to order we just defined as possible. After the first square is placed, the next square can be attached to its bottom or right side. If the second square is right of the first, then the third square can be attached to the top, right, or bottom edges of the second. The third square cannot be attached to the first square, because the first column must be completed before the second is begun. If the second square is below the first, the third square can be attached to the bottom or right sides of the second square, or the right side of the first way. In either case, there are 3 choices for the third square.

Suppose that m squares have been added, and that k squares are in the right-most column right now. If $k = 1$, then the next square can be attached to the top, right, or bottom side of this square. If $k > 1$, then the next square can be added to the bottom or right side of the last square in this column. The next square can also be added to the right of the top square in this column, so there are still 3 choices for the next square. Therefore, there are 3^{n-1} ways to build up the entire n -omino.

Almost all n -ominoes have no rotational symmetry, so they can be built in 4 different ways by this method. This correction gives us our estimate of $\frac{3^{n-1}}{4}$ for the number of n -ominoes. This reasoning is not exactly correct, because some n -ominoes are highly non-convex, and do not admit an in-order traversal like the one described above. If some columns in the n -ominoes have gaps, they are not enumerated by this construction, so $\frac{3^{n-1}}{4}$ is an underestimate. We estimate that there are 1195742 15-ominoes. The actual value of 3426576, which can be found in the Sloane Encyclopedia of Integer Sequences, differs from the estimate by less than a factor of 3.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010

Team Round A

1. You are trying to sink a submarine. Every second, you launch a missile at a point of your choosing on the x -axis. If the submarine is at that point at that time, you sink it. A *firing sequence* is a sequence of real numbers that specify where you will fire at each second. For example, the firing sequence $2, 3, 5, 6, \dots$ means that you will fire at 2 after one second, 3 after two seconds, 5 after three seconds, 6 after four seconds, and so on.

- (a) [5] Suppose that the submarine starts at the origin and travels along the positive x -axis with an (unknown) positive integer velocity. Show that there is a firing sequence that is guaranteed to hit the submarine eventually.

Solution: The firing sequence $1, 4, 9, \dots, n^2, \dots$ works. If the velocity of the submarine is v , then after v seconds it will be at $x = v^2$, the same location where the mine explodes at time v .

- (b) [10] Suppose now that the submarine starts at an unknown integer point on the non-negative x -axis and again travels with an unknown positive integer velocity. Show that there is still a firing sequence that is guaranteed to hit the submarine eventually.

Solution: Represent the submarine's motion by an ordered pair (a, b) , where a is the starting point of the submarine and b is its velocity. We want to find a way to map each positive integer to a possible ordered pair so that every ordered pair is covered. This way, if we fire at $b_n n + a_n$ at time n , where (a_n, b_n) is the point that n maps to, then we will eventually hit the submarine. (Keep in mind that $b_n n + a_n$ would be the location of the submarine at time n .) There are many such ways to map the positive integers to possible points; here is one way:

$$1 \rightarrow (1, 1), 2 \rightarrow (2, 1), 3 \rightarrow (1, 2), 4 \rightarrow (3, 1), 5 \rightarrow (2, 2), 6 \rightarrow (1, 3), 7 \rightarrow (4, 1), 8 \rightarrow (3, 2), \\ 9 \rightarrow (2, 3), 10 \rightarrow (1, 4), 11 \rightarrow (5, 1), 12 \rightarrow (4, 2), 13 \rightarrow (3, 3), 14 \rightarrow (2, 4), 15 \rightarrow (1, 5), \dots$$

(The path of points trace out diagonal lines that sweep every lattice point in the coordinate plane.) Since we cover every point, we will eventually hit the submarine.

Remark: The mapping shown above is known as a *bijection* between the positive integers and ordered pairs of integers (a, b) where $b > 0$.

2. [15] Consider the following two-player game. Player 1 starts with a number, N . He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number or 1, at which point that player loses. For which values of N does player 1 have a winning strategy?

Answer: All even numbers except for odd powers of 2. First we show that if you are stuck with an odd number, then you are guaranteed to lose. Suppose you have an odd number ab , where a and b are odd numbers, and you choose to subtract a . You pass your opponent the number $a(b-1)$. This cannot be a power of 2 (otherwise a is a power of 2 and hence $a = 1$, which is not allowed), so your opponent can find an odd proper divisor of $a(b-1)$, and you will have a smaller odd number. Eventually you will get to an odd prime and lose.

Now consider even numbers that aren't powers of 2. As with before, you can find an odd proper divisor of N and pass your opponent an odd number, so you are guaranteed to win.

Finally consider powers of 2. If you have the number $N = 2^k$, it would be unwise to choose a proper divisor other than 2^{k-1} ; otherwise you would give your opponent an even number that isn't a power of 2. Therefore if k is odd, you will end up with 2 and lose. If k is even, though, your opponent will end up with 2 and you will win.

Therefore player 1 has a winning strategy for all even numbers except for odd powers of 2.

3. [15] Call a positive integer in base 10 *k-good* if we can split it into two integers y and z , such that y is all digits on the left and z is all digits on the right, and such that $y = k \cdot z$. For example, 2010 is 2-good because we can split it into 20 and 10 and $20 = 2 \cdot 10$. 20010 is also 2-good, because we can split it into 20 and 010. In addition, it is 20-good, because we can split it into 200 and 10.

Show that there exists a 48-good perfect square.

Solution: We wish to find integers a, z such that $48z \cdot 10^a + z = z(48 \cdot 10^a + 1)$ a perfect square, where $z < 10^a$. This would prove that there exists a 48-good perfect square because we are pulling off the last a digits of the number and get two integers $48z$ and z . To make z small by keeping the product a perfect square, we'd like $48 \cdot 10^a + 1$ to be divisible by some reasonably large square. Take $a = 42 = \varphi(49)$. By Euler's theorem, $10^{42} \equiv 1 \pmod{49}$, so $48 \cdot 10^a + 1$ is a multiple of 49. Then we can take $z = \frac{48 \cdot 10^a + 1}{49}$. (Clearly $z < 10^a$, so we're fine.) Then we have $z(48 \cdot 10^a + 1) = \left(\frac{48 \cdot 10^{42} + 1}{7}\right)^2$.

4. [20] Let

$$\begin{aligned}e^x + e^y &= A \\xe^x + ye^y &= B \\x^2e^x + y^2e^y &= C \\x^3e^x + y^3e^y &= D \\x^4e^x + y^4e^y &= E.\end{aligned}$$

Prove that if A, B, C , and D are all rational, then so is E .

Solution: We can express $x + y$ in two ways:

$$\begin{aligned}x + y &= \frac{AD - BC}{AC - B^2} \\x + y &= \frac{AE - C^2}{AD - BC}\end{aligned}$$

(We have to be careful if $AC - B^2$ or $AD - BC$ is zero. We'll deal with that case later.) It is easy to check that these equations hold by substituting the expressions for A, B, C, D , and E . Setting these two expressions for $x + y$ equal to each other, we get

$$\frac{AD - BC}{AC - B^2} = \frac{AE - C^2}{AD - BC},$$

which we can easily solve for E as a rational function of A, B, C , and D . Therefore if A, B, C , and D are all rational, then E will be rational as well.

Now, we have to check what happens if $AC - B^2 = 0$ or $AD - BC = 0$. If $AC - B^2 = 0$, then writing down the expressions for A, B , and C gives us that $(x - y)^2 e^{x+y} = 0$, meaning that $x = y$. If $x = y$, and $x \neq 0$, A and D are also non-zero, and $\frac{B}{A} = \frac{E}{D} = x$. Since $\frac{B}{A}$ is rational and D is rational, this implies that E is rational. If $x = y = 0$, then $E = 0$ and so is certainly rational.

We finally must check what happens if $AD - BC = 0$. Since $AD - BC = (x + y)(AC - B^2)$, either $AC - B^2 = 0$ (a case we have already dealt with), or $x + y = 0$. But if $x + y = 0$ then $AE - C^2 = 0$, which implies that $E = \frac{C^2}{A}$ (we know that $A \neq 0$ because e^x and e^y are both positive). Since A and C are rational, this implies that E is also rational.

So, we have shown E to be rational in all cases, as desired.

5. [20] Show that, for every positive integer n , there exists a monic polynomial of degree n with integer coefficients such that the coefficients are decreasing and the roots of the polynomial are all integers.

Solution: We claim we can find values a and b such that $p(x) = (x - a)(x + b)^n$ is a polynomial of degree $n + 1$ that satisfies these constraints. We show that its coefficients are decreasing by finding a general formula for the coefficient of x^k .

The coefficient of x^k is $b^k \binom{n}{k} - ab^{k-1} \binom{n}{k-1}$, which can be seen by expanding out $(x+b)^n$ and then multiplying by $(x-a)$. Then we must prove that

$$b^{k+1} \binom{n}{k+1} - ab^k \binom{n}{k} < b^k \binom{n}{k} - ab^{k-1} \binom{n}{k-1},$$

or

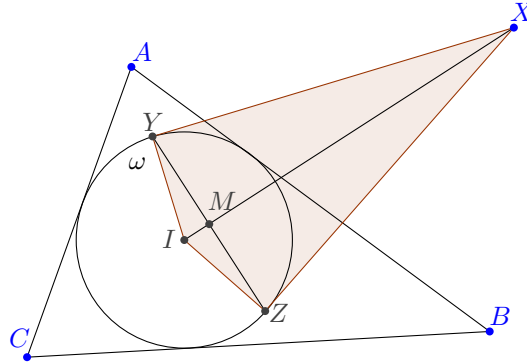
$$ab^{k-1} \left(b \binom{n}{k} - \binom{n}{k-1} \right) > b^k \left(b \binom{n}{k+1} - \binom{n}{k} \right).$$

Choose $b > \max \left(\frac{\binom{n}{k}}{\binom{n}{k-1}} \right)$ in order to make sure the right-hand term in each product on each side of the inequality sign is positive (we'll be dividing by it, so this makes things much easier), and choose $a > \max \left(\frac{b \left(\frac{b \binom{n}{k+1} - \binom{n}{k}}{b \binom{n}{k} - \binom{n}{k-1}} \right)}{\binom{n}{k} - \binom{n}{k-1}} \right)$ to make sure the inequality always holds. Since there are only finite values that k can take on given a fixed n (namely, integers between 0 and n inclusive), we can always find values of a and b that satisfy these constraints.

6. [20] Let S be a convex set in the plane with a finite area a . Prove that either $a = 0$ or S is bounded. Note: a set is bounded if it is contained in a circle of finite radius. Note: a set is *convex* if, whenever two points A and B are in the set, the line segment between them is also in the set.

Solution: If all points in S lie on a straight line, then $a = 0$.

Otherwise we may pick three points A, B , and C that are not collinear. Let ω be the incircle of $\triangle ABC$, with I its center and r its radius. Since S is convex, S must contain ω .



Suppose S also contains a point X at a distance d from I , with $d > R$. We will show that $d \leq \sqrt{r^2 + \frac{a^2}{r^2}}$, which implies that the S is bounded since all points are contained within the circle centered at I of radius $\sqrt{r^2 + \frac{a^2}{r^2}}$.

Let Y and Z be on ω such that \overline{XY} and \overline{XZ} are tangents to ω . Because S is convex, it must contain kite $IYXZ$, whose area we can compute in terms of d and r .

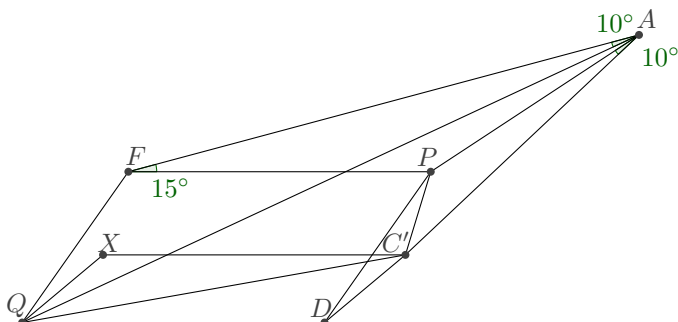
Let M be the midpoint of \overline{YZ} . Since $\triangle IYX \sim \triangle IMY$, we know that $\frac{IM}{IY} = \frac{IY}{IX}$, that is, $IM = \frac{(IY)^2}{IX} = \frac{r^2}{d}$. Then $MY = \sqrt{r^2 - \frac{r^4}{d^2}} = r \sqrt{1 - \left(\frac{r}{d}\right)^2} = \frac{1}{2}YZ$.

The area of $IYXZ$ is $\frac{1}{2}(YZ)(IX) = rd \sqrt{1 - \left(\frac{r}{d}\right)^2} = r \sqrt{d^2 - r^2}$. This must be less than or equal to a , the area of S . This yields $a^2 \geq r^2 d^2 - r^4$ or $d^2 \leq r^2 + \frac{a^2}{r^2}$. It follows that $d \leq \sqrt{r^2 + \frac{a^2}{r^2}}$, as desired.

7. [25] Point P lies inside a convex pentagon $AFQDC$ such that $FPDQ$ is a parallelogram. Given that $\angle FAQ = \angle PAC = 10^\circ$, and $\angle PFA = \angle PDC = 15^\circ$. What is $\angle AQC$?

Answer: $\boxed{\frac{\pi}{12}}$ Let C' be the point such that there is a spiral similarity between $\triangle AFP$ and $\triangle AQC'$. In other words, one triangle can be formed from the other by dilating and rotating about one of the

triangle's vertices (in this case, A). We will show that C' is C , so our answer will be $\angle AQC = \angle AQC' = \angle AFP = 15^\circ$. By the spiral similarity theorem, $\triangle AFQ \sim \triangle APC'$ (this is intuitive by looking at a diagram), so $\angle PAC = \angle FAQ = 10^\circ$, so to show that C' is C , it is sufficient to show that $\angle PDC' = 15^\circ$.



Let X be the fourth point of the parallelogram $FPC'X$ (see the above diagram). The angle between lines \overline{FP} and \overline{FA} is 15° . Since $\overline{XC'} \parallel \overline{FP}$, the angle between \overline{FA} and $\overline{XC'}$ is 15° as well. In addition, the angle between \overline{QA} and $\overline{QC'}$ is $\angle AQC' = 15^\circ$, so $\angle XC'Q = \angle FAQ$. Further, because $FPC'X$ is a parallelogram, $\frac{QC'}{XC'} = \frac{QP}{FP}$. By similar triangles $\triangle AFP$ and $\triangle AQC'$, $\frac{QC'}{FP} = \frac{QA}{FA}$. By SAS similarity, there is a spiral similarity between $\triangle XC'Q$ and $\triangle FAQ$, so $\angle FQX = \angle AQC' = 15^\circ$.

Note that the segments \overline{FP} , $\overline{XC'}$, and \overline{QD} are all parallel and equal in length. Therefore, $\triangle FQX \cong \triangle PDC'$ are congruent, and $\angle PDC' = 15^\circ$ as well. So C' is C , and $\angle AQC = 15^\circ$.

8. [30] A knight moves on a two-dimensional grid. From any square, it can move 2 units in one axis-parallel direction, then move 1 unit in an orthogonal direction, the way a regular knight moves in a game of chess. The knight starts at the origin. As it moves, it keeps track of a number t , which is initially 0. When the knight lands at the point (a, b) , the number is changed from x to $ax + b$.

Show that, for any integers a and b , it is possible for the knight to land at the points $(1, a)$ and $(-1, a)$ with t equal to b .

Solution: For convenience, we will refer to (a, b) as $[ax + b]$, the function it represents. This will make it easier to follow the trajectory of t over a given sequence of moves.

Suppose we start at $[x + 1]$ with $t = a$. Taking the path $[x + 1] \rightarrow [-x] \rightarrow [x - 1] \rightarrow [-x] \rightarrow [x + 1]$ will yield $t = a + 2$. So we can go from $t = a$ to $t = a + 2$ at $[x + 1]$.

We can also move until we get to $[-3]$, then go $[-3] \rightarrow [x - 1]$ to end up with $t = -4$ at $[x - 1]$. But going $[x - 1] \rightarrow [3x] \rightarrow [x - 1]$ means we can go from $t = a$ to $t = 3a - 1$ at $[x - 1]$. Since we can start with $t = -4$, this means we can therefore get arbitrarily small even and odd numbers at $[x - 1]$, hence also $[3x]$, hence also at $[x + 1]$.

This implies we can get any value of t we want at $[x + 1]$, so we can also get any value of t we want at $[-x]$, $[x - 1]$, $[-x - 2]$, $[x - 3]$, etc., as well as $[-x + 2]$, $[x + 3]$, $[-x + 4]$, $[x + 5]$, etc. We can do a similar thing starting at $[-x + 1]$ to get from $t = a$ to $t = a + 2$, and use the $[-x - 1] \rightarrow [-3x] \rightarrow [-x - 1]$ loop to get arbitrarily small integers of both parities. So we can get any value of t we want at all points of the form $[\pm x + k]$ for any integer k .

9. [30] Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$ be a polynomial with complex coefficients such that $a_i \neq 0$ for all i . Prove that $|r| \leq 2 \max_{i=0}^{n-1} \left| \frac{a_{i-1}}{a_i} \right|$ for all roots r of all such polynomials p . Here we let $|z|$ denote the absolute value of the complex number z .

Solution: If r is a root, then $-a_n r^n = a_{n-1} r^{n-1} + \dots + a_0$. By the Triangle Inequality, $|-a_n r^n| \leq |a_{n-1} r^{n-1}| + \dots + |a_0|$. Rearranging this inequality yields $|a_n r^n| - |a_{n-1} r^{n-1}| - \dots - |a_0| \leq 0$.

Now suppose $|r| = k \max_{i=0}^{n-1} \left| \frac{a_{i-1}}{a_i} \right|$. Applying this over values of i ranging from $m + 1$ to n (assuming $m + 1 \leq n$), we get $|a_m r^m| \leq \frac{|a_n r^n|}{k^{n-m}}$. This, along with the above equation, yields:

$$|a_n r^n| \cdot \left(1 - \frac{1}{k} - \frac{1}{k^2} - \frac{1}{k^3} - \dots - \frac{1}{k^n}\right) = 0$$

This is only true when $a_n = 0$, $r = 0$, or $(1 - \frac{1}{k} - \frac{1}{k^2} - \dots) = 0$. The first option is impossible by the constraints in the problem. The second option implies $k = 0$. The third option implies that $k < 2$; otherwise $(1 - \frac{1}{k} - \frac{1}{k^2} - \dots - \frac{1}{k^n})$ would always remain positive. Either way, $|r| \leq 2 \max_{i=0}^{n-1} |\frac{a_{i-1}}{a_i}|$.

10. Call an $2n$ -digit base-10 number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$ -digit number is special. (We allow leading zeros in $2n$ -digit numbers).

- (a) [20] The sequence p_n converges to a constant c . Find c .

Answer: $\boxed{\frac{1}{2}}$ We first claim that if a $2n$ -digit number x has at least eight 0's and at least eight 1's and the sum of its digits is even, then x is special.

Let A be a set of eight 0's and eight 1's and let B be the set of all the other digits. We split B arbitrarily into two sets Y and Z of equal size. If $|\sum_{y \in Y} y - \sum_{z \in Z} z| > 8$, then we swap the biggest element of the set with the bigger sum with the smallest element of the other set. This transposition always decreases the absolute value of the sum: in the worst case, a 9 from the bigger set is swapped for a 0 from the smaller set, which changes the difference by at most 18. Therefore, after a finite number of steps, we will have $|\sum_{y \in Y} y - \sum_{z \in Z} z| \leq 8$.

Note that this absolute value is even, since the sum of all the digits is even. Without loss of generality, suppose that $\sum_{y \in Y} y - \sum_{z \in Z} z = 2k$, where $0 \leq k \leq 4$. If we add k 0's and $8 - k$ 1's to Y , and we add the other elements of A to Z , then the two sets will balance, so x is special.

- (b) [45] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d . Find r and d .

Answer: $\boxed{r = \frac{1}{4}, d = -1}$ To get the next asymptotic term after the constant term of $\frac{1}{2}$, we need to consider what happens when the digit sum is even; we want to find the probability that such a number isn't balanced. We claim that the configuration that contributes the vast majority of unbalanced numbers is when all numbers are even and the sum is $2 \pmod 4$, or such a configuration with all numbers increased by 1. Clearly this gives q_n being asymptotic to $-\frac{1}{2} \cdot 2 \cdot (\frac{1}{2})^{2n} = -(\frac{1}{4})^n$, so $r = \frac{1}{4}$ and $d = -1$.

To prove the claim, first note that the asymptotic probability that there are at most 4 digits that occur more than 10 times is asymptotically much smaller than $(\frac{1}{2})^n$, so we can assume that there exist 5 digits that each occur at least 10 times. If any of those digits are consecutive, then the digit sum being even implies that the number is balanced (by an argument similar to part (a)).

So, we can assume that none of the numbers are consecutive. We would like to say that this implies that the numbers are either 0, 2, 4, 6, 8 or 1, 3, 5, 7, 9. However, we can't quite say this yet, as we need to rule out possibilities like 0, 2, 4, 7, 9. In this case, though, we can just pair 0 and 7 up with 2 and 4; by using the same argument as in part (a), except using 0 and 7 both (to get a sum of 7) and 2 and 4 both (to get a sum of 6) to balance out the two sets at the end.

In general, if there is ever a gap of size 3, consider the number right after it and the 3 numbers before it (so we have $k - 4, k - 2, k, k + 3$ for some k), and pair them up such that one pair has a sum that's exactly one more than the other (i.e. pair $k - 4$ with $k + 3$ and $k - 2$ with k). Since we again have pairs of numbers whose sums differ by 1, we can use the technique from part (a) of balancing out the sets at the end.

So, we can assume there is no gap of size 3, which together with the condition that no two numbers are adjacent implies that the 5 digits are either 0, 2, 4, 6, 8 or 1, 3, 5, 7, 9. For the remainder of the solution, we will deal with the 0, 2, 4, 6, 8 case, since it is symmetric with the other case under the transformation $x \mapsto 9 - x$.

If we can distribute the odd digits into two sets S_1 and S_2 such that (i) the difference in sums of S_1 and S_2 is small; and (ii) the difference in sums of S_1 and S_2 , plus the sum of the even digits, is divisible by 4, then the same argument as in part (a) implies that the number is good.

In fact, if there are *any* odd digits, then we can use them at the beginning to fix the parity mod 4 (by adding them all in such that the sums of the two sets remain close, and then potentially switching one with an even digit). Therefore, if there are any odd digits then the number is good. Also, even if there are no odd digits, if the sum of the digits is divisible by 4 then the number is good.

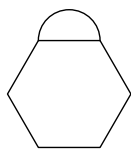
So, we have shown that almost all non-good numbers come from having all numbers being even with a digit sum that is $2 \pmod{4}$, or the analogous case under the mapping $x \mapsto 9 - x$. This formalizes the claim we made in the first paragraph, so $r = \frac{1}{4}$ and $d = -1$, as claimed.

13th Annual Harvard-MIT Mathematics Tournament
Saturday 20 February 2010
Team Round B

1. [10] How many ways are there to place pawns on an 8×8 chessboard, so that there is at most 1 pawn in each horizontal row? Express your answer in the form $p_1^{e_1} \cdot p_2^{e_2} \cdots$, where the p_i are distinct primes and the e_i are positive integers.

Answer: $\boxed{3^{16}}$ If there is at most 1 pawn in each row, then each row of the chessboard may have either 0 or 1 pawn somewhere in the row. There is 1 case if there are no pawns in the row. There are 8 possible cases if there is 1 pawn in the row, one case for each square in the row. Hence for each row, there are 9 possible pawn arrangements. There are 8 rows, thus we have $9^8 = 3^{16}$.

2. [10] In the following figure, a regular hexagon of side length 1 is attached to a semicircle of diameter 1. What is the longest distance between any two points in the figure?



Answer: $\boxed{\frac{1+\sqrt{13}}{2}}$ Inspection shows that one point must be on the semicircle and the other must be on the side of the hexagon directly opposite the edge with the semicircle, the bottom edge of the hexagon in the above diagram. Let O be the center of the semicircle and let M be the midpoint of the bottom edge.

We will determine the longest distance between points in the figure by comparing the lengths of all the segments with one endpoint on the bottom edge and the other endpoint on the semicircle. Fix a point A on the bottom edge of the hexagon. Suppose that B is chosen on the semicircle such that AB is as long as possible. Let C be the circle centered at A with radius AB . If C is not tangent to the semicircle, then part of the semicircle is outside C , so we could pick a B' on the semicircle such that AB' is longer than AB . So C must be tangent to the semicircle, and AB must pass through O .

Then OB is always $\frac{1}{2}$, no matter which A we choose on the bottom edge. All that remains is maximizing AO . This length is the hypotenuse of a right triangle with the fixed height MO , so it is maximized when AM is as large as possible - when A is an endpoint of the bottom edge. Note that $MO = 2 \cdot \frac{\sqrt{3}}{2}$, and that AM can be at most $\frac{1}{2}$, so AO can be at most $\frac{\sqrt{13}}{2}$. So the maximum distance between two points in the diagram is $AO + OB = \frac{1+\sqrt{13}}{2}$.

3. [15] Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, where each a_i is either 1 or -1 . Let r be a root of p . If $|r| > \frac{15}{8}$, what is the minimum possible value of n ?

Answer: $\boxed{4}$ We claim that $n = 4$ is the answer. First, we show that $n > 3$. Suppose that $n \leq 3$. Let r be the root of the polynomial with $|r| \geq \frac{15}{8}$. Then, by the Triangle Inequality, we have:

$$|a_n r^n| = |a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + \dots + a_0| \leq |a_{n-1} r^{n-1}| + |a_{n-2} r^{n-2}| + \dots + |a_0|$$

$$|r|^n \leq |r|^{n-1} + |r|^{n-2} + \dots + |1| = \frac{|r|^n - 1}{|r| - 1}$$

$$|r|^{n+1} - 2|r|^n + 1 \leq 0 \Rightarrow 1 \leq |r|^n (2 - |r|)$$

The right-hand side is increasing in n , for $|r| > 1$, so it is bounded by $|r|^3(2 - |r|)$. This expression is decreasing in r for $r \geq \frac{3}{2}$. When $|r| = \frac{15}{8}$, then the right-hand side is less than 1, which violates the inequalities. Therefore $n > 3$. Now, we claim that there is a 4th degree polynomial with a root r with $|r| \geq \frac{15}{8}$. Let $p(x) = x^4 - x^3 - x^2 - x - 1$. Then $p(\frac{15}{8}) < 0$ and $p(2) > 2$. By the Intermediate Value Theorem, $p(x)$ has such a root r .

4. [20] Find all 4-digit integers of the form $aabb$ (when written in base 10) that are perfect squares.

Answer: $\boxed{7744}$ Let x be an integer such that x^2 is of the desired form. Then $1100a + 11b = x^2$. Then x^2 is divisible by 11, which means x is divisible by 11. Then for some integer, y , $x = 11y$. Then $1100a + 11b = 11^2y^2 \Rightarrow 100a + b = 11y^2$. This means that $100a + b \equiv 0 \pmod{11} \Rightarrow a + b \equiv 0 \pmod{11}$. Because a and b must be nonzero digits, we have $2 \leq a, b \leq 9$, so we can write $b = 11 - a$.

Replacing b in the equation derived above, we obtain $99a + 11 = 11y^2 \Rightarrow 9a + 1 = y^2$. We check the possible values of a from 2 to 9, and only $a = 7$ yields a perfect square. When $a = 7$, $b = 4$, so the only perfect square of for $aabb$ is 7744.

5. [25] Compute

$$\sum_{n=1}^{98} \frac{2}{\sqrt{n} + \sqrt{n+2}} + \frac{1}{\sqrt{n+1} + \sqrt{n+2}}.$$

Answer: $\boxed{3\sqrt{11} - 2\sqrt{2} + 19}$ Rationalizing the denominator of both terms in the summation yields $\frac{2(\sqrt{n+2} - \sqrt{n})}{\sqrt{n+2} - \sqrt{n} + \sqrt{n+2} - \sqrt{n+1}} + \frac{1(\sqrt{n+2} - \sqrt{n+1})}{\sqrt{n+2} - \sqrt{n+1} + \sqrt{n+2} - \sqrt{n+1}}$. Then the sum $\sum_{n=1}^{98} 2\sqrt{n+2} - (\sqrt{n} + \sqrt{n+1})$ telescopes. All terms cancel except for $-(\sqrt{1} + \sqrt{2}) - \sqrt{2} + 2\sqrt{99} + 2\sqrt{100} - \sqrt{99} = 3\sqrt{11} - 2\sqrt{2} + 19$.

6. [25] Into how many regions can a circle be cut by 10 parabolas?

Answer: $\boxed{201}$ We will consider the general case of n parabolas, for which the answer is $2n^2 + 1$.

We will start with some rough intuition, then fill in the details afterwards. The intuition is that, if we make the parabolas steep enough, we can basically treat them as two parallel lines. Furthermore, the number of regions is given in terms of the number of intersections of the parabolas that occur within the circle, since every time two parabolas cross a new region is created. Since two pairs of parallel lines intersect in 4 points, and pairs of parabolas also intersect in 4 points, as long as we can always make all 4 points of intersection lie inside the circle, the parallel lines case is the best we can do.

In other words, the answer is the same as the answer if we were trying to add ten pairs of parallel lines. We can compute the answer for pairs of parallel lines as follows — when we add the k th set of parallel lines, there are already $2k - 2$ lines that the two new lines can intersect, meaning that each of the lines adds $2k - 1$ new regions¹. This means that we add $4k - 2$ regions when adding the k th set of lines, making the answer $1 + 2 + 6 + 10 + 14 + \dots + (4n - 2) = 1 + 2(1 + 3 + 5 + 7 + \dots + (2n - 1)) = 1 + 2 \cdot n^2 = 2n^2 + 1$.

Now that we have sketched out the solution, we will fill in the details more rigorously. First, if there are n parabolas inside the circle, and they intersect in K points total, then we claim that the number of regions the circle is divided into will be at most $K + n + r + 1$, where r is the number of parabolas that intersect the circle itself in exactly four points.

We will prove this by induction. In the base case of $n = 0$, we are just saying that the circle itself consists of exactly one region.

To prove the inductive step, suppose that we have n parabolas with K points of intersection. We want to show that if we add an additional parabola, and this parabola intersects the other parabolas in p points, then this new parabola adds either $p + 1$ or $p + 2$ regions to the circle, and that we get $p + 2$ regions if and only if the parabola intersects the circle in exactly four points.

We will do this by considering how many regions the parabola cuts through, following its path from when it initially enters the circle to when it exits the circle for the last time. When it initially enters the circle, it cuts through one region, thereby increasing the number of regions by one². Then, for each other parabola that this parabola crosses, we cut through one additional region. It is also possible for the parabola to leave and then re-enter the circle, which happens if and only if the parabola intersects

¹This is the maximum possible number of new regions, but it's not too hard to see that this is always attainable.

²While the fact that a curve going through a region splits it into two new regions is intuitively obvious, it is actually very difficult to prove. The proof relies on some deep results from algebraic topology and is known as the Jordan Curve Theorem. If you are interested in learning more about this, see http://en.wikipedia.org/wiki/Jordan_curve_theorem

the circle in four points, and also adds one additional region. Therefore, the number of regions is either $p + 1$ or $p + 2$, and it is $p + 2$ if and only if the parabola intersects the circle in four points. This completes the induction and proves the claim.

So, we are left with trying to maximize $K + n + r + 1$. Since a pair of parabolas intersects in at most 4 points, and there are $\binom{n}{2}$ pairs of parabolas, we have $K \leq 4\binom{n}{2} = 2n^2 - 2n$. Also, $r \leq n$, so $K + n + r + 1 \leq 2n^2 + 1$. On the other hand, as explained in the paragraphs giving the intuition, we can attain $2n^2 + 1$ by making the parabolas sufficiently steep that they act like pairs of parallel lines. Therefore, the answer is $2n^2 + 1$, as claimed.

7. [30] Evaluate

$$\sum_{k=1}^{2010} \cos^2(k)$$

Answer: $\boxed{1005 + \frac{\sin(4021) - \sin(1)}{4\sin(1)}}$ We use the identity $\cos^2(k) = \frac{1 + \cos(2k)}{2}$. Then our expression evaluates to $1005 + \frac{(\cos(2) + \dots + \cos(4020))}{2}$.

To evaluate $\cos(2) + \dots + \cos(4020)$, let $y = \cos(2) + \dots + \cos(4020) \Rightarrow y(\sin(1)) = \cos(2)\sin(1) + \dots + \cos(4020)\sin(1)$. Observe that for any x , $\cos(x)\sin(1) = \frac{\sin(x+1) - \sin(x-1)}{2}$. Then $y(\sin(1)) = \frac{\sin(3) - \sin(1)}{2} + \frac{\sin(5) - \sin(3)}{2} + \dots + \frac{\sin(4021) - \sin(4019)}{2}$. This is clearly a telescoping sum; we get $y(\sin(1)) = \frac{\sin(4021) - \sin(1)}{2}$. Then we have the desired $y = \frac{\sin(4021) - \sin(1)}{2\sin(1)}$. Then our original expression evaluates to $1005 + \frac{\sin(4021) - \sin(1)}{4\sin(1)}$.

8. [30] Consider the following two-player game. Player 1 starts with a number, N . He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number, at which point that player loses. For how many values of N between 2 and 100 inclusive does player 1 have a winning strategy?

Answer: $\boxed{47}$ We claim that player 1 has a winning strategy if and only if N is even and not an odd power of 2.

First we show that if you are stuck with an odd number, then you are guaranteed to lose. Suppose you have an odd number ab , where a and b are odd numbers, and you choose to subtract a . You pass your opponent the number $a(b - 1)$. This cannot be a power of 2 (otherwise a is a power of 2 and hence $a = 1$, which is not allowed), so your opponent can find an odd proper divisor of $a(b - 1)$ (such as a), and you will have a smaller odd number. Eventually you will get to an odd prime and lose.

Now consider even numbers that aren't powers of 2. As with before, you can find an odd proper divisor of N and pass your opponent an odd number, so you are guaranteed to win.

Finally consider powers of 2. If you have the number $N = 2^k$, it would be unwise to choose a proper divisor other than 2^{k-1} ; otherwise you would give your opponent an even number that isn't a power of 2. Therefore if k is odd, you will end up with 2 and lose. If k is even, though, your opponent will end up with 2 and you will win.

Therefore player 1 has a winning strategy for all even numbers except for odd powers of 2.

9. [35] Let S be the set of ordered pairs of integers (x, y) with $1 \leq x \leq 5$ and $1 \leq y \leq 3$. How many subsets R of S have the property that all the points of R lie on the graph of a single cubic? A cubic is a polynomial of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c , and d are real numbers (meaning that a is allowed to be 0).

Answer: $\boxed{796}$ We observe that R must contain at most 1 point from each column of S , because no function can contain more than 1 point with the same x -coordinate. Therefore, $|R| \leq 5$ ($|R|$ denotes the number of elements of R). Note that 4 points determine a cubic, so if R is any subset of points in distinct columns and $|R| \leq 4$, then R has the desired property. There are 4^5 ways to choose at most 1

point from each column and 3^5 ways to choose exactly 1 point from each column. There are therefore $4^5 - 3^5 = 781$ subsets R of S such that $|R| \leq 4$ and all points of R lie in distinct columns. As noted, these sets all automatically have the desired property.

Now we consider all sets R of size 5. As before, each point in R must come from a different column. Let us shift our origin to $(3, 2)$, and let p be the polynomial containing all 5 points of R . Then $R = \{(-2, p(-2)), (-1, p(-1)), (0, p(0)), (1, p(1)), (2, p(2))\}$.

By the method of finite differences³, or alternately by Lagrange Interpolation⁴, there is a unique polynomial p of degree less than 5 going through 5 specified points, and this polynomial is of degree less than 4 if and only if $p(-2) - 4p(-1) + 6p(0) - 4p(1) + p(2) = 0$.

Then $p(-2) + p(2) + 6p(0) = 4(p(-1) + p(1))$, where $p(-2) + p(2) \in \{-2 - 1, 0, 1, 2\}$, $p(-1) + p(1) \in \{-2, -1, 0, 1, 2\}$, and $p(0) \in \{-1, 0, 1\}$. We know that $6p(0)$ and $4(p(-1) + p(1))$ are necessarily even, thus we must have $p(-2) + p(2) \in \{-2, 0, 2\}$ in order for the equation to be satisfied.

Let $(a, b, c) = (p(-2) + p(2), 6p(0), 4(p(-1) + p(1)))$. The possible values of (a, b, c) that are solutions to $a + b = c$ are then $\{(-2, -6, -8), (-2, 6, 4), (0, 0, 0), (2, -6, -4), (2, 6, 8)\}$.

If $(a, b, c) = (-2, -6, -8)$, then we need $p(-2) + p(2) = -2$, $p(0) = -1$, $p(-1) + p(1) = -2$. There is only 1 possible solution to each of these equations: $(p(-2), p(2)) = (-1, -1)$ for the first one, $p(0) = -1$ for the second, and $(p(1)) = (-1, -1)$ for the third. Hence there is 1 possible subset R for the case $(a, b, c) = (-2, -6, -8)$.

If $(a, b, c) = (-2, 6, 4)$, then there is again 1 possible solution to $p(-2) + p(2) = 1$. There are two solutions to $p(-1) + p(1) = 1$: $(p(-1), p(1)) = (0, 1), (1, 0)$. Also, $p(0)$ can only be 1, so there are 2 possible subsets for this case.

If $(a, b, c) = (0, 0, 0)$, then there are 3 possible solutions to $p(-2) + p(2) = 0$: $(p(-2), p(2)) = (-1, 1), (0, 0), (1, -1)$. Similarly, there are 3 possible solutions to $p(-1) + p(1) = 0$. Also, $p(0)$ can only be 0, so there are 9 possible subsets for this case.

If $(a, b, c) = (2, -6, -4)$, then there is 1 possible solution to $p(-2) + p(2) = 2$: $(p(-2), p(2)) = (1, 1)$. There are 2 possible solutions to $p(-1) + p(1) = -1$: $(p(-1), p(1)) = (0, -1), (-1, 0)$. Also, $p(0)$ can only be -1, so there are 2 possible subsets for this case.

If $(a, b, c) = (2, 6, 8)$, then there is 1 possible solution to $p(-2) + p(2) = 2$, as shown above. There is 1 solution to $p(-1) + p(1) = 2$: $(p(-1), p(1)) = (1, 1)$. Also, $p(0)$ can only be 1, so there is 1 possible subset for this case.

Then there are $1 + 2 + 9 + 2 + 1 = 15$ total possible subsets of size 5 that can be fit to a polynomial of degree less than 4. Hence there are $781 + 15 = 796$ possible subsets total.

10. Call an $2n$ -digit number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$ -digit number is special (we will allow leading zeros in the number).

- (a) [25] The sequence p_n converges to a constant c . Find c .

Answer: $\frac{1}{2}$ We first claim that if a $2n$ -digit number x has at least eight 0's and at least eight 1's and the sum of its digits is even, then x is special.

Let A be a set of eight 0's and eight 1's and let B be the set of all the other digits. We split b arbitrarily into two sets Y and Z of equal size. If $|\sum_{y \in Y} y - \sum_{z \in Z} z| > 8$, then we swap the biggest element of the set with the bigger sum with the smallest element of the other set. This transposition always decreases the absolute value of the sum: in the worst case, a 9 from the bigger set is swapped for a 0 from the smaller set, which changes the difference by at most 18. Therefore, after a finite number of steps, we will have $|\sum_{y \in Y} y - \sum_{z \in Z} z| \leq 8$.

Note that this absolute value is even, since the sum of all the digits is even. Without loss of generality, suppose that $\sum_{y \in Y} y - \sum_{z \in Z} z$ is $2k$, where $0 \leq k \leq 4$. If we add k 0's and $8 - k$ 1's to Y , and we add the other elements of A to Z , then the two sets will balance, so x is special.

³See http://www.artofproblemsolving.com/Forum/weblog_entry.php?p=1263378.

⁴See http://en.wikipedia.org/wiki/Lagrange_polynomial.

- (b) [30] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d . Find r and d .

Answer: $\left(\frac{1}{4}, -1\right)$ To get the next asymptotic term after the constant term of $\frac{1}{2}$, we need to consider what happens when the digit sum is even; we want to find the probability that such a number isn't balanced. We claim that the configuration that contributes the vast majority of unbalanced numbers is when all numbers are even and the sum is $2 \pmod 4$, or such a configuration with all numbers increased by 1. Clearly this gives q_n being asymptotic to $-\frac{1}{2} \cdot 2 \cdot \left(\frac{1}{2}\right)^{2n} = -\left(\frac{1}{4}\right)^n$, so $r = \frac{1}{4}$ and $d = -1$.

To prove the claim, first note that the asymptotic probability that there are at most 4 digits that occur more than 10 times is asymptotically much smaller than $\left(\frac{1}{2}\right)^n$, so we can assume that there exist 5 digits that each occur at least 10 times. If any of those digits are consecutive, then the digit sum being even implies that the number is balanced (by an argument similar to part (a)).

So, we can assume that none of the numbers are consecutive. We would like to say that this implies that the numbers are either 0, 2, 4, 6, 8 or 1, 3, 5, 7, 9. However, we can't quite say this yet, as we need to rule out possibilities like 0, 2, 4, 7, 9. In this case, though, we can just pair 0 and 7 up with 2 and 4; by using the same argument as in part (a), except using 0 and 7 both (to get a sum of 7) and 2 and 4 both (to get a sum of 6) to balance out the two sets at the end.

In general, if there is ever a gap of size 3, consider the number right after it and the 3 numbers before it (so we have $k-4, k-2, k, k+3$ for some k), and pair them up such that one pair has a sum that's exactly one more than the other (i.e. pair $k-4$ with $k+3$ and $k-2$ with k). Since we again have pairs of numbers whose sums differ by 1, we can use the technique from part (a) of balancing out the sets at the end.

So, we can assume there is no gap of size 3, which together with the condition that no two numbers are adjacent implies that the 5 digits are either 0, 2, 4, 6, 8 or 1, 3, 5, 7, 9. For the remainder of the solution, we will deal with the 0, 2, 4, 6, 8 case, since it is symmetric with the other case under the transformation $x \mapsto 9 - x$.

If we can distribute the odd digits into two sets S_1 and S_2 such that (i) the difference in sums of S_1 and S_2 is small; and (ii) the difference in sums of S_1 and S_2 , plus the sum of the even digits, is divisible by 4, then the same argument as in part (a) implies that the number is good.

In fact, if there are *any* odd digits, then we can use them at the beginning to fix the parity mod 4 (by adding them all in such that the sums of the two sets remain close, and then potentially switching one with an even digit). Therefore, if there are any odd digits then the number is good.

Also, even if there are no odd digits, if the sum of the digits is divisible by 4 then the number is good.

So, we have shown that almost all non-good numbers come from having all numbers being even with a digit sum that is $2 \pmod 4$, or the analogous case under the mapping $x \mapsto 9 - x$. This formalizes the claim we made in the first paragraph, so $r = \frac{1}{4}$ and $d = -1$, as claimed.

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General Test

1. [2] Jacob flips five coins, exactly three of which land heads. What is the probability that the first two are both heads?

Answer: $\boxed{\frac{3}{10}}$ We can associate with each sequence of coin flips a unique word where H represents heads, and T represents tails. For example, the word HHTTH would correspond to the coin flip sequence where the first two flips were heads, the next two were tails, and the last was heads. We are given that exactly three of the five coin flips came up heads, so our word must be some rearrangement of HHHTT. To calculate the total number of possibilities, any rearrangement corresponds to a choice of three spots to place the H flips, so there are $\binom{5}{3} = 10$ possibilities. If the first two flips are both heads, then we can only rearrange the last three HTT flips, which corresponds to choosing one spot for the remaining H. This can be done in $\binom{3}{1} = 3$ ways. Finally, the probability is the quotient of these two, so we get the answer of $\frac{3}{10}$. Alternatively, since the total number of possibilities is small, we can write out all rearrangements: HHHTT, HHTHT, HHTTH, HTHHT, HTHTH, HTTHH, THHHT, THHTH, THTHH, TTHHH. Of these ten, only in the first three do we flip heads the first two times, so we get the same answer of $\frac{3}{10}$.

2. [3] How many sequences a_1, a_2, \dots, a_8 of zeroes and ones have $a_1a_2 + a_2a_3 + \dots + a_7a_8 = 5$?

Answer: $\boxed{9}$ First, note that we have seven terms in the left hand side, and each term can be either 0 or 1, so we must have five terms equal to 1 and two terms equal to 0. Thus, for $n \in \{1, 2, \dots, 8\}$, at least one of the a_n must be equal to 0. If we can find $i, j \in \{2, 3, \dots, 7\}$ such that $a_i = a_j = 0$ and $i < j$, then the terms $a_{i-1}a_i$, $a_i a_{i+1}$, and $a_j a_{j+1}$ will all be equal to 0. We did not count any term twice because $i - 1 < i < j$, so we would have three terms equal to 0, which cannot happen because we can have only two. Thus, we can find at most one $n \in \{2, 3, \dots, 7\}$ such that $a_n = 0$. We will do casework on which n in this range have $a_n = 0$.

If $n \in \{3, 4, 5, 6\}$, then we know that the terms $a_{n-1}a_n = a_n a_{n+1} = 0$, so all other terms must be 1, so $a_1a_2 = a_2a_3 = \dots = a_{n-2}a_{n-1} = 1$ and $a_{n+1}a_{n+2} = \dots = a_7a_8 = 1$. Because every a_i appears in one of these equations for $i \neq n$, then we must have $a_i = 1$ for all $i \neq n$, so we have 1 possibility for each choice of n and thus 4 possibilities total for this case.

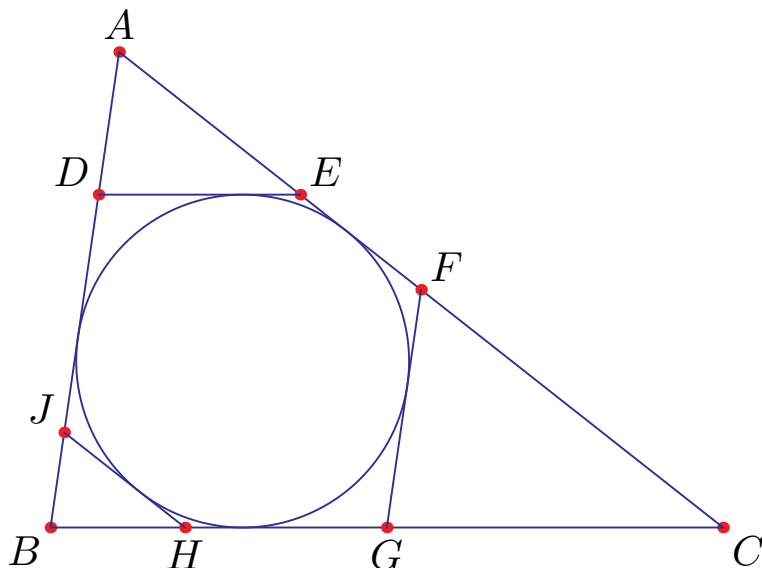
If $n = 2$, then again we have $a_1a_2 = a_2a_3 = 0$, so we must have $a_3a_4 = a_4a_5 = \dots = a_7a_8 = 1$, so $a_3 = a_4 = \dots = a_8 = 1$. However, this time a_1 is not fixed, and we see that regardless of our choice of a_1 the sum will still be equal to 5. Thus, since there are 2 choices for a_1 , then there are 2 possibilities total for this case. The case where $n = 7$ is analogous, with a_8 having 2 possibilities, so we have another 2 possibilities.

Finally, if $a_n = 1$ for $n \in \{2, 3, \dots, 7\}$, then we will have $a_2a_3 = a_3a_4 = \dots = a_6a_7 = 1$. We already have five terms equal to 1, so the remaining two terms a_1a_2 and a_7a_8 must be 0. Since $a_2 = 1$, then we must have $a_1 = 0$, and since $a_7 = 1$ then $a_8 = 0$. Thus, there is only 1 possibility for this case.

Summing, we have $4 + 2 + 2 + 1 = 9$ total sequences.

3. [3] Triangle ABC has $AB = 5$, $BC = 7$, and $CA = 8$. New lines not containing but parallel to AB , BC , and CA are drawn tangent to the incircle of ABC . What is the area of the hexagon formed by the sides of the original triangle and the newly drawn lines?

Answer: $\boxed{\frac{31}{5}\sqrt{3}}$



From the law of cosines we compute $\angle A = \cos^{-1}\left(\frac{5^2+8^2-7^2}{2(5)(8)}\right) = 60^\circ$. Using brackets to denote the area of a region, we find that

$$[ABC] = \frac{1}{2}AB \cdot AC \cdot \sin 60^\circ = 10\sqrt{3}.$$

The radius of the incircle can be computed by the formula

$$r = \frac{2[ABC]}{AB+BC+CA} = \frac{20\sqrt{3}}{20} = \sqrt{3}.$$

Now the height from A to BC is $\frac{2[ABC]}{BC} = \frac{20\sqrt{3}}{7}$. Then the height from A to DE is $\frac{20\sqrt{3}}{7} - 2r = \frac{6\sqrt{3}}{7}$. Then $[ADE] = \left(\frac{6\sqrt{3}/7}{20\sqrt{3}/7}\right)^2 [ABC] = \frac{9}{100}[ABC]$. Here, we use the fact that $\triangle ABC$ and $\triangle ADE$ are similar.

Similarly, we compute that the height from B to CA is $\frac{2[ABC]}{CA} = \frac{20\sqrt{3}}{8} = \frac{5\sqrt{3}}{2}$. Then the height from B to HJ is $\frac{5\sqrt{3}}{2} - 2r = \frac{\sqrt{3}}{2}$. Then $[BHJ] = \left(\frac{\sqrt{3}/2}{5\sqrt{3}/2}\right)^2 [ABC] = \frac{1}{25}[ABC]$.

Finally, we compute that the height from C to AB is $\frac{2[ABC]}{5} = \frac{20\sqrt{3}}{5} = 4\sqrt{3}$. Then the height from C to FG is $4\sqrt{3} - 2r = 2\sqrt{3}$. Then $[CFG] = \left(\frac{2\sqrt{3}}{4\sqrt{3}}\right)^2 [ABC] = \frac{1}{4}[ABC]$.

Finally we can compute the area of hexagon $DEFGHJ$. We have

$$[DEFGHJ] = [ABC] - [ADE] - [BHJ] - [CFG] = [ABC] \left(1 - \frac{9}{100} - \frac{1}{25} - \frac{1}{4}\right) = [ABC] \left(\frac{31}{50}\right) = 10\sqrt{3} \left(\frac{31}{50}\right) = \frac{31}{5}\sqrt{3}.$$

4. [4] An ant starts at the point $(1, 0)$. Each minute, it walks from its current position to one of the four adjacent lattice points until it reaches a point (x, y) with $|x| + |y| \geq 2$. What is the probability that the ant ends at the point $(1, 1)$?

Answer: $\boxed{\frac{7}{24}}$ From the starting point of $(1, 0)$, there is a $\frac{1}{4}$ chance we will go directly to $(1, 1)$, a $\frac{1}{2}$ chance we will end at $(2, 0)$ or $(1, -1)$, and a $\frac{1}{4}$ chance we will go to $(0, 0)$. Thus, if p is the probability that we will reach $(1, 1)$ from $(0, 0)$, then the desired probability is equal to $\frac{1}{4} + \frac{1}{4}p$, so we need only calculate p . Note that we can replace the condition $|x| + |y| \geq 2$ by $|x| + |y| = 2$, since in each iteration the quantity $|x| + |y|$ can increase by at most 1. Thus, we only have to consider the eight points $(2, 0)$, $(1, 1)$, $(0, 2)$, $(-1, 1)$, $(-2, 0)$, $(-1, -1)$, $(0, -2)$, $(1, -1)$. Let p_1, p_2, \dots, p_8 be the probability of

reaching each of these points from $(0, 0)$, respectively. By symmetry, we see that $p_1 = p_3 = p_5 = p_7$ and $p_2 = p_4 = p_6 = p_8$. We also know that there are two paths from $(0, 0)$ to $(1, 1)$ and one path from $(0, 0)$ to $(2, 0)$, thus $p_2 = 2p_1$. Because the sum of all probabilities is 1, we have $p_1 + p_2 + \dots + p_8 = 1$. Combining these equations, we see that $4p_1 + 4p_2 = 12p_1 = 1$, so $p_1 = \frac{1}{12}$ and $p_2 = \frac{1}{6}$. Since $p = p_2 = \frac{1}{6}$, then the final answer is $\frac{1}{4} + \frac{1}{4} \cdot \frac{1}{6} = \frac{7}{24}$.

5. [5] A polynomial P is of the form $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$. Given that $P(2) = 27$, what is $P(3)$?

Answer: 439 We use the following lemma:

Lemma. The sign of $\pm 2^n \pm 2^{n-1} \pm \dots \pm 2 \pm 1$ is the same as the sign of the 2^n term.

Proof. Without loss of generality, let 2^n be positive. (We can flip all signs.) Notice that $2^n \pm 2^{n-1} \pm 2^{n-2} \pm \dots \pm 2 \pm 1 \geq 2^n - 2^{n-1} - 2^{n-2} - \dots - 2 - 1 = 1$, which is positive.

We can use this lemma to uniquely determine the signs of P . Since our desired sum, 27, is positive, the coefficient of x^6 must be positive. Subtracting 64, we now have that $\pm 2^5 \pm 2^4 \pm \dots \pm 2 \pm 1 = -37$, so the sign of 2^5 must be negative. Continuing in this manner, we find that $P(x) = x^6 - x^5 - x^4 + x^3 + x^2 - x + 1$, so $P(3) = 3^6 - 3^5 - 3^4 + 3^3 + 3^2 - 3 + 1 = 439$.

6. [5] What is the sum of the positive solutions to $2x^2 - x[x] = 5$, where $[x]$ is the largest integer less than or equal to x ?

Answer: $\frac{3+\sqrt{41}+2\sqrt{11}}{4}$ We first note that $[x] \leq x$, so $2x^2 - x[x] \geq 2x^2 - x^2 = x^2$. Since this function is increasing on the positive reals, all solutions must be at most $\sqrt{5}$. This gives us 3 possible values of $[x]$: 0, 1, and 2.

If $[x] = 0$, then our equation becomes $2x^2 = 5$, which has positive solution $x = \sqrt{\frac{5}{2}}$. This number is greater than 1, so its floor is not 0; thus, there are no solutions in this case.

If $[x] = 1$, then our equation becomes $2x^2 - x = 5$. Using the quadratic formula, we find the positive solution $x = \frac{1+\sqrt{41}}{4}$. Since $3 < \sqrt{41} < 7$, this number is between 1 and 2, so it satisfies the equation.

If $[x] = 2$, then our equation becomes $2x^2 - 2x = 5$. We find the positive solution $x = \frac{1+\sqrt{11}}{2}$. Since $3 < \sqrt{11} < 5$, this number is between 2 and 3, so it satisfies the equation.

We then find that the sum of positive solutions is $\frac{1+\sqrt{41}}{4} + \frac{1+\sqrt{11}}{2} = \frac{3+\sqrt{41}+2\sqrt{11}}{4}$.

7. [6] What is the remainder when $(1+x)^{2010}$ is divided by $1+x+x^2$?

Answer: 1 We use polynomial congruence mod $1+x+x^2$ to find the desired remainder. Since $x^2 + x + 1 \mid x^3 - 1$, we have that $x^3 \equiv 1 \pmod{1+x+x^2}$. Now:

$$\begin{aligned} (1+x)^{2010} &\equiv (-x^2)^{2010} \pmod{1+x+x^2} \\ &\equiv x^{4020} \pmod{1+x+x^2} \\ &\equiv (x^3)^{1340} \pmod{1+x+x^2} \\ &\equiv 1^{1340} \pmod{1+x+x^2} \\ &\equiv 1 \pmod{1+x+x^2} \end{aligned}$$

Thus, the answer is 1.

8. [7] Two circles with radius one are drawn in the coordinate plane, one with center $(0, 1)$ and the other with center $(2, y)$, for some real number y between 0 and 1. A third circle is drawn so as to be tangent to both of the other two circles as well as the x axis. What is the smallest possible radius for this third circle?

Answer: $3 - 2\sqrt{2}$ Suppose that the smaller circle has radius r . Call the three circles (in order from left to right) O_1 , O_2 , and O_3 . The distance between the centers of O_1 and O_2 is $1+r$, and the distance in their y -coordinates is $1-r$. Therefore, by the Pythagorean theorem, the difference in x -coordinates

is $\sqrt{(1+r)^2 - (1-r)^2} = 2\sqrt{r}$, which means that O_2 has a center at $(2\sqrt{r}, r)$. But O_2 is also tangent to O_3 , which means that the difference in x -coordinate from the right-most point of O_2 to the center of O_3 is at most 1. Therefore, the center of O_3 has an x -coordinate of at most $2\sqrt{r} + r + 1$, meaning that $2\sqrt{r} + r + 1 \leq 2$. We can use the quadratic formula to see that this implies that $\sqrt{r} \leq \sqrt{2} - 1$, so $r \leq 3 - 2\sqrt{2}$. We can achieve equality by placing the center of O_3 at $(2, r)$ (which in this case is $(2, 3 - 2\sqrt{2})$).

9. [7] What is the sum of all numbers between 0 and 511 inclusive that have an even number of 1s when written in binary?

Answer: $\boxed{65408}$ Call a digit in the binary representation of a number a bit. We claim that for any given i between 0 and 8, there are 128 numbers with an even number of 1s that have a 1 in the bit representing 2^i . To prove this, we simply make that bit a 1, then consider all possible configurations of the other bits, excluding the last bit (or the second-last bit if our given bit is already the last bit). The last bit will then be restricted to satisfy the parity condition on the number of 1s. As there are 128 possible configurations of all the bits but two, we find 128 possible numbers, proving our claim.

Therefore, each bit is present as a 1 in 128 numbers in the sum, so the bit representing 2^i contributes $128 \cdot 2^i$ to our sum. Summing over all $0 \leq i \leq 8$, we find the answer to be $128(1 + 2 + \dots + 128) = 128 \cdot 511 = 65408$.

10. [8] You are given two diameters AB and CD of circle Ω with radius 1. A circle is drawn in one of the smaller sectors formed such that it is tangent to AB at E , tangent to CD at F , and tangent to Ω at P . Lines PE and PF intersect Ω again at X and Y . What is the length of XY , given that $AC = \frac{2}{3}$?

Answer: $\boxed{\frac{4\sqrt{2}}{3}}$ Let O denote the center of circle Ω . We first prove that $OX \perp AB$ and $OY \perp CD$. Consider the homothety about P which maps the smaller circle to Ω . This homothety takes E to X and also takes AB to the line tangent to circle Ω parallel to AB . Therefore, X is the midpoint of the arc AB , and so $OX \perp AB$. Similarly, $OY \perp CD$.

Let $\theta = \angle AOC$. By the Law of Sines, we have $AC = 2 \sin \frac{\theta}{2}$. Thus, $\sin \frac{\theta}{2} = \frac{1}{3}$, and $\cos \frac{\theta}{2} = \sqrt{1 - (\frac{1}{3})^2} = \frac{2\sqrt{2}}{3}$. Therefore,

$$\begin{aligned} XY &= 2 \sin \frac{\angle XOY}{2} \\ &= 2 \sin \left(90^\circ - \frac{\theta}{2} \right) \\ &= 2 \cos \frac{\theta}{2} \\ &= \frac{4\sqrt{2}}{3}. \end{aligned}$$

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Guts Round

1. [5] David, Delong, and Justin each showed up to a problem writing session at a random time during the session. If David arrived before Delong, what is the probability that he also arrived before Justin?

Answer: $\boxed{\frac{2}{3}}$ Let t_1 be the time that David arrives, let t_2 be the time that Delong arrives, and let t_3 be the time that Justin arrives. We can assume that all times are pairwise distinct because the probability of any two being equal is zero. Because the times were originally random and independent before we were given any information, then all orders $t_1 < t_2 < t_3$, $t_1 < t_3 < t_2$, $t_3 < t_1 < t_2$, $t_2 < t_1 < t_3$, $t_2 < t_3 < t_1$, $t_3 < t_2 < t_1$ must all be equally likely. Since we are given that $t_1 < t_2$, then we only have the first three cases to consider, and $t_1 < t_3$ in two cases of these three. Thus, the desired probability is $\frac{2}{3}$.

2. [5] A circle of radius 6 is drawn centered at the origin. How many squares of side length 1 and integer coordinate vertices intersect the interior of this circle?

Answer: $\boxed{132}$ By symmetry, the answer is four times the number of squares in the first quadrant. Let's identify each square by its coordinates at the bottom-left corner, (x, y) . When $x = 0$, we can have $y = 0 \dots 5$, so there are 6 squares. (Letting $y = 6$ is not allowed because that square intersects only the *boundary* of the circle.) When $x = 1$, how many squares are there? The equation of the circle is $y = \sqrt{36 - x^2} = \sqrt{36 - 1^2}$ is between 5 and 6, so we can again have $y = 0 \dots 5$. Likewise for $x = 2$ and $x = 3$. When $x = 4$ we have $y = \sqrt{20}$ which is between 4 and 5, so there are 5 squares, and when $x = 5$ we have $y = \sqrt{11}$ which is between 3 and 4, so there are 4 squares. Finally, when $x = 6$, we have $y = 0$, and no squares intersect the interior of the circle. This gives $6 + 6 + 6 + 6 + 5 + 4 = 33$. Since this is the number in the first quadrant, we multiply by four to get 132.

3. [5] Jacob flipped a fair coin five times. In the first three flips, the coin came up heads exactly twice. In the last three flips, the coin also came up heads exactly twice. What is the probability that the third flip was heads?

Answer: $\boxed{\frac{4}{5}}$ How many sequences of five flips satisfy the conditions, and have the third flip be heads? We have $_H_$, so exactly one of the first two flips is heads, and exactly one of the last two flips is heads. This gives $2 \times 2 = 4$ possibilities. How many sequences of five flips satisfy the conditions, and have the third flip be tails? Now we have $_T_$, so the first two and the last two flips must all be heads. This gives only 1 possibility. So the probability that the third flip was heads is $\frac{4}{4+1} = \frac{4}{5}$.

4. [6] Let x be a real number. Find the maximum value of $2^{x(1-x)}$.

Answer: $\boxed{\sqrt[4]{2}}$ Consider the function 2^y . This is monotonically increasing, so to maximize 2^y , you simply want to maximize y . Here, $y = x(1-x) = -x^2 + x$ is a parabola opening downwards. The vertex of the parabola occurs at $x = (-1)/(-2) = 1/2$, so the maximum value of the function is $2^{(1/2)(1/2)} = \sqrt[4]{2}$.

5. [6] An icosahedron is a regular polyhedron with twenty faces, all of which are equilateral triangles. If an icosahedron is rotated by θ degrees around an axis that passes through two opposite vertices so that it occupies exactly the same region of space as before, what is the smallest possible positive value of θ ?

Answer: $\boxed{72^\circ}$ Because this polyhedron is regular, all vertices must look the same. Let's consider just one vertex. Each triangle has a vertex angle of 60° , so we must have fewer than 6 triangles; if we had 6, there would be 360° at each vertex and you wouldn't be able to "fold" the polyhedron up (that is, it would be a flat plane). It's easy to see that we need at least 3 triangles at each vertex, and this gives a triangular pyramid with only 4 faces. Having 4 triangles meeting at each vertex gives an octahedron (two square pyramids with the squares glued together) with 8 faces. Therefore, an icosahedron has 5 triangles meeting at each vertex, so rotating by $\frac{360^\circ}{5} = 72^\circ$ gives another identical icosahedron.

Alternate solution: Euler's formula tells us that $V - E + F = 2$, where an icosahedron has V vertices, E edges, and F faces. We're told that $F = 20$. Each triangle has 3 edges, and every edge is common to 2 triangles, so $E = \frac{3(20)}{2} = 30$. Additionally, each triangle has 3 vertices, so if every vertex is common to n triangles, then $V = \frac{3(20)}{n} = \frac{60}{n}$. Plugging this into the formula, we have $\frac{60}{n} - 30 + 20 = 2$, so $\frac{60}{n} = 12$ and $n = 5$. Again this shows that the rotation is $\frac{360^\circ}{5} = 72^\circ$

6. [6] How many ordered pairs (S, T) of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are there whose union contains exactly three elements?

Answer: 3240 Let the three elements in the union be a, b , and c . We know that a can be only in S , only in T , or both, so there are 3 possibilities for placing it. (Recall that $S = \{a\}, T = \{b, c\}$ is different from $S = \{b, c\}, T = \{a\}$ because S and T are an *ordered* pair.) Likewise for b and c . The other 7 elements are in neither S nor T , so there is only 1 possibility for placing them. This gives $3^3 = 27$ ways to pick S and T once you've picked the union. There are $\binom{10}{3} = 120$ ways to pick the elements in the union, so we have $120 \times 27 = 3240$ ways total.

7. [7] Let $f(x, y) = x^2 + 2x + y^2 + 4y$. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$, and (x_4, y_4) be the vertices of a square with side length one and sides parallel to the coordinate axes. What is the minimum value of $f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4)$?

Answer: -18 The square's corners must be at $(x, y), (x + 1, y), (x + 1, y + 1)$, and $(x, y + 1)$ for some x and y . So,

$$\begin{aligned} f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4) &= 2(x^2 + 2x) + 2((x + 1)^2 + 2(x + 1)) + 2(y^2 + 4y) + 2((y + 1)^2 + 4(y + 1)) \\ &= 4x^2 + 12x + 6 + 4y^2 + 20y + 10 \\ &= (2x + 3)^2 - 3 + (2y + 5)^2 - 15 \\ &\geq -18 \end{aligned}$$

This attains its minimum value of -18 when $x = -\frac{3}{2}$ and $y = -\frac{5}{2}$.

8. [7] What is the sum of all four-digit numbers that are equal to the cube of the sum of their digits (leading zeros are not allowed)?

Answer: 10745 We want to find all integers x between 1000 and 9999 that are the cube of the sum of their digits. Of course, our search is only restricted to perfect cubes. The smallest such cube is $10^3 = 1000$ and the largest such cube is $21^3 = 9261$. This means we only have to check 12 different cubes, which is quite doable, but we can reduce the search even further with a little number theory.

Suppose we write our number as $x = 1000a + 100b + 10c + d$, where a, b, c , and d are the decimal digits of x . Then we have

$$(a + b + c + d)^3 \equiv 1000a + 100b + 10c + d \equiv a + b + c + d \pmod{9}$$

If we let $k = a + b + c + d$, then k must be a solution to the modular equation $k^3 \equiv k \pmod{9}$. A quick check of the values 0 through 8 shows that the only solutions are 0, 1, and 8.

Now, in our search, we only have to check values that are the cube of a number which is either 0, 1, or 8 mod 9.

$$\begin{aligned} 10^3 &= 1000, \text{ but } 1 + 0 + 0 + 0 \neq 10. \\ 17^3 &= 4913, \text{ and } 4 + 9 + 1 + 3 = 17. \\ 18^3 &= 5832, \text{ and } 5 + 8 + 3 + 2 = 18. \\ 19^3 &= 6859, \text{ but } 6 + 8 + 5 + 9 \neq 19. \end{aligned}$$

So the only solutions are 4913 and 5832, which sum to 10745.

9. [7] How many functions $f : \{1, 2, \dots, 10\} \rightarrow \{1, 2, \dots, 10\}$ satisfy the property that $f(i) + f(j) = 11$ for all values of i and j such that $i + j = 11$.

Answer: 100000 To construct such a function f , we just need to choose a value for $f(x)$ from $\{1, 2, \dots, 10\}$ for each $x \in \{1, 2, \dots, 10\}$. But the condition that $f(i) + f(j) = 11$ whenever $i + j = 11$ means that

$$\begin{aligned} f(10) &= 11 - f(1). \\ f(9) &= 11 - f(2). \\ &\vdots \\ f(6) &= 11 - f(5). \end{aligned}$$

This means that once we have chosen $f(1), f(2), f(3), f(4)$, and $f(5)$, the five remaining values of $f(6), f(7), f(8), f(9)$, and $f(10)$ are already determined. The answer is therefore just the number of ways to choose these first five values. Since there are 10 possibilities for each one, we get that the answer is $10^5 = 100000$.

10. [8] What is the smallest integer greater than 10 such that the sum of the digits in its base 17 representation is equal to the sum of the digits in its base 10 representation?

Answer: 153 We assume that the answer is at most three digits (in base 10). Then our desired number can be expressed in the form $\overline{abc}_{10} = \overline{def}_{17}$, where a, b, c are digits in base 10, and d, e, f are digits in base 17. These variables then satisfy the equations

$$\begin{aligned} 100a + 10b + c &= 289d + 17e + f, \\ a + b + c &= d + e + f. \end{aligned}$$

Subtracting the second equation from the first, we obtain $99a + 9b = 288d + 16e$, or $9(11a + b) = 16(18d + e)$. From this equation, we find that $11a + b$ must be divisible by 16, and $18d + e$ must be divisible by 9. To minimize \overline{abc} , we find the minimal possible value of a : If $a = 0$, then the only way for $11a + b = b$ to be divisible by 16 is to set $b = 0$; however, this is disallowed by the problem condition, which stipulates that the number must be greater than 10. If we try $a = 1$, then we find that the only possible value of b which lets $11a + b = b + 11$ be divisible by 16 is $b = 5$. Plugging these in and simplifying, we find that we must have $18d + e = 9$. The only possible solution to this is $d = 0, e = 9$. Now to satisfy $a + b + c = d + e + f$, we must have $1 + 5 + c = 0 + 9 + f$, or $c = f + 3$. The minimal possible solution to this is $c = 3, f = 0$. So our answer is $\overline{abc} = 153$, which is also equal to 090_{17} .

11. [8] How many nondecreasing sequences a_1, a_2, \dots, a_{10} are composed entirely of at most three distinct numbers from the set $\{1, 2, \dots, 9\}$ (so $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ and $2, 2, 2, 2, 5, 5, 5, 5, 5$ are both allowed)?

Answer: 3357 From any sequence a_1, a_2, \dots, a_{10} , construct a sequence b_1, b_2, \dots, b_9 , where b_i counts the number of times i occurs in the sequence. There is a correspondence from all possible sequences b_1, b_2, \dots, b_9 with at most 3 nonzero terms which add to 10, since any sequence of a_1, a_2, \dots, a_{10} will be converted to this form, and from any sequence b_1, b_2, \dots, b_9 , we can construct a unique sequence of a -s by listing i b_i times (for $1 \leq i \leq 9$) in nondecreasing order.

Our goal now is to count the number of possible sequences b_1, b_2, \dots, b_9 meeting our conditions. We casework on the number of nonzero terms in the sequence:

Case 1: The sequence has exactly one nonzero term.

Then exactly one of b_1, b_2, \dots, b_9 is equal to 10, and all the rest are equal to 0. This gives us 9 possible sequences in this case.

Case 2: The sequence has exactly two nonzero terms.

There are $\binom{9}{2} = 36$ ways to choose the two terms b_i, b_j ($i < j$) which are nonzero. From here, we have 9 choices for the value of b_i , namely 1 through 9 (since both b_i and b_j must be nonzero), and b_j will be fixed, so this case gives us $36 \cdot 9 = 324$ possible sequences.

Case 3: The sequence has exactly three nonzero terms.

There are $\binom{9}{3} = 84$ ways to choose the three terms b_i, b_j, b_k ($i < j < k$) which are nonzero. Letting

$c_i = b_i - 1, c_j = b_j - 1, c_k = b_k - 1$, we have that c_i, c_j, c_k are nonnegative integers which sum to 7. There are $\binom{9}{2} = 36$ solutions to this equation (consider placing two dividers in the nine spaces between the ten elements), giving $84 \cdot 36 = 3024$ possibilities in this case.

We then have $9 + 324 + 3024 = 3357$ possible sequences.

12. [8] An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point $(5, 5)$?

Answer: [84] We can change the ant's sequence of moves to a sequence a_1, a_2, \dots, a_{10} , with $a_i = 0$ if the i -th step is up, and $a_i = 1$ if the i -th step is right. We define a subsequence of moves a_i, a_{i+1}, \dots, a_j , ($i \leq j$) as an *up run* if all terms of the subsequence are equal to 0, and a_{i-1} and a_{j+1} either do not exist or are not equal to 0, and define a *right run* similarly. In a sequence of moves, up runs and right runs alternate, so the number of up rights can differ from the number of right runs by at most one.

Now let $f(n)$ denote the number of sequences a_1, a_2, \dots, a_n where $a_i \in \{1, 2\}$ for $1 \leq i \leq n$, and $a_1 + a_2 + \dots + a_n = 5$. (In essence, we are splitting the possible 5 up moves into up runs, and we are doing the same with the right moves). We can easily compute that $f(3) = 3, f(4) = 4, f(5) = 1$, and $f(n) = 0$ otherwise.

For each possible pair of numbers of up runs and right runs, we have two choices of which type of run is first. Our answer is then $2(f(3)^2 + f(3)f(4) + f(4)^2 + f(4)f(5) + f(5)^2) = 2(9 + 12 + 16 + 4 + 1) = 84$.

13. [8] How many sequences of ten binary digits are there in which neither two zeroes nor three ones ever appear in a row?

Answer: [28] Let a_n be the number of binary sequences of length n satisfying the conditions and ending in 0, let b_n be the number ending in 01, and let c_n be the number ending in 11. From the legal sequences of length 2 01, 11, 10, we find that $a_2 = b_2 = c_2 = 1$. We now establish a recursion by building sequences of length $n + 1$ from sequences of length n . We can add a 0 to a sequence of length n if and only if it ended with a 1, so $a_{n+1} = b_n + c_n$. We can have a sequence of length $n + 1$ ending with 01 only by adding a 1 to a sequence of length n ending in 0, so $b_{n+1} = a_n$. We can have a sequence of length $n + 1$ ending with 11 only by adding a 1 to a sequence of length n ending in 01, so $c_{n+1} = b_n$. We can now run the recursion:

n	a_n	b_n	c_n
2	1	1	1
3	2	1	1
4	2	2	1
5	3	2	2
6	4	3	2
7	5	4	3
8	7	5	4
9	9	7	5
10	12	9	7

Our answer is then $12 + 9 + 7 = 28$.

14. [8] The positive integer i is chosen at random such that the probability of a positive integer k being chosen is $\frac{3}{2}$ times the probability of $k + 1$ being chosen. What is the probability that the i^{th} digit after the decimal point of the decimal expansion of $\frac{1}{7}$ is a 2?

Answer: $\frac{108}{665}$ First we note that the probability that n is picked is $\frac{1}{2} \times \left(\frac{2}{3}\right)^n$, because this is the sequence whose terms decrease by a factor of $\frac{2}{3}$ each time and whose sum is 1 (recall that probabilities must sum to 1).

Now note that $\frac{1}{7} = .142857142857\dots$, meaning that 2 occurs at digits 3, 9, 15, 21, etc. We can then calculate the probability that we ever pick 2 as

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{2} \cdot \left(\frac{2}{3}\right)^{6k+3} &= \frac{4}{27} \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^{6k} \\
&= \frac{4}{27} \cdot \frac{1}{1 - \left(\frac{2}{3}\right)^6} \\
&= \frac{4}{27} \cdot \frac{729}{729 - 64} \\
&= \frac{4}{27} \cdot \frac{729}{665} \\
&= \frac{108}{665}.
\end{aligned}$$

15. [8] Distinct points A, B, C, D are given such that triangles ABC and ABD are equilateral and both are of side length 10. Point E lies inside triangle ABC such that $EA = 8$ and $EB = 3$, and point F lies inside triangle ABD such that $FD = 8$ and $FB = 3$. What is the area of quadrilateral $AEFD$?

Answer: $\boxed{\frac{91\sqrt{3}}{4}}$ Since $AEB \cong DFB$, we have $\angle EBA = \angle FBD$. Thus, $\angle EBF = \angle EBA + \angle ABF = \angle FBD + \angle ABF = \angle ABD = 60^\circ$. Since $EB = BF = 3$, this means that EBF is an equilateral triangle of side length 3. Now we have $[AEFD] = [AEBD] - [EBF] - [FBD] = [AEB] + [ABD] - [EBF] - [FBD] = [ABD] - [EBF] = \frac{\sqrt{3}}{4}(10^2 - 3^2) = \frac{91\sqrt{3}}{4}$.

16. [9] Triangle ABC is given in the plane. Let AD be the angle bisector of $\angle BAC$; let BE be the altitude from B to AD , and let F be the midpoint of AB . Given that $AB = 28, BC = 33, CA = 37$, what is the length of EF ?

Answer: $\boxed{14}$ $\triangle ABE$ is a right triangle, and F is the midpoint of the hypotenuse (and therefore the circumcenter), so $EF = BF = AF = 14$.

17. [9] A triangle with side lengths 5, 7, 8 is inscribed in a circle C . The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

Answer: $\boxed{\frac{49}{18}\pi}$ Let $\triangle PQR$ have sides $p = 7, q = 5, r = 8$. Of the four sectors determined by the diameters of C that are parallel to PQ and PR , two have angles equal to P and the other two have angles equal to $\pi - P$. We first find P using the law of cosines: $49 = 25 + 64 - 2(5)(8)\cos P$ implies $\cos P = \frac{1}{2}$ implies $P = \frac{\pi}{3}$. Thus the two smaller sectors will have angle $\frac{\pi}{3}$. Next we find the circumradius of $\triangle PQR$ using the formula $R = \frac{pqr}{4[PQR]}$, where $[PQR]$ is the area of $\triangle PQR$. By Heron's Formula we have $[PQR] = \sqrt{10(5)(3)(2)} = 10\sqrt{3}$; thus $R = \frac{5 \cdot 7 \cdot 8}{4(10\sqrt{3})} = \frac{7}{\sqrt{3}}$. The area of a smaller sector is thus $\frac{\pi/3}{2\pi} (\pi R^2) = \frac{\pi}{6} \left(\frac{7}{\sqrt{3}}\right)^2 = \frac{49}{18}\pi$.

18. [9] Jeff has a 50 point quiz at 11 am. He wakes up at a random time between 10 am and noon, then arrives at class 15 minutes later. If he arrives on time, he will get a perfect score, but if he arrives more than 30 minutes after the quiz starts, he will get a 0, but otherwise, he loses a point for each minute he's late (he can lose parts of one point if he arrives a nonintegral number of minutes late). What is Jeff's expected score on the quiz?

Answer: $\boxed{\frac{55}{2}}$ If Jeff wakes up between 10:00 and 10:45, he gets 50. If he wakes up between 10:45 and 11:15, and he wakes up k minutes after 10:45, then he gets $50 - k$ points. Finally, if he wakes up between 11:15 and 12:00 he gets 0 points. So he has a $\frac{3}{8}$ probability of 50, a $\frac{3}{8}$ probability of 0, and a $\frac{1}{4}$ probability of a number chosen uniformly between 20 and 50 (for an average of 35). Thus his expected score is $\frac{3}{8} \times 50 + \frac{1}{4} \times 35 = \frac{75+35}{4} = \frac{110}{4} = \frac{55}{2}$.

19. [11] How many 8-digit numbers begin with 1, end with 3, and have the property that each successive digit is either one more or two more than the previous digit, considering 0 to be one more than 9?

Answer: $\boxed{21}$ Given an 8-digit number a that satisfies the conditions in the problem, let a_i denote the difference between its $(i + 1)$ th and i th digit. Since $i \in \{1, 2\}$ for all $1 \leq i \leq 7$, we have $7 \leq a_1 + a_2 + \dots + a_7 \leq 14$. The difference between the last digit and the first digit of m is $3 - 1 \equiv 2 \pmod{10}$, which means $a_1 + \dots + a_7 = 12$. Thus, exactly five of the a_i s equal to 2 and the remaining two equal to 1. The number of permutations of five 2s and two 1s is $\binom{7}{2} = 21$.

20. [11] Given a permutation π of the set $\{1, 2, \dots, 10\}$, define a rotated cycle as a set of three integers i, j, k such that $i < j < k$ and $\pi(j) < \pi(k) < \pi(i)$. What is the total number of rotated cycles over all permutations π of the set $\{1, 2, \dots, 10\}$?

Answer: $\boxed{72576000}$ Let us consider a triple (i, j, k) with $i < j < k$ and determine how many permutations rotate it. There are $\binom{10}{3}$ choices for the values of $\pi(i), \pi(j), \pi(k)$ and the choice of this set of three determines the values of $\pi(i), \pi(j), \pi(k)$. The other 7 values then have $7!$ ways to be arranged (any permutation of them will work), so exactly $\binom{10}{3}7!$ permutations rotate (i, j, k) . Therefore, as there are $\binom{10}{3}$ such triples, the total number of rotated triples is $\binom{10}{3}^2 \cdot 7! = 72576000$.

21. [11] George, Jeff, Brian, and Travis decide to play a game of hot potato. They begin by arranging themselves clockwise in a circle in that order. George and Jeff both start with a hot potato. On his turn, a player gives a hot potato (if he has one) to a randomly chosen player among the other three (if a player has two hot potatoes on his turn, he only passes one). If George goes first, and play proceeds clockwise, what is the probability that Travis has a hot potato after each player takes one turn?

Answer: $\boxed{\frac{5}{27}}$ Notice that Travis can only have the hot potato at the end if he has two potatoes before his turn. A little bit of casework shows that this can only happen when

Case 1: George gives Travis his potato, while Jeff gives Brian his potato, which in then goes to Travis.

The probability of this occurring is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$

Case 2: George gives Travis his potato, while Jeff gives Travis his potato. The probability of this occurring is $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

Case 3: George gives Brian his potato, Jeff gives Travis his potato, and then Brian gives Travis his potato. The probability of this occurring is $\frac{1}{27}$

Because these events are all disjoint, the probability that Travis ends up with the hot potato is $\frac{5}{27}$

22. [12] Let $g_1(x) = \frac{1}{3}(1 + x + x^2 + \dots)$ for all values of x for which the right hand side converges. Let $g_n(x) = g_1(g_{n-1}(x))$ for all integers $n \geq 2$. What is the largest integer r such that $g_r(x)$ is defined for some real number x ?

Answer: $\boxed{5}$ Notice that the series is geometric with ratio x , so it converges if $-1 < x < 1$. Also notice that where $g_1(x)$ is defined, it is equal to $\frac{1}{3(1-x)}$. The image of $g_1(x)$ is then the interval $(\frac{1}{6}, \infty)$.

The image of $g_2(x)$ is simply the values of $g_1(x)$ for x in $(\frac{1}{6}, 1)$, which is the interval $(\frac{2}{5}, \infty)$. Similarly, the image of $g_3(x)$ is $(\frac{5}{9}, \infty)$, the image of $g_4(x)$ is $(\frac{3}{4}, \infty)$, and the image of $g_5(x)$ is $(\frac{4}{3}, \infty)$. As this does not intersect the interval $(-1, 1)$, $g_6(x)$ is not defined for any x , so the answer is 5.

23. [12] Let a_1, a_2, \dots be an infinite sequence of positive integers such that for integers $n > 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. How many such sequences $\{a_n\}$ are there such that $a_{2010} \leq 2^{2012}$?

Answer: $\boxed{36 \cdot 2^{2009} + 36}$ Consider the characteristic polynomial for the recurrence $a_{n+2} - 3a_{n+1} + 2a_n = 0$, which is $x^2 - 3x + 2$. The roots are at 2 and 1, so we know that numbers a_i must be of the form $a_i = a2^{i-1} + b$ for integers a and b . Therefore a_{2010} must equal to $a2^{2009} + b$, where a and b are both integers. If the expression is always positive, it is sufficient to say a_1 is positive and a is nonnegative, or $a + b > 0$, and $a \geq 0$.

For a given value of a , $1 - a \leq b \leq 2^{2012} - a2^{2009}$, so there are $2^{2012} - a2^{2009} + a$ possible values of b for each a (where the quantity is positive). a can take any value between 0 and 2^3 , we sum over all such a in this range, to attain $9 \cdot 2^{2012} - (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)2^{2009} + (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8)$, or $36(2^{2009}) + 36$, which is our answer.

24. [12] Let $P(x)$ be a polynomial of degree at most 3 such that $P(x) = \frac{1}{1+x+x^2}$ for $x = 1, 2, 3, 4$. What is $P(5)$?

Answer: $\boxed{\frac{-3}{91}}$ The forward difference of a polynomial P is $\Delta P(x) = P(x+1) - P(x)$, which is a new polynomial with degree reduced by one. Therefore, if we apply this operation three times we'll get a constant function, and we can work back up to get a value of $P(5)$. Practically, we create the following table of differences:

$$\begin{array}{cccccc} \frac{1}{3} & & \frac{1}{7} & & \frac{1}{13} & & \frac{1}{21} \\ & \frac{-4}{21} & & \frac{-6}{91} & & \frac{-8}{273} & \\ & & \frac{34}{273} & & \frac{10}{273} & & \\ & & & \frac{-24}{273} & & & \end{array}$$

Then extend it to be the following table:

$$\begin{array}{ccccccc} \frac{1}{3} & & \frac{1}{7} & & \frac{1}{13} & & \frac{1}{21} & & \frac{-9}{273} \\ & \frac{-4}{21} & & \frac{-6}{91} & & \frac{-8}{273} & & \frac{-22}{273} & \\ & & \frac{34}{273} & & \frac{10}{273} & & \frac{-14}{273} & & \\ & & & \frac{-24}{273} & & \frac{-24}{273} & & & \end{array}$$

So our answer is $\frac{-9}{273} = \frac{-3}{91}$

25. [14] Triangle ABC is given with $AB = 13$, $BC = 14$, $CA = 15$. Let E and F be the feet of the altitudes from B and C , respectively. Let G be the foot of the altitude from A in triangle AFE . Find AG .

Answer: $\boxed{\frac{396}{65}}$ By Heron's formula we have $[ABC] = \sqrt{21(8)(7)(6)} = 84$. Let D be the foot of the altitude from A to BC ; then $AD = 2 \cdot \frac{84}{14} = 12$. Notice that because $\angle BFC = \angle BEC$, $BFEC$ is cyclic, so $\angle AFE = 90 - \angle EFC = 90 - \angle EBC = \angle C$. Therefore, we have $\triangle AEF \sim \triangle ABC$, so $\frac{AG}{AD} = \frac{AE}{AB}$; $\frac{1}{2}(BE)(AC) = 84 \implies BE = \frac{56}{5} \implies AE = \sqrt{13^2 - \left(\frac{56}{5}\right)^2} = \sqrt{\frac{65^2 - 56^2}{5^2}} = \frac{33}{5}$. Then $AG = AD \cdot \frac{AE}{AB} = 12 \cdot \frac{33/5}{13} = \frac{396}{65}$.

26. [14] w, x, y, z are real numbers such that

$$\begin{aligned} w + x + y + z &= 5 \\ 2w + 4x + 8y + 16z &= 7 \\ 3w + 9x + 27y + 81z &= 11 \\ 4w + 16x + 64y + 256z &= 1 \end{aligned}$$

What is the value of $5w + 25x + 125y + 625z$?

Answer: $\boxed{-60}$ We note this system of equations is equivalent to evaluating the polynomial (in a) $P(a) = wa + xa^2 + ya^3 + za^4$ at 1, 2, 3, and 4. We know that $P(0) = 0$, $P(1) = 5$, $P(2) = 7$, $P(3) = 11$, $P(4) = 1$. The finite difference of a polynomial f is $f(n+1) - f(n)$, which is a polynomial with degree one less than the degree of f . The second, third, etc finite differences come from applying this operation repeatedly. The fourth finite difference of this polynomial is constant because this is a fourth degree polynomial. Repeatedly applying finite differences, we get

$$\begin{array}{cccccc} 0 & 5 & 7 & 11 & 1 & \\ & 5 & 2 & 4 & -10 & \\ & & -3 & 2 & -14 & \\ & & & 5 & -16 & \\ & & & & -21 & \end{array}$$

and we see that the fourth finite difference is -21 . We can extend this table, knowing that the fourth finite difference is always -21 , and we find that that $P(5) = -60$. The complete table is

0	5	7	11	1	-60
	5	2	4	-10	-61
		-3	2	-14	-51
			5	-16	-37
				-21	-21

27. [14] Let $f(x) = -x^2 + 10x - 20$. Find the sum of all 2^{2010} solutions to $\underbrace{f(f(\dots(x)\dots))}_{2010fs} = 2$.

Answer: $\boxed{5 \cdot 2^{2010}}$ Define $g(x) = f(f(\dots(x)\dots))$. We calculate:

$$f(10 - x) = -(10 - x)^2 + 10(10 - x) - 20 = -100 + 20x - x^2 + 100 - 10x - 20 = -x^2 + 10x - 20 = f(x).$$

This implies that $g(10 - x) = g(x)$. So if $g(x) = 2$, then $g(10 - x) = 2$. Moreover, we can calculate $f(5) = -25 + 50 - 20 = 5$, so $g(5) = 5 \neq 2$. Thus the possible solutions to $g(x) = 2$ can be grouped into pairs, $(x_1, 10 - x_1), (x_2, 10 - x_2), \dots$. The sum of the members of each pair is 10, and there are 2^{2009} pairs, so the sum is

$$10 \cdot 2^{2009} = 5 \cdot 2^{2010}.$$

28. [17] In the game of set, each card has four attributes, each of which takes on one of three values. A set deck consists of one card for each of the 81 possible four-tuples of attributes. Given a collection of 3 cards, call an attribute *good* for that collection if the three cards either all take on the same value of that attribute or take on all three different values of that attribute. Call a collection of 3 cards *two-good* if exactly two attributes are good for that collection. How many two-good collections of 3 cards are there? The order in which the cards appear does not matter.

Answer: $\boxed{25272}$ In counting the number of sets of 3 cards, we first want to choose which of our two attributes will be good and which of our two attributes will not be good. There are $\binom{4}{2} = 6$ such choices.

Now consider the two attributes which are not good, attribute X and attribute Y. Since these are not good, some value should appear exactly twice. Suppose the value a appears twice and b appears once for attribute X and that the value c appears twice and d appears once for attribute Y. There are three choices for a and then two choices for b ; similarly, there are three choices for c and then two choices for d . This gives $3 \cdot 2 \cdot 3 \cdot 2 = 36$ choices of a, b, c , and d .

There are two cases to consider. The first is that there are two cards which both have a and c , while the other card has both b and d . The second case is that only one card has both a and c , while one card has a and d and the other has b and c .

Case 1:

Card 1	Card 2	Card 3
— Good attribute 1 —		
— Good attribute 2 —		
a	a	b
c	c	d

The three cards need to be distinct. Card 3 is necessarily distinct from Card 1 and Card 2, but we need to ensure that Card 1 and Card 2 are distinct from each other. There are 9 choices for the two good attributes of Card 1, and then 8 choices for the two good attributes of Card 2. But we also want to divide by 2 since we do not care about the order of Card 1 and Card 2. So there are $\frac{9 \cdot 8}{2} = 36$ choices for the good attributes on Card 1 and Card 2. Then, the values of the good attributes of Card 1 and Card 2 uniquely determine the values of the good attributes of Card 3.

Case 2:

<i>Card 1</i>	<i>Card 2</i>	<i>Card 3</i>
—	—	—
Good attribute 1	Good attribute 1	Good attribute 1
—	—	—
Good attribute 2	Good attribute 2	Good attribute 2
—	—	—
a	a	b
c	d	c

Card 1, Card 2, and Card 3 will all be distinct no matter what the values of the good attributes are, because the values of attributes X and Y are unique to each card. So there are 9 possibilities for the values of the good attributes on card 1, and then there are 9 more possibilities for the values of the good attribute on Card 2. We do not have to divide by 2 this time, since Card 1 and Card 2 have distinct values in X and Y . So there are $9^2 = 81$ possibilities here.

So our final answer is $6 \cdot 6^2 \cdot (36 + 81) = 25272$.

29. [17] In the game of Galactic Dominion, players compete to amass cards, each of which is worth a certain number of points. Say you are playing a version of this game with only two kinds of cards, planet cards and hegemon cards. Each planet card is worth 2010 points, and each hegemon card is worth four points per planet card held. You start with no planet cards and no hegemon cards, and, on each turn, starting at turn one, you take either a planet card or a hegemon card, whichever is worth more points given the hand you currently hold. Define a sequence $\{a_n\}$ for all positive integers n by setting a_n to be 0 if on turn n you take a planet card and 1 if you take a hegemon card. What is the smallest value of N such that the sequence a_N, a_{N+1}, \dots is necessarily periodic (meaning that there is a positive integer k such that $a_{n+k} = a_n$ for all $n \geq N$)?

Answer: 503 If you have P planets and H hegemons, buying a planet gives you $2010 + 4H$ points while buying a hegemon gives you $4P$ points. Thus you buy a hegemon whenever $P - H \geq 502.5$, and you buy a planet whenever $P - H \leq 502.5$. Therefore $a_i = 1$ for $1 \leq i \leq 503$. Starting at $i = 504$ (at which point you have bought 503 planets) you must alternate buying planets and hegemons. The sequence $\{a_i\}_{i \geq 503}$ is periodic with period 2.

30. [17] In the game of projective set, each card contains some nonempty subset of six distinguishable dots. A projective set deck consists of one card for each of the 63 possible nonempty subsets of dots. How many collections of five cards have an even number of each dot? The order in which the cards appear does not matter.

Answer: 109368 We'll first count sets of cards where the order does matter. Suppose we choose the first four cards. Then there is exactly one card that can make each dot appear twice. However, this card could be empty or it could be one of the cards we've already chosen, so we have to subtract for these two cases. First, there are $63 \cdot 62 \cdot 61 \cdot 60$ ways to choose the first four cards. Let's now count how many ways there are that the fifth card could be empty.

The fifth card is empty if and only if the first four cards already have an even number of each dot. Suppose we choose the first two cards. There is a possible fourth card if the third card is not either of the first two or the card that completes a set. If that is the case, then the fourth card is unique. This comes to $63 \cdot 62 \cdot 60$ cases.

Now consider how many ways there are for the fifth card to be a duplicate. This is just the number of ways for three cards to have an even number of each dot, then have two copies of the same card in the other two slots, one of which needs to be the fifth slot. The number of ways for three cards to have an even number of each dot is just the number of ways to choose two cards. Therefore, we'll choose two cards ($63 \cdot 62$ ways), choose the slot in the first four positions for the duplicate card (4 ways), and the duplicate card, which can't be any of the nonduplicated cards, so there are 60 choices. Therefore, there are $63 \cdot 62 \cdot 4 \cdot 60$ ways for the fifth card to be the same as one of the first four.

This means that the number of five card sets where the order does matter is $63 \cdot 62 \cdot 61 \cdot 60 - 63 \cdot 62 \cdot 60 - 63 \cdot 62 \cdot 4 \cdot 60$, so our final answer is $\frac{63 \cdot 62 \cdot 61 \cdot 60 - 63 \cdot 62 \cdot 60 - 63 \cdot 62 \cdot 4 \cdot 60}{120} = \frac{63 \cdot 62 \cdot (61 - 1 - 4)}{2} = 63 \cdot 31 \cdot 56 = 109368$.

31. [20] What is the perimeter of the triangle formed by the points of tangency of the incircle of a 5-7-8 triangle with its sides?

Answer: $\boxed{\frac{9\sqrt{21}}{7} + 3}$ Let $\triangle ABC$ be a triangle with sides $a = 7$, $b = 5$, and $c = 8$. Let the incircle of $\triangle ABC$ be tangent to sides BC , CA , and AB at points D , E , and F . By the law of cosines (using the form $\cos(A) = \frac{b^2 + c^2 - a^2}{2bc}$), we have

$$\begin{aligned}\cos(A) &= \frac{8^2 + 5^2 - 7^2}{2(5)(8)} = \frac{1}{2} \\ \cos(B) &= \frac{8^2 + 7^2 - 5^2}{2(7)(8)} = \frac{11}{14} \\ \cos(C) &= \frac{5^2 + 7^2 - 8^2}{2(7)(5)} = \frac{1}{7}\end{aligned}$$

Now we observe that AEF , BDF , and CDE are all isosceles. Let us call the lengths of the legs of these triangles s , t , and u , respectively. Then we know that $s + t = 8$, $t + u = 7$, and $u + s = 5$, so $s = 3$, $t = 5$, and $u = 2$.

Our final observation is that an isosceles angle with legs of length l and whose non-equal angle is θ has a base of length $l\sqrt{2(1 - \cos(\theta))}$. This can be proven using the law of cosines or the Pythagorean theorem.

Using this, we can calculate that

$$\begin{aligned}DE &= 2\sqrt{2(1 - \cos(C))} \\ &= 2\sqrt{\frac{12}{7}} \\ EF &= 3\sqrt{2(1 - \cos(A))} \\ &= 3 \\ FD &= 5\sqrt{2(1 - \cos(B))} \\ &= 5\sqrt{\frac{3}{7}},\end{aligned}$$

and then

$$\begin{aligned}DE + EF + FD &= 2\sqrt{\frac{12}{7}} + 3 + 5\sqrt{\frac{3}{7}} \\ &= 3 + 9\sqrt{\frac{3}{7}} \\ &= 3 + 9\frac{\sqrt{21}}{7}.\end{aligned}$$

32. [20] Let T be the set of numbers of the form $2^a 3^b$ where a and b are integers satisfying $0 \leq a, b \leq 5$. How many subsets S of T have the property that if n is in S then all positive integer divisors of n are in S ?

Answer: $\boxed{924}$ Consider the correspondence $(a, b) \leftrightarrow 2^a 3^b$ for non-negative integers a and b . So we can view T as the square of lattice points (a, b) where $0 \leq a, b \leq 5$, and subsets of T as subsets of this square.

Notice then that the integer corresponding to (a_1, b_1) is a divisor of the integer corresponding to (a_2, b_2) if and only if $0 \leq a_1 \leq a_2$ and $0 \leq b_1 \leq b_2$. This means that subsets $S \subset T$ with the desired property,

correspond to subsets of the square where if a point is in the set, then so are all points to the left and south of it.

Consider any such subset S . For each $0 \leq x \leq 5$, let S_x be the maximum y value of any point $(x, y) \in S$, or -1 if there is no such point. We claim the values S_x uniquely characterize S . This is because each S_x characterizes the points of the form (x, y) in S . In particular, (x, z) will be in S if and only if $z \leq S_x$. If $(x, z) \in S$ with $z > S_x$, then S_x is not the maximum value, and if $(x, z) \notin S$ with $z \leq S_x$, then S fails to satisfy the desired property.

We now claim that $S_x \geq S_y$ for $x < y$, so the sequence S_0, \dots, S_1 is decreasing. This is because if (y, S_y) is in the set S , then so must be (x, S_y) . Conversely, it is easy to see that if S_0, \dots, S_1 is decreasing, then S is a set satisfying the desired property.

We now claim that decreasing sequences S_0, \dots, S_5 are in bijective correspondence with walks going only right and down from $(-1, 5)$ to $(5, -1)$. The sequence S_0, \dots, S_5 simply corresponds to the walk $(-1, 5) \rightarrow (-1, S_0) \rightarrow (0, S_0) \rightarrow (0, S_1) \rightarrow (1, S_1) \rightarrow \dots \rightarrow (4, S_5) \rightarrow (5, S_5) \rightarrow (5, -1)$. Geometrically, we are tracing out the outline of the set S .

The number of such walks is simply $\binom{12}{6}$, since we can view it as choosing the 6 of 12 steps at which to move right. Thus the number of subsets S of T with the desired property is $\binom{12}{6} = 924$.

33. [20] Convex quadrilateral $BCDE$ lies in the plane. Lines EB and DC intersect at A , with $AB = 2$, $AC = 5$, $AD = 200$, $AE = 500$, and $\cos \angle BAC = \frac{7}{9}$. What is the largest number of nonoverlapping circles that can lie in quadrilateral $BCDE$ such that all of them are tangent to both lines BE and CD ?

Answer: [5] Let $\theta = \angle BAC$, and $\cos \theta = \frac{7}{9}$ implies $\cos \frac{\theta}{2} = \sqrt{\frac{1+\frac{7}{9}}{2}} = \frac{2\sqrt{2}}{3}$; $\sin \frac{\theta}{2} = \frac{1}{3}$; $BC = \sqrt{4 + 25 - 2(2)(5)\frac{7}{9}} = \frac{11}{3}$. Let O_1 be the excircle of $\triangle ABC$ tangent to lines AB and AC , and let r_1 be its radius; let O_1 be tangent to line AB at point P_1 . Then $AP_1 = \frac{AB+BC+CA}{2}$ and $\frac{r_1}{AP_1} = \tan \frac{\theta}{2} = \frac{1}{2\sqrt{2}} \implies r_1 = \frac{16}{3 \cdot 2\sqrt{2}}$. Let O_n be a circle tangent to O_{n-1} and the lines AB and AC , and let r_n be its radius; let O_n be tangent to line AB at point P_n . Then $\frac{O_n P_n}{AO_n} = \sin \frac{\theta}{2} = \frac{1}{3}$; since $\triangle AP_n O_n \sim \triangle AP_{n-1} O_{n-1}$ and $O_n O_{n-1} = r_n + r_{n-1}$, we have $\frac{1}{3} = \frac{O_n P_n}{AO_n} = \frac{r_n}{AO_{n-1} + O_{n-1} O_n} = \frac{r_n}{3r_{n-1} + r_n + r_{n-1}} \implies r_n = 2r_{n-1} = 2^{n-1} \cdot \frac{16}{3 \cdot 2\sqrt{2}}$. We want the highest n such that O_n is contained inside $\triangle ADE$. Let the incircle of $\triangle ADE$ be tangent to AD at X ; then the inradius of $\triangle ADE$ is $\frac{AX}{\tan \frac{\theta}{2}} = \frac{500+200-\frac{1100}{3}}{2\sqrt{2}} = \frac{500}{3 \cdot 2\sqrt{2}}$. We want the highest n such that $r_n \leq \frac{500}{3 \cdot 2\sqrt{2}}$; thus $2^{n-1} \cdot 16 \leq 500 \implies n = 5$.

34. [25] Estimate the sum of all the prime numbers less than 1,000,000. If the correct answer is X and you write down A , your team will receive $\min(\lfloor \frac{25X}{A} \rfloor, \lfloor \frac{25A}{X} \rfloor)$ points, where $\lfloor x \rfloor$ is the largest integer less than or equal to x .

Answer: [37550402023] A decent approximation to the sum of all the primes can be obtained with the following two facts. First, there are approximately $\frac{n}{\ln n}$ primes less than n and second, the n^{th} prime is approximately $n \ln n$. We'll approximate $\ln 1000000$ as 15 (the actual number is 13.8), so there are approximately $\frac{10^6}{15}$ primes. Then we want $\sum_{n=1}^{\frac{10^6}{15}} n \ln n$. If you know calculus, this can be approximated by the integral $\int_1^{\frac{10^6}{15}} x \ln x dx$, which has the antiderivative $\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$, giving an answer of around $\frac{1}{2} \cdot \frac{10^{12}}{225} \cdot (15 - \ln 15) - \frac{1}{4} \cdot \frac{10^{12}}{225}$. Estimating $\ln 15$ as about 3, this is approximately $\frac{23 \cdot 10^{12}}{900} = 2.5 \cdot 10^{10} = 25,000,000,000$. The actual answer is 37,550,402,023, so an approximation of this accuracy would get 16 points. We can arrive at the same answer without calculus by approximating $\ln n$ as $\sum_{k=1}^n \frac{1}{k}$, so that our sum becomes $\sum_{n=1}^{\frac{10^6}{15}} \sum_{k=1}^n \frac{n}{k} = \sum_{k=1}^{\frac{10^6}{15}} \sum_{n=k}^{\frac{10^6}{15}} \frac{n}{k} \approx \sum_{k=1}^{\frac{10^6}{15}} \frac{1}{2} \cdot \frac{\frac{10^{12}}{225} - k^2}{k} = \frac{1}{2} \cdot \frac{10^{12}}{225} \cdot \ln\left(\frac{10^6}{15}\right) - \frac{1}{2} \sum_{k=1}^{\frac{10^6}{15}} k \approx \frac{1}{2} \cdot \frac{10^{12}}{225} \cdot \ln\left(\frac{10^6}{15}\right) - \frac{1}{4} \frac{10^{12}}{225} \approx 2.5 \cdot 10^{10}$.

35. [25] A mathematician M' is called a descendent of mathematician M if there is a sequence of mathematicians $M = M_1, M_2, \dots, M_k = M'$ such that M_i was M_{i+1} 's doctoral advisor for all i . Estimate

the number of descendants that the mathematician who has had the largest number of descendants has had, according to the Mathematical Genealogy Project. Note that the Mathematical Genealogy Project has records dating back to the 1300s. If the correct answer is X and you write down A , your team will receive $\max\left(25 - \lfloor \frac{|X-A|}{100} \rfloor, 0\right)$ points, where $\lfloor x \rfloor$ is the largest integer less than or equal to x .

Answer: 82310 First let's estimate how many "generations" of mathematicians there have been since 1300. If we suppose that a mathematician gets his PhD around age 30 and becomes a PhD advisor around age 60, then we'll get a generation length of approximately 30 years. However, not all mathematicians will train more than one PhD. Let's say that only 40% of mathematicians train at least 2 PhDs. Then effectively we have only 40% of the generations, or in other words each effective generation takes 75 years. Then we have $\frac{22}{3}$ branching generations. If we assume that all of these only train 2 PhDs, then we get an answer of $2^{\frac{22}{3}} \approx 1625$. But we can ensure that our chain has at least a single person who trained 100 PhDs (this is approximately the largest number of advisees for a single mathematician), allowing us to change one factor of 2 into a factor of 100. That gives us an answer of $1625 \cdot 50 = 81250$, which is very close to the actual value of 82310.

36. **[25]** Paul Erdős was one of the most prolific mathematicians of all time and was renowned for his many collaborations. The Erdős number of a mathematician is defined as follows. Erdős has an Erdős number of 0, a mathematician who has coauthored a paper with Erdős has an Erdős number of 1, a mathematician who has not coauthored a paper with Erdős, but has coauthored a paper with a mathematician with Erdős number 1 has an Erdős number of 2, etc. If no such chain exists between Erdős and another mathematician, that mathematician has an Erdős number of infinity. Of the mathematicians with a finite Erdős number (including those who are no longer alive), what is their average Erdős number according to the Erdős Number Project? If the correct answer is X and you write down A , your team will receive $\max(25 - \lfloor 100|X - A| \rfloor, 0)$ points where $\lfloor x \rfloor$ is the largest integer less than or equal to x .

Answer: 4.65 We'll suppose that each mathematician collaborates with approximately 20 people (except for Erdős himself, of course). Furthermore, if a mathematician has Erdős number k , then we'd expect him to be the cause of approximately $\frac{1}{2^k}$ of his collaborators' Erdős numbers. This is because as we get to higher Erdős numbers, it is more likely that a collaborator has a lower Erdős number already. Therefore, we'd expect about 10 times as many people to have an Erdős number of 2 than with an Erdős number of 1, then a ratio of 5, 2.5, 1.25, and so on. This tells us that more mathematicians have an Erdős number of 5 than any other number, then 4, then 6, and so on. If we use this approximation, we have a ratio of mathematicians with Erdős number 1, 2, and so on of about $1 : 10 : 50 : 125 : 156 : 97 : 30 : 4 : 0.3$, which gives an average Erdős number of 4.8. This is close to the actual value of 4.65.

3rd Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Team Round

Polyhedron Hopping

1. [3] Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?

Answer: $\boxed{\frac{7}{27}}$ Let the cube have vertices all 0 or 1 in the x, y, z , coordinate system. Travis starts at $(0, 0, 0)$. If after 3 moves he is at $(1, 1, 1)$ he cannot get back to $(0, 0, 0)$. From any other vertex he has a $\frac{1}{3}$ chance of getting back on the final move. There is a $\frac{2}{9}$ chance he ends up at $(1, 1, 1)$, and thus a $\frac{7}{9}$ chance he does not end up there, and thus a $\frac{7}{27}$ chance he ends up at $(0, 0, 0)$.

2. [6] In terms of k , for $k > 0$ how likely is he to be back where he started after $2k$ minutes?

Answer: $\boxed{\frac{1}{4} + \frac{3}{4} \left(\frac{1}{9}\right)^k}$ Again, Travis starts at $(0, 0, 0)$. At each step, exactly one of the three coordinates will change. The parity of the sum of the three coordinates will change at each step, so after $2k$ steps, the sum of the coordinates must be even. There are only four possibilities for Travis's position: $(0, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$. Let p_k be the probability that Travis is at $(0, 0, 0)$ after $2k$ steps. Then $1 - p_k$ is the probability that he is on $(1, 1, 0)$, $(1, 0, 1)$, or $(0, 1, 1)$. Suppose we want to compute p_{k+1} . There are two possibilities: we were either at $(0, 0, 0)$ after $2k$ steps or not. If we were, then there is a $\frac{1}{3}$ probability that we will return (since our $(2k + 1)^{\text{th}}$ step can be arbitrary, but there is a $\frac{1}{3}$ chance that we will reverse that as our $(2k + 2)^{\text{th}}$ step). If we were not at $(0, 0, 0)$ after our $2k^{\text{th}}$ steps, then two of our coordinates must have been ones. There is a $\frac{2}{3}$ probability that the $(2k + 1)^{\text{th}}$ step will change one of those to a zero, and there is a $\frac{1}{3}$ step that that the $(2k + 2)^{\text{th}}$ step will change the remaining one. Hence, in this case, there is a $\left(\frac{2}{3}\right) \left(\frac{1}{3}\right) = \frac{2}{9}$ probability that Travis ends up $(0, 0, 0)$ in this case. So we have:

$$\begin{aligned} p_{k+1} &= p_k \left(\frac{1}{3}\right) + (1 - p_k) \left(\frac{2}{9}\right) \\ p_{k+1} &= \frac{1}{9}p_k + \frac{2}{9} \\ \left(p_{k+1} - \frac{1}{4}\right) &= \frac{1}{9} \left(p_k - \frac{1}{4}\right) \end{aligned}$$

(We get the value $\frac{1}{4}$ either by guessing that the sequence p_0, p_1, p_2, \dots should converge to $\frac{1}{4}$ or simply by solving the equation $-\frac{1}{9}x + x = \frac{2}{9}$.) This shows that $p_0 - \frac{1}{4}, p_1 - \frac{1}{4}, \dots$ is a geometric series with ratio $\frac{1}{9}$. Since $p_0 - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4}$, we get that $p_k - \frac{1}{4} = \frac{3}{4} \left(\frac{1}{9}\right)^k$, or that $p_k = \frac{1}{4} + \frac{3}{4} \left(\frac{1}{9}\right)^k$.

3. [3] While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After five minutes, how likely is Sherry to be one edge away from where she started?

Answer: $\boxed{\frac{11}{16}}$ Let the starting vertex be the 'bottom' one. Then there is a 'top' vertex, and 4 'middle' ones. If $p(n)$ is the probability that Sherry is on a middle vertex after n minutes, $p(0) = 0$,

$p(n+1) = (1 - p(n)) + p(n) \cdot \frac{1}{2}$. This recurrence gives us the following equations.

$$\begin{aligned} p(n+1) &= 1 - \frac{p(n)}{2} \\ p(0) &= 0 \\ p(1) &= 1 \\ p(2) &= \frac{1}{2} \\ p(3) &= \frac{3}{4} \\ p(4) &= \frac{5}{8} \\ p(5) &= \frac{11}{16} \end{aligned}$$

4. [6] In terms of k , for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?

Answer: $\boxed{\frac{1}{6} + \frac{1}{3(-2)^k}}$ Take $p(n)$ from the last problem. By examining the last move of Sherry, the probability that she ends up on the original vertex is equal to the probability that she ends up on the top vertex, and both are equal to $\frac{1-p(n)}{2}$ for $n \geq 1$.

From the last problem,

$$\begin{aligned} p(n+1) &= 1 - \frac{p(n)}{2} \\ p(n+1) - \frac{2}{3} &= -\frac{1}{2} \left(p(n) - \frac{2}{3} \right) \end{aligned}$$

and so $p(n) - \frac{2}{3}$ is a geometric series with ratio $-\frac{1}{2}$. Since $p(0) = 0$, we get $p(n) - \frac{2}{3} = -\frac{2}{3} \left(-\frac{1}{2}\right)^n$, or that $p(n) = \frac{2}{3} - \frac{2}{3} \left(-\frac{1}{2}\right)^n$.

Now, for $k \geq 1$, we have that the probability of ending up on the vertex opposite Sherry's initial vertex after k minutes is $\frac{1-p(k)}{2} = \frac{1}{2} - \frac{1}{2} \left(\frac{2}{3} - \frac{2}{3} \left(-\frac{1}{2}\right)^k \right) = \frac{1}{6} + \frac{1}{3} \left(-\frac{1}{2}\right)^k = \frac{1}{6} + \frac{1}{3(-2)^k}$.

Circles in Circles

5. [4] Circle O has chord AB . A circle is tangent to O at T and tangent to AB at X such that $AX = 2XB$. What is $\frac{AT}{BT}$?

Answer: $\boxed{2}$ Let TX meet circle O again at Y . Since the homothety centered at T takes X to Y also takes AB to the tangent line of circle O passing through Y , we have Y is the midpoint of arc AB . This means that $\angle ATY = \angle YTB$. By the Angle Bisector Theorem, $\frac{AT}{BT} = \frac{AX}{BX} = 2$.

6. [6] AB is a diameter of circle O . X is a point on AB such that $AX = 3BX$. Distinct circles ω_1 and ω_2 are tangent to O at T_1 and T_2 and to AB at X . The lines T_1X and T_2X intersect O again at S_1 and S_2 . What is the ratio $\frac{T_1T_2}{S_1S_2}$?

Answer: $\boxed{\frac{3}{5}}$ Since the problem only deals with ratios, we can assume that the radius of O is 1. As we have proven in Problem 5, points S_1 and S_2 are midpoints of arc AB . Since AB is a diameter, S_1S_2 is also a diameter, and thus $S_1S_2 = 2$.

Let O_1 , O_2 , and P denote the center of circles ω_1 , ω_2 , and O . Since ω_1 is tangent to O , we have $PO_1 + O_1X = 1$. But $O_1X \perp AB$. So $\triangle PO_1X$ is a right triangle, and $O_1X^2 + XP^2 = O_1P^2$. Thus, $O_1X^2 + 1/4 = (1 - O_1X)^2$, which means $O_1X = \frac{3}{8}$ and $O_1P = \frac{5}{8}$.

Since $T_1T_2 \parallel O_1O_2$, we have $T_1T_2 = O_1O_2 \cdot \frac{PT_1}{PO_1} = 2O_1X \cdot \frac{PT_1}{PO_1} = 2 \left(\frac{3}{8}\right) \frac{1}{5/8} = \frac{6}{5}$. Thus $\frac{T_1T_2}{S_1S_2} = \frac{6/5}{2} = \frac{3}{5}$.

7. [7] ABC is a right triangle with $\angle A = 30^\circ$ and circumcircle O . Circles ω_1 , ω_2 , and ω_3 lie outside ABC and are tangent to O at T_1 , T_2 , and T_3 respectively and to AB , BC , and CA at S_1 , S_2 , and S_3 , respectively. Lines T_1S_1 , T_2S_2 , and T_3S_3 intersect O again at A' , B' , and C' , respectively. What is the ratio of the area of $A'B'C'$ to the area of ABC ?

Answer: $\frac{\sqrt{3}+1}{2}$ Let $[PQR]$ denote the area of $\triangle PQR$. The key to this problem is following fact:
 $[PQR] = \frac{1}{2}PQ \cdot PR \sin \angle QPR$.

Assume that the radius of O is 1. Since $\angle A = 30^\circ$, we have $BC = 1$ and $AB = \sqrt{3}$. So $[ABC] = \frac{\sqrt{3}}{2}$. Let K denote the center of O . Notice that $\angle B'KA' = 90^\circ$, $\angle AKC' = 90^\circ$, and $\angle B'KA = \angle KAB = 30^\circ$. Thus, $\angle B'KC' = \angle B'KA + \angle AKC' = 120^\circ$ and consequently $\angle C'KA' = 150^\circ$.

Therefore, $[A'B'C'] = [A'KB'] + [B'KC'] + [C'KA'] = \frac{1}{2} + \frac{1}{2} \sin 120^\circ + \frac{1}{2} \sin 150^\circ = \frac{3}{4} + \frac{\sqrt{3}}{4}$. This gives the desired result that $[A'B'C'] = \frac{\sqrt{3}+1}{2}[ABC]$.

Linear? What's The Problem?

A function $f(x_1, x_2, \dots, x_n)$ is said to be linear in each of its variables if it is a polynomial such that no variable appears with power higher than one in any term. For example, $1 + x + xy$ is linear in x and y , but $1 + x^2$ is not. Similarly, $2x + 3yz$ is linear in x , y , and z , but xyz^2 is not.

8. [4] A function $f(x, y)$ is linear in x and in y . $f(x, y) = \frac{1}{xy}$ for $x, y \in \{3, 4\}$. What is $f(5, 5)$?

Answer: $\frac{1}{36}$ The main fact that we will use in solving this problem is that $f(x+2, y) - f(x+1, y) = f(x+1, y) - f(x, y)$ whenever f is linear in x and y . Suppose that $f(x, y) = axy + by + cx + d = x(ay + c) + (by + d)$ for some constants a , b , c , and d . Then it is easy to see that

$$\begin{aligned} f(x+2, y) - f(x+1, y) &= (x+2)(ay+c) + (by+d) - (x+1)(ay+c) - (by+d) = ay+c \\ f(x+1, y) - f(x, y) &= (x+1)(ay+c) + (by+d) - x(ay+c) - (by+d) = ay+c, \end{aligned}$$

which implies that $f(x+2, y) - f(x+1, y) = f(x+1, y) - f(x, y)$. In particular, $f(5, y) - f(4, y) = f(4, y) - f(3, y)$, so $f(5, y) = 2f(4, y) - f(3, y)$. Similarly, $f(x, 5) = 2f(x, 4) - f(x, 3)$. Now we see that:

$$\begin{aligned} f(5, 5) &= 2f(5, 4) - f(5, 3) \\ &= 2[2f(4, 4) - f(3, 4)] - [2f(4, 3) - f(3, 3)] \\ &= 4f(4, 4) - 2f(3, 4) - 2f(4, 3) + f(3, 3) \\ &= \frac{4}{16} - \frac{4}{12} + \frac{1}{9} \\ &= \frac{1}{4} - \frac{1}{3} + \frac{1}{9} \\ &= \frac{1}{9} - \frac{1}{12} \\ &= \frac{1}{36}, \end{aligned}$$

so the answer is $\frac{1}{36}$.

9. [5] A function $f(x, y, z)$ is linear in x , y , and z such that $f(x, y, z) = \frac{1}{xyz}$ for $x, y, z \in \{3, 4\}$. What is $f(5, 5, 5)$?

Answer: $\frac{1}{216}$ We use a similar method to the previous problem. Notice that $f(x, y, 5) = 2f(x, y, 4) - f(x, y, 3)$. Let f_2 denote the function from the previous problem and f_3 the function from this problem.

Since $3f_3(x, y, 3)$ is linear in x and y , and $3f_3(x, y, 3) = \frac{1}{xy}$ for all $x, y \in \{3, 4\}$, the previous problem implies that $3f_3(5, 5, 3) = \frac{1}{36} = f_2(5, 5)$. Similarly, $4f_3(5, 5, 4) = f_2(5, 5)$. Now we have

$$\begin{aligned} f_3(5, 5, 5) &= 2f_3(5, 5, 4) - f_3(5, 5, 3) \\ &= \frac{1}{2}f_2(5, 5) - \frac{1}{3}f_2(5, 5) \\ &= \frac{1}{6}f_2(5, 5) \\ &= \frac{1}{6 \cdot 36} \\ &= \frac{1}{216}. \end{aligned}$$

10. [6] A function $f(x_1, x_2, \dots, x_n)$ is linear in each of the x_i and $f(x_1, x_2, \dots, x_n) = \frac{1}{x_1 x_2 \dots x_n}$ when $x_i \in \{3, 4\}$ for all i . In terms of n , what is $f(5, 5, \dots, 5)$?

Answer: $\boxed{\frac{1}{6^n}}$ Let $f_n(x_1, x_2, \dots, x_n)$ denote the n -variable version of the function. We will prove that $f_n(5, \dots, 5) = \frac{1}{6^n}$ by induction. The base case was done in the two previous problems. Suppose we know that $f_{n-1}(5, 5, \dots, 5) = \frac{1}{6^{n-1}}$. Let $g(x_1, \dots, x_{n-1}) = 3f_n(x_1, \dots, x_{n-1}, 3)$. We have that g is linear in x_1, \dots, x_{n-1} and $g(x_1, \dots, x_{n-1}) = \frac{1}{x_1 \dots x_{n-1}}$ for all $x_1, \dots, x_{n-1} \in \{3, 4\}$. By the inductive hypothesis, we have $g(5, \dots, 5) = \frac{1}{6^{n-1}} = f_{n-1}(5, \dots, 5)$. Therefore, $f_n(5, \dots, 5, 3) = \frac{f_{n-1}(5, \dots, 5)}{3}$. Similarly, $f_n(5, \dots, 5, 4) = \frac{f_{n-1}(5, \dots, 5)}{4}$.

$$\begin{aligned} f_n(5, 5, \dots, 5, 5) &= 2f_n(5, 5, \dots, 5, 4) - f_n(5, 5, \dots, 5, 3) \\ &= \frac{1}{2}f_{n-1}(5, 5, \dots, 5) - \frac{1}{3}f_{n-1} \\ &= \frac{1}{6}f_{n-1}(5, 5, \dots, 5) \\ &= \frac{1}{6 \cdot 6^{n-1}} \\ &= \frac{1}{6^n}, \end{aligned}$$

and this proves our conjecture by induction.

3rd Annual Harvard-MIT November Tournament
Sunday 7 November 2010

Theme Round

StarCraft

1. [3] 16 progamers are playing in a single elimination tournament. Each player has a different skill level and when two play against each other the one with the higher skill level will always win. Each round, each progamer plays a match against another and the loser is eliminated. This continues until only one remains. How many different progamers can reach the round that has 2 players remaining?

Answer: [9] Each finalist must be better than the person he beat in the semifinals, both of the people they beat in the second round, and all 4 of the people any of those people beat in the first round. So, none of the 7 worst players can possibly make it to the finals. Any of the 9 best players can make it to the finals if the other 8 of the best 9 play each other in all rounds before the finals. So, exactly 9 people are capable of making it to the finals.

2. [4] 16 progamers are playing in another single elimination tournament. Each round, each of the remaining progamers plays against another and the loser is eliminated. Additionally, each time a progamer wins, he will have a ceremony to celebrate. A player's first ceremony is ten seconds long, and afterward each ceremony is ten seconds longer than the last. What is the total length in seconds of all the ceremonies over the entire tournament?

Answer: [260] At the end of the first round, each of the 8 winners has a 10 second ceremony. After the second round, the 4 winners have a 20 second ceremony. The two remaining players have 30 second ceremonies after the third round, and the winner has a 40 second ceremony after the finals. So, all of the ceremonies combined take $8 \cdot 10 + 4 \cdot 20 + 2 \cdot 30 + 40 = 260$ seconds.

3. [5] Dragons take up 1×1 squares in the plane with sides parallel to the coordinate axes such that the interiors of the squares do not intersect. A dragon can fire at another dragon if the difference in the x -coordinates of their centers and the difference in the y -coordinates of their centers are both at most 6, regardless of any dragons in between. For example, a dragon centered at $(4, 5)$ can fire at a dragon centered at the origin, but a dragon centered at $(7, 0)$ can not. A dragon cannot fire at itself. What is the maximum number of dragons that can fire at a single dragon simultaneously?

Answer: [168] Assign coordinates in such a way that the dragon being fired on is centered at $(0, 0)$. Any dragon firing at it must have a center with x -coordinates and y -coordinates that are no smaller than -6 and no greater than 6 . That means that every dragon firing at it must lie entirely in the region bounded by the lines $x = -6.5$, $x = 6.5$, $y = -6.5$, and $y = 6.5$. This is a square with sides of length 13 , so there is room for exactly 169 dragons in it. One of them is the dragon being fired on, so there are at most 168 dragons firing at it.

4. [5] A zerg player can produce one zergling every minute and a protoss player can produce one zealot every 2.1 minutes. Both players begin building their respective units immediately from the beginning of the game. In a fight, a zergling army overpowers a zealot army if the ratio of zerglings to zealots is more than 3. What is the total amount of time (in minutes) during the game such that at that time the zergling army would overpower the zealot army?

Answer: [1.3] At the end of the first minute, the zerg player produces a zergling and has a superior army for the 1.1 minutes before the protoss player produces the first zealot. At this point, the zealot is at least a match for the zerglings until the fourth is produced 4 minutes into the game. Then, the zerg army has the advantage for the .2 minutes before a second zealot is produced. A third zealot will be produced 6.3 minutes into the game, which will be before the zerg player accumulates the 7 zerglings needed to overwhelm the first 2 zealots. After this, the zerglings will never regain the advantage because the zerg player can never produce 3 more zerglings to counter the last zealot before another one is produced. So, the zerg player will have the military advantage for $1.1 + .2 = 1.3$ minutes.

5. [7] There are 111 StarCraft progamers. The StarCraft team SKT starts with a given set of eleven progamers on it, and at the end of each season, it drops a progamer and adds a progamer (possibly the same one). At the start of the second season, SKT has to field a team of five progamers to play the opening match. How many different lineups of five players could be fielded if the order of players on the lineup matters?

Answer: $\boxed{4015440}$ We disregard the order of the players, multiplying our answer by $5! = 120$ at the end to account for it. Clearly, SKT will be able to field at most 1 player not in the original set of eleven players. If it does not field a new player, then it has $\binom{11}{5} = 462$ choices. If it does field a new player, then it has 100 choices for the new player, and $\binom{11}{4} = 330$ choices for the 4 other players, giving 33000 possibilities. Thus, SKT can field at most $33000 + 462 = 33462$ unordered lineups, and multiplying this by 120, we find the answer to be $\boxed{4015440}$.

Unfair Coins

6. [4] When flipped, a coin has a probability p of landing heads. When flipped twice, it is twice as likely to land on the same side both times as it is to land on each side once. What is the larger possible value of p ?

Answer: $\boxed{\frac{3+\sqrt{3}}{6}}$ The probability that the coin will land on the same side twice is $p^2 + (1-p)^2 = 2p^2 - 2p + 1$. The probability that the coin will land on each side once is $p(p-1) + (p-1)p = 2p(1-p) = 2p - 2p^2$. We are told that it is twice as likely to land on the same side both times, so $2p^2 - 2p + 1 = 2(2p - 2p^2) = 4p - 4p^2$. Solving, we get $6p^2 - 6p + 1 = 0$, so $p = \frac{3 \pm \sqrt{3}}{6}$. The larger solution is $p = \frac{3+\sqrt{3}}{6}$.

7. [4] George has two coins, one of which is fair and the other of which always comes up heads. Jacob takes one of them at random and flips it twice. Given that it came up heads both times, what is the probability that it is the coin that always comes up heads?

Answer: $\boxed{\frac{4}{5}}$ In general, $P(A|B) = \frac{P(A \cap B)}{P(B)}$, where $P(A|B)$ is the probability of A given B and $P(A \cap B)$ is the probability of A and B (See http://en.wikipedia.org/wiki/Conditional_probability for more information). If A is the event of selecting the “double-headed” coin and B is the event of Jacob flipping two heads, then $P(A \cap B) = (\frac{1}{2})(1)$, since there is a $\frac{1}{2}$ chance of picking the double-headed coin and Jacob will always flip two heads when he has it. By conditional probability, $P(B) = (\frac{1}{2})(1) + (\frac{1}{2})(\frac{1}{4}) = \frac{5}{8}$, so $P(A|B) = \frac{1/2}{5/8} = \frac{4}{5}$.

8. [5] Allison has a coin which comes up heads $\frac{2}{3}$ of the time. She flips it 5 times. What is the probability that she sees more heads than tails?

Answer: $\boxed{\frac{64}{81}}$ The probability of flipping more heads than tails is the probability of flipping 3 heads, 4 heads, or 5 heads. Since 5 flips will give n heads with probability $\binom{5}{n}(\frac{2}{3})^n(\frac{1}{3})^{5-n}$, our answer is $\binom{5}{3}(\frac{2}{3})^3(\frac{1}{3})^2 + \binom{5}{4}(\frac{2}{3})^4(\frac{1}{3})^1 + \binom{5}{5}(\frac{2}{3})^5(\frac{1}{3})^0 = \frac{64}{81}$.

9. [6] Newton and Leibniz are playing a game with a coin that comes up heads with probability p . They take turns flipping the coin until one of them wins with Newton going first. Newton wins if he flips a heads and Leibniz wins if he flips a tails. Given that Newton and Leibniz each win the game half of the time, what is the probability p ?

Answer: $\boxed{\frac{3-\sqrt{5}}{2}}$ The probability that Newton will win on the first flip is p . The probability that Newton will win on the third flip is $(1-p)p^2$, since the first flip must be tails, the second must be heads, and the third flip must be heads. By the same logic, the probability Newton will win on the $(2n+1)$ st flip is $(1-p)^n(p)^{n+1}$. Thus, we have an infinite geometric sequence $p + (1-p)p^2 + (1-p)^2p^3 + \dots$ which equals $\frac{p}{1-p(1-p)}$. We are given that this sum must equal $\frac{1}{2}$, so $1-p+p^2 = 2p$, so $p = \frac{3-\sqrt{5}}{2}$ (the other solution is greater than 1).

10. [7] Justine has a coin which will come up the same as the last flip $\frac{2}{3}$ of the time and the other side $\frac{1}{3}$ of the time. She flips it and it comes up heads. She then flips it 2010 more times. What is the probability that the last flip is heads?

Answer: $\frac{3^{2010}+1}{2 \cdot 3^{2010}}$ Let the “value” of a flip be 1 if the flip is different from the previous flip and let it be 0 if the flip is the same as the previous flip. The last flip will be heads if the sum of the values of all 2010 flips is even. The probability that this will happen is $\sum_{i=0}^{1005} \binom{2010}{2i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{2010-2i}$.

We know that

$$\begin{aligned} \sum_{i=0}^{1005} \binom{2010}{2i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{2010-2i} + \binom{2010}{2i+1} \left(\frac{1}{3}\right)^{2i+1} \left(\frac{2}{3}\right)^{2010-(2i+1)} &= \\ \sum_{k=0}^{2010} \binom{2010}{k} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{2010-k} &= \left(\frac{1}{3} + \frac{2}{3}\right)^{2010} = 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{1005} \binom{2010}{2i} \left(\frac{1}{3}\right)^{2i} \left(\frac{2}{3}\right)^{2010-2i} - \binom{2010}{2i+1} \left(\frac{1}{3}\right)^{2i+1} \left(\frac{2}{3}\right)^{2010-(2i+1)} &= \\ \sum_{k=0}^{2010} \binom{2010}{k} \left(\frac{-1}{3}\right)^k \left(\frac{2}{3}\right)^{2010-k} &= \left(\frac{-1}{3} + \frac{2}{3}\right)^{2010} = \left(\frac{1}{3}\right)^{2010} \end{aligned}$$

Summing these two values and dividing by 2 gives the answer $\frac{1+(\frac{1}{3})^{2010}}{2} = \frac{3^{2010}+1}{2 \cdot 3^{2010}}$.

12th Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Algebra Test

1. [3] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

Answer:

Solution: We can factor the equation as $(a - b^2)(a + b^2) = 41 \cdot 49$, from which it is evident that $a = 45$ and $b = 2$ is a possible solution. By examining the factors of 2009, one can see that there are no other solutions.

2. [3] Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?

Answer:

Solution: The sum of all the coefficients is $(1 + i)^{2009}$, and the sum of the real coefficients is the real part of this, which is $\frac{1}{2} \left((1 + i)^{2009} + (1 - i)^{2009} \right) = 2^{1004}$. Thus $\log_2(S) = 1004$.

3. [4] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

Answer:

Solution: We have $\cot x + \cot y = \frac{\tan x + \tan y}{\tan x \tan y}$, so $\tan x \tan y = \frac{4}{5}$. Thus, by the tan sum formula, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = 20$.

4. [4] Suppose a , b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the ring of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $x^3 - 4x^2 + x + 6$. Find $a + b + c$.

Answer:

Solution: Since $x + 1$ divides $x^2 + ax + b$ and the constant term is b , we have $x^2 + ax + b = (x + 1)(x + b)$, and similarly $x^2 + bx + c = (x + 1)(x + c)$. Therefore, $a = b + 1 = c + 2$. Furthermore, the least common multiple of the two polynomials is $(x + 1)(x + b)(x + c) = x^3 - 4x^2 + x + 6$, so $b = -2$. Thus $a = -1$ and $c = -3$, and $a + b + c = -6$.

5. [4] Let a , b , and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

Answer:

Solution: We can substitute $x = y - 1$ to obtain a polynomial having roots $a + 1$, $b + 1$, $c + 1$, namely, $(y - 1)^3 - (y - 1) + 1 = y^3 - 3y^2 + 2y + 1$. The sum of the reciprocals of the roots of this polynomial is, by Viète's formulas, $\frac{2}{-1} = -2$.

6. [5] Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n . Suppose

$$\frac{\sin \theta}{x} = \frac{\cos \theta}{y}$$

and

$$\frac{\cos^4 \theta}{x^4} + \frac{\sin^4 \theta}{y^4} = \frac{97 \sin 2\theta}{x^3 y + y^3 x}.$$

Compute $\frac{x}{y} + \frac{y}{x}$.

Answer: $\boxed{4}$

Solution: From the first relation, there exists a real number k such that $x = k \sin \theta$ and $y = k \cos \theta$. Then we have

$$\frac{\cos^4 \theta}{\sin^4 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta} = \frac{194 \sin \theta \cos \theta}{\sin \theta \cos \theta (\cos^2 \theta + \sin^2 \theta)} = 194.$$

Notice that if $t = \frac{x}{y} + \frac{y}{x}$ then $(t^2 - 2)^2 - 2 = \frac{\cos^4 \theta}{\sin^4 \theta} + \frac{\sin^4 \theta}{\cos^4 \theta} = 194$ and so $t = 4$.

7. [5] Simplify the product

$$\prod_{m=1}^{100} \prod_{n=1}^{100} \frac{x^{n+m} + x^{n+m+2} + x^{2n+1} + x^{2m+1}}{x^{2n} + 2x^{n+m} + x^{2m}}.$$

Express your answer in terms of x .

Answer: $x^{9900} \left(\frac{1+x^{100}}{2} \right)^2$ (OR $\frac{1}{4}x^{9900} + \frac{1}{2}x^{10000} + \frac{1}{4}x^{10100}$)

Solution: We notice that the numerator and denominator of each term factors, so the product is equal to

$$\prod_{m=1}^{100} \prod_{n=1}^{100} \frac{(x^m + x^{n+1})(x^{m+1} + x^n)}{(x^m + x^n)^2}.$$

Each term of the numerator cancels with a term of the denominator except for those of the form $(x^m + x^{101})$ and $(x^{101} + x^n)$ for $m, n = 1, \dots, 100$, and the terms in the denominator which remain are of the form $(x^1 + x^n)$ and $(x^1 + x^m)$ for $m, n = 1, \dots, 100$. Thus the product simplifies to

$$\left(\prod_{m=1}^{100} \frac{x^m + x^{101}}{x^1 + x^m} \right)^2$$

Reversing the order of the factors of the numerator, we find this is equal to

$$\begin{aligned} \left(\prod_{m=1}^{100} \frac{x^{101-m} + x^{101}}{x^1 + x^m} \right)^2 &= \left(\prod_{m=1}^{100} x^{100-m} \frac{x^1 + x^{m+1}}{x^1 + x^m} \right)^2 \\ &= \left(\frac{x^1 + x^{101}}{x^1 + x^1} \prod_{m=1}^{100} x^{100-m} \right)^2 \\ &= \left(x^{\frac{99 \cdot 100}{2}} \left(\frac{1 + x^{100}}{2} \right)^2 \right)^2 \end{aligned}$$

as desired.

8. [7] If a, b, x , and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.

Answer: $\boxed{20}$.

Solution: We have $ax^3 + by^3 = 16$, so $(ax^3 + by^3)(x + y) = 16(x + y)$ and thus

$$ax^4 + by^4 + xy(ax^2 + by^2) = 16(x + y)$$

It follows that

$$42 + 7xy = 16(x + y) \tag{1}$$

From $ax^2 + by^2 = 7$, we have $(ax^2 + by^2)(x + y) = 7(x + y)$ so $ax^3 + by^3 + xy(ax^2 + by^2) = 7(x + y)$. This simplifies to

$$16 + 3xy = 7(x + y) \tag{2}$$

We can now solve for $x + y$ and xy from (1) and (2) to find $x + y = -14$ and $xy = -38$. Thus we have $(ax^4 + by^4)(x + y) = 42(x + y)$, and so $ax^5 + by^5 + xy(ax^3 + by^3) = 42(x + y)$. Finally, it follows that $ax^5 + by^5 = 42(x + y) - 16xy = 20$ as desired.

9. [7] Let $f(x) = x^4 + 14x^3 + 52x^2 + 56x + 16$. Let z_1, z_2, z_3, z_4 be the four roots of f . Find the smallest possible value of $|z_a z_b + z_c z_d|$ where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

Answer: 8

Solution: Note that $\frac{1}{16}f(2x) = x^4 + 7x^3 + 13x^2 + 7x + 1$. Because the coefficients of this polynomial are symmetric, if r is a root of $f(x)$ then $\frac{4}{r}$ is as well. Further, $f(-1) = -1$ and $f(-2) = 16$ so $f(x)$ has two distinct roots on $(-2, 0)$ and two more roots on $(-\infty, -2)$. Now, if σ is a permutation of $\{1, 2, 3, 4\}$:

$$|z_{\sigma(1)}z_{\sigma(2)} + z_{\sigma(3)}z_{\sigma(4)}| \leq \frac{1}{2}(z_{\sigma(1)}z_{\sigma(2)} + z_{\sigma(3)}z_{\sigma(4)} + z_{\sigma(4)}z_{\sigma(3)} + z_{\sigma(2)}z_{\sigma(1)})$$

Let the roots be ordered $z_1 \leq z_2 \leq z_3 \leq z_4$, then by rearrangement the last expression is at least:

$$\frac{1}{2}(z_1z_4 + z_2z_3 + z_3z_2 + z_4z_1)$$

Since the roots come in pairs $z_1z_4 = z_2z_3 = 4$, our expression is minimized when $\sigma(1) = 1, \sigma(2) = 4, \sigma(3) = 3, \sigma(4) = 2$ and its minimum value is 8.

10. [8] Let $f(x) = 2x^3 - 2x$. For what positive values of a do there exist distinct b, c, d such that $(a, f(a)), (b, f(b)), (c, f(c)), (d, f(d))$ is a rectangle?

Answer: $[\frac{\sqrt{3}}{3}, 1]$

Solution: Say we have four points $(a, f(a)), (b, f(b)), (c, f(c)), (d, f(d))$ on the curve which form a rectangle. If we interpolate a cubic through these points, that cubic will be symmetric around the center of the rectangle. But the unique cubic through the four points is $f(x)$, and $f(x)$ has only one point of symmetry, the point $(0, 0)$.

So every rectangle with all four points on $f(x)$ is of the form $(a, f(a)), (b, f(b)), (-a, f(-a)), (-b, f(-b))$, and without loss of generality we let $a, b > 0$. Then for any choice of a and b these points form a parallelogram, which is a rectangle if and only if the distance from $(a, f(a))$ to $(0, 0)$ is equal to the distance from $(b, f(b))$ to $(0, 0)$. Let $g(x) = x^2 + (f(x))^2 = 4x^6 - 8x^4 + 5x^2$, and consider $g(x)$ restricted to $x \geq 0$. We are looking for all the values of a such that $g(x) = g(a)$ has solutions other than a .

Note that $g(x) = h(x^2)$ where $h(x) = 4x^3 - 8x^2 + 5x$. This polynomial $h(x)$ has a relative maximum of 1 at $x = \frac{1}{2}$ and a relative minimum of $\frac{25}{27}$ at $x = \frac{5}{6}$. Thus the polynomial $h(x) - h(1/2)$ has the double root $1/2$ and factors as $(4x^2 - 4x + 1)(x - 1)$, the largest possible value of a^2 for which $h(x^2) = h(a^2)$ is $a^2 = 1$, or $a = 1$. The smallest such value is that which evaluates to $\frac{25}{27}$ other than $5/6$, which is similarly found to be $a^2 = 1/3$, or $a = \frac{\sqrt{3}}{3}$. Thus, for a in the range $\frac{\sqrt{3}}{3} \leq a \leq 1$ the equation $g(x) = g(a)$ has nontrivial solutions and hence an inscribed rectangle exists.

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Individual Round: Calculus Test Solutions

1. [3] Let f be a differentiable real-valued function defined on the positive real numbers. The tangent lines to the graph of f always meet the y -axis 1 unit lower than where they meet the function. If $f(1) = 0$, what is $f(2)$?

Answer: $\boxed{\ln 2}$

Solution: The tangent line to f at x meets the y -axis at $f(x) - 1$ for any x , so the slope of the tangent line is $f'(x) = \frac{1}{x}$, and so $f(x) = \ln(x) + C$ for some a . Since $f(1) = 0$, we have $C = 0$, and so $f(x) = \ln(x)$. Thus $f(2) = \ln(2)$.

2. [3] The differentiable function $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $F(0) = -1$ and

$$\frac{d}{dx}F(x) = \sin(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x).$$

Find $F(x)$ as a function of x .

Answer: $\boxed{-\cos(\sin(\sin(\sin(x))))}$

Solution: Substituting $u = \sin(\sin(\sin(x)))$, we find

$$F(x) = \int \sin(u) du = -\cos(u) + C.$$

for some C . Since $F(0) = -1$ we find $C = 0$.

3. [4] Compute e^A where A is defined as

$$\int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1} dx.$$

Answer: $\boxed{\frac{16}{9}}$

Solution: We can use partial fractions to decompose the integrand to $\frac{1}{x+1} + \frac{x}{x^2+1}$, and then integrate the addends separately by substituting $u = x + 1$ for the former and $u = x^2 + 1$ for latter, to obtain $\ln(x+1) + \frac{1}{2} \ln(x^2+1) \Big|_{3/4}^{4/3} = \ln((x+1)\sqrt{x^2+1}) \Big|_{3/4}^{4/3} = \ln \frac{16}{9}$. Thus $e^A = 16/9$.

Alternate solution: Substituting $u = 1/x$, we find

$$A = \int_{4/3}^{3/4} \frac{2u + u^2 + u^3}{1 + u + u^2 + u^3} \left(-\frac{1}{u^2}\right) du = \int_{3/4}^{4/3} \frac{2/u + 1 + u}{1 + u + u^2 + u^3} du$$

Adding this to the original integral, we find

$$2A = \int_{3/4}^{4/3} \frac{2/u + 2 + 2u + 2u^2}{1 + u + u^2 + u^3} du = \int_{3/4}^{4/3} \frac{2}{u} du$$

Thus $A = \ln \frac{16}{9}$ and $e^A = \frac{16}{9}$.

4. [4] Let P be a fourth degree polynomial, with derivative P' , such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

Answer: $\boxed{\frac{89}{11}}$

Solution: Observe that 7 is not a root of P . If r_1, r_2, r_3, r_4 are the roots of P , then $\frac{P'(7)}{P(7)} = \sum_i \frac{1}{7-r_i} = 0$. Thus $r_4 = 7 - \left(\sum_{i \neq 4} \frac{1}{7-r_i}\right)^{-1} = 7 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{2}\right)^{-1} = 7 + 12/11 = 89/11$.

5. [4] Compute

$$\lim_{h \rightarrow 0} \frac{\sin\left(\frac{\pi}{3} + 4h\right) - 4\sin\left(\frac{\pi}{3} + 3h\right) + 6\sin\left(\frac{\pi}{3} + 2h\right) - 4\sin\left(\frac{\pi}{3} + h\right) + \sin\left(\frac{\pi}{3}\right)}{h^4}$$

Answer: $\boxed{\frac{\sqrt{3}}{2}}$

Solution: The derivative of a function is defined as $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$. Iterating this formula four times yields

$$f^{(4)}(x) = \lim_{h \rightarrow 0} \frac{f(x+4h) - 4f(x+3h) + 6f(x+2h) - 4f(x+h) + f(x)}{h^4}.$$

Substituting $f = \sin$ and $x = \pi/3$, the expression is equal to $\sin^{(4)}(\pi/3) = \sin(\pi/3) = \frac{\sqrt{3}}{2}$.

6. [5] Let $p_0(x), p_1(x), p_2(x), \dots$ be polynomials such that $p_0(x) = x$ and for all positive integers n , $\frac{d}{dx} p_n(x) = p_{n-1}(x)$. Define the function $p(x) : [0, \infty) \rightarrow \mathbb{R}$ by $p(x) = p_n(x)$ for all $x \in [n, n+1]$. Given that $p(x)$ is continuous on $[0, \infty)$, compute

$$\sum_{n=0}^{\infty} p_n(2009).$$

Answer: $\boxed{e^{2010} - e^{2009} - 1}$

Solution: By writing out the first few polynomials, one can guess and then show by induction that $p_n(x) = \frac{1}{(n+1)!} (x+1)^{n+1} - \frac{1}{n!} x^n$. Thus the sum evaluates to $e^{2010} - e^{2009} - 1$ by the series expansion of e^x .

7. [5] A line in the plane is called *strange* if it passes through $(a, 0)$ and $(0, 10 - a)$ for some a in the interval $[0, 10]$. A point in the plane is called *charming* if it lies in the first quadrant and also lies below some strange line. What is the area of the set of all charming points?

Answer: $\boxed{50/3}$

Solution: The strange lines form an envelope (set of tangent lines) of a curve $f(x)$, and we first find the equation for f on $[0, 10]$. Assuming the derivative f' is continuous, the point of tangency of the line ℓ through $(a, 0)$ and $(0, b)$ to f is the limit of the intersection points of this line with the lines ℓ_ϵ passing through $(a + \epsilon, 0)$ and $(0, b - \epsilon)$ as $\epsilon \rightarrow 0$. If these limits exist, then the derivative is indeed continuous and we can calculate the function from the points of tangency.

The intersection point of ℓ and ℓ_ϵ can be calculated to have x -coordinate $\frac{a(a-\epsilon)}{a+b}$, so the tangent point of ℓ has x -coordinate $\lim_{\epsilon \rightarrow 0} \frac{a(a-\epsilon)}{a+b} = \frac{a^2}{a+b} = \frac{a^2}{10}$. Similarly, the y -coordinate is $\frac{b^2}{10} = \frac{(10-a)^2}{10}$. Thus,

solving for the y coordinate in terms of the x coordinate for $a \in [0, 10]$, we find $f(x) = 10 - 2\sqrt{10}\sqrt{x} + x$, and so the area of the set of charming points is

$$\int_0^{10} (10 - 2\sqrt{10}\sqrt{x} + x) dx = 50/3.$$

8. [7] Compute

$$\int_1^{\sqrt{3}} x^{2x^2+1} + \ln(x^{2x^{2x^2+1}}) dx.$$

Answer: $\boxed{13}$

Solution: Using the fact that $x = e^{\ln(x)}$, we evaluate the integral as follows:

$$\begin{aligned} \int x^{2x^2+1} + \ln(x^{2x^{2x^2+1}}) dx &= \int x^{2x^2+1} + x^{2x^2+1} \ln(x^2) dx \\ &= \int e^{\ln(x)(2x^2+1)} (1 + \ln(x^2)) dx \\ &= \int x e^{x^2 \ln(x^2)} (1 + \ln(x^2)) dx \end{aligned}$$

Noticing that the derivative of $x^2 \ln(x^2)$ is $2x(1 + \ln(x^2))$, it follows that the integral evaluates to

$$\frac{1}{2} e^{x^2 \ln(x^2)} = \frac{1}{2} x^{2x^2}.$$

Evaluating this from 1 to $\sqrt{3}$ we obtain the answer.

9. [7] let \mathcal{R} be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving \mathcal{R} around the line $y = x$.

Answer: $\boxed{\frac{\sqrt{2}\pi}{60}}$

Solution: We integrate from 0 to 1 using the method of washers. Fix d between 0 and 1. Let the line $x = d$ intersect the graph of $y = x^2$ at Q , and let the line $x = d$ intersect the graph of $y = x$ at P . Then $P = (d, d)$, and $Q = (d, d^2)$. Now drop a perpendicular from Q to the line $y = x$, and let R be the foot of this perpendicular. Because PQR is a $45 - 45 - 90$ triangle, $QR = (d - d^2)/\sqrt{2}$. So the differential washer has a radius of $(d - d^2)/\sqrt{2}$ and a height of $\sqrt{2}dx$. So we integrate (from 0 to 1) the expression $[(x - x^2)/\sqrt{2}]^2 \sqrt{2} dx$, and the answer follows.

10. [8] Let a and b be real numbers satisfying $a > b > 0$. Evaluate

$$\int_0^{2\pi} \frac{1}{a + b \cos(\theta)} d\theta.$$

Express your answer in terms of a and b .

Answer: $\boxed{\frac{2\pi}{\sqrt{a^2 - b^2}}}$

Solution: Using the geometric series formula, we can expand the integral as follows:

$$\begin{aligned}
 \int_0^{2\pi} \frac{1}{a + b \cos(\theta)} d\theta &= \frac{1}{a} \int_0^{2\pi} 1 + \frac{b}{a} \cos(\theta) + \left(\frac{b}{a}\right)^2 \cos^2(\theta) d\theta \\
 &= \frac{1}{a} \sum_{n=0}^{\infty} \int_0^{2\pi} \left(\frac{b}{a}\right)^n \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^n d\theta \\
 &= \frac{2\pi}{a} \sum_{n=0}^{\infty} \left(\frac{b^2}{a^2}\right)^n \frac{\binom{2n}{n}}{2^{2n}} d\theta
 \end{aligned}$$

To evaluate this sum, recall that $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number. The generating function for the Catalan numbers is

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1 - 4x}}{2x},$$

and taking the derivative of x times this generating function yields $\sum \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$. Thus the integral evaluates to $\frac{2\pi}{\sqrt{a^2-b^2}}$, as desired.

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Individual Round: Combinatorics Test

1. [3] How many ways can the integers from -7 to 7 be arranged in a sequence such that the absolute value of the numbers in the sequence is nondecreasing?

Answer:

Solution: Each of the pairs $a, -a$ must occur in increasing order of a for $a = 1, \dots, 7$, but a can either occur before or after $-a$, for a total of $2^7 = 128$ possible sequences.

2. [3] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be between the two jokers?

Answer:

Solution: Each card has an equal likelihood of being either on top of the jokers, in between them, or below the jokers. Thus, on average, $1/3$ of them will land between the two jokers.

3. [4] In how many ways can you rearrange the letters of "HMMTHMMT" such that the consecutive substring "HMMT" does not appear?

Answer:

Solution: There are $8!/(4!2!2!) = 420$ ways to order the letters. If the permuted letters contain "HMMT", there are $5 \cdot 4!/2! = 60$ ways to order the other letters, so we subtract these. However, we have subtracted "HMMTHMMT" twice, so we add it back once to obtain 361 possibilities.

4. [4] How many functions $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

Answer:

Solution: A *fixed point* of a function f is an element a such that $f(a) = a$. The condition is equivalent to the property that f maps every number to a fixed point. Counting by the number of fixed points of f , the total number of such functions is

$$\begin{aligned} \sum_{k=1}^5 \binom{5}{k} k^{5-k} &= 1 \cdot (5^0) + 5 \cdot (1^4 + 4^1) + 10 \cdot (2^3 + 3^2) \\ &= 1 + 25 + 10 \cdot 17 \\ &= 196. \end{aligned}$$

5. [4] Let $s(n)$ denote the number of 1's in the binary representation of n . Compute

$$\frac{1}{255} \sum_{0 \leq n < 16} 2^n (-1)^{s(n)}.$$

Answer:

Solution: Notice that if $n < 8$, $(-1)^{s(n)} = (-1) \cdot (-1)^{s(n+8)}$ so the sum becomes $\frac{1}{255} (1-2^8) \sum_{0 \leq n < 8} 2^n (-1)^{s(n)} = 45$.

6. [5] How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \leq a + b + c + d + e \leq 10$?

Answer: 116

Solution: We count based on how many 1's the sequence contains. If $a = b = c = d = e = 1$ then this gives us 1 possibility. If $a = b = c = d = 1$ and $e \neq 1$, e can be 2, 3, 4, 5, 6. Each such sequence $(1, 1, 1, 1, e)$ can be arranged in 5 different ways, for a total of $5 \cdot 5 = 25$ ways in this case.

If three of the numbers are 1, the last two can be $(2, 2)$, $(3, 3)$, $(2, 3)$, $(2, 4)$, or $(2, 5)$. Counting ordering, this gives a total of $2 \cdot 10 + 3 \cdot 20 = 80$ possibilities.

If two of the numbers are 1, the other three must be equal to 2 for the product to be under 10, and this yields 10 more possibilities.

Thus there are $1 + 25 + 80 + 10 = 116$ such sequences.

7. [7] Paul fills in a 7×7 grid with the numbers 1 through 49 in a random arrangement. He then erases his work and does the same thing again (to obtain two different random arrangements of the numbers in the grid). What is the expected number of pairs of numbers that occur in either the same row as each other or the same column as each other in both of the two arrangements?

Answer: 147/2

Solution: Each of the $\binom{49}{2}$ pairs of numbers has a probability of $\frac{14 \cdot \binom{7}{2}}{\binom{49}{2}} = 1/4$ of being in the same row or column in **one** of the arrangements, so the expected number that are in the same row or column in **both** arrangements is

$$\binom{49}{2} \cdot (1/4)^2 = \frac{147}{2}.$$

8. [7] There are 5 students on a team for a math competition. The math competition has 5 subject tests. Each student on the team must choose 2 distinct tests, and each test must be taken by exactly two people. In how many ways can this be done?

Answer: 2040

Solution: We can model the situation as a bipartite graph on 10 vertices, with 5 nodes representing the students and the other 5 representing the tests. We now simply want to count the number of bipartite graphs on these two sets such that there are two edges incident on each vertex.

Notice that in such a graph, we can start at any vertex and follow one of the edges emanating from it, then follow the other edge emanating from the second vertex, etc, and in this manner we must eventually end up back at the starting vertex, so the graph is partitioned into even cycles. Since each vertex has degree two, we cannot have a 2-cycle, so we must have either a 10-cycle or a 4-cycle and a 6-cycle.

In the former case, starting with Person A , there are 5 ways to choose one of his tests. This test can be taken by one of 4 other people, who can take one of 4 other tests, which can be taken by one of 3 other people, etc, so the number of 10-cycles we obtain in this way is $5! \cdot 4!$. However, it does not matter which of the first person's tests we choose first in a given 10-cycle, so we overcounted by a factor of 2. Thus there are $5! \cdot 4!/2 = 1440$ possibilities in this case.

In the latter case, there are $\binom{5}{3}^2 = 100$ ways to choose which three people and which three tests are in the 6-cycle. After choosing this, a similar argument to that above shows there are $2! \cdot 1!/2$ possible 4-cycles and $3! \cdot 2!/2$ possible 6-cycles, for a total of $100 \cdot 1 \cdot 6 = 600$ possibilities in this case.

Thus there are a total of 2040 ways they can take the tests.

9. [5] The squares of a 3×3 grid are filled with positive integers such that 1 is the label of the upper-leftmost square, 2009 is the label of the lower-rightmost square, and the label of each square divides the one directly to the right of it and the one directly below it. How many such labelings are possible?

Answer: 2448

Solution: We factor 2009 as $7^2 \cdot 41$ and place the 41's and the 7's in the squares separately. The number of ways to fill the grid with 1's and 41's so that the divisibility property is satisfied is equal to the number of nondecreasing sequences a_1, a_2, a_3 where each $a_i \in \{0, 1, 2, 3\}$ and the sequence is not $0, 0, 0$ and not $1, 1, 1$ (here a_i corresponds to the number of 41's in the i th column.) Thus there are $\binom{3+4-1}{3} - 2 = 18$ ways to choose which squares are divisible by 41.

To count the arrangements of divisibility by 7 and 49, we consider three cases.

If 49 divides the middle square, then each of the squares to the right and below it are divisible 49. The two squares in the top row (besides the upper left) can be $(1, 1)$, $(1, 7)$, $(1, 49)$, $(7, 7)$, $(7, 49)$, or $(49, 49)$ (in terms of the highest power of 7 dividing the square). The same is true, independently, for the two blank squares on the left column, for a total of $6^2 = 36$ possibilities in this case.

If 1 is the highest power of 7 dividing the middle square, there are also 36 possibilities by a similar argument.

If 7 is the highest power of 7 dividing the middle square, there are 8 possibilities for the upper right three squares. Thus there are 64 possibilities in this case.

Thus there are a total of 136 options for the divisibility of each number by 7 and 7^2 , and 18 options for the divisibility of the numbers by 41. Since each number divides 2009, this uniquely determines the numbers, and so there are a total of $18 \cdot 136 = 2448$ possibilities.

10. [8] Given a rearrangement of the numbers from 1 to n , each pair of consecutive elements a and b of the sequence can be either increasing (if $a < b$) or decreasing (if $b < a$). How many rearrangements of the numbers from 1 to n have exactly two increasing pairs of consecutive elements?

Answer: $3^n - (n + 1) \cdot 2^n + n(n + 1)/2$ or equivalent

Solution: Notice that each such permutation consists of 3 disjoint subsets of $\{1, \dots, n\}$ whose union is $\{1, \dots, n\}$, each arranged in decreasing order. For instance, if $n = 6$, in the permutation 415326 (which has the two increasing pairs 15 and 26), the three sets are $\{4, 1\}$, $\{5, 3, 2\}$, and 6. There are 3^n ways to choose which of the first, second, or third set each element is in. However, we have overcounted: some choices of these subsets result in permutations with 1 or 0 increasing pairs, such as $\{6, 5, 4\}$, $\{3, 2\}$, $\{1\}$.

Thus, we must subtract the number of ordered partitions of $\{1, 2, \dots, n\}$ into 3 subsets for which the minimum value of the first is not less than the maximum of the second, or the minimum value of the second is not less than the maximum of the third.

We first prove that the number of permutations having exactly **one** increasing consecutive pair of elements is $2^n - (n + 1)$. To do so, note that there are 2^n ways to choose which elements occur before the increasing pair, and upon choosing this set we must arrange them in decreasing order, followed by the remaining elements arranged in decreasing order. The resulting permutation will have either one increasing pair or none. There are exactly $n + 1$ subsets for which the resulting permutation has none, namely, $\{\}$, $\{n\}$, $\{n, n - 1\}$, $\{n, n - 1, n - 2\}$, etc. Thus the total number of permutations having one increasing pair is $2^n - (n + 1)$ as desired.

We now count the partitions of $\{1, 2, \dots, n\}$ whose associated permutation has exactly one increasing pair. For each of the $2^n - (n + 1)$ permutations p having exactly one increasing pair, there are $n + 1$ partitions of $\{1, 2, \dots, n\}$ into 3 subsets whose associated permutation is p . This is because there are $n + 1$ ways to choose the "breaking point" to split one of the subsets into two. Thus there are a total of $(n + 1)(2^n - (n + 1))$ partitions whose associated permutation has exactly one increasing pair.

Finally, we must count the number of partitions whose associated permutation is $n, n - 1, \dots, 3, 2, 1$, i.e. has no increasing pair. There are $\frac{n+2}{2}$ ways of placing two barriers between these elements to split the numbers into three subsets, and so there are $\frac{n+2}{2}$ such partitions of $\{1, 2, \dots, n\}$ into three subsets.

Thus, subtracting off the partitions we did not want to count, the answer is $3^n - (n+1)(2^n - (n+1)) - \binom{n+2}{2} = 3^n - (n+1) \cdot 2^n + n(n+1)/2$.

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Individual Round: General Test, Part 1

1. [2] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

Answer:

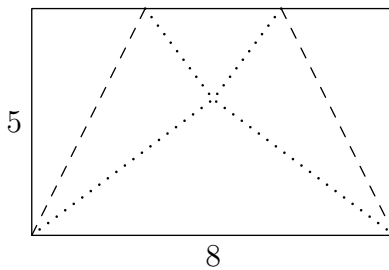
Solution: We can factor the equation as $(a - b^2)(a + b^2) = 41 \cdot 49$, from which it is evident that $a = 45$ and $b = 2$ is a possible solution. By examining the factors of 2009, one can see that there are no other solutions.

2. [2] Suppose N is a 6-digit number having base-10 representation $\underline{a}\underline{b}\underline{c}\underline{d}\underline{e}\underline{f}$. If N is $6/7$ of the number having base-10 representation $\underline{d}\underline{e}\underline{f}\underline{a}\underline{b}\underline{c}$, find N .

Answer:

Solution: We have $7(abcdef)_{10} = 6(defabc)_{10}$, so $699400a + 69940b + 6994c = 599300d + 59930e + 5993f$. We can factor this equation as $6994(100a + 10b + c) = 5993(100d + 10e + f)$, which yields $538(abc)_{10} = 461(def)_{10}$. Since $\gcd(538, 461) = 1$, we must have $(abc)_{10} = 461$ and $(def)_{10} = 538$.

3. [3] A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



Answer:

Solution: Drawing the perpendiculars from the point of intersection of the corners to the bases of the trapezoid, we see that we have similar $3-4-5$ right triangles, and we can calculate that the length of the smaller base is 3. Thus the area of the trapezoid is $\frac{8+3}{2} \cdot 5 = 55/2$.

4. [3] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

Answer:

Solution: We have $\cot x + \cot y = \frac{\tan x + \tan y}{\tan x \tan y}$, so $\tan x \tan y = \frac{4}{5}$. Thus, by the tan addition formula, $\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = 20$.

5. [4] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

Answer:

Solution: Each card has an equal likelihood of being either on top of the jokers, in between them, or below the jokers. Thus, on average, $1/3$ of them will land between the two jokers.

6. [4] The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

Answer: $\boxed{2\sqrt{3}/3}$

Solution: The major diagonal has a length of $\sqrt{3}$. The volume of the pyramid is $1/6$, and so its height h satisfies $\frac{1}{3} \cdot h \cdot \frac{\sqrt{3}}{4} (\sqrt{2})^2 = 1/6$ since the freshly cut face is an equilateral triangle of side length $\sqrt{2}$. Thus $h = \sqrt{3}/3$, and the answer follows.

7. [5] Let $s(n)$ denote the number of 1's in the binary representation of n . Compute

$$\frac{1}{255} \sum_{0 \leq n < 16} 2^n (-1)^{s(n)}.$$

Answer: $\boxed{45}$

Solution: Notice that if $n < 8$, $(-1)^{s(n)} = (-1) \cdot (-1)^{s(n+8)}$ so the sum becomes $\frac{1}{255} (1-2^8) \sum_{0 \leq n < 8} 2^n (-1)^{s(n)} = 45$.

8. [5] Let a , b , and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

Answer: $\boxed{-2}$

Solution: We can substitute $x = y - 1$ to obtain a polynomial having roots $a + 1$, $b + 1$, $c + 1$, namely, $(y - 1)^3 - (y - 1) + 1 = y^3 - 3y^2 + 2y + 1$. The sum of the reciprocals of the roots of this polynomial is, by Viète's formulas, $\frac{2}{-1} = -2$.

9. [6] How many functions $f : \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

Answer: $\boxed{196}$

Solution: A *fixed point* of a function f is an element a such that $f(a) = a$. The condition is equivalent to the property that f maps every number to a fixed point. Counting by the number of fixed points of f , the total number of such functions is

$$\begin{aligned} \sum_{k=1}^5 \binom{5}{k} k^{5-k} &= 1 \cdot (0^5 + 5^0) + 5 \cdot (1^4 + 4^1) + 10 \cdot (2^3 + 3^2) \\ &= 1 + 25 + 10 \cdot 17 \\ &= 196. \end{aligned}$$

10. [6] A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD . Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω . Let N' be the point of tangency of $B'C'$ to ω . If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

Answer: $\boxed{\frac{1+\sqrt{5}}{2}}$

Solution: Let's focus on the right triangle ABC and the semicircle inscribed in it since the situation is symmetric about AC . First we find the radius a of circle O . Let $AB = x$ and $BC = y$. Drawing the radii OM and ON , we see that $AM = x - a$ and $\triangle AMO \sim \triangle ABC$. In other words,

$$\begin{aligned}\frac{AM}{MO} &= \frac{AB}{BC} \\ \frac{x-a}{a} &= \frac{x}{y} \\ a &= \frac{xy}{x+y}.\end{aligned}$$

Now we notice that the situation is homothetic about A . In particular,

$$\triangle AMO \sim \triangle ONC \sim \triangle CN'C'.$$

Also, CB and CN' are both radii of circle C . Thus, when $MN' \parallel AC'$, we have

$$\begin{aligned}AM &= CN' = CB \\ x - a &= y \\ a &= \frac{xy}{x+y} = x - y \\ x^2 - xy - y^2 &= 0 \\ x &= \frac{y}{2} \pm \sqrt{\frac{y^2}{4} + y^2} \\ \frac{AB}{BC} &= \frac{x}{y} = \frac{1 + \sqrt{5}}{2}.\end{aligned}$$

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Individual Round: General Test, Part 2

1. [2] How many ways can the integers from -7 to 7 be arranged in a sequence such that the absolute values of the numbers in the sequence are nonincreasing?

Answer:

Solution: Each of the pairs $a, -a$ must occur in increasing order of a for $a = 1, \dots, 7$, but a can either occur before or after $-a$, for a total of $2^7 = 128$ possible sequences.

2. [2] How many ways can you tile the white squares of following 2×24 grid with dominoes? (A domino covers two adjacent squares, and a tiling is a non-overlapping arrangement of dominoes that covers every white square and does not intersect any black square.)



Answer:

Solution: Divide the rectangle into three 3×8 sub-rectangles. It is easy to count that there are 3 ways of tiling each of these sub-rectangles independently, for a total of $3^3 = 27$ possibilities.

3. [3] Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?

Answer:

Solution: The sum of all the coefficients is $(1 + i)^{2009}$, and the sum of the real coefficients is the real part of this, which is $\frac{1}{2} \left((1 + i)^{2009} + (1 - i)^{2009} \right) = 2^{1004}$. Thus $\log_2(S) = 1004$.

4. [3] A torus (donut) having inner radius 2 and outer radius 4 sits on a flat table. What is the radius of the largest spherical ball that can be placed on top of the center torus so that the ball still touches the horizontal plane? (If the $x - y$ plane is the table, the torus is formed by revolving the circle in the $x - z$ plane centered at $(3, 0, 1)$ with radius 1 about the z axis. The spherical ball has its center on the z -axis and rests on either the table or the donut.)

Answer:

Solution: Let r be the radius of the sphere. One can see that it satisfies $(r + 1)^2 = (r - 1)^2 + 3^2$ by the Pythagorean Theorem, so $r = 9/4$.

5. [4] Suppose a, b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $x^3 - 4x^2 + x + 6$. Find $a + b + c$.

Answer:

Solution: Since $x + 1$ divides $x^2 + ax + b$ and the constant term is b , we have $x^2 + ax + b = (x + 1)(x + b)$, and similarly $x^2 + bx + c = (x + 1)(x + c)$. Therefore, $a = b + 1 = c + 2$. Furthermore, the least common

multiple of the two polynomials is $(x+1)(x+b)(x+b-1) = x^3 - 4x^2 + x + 6$, so $b = -2$. Thus $a = -1$ and $c = -3$, and $a + b + c = -6$.

6. [4] In how many ways can you rearrange the letters of “HMMTHMMT” such that the consecutive substring “HMMT” does not appear?

Answer: 361

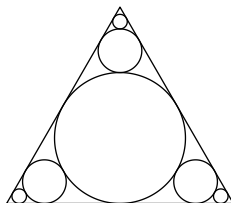
Solution: There are $8!/(4!2!2!) = 420$ ways to order the letters. If the permuted letters contain “HMMT”, there are $5 \cdot 4!/2! = 60$ ways to order the other letters, so we subtract these. However, we have subtracted “HMMTHMMT” twice, so we add it back once to obtain 361 possibilities.

7. [5] Let F_n be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Compute $\sum_{n=0}^{\infty} F_n/10^n$.

Answer: 10/89

Solution: Write $F(x) = \sum_{n=0}^{\infty} F_n x^n$. Then the Fibonacci recursion tells us that $F(x) - xF(x) - x^2F(x) = x$, so $F(x) = x/(1 - x - x^2)$. Plugging in $x = 1/10$ gives the answer.

8. [5] The incircle ω of equilateral triangle ABC has radius 1. Three smaller circles are inscribed tangent to ω and the sides of ABC , as shown. Three smaller circles are then inscribed tangent to the previous circles and to each of two sides of ABC . This process is repeated an infinite number of times. What is the total length of the circumferences of all the circles?



Answer: 5π

Solution: One can find using the Pythagorean Theorem that, in each iteration, the new circles have radius $1/3$ of that of the previously drawn circles. Thus the total circumference is $2\pi + 3 \cdot 2\pi(\frac{1}{1-1/3} - 1) = 5\pi$.

9. [6] How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \leq a + b + c + d + e \leq 10$?

Answer: 116

Solution: We count based on how many 1's the sequence contains. If $a = b = c = d = e = 1$ then this gives us 1 possibility. If $a = b = c = d = 1$ and $e \neq 1$, e can be 2, 3, 4, 5, 6. Each such sequence $(1, 1, 1, 1, e)$ can be arranged in 5 different ways, for a total of $5 \cdot 5 = 25$ ways in this case.

If three of the numbers are 1, the last two can be $(2, 2)$, $(3, 3)$, $(2, 3)$, $(2, 4)$, or $(2, 5)$. Counting ordering, this gives a total of $2 \cdot 10 + 3 \cdot 20 = 80$ possibilities.

If two of the numbers are 1, the other three must be equal to 2 for the product to be under 10, and this yields 10 more possibilities.

Thus there are $1 + 25 + 80 + 10 = 116$ such sequences.

10. [6] Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called *awesome* if P is the center of a parallelogram whose vertices all lie on the boundary of T . What is the area of the set of awesome points?

Answer: $\boxed{3/2}$

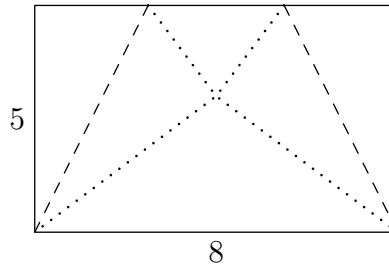
Solution: The set of awesome points is the medial triangle, which has area $6/4 = 3/2$.

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Individual Round: Geometry Test

1. [3] A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



Answer: $\boxed{55/2}$

Solution: Drawing the perpendiculars from the point of intersection of the corners to the bases of the trapezoid, we see that we have similar 3–4–5 right triangles, and we can calculate that the length of the smaller base is 3. Thus the area of the trapezoid is $\frac{8+3}{2} \cdot 5 = 55/2$.

2. [3] The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

Answer: $\boxed{2\sqrt{3}/3}$

Solution: The major diagonal has a length of $\sqrt{3}$. The volume of the pyramid is $1/6$, and so its height h satisfies $\frac{1}{3} \cdot h \cdot \frac{\sqrt{3}}{4} (\sqrt{2})^2 = 1/6$ since the freshly cut face is an equilateral triangle of side length $\sqrt{2}$. Thus $h = \sqrt{3}/3$, and the answer follows.

3. [4] Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called *awesome* if P is the center of a parallelogram whose vertices all lie on the boundary of T . What is the area of the set of awesome points?

Answer: $\boxed{3/2}$

Solution: The set of awesome points is the medial triangle, which has area $6/4 = 3/2$.

4. [4] A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD . Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω . Let N' be the point of tangency of $B'C'$ to ω . If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

Answer: $\boxed{\frac{1+\sqrt{5}}{2}}$

Solution: Let's focus on the right triangle ABC and the semicircle inscribed in it since the situation is symmetric about AC . First we find the radius a of circle O . Let $AB = x$ and $BC = y$. Drawing the radii OM and ON , we see that $AM = x - a$ and $\triangle AMO \sim \triangle ABC$. In other words,

$$\begin{aligned}\frac{AM}{MO} &= \frac{AB}{BC} \\ \frac{x-a}{a} &= \frac{x}{y} \\ a &= \frac{xy}{x+y}.\end{aligned}$$

Now we notice that the situation is homothetic about A . In particular,

$$\triangle AMO \sim \triangle ONC \sim \triangle CN'C'.$$

Also, CB and CN' are both radii of circle C . Thus, when $MN' \parallel AC'$, we have

$$\begin{aligned}AM &= CN' = CB \\ x - a &= y \\ a &= \frac{xy}{x+y} = x - y \\ x^2 - xy - y^2 &= 0 \\ x &= \frac{y}{2} \pm \sqrt{\frac{y^2}{4} + y^2} \\ \frac{AB}{BC} = \frac{x}{y} &= \frac{1 + \sqrt{5}}{2}.\end{aligned}$$

5. [4] Circle B has radius $6\sqrt{7}$. Circle A , centered at point C , has radius $\sqrt{7}$ and is contained in B . Let L be the locus of centers C such that there exists a point D on the boundary of B with the following property: if the tangents from D to circle A intersect circle B again at X and Y , then XY is also tangent to A . Find the area contained by the boundary of L .

Answer: $\boxed{168\pi}$

Solution: The conditions imply that there exists a triangle such that B is the circumcircle and A is the incircle for the position of A . The distance between the circumcenter and incenter is given by $\sqrt{(R-2r)R}$, where R, r are the circumradius and inradius, respectively. Thus the locus of C is a circle concentric to B with radius $2\sqrt{42}$. The conclusion follows.

6. [4] Let ABC be a triangle in the coordinate plane with vertices on lattice points and with $AB = 1$. Suppose the perimeter of ABC is less than 17. Find the largest possible value of $1/r$, where r is the inradius of ABC .

Answer: $\boxed{1 + 5\sqrt{2} + \sqrt{65}}$

Solution: Let a denote the area of the triangle, r the inradius, and p the perimeter. Then $a = rp/2$, so $r = 2a/p > 2a/17$. Notice that $a = h/2$ where h is the height of the triangle from C to AB , and h is an integer since the vertices are lattice points. Thus we first guess that the inradius is minimized when $h = 1$ and the area is $1/2$. In this case, we can now assume WLOG that $A = (0, 0)$, $B = (1, 0)$, and $C = (n + 1, 1)$ for some nonnegative integer n . The perimeter of ABC is $\sqrt{n^2 + 2n + 2} + \sqrt{n^2 + 1} + 1$. Since $n = 8$ yields a perimeter greater than 17, the required triangle has $n = 7$ and inradius $r = 1/p = \frac{1}{1+5\sqrt{2}+\sqrt{65}}$ which yields the answer of $1/r = 1 + 5\sqrt{2} + \sqrt{65}$. We can now verify that this is indeed

minimal over all h by noting that its perimeter is greater than $17/2$, which is the upper bound in the case $h \geq 2$.

7. [5] In $\triangle ABC$, D is the midpoint of BC , E is the foot of the perpendicular from A to BC , and F is the foot of the perpendicular from D to AC . Given that $BE = 5$, $EC = 9$, and the area of triangle ABC is 84, compute $|EF|$.

Answer: $\boxed{\frac{6\sqrt{37}}{5}, \frac{21}{205}\sqrt{7585}}$

Solution: There are two possibilities for the triangle ABC based on whether E is between B and C or not. We first consider the former case.

We find from the area and the Pythagorean theorem that $AE = 12$, $AB = 13$, and $AC = 15$. We can then use Stewart's theorem to obtain $AD = 2\sqrt{37}$.

Since the area of $\triangle ADC$ is half that of ABC , we have $\frac{1}{2}AC \cdot DF = 42$, so $DF = 14/5$. Also, $DC = 14/2 = 7$ so $ED = 9 - 7 = 2$.

Notice that $AEDF$ is a cyclic quadrilateral. By Ptolemy's theorem, we have $EF \cdot 2\sqrt{37} = (28/5) \cdot 12 + 2 \cdot (54/5)$. Thus $EF = \frac{6\sqrt{37}}{5}$ as desired.

The latter case is similar.

8. [7] Triangle ABC has side lengths $AB = 231$, $BC = 160$, and $AC = 281$. Point D is constructed on the opposite side of line AC as point B such that $AD = 178$ and $CD = 153$. Compute the distance from B to the midpoint of segment AD .

Answer: $\boxed{208}$

Solution: Note that $\angle ABC$ is right since

$$BC^2 = 160^2 = 50 \cdot 512 = (AC - AB) \cdot (AC + AB) = AC^2 - AB^2.$$

Construct point B' such that $ABCB'$ is a rectangle, and construct D' on segment $B'C$ such that $AD = AD'$. Then

$$B'D'^2 = AD'^2 - AB'^2 = AD^2 - BC^2 = (AD - BC)(AD + BC) = 18 \cdot 338 = 78^2.$$

It follows that $CD' = B'C - B'D' = 153 = CD$; thus, points D and D' coincide, and $AB \parallel CD$. Let M denote the midpoint of segment AD , and denote the orthogonal projections M to lines AB and BC by P and Q respectively. Then Q is the midpoint of BC and $AP = 39$, so that $PB = AB - AP = 192$ and

$$BM = PQ = \sqrt{80^2 + 192^2} = 16\sqrt{5^2 + 12^2} = 208.$$

9. [7] Let ABC be a triangle with $AB = 16$ and $AC = 5$. Suppose the bisectors of angles $\angle ABC$ and $\angle BCA$ meet at point P in the triangle's interior. Given that $AP = 4$, compute BC .

Answer: $\boxed{14}$

Solution: As the incenter of triangle ABC , point P has many properties. Extend AP past P to its intersection with the circumcircle of triangle ABC , and call this intersection M . Now observe that

$$\angle PBM = \angle PBC + \angle CBM = \angle PBC + \angle CAM = \beta + \alpha = 90 - \gamma,$$

where α , β , and γ are the half-angles of triangle ABC . Since

$$\angle BMP = \angle BMA = \angle BCA = 2\gamma,$$

it follows that $BM = MP = CM$. Let Q denote the intersection of AM and BC , and observe that $\triangle AQB \sim \triangle CQM$ and $\triangle AQC \sim \triangle BQM$; some easy algebra gives

$$AM/BC = (AB \cdot AC + BM \cdot CM)/(AC \cdot CM + AB \cdot BM).$$

Writing $(a, b, c, d, x) = (BC, AC, AB, MP, AP)$, this is $(x + d)/a = (bc + d^2)/((b + c)d)$. Ptolemy's theorem applied to $ABCD$ gives $a(d + x) = d(b + c)$. Multiplying the two gives $(d + x)^2 = bc + d^2$. We easily solve for $d = (bc - x^2)/(2x) = 8$ and $a = d(b + c)/(d + x) = 14$.

10. [8] Points A and B lie on circle ω . Point P lies on the extension of segment AB past B . Line ℓ passes through P and is tangent to ω . The tangents to ω at points A and B intersect ℓ at points D and C respectively. Given that $AB = 7$, $BC = 2$, and $AD = 3$, compute BP .

Answer: 9

Solution: Say that ℓ be tangent to ω at point T . Observing equal tangents, write

$$CD = CT + DT = BC + AD = 5.$$

Let the tangents to ω at A and B intersect each other at Q . Working from Menelaus applied to triangle CDQ and line AB gives

$$\begin{aligned} -1 &= \frac{DA}{AQ} \cdot \frac{QB}{BC} \cdot \frac{CP}{PD} \\ &= \frac{DA}{BC} \cdot \frac{CP}{PC + CD} \\ &= \frac{3}{2} \cdot \frac{CP}{PC + 5}, \end{aligned}$$

from which $PC = 10$. By power of a point, $PT^2 = AP \cdot BP$, or $12^2 = BP \cdot (BP + 7)$, from which $BP = 9$.

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Guts Round

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1. [5] Compute

$$1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + 19 \cdot 20^2.$$

Answer: $\boxed{41230}$ **Solution:** We can write this as $(1^3 + 2^3 + \cdots + 20^3) - (1^2 + 2^2 + \cdots + 20^2)$, which is equal to $44100 - 2870 = 41230$.

2. [5] Given that $\sin A + \sin B = 1$ and $\cos A + \cos B = 3/2$, what is the value of $\cos(A - B)$?

Answer: $\boxed{5/8}$

Solution: Squaring both equations and add them together, one obtains $1 + 9/4 = 2 + 2(\cos(A) \cos(B) + \sin(A) \sin(B)) = 2 + 2 \cos(A - B)$. Thus $\cos A - B = 5/8$.

3. [5] Find all pairs of integer solutions (n, m) to

$$2^{3^n} = 3^{2^m} - 1.$$

Answer: $\boxed{(0, 0) \text{ and } (1, 1)}$

Solution: We find all solutions of $2^x = 3^y - 1$ for positive integers x and y . If $x = 1$, we obtain the solution $x = 1, y = 1$, which corresponds to $(n, m) = (0, 0)$ in the original problem. If $x > 1$, consider the equation modulo 4. The left hand side is 0, and the right hand side is $(-1)^y - 1$, so y is even. Thus we can write $y = 2z$ for some positive integer z , and so $2^x = (3^z - 1)(3^z + 1)$. Thus each of $3^z - 1$ and $3^z + 1$ is a power of 2, but they differ by 2, so they must equal 2 and 4 respectively. Therefore, the only other solution is $x = 3$ and $y = 2$, which corresponds to $(n, m) = (1, 1)$ in the original problem.
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4. [6] Simplify: $i^0 + i^1 + \cdots + i^{2009}$.

Answer: $\boxed{1 + i}$

Solution: By the geometric series formula, the sum is equal to $\frac{i^{2010} - 1}{i - 1} = \frac{-2}{i - 1} = 1 + i$.

5. [6] In how many distinct ways can you color each of the vertices of a tetrahedron either red, blue, or green such that no face has all three vertices the same color? (Two colorings are considered the same if one coloring can be rotated in three dimensions to obtain the other.)

Answer: $\boxed{6}$

Solution: If only two colors are used, there is only one possible arrangement up to rotation, so this gives 3 possibilities. If all three colors are used, then one is used twice. There are 3 ways to choose the color that is used twice. Say this color is red. Then the red vertices are on a common edge, and the green and blue vertices are on another edge. We see that either choice of arrangement of the green and blue vertices is the same up to rotation. Thus there are 6 possibilities total.

6. [6] Let ABC be a right triangle with hypotenuse AC . Let B' be the reflection of point B across AC , and let C' be the reflection of C across AB' . Find the ratio of $[BCB']$ to $[BC'B']$.

Answer: 1 **Solution:** Since C , B' , and C' are collinear, it is evident that $[BCB'] = \frac{1}{2}[BCC']$. It immediately follows that $[BCB'] = [BC'B']$. Thus, the ratio is 1.

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7. [6] How many perfect squares divide $2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9$?

Answer: 120

Solution: The number of such perfect squares is $2 \cdot 3 \cdot 4 \cdot 5$, since the exponent of each prime can be any nonnegative even number less than the given exponent.

8. [6] Which is greater, $\log_{2008}(2009)$ or $\log_{2009}(2010)$?

Answer: $\log_{2008} 2009$.

Solution: Let $f(x) = \log_x(x+1)$. Then $f'(x) = \frac{x \ln x - (x+1) \ln(x+1)}{x(x+1) \ln^2 x} < 0$ for any $x > 1$, so f is decreasing. Thus $\log_{2008}(2009)$ is greater.

9. [6] An icosidodecahedron is a convex polyhedron with 20 triangular faces and 12 pentagonal faces. How many vertices does it have?

Answer: 30

Solution: Since every edge is shared by exactly two faces, there are $(20 \cdot 3 + 12 \cdot 5)/2 = 60$ edges. Using Euler's formula $v - e + f = 2$, we see that there are 30 vertices.

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10. [7] Let a , b , and c be real numbers. Consider the system of simultaneous equations in variables x and y :

$$\begin{aligned} ax + by &= c - 1 \\ (a + 5)x + (b + 3)y &= c + 1 \end{aligned}$$

Determine the value(s) of c in terms of a such that the system always has a solution for any a and b .

Answer: $2a/5 + 1$. (or $\frac{2a+5}{5}$)

Solution: We have to only consider when the determinant of $\begin{pmatrix} a & b \\ a+5 & b+3 \end{pmatrix}$ is zero. That is, when $b = 3a/5$. Plugging in $b = 3a/5$, we find that $(a + 5)(c - 1) = a(c + 1)$ or that $c = 2a/5 + 1$.

11. [7] There are 2008 distinct points on a circle. If you connect two of these points to form a line and then connect another two points (distinct from the first two) to form another line, what is the probability that the two lines intersect inside the circle?

Answer: 1/3

Solution: Given four of these points, there are 3 ways in which to connect two of them and then connect the other two, and of these possibilities exactly one will intersect inside the circle. Thus $1/3$ of all the ways to connect two lines and then connect two others have an intersection point inside the circle.

12. [7] Bob is writing a sequence of letters of the alphabet, each of which can be either uppercase or lowercase, according to the following two rules:
- If he had just written an **uppercase** letter, he can either write the same letter in **lowercase** after it, or the **next** letter of the alphabet in **uppercase**.
 - If he had just written a **lowercase** letter, he can either write the same letter in **uppercase** after it, or the **preceding** letter of the alphabet in **lowercase**.

For instance, one such sequence is $aAaABCDdcbBC$. How many sequences of 32 letters can he write that start at (lowercase) a and end at (lowercase) z ? (The alphabet contains 26 letters from a to z .)

Answer: 376

Solution: The smallest possible sequence from a to z is $aABCD\dots Zz$, which has 28 letters. To insert 4 more letters, we can either switch two (not necessarily distinct) letters to lowercase and back again (as in $aABCcCDEFfFGH\dots Zz$), or we can insert a lowercase letter after its corresponding uppercase letter, insert the previous letter of the alphabet, switch back to uppercase, and continue the sequence (as in $aABCcbBCDE\dots Zz$). There are $\binom{27}{2} = 13 \cdot 27$ sequences of the former type and 25 of the latter, for a total of 376 such sequences.

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13. [8] How many ordered quadruples (a, b, c, d) of four distinct numbers chosen from the set $\{1, 2, 3, \dots, 9\}$ satisfy $b < a$, $b < c$, and $d < c$?

Answer: 630

Solution: Given any 4 elements $p < q < r < s$ of $\{1, 2, \dots, 9\}$, there are 5 ways of rearranging them to satisfy the inequality: $prqs$, $psqr$, $qspr$, $qrps$, and $rspq$. This gives a total of $\binom{9}{4} \cdot 5 = 630$ quadruples.

14. [8] Compute

$$\sum_{k=1}^{2009} k \left(\left\lfloor \frac{2009}{k} \right\rfloor - \left\lfloor \frac{2008}{k} \right\rfloor \right).$$

Answer: 2394

Solution: The summand is equal to k if k divides 2009 and 0 otherwise. Thus the sum is equal to the sum of the divisors of 2009, or 2394.

15. [8] Stan has a stack of 100 blocks and starts with a score of 0, and plays a game in which he iterates the following two-step procedure:

- Stan picks a stack of blocks and splits it into 2 smaller stacks each with a positive number of blocks, say a and b . (The order in which the new piles are placed does not matter.)
- Stan adds the product of the two piles' sizes, ab , to his score.

The game ends when there are only 1-block stacks left. What is the expected value of Stan's score at the end of the game?

Answer: 4950

Solution: Let $E(n)$ be the expected value of the score for an n -block game. It suffices to show that the score is invariant regardless of how the game is played. We proceed by induction. We have $E(1) = 0$ and $E(2) = 1$. We require that $E(n) = E(n-k) + E(k) + (n-k)k$ for all k . Setting $k = 1$, we hypothesize that $E(n) = n(n-1)/2$. This satisfies the recursion and base cases so $E(100) = 100 \cdot 99/2 = 4950$.

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16. [9] A spider is making a web between $n > 1$ distinct leaves which are equally spaced around a circle. He chooses a leaf to start at, and to make the base layer he travels to each leaf one at a time, making a straight line of silk between each consecutive pair of leaves, such that no two of the lines of silk cross each other and he visits every leaf exactly once. In how many ways can the spider make the base layer of the web? Express your answer in terms of n .

Answer: $n2^{n-2}$

Solution: There are n ways to choose a starting vertex, and at each vertex he has only two choices for where to go next: the nearest untouched leaf in the clockwise direction, and the nearest untouched leaf in the counterclockwise direction. For, if the spider visited a leaf which is not nearest in some direction, there are two untouched leaves which are separated by this line of silk, and so the silk would eventually cross itself. Thus, for the first $n - 2$ choices there are 2 possibilities, and the $(n - 1)$ st choice is then determined.

Note: This formula can also be derived recursively.

17. [9] How many positive integers $n \leq 2009$ have the property that $\lfloor \log_2(n) \rfloor$ is odd?

Answer: 682

Solution: We wish to find n such that there is some natural number k for which $2k - 1 \leq \log_2 n < 2k$. Since $n \leq 2009$ we must have $k \leq 5$. This is equivalent to finding the number of positive integers $n \leq 2009$ satisfying $2^{2k-1} \leq n < 2^{2k}$ for some $k \leq 5$, so the number of such integers is $2 + 2^3 + 2^5 + 2^7 + 2^9 = 682$.

18. [9] If n is a positive integer such that $n^3 + 2n^2 + 9n + 8$ is the cube of an integer, find n .

Answer: 7

Solution: Since $n^3 < n^3 + 2n^2 + 9n + 8 < (n + 2)^3$, we must have $n^3 + 2n^2 + 9n + 8 = (n + 1)^3$. Thus $n^2 = 6n + 7$, so $n = 7$.

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19. [10] Shelly writes down a vector $v = (a, b, c, d)$, where $0 < a < b < c < d$ are integers. Let $\sigma(v)$ denote the set of 24 vectors whose coordinates are a, b, c , and d in some order. For instance, $\sigma(v)$ contains (b, c, d, a) . Shelly notes that there are 3 vectors in $\sigma(v)$ whose sum is of the form (s, s, s, s) for some s . What is the smallest possible value of d ?

Answer: 6

Solution: If $k = a + b + c + d$, first you notice $4 \mid 3k$, and $k \geq 10$. So we try $k = 12$, which works with $a, b, c, d = 1, 2, 3, 6$ and not $1, 2, 4, 5$.

20. [10] A positive integer is called *jubilant* if the number of 1's in its binary representation is even. For example, $6 = 110_2$ is a jubilant number. What is the 2009th smallest jubilant number?

Answer: 4018

Solution: Notice that for each pair of consecutive positive integers $2k$ and $2k + 1$, their binary representation differs by exactly one 1 (in the units digit), so exactly one of 2 and 3 is jubilant, exactly one of 4 and 5 is jubilant, etc. It follows that there are exactly 2009 jubilant numbers less than or equal to 4019. We now simply need to check whether 4018 or 4019 is jubilant. Since the binary representation of 4018 is 111110110010, 4018 is the 2009th jubilant number.

21. [10] Simplify

$$2 \cos^2(\ln(2009)i) + i \sin(\ln(4036081)i).$$

Answer: $\boxed{\frac{4036082}{4036081}}$ **Solution:** We have

$$\begin{aligned}
2 \cos^2(\ln(2009)i) + i \sin(\ln(4036081)i) &= 1 + \cos(2 \ln(2009)i) + i \sin(\ln(4036081)i) \\
&= 1 + \cos(\ln(4036081)i) + i \sin(\ln(4036081)i) \\
&= 1 + e^{i^2 \ln(4036081)} \\
&= 1 + \frac{1}{4036081} \\
&= \frac{4036082}{4036081}
\end{aligned}$$

as desired.

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22. [10] A circle having radius r_1 centered at point N is tangent to a circle of radius r_2 centered at M . Let l and j be the two common external tangent lines to the two circles. A circle centered at P with radius r_2 is externally tangent to circle N at the point at which l coincides with circle N , and line k is externally tangent to P and N such that points M , N , and P all lie on the same side of k . For what ratio r_1/r_2 are j and k parallel?

Answer: $\boxed{3}$

Solution: Suppose the lines are parallel. Draw the other tangent line to N and P - since M and P have the same radius, it is tangent to all three circles. Let j and k meet circle N at A and B , respectively. Then by symmetry we see that $\angle ANM = \angle MNP = \angle PNB = 60^\circ$ since A , N , and B are collinear (perpendicular to j and k). Let D be the foot of the perpendicular from M to AN . In $\triangle MDN$, we have $MN = 2DN$, so $r_1 + r_2 = 2(r_1 - r_2)$, and so $r_1/r_2 = 3$.

23. [10] The roots of $z^6 + z^4 + z^2 + 1 = 0$ are the vertices of a convex polygon in the complex plane. Find the sum of the squares of the side lengths of the polygon.

Answer: $\boxed{12 - 4\sqrt{2}}$

Solution: Factoring the polynomial as $(z^4 + 1)(z^2 + 1) = 0$, we find that the 6 roots are $e^{\pm i\pi/4}$, $e^{\pm i\pi/2}$, $e^{\pm i3\pi/4}$. The calculation then follows from the Law of Cosines or the distance formula.

24. [10] Compute, in terms of n ,

$$\sum_{k=0}^n \binom{n-k}{k} 2^k.$$

Note that whenever $s < t$, $\binom{s}{t} = 0$.

Answer: $\boxed{\frac{2 \cdot 2^n + (-1)^n}{3}}$

Solution: Let $T_n = \sum_{k=0}^n \binom{n-k}{k} 2^k$. From Pascal's recursion for binomial coefficients, we can find $T_n = 2T_{n-2} + T_{n-1}$, with $T_0 = 1$ and $T_1 = 1$. The characteristic polynomial of this recursion is $x^2 - x - 2 = 0$, which has roots 2 and -1 . Thus $T_n = a \cdot 2^n + b \cdot (-1)^n$ for some a and b . From the initial conditions we have $a + b = 1$ and $2a - b = 1$. It follows that $a = 2/3$ and $b = 1/3$, from which the conclusion follows.

25. [12] Four points, A, B, C , and D , are chosen randomly on the circumference of a circle with independent uniform probability. What is the expected number of sides of triangle ABC for which the projection of D onto the line containing the side lies between the two vertices?

Answer: $\boxed{3/2}$

Solution: By linearity of expectations, the answer is exactly 3 times the probability that the orthogonal projection of D onto AB lies interior to the segment. This happens exactly when either $\angle DAB$ or $\angle DBA$ is obtuse, which is equivalent to saying that A and B lie on the same side of the diameter through D . This happens with probability $1/2$. Therefore, desired answer is $3/2$.

26. [12] Define the sequence $\{x_i\}_{i \geq 0}$ by $x_0 = 2009$ and $x_n = -\frac{2009}{n} \sum_{k=0}^{n-1} x_k$ for all $n \geq 1$. Compute the value of $\sum_{n=0}^{2009} 2^n x_n$.

Answer: $\boxed{2009}$

Solution: We have

$$-\frac{nx_n}{2009} = x_{n-1} + x_{n-2} + \dots + x_0 = x_{n-1} + \frac{(n-1)x_{n-1}}{2009}$$

, which yields the recursion $x_n = \frac{n-2010}{n}x_{n-1}$. Unwinding this recursion, we find $x_n = (-1)^n \cdot 2009 \cdot \binom{2008}{n}$. Thus

$$\begin{aligned} \sum_{k=0}^{2009} 2^k x_k &= \sum_{k=0}^{2009} (-2)^k \cdot 2009 \cdot \binom{2008}{k} \\ &= 2009 \sum_{k=0}^{2008} (-2)^k \binom{2008}{k} \\ &= 2009(-2+1)^{2008} \end{aligned}$$

as desired.

27. [12] Circle Ω has radius 5. Points A and B lie on Ω such that chord AB has length 6. A unit circle ω is tangent to chord AB at point T . Given that ω is also internally tangent to Ω , find $AT \cdot BT$.

Answer: $\boxed{2}$

Solution: Let M be the midpoint of chord AB and let O be the center of Ω . Since $AM = BM = 3$, Pythagoras on triangle AMO gives $OM = 4$. Now let ω be centered at P and say that ω and Ω are tangent at Q . Because the diameter of ω exceeds 1, points P and Q lie on the same side of AB . By tangency, O, P , and Q are collinear, so that $OP = OQ - PQ = 4$. Let H be the orthogonal projection of P onto OM ; then $OH = OM - HM = OM - PT = 3$. Pythagoras on OHP gives $HP^2 = 7$. Finally,

$$AT \cdot BT = AM^2 - MT^2 = AM^2 - HP^2 = 9 - 7 = 2.$$

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28. [15] The vertices of a regular hexagon are labeled $\cos(\theta), \cos(2\theta), \dots, \cos(6\theta)$. For every pair of vertices, Bob draws a blue line through the vertices if one of these functions can be expressed as a polynomial function of the other (that holds for all real θ), and otherwise Roberta draws a red line through the

vertices. In the resulting graph, how many triangles whose vertices lie on the hexagon have at least one red and at least one blue edge?

Answer: 14 **Solution:** The existence of the Chebyshev polynomials, which express $\cos(n\theta)$ as a polynomial in $\cos(\theta)$, imply that Bob draws a blue line between $\cos(\theta)$ and each other vertex, and also between $\cos(2\theta)$ and $\cos(4\theta)$, between $\cos(2\theta)$ and $\cos(6\theta)$, and between $\cos(3\theta)$ and $\cos(6\theta)$ (by substituting $\theta' = 2\theta$ or 3θ as necessary). We now show that Roberta draws a red line through each other pair of vertices.

Let m and n be positive integers. Notice that $\cos(n\theta)$ is a periodic function with period $\frac{2\pi}{n}$, and $\cos(m\theta)$ is periodic with period $\frac{2\pi}{m}$. Thus, any polynomial in $\cos(m\theta)$ is also periodic of period $\frac{2\pi}{m}$. This may not be the minimum period of the polynomial, however, so the minimum period is $\frac{2\pi}{mk}$ for some k . Therefore, if $\cos(n\theta)$ can be expressed as a polynomial in $\cos(m\theta)$ then $\frac{2\pi}{n} = \frac{2\pi}{mk}$ for some k , so $m \mid n$. This shows that there is a blue line between two vertices $\cos(a\theta)$ and $\cos(b\theta)$ if and only if one of a or b divides the other.

Drawing the graph, one can easily count that there are 3 triangles with all blue edges, 3 triangles with all red edges, and $\binom{6}{3} = 20$ triangles total. Thus there are $20 - 3 - 3 = 14$ triangles having at least one red and at least one blue edge.

29. [15] The average of a set of distinct primes is 27. What is the largest prime that can be in this set?

Answer: 139 **Solution:** Denote the set of these primes by A and the number of elements in A by n . There are 9 primes smaller than 27, namely 2,3,5,7,11,13,17,19 and 23. Since 27 is odd and all primes except 2 are odd, $2 \notin A$. Thus the largest prime p is at most $27 \cdot 9 - 3 - 5 - 7 - 11 - 13 - 17 - 19 - 23 = 145$, so $p \leq 141$. When the primes are 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 and 139, their average is 27. Therefore $p = 139$.

30. [15] Let f be a polynomial with integer coefficients such that the greatest common divisor of all its coefficients is 1. For any $n \in \mathbb{N}$, $f(n)$ is a multiple of 85. Find the smallest possible degree of f .

Answer: 17 **Solution:** Notice that, if p is a prime and g is a polynomial with integer coefficients such that $g(n) \equiv 0 \pmod{p}$ for some n , then $g(n + mp)$ is divisible by p as well for any integer multiple mp of p . Therefore, it suffices to find the smallest possible degree of a polynomial f for which $f(0), f(1), f(2), \dots, f(16)$ are divisible by 17 and by 5.

There is a polynomial of degree 17 with integer coefficients having $f(0) = f(1) = \dots = f(16) = 0$, namely $f(x) = (x)(x-1)(x-2)\dots(x-16)$. Thus the minimal degree is no larger than 17.

Now, let f be such a polynomial and consider f modulo 17. The polynomial has 17 roots, so it must be at least degree 17 when taken modulo 17. Thus f has degree at least 17 as well.

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31. [18] How many ways are there to win tic-tac-toe in \mathbb{R}^n ? (That is, how many lines pass through three of the lattice points (a_1, \dots, a_n) in \mathbb{R}^n with each coordinate a_i in $\{1, 2, 3\}$?) Express your answer in terms of n .

Answer: $(5^n - 3^n)/2$ **Solution:** A line consists of three points. Each coordinate can do one of three things passing from the first point to the last point: increase by 1 each time, stay the same, or decrease by 1 each time. There are three ways to stay the same (three coordinates), one way to increase by 1, and one way to decrease by 1, so there are 5^n possible types of behavior. Determining this behavior uniquely determines the end point and start point except that we have traced every line exactly twice (forwards and backwards) and incorrectly counted the 3^n “lines” where each coordinate stays the same, so we subtract 3^n and divide by 2.

32. [18] Circle Ω has radius 13. Circle ω has radius 14 and its center P lies on the boundary of circle Ω . Points A and B lie on Ω such that chord AB has length 24 and is tangent to ω at point T . Find $AT \cdot BT$.

Answer: 56 **Solution:** Let M be the midpoint of chord AB ; then $AM = BM = 12$ and Pythagoras on triangle AMO gives $MO = 5$. Note that $\angle AOM = \angle AOB/2 = \angle APB = \angle APT + \angle TPB$ or $\tan(\angle AOM) = \tan(\angle APT + \angle TPB)$. Applying the tangent addition formula,

$$\begin{aligned} \frac{AM}{MO} &= \frac{\frac{AT}{TP} + \frac{BT}{TP}}{1 - \frac{AT}{TP} \cdot \frac{BT}{TP}} \\ &= \frac{AB \cdot TP}{TP^2 - AT \cdot BT}, \end{aligned}$$

from which $AT \cdot BT = TP^2 - AB \cdot TP \cdot MO/AM = 14^2 - 24 \cdot 14 \cdot 5/12 = 56$.

33. [18] Let m be a positive integer. Let $d(n)$ denote the number of divisors of n , and define the function

$$F(x) = \sum_{n=1}^{105^m} \frac{d(n)}{n^x}.$$

Define the numbers $a(n)$ to be the positive integers for which

$$F(x)^2 = \sum_{n=1}^{105^{2m}} \frac{a(n)}{n^x}$$

for all real x . Express $a(105^m)$ in terms of m .

Answer: $\left(\frac{m^3+6m^2+11m+6}{6}\right)^3$ OR $\binom{m+3}{3}^3$

(The expanded polynomial $\frac{1}{216}(216 + 1188m + 2826m^2 + 3815m^3 + 3222m^4 + 1767m^5 + 630m^6 + 141m^7 + 18m^8 + m^9)$ is also an acceptable answer.)

Solution: The denominator of a term in the expansion of $F(x)^2$ is equal to n^x if and only if it is a product of two terms of F of the form $\frac{d(n/k)}{(n/k)^x}$ and $\frac{d(k)}{k^x}$ for some divisor k of n . Thus $a(105^m) = \sum_{k|105^m} d(k)d(\frac{105^m}{k})$. We can write $k = 3^a 5^b 7^c$ with $a, b, c \leq m$ for any divisor k of 105^m , and in this case $d(k) = (a+1)(b+1)(c+1)$. Thus the sum becomes

$$\sum_{0 \leq a, b, c \leq m} (a+1)(b+1)(c+1)(m-a+1)(m-b+1)(m-c+1).$$

For a fixed b and c , we can factor out $(b+1)(c+1)(m-b+1)(m-c+1)$ from the terms having this b and c and find that the sum is equal to

$$\begin{aligned} a(105^m) &= \sum_{0 \leq b, c \leq m} (b+1)(c+1)(m-b+1)(m-c+1) \left(\sum_{a=1}^{m+1} a(m-a+2) \right) \\ &= \sum_{0 \leq b, c \leq m} (b+1)(c+1)(m-b+1)(m-c+1) \left((m+2) \frac{(m+1)(m+2)}{2} - \frac{(m+1)(m+2)(2m+3)}{6} \right) \\ &= \sum_{0 \leq b, c \leq m} (b+1)(c+1)(m-b+1)(m-c+1) \left(\frac{(3m+6-2m-3)(m+1)(m+2)}{6} \right) \\ &= \sum_{0 \leq b, c \leq m} (b+1)(c+1)(m-b+1)(m-c+1) \binom{m+3}{3} \end{aligned}$$

Fixing c and factoring out terms again, we find by a similar argument that $a(105^m) = \binom{m+3}{3}^3$.

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34. [≤ 25] **Descartes's Blackjack:** How many integer lattice points (points of the form (m, n) for integers m and n) lie inside or on the boundary of the disk of radius 2009 centered at the origin?

If your answer is higher than the correct answer, you will receive 0 points. If your answer is d less than the correct answer, your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$.

Answer:

35. [≤ 25] **Von Neumann's Poker:** The first step in Von Neumann's game is selecting a random number on $[0, 1]$. To generate this number, Chebby uses the factorial base: the number $0.A_1A_2A_3A_4\dots$ stands for $\sum_{n=0}^{\infty} \frac{A_n}{(n+1)!}$, where each A_n is an integer between 0 and n , inclusive.

Chebby has an infinite number of cards labeled $0, 1, 2, \dots$. He begins by putting cards 0 and 1 into a hat and drawing randomly to determine A_1 . The card assigned A_1 does not get reused. Chebby then adds in card 2 and draws for A_2 , and continues in this manner to determine the random number. At each step, he only draws one card from two in the hat.

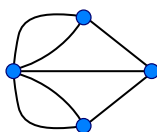
Unfortunately, this method does not result in a uniform distribution. What is the expected value of Chebby's final number?

Your score on this problem will be the larger of 0 and $\lfloor 25(1 - d) \rfloor$, where d is the positive difference between your answer and the correct answer.

Answer:

36. [≤ 25] **Euler's Bridge:** The following figure is the graph of the city of Königsburg in 1736 - vertices represent sections of the cities, edges are bridges. An *Eulerian path* through the graph is a path which moves from vertex to vertex, crossing each edge exactly once. How many ways could World War II bombers have knocked out some of the bridges of Königsburg such that the Allied victory parade could trace an Eulerian path through the graph? (The order in which the bridges are destroyed matters.)

Your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$, where d is the positive difference between your answer and the correct answer.



Answer:

12th Annual Harvard-MIT Math Tournament

Saturday 21 February 2009

Solutions: Team Round - Division A

1. [8] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n -gon is a set of $n - 3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n - 2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n - 3$ diagonals are the edges.

For a fixed n -gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

Solution: We will show that all triangulations have chromatic number 3, by induction on n . As a base case, if $n = 3$, a triangle has chromatic number 3. Now, given a triangulation of an n -gon for $n > 3$, every edge is either a side or a diagonal of the polygon. There are n sides and only $n - 3$ diagonals in the edge-set, so the Pigeonhole Principle guarantees a triangle with two side edges. These two sides must be adjacent, so we can remove this triangle to leave a triangulation of an $(n - 1)$ -gon, which has chromatic number 3 by the inductive hypothesis. Adding the last triangle adds only one new vertex with two neighbors, so we can color this vertex with one of the three colors not used on its neighbors.

2. (a) [6] Let P be a graph with one vertex v_n for each positive integer n . If $a < b$, then an edge connects vertices v_a and v_b if and only if $\frac{b}{a}$ is a prime number. What is the chromatic number of P ? Prove your answer.

Answer: $\boxed{2}$

Solution: At least two colors are needed in a good coloring of P . We show that two is sufficient. Write the positive integer n as $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, for distinct primes p_1, p_2, \dots, p_k , and let $f(n) = e_1 + e_2 + \dots + e_k$. Notice that if v_a and v_b are connected, then $f(a)$ and $f(b)$ have opposite parity. So, if we color v_n red if $f(n)$ is odd and blue otherwise, the two-coloring is good.

- (b) [6] Let T be a graph with one vertex v_n for every integer n . An edge connects v_a and v_b if $|a - b|$ is a power of two. What is the chromatic number of T ? Prove your answer.

Answer: $\boxed{3}$

Solution: Since v_0, v_1 , and v_2 are all connected to each other, three colors is necessary. Now, color v_n red if $n \equiv 0 \pmod{3}$, blue if $n \equiv 1 \pmod{3}$, and green otherwise. Since v_a and v_b are the same color only if $3|(a - b)$, no two connected vertices are the same color.

3. A graph is *finite* if it has a finite number of vertices.

- (a) [6] Let G be a finite graph in which every vertex has degree k . Prove that the chromatic number of G is at most $k + 1$.

Solution: We find a good coloring with $k + 1$ colors. Order the vertices and color them one by one. Since each vertex has at most k neighbors, one of the $k + 1$ colors has not been used on a neighbor, so there is always a good color for that vertex. In fact, we have shown that any graph in which every vertex has degree *at most* k can be colored with $k + 1$ colors.

- (b) [10] In terms of n , what is the minimum number of edges a finite graph with chromatic number n could have? Prove your answer.

Answer: $\boxed{\frac{n(n-1)}{2}}$

Solution: We prove this claim by induction - it holds for $n = 1$. Now assume the claim holds for n , and consider a graph of chromatic number $n + 1$. This graph must have at least one vertex of degree n , or else, by part a), it could be colored with only n colors.

Now, if we remove this vertex, the remaining graph must have chromatic number n or $n + 1$ - if the chromatic number is $n - 1$ or less, we can add the vertex back and give it a new color, creating a good coloring with only n colors. By the inductive hypothesis, the new graph has at least $\frac{n(n-1)}{2}$ edges, so the original graph had at least $\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$ edges.

The complete graph on $n + 1$ vertices has exactly $\frac{n(n+1)}{2}$ edges, so the lower bound is tight and the inductive step is complete.

4. A k -clique of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

- (a) [4] Find a graph with chromatic number 3 that does not contain any 3-cliques.

Solution: Consider a graph with 5 vertices arranged in a circle, with each vertex connected to its two neighbors. If only two colors are used, it is impossible to alternate colors to avoid using the same color on two adjacent vertices, so the chromatic number is 3.

- (b) [10] Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n -cliques.

Solution: We prove the claim by induction on n . The case $n = 3$ was addressed in (a). Let $n \geq 3$ and suppose G is a graph with chromatic number n containing no n -cliques. We produce a graph G' with chromatic number $n + 1$ containing no $(n + 1)$ -cliques as follows. Add a vertex v to G , and add an edge from v to each vertex of G .

To see this graph has chromatic number $n + 1$, observe that any coloring of the vertices of G' restricts to a valid coloring of the vertices of G . So at least n distinct colors must be used among the vertices of G . In addition, another color must be used for v . By coloring v a new color, we have constructed a coloring of G' having $n + 1$ colors.

Lastly, any $(n + 1)$ -clique in G' must have at least n vertices in G which form an n -clique, which is impossible. Therefore, G' has no $(n + 1)$ -cliques.

5. The *size* of a finite graph is the number of vertices in the graph.

- (a) [15] Show that, for any $n > 2$, and any positive integer N , there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

Solution: Let $k > 1$ be an odd number, and let G be a graph with k vertices arranged in a circle, with each vertex connected to its two neighbors. If $n = 3$, these graphs can be arbitrarily large, and are the graphs we need. If $n > 3$, let H be a complete graph on $n - 3$ vertices, and let J be the graph created by adding an edge from every vertex in G to every vertex in H . Then $n - 3$ colors are needed to color H and another 3 are

needed to color G , so n colors is both necessary and sufficient for a good coloring of J . Now, say a vertex is removed from J . There are two cases:

If the vertex was removed from G , then the remaining vertices in G can be colored with 2 colors, because the cycle has been broken. A set of $n - 3$ different colors can be used to color H , so only $n - 1$ colors are needed to color the reduced graph. On the other hand, if the vertex was removed from H , then $n - 4$ colors are used to color H and 3 used to color G . So removing any vertex decreases the chromatic number of J .

- (b) [15] Show that, for any positive integers n and r , there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n , it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n - 1$.

Solution: We claim that $N = nr$ is large enough. Take a graph with at least nr vertices and chromatic number n , and take a good n -coloring of the graph. Then by Pigeonhole, at least r of the vertices are the same color, which means that no pair of these r vertices is adjacent.

Remove this r vertices. If the resulting graph can be colored with only $n - 2$ colors, then we can add the r vertices back in and color them with a new $(n - 1)$ st color, creating a good coloring of the graph with only $n - 1$ colors. Since the original graph has chromatic number n , it must be impossible to color the smaller graph with $n - 2$ colors, so we have removed r vertices without decreasing the chromatic number by 2 or more.

6. For any set of graphs G_1, G_2, \dots, G_n all having the same set of vertices V , define their *overlap*, denoted $G_1 \cup G_2 \cup \dots \cup G_n$, to be the graph having vertex set V for which two vertices are adjacent in the overlap if and only if they are adjacent in at least one of the graphs G_i .

- (a) [10] Let G and H be graphs having the same vertex set and let a be the chromatic number of G and b the chromatic number of H . Find, in terms of a and b , the largest possible chromatic number of $G \cup H$. Prove your answer.

Answer: \boxed{ab}

Solution: First, we show that we can always color $G \cup H$ using ab colors. Given a good coloring of G in a colors c_1, \dots, c_a and a good coloring of H using b colors d_1, \dots, d_b , define ab new colors to be the ordered pairs (c_i, d_j) . Label a vertex of $G \cup H$ with the color (c_i, d_j) if it is colored c_i in G and d_j in H . This gives a good coloring of $G \cup H$.

Now, it only remains to find graphs G and H such that $G \cup H$ has chromatic number ab . Consider the complete graph K_{ab} having ab vertices v_1, \dots, v_{ab} (every pair of vertices is adjacent). Let G be the graph with vertices v_1, \dots, v_{ab} such that v_i is connected to v_j if and only if $i - j$ is a multiple of a . Also, let H be the graph on v_1, \dots, v_{ab} such that two vertices are adjacent if and only if they are not adjacent in G . Then $G \cup H = K_{ab}$. Note that G has chromatic number a since it is the disjoint union of b complete graphs on a vertices. Also, H has chromatic number b since we can color each set of vertices v_i with a color corresponding to i modulo b to obtain a good coloring. The chromatic number of $G \cup H$ is clearly ab , and so we have found such a pair of graphs.

- (b) [10] Suppose G is a graph with chromatic number n . Suppose there exist k graphs G_1, G_2, \dots, G_k having the same vertex set as G such that $G_1 \cup G_2 \cup \dots \cup G_k = G$ and each G_i has chromatic number at most 2. Show that $k \geq \lceil \log_2(n) \rceil$, and show that one can always find such a decomposition of G into $\lceil \log_2(n) \rceil$ graphs.

Solution: [NOTE: This problem differs from the problem statement in the test as administered at the 2009 HMMT. The reader is encouraged to try it before reading the solution.]

The bound on k follows from iterating part (a).

Let G be a graph with chromatic number n . Consider a coloring of G using n colors labeled $1, 2, \dots, n$. For i from 1 to $\lceil \log_2(n) \rceil$, define G_i to be the graph on the vertices of G for which two vertices are connected by an edge if and only if the i th digit from the right in the binary expansions of their colors do not match. Clearly each of the graphs G_i have chromatic number at most 2, by coloring each node with the i th digit of the binary expansion of their color in G . Moreover, each edge occurs in some G_i , since if two vertices match in every digit they are not connected by an edge. Therefore $G_1 \cup G_2 \cup \dots \cup G_{\lceil \log_2(n) \rceil} = G$, and so we have found such a decomposition of G .

7. [20] Let n be a positive integer. Let V_n be the set of all sequences of 0's and 1's of length n . Define G_n to be the graph having vertex set V_n , such that two sequences are adjacent in G_n if and only if they differ in either 1 or 2 places. For instance, if $n = 3$, the sequences $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ are mutually adjacent, but $(1, 0, 0)$ is not adjacent to $(0, 1, 1)$.

Show that, if $n + 1$ is not a power of 2, then the chromatic number of G_n is at least $n + 2$.

Solution: We will assume that there is a coloring with $n+1$ colors and derive a contradiction. For each string s , let T_s be the set consisting of all strings that differ from s in at most 1 place. Thus T_s has size $n + 1$ and all vertices in T_s are adjacent. In particular, if there is an $(n + 1)$ -coloring, then each color is used exactly once in T_s . Let c be one of the colors that we used. We will determine how many vertices are colored with c . We will do this by counting in two ways.

Let k be the number of vertices colored with color c . Each such vertex is part of T_s for exactly $n + 1$ values of s . On the other hand, each T_s contains exactly one vertex with color c . It follows that $k(n + 1) = 2^n$. In particular, since k is an integer, $n + 1$ divides 2^n . This is a contradiction since $n + 1$ is now a power of 2 by assumption, so actually there can be no $n + 1$ -coloring, as claimed.

8. [30] Two colorings are *distinct* if there is no way to relabel the colors to transform one into the other. Equivalently, they are distinct if and only if there is some pair of vertices which are the same color in one coloring but different colors in the other. For what pairs (n, k) of positive integers does there exist a finite graph with chromatic number n which has exactly k distinct good colorings using n colors?

Answer: $(1, 1), (2, 2^k)$ for integers $k \geq 0$, and (n, k) for $n > 2, k > 0$

Solution: If $n = 1$, there is only one coloring. If $n = 2$, then each connected component of the graph can be colored in two ways, because the color of any vertex in the graph determines the colors of all vertices connected to it. If the color scheme in one component is fixed, and there are k components, then there are 2^{k-1} ways to finish the coloring.

Now say $n > 2$. We construct a graph with k different colorings. We begin with a complete graph G on n vertices, which can be colored in exactly one way. Let v_1, v_2 , and v_3 be three vertices in the complete graph. If $k > 1$, add to the graph a row of vertices w_1, w_2, \dots, w_{k-1} , such that w_i is connected to w_{i+1} for $1 \leq k - 2$. Now, if $i \equiv 0 \pmod{3}$, connect w_i to all the

vertices in G except v_1 and v_2 . If $i \equiv 1 \pmod{3}$, connect w_i to all the vertices in G except v_2 and v_3 , and if $i \equiv 2 \pmod{3}$, connect w_i to all the vertices in G except v_1 and v_3 .

We need to show that this graph can be colored with n colors in exactly k different ways. Say that v_1 is colored red, v_2 blue, and v_3 green. Then each of the w_i can be colored one of exactly two colors. Further, there is exactly one possible color that w_i and w_{i+1} could both be. Call the color w_i and w_{i+1} could both be w_i 's *leading* color, and call w_i 's other color its *lagging* color. Notice that w_i 's lagging color is w_{i+1} 's leading color. So, if any w_i is colored with its lagging color, then all w_j with $j > i$ are also colored with their lagging colors.

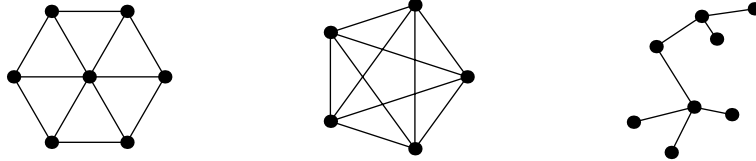
So one possibility is that all the w_i are colored with their leading colors. Otherwise, some of them are colored with their lagging colors - these colorings are completely defined by which one of the $k - 1$ w_i is the first vertex colored with its lagging color. So there are $1 + k - 1$ or k colorings of this graph, as needed.

12th Annual Harvard-MIT Math Tournament

Saturday 21 February 2009

Solutions: Team Round - Division B

1. [6] What are the chromatic numbers of each of the three graphs shown below? Draw a coloring having the minimum number of colors for each. Label the vertices with symbols to indicate the color of each vertex. (For example, you may mark a vertex “R” if you wish to indicate that it is colored red.)



Solution: The chromatic numbers of the three graphs are 3, 5, and 2, respectively.

2. [6] In a *connected* graph, it is possible to reach any vertex from any other vertex by following the edges. A *tree* is a connected graph with n vertices and $n - 1$ edges for some positive integer n . Suppose $n \geq 2$. What is the chromatic number of a tree having n vertices? Prove your answer.

Solution: The chromatic number of any tree is 2. We show this by induction on the size of the tree. A tree with 2 nodes can clearly be 2-colored. Now, suppose a tree of size $n - 1$ can be colored in 2 colors. Given a tree of size n , choose any *leaf* (node with only one edge coming out of it), say l , of the tree, and color it red. The node it is adjacent to, say x , must be colored a different color, say blue. Using the inductive hypothesis, we can 2-color the tree formed by removing l , and we can choose the color of x to be blue and the other color to be red. Thus the entire tree on n vertices can be two-colored, completing the induction.

3. [8] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n -gon is a set of $n - 3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n - 2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n - 3$ diagonals are the edges.

For a fixed n -gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

Solution: We will show that all triangulations have chromatic number 3, by induction on n . As a base case, if $n = 3$, a triangle has chromatic number 3. Now, given a triangulation of an n -gon for $n > 3$, every edge is either a side or a diagonal of the polygon. There are n sides and only $n - 3$ diagonals in the edge-set, so the Pigeonhole Principle guarantees a triangle with two side edges. These two sides must be adjacent, so we can remove this triangle to leave a triangulation of an $n - 1$ -gon, which has chromatic number 3 by the inductive hypothesis. Adding the last triangle adds only one new vertex with two neighbors, so we can color this vertex with one of the three colors not used on its neighbors.

4. [10] Let G be a finite graph in which every vertex has degree less than or equal to k . Prove that the chromatic number of G is less than or equal to $k + 1$.

Solution: Using a greedy algorithm we find a good coloring with $k + 1$ colors. Order the vertices and color them one by one - since each vertex has at most k neighbors, one of the $k + 1$ colors has not been used on a neighbor, so there is always a good color for that vertex. In fact, we have shown that any graph in which every vertex has degree *at most* k can be colored with $k + 1$ colors.

5. [10] A *k-clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent. The *clique number* of a graph is the size of the largest clique in the graph. Does there exist a graph which has a clique number smaller than its chromatic number?

Solution: Consider a graph with 5 vertices arranged in a circle, with each vertex connected to its two neighbors. If only two colors are used, it is impossible to alternate colors to avoid using the same color on two adjacent vertices, so the chromatic number is 3. Its clique number is 2, so we have found such a graph.

6. (a) [5] If a single vertex (and all its incident edges) is removed from a finite graph, show that the graph's chromatic number cannot decrease by more than 1.

Solution: Suppose the chromatic number of the graph was C , and removing a single vertex resulted in a graph with chromatic number at most $C - 2$. Then we can color the remaining graph with at most $C - 2$ colors. Replacing the vertex and its edges, we can then choose any color not already used to form a coloring of the original graph using at most $C - 1$ colors, contradicting the fact that C is the chromatic number of the graph.

- (b) [15] Show that, for any $n > 2$, there are infinitely many graphs with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

Solution: Let $k > 0$ be an odd number, and let G be a graph with k vertices arranged in a circle, with each vertex connected to its two neighbors. If $n = 3$, these graphs can be arbitrarily large, and are the graphs we need. If $n > 3$, let H be a complete graph on $n - 3$ vertices, and let J be the graph created by adding an edge from every vertex in G to every vertex in H . Then $n - 3$ colors are needed to color H and another 3 are needed to color G , so n colors is both necessary and sufficient for a good coloring of J . Now, say a vertex is removed from J . There are two cases:

If the vertex was removed from G , then the remaining vertices in G can be colored with 2 colors, because the cycle has been broken. A set of $n - 3$ different colors can be used to color H , so only $n - 1$ colors are needed to color the reduced graph. On the other hand, if the vertex was removed from H , then $n - 4$ colors are used to color H and 3 used to color G . So removing any vertex decreases the chromatic number of J .

2nd Annual Harvard-MIT November Tournament
Saturday 7 November 2009
General Test

1. [2] Evaluate the sum:

$$11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \dots + 20^2 - 10^2.$$

Answer: $\boxed{2100}$ This sum can be written as $\sum_{a=1}^{10} (a+10)^2 - a^2 = \sum_{a=1}^{10} 10(2a+10) = 10 * 10 * 11 + 10 * 10 * 10 = 2100$.

2. [3] Given that $a+b+c=5$ and that $1 \leq a, b, c \leq 2$, what is the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$?

Answer: $\boxed{\frac{4}{7}}$ If $a > 1$ and $b < 2$, we can decrease the sum by decreasing a and increasing b . You can follow a similar procedure if $c > 1$ and $b < 2$. Therefore, the sum is minimized when $b = 2$. We can then cross-multiply the two fractions and see that we are trying to minimize $\frac{a+c+4}{(a+2)(c+2)} = \frac{7}{(a+2)(c+2)}$. The product of two numbers with a fixed sum is maximized when those two numbers are equal, so $\frac{7}{(a+2)(c+2)}$ is minimized for $a = c = \frac{3}{2}$, which gives us an answer of $\frac{4}{7}$.

3. [3] What is the period of the function $f(x) = \cos(\cos(x))$?

Answer: $\boxed{\pi}$ Since $f(x)$ never equals $\cos(1)$ for $x \in (0, \pi)$ but $f(0) = \cos(1)$, the period is at least π . However, $\cos(x + \pi) = -\cos(x)$, so $\cos(\cos(x + \pi)) = \cos(\cos(x))$.

4. [4] How many subsets A of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ have the property that no two elements of A sum to 11?

Answer: $\boxed{243}$ For each element listed, there is exactly one other element such that the two elements sum to 11. Thus, we can list all the 10 numbers above as 5 pairs of numbers, such that each pair sums to 11. The problem then can be solved as follows: in any given subset with no two elements summing to 11, at most one element from each pair can be present. Thus, there are 3 ways in which each pair can contribute to a given subset (no element, the first element in the pair, or the second element in the pair). Since there are 5 pairs, the total number of ways to construct a subset with no two elements summing to 11 is $3^5 = 243$.

5. [5] A polyhedron has faces that are all either triangles or squares. No two square faces share an edge, and no two triangular faces share an edge. What is the ratio of the number of triangular faces to the number of square faces?

Answer: $\boxed{\frac{4}{3}}$ Let s be the number of square faces and t be the number of triangular faces. Every edge is adjacent to exactly one square face and one triangular face. Therefore, the number of edges is equal to $4s$, and it is also equal to $3t$. Thus $4s = 3t$ and $\frac{t}{s} = \frac{4}{3}$.

6. [5] Find the maximum value of $x + y$, given that $x^2 + y^2 - 3y - 1 = 0$.

Answer: $\boxed{\frac{\sqrt{26}+3}{2}}$ We can rewrite $x^2 + y^2 - 3y - 1 = 0$ as $x^2 + (y - \frac{3}{2})^2 = \frac{13}{4}$. We then see that the set of solutions to $x^2 - y^2 - 3y - 1 = 0$ is the circle of radius $\frac{\sqrt{13}}{2}$ and center $(0, \frac{3}{2})$. This can be written as $x = \frac{\sqrt{13}}{2} \cos(\theta)$ and $y = \frac{\sqrt{13}}{2} \sin(\theta) + \frac{3}{2}$. Thus, $x + y = \frac{3}{2} + \frac{\sqrt{13}}{2}(\cos(\theta) + \sin(\theta)) = \frac{3}{2} + \frac{\sqrt{13}}{2}\sqrt{2} \sin(\theta + 45^\circ)$, which is maximized for $\theta = 45^\circ$ and gives $\frac{\sqrt{26}+3}{2}$. (We could also solve this geometrically by noting that if $x + y$ attains a maximum value of s then the line $x + y = s$ is tangent to the circle.)

7. [6] There are 15 stones placed in a line. In how many ways can you mark 5 of these stones so that there are an odd number of stones between any two of the stones you marked?

Answer: $\boxed{77}$ Number the stones 1 through 15 in order. We note that the condition is equivalent to stipulating that the stones have either all odd numbers or all even numbers. There are $\binom{8}{5}$ ways to choose 5 odd-numbered stones, and $\binom{7}{5}$ ways to choose all even-numbered stones, so the total number

of ways to pick the stones is $\binom{8}{5} + \binom{7}{5} = 77$. ($\binom{n}{k}$ is the number of ways to choose k out of n items. It equals $\frac{n!}{k!(n-k)!}$).

8. [7] Let $\triangle ABC$ be an equilateral triangle with height 13, and let O be its center. Point X is chosen at random from all points inside $\triangle ABC$. Given that the circle of radius 1 centered at X lies entirely inside $\triangle ABC$, what is the probability that this circle contains O ?

Answer: $\frac{\sqrt{3}\pi}{100}$ The set of points X such that the circle of radius 1 centered at X lies entirely inside $\triangle ABC$ is itself a triangle, $A'B'C'$, such that AB is parallel to $A'B'$, BC is parallel to $B'C'$, and CA is parallel to $C'A'$, and furthermore AB and $A'B'$, BC and $B'C'$, and CA and $C'A'$ are all 1 unit apart. We can use this to calculate that $A'B'C'$ is an equilateral triangle with height 10, and hence has area $\frac{100}{\sqrt{3}}$. On the other hand, the set of points X such that the circle of radius 1 centered at X contains O is a circle of radius 1, centered at O , and hence has area π . The probability that the circle centered at X contains O given that it also lies in ABC is then the ratio of the two areas, that is, $\frac{\pi}{\frac{100}{\sqrt{3}}} = \frac{\sqrt{3}\pi}{100}$.

9. [7] A set of points is *convex* if the points are the vertices of a convex polygon (that is, a non-self-intersecting polygon with all angles less than or equal to 180°). Let S be the set of points (x, y) such that x and y are integers and $1 \leq x, y \leq 26$. Find the number of ways to choose a convex subset of S that contains exactly 98 points.

Answer: $\boxed{4958}$ For this problem, let $n = 26$. A convex set may be divided into four subsets: a set of points with maximal y coordinate, a set of points with minimal y coordinate, the points to the left of one of these subsets, and the points to the right of one of these subsets (the left, top, right, and bottom of the corresponding convex polygon). Each of these four parts contains at most n points. (All points in the top or bottom have distinct x coordinates while all points in the left or right have distinct y coordinates.) Moreover, there are four corners each of which is contained in two of these regions. This implies that at most $4n - 4$ distinct points are in any convex set. To find a set of size $4n - 6$ we can remove 2 additional points. Either exactly one of the top, bottom, left, or right contains exactly $n - 2$ points or some two of them each contain exactly $n - 1$ points.

Any of the $\binom{100}{98} = 4950$ sets of 98 points with either x or y coordinate either 1 or 26 have this property. Suppose instead that some of the points have x coordinate and y coordinate both different from 1 and from 26. In this case we can check that it is impossible for one side to have $n - 2$ points. If two opposite sides (top/bottom or left/right) have $n - 1$ points, then we obtain all the points on the boundary of an $n - 1$ by n rectangle (of which there are four). If two adjacent sides (any of the other pairs) have $n - 1$ points, then we obtain the points on the boundary of an n by n square with the points $(1, 1)$, $(1, 2)$, $(2, 1)$ missing and the point $(2, 2)$ added (or one of its rotations). There are an additional 4 such sets, for a total of 4958.

10. [8] Compute

$$\prod_{n=0}^{\infty} \left(1 - \left(\frac{1}{2}\right)^{3^n} + \left(\frac{1}{4}\right)^{3^n} \right).$$

Answer: $\frac{2}{3}$ We can rewrite each term as $\frac{1 + (\frac{1}{2})^{3^{n+1}}}{1 + (\frac{1}{2})^{3^n}}$. In the infinite product, each term of the form $1 + (\frac{1}{2})^{3^n}$ with $n > 0$ appears once in the numerator and once in the denominator. The only remaining term is $1 + (\frac{1}{2})^1$ in the first denominator.

2nd Annual Harvard-MIT November Tournament
Saturday 7 November 2009
Guts Round

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

1. [5] If $f(x) = x/(x + 1)$, what is $f(f(f(f(2009))))$?

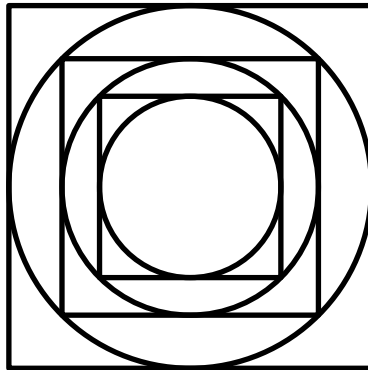
Answer: $\boxed{\frac{2009}{8037}}$ $f(f(x)) = \frac{(x/(x+1))}{(x/(x+1))+1} = x/2x + 1, f(f(f(f(x)))) = x/4x + 1 = \frac{2009}{8037}$

2. [5] A knight begins on the lower-left square of a standard chessboard. How many squares could the knight end up at after exactly 2009 legal knight's moves? (A knight's move is 2 squares either horizontally or vertically, followed by 1 square in a direction perpendicular to the first.)

Answer: $\boxed{32}$ The knight goes from a black square to a white square on every move, or vice versa, so after 2009 moves he must be on a square whose color is opposite of what he started on. So he can only land on half the squares after 2009 moves. Note that he can access any of the 32 squares (there are no other parity issues) because any single jump can also be accomplished in 3 jumps, so with 2009 jumps, he can land on any of the squares of the right color.

3. [5] Consider a square, inside which is inscribed a circle, inside which is inscribed a square, inside which is inscribed a circle, and so on, with the outermost square having side length 1. Find the difference between the sum of the areas of the squares and the sum of the areas of the circles.

Answer: $\boxed{2 - \frac{\pi}{2}}$



The ratio of the area of each square and the circle immediately inside it is $\frac{4}{\pi}$. The total sum of the areas of the squares is $1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$. Difference in area is then $2 - 2 \cdot \frac{4}{\pi}$.

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

4. [6] A cube has side length 1. Find the product of the lengths of the diagonals of this cube (a diagonal is a line between two vertices that is not an edge).

Answer: $\boxed{576}$ There are 12 diagonals that go along a face and 4 that go through the center of the cube, so the answer is $\sqrt{2}^{12} \cdot \sqrt{3}^4 = 576$.

5. [6] Tanks has a pile of 5 blue cards and 5 red cards. Every morning, he takes a card and throws it down a well. What is the probability that the first card he throws down and the last card he throws down are the same color?

Answer: $\boxed{\frac{4}{9}}$ Once he has thrown the first card down the well, there are 9 remaining cards, and only 4 have the same color as the card that was thrown down. Therefore, the probability that the last card he throws down has the same color is $\frac{4}{9}$.

6. [6] Find the last two digits of 1032^{1032} . Express your answer as a two-digit number.

Answer: $\boxed{76}$ The last two digits of 1032^{1032} is the same as the last two digits of 32^{1032} . The last two digits of 32^n repeat with a period of four as 32, 24, 68, 76, 32, 24, 68, 76, ...

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

7. [7] A *computer program* is a function that takes in 4 bits, where each bit is either a 0 or a 1, and outputs TRUE or FALSE. How many computer programs are there?

Answer: $\boxed{65536}$ The function has 2^4 inputs and 2 outputs for each possible input, so the answer is $2^{2^4} = 2^{16} = 65536$.

8. [7] The angles of a convex n -sided polygon form an arithmetic progression whose common difference (in degrees) is a non-zero integer. Find the largest possible value of n for which this is possible. (A polygon is convex if its interior angles are all less than 180° .)

Answer: $\boxed{27}$ The exterior angles form an arithmetic sequence too (since they are each 180° minus the corresponding interior angle). The sum of this sequence must be 360° . Let the smallest exterior angle be x and the common difference be d . The sum of the exterior angles is then $x + (x + d) + (x + 2d) + \dots + (x + (n-1)d) = \frac{n(n-1)}{2} \cdot d + nx$. Setting this to 360, and using $nx > 0$, we get $n(n-1) < 720$, so $n \leq 27$.

9. [7] Daniel wrote all the positive integers from 1 to n inclusive on a piece of paper. After careful observation, he realized that the sum of all the digits that he wrote was exactly 10,000. Find n .

Answer: $\boxed{799}$ Let $S(n)$ denote the sum of the digits of n , and let $f(x) = \sum_{n=0}^x S(n)$. (We may add $n = 0$ because $S(0) = 0$.) Observe that:

$$f(99) = \sum_{a=0}^9 \left(\sum_{b=0}^9 (a+b) \right) = 10 \sum_{b=0}^9 b + 10 \sum_{a=0}^9 a = 900$$

If a is an integer between 1 and 9 inclusive, then:

$$\sum_{n=100a}^{100a+99} S(n) = \sum_{n=100a}^{100a+99} (a + S(n - 100a)) = 100a + f(99) = 100a + 900$$

Summing, we get:

$$f(100a + 99) = \sum_{n=0}^a (100n + 900) = 900(a + 1) + 50a(a + 1)$$

This formula can be used to find benchmarks. However, it turns out that this formula alone will suffice, as things turn out rather nicely:

$$\begin{aligned} 900(a + 1) + 50a(a + 1) &= 10000 \\ 50a^2 + 950a + 900 &= 10000 \\ 50a^2 + 950a - 9100 &= 0 \\ 50(a + 26)(a - 7) &= 0 \\ a &= 7 \end{aligned}$$

Therefore $f(799) = 10000$, and our answer is 799.

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

10. [8] Admiral Ackbar needs to send a 5-character message through hyperspace to the Rebels. Each character is a lowercase letter, and the same letter may appear more than once in a message. When the message is beamed through hyperspace, the characters come out in a random order. Ackbar chooses his message so that the Rebels have at least a $\frac{1}{2}$ chance of getting the same message he sent. How many distinct messages could he send?

Answer: 26 If there is more than one distinct letter sent in the message, then there will be at most a $1/5$ chance of transmitting the right message. So the message must consist of one letter repeated five times, so there are 26 possible messages.

11. [8] Lily and Sarah are playing a game. They each choose a real number at random between -1 and 1. They then add the squares of their numbers together. If the result is greater than or equal to 1, Lily wins, and if the result is less than 1, Sarah wins. What is the probability that Sarah wins?

Answer: $\frac{\pi}{4}$ If we let x denote Lily's choice of number and y denote Sarah's, then all possible outcomes are represented by the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, -1)$, and $(1, 1)$. Sarah wins if $x^2 + y^2 \leq 1$, which is the area inside the unit circle. Since this has an area of π and the entire square has an area of 4, the probability that Sarah wins is $\frac{\pi}{4}$.

12. [8] Let ω be a circle of radius 1 centered at O . Let B be a point on ω , and let l be the line tangent to ω at B . Let A be on l such that $\angle AOB = 60^\circ$. Let C be the foot of the perpendicular from B to OA . Find the length of line segment OC .

Answer: $\frac{1}{2}$ We have $OC/OB = \cos(60^\circ)$. Since $OB = 1$, $OC = \frac{1}{2}$.

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

13. [8] 8 students are practicing for a math contest, and they divide into pairs to take a practice test. In how many ways can they be split up?

Answer: 105 We create the pairs one at a time. The first person has 7 possible partners. Set this pair aside. Of the remaining six people, pick a person. He or she has 5 possible partners. Set this pair aside. Of the remaining four people, pick a person. He or she has 3 possible partners. Set this pair aside. Then the last two must be partners. So there are $7 \cdot 5 \cdot 3 = 105$ possible groupings. Alternatively, we can consider the $8!$ permutations of the students in a line, where the first two are a pair, the next two are a pair, etc. Given a grouping, there are $4!$ ways to arrange the four pairs in order, and in each pair, 2 ways to order the students. So our answer is $\frac{8!}{4!2^4} = 7 \cdot 5 \cdot 3 = 105$.

14. [8] Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial whose roots are all negative integers. If $a + b + c + d = 2009$, find d .

Answer: 528 Call the roots $-x_1, -x_2, -x_3$, and $-x_4$. Then $f(x)$ must factor as $(x + x_1)(x + x_2)(x + x_3)(x + x_4)$. If we evaluate f at 1, we get $(1 + x_1)(1 + x_2)(1 + x_3)(1 + x_4) = a + b + c + d + 1 = 2009 + 1 = 2010$. $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. d is the product of the four roots, so $d = (-1) \cdot (-2) \cdot (-4) \cdot (-66)$.

15. [8] The curves $x^2 + y^2 = 36$ and $y = x^2 - 7$ intersect at four points. Find the sum of the squares of the x -coordinates of these points.

Answer: 26 If we use the system of equations to solve for y , we get $y^2 + y - 29 = 0$ (since $x^2 = y + 7$). The sum of the roots of this equation is -1 . Combine this with $x^2 = y + 7$ to see that the sum of the square of the possible values of x is $2 \cdot (-1 + 7 \cdot 2) = 26$.

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16. [9] Pick a random digit in the decimal expansion of $\frac{1}{99999}$. What is the probability that it is 0?

Answer: $\boxed{\frac{4}{5}}$ The decimal expansion of $\frac{1}{99999}$ is $0.\overline{00001}$.

17. [9] A circle passes through the points $(2, 0)$ and $(4, 0)$ and is tangent to the line $y = x$. Find the sum of all possible values for the y -coordinate of the center of the circle.

Answer: $\boxed{-6}$ First, we see that the x coordinate must be 3. Let the y coordinate be y . Now, we see that the radius is $r = \sqrt{1 + y^2}$. The line from the center of the circle to the point of tangency with the line $x = y$ is perpendicular to the line $x = y$. Hence, the distance from the center of the circle to the line $x = y$ is $d = \sqrt{2}(3 - y)$. Letting $r = d$ we see that the two possible values for y are 1 and -7 , which sum to -6 .

18. [9] Let f be a function that takes in a triple of integers and outputs a real number. Suppose that f satisfies the equations

$$\begin{aligned} f(a, b, c) &= \frac{f(a + 1, b, c) + f(a - 1, b, c)}{2} \\ f(a, b, c) &= \frac{f(a, b + 1, c) + f(a, b - 1, c)}{2} \\ f(a, b, c) &= \frac{f(a, b, c + 1) + f(a, b, c - 1)}{2} \end{aligned}$$

for all integers a, b, c . What is the minimum number of triples at which we need to evaluate f in order to know its value everywhere?

Answer: $\boxed{8}$ Note that if we have the value of f at the 8 points: $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)$, we can calculate the value for any triple of points because we have that $f(a + 1, b, c) - f(a, b, c)$ constant for any a , if b and c are fixed (and similarly for the other coordinates). To see why we cannot do this with less points, notice that we need to determine what the value of these 8 points anyways, and there is no "more efficient" way to determine them all in fewer evaluations.

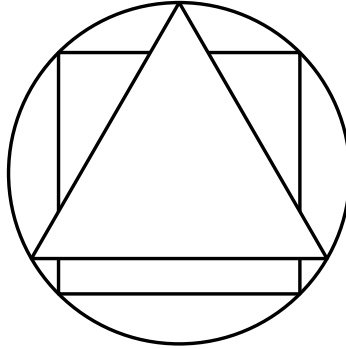
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19. [11] You are trapped in ancient Japan, and a giant enemy crab is approaching! You must defeat it by cutting off its two claws and six legs and attacking its weak point for massive damage. You cannot cut off any of its claws until you cut off at least three of its legs, and you cannot attack its weak point until you have cut off all of its claws and legs. In how many ways can you defeat the giant enemy crab? (Note that the legs are distinguishable, as are the claws.)

Answer: $\boxed{14400}$ The answer is given by $6!2\binom{5}{2}$, because we can cut off the claws and legs in any order and there are $\binom{5}{2}$ ways to decide when to cut off the two claws (since we can do it at any time among the last 5 cuts).

20. [11] Consider an equilateral triangle and a square both inscribed in a unit circle such that one side of the square is parallel to one side of the triangle. Compute the area of the convex heptagon formed by the vertices of both the triangle and the square.

Answer: $\boxed{\frac{3+\sqrt{3}}{2}}$



Consider the diagram above. We see that the shape is a square plus 3 triangles. The top and bottom triangles have base $\sqrt{2}$ and height $\frac{1}{2}(\sqrt{3} - \sqrt{2})$, and the triangle on the side has the same base and height $1 - \frac{\sqrt{2}}{2}$. Adding their areas, we get the answer.

21. [11] Let $f(x) = x^2 + 2x + 1$. Let $g(x) = f(f(\cdots f(x)))$, where there are 2009 f s in the expression for $g(x)$. Then $g(x)$ can be written as

$$g(x) = x^{2^{2009}} + a_{2^{2009}-1}x^{2^{2009}-1} + \cdots + a_1x + a_0,$$

where the a_i are constants. Compute $a_{2^{2009}-1}$.

Answer: $\boxed{2^{2009}}$ $f(x) = (x+1)^2$, so $f(x^n + cx^{n-1} + \cdots) = (x^n + cx^{n-1} + \cdots + 1)^2 = x^{2n} + 2cx^{2n-1} + \cdots$. Applying the preceding formula repeatedly shows us that the coefficient of the term of second highest degree in the polynomial doubles each time, so after 2009 applications of f it is 2^{2009} .

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22. [12] Five cards labeled A, B, C, D, and E are placed consecutively in a row. How many ways can they be re-arranged so that no card is moved more than one position away from where it started? (Not moving the cards at all counts as a valid re-arrangement.)

Answer: $\boxed{8}$ The only things we can do is leave cards where they are or switch them with adjacent cards. There is 1 way to leave them all where they are, 4 ways to switch just one adjacent pair, and 3 ways to switch two different adjacent pairs, for 8 possibilities total.

23. [12] Let a_0, a_1, \dots be a sequence such that $a_0 = 3$, $a_1 = 2$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Find

$$\sum_{n=0}^8 \frac{a_n}{a_{n+1}a_{n+2}}.$$

Answer: $\boxed{\frac{105}{212}}$ We can re-write $\frac{a_n}{a_{n+1}a_{n+2}}$ as $\frac{a_{n+2} - a_{n+1}}{a_{n+1}a_{n+2}} = \frac{1}{a_{n+1}} - \frac{1}{a_{n+2}}$. We can thus re-write the sum as

$$\left(\frac{1}{a_1} - \frac{1}{a_2}\right) + \left(\frac{1}{a_2} - \frac{1}{a_3}\right) + \left(\frac{1}{a_3} - \frac{1}{a_4}\right) + \cdots + \left(\frac{1}{a_9} - \frac{1}{a_{10}}\right) = \frac{1}{a_1} - \frac{1}{a_{10}} = \frac{1}{2} - \frac{1}{212} = \frac{105}{212}.$$

24. [12] Penta chooses 5 of the vertices of a unit cube. What is the maximum possible volume of the figure whose vertices are the 5 chosen points?

Answer: $\boxed{\frac{1}{2}}$ Label the vertices of the cube A, B, C, D, E, F, G, H , such that $ABCD$ is the top face of the cube, E is directly below A , F is directly below B , G is directly below C , and H is directly

below D . We can obtain a volume of $\frac{1}{2}$ by taking the vertices A, B, C, F , and H . To compute the volume of $ACBFH$, we will instead compute the volume of the parts of the cube that are not part of $ABCFH$. This is just the three tetrahedrons $CFGH$, $AEFH$, and $ACDH$, which each have volume $\frac{1}{6}$ (by using the $\frac{1}{3}bh$ formula for the area of a pyramid). Therefore, the volume not contained in $ACBFH$ is $3 \cdot \frac{1}{6} = \frac{1}{2}$, so the volume contained in $ACBFH$ is $1 - \frac{1}{2} = \frac{1}{2}$.

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25. [14] Find all solutions to $x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$ (including non-real solutions).

Answer: $\boxed{-1, i, -i}$ We can factor the polynomial as $(x + 1)^2(x^2 + 1)$.

26. [14] In how many ways can the positive integers from 1 to 100 be arranged in a circle such that the sum of every two integers placed opposite each other is the same? (Arrangements that are rotations of each other count as the same.) Express your answer in the form $a! \cdot b^c$.

Answer: $\boxed{49! \cdot 2^{49}}$ Split the integers up into pairs of the form $(x, 101 - x)$. In the top half of the circle, exactly one element from each pair occurs, and there are thus $50!$ ways to arrange them. and also 2^{50} ways to decide whether the larger or smaller number in each pair occurs in the top half of the circle. We then need to divide by 100 since rotations are not considered distinct, so we get $\frac{50!2^{50}}{100} = 49! \cdot 2^{49}$.

27. [14] $ABCD$ is a regular tetrahedron of volume 1. Maria glues regular tetrahedra $A'BCD$, $AB'CD$, $ABC'D$, and $ABCD'$ to the faces of $ABCD$. What is the volume of the tetrahedron $A'B'C'D'$?

Answer: $\boxed{\frac{125}{27}}$ Consider the tetrahedron with vertices at $W = (1, 0, 0)$, $X = (0, 1, 0)$, $Y = (0, 0, 1)$, and $Z = (1, 1, 1)$. This tetrahedron is similar to $ABCD$. It has center $O = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. We can construct a tetrahedron $W'X'Y'Z'$ in the same way that $A'B'C'D'$ was constructed by letting W' be the reflection of W across XYZ and so forth. Then we see that $Z' = (-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, so OZ' has length $\frac{5}{6}\sqrt{3}$, whereas OZ has length $\frac{1}{2}\sqrt{3}$. We thus see that $W'X'Y'Z'$ has a side length $\frac{\frac{5}{6}\sqrt{3}}{\frac{1}{2}\sqrt{3}} = \frac{5}{3}$ that of $WXYZ$, so by similarity the same is true of $A'B'C'D'$ and $ABCD$. In particular, the volume of $A'B'C'D'$ is $(\frac{5}{3})^3$ that of $ABCD$, so it is $\frac{125}{27}$.

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28. [17] Six men and their wives are sitting at a round table with 12 seats. These men and women are very jealous — no man will allow his wife to sit next to any man except for himself, and no woman will allow her husband to sit next to any woman except for herself. In how many distinct ways can these 12 people be seated such that these conditions are satisfied? (Rotations of a valid seating are considered distinct.)

Answer: $\boxed{288000}$ Think of this problem in terms of “blocks” of men and women, that is, groups of men and women sitting together. Each block must contain at least two people; otherwise you have a man sitting next to two women (or vice-versa).

We will define the notation $[a_1, b_1, a_2, b_2, \dots]$ to mean a seating arrangement consisting of, going in order, a_1 men, b_1 women, a_2 men, b_2 women, and so on.

Split the problem into three cases, each based on the number of blocks of men and women:

Case 1: One block of each, $[6, 6]$.

There are 12 ways to choose the seats where the men sit, and $6!$ ways to arrange those men. The two women on either side of the men are uniquely determined by the men they sit next to. There are $4!$ ways to arrange the other four women. This gives $6! \cdot 288$ ways.

Case 2: Two blocks of each.

The arrangement is $[a, b, c, d]$, where $a + c = b + d = 6$. There are five distinct block schematics: $[2, 2, 4, 4]$, $[2, 3, 4, 3]$, $[2, 4, 4, 2]$, $[3, 2, 3, 4]$, and $[3, 3, 3, 3]$. (The others are rotations of one of the above.)

For each of these, there are $6!$ ways to arrange the men. In addition, four women are uniquely determined because they sit next to a man. There are 2 ways to arrange the other two women. Each of the first four possible block schematics gives 12 distinct rotations, while the fifth one gives 6. This gives $6!(2)(12 + 12 + 12 + 12 + 6) = 6! \cdot 108$ ways.

Case 3: Three blocks of each, $[2, 2, 2, 2, 2, 2]$.

There are 4 ways to choose where the men sit and $6!$ ways to arrange those men. Each placement of men will uniquely determine the placement of each women. This gives $6! \cdot 4$ ways.

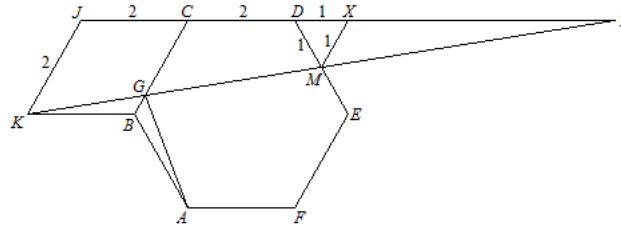
Then we have a grand total of $6! \cdot (288 + 108 + 4) = 6! \cdot 400 = 288000$ seating arrangements.

29. [17] For how many integer values of b does there exist a polynomial function with integer coefficients such that $f(2) = 2010$ and $f(b) = 8$?

Answer: 32 We can take $f(x) = -\frac{2002}{d}(x - b) + 2010$ for all divisors d of -2002 . To see that we can't get any others, note that $b - 2$ must divide $f(b) - f(2)$, so $b - 2$ divides -2002 (this is because $b - 2$ divides $b^n - 2^n$ and hence any sum of numbers of the form $b^n - 2^n$).

30. [17] Regular hexagon $ABCDEF$ has side length 2. A laser beam is fired inside the hexagon from point A and hits \overline{BC} at point G . The laser then reflects off \overline{BC} and hits the midpoint of \overline{DE} . Find BG .

Answer: $\frac{2}{5}$ Look at the diagram below, in which points $J, K, M, T,$ and X have been defined. M is the midpoint of \overline{DE} , $BCJK$ is a rhombus with J lying on the extension of \overline{CD} , T is the intersection of lines \overline{CD} and \overline{GM} when extended, and X is on \overline{JT} such that $\overline{XM} \parallel \overline{JK}$.



It can be shown that $m\angle MDX = m\angle MXD = 60^\circ$, so $\triangle DMX$ is equilateral, which yields $XM = 1$. The diagram indicates that $JX = 5$. One can show by angle-angle similarity that $\triangle TXM \sim \triangle TJK$, which yields $TX = 5$.

One can also show by angle-angle similarity that $\triangle TJK \sim \triangle TCG$, which yields the proportion $\frac{TJ}{JK} = \frac{TC}{CG}$. We know everything except CG , which we can solve for. This yields $CG = \frac{8}{5}$, so $BG = \frac{2}{5}$.

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31. [20] There are two buildings facing each other, each 5 stories high. How many ways can Kevin string ziplines between the buildings so that:
- (a) each zipline starts and ends in the middle of a floor.
 - (b) ziplines can go up, stay flat, or go down, but can't touch each other (this includes touching at their endpoints).

Note that you can't string a zipline between two floors of the same building.

Answer: 252 Associate with each configuration of ziplines a path in the plane as follows: Suppose there are k ziplines. Let a_0, \dots, a_k be the distances between consecutive ziplines on the left building

(a_0 is the floor on which the first zipline starts, and a_k is the distance from the last zipline to the top of the building). Define b_0, \dots, b_k analogously for the right building. The path in the plane consists of starting at $(0, 0)$ and going a distance a_0 to the right, b_0 up, a_1 to the right, b_1 up, etc. We thus go from $(0, 0)$ to $(5, 5)$ while traveling only up and to the right between integer coordinates. We can check that there is exactly one configuration of ziplines for each such path, so the answer is the number of paths from $(0, 0)$ to $(5, 5)$ where you only travel up and to the right. This is equal to $\binom{10}{5} = 252$, since there are 10 total steps to make, and we must choose which 5 of them go to the right.

32. [20] A circle ω_1 of radius 15 intersects a circle ω_2 of radius 13 at points P and Q . Point A is on line PQ such that P is between A and Q . R and S are the points of tangency from A to ω_1 and ω_2 , respectively, such that the line AS does not intersect ω_1 and the line AR does not intersect ω_2 . If $PQ = 24$ and $\angle RAS$ has a measure of 90° , compute the length of AR .

Answer: $\boxed{14 + \sqrt{97}}$ Let O_1 be the center of ω_1 and O_2 be the center of ω_2 . Then O_1O_2 and PQ are perpendicular. Let their point of intersection be X . Using the Pythagorean theorem, the fact that $PQ = 24$, and our knowledge of the radii of the circles, we can compute that $O_1X = 9$ and $O_2X = 5$, so $O_1O_2 = 14$. Let SO_1 and RO_2 meet at Y . Then $SARY$ is a square, say of side length s . Then $O_1Y = s - 15$ and $O_2Y = s - 13$. So, O_1O_2Y is a right triangle with sides 14, $s - 15$, and $s - 13$. By the Pythagorean theorem, $(s - 13)^2 + (s - 15)^2 = 14^2$. We can write this as $2s^2 - 4 \cdot 14s + 198 = 0$, or $s^2 - 28s + 99 = 0$. The quadratic formula then gives $s = \frac{28 \pm \sqrt{388}}{2} = 14 \pm \sqrt{97}$. Since $14 - \sqrt{97} < 15$ and $YO_1 > 15$, we can discard the root of $14 - \sqrt{97}$, and the answer is therefore $14 + \sqrt{97}$.

33. [20] Compute

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}}.$$

Note that $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$.

Answer: $\boxed{\frac{2009}{2008}}$ Observe that

$$\begin{aligned} \frac{k+1}{k} \left(\frac{1}{\binom{n-1}{k}} - \frac{1}{\binom{n}{k}} \right) &= \frac{k+1}{k} \frac{\binom{n}{k} - \binom{n-1}{k}}{\binom{n}{k} \binom{n-1}{k}} \\ &= \frac{k+1}{k} \frac{\binom{n-1}{k-1}}{\binom{n}{k} \binom{n-1}{k}} \\ &= \frac{k+1}{k} \frac{(n-1)!k!(n-k-1)!(n-k)!}{n!(n-1)!(k-1)!(n-k)!} \\ &= \frac{k+1}{k} \frac{k \cdot k!(n-k-1)!}{n!} \\ &= \frac{(k+1)!(n-k-1)!}{n!} \\ &= \frac{1}{\binom{n}{k+1}} \end{aligned}$$

Now apply this with $k = 2008$ and sum across all n from 2009 to ∞ . We get

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}} = \frac{2009}{2008} \sum_{n=2009}^{\infty} \frac{1}{\binom{n-1}{2008}} - \frac{1}{\binom{2008}{2008}}.$$

All terms from the sum on the right-hand-side cancel, except for the initial $\frac{1}{\binom{2008}{2008}}$, which is equal to 1, so we get $\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}} = \frac{2009}{2008}$.

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34. [25] How many hits does “3.1415” get on Google? Quotes are for clarity only, and not part of the search phrase. Also note that Google does not search substrings, so a webpage with 3.14159 on it will not match 3.1415. If A is your answer, and S is the correct answer, then you will get $\max(25 - |\ln(A) - \ln(S)|, 0)$ points, rounded to the nearest integer.

Answer:

35. [25] Call an integer $n > 1$ *radical* if $2^n - 1$ is prime. What is the 20th smallest radical number? If A is your answer, and S is the correct answer, you will get $\max\left(25\left(1 - \frac{|A-S|}{S}\right), 0\right)$ points, rounded to the nearest integer.

Answer:

36. [25] Write down a pair of integers (a, b) , where $-100000 < a < b < 100000$. You will get $\max(25, k)$ points, where k is the number of other teams' pairs that you interleave. (Two pairs (a, b) and (c, d) of integers *interleave* each other if $a < c < b < d$ or $c < a < d < b$.)

2nd Annual Harvard-MIT November Tournament
Saturday 7 November 2009
Team Round

Down the Infinite Corridor

Consider an isosceles triangle T with base 10 and height 12. Define a sequence $\omega_1, \omega_2, \dots$ of circles such that ω_1 is the incircle of T and ω_{i+1} is tangent to ω_i and both legs of the isosceles triangle for $i > 1$.

1. [3] Find the radius of ω_1 .

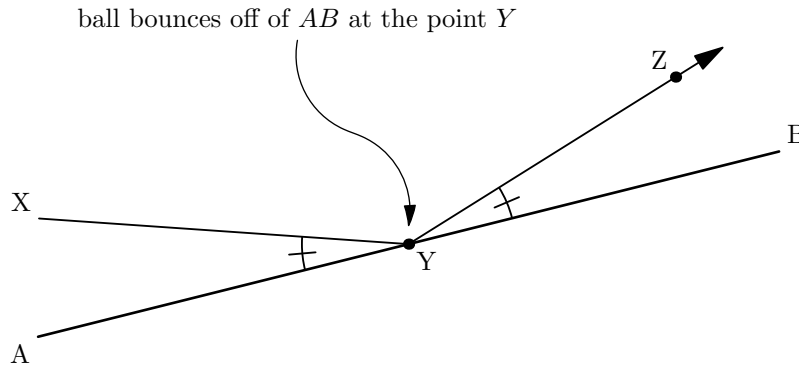
Answer: $\boxed{\frac{10}{3}}$ Using the Pythagorean theorem, we see that the legs of T each have length 13. Let r be the radius of ω_1 . We can divide T into three triangles, each with two vertices at vertices of T and one vertex at the center of ω_1 . These triangles all have height r and have bases 13, 13, and 10. Thus their total area is $\frac{13r}{2} + \frac{13r}{2} + \frac{10r}{2} = 18r$. However, T has height 12 and base 10, so its area is 60. Thus $18r = 60$, so $r = \frac{10}{3}$.

2. [3] Find the ratio of the radius of ω_{i+1} to the radius of ω_i .
 3. [3] Find the total area contained in all the circles.

Answer: $\boxed{\frac{180\pi}{13}}$ Using the notation from the previous solution, the area contained in the i th circle is equal to πr_i^2 . Since the radii form a geometric sequence, the areas do as well. Specifically, the areas form a sequence with initial term $\pi \cdot \frac{100}{9}$ and common ratio $\frac{16}{81}$, so their sum is then $\pi \cdot \frac{100}{\frac{65}{81}} = \frac{180\pi}{13}$.

Bouncy Balls

In the following problems, you will consider the trajectories of balls moving and bouncing off of the boundaries of various containers. The balls are small enough that you can treat them as points. Let us suppose that a ball starts at a point X , strikes a boundary (indicated by the line segment AB) at Y , and then continues, moving along the ray YZ . Balls always bounce in such a way that $\angle XYA = \angle BYZ$. This is indicated in the above diagram.



Balls bounce off of boundaries in the same way light reflects off of mirrors - if the ball hits the boundary at point P , the trajectory after P is the reflection of the trajectory before P through the perpendicular to the boundary at P .

A ball inside a rectangular container of width 7 and height 12 is launched from the lower-left vertex of the container. It first strikes the right side of the container after traveling a distance of $\sqrt{53}$ (and strikes no other sides between its launch and its impact with the right side).

4. [2] Find the height at which the ball first contacts the right side.

Answer: [2] Let h be this height. Then, using the Pythagorean theorem, we see that $h^2 + 7^2 = 53$, so $h = 2$.

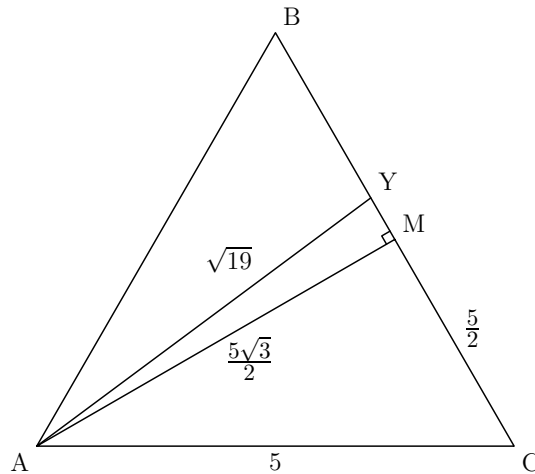
5. [3] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Answer: [5] Every segment the ball traverses between bounces takes it 7 units horizontally and 2 units up. Thus, after 5 bounces it has traveled up 10 units, and the final segment traversed takes it directly to the upper right vertex of the rectangle.

Now a ball is launched from a vertex of an equilateral triangle with side length 5. It strikes the opposite side after traveling a distance of $\sqrt{19}$.

6. [4] Find the distance from the ball's point of first contact with a wall to the nearest vertex.

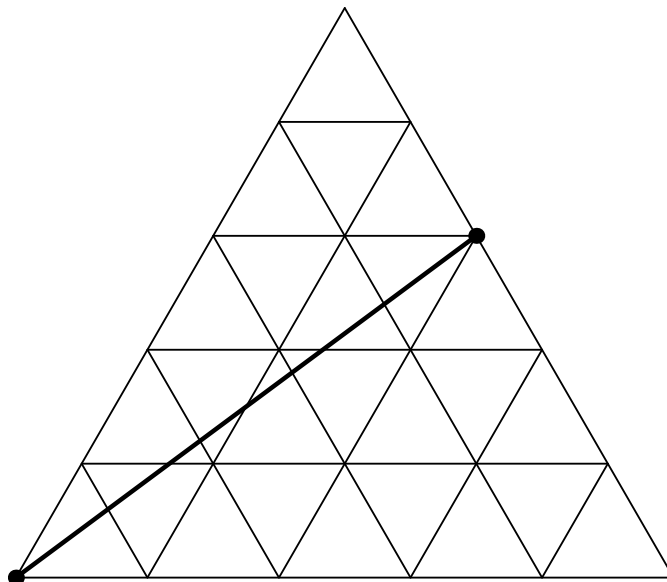
Answer: [2]



Consider the diagram above, where M is the midpoint of BC . Then AM is perpendicular to BC since ABC is equilateral, so by the Pythagorean theorem $AM = \frac{5\sqrt{3}}{2}$. Then, using the Pythagorean theorem again, we see that $MY = \frac{1}{2}$, so that $BY = 2$.

7. [4] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Answer: [7] The key idea is that, instead of reflecting the line AY off of BC , we will reflect ABC about BC and extend AY beyond $\triangle ABC$. We keep doing this until the extension of AY hits a vertex of one of our reflected triangles. This is illustrated in the diagram below:



We can calculate that the line AY has slope $\frac{\frac{3\sqrt{3}}{2}}{\frac{3}{2}} = \frac{3\sqrt{3}}{7}$, so that (as indicated in the diagram), AY first intersects a vertex at the point $\left(\frac{35}{2}, \frac{15\sqrt{3}}{2}\right)$. To get there, it has to travel through 2 horizontal lines, 1 upward sloping line, and 4 downward sloping lines, so it bounces $2 + 1 + 4 = 7$ times total.

In this final problem, a ball is again launched from the vertex of an equilateral triangle with side length 5.

8. [6] In how many ways can the ball be launched so that it will return again to a vertex for the first time after 2009 bounces?

Answer: [502] We will use the same idea as in the previous problem. We first note that every vertex of a triangle can be written uniquely in the form $a(5, 0) + b\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$, where a and b are non-negative integers. Furthermore, if a ball ends at $a(5, 0) + b\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$, then it bounces off of a wall $2(a + b) - 3$ times. Therefore, the possible directions that you can launch the ball in correspond to solutions to $2(a + b) - 3 = 2009$, or $a + b = 1006$. However, if a and b have a common factor, say, k , then the ball will pass through the vertex corresponding to $\frac{a}{k}$ and $\frac{b}{k}$ before it passes through the vertex corresponding to a and b . Therefore, we must discount all such pairs a, b . This corresponds to when a is even or $a = 503$, so after removing these we are left with 502 remaining possible values of a , hence 502 possible directions in which to launch the ball.

Super Mario 64!

Mario is once again on a quest to save Princess Peach. Mario enters Peach's castle and finds himself in a room with 4 doors. This room is the first in a sequence of 2 indistinguishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowser's level), while the other 3 doors lead to the first room.

9. [3] Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Mario's initial entrance to the first room) through which Mario will pass before he reaches Bowser's level?

Answer: [20] Let E_i be the expected number of doors through which Mario will pass in the future if he is currently in room i for $i = 1, 2, 3$ (we will set $E_3 = 0$). We claim that $E_i = 1 + \frac{3}{4}E_1 + \frac{1}{4}E_{i+1}$. Indeed, the 1 at the beginning comes from the fact that we need to pass through a door to leave the room, the $\frac{3}{4}E_1$ comes from the fact that there is a $\frac{3}{4}$ chance of ending up in room 1, and the $\frac{1}{4}E_{i+1}$ corresponds to the fact that there is a $\frac{1}{4}$ chance of ending up in E_{i+1} . Using this, we get $E_1 = 1 + \frac{3}{4}E_1 + \frac{1}{4}E_2$, or $E_1 = 4 + E_2$. We also get $E_2 = 1 + \frac{3}{4}E_1$. Solving this system of equations yields $E_1 = 20$.

10. [4] Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowser's level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

Answer: $\boxed{5460}$ This problem works in the same general way as the last problem, but it can be more succinctly solved using the general formula, which is provided below in the solution to the next problem.

11. [5] In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowser's level, what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

Answer: $\boxed{\frac{d(d^r - 1)}{d - 1}}$ Let E_i be the expected number of doors through which Mario will pass in the future if he is currently in room i for $i = 1, 2, \dots, r + 1$ (we will set $E_{r+1} = 0$). We claim that $E_i = 1 + \frac{d-1}{d}E_1 + \frac{1}{d}E_{i+1}$. This is because, as before, there is a $\frac{d-1}{d}$ chance of ending up in room 1, and a $\frac{1}{d}$ chance of ending up in room $i + 1$. Note that we can re-write this equation as $dE_i = d + (d - 1)E_1 + E_{i+1}$.

We can solve this system of equation as follows: let $E_1 = E_i + c_i$. Then we can re-write each equation as $dE_1 - d \cdot c_i = d + (d - 1)E_1 + E_1 - c_{i+1}$, so $c_{i+1} = d \cdot c_i + d$, and $c_1 = 0$. We can then see that $c_i = d \frac{d^{i-1} - 1}{d - 1}$ (either by finding the pattern or by using more advanced techniques, such as those given at http://en.wikipedia.org/wiki/Recurrence_relation). Solving for E_1 using $E_1 = E_r + c_r$ and $E_r = 1 + \frac{d-1}{d}E_1$, we get $E_1 = \frac{d(d^r - 1)}{d - 1}$.

2nd Annual Harvard-MIT November Tournament
Saturday 7 November 2009
Theme Round

Shortest Paths

1. [3] Paul starts with the number 19. In one step, he can add 1 to his number, divide his number by 2, or divide his number by 3. What is the minimum number of steps Paul needs to get to 1?

Answer: 6 One possible path is $19 \rightarrow 20 \rightarrow 10 \rightarrow 5 \rightarrow 6 \rightarrow 2 \rightarrow 1$.

2. [4] You start with a number. Every second, you can add or subtract any number of the form $n!$ to your current number to get a new number. In how many ways can you get from 0 to 100 in 4 seconds? ($n!$ is defined as $n \times (n - 1) \times (n - 2) \times \dots \times 2 \times 1$, so $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc.)

Answer: 36 To get to 100, you have to use one number which is at least $5! = 120$, because $24 \times 4 = 96$, which is less than 100. If you use $6! = 720$ or anything larger, you need to get back from 720 to 100 (or further) in three seconds. Since $3 \cdot 5! < 620$, there is no way to do this in 3 seconds. This means you have to use $5!$ at least once. The remaining numbers must get you from 120 to 100. If you use three numbers all at most $3!$, you can move by at most $3 \cdot 3! = 18 < 120 - 100$. This means you have to use $4!$. From $120 - 24 = 96$, there are two ways to get to 100: adding 6 then subtracting 2, or adding 2 twice. So, to get to 100 from 0 in four seconds, you must either add 120, subtract 24, add 6, and subtract 2, or add 120, subtract 24, and add 2 twice. You can do these steps in any order, so the first sequence yields 24 paths and the second sequence yields 12.

3. [5] Let C be the circle of radius 12 centered at $(0, 0)$. What is the length of the shortest path in the plane between $(8\sqrt{3}, 0)$ and $(0, 12\sqrt{2})$ that does not pass through the interior of C ?

Answer: $12 + 4\sqrt{3} + \pi$ The shortest path consists of a tangent to the circle, a circular arc, and then another tangent. The first tangent, from $(8\sqrt{3}, 0)$ to the circle, has length $4\sqrt{3}$, because it is a leg of a 30-60-90 right triangle. The 15° arc has length $\frac{15}{360}(24\pi)$, or π , and the final tangent, to $(0, 12\sqrt{2})$, has length 12.

4. [6] You are given a 5×6 checkerboard with squares alternately shaded black and white. The bottom-left square is white. Each square has side length 1 unit. You can normally travel on this board at a speed of 2 units per second, but while you travel through the interior (not the boundary) of a black square, you are slowed down to 1 unit per second. What is the shortest time it takes to travel from the bottom-left corner to the top-right corner of the board?

Answer: $\frac{1+5\sqrt{2}}{2}$ It is always faster to take a path around a black square than through it, since the length of the hypotenuse of any right triangle is greater than half the sum of the length of its legs. Therefore, an optimal path always stays on white squares or on boundaries, and the shortest such path has length $1 + 5\sqrt{2}$.

5. [7] The following grid represents a mountain range; the number in each cell represents the height of the mountain located there. Moving from a mountain of height a to a mountain of height b takes $(b - a)^2$ time. Suppose that you start on the mountain of height 1 and that you can move up, down, left, or right to get from one mountain to the next. What is the minimum amount of time you need to get to the mountain of height 49?

1	3	6	10	15	21	28
2	5	9	14	20	27	34
4	8	13	19	26	33	39
7	12	18	25	32	38	43
11	17	24	31	37	42	46
16	23	30	36	41	45	48
22	29	35	40	44	47	49

Answer: 212 Consider the diagonals of the board running up and to the right - so the first diagonal is the square 1, the second diagonal is the squares 2 and 3, and so on. The i th ascent is the largest step taken from a square in the i th diagonal to a square in the $i + 1$ st. Since you must climb from square 1 to square 49, the sum of the ascents is at least 48. Since there are 12 ascents, the average ascent is at least 4.

The 1st and 12th ascents are at most 2, and the 2nd and 11th ascents are at most 3. The 6th and 7th ascents are at least 6, and the 5th and 8th ascents are at least 5. Because $f(x) = x^2$ is convex, the sum of squares of the ascents is minimized when they are as close together as possible. One possible shortest path is then $1 \rightarrow 3 \rightarrow 6 \rightarrow 10 \rightarrow 14 \rightarrow 19 \rightarrow 25 \rightarrow 31 \rightarrow 36 \rightarrow 40 \rightarrow 44 \rightarrow 47 \rightarrow 49$, which has ascents of size 2, 3, 4, 4, 5, 6, 6, 5, 4, 4, 3, and 2. Thus, our answer is 212, the sums of the squares of these ascents. There are other solutions to this problem. One alternative problem involves computing the shortest path to each square of the graph, recursively, starting from squares 2 and 3.

Five Guys

6. [3] There are five guys named Alan, Bob, Casey, Dan, and Eric. Each one either always tells the truth or always lies. You overhear the following discussion between them:

Alan: "All of us are truth-tellers."
 Bob: "No, only Alan and I are truth-tellers."
 Casey: "You are both liars."
 Dan: "If Casey is a truth-teller, then Eric is too."
 Eric: "An odd number of us are liars."

Who are the liars?

Answer: Alan, Bob, Dan, and Eric Alan and Bob each claim that both of them are telling the truth, but they disagree on the others. Therefore, they must both be liars, and Casey must be a truth-teller. If Dan is a truth-teller, then so is Eric, but then there would only be two truth-tellers, contradicting Eric's claim. Therefore, Dan is a liar, and so is Eric.

7. [4] Five guys are eating hamburgers. Each one puts a top half and a bottom half of a hamburger bun on the grill. When the buns are toasted, each guy randomly takes two pieces of bread off of the grill. What is the probability that each guy gets a top half and a bottom half?

Answer: $\frac{8}{63}$ Say a guy is *content* if he gets a top half and a bottom half. Suppose, without loss of generality, that the first guy's first piece of bread is a top. Then there is a $\frac{5}{9}$ chance that his second piece of bread is a bottom. By the same reasoning, given that the first guy is content, there is a $\frac{4}{7}$ chance that the second guy is content. Given that the first two guys are content, there is a $\frac{3}{5}$ chance that the third guy is content, and so on. Our final answer is $\frac{5}{9} \cdot \frac{4}{7} \cdot \frac{3}{5} \cdot \frac{2}{3} \cdot \frac{1}{1} = \frac{8}{63}$.

8. [5] A single burger is not enough to satisfy a guy's hunger. The five guys go to Five Guys' Restaurant, which has 20 different meals on the menu. Each meal costs a different integer dollar amount between \$1 and \$20. The five guys have \$20 to split between them, and they want to use all the money to order five different meals. How many sets of five meals can the guys choose?

Answer: 7 Suppose the meals, sorted in descending order, cost $5 + x_1, 4 + x_2, \dots, 1 + x_5$. To satisfy the conditions in the problem, the x_i must be a non-increasing sequence of non-negative integers which sums to 5. Therefore, there is exactly one order for each partition of 5: order the elements of the partition from largest to smallest and use these parts as the x_i . For example, the partition $3 + 2$ corresponds to the order $5 + 3, 4 + 2, 3, 2, 1$. There are thus 7 orders, corresponding to the 7 partitions of 5 below.

$$1 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 2, 1 + 2 + 2, 1 + 1 + 3, 2 + 3, 1 + 4, 5$$

These partitions yield the following seven orders:

$$(2, 3, 4, 5, 6), (1, 3, 4, 5, 7), (1, 2, 4, 6, 7), (1, 2, 3, 5, 7), \\ (1, 2, 3, 6, 8), (1, 2, 3, 5, 9), (1, 2, 3, 4, 10)$$

9. [6] Five guys each have a positive integer (the integers are not necessarily distinct). The greatest common divisor of any two guys' numbers is always more than 1, but the greatest common divisor of all the numbers is 1. What is the minimum possible value of the product of the numbers?

Answer: 32400 Let $\omega(n)$ be the number of distinct prime divisors of a number. Each of the guys' numbers must have $\omega(n) \geq 2$, since no prime divides all the numbers. Therefore, if the answer has prime factorization $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, then $e_1 + e_2 + \dots + e_k \geq 10$. If p^2 divided any of the guys' numbers, we could divide their number by p to reduce the product. Therefore we may assume $e_i \leq 4$ for each i , so the smallest possible product is $2^4 3^4 5^2$. This bound is achievable: give the guys the numbers 10, 6, 6, 6, and 15.

10. [7] Five guys join five girls for a night of bridge. Bridge games are always played by a team of two guys against a team of two girls. The guys and girls want to make sure that every guy and girl play against each other an equal number of times. Given that at least one game is played, what is the least number of games necessary to accomplish this?

Answer: 25 Suppose that each guy plays each girl t times. Since each guy plays against two girls in one game, the total number of games each guy plays is $\frac{5t}{2}$. Then the total number of games is $\frac{25t}{4}$, which is a multiple of 25 and therefore at least 25. To check that 25 games is enough, we arrange the guys and girls in two circles. A *good pair of guys* is a pair of guys who are adjacent in the circle; a *good pair of girls* is defined similarly. There are 5 good pairs of guys and girls — making each good pair of guys play each good pair of girls works.

11th Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Algebra Test

1. [3] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy .

Answer: $\boxed{\frac{1}{6}}$ We have $2x^2y^2 = (x^2 + y^2)^2 - (x^4 + y^4) = \frac{1}{18}$, so $xy = \frac{1}{6}$.

2. [3] Let $f(n)$ be the number of times you have to hit the $\sqrt{\quad}$ key on a calculator to get a number less than 2 starting from n . For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

Answer: $\boxed{242}$ This is $[2^1, 2^2) \cup [2^4, 2^8) \cup [2^{16}, 2^{32}) \dots$, and $2^8 < 2008 < 2^{16}$ so we have exactly the first two intervals.

3. [4] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x .

Answer: $\boxed{1, 2}$ Let $f(x) = x^2 + 2ax + 3a$. Note that $f(-3/2) = 9/4$, so the graph of f is a parabola that goes through $(-3/2, 9/4)$. Then, the condition that $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution means that the parabola has exactly one point in the strip $-1 \leq y \leq 1$, which is possible if and only if the parabola is tangent to $y = 1$. That is, $x^2 + 2ax + 3a = 2$ has exactly one solution. Then, the discriminant $\Delta = 4a^2 - 4(3a - 2) = 4a^2 - 12a + 8$ must be zero. Solving the equation yields $a = 1, 2$.

4. [4] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers x, y . Determine the value of $f(10)$.

Answer: $\boxed{-49}$ Setting $x = 10$ and $y = 5$ gives $f(10) + f(25) + 250 = f(25) + 200 + 1$, from which we get $f(10) = -49$.

Remark: By setting $y = \frac{x}{2}$, we see that the function is $f(x) = -\frac{1}{2}x^2 + 1$, and it can be checked that this function indeed satisfies the given equation.

5. [5] Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f . Find $g(9)$.

Answer: $\boxed{899}$ Let a, b, c be the zeros of f . Then $f(x) = (x - a)(x - b)(x - c)$. Then, the roots of g are a^2, b^2, c^2 , so $g(x) = k(x - a^2)(x - b^2)(x - c^2)$ for some constant k . Since $abc = -f(0) = -1$, we have $k = ka^2b^2c^2 = -g(0) = 1$. Thus,

$$g(x^2) = (x^2 - a^2)(x^2 - b^2)(x^2 - c^2) = (x - a)(x - b)(x - c)(x + a)(x + b)(x + c) = -f(x)f(-x).$$

Setting $x = 3$ gives $g(9) = -f(3)f(-3) = -(31)(-29) = 899$.

6. [5] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n . Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b .

Answer: $\boxed{8}$ The only real roots of unity are 1 and -1 . If ζ is a complex root of unity that is also a root of the equation $z^2 + az + b$, then its conjugate $\bar{\zeta}$ must also be a root. In this case, $|a| = |\zeta + \bar{\zeta}| \leq |\zeta| + |\bar{\zeta}| = 2$ and $b = \zeta\bar{\zeta} = 1$. So we only need to check the quadratics $z^2 + 2z + 1, z^2 + z + 1, z^2 + 1, z^2 - z + 1, z^2 - 2z + 1$. We find 8 roots of unity: $\pm 1, \pm i, \frac{1}{2}(\pm 1 \pm \sqrt{3}i)$.

7. [5] Compute $\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.

Answer: $\boxed{\frac{4}{9}}$ We change the order of summation:

$$\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}} = \sum_{k=1}^{\infty} \frac{k}{2^k} \sum_{n=k+1}^{\infty} \frac{1}{2^n} = \sum_{k=1}^{\infty} \frac{k}{4^k} = \frac{4}{9}.$$

(The last two steps involve the summation of an infinite geometric series, and what is sometimes called an infinite arithmetico-geometric series. These summations are quite standard, and thus we omit the details here.)

8. [6] Compute $\arctan(\tan 65^\circ - 2 \tan 40^\circ)$. (Express your answer in degrees as an angle between 0° and 180° .)

Answer: $\boxed{25^\circ}$ **First Solution:** We have

$$\tan 65^\circ - 2 \tan 40^\circ = \cot 25^\circ - 2 \cot 50^\circ = \cot 25^\circ - \frac{\cot^2 25^\circ - 1}{\cot 25^\circ} = \frac{1}{\cot 25^\circ} = \tan 25^\circ.$$

Therefore, the answer is 25° .

Second Solution: We have

$$\tan 65^\circ - 2 \tan 40^\circ = \frac{1 + \tan 20^\circ}{1 - \tan 20^\circ} - \frac{4 \tan 20^\circ}{1 - \tan^2 20^\circ} = \frac{(1 - \tan 20^\circ)^2}{(1 - \tan 20^\circ)(1 + \tan 20^\circ)} = \tan(45^\circ - 20^\circ) = \tan 25^\circ.$$

Again, the answer is 25° .

9. [7] Let S be the set of points (a, b) with $0 \leq a, b \leq 1$ such that the equation

$$x^4 + ax^3 - bx^2 + ax + 1 = 0$$

has at least one real root. Determine the area of the graph of S .

Answer: $\boxed{\frac{1}{4}}$ After dividing the equation by x^2 , we can rearrange it as

$$\left(x + \frac{1}{x}\right)^2 + a\left(x + \frac{1}{x}\right) - b - 2 = 0$$

Let $y = x + \frac{1}{x}$. We can check that the range of $x + \frac{1}{x}$ as x varies over the nonzero reals is $(-\infty, -2] \cup [2, \infty)$. Thus, the following equation needs to have a real root:

$$y^2 + ay - b - 2 = 0.$$

Its discriminant, $a^2 + 4(b + 2)$, is always positive since $a, b \geq 0$. Then, the maximum absolute value of the two roots is

$$\frac{a + \sqrt{a^2 + 4(b + 2)}}{2}.$$

We need this value to be at least 2. This is equivalent to

$$\sqrt{a^2 + 4(b + 2)} \geq 4 - a.$$

We can square both sides and simplify to obtain

$$2a \geq 2 - b$$

This equation defines the region inside $[0, 1] \times [0, 1]$ that is occupied by S , from which we deduce that the desired area is $1/4$.

10. [8] Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

Answer: $\boxed{\sqrt{5}}$ **First Solution:** Note that

$$\begin{aligned} \binom{2n}{n} &= \frac{(2n)!}{n! \cdot n!} = \frac{(2n)(2n-2)(2n-4)\cdots(2)}{n!} \cdot \frac{(2n-1)(2n-3)(2n-5)\cdots(1)}{n!} \\ &= 2^n \cdot \frac{(-2)^n}{n!} \left(-\frac{1}{2}\right) \left(-\frac{1}{2}-1\right) \left(-\frac{1}{2}-2\right) \cdots \left(-\frac{1}{2}-n+1\right) \\ &= (-4)^n \binom{-\frac{1}{2}}{n}. \end{aligned}$$

Then, by the binomial theorem, for any real x with $|x| < \frac{1}{4}$, we have

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-4x)^n = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

Therefore,

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left(\frac{1}{5}\right)^n = \frac{1}{\sqrt{1-\frac{4}{5}}} = \sqrt{5}.$$

Second Solution: Consider the generating function

$$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} x^n.$$

It has formal integral given by

$$g(x) = I(f(x)) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^{n+1} = \sum_{n=0}^{\infty} C_n x^{n+1} = x \sum_{n=0}^{\infty} C_n x^n,$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the n th Catalan number. Let $h(x) = \sum_{n=0}^{\infty} C_n x^n$; it suffices to compute this generating function. Note that

$$1 + xh(x)^2 = 1 + x \sum_{i,j \geq 0} C_i C_j x^{i+j} = 1 + x \sum_{k \geq 0} \left(\sum_{i=0}^k C_i C_{k-i} \right) x^k = 1 + \sum_{k \geq 0} C_{k+1} x^{k+1} = h(x),$$

where we've used the recurrence relation for the Catalan numbers. We now solve for $h(x)$ with the quadratic equation to obtain

$$h(x) = \frac{1/x \pm \sqrt{1/x^2 - 4/x}}{2} = \frac{1 \pm \sqrt{1-4x}}{2x}.$$

Note that we must choose the $-$ sign in the \pm , since the $+$ would lead to a leading term of $\frac{1}{x}$ for h (by expanding $\sqrt{1-4x}$ into a power series). Therefore, we see that

$$f(x) = D(g(x)) = D(xh(x)) = D\left(\frac{1 - \sqrt{1-4x}}{2}\right) = \frac{1}{\sqrt{1-4x}}$$

and our answer is hence $f(1/5) = \sqrt{5}$.

11th Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Calculus Test

1. [3] Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.

Answer: $\boxed{5050}$ Note that $f(x) = 1 + 2x + 3x^2 + \cdots + 100x^{99}$, so $f'(1) = 1 + 2 + \cdots + 100 = \frac{100 \cdot 101}{2} = 5050$.

2. [3] Let ℓ be the line through $(0, 0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ .

Answer: $\boxed{13}$ Let the point of tangency be $(t, t^3 + t + 16)$, then the slope of ℓ is $(t^3 + t + 16)/t$. On the other hand, since $dy/dx = 3x^2 + 1$, the slope of ℓ is $3t^2 + 1$. Therefore,

$$\frac{t^3 + t + 16}{t} = 3t^2 + 1.$$

Simplifying, we get $t^3 = 8$, so $t = 2$. It follows that the slope is $3(2)^2 + 1 = 13$.

3. [4] Find all $y > 1$ satisfying $\int_1^y x \ln x \, dx = \frac{1}{4}$.

Answer: $\boxed{\sqrt{e}}$ Applying integration by parts with $u = \ln x$ and $v = \frac{1}{2}x^2$, we get

$$\int_1^y x \ln x \, dx = \frac{1}{2}x^2 \ln x \Big|_1^y - \frac{1}{2} \int_1^y x \, dx = \frac{1}{2}y^2 \ln y - \frac{1}{4}y^2 + \frac{1}{4}.$$

So $y^2 \ln y = \frac{1}{2}y^2$. Since $y > 1$, we obtain $\ln y = \frac{1}{2}$, and thus $y = \sqrt{e}$.

4. [4] Let a, b be constants such that $\lim_{x \rightarrow 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = 1$. Determine the pair (a, b) .

Answer: $\boxed{(-2, 1)}$ When $x = 1$, the numerator is 0, so the denominator must be zero as well, so $1 + a + b = 0$. Using l'Hôpital's rule, we must have

$$1 = \lim_{x \rightarrow 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = \lim_{x \rightarrow 1} \frac{2 \ln(2-x)}{(x-2)(2x+a)},$$

and by the same argument we find that $2 + a = 0$. Thus, $a = -2$ and $b = 1$. This is indeed a solution, as can be seen by finishing the computation.

5. [4] Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers x . Determine $f^{(2008)}(0)$ (i.e., f differentiated 2008 times and then evaluated at $x = 0$).

Answer: $\boxed{\frac{3}{8}}$ We have

$$\begin{aligned} \sin^6 x + \cos^6 x &= (\sin^2 x + \cos^2 x)^3 - 3 \sin^2 x \cos^2 x (\sin^2 x + \cos^2 x) \\ &= 1 - 3 \sin^2 x \cos^2 x = 1 - \frac{3}{4} \sin^2 2x = 1 - \frac{3}{4} \left(\frac{1 - \cos 4x}{2} \right) \\ &= \frac{5}{8} + \frac{3}{8} \cos 4x. \end{aligned}$$

It follows that $f(x) = \frac{5}{8} + \frac{3}{8} \cos x$. Thus $f^{(2008)}(x) = \frac{3}{8} \cos x$. Evaluating at $x = 0$ gives $\frac{3}{8}$.

6. [5] Determine the value of $\lim_{n \rightarrow \infty} \sum_{k=0}^n \binom{n}{k}^{-1}$.

Answer: $\boxed{2}$ Let S_n denote the sum in the limit. For $n \geq 1$, we have $S_n \geq \binom{n}{0}^{-1} + \binom{n}{n}^{-1} = 2$. On the other hand, for $n \geq 3$, we have

$$S_n = \binom{n}{0}^{-1} + \binom{n}{1}^{-1} + \binom{n}{n-1}^{-1} + \binom{n}{n}^{-1} + \sum_{k=2}^{n-2} \binom{n}{k}^{-1} \leq 2 + \frac{2}{n} + (n-3) \binom{n}{2}^{-1}$$

which goes to 2 as $n \rightarrow \infty$. Therefore, $S_n \rightarrow 2$.

7. [5] Find p so that $\lim_{x \rightarrow \infty} x^p (\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x})$ is some non-zero real number.

Answer: $\boxed{\frac{5}{3}}$ Make the substitution $t = \frac{1}{x}$. Then the limit equals to

$$\lim_{t \rightarrow 0} t^{-p} \left(\sqrt[3]{\frac{1}{t} + 1} + \sqrt[3]{\frac{1}{t} - 1} - 2\sqrt[3]{\frac{1}{t}} \right) = \lim_{t \rightarrow 0} t^{-p-\frac{1}{3}} (\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2).$$

We need the degree of the first nonzero term in the MacLaurin expansion of $\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2$. We have

$$\sqrt[3]{1+t} = 1 + \frac{1}{3}t - \frac{1}{9}t^2 + o(t^2), \quad \sqrt[3]{1-t} = 1 - \frac{1}{3}t - \frac{1}{9}t^2 + o(t^2).$$

It follows that $\sqrt[3]{1+t} + \sqrt[3]{1-t} - 2 = -\frac{2}{9}t^2 + o(t^2)$. By consider the degree of the leading term, it follows that $-p - \frac{1}{3} = -2$. So $p = \frac{5}{3}$.

8. [7] Let $T = \int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1} dx$. Evaluate e^T .

Answer: $\boxed{\frac{11}{4}}$ Divide the top and bottom by e^x to obtain that

$$T = \int_0^{\ln 2} \frac{2e^{2x} + e^x - e^{-x}}{e^{2x} + e^x - 1 + e^{-x}} dx$$

Notice that $2e^{2x} + e^x - e^{-x}$ is the derivative of $e^{2x} + e^x - 1 + e^{-x}$, and so

$$T = \left[\ln |e^{2x} + e^x - 1 + e^{-x}| \right]_0^{\ln 2} = \ln \left(4 + 2 - 1 + \frac{1}{2} \right) - \ln 2 = \ln \left(\frac{11}{4} \right)$$

Therefore, $e^T = \frac{11}{4}$.

9. [7] Evaluate the limit $\lim_{n \rightarrow \infty} n^{-\frac{1}{2}(1+\frac{1}{n})} (1^1 \cdot 2^2 \cdot \dots \cdot n^n)^{\frac{1}{n^2}}$.

Answer: $\boxed{e^{-1/4}}$ Taking the logarithm of the expression inside the limit, we find that it is

$$-\frac{1}{2} \left(1 + \frac{1}{n} \right) \ln n + \frac{1}{n^2} \sum_{k=1}^n k \ln k = \frac{1}{n} \sum_{k=1}^n \frac{k}{n} \ln \left(\frac{k}{n} \right).$$

We can recognize this as the as Riemann sum expansion for the integral $\int_0^1 x \ln x dx$, and thus the limit of the above sum as $n \rightarrow \infty$ equals to the value of this integral. Evaluating this integral using integration by parts, we find that

$$\int_0^1 x \ln x dx = \frac{1}{2} x^2 \ln x \Big|_0^1 - \int_0^1 \frac{x}{2} dx = -\frac{1}{4}.$$

Therefore, the original limit is $e^{-1/4}$.

10. [8] Evaluate the integral $\int_0^1 \ln x \ln(1-x) dx$.

Answer: $\boxed{2 - \frac{\pi^2}{6}}$ We have the MacLaurin expansion $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots$. So

$$\int_0^1 \ln x \ln(1-x) dx = - \int_0^1 \ln x \sum_{n=1}^{\infty} \frac{x^n}{n} dx = - \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^n \ln x dx.$$

Using integration by parts, we get

$$\int_0^1 x^n \ln x dx = \frac{x^{n+1} \ln x}{n+1} \Big|_0^1 - \int_0^1 \frac{x^n}{n+1} dx = -\frac{1}{(n+1)^2}.$$

(We used the fact that $\lim_{x \rightarrow 0} x^n \ln x = 0$ for $n > 0$, which can be proven using l'Hôpital's rule.) Therefore, the original integral equals to

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2} \right).$$

Telescoping the sum and using the well-known identity $\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we see that the above sum is equal to $2 - \frac{\pi^2}{6}$.

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Individual Round: Combinatorics Test

1. [3] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

Answer: 60 There are 6 such slices, and each slice gives 10 valid pairs (with no overcounting). Therefore, there are 60 such pairs.

2. [3] Let $S = \{1, 2, \dots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S .

Answer: $\frac{2009}{2}$ For any subset A , we can define the “reflected subset” $A' = \{i \mid 2009 - i \in A\}$. Then $m(A) = 2009 - m(A')$. Note that as A is taken over all nonempty subsets of S , A' goes through all the nonempty subsets of S as well. Thus, the average of $m(A)$ is equal to the average of $\frac{m(A)+m(A')}{2}$, which is the constant $\frac{2009}{2}$.

Remark: : This argument is very analogous to the famous argument that Gauss used to sum the series $1 + 2 + \dots + 100$.

3. [4] Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? Assume that all animals are distinguishable from each other. (Please write your answer as an integer, without any incomplete computations.)

Answer: 100800 Since there are 9 cow and pigs combined and 7 horses, there must be a pair with 1 cow and 1 pig, and all the other pairs must contain a horse. There are 4×5 ways of selecting the cow-pig pair, and $7!$ ways to select the partners for the horses. It follows that the answer is $4 \times 5 \times 7! = 100800$.

4. [4] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

Answer: 10201

It is easy to see that the coordinates of the frog’s final position must have the same parity. Suppose that the frog went to sleep at (x, y) . Then, we have that $-100 \leq y \leq 100$ and $|x| \leq 100 - |y|$, so x can take on the values $-100 + |y|, -98 + |y|, \dots, 100 - |y|$. There are $101 - |y|$ such values, so the total number of such locations is

$$\sum_{y=-100}^{100} 101 - |y| = 201 \cdot 101 - 2 \cdot \frac{100(100 + 1)}{2} = 101^2 = 10201.$$

5. [5] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in S less than 2008.

Answer: 128 Write the elements of S in their ternary expansion (i.e. base 3). Then the second condition translates into, if $\overline{d_1 d_2 \dots d_k} \in S$, then $\overline{d_1 d_2 \dots d_k 0}$ and $\overline{d_1 d_2 \dots d_k 1}$ are also in S . It follows that S is the set of nonnegative integers whose tertiary representation contains only the digits 0 and 1. Since $2 \cdot 3^6 < 2008 < 3^7$, there are $2^7 = 128$ such elements less than 2008. Therefore, there are 128 such non-negative elements.

6. [5] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \dots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

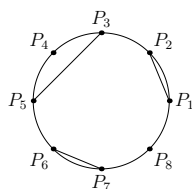
1								
	2							
		?						

Answer: $\frac{2}{21}$ The third row must contain the digit 1, and it cannot appear in the leftmost three squares. Therefore, the digit 1 must fall into one of the six squares shown below that are marked with \star . By symmetry, each starred square has an equal probability of containing the digit 1 (To see this more precisely, note that swapping columns 4 and 5 gives another Sudoku matrix, so the probability that the 4th column \star has the 1 is the same as the probability that the 5th column \star has the 1. Similarly, switching the 4-5-6th columns with the 7-8-9th columns yields another Sudoku matrix, which implies in particular that the probability that the 4th column \star has the 1 is the same as the probability that the 7th column \star has the 1. The rest of the argument follows analogously.) Therefore, the probability that the ? square contains 1 is $1/6$.

1								
	2							
			\star	\star	\star	\star	\star	\star

Similarly the probability that the digit 2 appears at ? is also $1/6$. By symmetry, the square ? has equal probability of containing the digits 3, 4, 5, 6, 7, 8, 9. It follows that this probability is $(1 - \frac{1}{6} - \frac{1}{6}) / 7 = \frac{2}{21}$.

7. [6] Let P_1, P_2, \dots, P_8 be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each P_i is the endpoint of at most one segment and (2) two no segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)



Answer: 323 Let $f(n)$ denote the number of valid configurations when there are n points on the circle. Let P be one of the points. If P is not the end point of an edge, then there are $f(n - 1)$ ways to connect the remaining $n - 1$ points. If P belongs to an edge that separates the circle so that there are k points on one side and $n - k - 2$ points on the other side, then there are $f(k)f(n - k - 2)$ ways of finishing the configuration. Thus, $f(n)$ satisfies the recurrence relation

$$f(n) = f(n - 1) + f(0)f(n - 2) + f(1)f(n - 3) + f(2)f(n - 4) + \dots + f(n - 2)f(0), n \geq 2.$$

The initial conditions are $f(0) = f(1) = 1$. Using the recursion, we find that $f(2) = 2, f(3) = 4, f(4) = 9, f(5) = 21, f(6) = 51, f(7) = 127, f(8) = 323$.

Remark: These numbers are known as the *Motzkin numbers*. This is sequence A001006 in the On-Line Encyclopedia of Integer Sequences (<http://www.research.att.com/~njas/sequences/A001006>). In

Richard Stanley's Enumerative Combinatorics Volume 2, one can find 13 different interpretations of Motzkin numbers in exercise 6.38.

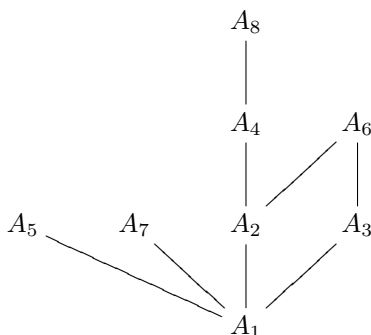
8. [6] Determine the number of ways to select a sequence of 8 sets A_1, A_2, \dots, A_8 , such that each is a subset (possibly empty) of $\{1, 2\}$, and A_m contains A_n if m divides n .

Answer: 2025 Consider an arbitrary $x \in \{1, 2\}$, and let us consider the number of ways for x to be in some of the sets so that the constraints are satisfied. We divide into a few cases:

- **Case:** $x \notin A_1$. Then x cannot be in any of the sets. So there is one possibility.
- **Case:** $x \in A_1$ but $x \notin A_2$. Then the only other sets that x could be in are A_3, A_5, A_7 , and x could be in some collection of them. There are 8 possibilities in this case.
- **Case:** $x \in A_2$. Then $x \in A_1$ automatically. There are 4 independent choices to be made here: (1) whether $x \in A_5$; (2) whether $x \in A_7$; (3) whether $x \in A_3$, and if yes, whether $x \in A_6$; (4) whether $x \in A_4$, and if yes, whether $x \in A_8$. There are $2 \times 2 \times 3 \times 3 = 36$ choices here.

Therefore, there are $1 + 8 + 36 = 45$ ways to place x into some of the sets. Since the choices for $x = 1$ and $x = 2$ are made independently, we see that the total number of possibilities is $45^2 = 2025$.

Remark: The solution could be guided by the following diagram. Set A is above B and connected to B if and only if $A \subset B$. Such diagrams are known as *Hasse diagrams*, which are used to depict *partially ordered sets*.



9. [7] On an infinite chessboard (whose squares are labeled by (x, y) , where x and y range over all integers), a king is placed at $(0, 0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

Answer: $\frac{1}{4} + \frac{3}{4 \cdot 5^{2008}}$ Since only the parity of the coordinates are relevant, it is equivalent to consider a situation where the king moves $(1, 0)$ with probability 0.2, moves $(0, 1)$ with probability 0.2, moves $(1, 1)$ with probability 0.2, and stays put with probability 0.4. This can be analyzed using the generating function

$$f(x, y) = (0.4 + 2 \times 0.1x + 2 \times 0.1y + 4 \times 0.05xy)^{2008} = \frac{(2 + x + y + xy)^{2008}}{5^{2008}}.$$

We wish to find the sum of the coefficients of the terms $x^a y^b$, where both a and b are even. This is simply equal to $\frac{1}{4}(f(1, 1) + f(1, -1) + f(-1, 1) + f(-1, -1))$. We have $f(1, 1) = 1$ and $f(1, -1) = f(-1, 1) = f(-1, -1) = 1/5^{2008}$. Therefore, the answer is

$$\frac{1}{4}(f(1, 1) + f(1, -1) + f(-1, 1) + f(-1, -1)) = \frac{1}{4} \left(1 + \frac{3}{5^{2008}} \right) = \frac{1}{4} + \frac{3}{4 \cdot 5^{2008}}.$$

10. [7] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

Answer: 1540 For each $k = 1, 2, 3, 4$, note that set of pairs (a_k, b_k) with $0 \leq a_k \leq k$ maps bijectively to the set of nonnegative integers through the map $(a_k, b_k) \mapsto a_k + (k+1)b_k$, as a_k is simply the remainder of $a_k + (k+1)b_k$ upon division by $k+1$. By letting $x_k = a_k + (k+1)b_k$, we see that the problem is equivalent to finding the number of quadruples of nonnegative integers (x_1, x_2, x_3, x_4) such that $x_1 + x_2 + x_3 + x_4 = 19$. This is the same as finding the number of quadruples of positive integers $(x_1 + 1, x_2 + 1, x_3 + 1, x_4 + 1)$ such that $x_1 + x_2 + x_3 + x_4 = 23$. By a standard “dots and bars” argument, we see that the answer is $\binom{22}{3} = 1540$.

A generating functions solution is also available. It’s not hard to see that the answer is the coefficient of x^{19} in

$$\begin{aligned} & (1+x)(1+x+x^2)(1+x+x^2+x^3)(1+x+x^2+x^3+x^4) \\ & \quad (1+x^2+x^4+\dots)(1+x^3+x^6+\dots)(1+x^4+x^8+\dots)(1+x^5+x^{10}+\dots) \\ = & \left(\frac{1-x^2}{1-x}\right)\left(\frac{1-x^3}{1-x}\right)\left(\frac{1-x^4}{1-x}\right)\left(\frac{1-x^5}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^5}\right) \\ = & \frac{1}{(1-x)^4} = (1-x)^{-4}. \end{aligned}$$

Using binomial theorem, we find that the coefficient of x^{19} in $(1-x)^{-4}$ is $(-1)^{19}\binom{-4}{19} = \binom{22}{19} = 1540$.

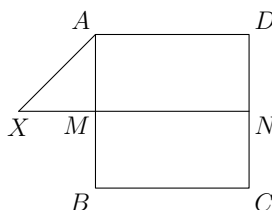
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Individual Round: General Test, Part 1

1. [2] Let $ABCD$ be a unit square (that is, the labels A, B, C, D appear in that order around the square). Let X be a point outside of the square such that the distance from X to AC is equal to the distance from X to BD , and also that $AX = \frac{\sqrt{2}}{2}$. Determine the value of CX^2 .

Answer: $\boxed{\frac{5}{2}}$



Since X is equidistant from AC and BD , it must lie on either the perpendicular bisector of AB or the perpendicular bisector of AD . It turns that the two cases yield the same answer, so we will just assume the first case. Let M be the midpoint of AB and N the midpoint of CD . Then, XM is perpendicular to AB , so $XM = \frac{1}{2}$ and thus $XN = \frac{3}{2}$, $NC = \frac{1}{2}$. By the Pythagorean Theorem we find $XC = \frac{\sqrt{10}}{2}$ and the answer follows.

2. [3] Find the smallest positive integer n such that $107n$ has the same last two digits as n .

Answer: $\boxed{50}$ The two numbers have the same last two digits if and only if 100 divides their difference $106n$, which happens if and only if 50 divides n .

3. [3] There are 5 dogs, 4 cats, and 7 bowls of milk at an animal gathering. Dogs and cats are distinguishable, but all bowls of milk are the same. In how many ways can every dog and cat be paired with either a member of the other species or a bowl of milk such that all the bowls of milk are taken?

Answer: $\boxed{20}$ Since there are 9 dogs and cats combined and 7 bowls of milk, there can only be one dog-cat pair, and all the other pairs must contain a bowl of milk. There are 4×5 ways of selecting the dog-cat pair, and only one way of picking the other pairs, since the bowls of milk are indistinguishable, so the answer is $4 \times 5 = 20$.

4. [3] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy .

Answer: $\boxed{\frac{1}{6}}$ Same as Algebra Test problem 1.

5. [4] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers x, y . Determine the value of $f(10)$.

Answer: $\boxed{-49}$ Same as Algebra Test problem 4.

6. [4] In a triangle ABC , take point D on BC such that $DB = 14$, $DA = 13$, $DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC . What is the area of triangle ABC ?

Answer: $\boxed{108}$ Same as Geometry Test problem 4.

7. [5] The equation $x^3 - 9x^2 + 8x + 2 = 0$ has three real roots p, q, r . Find $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.

Answer: $\boxed{25}$ From Vieta's relations, we have $p + q + r = 9$, $pq + qr + pr = 8$ and $pqr = -2$. So

$$\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2} = \frac{(pq + qr + rp)^2 - 2(p + q + r)(pqr)}{(pqr)^2} = \frac{8^2 - 2 \cdot 9 \cdot (-2)}{(-2)^2} = \boxed{25}.$$

8. [5] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of positive integers in S less than 2008.

Answer: $\boxed{127}$ Same as Combinatorics Problem 5.

9. [5] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \dots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?

1								
	2							
			?					

Answer: $\boxed{\frac{2}{21}}$ Same as Combinatorics Test problem 6.

10. [6] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ , visits all three sides of ABC , and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

Answer: $\boxed{\sqrt{\frac{28}{3}} - 1}$ Same as Geometry Test problem 8.

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Individual Round: General Test, Part 2

1. [2] Four students from Harvard, one of them named Jack, and five students from MIT, one of them named Jill, are going to see a Boston Celtics game. However, they found out that only 5 tickets remain, so 4 of them must go back. Suppose that at least one student from each school must go see the game, and at least one of Jack and Jill must go see the game, how many ways are there of choosing which 5 people can see the game?

Answer: 104 Let us count the number of way of distributing the tickets so that one of the conditions is violated. There is 1 way to give all the tickets to MIT students, and $\binom{7}{5}$ ways to give all the tickets to the 7 students other than Jack and Jill. Therefore, the total number of valid ways is $\binom{9}{5} - 1 - \binom{7}{5} = 104$.

2. [2] Let ABC be an equilateral triangle. Let Ω be a circle inscribed in ABC and let ω be a circle tangent externally to Ω as well as to sides AB and AC . Determine the ratio of the radius of Ω to the radius of ω .

Answer: 3 Same as Geometry Test problem 2.

3. [3] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes (order is irrelevant) from a $3 \times 3 \times 1$ block with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

Answer: 60 Same as Combinatorics Test problem 1.

4. [3] Suppose that a, b, c, d are real numbers satisfying $a \geq b \geq c \geq d \geq 0$, $a^2 + d^2 = 1$, $b^2 + c^2 = 1$, and $ac + bd = 1/3$. Find the value of $ab - cd$.

Answer: $\frac{2\sqrt{2}}{3}$ We have

$$(ab - cd)^2 = (a^2 + d^2)(b^2 + c^2) - (ac + bd)^2 = (1)(1) - \left(\frac{1}{3}\right)^2 = \frac{8}{9}.$$

Since $a \geq b \geq c \geq d \geq 0$, $ab - cd \geq 0$, so $ab - cd = \frac{2\sqrt{2}}{3}$.

Comment: Another way to solve this problem is to use the trigonometric substitutions $a = \sin \theta$, $b = \sin \phi$, $c = \cos \phi$, $d = \cos \theta$.

5. [4] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

Answer: 10201 Same as Combinatorics Test problem 4.

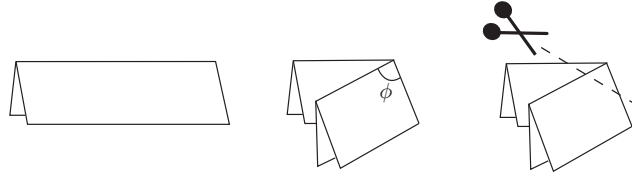
6. [4] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x .

Answer: 1, 2 Same as Algebra Test problem 3.

7. [5] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n . Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b .

Answer: 8 Same as Algebra Test problem 6.

8. [5] A piece of paper is folded in half. A second fold is made such that the angle marked below has measure ϕ ($0^\circ < \phi < 90^\circ$), and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O , and also that $\angle XOY = 144^\circ$, where X and Y are the vertices of the hole adjacent to O . Find the value(s) of ϕ (in degrees).

Answer: $\boxed{81^\circ}$ Same as Geometry Test problem 5.

9. [6] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D , and let lines BI and CI meet the circle with diameter AI at points P and Q , respectively. Given $BI = 6$, $CI = 5$, $DI = 3$, determine the value of $(DP/DQ)^2$.

Answer: $\boxed{\frac{75}{64}}$ Same as Geometry Test problem 9.

10. [6] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

Answer: $\boxed{1540}$ Same as Combinatorics Test problem 10.

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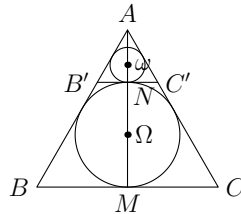
Individual Round: Geometry Test

1. [3] How many different values can $\angle ABC$ take, where A, B, C are distinct vertices of a cube?

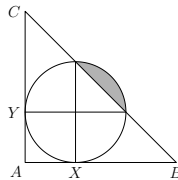
Answer: 5. In a unit cube, there are 3 types of triangles, with side lengths $(1, 1, \sqrt{2})$, $(1, \sqrt{2}, \sqrt{3})$ and $(\sqrt{2}, \sqrt{2}, \sqrt{2})$. Together they generate 5 different angle values.

2. [3] Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC . Determine the ratio of the radius of Ω to the radius of ω .

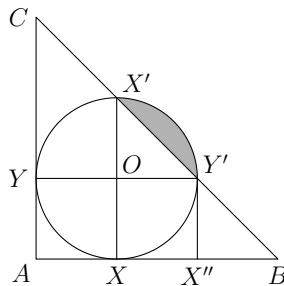
Answer: 3 Label the diagram as shown below, where Ω and ω also denote the center of the corresponding circles. Note that AM is a median and Ω is the centroid of the equilateral triangle. So $AM = 3M\Omega$. Since $M\Omega = N\omega$, it follows that $AM/AN = 3$, and triangle ABC is the image of triangle $AB'C'$ after a scaling by a factor of 3, and so the two incircles must also be related by a scale factor of 3.



3. [4] Let ABC be a triangle with $\angle BAC = 90^\circ$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC . Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



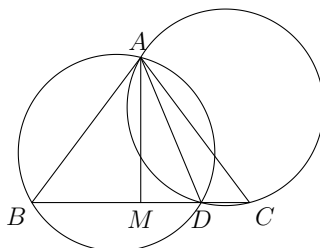
Answer: $\pi - 2$ Let O be the center of the circle, and r its radius, and let X' and Y' be the points diametrically opposite X and Y , respectively. We have $OX' = OY' = r$, and $\angle X'OY' = 90^\circ$. Since triangles $X'OY'$ and BAC are similar, we see that $AB = AC$. Let X'' be the projection of Y' onto AB . Since $X''BY'$ is similar to ABC , and $X''Y' = r$, we have $X''B = r$. It follows that $AB = 3r$, so $r = 2$.



Then, the desired area is the area of the quarter circle minus that of the triangle $X'OY'$. And the answer is $\frac{1}{4}\pi r^2 - \frac{1}{2}r^2 = \pi - 2$.

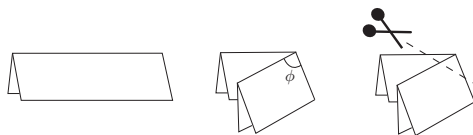
4. [4] In a triangle ABC , take point D on BC such that $DB = 14$, $DA = 13$, $DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC . What is the area of triangle ABC ?

Answer: 108



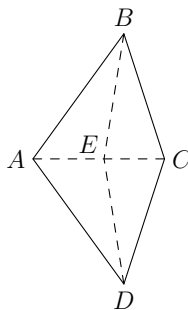
The fact that the two circumcircles are congruent means that the chord AD must subtend the same angle in both circles. That is, $\angle ABC = \angle ACB$, so ABC is isosceles. Drop the perpendicular M from A to BC ; we know $MC = 9$ and so $MD = 5$ and by Pythagoras on AMD , $AM = 12$. Therefore, the area of ABC is $\frac{1}{2}(AM)(BC) = \frac{1}{2}(12)(18) = 108$.

5. [5] A piece of paper is folded in half. A second fold is made such that the angle marked below has measure ϕ ($0^\circ < \phi < 90^\circ$), and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O , and also that $\angle XOY = 144^\circ$, where X and Y are the the vertices of the hole adjacent to O . Find the value(s) of ϕ (in degrees).

Answer: 81° Try actually folding a piece of paper. We see that the cut out area is a kite, as shown below. The fold was made on AC , and then BE and DE . Since DC was folded onto DA , we have $\angle ADE = \angle CDE$.



Either A or C is the center of the circle. If it's A , then $\angle BAD = 144^\circ$, so $\angle CAD = 72^\circ$. Using $CA = DA$, we see that $\angle ACD = \angle ADC = 54^\circ$. So $\angle EDA = 27^\circ$, and thus $\phi = 72^\circ + 27^\circ = 99^\circ$, which is inadmissible, as $\phi < 90^\circ$.

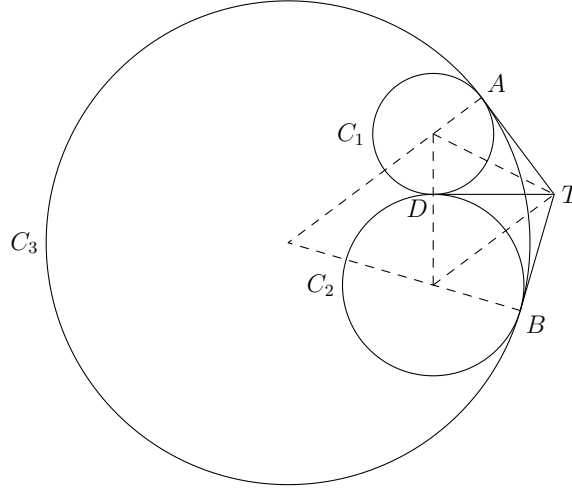
So C is the center of the circle. Then, $\angle CAD = \angle CDA = 54^\circ$, $\angle ADE = 27^\circ$, and $\phi = 54^\circ + 27^\circ = 81^\circ$.

6. [5] Let ABC be a triangle with $\angle A = 45^\circ$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC . Determine the length OP .

Answer: $\sqrt{17}$ Using extended Sine law, we find the circumradius of ABC to be $R = \frac{BC}{2 \sin A} = 4\sqrt{2}$. By considering the power of point P , we find that $R^2 - OP^2 = PB \cdot PC = 15$. So $OP = \sqrt{R^2 - 15} = \sqrt{16 \cdot 2 - 15} = \sqrt{17}$.

7. [6] Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B , respectively. The tangents to C_3 at A and B meet at T , and $TA = 4$. Determine the radius of C_3 .

Answer: [8] Let D be the point of tangency between C_1 and C_2 . We see that T is the radical center of the three circles, and so it must lie on the radical axis of C_1 and C_2 , which happens to be their common tangent TD . So $TD = 4$.



We have

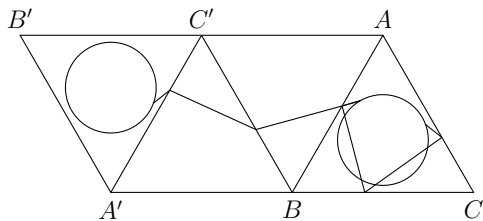
$$\tan \frac{\angle ATD}{2} = \frac{2}{TD} = \frac{1}{2}, \quad \text{and} \quad \tan \frac{\angle BTD}{2} = \frac{3}{TD} = \frac{3}{4}.$$

Thus, the radius of C_3 equals to

$$\begin{aligned} TA \tan \frac{\angle ATB}{2} &= 4 \tan \left(\frac{\angle ATD + \angle BTD}{2} \right) \\ &= 4 \cdot \frac{\tan \frac{\angle ATD}{2} + \tan \frac{\angle BTD}{2}}{1 - \tan \frac{\angle ATD}{2} \tan \frac{\angle BTD}{2}} \\ &= 4 \cdot \frac{\frac{1}{2} + \frac{3}{4}}{1 - \frac{1}{2} \cdot \frac{3}{4}} \\ &= 8. \end{aligned}$$

8. [6] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ , visits all three sides of ABC , and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

Answer: $\sqrt{\frac{28}{3}} - 1$ Suppose that the path visits sides AB, BC, CA in this order. Construct points A', B', C' so that C' is the reflection of C across AB , A' is the reflection of A across BC' , and B' is the reflection of B across $A'C'$. Finally, let Γ' be the circle with radius $\frac{1}{2}$ centered at the center of $A'B'C'$. Note that Γ' is the image of Γ after the three reflections: $AB, BC', C'A'$.

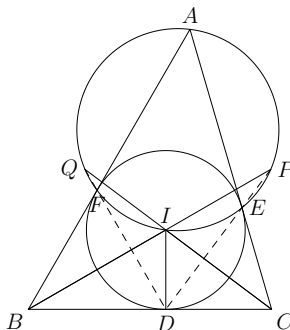


When the path hits AB , let us reflect the rest of the path across AB and follow this reflected path. When we hit BC' , let us reflect the rest of the path across BC' , and follow the new path. And when we hit $A'C'$, reflect the rest of the path across $A'C'$ and follow the new path. We must eventually end up at Γ' .

It is easy to see that the shortest path connecting some point on Γ to some point on Γ' lies on the line connecting the centers of the two circles. We can easily find the distance between the two centers to be $\sqrt{3^2 + \left(\frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{28}{3}}$. Therefore, the length of the shortest path connecting Γ to Γ' has length $\sqrt{\frac{28}{3}} - 1$. By reflecting this path three times back into ABC , we get a path that satisfies our conditions.

9. [7] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D , and let lines BI and CI meet the circle with diameter AI at points P and Q , respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

Answer: $\boxed{\frac{75}{64}}$

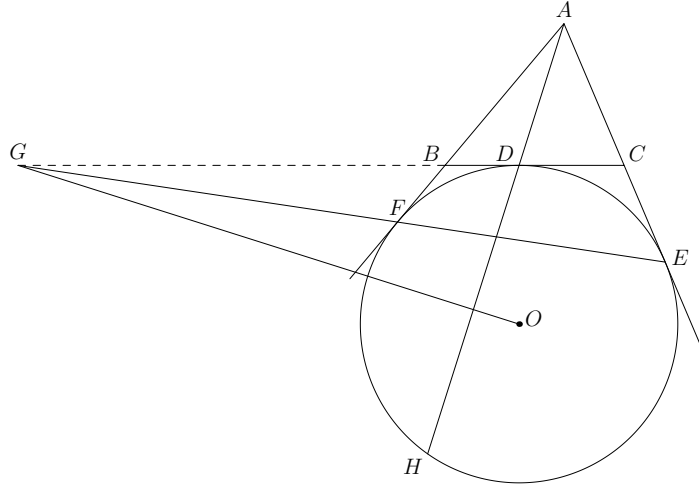


Let the incircle touch sides AC and AB at E and F respectively. Note that E and F both lie on the circle with diameter AI since $\angle AEI = \angle AFI = 90^\circ$. The key observation is that D, E, P are collinear. To prove this, suppose that P lies outside the triangle (the other case is analogous), then $\angle PEA = \angle PIA = \angle IBA + \angle IAB = \frac{1}{2}(\angle B + \angle A) = 90^\circ - \frac{1}{2}\angle C = \angle DEC$, which implies that D, E, P are collinear. Similarly D, F, Q are collinear. Then, by Power of a Point, $DE \cdot DP = DF \cdot DQ$. So $DP/DQ = DF/DE$.

Now we compute DF/DE . Note that $DF = 2DB \sin \angle DBI = 2\sqrt{6^2 - 3^2} \left(\frac{3}{6}\right) = 3\sqrt{3}$, and $DE = 2DC \sin \angle DCI = 2\sqrt{5^2 - 3^2} \left(\frac{3}{5}\right) = \frac{24}{5}$. Therefore, $DF/DE = \frac{5\sqrt{3}}{8}$.

10. [7] Let ABC be a triangle with $BC = 2007, CA = 2008, AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D , and touches extensions of lines AC and AB at E and F , respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω . Let ℓ be the line through O perpendicular to AD . Let ℓ meet line EF at G . Compute the length DG .

Answer: $\boxed{2014024}$ Let line AD meet ω again at H . Since AF and AE are tangents to ω and ADH is a secant, we see that $DEHF$ is a harmonic quadrilateral. This implies that the pole of AD with respect to ω lies on EF . Since $\ell \perp AD$, the pole of AD lies on ℓ . It follows that the pole of AD is G .



Thus, G must lie on the tangent to ω at D , so C, D, B, G are collinear. Furthermore, since the pencil of lines $(AE, AF; AD, AG)$ is harmonic, by intersecting it with the line BC , we see that $(C, B; D, G)$ is harmonic as well. This means that

$$\frac{BD}{DC} \cdot \frac{CG}{GB} = -1.$$

(where the lengths are directed.) The semiperimeter of ABC is $s = \frac{1}{2}(2007 + 2008 + 2009) = 3012$. So $BD = s - 2009 = 1003$ and $CD = s - 2008 = 1004$. Let $x = DG$, then the above equations gives

$$\frac{1003}{1004} \cdot \frac{x + 1004}{x - 1003} = 1.$$

Solving gives $x = 2014024$.

Remark: If you are interested to learn about projective geometry, check out the last chapter of *Geometry Revisited* by Coxeter and Greitzer or *Geometric Transformations III* by Yaglom.

11th Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Guts Round

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11th HARVARD-MIT MATHEMATICS TOURNAMENT, 23 FEBRUARY 2008 — GUTS ROUND

1. [5] Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.

Answer: $\boxed{(4, -2), (\frac{5}{2}, -5)}$ Since $10, a, b$ is an arithmetic progression, we have $a = \frac{1}{2}(10 + b)$. Also, we have $a + ab = 2b$, and so $a(1 + b) = 2b$. Substituting the expression for a gives $(10 + b)(1 + b) = 4b$. Solving this quadratic equation gives the solutions $b = -2$ and $b = -5$. The corresponding values for a can be found by $a = \frac{1}{2}(10 + b)$.

2. [5] Given right triangle ABC , with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C . What is the radius of ω ?

Answer: $\boxed{\frac{25}{8}}$ Let O be the center of ω , and let M be the midpoint of AC . Since $OA = OC$, $OM \perp AC$. Also, $\angle OCM = \angle BAC$, and so triangles ABC and CMO are similar. Then, $CO/CM = AC/AB$, from which we obtain that the radius of ω is $CO = \frac{25}{8}$.

3. [5] How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

Answer: $\boxed{2 \cdot 3^{2008}}$ Denote the colors A, B, C . The left-most column can be colored in 6 ways. For each subsequent column, if the k th column is colored with AB , then the $(k + 1)$ th column can only be colored with one of BA, BC, CA . That is, if we have colored the first k columns, then there are 3 ways to color the $(k + 1)$ th column. It follows that the number of ways of coloring the board is 6×3^{2007} .

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4. [6] Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.

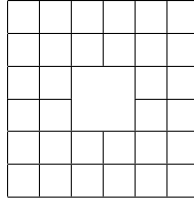
Answer: $\boxed{(-1, 1)}$ Set $p = x + 1$ and $q = y - 1$, then we get $(p + q)^2 = pq$, which simplifies to $p^2 + pq + q^2 = 0$. Then we have $(p + \frac{q}{2})^2 + \frac{3q^2}{4}$, and so $p = q = 0$. Thus $(x, y) = (-1, 1)$.

5. [6] A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?

Answer: $\boxed{\frac{416}{729}}$ Consider the error that was introduced on day 1. The probability that the Moderator misses this error on all three checks is $1/3^3$, so the probability that this error gets removed is $1 - \frac{1}{3^3}$. Similarly, the probability that the moderator misses the other two errors are $1 - \frac{1}{3^2}$ and $1 - \frac{1}{3}$. So the probability that the article is error-free is

$$\left(1 - \frac{1}{3^3}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{3}\right) = \frac{416}{729}.$$

6. [6] Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



Answer: $\boxed{297}$ First, let us count the total number of rectangles in the grid without the hole in the middle. There are $\binom{7}{2} = 21$ ways to choose the two vertical boundaries of the rectangle, and there are 21 ways to choose the two horizontal boundaries of the rectangles. This makes $21^2 = 441$ rectangles. However, we must exclude those rectangles whose boundary passes through the center point. We can count these rectangles as follows: the number of rectangles with the center of the grid lying in the interior of its south edge is $3 \times 3 \times 3 = 27$ (there are three choices for each of the three other edges); the number of rectangles whose south-west vertex coincides with the center is $3 \times 3 = 9$. Summing over all 4 orientations, we see that the total number of rectangles to exclude is $4(27 + 9) = 144$. Therefore, the answer is $441 - 144 = 297$.

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7. [6] Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \leq y \leq \pi/2$, find the value of $x + y$.

Answer: $\boxed{2007 + \frac{\pi}{2}}$ Subtracting the two equations gives $\sin y - 2008 \cos y = 1$. But since $0 \leq y \leq \pi/2$, the maximum of $\sin y$ is 1 and the minimum of $\cos y$ is 0, so we must have $\sin y = 1$, so $y = \pi/2$ and $x + y = 2007 + \frac{\pi}{2}$.

8. [6] Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?

Answer: $\boxed{1920}$ We look at the number of cottages after each wave of peasants. Let A_n be the number of cottages remaining after $8n$ minutes. During each 8 minute interval, Trodgor burns a total of $480/5 = 96$ peasants and cottages. Trodgor first burns A_n peasants and spends the remaining time burning $96 - A_n$ cottages. Therefore, as long as we do not reach negative cottages, we have the recurrence relation $A_{n+1} = A_n - (96 - A_n)$, which is equivalent to $A_{n+1} = 2A_n - 96$. Computing the first few terms of the series, we get that $A_1 = 84$, $A_2 = 72$, $A_3 = 48$, and $A_4 = 0$. Therefore, it takes Trodgor 32 minutes, which is 1920 seconds.

9. [6] Consider a circular cone with vertex V , and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV .

Answer: $\boxed{\frac{\pi}{4-\pi}}$ Let R be the radius of the base, H the height of the cone, h the height of the pyramid and let $BL/LV = x/y$. Let $[\cdot]$ denote volume. Then $[\text{cone}] = \frac{1}{3}\pi R^2 H$ and $[ABCL] = \frac{1}{3}\pi R^2 h$ and $h = \frac{x}{x+y}H$. We are given that $[\text{cone}] = 4[ABCL]$, so $x/y = \frac{\pi}{4-\pi}$.

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10. [7] Find the number of subsets S of $\{1, 2, \dots, 63\}$ the sum of whose elements is 2008.

Answer: $\boxed{6}$ Note that $1 + 2 + \dots + 63 = 2016$. So the problem is equivalent to finding the number of subsets of $\{1, 2, \dots, 63\}$ whose sum of elements is 8. We can count this by hand: $\{8\}$, $\{1, 7\}$, $\{2, 6\}$, $\{3, 5\}$, $\{1, 2, 5\}$, $\{1, 3, 4\}$.

11. [7] Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \dots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

Answer: $\boxed{\frac{2007}{2008}}$ We change the order of summation:

$$\sum_{k=2}^{\infty} \sum_{j=2}^{2008} \frac{1}{j^k} = \sum_{j=2}^{2008} \sum_{k=2}^{\infty} \frac{1}{j^k} = \sum_{j=2}^{2008} \frac{1}{j^2(1 - \frac{1}{j})} = \sum_{j=2}^{2008} \frac{1}{j(j-1)} = \sum_{j=2}^{2008} \left(\frac{1}{j-1} - \frac{1}{j} \right) = 1 - \frac{1}{2008} = \frac{2007}{2008}.$$

12. [7] Suppose we have an (infinite) cone \mathcal{C} with apex A and a plane π . The intersection of π and \mathcal{C} is an ellipse \mathcal{E} with major axis BC , such that B is closer to A than C , and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of \mathcal{C} cut up by \mathcal{E} with both spheres tangent to \mathcal{E} . What is the ratio of the radii of the spheres (smaller to larger)?

Answer: $\boxed{\frac{1}{3}}$ It can be seen that the points of tangency of the spheres with E must lie on its major axis due to symmetry. Hence, we consider the two-dimensional cross-section with plane ABC . Then the two spheres become the incentre and the excentre of the triangle ABC , and we are looking for the ratio of the inradius to the exradius. Let s , r , r_a denote the semiperimeter, inradius, and exradius (opposite to A) of the triangle ABC . We know that the area of ABC can be expressed as both rs and $r_a(s - |BC|)$, and so $\frac{r}{r_a} = \frac{s - |BC|}{s}$. For the given triangle, $s = 6$ and $a = 4$, so the required ratio is $\frac{1}{3}$.

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13. [8] Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \dots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

Answer: $\boxed{2008! - 1}$ Consider the polynomial $Q(x) = P(x) + x - 2007$. The given conditions tell us that $Q(x) = 0$ for $x = 0, 1, 2, \dots, 2007$, so these are the roots of $Q(x)$. On the other hand, we know that $Q(x)$ is also a polynomial with degree 2008 and leading coefficient 1. It follows that $Q(x) = x(x-1)(x-2)(x-3)\dots(x-2007)$. Thus

$$P(x) = x(x-1)(x-2)(x-3)\dots(x-2007) - x + 2007.$$

Setting $x = 2008$ gives the answer.

14. [8] Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

Answer: $\boxed{\frac{3}{8}}$ We have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n}{n^4+4} &= \sum_{n=1}^{\infty} \frac{n}{(n^2+2n+2)(n^2-2n+2)} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{n^2-2n+2} - \frac{1}{n^2+2n+2} \right) \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{(n-1)^2+1} - \frac{1}{(n+1)^2+1} \right). \end{aligned}$$

Observe that the sum telescopes. From this we find that the answer is $\frac{1}{4} \left(\frac{1}{0^2+1} + \frac{1}{1^2+1} \right) = \frac{3}{8}$.

15. [8] In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

Answer: $\frac{5}{21}$ If Bob initially chooses a door with a prize, then he will not find a prize by switching. With probability $\frac{5}{7}$ his original door does not hide the prize. After the host opens the three doors, the remaining three doors have equal probability of hiding the prize. Therefore, the probability that Bob finds the prize is $\frac{5}{7} \times \frac{1}{3} = \frac{5}{21}$.

Remark: This problem can be easily recognized as a variation of the classic Monty Hall problem.

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16. [9] Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Find the x -coordinate of the point X on the x -axis maximizing $\angle AXB$.

Answer: $5\sqrt{2} - 3$ Let X be a point on the x -axis and let $\theta = \angle AXB$. We can easily see that the circle with diameter AB does not meet the x -axis, so $\theta \leq \pi$. Thus, maximizing θ is equivalent to maximizing $\sin \theta$. By the Law of Sines, this in turn is equivalent to minimizing the circumradius of triangle ABX . This will occur when the circumcircle of ABX is the smaller of the two circles through A and B tangent to the x -axis. So let X now be this point of tangency. Extend line AB to meet the x -axis at $C = (-3, 0)$; by Power of a Point $CX^2 = CA \cdot CB = 50$ so $CX = 5\sqrt{2}$. Clearly X has larger x -coordinate than C , so the x -coordinate of X is $5\sqrt{2} - 3$.

17. [9] Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\dots + \sqrt{4^{2008}x + 3}}}}} - \sqrt{x} = 1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

Answer: $\frac{1}{2^{4016}}$ Rewrite the equation to get

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\dots + \sqrt{4^{2008}x + 3}}}}} = \sqrt{x} + 1.$$

Squaring both sides yields

$$\sqrt{4x + \sqrt{\dots + \sqrt{4^{2008}x + 3}}} = 2\sqrt{x} + 1.$$

Squaring again yields

$$\sqrt{16x + \sqrt{\dots + \sqrt{4^{2008}x + 3}}} = 4\sqrt{x} + 1.$$

One can see that by continuing this process one gets

$$\sqrt{4^{2008}x + 3} = 2^{2008}\sqrt{x} + 1,$$

so that $2 \cdot 2^{2008}\sqrt{x} = 2$. Hence $x = 4^{-2008}$. It is also easy to check that this is indeed a solution to the original equation.

18. [9] Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F . If $\angle BFC = 135^\circ$, determine BC/AB .

Answer: $\frac{\sqrt{13}}{2}$ Let $\alpha = \angle ADC$ and $\beta = \angle ABE$. By exterior angle theorem, $\alpha = \angle BFD + \beta = 45^\circ + \beta$. Also, note that $\tan \beta = AE/AB = AD/AB = 1/2$. Thus,

$$1 = \tan 45^\circ = \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} = \frac{\tan \alpha - \frac{1}{2}}{1 + \frac{1}{2} \tan \alpha}.$$

Solving for $\tan \alpha$ gives $\tan \alpha = 3$. Therefore, $AC = 3AD = \frac{3}{2}AB$. Using Pythagorean Theorem, we find that $BC = \frac{\sqrt{13}}{2}AB$. So the answer is $\frac{\sqrt{13}}{2}$.

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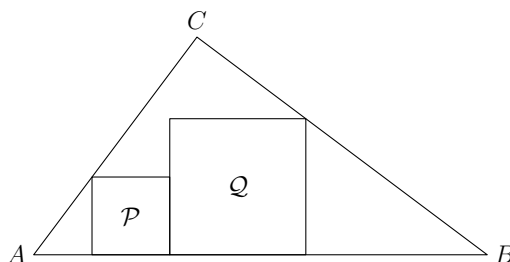
19. [10] Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD . Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin \angle PBO$.

Answer: $\boxed{\frac{1}{6}}$ We translate the problem into one about 2-D geometry. Consider the right triangle ABO , and P is some point on AO . Then, the choice of P minimizes $PA + 6PB$. Construct the line ℓ through A but outside the triangle ABO so that $\sin \angle(AO, \ell) = \frac{1}{6}$. For whichever P chosen, let Q be the projection of P onto ℓ , then $PQ = \frac{1}{6}AP$. Then, since $PA + 6PB = 6(PQ + PB)$, it is equivalent to minimize $PQ + PB$. Observe that this sum is minimized when B, P, Q are collinear and the line through them is perpendicular to ℓ (so that $PQ + PB$ is simply the distance from B to ℓ). Then, $\angle AQB = 90^\circ$, and since $\angle AOB = 90^\circ$ as well, we see that A, Q, P, B are concyclic. Therefore, $\angle PBO = \angle OPA = \angle(AO, \ell)$, and the sine of this angle is therefore $\frac{1}{6}$.

20. [10] For how many ordered triples (a, b, c) of positive integers are the equations $abc + 9 = ab + bc + ca$ and $a + b + c = 10$ satisfied?

Answer: $\boxed{21}$ Subtracting the first equation from the second, we obtain $1 - a - b - c + ab + bc + ca - abc = (1 - a)(1 - b)(1 - c) = 0$. Since a, b , and c are positive integers, at least one must equal 1. Note that $a = b = c = 1$ is not a valid triple, so it suffices to consider the cases where exactly two or one of a, b, c are equal to 1. If $a = b = 1$, we obtain $c = 8$ and similarly for the other two cases, so this gives 3 ordered triples. If $a = 1$, then we need $b + c = 9$, which has 6 solutions for $b, c \neq 1$; a similar argument for b and c gives a total of 18 such solutions. It is easy to check that all the solutions we found are actually solutions to the original equations. Adding, we find $18 + 3 = 21$ total triples.

21. [10] Let ABC be a triangle with $AB = 5, BC = 4$ and $AC = 3$. Let \mathcal{P} and \mathcal{Q} be squares inside ABC with disjoint interiors such that they both have one side lying on AB . Also, the two squares each have an edge lying on a common line perpendicular to AB , and \mathcal{P} has one vertex on AC and \mathcal{Q} has one vertex on BC . Determine the minimum value of the sum of the areas of the two squares.



Answer: $\boxed{\frac{144}{49}}$ Let the side lengths of \mathcal{P} and \mathcal{Q} be a and b , respectively. Label two of the vertices of \mathcal{P} as D and E so that D lies on AB and E lies on AC , and so that DE is perpendicular to AB . The triangle ADE is similar to ACB . So $AD = \frac{3}{4}a$. Using similar arguments, we find that

$$\frac{3a}{4} + a + b + \frac{4b}{3} = AB = 5$$

so

$$\frac{a}{4} + \frac{b}{3} = \frac{5}{7}.$$

Using Cauchy-Schwarz inequality, we get

$$(a^2 + b^2) \left(\frac{1}{4^2} + \frac{1}{3^2} \right) \geq \left(\frac{a}{4} + \frac{b}{3} \right)^2 = \frac{25}{49}.$$

It follows that

$$a^2 + b^2 \geq \frac{144}{49}.$$

Equality occurs at $a = \frac{36}{35}$ and $b = \frac{48}{35}$.

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22. [10] For a positive integer n , let $\theta(n)$ denote the number of integers $0 \leq x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

Answer: $\boxed{335}$ Let us consider the sum $\sum_{n=0}^{2009} n \cdot \theta(n) \pmod{2010}$ in another way. Consider the sum $0^2 + 1^2 + 2^2 + \dots + 2007^2 \pmod{2010}$. For each $0 \leq n < 2010$, in the latter sum, the term n appears $\theta(n)$ times, so the sum is congruent to $\sum_{n=0}^{2009} n \cdot \theta(n)$. In other words,

$$\sum_{n=0}^{2009} n \cdot \theta(n) = \sum_{n=0}^{2009} n^2 = \frac{(2009)(2009+1)(2 \cdot 2009+1)}{6} \equiv (-1) \cdot \frac{2010}{6} \cdot (-1) = 335 \pmod{2010}.$$

23. [10] Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n , and it tells Jason the number of divisors of n . Then, Kelly and Jason have the following dialogue:

Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

Jason: Oh, then I know what the value of n is.

Kelly: Now I also know what n is.

Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n ?

Answer: $\boxed{10}$ The only way in which Kelly can know that n is divisible by at least two different primes is if she is given 0 as the unit digit of n , since if she received anything else, then there is some number with that unit digit and not divisible by two primes (i.e., 1, 2, 3, 4, 5, 16, 7, 8, 9). Then, after Kelly says the first line, Jason too knows that n is divisible by 10.

The number of divisors of 10, 20, 30, 40, 50 are 4, 6, 8, 8, 6, respectively. So unless Jason received 4, he cannot otherwise be certain of what n is. It follows that Jason received 4, and thus $n = 10$.

24. [10] Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC .

Answer: $\boxed{\frac{9}{10}}$ Let Q be the point on AB so that $AQ = 2BQ$, and let X be the intersection of BP and CQ . The key observation that, as we will show, BX and CX are fixed lengths, and the ratio of areas $[ABC]/[BCX]$ is constant. So, to maximize $[ABC]$, it is equivalent to maximize $[BCX]$.

Using Menelaus' theorem on ABP , we have

$$\frac{BX \cdot PC \cdot AQ}{XP \cdot CA \cdot QB} = 1.$$

Since $PC/CA = 1/3$ and $AQ/QB = 2$, we get $BX/XP = 3/2$. It follows that $BX = 3/5$. By symmetry, $CX = 3/5$.

Also, we have

$$[ABC] = 3[BPC] = 3 \cdot \frac{5}{3}[BXC] = 5[BXC].$$

Note that $[BXC]$ is maximized when $\angle BXC = 90^\circ$ (one can check that this configuration is indeed possible). Thus, the maximum value of $[BXC]$ is $\frac{1}{2}BX \cdot CX = \frac{1}{2} \left(\frac{3}{5}\right)^2 = \frac{9}{50}$. It follows that the maximum value of $[ABC]$ is $\frac{9}{10}$.

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25. [12] Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

Answer: [35] Consider the number of beans Alice has in base 5. Note that $2008 = 31013_5$, $42 = 132_5$, and $100 = 400_5$. Now, suppose Alice has $d_k \cdots d_2 d_1$ beans when she wins; the conditions for winning mean that these digits must satisfy $d_2 d_1 = 32$, $d_k \cdots d_3 \geq 310$, and $d_k \cdots d_3 = 4i + 1$ for some i . To gain these $d_k \cdots d_2 d_1$ beans, Alice must spend at least $5(d_1 + d_2 + \cdots + d_k) + k - 1$ cents (5 cents to get each bean in the “units digit” and $k - 1$ cents to promote all the beans). We now must have $k \geq 5$ because $d_k \cdots d_2 d_1 > 2008$. If $k = 5$, then $d_k \geq 3$ since $d_k \cdots d_3 \geq 3100$; otherwise, we have $d_k \geq 1$. Therefore, if $k = 5$, we have $5(d_1 + d_2 + \cdots + d_k) + k - 1 \geq 44 > 36$; if $k > 5$, we have $5(d_1 + d_2 + \cdots + d_k) + k - 1 \geq 30 + k - 1 \geq 35$. But we can attain 36 cents by taking $d_k \cdots d_3 = 1000$, so this is indeed the minimum.

26. [12] Let \mathcal{P} be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on \mathcal{P} so that $\angle AV_1 B = 90^\circ$. Let \mathcal{Q} be the locus of the midpoint of AB . It turns out that \mathcal{Q} is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio $F_1 F_2 / V_1 V_2$.

Answer: [7/8] Since all parabolas are similar, we may assume that \mathcal{P} is the curve $y = x^2$. Then, if $A = (a, a^2)$ and $B = (b, b^2)$, the condition that $\angle AV_1 B = 90^\circ$ gives $ab + a^2 b^2 = 0$, or $ab = -1$. Then, the midpoint of AB is

$$\frac{A+B}{2} = \left(\frac{a+b}{2}, \frac{a^2+b^2}{2} \right) = \left(\frac{a+b}{2}, \frac{(a+b)^2 - 2ab}{2} \right) = \left(\frac{a+b}{2}, \frac{(a+b)^2}{2} + 1 \right).$$

(Note that $a + b$ can range over all real numbers under the constraint $ab = -1$.) It follows that the locus of the midpoint of AB is the curve $y = 2x^2 + 1$.

Recall that the focus of $y = ax^2$ is $(0, \frac{1}{4a})$. We find that $V_1 = (0, 0)$, $V_2 = (0, 1)$, $F_1 = (0, \frac{1}{4})$, $F_2 = (0, 1 + \frac{1}{8})$. Therefore, $F_1 F_2 / V_1 V_2 = \frac{7}{8}$.

27. [12] Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90^\circ$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD .

Answer: [7] By Pythagoras, $AC = 25$. Since \overline{AC} is a diameter, angles $\angle ADC$ and $\angle AEC$ are also right, so that $CE = 20$ and $AD^2 + CD^2 = AC^2$ as well. Beginning with Ptolemy’s theorem,

$$\begin{aligned} (AE \cdot CD + AC \cdot DE)^2 &= AD^2 \cdot EC^2 = (AC^2 - CD^2) EC^2 \\ &\implies CD^2 (AE^2 + EC^2) + 2 \cdot CD \cdot AE^2 \cdot AC + AC^2 (DE^2 - EC^2) = 0 \\ &\implies CD^2 + 2CD \left(\frac{AE^2}{AC} \right) + DE^2 - EC^2 = 0. \end{aligned}$$

It follows that $CD^2 + 18CD - 175 = 0$, from which $CD = 7$.

Remark: A simple trigonometric solution is possible. One writes $\alpha = \angle ACE = \angle ECD \implies \angle DAC = 90^\circ - 2\alpha$ and applies double angle formula.

28. [15] Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

Answer: $\boxed{\frac{\sqrt{2}}{3}}$ Observe that P is a truncated octahedron, formed by cutting off the corners from a regular octahedron with edge length 3. So, to compute the value of P , we can find the volume of the octahedron, and then subtract off the volume of truncated corners. Given a square pyramid where each triangular face an equilateral triangle, and whose side length is s , the height of the pyramid is $\frac{\sqrt{2}}{2}s$, and thus the volume is $\frac{1}{3} \cdot s^2 \cdot \frac{\sqrt{2}}{2}s = \frac{\sqrt{2}}{6}s^3$. The side length of the octahedron is 3, and noting that the octahedron is made up of two square pyramids, its volume must be $2 \cdot \frac{\sqrt{2}(3)^3}{6} = 9\sqrt{2}$. The six “corners” that we remove are all square pyramids, each with volume $\frac{\sqrt{2}}{6}$, and so the resulting polyhedron P has volume $9\sqrt{2} - 6 \cdot \frac{\sqrt{2}}{6} = 8\sqrt{2}$.

Finally, to find the volume of all points closer to one particular vertex than any other vertex, note that due to symmetry, every point in P (except for a set with zero volume), is closest to one of the 24 vertices. Due to symmetry, it doesn't matter which V is picked, so we can just divide the volume of P by 24 to obtain the answer $\frac{\sqrt{2}}{3}$.

29. [15] Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2 + y^2}, \quad \text{and} \quad 33x - 56y = \frac{x}{x^2 + y^2}.$$

Determine the value of $|x| + |y|$.

Answer: $\boxed{\frac{11}{65}}$ Observe that

$$\frac{1}{x + yi} = \frac{x - yi}{x^2 + y^2} = 33x - 56y + (56x + 33y)i = (33 + 56i)(x + yi).$$

So

$$(x + yi)^2 = \frac{1}{33 + 56i} = \frac{1}{(7 + 4i)^2} = \left(\frac{7 - 4i}{65}\right)^2.$$

It follows that $(x, y) = \pm \left(\frac{7}{65}, -\frac{4}{65}\right)$.

30. [15] Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120^\circ$. Points P and Q lie on segment BC such that

$$\begin{aligned} AB^2 + BC \cdot CP &= BC^2 \\ 3AC^2 + 2BC \cdot CQ &= BC^2 \end{aligned}$$

Find $\angle PAQ$ in degrees.

Answer: $\boxed{40^\circ}$ We have $AB^2 = BC(BC - CP) = BC \cdot BP$, so triangle ABC is similar to triangle PBA . Also, $AB^2 = BC(BC - 2CQ) + AC^2 = (BC - CQ)^2 - CQ^2 + AC^2$, which rewrites as $AB^2 + CQ^2 = BQ^2 + AC^2$. We deduce that Q is the foot of the altitude from A . Thus, $\angle PAQ = 90^\circ - \angle QPA = 90^\circ - \angle ABP - \angle BAP$. Using the similar triangles, $\angle PAQ = 90^\circ - \angle ABC - \angle BCA = \angle BAC - 90^\circ = 40^\circ$.

31. [18] Let \mathcal{C} be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x -axis, we construct a sequence of points (P_n) on the x -axis in the following manner: let ℓ_n be the line with slope 1 passing passing

through P_n , then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and \mathcal{C} onto the x -axis. (If $P_n = 0$, then the sequence simply terminates.)

Let N be the number of starting positions P_0 on the x -axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

Answer: 254 Let $P_n = (x_n, 0)$. Then the ℓ_n meet \mathcal{C} at $(x_{n+1}, x_{n+1} - x_n)$. Since this point lies on the hyperbola, we have $(x_{n+1} - x_n)^2 - x_{n+1}^2 = 1$. Rearranging this equation gives

$$x_{n+1} = \frac{x_n^2 - 1}{2x_n}.$$

Choose a $\theta_0 \in (0, \pi)$ with $\cot \theta_0 = x_0$, and define $\theta_n = 2^n \theta_0$. Using the double-angle formula, we have

$$\cot \theta_{n+1} = \cot(2\theta_n) = \frac{\cot^2 \theta_n - 1}{2 \cot \theta_n}.$$

It follows by induction that $x_n = \cot \theta_n$. Then, $P_0 = P_{2008}$ corresponds to $\cot \theta_0 = \cot(2^{2008} \theta_0)$ (assuming that P_0 is never at the origin, or equivalently, $2^n \theta$ is never an integer multiple of π). So, we need to find the number of $\theta_0 \in (0, \pi)$ with the property that $2^{2008} \theta_0 - \theta_0 = k\pi$ for some integer k . We have $\theta_0 = \frac{k\pi}{2^{2008}-1}$, so k can be any integer between 1 and $2^{2008} - 2$ inclusive (and note that since the denominator is odd, the sequence never terminates). It follows that the number of starting positions is $N = 2^{2008} - 2$.

Finally, we need to compute the remainder when N is divided by 2008. We have $2008 = 2^3 \times 251$. Using Fermat's Little Theorem with 251, we get $2^{2008} \equiv (2^{250})^4 \cdot 256 \equiv 1^4 \cdot 5 = 5 \pmod{251}$. So we have $N \equiv 3 \pmod{251}$ and $N \equiv -2 \pmod{8}$. Using Chinese Remainder Theorem, we get $N \equiv 254 \pmod{2008}$.

32. [18] Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

Answer: $\frac{225\sqrt{11}}{88}$ Let C' be the point on minor arc BCD such that $BC' = 12$ and $C'D = 5$, and write $AC' = BD = C'E = x$, $AD = y$, and $BE = z$. Ptolemy applied to quadrilaterals $ABC'D$, $BC'DE$, and $ABDE$ gives

$$\begin{aligned} x^2 &= 12y + 5^2 \\ x^2 &= 5z + 12^2 \\ yz &= 14x + 5 \cdot 12 \end{aligned}$$

Then

$$(x^2 - 5^2)(x^2 - 12^2) = 5 \cdot 12yz = 5 \cdot 12 \cdot 14x + 5^2 \cdot 12^2,$$

from which $x^3 - 169x - 5 \cdot 12 \cdot 14 = 0$. Noting that $x > 13$, the rational root theorem leads quickly to the root $x = 15$. Then triangle BCD has area $\sqrt{16 \cdot 1 \cdot 4 \cdot 11} = 8\sqrt{11}$ and circumradius $R = \frac{5 \cdot 12 \cdot 15}{4 \cdot 8\sqrt{11}} = \frac{225\sqrt{11}}{88}$.

33. [18] Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.

Answer: $\frac{6}{5}$ Let $\sigma_1 = a + b + c$, $\sigma_2 = ab + bc + ca$ and $\sigma_3 = abc$ be the three elementary symmetric polynomials. Since $a^3 + b^3 + c^3$ is a symmetric polynomial, it can be written as a polynomial in σ_1 , σ_2 and σ_3 . Now, observe that $\sigma_1 = 0$, and so we only need to worry about the terms not containing σ_1 . By considering the degrees of the terms, we see that the only possibility is σ_3 . That is, $a^3 + b^3 + c^3 = k\sigma_3$ for some constant k . By setting $a = b = 1$, $c = -2$, we see that $k = 3$.

By similar reasoning, we find that $a^5 + b^5 + c^5 = h\sigma_2\sigma_3$ for some constant h . By setting $a = b = 1$ and $c = -2$, we get $h = -5$.

So, we now know that $a + b + c = 0$ implies

$$a^3 + b^3 + c^3 = 3abc \quad \text{and} \quad a^5 + b^5 + c^5 = -5abc(ab + bc + ca)$$

Then $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$ implies that $3abc = -5abc(ab + bc + ca)$. Given that a, b, c are nonzero, we get $ab + bc + ca = -\frac{3}{5}$. Then, $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = \frac{6}{5}$.

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34. **Who Wants to Be a Millionaire.** In 2000, the Clay Mathematics Institute named seven *Millennium Prize Problems*, with each carrying a prize of \$1 Million for its solution. Write down the name of ONE of the seven Clay Millennium Problems. If your submission is incorrect or misspelled, then your submission is disqualified. If another team wrote down the same Millennium Problem as you, then you get 0 points, otherwise you get 20 points.

Solution: The seven Millennium Prize Problems are:

- (a) Birch and Swinnerton-Dyer Conjecture
- (b) Hodge Conjecture
- (c) Navier-Stokes Equations
- (d) P vs NP
- (e) Poincaré Conjecture
- (f) Riemann Hypothesis
- (g) Yang-Mills Theory

More information can be found on its official website <http://www.claymath.org/millennium/>.

As far as this as an HMMT problem goes, it's probably a good idea to submit something that you think is least likely for another team to think of (or to spell correctly). Though, this may easily turn into a contest of who can still remember the names of the user ranks from the Art of Problem Solving forum.

35. **NUMB3RS.** The RSA Factoring Challenge, which ended in 2007, challenged computational mathematicians to factor extremely large numbers that were the product of two prime numbers. The largest number successfully factored in this challenge was RSA-640, which has 193 decimal digits and carried a prize of \$20,000. The next challenge number carried prize of \$30,000, and contains N decimal digits. Your task is to submit a guess for N . Only the team(s) that have the closest guess(es) receives points. If k teams all have the closest guesses, then each of them receives $\lceil \frac{20}{k} \rceil$ points.

Answer: 212 For more information, see the Wikipedia entry at http://en.wikipedia.org/wiki/RSA_Factoring_Challenge.

RSA-640 was factored in November 2005, and the effort took approximately 30 2.2GHz-Opteron-CPU years over five months of calendar time.

36. **The History Channel.** Below is a list of famous mathematicians. Your task is to list a subset of them in the chronological order of their birth dates. Your submission should be a sequence of letters. If your sequence is not in the correct order, then you get 0 points. Otherwise your score will be $\min\{\max\{5(N - 4), 0\}, 25\}$, where N is the number of letters in your sequence.

(A) Niels Abel (B) Arthur Cayley (C) Augustus De Morgan (D) Gustav Dirichlet (E) Leonhard Euler (F) Joseph Fourier (G) Évariste Galois (H) Carl Friedrich Gauss (I) Marie-Sophie Germain (J) Joseph Louis Lagrange (K) Pierre-Simon Laplace (L) Henri Poincaré (N) Bernhard Riemann

Answer: any subsequence of EJKFIHADCGBNL The corresponding birth dates are listed below:

- (A) Niels Abel (1802–1829)

- (B) Arthur Cayley (1821–1895)
- (C) Augustus De Morgan (1806–1871)
- (D) Gustav Dirichlet (1805–1859)
- (E) Leonhard Euler (1707–1783)
- (F) Joseph Fourier (1768–1830)
- (G) Évariste Galois (1811–1832)
- (H) Carl Friedrich Gauss (1777–1855)
- (I) Marie-Sophie Germain (1776–1831)
- (J) Joseph Louis Lagrange (1736–1813)
- (K) Pierre-Simon Laplace (1749–1827)
- (L) Henri Poincaré (1854–1912)
- (N) Bernhard Riemann (1826–1866)

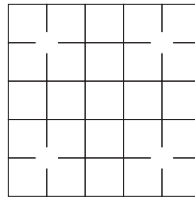
11th Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: A Division

Lattice Walks [90]

1. [20] Determine the number of ways of walking from $(0, 0)$ to $(5, 5)$ using only up and right unit steps such that the path does not pass through any of the following points: $(1, 1)$, $(1, 4)$, $(4, 1)$, $(4, 4)$.



Answer: 34

Solution: In the following figure, each lattice point (with the bottom-left-most point $(0, 0)$) is labeled with the number of ways of reaching there from $(0, 0)$. With the exception of the forbidden points, the labels satisfy the recursion formula $f(x, y) = f(x - 1, y) + f(x, y - 1)$. We see from the diagram that there are 34 ways to reach $(5, 5)$.

	1	1	5	17	17	34
1	0	4	12	0	17	
1	2	4	8	12	17	
1	1	2	4	4	5	
1	0	1	2	0	1	
1	1	1	1	1	1	

2. [20] Let $n > 2$ be a positive integer. Prove that there are $\frac{1}{2}(n - 2)(n + 1)$ ways to walk from $(0, 0)$ to $(n, 2)$ using only up and right unit steps such that the walk never visits the line $y = x$ after it leaves the origin.

Solution: The first two steps can only go to the right. Then we need to compute the number of ways of walking from $(2, 0)$ to $(n, 2)$ which does not pass through the point $(2, 2)$. There are $\binom{n}{2}$ ways to walk from $(2, 0)$ to $(n, 2)$, and exactly one of those paths passes through the point $(2, 2)$. So the number of valid paths is $\binom{n}{2} - 1 = \frac{1}{2}n(n - 1) - 1 = \frac{1}{2}(n - 2)(n + 1)$.

Remark: We used the well-known fact that there are $\binom{a+b}{a}$ ways to walk from $(0, 0)$ to (a, b) using only up and right unit steps. This is true because there are $a + b$ steps, and we need to choose a of them to be right steps, and the rest up steps.

3. [20] Let $n > 4$ be a positive integer. Determine the number of ways to walk from $(0, 0)$ to $(n, 2)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 2$ except at the start and at the end.

Answer: $\boxed{\frac{1}{2}(n^2 - 5n + 2)}$

Solution: It is easy to see the the first two steps and the last two steps must all be right steps. So we need to compute the number of walks from $(2, 0)$ to $(n - 2, 0)$ that does not pass through $(2, 2)$ and $(n - 2, 0)$. There are $\binom{n-2}{2}$ paths from $(2, 0)$ to $(n - 2, 0)$, and exactly two of them are invalid. So the answer is $\binom{n-2}{2} - 2 = \frac{1}{2}(n - 2)(n - 3) - 2 = \frac{1}{2}(n^2 - 5n + 2)$.

4. [30] Let $n > 6$ be a positive integer. Determine the number of ways to walk from $(0, 0)$ to $(n, 3)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 3$ except at the start and at the end.

Answer: $\boxed{\frac{1}{6}(n - 6)(n - 1)(n + 1)}$ Consider the first point of the path that lies on $x = 3$. There are two possibilities for this point: $(3, 0)$ and $(3, 1)$, and there is exactly one valid way of getting to each point from the origin. Similarly, consider the last point of the path that lies on $x = n - 3$. There are two possibilities: $(n - 3, 2)$ and $(n - 3, 3)$, and there is exactly one valid way of getting to the destination from each of the two points. Now we count the number of valid paths from each of $(3, 0)$ and $(3, 1)$, to each of $(n - 3, 2)$ and $(n - 3, 3)$, and the answer will be the sum.

- From $(3, 1)$ to $(n - 3, 2)$: there are no forbidden points along the way, so there are $n - 5$ ways.
- From $(3, 0)$ to $(n - 3, 2)$: the path must not pass through $(n - 3, 0)$, and there is exactly one invalid path. So there are $\binom{n-4}{2} - 1$ ways.
- From $(3, 1)$ to $(n - 3, 3)$: the path must not pass through $(3, 3)$, and there is exactly one invalid path. So there are $\binom{n-4}{2} - 1$ ways.
- From $(3, 0)$ to $(n - 3, 3)$: the path must not pass through $(n - 3, 0)$ and $(3, 3)$, and there are exactly two invalid paths. So there are $\binom{n-3}{3} - 2$ ways.

Summing, we obtain the answer:

$$n - 5 + \binom{n - 4}{2} - 1 + \binom{n - 4}{2} - 1 + \binom{n - 3}{3} - 2 = \frac{n^3 - 6n^2 - n + 6}{6} = \frac{(n - 6)(n - 1)(n + 1)}{6}.$$

Lattice and Centroids [130]

A d -dimensional *lattice point* is a point of the form (x_1, x_2, \dots, x_d) where x_1, x_2, \dots, x_d are all integers. For a set of d -dimensional points, their *centroid* is the point found by taking the coordinate-wise average of the given set of points.

Let $f(n, d)$ denote the minimal number f such that any set of f lattice points in the d -dimensional Euclidean space contains a subset of size n whose centroid is also a lattice point.

5. [10] Let S be a set of 5 points in the 2-dimensional lattice. Show that we can always choose a pair of points in S whose midpoint is also a lattice point.

Solution: Consider the parities of the coordinates. There are four possibilities: (odd, odd), (odd, even), (even, odd), (even, even). By the pigeonhole principle, two of the points must have the same parity in both coordinates (i.e., they are congruent in mod 2). Then, the midpoint of these two points must be a lattice point.

6. [10] Construct a set of 2^d d -dimensional lattice points so that for any two chosen points A, B , the line segment AB does not pass through any other lattice point.

Solution: The simplest example is the set of 2^d points of the form (a_1, a_2, \dots, a_d) , where $a_k \in \{0, 1\}$ for each k . This is the set of vertices of a d -dimensional cube.

7. [35] Show that for positive integers n and d ,

$$(n-1)2^d + 1 \leq f(n, d) \leq (n-1)n^d + 1.$$

Solution: Note that taking the set of points to be a multiset does not affect $f(n, d)$ as adding multiples of n to any of the coordinate values does not change the result. The lower bound is obtained by considering the multiset consisting of $n-1$ copies of each of the 2^d 0, 1-vectors of length d , as it contains no submultiset of size n whose centroid is also a lattice point. By the pigeonhole principle, any multiset of $(n-1)n^d + 1$ lattice points must contain n points whose coordinates are congruent modulo n . The centroid of these n points is also a lattice point, thus proving the upper bound.

8. [40] Show that for positive integers n_1, n_2 and d ,

$$f(n_1 n_2, d) \leq f(n_1, d) + n_1 (f(n_2, d) - 1).$$

Solution: Given a multiset of $f(n_1, d) + n_1 (f(n_2, d) - 1)$ lattice points, we may select $l = f(n_2, d)$ pairwise disjoint submultisets S_1, S_2, \dots, S_l , each consisting of n_1 points, whose centroid is a lattice point. Let φ map each multiset S_i to its centroid g_i . By the definition of $f(n_2, d)$, there exists a submultiset $T \subset \{g_1, g_2, \dots, g_l\}$ satisfying $|T| = n_2$ whose centroid is a lattice point. Then $\bigcup_{i \in T} \varphi^{-1}(g_i)$ is a multiset of $n_1 n_2$ lattice points whose centroid is also a lattice point.

9. [35] Determine, with proof, a simple closed-form expression for $f(2^a, d)$.

Answer: $(2^a - 1)2^d + 1$

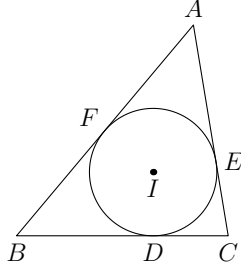
Solution: From Problem ??, $f(2^a, d) \geq (2^a - 1)2^d + 1$. We prove by induction on a that $f(2^a, d) \leq (2^a - 1)2^d + 1$. When $a = 1$, Problem ?? shows that $f(2, d) \leq 2^d + 1$. Fix $a > 1$ and suppose that the assertion holds for smaller values of a . Using Problem ??,

$$\begin{aligned} f(2^a, d) &\leq f(2, d) + 2 (f(2^{a-1}, d) - 1) \\ &\leq 2^d + 1 + 2 \cdot (2^{a-1} - 1)2^d \\ &= (2^a - 1)2^d + 1. \end{aligned}$$

Thus $f(2^a, d) = (2^a - 1)2^d + 1$.

Incircles [180]

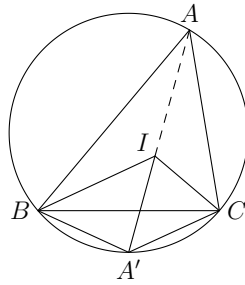
In the following problems, ABC is a triangle with incenter I . Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB , respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. On the circumcircle of ABC , let A' be the midpoint of arc BC (not containing A).
- (a) [10] Show that A, I, A' are collinear.
 - (b) [20] Show that A' is the circumcenter of BIC .

Solution:



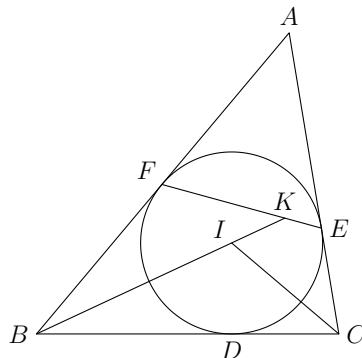
- (a) Since A' bisectors the arc BC , the two arcs $A'B$ and $A'C$ are equal, and so $\angle BAA' = \angle CAA'$. Thus, A' lies on the angle bisector of BAC . Since I also lies on the angle bisector of BAC , we see that A, I, A' are collinear.
- (b) We have

$$\angle CIA' = \angle A'AC + \angle ICA = \angle A'AB + \angle ICB = \angle A'CB + \angle ICB = \angle ICA'.$$

Therefore, $A'I = A'C$. By similar arguments, $A'I = A'B$. So, A' is equidistant from B, I, C , and thus is its circumcenter.

11. [30] Let lines BI and EF meet at K . Show that I, K, E, C, D are concyclic.

Solution: First, note that there are two possible configurations, as K could lie inside segment EF , or on its extension. The following proof works for both cases. We have

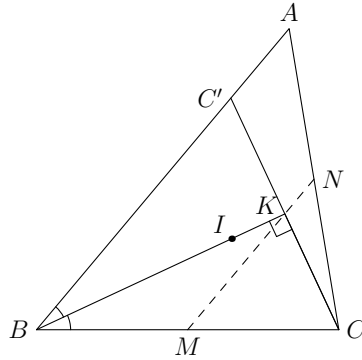


$$\angle KIC = \angle IBC + \angle ICB = \frac{1}{2}\angle ABC + \frac{1}{2}\angle ACB = 90^\circ - \frac{1}{2}\angle BAC = \angle AEF.$$

It follows that I, K, E, C are concyclic. The point D also lies on this circle because $\angle IDC = \angle IEC = 90^\circ$. Thus, all five points are concyclic.

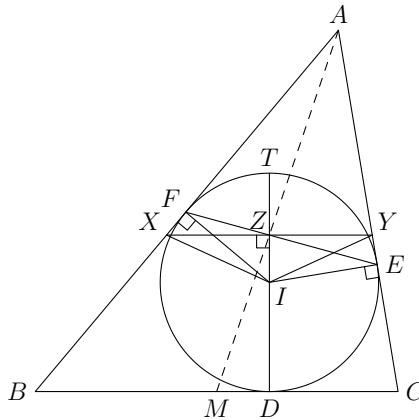
12. [40] Let K be as in the previous problem. Let M be the midpoint of BC and N the midpoint of AC . Show that K lies on line MN .

Solution: Since I, K, E, C are concyclic, we have $\angle IKC = \angle IEC = 90^\circ$. Let C' be the reflection of C across BI , then C' must lie on AB . Then, K is the midpoint of CC' . Consider a dilation centered at C with factor $\frac{1}{2}$. Since C' lies on AB , it follows that K lies on MN .



13. [40] Let M be the midpoint of BC , and T diametrically opposite to D on the incircle of ABC . Show that DT, AM, EF are concurrent.

Solution: If $AB = AC$, then the result is clear as AM and DT coincide. So, assume that $AB \neq AC$.



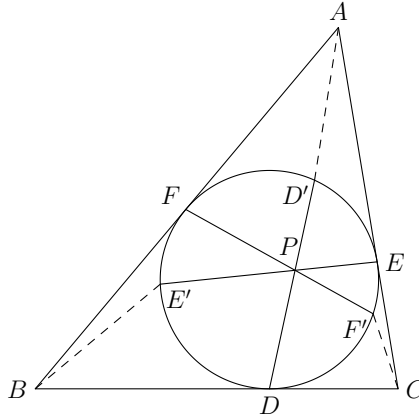
Let lines DT and EF meet at Z . Construct a line through Z parallel to BC , and let it meet AB and AC at X and Y , respectively. We have $\angle XZI = 90^\circ$, and $\angle XFI = 90^\circ$. Therefore, F, Z, I, X are concyclic, and thus $\angle IXZ = \angle IFZ$. By similar arguments, we also have $\angle IYZ = \angle IEZ$. Thus, triangles IFE and IXY are similar. Since $IE = IF$, we must also have $IX = IY$. Since IZ is an altitude of the isosceles triangle IXY , Z is the midpoint of XY .

Since XY and BC are parallel, there is a dilation centered at A that sends XY to BC . So it must send the midpoint Z to the midpoint M . Therefore, A, Z, M are collinear. It follows that DT, AM, EF are concurrent.

14. [40] Let P be a point inside the incircle of ABC . Let lines DP, EP, FP meet the incircle again at D', E', F' . Show that AD', BE', CF' are concurrent.

Solution: Using the trigonometric version of Ceva's theorem, it suffices to prove that

$$\frac{\sin \angle BAD'}{\sin \angle D'AC} \cdot \frac{\sin \angle CBE'}{\sin \angle E'BA} \cdot \frac{\sin \angle ACF'}{\sin \angle F'CB} = 1. \quad (\dagger)$$



Using sine law, we have

$$\sin \angle BAD' = \frac{FD'}{AD'} \cdot \sin \angle AFD' = \frac{FD'}{AD'} \cdot \sin \angle FDD'$$

Let r be the inradius of ABC . Using the extended sine law, we have $FD' = 2r \sin \angle FDD'$. Therefore,

$$\sin \angle BAD' = \frac{2r}{AD'} \cdot \sin^2 \angle FDD'.$$

Do this for all the factors in (\dagger) , and we get

$$\frac{\sin \angle BAD'}{\sin \angle D'AC} \cdot \frac{\sin \angle CBE'}{\sin \angle E'BA} \cdot \frac{\sin \angle ACF'}{\sin \angle F'CB} = \left(\frac{\sin \angle FDD'}{\sin \angle D'DE} \cdot \frac{\sin \angle DEE'}{\sin \angle E'EF} \cdot \frac{\sin \angle EFF'}{\sin \angle F'FD} \right)^2$$

Since DD', EE', FF' are concurrent, the above expression equals to 1 by using trig Ceva on triangle DEF . The result follows.

Remark: This result is known as *Steinbart Theorem*. Beware that its converse is not completely true. For more information and discussion, see Darij Grinberg's paper "Variations of the Steinbart Theorem" at http://de.geocities.com/darij_grinberg/.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point. A set of points are *conyclic* if they lie on a common circle.

- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC , tangent to all three sides.
- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB , respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

- "*Trig*" *Ceva* states that given ABC a triangle, and points X, Y, Z inside the triangle, the lines AX, BY, CZ are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$

11th Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: B Division

Tropical Mathematics [95]

For real numbers x and y , let us consider the two operations \oplus and \odot defined by

$$x \oplus y = \min(x, y) \quad \text{and} \quad x \odot y = x + y.$$

We also include ∞ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all x . When unspecified, \odot precedes \oplus in the order of operations.

1. [10] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.

Solution: This is equivalent to proving that

$$\min(x, y) + z = \min(x + z, y + z).$$

Consider two cases. If $x \leq y$, then $LHS = x + z$ and $RHS = x + z$. If $x > y$, then $LHS = y + z$ and $RHS = y + z$. It follows that $LHS = RHS$.

2. [10] (Freshman's Dream) Let z^n denote $z \odot z \odot z \odot \cdots \odot z$ with z appearing n times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer n .

Solution: Without loss of generality, suppose that $x \leq y$, then $LHS = \min(x, y)^n = x^n = nx$, and $RHS = \min(x^n, y^n) = \min(nx, ny) = nx$.

3. [35] By a *tropical polynomial* we mean a function of the form

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

where exponentiation is as defined in the previous problem.

Let p be a tropical polynomial. Prove that

$$p\left(\frac{x+y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)

Solution: First, note that for any $x_1, \dots, x_n, y_1, \dots, y_n$, we have

$$\min\{x_1 + y_1, x_2 + y_2, \dots, x_n + y_n\} \geq \min\{x_1, x_2, \dots, x_n\} + \min\{y_1, y_2, \dots, y_n\}.$$

Indeed, suppose that $x_m + y_m = \min_i\{x_i + y_i\}$, then $x_m \geq \min_i x_i$ and $y_m \geq \min_i y_i$, and so $\min_i\{x_i + y_i\} = x_m + y_m \geq \min_i x_i + \min_i y_i$.

Now, let us write a tropical polynomial in a more familiar notation. We have

$$p(x) = \min_{0 \leq k \leq n} \{a_k + kx\}.$$

So

$$\begin{aligned}
p\left(\frac{x+y}{2}\right) &= \min_{0 \leq k \leq n} \left\{ a_k + k \left(\frac{x+y}{2}\right) \right\} \\
&= \frac{1}{2} \min_{0 \leq k \leq n} \{(a_k + kx) + (a_k + ky)\} \\
&\geq \frac{1}{2} \left(\min_{0 \leq k \leq n} \{a_k + kx\} + \min_{0 \leq k \leq n} \{a_k + ky\} \right) \\
&= \frac{1}{2} (p(x) + p(y)).
\end{aligned}$$

4. [40] (Fundamental Theorem of Algebra) Let p be a tropical polynomial:

$$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0, \quad a_n \neq \infty$$

Prove that we can find $r_1, r_2, \dots, r_n \in \mathbb{R} \cup \{\infty\}$ so that

$$p(x) = a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n)$$

for all x .

Solution: Again, we have

$$p(x) = \min_{0 \leq k \leq n} \{a_k + kx\}.$$

So the graph of $y = p(x)$ can be drawn as follows: first, draw all the lines $y = a_k + kx$, $k = 0, 1, \dots, n$, then trace out the lowest broken line, which then is the graph of $y = p(x)$.

So $p(x)$ is piecewise linear and continuous, and has slopes from the set $\{0, 1, 2, \dots, n\}$. We know from the previous problem that $p(x)$ is concave, and so its slope must be decreasing (this can also be observed simply from the drawing of the graph of $y = p(x)$). Then, let r_k denote the x -coordinate of the leftmost kink such that the slope of the graph is less than k to the right of this kink. Then, $r_n \leq r_{n-1} \leq \cdots \leq r_1$, and for $r_{k-1} \leq x \leq r_k$, the graph of p is linear with slope k . Note that is if possible that $r_{k-1} = r_k$, if no segment of p has slope k . Also, since $a_n \neq \infty$, the leftmost piece of $p(x)$ must have slope n , and thus r_n exists, and thus all r_i exist.

Now, compare $p(x)$ with

$$\begin{aligned}
q(x) &= a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n) \\
&= a_n + \min(x, r_1) + \min(x, r_2) + \cdots + \min(x, r_n).
\end{aligned}$$

For $r_{k-1} \leq x \leq r_k$, the slope of $q(x)$ is k , and for $x \leq r_n$ the slope of q is n and for $x \geq r_1$ the slope of q is 0. So q is piecewise linear, and of course it is continuous. It follows that the graph of q coincides with that of p up to a translation. By taking any $x < r_n$, we see that $q(x) = a_n + nx = p(x)$, we see that the graphs of p and q coincide, and thus they must be the same function.

Juggling [125]

A *juggling sequence* of length n is a sequence $j(\cdot)$ of n nonnegative integers, usually written as a string

$$j(0)j(1)\dots j(n-1)$$

such that the mapping $f : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by

$$f(t) = t + j(\bar{t})$$

is a permutation of the integers. Here \bar{t} denotes the remainder of t when divided by n . In this case, we say that f is the corresponding *juggling pattern*.

For a juggling pattern f (or its corresponding juggling sequence), we say that it has b *balls* if the permutation induces b infinite orbits on the set of integers. Equivalently, b is the maximum number such that we can find a set of b integers $\{t_1, t_2, \dots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \dots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \dots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time t , we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no throw takes place at time t , which could correspond to an empty hand. Then, b is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from t to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then b is simply the number of contiguous “paths” drawn, which is 3 in this case.

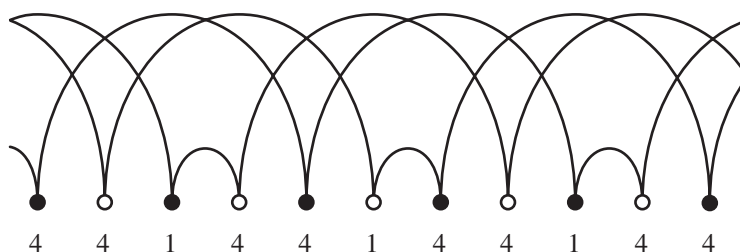


Figure 1: Juggling diagram of 441.

5. [10] Prove that 572 is not a juggling sequence.

Solution: We are given $j(0) = 5$, $j(1) = 7$ and $j(2) = 2$. So $f(3) = 3 + j(0) = 8$ and $f(1) = 1 + j(1) = 8$. Thus $f(3) = f(1)$ and so f is not a permutation of \mathbb{Z} , and hence 572 is not a juggling pattern. (In other words, there is a “collision” at times $t \equiv 2 \pmod{3}$.)

6. [40] Suppose that $j(0)j(1)\cdots j(n-1)$ is a valid juggling sequence. For $i = 0, 1, \dots, n-1$, let a_i denote the remainder of $j(i) + i$ when divided by n . Prove that $(a_0, a_1, \dots, a_{n-1})$ is a permutation of $(0, 1, \dots, n-1)$.

Solution: Suppose that $a_i = j(i) + i - b_i n$, where b_i is an integer. Note that $f(i - b_i n) = i - b_i n + j(i) = a_i$. Since $\{i - b_i n \mid i = 0, 1, \dots, n-1\}$ contains n distinct integers (as their residue mod n are all distinct), and f is a permutation, we see that after applying the map f , the resulting set $\{a_0, a_1, \dots, a_{n-1}\}$ is a set of n distinct integers. Since $0 \leq a_i < n$ from definition, we see that $(a_0, a_1, \dots, a_{n-1})$ is a permutation of $(0, 1, \dots, n-1)$.

7. [30] Determine the number of juggling sequences of length n with exactly 1 ball.

Answer: $2^n - 1$. **Solution:** With 1 ball, we simply need to decide at times should the ball land in our hand. That is, we need to choose a non-empty subset of $\{0, 1, 2, \dots, n-1\}$ where the ball lands. It follows that the answer is $2^n - 1$.

8. [40] Prove that the number of balls b in a juggling sequence $j(0)j(1)\cdots j(n-1)$ is simply the average

$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n}.$$

Solution: Consider the corresponding juggling diagram. Say the *length* of an curve from t to $f(t)$ is $f(t) - t$. Let us draw only the curves whose left endpoint lies inside $[0, Mn - 1]$. For every single ball, the sum of the lengths of the arrows drawn corresponding to that ball is between $Mn - J$ and $Mn + J$, where $J = \max\{j(0), j(1), \dots, j(n-1)\}$. It follows that the sum of the lengths of the arrows drawn is between $b(Mn - J)$ and $b(Mn + J)$. Since the arrow drawn at t has length $j(\bar{t})$, the sum of the lengths of the arrows drawn is $M(j(0) + j(1) + \cdots + j(n-1))$. It follows that

$$b(Mn - J) \leq M(j(0) + j(1) + \cdots + j(n-1)) \leq b(Mn + J).$$

Dividing by Mn , we get

$$b \left(1 - \frac{J}{nM}\right) \leq \frac{j(0) + j(1) + \cdots + j(n-1)}{n} \leq b \left(1 + \frac{J}{nM}\right).$$

Since we can take M to be arbitrarily large, we must have

$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n},$$

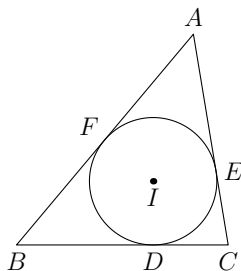
as desired.

9. [5] Show that the converse of the previous statement is false by providing a non-juggling sequence $j(0)j(1)j(2)$ of length 3 where the average $\frac{1}{3}(j(0) + j(1) + j(2))$ is an integer. Show that your example works.

Solution: One such example is 210. This is not a juggling sequence since $f(0) = f(1) = 2$.

Incircles [180]

In the following problems, ABC is a triangle with incenter I . Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB , respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. [15] Let a, b, c denote the side lengths of BC, CA, AB . Find the lengths of AE, BF, CD in terms of a, b, c .

Solution: Let $x = AE = AF$, $y = BD = BF$, $z = CD = CE$. Since $BC = BD + CD$, we have $a = x + y$. Similarly with the other sides, we arrive at the following system of equations:

$$a = y + z, \quad b = x + z, \quad c = x + y.$$

Solving this system gives us

$$\begin{aligned} AE = x &= \frac{b + c - a}{2}, \\ BF = y &= \frac{a + c - b}{2}, \\ CD = z &= \frac{a + b - c}{2}. \end{aligned}$$

11. [15] Show that lines AD, BE, CF pass through a common point.

Solution: Using Ceva's theorem on triangle ABC , we see that it suffices to show that

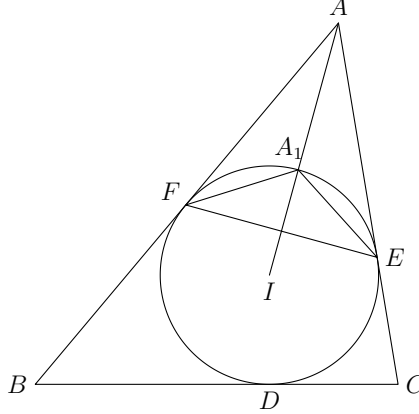
$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

Since $AF = AE$, $BD = BF$, and $CD = CE$ (due to equal tangents), we see that the LHS is indeed 1.

Remark: The point of concurrency is known as the *Gergonne point*.

12. [35] Show that the incenter of triangle AEF lies on the incircle of ABC .

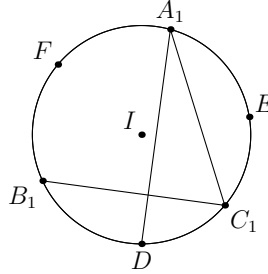
Solution: Let segment AI meet the incircle at A_1 . Let us show that A_1 is the incenter of AEF .



Since $AE = AF$ and AA' is the angle bisector of $\angle EAF$, we find that $A_1E = A_1F$. Using tangent-chord, we see that $\angle AFA_1 = \angle A_1EF = \angle A_1FE$. Therefore, A_1 lies on the angle bisector of $\angle AFE$. Since A_1 also lies on the angle bisector of $\angle EAF$, A_1 must be the incenter of AEF , as desired.

13. [35] Let A_1, B_1, C_1 be the incenters of triangle AEF, BDF, CDE , respectively. Show that A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

Solution: Using the result from the previous problem, we see that A_1, B_1, C_1 are respectively the midpoints of the arc FE, FD, DF of the incircle. We have



$$\begin{aligned} \angle DA_1C_1 + \angle B_1C_1A_1 &= \frac{1}{2}\angle DIC_1 + \frac{1}{2}\angle B_1IF + \frac{1}{2}\angle FIA_1 \\ &= \frac{1}{4}(\angle EID + \angle DIF + \angle FIE) \\ &= \frac{1}{4} \cdot 360^\circ \\ &= 90^\circ. \end{aligned}$$

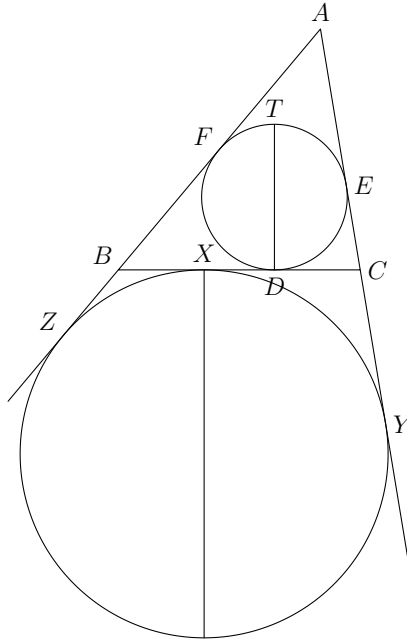
It follows that A_1D is perpendicular to B_1C_1 , and thus A_1D passes through the orthocenter of $A_1B_1C_1$. Similarly, A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

14. [40] Let X be the point on side BC such that $BX = CD$. Show that the excircle ABC opposite of vertex A touches segment BC at X .

Solution: Let the excircle touch lines BC, AC and AB at X', Y and Z , respectively. Using the equal tangent property repeatedly, we have

$$BX' - X'C = BZ - CY = (EY - CY) - (FZ - BZ) = CE - BF = CD - BD.$$

It follows that $BX' = CD$, and thus $X' = X$. So the excircle touches BC at X .

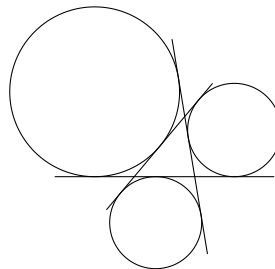


15. [40] Let X be as in the previous problem. Let T be the point diametrically opposite to D on the incircle of ABC . Show that A, T, X are collinear.

Solution: Consider a dilation centered at A that carries the incircle to the excircle. This dilation must send the diameter DT to some the diameter of excircle that is perpendicular to BC . The only such diameter is the one goes through X . It follows that T gets carried to X . Therefore, A, T, X are collinear.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC , tangent to all three sides.
- The *excircles* of a triangle ABC are the three circles on the exterior the triangle but tangent to all three lines AB, BC, CA .



- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.

- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB , respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

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Individual Round

1. [2] Find the minimum of $x^2 - 2x$ over all real numbers x .

Answer: $\boxed{-1}$ Write $x^2 - 2x = x^2 - 2x + 1 - 1 = (x - 1)^2 - 1$. Since $(x - 1)^2 \geq 0$, it is clear that the minimum is -1 .

Alternate method: The graph of $y = x^2 - 2x$ is a parabola that opens up. Therefore, the minimum occurs at its vertex, which is at $\frac{-b}{2a} = \frac{-(-2)}{2} = 1$. But $1^2 - 2 \cdot 1 = -1$, so the minimum is -1 .

2. [3] What is the units digit of 7^{2009} ?

Answer: $\boxed{7}$ Note that the units digits of $7^1, 7^2, 7^3, 7^4, 7^5, 7^6, \dots$ follows the pattern $7, 9, 3, 1, 7, 9, 3, 1, \dots$. The 2009th term in this sequence should be 7.

Alternate method: Note that the units digit of 7^4 is equal to 1, so the units digit of $(7^4)^{502}$ is also 1. But $(7^4)^{502} = 7^{2008}$, so the units digit of 7^{2008} is 1, and therefore the units digit of 7^{2009} is 7.

3. [3] How many diagonals does a regular undecagon (11-sided polygon) have?

Answer: $\boxed{44}$ There are 8 diagonals coming from the first vertex, 8 more from the next, 7 from the next, 6 from the next, 5 from the next, etc., and 1 from the last, for $8 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 44$ total.

Third method: Each vertex has 8 diagonals touching it. There are 11 vertices. Since each diagonal touches two vertices, this counts every diagonal twice, so there are $\frac{8 \cdot 11}{2} = 44$ diagonals.

4. [4] How many numbers between 1 and 1,000,000 are perfect squares but not perfect cubes?

Answer: $\boxed{990}$ $1000000 = 1000^2 = 10^6$. A number is both a perfect square and a perfect cube if and only if it is exactly a perfect sixth power. So, the answer is the number of perfect squares, minus the number of perfect sixth powers, which is $1000 - 10 = 990$.

5. [5] Joe has a triangle with area $\sqrt{3}$. What's the smallest perimeter it could have?

Answer: $\boxed{6}$ The minimum occurs for an equilateral triangle. The area of an equilateral triangle with side-length s is $\frac{\sqrt{3}}{4}s^2$, so if the area is $\sqrt{3}$ then $s = \sqrt{\sqrt{3} \frac{4}{\sqrt{3}}} = 2$. Multiplying by 3 to get the perimeter yields the answer.

6. [5] We say " s grows to r " if there exists some integer $n > 0$ such that $s^n = r$. Call a real number r "sparse" if there are only finitely many real numbers s that grow to r . Find all real numbers that are sparse.

Answer: $\boxed{-1, 0, 1}$ For any number x , other than these 3, $x, \sqrt[3]{x}, \sqrt[5]{x}, \sqrt[7]{x}, \dots$ provide infinitely many possible values of s , so these are the only possible sparse numbers. On the other hand, -1 is the only possible value of s for $r = -1$, 0 is the only value for $r = 0$, and -1 and 1 are the only values for $r = 1$. Therefore, $-1, 0$, and 1 are all sparse.

7. [6] Find all ordered pairs (x, y) such that

$$(x - 2y)^2 + (y - 1)^2 = 0.$$

Answer: $\boxed{(2, 1)}$ The square of a real number is always at least 0, so to have equality we must have $(x - 2y)^2 = 0$ and $(y - 1)^2 = 0$. Then $y = 1$ and $x = 2y = 2$.

8. [7] How many integers between 2 and 100 inclusive *cannot* be written as $m \cdot n$, where m and n have no common factors and neither m nor n is equal to 1? Note that there are 25 primes less than 100.

Answer: [35] A number cannot be written in the given form if and only if it is a power of a prime. We can see this by considering the prime factorization. Suppose that $k = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$, with p_1, \dots, p_n primes. Then we can write $m = p_1^{e_1}$ and $n = p_2^{e_2} \cdots p_n^{e_n}$. So, we want to find the powers of primes that are less than or equal to 100. There are 25 primes, as given in the problem statement. The squares of primes are $2^2, 3^2, 5^2, 7^2$. The cubes of primes are $2^3, 3^3$. The fourth powers of primes are $2^4, 3^4$. The fifth powers of primes are 2^5 , The sixth powers of primes are 2^6 . There are no seventh or higher powers of primes between 2 and 100. This adds 10 non-primes to the list, so that in total there are $10 + 25 = 35$ such integers.

9. [7] Find the product of all real x for which

$$2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3} = 0.$$

Answer: [-3] We can re-write the equation as $2^x(2 \cdot (2^x)^2 - 17 \cdot (2^x) + 8) = 0$, or $2 \cdot (2^x)^2 - 17 \cdot (2^x) + 8 = 0$. Make the substitution $y = 2^x$. Then we have $2y^2 - 17y + 8 = 0$, which has solutions (by the quadratic formula) $y = \frac{17 \pm \sqrt{289 - 64}}{4} = \frac{17 \pm 15}{4} = 8, \frac{1}{2}$, so $2^x = 8, \frac{1}{2}$ and $x = 3, -1$. The product of these numbers is -3 .

10. [8] Find the largest positive integer n such that $n^3 + 4n^2 - 15n - 18$ is the cube of an integer.

Answer: [19] Note that the next cube after n^3 is $(n+1)^3 = n^3 + 3n^2 + 3n + 1$. After that, it is $(n+2)^3 = n^3 + 6n^2 + 12n + 8$. $n^3 + 6n^2 + 12n + 8$ is definitely bigger than $n^3 + 4n^2 - 15n - 18$, so the largest cube that $n^3 + 4n^2 - 15n - 18$ could be is $(n+1)^3$. On the other hand, for $n \geq 4$, $n^3 + 4n^2 - 15n - 18$ is larger than $(n-2)^3 = n^3 - 6n^2 + 12n - 8$ (as $4n^2 - 15n - 18 > -6n^2 + 12n - 8 \iff 10n^2 - 27n - 10 > 0$, which is true for $n \geq 4$).

So, we will check for all solutions to $n^3 + 4n^2 - 15n - 18 = (n-1)^3, n^3, (n+1)^3$. The first case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 - 3n^2 + 3n - 1 \iff 7n^2 - 18n - 17 = 0$$

which has no integer solutions. The second case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 \iff 4n^2 - 15n - 18 = 0$$

which also has no integer solutions. The final case yields

$$n^3 + 4n^2 - 15n - 18 = n^3 + 3n^2 + 3n + 1 \iff n^2 - 18n - 19 = 0$$

which has integer solutions $n = -1, 19$. So, the largest possible n is 19.

Remark: The easiest way to see that the first two polynomials have no integer solutions is using the *Rational Root Theorem*, which states that the rational solutions of a polynomial $ax^n + \dots + b$ are all of the form $\pm \frac{b'}{a'}$, where b' divides b and a' divides a .

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Guts Round

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1. [5] Find the sum of all solutions for x :

$$\begin{aligned}xy &= 1 \\x + y &= 3\end{aligned}$$

Answer: [3] Substitute $3 - x$ in for y into the first equation:

$$x(3 - x) = 1 \Leftrightarrow x^2 - 3x + 1 = 0$$

This equation has two distinct roots, each of which corresponds to a possible solution x . The sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is $-\frac{b}{a}$, which in this case is 3.

2. [5] Evaluate the sum

$$1 - 2 + 3 - 4 + \cdots + 2007 - 2008.$$

Answer: [-1004] Every odd integer term can be paired with the next even integer, and this pair sums to -1 . There are 1004 such pairs, so the total sum is -1004 .

3. [5] What is the largest x such that x^2 divides $24 \cdot 35 \cdot 46 \cdot 57$?

Answer: [12] We factor the product as $2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 19 \cdot 23$. If x^2 divides this product, x can have at most 2 factors of 2, 1 factor of 3, and no factors of any other prime. So $2^2 \cdot 3 = 12$ is the largest value of x .

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4. [6] What is the smallest prime divisor of $5^{7^{10^7}} + 1$?

Answer: [2] Notice that 5 to any power is odd, so this number is even. Then 2 is a prime divisor. It also happens to be the smallest prime.

5. [6] What is the sum of all integers x such that $|x + 2| \leq 10$?

Answer: [-42] The inequality $|x + 2| \leq 10$ holds if and only if $x + 2 \leq 10$ and $x + 2 \geq -10$. So x must be in the range $-12 \leq x \leq 8$. If we add up the integers in this range, each positive integer cancels with its additive inverse, so the sum is equal to $-12 - 11 - 10 - 9 = -42$.

6. [6] Sarah is deciding whether to visit Russia or Washington, DC for the holidays. She makes her decision by rolling a regular 6-sided die. If she gets a 1 or 2, she goes to DC. If she rolls a 3, 4, or 5, she goes to Russia. If she rolls a 6, she rolls again. What is the probability that she goes to DC?

Answer: [$\frac{2}{5}$] On each roll, the probability that Sarah decides to go to Russia is $3/2$ times the probability she decides to go to DC. So, the total probability that she goes to Russia is $3/2$ times the total probability that she goes to DC. Since these probabilities sum to 1 (they are the only two eventual outcomes) Sarah goes to DC with probability $\frac{2}{5}$ and Russia with probability $\frac{3}{5}$.

7. [7] Compute $2009^2 - 2008^2$.

Answer: $\boxed{4017}$ Factoring this product with difference of squares, we find it equals:

$$(2009 + 2008)(2009 - 2008) = (4017)(1) = 4017$$

8. [7] Alice rolls two octahedral dice with the numbers 2, 3, 4, 5, 6, 7, 8, 9. What's the probability the two dice sum to 11?

Answer: $\boxed{\frac{1}{8}}$ No matter what comes up on the first die, there is exactly one number that could appear on the second die to make the sum 11, because 2 can be paired with 9, 3 with 8, and so on. So, there is a $\frac{1}{8}$ chance of getting the correct number on the second die.

9. [7] Let $a_0 = \frac{6}{7}$, and

$$a_{n+1} = \begin{cases} 2a_n & \text{if } a_n < \frac{1}{2} \\ 2a_n - 1 & \text{if } a_n \geq \frac{1}{2}. \end{cases}$$

Find a_{2008} .

Answer: $\boxed{\frac{5}{7}}$ We calculate the first few a_i :

$$a_1 = \frac{5}{7}, a_2 = \frac{3}{7}, a_3 = \frac{6}{7} = a_0$$

So this sequence repeats every three terms, so $a_{2007} = a_0 = \frac{6}{7}$. Then $a_{2008} = \frac{5}{7}$.

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10. [8] Find the sum of all positive integers n such that n divides $n^2 + n + 2$.

Answer: $\boxed{3}$ Since n always divides $n^2 + n$, the only n that work are divisors of 2, because if n divides a and n divides b , then n divides $a + b$. So the solutions are 1 and 2 which sum to 3.

11. [8] Al has a rectangle of integer side lengths a and b , and area 1000. What is the smallest perimeter it could have?

Answer: $\boxed{130}$ To minimize the sum of the side lengths, we need to keep the height and width as close as possible, because the square has the smallest perimeter of all rectangles with a fixed area. So, 40 and 25 multiply to 1000 and are as close as possible - the 40x25 rectangle has perimeter 130.

12. [8] Solve the following system of equations for w .

$$\begin{aligned} 2w + x + y + z &= 1 \\ w + 2x + y + z &= 2 \\ w + x + 2y + z &= 2 \\ w + x + y + 2z &= 1. \end{aligned}$$

Answer: $\boxed{\frac{-1}{5}}$ Add all the equations together to find that $5x + 5y + 5z + 5w = 6$, or $x + y + z + w = \frac{6}{5}$. We can now subtract this equation from the first equation to see that $w = \frac{-1}{5}$.

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13. [9] Find the number of distinct primes dividing $1 \cdot 2 \cdot 3 \cdots 9 \cdot 10$.

Answer: [4] A prime divides this product if and only if it divides one of the multiplicands, so prime divisors of this product must be less than 10. There are 4 primes less than 10, namely, 2, 3, 5, and 7.

14. [9] You have a 2×3 grid filled with integers between 1 and 9. The numbers in each row and column are distinct, the first row sums to 23, and the columns sum to 14, 16, and 17 respectively.

	14	16	17
23	a	b	c
	x	y	z

What is $x + 2y + 3z$?

Answer: [49] The sum of all 6 numbers is $14 + 16 + 17 = 47$, so $x + y + z = 47 - 23 = 24$. If three distinct digits sum to 24, they must be 7, 8, and 9, because any other triple of digits would have a smaller sum. So, we try placing these digits in for x , y , and z , and the only arrangement that does not force equal digits in any row or column is $x = 8, y = 7, z = 9$. In this case, $x + 2y + 3z = 49$.

15. [9] A cat is going up a stairwell with ten stairs. However, instead of walking up the stairs one at a time, the cat jumps, going either two or three stairs up at each step (though if necessary, it will just walk the last step). How many different ways can the cat go from the bottom to the top?

Answer: [12] The number of ways for the cat to get to the i th step is the number of ways for the cat to get to step $i - 2$ plus the number of ways to get to step $i - 3$, because for each way to get to step i , we can undo the last move the cat made to go back to one of these two steps. The cat can get to step 1 in 0 ways, to step 2 in 1 way, and to step 3 in 1 way. Now we repeatedly use our formula for calculating the number of ways to get to the i th step to see that the cat gets to:

<i>Step</i>	1	2	3	4	5	6	7	8	9	10
Number of ways	0	1	1	1	2	2	3	4	5	7

So our answer is $5 + 7 = 12$.

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16. [10] If p and q are positive integers and $\frac{2008}{2009} < \frac{p}{q} < \frac{2009}{2010}$, what is the minimum value of p ?

Answer: [4017] By multiplying out the fraction inequalities, we find that $2008q + 1 \leq 2009p$ and $2010p + 1 \leq 2009q$. Adding 2009 times the first inequality to 2008 times the second, we find that $2008 \cdot 2009q + 4017 \leq 2008 \cdot 2009q + p$, or $p \geq 4017$. This minimum is attained when $\frac{p}{q} = \frac{4017}{4019}$.

17. [10] Determine the last two digits of 17^{17} , written in base 10.

Answer: [77] We are asked to find the remainder when 17^{17} is divided by 100. Write the power as $(7 + 10)^{17}$ and expand with the binomial theorem:

$$(7 + 10)^{17} = 7^{17} + 17 \cdot 7^{16} \cdot 10 + \dots$$

We can ignore terms with more than one factor of 10 because these terms are divisible by 100, so adding them does not change the last two digits. Now, $7^4 = 2401$, which has remainder 1 mod 100, so 7^{17} has last two digits 07 and $7^{16} \cdot 10$ has last two digits 70. We add these together.

18. [10] Find the coefficient of x^6 in the expansion of

$$(x + 1)^6 \cdot \sum_{i=0}^6 x^i$$

Answer: $\boxed{64}$ Each term of $(x + 1)^6$ can be multiplied by a unique power x^i , $0 \leq i \leq 6$ to get a sixth degree term. So the answer is the sum of the coefficients of the terms of $(x + 1)^6$, which is the same as substituting $x = 1$ into this power to get $2^6 = 64$.

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19. [11] Let P be a polynomial with $P(1) = P(2) = \dots = P(2007) = 0$ and $P(0) = 2009!$. $P(x)$ has leading coefficient 1 and degree 2008. Find the largest root of $P(x)$.

Answer: $\boxed{4034072}$ $P(0)$ is the constant term of $P(x)$, which is the product of all the roots of the polynomial, because its degree is even. So the product of all 2008 roots is $2009!$ and the product of the first 2007 is $2007!$, which means the last root is $\frac{2009!}{2007!} = 2009 \cdot 2008 = 4034072$.

20. [11] You have a die with faces labelled 1 through 6. On each face, you draw an arrow to an adjacent face, such that if you start on a face and follow the arrows, after 6 steps you will have passed through every face once and will be back on your starting face. How many ways are there to draw the arrows so that this is true?

Answer: $\boxed{32}$ There are 4 choices for where to go from face 1. Consider the 4 faces adjacent to 1. We can visit either 1, 2, or 3 of them before visiting the face opposite 1. If we only visit one of these adjacent faces, we have 4 choices for which one, then we visit face 6, opposite face 1, then we visit the remaining 3 faces in one of two orders - for a total of 8 ways. If we visit 2 adjacent faces first, there is 8 choices for these two faces, then 2 choices for the path back from face 6 to face 1. Lastly, there are 8 ways to visit three of the adjacent faces before visiting the opposite face. These choices give 32 total.

21. [11] Call a number *overweight* if it has at least three positive integer divisors (including 1 and the number), and call a number *obese* if it has at least four positive integer divisors (including 1 and the number). How many positive integers between 1 and 200 are overweight, but not obese?

Answer: $\boxed{6}$ A positive integer is overweight, but not obese, if it has exactly 3 factors - this can only happen if that integer is the square of a prime. (If two primes, p and q , divide the number, then p , q , pq , and 1 all divide it, making it at least obese). So, the integers less than 200 which are squares of a prime are the squares of 2, 3, 5, 7, 11, and 13.

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22. [12] Sandra the Maverick has 5 pairs of shoes in a drawer, each pair a different color. Every day for 5 days, Sandra takes two shoes out and throws them out the window. If they are the same color, she treats herself to a practice problem from a past HMMT. What is the expected value (average number) of practice problems she gets to do?

Answer: $\boxed{\frac{5}{9}}$ On any given day, there is a $\frac{1}{9}$ chance that the second shoe that Sandra chooses makes a pair with the first shoe she chose. Thus the average number of problems she does in a day is $\frac{1}{9}$, so, by the linearity of expectation, she does $\frac{5}{9}$ problems total, on average.

23. [12] If x and y are real numbers such that $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, find the largest possible value of $\frac{x^2}{4} + \frac{y^2}{9}$.

Answer: $\boxed{9}$ The first equation is an ellipse with major axis parallel to the y-axis. If the second expression is set equal to a certain value c , then it is also the equation of an ellipse with major axis parallel to the y-axis; further, it is similar to the first ellipse. So the largest value of c occurs when both ellipse are tangent on the x-axis, at $x = 6, y = 0$, which gives 9 as the largest value of c .

24. [12] Let $f(x) = \frac{1}{1-x}$. Let $f^{k+1}(x) = f(f^k(x))$, with $f^1(x) = f(x)$. What is $f^{2008}(2008)$?

Answer: $\boxed{\frac{-1}{2007}}$ Notice that, if $x \neq 0, 1$, then $f^2(x) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$, which means that $f^3(x) = \frac{1}{1-\frac{x-1}{x}} = x$. So f^n is periodic with period $n = 3$, which means that $f^{2007}(x) = x$ so $f^{2008}(2008) = f(2008) = \frac{-1}{2007}$.

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25. [13] Evaluate the sum

$$\cos\left(\frac{2\pi}{18}\right) + \cos\left(\frac{4\pi}{18}\right) + \cdots + \cos\left(\frac{34\pi}{18}\right).$$

Answer: $\boxed{-1}$ If $k < 18$, then we can pair $\cos\left(\frac{k\pi}{18}\right)$ with $\cos\left(\frac{(18-k)\pi}{18}\right)$, and these two terms sum to 0. If $k > 18$, then the pair $\cos\left(\frac{k\pi}{18}\right)$ and $\cos\left(\frac{(36-k)\pi}{18}\right)$ also sums to 0. So, the only term in this series that is left over is $\cos\left(\frac{18\pi}{18}\right) = -1$.

26. [13] John M. is sitting at $(0, 0)$, looking across the aisle at his friends sitting at (i, j) for each $1 \leq i \leq 10$ and $0 \leq j \leq 5$. Unfortunately, John can only see a friend if the line connecting them doesn't pass through any other friend. How many friends can John see?

Answer: $\boxed{36}$ The simplest method is to draw a picture and count which friends he can see. John can see the friend on point (i, j) if and only if i and j are relatively prime.

27. [13] $ABCDE$ is a regular pentagon inscribed in a circle of radius 1. What is the area of the set of points inside the circle that are farther from A than they are from any other vertex?

Answer: $\boxed{\frac{\pi}{5}}$ Draw the perpendicular bisectors of all the sides and diagonals of the pentagon with one endpoint at A . These lines all intersect in the center of the circle, because they are the set of points equidistant from two points on the circle. Now, a given point is farther from A than from point X if it is on the X side of the perpendicular bisector of segment AX . So, we want to find the area of the set of all points which are separated from A by all of these perpendicular bisectors, which turns out to be a single 72° sector of the circle, which has area $\frac{\pi}{5}$.

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28. [14] Johnny the grad student is typing all the integers from 1 to ∞ , in order. The 2 on his computer is broken however, so he just skips any number with a 2. What's the 2008th number he types?

Answer: $\boxed{3781}$ Write 2008 in base 9 as 2671, and interpret the result as a base 10 number such that the base 9 digits 2, 3, ... 8 correspond to the base 10 digits 3, 4, ... 9. This gives an answer of 3781.

29. [14] Let $p(x)$ be the polynomial of degree 4 with roots 1, 2, 3, 4 and leading coefficient 1. Let $q(x)$ be the polynomial of degree 4 with roots $1, \frac{1}{2}, \frac{1}{3},$ and $\frac{1}{4}$ and leading coefficient 1. Find $\lim_{x \rightarrow 1} \frac{p(x)}{q(x)}$.

Answer: $\boxed{-24}$ Consider the polynomial $f(x) = x^4 q\left(\frac{1}{x}\right)$ - it has the same roots, 1, 2, 3, and 4, as $p(x)$. But this polynomial also has the same coefficients as $q(x)$, just in reverse order. Its leading coefficient is $q(0) = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{24}$. So $f(x)$ is $p(x)$ scaled by $\frac{1}{24}$, which means that $p(x)/f(x)$ goes to 24 as x goes to 1, and $f(x)/q(x)$ goes to -1 .

30. [14] Alice has an equilateral triangle ABC of area 1. Put D on BC , E on CA , and F on AB , with $BD = DC$, $CE = 2EA$, and $2AF = FB$. Note that $AD, BE,$ and CF pass through a single point M . What is the area of triangle EMC ?

Answer: $\boxed{\frac{1}{6}}$ Triangles ACF and BCF share a height, so the ratio of their areas is $AF/BF = 1/2$. By the same method, the ratio of the areas of AMF and BMF is $1/2$. So, the ratio of the areas of

ACM and BCM is also $1/2$. Similarly, the ratio of the areas of ABM and BCM is $1/2$. But the sum of the areas of ACM , BCM , and ABM is 1, so the area of ACM is $\frac{1}{4}$. Then the area of EMC is $2/3$ the area of ACM , because they share heights, so their areas are in the same ratio as their bases. The area of EMC is then $\frac{2 \cdot 1}{3 \cdot 4} = \frac{1}{6}$.

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31. [15] Find the sum of all primes p for which there exists a prime q such that $p^2 + pq + q^2$ is a square.

Answer: $\boxed{8}$ 3 and 5 both work, because $3^2 + 3 \cdot 5 + 5^2 = 49$. Now, say $p^2 + pq + q^2 = k^2$, for a positive integer k . Then $(p + q)^2 - k^2 = pq$, or:

$$(p + q + k)(p + q - k) = pq$$

Since $p + q + k$ is a divisor of pq , and it is greater than p and q , $p + q + k = pq$. Then $p + q - k = 1$. So:

$$2p + 2q = pq + 1 \Leftrightarrow pq - 2p - 2q + 4 = 3 \Leftrightarrow (p - 2)(q - 2) = 3$$

This shows that one of p and q is 3 and the other is 5.

32. [15] Pirate ships Somy and Lia are having a tough time. At the end of the year, they are both one pillage short of the minimum required for maintaining membership in the Pirate Guild, so they decide to pillage each other to bring their counts up. Somy by tradition only pillages $28 \cdot 3^k$ coins for integers k , and Lia by tradition only pillages $82 \cdot 3^j$ coins for integers j . Note that each pillage can have a different k or j . Soma and Lia work out a system where Somy pillages Lia n times, Lia pillages Somy n times, and after both sets of pillages Somy and Lia are financially even.

What is the smallest n can be?

Answer: $\boxed{2}$ Clearly, $n = 1$ cannot be achieved, because $28 \cdot 3^k$ is never a multiple of 82. However, two pillages is enough: Somy pillages 28 and $28 \cdot 81$ from Lia, and Lia pillages 81 and $81 \cdot 27$ from Somy. As is easily checked, both pillage $28 \cdot 82$.

33. [15] The polynomial $ax^2 - bx + c$ has two distinct roots p and q , with a , b , and c positive integers and with $0 < p, q < 1$. Find the minimum possible value of a .

Answer: $\boxed{5}$ Let x and y be the roots. Then:

$$\frac{b}{a} = x + y < 2 \Rightarrow b < 2a$$

$$\frac{c}{a} = xy < 1 \Rightarrow c < a \Rightarrow a > 1$$

$$b^2 > 4ac > 4c^2 \Rightarrow b > 2c$$

Evaluated at 1, the polynomial must be greater than 0, so $a + c > b$. Then:

$$2c < b < a + c$$

$$2c + 1 \leq b \leq a + c - 1$$

$$a \geq c + 2 \geq 3$$

If $a = 3$, then $c = 1$ and $b = 3$, by the above bounds, but this polynomial has complex roots. Similarly, if $a = 4$, then $c = 1$ and b is forced to be either 3 or 4, again giving either 0 or 1 distinct real roots. So $a \geq 5$. But the polynomial $5x^2 - 5x + 1$ satisfies the condition, so 5 is the answer.

34. [20] How many hits did “math tournament” get on Google the morning of November 8, 2008? If you submit integer N , and the correct answer is A , you will receive $\lfloor 20 \cdot \min\{\frac{N}{A}, \frac{A}{N}\} \rfloor$ points for this problem.
35. [25] Find $\max\{\text{Perimeter}(T)\}$ for T a triangle contained in a regular septagon (7-sided figure) of unit edge length. Write your answer N to 2 places after the decimal. If the correct answer rounded to 2 decimal places is A , you will receive 0 points if $N < A$ and $\lfloor \max\{0, 25 - 50 \cdot (N - A)\} \rfloor$ points otherwise.

Answer: 5.85086 Let the septagon be $A_0A_1 \dots A_6$.

If x is a point that can move along the x-axis, the distance from x to a fixed point p is a convex function in the x-coordinate. Therefore, the sum of the distances from x to two other points is convex too, so if x is constrained to lie on a closed line segment, its maximum value is attained at an endpoint. Therefore, the triangle of maximal perimeter has vertices at the vertices of the pentagon. The triangle with the largest such perimeter has almost evenly spaced vertices, so triangle $A_0A_2A_4$ has the maximal perimeter. It has area 5.85 . . .

36. [25] How many numbers less than 1,000,000 are the product of exactly 2 distinct primes? You will receive $\lfloor 25 - 50 \cdot |\frac{N}{A} - 1| \rfloor$ points, if you submit N and the correct answer is A .

Answer: 209867 While it is difficult to compute this answer without writing a program or using a calculator, it can be approximated using the fact that the number of primes less than a positive integer n is about $\frac{n}{\log n}$.

1st Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Team Round

Unit Fractions [100]

A *unit fraction* is a fraction of the form $\frac{1}{n}$, where n is a positive integer. In this problem, you will find out how rational numbers can be expressed as sums of these unit fractions. Even if you do not solve a problem, you may apply its result to later problems.

We say we *decompose* a rational number q into unit fractions if we write q as a sum of 2 or more **distinct** unit fractions. In particular, if we write q as a sum of k distinct unit fractions, we say we have decomposed q into k fractions. As an example, we can decompose $\frac{2}{3}$ into 3 fractions: $\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$.

1. (a) Decompose 1 into unit fractions.

Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}}$

- (b) Decompose $\frac{1}{4}$ into unit fractions.

Answer: $\boxed{\frac{1}{8} + \frac{1}{12} + \frac{1}{24}}$

- (c) Decompose $\frac{2}{5}$ into unit fractions.

Answer: $\boxed{\frac{1}{5} + \frac{1}{10} + \frac{1}{15} + \frac{1}{30}}$

2. Explain how any unit fraction $\frac{1}{n}$ can be decomposed into other unit fractions.

Answer: $\boxed{\frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}}$

3. (a) Write 1 as a sum of 4 distinct unit fractions.

Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}}$

- (b) Write 1 as a sum of 5 distinct unit fractions.

Answer: $\boxed{\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{43 \cdot 42}}$

- (c) Show that, for any integer $k > 3$, 1 can be decomposed into k unit fractions.

Solution: If we can do it for k fractions, simply replace the last one (say $\frac{1}{n}$) with $\frac{1}{n+1} + \frac{1}{n(n+1)}$. Then we can do it for $k+1$ fractions. So, since we can do it for $k=3$, we can do it for any $k > 3$.

4. Say that $\frac{a}{b}$ is a positive rational number in simplest form, with $a \neq 1$. Further, say that n is an integer such that:

$$\frac{1}{n} > \frac{a}{b} > \frac{1}{n+1}$$

Show that when $\frac{a}{b} - \frac{1}{n+1}$ is written in simplest form, its numerator is smaller than a .

Solution: $\frac{a}{b} - \frac{1}{n+1} = \frac{a(n+1)-b}{b(n+1)}$. Therefore, when we write it in simplest form, its numerator will be at most $a(n+1) - b$. We claim that $a(n+1) - b < a$. Indeed, this is the same as $an - b < 0 \iff an < b \iff \frac{b}{a} > n$, which is given.

5. **An aside: the sum of all the unit fractions**

It is possible to show that, given any real M , there exists a positive integer k large enough that:

$$\sum_{n=1}^k \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \dots > M$$

Note that this statement means that the infinite harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, grows without bound, or diverges. For the specific example $M = 5$, find a value of k , *not necessarily the smallest*, such that the inequality holds. Justify your answer.

Solution: Note that $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{1}{2n} + \dots + \frac{1}{2n} = \frac{1}{2}$. Therefore, if we apply this to $n = 1, 2, 4, 8, 16, 32, 64, 128$, we get

$$\left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{129} + \dots + \frac{1}{256}\right) > \frac{1}{2} + \dots + \frac{1}{2} = 4$$

so, adding in $\frac{1}{1}$, we get

$$\sum_{n=1}^{256} \frac{1}{n} > 5$$

so $k = 256$ will suffice.

6. Now, using information from problems 4 and 5, prove that the following method to decompose any positive rational number will always terminate:

Step 1. Start with the fraction $\frac{a}{b}$. Let t_1 be the largest unit fraction $\frac{1}{n}$ which is less than or equal to $\frac{a}{b}$.

Step 2. If we have already chosen t_1 through t_k , and if $t_1 + t_2 + \dots + t_k$ is still less than $\frac{a}{b}$, then let t_{k+1} be the largest unit fraction less than both t_k and $\frac{a}{b}$.

Step 3. If $t_1 + \dots + t_{k+1}$ equals $\frac{a}{b}$, the decomposition is found. Otherwise, repeat step 2.

Why does this method never result in an infinite sequence of t_i ?

Solution: Let $\frac{a_k}{b_k} = \frac{a}{b} - t_1 - \dots - t_k$, where $\frac{a_k}{b_k}$ is a fraction in simplest terms. Initially, this algorithm will have $t_1 = \frac{1}{1}$, $t_2 = \frac{1}{2}$, $t_3 = \frac{1}{3}$, etc. until $\frac{a_k}{b_k} < \frac{1}{k+1}$. This will eventually happen by problem 5, since there exists a k such that $\frac{1}{1} + \dots + \frac{1}{k+1} > \frac{a_k}{b_k}$. At that point, there is some n with $\frac{1}{n} < t_k$ such that $\frac{1}{n} > \frac{a_k}{b_k} > \frac{1}{n+1}$. In this case, $t_{k+1} = \frac{1}{n+1}$.

Suppose that there exists n_k such that $\frac{1}{n_k} > \frac{a_k}{b_k} > \frac{1}{n_k+1}$ for some k . Then we have $t_{k+1} = \frac{1}{n_k+1}$ and $\frac{a_{k+1}}{b_{k+1}} < \frac{1}{n_k(n_k+1)}$. This shows that once we have found n_k such that $\frac{1}{n_k} > \frac{a_k}{b_k} > \frac{1}{n_k+1}$ and $\frac{1}{n_k} \leq t_k$, we no longer have to worry about t_{k+1} being less than t_k , since $t_{k+1} = \frac{1}{n_k+1} < \frac{1}{n_k} < t_k$, and also $n_{k+1} \geq n_k(n_k + 1)$ while $\frac{1}{n_k(n_k+1)} \leq \frac{1}{n_k+1} = t_{k+1}$.

On the other hand, once we have found such an n_k , the sequence $\{a_k\}$ must be decreasing by problem 4. Since the a_k are all integers, we eventually have to get to 0 (as there is no infinite decreasing sequence of positive integers). Therefore, after some finite number of steps the algorithm terminates with $a_{k+1} = 0$, so $0 = \frac{a_k}{b_k} = \frac{a}{b} - t_1 - \dots - t_k$, so $\frac{a}{b} = t_1 + \dots + t_k$, which is what we wanted.

Juicy Numbers [100]

A *juicy number* is an integer $j > 1$ for which there is a sequence $a_1 < a_2 < \dots < a_k$ of positive integers such that $a_k = j$ and such that the sum of the reciprocals of all the a_i is 1. For example, 6 is a juicy number because $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, but 2 is not juicy.

In this part, you will investigate some of the properties of juicy numbers. Remember that if you do not solve a question, you can still use its result on later questions.

1. Explain why 4 is not a juicy number.

Solution: If 4 were juicy, then we would have $1 = \dots + \frac{1}{4}$. The \dots can only possibly contain $\frac{1}{2}$ and $\frac{1}{3}$, but it is clear that $\frac{1}{2} + \frac{1}{4}$, $\frac{1}{3} + \frac{1}{4}$, and $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ are all not equal to 1.

2. It turns out that 6 is the smallest juicy integer. Find the next two smallest juicy numbers, and show a decomposition of 1 into unit fractions for each of these numbers. You do not need to prove that no smaller numbers are juicy.

Answer: 12 and 15 $1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12}$, $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{10} + \frac{1}{15}$.

3. Let p be a prime. Given a sequence of positive integers b_1 through b_n , exactly one of which is divisible by p , show that when

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_n}$$

is written as a fraction in lowest terms, then its denominator is divisible by p . Use this fact to explain why no prime p is ever juicy.

Solution: We can assume that b_n is the term divisible by p (i.e. $b_n = kp$) since the order of addition doesn't matter. We can then write

$$\frac{1}{b_1} + \frac{1}{b_2} + \dots + \frac{1}{b_{n-1}} = \frac{a}{b}$$

where b is not divisible by p (since none of the b_i are). But then $\frac{a}{b} + \frac{1}{kp} = \frac{kpa+b}{kpb}$. Since b is not divisible by p , $kpa + b$ is not divisible by p , so we cannot remove the factor of p from the denominator. In particular, p cannot be juicy as 1 can be written as $\frac{1}{1}$, which has a denominator not divisible by p , whereas being juicy means we have a sum $\frac{1}{b_1} + \dots + \frac{1}{b_n} = 1$, where $b_1 < b_2 < \dots < b_n = p$, and so in particular none of the b_i with $i < n$ are divisible by p .

4. Show that if j is a juicy integer, then $2j$ is juicy as well.

Solution: Just replace $\frac{1}{b_1} + \dots + \frac{1}{b_n}$ with $\frac{1}{2} + \frac{1}{2b_1} + \frac{1}{2b_2} + \dots + \frac{1}{2b_n}$. Since $n > 1$, $2b_1 > 2$.

5. Prove that the product of two juicy numbers (not necessarily distinct) is always a juicy number. Hint: if j_1 and j_2 are the two numbers, how can you change the decompositions of 1 ending in $\frac{1}{j_1}$ or $\frac{1}{j_2}$ to make them end in $\frac{1}{j_1 j_2}$?

Solution: Let $1 = \frac{1}{b_1} + \dots + \frac{1}{b_n} = \frac{1}{c_1} + \dots + \frac{1}{c_m}$, where $b_n = j_1$ and $c_m = j_2$. Then

$$1 = \frac{1}{b_1} + \dots + \frac{1}{b_{n-1}} + \left(\frac{1}{b_n c_1} + \frac{1}{b_n c_2} + \dots + \frac{1}{b_n c_m} \right)$$

and so $j_1 j_2$ is juicy.

1st Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Theme Round

Triangular Areas [25]

Triangles have many properties. Calculate some things about their area.

1. [3] A triangle has sides of length 9, 40, and 41. What is its area?

Answer: $\boxed{180}$ Observe that $9^2 + 40^2 = 41^2$, so this triangle is right and therefore has area $\frac{1}{2} \cdot 9 \cdot 40 = 180$.

2. [4] Let ABC be a triangle, and let M be the midpoint of side AB . If AB is 17 units long and CM is 8 units long, find the maximum possible value of the area of ABC .

Answer: $\boxed{68}$ Let h be the length of the altitude from C to AB . Observe that $K = \frac{1}{2} \cdot h \cdot AB \leq \frac{1}{2} \cdot CM \cdot AB = 68$ and that equality is achieved when $CM \perp AB$.

3. [5] Let DEF be a triangle and H the foot of the altitude from D to EF . If $DE = 60$, $DF = 35$, and $DH = 21$, what is the difference between the minimum and the maximum possible values for the area of DEF ?

Answer: $\boxed{588}$ Observe that the two possible configurations come from DEF obtuse and DEF acute. In either case, we have that $HF = \sqrt{35^2 - 21^2} = 28$ and $EH = \sqrt{60^2 - 21^2} = 9\sqrt{39}$. This means that $HF > EH$, so in the two cases the values of EF are $FH + EH$ and $FH - EH$. The difference in area is hence $\frac{1}{2} \cdot 2 \cdot FH \cdot DH = 28 \cdot 21 = 588$.

4. [6] Right triangle XYZ , with hypotenuse YZ , has an incircle of radius $\frac{3}{8}$ and one leg of length 3. Find the area of the triangle.

Answer: $\boxed{\frac{21}{16}}$ Let the other leg have length x . Then the tangents from Y and Z to the incircle have length $x - \frac{3}{8}$ and $3 - \frac{3}{8}$. So the hypotenuse has length $x + \frac{9}{4}$, the semiperimeter of the triangle is $x + \frac{21}{8}$, and the area of the triangle is $\frac{3}{8}(x + \frac{21}{8})$. But the area can also be calculated as $\frac{3x}{2}$. Setting these expressions equal, we find $x = \frac{7}{8}$ and the area is $\frac{21}{16}$.

5. [7] A triangle has altitudes of length 15, 21, and 35. Find its area.

Answer: $\boxed{245\sqrt{3}}$ If A is the area of the triangle, the sides are $\frac{2A}{15}$, $\frac{2A}{21}$, and $\frac{2A}{35}$. So the triangle is similar to a $\frac{1}{15}, \frac{1}{21}, \frac{1}{35}$ triangle, which is similar to a 3, 5, 7 triangle. Let the sides be $3k$, $5k$, and $7k$. Then the angle between the sides of length $3k$ and $7k$ is 120° , so the area is $\frac{15\sqrt{3}}{4}k^2$. But the area can also be calculated as $\frac{(3k)(35)}{2} = \frac{105k}{2}$. Setting these values equal, $k = \frac{14\sqrt{3}}{3}$ and the area is $245\sqrt{3}$.

Chessboards [25]

Joe B. is playing with some chess pieces on a 6×6 chessboard. Help him find out some things.

1. [3] Joe B. first places the black king in one corner of the board. In how many of the 35 remaining squares can he place a white bishop so that it does not check the black king?

Answer: $\boxed{30}$ Any square not on the diagonal containing the corner is a possible location, and there are $36 - 6 = 30$ such squares.

2. [4] Joe B. then places a white king in the opposite corner of the board. How many total ways can he place one black bishop and one white bishop so that neither checks the king of the opposite color?

Answer: $\boxed{876}$ Observe that either both bishops are on the diagonal containing both kings or neither are. If both are on the diagonal, each of the $\binom{4}{2} = 6$ choices of pairs of squares yields one possible configuration, so there are 6 possibilities in this case. Off the diagonal, any pair of locations works, giving $30 \cdot 29 = 870$ possibilities in this case. Summing, we obtain $870 + 6 = 876$ total possibilities.

3. [5] Joe B. now clears the board. How many ways can he place 3 white rooks and 3 black rooks on the board so that no two rooks of **opposite** color can attack each other?

Answer: $\boxed{608}$ Consider first placing the white rooks. They will occupy either 3 columns and 1 row, 3 columns and 2 rows, 3 columns and 3 rows, 2 rows and 2 columns, 2 columns and 3 rows, or 1 column and 3 rows. First note that placing the black rooks is impossible in the second, third, and fifth cases. The first case can happen in $4 \cdot 4$ ways, and each leads to a unique way to place the black rooks. In the fourth case, we can choose the row with 2 rooks in 4 ways, place the rooks in $\binom{4}{2}$ ways, choose the column of the other rook in 2 ways, and place it in 3 ways, for a total of $4 \cdot \binom{4}{2} \cdot 2 \cdot 3 = 144$ ways to place the white rooks in this configuration; the black rooks can then be placed in any of 4 possible locations, and there are 4 ways to do this, leading to 576 possibilities in this case. Finally, the sixth case is analogous to the first, giving 16 possibilities. Summing, we find $16 + 576 + 16 = 608$ total placements.

4. [6] Joe B. is frustrated with chess. He breaks the board, leaving a 4×4 board, and throws 3 black knights and 3 white kings at the board. Miraculously, they all land in distinct squares! What is the expected number of checks in the resulting position? (Note that a knight can administer multiple checks and a king can be checked by multiple knights.)

Answer: $\boxed{\frac{9}{5}}$ We first compute the expected number of checks between a single knight-king pair. If the king is located at any of the 4 corners, the knight has 2 possible checks. If the king is located in one of the 8 squares on the side of the board but not in the corner, the knight has 3 possible checks. If the king is located in any of the 4 central squares, the knight has 4 possible checks. Summing up, $4 \cdot 2 + 8 \cdot 3 + 4 \cdot 4 = 48$ of the $16 \cdot 15$ knight-king positions yield a single check, so each pair yields $\frac{48}{16 \cdot 15} = \frac{1}{5}$ expected checks. Now, note that each of the 9 knight-king pairs is in each of $16 \cdot 15$ possible positions with equal probability, so by linearity of expectation the answer is $9 \cdot \frac{1}{5} = \frac{9}{5}$.

5. [7] Suppose that at some point Joe B. has placed 2 black knights on the original board, but gets bored of chess. He now decides to cover the 34 remaining squares with 17 dominos so that no two overlap and the dominos cover the entire rest of the board. For how many initial arrangements of the two pieces is this possible?

Answer: $\boxed{324}$ Color the squares of the board red and blue in a checkerboard pattern, and observe that any domino will cover exactly one red square and one blue square. Therefore, if the two knights cover squares of the same color, this is impossible. We now claim that it is always possible if they cover squares of opposite colors, which will give an answer of $18^2 = 324$. Consider the rectangle R with the knights at its corners. Because the knights cover differently colored squares, R must have one side length odd and one side length even. Therefore, the 4 lines bounding R cut the original board into R and up to 8 other rectangles, which can be put together into rectangles with at least one side even. These rectangles can be tiled, and it is easy to see that R can be tiled, proving the claim.

Note: Chess is a game played with pieces of two colors, black and white, that players can move between squares on a rectangular grid. Some of the pieces move in the following ways:

- **Bishop:** This piece can move any number of squares diagonally if there are no other pieces along its path.
- **Rook:** This piece can move any number of squares either vertically or horizontally if there are no other pieces along its path.
- **Knight:** This piece can move either two squares along a row and one square along a column or two squares along a column and one square along a row.
- **King:** This piece can move to any open adjacent square (including diagonally).

If a piece can move to a square occupied by a king of the opposite color, we say that it is *checking* the king.

If a piece moves to a square occupied by another piece, this is called *attacking*.

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Individual Round: Algebra Test

1. [3] Compute

$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$

(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .)

Answer: 2006. We have

$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor = \left\lfloor \frac{(2007 \cdot 2006 + \frac{1}{2005}) \cdot 2005!}{(2006 + 1) \cdot 2005!} \right\rfloor = \left\lfloor \frac{2007 \cdot 2006 + \frac{1}{2005}}{2007} \right\rfloor = \left\lfloor 2006 + \frac{1}{2005 \cdot 2007} \right\rfloor.$$

2. [3] Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy .

Answer: -3. We have $28 = x^3 - y^3 = (x - y)(x^2 + xy + y^2) = (x - y)((x - y)^2 + 3xy) = 4 \cdot (16 + 3xy)$, from which $xy = -3$.

3. [4] Three real numbers x, y , and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y .

Answer: 1/2. Because the first and third fractions are equal, adding their numerators and denominators produces another fraction equal to the others: $((x+4) + (x+5))/(2 + (z-5)) = (2x+9)/(z-3)$. Then $y+9 = 2x+9$, etc.

4. [4] Compute

$$\frac{2^3 - 1}{2^3 + 1} \cdot \frac{3^3 - 1}{3^3 + 1} \cdot \frac{4^3 - 1}{4^3 + 1} \cdot \frac{5^3 - 1}{5^3 + 1} \cdot \frac{6^3 - 1}{6^3 + 1}.$$

Answer: 43/63. Use the factorizations $n^3 - 1 = (n - 1)(n^2 + n + 1)$ and $n^3 + 1 = (n + 1)(n^2 - n + 1)$ to write

$$\frac{1 \cdot 7}{3 \cdot 3} \cdot \frac{2 \cdot 13}{4 \cdot 7} \cdot \frac{3 \cdot 21}{5 \cdot 13} \cdot \frac{4 \cdot 31}{6 \cdot 21} \cdot \frac{5 \cdot 43}{7 \cdot 31} = \frac{1 \cdot 2 \cdot 43}{3 \cdot 6 \cdot 7} = \frac{43}{63}.$$

5. [5] A convex quadrilateral is determined by the points of intersection of the curves $x^4 + y^4 = 100$ and $xy = 4$; determine its area.

Answer: 4√17. By symmetry, the quadrilateral is a rectangle having $x = y$ and $x = -y$ as axes of symmetry. Let (a, b) with $a > b > 0$ be one of the vertices. Then the desired area is

$$\left(\sqrt{2}(a - b)\right) \cdot \left(\sqrt{2}(a + b)\right) = 2(a^2 - b^2) = 2\sqrt{a^4 - 2a^2b^2 + b^4} = 2\sqrt{100 - 2 \cdot 4^2} = 4\sqrt{17}.$$

6. [5] Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.

Answer: $r > 3, r < \frac{49}{27}$. Because such roots to polynomial equations come in conjugate pairs, we seek the values r such that $P(x) = r$ has just one real root x . Considering the shape of a cubic, we are interested in the boundary values r such that $P(x) - r$ has a repeated zero. Thus, we write

$$P(x) - r = x^3 + x^2 - x + (2 - r) = (x - p)^2(x - q) = x^3 - (2p + q)x^2 + p(p + 2q)x - p^2q.$$

Then $q = -2p - 1$ and $1 = p(p + 2q) = p(-3p - 2)$ so that $p = 1/3$ or $p = -1$. It follows that the graph of $P(x)$ is horizontal at $x = 1/3$ (a maximum) and $x = -1$ (a minimum), so the desired values r are $r > P(-1) = 3$ and $r < P(1/3) = 1/27 + 1/9 - 1/3 + 2 = 49/27$.

7. [5] An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \dots$. Find the possible value(s) of a_{2007} .

Answer: $\boxed{2^{2007}}$. The characteristic equation of the linear homogeneous equation is $m^2 + m - 6 = (m + 3)(m - 2) = 0$ with solutions $m = -3$ and $m = 2$. Hence the general solution is given by $a_n = A(2)^n + B(-3)^n$ where A and B are constants to be determined. Then we have $a_n > 0$ for $n \geq 0$, so necessarily $B = 0$, and $a_0 = 1 \Rightarrow A = 1$. Therefore, the unique solution to the recurrence is $a_n = 2^n$ for all n .

8. [6] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \rightarrow \mathbb{R}$ has the property that for all $x \in A$,

$$f(x) + f\left(1 - \frac{1}{x}\right) = \log|x|.$$

Compute the value of $f(2007)$.

Answer: $\boxed{\log(2007/2006)}$. Let $g : A \rightarrow A$ be defined by $g(x) := 1 - 1/x$; the key property is that

$$g(g(g(x))) = 1 - \frac{1}{1 - \frac{1}{1 - \frac{1}{x}}} = x.$$

The given equation rewrites as $f(x) + f(g(x)) = \log|x|$. Substituting $x = g(y)$ and $x = g(g(z))$ gives the further equations $f(g(y)) + f(g(g(y))) = \log|g(y)|$ and $f(g(g(z))) + f(z) = \log|g(g(z))|$. Setting y and z to x and solving the system of three equations for $f(x)$ gives

$$f(x) = \frac{1}{2} \cdot (\log|x| - \log|g(x)| + \log|g(g(x))|).$$

For $x = 2007$, we have $g(x) = \frac{2006}{2007}$ and $g(g(x)) = \frac{-1}{2006}$, so that

$$f(2007) = \frac{\log|2007| - \log\left|\frac{2006}{2007}\right| + \log\left|\frac{-1}{2006}\right|}{2} = \log(2007/2006).$$

9. [7] The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Answer: $\boxed{\{1 \pm \sqrt{5}, -2\}}$. Employing the elementary symmetric polynomials ($s_1 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = -2$, $s_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4 = 0$, $s_3 = \alpha_1\alpha_2\alpha_3 + \alpha_2\alpha_3\alpha_4 + \alpha_3\alpha_4\alpha_1 + \alpha_4\alpha_1\alpha_2 = 0$, and $s_4 = \alpha_1\alpha_2\alpha_3\alpha_4 = 2$) we consider the polynomial

$$P(x) = (x - (\alpha_1\alpha_2 + \alpha_3\alpha_4))(x - (\alpha_1\alpha_3 + \alpha_2\alpha_4))(x - (\alpha_1\alpha_4 + \alpha_2\alpha_3))$$

Because P is symmetric with respect to $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, we can express the coefficients of its expanded form in terms of the elementary symmetric polynomials. We compute

$$\begin{aligned} P(x) &= x^3 - s_2x^2 + (s_3s_1 - 4s_4)x + (-s_3^2 - s_4s_1^2 + s_4s_2) \\ &= x^3 - 8x - 8 \\ &= (x + 2)(x^2 - 2x - 4) \end{aligned}$$

The roots of $P(x)$ are -2 and $1 \pm \sqrt{5}$, so the answer is $\{1 \pm \sqrt{5}, -2\}$.

Remarks. It is easy to find the coefficients of x^2 and x by expansion, and the constant term can be computed without the complete expansion and decomposition of $(\alpha_1\alpha_2 + \alpha_3\alpha_4)(\alpha_1\alpha_3 + \alpha_2\alpha_4)(\alpha_1\alpha_4 + \alpha_2\alpha_3)$ by noting that the only nonzero 6th degree expressions in s_1, s_2, s_3 , and s_4 are s_1^6 and $s_4s_1^2$. The general polynomial P constructed here is called the *cubic resolvent* and arises in Galois theory.

10. [8] The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes r_1, \dots, r_{2007} . A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \dots, 2007$. Determine the value of $P(1)/P(-1)$.

Answer: $\boxed{\frac{289}{259}}$. For some constant k , we have

$$P(z) = k \prod_{j=1}^{2007} \left(z - \left(r_j + \frac{1}{r_j} \right) \right).$$

Now writing $\omega^3 = 1$ with $\omega \neq 1$, we have $\omega^2 + \omega = -1$. Then

$$\begin{aligned} P(1)/P(-1) &= \frac{k \prod_{j=1}^{2007} \left(1 - \left(r_j + \frac{1}{r_j} \right) \right)}{k \prod_{j=1}^{2007} \left(-1 - \left(r_j + \frac{1}{r_j} \right) \right)} = \prod_{j=1}^{2007} \frac{r_j^2 - r_j + 1}{r_j^2 + r_j + 1} = \prod_{j=1}^{2007} \frac{(-\omega - r_j)(-\omega^2 - r_j)}{(\omega - r_j)(\omega^2 - r_j)} \\ &= \frac{f(-\omega)f(-\omega^2)}{f(\omega)f(\omega^2)} = \frac{(-\omega^{2007} + 17\omega^{2006} + 1)(-\omega^2)^{2007} + 17(\omega^2)^{2006} + 1}{(\omega^{2007} + 17\omega^{2006} + 1)((\omega^2)^{2007} + 17(\omega^2)^{2006} + 1)} = \frac{(17\omega^2)(17\omega)}{(2+17\omega^2)(2+17\omega)} \\ &= \frac{289}{4+34(\omega+\omega^2)+289} = \frac{289}{259}. \end{aligned}$$

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Individual Round: Calculus Test

1. [3] Compute:

$$\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)}$$

Answer: [2.] Since $\sin^2(x) = 1 - \cos^2(x)$, we multiply the numerator and denominator by $1 + \cos(x)$ and use the fact that $x/\sin(x) \rightarrow 1$, obtaining

$$\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \rightarrow 0} \frac{x^2(1 + \cos(x))}{1 - \cos^2(x)} = \lim_{x \rightarrow 0} \left(\frac{x}{\sin(x)} \right)^2 \cdot 2 = 2$$

Remarks. Another solution, using *L'Hôpital's rule*, is possible: $\lim_{x \rightarrow 0} \frac{x^2}{1 - \cos(x)} = \lim_{x \rightarrow 0} \frac{2x}{\sin(x)} = 2$.

2. [3] Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

Answer: [1.] Being a relative minimum, we have $0 = f'(a) = 4a^3 - 3a^2 - 2a + a = a(4a + 1)(a - 1)$. Then $a = 0, 1, -1/4$ are the only possibilities. However, it is easily seen that $a = 1$ is the only value satisfying $f(a) = a$.

3. [4] Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{dx}{x + \sqrt{x}}$$

achieves its smallest possible value.

Answer: [3 - 2√2.] Let $F(a)$ denote the given definite integral. Then

$$F'(a) = \frac{d}{da} \int_a^{a^2} \frac{dx}{x + \sqrt{x}} = 2a \cdot \frac{1}{a^2 + \sqrt{a^2}} - \frac{1}{a + \sqrt{a}}.$$

Setting $F'(a) = 0$, we find that $2a + 2\sqrt{a} = a + 1$ or $(\sqrt{a} + 1)^2 = 2$. We find $\sqrt{a} = \pm\sqrt{2} - 1$, and because $\sqrt{a} > 0$, $a = (\sqrt{2} - 1)^2 = 3 - 2\sqrt{2}$.

4. [4] Find the real number α such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = \alpha x^2$.

Answer: [e²/4.] Suppose tangency occurs at $x = x_0$. Then $e^{x_0} = \alpha x_0^2$ and $f'(x_0) = 2\alpha x_0$. On the other hand, $f'(x) = f(x)$, so $\alpha x_0^2 = 2\alpha x_0$. Clearly, $\alpha = 0$ and $x_0 = 0$ are impossible, so it must be that $x_0 = 2$. Then $\alpha = e^{x_0}/(x_0^2) = e^2/4$.

5. [5] The function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies $f(x^2)f''(x) = f'(x)f'(x^2)$ for all real x . Given that $f(1) = 1$ and $f'''(1) = 8$, determine $f'(1) + f''(1)$.

Answer: [6.] Let $f'(1) = a$ and $f''(1) = b$. Then setting $x = 1$ in the given equation, $b = a^2$. Differentiating the given yields

$$2xf'(x^2)f''(x) + f(x^2)f'''(x) = f''(x)f'(x^2) + 2xf'(x)f''(x^2).$$

Plugging $x = 1$ into this equation gives $2ab + 8 = ab + 2ab$, or $ab = 8$. Then because a and b are real, we obtain the solution $(a, b) = (2, 4)$.

Remarks. A priori, the function needn't exist, but one possibility is $f(x) = e^{2x-2}$.

6. [5] The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point (x_0, y_0) . Determine the sum of all possible values of x_0 .

Answer: $\boxed{\frac{1}{3}}$. Note that $y^2 \geq 0$, so $x^3 \geq -1$ and $x \geq -1$. Let the circle be defined by $(x - 4)^2 + y^2 = c$ for some $c \geq 0$. Now differentiate the equations with respect to x , obtaining $2y \frac{dy}{dx} = 3x^2$ from the given and $2y \frac{dy}{dx} = -2x + 8$ from the circle. For tangency, the two expressions $\frac{dy}{dx}$ must be equal if they are well-defined, and this is almost always the case. Thus, $-2x_0 + 8 = 3x_0^2$ so $x_0 = -2$ or $x_0 = 4/3$, but only the latter corresponds to a point on $y^2 = x^3 + 1$. Otherwise, $y_0 = 0$, and this gives the trivial solution $x_0 = -1$.

7. [5] Compute

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}.$$

Answer: $\boxed{3 - e}$. We write

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) \frac{1}{(n+1)!} = \sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)!} - \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+1)!} \\ &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{n \cdot (n+1)!} - \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+1)!} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+1) \cdot (n+2)!} - \frac{1}{(n+1) \cdot (n+1)!} \\ &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 - (n+2)}{(n+1) \cdot (n+2)!} = \frac{1}{2} - \left(\frac{1}{3!} + \frac{1}{4!} + \dots \right) = 3 - \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots \right) = 3 - e. \end{aligned}$$

Alternatively, but with considerably less motivation, we can induce telescoping by adding and subtracting $e - 2 = 1/2! + 1/3! + \dots$, obtaining

$$\begin{aligned} 2 - e + \sum_{n=1}^{\infty} \frac{n(n+1) + 1}{n \cdot (n+1) \cdot (n+1)!} &= 2 - e + \sum_{n=1}^{\infty} \frac{(n+1)^2 - n}{n \cdot (n+1) \cdot (n+1)!} \\ 2 - e + \sum_{n=1}^{\infty} \frac{1}{n \cdot n!} - \frac{1}{(n+1) \cdot (n+1)!} &= 3 - e. \end{aligned}$$

8. [6] Suppose that ω is a primitive 2007th root of unity. Find $(2^{2007} - 1) \sum_{j=1}^{2006} \frac{1}{2 - \omega^j}$.

For this problem only, you may express your answer in the form $m \cdot n^k + p$, where m, n, k , and p are positive integers. Note that a number z is a *primitive n^{th} root of unity* if $z^n = 1$ and n is the smallest number amongst $k = 1, 2, \dots, n$ such that $z^k = 1$.

Answer: $\boxed{2005 \cdot 2^{2006} + 1}$. Note that

$$\begin{aligned} \frac{1}{z - \omega} + \dots + \frac{1}{z - \omega^{2006}} &= \frac{\sum_{j=1}^{2006} \prod_{i \neq j} (z - \omega^i)}{(z - \omega) \dots (z - \omega^{2006})} \\ &= \frac{\frac{d}{dz} [z^{2006} + z^{2005} + \dots + 1]}{z^{2006} + z^{2005} + \dots + 1} = \frac{2006z^{2005} + 2005z^{2004} + \dots + 1}{z^{2006} + z^{2005} + \dots + 1} \cdot \frac{z - 1}{z - 1} \\ &= \frac{2006z^{2006} - z^{2005} - z^{2004} - \dots - 1}{z^{2007} - 1} \cdot \frac{z - 1}{z - 1} = \frac{2006z^{2007} - 2007z^{2006} + 1}{(z^{2007} - 1)(z - 1)}. \end{aligned}$$

Plugging in $z = 2$ gives $\frac{2005 \cdot 2^{2006} + 1}{2^{2007} - 1}$; whence the answer.

9. [7] g is a twice differentiable function over the positive reals such that

$$g(x) + 2x^3 g'(x) + x^4 g''(x) = 0 \quad \text{for all positive reals } x. \quad (1)$$

$$\lim_{x \rightarrow \infty} xg(x) = 1 \quad (2)$$

Find the real number $\alpha > 1$ such that $g(\alpha) = 1/2$.

Answer: $\frac{6}{\pi}$. In the first equation, we can convert the expression $2x^3g'(x) + x^4g''(x)$ into the derivative of a product, and in fact a second derivative, by writing $y = 1/x$. Specifically,

$$\begin{aligned} 0 = g(x) + 2x^3g'(x) + x^4g''(x) &= g\left(\frac{1}{y}\right) + 2y^{-3}g'\left(\frac{1}{y}\right) + y^{-4}g''\left(\frac{1}{y}\right) \\ &= g\left(\frac{1}{y}\right) + \frac{d}{dy}\left[-y^{-2}g'\left(\frac{1}{y}\right)\right] \\ &= g\left(\frac{1}{y}\right) + \frac{d^2}{dy^2}\left[g\left(\frac{1}{y}\right)\right] \end{aligned}$$

Thus $g\left(\frac{1}{y}\right) = c_1 \cos(y) + c_2 \sin(y)$ or $g(x) = c_1 \cos(1/x) + c_2 \sin(1/x)$. Now the second condition gives

$$1 = \lim_{x \rightarrow \infty} c_1 x + c_2 \cdot \frac{\sin(1/x)}{1/x} = c_2 + \lim_{x \rightarrow \infty} c_1 x$$

It must be that $c_1 = 0, c_2 = 1$. Now since $0 < 1/\alpha < 1$, the value of α such that $g(\alpha) = \sin(1/\alpha) = 1/2$ is given by $1/\alpha = \pi/6$ and so $\alpha = 6/\pi$.

10. [8] Compute

$$\int_0^{\infty} \frac{e^{-x} \sin(x)}{x} dx$$

Answer: $\frac{\pi}{4}$. We can compute the integral by introducing a parameter and exchanging the order of integration:

$$\begin{aligned} \int_0^{\infty} e^{-x} \left(\frac{\sin(x)}{x}\right) dx &= \int_0^{\infty} e^{-x} \left(\int_0^1 \cos(ax) da\right) dx = \int_0^1 \left(\int_0^{\infty} e^{-x} \cos(ax) dx\right) da \\ &= \int_0^1 \operatorname{Re} \left[\int_0^{\infty} e^{(-1+ai)x} dx\right] da = \int_0^1 \operatorname{Re} \left[\frac{e^{(-1+ai)x}}{-1+ai} \Big|_{x=0}^{\infty}\right] da \\ &= \int_0^1 \operatorname{Re} \left[\frac{1}{1-ai}\right] da = \int_0^1 \operatorname{Re} \left[\frac{1+ai}{1+a^2}\right] da \\ &= \int_0^1 \frac{1}{1+a^2} da = \tan^{-1}(a) \Big|_{a=0}^1 = \frac{\pi}{4} \end{aligned}$$

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Individual Round: Combinatorics Test

1. [3] A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

Answer: 41. If Biff and Jacob are on the committee, there are $\binom{7}{3} = 35$ ways for the other members to be chosen. Amongst these 35 possibilities, we reject the $\binom{5}{1} = 5$ choices where both Alice and Jane are also serving. If Biff and Jacob are not serving, then there are $\binom{7}{5} = 21$ ways to choose the remaining 5 members. Again, we reject the $\binom{5}{3} = 10$ instances where Alice and Jane are chosen, so the total is $(35 - 5) + (21 - 10) = 41$.

2. [3] How many 5-digit numbers \overline{abcde} exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

Answer: 330. Note that $a > 0$, so that $b > c$, and $e \geq 0$ so that $d \geq c$. Conversely, for each choice of (b, c, d) with $b > c$ and $d \geq c$, there exists a unique pair (a, e) such that \overline{abcde} is a number having the desired property. Thus, we compute

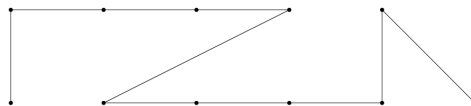
$$\sum_{c=0}^9 (9-c)(10-c) = \sum_{c=0}^9 c^2 - 19c + 90 = 330.$$

3. [4] Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

Answer: $\frac{12}{37}$. The desired probability is the relative probability that Jill draws the spade. In the first round, Jack, Jill, and John draw a spade with probability $1/4$, $3/4 \cdot 1/4$, and $(3/4)^2 \cdot 1/4$ respectively. Thus, the probability that Jill draws the spade is

$$\frac{3/4 \cdot 1/4}{1/4 + 3/4 \cdot 1/4 + (3/4)^2 \cdot 1/4} = \frac{12}{37}.$$

4. [4] On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \leq i \leq 1, 0 \leq j \leq 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?



Answer: 252. Observe that Johnny needs to pass through the points $(0, 0), (1, 0), (2, 0), \dots, (5, 0)$ in that order, and he needs to pass through $(0, 1), (1, 1), (2, 1), \dots, (5, 1)$ in that order, or else he will intersect his own path. Then, the problem is equivalent to interlacing those two sequence together, so that the first term is $(0, 0)$ and the final term is $(5, 1)$. To do this, we need to select 5 positions out of 10 to have points with x -coordinate 0. Hence the answer is $\binom{10}{5} = 252$.

5. [5] Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

Answer: $\boxed{12869 = \binom{16}{8} - 1}$. If k is the number of squares chosen, then there are $\binom{8}{k}$ ways to choose k columns, and $\binom{8}{k}$ ways to choose k rows, and this would uniquely determine the set of squares selected. Thus the answer is

$$\sum_{k=1}^8 \binom{8}{k} \binom{8}{k} = -1 + \sum_{k=0}^8 \binom{8}{k} \binom{8}{k} = -1 + \binom{16}{8} = 12869$$

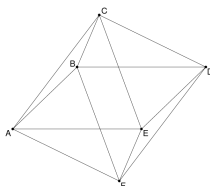
6. [5] Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

Answer: $\boxed{\frac{7}{33}}$. Select any blue marble and consider the remaining eleven marbles, arranged in a line. The proportion of arrangement for which no two red marbles are adjacent will be the same as for the original twelve marbles, arranged in a ring. The total number of ways of arranging 4 red marbles out of 11 is $\binom{11}{4} = 330$. To count the number of arrangements such that no two red marbles are adjacent, there must be one red marble between each two would-be adjacent red marbles. Having fixed the positions of three blue marbles we have four blue marbles to play with. So that we can arrange the remaining four marbles is $\binom{8}{4} = 70$ ways. This yields a probability of $70/330 = 7/33$ as our final answer.

7. [5] Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off of the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average?

Answer: $\boxed{\frac{43}{14}}$. Note that there are 13 prime numbers amongst the cards. We may view these as separating the remaining 29 cards into 14 groups of nonprimes - those appearing before the first prime, between the first and second, etc. Each of these groups is equally likely to appear first, so $29/14$ nonprimes are removed on average. We are done since exactly one prime is always drawn.

8. [6] A set of six edges of a regular octahedron is called *Hamiltonian cycle* if the edges in some order constitute a single continuous loop that visits each vertex exactly once. How many ways are there to partition the twelve edges into two Hamiltonian cycles?



Answer: $\boxed{6}$. Call the octahedron $ABCDEF$, where A, B , and C are opposite D, E , and F , respectively. Note that each Hamiltonian cycle can be described in terms of the order it visits vertices in exactly 12 different ways. Conversely, listing the six vertices in some order determines a Hamiltonian cycle precisely when no pair of opposite vertices are listed consecutively or first-and-last. Suppose we begin with AB . If D is listed third, then the final three letters are CEF or FEC . Otherwise, C or F is listed next, and each gives three possibilities for the final three. For example ABC is followed by DEF, DFE , or EDF . Thus, there are $6 \cdot 4 \cdot (2 + 3 + 3) = 192$ listings. These correspond to $192/12 = 16$ Hamiltonian cycles. Finally, the complement of all but four Hamiltonian cycles is a Hamiltonian cycle. For, each vertex has degree four, so is an endpoint of two edges in the complement of a Hamiltonian cycle, so is also a Hamiltonian cycle unless it describes two opposite faces. It follows that there are six pairs of disjoint Hamiltonian cycles.

9. [7] Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute

$$\sum_{(i,j,k) \in S} ijk.$$

Answer: $\boxed{11628 = \binom{19}{5}}$. We view choosing five objects from a row of 19 objects in an unusual way. First, remove two of the chosen objects, the second and fourth, which are not adjacent nor at either end, forming three nonempty groups of consecutive objects. We then have i, j , and k choices for the first, third, and fifth objects. Because this is a reversible process taking a triple (i, j, k) to ijk choices, the answer is $\binom{19}{5} = 11628$.

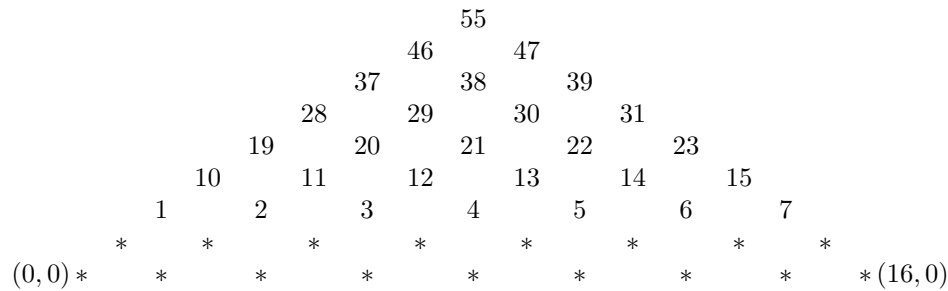
A simple generating functions argument is also possible. Let $s_n = \sum_{i+j+k=n} ijk$. Then

$$\sum_{n \geq 0} s_n x^n = \left(\sum_{n \geq 0} n x^n \right)^3 = \left(\frac{x}{(1-x)^2} \right)^3 = \frac{x^3}{(1-x)^6},$$

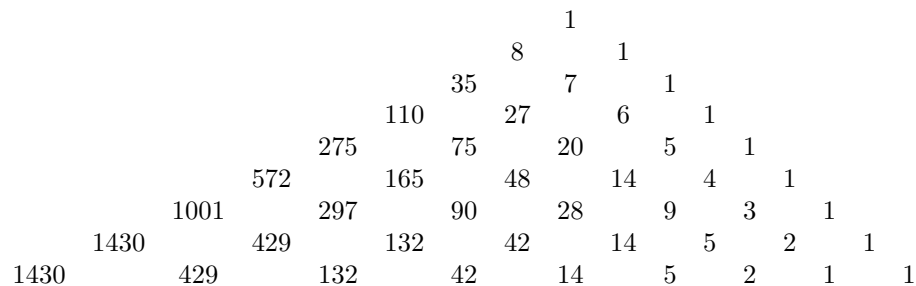
and so $s_n = \binom{\binom{6}{n-3}}{\binom{n+2}{5}}$, yielding $s_{17} = \binom{19}{5}$.

10. [8] A subset S of the nonnegative integers is called *supported* if it contains 0, and $k+8, k+9 \in S$ for all $k \in S$. How many supported sets are there?

Answer: $\boxed{1430}$. Note that every supported set S contains 0, 8, 9, 16, 17, 18, 24-27, 32-36, 40-45, 48-54, and all $n \geq 55$. Now define $\bar{S} := \mathbb{Z}^+ \setminus S$, which is a subset of $\{1-7, 10-15, 19-23, 28-31, 37, 38, 39, 46, 47, 55\}$ satisfying the opposite property that $k \in \bar{S} \implies k-8, k-9 \in \bar{S}$.



Consider the above arrangement after removing the numbers not in \bar{S} . The condition that S be supported ensures that sets \bar{S} are in bijective correspondence with paths from $(0,0)$ to $(16,0)$ consisting of discrete steps of $\langle 1, 1 \rangle$ and $\langle 1, -1 \rangle$ and lying above the x -axis: from the modified version of the above diagram, a unique path passes through the top items left in each column. The number of such paths is the 8th Catalan number, so the answer is $C_8 = \frac{1}{8+1} \binom{8 \cdot 2}{8} = \frac{12870}{9} = 1430$. (Incidentally, 16 choose 8 was computed in an earlier problem.) Without the explicit formula for Catalan numbers, the answer can be computed recursively by filling in the number of ways a path can reach $(16,0)$ from each position in the figure. One works right to left, obtaining the following:



One can exploit symmetry and, having determined the middle column, sum the squares: $1^2 + 7^2 + 20^2 + 28^2 + 14^2 = 1430$.

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Individual Round: General Test, Part 1

1. [2] Michael has 16 white socks, 3 blue socks, and 6 red socks in a drawer. Ever the lazy college student, he has overslept and is late for his favorite team's season-opener. Because he is now in such a rush to get from Harvard to Foxborough, he randomly takes socks from the drawer (one at a time) until he has a pair of the same color. What is the largest number of socks he could possibly withdraw in this fashion?

Answer: [4.] It is possible for him to begin with three socks of different colors, but an instance of the Pigeon Hole Principle is that among any four objects of three types some two are the same type.

2. [2] Rectangle $ABCD$ has side lengths $AB = 12$ and $BC = 5$. Let P and Q denote the midpoints of segments AB and DP , respectively. Determine the area of triangle CDQ .

Answer: [15.] Note that $[CDP] = \frac{1}{2} \cdot 5 \cdot 12 = 30$, while the area of triangle CDQ is half of the area of triangle CDP .

3. [3] A, B, C , and D are points on a circle, and segments \overline{AC} and \overline{BD} intersect at P , such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP , given that $BP < DP$.

Answer: [2.] Same as Geometry #2.

4. [3] Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

Answer: [17.] First group as $17(a - b) + 23b = 13$. Taking this equation modulo 23, we get $-6(a - b) \equiv -10 \pmod{23}$. Since -4 is an inverse of -6 modulo 23, then we multiply to get $(a - b) \equiv 17 \pmod{23}$. Therefore, the smallest possible positive value for $(a - b)$ is 17. This can be satisfied by $a = 5, b = -12$.

5. [4] Find the smallest positive integer that is twice a perfect square and three times a perfect cube.

Answer: [648.] Let n be such a number. If n is divisible by 2 and 3 exactly e_2 and e_3 times, then e_2 is odd and a multiple of three, and e_3 is even and one more than a multiple of three. The smallest possible exponents are $n_2 = 3$ and $n_3 = 4$. The answer is then $2^3 \cdot 3^4 = 648$.

6. [4] The positive integer n is such that the numbers 2^n and 5^n start with the same digit when written in decimal notation; determine this common leading digit.

Answer: [3.] Note $1 = 1^2 < 2^2 < 3^2 < 10 < 4^2 < \dots < 9^2 < 10^2 = 100$. Divide 2^n and 5^n by 10 repeatedly until each is reduced to a decimal number less than 10 but at least 1; call the resulting numbers x and y . Since $(5^n)(2^n) = 10^n$, either $xy = 1$ or $xy = 10$. Because 2^n and 5^n begin with the same digit, x and y are bounded by the same pair of adjacent integers. It follows that either $x = y = 1$ or $3 \leq x, y < 4$. Because n is positive, neither 2^n nor 5^n is a perfect power of 10, so the former is impossible.

7. [4] Jack, Jill, and John play a game in which each randomly picks and then replaces a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

Answer: [$\frac{12}{37}$.] Same as Combo #3.

8. [5] Determine the largest positive integer n such that there exist positive integers x, y, z so that

$$n^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 3x + 3y + 3z - 6$$

Answer: [8.] The given equation rewrites as $n^2 = (x + y + z + 1)^2 + (x + y + z + 1) - 8$. Writing $r = x + y + z + 1$, we have $n^2 = r^2 + r - 8$. Clearly, one possibility is $n = r = 8$, which is realized by $x = y = 1, z = 6$. On the other hand, for $r > 8$, we have $r^2 < r^2 + r - 8 < (r + 1)^2$.

9. [6] I have four distinct rings that I want to wear on my right hand hand (five distinct fingers.) One of these rings is a Canadian ring that must be worn on a finger by itself, the rest I can arrange however I want. If I have two or more rings on the same finger, then I consider different orders of rings along the same finger to be different arrangements. How many different ways can I wear the rings on my fingers?

Answer: 600. First we pick the finger for the Canadian ring. This gives a multiplicative factor of 5. For distributing the remaining 3 rings among 4 fingers, they can either be all on the same finger ($4 \cdot 3!$ ways), all on different fingers ($\binom{4}{3} \cdot 3!$ ways), or two on one finger and one on another ($4 \cdot \binom{3}{2} \cdot 2! \cdot 3$ ways.) Therefore, I have $5 \cdot (24 + 24 + 72) = 600$ choices.

10. [7] $\alpha_1, \alpha_2, \alpha_3,$ and α_4 are the complex roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

Answer: $\{1 \pm \sqrt{5}, -2\}$. Same as Algebra #9.

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Individual Round: General Test, Part 2

1. [2] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s .

Answer: 1, 5. Same as Geometry #1.

2. [2] A parallelogram has 3 of its vertices at $(1,2)$, $(3,8)$, and $(4,1)$. Compute the sum of all possible x coordinates of the 4th vertex.

Answer: 8. There are three possibilities: the 4th vertex must be opposite one of the three given vertices. These three possibilities have as a medial triangle the three given vertices, so the sum of their x coordinates is the same as the sum of the x coordinates of the given triangle.

3. [3] Compute

$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$

(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .)

Answer: 2006. Same as Algebra #1.

4. [3] Three brothers Abel, Banach, and Gauss each have portable music players that can share music with each other. Initially, Abel has 9 songs, Banach has 6 songs, and Gauss has 3 songs, and none of these songs are the same. One day, Abel flips a coin to randomly choose one of his brothers and he adds all of that brother's songs to his collection. The next day, Banach flips a coin to randomly choose one of his brothers and he adds all of that brother's collection of songs to his collection. Finally, each brother randomly plays a song from his collection with each song in his collection being equally likely to be chosen. What is the probability that they all play the same song?

Answer: $\frac{1}{288}$. If Abel copies Banach's songs, this can never happen. Therefore, we consider only the cases where Abel copies Gauss's songs. Since all brothers have Gauss's set of songs, the probability that they play the same song is equivalent to the probability that they independently match whichever song Gauss chooses. Case 1: Abel copies Gauss and Banach copies Gauss ($1/4$ chance) – The probability of songs matching is then $1/12 \cdot 1/9$. Case 2: Abel copies Gauss and Banach copies Abel ($1/4$ probability) – The probability of songs matching is then $1/12 \cdot 1/18$. We add the two probabilities together to get $1/4 \cdot 1/12 \cdot (1/9 + 1/18) = 1/288$.

5. [4] A best of 9 series is to be played between two teams. That is, the first team to win 5 games is the winner. One of the teams, the Mathletes, has a $2/3$ chance of winning any given game. What is the probability that the winner is determined in the 7th game?

Answer: $\frac{20}{81}$. If the Mathletes are the winners, they must win the 7th game and have won exactly four of the previous 6 games. The probability of this occurring is

$$\left((2/3)^4 \cdot (1/3)^2 \cdot \binom{6}{2} \right) \cdot (2/3).$$

Analogously, the other team wins with probability $\left((1/3)^4 \cdot (2/3)^2 \cdot \binom{6}{2} \right) \cdot (1/3)$. Summing, the probability is

$$\frac{\binom{6}{2} \cdot 2^2 \cdot (2^3 \cdot 1^2 + 1^5)}{3^7} = \frac{5 \cdot 4}{3^4} = \frac{20}{81}.$$

6. [4] Circle ω has radius 5 and is centered at O . Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each

tangent), such that line BC is tangent to ω and ω lies outside triangle ABC . Compute $AB + AC$ given that $BC = 7$.

Answer: 17. Same as Geometry #4.

7. [4] My friend and I are playing a game with the following rules: If one of us says an integer n , the opponent then says an integer of their choice between $2n$ and $3n$, inclusive. Whoever first says 2007 or greater loses the game, and their opponent wins. I must begin the game by saying a positive integer less than 10. With how many of them can I guarantee a win?

Answer: 6. We assume optimal play and begin working backward. I win if I say any number between 1004 and 2006. Thus, by saying such a number, my friend can force a win for himself if I ever say a number between 335 and 1003. Then I win if I say any number between 168 and 334, because my friend must then say one of the losing numbers just considered. Similarly, I lose by saying 56 through 167, win by saying 28 through 55, lose with 10 through 17, win with 5 through 9, lose with 2 through 4, and win with 1.

8. [5] Compute the number of sequences of numbers a_1, a_2, \dots, a_{10} such that

- I. $a_i = 0$ or 1 for all i
- II. $a_i \cdot a_{i+1} = 0$ for $i = 1, 2, \dots, 9$
- III. $a_i \cdot a_{i+2} = 0$ for $i = 1, 2, \dots, 8$.

Answer: 60. Call such a sequence “good,” and let A_n be the number of good sequences of length n . Let a_1, a_2, \dots, a_n be a good sequence. If $a_1 = 0$, then a_1, a_2, \dots, a_n is a good sequence if and only if a_2, \dots, a_n is a good sequence, so there are A_{n-1} of them. If $a_1 = 1$, then we must have $a_2 = a_3 = 0$, and in this case, a_1, a_2, \dots, a_n is a good sequence if and only if a_4, a_5, \dots, a_n is a good sequence, so there are A_{n-3} of them. We thus obtain the recursive relation $A_n = A_{n-1} + A_{n-3}$. Note that $A_1 = 2, A_2 = 3, A_3 = 4$. Plugging these into the recursion eventually yields $A_{10} = 60$.

9. [6] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \rightarrow \mathbb{R}$ has the property that for all $x \in A$,

$$f(x) + f\left(1 - \frac{1}{x}\right) = \log|x|.$$

Compute the value of $f(2007)$.

Answer: $\log(2007/2006)$. Same as Algebra #8.

10. [7] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.

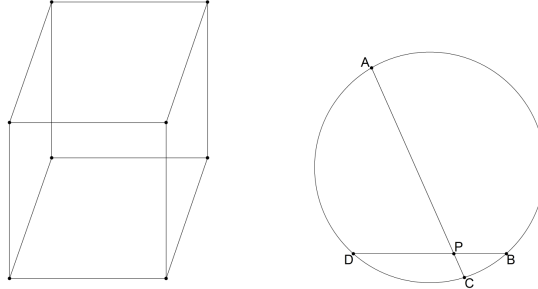
Answer: $\frac{1+\sqrt{13}}{3}$. Same as Geometry #10.

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Individual Round: Geometry Test

1. [3] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s .

Answer: 1, 5. The volume of the cube is s^3 and its surface area is $6s^2$, so we have $6s^2 = s^3 + 5s$, or $0 = s^3 - 6s^2 + 5s = s(s-1)(s-5)$.

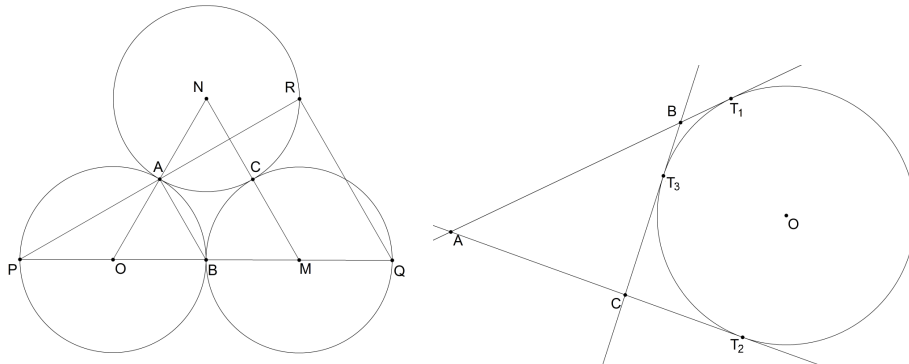


2. [3] $A, B, C,$ and D are points on a circle, and segments \overline{AC} and \overline{BD} intersect at P , such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP , given that $BP < DP$.

Answer: 2. Writing $BP = x$ and $PD = 6 - x$, we have that $BP < 3$. Power of a point at P gives $AP \cdot PC = BP \cdot PD$ or $8 = x(6 - x)$. This can be solved for $x = 2$ and $x = 4$, and we discard the latter.

3. [4] Circles $\omega_1, \omega_2,$ and ω_3 are centered at $M, N,$ and O , respectively. The points of tangency between ω_2 and ω_3, ω_3 and $\omega_1,$ and ω_1 and ω_2 are tangent at $A, B,$ and C , respectively. Line MO intersects ω_3 and ω_1 again at P and Q respectively, and line AP intersects ω_2 again at R . Given that ABC is an equilateral triangle of side length 1, compute the area of PQR .

Answer: $2\sqrt{3}$. Note that ONM is an equilateral triangle of side length 2, so $m\angle BPA = m\angle BOA/2 = \pi/6$. Now BPA is a 30-60-90 triangle with short side length 1, so $AP = \sqrt{3}$. Now A and B are the midpoints of segments PR and PQ , so $[PQR] = \frac{PR}{PA} \cdot \frac{PQ}{PB} [PBA] = 2 \cdot 2 [PBA] = 2\sqrt{3}$.

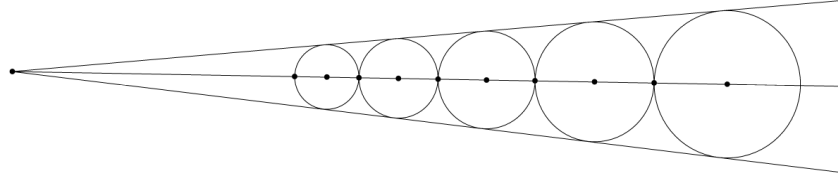


4. [4] Circle ω has radius 5 and is centered at O . Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC . Compute $AB + AC$ given that $BC = 7$.

Answer: 17. Let T_1, T_2 , and T_3 denote the points of tangency of AB, AC , and BC with ω , respectively. Then $7 = BC = BT_3 + T_3C = BT_1 + CT_2$. By Pythagoras, $AT_1 = AT_2 = \sqrt{13^2 - 5^2} = 12$. Now note that $24 = AT_1 + AT_2 = AB + BT_1 + AC + CT_2 = AB + AC + 7$.

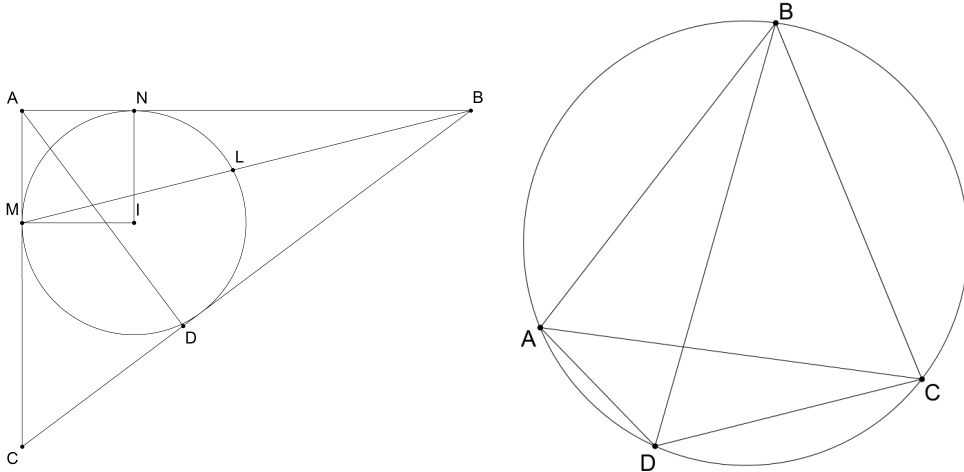
5. [5] Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall. The smallest marble has a radius of 8, and the largest marble has a radius of 18. What is the radius of the middle marble?

Answer: 12. One can either go through all of the algebra, find the slope of the funnel wall and go from there to figure out the radius of the middle marble. Or one can notice that the answer will just be the geometric mean of 18 and 8 which is 12.



6. [5] Triangle ABC has $\angle A = 90^\circ$, side $BC = 25$, $AB > AC$, and area 150. Circle ω is inscribed in ABC , with M its point of tangency on AC . Line BM meets ω a second time at point L . Find the length of segment BL .

Answer: $45\sqrt{17}/17$. Let D be the foot of the altitude from A to side BC . The length of AD is $2 \cdot 150/25 = 12$. Triangles ADC and BDA are similar, so $CD \cdot DB = AD^2 = 144 \Rightarrow BD = 16$ and $CD = 9 \Rightarrow AB = 20$ and $AC = 15$. Using equal tangents or the formula inradius as area divided by semiperimeter, we can find the radius of ω to be 5. Now, let N be the tangency point of ω on AB . By power of a point, we have $BL \cdot BM = BN^2$. Since the center of ω together with M, A , and N determines a square, $BN = 15$ and $BM = 5\sqrt{17}$, and we have $BL = 45\sqrt{17}/17$.

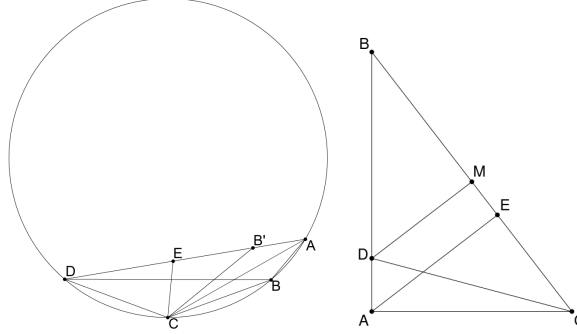


7. [5] Convex quadrilateral $ABCD$ has sides $AB = BC = 7$, $CD = 5$, and $AD = 3$. Given additionally that $m\angle ABC = 60^\circ$, find BD .

Answer: 8. Triangle ABC is equilateral, so $AC = 7$ as well. Now the law of cosines shows that $m\angle CDA = 120^\circ$; i.e., $ABCD$ is cyclic. Ptolemy's theorem now gives $AC \cdot BD = AB \cdot CD + AD \cdot BC$, or simply $BD = CD + AD = 8$.

8. [6] $ABCD$ is a convex quadrilateral such that $AB < AD$. The diagonal \overline{AC} bisects $\angle BAD$, and $m\angle ABD = 130^\circ$. Let E be a point on the interior of \overline{AD} , and $m\angle BAE = 40^\circ$. Given that $BC = CD = DE$, determine $m\angle ACE$ in degrees.

Answer: $\boxed{55^\circ}$. First, we check that $ABCD$ is cyclic. Reflect B over \overline{AC} to B' on \overline{AD} , and note that $B'C = CD$. Therefore, $m\angle ADC = m\angle B'DC = m\angle CB'D = 180^\circ - m\angle AB'C = 180^\circ - m\angle CBA$. Now $m\angle CBD = m\angle CAD = 20^\circ$ and $m\angle ADC = 180^\circ - m\angle CBA = 30^\circ$. Triangle CDE is isosceles, so $m\angle CED = 75^\circ$ and $m\angle AEC = 105^\circ$. It follows that $m\angle ECA = 180^\circ - m\angle AEC - m\angle CAE = 55^\circ$.



9. [7] $\triangle ABC$ is right angled at A . D is a point on AB such that $CD = 1$. AE is the altitude from A to BC . If $BD = BE = 1$, what is the length of AD ?

Answer: $\boxed{\sqrt[3]{2} - 1}$. Let $AD = x$, angle $ABC = t$. We also have $\angle BCA = 90 - t$ and $\angle DCA = 90 - 2t$ so that $\angle ADC = 2t$. Considering triangles ABE and ADC , we obtain, respectively,

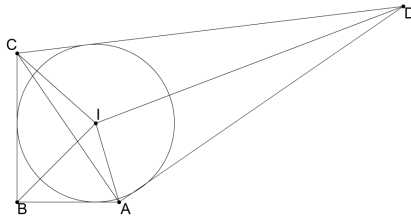
$$\cos(t) = 1/(1+x) \text{ and } \cos(2t) = x. \text{ By the double angle formula we get, } (1+x)^3 = 2.$$

Alternatively, construct M , the midpoint of segment BC , and note that triangles ABC, EBA , and MBD are similar. Thus, $AB^2 = BC \cdot BE = BC$. In particular,

$$AB = \frac{BC}{AB} = \frac{AB}{BE} = \frac{BD}{BM} = \frac{2BD}{BC} = \frac{2}{AB^2},$$

from which $AB = \sqrt[3]{2}$ and $AD = \sqrt[3]{2} - 1$.

10. [8] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.



Answer: $\boxed{\frac{1+\sqrt{13}}{3}}$. Note that $AC^2 = AB^2 + BC^2 = 13 = CD^2 - DA^2$. It follows that $\angle DAC$ is right, and so

$$[ABCD] = [ABC] + [DAC] = 2 \cdot 3/2 + 6 \cdot \sqrt{13}/2 = 3 + 3\sqrt{13}$$

On the other hand, if I denotes the incenter and r denotes the inradius,

$$[ABCD] = [AIB] + [BIC] + [CID] + [DIA] = AB \cdot r/2 + BC \cdot r/2 + CD \cdot r/2 + DA \cdot r/2 = 9r$$

Therefore, $r = (3 + 3\sqrt{13})/9 = \frac{1+\sqrt{13}}{3}$.

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Guts Round

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Note that there are just 36 problems in the Guts round this year.

1. [5] Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007} .

Answer: 43. Note that the last two digits of 7^4 are 01. Also, $a_{2006} = 7^{a_{2005}} = (-1)^{a_{2005}} = -1 = 3 \pmod{4}$ since a_{2005} is odd. Therefore, $a_{2007} = 7^{a_{2006}} = 7^3 = 43 \pmod{100}$.

2. [5] A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

Answer: $\frac{12}{125}$. There are $\binom{5}{2} = 10$ possible pairs of colors. Each pair of colors contributes $2^5 - 2 = 30$ sequences of beans that use *both* colors. Thus, the answer is $10 \cdot 30/5^5 = 12/125$.

3. [5] The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.

Answer: $\frac{1-\sqrt{2}}{2}$. Complete the square by adding 1 to each side. Then $(x+1)^2 = 1+i = e^{i\pi/4}\sqrt{2}$, so $x+1 = \pm e^{i\pi/8}\sqrt[4]{2}$. The desired product is then

$$\left(-1 + \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right)\left(-1 - \cos\left(\frac{\pi}{8}\right)\sqrt[4]{2}\right) = 1 - \cos^2\left(\frac{\pi}{8}\right)\sqrt{2} = 1 - \frac{(1 + \cos(\frac{\pi}{4}))}{2}\sqrt{2} = \frac{1 - \sqrt{2}}{2}.$$

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4. [6] A sequence consists of the digits 122333444455555... such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.

Answer: 13. Note that n contributes $n \cdot d(n)$ digits, where $d(n)$ is the number of digits of n . Then because $1 + \dots + 99 = 4950$, we know that the digits of interest appear amongst copies of two digit numbers. Now for $10 \leq n \leq 99$, the number of digits in the subsequence up to the last copy of n is

$$1 + 2 + 3 + \dots + 9 + 2 \cdot (10 + \dots + n) = 2 \cdot (1 + \dots + n) - 45 = n^2 + n - 45.$$

Since $67^2 + 67 - 45 = 4511$, the two digits are 6 and 7 in some order, so have sum 13.

5. [6] Compute the largest positive integer such that $\frac{2007!}{2007^n}$ is an integer.

Answer: 9. Note that $2007 = 3^2 \cdot 223$. Using the fact that the number of times a prime p divides $n!$ is given by

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots,$$

it follows that the answer is 9.

6. [6] There are three video game systems: the Paystation, the WHAT, and the ZBoz2 π , and none of these systems will play games for the other systems. Uncle Riemann has three nephews: Bernoulli, Galois, and Dirac. Bernoulli owns a Paystation and a WHAT, Galois owns a WHAT and a ZBoz2 π , and Dirac owns a ZBoz2 π and a Paystation. A store sells 4 different games for the Paystation, 6 different games for the WHAT, and 10 different games for the ZBoz2 π . Uncle Riemann does not understand the

difference between the systems, so he walks into the store and buys 3 random games (not necessarily distinct) and randomly hands them to his nephews. What is the probability that each nephew receives a game he can play?

Answer: $\frac{7}{25}$. Since the games are not necessarily distinct, probabilities are independent. Multiplying the odds that each nephew receives a game he can play, we get $10/20 \cdot 14/20 \cdot 16/20 = 7/25$.

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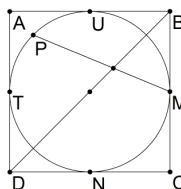
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7. [7] A student at Harvard named Kevin
 Was counting his stones by 11
 He messed up n times
 And instead counted 9s
 And wound up at 2007.

How many values of n could make this limerick true?

Answer: $\boxed{21}$. The mathematical content is that $9n + 11k = 2007$, for some nonnegative integers n and k . As $2007 = 9 \cdot 223$, k must be divisible by 9. Using modulo 11, we see that n is 3 more than a multiple of 11. Thus, the possibilities are $n = 223, 212, 201, \dots, 3$, which are 21 in number.

8. [7] A circle inscribed in a square,
 Has two chords as shown in a pair.
 It has radius 2,
 And P bisects TU .
 The chords' intersection is where?



Answer the question by giving the distance of the point of intersection from the center of the circle.

Answer: $\boxed{2\sqrt{2} - 2}$. Let OB intersect the circle at X and Y , and the chord PM at Q , such that O lies between X and Q . Then $MNXQ$ is a parallelogram. For, $OB \parallel NM$ by homothety at C and $PM \parallel NX$ because $MNXQ$ is an isosceles trapezoid. It follows that $QX = MN$. Considering that the center of the circle together with points M, C , and N determines a square of side length 2, it follows that $MN = 2\sqrt{2}$, so the answer is $2\sqrt{2} - 2$.

9. [7] I ponder some numbers in bed,
 All products of three primes I've said,
 Apply ϕ they're still fun:
 now Elev'n cubed plus one.
 What numbers could be in my head?
- $$n = 37^2 \cdot 3 \dots$$
- $$\phi(n) = 11^3 + 1?$$

Answer: $\boxed{2007, 2738, 3122}$. The numbers expressible as a product of three primes are each of the form p^3 , p^2q , or pqr , where p, q , and r are distinct primes. Now, $\phi(p^3) = p^2(p-1)$, $\phi(p^2q) = p(p-1)(q-1)$, and $\phi(pqr) = (p-1)(q-1)(r-1)$. We require $11^3 + 1 = 12 \cdot 111 = 2^2 \cdot 3^2 \cdot 37$. The first case is easy to rule out, since necessarily $p = 2$ or $p = 3$, which both fail. The second case requires $p = 2$, $p = 3$, or $p = 37$. These give $q = 667, 223$, and 2 , respectively. As $667 = 23 \cdot 29$, we reject $2^2 \cdot 667$, but $3^2 \cdot 223 = 2007$ and $37^2 \cdot 2 = 2738$. In the third case, exactly one of the primes is 2, since all other primes are odd. So say $p = 2$. There are three possibilities for (q, r) : $(2 \cdot 1 + 1, 2 \cdot 3^2 \cdot 37 + 1)$, $(2 \cdot 3 + 1, 2 \cdot 3 \cdot 37 + 1)$, and $(2 \cdot 3^2 + 1, 2 \cdot 37 + 1)$. Those are $(3, 667)$, $(7, 223)$, and $(19, 75)$, respectively, of which only $(7, 223)$ is a pair of primes. So the third and final possibility is $2 \cdot 7 \cdot 223 = 3122$.

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10. [8] Let A_{12} denote the answer to problem 12. There exists a unique triple of digits (B, C, D) such that $10 > A_{12} > B > C > D > 0$ and

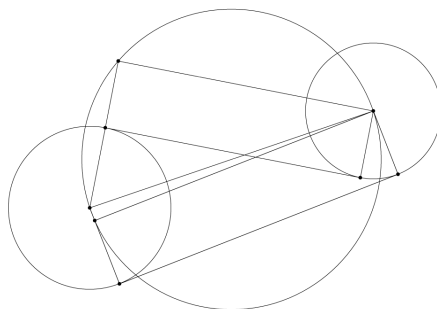
$$\overline{A_{12}BCD} - \overline{DCBA_{12}} = \overline{BDA_{12}C},$$

where $\overline{A_{12}BCD}$ denotes the four digit base 10 integer. Compute $B + C + D$.

Answer: [11.] Since $D < A_{12}$, when A is subtracted from D we must carry over from C . Thus, $D + 10 - A_{12} = C$. Next, since $C - 1 < C < B$, we must carry over from the tens digit, so that $(C - 1 + 10) - B = A_{12}$. Now $B > C$ so $B - 1 \geq C$, and $(B - 1) - C = D$. Similarly, $A_{12} - D = B$. Solving this system of four equations produces $(A_{12}, B, C, D) = (7, 6, 4, 1)$.

11. [8] Let A_{10} denote the answer to problem 10. Two circles lie in the plane; denote the lengths of the internal and external tangents between these two circles by x and y , respectively. Given that the product of the radii of these two circles is $15/2$, and that the distance between their centers is A_{10} , determine $y^2 - x^2$.

Answer: [30.] Suppose the circles have radii r_1 and r_2 . Then using the tangents to build right triangles, we have $x^2 + (r_1 + r_2)^2 = A_{10}^2 = y^2 + (r_1 - r_2)^2$. Thus, $y^2 - x^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2 = 4r_1r_2 = 30$.



12. [8] Let A_{11} denote the answer to problem 11. Determine the smallest prime p such that the arithmetic sequence $p, p + A_{11}, p + 2A_{11}, \dots$ begins with the largest possible number of primes.

Answer: [7.] First, note that the maximal number of initial primes is bounded above by the smallest prime not dividing A_{11} , with equality possible only if p is this prime. For, if q is the smallest prime not dividing A_{11} , then the first q terms of the arithmetic sequence determine a complete residue class modulo q , and the multiple of q is nonprime unless it equals q . If $q < A_{11}$, then q must appear first in the sequence, and thus divide the $(q + 1)$ st term. If $q > A_{11}$, then $A_{11} = 2$ and $q = 3$ by Bertrand's postulate, so q must appear first by inspection.

Now since $A_{11} = 30$, the bound is 7. In fact, 7, 37, 67, 97, 127, and 157 are prime, but 187 is not. Then on the one hand, our bound of seven initial primes is not realizable. On the other hand, this implies an upper bound of six, and this bound is achieved by $p = 7$. Smaller primes p yield only one initial prime, so 7 is the answer.

Remarks. A number of famous theorems are concerned with the distribution of prime numbers. For two relatively prime positive integers a and b , the arithmetic progression $a, a + b, a + 2b, \dots$ contains infinitely many primes, a result known as *Dirichlet's theorem*. It was shown recently (c. 2004) that there exist arbitrarily long arithmetic progressions consisting of primes only. *Bertrand's postulate* states that for any positive integer n , there exists a prime p such that $n < p \leq 2n$. This is an unfortunate misnomer, as the statement is known to be true. As with many theorems concerning the distributions of primes, these results are easily stated in elementary terms, concealing elaborate proofs.

There is just one triple of possible (A_{10}, A_{11}, A_{12}) of answers to these three problems. Your team will receive credit only for answers matching these. (So, for example, submitting a wrong answer for problem 11 will not alter the correctness of your answer to problem 12.)

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13. [9] Determine the largest integer n such that $7^{2048} - 1$ is divisible by 2^n .

Answer: 14. We have

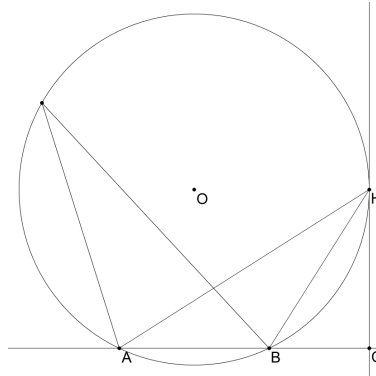
$$7^{2048} - 1 = (7 - 1)(7 + 1)(7^2 + 1)(7^4 + 1) \cdots (7^{1024} + 1).$$

In the expansion, the eleven terms other than $7 + 1$ are divisible by 2 exactly once, as can be checked easily with modulo 4.

14. [9] We are given some similar triangles. Their areas are $1^2, 3^2, 5^2, \dots$, and 49^2 . If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

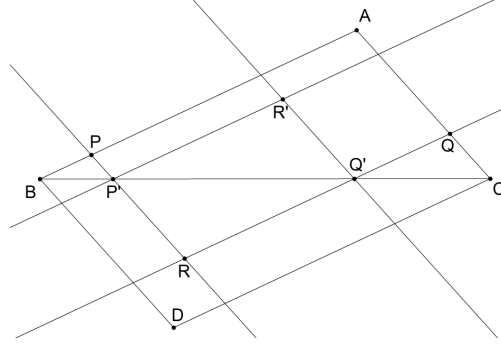
Answer: 2500. Because the triangles are all similar, they all have the same ratio of perimeter squared to area, or, equivalently, the same ratio of perimeter to the square root of area. Because the latter ratio is 4 for the smallest triangle, it is 4 for all the triangles, and thus their perimeters are $4 \cdot 1, 4 \cdot 3, 4 \cdot 5, \dots, 4 \cdot 49$, and the sum of these numbers is $4(1 + 3 + 5 + \dots + 49) = 4(25^2) = 2500$.

15. [9] Points A, B , and C lie in that order on line ℓ , such that $AB = 3$ and $BC = 2$. Point H is such that CH is perpendicular to ℓ . Determine the length CH such that $\angle AHB$ is as large as possible.



Answer: $\sqrt{10}$. Let ω denote the circumcircle of triangle ABH . Since AB is fixed, the smaller the radius of ω , the bigger the angle AHB . If ω crosses the line CH in more than one point, then there exists a smaller circle that goes through A and B that crosses CH at a point H' . But angle $AH'B$ is greater than AHB , contradicting our assumption that H is the optimal spot. Thus the circle ω crosses the line CH at exactly one spot: ie, ω is tangent to CH at H . By Power of a Point, $CH^2 = CA \cdot CB = 5 \cdot 2 = 10$, so $CH = \sqrt{10}$.

16. [10] Let ABC be a triangle with $AB = 7, BC = 9$, and $CA = 4$. Let D be the point such that $AB \parallel CD$ and $CA \parallel BD$. Let R be a point within triangle BCD . Lines ℓ and m going through R are parallel to CA and AB respectively. Line ℓ meets AB and BC at P and P' respectively, and m meets CA and BC at Q and Q' respectively. If S denotes the largest possible sum of the areas of triangles $BPP', RP'Q'$, and CQQ' , determine the value of S^2 .



Answer: 180. Let R' denote the intersection of the lines through Q' and P' parallel to ℓ and m respectively. Then $[RP'Q'] = [R'P'Q']$. Triangles BPP' , $R'P'Q'$, and CQQ' lie in ABC without overlap, so that on the one hand, $S \leq ABC$. On the other, this bound is realizable by taking R to be a vertex of triangle BCD . We compute the square of the area of ABC to be $10 \cdot (10-9) \cdot (10-7) \cdot (10-4) = 180$.

17. [10] During the regular season, Washington Redskins achieve a record of 10 wins and 6 losses. Compute the probability that their wins came in three streaks of consecutive wins, assuming that all possible arrangements of wins and losses are equally likely. (For example, the record LLWWWWLWLLWWLL contains three winning streaks, while WWWWWLWLLLLLWWW has just two.)

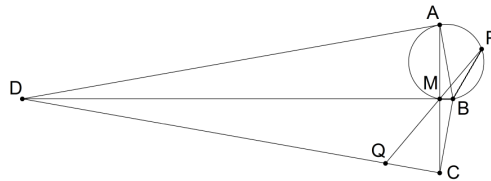
Answer: $\frac{45}{286}$. Suppose the winning streaks consist of w_1, w_2 , and w_3 wins, in chronological order, where the first winning streak is preceded by l_0 consecutive losses and the i winning streak is immediately succeeded by l_i losses. Then $w_1, w_2, w_3, l_1, l_2 > 0$ are positive and $l_0, l_3 \geq 0$ are nonnegative. The equations

$$w_1 + w_2 + w_3 = 10 \quad \text{and} \quad (l_0 + 1) + l_1 + l_2 + (l_3 + 1) = 8,$$

are independent, and have $\binom{9}{2}$ and $\binom{7}{3}$ solutions, respectively. It follows that the answer is

$$\frac{\binom{9}{2} \binom{7}{3}}{\binom{16}{6}} = \frac{315}{2002}.$$

18. [10] Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M . Points P and Q lie on the circumcircle of triangle AMB and segment CD , respectively, such that points P, M , and Q are collinear. Suppose that $m\angle ABC = 160^\circ$ and $m\angle QMC = 40^\circ$. Find $MP \cdot MQ$, given that $MC = 6$.



Answer: 36. Note that $m\angle QPB = m\angle MPB = m\angle MAB = m\angle CAB = \angle BCA = \angle CDB$. Thus, $MP \cdot MQ = MB \cdot MD$. On the other hand, segment CM is an altitude of right triangle BCD , so $MB \cdot MD = MC^2 = 36$.

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19. [10] Define $x \star y = \frac{\sqrt{x^2 + 3xy + y^2 - 2x - 2y + 4}}{xy + 4}$. Compute

$$((\dots((2007 \star 2006) \star 2005) \star \dots) \star 1).$$

Answer: $\frac{\sqrt{15}}{9}$. Note that $x \star 2 = \frac{\sqrt{x^2+6x+4}-2x-4+4}{2x+4} = \frac{\sqrt{(x+2)^2}}{2(x+2)} = \frac{1}{2}$ for $x > -2$. Because $x \star y > 0$ if $x, y > 0$, we need only compute $\frac{1}{2} \star 1 = \frac{\sqrt{\frac{1}{4}+\frac{3}{2}+1-3+4}}{\frac{1}{2}+4} = \frac{\sqrt{15}}{9}$.

20. [10] For a a positive real number, let x_1, x_2, x_3 be the roots of the equation $x^3 - ax^2 + ax - a = 0$. Determine the smallest possible value of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$.

Answer: -4 . Note that $x_1 + x_2 + x_3 = x_1x_2 + x_2x_3 + x_3x_1 = a$. Then

$$\begin{aligned} x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3 &= (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2 - (x_1x_2 + x_2x_3 + x_3x_1)) \\ &= (x_1 + x_2 + x_3) \left((x_1 + x_2 + x_3)^2 - 3(x_1x_2 + x_2x_3 + x_3x_1) \right) = a \cdot (a^2 - 3a) = a^3 - 3a^2. \end{aligned}$$

The expression is negative only where $0 < a < 3$, so we need only consider these values of a . Finally, AM-GM gives $\sqrt[3]{(6-2a)(a)(a)} \leq \frac{(6-2a)+a+a}{3} = 2$, with equality where $a = 2$, and this rewrites as $(a-3)a^2 \geq -4$.

21. [10] Bob the bomb-defuser has stumbled upon an active bomb. He opens it up, and finds the red and green wires conveniently located for him to cut. Being a seasoned member of the bomb-squad, Bob quickly determines that it is the green wire that he should cut, and puts his wirecutters on the green wire. But just before he starts to cut, the bomb starts to count down, ticking every second. Each time the bomb ticks, starting at time $t = 15$ seconds, Bob panics and has a certain chance to move his wirecutters to the other wire. However, he is a rational man even when panicking, and has a $\frac{1}{2t^2}$ chance of switching wires at time t , regardless of which wire he is about to cut. When the bomb ticks at $t = 1$, Bob cuts whatever wire his wirecutters are on, without switching wires. What is the probability that Bob cuts the green wire?

Answer: $\frac{23}{30}$. Suppose Bob makes n independent decisions, with probabilities of switching p_1, p_2, \dots, p_n . Then in the expansion of the product

$$P(x) = (p_1 + (1 - p_1)x)(p_2 + (1 - p_2)x) \cdots (p_n + (1 - p_n)x),$$

the sum of the coefficients of even powers of x gives the probability that Bob makes his original decision. This is just $(P(1) + P(-1))/2$, so the probability is just

$$\frac{1 + \left(1 - \frac{1}{15 \cdot 15}\right) \left(1 - \frac{1}{14 \cdot 14}\right) \cdots \left(1 - \frac{1}{2 \cdot 2}\right)}{2} = \frac{1 + \frac{14}{15} \frac{16}{15} \frac{13}{14} \frac{15}{14} \cdots \frac{1}{2} \frac{3}{2}}{2} = \frac{1 + \frac{8}{15}}{2} = \frac{23}{30}.$$

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22. [12] The sequence $\{a_n\}_{n \geq 1}$ is defined by $a_{n+2} = 7a_{n+1} - a_n$ for positive integers n with initial values $a_1 = 1$ and $a_2 = 8$. Another sequence, $\{b_n\}$, is defined by the rule $b_{n+2} = 3b_{n+1} - b_n$ for positive integers n together with the values $b_1 = 1$ and $b_2 = 2$. Find $\gcd(a_{5000}, b_{501})$.

Answer: 89 . We show by induction that $a_n = F_{4n-2}$ and $b_n = F_{2n-1}$, where F_k is the k th Fibonacci number. The base cases are clear. As for the inductive steps, note that

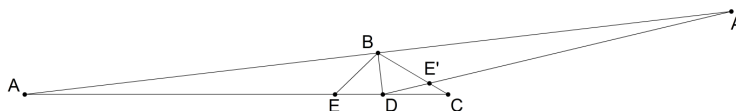
$$F_{k+2} = F_{k+1} + F_k = 2F_k + F_{k-1} = 3F_k - F_{k-2}$$

and

$$F_{k+4} = 3F_{k+2} - F_k = 8F_k + 3F_{k-2} = 7F_k - F_{k-4}.$$

We wish to compute the greatest common denominator of F_{19998} and F_{1001} . The Fibonacci numbers satisfy the property that $\gcd(F_m, F_n) = F_{\gcd(m,n)}$, which can be proven by noting that they are periodic modulo any positive integer. So since $\gcd(19998, 1001) = 11$, the answer is $F_{11} = 89$.

23. [12] In triangle ABC , $\angle ABC$ is obtuse. Point D lies on side AC such that $\angle ABD$ is right, and point E lies on side AC between A and D such that BD bisects $\angle EBC$. Find CE , given that $AC = 35$, $BC = 7$, and $BE = 5$.



Answer: 10. Reflect A and E over BD to A' and E' respectively. Note that the angle conditions show that A' and E' lie on AB and BC respectively. B is the midpoint of segment AA' and $CE' = BC - BE' = 2$. Menelaus' theorem now gives

$$\frac{CD}{DA} \cdot \frac{AA'}{A'B} \cdot \frac{BE'}{E'C} = 1,$$

from which $DA = 5CD$ or $CD = AC/6$. By the angle bisector theorem, $DE = 5CD/7$, so that $CE = 12CD/7 = 10$.

24. [12] Let x, y, n be positive integers with $n > 1$. How many ordered triples (x, y, n) of solutions are there to the equation $x^n - y^n = 2^{100}$?

Answer: 49. Break all possible values of n into the four cases: $n = 2$, $n = 4$, $n > 4$ and n odd. By Fermat's theorem, no solutions exist for the $n = 4$ case because we may write $y^4 + (2^{25})^4 = x^4$.

We show that for n odd, no solutions exist to the more general equation $x^n - y^n = 2^k$ where k is a positive integer. Assume otherwise for contradiction's sake, and suppose on the grounds of well ordering that k is the least exponent for which a solution exists. Clearly x and y must both be even or both odd. If both are odd, we have $(x - y)(x^{n-1} + \dots + y^{n-1})$. The right factor of this expression contains an odd number of odd terms whose sum is an odd number greater than 1, impossible. Similarly if x and y are even, write $x = 2u$ and $y = 2v$. The equation becomes $u^n - v^n = 2^{k-n}$. If $k - n$ is greater than 0, then our choice k could not have been minimal. Otherwise, $k - n = 0$, so that two consecutive positive integers are perfect n th powers, which is also absurd.

For the case that n is even and greater than 4, consider the same generalization and hypotheses. Writing $n = 2m$, we find $(x^m - y^m)(x^m + y^m) = 2^k$. Then $x^m - y^m = 2^a < 2^k$. By our previous work, we see that m cannot be an odd integer greater than 1. But then m must also be even, contrary to the minimality of k .

Finally, for $n = 2$ we get $x^2 - y^2 = 2^{100}$. Factoring the left hand side gives $x - y = 2^a$ and $x + y = 2^b$, where implicit is $a < b$. Solving, we get $x = 2^{b-1} + 2^{a-1}$ and $y = 2^{b-1} - 2^{a-1}$, for a total of 49 solutions. Namely, those corresponding to $(a, b) = (1, 99), (2, 98), \dots, (49, 51)$.

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25. [12] Two real numbers x and y are such that $8y^4 + 4x^2y^2 + 4xy^2 + 2x^3 + 2y^2 + 2x = x^2 + 1$. Find all possible values of $x + 2y^2$.

Answer: $\frac{1}{2}$. Writing $a = x + 2y^2$, the given quickly becomes $4y^2a + 2x^2a + a + x = x^2 + 1$. We can rewrite $4y^2a$ for further reduction to $a(2a - 2x) + 2x^2a + a + x = x^2 + 1$, or

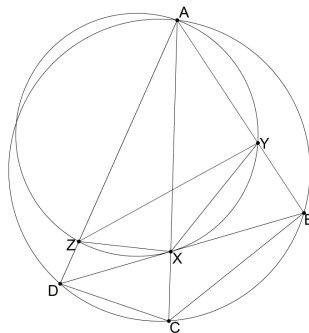
$$2a^2 + (2x^2 - 2x + 1)a + (-x^2 + x - 1) = 0. \quad (*)$$

The quadratic formula produces the discriminant

$$(2x^2 - 2x + 1)^2 + 8(x^2 - x + 1) = (2x^2 - 2x + 3)^2,$$

an identity that can be treated with the difference of squares, so that $a = \frac{-2x^2 + 2x - 1 \pm (2x^2 - 2x + 3)}{4} = \frac{1}{2}, -x^2 + x - 1$. Now a was constructed from x and y , so is not free. Indeed, the second expression flies in the face of the trivial inequality: $a = -x^2 + x - 1 < -x^2 + x \leq x + 2y^2 = a$. On the other hand, $a = 1/2$ is a bona fide solution to (*), which is identical to the original equation.

26. [12] $ABCD$ is a cyclic quadrilateral in which $AB = 4, BC = 3, CD = 2$, and $AD = 5$. Diagonals AC and BD intersect at X . A circle ω passes through A and is tangent to BD at X . ω intersects AB and AD at Y and Z respectively. Compute YZ/BD .



Answer: $\frac{115}{143}$. Denote the lengths AB, BC, CD , and DA by a, b, c , and d respectively. Because $ABCD$ is cyclic, $\triangle ABX \sim \triangle DCX$ and $\triangle ADX \sim \triangle BCX$. It follows that $\frac{AX}{DX} = \frac{BX}{CX} = \frac{a}{c}$ and $\frac{AX}{BX} = \frac{DX}{CX} = \frac{d}{b}$. Therefore we may write $AX = adk, BX = abk, CX = bck$, and $DX = cdk$ for some k . Now, $\angle XDC = \angle BAX = \angle YXB$ and $\angle DCX = \angle XBY$, so $\triangle BXY \sim \triangle CDX$. Thus, $XY = DX \cdot \frac{BX}{CD} = cdk \cdot \frac{abk}{c} = abdk^2$. Analogously, $XY = acdk^2$. Note that $XY/XZ = CB/CD$. Since $\angle YXZ = \pi - \angle ZAY = \angle BCD$, we have that $\triangle XYZ \sim \triangle CBD$. Thus, $YZ/BD = XY/CB = adk^2$. Finally, Ptolemy's theorem applied to $ABCD$ gives

$$(ad + bc)k \cdot (ab + cd)k = ac + bd$$

It follows that the answer is

$$\frac{ad(ac + bd)}{(ab + cd)(ad + bc)} = \frac{20 \cdot 23}{22 \cdot 26} = \frac{115}{143}$$

27. [12] Find the number of 7-tuples (n_1, \dots, n_7) of integers such that

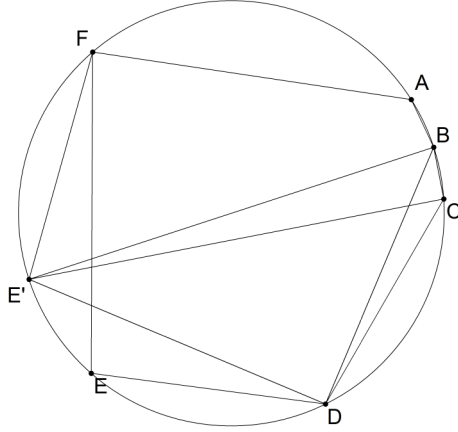
$$\sum_{i=1}^7 n_i^6 = 96957.$$

Answer: $\mathbf{2688}$. Consider the equation in modulo 9. All perfect 6th powers are either 0 or 1. Since 9 divides 96957, it must be that each n_i is a multiple of 3. Writing $3a_i = n_i$ and dividing both sides by 3^6 , we have $a_1^6 + \dots + a_7^6 = 133$. Since sixth powers are nonnegative, $|a_i| \leq 2$. Again considering modulo 9, we see that $a_i \neq 0$. Thus, $a_i^6 \in \{1, 64\}$. The only possibility is $133 = 64 + 64 + 1 + 1 + 1 + 1 + 1$, so $|a_1|, \dots, |a_7|$ consists of 2 2's and 5 1's. It follows that the answer is $\binom{7}{2} \cdot 2^7 = 2688$.

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28. [15] Compute the circumradius of cyclic hexagon $ABCDEF$, which has side lengths $AB = BC = 2, CD = DE = 9$, and $EF = FA = 12$.



Answer: 8. Construct point E' on the circumcircle of $ABCDEF$ such that $DE' = EF = 12$ and $E'F = DE = 9$; then $\overline{BE'}$ is a diameter. Let $BE' = d$. Then $CE' = \sqrt{BE'^2 - BC^2} = \sqrt{d^2 - 4}$ and $BD = \sqrt{BE'^2 - DE'^2} = \sqrt{d^2 - 144}$. Applying Ptolemy's theorem to $BCDE'$ now yields

$$9 \cdot d + 2 \cdot 12 = \sqrt{(d^2 - 4)(d^2 - 144)}$$

Squaring and rearranging, we find $0 = d^4 - 229d^2 - 432d = d(d - 16)(d^2 + 16d + 27)$. Since d is a positive real number, $d = 16$, and the circumradius is 8.

29. **[15]** A sequence $\{a_n\}_{n \geq 1}$ of positive reals is defined by the rule $a_{n+1}a_{n-1}^5 = a_n^4a_{n-2}^2$ for integers $n > 2$ together with the initial values $a_1 = 8$ and $a_2 = 64$ and $a_3 = 1024$. Compute

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

Answer: $3\sqrt{2}$. Taking the base-2 log of the sequence $\{a_n\}$ converts the multiplicative rule to a more familiar additive rule: $\log_2(a_{n+1}) - 4\log_2(a_n) + 5\log_2(a_{n-1}) - 2\log_2(a_{n-2}) = 0$. The characteristic equation is $0 = x^3 - 4x^2 + 5x - 2 = (x - 1)^2(x - 2)$, so $\log_2(a_n)$ is of the form $a \cdot n + b + c \cdot 2^n$ and we find $a_n = 2^{2n+2^{n-1}}$. Now,

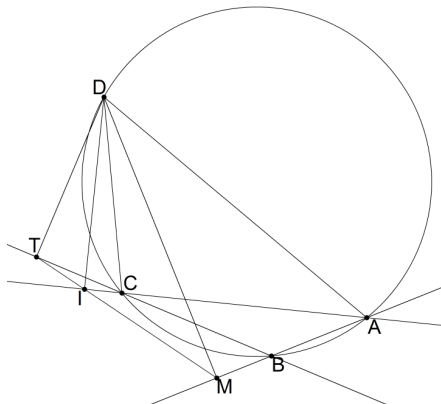
$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}} = \sqrt{2} \cdot \sqrt{4 + \sqrt{16 + \sqrt{64 + \cdots}}}$$

We can estimate the new nested radical expression as 3, which expands thus

$$3 = \sqrt{4 + 5} = \sqrt{4 + \sqrt{16 + 9}} = \sqrt{4 + \sqrt{16 + \sqrt{64 + 17}}} = \cdots$$

As a rigorous confirmation, we have $2^k + 1 = \sqrt{4^k + (2^{k+1} + 1)}$, as desired. It follows that the answer is $3\sqrt{2}$.

30. **[15]** $ABCD$ is a cyclic quadrilateral in which $AB = 3$, $BC = 5$, $CD = 6$, and $AD = 10$. M , I , and T are the feet of the perpendiculars from D to lines AB , AC , and BC respectively. Determine the value of MI/IT .



Answer: $\boxed{\frac{25}{9}}$. Quadrilaterals $AMID$ and $DICT$ are cyclic, having right angles $\angle AMD, \angle AID,$ and $\angle CID, \angle CTD$ respectively. We see that $M, I,$ and T are collinear. For, $m\angle MID = \pi - m\angle DAM = \pi - m\angle DAB = m\angle BCD = \pi - m\angle DCT = \pi - m\angle DIT$. Therefore, Menelaus' theorem applied to triangle MTB and line ICA gives

$$\frac{MI}{IT} \cdot \frac{TC}{CB} \cdot \frac{BA}{AM} = 1$$

On the other hand, triangle ADM is similar to triangle CDT since $\angle AMD \cong \angle CTD$ and $\angle DAM \cong \angle DCT$ and thus $AM/CT = AD/CD$. It follows that

$$\frac{MI}{IT} = \frac{BC \cdot AM}{AB \cdot CT} = \frac{BC \cdot AD}{AB \cdot CD} = \frac{5 \cdot 10}{3 \cdot 6} = \frac{25}{9}$$

Remarks. The line MIT , constructed in this problem by taking perpendiculars from a point on the circumcircle of ABC , is known as the *Simson line*. It is often helpful for us to use *directed angles* while angle chasing to avoid supplementary configuration issues, such as those arising while establishing the collinearity of $M, I,$ and T .

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31. [18] A sequence $\{a_n\}_{n \geq 0}$ of real numbers satisfies the recursion $a_{n+1} = a_n^3 - 3a_n^2 + 3$ for all positive integers n . For how many values of a_0 does $a_{2007} = a_0$?

Answer: $\boxed{3^{2007}}$. If x appears in the sequence, the next term $x^3 - 3x^2 + 3$ is the same if and only if $0 = x^3 - 3x^2 - x + 3 = (x - 3)(x - 1)(x + 1)$. Moreover, that next term is strictly larger if $x > 3$ and strictly smaller if $x < -1$. It follows that no values of a_0 with $|a_0 - 1| > 2$ yield $a_0 = a_{2007}$.

Now suppose $a_0 = a_{2007}$ and write $a_0 = 1 + e^{\alpha i} + e^{-\alpha i}$; the values a_0 we seek will be in bijective correspondence with solutions α where $0 \leq \alpha \leq \pi$. Then

$$a_1 = (a_0 - 1)^3 - 3a_0 + 4 = e^{3\alpha i} + 3e^{\alpha i} + 3e^{-\alpha i} + e^{-3\alpha i} - 3e^{\alpha i} - 3e^{-\alpha i} - 3 + 4 = e^{3\alpha i} + e^{-3\alpha i} + 1,$$

and an easy inductive argument gives $a_{2007} = e^{3^{2007}\alpha i} + e^{-3^{2007}\alpha i} + 1$. It follows that $a_0 = a_{2007}$ is equivalent to $\cos(\alpha) = \cos(3^{2007}\alpha)$. Now,

$$\cos(3^{2007}\alpha) - \cos(\alpha) = 2 \sin\left(\left(\frac{3^{2007} + 1}{2}\right)\alpha\right) \sin\left(\left(\frac{3^{2007} - 1}{2}\right)\alpha\right),$$

so since $\sin(kx) = 0$ for a positive integer k if and only if x is a multiple of $\frac{\pi}{k}$, the solutions α are $\{0, \frac{2\pi}{3^{2007}-1}, \frac{4\pi}{3^{2007}-1}, \dots, \pi\} \cup \{0, \frac{2\pi}{3^{2007}+1}, \dots, \pi\}$. Because our values k are consecutive, these sets overlap only at 0 and π , so there are 3^{2007} distinct α .

32. [18] Triangle ABC has $AB = 4, BC = 6$, and $AC = 5$. Let O denote the circumcenter of ABC . The circle Γ is tangent to and surrounds the circumcircles of triangles AOB, BOC , and AOC . Determine the diameter of Γ .

Answer: $\frac{256\sqrt{7}}{17}$. Denote by $\omega, \Gamma_1, \Gamma_2$, and Γ_3 the circumcenters of triangles ABC, BOC, COA , and AOB , respectively. An inversion about ω interchanges Γ_1 and line BC , Γ_2 and line CA , and Γ_3 and line AB . This inversion also preserves tangency between generalized circles, so the image of Γ is a circle tangent to AB, BC , and CA . It is the incircle of ABC because it is closer to O than these lines and ABC is acute.

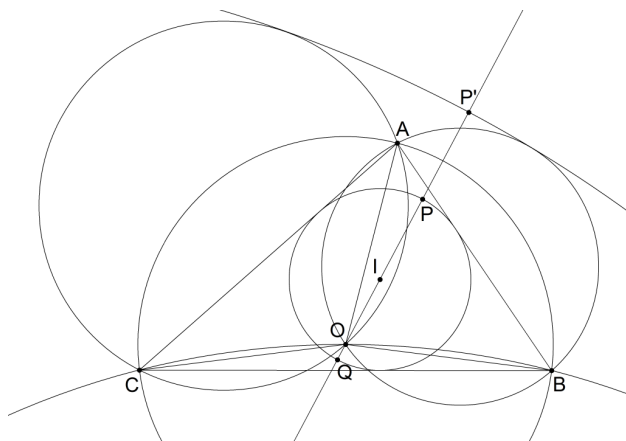
Now we run a few standard calculations. Where s, r , and R denote the semiperimeter, inradius, and circumradius of ABC , respectively, we have the following:

$$\begin{aligned} [ABC] &= \sqrt{s(s-a)(s-b)(s-c)} = \frac{15\sqrt{7}}{4}; \\ r &= [ABC]/s = \sqrt{7}/2; \\ R &= \frac{abc}{4[ABC]} = \frac{8}{\sqrt{7}} \\ OI^2 &= R(R-2r) = \frac{8}{7}. \end{aligned}$$

Let OI intersect the incircle of ABC at P and Q , with I between P and O . Then $OP = r + OI$ and $OQ = r - OI$, and \overline{PQ} is a diameter. Under the inversion, P and Q map to P' and Q' respectively. Because P, I, O , and Q are collinear in that order, P' and Q' are diametrically opposed on Γ . It follows that the diameter of Γ is

$$P'Q' = OP' + OQ' = \frac{R^2}{OP} + \frac{R^2}{OQ} = R^2 \left(\frac{1}{r+OI} + \frac{1}{r-OI} \right) = \frac{2rR^2}{r^2 - OI^2}.$$

We plug in the values found above to arrive at $\frac{256\sqrt{7}}{17}$.



33. [18] Compute

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x} dx.$$

(No, your TI-89 doesn't know how to do this one. Yes, the end is near.)

Answer: $\ln \frac{80}{23}$. We break the given integral into two pieces:

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x} dx = 5 \int_1^2 \frac{x^4+3x+1}{x^5+3x^2+x} dx - \int_1^2 \frac{5x^4+6x+1}{x^5+3x^2+x} dx.$$

These two new integrals are easily computed; for, the first integrand reduces to $1/x$ and the second is of the form $f'(x)/f(x)$. We obtain

$$\left[5 \ln |x| - \ln |x^5 + 3x^2 + x|\right]_1^2 = \ln 32 - \ln 46 + \ln 5 = \ln \frac{80}{23}$$

Motivation. Writing $f(x) = 9x + 4$ and $g(x) = x^5 + 3x^2 + x = x(x^4 + 3x + 1)$, we wish to find the antiderivative of $f(x)/g(x)$. It makes sense to consider other rational functions with denominator $g(x)$ that have an exact antiderivative. Clearly, if the numerator were $f_1(x) = x^4 + 3x + 1$ or a constant multiple, then we can integrate the function. Another trivial case is if the numerator were $f_2(x) = g'(x) = 5x^4 + 6x + 1$ or a constant multiple. Guessing that $f(x)$ is a linear combination of $f_1(x)$ and $f_2(x)$, we easily find that $f(x) = 9x + 4 = 5f_1(x) - f_2(x)$.

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34. [?] *The Game.* Eric and Greg are watching their new favorite TV show, *The Price is Right*. Bob Barker recently raised the intellectual level of his program, and he begins the latest installment with bidding on following question: How many Carmichael numbers are there less than 100,000?

Each team is to list one nonnegative integer not greater than 100,000. Let X denote the answer to Bob's question. The teams listing N , a maximal bid (of those submitted) not greater than X , will receive N points, and all other teams will neither receive nor lose points. (A Carmichael number is an odd composite integer n such that n divides $a^{n-1} - 1$ for all integers a relatively prime to n with $1 < a < n$.)

Answer: 16? There are 16 such numbers: 561, 1105, 1729, 2465, 2821, 6601, 8911, 10585, 15841, 29341, 41041, 46657, 52633, 62745, 63973, and 75361. The next, 101101, is too large to be counted. Their distribution is considerably more subtle than that of the primes, and it was only recently proven that there are infinitely many Carmichael numbers. For sufficiently large n , $C(n) > n^{2/7}$, although this bound has been subsequently improved. (For details, see Alford, W. R.; Granville, A.; and Pomerance, C. "There are Infinitely Many Carmichael Numbers." *Annals of Mathematics*. **139** (1994), 703-722.) The expectation is that teams are unable to determine that $X = 16$; otherwise, the obvious dominant play is listing 16. The Problem Czar, anticipating that teams will attempt to deduce X by considering the point values of the other problems in the triplet, has chosen a value X that is considerably lower. Teams may of course speculate whether this action has been taken, and to what extent, etc... On the actual TV show, many contestants win by guessing prices of 1, or other numbers dramatically lower than the actual price. This strategy is enhanced because of the show's ordered bidding, and will be more difficult here. It will be interesting to see the submissions.

35. [≤ 25] *The Algorithm.* There are thirteen broken computers situated at the following set S of thirteen points in the plane:

$A = (1, 10)$	$B = (976, 9)$	$C = (666, 87)$
$D = (377, 422)$	$E = (535, 488)$	$F = (775, 488)$
$G = (941, 500)$	$H = (225, 583)$	$I = (388, 696)$
$J = (3, 713)$	$K = (504, 872)$	$L = (560, 934)$
	$M = (22, 997)$	

At time $t = 0$, a repairman begins moving from one computer to the next, traveling continuously in straight lines at unit speed. Assuming the repairman begins at A and fixes computers instantly, what path does he take to minimize the *total downtime* of the computers? List the points he visits in order. Your score will be $\lfloor \frac{N}{40} \rfloor$, where

$$N = 1000 + \lfloor \text{the optimal downtime} \rfloor - \lfloor \text{your downtime} \rfloor,$$

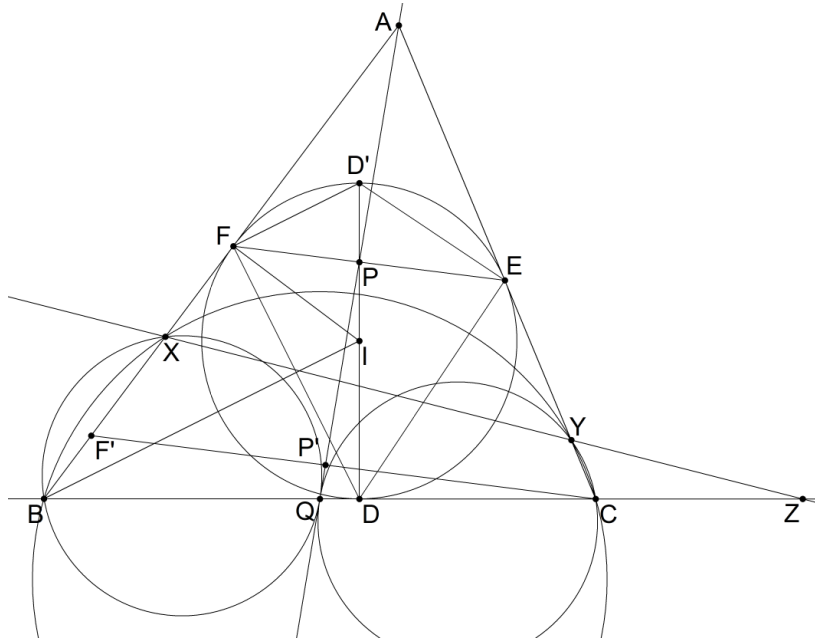
or 0, whichever is greater. By total downtime we mean the sum

$$\sum_{P \in S} t_P,$$

where t_P is the time at which the repairman reaches P .

Answer: ADHIKLEFGBCJM. This is an instance of the *minimum-latency* problem, which is at least NP-hard. There is an easy $O(n!)$ algorithm, but this is unavailable to teams on computational grounds (100MHz calculators used to seem fast...) The best strategy may be drawing an accurate picture and exercising geometric intuition. The distribution of the points somewhat resembles a short, four-pronged fork with its outermost prongs bent apart; it is plausible to assume that the optimal order respects this shape. The optimal downtime is 24113.147907, realized by ADHIKLEFGBCJM, though a number of others also receive positive marks.

36. [25] *The Marathon*. Let ω denote the incircle of triangle ABC . The segments BC, CA , and AB are tangent to ω at D, E , and F , respectively. Point P lies on EF such that segment PD is perpendicular to BC . The line AP intersects BC at Q . The circles ω_1 and ω_2 pass through B and C , respectively, and are tangent to AQ at Q ; the former meets AB again at X , and the latter meets AC again at Y . The line XY intersects BC at Z . Given that $AB = 15, BC = 14$, and $CA = 13$, find $\lfloor XZ \cdot YZ \rfloor$.



Answer: 101. Construct D' diametrically opposed to D , so that $\angle DFD'$ and $\angle DED'$ are right, and note that P lies on DD' . By standard angle chasing, $m\angle FDD' = \beta$ (half angle B) and $m\angle D'DE = \gamma$. Thus, $m\angle DD'F = 90^\circ - \beta$ and $m\angle ED'D = 90^\circ - \gamma$. Then by the law of sines, $DE : ED' : D'F : FD = \cos(\gamma) : \sin(\gamma) : \sin(\beta) : \sin(\gamma)$. Now using $\triangle DEP \sim \triangle FD'P$ and $\triangle DFP \sim \triangle ED'P$, we have

$$\frac{EP}{PF} = \frac{ED \cdot ED'}{FD \cdot FD'} = \frac{\sin(\gamma) \cos(\gamma)}{\sin(\beta) \sin(\beta)} = \frac{c}{b}.$$

Let the dilation centered at A sending E to C map P and F to P' and F' , respectively. Note that $AF' = AC$ as AE and AF are equal tangents, and $CP' : P'F' = EP : PF = c : b$ by similarity. Then by Menelaus' theorem,

$$1 = \frac{BQ}{QC} \frac{CP'}{P'F'} \frac{F'A}{AB} = \frac{BQ}{QC} \frac{cb}{bc},$$

so that $BQ = QC$ and AQ is actually a median. So, $AQ^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2) = 148$. Now by Power of a Point, $AB \cdot AX = AQ^2 = AC \cdot AY$, so $AX = 148/15$ and $AY = 148/13$. Moreover, $BXC Y$ is cyclic as $\triangle ABC \sim \triangle AYX$. Thus, $XZ \cdot YZ = BZ \cdot CZ$, and it suffices to compute BZ/CZ . Menelaus once more gives

$$1 = \frac{BZ}{ZC} \frac{CY}{YA} \frac{AX}{XB},$$

whence, $BZ/CZ = (AY/AX)(BX/CY) = (15/13)((77 \cdot 13)/(21 \cdot 15)) = 11/3$. We write $CZ = 3d$ and $BZ = 11d$. Because $AX < AB$ and $AY < AC$, Z does not lie on segment BC . Given the configuration information, $BC = 8d = 14$, so $d = 7/4$, and finally $\lfloor BZ \cdot CZ \rfloor = \lfloor 33d^2 \rfloor = \lfloor \frac{1617}{16} \rfloor = 101$.

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Team Round: A Division

Σ, τ , and You: Fun at Fraternities? [270]

A *number theoretic function* is a function whose domain is the set of positive integers. A *multiplicative number theoretic function* is a number theoretic function f such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n . Examples of multiplicative number theoretic functions include σ, τ, ϕ , and μ , defined as follows. For each positive integer n ,

- The *sum-of-divisors function*, $\sigma(n)$, is the sum of all positive integer divisors of n . If p_1, \dots, p_i are distinct primes and e_1, \dots, e_i are positive integers,

$$\sigma(p_1^{e_1} \cdots p_i^{e_i}) = \prod_{k=1}^i (1 + p_k + \cdots + p_k^{e_k}) = \prod_{k=1}^i \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

- The *divisor function*, $\tau(n)$, is the number of positive integer divisors of n . It can be computed by the formula

$$\tau(p_1^{e_1} \cdots p_i^{e_i}) = (e_1 + 1) \cdots (e_i + 1),$$

where p_1, \dots, p_i and e_1, \dots, e_i are as above.

- Euler's *totient function*, $\phi(n)$, is the number of positive integers $k \leq n$ such that k and n are relatively prime. For p_1, \dots, p_i and e_1, \dots, e_i as above, the phi function satisfies

$$\phi(p_1^{e_1} \cdots p_i^{e_i}) = \prod_{k=1}^i p_k^{e_k-1} (p_k - 1).$$

- The *Möbius function*, $\mu(n)$, is equal to either 1, -1, or 0. An integer is called *square-free* if it is not divisible by the square of any prime. If n is a square-free positive integer having an even number of distinct prime divisors, $\mu(n) = 1$. If n is a square-free positive integer having an odd number of distinct prime divisors, $\mu(n) = -1$. Otherwise, $\mu(n) = 0$.

The Möbius function has a number of peculiar properties. For example, if f and g are number theoretic functions such that

$$g(n) = \sum_{d|n} f(d),$$

for all positive integers n , then

$$f(n) = \sum_{d|n} g(d) \mu\left(\frac{n}{d}\right).$$

This is known as *Möbius inversion*. In proving the following problems, *you may use any of the preceding assertions without proving them. You may also cite the results of previous problems, even if you were unable to prove them.*

1. [15] Evaluate the functions $\phi(n), \sigma(n)$, and $\tau(n)$ for $n = 12, n = 2007$, and $n = 2^{2007}$.

Solution. For $n = 12 = 2^2 \cdot 3^1$,

$$\phi(12) = 2(2-1)(3-1) = 4, \quad \sigma(12) = (1+2+4)(1+3) = 28, \quad \tau(12) = (2+1)(1+1) = 6;$$

for $n = 2007 = 3^2 \cdot 223$,

$$\phi(2007) = 3(3-1)(223-1) = 1332, \quad \sigma(2007) = (1+3+9)(1+223) = 2912, \quad \tau(2007) = (2+1)(1+1) = 6;$$

and for $n = 2^{2007}$,

$$\phi(2^{2007}) = 2^{2006}, \quad \sigma(2^{2007}) = (1+2+\cdots+2^{2007}) = 2^{2008} - 1, \quad \tau(2^{2007}) = 2007 + 1 = 2008. \square$$

2. [20] Solve for the positive integer(s) n such that $\phi(n^2) = 1000\phi(n)$.

Answer: 1000.

Solution. The unique solution is $n = 1000$. For, $\phi(pn) = p\phi(n)$ for every prime p dividing n , so that $\phi(n^2) = n\phi(n)$ for all positive integers n . \square

3. [25] Prove that for every integer n greater than 1,

$$\sigma(n)\phi(n) \leq n^2 - 1.$$

When does equality hold?

Solution. Note that

$$\sigma(mn)\phi(mn) = \sigma(m)\phi(m)\sigma(n)\phi(n) \leq (m^2 - 1)(n^2 - 1) = (mn)^2 - (m^2 + n^2 - 1) < (mn)^2 - 1$$

for any pair of relatively prime positive integers (m, n) other than $(1, 1)$. Now, for p a prime and k a positive integer, $\sigma(p^k) = 1 + p + \dots + p^k = \frac{p^{k+1} - 1}{p - 1}$ and $\phi(p^k) = p^k - \frac{1}{p} \cdot p^k = (p - 1)p^{k-1}$. Thus,

$$\sigma(p^k)\phi(p^k) = \frac{p^{k+1} - 1}{p - 1} \cdot (p - 1)p^{k-1} = (p^{k+1} - 1)p^{k-1} = p^{2k} - p^{k-1} \leq p^{2k} - 1,$$

with equality where $k = 1$. It follows that equality holds in the given inequality if and only if n is prime. \square

4. [25] Let F and G be two multiplicative functions, and define for positive integers n ,

$$H(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right).$$

The number theoretic function H is called the *convolution* of F and G . Prove that H is multiplicative.

Solution. Let m and n be relatively prime positive integers. We have

$$\begin{aligned} H(m)H(n) &= \left(\sum_{d|m} F(d)G\left(\frac{m}{d}\right) \right) \left(\sum_{d'|n} F(d')G\left(\frac{n}{d'}\right) \right) \\ &= \sum_{d|m, d'|n} F(d)F(d')G\left(\frac{m}{d}\right)G\left(\frac{n}{d'}\right) = \sum_{d|m, d'|n} F(dd')G\left(\frac{mn}{dd'}\right) \\ &= \sum_{d|mn} F(d)G\left(\frac{mn}{d}\right) = H(mn). \square \end{aligned}$$

5. [30] Prove the identity

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d) \right)^2.$$

Solution. Note that τ^3 is multiplicative; in light of the convolution property just shown, it follows that both sides of the posed equality are multiplicative. Thus, it would suffice to prove the claim for n a power of a prime. So, write $n = p^k$ where p is a prime and k is a nonnegative integer. Then

$$\begin{aligned} \sum_{d|n} \tau(d)^3 &= \sum_{i=0}^k \tau(p^i)^3 = \sum_{i=0}^k (i+1)^3 \\ &= 1^3 + \dots + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4} = \left(\frac{(k+1)(k+2)}{2} \right)^2 \\ &= \left(\sum_{i=0}^k \tau(p^i) \right)^2 = \left(\sum_{d|n} \tau(d) \right)^2, \end{aligned}$$

as required. \square

6. [25] Show that for positive integers n ,

$$\sum_{d|n} \phi(d) = n.$$

Solution. Both sides are multiplicative functions of n , the right side trivially and the left because for relatively prime positive integers n and n' ,

$$\left(\sum_{d|n} \phi(d) \right) \left(\sum_{d'|n'} \phi(d') \right) = \sum_{d|n, d'|n'} \phi(d)\phi(d'),$$

and $\phi(d)\phi(d') = \phi(dd')$. The identity is then easy to check; since $\phi(p^k) = p^{k-1}(p-1)$ for positive integers k and $\phi(1) = 1$, we have $\phi(1) + \phi(p) + \cdots + \phi(p^k) = 1 + (p-1) + (p^2-p) + \cdots + (p^k - p^{k-1}) = p^k$, as desired. \square

7. [25] Show that for positive integers n ,

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

Solution. On the grounds of the previous problem, Möbius inversion with $f(k) = \phi(k)$ and $g(k) = k$ gives:

$$\phi(n) = f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right) = \sum_{d'|n} g\left(\frac{n}{d'}\right)\mu(d') = \sum_{d'|n} \frac{n}{d'}\mu(d'). \square$$

Alternatively, one uses the convolution of the functions $f(k) = n$ and $g(k) = \frac{\mu(d)}{d}$. The strategy is the same as the previous convolution proof. For $n = p^k$ with k a positive integer, we have $\phi(n) = p^k - p^{k-1}$, while the series reduces to $p^k \cdot \mu(1) + p^k \cdot \mu(p)/p = p^k - p^{k-1}$.

8. [30] Determine with proof, a simple closed form expression for

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right).$$

Solution. We claim the series reduces to $\sigma(n)$. The series counts the ordered triples (d, x, y) with $d|n; x|d; 0 < y \leq n/d$; and $(y, n/d) = 1$. To see this, write

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right) = \sum_{d'|n} \phi\left(\frac{n}{d'}\right)\tau(d'),$$

so that for a given $d'|n$ we may choose x and y as described above. On the other hand, we can count these triples by groups sharing a given x . Fixing x as a divisor of n fixes an integer $\frac{n}{x}$. Then d varies such that $\frac{n}{d}$ is a divisor of $\frac{n}{x}$. For each divisor $\frac{n}{d}$ of $\frac{n}{x}$ there are precisely $\phi\left(\frac{n}{d}\right)$ choices y , so that by the lemma from the previous problem, there are $\frac{n}{x}$ triples (d, x, y) for a given x . It follows that there are precisely $\sigma(n)$ such triples (d, x, y) . \square

Again, an alternative is to use the multiplicativity of the convolution, although it is now a little more difficult. Write $n = p^k$ so that

$$\begin{aligned} \sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right) &= \sum_{m=0}^k \phi(p^m)\tau(p^{k-m}) = k+1 + \sum_{m=1}^k p^{m-1}(p-1)(k-m+1) \\ &= k+1 + \left(\sum_{m=1}^k p^m(k-m+1) \right) - \left(\sum_{m=1}^k p^{m-1}(k-m+1) \right) = k+1 + p^k - k + \sum_{m'=1}^{k-1} p^{m'} \\ &= 1 + p + \cdots + p^k = \sigma(p^k). \end{aligned}$$

9. [35] Find all positive integers n such that

$$\sum_{k=1}^n \phi(k) = \frac{3n^2 + 5}{8}.$$

Answer: $\boxed{1, 3, 5}.$

Solution. We contend that the proper relation is

$$\sum_{k=1}^n \phi(k) \leq \frac{3n^2 + 5}{8}. \quad (*)$$

Let $\Phi(k)$ denote the left hand side of $(*)$. It is trivial to see that for $n \leq 7$ the posed inequality holds, has equality where $n = 1, 3, 5$, and holds strictly for $n = 7$. Note that $\phi(2k) \leq k$ and $\phi(2k+1) \leq 2k$, the former because $2, 4, \dots, 2k$ share a common divisor. It follows that $\phi(2k) + \phi(2k+1) \leq 3k$. Suppose for the sake of induction that $\Phi(2k-1) < \frac{3(2k-1)^2 + 5}{8}$. Then

$$\Phi(2k+1) = \Phi(2k-1) + \phi(2k) + \phi(2k+1) < \frac{3(2k-1)^2 + 5}{8} + 3k = \frac{3(2k+1)^2 + 5}{8}.$$

To complete the proof, it is enough to note that for a positive integer k ,

$$\frac{3(2k-1)^2 + 5}{8} + k < \frac{3(2k)^2 + 5}{8}. \quad \square$$

10. [40] Find all pairs (n, k) of positive integers such that

$$\sigma(n)\phi(n) = \frac{n^2}{k}.$$

Answer: $\boxed{(1, 1)}.$

Solution. It is clear that for a given integer n , there is at most one integer k for which the equation holds. For $n = 1$ this is $k = 1$. But, for $n > 1$, problem 1 asserts that $\sigma(n)\phi(n) \leq n^2 - 1 < n^2$, so that $k \geq 2$. We now claim that $2 > \frac{n^2}{\sigma(n)\phi(n)}$. Write $n = p_1^{e_1} \cdots p_k^{e_k}$, where the p_i are distinct primes and $e_i \geq 1$ for all i , and let $q_1 < q_2 < \cdots$ be the primes in ascending order. Then

$$\begin{aligned} \frac{n^2}{\sigma(n)\phi(n)} &= \prod_{i=1}^k \frac{p_i^{2e_i}}{\frac{p_i^{e_i+1}-1}{p_i-1} \cdot (p_i-1)p_i^{e_i-1}} = \prod_{i=1}^k \frac{p_i^{2e_i}}{p_i^{2e_i} - p_i^{e_i-1}} \\ &= \prod_{i=1}^k \frac{1}{1 - p_i^{-1-e_i}} \leq \prod_{i=1}^k \frac{1}{1 - p_i^{-2}} < \prod_{i=1}^{\infty} \frac{1}{1 - q_i^{-2}} \\ &= \prod_{i=1}^{\infty} \left(\sum_{j=0}^{\infty} \frac{1}{q_i^{2j}} \right) = \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &< 1 + \sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = 1 + \frac{1}{2} \left(\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \cdots \right) = \frac{7}{4} < 2. \end{aligned}$$

It follows that there can be no solutions to $k = \frac{n^2}{\sigma(n)\phi(n)}$ other than $n = k = 1$. \square

Grab Bag - Miscellaneous Problems [130]

11. [30] Find all functions $f : \mathbb{Q} \rightarrow \mathbb{Q}$ such that

$$\begin{aligned} f(x)f(y) &= f(x) + f(y) - f(xy) \\ 1 + f(x+y) &= f(xy) + f(x)f(y) \end{aligned}$$

for all rational numbers x, y .

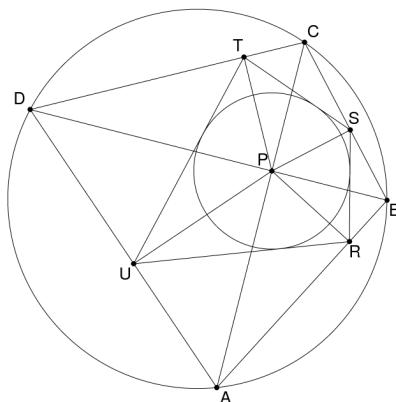
Answer: $\boxed{f(x) = 1 \forall x, \text{ and } f(x) = 1 - x \forall x.}$

Solution. Considering the first equation, either side of the second equation is equal to $f(x) + f(y)$. Now write $g(x) = 1 - f(x)$, so that

$$\begin{aligned} g(xy) &= 1 - f(xy) = 1 - f(x) - f(y) + f(x)f(y) = (1 - f(x))(1 - f(y)) = g(x)g(y) \\ g(x+y) &= 1 - f(x+y) = 1 - f(x) + 1 - f(y) = g(x) + g(y), \end{aligned}$$

By induction, $g(nx) = ng(x)$ for all integers n , so that $g(p/q) = (p/q)g(1)$ for integers p and q with q nonzero; i.e., $g(x) = xg(1)$. As g is multiplicative, $g(1) = g(1)^2$, so the only possibilities are $g(1) = 1$ and $g(1) = 0$. These give $g(x) = x$ and $g(x) = 0$, or $f(x) = 1 - x$ and $f(x) = 1$, respectively. One easily checks that these functions are satisfactory. \square

12. [30] Let $ABCD$ be a cyclic quadrilateral, and let P be the intersection of its two diagonals. Points R, S, T , and U are feet of the perpendiculars from P to sides AB, BC, CD , and AD , respectively. Show that quadrilateral $RSTU$ is bicentric if and only if $AC \perp BD$. (Note that a quadrilateral is called *inscriptible* if it has an incircle; a quadrilateral is called *bicentric* if it is both cyclic and inscriptible.)



Solution. First we show that $RSTU$ is always inscriptible. Note that in addition to $ABCD$, we have cyclic quadrilaterals $ARPU$ and $BSPR$. Thus,

$$\angle PRU = \angle PAU = \angle CAD = \angle CBD = \angle SBP = \angle SRP,$$

and it follows that P lies on the bisector of $\angle SRU$. Analogously, P lies on the bisectors of $\angle TSR$ and $\angle UTS$, so is equidistant from lines UR, RS, ST , and TU , and $RSTU$ is inscriptible having incenter P . Now we show that $RSTU$ is cyclic if and only if the diagonals of $ABCD$ are orthogonal. We have

$$\begin{aligned} \angle APB &= \pi - \angle BAP - \angle PBA = \pi - \angle RAP - \angle PBR = \pi - \angle RUP - \angle PSR \\ &= \pi - \frac{1}{2} (\angle RUT + \angle TSR). \end{aligned}$$

It follows that $\angle APB = \frac{\pi}{2}$ if and only if $\angle RUT + \angle TSR = \pi$, as desired. \square

13. [30] Find all nonconstant polynomials $P(x)$, with real coefficients and having only real zeros, such that $P(x+1)P(x^2-x+1) = P(x^3+1)$ for all real numbers x .

Answer: $\{\mathbf{P}(\mathbf{x}) = \mathbf{x}^{\mathbf{k}} \mid \mathbf{k} \in \mathbb{Z}^+\}$.

Solution. Note that if $P(\alpha) = 0$, then by setting $x = \alpha - 1$ in the given equation, we find $0 = P(x^3 + 1) = P(\alpha^3 - 3\alpha^2 + 3\alpha)$. Because P is nonconstant, it has at least one zero. Because P has finite degree, there exist minimal and maximal roots of P . Writing $\alpha^3 - 3\alpha^2 + 3\alpha \geq \alpha \iff \alpha(\alpha - 1)(\alpha - 2) \geq 0$, we see that the largest zero of P cannot exceed 2. Likewise, the smallest zero cannot be negative, so all of the zeroes of P lie in $[0, 2]$. Moreover, if $\alpha \notin \{0, 1, 2\}$ is a zero of P , then $\alpha' = \alpha^3 - 3\alpha^2 + 3\alpha$ is another zero of P that lies strictly between α and 1. Because P has only finitely many zeroes, all of its zeroes must lie in $\{0, 1, 2\}$. Now write $P(x) = kx^p(x-1)^q(x-2)^r$ for nonnegative integers p, q and r having a positive sum. The given equation becomes

$$\begin{aligned} k^2(x+1)^p x^q (x-1)^r (x^2-x+1)^p (x^2-x)^q (x^2-x-1)^r &= P(x+1)P(x^2-x+1) \\ &= P(x^3+1) = k(x^3+1)^p x^{3q} (x^3-1)^r. \quad (*) \end{aligned}$$

For the leading coefficients to agree, we require $k = k^2$. Because the leading coefficient is nonzero, P must be monic. In $(*)$, r must be zero lest $P(x^3+1) = 0$ have complex roots. Then q must be zero as well. For, if q is positive, then $P(x^2-x+1) = 0$ has 1 as a root while $P(x^3+1)$ does not. Finally, the remaining possibilities are $P(x) = x^p$ for p an arbitrary positive integer. It is easily seen that these polynomials are satisfactory. \square

14. [40] Find an explicit, closed form formula for

$$\sum_{k=1}^n \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n+k+1}.$$

Answer: $\frac{-1}{\binom{2n+1}{n}}$ or $-\frac{n!(n+1)!}{(2n+1)!}$ or obvious equivalent.

Solution. Consider the interpolation of the polynomial $P(x) = x \cdot n!$ at $x = 0, 1, \dots, n$. We obtain the identity

$$\begin{aligned} P(x) = x \cdot n! &= \sum_{k=0}^n k \cdot n! \prod_{j \neq k} \frac{x-j}{k-j} \\ &= \sum_{k=0}^n k \cdot n! \cdot \frac{x(x-1) \cdots (x-k+1)(x-k-1) \cdots (x-n)}{k!(n-k)!(-1)^{n-k}} \\ &= \sum_{k=1}^n k \cdot (-1)^{n-k} \cdot \binom{n}{k} \cdot x(x-1) \cdots (x-k+1)(x-k-1) \cdots (x-n). \end{aligned}$$

This identity is valid for all complex numbers x , but, to extract a factor $\frac{1}{n+k+1}$ from the valid product of each summand, we set $x = -n-1$, so that

$$-(n+1)! = \sum_{k=1}^n k(-1)^{n-k} \binom{n}{k} (-n-1) \cdots (-n-k)(-n-k-2) \cdots (-2n-1) = \sum_{k=1}^n \frac{k(-1)^k \binom{n}{k} (2n+1)!}{n!(n+k+1)}.$$

Finally,

$$\sum_{k=1}^n \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n+k+1} = \frac{-n!(n+1)!}{(2n+1)!} = \frac{-1}{\binom{2n+1}{n}}. \square$$

10th Annual Harvard-MIT Mathematics Tournament
Saturday 24 February 2007

Team Round: B Division

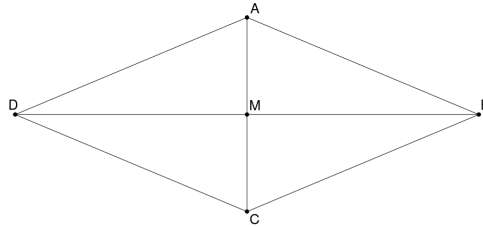
Compute $(x - a)(x - b) \cdots (x - z)$ - Short Answer [200]

For this section, your team should give only the answers to the problems.

1. [20] Find the sum of the positive integer divisors of 2^{2007} .

Answer: $2^{2008} - 1$. The divisors are the powers of two not exceeding 2^{2007} . So the sum is
 $1 + 2 + 2^2 + \cdots + 2^{2007} = -1 + 2 + 2 + 2^2 + \cdots + 2^{2007} = -1 + 2^2 + 2^2 + \cdots + 2^{2007} = \cdots = -1 + 2^{2008}$.

2. [20] The four sides of quadrilateral $ABCD$ are equal in length. Determine the perimeter of $ABCD$ given that it has area 120 and $AC = 10$.



Answer: **52.** Let M be the midpoint of AC . Then triangles AMB , BMC , CMD , and DMA are all right triangles having legs 5 and h for some h . The area of $ABCD$ is 120, but also $4 \cdot (\frac{1}{2} \cdot 5 \cdot h) = 10h$, so $h = 12$. Then $AB = BC = CD = DA = \sqrt{12^2 + 5^2} = 13$, and the perimeter of $ABCD$ is 52.

3. [20] Five people are crowding into a booth against a wall at a noisy restaurant. If at most three can fit on one side, how many seating arrangements accommodate them all?

Answer: **240.** Three people will sit on one side and two sit on the other, giving a factor of two. Then there are $5!$ ways to permute the people.

4. [20] Thomas and Michael are just two people in a large pool of well qualified candidates for appointment to a problem writing committee for a prestigious college math contest. It is 40 times more likely that both will serve if the size of the committee is increased from its traditional 3 members to a whopping n members. Determine n . (Each person in the pool is equally likely to be chosen.)

Answer: **16.** Suppose there are k candidates. Then the probability that both serve on a 3 membered committee is $(k - 2)/\binom{k}{3}$, and the odds that both serve on an n membered committee are $\binom{k-2}{n-2}/\binom{k}{n}$. The ratio of the latter to the former is

$$\frac{\binom{k}{3} \binom{k-2}{n-2}}{(k-2) \binom{k}{n}} = \frac{k!(k-2)!1!(k-3)!n!(k-n)!}{k!(k-2)!(n-2)!(k-n)!3!(k-3)!} = \frac{n \cdot (n-1)}{3!}.$$

Solving $n \cdot (n - 1) = 240$ produces $n = 16, -15$, and we discard the latter.

5. [20] The curves $y = x^2(x - 3)^2$ and $y = (x^2 - 1)(x - 2)$ intersect at a number of points in the real plane. Determine the sum of the x -coordinates of these points of intersection.

Answer: **7.** Because the first curve touches the x -axis at $x = 0$ and $x = 3$ while the second curve crosses the x -axis at $x = \pm 1$ and $x = 2$, there are four points of intersection. In particular, the points of intersection have x -coordinates determined by the difference of the two curves:

$$0 = x^2(x - 3)^2 - (x^2 - 1)(x - 2) = (x^4 - 6x^3 + \cdots) - (x^3 + \cdots) = x^4 - 7x^3 + \cdots.$$

We need only the first two coefficients to determine $x_1 + x_2 + x_3 + x_4 = -(\frac{-7}{1}) = 7$.

6. [20] Andrew has a fair six sided die labeled with 1 through 6 as usual. He tosses it repeatedly, and on every third roll writes down the number facing up as long as it is not the 6. He stops as soon as the last two numbers he has written down are squares or one is a prime and the other is a square. What is the probability that he stops after writing squares consecutively?

Answer: $\boxed{4/25}$. We can safely ignore all of the rolls he doesn't record. The probability that he stops after writing two squares consecutively is the same as the probability that he never rolls a prime. For, as soon as the first prime is written, either it must have been preceded by a square or it will be followed by a nonnegative number of additional primes and then a square. So we want the probability that two numbers chosen uniformly with replacement from $\{1, 2, 3, 4, 5\}$ are both squares, which is $(2/5)^2$.

7. [20] Three positive reals x, y , and z are such that

$$\begin{aligned}x^2 + 2(y - 1)(z - 1) &= 85 \\y^2 + 2(z - 1)(x - 1) &= 84 \\z^2 + 2(x - 1)(y - 1) &= 89.\end{aligned}$$

Compute $x + y + z$.

Answer: $\boxed{18}$. Add the three equations to obtain

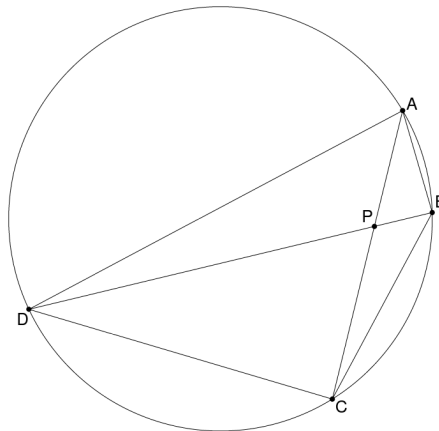
$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx - 4x - 4y - 4z + 6 = 258,$$

which rewrites as $(x + y + z - 2)^2 = 256$. Evidently, $x + y + z = 2 \pm 16$. Since x, y , and z are positive, $x + y + z > 0$ so $x + y + z = 2 + 16 = 18$.

8. [20] Find the *positive* real number(s) x such that $\frac{1}{2}(3x^2 - 1) = (x^2 - 50x - 10)(x^2 + 25x + 5)$.

Answer: $\boxed{25 + 2\sqrt{159}}$. Write $a = x^2 - 50x - 10$ and $b = x^2 + 25x + 5$; the given becomes $\frac{a+2b-1}{2} = ab$, so $0 = 2ab - a - 2b + 1 = (a - 1)(2b - 1)$. Then $a - 1 = x^2 - 50x - 11 = 0$ or $2b - 1 = 2x^2 + 50x + 9 = 0$. The former has a positive root, $x = 25 + 2\sqrt{159}$, while the latter cannot, for obvious reasons.

9. [20] Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$, and $AD = 4$. Determine AC/BD .



Answer: $\boxed{5/7}$. Let the diagonals intersect at P . Note that triangles ABP and DCP are similar, so that $3AP = DP$ and $3BP = CP$. Additionally, triangles BCP and ADP are similar, so that $2BP = AP$. It follows that

$$\frac{AC}{BD} = \frac{AP + PC}{BP + PD} = \frac{2BP + 3BP}{BP + 6BP} = \frac{5}{7}.$$

10. [20] A positive real number x is such that

$$\sqrt[3]{1-x^3} + \sqrt[3]{1+x^3} = 1.$$

Find x^2 .

Answer: $\boxed{\frac{\sqrt[3]{28}}{3}}$. Cubing the given equation yields

$$1 = (1-x^3) + 3\sqrt[3]{(1-x^3)(1+x^3)} \left(\sqrt[3]{1-x^3} + \sqrt[3]{1+x^3} \right) + (1+x^3) = 2 + 3\sqrt[3]{1-x^6}.$$

Then $\frac{-1}{3} = \sqrt[3]{1-x^6}$, so $\frac{-1}{27} = 1-x^6$ and $x^6 = \frac{28}{27}$ and $x^2 = \frac{\sqrt[3]{28}}{3}$.

Adult Acorns - Gee, I'm a Tree! [200]

In this section of the team round, your team will derive some basic results concerning *tangential* quadrilaterals. Tangential quadrilaterals have an *incircle*, or a circle lying within them that is tangent to all four sides. If a quadrilateral has an incircle, then the center of this circle is the *incenter* of the quadrilateral. As you shall see, tangential quadrilaterals are related to cyclic quadrilaterals. For reference, a review of cyclic quadrilaterals is given at the end of this section.

Your answers for this section of the team test should be proofs. Note that you may use any standard facts about cyclic quadrilaterals, such as those listed at the end of this test, without proving them. Additionally, you may cite the results of previous problems, even if you were unable to prove them.

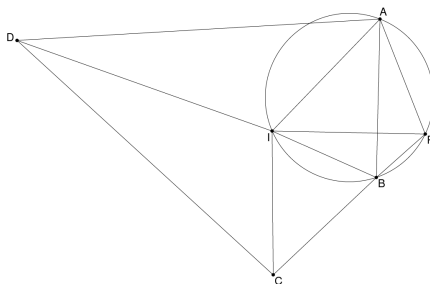
For these problems, $ABCD$ is a tangential quadrilateral having incenter I . For the first three problems, the point P is constructed such that triangle PAB is similar to triangle IDC and lies outside $ABCD$.

1. [30] Show that $PAIB$ is cyclic by proving that $\angle IAP$ is supplementary to $\angle PBI$.

Solution. Note that I lies on the angle bisectors of the angles of quadrilateral $ABCD$. So writing $\angle DAB = 2\alpha$, $\angle ABC = 2\beta$, $\angle BCD = 2\gamma$, and $\angle CDA = 2\delta$, we have

$$\begin{aligned} \angle IAP + \angle PBI &= \angle IAB + \angle BAP + \angle PBA + \angle ABI \\ &= \angle IAB + \angle CDI + \angle ICD + \angle ABI \\ &= \alpha + \beta + \gamma + \delta. \end{aligned}$$

We are done because the angles in quadrilateral $ABCD$ add up to 360° . \square



2. [40] Show that triangle PAI is similar to triangle BIC . Then conclude that

$$PA = \frac{PI}{BC} \cdot BI.$$

Solution. We have $\angle IBC = \angle ABI$ because I lies on the angle bisector, and $\angle ABI = \angle API$ because $PAIB$ is cyclic. Additionally,

$$\angle BCI = \angle ICD = \angle PBA = \angle PIA,$$

by the angle bisector CI , that triangles PAB and IDC are similar, and the fact that $PAIB$ is cyclic, respectively. It follows that triangles PAI and BIC are similar. In particular, it follows that $IP/PA = BC/BI$, as required. \square

3. [25] Deduce from the above that

$$\frac{BC}{AD} \cdot \frac{AI}{BI} \cdot \frac{DI}{CI} = 1.$$

Solution. Exchanging the roles of A and D with B and C , respectively, converts the formula from problem 2 into another formula:

$$PB = \frac{PI}{AD} \cdot AD.$$

Then on the one hand, dividing the two gives $PA/PB = (AD \cdot BI)/(BC \cdot AI)$. On the other hand, $PA/PB = DI/CI$ because triangles PAB and IDC are similar. Clearing the denominators in the equation

$$\frac{DI}{CI} = \frac{AD \cdot BI}{BC \cdot AI}$$

yields the desired form. \square

4. [25] Show that $AB + CD = AD + BC$. Use the above to conclude that for some positive number α ,

$$\begin{aligned} AB &= \alpha \cdot \left(\frac{AI}{CI} + \frac{BI}{DI} \right) & BC &= \alpha \cdot \left(\frac{BI}{DI} + \frac{CI}{AI} \right) \\ CD &= \alpha \cdot \left(\frac{CI}{AI} + \frac{DI}{BI} \right) & DA &= \alpha \cdot \left(\frac{DI}{BI} + \frac{AI}{CI} \right). \end{aligned}$$

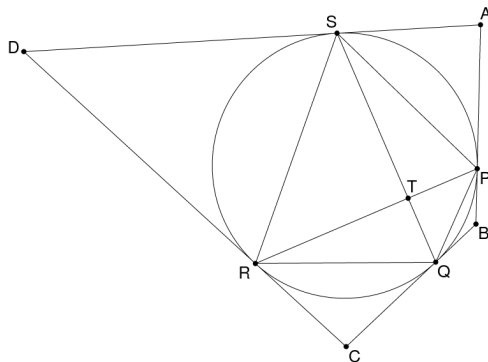
Solution. Draw in the points of tangency P, Q, R , and S , of the incircle with sides AB, BC, CD , and AD , as shown. Then we have equal tangents $AP = AS, BP = BQ, CQ = CR$, and $DR = DS$. Then

$$AB + CD = AP + BP + CR + DR = AS + (BQ + CQ) + DS = BC + AD.$$

Using the result of problem 3, we set $BC = x \cdot BI \cdot CI$ and $AD = x \cdot AI \cdot DI$ for some x , and $AB = y \cdot AI \cdot BI$ and $CD = y \cdot CI \cdot DI$ for some y . Now because $AB + CD = BC + AD$, we obtain

$$y(AI \cdot BI + CI \cdot DI) = x(BI \cdot CI + AI \cdot DI).$$

So it follows that the ratio $AB : BC : CD : DA$ is uniquely determined. One easily checks that the posed ratio satisfies the three required relations. \square



5. [40] Show that

$$AB \cdot BC = BI^2 + \frac{AI \cdot BI \cdot CI}{DI}.$$

Solution. Returning to the original set up, Ptolemy's theorem applied to quadrilateral $PAIB$ gives $AB \cdot PI = PA \cdot BI + PB \cdot AI$. Substituting equation $PA = \frac{PI}{BC} \cdot BI$ from problem 2 and its cousin $PB = \frac{PI}{AD} \cdot AI$ allows us to write

$$AB \cdot PI = \frac{PI}{BC} \cdot BI^2 + \frac{PI}{AD} \cdot AI^2,$$

or

$$AB \cdot BC = BI^2 + \frac{BC}{AD} \cdot AI^2.$$

Substituting the formula $BC/AD = \frac{BI \cdot CI}{AI \cdot DI}$ from problem 3 finishes the problem. \square

6. [40] Let the incircle of $ABCD$ be tangent to sides AB, BC, CD , and AD at points P, Q, R , and S , respectively. Show that $ABCD$ is cyclic if and only if $PR \perp QS$.

Solution. Let the diagonals of $PQRS$ intersect at T . Because \overline{AP} and \overline{AS} are tangent to ω at P and S , we may write $\alpha = \angle ASP = \angle SPA = \angle SQP$ and $\beta = \angle CQR = \angle QRC = \angle QPR$. Then $\angle PTQ = \pi - \alpha - \beta$. On the other hand, $\angle PAS = \pi - 2\alpha$ and $\angle RCQ = \pi - 2\beta$, so that $ABCD$ is cyclic if and only if

$$\pi = \angle BAD + \angle DCB = 2\pi - 2\alpha - 2\beta,$$

or simply

$$\pi/2 = \pi - \alpha - \beta = \angle PTQ,$$

as desired. \square

A brief review of cyclic Quadrilaterals.

The following discussion of cyclic quadrilaterals is included for reference. Any of the results given here may be cited without proof in your writeups.

A *cyclic quadrilateral* is a quadrilateral whose four vertices lie on a circle called the *circumcircle* (the circle is unique if it exists.) If a quadrilateral has a circumcircle, then the center of this circumcircle is called the *circumcenter* of the quadrilateral. For a convex quadrilateral $ABCD$, the following are equivalent:

- Quadrilateral $ABCD$ is cyclic;
- $\angle ABD = \angle ACD$ (or $\angle BCA = \angle BDA$, etc.);
- Angles $\angle ABC$ and $\angle CDA$ are *supplementary*, that is, $m\angle ABC + m\angle CDA = 180^\circ$ (or angles $\angle BCD$ and $\angle BAD$ are supplementary);

Cyclic quadrilaterals have a number of interesting properties. A cyclic quadrilateral $ABCD$ satisfies

$$AC \cdot BD = AB \cdot CD + AD \cdot BC,$$

a result known as *Ptolemy's theorem*. Another result, typically called *Power of a Point*, asserts that given a circle ω , a point P anywhere in the plane of ω , and a line ℓ through P intersecting ω at points A and B , the value of $AP \cdot BP$ is independent of ℓ ; i.e., if a second line ℓ' through P intersects ω at A' and B' , then $AP \cdot BP = A'P \cdot B'P$. This second theorem is proved via similar triangles. Say P lies outside of ω , that ℓ and ℓ' are as before and that A and A' lie on segments BP and $B'P$ respectively. Then triangle $AA'P$ is similar to triangle $B'B'P$ because the triangles share an angle at P and we have

$$m\angle AA'P = 180^\circ - m\angle B'A'A = m\angle ABB' = m\angle PBB'.$$

The case where $A = B$ is valid and describes the tangents to ω . A similar proof works for P inside ω .

IXth Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Algebra Test: Solutions

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to row from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

Answer: 14:24

Solution: Let the distance between Harvard and MIT be 1, and let c, s, r denote the speeds of the current and Larry's swimming and rowing, respectively. Then we are given

$$s + c = \frac{1}{40} = \frac{9}{360}, \quad s - c = \frac{1}{45} = \frac{8}{360}, \quad r - c = \frac{1}{15} = \frac{24}{360},$$

so

$$r + c = (s + c) - (s - c) + (r - c) = \frac{9 - 8 + 24}{360} = \frac{25}{360},$$

and it takes Larry $360/25 = 14.4$ minutes, or 14:24, to row from Harvard to MIT.

2. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

Answer: $(3, 3), (-4, -4), \left(\frac{1 + 3\sqrt{5}}{2}, \frac{1 - 3\sqrt{5}}{2}\right), \left(\frac{1 - 3\sqrt{5}}{2}, \frac{1 + 3\sqrt{5}}{2}\right)$

Solution: We have $x^2 + y = y^2 + x$ which can be written as $(x - y)(x + y - 1) = 0$. The case $x = y$ yields $x^2 + x - 12 = 0$, hence $(x, y) = (3, 3)$ or $(-4, -4)$. The case $y = 1 - x$ yields $x^2 + 1 - x - 12 = x^2 - x - 11 = 0$ which has solutions $x = \frac{1 \pm \sqrt{1 + 44}}{2} = \frac{1 \pm 3\sqrt{5}}{2}$. The other two solutions follow.

3. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

Answer: $\frac{13}{48}$

Solution: Suppose we fix the time at which Train B arrives at Intersection X; then call the interval during which Train A could arrive (given its schedule) and collide with Train B the "disaster window."

We consider two cases:

- (i) *Train B enters Intersection X between 9:30 and 9:45.* If Train B arrives at 9:30, the disaster window is from 9:00 to 10:15, an interval of $1\frac{1}{4}$ hours. If Train B arrives at 9:45, the disaster window is $1\frac{1}{2}$ hours long. Thus, the disaster window has an average length of $(1\frac{1}{4} + 1\frac{1}{2}) \div 2 = \frac{11}{8}$. From 9:00 to 2:30 is $5\frac{1}{2}$ hours. The probability of a collision is thus $\frac{11}{8} \div 5\frac{1}{2} = \frac{1}{4}$.

(ii) *Train B enters Intersection X between 9:45 and 12:30.* Here the disaster window is always $1\frac{1}{2}$ hours long, so the probability of a collision is $1\frac{1}{2} \div 5\frac{1}{2} = \frac{3}{11}$.

From 9:30 to 12:30 is 3 hours. Now case (i) occurs with probability $\frac{1}{4} \div 3 = \frac{1}{12}$, and case (ii) occurs with probability $\frac{11}{12}$. The overall probability of a collision is therefore $\frac{1}{12} \cdot \frac{1}{4} + \frac{11}{12} \cdot \frac{3}{11} = \frac{1}{48} + \frac{1}{4} = \frac{13}{48}$.

4. Let a_1, a_2, \dots be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

Answer: $\frac{1}{11}$

Solution: Let X denote the desired sum. Note that

$$\begin{aligned} X &= \frac{1}{4^2} + \frac{1}{4^3} + \frac{2}{4^4} + \frac{3}{4^5} + \frac{5}{4^6} + \dots \\ 4X &= \frac{1}{4^1} + \frac{1}{4^2} + \frac{2}{4^3} + \frac{3}{4^4} + \frac{5}{4^5} + \frac{8}{4^6} + \dots \\ 16X &= \frac{1}{4^0} + \frac{1}{4^1} + \frac{2}{4^2} + \frac{3}{4^3} + \frac{5}{4^4} + \frac{8}{4^5} + \frac{13}{4^6} + \dots \end{aligned}$$

so that $X + 4X = 16X - 1$, and $X = 1/11$.

5. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

Answer: 33

Solution: A tricky thing about this problem may be that the angles on the two clocks might be reversed and would still count as being the same (for example, both angles could be 90° , but the hour hand may be ahead of the minute hand on one clock and behind on the other).

Let x , $-12 \leq x < 12$, denote the number of hours since noon. If we take 0° to mean upwards to the “XII” and count angles clockwise, then the hour and minute hands of the correct clock are at $30x^\circ$ and $360x^\circ$, and those of the slow clock are at $15x^\circ$ and $180x^\circ$. The two angles are thus $330x^\circ$ and $165x^\circ$, of course after removing multiples of 360° and possibly flipping sign; we are looking for solutions to

$$330x^\circ \equiv 165x^\circ \pmod{360^\circ} \text{ or } 330x^\circ \equiv -165x^\circ \pmod{360^\circ}.$$

In other words,

$$360 \mid 165x \text{ or } 360 \mid 495x.$$

Or, better yet,

$$\frac{165}{360}x = \frac{11}{24}x \text{ and/or } \frac{495}{360}x = \frac{11}{8}x$$

must be an integer. Now x is any *real* number in the range $[-12, 12)$, so $11x/8$ ranges in $[-16.5, 16.5)$, an interval that contains 33 integers. For any value of x such that $11x/24$ is an integer, of course $11x/8 = 3 \times (11x/24)$ is also an integer, so the answer is just 33.

6. Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.

Answer: -37

Solution: First of all, as the left side of the first given equation takes values $-1, 2, -7$, and 32 when $x = 0, 1, 2$, and 3 , respectively, we know that a, b , and c are distinct positive reals. Let $t = \sqrt{ab} + \sqrt{bc} + \sqrt{ca}$, and note that

$$\begin{aligned} s^2 &= a + b + c + 2t = 9 + 2t, \\ t^2 &= ab + bc + ca + 2\sqrt{abcs} = 11 + 2s, \\ s^4 &= (9 + 2t)^2 = 81 + 36t + 4t^2 = 81 + 36t + 44 + 8s = 125 + 36t + 8s, \\ 18s^2 &= 162 + 36t, \end{aligned}$$

so that $s^4 - 18s^2 - 8s = -37$.

7. Let

$$f(x) = x^4 - 6x^3 + 26x^2 - 46x + 65.$$

Let the roots of $f(x)$ be $a_k + ib_k$ for $k = 1, 2, 3, 4$. Given that the a_k, b_k are all integers, find $|b_1| + |b_2| + |b_3| + |b_4|$.

Answer: 10

Solution: The roots of $f(x)$ must come in complex-conjugate pairs. We can then say that $a_1 = a_2$ and $b_1 = -b_2$; $a_3 = a_4$ and $b_3 = -b_4$. The constant term of $f(x)$ is the product of these, so $5 \cdot 13 = (a_1^2 + b_1^2)(a_3^2 + b_3^2)$. Since a_k and b_k are integers for all k , and it is simple to check that 1 and i are not roots of $f(x)$, we must have $a_1^2 + b_1^2 = 5$ and $a_3^2 + b_3^2 = 13$. The only possible ways to write these sums with positive integers is $1^2 + 2^2 = 5$ and $2^2 + 3^2 = 13$, so the values of a_1 and b_1 *up to sign* are 1 and 2 ; and a_3 and b_3 *up to sign* are 2 and 3 . From the x^3 coefficient of $f(x)$, we get that $a_1 + a_2 + a_3 + a_4 = 6$, so $a_1 + a_3 = 3$. From the limits we already have, this tells us that $a_1 = 1$ and $a_3 = 2$. Therefore $b_1, b_2 = \pm 2$ and $b_3, b_4 = \pm 3$, so the required sum is $2 + 2 + 3 + 3 = 10$.

8. Solve for all complex numbers z such that $z^4 + 4z^2 + 6 = z$.

Answer: $\frac{1 \pm i\sqrt{7}}{2}, \frac{-1 \pm i\sqrt{11}}{2}$

Solution: Rewrite the given equation as $(z^2 + 2)^2 + 2 = z$. Observe that a solution to $z^2 + 2 = z$ is a solution of the quartic by substitution of the left hand side into itself. This gives $z = \frac{1 \pm i\sqrt{7}}{2}$. But now, we know that $z^2 - z + 2$ divides into $(z^2 + 2)^2 - z + 2 = z^4 + 4z^2 - z + 6$. Factoring it out, we obtain $(z^2 - z + 2)(z^2 + z + 3) = z^4 + 4z^2 - z + 6$. Finally, the second term leads to the solutions $z = \frac{-1 \pm i\sqrt{11}}{2}$.

9. Compute the value of the infinite series

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)}$$

Answer: $\frac{11}{10}$

Solution: We employ the difference of squares identity, uncovering the factorization of the denominator: $n^4 + 4 = (n^2 + 2)^2 - (2n)^2 = (n^2 - 2n + 2)(n^2 + 2n + 2)$. Now,

$$\begin{aligned} \frac{n^4 + 3n^2 + 10n + 10}{n^4 + 4} &= 1 + \frac{3n^2 + 10n + 6}{n^4 + 4} \\ &= 1 + \frac{4}{n^2 - 2n + 2} - \frac{1}{n^2 + 2n + 2} \\ \implies \sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)} &= \sum_{n=2}^{\infty} \frac{1}{2^n} + \frac{4}{2^n \cdot (n^2 - 2n + 2)} - \frac{1}{2^n \cdot (n^2 + 2n + 2)} \\ &= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1}{2^{n-2} \cdot ((n-1)^2 + 1)} - \frac{1}{2^n \cdot ((n+1)^2 + 1)} \end{aligned}$$

The last series telescopes to $\frac{1}{2} + \frac{1}{10}$, which leads to an answer of $\frac{1}{2} + \frac{1}{2} + \frac{1}{10} = \frac{11}{10}$.

10. Determine the maximum value attained by

$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$

over real numbers $x > 1$.

Answer: $\frac{1}{6}$

Solution: We have the following algebra:

$$\begin{aligned} \frac{x^4 - x^2}{x^6 + 2x^3 - 1} &= \frac{x - \frac{1}{x}}{x^3 + 2 - \frac{1}{x^3}} \\ &= \frac{x - \frac{1}{x}}{\left(x - \frac{1}{x}\right)^3 + 2 + 3\left(x - \frac{1}{x}\right)} \\ &\leq \frac{x - \frac{1}{x}}{3\left(x - \frac{1}{x}\right) + 3\left(x - \frac{1}{x}\right)} = \frac{1}{6} \end{aligned}$$

where $\left(x - \frac{1}{x}\right)^3 + 1 + 1 \geq 3\left(x - \frac{1}{x}\right)$ in the denominator was deduced by the AM-GM inequality. As a quick check, equality holds where $x - \frac{1}{x} = 1$ or when $x = \frac{1+\sqrt{5}}{2}$.

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Calculus Test: Solutions

1. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

Answer: $\frac{1}{18}$

Solution: Suppose that the leading term of $f(x)$ is cx^n , where $c \neq 0$. Then the leading terms of $f'(x)$ and of $f''(x)$ are cnx^{n-1} and $cn(n-1)x^{n-2}$, respectively, so $cx^n = cnx^{n-1} \cdot cn(n-1)x^{n-2}$, which implies that $n = (n-1) + (n-2)$, or $n = 3$, and $c = cn \cdot cn(n-1) = 18c^2$, or $c = \frac{1}{18}$.

2. Compute $\lim_{x \rightarrow 0} \frac{e^{x \cos x} - 1 - x}{\sin(x^2)}$.

Answer: $\frac{1}{2}$

Solution: Let's compute all the relevant Maclaurin series expansions, up to the quadratic terms:

$$x \cos x = x + \dots, \quad e^{x \cos x} = 1 + x + \frac{1}{2}x^2 + \dots, \quad \sin(x^2) = x^2 + \dots,$$

so

$$\lim_{x \rightarrow 0} \frac{e^{x \cos x} - 1 - x}{\sin(x^2)} = \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2 + \dots}{x^2 + \dots} = \frac{1}{2}.$$

3. At time 0, an ant is at $(1,0)$ and a spider is at $(-1,0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x -axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

Answer: $\frac{\sqrt{14}}{4}$

Solution: Picture an instant in time where the ant and spider have x -coordinates a and s , respectively. If $1 \leq s \leq 3$, then $a \leq 0$, and the distance between the bugs is at least 1. If $s > 3$, then, needless to say the distance between the bugs is at least 2. If $-1 \leq s \leq 1$, then $s = 1 - 2a$, and the distance between the bugs is

$$\sqrt{(a - (1 - 2a))^2 + (1 - a^2)} = \sqrt{8a^2 - 6a + 2} = \sqrt{\frac{(8a - 3)^2 + 7}{8}},$$

which attains the minimum value of $\sqrt{7/8}$ when $a = 3/8$.

4. Compute $\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

Answer: $15e$

Solution: Define, for non-negative integers n ,

$$S_n := \sum_{k=0}^{\infty} \frac{k^n}{k!},$$

where $0^0 = 1$ when it occurs. Then $S_0 = e$, and, for $n \geq 1$,

$$S_n = \sum_{k=0}^{\infty} \frac{k^n}{k!} = \sum_{k=1}^{\infty} \frac{k^n}{k!} = \sum_{k=0}^{\infty} \frac{(k+1)^n}{(k+1)!} = \sum_{k=0}^{\infty} \frac{(k+1)^{n-1}}{k!} = \sum_{i=0}^{n-1} \binom{n-1}{i} S_i,$$

so we can compute inductively that $S_1 = e$, $S_2 = 2e$, $S_3 = 5e$, and $S_4 = 15e$.

5. Compute

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

Answer: $5 - 6 \ln 2$

Solution: Writing $x = u^6$ so that $dx = 6u^5 du$, we have

$$\begin{aligned} \int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}} &= \int_0^1 \frac{6u^5 du}{u^3 + u^2} \\ &= 6 \int_0^1 \frac{u^3 du}{u + 1} \\ &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u + 1} \right) du \\ &= 6 \left(\frac{u^3}{3} - \frac{u^2}{2} + u - \ln |u + 1| \Big|_0^1 \right) = 5 - 6 \ln(2) \end{aligned}$$

6. A triangle with vertices at $(1003, 0)$, $(1004, 3)$, and $(1005, 1)$ in the xy -plane is revolved all the way around the y -axis. Find the volume of the solid thus obtained.

Answer: 5020π

Solution: Let $T \subset \mathbb{R}^2$ denote the triangle, including its interior. Then T 's area is $5/2$, and its centroid is $(1004, 4/3)$, so

$$\int_{(x,y) \in T} x \, dx \, dy = \frac{5}{2} \cdot 1004 = 2510.$$

We are interested in the volume

$$\int_{(x,y) \in T} 2\pi x \, dx \, dy,$$

but this is just $2\pi \cdot 2510 = 5020\pi$.

7. Find all positive real numbers c such that the graph of $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x -axis.

Answer: $\frac{\sqrt{3}}{2}$

Solution: The equation $0 = f'(x) = 3x^2 - c$ has two real roots: $\pm\sqrt{c/3}$. Let $a := \sqrt{c/3}$. As $f''(-a) = -6\sqrt{c/3} < 0$, f has a unique local maximum at $x = -a$.

Because f has half-turn symmetry about the origin, it suffices to consider this local extremum. The radius of curvature at any local extremum is

$$r(x) = \frac{1}{|f''(x)|} = \frac{1}{6|x|},$$

so the condition in the problem is equivalent to

$$\begin{aligned} r(-a) &= f(-a) \\ \frac{1}{6a} &= -a(a^2 - c) \\ 1 &= 6a^2(c - a^2) = 2c(2c/3) \\ c &= \sqrt{3}/2. \end{aligned}$$

8. Compute

$$\int_0^{\pi/3} x \tan^2(x) dx$$

Answer: $\frac{\pi\sqrt{3}}{3} - \frac{\pi^2}{18} - \ln 2$

Solution: We have

$$\begin{aligned} \int_0^{\pi/3} x \tan^2(x) dx &= \int_0^{\pi/3} x \left(-1 + \frac{1}{\cos^2(x)} \right) dx \\ &= -\frac{x^2}{2} \Big|_0^{\pi/3} + \int_0^{\pi/3} \frac{x dx}{\cos^2(x)} \\ &= -\frac{x^2}{2} \Big|_0^{\pi/3} + \left(x \tan(x) \Big|_0^{\pi/3} - \int_0^{\pi/3} \tan(x) dx \right) \quad (u = x; dv = \frac{dx}{\cos^2(x)}) \\ &= -\frac{x^2}{2} + x \tan(x) + \ln |\cos(x)| \Big|_0^{\pi/3} = -\frac{\pi^2}{18} + \frac{\pi\sqrt{3}}{3} - \ln(2) \end{aligned}$$

9. Compute the sum of all real numbers x such that

$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0$$

Answer: $-\frac{1}{2}$

Solution: The carefully worded problem statement suggests that repeated roots might be involved (not to be double counted), as well as complex roots (not to be counted). Let $P(x) = 2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1$. Now, a is a double root of the polynomial $P(x)$ if and only if $P(a) = P'(a) = 0$. Hence, we consider the system

$$\begin{aligned} P(a) &= 2a^6 - 3a^5 + 3a^4 + a^3 - 3a^2 + 3a - 1 = 0 \\ P'(a) &= 12a^5 - 15a^4 + 12a^3 + 3a^2 - 6a + 3 = 0 \\ &\implies 3a^4 + 8a^3 - 15a^2 + 18a - 7 = 0 \\ &\quad 37a^3 - 57a^2 + 57a - 20 = 0 \\ &\quad a^2 - a + 1 = 0 \end{aligned}$$

We have used polynomial long division to deduce that any double root must be a root of $a^2 - a + 1$! With this information, we can see that $P(x) = (x^2 - x + 1)^2(2x^2 + x - 1)$. The real roots are easily computed via the quadratic formula, leading to an answer of $-\frac{1}{2}$. In fact the repeated roots were complex.

10. Suppose f and g are differentiable functions such that

$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$

for all real x . Moreover, f is nonnegative and g is positive. Furthermore,

$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$

for all reals a . Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

Answer: e^{-16} or $\frac{1}{e^{16}}$

Solution: Differentiating the given integral with respect to a gives $f(g(a)) = e^{-2a}$. Now

$$x \frac{d[\ln(f(g(x)))]}{dx} = x \frac{f'(g(x))g'(x)}{f(g(x))} = \frac{g'(f(x))f'(x)}{g(f(x))} = \frac{d[\ln(g(f(x)))]}{dx}$$

where the second equals sign follows from the given. Since $\ln(f(g(x))) = -2x$, we have $-x^2 + C = \ln(g(f(x)))$, so $g(f(x)) = Ke^{-x^2}$. It follows that $K = 1$ and $g(f(4)) = e^{-16}$.

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Combinatorics Test: Solutions

Combinatorics Test

1. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

Answer: 11

Solution: Suppose that n seniors play all three sports and that $2n$ are on the lacrosse team. Then, by the principle of inclusion-exclusion, $85 = (74 + 26 + 2n) - (17 + 18 + 13) + (n) = 100 + 2n - 48 + n = 52 + 3n$. It is easily seen that $n = 11$.

2. Compute

$$\sum_{n_{60}=0}^2 \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

Answer: 1953

Solution: The given sum counts the number of non-decreasing 61-tuples of integers (n_0, \dots, n_{60}) from the set $\{0, 1, 2\}$. Such 61-tuples are in one-to-one correspondence with strictly increasing 61-tuples of integers (m_0, \dots, m_{60}) from the set $\{0, 1, 2, \dots, 62\}$: simply let $m_k = n_k + k$. But the number of such (m_0, \dots, m_{60}) is almost by definition $\binom{63}{61} = \binom{63}{2} = 1953$.

3. A moth starts at vertex A of a certain cube and is trying to get to vertex B , which is opposite A , in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B . How many ways can the moth achieve its objective?

Answer: 48

Solution: Let X, Y, Z be the three directions in which the moth can initially go. We can symbolize the trajectory of the moth by a sequence of stuff from X s, Y s, and Z s in the obvious way: whenever the moth takes a step in a direction parallel or opposite to X , we write down X , and so on.

The moth can reach B in either exactly 3 or exactly 5 steps. A path of length 3 must be symbolized by XYZ in some order. There are $3! = 6$ such orders. A trajectory of length 5 must be symbolized by $XYZXX$, $XYZYY$, or $XYZZZ$, in some order. There are $3 \cdot \frac{5!}{3!1!1!} = 3 \cdot 20 = 60$ possibilities here. However, we must remember to subtract out those trajectories that already arrive at B by the 3rd step: there are $3 \cdot 6 = 18$ of those. The answer is thus $60 - 18 + 6 = 48$.

4. A dot is marked at each vertex of a triangle ABC . Then, 2, 3, and 7 more dots are marked on the sides AB , BC , and CA , respectively. How many triangles have their vertices at these dots?

Answer: 357

Solution: Altogether there are $3 + 2 + 3 + 7 = 15$ dots, and thus $\binom{15}{3} = 455$ combinations of 3 dots. Of these combinations, $\binom{2+2}{3} + \binom{2+3}{3} + \binom{2+7}{3} = 4 + 10 + 84 = 98$ do not give triangles because they are collinear (the rest do give triangles). Thus $455 - 98 = 357$ different triangles can be formed.

5. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

Answer: 125

Solution: Suppose that you are one of the freshmen; then there's a $6/15$ chance that you'll get one of the handouts. We may ask, given that you do get a handout, how many ways are there to distribute the rest? We need only multiply the answer to that question by $15/6$ to answer the original question.

Going clockwise around the table from you, one might write down the sizes of the gaps between people with handouts. There are six such gaps, each of size 0–2, and the sum of their sizes must be $15 - 6 = 9$. So the gap sizes are either 1, 1, 1, 2, 2, 2 in some order, or 0, 1, 2, 2, 2, 2 in some order. In the former case, $\frac{6!}{3!3!} = 20$ orders are possible; in the latter, $\frac{6!}{1!1!4!} = 30$ are. Altogether, then, there are $20 + 30 = 50$ possibilities.

Multiplying this by $15/6$, or $5/2$, gives 125.

6. For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

Answer: 102

Solution: One number must be 5. The other two must have a product divisible by 4. Either both are even, or one is divisible by 4 and the other is odd. In the former case, there are $48 = 3 \times 4 \times 4$ possibilities: 3 positions for the 5, and any of 4 even numbers to fill the other two. In the latter case, there are $54 = 3 \times 2 \times 9$ possibilities: 3 positions and 2 choices for the multiple of 4, and 9 ways to fill the other two positions using at least one 5.

7. Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n + 1$ rounds. In each round, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n .

Answer: $\frac{2^n - 1}{8^n}$

Solution: At any point during this competition, we shall say that the situation is *living* if both players 1 and 2^n are still in the running. A living situation is *far* if those two players are diametrically opposite each other, and *near* otherwise, in which case (as one can check inductively) they must be just one person shy of that maximal separation. At the start of the tournament, the situation is living and near. In each of rounds 1 to n , a far situation can never become near, and a near situation can stay near or become far with equal likelihood.

In each of rounds 1 to $n - 1$, a living situation has a $1/4$ probability of staying living. Therefore, at the end of round k , where $1 \leq k \leq n - 1$, the situation is near with probability $1/8^k$, and far with probability $1/4^k - 1/8^k$. In round n , a far situation has a $1/4$ probability of staying living, whereas a near situation has only a $1/8$ probability of staying living. But if the situation is living at the beginning of the last round, it can only be far, so we can say with complete generality that, at the end of round k , where $1 \leq k \leq n$, the situation is living and far with probability $1/4^k - 1/8^k$. We are interested in finding the probability that the situation is living at the end of round n (and hence far); that probability is thus $\frac{1}{4^n} - \frac{1}{8^n} = \frac{2^n - 1}{8^n}$.

8. In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?
- (a) Each row contains all four numbers.
 - (b) Each column contains all four numbers.
 - (c) Each “quadrant” contains all four numbers. (The quadrants are the four corner 2×2 squares.)

Answer: 288

Solution: Call a filled 4×4 array satisfying the given conditions *cool*. There are $4!$ possibilities for the first row; WLOG, let it be 1 2 3 4. Since each quadrant has to contain all four numbers, we have exactly four possibilities for the second row, namely:

- (i) 3 4 1 2
- (ii) 3 4 2 1
- (iii) 4 3 1 2
- (iv) 4 3 2 1

I claim that the number of cool arrays with (i) is equal to those with (iv), and that the number of cool arrays with (ii) is equal to those with (iii). Let's first consider (i) and (iv). Now, (i) is

1 2 3 4
3 4 1 2

while (iv) is

1 2 3 4
4 3 2 1

In (iv), switch 3 and 4 (relabeling doesn't affect the coolness of the array); then, it becomes

$$\begin{array}{cccc} 1 & 2 & 4 & 3 \\ 3 & 4 & 2 & 1 \end{array}$$

Now, interchange the last two columns, which also does not affect the coolness. This gives us (i). Hence, the cool arrays with (i) and the cool arrays with (iv) have a 1:1 correspondence. Using the exact same argument, we can show that the number of cool arrays with (ii) equals those with (iii).

So we only need consider cases (i) and (ii). It is easy to verify that there are four cool arrays with (i), determined precisely by, say, the first two entries of the third row; and two with (ii), determined precisely by, say the first entry of the third row. Hence, the answer is $4! \times (4 + 2) \times 2 = 288$.

9. Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

Answer: 3507

Solution: Let the celebrities get into one or more circles so that each circle has at least three celebrities, and each celebrity shook hands precisely with his or her neighbors in the circle.

Let's consider the possible circle sizes:

- *There's one big circle with all 8 celebrities.* Depending on the ordering of the people in the circle, the fan's list can still vary. Literally speaking, there are $7!$ different circles 8 people can make: fix one of the people, and then there are 7 choices for the person to the right, 6 for the person after that, and so on. But this would be double-counting because, as far as the fan's list goes, it makes no difference if we "reverse" the order of all the people. Thus, there are $7!/2 = 2520$ different possible lists here.
- *5+3.* In this case there are $\binom{8}{5}$ ways to split into the two circles, $\frac{4!}{2}$ essentially different ways of ordering the 5-circle, and $\frac{2!}{2}$ ways for the 3-circle, giving a total count of $56 \cdot 12 \cdot 1 = 672$.
- *4+4.* In this case there are $\binom{8}{4}/2 = 35$ ways to split into the two circles (we divide by 2 because here, unlike in the 5 + 3 case, it does not matter which circle is which), and $\frac{3!}{2} = 3$ ways of ordering each, giving a total count of $35 \cdot 3 \cdot 3 = 315$.

Adding them up, we get $2520 + 672 + 315 = 3507$.

10. Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.

Answer: 253

Solution: Let $c_{i,j}$ denote the number of students correctly answering questions i and j ($1 \leq i < j \leq 10$), and let $w_{i,j}$ denote the number of students getting both questions

wrong. An individual student answers k questions correctly and $10 - k$ questions incorrectly. This student answers $\binom{k}{2}$ pairs of questions correctly and $\binom{10-k}{2}$ pairs of questions incorrectly. Now observe that

$$\binom{k}{2} + \binom{10-k}{2} = k^2 - 10k + 45 = (k-5)^2 + 20 \geq 20$$

Therefore,

$$\sum_{1 \leq i < j \leq 10} c_{i,j} + w_{i,j} \geq 20n$$

Now if the performance is not laughable, then $c_{i,j} \leq 56$ and $w_{i,j} \leq 56$ for all $1 \leq i < j \leq 10$. Observe that there are $2\binom{10}{2} = 90$ of these variables. Hence, in a boring performance,

$$20n \leq \sum_{1 \leq i < j \leq 10} c_{i,j} + w_{i,j} \leq 90 \cdot 56 = 5040$$

or $n \leq 252$. In particular this implies that if $n \geq 253$, the performance is laughable. This is the best bound because $\binom{10}{5} = 252$, and if each of 252 students correctly answers a different 5 element subset of the 10 questions, then $c_{i,j} = w_{i,j} = 56$ for all $1 \leq i < j \leq 10$.

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General Test, Part 1: Solutions

1. How many positive integers x are there such that $3x$ has 3 digits and $4x$ has four digits?

Answer: 84

Solution: Note that x must be between 250 and 333, inclusive. There are 84 integers in that interval.

2. What is the probability that two cards randomly selected (without replacement) from a standard 52-card deck are neither of the same value nor the same suit?

Answer: $\frac{12}{17}$

Solution: After choosing a first card, the second needs to be in one of the other three suits and of a different value. Hence, the answer is $\frac{3 \cdot 12}{52-1} = \frac{12}{17}$.

3. A square and an equilateral triangle together have the property that the area of each is the perimeter of the other. Find the square's area.

Answer: $12\sqrt[3]{4}$

Solution: Let s, t be the side lengths of the square and triangle, respectively. Then $4s = \frac{\sqrt{3}t^2}{4}$, and so

$$3t = s^2 = \left(\frac{\sqrt{3}t^2}{16} \right)^2 = \frac{3t^4}{2^8},$$

so that $t^3 = 2^8$ and $s^2 = 3t = 3\sqrt[3]{2^8} = 12\sqrt[3]{4}$.

4. Find

$$\frac{\sqrt{31 + \sqrt{31 + \sqrt{31 + \dots}}}}{\sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}.$$

Answer: $6 - \sqrt{5}$

Solution: Let the numerator be x and the denominator y . Then $x^2 = 31 + x$, so, as $x > 0$, we have

$$x = \frac{1 + \sqrt{1 + 4 \cdot 31}}{2} = \frac{1 + 5\sqrt{5}}{2}.$$

Similarly we compute that

$$y = \frac{1 + \sqrt{1 + 4 \cdot 1}}{2} = \frac{1 + \sqrt{5}}{2},$$

so that

$$\frac{x}{y} = \frac{1 + 5\sqrt{5}}{1 + \sqrt{5}} = \frac{1 + 5\sqrt{5}}{1 + \sqrt{5}} \cdot \frac{1 - \sqrt{5}}{1 - \sqrt{5}} = \frac{-24 + 4\sqrt{5}}{-4} = 6 - \sqrt{5}.$$

5. In the plane, what is the length of the shortest path from $(-2, 0)$ to $(2, 0)$ that avoids the interior of the unit circle (i.e., circle of radius 1) centered at the origin?

Answer: $2\sqrt{3} + \frac{\pi}{3}$

Solution: The path goes in a line segment tangent to the circle, then an arc of the circle, then another line segment tangent to the circle. Since one of these tangent lines and a radius of the circle give two legs of a right triangle with hypotenuse the line from $(0,0)$ to $(-2,0)$ or $(2,0)$, the length of each tangent line is $\sqrt{2^2 - 1^2} = \sqrt{3}$. Also, because these are 30° - 60° - 90° right triangles, the angle of the arc is 60° and has length $\pi/3$.

6. Six celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

Answer: 70

Solution: Let the celebrities get into one or more circles so that each circle has at least three celebrities, and each celebrity shook hands precisely with his or her neighbors in the circle.

Either there is one big circle of all 6 celebrities or else there are two small circles of 3 celebrities each.

If there is one big circle of 6, then depending on the ordering of the people in the circle, the fan's list can still vary. Literally speaking, there are $5!$ different circles 6 people can make: fix one of the people, and then there are 5 choices for the person to the right, 4 for the person after that, and so on. But this would be double-counting because, as far as the fan's list goes, it makes no difference if we "reverse" the order of all the people. There are thus $5!/2 = 60$ different possible lists here.

If there are two small circles of 3, then there are $\binom{6}{3}$ different ways the members of the "first" small circle may be selected. But this, too, is double-counting, because it makes no difference which circle is termed the "first" and which the "second." There are therefore $\binom{6}{3}/2 = 10$ essentially different ways to split up the people into the two circles. In a circle of just three, each person shakes hands with both the others, so naturally the order of people in the circle doesn't matter. There are thus 10 different possible lists here.

(Note that, translated into the language of graph theory, the problem is asking for the number of graphs on six labeled vertices such that each vertex has degree two.)

7. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

Answer: $\frac{13}{48}$

Solution: Suppose we fix the time at which Train B arrives at Intersection X; then call the interval during which Train A could arrive (given its schedule) and collide with Train B the "disaster window."

We consider two cases:

- (i) *Train B enters Intersection X between 9:30 and 9:45.* If Train B arrives at 9:30, the disaster window is from 9:00 to 10:15, an interval of $1\frac{1}{4}$ hours. If Train B arrives at 9:45, the disaster window is $1\frac{1}{2}$ hours long. Thus, the disaster window has an average length of $(1\frac{1}{4} + 1\frac{1}{2}) \div 2 = \frac{11}{8}$. From 9:00 to 2:30 is $5\frac{1}{2}$ hours. The probability of a collision is thus $\frac{11}{8} \div 5\frac{1}{2} = \frac{1}{4}$.
- (ii) *Train B enters Intersection X between 9:45 and 12:30.* Here the disaster window is always $1\frac{1}{2}$ hours long, so the probability of a collision is $1\frac{1}{2} \div 5\frac{1}{2} = \frac{3}{11}$.

From 9:30 to 12:30 is 3 hours. Now case (i) occurs with probability $\frac{1}{4} \div 3 = \frac{1}{12}$, and case (ii) occurs with probability $\frac{11}{12}$. The overall probability of a collision is therefore $\frac{1}{12} \cdot \frac{1}{4} + \frac{11}{12} \cdot \frac{3}{11} = \frac{1}{48} + \frac{1}{4} = \frac{13}{48}$.

8. A dot is marked at each vertex of a triangle ABC . Then, 2, 3, and 7 more dots are marked on the sides AB , BC , and CA , respectively. How many triangles have their vertices at these dots?

Answer: 357

Solution: Altogether there are $3 + 2 + 3 + 7 = 15$ dots, and thus $\binom{15}{3} = 455$ combinations of 3 dots. Of these combinations, $\binom{2+2}{3} + \binom{2+3}{3} + \binom{2+7}{3} = 4 + 10 + 84 = 98$ do not give triangles because they are collinear (the rest do give triangles). Thus $455 - 98 = 357$ different triangles can be formed.

9. Take a unit sphere S , i.e., a sphere with radius 1. Circumscribe a cube C about S , and inscribe a cube D in S , so that every edge of cube C is parallel to some edge of cube D . What is the shortest possible distance from a point on a face of C to a point on a face of D ?

Answer: $\frac{3 - \sqrt{3}}{3} = 1 - \frac{\sqrt{3}}{3}$, or equivalent

Solution: Using the Pythagorean theorem, we know that the length of a diagonal of a cube of edge length s is $s\sqrt{3}$. Since D is inscribed in a sphere that has diameter 2, this means that its side length is $2/\sqrt{3}$.

The distance from a face of D to a face of C will be the distance between them along any line perpendicular to both of them; take such a line passing through the center of S . The distance from the center to any face of D along this line will be half the side length of D , or $1/\sqrt{3}$. The distance from the center to the edge of C is the radius of S , which is 1. Therefore the desired distance is $1 - 1/\sqrt{3} = 1 - \frac{\sqrt{3}}{3}$.

10. A positive integer n is called “flippant” if n does not end in 0 (when written in decimal notation) and, moreover, n and the number obtained by reversing the digits of n are both divisible by 7. How many flippant integers are there between 10 and 1000?

Answer: 17

Solution: We use the notation “ $|$ ” to mean “divides.”

There is only one flippant 2-digit number, namely 77. Indeed, if $10a + b$ is flippant (where a, b are integers 1–9), then $7 | 10a + b$ and $7 | 10b + a$. Thus,

$$7 | 3(10a + b) - (10b + a) = 29a - 7b = a + 7(4a - b),$$

so that $7 | a$, and similarly $7 | b$, so we’d better have $a = b = 7$.

There are 16 flippant 3-digit numbers. First consider the 12 palindromic ones (ones where the hundreds and units digits are the same): 161, 252, 343, 434, 525, 595, 616, 686, 707, 777, 868, and 959. Now consider the general case: suppose $100a + 10b + c$ is flippant, where a, b, c are integers 1–9. Then $7 \mid 100a + 10b + c$ and $7 \mid 100c + 10b + a$, so $7 \mid (100a + 10b + c) - (100c + 10b + a) = 99(a - c)$, and so $7 \mid a - c$. In order for this not to result in a *palindromic* integer, we must have $a - c = \pm 7$ and, moreover, both $100a + 10b + a$ and $100c + 10b + c$ must be palindromic flippant integers. Consulting our list above, we find 4 more flippant integers: 168, 259, 861, and 952.

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General Test, Part 2: Solutions

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

Answer: 14:24

Solution: Let the distance between Harvard and MIT be 1, and let c, s, r denote the speeds of the current and Larry's swimming and rowing, respectively. Then we are given

$$s + c = \frac{1}{40} = \frac{9}{360}, \quad s - c = \frac{1}{45} = \frac{8}{360}, \quad r - c = \frac{1}{15} = \frac{24}{360},$$

so

$$r + c = (s + c) - (s - c) + (r - c) = \frac{9 - 8 + 24}{360} = \frac{25}{360},$$

and it takes Larry $360/25 = 14.4$ minutes, or 14:24, to row from Harvard to MIT.

2. Find

$$\frac{2^2}{2^2 - 1} \cdot \frac{3^2}{3^2 - 1} \cdot \frac{4^2}{4^2 - 1} \cdots \frac{2006^2}{2006^2 - 1}.$$

Answer: $\frac{4012}{2007}$

Solution:

$$\prod_{k=2}^{2006} \frac{k^2}{k^2 - 1} = \prod_{k=2}^{2006} \frac{k^2}{(k-1)(k+1)} = \prod_{k=2}^{2006} \frac{k}{k-1} \prod_{k=2}^{2006} \frac{k}{k+1} = \frac{2006}{1} \cdot \frac{2}{2007} = \frac{4012}{2007}.$$

3. Let C be the unit circle. Four distinct, smaller congruent circles C_1, C_2, C_3, C_4 are internally tangent to C such that C_i is externally tangent to C_{i-1} and C_{i+1} for $i = 1, \dots, 4$ where C_5 denotes C_1 and C_0 represents C_4 . Compute the radius of C_1 .

Answer: $\sqrt{2} - 1$

Solution: Let O and O' be the centers of C and C_1 respectively, and let C_1 be tangent to C, C_2, C_4 at $P, Q,$ and R respectively. Observe that $QORO'$ is a square and that $P, O',$ and O are collinear. Thus, if r is the desired radius, $1 = r + OO' = r + r\sqrt{2}$, so that $r = \frac{1}{\sqrt{2}+1} = \sqrt{2} - 1$.

4. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

Answer: 11

Solution: Suppose that n seniors play all three sports and that $2n$ are on the lacrosse team. Then, by the principle of inclusion-exclusion, $85 = (74 + 26 + 2n) - (17 + 18 + 13) + (n) = 100 + 2n - 48 + n = 52 + 3n$. It is easily seen that $n = 11$.

5. If a, b are nonzero real numbers such that $a^2 + b^2 = 8ab$, find the value of $\left| \frac{a+b}{a-b} \right|$.

Answer: $\frac{\sqrt{15}}{3}$

Solution: Note that

$$\left| \frac{a+b}{a-b} \right| = \sqrt{\frac{(a+b)^2}{(a-b)^2}} = \sqrt{\frac{a^2 + b^2 + 2ab}{a^2 + b^2 - 2ab}} = \sqrt{\frac{10ab}{6ab}} = \frac{\sqrt{15}}{3}.$$

6. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

Answer: $20 + \sqrt{2}$

Solution: Extend sides AB, CD, EF, GH to form a rectangle: let X be the intersection of lines GH and AB ; Y that of AB and CD ; Z that of CD and EF ; and W that of EF and GH .

As $BC = 2$, we have $BY = YC = \sqrt{2}$. As $DE = 4$, we have $DZ = ZE = 2\sqrt{2}$. As $FG = 2$, we have $FW = WG = \sqrt{2}$.

We can compute the dimensions of the rectangle: $WX = YZ = YC + CD + DZ = 3 + 3\sqrt{2}$, and $XY = ZW = ZE + EF + FW = 2 + 3\sqrt{2}$. Thus, $HX = XA = XY - AB - BY = 1 + 2\sqrt{2}$, and so $AH = \sqrt{2}HX = 4 + \sqrt{2}$, and $GH = WX - WG - HX = 2$. The perimeter of the octagon can now be computed by adding up all its sides.

7. What is the smallest positive integer n such that n^2 and $(n+1)^2$ both contain the digit 7 but $(n+2)^2$ does not?

Answer: 27

Solution: The last digit of a square is never 7. No two-digit squares begin with 7. There are no 3-digit squares beginning with the digits 17, 27, 37, or 47. In fact, the smallest square containing the digit 7 is $576 = 24^2$. Checking the next few numbers, we see that $25^2 = 625$, $26^2 = 676$, $27^2 = 729$, $28^2 = 784$, and $29^2 = 841$, so the answer is 27.

8. Six people, all of different weights, are trying to build a human pyramid: that is, they get into the formation

A
B C
D E F

We say that someone not in the bottom row is “supported by” each of the two closest people beneath her or him. How many different pyramids are possible, if nobody can be supported by anybody of lower weight?

Answer: 16

Solution: Without loss of generality, let the weights of the people be 1, 2, 3, 4, 5, and 6. Clearly we must have $A = 1$. Then, equally clearly, either B or C must be 2.

Suppose $B = 2$: Then either C or D must be 3. If $C = 3$, we have $3! = 6$ possibilities to fill the bottom row. If $D = 3$, then $C = 4$ and we have $2! = 2$ possibilities to fill E and F . Altogether there are $6 + 2 = 8$ possibilities in this case.

Suppose $C = 2$: then, similarly, there are 8 possibilities here.

Altogether there are $8 + 8 = 16$ possibilities.

9. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

Answer: 33

Solution: A tricky thing about this problem may be that the angles on the two clocks might be reversed and would still count as being the same (for example, both angles could be 90° , but the hour hand may be ahead of the minute hand on one clock and behind on the other).

Let x , $-12 \leq x < 12$, denote the number of hours since noon. If we take 0° to mean upwards to the “XII” and count angles clockwise, then the hour and minute hands of the correct clock are at $30x^\circ$ and $360x^\circ$, and those of the slow clock are at $15x^\circ$ and $180x^\circ$. The two angles are thus $330x^\circ$ and $165x^\circ$, of course after removing multiples of 360° and possibly flipping sign; we are looking for solutions to

$$330x^\circ \equiv 165x^\circ \pmod{360^\circ} \text{ or } 330x^\circ \equiv -165x^\circ \pmod{360^\circ}.$$

In other words,

$$360 \mid 165x \text{ or } 360 \mid 495x.$$

Or, better yet,

$$\frac{165}{360}x = \frac{11}{24}x \text{ and/or } \frac{495}{360}x = \frac{11}{8}x$$

must be an integer. Now x is any *real* number in the range $[-12, 12)$, so $11x/8$ ranges in $[-16.5, 16.5)$, an interval that contains 33 integers. For any value of x such that $11x/24$ is an integer, of course $11x/8 = 3 \times (11x/24)$ is also an integer, so the answer is just 33.

10. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today’s handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor’s. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

Answer: 125

Solution: Suppose that you are one of the freshmen; then there’s a $6/15$ chance that you’ll get one of the handouts. We may ask, given that you do get a handout, how

many ways are there to distribute the rest? We need only multiply the answer to that question by $15/6$ to answer the original question.

Going clockwise around the table from you, one might write down the sizes of the gaps between people with handouts. There are six such gaps, each of size 0–2, and the sum of their sizes must be $15 - 6 = 11$. So the gap sizes are either 1, 1, 1, 2, 2, 2 in some order, or 0, 1, 2, 2, 2, 2 in some order. In the former case, $\frac{6!}{3!3!} = 20$ orders are possible; in the latter, $\frac{6!}{1!1!4!} = 30$ are. Altogether, then, there are $20 + 30 = 50$ possibilities.

Multiplying this by $15/6$, or $5/2$, gives 125.

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Geometry Test: Solutions

1. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

Answer: $20 + \sqrt{2}$

Solution: Extend sides AB, CD, EF, GH to form a rectangle: let X be the intersection of lines GH and AB ; Y that of AB and CD ; Z that of CD and EF ; and W that of EF and GH .

As $BC = 2$, we have $BY = YC = \sqrt{2}$. As $DE = 4$, we have $DZ = ZE = 2\sqrt{2}$. As $FG = 2$, we have $FW = WG = \sqrt{2}$.

We can compute the dimensions of the rectangle: $WX = YZ = YC + CD + DZ = 3 + 3\sqrt{2}$, and $XY = ZW = ZE + EF + FW = 2 + 3\sqrt{2}$. Thus, $HX = XA = XY - AB - BY = 1 + 2\sqrt{2}$, and so $AH = \sqrt{2}HX = 4 + \sqrt{2}$, and $GH = WX - WG - HX = 2$. The perimeter of the octagon can now be computed by adding up all its sides.

2. Suppose ABC is a scalene right triangle, and P is the point on hypotenuse \overline{AC} such that $\angle ABP = 45^\circ$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC .

Answer: $\frac{9}{5}$

Solution: Notice that \overline{BP} bisects the right angle at B . Thus, we write $AB = 2x$, $BC = x$. By the Pythagorean theorem, $5x^2 = 9$, from which the area $\frac{1}{2}(x)(2x) = x^2 = \frac{9}{5}$.

3. Let A, B, C , and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y . If $PQ = 27$, find XY .

Answer: 31

Solution: Suppose X, P, Q, Y lie in that order. Let $PX = x$ and $QY = y$. By power of a point from P , $x \cdot (27 + y) = 30$, and by power of a point from Q , $y \cdot (27 + x) = 84$. Subtracting the first from the second, $27 \cdot (y - x) = 54$, so $y = x + 2$. Now, $x \cdot (29 + x) = 30$, and we find $x = 1, -30$. Since -30 makes no sense, we take $x = 1$ and obtain $XY = 1 + 27 + 3 = 31$.

4. Let ABC be a triangle such that $AB = 2, CA = 3$, and $BC = 4$. A semicircle with its diameter on \overline{BC} is tangent to \overline{AB} and \overline{AC} . Compute the area of the semicircle.

Answer: $\frac{27\pi}{40}$

Solution: Let O, D , and E be the midpoint of the diameter and the points of tangency with \overline{AB} and \overline{AC} respectively. Then $[ABC] = [AOB] + [AOC] = \frac{1}{2}(AB + AC)r$, where r is the radius of the semicircle. Now by Heron's formula, $[ABC] = \sqrt{\frac{9}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}} = \frac{3\sqrt{15}}{4}$. We solve for $r = \frac{3\sqrt{15}}{10}$ and compute $\frac{1}{2}\pi r^2 = \frac{27\pi}{40}$.

5. Triangle ABC has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point D is on side AC such that $CD = 1$, and F is a point such that $BF = 2$ and $CF = 3$. Let E be the intersection of lines AB and DF . Find the area of $CDEB$.

Answer: $\frac{22}{35}$

Solution: Draw segment AF . Then notice $AF = 4$, and we have a right triangle. Now draw line CE , let it intersect AF at G . By Ceva, $FG = \frac{4}{3}$ and $AG = \frac{8}{3}$. Using mass points we find that $\frac{AE}{EB} = 6$ so $\frac{[AEF]}{[BEF]} = 6$, and since $[ABF] = 4$, $[BEF] = \frac{4}{7}$. It's easy to see that $[CDF] = \frac{1}{5}[ACF] = \frac{6}{5}$, so

$$[BCDE] = [CDF] - [BEF] = \frac{6}{5} - \frac{4}{7} = \frac{22}{35}.$$

6. A circle of radius t is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin \frac{\pi}{8}$. Find rt .

Answer: $\frac{2 + \sqrt{2}}{4}$

Solution: The distance between the point of tangency of the two circles and the nearest vertex of the triangle is seen to be both $r(\csc \frac{\pi}{8} - 1)$ and $t(\csc \frac{\pi}{8} + 1)$, so

$$\begin{aligned} rt &= \frac{r^2(\csc \frac{\pi}{8} - 1)}{\csc \frac{\pi}{8} + 1} = \frac{(1 + \sin \frac{\pi}{8})^2(1 - \sin \frac{\pi}{8})}{1 + \sin \frac{\pi}{8}} = 1 - \sin^2 \frac{\pi}{8} \\ &= \frac{1}{2} + \frac{1 - 2\sin^2 \frac{\pi}{8}}{2} = \frac{1}{2} + \frac{\cos \frac{\pi}{4}}{2} = \frac{1}{2} + \frac{\sqrt{2}}{4} = \frac{2 + \sqrt{2}}{4}. \end{aligned}$$

7. Suppose $ABCD$ is an isosceles trapezoid in which $\overline{AB} \parallel \overline{CD}$. Two mutually externally tangent circles ω_1 and ω_2 are inscribed in $ABCD$ such that ω_1 is tangent to \overline{AB} , \overline{BC} , and \overline{CD} while ω_2 is tangent to \overline{AB} , \overline{DA} , and \overline{CD} . Given that $AB = 1$, $CD = 6$, compute the radius of either circle.

Answer: $\frac{3}{7}$

Solution: Let the radius of both circles be r , and let ω_1 be centered at O_1 . Let ω_1 be tangent to \overline{AB} , \overline{BC} , and \overline{CD} at P , Q , and R respectively. Then, by symmetry, $PB = \frac{1}{2} - r$ and $RC = 3 - r$. By equal tangents from B and C , $BQ = \frac{1}{2} - r$ and $QC = 3 - r$. Now, $\angle BO_1C$ is right because $m\angle O_1BC + m\angle BCO_1 = \frac{1}{2}(m\angle PBC + m\angle BCR) = 90^\circ$. Since $\overline{O_1Q} \perp \overline{BC}$, $r^2 = O_1Q^2 = BQ \cdot QC = (\frac{1}{2} - r)(3 - r) = r^2 - \frac{7}{2}r + \frac{3}{2}$. Solving, we find $r = \frac{3}{7}$.

8. Triangle ABC has a right angle at B . Point D lies on side BC such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute BD .

Answer: $\frac{3}{8}$

Solution: Let $BD = x$. We reflect D over AB to D' . Then $DD' = 2x$, but AD bisects CAD' , so $4x = AD' = AD$. Also, $AD = \sqrt{x^2 + AB^2} = \sqrt{x^2 + AC^2 - BC^2} = \sqrt{x^2 + 4 - (x+1)^2} = \sqrt{3 - 2x}$. We have the quadratic $16x^2 = 3 - 2x$ which gives $x = 3/8$.

9. Four spheres, each of radius r , lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r .

Answer: $\frac{\sqrt{6} - 1}{10}$

Solution: Let O be the center of the sphere that is tangent to the faces ABC , ABD , and BCD . Let P , Q be the feet of the perpendiculars from O to ABC and ABD respectively. Let R be the foot of the perpendicular from P to AB . Then, $OPRQ$ is a quadrilateral such that $\angle P$, $\angle Q$ are right angles and $OP = OQ = r$. Also, $\angle R$ is the dihedral angle between faces ABC and ABD , so $\cos \angle R = 1/3$. We can then compute $QR = \sqrt{2}r$, so $BR = \sqrt{6}r$. Hence, $1 = AB = 2(\sqrt{6}r) + 2r = 2r(\sqrt{6} + 1)$, so $r = (\sqrt{6} - 1)/10$.

10. Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC .

Answer: 24

Solution: Let Γ be the circumcircle of triangle ABC , and let E be the center of the circle tangent to Γ and the sides AC and BC . Notice that $\angle C = 90^\circ$ because $33^2 + 56^2 = 65^2$. Let D be the second intersection of line CE with Γ , so that D is the midpoint of the arc AB away from C . Because $\angle BCD = 45^\circ$, one can easily calculate $CD = 89\sqrt{2}/2$. The power of E with respect to Γ is both $r(65 - r)$ and $r\sqrt{2} \cdot (89\sqrt{2}/2 - r\sqrt{2}) = r(89 - 2r)$, so $r = 89 - 65 = 24$.

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Guts Round Solutions

1. A bear walks one mile south, one mile east, and one mile north, only to find itself where it started. Another bear, more energetic than the first, walks two miles south, two miles east, and two miles north, only to find itself where it started. However, the bears are *not* white and did *not* start at the north pole. At most how many miles apart, to the nearest .001 mile, are the two bears' starting points?

Answer: 3.477

Solution: Say the first bear walks a mile south, an integer $n > 0$ times around the south pole, and then a mile north. The middle leg of the first bear's journey is a circle of circumference $1/n$ around the south pole, and therefore about $\frac{1}{2n\pi}$ miles north of the south pole. (This is not exact even if we assume the Earth is perfectly spherical, but it is correct to about a micron.) Adding this to the mile that the bear walked south/north, we find that it started about $1 + \frac{1}{2n\pi}$ miles from the south pole. Similarly, the second bear started about $2 + \frac{2}{2m\pi}$ miles from the south pole for some integer $m > 0$, so they must have started at most

$$3 + \frac{1}{2n\pi} + \frac{2}{2m\pi} \leq 3 + \frac{3}{2\pi} \approx 3.477$$

miles apart.

2. Compute the positive integer less than 1000 which has exactly 29 positive proper divisors. (Here we refer to positive integer divisors other than the number itself.)

Answer: 720

Solution: Recall that the number $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where the p_i are distinct primes) has exactly $(e_1 + 1)(e_2 + 1) \cdots (e_k + 1)$ positive integer divisors including itself. We seek $N < 1000$ such that this expression is 30. Since $30 = 2 \cdot 3 \cdot 5$, we take $e_1 = 1, e_2 = 2, e_3 = 4$. Then we see that $N = 5^1 3^2 4^2 = 720$ is satisfactory.

3. At a nursery, 2006 babies sit in a circle. Suddenly each baby pokes the baby immediately to either its left or its right, with equal probability. What is the expected number of unpoked babies?

Answer: $\frac{1003}{2}$

Solution: The probability that any given baby goes unpoked is $1/4$. So the answer is $2006/4 = 1003/2$.

4. Ann and Anne are in bumper cars starting 50 meters apart. Each one approaches the other at a constant ground speed of 10 km/hr. A fly starts at Ann, flies to Anne, then back to Ann, and so on, back and forth until it gets crushed when the two bumper cars collide. When going from Ann to Anne, the fly flies at 20 km/hr; when going in the opposite direction the fly flies at 30 km/hr (thanks to a breeze). How many meters does the fly fly?

Answer: 55

Solution: Suppose that at a given instant the fly is at Ann and the two cars are $12d$ apart. Then, while each of the cars travels $4d$, the fly travels $8d$ and meets Anne. Then the fly turns around, and while each of the cars travels d , the fly travels $3d$ and meets Ann again. So, in this process described, each car travels a total of $5d$ while the fly travels $11d$. So the fly will travel $\frac{11}{5}$ times the distance traveled by each bumper car: $\frac{11}{5} \cdot \frac{50}{2} = 55$ meters.

5. Find the number of solutions in positive integers $(k; a_1, a_2, \dots, a_k; b_1, b_2, \dots, b_k)$ to the equation

$$a_1(b_1) + a_2(b_1 + b_2) + \dots + a_k(b_1 + b_2 + \dots + b_k) = 7.$$

Answer: 15

Solution: Let $k, a_1, \dots, a_k, b_1, \dots, b_k$ be a solution. Then $b_1, b_1 + b_2, \dots, b_1 + \dots + b_k$ is just some increasing sequence of positive integers. Considering the a_i as multiplicities, the a_i 's and b_i 's uniquely determine a partition of 7. Likewise, we can determine a_i 's and b_i 's from any partition of 7, so the number of solutions is $p(7) = 15$.

6. Suppose ABC is a triangle such that $AB = 13, BC = 15$, and $CA = 14$. Say D is the midpoint of \overline{BC} , E is the midpoint of \overline{AD} , F is the midpoint of \overline{BE} , and G is the midpoint of \overline{DF} . Compute the area of triangle EFG .

Answer: $\frac{21}{4}$

Solution: By Heron's formula, $[ABC] = \sqrt{21(21-15)(21-14)(21-13)} = 84$. Now, unwinding the midpoint conditions yields $[EFG] = \frac{[DEF]}{2} = \frac{[BDE]}{4} = \frac{[ABD]}{8} = \frac{[ABC]}{16} = \frac{84}{16} = \frac{21}{4}$.

7. Find all real numbers x such that

$$x^2 + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = 10.$$

Answer: $-\sqrt{14}$

Solution: Evidently x^2 must be an integer. Well, there aren't that many things to check, are there? Among positive x , $\sqrt{8}$ is too small and $\sqrt{9}$ is too big; among negative x , $-\sqrt{15}$ is too small and $-\sqrt{13}$ is too big.

8. How many ways are there to label the faces of a regular octahedron with the integers 1–8, using each exactly once, so that any two faces that share an edge have numbers that are relatively prime? Physically realizable rotations are considered indistinguishable, but physically unrealizable reflections are considered different.

Answer: 12

Solution: Well, instead of labeling the faces of a regular octahedron, we may label the vertices of a cube. Then, as no two even numbers may be adjacent, the even numbers better form a regular tetrahedron, which can be done in 2 ways (because

rotations are indistinguishable but reflections are different). Then 3 must be opposite 6, and the remaining numbers — 1, 5, 7 — may be filled in at will, in $3! = 6$ ways. The answer is thus $2 \times 6 = 12$.

9. Four unit circles are centered at the vertices of a unit square, one circle at each vertex. What is the area of the region common to all four circles?

Answer: $\frac{\pi}{3} + 1 - \sqrt{3}$

Solution: The desired region consists of a small square and four “circle segments,” i.e. regions of a circle bounded by a chord and an arc. The side of this small square is just the chord of a unit circle that cuts off an angle of 30° , and the circle segments are bounded by that chord and the circle. Using the law of cosines (in an isosceles triangle with unit leg length and vertex angle 30°), we find that the square of the length of the chord is equal to $2 - \sqrt{3}$. We can also compute the area of each circle segment, namely $\frac{\pi}{12} - \frac{1}{2}(1)(1) \sin 30^\circ = \frac{\pi}{12} - \frac{1}{4}$. Hence, the desired region has area $2 - \sqrt{3} + 4 \left(\frac{\pi}{12} - \frac{1}{4} \right) = \frac{\pi}{3} + 1 - \sqrt{3}$.

10. Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

Answer: 9

Solution: We see the size of the set $f^{-1}(f^{-1}(f^{-1}(f^{-1}(3))))$. Note that $f(x) = (x - 1)^2 - 1 = 3$ has two solutions: $x = 3$ and $x = -1$, and that the fixed points $f(x) = x$ are $x = 3$ and $x = 0$. Therefore, the number of real solutions is equal to the number of distinct real numbers c such that $c = 3$, $c = -1$, $f(c) = -1$ or $f(f(c)) = -1$, or $f(f(f(c))) = -1$. The equation $f(x) = -1$ has exactly one root $x = 1$. Thus, the last three equations are equivalent to $c = 1$, $f(c) = 1$, and $f(f(c)) = 1$. $f(c) = 1$ has two solutions, $c = 1 \pm \sqrt{2}$, and for each of these two values c there are two preimages. It follows that the answer is $1 + 1 + 1 + 2 + 4 = 9$.

11. Find all positive integers $n > 1$ for which $\frac{n^2+7n+136}{n-1}$ is the square of a positive integer.

Answer: 5, 37

Solution: Write $\frac{n^2+7n+136}{n-1} = n + \frac{8n+136}{n-1} = n + 8 + \frac{144}{n-1} = 9 + (n-1) + \frac{144}{(n-1)}$. We seek to find p and q such that $pq = 144$ and $p + q + 9 = k^2$. The possibilities are seen to be $1 + 144 + 9 = 154$, $2 + 72 + 9 = 83$, $3 + 48 + 9 = 60$, $4 + 36 + 9 = 49$, $6 + 24 + 9 = 39$, $8 + 18 + 9 = 35$, $9 + 16 + 9 = 34$, and $12 + 12 + 9 = 33$. Of these, $\{p, q\} = \{4, 36\}$ is the only solution to both equations. Hence $n - 1 = 4, 36$ and $n = 5, 37$.

12. For each positive integer n let S_n denote the set $\{1, 2, 3, \dots, n\}$. Compute the number of triples of subsets A, B, C of S_{2006} (not necessarily nonempty or proper) such that A is a subset of B and $S_{2006} - A$ is a subset of C .

Answer: 2^{4012}

Solution: Let A_o, B_o, C_o be sets satisfying the said conditions. Note that $1 \in A_o$ implies that $1 \in B_o$ and $1 \notin S_{2006} - A_o$ so that 1 may or may not be in C_o . Also,

$1 \notin A_o$ implies that $1 \in S_{2006} - A_o \subset C_o$ while 1 may or may not be in B_o . Thus there are four possibilities for the distribution of 1, and since the same argument holds independently for 2, 3, ..., 2006, the answer is 4^{2006} or 2^{4012} .

13. Let Z be as in problem 15. Let X be the greatest integer such that $|XZ| \leq 5$. Find X .

Answer: 2

Solution: Problems 13–15 go together. See below.

14. Let X be as in problem 13. Let Y be the number of ways to order X crimson flowers, X scarlet flowers, and X vermilion flowers in a row so that no two flowers of the same hue are adjacent. (Flowers of the same hue are mutually indistinguishable.) Find Y .

Answer: 30

Solution: Problems 13–15 go together. See below.

15. Let Y be as in problem 14. Find the maximum Z such that three circles of radius \sqrt{Z} can simultaneously fit inside an equilateral triangle of area Y without overlapping each other.

Answer: $10\sqrt{3} - 15$

Solution: We first find that, in problem 15, each of the circles of radius \sqrt{Z} is the incircle of a 30° - 60° - 90° triangle formed by cutting the equilateral one in half. The equilateral triangle itself has sidelength $\frac{2\sqrt{Y}}{\sqrt{3}}$, so the said inradius is

$$\sqrt{Z} = \frac{1 + \sqrt{3} - 2}{2} \cdot \frac{1}{2} \cdot \frac{2\sqrt{Y}}{\sqrt{3}},$$

so that

$$Z = \frac{(-1 + \sqrt{3})^2}{4\sqrt{3}} Y = \frac{4 - 2\sqrt{3}}{4\sqrt{3}} Y = \frac{2\sqrt{3} - 3}{6} Y.$$

Now we guess that $X = 2$ and see that, miraculously, everything works: in the problem 14, say a crimson flower is placed first. Then there are 2 possibilities for $C_C_ _ _$, 4 for $C_ _ C_ _ _$, 2 for $C_ _ _ C_ _$, and 2 for $C_ _ _ _ C$, giving a total of 10. Of course, the first flower can be of any of the three hues, so $Y = 3 \cdot 10 = 30$. We compute Z and check X in a straightforward manner.

If $X > 2$, then $Y > 30$, and $Z > 10\sqrt{3} - 15$, with the result that $X \leq 2$, a contradiction. Assuming $X < 2$ results in a similar contradiction.

16. A sequence a_1, a_2, a_3, \dots of positive reals satisfies $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine all a_1 such that $a_i = \frac{\sqrt{6} + \sqrt{2}}{4}$ for some positive integer i .

Answer: $\frac{\sqrt{2} + \sqrt{6}}{2}, \frac{\sqrt{3}}{2}, \frac{1}{2}$

Solution: Clearly $a_1 < 1$, or else $1 \leq a_1 \leq a_2 \leq a_3 \leq \dots$. We can therefore write $a_1 = \cos \theta$ for some $0 < \theta < 90^\circ$. Note that $\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}$, and $\cos 15^\circ =$

$\frac{\sqrt{6} + \sqrt{2}}{4}$. Hence, the possibilities for a_1 are $\cos 15^\circ$, $\cos 30^\circ$, and $\cos 60^\circ$, which are $\frac{\sqrt{2} + \sqrt{6}}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{1}{2}$.

17. Beginning at a vertex, an ant is crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities equally likely.) What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

Answer: $\frac{2^{2005}+1}{3 \cdot 2^{2006}}$

Solution: For each nonnegative integer n , let a_n , b_n , and c_n denote the respective probabilities that the ant is where it began, at a neighbor of where it began, or is opposite where it began after moving along n edges. We seek a_{2006} . We have $a_0 = 1$ and $b_0 = c_0 = 0$. We also have the recursive system

$$\begin{aligned} a_n &= \frac{b_{n-1}}{4} \\ b_n &= a_{n-1} + \frac{b_{n-1}}{2} + c_{n-1} \\ c_n &= \frac{b_{n-1}}{4} \end{aligned}$$

for integers $n \geq 1$. Substituting into the second equation we have $b_n = \frac{b_{n-1}}{2} + \frac{b_{n-2}}{2}$ for $n \geq 2$. Solving the characteristic equation $x^2 - \frac{x}{2} - \frac{1}{2} = 0$ for $x = 1, -\frac{1}{2}$, we write $b_n = a \cdot 1^n + b(-1/2)^n$. Using $b_0 = 0, b_1 = 1$, we compute

$$b_n = \frac{2}{3} \cdot (1 - (-1/2)^n)$$

From which we find $a_{2006} = \frac{b_{2005}}{4} = \frac{1}{6} \left(1 + \frac{1}{2^{2005}}\right) = \frac{2^{2005}+1}{3 \cdot 2^{2006}}$.

18. Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Points P and Q are the midpoints of \overline{BC} and \overline{DA} . Compute PQ^2 .

Answer: $\frac{116}{35}$

Solution: Construct $\overline{AC}, \overline{AQ}, \overline{BQ}, \overline{BD}$, and let R denote the intersection of \overline{AC} and \overline{BD} . Because $ABCD$ is cyclic, we have that $\triangle ABR \sim \triangle DCR$ and $\triangle ADR \sim \triangle BCR$. Thus, we may write $AR = 4x, BR = 2x, CR = 6x, DR = 12x$. Now, Ptolemy applied to $ABCD$ yields $140x^2 = 1 \cdot 3 + 2 \cdot 4 = 11$. Now \overline{BQ} is a median in triangle ABD . Hence, $BQ^2 = \frac{2BA^2+2BD^2-AD^2}{4}$. Likewise, $CQ^2 = \frac{2CA^2+2CD^2-DA^2}{4}$. But PQ is a median in triangle BQC , so $PQ^2 = \frac{2BQ^2+2CQ^2-BC^2}{4} = \frac{AB^2+BD^2+CD^2+CA^2-BC^2-AD^2}{4} = \frac{(196+100)x^2+1^2+3^2-2^2-4^2}{4} = \frac{148x^2-5}{2} = \frac{148 \cdot \frac{11}{140} - 5}{2} = \frac{116}{35}$.

Another solution is possible. Extend \overline{AD} and \overline{BC} past A and B to their intersection S . Use similar triangles SAB and SCD , and similar triangles SAC and SBD to compute SA and SB , then apply the Law of Cosines twice, first to compute the cosine of $\angle A$ and then to compute PQ^2 .

19. Let ABC be a triangle with $AB = 2, CA = 3, BC = 4$. Let D be the point diametrically opposite A on the circumcircle of ABC , and let E lie on line AD such that D is the midpoint of \overline{AE} . Line l passes through E perpendicular to \overline{AE} , and F and G are the intersections of the extensions of \overline{AB} and \overline{AC} with l . Compute FG .

Answer: $\frac{1024}{45}$

Solution: Using Heron's formula we arrive at $[ABC] = \frac{3\sqrt{15}}{4}$. Now invoking the relation $[ABC] = \frac{abc}{4R}$ where R is the circumradius of ABC , we compute $R^2 = \left(\frac{2 \cdot 3}{[ABC]^2}\right) = \frac{64}{15}$. Now observe that $\angle ABD$ is right, so that $BDEF$ is a cyclic quadrilateral. Hence $AB \cdot AF = AD \cdot AE = 2R \cdot 4R = \frac{512}{15}$. Similarly, $AC \cdot AG = \frac{512}{15}$. It follows that $BCGF$ is a cyclic quadrilateral, so that triangles ABC and AGF are similar. Then $FG = BC \cdot \frac{AF}{AC} = 4 \cdot \frac{512}{2 \cdot 15 \cdot 3} = \frac{1024}{45}$

20. Compute the number of real solutions (x, y, z, w) to the system of equations:

$$\begin{aligned} x &= z + w + zwx & z &= x + y + xyz \\ y &= w + x + wxy & w &= y + z + yzw \end{aligned}$$

Answer: 5

Solution: The first equation rewrites as $x = \frac{w+z}{1-wz}$, which is a fairly strong reason to consider trigonometric substitution. Let $x = \tan(a), y = \tan(b), z = \tan(c)$, and $w = \tan(d)$, where $-90^\circ < a, b, c, d < 90^\circ$. Under modulo 180° , we find $a \equiv c + d; b \equiv d + a; c \equiv a + b; d \equiv b + c$. Adding all of these together yields $a + b + c + d \equiv 0$. Then $a \equiv c + d \equiv -a - b$ so $b \equiv -2a$. Similarly, $c \equiv -2b; d \equiv -2c; d \equiv -2a$. Hence, $c \equiv -2b \equiv 4a, d \equiv -2c \equiv -8a$, and $a \equiv -2d \equiv 16a$, so the only possible solutions are $(a, b, c, d) \equiv (t, -2t, 4t, -8t)$ where $15t \equiv 0$. Checking, these, we see that actually $5t \equiv 0$, which yields 5 solutions. Our division by $1 - yz$ is valid since $1 - yz = 0$ iff $yz = 1$, but $x = y + z + xyz$ so $y = -z$, which implies that $yz \leq 0 < 1$, which is impossible. (The solutions we have computed are in fact $(0,0,0,0)$ and the cyclic permutations of $(\tan(36^\circ), \tan(-72^\circ), \tan(-36^\circ), \tan(72^\circ))$.)

21. Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

Answer: 84

Solution: Let $Q(z)$ denote the polynomial divisor. We need that the roots of Q are k -th roots of unity. With this in mind, we might observe that solutions to $z^7 = 1$ and $z \neq 1$ are roots of Q , which leads to its factorization. Alternatively, we note that

$$(z - 1)Q(z) = z^{11} - z^9 + z^7 - z^4 + z^2 - 1 = (z^4 - z^2 + 1)(z^7 - 1)$$

Solving for the roots of the first factor, $z^2 = \frac{1+i\sqrt{3}}{2} = \pm \text{cis}\pi/3$ (we use the notation $\text{cis}(x) = \cos(x) + i\sin(x)$) so that $z = \pm \text{cis}(\pm\pi/6)$. These are primitive 12-th roots of unity. The other roots of $Q(z)$ are the primitive 7-th roots of unity (we introduced $z = 1$ by multiplication.) It follows that the answer is $\text{lcm}[12, 7] = 84$.

22. Let $f(x)$ be a degree 2006 polynomial with complex roots $c_1, c_2, \dots, c_{2006}$, such that the set

$$\{|c_1|, |c_2|, \dots, |c_{2006}|\}$$

consists of exactly 1006 distinct values. What is the minimum number of real roots of $f(x)$?

Answer: 6

Solution: The complex roots of the polynomial must come in pairs, c_i and \bar{c}_i , both of which have the same absolute value. If n is the number of distinct absolute values $|c_i|$ corresponding to those of non-real roots, then there are at least $2n$ non-real roots of $f(x)$. Thus $f(x)$ can have at most $2006 - 2n$ real roots. However, it must have at least $1006 - n$ real roots, as $|c_i|$ takes on $1006 - n$ more values. By definition of n , these all correspond to real roots. Therefore $1006 - n \leq \# \text{ real roots} \leq 2006 - 2n$, so $n \leq 1000$, and $\# \text{ real roots} \geq 1006 - n \geq 6$. It is easy to see that equality is attainable.

23. Let a_0, a_1, a_2, \dots be a sequence of real numbers defined by $a_0 = 21, a_1 = 35$, and $a_{n+2} = 4a_{n+1} - 4a_n + n^2$ for $n \geq 2$. Compute the remainder obtained when a_{2006} is divided by 100.

Answer: 0

Solution: No pattern is evident in the first few terms, so we look for a formula for a_n . If we write $a_n = An^2 + Bn + C + b_n$ and put $b_{n+2} = 4b_{n+1} - 4b_n$. Rewriting the original recurrence, we find

$$\begin{aligned} An^2 + (4A + B)n + (4A + 2B + C) + b_{n+2} \\ = 4(An^2 + (2A + B)n + (A + B + C) + b_{n+1}) - 4(An^2 + Bn + C + b_n) + n^2 \\ = n^2 + 8An + (4A + 4B) + 4b_{n+1} - 4b_n \end{aligned}$$

Solving, $A = 1, B = 4, C = 8$. With this information, we can solve for $b_0 = 1$ and $b_1 = 6$. Since the characteristic equation of the recurrence of the b_i is $x^2 - 4x + 4 = (x - 2)^2 = 0$, we have $b_n = (Dn + E) \cdot 2^n$ for some constants D and E . Using the known values b_0 and b_1 , we compute $D = 2$ and $E = 1$, and finally

$$a_n = n^2 + 4n + 8 + (2n + 1) \cdot 2^n$$

Now, taking modulo 100, we have $a_{2006} \equiv 6^2 + 4 \cdot 6 + 8 + 13 \cdot 2^{2006} \pmod{100}$. Evidently $2^{2006} \equiv 0 \pmod{4}$, but by Euler's theorem $2^{\phi(25)} \equiv 2^{20} \equiv 1 \pmod{25}$, and so $2^{2006} \equiv 2^6 \equiv 14 \pmod{25}$. Now the Chinese remainder theorem yields $2^{2006} \equiv 64 \pmod{100}$, and we compute $a_{2006} \equiv 36 + 24 + 8 + 13 \cdot 64 \equiv 0 \pmod{100}$.

24. Two 18-24-30 triangles in the plane share the same circumcircle as well as the same incircle. What's the area of the region common to both the triangles?

Answer: 132

Solution: Notice, first of all, that 18-24-30 is 6 times 3-4-5, so the triangles are right. Thus, the midpoint of the hypotenuse of each is the center of their common circumcircle, and the inradius is $\frac{1}{2}(18 + 24 - 30) = 6$. Let one of the triangles be ABC ,

where $\angle A < \angle B < \angle C = 90^\circ$. Now the line ℓ joining the midpoints of sides AB and AC is tangent to the incircle, because it is the right distance (12) from line BC . So, the hypotenuse of the other triangle lies along ℓ . We may formulate this thus: *The hypotenuse of each triangle is parallel to the shorter leg, and therefore perpendicular to the longer leg, of the other.* Now it is not hard to see, as a result of these parallel- and perpendicularisms, that the other triangle “cuts off” at each vertex of $\triangle ABC$ a smaller, similar right triangle. If we compute the dimensions of these smaller triangles, we find that they are as follows: 9-12-15 at A , 6-8-10 at B , and 3-4-5 at C . The total area chopped off of $\triangle ABC$ is thus

$$\frac{9 \cdot 12}{2} + \frac{6 \cdot 8}{2} + \frac{3 \cdot 4}{2} = 54 + 24 + 6 = 84.$$

The area of $\triangle ABC$ is $18 \cdot 24/2 = 216$. The area of the region common to both the original triangles is thus $216 - 84 = 132$.

25. Points A , C , and B lie on a line in that order such that $AC = 4$ and $BC = 2$. Circles ω_1 , ω_2 , and ω_3 have \overline{BC} , \overline{AC} , and \overline{AB} as diameters. Circle Γ is externally tangent to ω_1 and ω_2 at D and E respectively, and is internally tangent to ω_3 . Compute the circumradius of triangle CDE .

Answer: $\frac{2}{3}$

Solution: Let the center of ω_i be O_i for $i = 1, 2, 3$ and let O denote the center of Γ . Then O, D , and O_1 are collinear, as are O, E , and O_2 . Denote by F the point of tangency between Γ and ω_3 ; then F, O , and O_3 are collinear. Writing r for the radius of Γ we have $OO_1 = r + 2$, $OO_2 = r + 1$, $OO_3 = 3 - r$. Now since $O_1O_3 = 1$ and $O_3O_2 = 2$, we apply Stewart’s theorem:

$$\begin{aligned} OO_1^2 \cdot O_2O_3 + OO_2^2 \cdot O_1O_3 &= OO_3^2 \cdot O_1O_2 + O_1O_3 \cdot O_3O_2 \cdot O_1O_2 \\ 2(r+2)^2 + (r+1)^2 &= 3(3-r)^2 + 1 \cdot 2 \cdot 3 \end{aligned}$$

We find $r = \frac{6}{7}$. Now the key observation is that the circumcircle of triangle CDE is the incircle of triangle OO_1O_2 . We easily compute the sides of OO_1O_2 to be $\frac{13}{7}$, $\frac{20}{7}$, and 3. By Heron’s formula, the area of OO_1O_2 is $\frac{18}{7}$, but the semiperimeter is $\frac{27}{7}$, so the desired radius is $\frac{2}{3}$.

26. Let $a \geq b \geq c$ be real numbers such that

$$\begin{aligned} a^2bc + ab^2c + abc^2 + 8 &= a + b + c \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 &= 2 + ab + bc + ca \end{aligned}$$

If $a + b + c > 0$, then compute the integer nearest to a^5 .

Answer: 1279

Solution: We factor the first and third givens, obtaining the system

$$\begin{aligned} a^2bc + ab^2c + abc^2 - a - b - c &= (abc - 1)(a + b + c) = -8 \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= (ab + bc + ca)(a + b + c) = -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 - ab - bc - ca &= (abc - 1)(ab + bc + ca) = 2 \end{aligned}$$

Writing $X = a+b+c$, $Y = ab+bc+ca$, $Z = abc-1$, we have $XZ = -8$, $XY = -4$, $YZ = 2$. Multiplying the three yields $(XYZ)^2 = 64$ from which $XYZ = \pm 8$. Since we are given $X > 0$, multiplying the last equation by X we have $2X = XYZ = \pm 8$. Evidently $XYZ = 8$ from which $X = 4$, $Y = -1$, $Z = -2$. We conclude that a, b, c are the roots of the polynomial $P(t) = t^3 - 4t^2 - t + 1$. Thus, $P(a) = a^3 - 4a^2 - a + 1 = 0$, and also $P(b) = P(c) = 0$. Now since $P(1/2) = -\frac{3}{8}$, $P(0) = 1$ and $P(-2/3) = -\frac{11}{27}$, we conclude that $-2/3 < c < 0 < b < 1/2 < a$. It follows that $|b^5 + c^5| < \frac{1}{2}$. Thus, we compute $a^5 + b^5 + c^5$.

Defining $S_n = a^n + b^n + c^n$, we have $S_{n+3} = 4S_{n+2} + S_{n+1} - S_n$ for $n \geq 0$. Evidently $S_0 = 3$, $S_1 = 4$, $S_2 = (a+b+c)^2 - 2(ab+bc+ca) = 18$. Then $S_3 = 4 \cdot 18 + 4 - 3 = 73$, $S_4 = 4 \cdot 73 + 18 - 4 = 306$, and $S_5 = 4 \cdot 306 + 73 - 18 = 1279$. Since $|b^5 + c^5| < \frac{1}{2}$, we conclude that $|S_5 - a^5| < \frac{1}{2}$, or that 1279 is the integer nearest to a^5 .

27. Let N denote the number of subsets of $\{1, 2, 3, \dots, 100\}$ that contain more prime numbers than multiples of 4. Compute the largest integer k such that 2^k divides N .

Answer: 52

Solution: Let S denote a subset with the said property. Note that there are 25 multiples of 4 and 25 primes in the set $\{1, 2, 3, \dots, 100\}$, with no overlap between the two. Let T denote the subset of 50 numbers that are neither prime nor a multiple of 4, and let U denote the 50 other numbers. Elements of T can be arbitrarily included in or excluded by S . Consider $S \cap U = S_1$ and $U - S = S_2$ (the set difference is defined to be all elements of U that are not in S .) S_1 and S_2 are two disjoint sets such that $U = S_1 \cup S_2$. If S_1 contains more multiples of 4 than primes, then S_2 contains more primes than multiples of 4, and conversely. Furthermore, S_1 contains an equal number of primes and multiples of 4 if and only if S_2 contains equal numbers as well. Let V denote an arbitrary subset of T . It follows from examining pairs of sets $V \cup S_1$ and $V \cup S_2$ that

$$\begin{aligned} N &= 2^{50} \cdot \frac{1}{2} \left(2^{50} - \sum_{k=0}^{25} \binom{25}{k}^2 \right) \\ &= 2^{49} \cdot \left(2^{50} - \binom{50}{25} \right) \end{aligned}$$

Since $50!$ is divisible by 2 exactly $25 + 12 + 6 + 3 + 1 = 47$ times while $25!$ is divisible by 2 exactly $12 + 6 + 3 + 1 = 22$ times, it follows that $\binom{50}{25}$ is divisible by 2 exactly 3 times, so the answer is $49 + 3 = 52$.

28. A pebble is shaped as the intersection of a cube of side length 1 with the solid sphere tangent to all of the cube's edges. What is the surface area of this pebble?

Answer: $\frac{6\sqrt{2} - 5}{2}\pi$

Solution: Imagine drawing the sphere and the cube. Take a cross section, with a plane parallel to two of the cube's faces, passing through the sphere's center. In this cross section, the sphere looks like a circle, and the cube looks like a square (of side length 1) inscribed in that circle. We can now calculate that the sphere has diameter

$d := \sqrt{2}$ and surface area $S := \pi d^2 = 2\pi$, and that the sphere protrudes a distance of $x := \frac{\sqrt{2}-1}{2}$ out from any given face of the cube.

It is known that the surface area chopped off from a sphere by any plane is proportional to the perpendicular distance thus chopped off. Thus, each face of the cube chops off a fraction $\frac{x}{d}$ of the sphere's surface. The surface area of the pebble contributed by the sphere is thus $S \cdot (1 - 6 \cdot \frac{x}{d})$, whereas the cube contributes 6 circles of radius $\frac{1}{2}$, with total area $6 \cdot \pi (\frac{1}{2})^2 = \frac{3}{2}\pi$. The pebble's surface area is therefore

$$S \cdot \left(1 - 6 \cdot \frac{x}{d}\right) + \frac{3}{2}\pi = 2\pi \cdot \left(1 - 6 \cdot \frac{\sqrt{2}-1}{2\sqrt{2}}\right) + \frac{3}{2}\pi = \frac{6\sqrt{2}-5}{2}\pi.$$

29. Find the area in the first quadrant bounded by the hyperbola $x^2 - y^2 = 1$, the x -axis, and the line $3x = 4y$.

Answer: $\frac{\ln 7}{4}$

Solution: Convert to polar coordinates: the hyperbola becomes

$$1 = r^2(\cos^2 \theta - \sin^2 \theta) = r^2 \cos(2\theta),$$

so, letting $\alpha := \arctan(3/4)$, the area is

$$S := \int_0^\alpha \frac{r^2}{2} d\theta = \frac{1}{2} \int_0^\alpha \sec(2\theta) d\theta = \frac{1}{4} \ln |\sec(2\theta) + \tan(2\theta)| \Big|_0^\alpha.$$

Now

$$\begin{aligned} \tan(2\alpha) &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} = \frac{3/2}{7/16} = \frac{24}{7}, \\ \sec(2\alpha) &= \sqrt{1 + \tan^2(2\alpha)} = \frac{25}{7}, \end{aligned}$$

so

$$S = \frac{1}{4} \left(\ln \left| \frac{25}{7} + \frac{24}{7} \right| - \ln |1 + 0| \right) = \frac{\ln 7}{4}.$$

30. ABC is an acute triangle with incircle ω . ω is tangent to sides \overline{BC} , \overline{CA} , and \overline{AB} at D , E , and F respectively. P is a point on the altitude from A such that Γ , the circle with diameter \overline{AP} , is tangent to ω . Γ intersects \overline{AC} and \overline{AB} at X and Y respectively. Given $XY = 8$, $AE = 15$, and that the radius of Γ is 5, compute $BD \cdot DC$.

Answer: $\frac{675}{4}$

Solution: By the Law of Sines we have $\sin \angle A = \frac{XY}{AP} = \frac{4}{5}$. Let I , T , and Q denote the center of ω , the point of tangency between ω and Γ , and the center of Γ respectively. Since we are told ABC is acute, we can compute $\tan \angle \frac{A}{2} = \frac{1}{2}$. Since $\angle EAI = \frac{A}{2}$ and \overline{AE} is tangent to ω , we find $r = \frac{AE}{2} = \frac{15}{2}$. Let H be the foot of the altitude from A to \overline{BC} . Define h_T to be the homothety about T which sends Γ to ω . We have $h_T(\overline{AQ}) = \overline{DI}$, and conclude that A, T , and D are collinear. Now since \overline{AP} is a diameter of Γ , $\angle PAT$ is right, implying that $DTHP$ is cyclic. Invoking Power of

a Point twice, we have $225 = AE^2 = AT \cdot AD = AP \cdot AH$. Because we are given radius of Γ we can find $AP = 10$ and $AH = \frac{45}{2} = h_a$. If we write a, b, c, s in the usual manner with respect to triangle ABC , we seek $BD \cdot DC = (s - b)(s - c)$. But recall that Heron's formula gives us

$$\sqrt{s(s - a)(s - b)(s - c)} = K$$

where K is the area of triangle ABC . Writing $K = rs$, we have $(s - b)(s - c) = \frac{r^2 s}{s - a}$. Knowing $r = \frac{15}{2}$, we need only compute the ratio $\frac{s}{a}$. By writing $K = \frac{1}{2}ah_a = rs$, we find $\frac{s}{a} = \frac{h_a}{2r} = \frac{3}{2}$. Now we compute our answer, $\frac{r^2 s}{s - a} = \left(\frac{15}{2}\right)^2 \cdot \frac{\frac{3}{2}}{\frac{3}{2} - 1} = \frac{675}{4}$.

31. Let A be as in problem 33. Let W be the sum of all positive integers that divide A . Find W .

Answer: 8

Solution: Problems 31–33 go together. See below.

32. In the alphametic $WE \times EYE = SCENE$, each different letter stands for a different digit, and no word begins with a 0. The W in this problem has the same value as the W in problem 31. Find S .

Answer: 5

Solution: Problems 31–33 go together. See below.

33. Let W, S be as in problem 32. Let A be the least positive integer such that an acute triangle with side lengths S, A , and W exists. Find A .

Answer: 7

Solution: There are two solutions to the alphametic in problem 32: $36 \times 686 = 24696$ and $86 \times 636 = 54696$. So (W, S) may be $(3, 2)$ or $(8, 5)$. If $(W, S) = (3, 2)$, then by problem (3) $A = 3$, but then by problem 31 $W = 4$, a contradiction. So, (W, S) must be $(8, 5)$. By problem 33, $A = 7$, and this indeed checks in problem 31.

34. In bridge, a standard 52-card deck is dealt in the usual way to 4 players. By convention, each hand is assigned a number of “points” based on the formula

$$4 \times (\# \text{ A's}) + 3 \times (\# \text{ K's}) + 2 \times (\# \text{ Q's}) + 1 \times (\# \text{ J's}).$$

Given that a particular hand has exactly 4 cards that are A, K, Q, or J, find the probability that its point value is 13 or higher.

Answer: $\frac{197}{1820}$

Solution: Obviously, we can ignore the cards lower than J. Simply enumerate the ways to get at least 13 points: AAAA (1), AAAK (16), AAAQ (16), AA AJ (16), AAKK (36), AAKQ (96), AKKK (16). The numbers in parentheses represent the number of ways to choose the suits, given the choices for the values. We see that there are a total of $1 + 16 + 16 + 16 + 36 + 96 + 16 = 197$ ways to get at least 13. There are a total of $\binom{16}{4} = 1820$ possible ways to choose 4 cards from the 16 total A's, K's, Q's, and J's. Hence the answer is $197/1820$.

35. A sequence is defined by $A_0 = 0, A_1 = 1, A_2 = 2$, and, for integers $n \geq 3$,

$$A_n = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

Compute $\lim_{N \rightarrow \infty} A_N$.

Answer: $\frac{13}{6} - \frac{\pi^2}{12}$.

Solution: If we sum the given equation for $n = 3, 4, 5, \dots, N$, we obtain

$$\sum_{n=3}^N A_n = \sum_{n=3}^N \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

This reduces dramatically to

$$A_N + \frac{2A_{N-1}}{3} + \frac{A_{N-2}}{3} = A_2 + \frac{2A_1}{3} + \frac{A_0}{3} + \sum_{n=3}^N \frac{1}{n^4 - n^2} \quad (*)$$

Let $\lim_{N \rightarrow \infty} A_N = L$. Under this limit, the left hand side of (*) is simply $2L$. We compute the sum on the right with the help of partial fractions

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=3}^N \frac{1}{n^4 - n^2} &= \sum_{n=3}^{\infty} \frac{1}{n^2 - 1} - \frac{1}{n^2} \\ &= \left(\sum_{n=3}^{\infty} \frac{1}{2} \left(\frac{1}{n-1} - \frac{1}{n+1} \right) \right) + \frac{1}{1^2} + \frac{1}{2^2} - \sum_{n=1}^{\infty} \frac{1}{n^2} \\ &= \frac{1}{2} \left(\frac{1}{2} + \frac{1}{3} \right) + \frac{5}{4} - \frac{\pi^2}{6} \\ &= \frac{5}{3} - \frac{\pi^2}{6} \end{aligned}$$

With this we easily find $L = \frac{1}{2} \cdot \left(2 + \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 0 + \frac{5}{3} - \frac{\pi^2}{6} \right) = \frac{13}{6} - \frac{\pi^2}{12}$, and we are done.

36. Four points are independently chosen uniformly at random from the interior of a regular dodecahedron. What is the probability that they form a tetrahedron whose interior contains the dodecahedron's center?

Answer: $\frac{1}{8}$

Solution: Situate the origin O at the dodecahedron's center, and call the four random points P_i , where $1 \leq i \leq 4$.

To any tetrahedron $P_1P_2P_3P_4$ we can associate a quadruple $(\epsilon_{(ijk)})$, where (ijk) ranges over all conjugates of the cycle (123) in the alternating group A_4 : ϵ_{ijk} is the sign of the directed volume $[OP_iP_jP_k]$. Assume that, for a given tetrahedron $P_1P_2P_3P_4$, all members of its quadruple are nonzero (this happens with probability 1). For $1 \leq i \leq 4$, if we replace P_i with its reflection through the origin, the three members of the tetrahedron's quadruple that involve P_i all flip sign, because each $[OP_iP_jP_k]$ is a linear

function of the vector $\overrightarrow{OP_i}$. Thus, if we consider the 16 sister tetrahedra obtained by choosing independently whether to flip each P_i through the origin, the quadruples range through all 16 possibilities (namely, all the quadruples consisting of ± 1 s). Two of these 16 tetrahedra, namely those with quadruples $(1, 1, 1, 1)$ and $(-1, -1, -1, -1)$, will contain the origin.

So the answer is $2/16 = 1/8$.

37. Compute

$$\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3 + 7n^2 + 14n + 8)}$$

Answer: $\frac{137}{24} - 8 \ln 2$

Solution: First, we manipulate using partial fractions and telescoping:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3 + 7n^2 + 14n + 8)} &= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{2}{n+1} - \frac{1}{n+2} - \frac{1}{n+4} \right) \\ &= \frac{1}{4} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n \cdot (n+4)} \end{aligned}$$

Now, consider the function $f(r, k) := \sum_{n=1}^{\infty} \frac{r^n}{n^k}$. We have

$$\begin{aligned} \frac{\partial f(r, k)}{\partial r} &= \frac{\partial}{\partial r} \sum_{n=1}^{\infty} \frac{r^n}{n^k} = \sum_{n=1}^{\infty} \frac{\partial}{\partial r} \left[\frac{r^n}{n^k} \right] = \sum_{n=1}^{\infty} \frac{r^{n-1}}{n^{k-1}} = \frac{1}{r} f(r, k-1) \\ \frac{df(r, 1)}{dr} &= \frac{1}{r} \sum_{n=1}^{\infty} \frac{r^n}{n^0} = \frac{1}{r} \cdot \frac{r}{1-r} = \frac{1}{1-r} \\ f(r, 1) &= \int \frac{dr}{1-r} = -\ln(1-r) + f(0, 1) \end{aligned}$$

By inspection, $f(0, 1) = 0$, so $f\left(\frac{1}{2}, 1\right) = \sum_{n=1}^{\infty} \frac{1}{n \cdot 2^n} = \ln(2)$. It is easy to compute the desired sum in terms of $f\left(\frac{1}{2}, 1\right)$, and we find $\sum_{n=1}^{\infty} \frac{1}{2^n(n+4)} = 16 \ln(2) - \frac{131}{12}$. Hence, our final answer is $\frac{137}{24} - 8 \ln(2)$.

38. Suppose ABC is a triangle with incircle ω , and ω is tangent to \overline{BC} and \overline{CA} at D and E respectively. The bisectors of $\angle A$ and $\angle B$ intersect line DE at F and G respectively, such that $BF = 1$ and $FG = GA = 6$. Compute the radius of ω .

Answer: $\frac{2\sqrt{5}}{5}$

Solution: Let α, β, γ denote the measures of $\frac{1}{2}\angle A, \frac{1}{2}\angle B, \frac{1}{2}\angle C$, respectively. We have $m\angle CEF = 90^\circ - \gamma, m\angle FEA = 90^\circ + \gamma, m\angle AFG = m\angle AFE = 180^\circ - \alpha - (90^\circ + \gamma) = \beta = m\angle ABG$, so $ABFG$ is cyclic. Now $AG = GF$ implies that \overline{BG} bisects $\angle ABF$. Since \overline{BG} by definition bisects $\angle ABC$, we see that F must lie on \overline{BC} . Hence, $F = D$. If I denotes the incenter of triangle ABC , then \overline{ID} is perpendicular to \overline{BC} , but since A, I, F are collinear, we have that $\overline{AD} \perp \overline{BC}$. Hence, ABC is isosceles with $AB = AC$. Furthermore, $BC = 2BF = 2$. Moreover, since $ABFG$ is cyclic, $\angle BGA$ is a right angle. Construct F' on minor arc GF such that $BF' = 6$ and $F'G = 1$, and let $AB = x$. By

the Pythagorean theorem, $AF' = BG = \sqrt{x^2 - 36}$, so that Ptolemy applied to $ABF'G$ yields $x^2 - 36 = x + 36$. We have $(x - 9)(x + 8) = 0$. Since x is a length we find $x = 9$. Now we have $AB = AC = 9$. Pythagoras applied to triangle ABD now yields $AD = \sqrt{9^2 - 1^2} = 4\sqrt{5}$, which enables us to compute $[ABC] = \frac{1}{2} \cdot 2 \cdot 4\sqrt{5} = 4\sqrt{5}$. Since the area of a triangle is also equal to its semiperimeter times its inradius, we have $4\sqrt{5} = 10r$ or $r = \frac{2\sqrt{5}}{5}$.

REMARK. In fact, $ABFG$ is always a cyclic quadrilateral for which \overline{AB} plays a diameter. That is, we could have proven this fact without using $FG = GA$.

39. A *fat coin* is one which, when tossed, has a $2/5$ probability of being heads, $2/5$ of being tails, and $1/5$ of landing on its edge. Mr. Fat starts at 0 on the real line. Every minute, he tosses a fat coin. If it's heads, he moves left, decreasing his coordinate by 1; if it's tails, he moves right, increasing his coordinate by 1. If the coin lands on its edge, he moves back to 0. If Mr. Fat does this *ad infinitum*, what fraction of his time will he spend at 0?

Answer: $\frac{1}{3}$

Solution: For $n \in \mathbb{Z}$, let a_n be the fraction of the time Mr. Fat spends at n . By symmetry, $a_n = a_{-n}$ for all n .

For $n > 0$, we have $a_n = \frac{2}{5}a_{n-1} + \frac{2}{5}a_{n+1}$, or $a_{n+1} = \frac{5}{2}a_n - a_{n-1}$. This Fibonacci-like recurrence can be solved explicitly to obtain

$$a_n = \alpha \cdot 2^{|n|} + \beta \cdot 2^{-|n|}$$

for all $n \in \mathbb{Z}$. Now we also have

$$\sum_{n \in \mathbb{Z}} a_n = 1,$$

so we better have $\alpha = 0$, so that $a_0 = \beta$ and $a_{\pm 1} = \frac{\beta}{2}$. Now we also have $a_0 = \frac{2}{5}a_{-1} + \frac{2}{5}a_1 + \frac{1}{5}$, so $\beta = \frac{1}{3}$. This matches perfectly with $\sum_{n \in \mathbb{Z}} a_n = 1$.

40. Compute

$$\sum_{k=1}^{\infty} \frac{3k+1}{2k^3+k^2} \cdot (-1)^{k+1}.$$

Answer: $\frac{\pi^2}{12} + \frac{\pi}{2} - 2 + \ln 2$

Solution: Via partial fraction decomposition we write the sum as

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{2}{1+2k} + \frac{1}{k^2} \right) (-1)^{k+1}$$

Now recall that

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{k^2} &= \frac{\pi^2}{6} = S_1 \\ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} &= \ln(2) = S_2 \\ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} &= \frac{\pi}{4} = S_3 \end{aligned}$$

Manipulating (1), we deduce

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} &= \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) - 2 \cdot \sum_{k=1}^{\infty} \frac{1}{(2k)^2} \\ &= \frac{\pi^2}{6} - 2/4 \cdot \frac{\pi^2}{6} = \frac{\pi^2}{12} = S_4 \end{aligned}$$

It is then easily seen that the answer is equal to $S_2 + 2 \cdot S_3 - 2 + S_4$.

41. Let Γ denote the circumcircle of triangle ABC . Point D is on \overline{AB} such that \overline{CD} bisects $\angle ACB$. Points P and Q are on Γ such that \overline{PQ} passes through D and is perpendicular to \overline{CD} . Compute PQ , given that $BC = 20, CA = 80, AB = 65$.

Answer: $4\sqrt{745}$

Solution: Suppose that P lies between A and B and Q lies between A and C , and let line PQ intersect lines AC and BC at E and F respectively. As usual, we write a, b, c for the lengths of BC, CA, AB . By the angle bisector theorem, $AD/DB = AC/CB$ so that $AD = \frac{bc}{a+b}$ and $BD = \frac{ac}{a+b}$. Now by Stewart's theorem, $c \cdot CD^2 + \left(\frac{ac}{a+b}\right) \left(\frac{bc}{a+b}\right) c = \frac{a^2bc}{a+b} + \frac{ab^2c}{a+b}$ from which $CD^2 = \frac{ab((a+b)^2 - c^2)}{(a+b)^2}$. Now observe that triangles CDE and CDF are congruent, so $ED = DF$. By Menelaus' theorem, $\frac{CA}{AE} \frac{ED}{DF} \frac{FB}{BC} = 1$ so that $\frac{CA}{BC} = \frac{AE}{FB}$. Since $CF = CE$ while $b > a$, it follows that $AE = \frac{b(b-a)}{a+b}$ so that $EC = \frac{2ab}{a+b}$. Finally, $DE = \sqrt{CE^2 - CD^2} = \frac{\sqrt{ab(c^2 - (a-b)^2)}}{a+b}$. Plugging in $a = 20, b = 80, c = 65$, we see that $AE = 48, EC = 32, DE = 10$ as well as $AD = 52, BD = 13$. Now let $PD = x, QE = y$. By power of a point about D and E , we have $x(y+10) = 676$ and $y(x+10) = 1536$. Subtracting one from the other, we see that $y = x + 86$. Therefore, $x^2 + 96x - 676 = 0$, from which $x = -48 + 2\sqrt{745}$. Finally, $PQ = x + y + 10 = 4\sqrt{745}$.

42. Suppose hypothetically that a certain, very corrupt political entity in a universe holds an election with two candidates, say A and B . A total of 5,825,043 votes are cast, but, in a sudden rainstorm, all the ballots get soaked. Undaunted, the election officials decide to guess what the ballots say. Each ballot has a 51% chance of being deemed a vote for A , and a 49% chance of being deemed a vote for B . The probability that B will win is 10^{-X} . What is X rounded to the nearest 10?

Answer: 510

Solution: Let $N = 2912521$, so that the number of ballots cast is $2N + 1$. Let P be the probability that B wins, and let $\alpha = 51\%$ and $\beta = 49\%$ and $\gamma = \beta/\alpha < 1$. We have

$$10^{-X} = P = \sum_{i=0}^N \binom{2N+1}{N-i} \alpha^{N-i} \beta^{N+1+i} = \alpha^N \beta^{N+1} \sum_{i=0}^N \binom{2N+1}{N-i} \gamma^i$$

(think of $2i + 1$ as representing B 's margin of victory). Now

$$\frac{2^{2N+1}}{2N+1} < \binom{2N+1}{N} < \sum_{i=0}^N \binom{2N+1}{N-i} \gamma^i < 2^{2N+1},$$

So

$$-X = \log P = N \log \alpha + (N+1) \log \beta + (2N+1) \log 2 - \epsilon = N \log(2\alpha) + (N+1) \log(2\beta) - \epsilon,$$

where $0 < \epsilon < \log(2N+1) < 7$. With a calculator, we find that

$$-X \approx 25048.2 - 25554.2 - \epsilon = -506.0 - \epsilon,$$

so $X \approx 510$.

43. Write down at least one, and up to ten, different 3-digit prime numbers. If you somehow fail to do this, we will ignore your submission for this problem. Otherwise, you're entered into a game with other teams. In this game, you start with 10 points, and each number you write down is like a bet: if no one else writes that number, you gain 1 point, but if anyone else writes that number, you lose 1 point. Thus, your score on this problem can be anything from 0 to 20.

Solution: There are 143 three-digit primes. None of the following necessarily applies to the actual contest, but it might be useful to think about. Suppose that you're trying to maximize your expected score on this problem. Then you should write down a number if you think the probability that someone else is writing it is less than $1/2$ (of course, limit yourself to 10 numbers). You should avoid writing down any number if you think the probability that someone else is writing it is more than $1/2$ (of course, write down at least 1 number). Suppose that you expect a total of M different numbers are going to be written down, but have no idea what numbers they might be. If you think $M \geq 72$, you should write down 10 numbers at random; if $M \leq 71$, you should write just 1 number.

44. On the Euclidean plane are given 14 points:

$$\begin{array}{llll} A = (0, 428) & B = (9, 85) & C = (42, 865) & D = (192, 875) \\ E = (193, 219) & F = (204, 108) & G = (292, 219) & H = (316, 378) \\ I = (375, 688) & J = (597, 498) & K = (679, 766) & L = (739, 641) \\ & M = (772, 307) & N = (793, 0) & \end{array}$$

A fly starts at A , visits all the other points, and comes back to A in such a way as to minimize the total distance covered. What path did the fly take? Give the names of the points it visits in order. Your score will be

$$20 + \lfloor \text{the optimal distance} \rfloor - \lfloor \text{your distance} \rfloor$$

or 0, whichever is greater.

Answer: The optimal path is $ACDIKLJMNHGEFB(A)$, or the reverse, of course. In this way the total distance covered by the fly is just over 3591.22.

Solution: This problem is an instance of the Traveling Salesman Problem, which is NP-hard. There is an obvious algorithm in $O(n!)$ time (where n is the number of points), but faster algorithms exist. Nonetheless, the best strategy for solving this problem is probably to draw the points and exercise your geometric intuition.

45. On your answer sheet, *clearly* mark at least seven points, as long as
- (i) No three are collinear.
 - (ii) No seven form a convex heptagon.

Please do not cross out any points; erase if you can do so neatly. If the graders deem that your paper is too messy, or if they determine that you violated one of those conditions, your submission for this problem will be disqualified. Otherwise, your score will be the number of points you marked minus 6, even if you actually violated one of the conditions but were able to fool the graders.

Solution: This is the heptagon case of what is known as the “Happy Ending” or “Erdős-Szekeres” problem, which in general asks, *For any integer $n \geq 3$, what is the smallest $N(n)$, such that any $N(n)$ points in the plane in general position determine a convex n -gon?* It is known that such an $N(n)$ always exists and is finite (in fact a specific upper bound has been found). The best known lower bound is $N(n) \geq 2^{n-2} + 1$; Erdős and Szekeres conjectured that this bound is tight. The $n \leq 5$ cases have been known for some time. According to the Wikipedia, the $n = 6$ case is solved but unpublished, and for $n \geq 7$, the problem remains open.

For a discussion, see

W. Morris and V. Soltan. The Erdős-Szekeres Problem on Points in Convex Position—A Survey, *Bulletin of the American Math Monthly*. **37** (2000), 437–458.

This article is available at

<http://www.ams.org/bull/2000-37-04/S0273-0979-00-00877-6/home.html>.

If $N(7) = 33$, the highest sure score on this problem would be $32 - 6 = 26$. It is not known whether there exist arbitrarily large sets of points that will fool the graders.

The unexamined life is not worth living.

IXth Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Team Round A: Solutions

Mobotics [120]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged at the points of a certain grid (to be specified later). Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction. At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot start on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

1. [15] For this problem, our lawn consists of a row of n clumps of grass. This row runs in an east-west direction. In our formation, each mobot may be oriented toward the north, south, east, or west. One example of an allowable formation if $n = 6$ is symbolized below:

. . ← ↑ ↑ ↓

(The mobot on the third clump will move westward, mowing the first three clumps. Each of the last three clumps is mowed by a different mobot.) Here's another allowable formation for $n = 6$, considered different from the first:

. . ← ↑ ↑ →

Compute the number of different allowable formations for any given n .

Solution: Let a be the number of clumps mowed by a mobot oriented to go west, and let b be the number of clumps mowed by a mobot oriented to go east. (So in the two examples given, (a, b) would be $(3, 0)$ and $(3, 1)$, respectively.) We may ask how many allowable formations with a given ordered pair (a, b) there are, and then sum our answers up over all possible ordered pairs (a, b) .

Given any particular (a, b) , first of all, a and b had better be non-negative integers summing to at most n . As long as that's true, there will be $n - a - b$ clumps of grass which are each mowed by a single mobot oriented to go either north or south: there

are thus 2^{n-a-b} possibilities. So our answer is

$$\begin{aligned} \sum_{a=0}^n \sum_{b=0}^{n-a} 2^{n-a-b} &= \sum_{a=0}^n (2^{n-a} + 2^{n-a-1} + \dots + 2^0) = \sum_{a=0}^n (2^{n-a+1} - 1) \\ &= -(n+1) + \sum_{a=0}^n 2^{n-a+1} = -(n+1) + (2^{n+1} + 2^n + 2^{n-1} + \dots + 2^1) \\ &= -(n+1) + (2^{n+2} - 2) = 2^{n+2} - n - 3. \end{aligned}$$

2. [25] For this problem, our lawn is an $m \times n$ rectangular grid of clumps, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

However, mobots are now allowed to be oriented to go either north or east only. So one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

Prove that the number of allowable formations for given m and n is $\frac{(m+n)!}{m!n!}$.

Solution: There is a one-to-one correspondence between allowable formations and paths from $(0, 0)$ to (n, m) made up of n moves 1 unit to the right and m moves 1 unit up. The correspondence works as follows: There must be a mobot at $(0, 0)$, so start the path there. If that mobot is oriented to move up, then move to the right; if that mobot is oriented to move to the right, then move up. In doing so, you will meet another mobot, upon which you can repeat the above process, until you leave the lawn. Once you leave the lawn, there will be only one way to proceed to (n, m) — either keep going right, or keep going up. (For the example in the problem, our path would be $(0, 0) \rightarrow (1, 0) \rightarrow (1, 1) \rightarrow (1, 2) \rightarrow (2, 2) \rightarrow (3, 2)$.)

Conversely, given any path from $(0, 0)$ to (n, m) , we can derive back a mobot formation: place a mobot at every lattice point *of the lawn* that the path touches, and don't orient that mobot in the same direction as the path takes when leaving that point.

It is easy to check that this is indeed a one-to-one correspondence as claimed. Every path from $(0, 0)$ to (n, m) consists of n moves to the right and m moves up done in an arbitrary order, and there are precisely $\binom{m+n}{m} = \frac{(m+n)!}{m!n!}$ orders.

3. [40] In this problem, we stipulate that $m \geq n$, and the lawn is shaped differently. The clumps are now at the lattice points in a trapezoid:

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m + 1 - n + x\},$$

As in problem 2, mobots can be set to move either north or east. For given m and n , determine with proof the number of allowable formations.

Solution: For exactly the same reasons as in problem 2, we have a one-to-one correspondence between the allowable formations and paths (going 1 unit up or right at a time) from $(0, 0)$ to (n, m) avoiding points (x, y) with $y > m + 1 - n + x$.

The number of these paths equals the total number of up/right paths from $(0, 0)$ to (n, m) minus the number of up/right paths from $(0, 0)$ to (n, m) that *do* pass through at least one point (x, y) with $y > m + 1 - n + x$. The first of these numbers is $\binom{m+n}{m}$, as before. It remains to calculate the second of these numbers.

To that end, we first note that the reflection of (n, m) across the line $y = m + 2 - n + x$ is $(n - 2, m + 2)$. Now there is a one-to-one correspondence between up/right paths from $(0, 0)$ to (m, n) that pass through something with $y \geq m + 2 - n + x$ with up/right paths from $(0, 0)$ to $(n - 2, m + 2)$: indeed, taking a path of one kind, we may isolate the first point on it that lies on the line $y = m + 2 - n + x$, and reflect the rest of the path through that line, to obtain a path of the other kind. There are therefore $\binom{m+n}{m+2}$ paths of either kind.

The answer is therefore

$$\binom{m+n}{m} - \binom{m+n}{m+2}$$

or, if you happen to prefer,

$$\frac{(m+n)! [(m+2)(m+1) - n(n-1)]}{(m+2)! n!}.$$

4. [15] In this problem and the next, the lawn consists of points in a triangular grid of size n , so that for $n = 3$ the lawn looks like

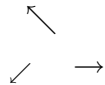


Mobots are allowed to be oriented to the east, 30° west of north, or 30° west of south. Under these conditions, for any given n , what is the minimum number of mobots needed to mow the lawn?

Answer: n

Solution: It is evident that n mobots are enough; just place one at the left edge of each row, and set them to move to the right. Suppose for the sake of contradiction that, for some n , it is possible to mow the lawn with fewer than n mobots. Consider the minimum n for which this is the case. Clearly $n > 1$. Now consider the mobot that mows the southwest corner of the lawn. This mobot is confined to either the bottom row or the left edge of the lawn; let's assume it's the former (the latter case is very similar). This mobot starts in the bottom row. Let's throw it away. Now if any remaining mobots start in the bottom row, they must not be set to move east, and we may advance them by one step in the direction they're set to go in, so that they land either off the lawn or in the second-to-last row. In doing so we have produced a starting formation of fewer than $n - 1$ mobots in the first $n - 1$ rows that will mow those rows. This contradicts the assumed minimality of n .

5. [25] With the same lawn and the same allowable mobot orientations as in the previous problem, let us call a formation “happy” if it is invariant under 120° rotations. (A rotation applies both to the positions of the mobots and to their orientations.) An example of a happy formation for $n = 2$ might be



Find the number of happy formations for a given n .

Solution: If $n \equiv 1 \pmod{3}$, then there is a clump of grass at the center of the lawn; otherwise there are 3 blades of grass equally closest to the center. In the former case, whatever mobot mows this center blade of grass cannot possibly have a counterpart under a 120° rotation: if this mobot starts in the center, it must be oriented a certain way, and whatever way that is, cannot remain invariant under such a rotation; but if the mobot does not start in the center, then it will collide in the center with its counterparts under the 120° rotations.

In the case that $n \not\equiv 1 \pmod{3}$, by considering how the three central blades of grass can be mowed, we easily see that there are two possibilities, both of which involve starting one mobot at each of these three positions. Both possibilities distinctly have the effect of cutting up the rest of the lawn (i.e., the part of the lawn not mowed by any of these three mobots) into three congruent pieces. The point is that these pieces are equivalent to trapezoids as in problem 3, both in their shape, and in the allowed directions of mobot motion. Luckily, we know how to count the ways to configure each such trapezoid. Call the trapezoid defined in problem 3 an “ (m, n) -trapezoid.”

If $n \equiv 0 \pmod{3}$, then these pieces are either all $(\frac{2n}{3}, \frac{n-3}{3})$ -trapezoids or all $(\frac{2n-3}{3}, \frac{n}{3})$ -trapezoids, depending on how the central mobots are configured. In this case, the answer is

$$\left(\binom{n-1}{\frac{2n}{3}} - \binom{n-1}{\frac{2n+6}{3}} \right)^3 + \left(\binom{n-1}{\frac{2n-3}{3}} - \binom{n-1}{\frac{2n+3}{3}} \right)^3.$$

If, on the other hand, $n \equiv 2 \pmod{3}$, then these pieces are either all $(\frac{2n-1}{3}, \frac{n-2}{3})$ -trapezoids or all $(\frac{2n-4}{3}, \frac{n+1}{3})$ -trapezoids, again depending on how the central mobots are configured. In this case, the answer is

$$\left(\binom{n-1}{\frac{2n-1}{3}} - \binom{n-1}{\frac{2n+1}{3}} \right)^3 + \left(\binom{n-1}{\frac{2n-4}{3}} - \binom{n-1}{\frac{2n-2}{3}} \right)^3.$$

Polygons [110]

6. [15] Let n be an integer at least 5. At most how many diagonals of a regular n -gon can be simultaneously drawn so that no two are parallel? Prove your answer.

Answer: n

Solution: Let O be the center of the n -gon. Let us consider two cases, based on the parity of n :

- *n is odd.* In this case, for each diagonal d , there is exactly one vertex D of the n -gon, such that d is perpendicular to line OD ; and of course, for each vertex D , there is at least one diagonal d perpendicular to OD , because $n \geq 5$. The problem of picking a bunch of d 's so that no two are parallel is thus transmuted into one of picking a bunch of d 's so that none of the corresponding D 's are the same. Well, go figure.
- *n is even.* What can I say? For each diagonal d , the perpendicular dropped from O to d either passes through two opposite vertices of the n -gon, or else bisects two opposite sides. Conversely, for each line joining opposite vertices or bisecting opposite sides, there is at least one diagonal perpendicular to it, because $n \geq 6$. By reasoning similar to the odd case, we find the answer to be n .

7. [25] Given a convex n -gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal either in the interior of the n -gon or at a vertex? Prove your answer.

Answer: If $n = 4$, then 2; otherwise, n .

Solution: First of all, assume without loss of generality that the n -gon is regular (this has no effect as far as diagonal intersection is concerned). Also, treat $n = 4$ as a special case; obviously the answer is 2 here.

If n is odd, simply draw n diagonals, connecting each vertex to the ones $(n - 1)/2$ vertices away (in either direction).

If n is even, first draw the $n/2$ diagonals connecting pairs of vertices $n/2$ vertices apart. Then, there are n diagonals connecting pairs of vertices $n/2 - 1$ vertices apart; they come in $n/2$ pairs of parallel diagonals; from each such pair, randomly pick one diagonal and draw it.

To see that these constructions work, note that two diagonals, each connecting pairs of vertices at least $n/2 - 1$ vertices apart, can fail to intersect or share a vertex only if they are parallel.

The previous problem shows that these constructions are optimal.

8. [15] Given a regular n -gon with sides of length 1, what is the smallest radius r such that there is a non-empty intersection of n circles of radius r centered at the vertices of the n -gon? Give r as a formula in terms of n . Be sure to prove your answer.

Answer: $r = \frac{1}{2} \csc \frac{180^\circ}{n}$

Solution: It is easy to see that, with this r , all the circles pass through the center of the n -gon. The following proves that this r is necessary even if the word “circle” is replaced by the word “disk.”

For n even, it is easy to see using symmetry that containing the center point is necessary and sufficient. For n odd, there is more work to do. Again, containing the center point is sufficient. To see its necessity, consider three circles: a circle at a vertex A , and the two circles on the segment BC opposite A . Circles B and C intersect in a region R symmetric about the perpendicular bisector of BC , with the closest point of R to A being on this line. Hence, circle A must intersect R at some point on the perpendicular bisector of BC ; and thus we see the entire perpendicular bisector of BC

inside of the n -gon is contained in the circles. Now, this bisector contains the center of the n -gon, so some circle must contain the center. But by symmetry, if one circle contains the center, all do. Thus, in any case, it is necessary and sufficient for r to be large enough so the center is contained in a circle. Basic trigonometry gives the answer, which equals the distance between a vertex and the center.

9. [40] Let $n \geq 3$ be a positive integer. Prove that given any n angles $0 < \theta_1, \theta_2, \dots, \theta_n < 180^\circ$, such that their sum is $180(n - 2)$ degrees, there exists a convex n -gon having exactly those angles, in that order.

Solution: We induct on n . The statement holds trivially for $n = 3$, as all triangles are convex. Now, suppose that the statement is true for $n - 1$, where $n \geq 4$. Let $\theta_1, \theta_2, \dots, \theta_n$ be n angles less than 180° whose sum equals $180(n - 2)$ degrees. The statement is clearly true if $n = 4$ and $\theta_1 = \theta_3 = 180^\circ - \theta_2 = 180^\circ - \theta_4$ since we can easily form a parallelogram, so assume otherwise.

I claim that there exist two adjacent angles whose sum is greater than 180° . Assume otherwise. Then, we have $\theta_i + \theta_{i+1} \leq 180$ for $i = 1, 2, \dots, n$, where $\theta_{n+1} = \theta_1$. Summing these inequalities over all i yields $2 \cdot 180(n - 2) \leq 180n$, which is equivalent to $n \leq 4$. Of course, we can have $n = 4$ if and only if we have equality in each of the above inequalities, forcing us to have a parallelogram contrary to our assumption.

Hence, we have two adjacent angles with sum greater than 180° . Without loss of generality, let these angles be θ_{n-1} and θ_n , relabeling if necessary. By the inductive hypothesis, we may construct an $(n-1)$ -gon with angles $\theta_1, \theta_2, \dots, \theta_{n-2}, \theta_{n-1} + \theta_n - 180^\circ$, as these angles are each less than 180° and their sum equals $180(n-3)$ degrees. Consider the vertex with angle $\theta_{n-1} + \theta_n - 180^\circ$. Note that we can “clip off” a triangle with angles $\theta_{n-1} + \theta_n - 180^\circ$, $180^\circ - \theta_{n-1}$, and $180^\circ - \theta_n$ at this vertex, yielding an n -gon with the desired angles, completing the inductive step.

10. [15] Suppose we have an n -gon such that each interior angle, measured in degrees, is a positive integer. Suppose further that all angles are less than 180° , and that all angles are different sizes. What is the maximum possible value of n ? Prove your answer.

Answer: 26

Solution: Let's work with the exterior angles (each is 180 minus the interior angle). Then the conditions on the exterior angles are identical to the conditions on the interior angles: each is a positive integer between 1 and 179 inclusive. The sum of the exterior angles is exactly 360. However, the sum of 1 through 27 is $27 \cdot 28/2 = 378$, which is too large. We can get 26 using angles of 1 through 25 (sum 325) and an angle of 35. The previous problem shows that this is actually possible.

What do the following problems have in common? [170]

11. [15] The lottery cards of a certain lottery contain all nine-digit numbers that can be formed with the digits 1, 2 and 3. There is exactly one number on each lottery card. There are only red, yellow and blue lottery cards. Two lottery numbers that differ from each other in all nine digits always appear on cards of different color. Someone

draws a red card and a yellow card. The red card has the number 122 222 222 and the yellow card has the number 222 222 222. The first prize goes to the lottery card with the number 123 123 123. What color(s) can it possibly have? Prove your answer.

Answer: The card with the number 123 123 123 is red.

Solution: First, it can in fact be red, if, say, cards are colored based on the first digit only (1 = red, 2 = yellow, 3 = blue). We now endeavor to show it must be red.

Consider the cards 333 133 133 and 331 331 331: they each differ in all their digits from 122 222 222 and from 222 222 222, so they must both be blue. Now 211 311 311 differs in all its digits from both 122 222 222 and 333 133 133, so it must be yellow. Finally, 123 123 123 differs in all its digits from both 331 331 331 and 211 311 311, so it must be red.

12. [25] A $3 \times 3 \times 3$ cube is built from 27 unit cubes. Suddenly five of those cubes mysteriously teleport away. What is the minimum possible surface area of the remaining solid? Prove your answer.

Answer: 50

Solution: Orient the cube so that its edges are parallel to the x -, y -, and z -axes. A set of three unit cubes whose centers differ only in their x -coordinate will be termed an “ x -row”; there are thus nine x -rows. Define “ y -row” and “ z -row” similarly.

To achieve 50, simply take away one x -row and one y -row (their union consists of precisely five unit cubes).

To show that 50 is the minimum: Note that there cannot be two x -rows that are both completely removed, as that would imply removing six unit cubes. (Similar statements apply for y - and z -rows, of course.) It is also impossible for there to be one x -row, one y -row, and one z -row that are all removed, as that would imply removing seven unit cubes. Every x -, y -, or z -row that is not completely removed contributes at least 2 square units to the surface area. Thus, the total surface area is at least $9 \cdot 2 + 8 \cdot 2 + 8 \cdot 2 = 50$.

13. [40] Having lost a game of checkers and my temper, I dash all the pieces to the ground but one. This last checker, which is perfectly circular in shape, remains completely on the board, and happens to cover equal areas of red and black squares. Prove that the center of this piece must lie on a boundary between two squares (or at a junction of four).

Solution: Suppose, for the sake of contradiction, that the problem is false. Evidently, at least one boundary between adjacent squares must lie within our checker, or else the checker would exist entirely within one square, meaning it would cover only one color. Note also that a checker’s diameter is smaller than the side of any square of the board, so there are at most two such boundaries within our checker (one in each direction). Let ℓ be this, or one of these, boundaries. Draw a diameter d of the checker parallel to ℓ . Presumably, the strip of the checker between ℓ and d is part red, part black. These red and black areas are unequal, however, because the center of the checker does not lie on *any* boundary between squares. But, if we were to swap colors within this strip, *then* the checker *would* have equal red and black areas, because then it would be

colored in a way such that flipping it across d swaps the colors. This shows that, the way it is *currently* colored, the checker does *not* have equal red and black areas. This gives us the desired contradiction.

14. [40] A number n is called *bumped out* if there is exactly one ordered pair of positive integers (x, y) such that

$$\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n.$$

Find all bumped out numbers.

Answer: 2, 6, 8, 10

Solution: Suppose n is bumped out. If (a, b) is one solution for (x, y) to the given equation $\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n$, then (b, a) is another, so the unique solution (a, b) better have the property that $a = b$ and $n = 2a \geq 2$. In particular, n is an even positive integer.

Now, if $n = 2a \geq 12$, then setting $x = a - 1 \geq 5$, $y = a + 1 \geq 7$, we have

$$\left\lfloor \frac{x^2}{y} \right\rfloor + \left\lfloor \frac{y^2}{x} \right\rfloor = \left\lfloor a - 3 + \frac{4}{a+1} \right\rfloor + \left\lfloor a + 3 + \frac{4}{a-1} \right\rfloor = 2a = n,$$

so n cannot be bumped out.

Moreover, $\lfloor 1^2/2 \rfloor + \lfloor 2^2/1 \rfloor = 4$, so 4 is not bumped out. The only possibilities left are 2, 6, 8, and 10.

To check these, note that

$$n = \left\lfloor \frac{x^2}{y} \right\rfloor + \left\lfloor \frac{y^2}{x} \right\rfloor > -2 + \frac{x^2}{2y} + \frac{x^2}{2y} + \frac{y^2}{x} \geq -2 + \frac{3x}{\sqrt[3]{4}}$$

so

$$x < \frac{\sqrt[3]{4}}{3}(n+2) < .53(n+2),$$

and similarly for y . So we only have to check $x, y \leq \lfloor .53(10+2) \rfloor = 6$:

$x \setminus y$	1	2	3	4	5	6
1	2	4	9	16	25	36
2	4	4	5	9	12	18
3	9	5	6	7	9	13
4	16	9	7	8	9	11
5	25	12	9	9	10	11
6	36	18	13	11	11	12

15. [50] Find, with proof, all positive integer palindromes whose square is also a palindrome.

Answer: A palindrome satisfies the requirement if and only if the sum of the squares of its digits is less than 10. We may categorize these numbers this way:

- 3
- Any palindromic combination of 1s and 0s with at most nine 1s.

- Any palindrome consisting of a single 2 in the middle and 1s and 0s elsewhere, with at most four 1s.
- 2000...0002
- 2000...0001000...0002

Solution: Let $n := \sum_{i=0}^d a_i \cdot 10^i$ be a palindrome, where the a_i are digits with $a_i = a_{d-i}$ and $a_d \neq 0$. Then, if we let

$$b_k := \sum_{i+j=k} a_i a_j$$

for all $0 \leq k \leq 2d$, then

$$n^2 = \sum_{k=0}^{2d} b_k \cdot 10^k$$

(this is not necessarily the decimal expansion of n^2 , however). We have to show that $\sum_{i=0}^d a_i^2 < 10$ if and only if n^2 is a palindrome.

Suppose $\sum_{i=0}^d a_i^2 < 10$. Then, by the AM-GM inequality, we have

$$b_k = \sum_{i+j=k} a_i a_j \leq \sum_{i+j=k} \frac{a_i^2 + a_j^2}{2} \leq \sum_{i=0}^d \frac{a_i^2}{2} + \sum_{j=0}^d \frac{a_j^2}{2} < \frac{10}{2} + \frac{10}{2} = 10.$$

Thus, loosely speaking, no carrying is ever done in computing $n \times n$ by long multiplication, so the digit in the 10^k place in n^2 is precisely b_k , and it's easy to see that $b_k = b_{2d-k}$ and that $b_{2d} = a_d^2 \neq 0$. So n^2 is indeed a palindrome, as desired.

Now suppose $\sum_{i=0}^d a_i^2 \geq 10$. Here note that

$$b_d = \sum_{i+j=d} a_i a_j = \sum_{i=0}^d a_i a_{d-i} = \sum_{i=0}^d a_i^2 \geq 10.$$

Thus, it *cannot* be true that, for all k , b_k represents the 10^k digit of n^2 , because no digit can be greater than or equal to 10. Let ℓ be the greatest such that b_ℓ does not represent the 10^ℓ digit of n^2 . We are trying to prove that n^2 cannot be a palindrome. Consider three cases:

- $a_d = a_0 \geq 4$. In this case we must have $\ell \geq 2d$, because $b_{2d} = a_d^2 > 10$.
If $a_0 = 4$, then n^2 ends in the digit 6, but lies in the interval $[16 \cdot 10^{2d}, 25 \cdot 10^{2d})$, and so starts with either a 1 or a 2; thus, n^2 cannot be a palindrome. Similarly, if $a_0 = 5$, then n^2 ends in 5 but starts with 2 or 3; if $a_0 = 6$, then n^2 ends in 6 but starts with 3 or 4; if $a_0 = 7$, then n^2 ends in 9 but starts with 4, 5, or 6; if $a_0 = 8$, then n^2 ends in 4 but starts with 6, 7 or 8; if $a_0 = 9$, then n^2 ends in 1 but starts with 8 or 9.
- $\ell \geq 2d$ and $a_d = a_0 \leq 3$.

Here we do something similar, but with a slight twist. The units digit of n^2 is a_0^2 . Because $\ell \geq 2d$, n^2 must be in the interval $[(a_0^2 + 1) \cdot 10^{2d}, (a_0 + 1)^2 \cdot 10^{2d})$,

which is certainly a subset of the interval $[(a_0^2 + 1) \cdot 10^{2d}, a_0^2 \cdot 10^{2d+1})$. No integer in even this larger interval manages to start with the digit a_0^2 , so n^2 cannot be palindromic.

- $\ell < 2d$.

Here we can rest assured that n^2 does have $(2d + 1)$ digits — that is, the first digit is in the 10^{2d} place. In order for n^2 to be a palindrome, the digits in the 10^k and 10^{2d-k} places must always be the same.

Now $b_\ell, b_{\ell+1}, \dots, b_{2d}$ had all better be less than 10, or else ℓ would be greater than what it is. Thus, the numbers just listed *do* appear as the *lowest* digits of n^2 in left-to-right order, although they *don't* appear as the *highest* $(2d + 1 - \ell)$ digits of n^2 in right-to-left order. Thus, n^2 cannot be a palindrome.

IXth Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Team Round B: Solutions

Mobotics [135]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged in an $m \times n$ rectangular grid, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

Our machinery consists of a fleet of identical *mowbots* (or “mobots” for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction, either north or east (*not* south or west). At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered “different” if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot start on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

As an example, one allowable formation for $m = 2$, $n = 3$ might be as follows:

$$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

1. [25] Prove that the maximum number of mobots you need to mow your lawn is $m+n-1$.

Solution: This is attainable if we place one mobot at each clump in the first row, oriented north, and one mobot in the first column of each row except the first, oriented east.

To show that at most $m+n-1$ mobots can be used, note that each mobot must mow at least one of the $m+n-1$ clumps in union of the first row and last column.

2. [40] Prove that the minimum number of mobots you need to mow your lawn is $\min\{m, n\}$.

Solution: This is attainable if we place one mobot at each clump in the first column, oriented toward the east, or one mobot at each clump in the last row, oriented toward the north (whichever of the two is more efficient).

To show that at least $\min\{m, n\}$ mobots must be used, note that each of the clumps $(0, 0)$, $(1, 1)$, $(2, 2)$, \dots , $(\min\{m, n\} - 1, \min\{m, n\} - 1)$ must be mowed by a different mobot.

3. [15] Prove that, given any formation, each mobot may be colored in one of three colors — say, white, black, and blue — such that no two adjacent clumps of grass are mowed by different mobots of the same color. Two clumps of grass are adjacent if the distance between them is 1. In your proof, you may use the Four-Color Theorem if you're familiar with it.

Solution: We can divide the coordinate plane into regions: Let's say a point belongs to Region 0 if the closest lattice point to it is not on the lawn, and each mobot M owns a region that is the set of points for which the closest lattice point is on the lawn and mowed by M . Applying the Four-Color Theorem to these regions, we note that all the conditions demanded in the problem are satisfied. In particular at most 3 colors are used on the mobots because every mobot region borders Region 0 and hence is not colored the same color as Region 0.

4. [15] For $n = m = 4$, find a formation with 6 mobots for which there are exactly 12 ways to color the mobots in three colors as in problem 3. (No proof is necessary.)

Solution: Place north-oriented mobots at $(0, 0)$, $(1, 0)$, $(2, 2)$, and $(3, 2)$, and east-oriented mobots at $(2, 0)$ and $(2, 1)$. Other formations are possible.

5. [40] For $n, m \geq 3$, prove that a formation has exactly six possible colorings satisfying the conditions in problem 3 if and only if there is a mobot that starts at $(1, 1)$.

Solution: Let's trace a path through all the mobots. There must be a mobot at $(0, 0)$, so start the path there. If that mobot is oriented to move up, then move to the right; if that mobot is oriented to move to the right, then move up. In doing so, you will meet another mobot, upon which you can repeat the above process, until you leave the lawn.

In tracing this path, we've effectively ordered the mobots, starting with $(0, 0)$. At any point in this ordering, we have a number of choices for how to color the current mobot without violating the condition with any previously colored mobot. Specifically, there are three ways to color the mobot at $(0, 0)$. If a mobot has x - or y -coordinate 0, there are 2 ways to color it: it need only not be the same color as the previous mobot on our trail. If a mobot has both x - and y -coordinates positive, then there is only 1 way to color it: it cannot be the same color as whatever mobots mows the clump directly south or directly west of its starting point.

Thus, if there is a mobot at $(1, 1)$, our path must begin with either $(0, 0)$ – $(0, 1)$ – $(1, 1)$ or $(0, 0)$ – $(1, 0)$ – $(1, 1)$, and in either case there are only 6 ways to do the coloring. Otherwise, if there is no mobot at $(1, 1)$, our path must begin with either $(0, 0)$ – $(0, 1)$ – $(0, 2)$ or $(0, 0)$ – $(1, 0)$ – $(2, 0)$, and in either case there are already $3 \times 2 \times 2 = 12$ ways to do the coloring thus far.

Polygons [130]

6. [15] Suppose we have a regular hexagon and draw all its sides and diagonals. Into how many regions do the segments divide the hexagon? (No proof is necessary.)

Answer: 24

Solution: An accurate diagram and a careful count yields the answer.

7. [25] Suppose we have an octagon with all angles of 135° , and consecutive sides of alternating length 1 and $\sqrt{2}$. We draw all its sides and diagonals. Into how many regions do the segments divide the octagon? (No proof is necessary.)

Answer: 84

Solution: The easiest way to see the answer is to view the octagon as five unit squares in a cross arrangement, with four half-squares wedged at the corners. The center square is divided into 8 regions. The other 4 squares are each divided into 15 regions. The 4 half-squares are each divided into 4 regions. The answer is thus $8 + 4 \times 15 + 4 \times 4 = 84$.

8. [25] A regular 12-sided polygon is inscribed in a circle of radius 1. How many chords of the circle that join two of the vertices of the 12-gon have lengths whose squares are rational? (No proof is necessary.)

Answer: 42

Solution: The chords joining vertices subtend minor arcs of 30° , 60° , 90° , 120° , 150° , or 180° . There are 12 chords of each of the first five kinds and 6 diameters. For a chord with central angle θ , we can draw radii from the two endpoints of the chord to the center of the circle. By the law of cosines, the square of the length of the chord is $1 + 1 - 2 \cos \theta$, which is rational when θ is 60° , 90° , 120° , or 180° . The answer is thus $12 + 12 + 12 + 6 = 42$.

9. [25] Show a way to construct an equiangular hexagon with side lengths 1, 2, 3, 4, 5, and 6 (not necessarily in that order).

Solution: The trick is to view an equiangular hexagon as an equilateral triangle with its corners cut off. Consider an equilateral triangle with side length 9, and cut off equilateral triangles of side length 1, 2, and 3 from its corners. This yields an equiangular hexagon with sides of length 1, 6, 2, 4, 3, 5 in that order.

10. [40] Given a convex n -gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal strictly in the interior of the n -gon? Prove that your answer is correct.

Answer: $\lfloor n/2 \rfloor$

Solution: If n is even, simply draw all $n/2$ diagonals connecting a vertex to the one $n/2$ vertices away. If n is odd, pretend one of the vertices does not exist, and do the above for the $(n - 1)$ -gon remaining.

To show this is optimal, consider any given drawn diagonal: it divides the remaining $n - 2$ vertices into two camps, one of which therefore has size at most $\lfloor n/2 \rfloor - 1$, and one cannot draw two diagonals sharing a vertex.

What do the following problems have in common? [135]

11. [15] Find the largest positive integer n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square. Prove that your answer is correct.

Answer: 3

Solution: Clearly $1! + 2! + 3! = 9$ works. For $n \geq 4$, we have

$$1! + 2! + 3! + \cdots + n! \equiv 1! + 2! + 3! + 4! \equiv 3 \pmod{5},$$

but there are no squares congruent to 3 modulo 5.

12. [15] Find all ordered triples (x, y, z) of positive reals such that $x + y + z = 27$ and $x^2 + y^2 + z^2 - xy - yz - zx = 0$. Prove that your answer is correct.

Answer: $(9, 9, 9)$

Solution: We have $x^2 + y^2 + z^2 - xy - yz - zx = \frac{(x - y)^2 + (y - z)^2 + (z - x)^2}{2} = 0$, which implies $x = y = z = \frac{27}{3} = 9$.

13. [25] Four circles with radii 1, 2, 3, and r are externally tangent to one another. Compute r . (No proof is necessary.)

Answer: $6/23$

Solution: Let A, B, C, P be the centers of the circles with radii 1, 2, 3, and r , respectively. Then, ABC is a 3-4-5 right triangle. Using the law of cosines in $\triangle PAB$ yields

$$\cos \angle PAB = \frac{3^2 + (1 + r)^2 - (2 + r)^2}{2 \cdot 3 \cdot (1 + r)} = \frac{3 - r}{3(1 + r)}$$

Similarly,

$$\cos \angle PAC = \frac{4^2 + (1 + r)^2 - (3 + r)^2}{2 \cdot 4 \cdot (1 + r)} = \frac{2 - r}{2(1 + r)}$$

We can now use the equation $(\cos \angle PAB)^2 + (\cos \angle PAC)^2 = 1$, which yields $0 = 23r^2 + 132r - 36 = (23r - 6)(r + 6)$, or $r = 6/23$.

14. [40] Find the prime factorization of

$$2006^2 \cdot 2262 - 669^2 \cdot 3599 + 1593^2 \cdot 1337.$$

(No proof is necessary.)

Answer: $2 \cdot 3 \cdot 7 \cdot 13 \cdot 29 \cdot 59 \cdot 61 \cdot 191$

Solution: Upon observing that $2262 = 669 + 1593$, $3599 = 1593 + 2006$, and $1337 = 2006 - 669$, we are inspired to write $a = 2006, b = 669, c = -1593$. The expression in question then rewrites as $a^2(b - c) + b^2(c - a) + c^2(a - b)$. But, by experimenting in the general case (e.g. setting $a = b$), we find that this polynomial is zero when two of a, b, c are equal. Immediately we see that it factors as $(b - a)(c - b)(a - c)$, so the original expression is a way of writing $(-1337) \cdot (-2262) \cdot (3599)$. Now, $1337 = 7 \cdot 191$, $2262 = 2 \cdot 3 \cdot 13 \cdot 29$, and $3599 = 60^2 - 1^2 = 59 \cdot 61$.

15. [40] Let a, b, c, d be real numbers so that c, d are not both 0. Define the function

$$m(x) = \frac{ax + b}{cx + d}$$

on all real numbers x except possibly $-d/c$, in the event that $c \neq 0$. Suppose that the equation $x = m(m(x))$ has at least one solution that is not a solution of $x = m(x)$. Find all possible values of $a + d$. Prove that your answer is correct.

Answer: 0

Solution: That 0 is a possible value of $a + d$ can be seen by taking $m(x) = -x$, i.e., $a = -d = 1$, $b = c = 0$. We will now show that 0 is the only possible value of $a + d$.

The equation $x = m(m(x))$ implies $x = \frac{(a^2 + bc)x + (a + d)b}{(a + d)cx + (bc + d^2)}$, which in turn implies

$$(a + d)[cx^2 + (-a + d)x - b] = 0.$$

Suppose for the sake of contradiction that $a + d \neq 0$. Then the above equation would further imply

$$cx^2 + (-a + d)x - b = 0, x(cx + d) = ax + b,$$

which would imply $x = m(x)$ for any x except possibly $-d/c$. But of course $-d/c$ is not a root of $x = m(m(x))$ anyway, so in this case, all solutions of $x = m(m(x))$ are also solutions of $x = m(x)$, a contradiction. So our assumption was wrong, and in fact $a + d = 0$, as claimed.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Algebra Subject Test — Solutions

1. How many real numbers x are solutions to the following equation?

$$|x - 1| = |x - 2| + |x - 3|$$

Solution: $\boxed{2}$

If $x < 1$, the equation becomes $(1 - x) = (2 - x) + (3 - x)$ which simplifies to $x = 4$, contradicting the assumption $x < 1$. If $1 \leq x \leq 2$, we get $(x - 1) = (2 - x) + (3 - x)$, which gives $x = 2$. If $2 \leq x \leq 3$, we get $(x - 1) = (x - 2) + (3 - x)$, which again gives $x = 2$. If $x \geq 3$, we get $(x - 1) = (x - 2) + (x - 3)$, or $x = 4$. So 2 and 4 are the only solutions, and the answer is 2.

2. How many real numbers x are solutions to the following equation?

$$2003^x + 2004^x = 2005^x$$

Solution: $\boxed{1}$

Rewrite the equation as $(2003/2005)^x + (2004/2005)^x = 1$. The left side is strictly decreasing in x , so there cannot be more than one solution. On the other hand, the left side equals $2 > 1$ when $x = 0$ and goes to 0 when x is very large, so it must equal 1 somewhere in between. Therefore there is one solution.

3. Let x , y , and z be distinct real numbers that sum to 0. Find the maximum possible value of

$$\frac{xy + yz + zx}{x^2 + y^2 + z^2}.$$

Solution: $\boxed{-1/2}$

Note that $0 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$. Rearranging, we get that $xy + yz + zx = -\frac{1}{2}(x^2 + y^2 + z^2)$, so that in fact the quantity is always equal to $-1/2$.

4. If $a, b, c > 0$, what is the smallest possible value of $\lfloor \frac{a+b}{c} \rfloor + \lfloor \frac{b+c}{a} \rfloor + \lfloor \frac{c+a}{b} \rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .)

Solution: $\boxed{4}$

Since $\lfloor x \rfloor > x - 1$ for all x , we have that

$$\begin{aligned} \left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor &> \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - 3 \\ &= \left(\frac{a}{b} + \frac{b}{a} \right) + \left(\frac{b}{c} + \frac{c}{b} \right) + \left(\frac{c}{a} + \frac{a}{c} \right) - 3. \end{aligned}$$

But by the AM-GM inequality, each of the first three terms in the last line is at least 2. Therefore, the lefthand side is greater than $2 + 2 + 2 - 3 = 3$. Since it is an integer, the smallest value it can be is 4. This is in fact attainable by letting $(a, b, c) = (6, 8, 9)$.

5. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

Solution: $\boxed{28}$

First note that all the integers must be at least 2, because the greatest common divisor of any two positive integers is at least 1. Let n be the largest integer in the circle. The greatest common divisor of its two neighbors is $n - 1$. Therefore, each of the two neighbors is at least $n - 1$ but at most n , so since $n - 1 \nmid n$ for $n - 1 \geq 2$, they must both be equal to $n - 1$. Let m be one of the numbers on the other side of $n - 1$ from n . Then $\gcd(n, m) = n - 2$. Since $n - 2 \geq 0$, $n - 2 \mid n$ only for $n = 3$ or 4 . If $n = 3$, each number must be 2 or 3, and it is easy to check that there is no solution. If $n = 4$, then it is again not hard to find that there is a unique solution up to rotation, namely 4322343223. The only possible sum is therefore 28.

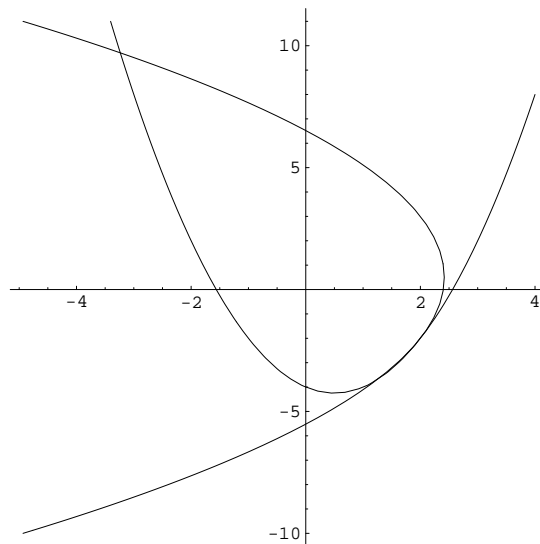
6. Find the sum of the x -coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

Solution: $\boxed{0}$

Substituting $y = x^2 - x - 4$ into the second equation yields

$$\begin{aligned} 0 &= (x^2 - x - 4)^2 - (x^2 - x - 4) + 15x - 36 \\ &= x^4 - 2x^3 - 7x^2 + 8x + 16 - x^2 + x + 4 + 15x - 36 \\ &= x^4 - 2x^3 - 8x^2 + 24x - 16 \\ &= (x - 2)(x^3 - 8x + 8) = (x - 2)^2(x^2 + 2x - 4). \end{aligned}$$

This quartic has three distinct real roots at $x = 2, -1 \pm \sqrt{5}$. Each of these yields a distinct point of intersection, so the answer is their sum, 0.



7. Let x be a positive real number. Find the maximum possible value of

$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}.$$

Solution: $\boxed{2\sqrt{2} - 2}$

Rationalizing the numerator, we get

$$\begin{aligned} \frac{x^2 + 2 - \sqrt{x^4 + 4}}{x} \cdot \frac{x^2 + 2 + \sqrt{x^4 + 4}}{x^2 + 2 + \sqrt{x^4 + 4}} &= \frac{(x^2 + 2)^2 - (x^4 + 4)}{x(x^2 + 2 + \sqrt{x^4 + 4})} \\ &= \frac{4x^2}{x(x^2 + 2 + \sqrt{x^4 + 4})} \\ &= \frac{4}{\frac{1}{x}(x^2 + 2 + \sqrt{x^4 + 4})} \\ &= \frac{4}{x + \frac{2}{x} + \sqrt{x^2 + \frac{4}{x^2}}}. \end{aligned}$$

Since we wish to maximize this quantity, we wish to minimize the denominator. By AM-GM, $x + \frac{2}{x} \geq 2\sqrt{2}$ and $x^2 + \frac{4}{x^2} \geq 4$, so that the denominator is at least $2\sqrt{2} + 2$. Therefore,

$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x} \leq \frac{4}{2\sqrt{2} + 2} = 2\sqrt{2} - 2,$$

with equality when $x = \sqrt{2}$.

8. Compute

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

Solution: $\boxed{1/2}$

Note that

$$n^4 + n^2 + 1 = (n^4 + 2n^2 + 1) - n^2 = (n^2 + 1)^2 - n^2 = (n^2 + n + 1)(n^2 - n + 1).$$

Decomposing into partial fractions, we find that

$$\frac{n}{n^4 + n^2 + 1} = \frac{1}{2} \left(\frac{1}{n^2 - n + 1} - \frac{1}{n^2 + n + 1} \right).$$

Now, note that if $f(n) = \frac{1}{n^2 - n + 1}$, then $f(n + 1) = \frac{1}{(n+1)^2 - (n+1) + 1} = \frac{1}{n^2 + n + 1}$. It follows that

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1} = \frac{1}{2} \left((f(0) - f(1)) + (f(1) - f(2)) + (f(2) - f(3)) + \dots \right).$$

Since $f(n)$ tends towards 0 as n gets large, this sum telescopes to $f(0)/2 = 1/2$.

9. The number 27,000,001 has exactly four prime factors. Find their sum.

Solution: $\boxed{652}$

First, we factor

$$\begin{aligned}27x^6 + 1 &= (3x^2)^3 + 1 \\&= (3x^2 + 1)(9x^4 - 3x^2 + 1) \\&= (3x^2 + 1)((9x^4 + 6x^2 + 1) - 9x^2) \\&= (3x^2 + 1)((3x^2 + 1)^2 - (3x)^2) \\&= (3x^2 + 1)(3x^2 + 3x + 1)(3x^2 - 3x + 1).\end{aligned}$$

Letting $x = 10$, we get that $27000001 = 301 \cdot 331 \cdot 271$. A quick check shows that $301 = 7 \cdot 43$, so that $27000001 = 7 \cdot 43 \cdot 271 \cdot 331$. Each factor here is prime, and their sum is 652.

10. Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.

Solution: $\boxed{2 + 2\sqrt{2} + 2\sqrt{3}}$

$$\begin{aligned}x^4 - 4x^3 - 4x^2 + 16x - 8 &= (x^4 - 4x^3 + 4x^2) - (8x^2 - 16x + 8) \\&= x^2(x - 2)^2 - 8(x - 1)^2 \\&= (x^2 - 2x)^2 - (2\sqrt{2}x - 2\sqrt{2})^2 \\&= (x^2 - (2 + 2\sqrt{2})x + 2\sqrt{2})(x^2 - (2 - 2\sqrt{2})x - 2\sqrt{2}).\end{aligned}$$

But noting that $(1 + \sqrt{2})^2 = 3 + 2\sqrt{2}$ and completing the square,

$$\begin{aligned}x^2 - (2 + 2\sqrt{2})x + 2\sqrt{2} &= x^2 - (2 + 2\sqrt{2})x + 3 + 2\sqrt{2} - 3 \\&= (x - (1 + \sqrt{2}))^2 - (\sqrt{3})^2 \\&= (x - 1 - \sqrt{2} + \sqrt{3})(x - 1 - \sqrt{2} - \sqrt{3}).\end{aligned}$$

Likewise,

$$x^2 - (2 - 2\sqrt{2})x - 2\sqrt{2} = (x - 1 + \sqrt{2} + \sqrt{3})(x - 1 + \sqrt{2} - \sqrt{3}),$$

so the roots of the quartic are $1 \pm \sqrt{2} \pm \sqrt{3}$. Only one of these is negative, namely $1 - \sqrt{2} - \sqrt{3}$, so the sum of the absolute values of the roots is

$$(1 + \sqrt{2} + \sqrt{3}) + (1 + \sqrt{2} - \sqrt{3}) + (1 - \sqrt{2} + \sqrt{3}) - (1 - \sqrt{2} - \sqrt{3}) = 2 + 2\sqrt{2} + 2\sqrt{3}.$$

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Calculus Subject Test — Solutions

1. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

Solution: $\boxed{1}$

Since $f'(x) = 3x^2 + a$, we must have $3a^2 + a = 3b^2 + a$. Then $a^2 = b^2$, and since $a \neq b$, $a = -b$. Thus $f(1) = 1 + a + b = 1$.

2. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u} du$ and $y(t) = \int_t^\infty \frac{\sin u}{u} du$ for $1 \leq t \leq 2$. What is the length of the curve?

Solution: $\boxed{\ln 2}$

By the Second Fundamental Theorem of Calculus, $\frac{dx}{dt} = -\frac{\cos t}{t}$ and $\frac{dy}{dt} = -\frac{\sin t}{t}$. Therefore, the length of the curve is

$$\int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_1^2 \sqrt{\frac{\cos^2 t}{t^2} + \frac{\sin^2 t}{t^2}} dt = \int_1^2 \frac{1}{t} dt = [\ln t]_1^2 = \ln 2.$$

3. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

Solution: $\boxed{486/5}$

$$0 = \int_0^1 f(x)f'(x)dx = \int_{f(0)}^{f(1)} u du = \frac{1}{2}(f(1)^2 - f(0)^2), \text{ and}$$

$$18 = \int_0^1 f(x)^2 f'(x)dx = \int_{f(0)}^{f(1)} u^2 du = \frac{1}{3}(f(1)^3 - f(0)^3).$$

The first equation implies $f(0) = \pm f(1)$. The second equation shows that $f(0) \neq f(1)$, and in fact $54 = f(1)^3 - f(0)^3 = 2f(1)^3$, so $f(1) = 3$ and $f(0) = -3$. Then

$$\int_0^1 f(x)^4 f'(x)dx = \int_{f(0)}^{f(1)} u^4 du = \frac{1}{5}(f(1)^5 - f(0)^5) = \frac{1}{5}(243 + 243) = \frac{486}{5}.$$

4. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x)f''(x)$ for all x . Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

Solution: $\boxed{\pm\sqrt{3}}$

Let $f'(0) = a$. Then the equation gives $f''(0) = a^2$. Differentiating the given equation gives

$$2f'(x)f''(x) = f(x)f'''(x) + f'(x)f''(x),$$

or $f'(x)f''(x) = f(x)f'''(x)$. Differentiating once more gives

$$f'(x)f'''(x) + f''(x)^2 = f(x)f^{(4)}(x) + f'(x)f''(x)$$

or $f''(x)^2 = f(x)f^{(4)}(x)$, giving $9 = f^{(4)}(0) = a^4$. Thus $a = \pm\sqrt{3}$. These are indeed both attainable by $f(x) = e^{\pm x\sqrt{3}}$.

Alternative Solution: Rewrite the given equation as $\frac{f''(x)}{f'(x)} = \frac{f'(x)}{f(x)}$. Integrating both sides gives $\ln f'(x) = \ln f(x) + C_1$, and exponentiating gives $f'(x) = Cf(x)$. This has solution $f(x) = Ae^{Cx}$ for constants A and C . Since $f(0) = 1$, $A = 1$, and differentiating we find that $C^4 = f^{(4)}(0) = 9$, yielding $f'(0) = C = \pm\sqrt{3}$.

5. Calculate

$$\lim_{x \rightarrow 0^+} (x^{x^x} - x^x).$$

Solution: -1

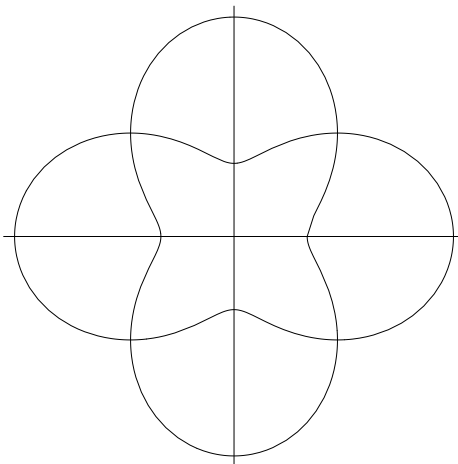
We first calculate $\lim_{x \rightarrow 0^+} x^x$: it is just $\exp(\lim_{x \rightarrow 0^+} x \ln x)$. But

$$\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} -x = 0$$

by L'Hôpital's Rule. Therefore $\lim_{x \rightarrow 0^+} x^x = 1$. Then $\lim_{x \rightarrow 0^+} x^{x^x} = 0^1 = 0$, so $\lim_{x \rightarrow 0^+} (x^{x^x} - x^x) = -1$.

6. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

Solution: $9\pi/2 - 8$



The original graph is closer to the origin than its reflection for $\theta \in (\frac{\pi}{4}, \frac{3\pi}{4}) \cup (\frac{5\pi}{4}, \frac{7\pi}{4})$, and the region is symmetric about the origin. Therefore the area we wish to find is the polar integral

$$\begin{aligned} 4 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{2} (2 + \cos 2\theta)^2 d\theta &= 2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (4 + 4 \cos 2\theta + \cos^2 2\theta) d\theta \\ &= 2 \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \left(4 + 4 \cos 2\theta + \frac{1}{2} (1 + \cos 4\theta) \right) d\theta \\ &= \left[9\theta + 4 \sin 2\theta + \frac{1}{4} \sin 4\theta \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \left(\frac{27\pi}{4} - 4 \right) - \left(\frac{9\pi}{4} + 4 \right) = \frac{9\pi}{2} - 8. \end{aligned}$$

7. Two ants, one starting at $(-1, 1)$, the other at $(1, 1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

Solution: $\boxed{3\sqrt{3} - 3}$

When the left ant first hits the line $y = \frac{1}{2}$, the right ant hits the line $y = \frac{3}{2}$. The left ant is then at $(-\frac{\sqrt{2}}{2}, \frac{1}{2})$, and the right ant is at $(\frac{\sqrt{6}}{2}, \frac{3}{2})$. Let the left ant have velocity with components v_x and v_y , the right ant velocity with components w_x and w_y . Since $\frac{dy}{dx} = 2x$, $\frac{v_y}{v_x} = -\sqrt{2}$ and $\frac{w_y}{w_x} = \sqrt{6}$. Since the midpoint of the ants moves at speed 1 along the line $y = 1$, $\frac{1}{2}(v_x + w_x) = 1$ and $\frac{1}{2}(v_y + w_y) = 0$. Therefore, $\sqrt{2}v_x = -v_y = w_y = \sqrt{6}w_x = \sqrt{6}(2 - v_x)$. Solving for v_x gives $\frac{2\sqrt{6}}{\sqrt{6} + \sqrt{2}} = 3 - \sqrt{3}$. Then the speed of the left ant is

$$\sqrt{v_x^2 + v_y^2} = \sqrt{v_x^2 + (-\sqrt{2}v_x)^2} = \sqrt{3}|v_x| = 3\sqrt{3} - 3.$$

8. If f is a continuous real function such that $f(x - 1) + f(x + 1) \geq x + f(x)$ for all x , what is the minimum possible value of $\int_1^{2005} f(x) dx$?

Solution: $\boxed{2010012}$

Let $g(x) = f(x) - x$. Then

$$g(x - 1) + x - 1 + g(x + 1) + x + 1 \geq x + g(x) + x,$$

or $g(x - 1) + g(x + 1) \geq g(x)$. But now,

$$g(x + 3) \geq g(x + 2) - g(x + 1) \geq -g(x).$$

Therefore

$$\begin{aligned} \int_a^{a+6} g(x) dx &= \int_a^{a+3} g(x) dx + \int_{a+3}^{a+6} g(x) dx \\ &= \int_a^{a+3} (g(x) + g(x + 3)) dx \geq 0. \end{aligned}$$

It follows that

$$\int_1^{2005} g(x) dx = \sum_{n=0}^{333} \int_{6n+1}^{6n+7} g(x) dx \geq 0,$$

so that

$$\int_1^{2005} f(x) dx = \int_1^{2005} (g(x) + x) dx \geq \int_1^{2005} x dx = \left[\frac{x^2}{2} \right]_1^{2005} = \frac{2005^2 - 1}{2} = 2010012.$$

Equality holds for $f(x) = x$.

9. Compute

$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}.$$

Solution: $\boxed{e + 1/e + 2 \cos 1}$

This is the power series

$$4 + \frac{4x^4}{4!} + \frac{4x^8}{8!} + \dots$$

evaluated at $x = 1$. But this power series can be written as the sum

$$\begin{aligned} & \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots \right) \\ + & \left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \frac{x^7}{7!} + \dots \right) \\ + & 2 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) \\ = & e^x + e^{-x} + 2 \cos x. \end{aligned}$$

It follows that the quantity is $e + 1/e + 2 \cos 1$.

10. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ be a smooth function such that $f'(x) = f(1-x)$ for all x and $f(0) = 1$. Find $f(1)$.

Solution: $\sec 1 + \tan 1$

Differentiating the given equation gives $f''(x) = -f(x)$. This has solutions of the form $A \cos(x) + B \sin(x)$. Since $f(0) = 1$, $A = 1$. Then $f'(x) = B \cos(x) - \sin(x)$ and

$$\begin{aligned} f(1-x) &= \cos(1-x) + B \sin(1-x) \\ &= \cos 1 \cos x + \sin 1 \sin x + B \sin 1 \cos x - B \cos 1 \sin x \\ &= (\cos 1 + B \sin 1) \cos x + (\sin 1 - B \cos 1) \sin x. \end{aligned}$$

Therefore, $B = \cos 1 + B \sin 1$ and $-1 = \sin 1 - B \cos 1$, both of which yield as solutions

$$B = \frac{\cos 1}{1 - \sin 1} = \frac{1 + \sin 1}{\cos 1} = \sec 1 + \tan 1.$$

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Combinatorics Subject Test — Solutions

1. A true-false test has ten questions. If you answer five questions “true” and five “false,” your score is guaranteed to be at least four. How many answer keys are there for which this is true?

Solution: 22

Suppose that either nine or ten of the questions have the same answer. Then no matter which five questions we pick to have this answer, we will be right at least four times. Conversely, suppose that there are at least two questions with each answer; we will show that we can get a score less than four. By symmetry, assume there are at least five questions whose answer is true. Then if we label five of these false, not only will we get these five wrong, but we will also have answered all the false questions with true, for a total of at least seven incorrect. There are 2 ways for all the questions to have the same answer, and $2 \cdot 10 = 20$ ways for one question to have a different answer from the others, for a total of 22 ways.

2. How many nonempty subsets of $\{1, 2, 3, \dots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

Solution: 1365

If a is the smallest element of such a set, then $13 - a$ is the largest element, and for the remaining elements we may choose any (or none) of the $12 - 2a$ elements $a + 1, a + 2, \dots, (13 - a) - 1$. Thus there are 2^{12-2a} such sets whose smallest element is a . Also, $13 - a \geq a$ clearly implies $a < 7$. Summing over all $a = 1, 2, \dots, 6$, we get a total of

$$2^{10} + 2^8 + 2^6 + \dots + 2^0 = 4^5 + 4^4 + \dots + 4^0 = (4^6 - 1)/(4 - 1) = 4095/3 = 1365$$

possible sets.

3. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

Solution: 1/2

Note that if we imagine that the series always continues to seven games even after one team has won four, this will never change the winner of the series. Notice also that the probability that the Red Sox will win Game n is precisely the probability that the Yankees will win Game $8 - n$. Therefore, the probability that the Yankees win at least four games is the same as the probability that the Red Sox win at least four games, namely $1/2$.

4. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

Solution: 216

There are $4! = 24$ ways to place the four purple balls into the grid. Choose any purple ball, and place two green balls, one in its row and the other in its column. There are four boxes that do not yet lie in the same row or column as a green ball, and at least one of these contains a purple ball (otherwise the two rows containing green balls would contain the original purple ball as well as the two in the columns not containing green balls). It is then easy to see that there is a unique way to place the remaining green balls. Therefore, there are a total of $24 \cdot 9 = 216$ ways.

5. Doug and Ryan are competing in the 2005 Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $1/3$. When Ryan swings, the probability that he will hit a home run is $1/2$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

Solution: 1/5

Denote this probability by p . Doug hits more home runs if he hits a home run on his first try when Ryan does not, or if they both hit home runs on their first try and Doug hits more home runs thereafter. The probability of the first case occurring is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$, and the probability of the second case occurring is $\frac{1}{3} \cdot \frac{1}{2} \cdot p = \frac{p}{6}$. Therefore $p = \frac{1}{6} + \frac{p}{6}$, which we solve to find $p = \frac{1}{5}$.

6. Three fair six-sided dice, each numbered 1 through 6, are rolled. What is the probability that the three numbers that come up can form the sides of a triangle?

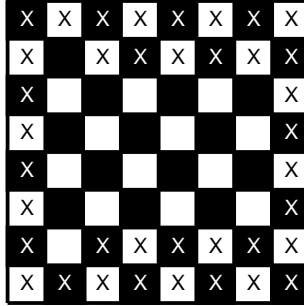
Solution: 37/72

Denote this probability by p , and let the three numbers that come up be x , y , and z . We will calculate $1-p$ instead: $1-p$ is the probability that $x \geq y+z$, $y \geq z+x$, or $z \geq x+y$. Since these three events are mutually exclusive, $1-p$ is just 3 times the probability that $x \geq y+z$. This happens with probability $(0+1+3+6+10+15)/216 = 35/216$, so the answer is $1 - 3 \cdot (35/216) = 1 - 35/72 = 37/72$.

7. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

Solution: 38

If the chessboard is colored black and white as usual, then any diagonal is a solid color, so we may consider bishops on black and white squares separately. In one direction, the lengths of the black diagonals are 2, 4, 6, 8, 6, 4, and 2. Each of these can have at most three bishops, except the first and last which can have at most two, giving a total of at most $2 + 3 + 3 + 3 + 3 + 3 + 2 = 19$ bishops on black squares. Likewise there can be at most 19 bishops on white squares for a total of at most 38 bishops. This is indeed attainable as in the diagram below.



8. Every second, Andrea writes down a random digit uniformly chosen from the set $\{1, 2, 3, 4\}$. She stops when the last two numbers she has written sum to a prime number. What is the probability that the last number she writes down is 1?

Solution: $\boxed{15/44}$

Let p_n be the probability that the last number she writes down is 1 when the first number she writes down is n . Suppose she starts by writing 2 or 4. Then she can continue writing either 2 or 4, but the first time she writes 1 or 3, she stops. Therefore $p_2 = p_4 = \frac{1}{2}$. Suppose she starts by writing 1. Then she stops if she writes 1, 2, or 4, but continues if she writes 3. Therefore $p_1 = \frac{1}{4}(1 + p_3)$. If she starts by writing 3, then she stops if she writes 2 or 4 and otherwise continues. Therefore $p_3 = \frac{1}{4}(p_1 + p_3) = \frac{1}{16}(1 + 5p_3)$. Solving gives $p_3 = \frac{1}{11}$ and $p_1 = \frac{3}{11}$. The probability we want to find is therefore $\frac{1}{4}(p_1 + p_2 + p_3 + p_4) = \frac{15}{44}$.

9. Eight coins are arranged in a circle heads up. A move consists of flipping over two adjacent coins. How many different sequences of six moves leave the coins alternating heads up and tails up?

Solution: $\boxed{7680}$

Imagine we flip over two adjacent coins by pushing a button halfway between them. Then the outcome depends only on the parities of the number of times that each button is pushed. To flip any coin, we must push the two buttons adjacent to that coin a total of an odd number of times. To flip every other coin, the parities must then progress around the circle as even, even, odd, odd, even, even, odd, odd. There are 4 ways to assign these parities. If we assume each button is pressed either once or not at all, this accounts for only four presses, so some button is also pressed twice more. Suppose this button was already pushed once. There are 4 of these, and the number of possible sequences of presses is then $6!/3! = 120$. Suppose it has not already been pressed. There are 4 of these as well, and the number of possible sequences is $6!/2! = 360$. The total number of sequences is then $4(4 \cdot 120 + 4 \cdot 360) = 7680$.

10. You start out with a big pile of 3^{2004} cards, with the numbers $1, 2, 3, \dots, 3^{2004}$ written on them. You arrange the cards into groups of three any way you like; from each group, you keep the card with the largest number and discard the other two. You now again arrange these 3^{2003} remaining cards into groups of three any way you like, and in each group, keep the card with the smallest number and discard the other two. You now have 3^{2002} cards, and you again arrange these into groups of three and keep the largest number in each group. You proceed in this manner, alternating between keeping the largest number and keeping the smallest number in each group, until you have just one card left.

How many different values are possible for the number on this final card?

Solution: $\boxed{3^{2004} - 2 \cdot 3^{1002} + 2}$

We claim that if you have cards numbered $1, 2, \dots, 3^{2n}$ and perform $2n$ successive grouping operations, then c is a possible value for your last remaining card if and only if

$$3^n \leq c \leq 3^{2n} - 3^n + 1.$$

This gives $3^{2n} - 2 \cdot 3^n + 2$ possible values of c , for a final answer of $3^{2004} - 2 \cdot 3^{1002} + 2$.

Indeed, notice that the last remaining card c must have been the largest of some set of three at the $(2n - 1)$ th step; each of these was in turn the largest of some set of three (and so c was the largest of some set of 9 cards) remaining at the $(2n - 3)$ th step; each of these was in turn the largest of some set of three (and so c was the largest of some set of 27) remaining at the $(2n - 5)$ th step; continuing in this manner, we get that c was the largest of some 3^n cards at the first step, so $c \geq 3^n$. A similar analysis of all of the steps in which we save the smallest card gives that c is the smallest of some set of 3^n initial cards, so $c \leq 3^{2n} - 3^n + 1$.

To see that any c in this interval is indeed possible, we will carry out the groupings inductively so that, after $2i$ steps, the following condition is satisfied: if the numbers remaining are $a_1 < a_2 < \dots < a_{3^{2(n-i)}}$, then c is one of these, and there are at least $3^{n-i} - 1$ numbers smaller than c and at least $3^{n-i} - 1$ numbers larger than c . This is certainly true when $i = 0$, so it suffices to show that if it holds for some $i < n$, we can perform the grouping so that the condition will still hold for $i + 1$. Well, we first group the smallest numbers as $\{a_1, a_2, a_3\}, \{a_4, a_5, a_6\}, \dots, \{a_{3^{n-i}-5}, a_{3^{n-i}-4}, a_{3^{n-i}-3}\}$. We then group the remaining numbers in such a way that c and the largest $3^{n-i} - 1$ numbers are each the largest in its respective group; it is easy to see that we can do this. After retaining the largest number in each group, we will then have at least $3^{n-i-1} - 1$ numbers smaller than c remaining and at least $3^{n-i} - 1$ numbers larger than c remaining. And for the next grouping, we similarly group the largest $3^{n-i} - 3$ numbers into $3^{n-i-1} - 1$ groups, and arrange the remaining numbers so that the smallest $3^{n-i-1} - 1$ numbers and c are all the smallest in their groups. After this round of discarding, then c will be retained, and we will still have at least $3^{n-i-1} - 1$ numbers larger than c and 3^{n-i-1} numbers smaller than c . This proves the induction step, and now the solution is complete.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 1 — Solutions

1. How many real numbers x are solutions to the following equation?

$$|x - 1| = |x - 2| + |x - 3|$$

Solution: $\boxed{2}$

If $x < 1$, the equation becomes $(1 - x) = (2 - x) + (3 - x)$ which simplifies to $x = 4$, contradicting the assumption $x < 1$. If $1 \leq x \leq 2$, we get $(x - 1) = (2 - x) + (3 - x)$, which gives $x = 2$. If $2 \leq x \leq 3$, we get $(x - 1) = (x - 2) + (3 - x)$, which again gives $x = 2$. If $x \geq 3$, we get $(x - 1) = (x - 2) + (x - 3)$, or $x = 4$. So 2 and 4 are the only solutions, and the answer is 2.

2. A true-false test has ten questions. If you answer five questions “true” and five “false,” your score is guaranteed to be at least four. How many answer keys are there for which this is true?

Solution: $\boxed{22}$

Suppose that either nine or ten of the questions have the same answer. Then no matter which five questions we pick to have this answer, we will be right at least four times. Conversely, suppose that there are at least two questions with each answer; we will show that we can get a score less than four. By symmetry, assume there are at least five questions whose answer is true. Then if we label five of these false, not only will we get these five wrong, but we will also have answered all the false questions with true, for a total of at least seven incorrect. There are 2 ways for all the questions to have the same answer, and $2 \cdot 10 = 20$ ways for one question to have a different answer from the others, for a total of 22 ways.

3. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

Solution: $\boxed{1 + 2\sqrt{3}}$

The plane intersects each face of the tetrahedron in a midline of the face; by symmetry it follows that the intersection of the plane with the tetrahedron is a square of side length 1. The surface area of each piece is half the total surface area of the tetrahedron plus the area of the square, that is, $\frac{1}{2} \cdot 4 \cdot \frac{2^2\sqrt{3}}{4} + 1 = 1 + 2\sqrt{3}$.

4. Find all real solutions to $x^3 + (x + 1)^3 + (x + 2)^3 = (x + 3)^3$.

Solution: $\boxed{3}$

The equation simplifies to $3x^3 + 9x^2 + 15x + 9 = x^3 + 9x^2 + 27x + 27$, or equivalently, $2x^3 - 12x - 18 = 2(x - 3)(x^2 + 3x + 3) = 0$. The discriminant of $x^2 + 3x + 3$ is $-3 < 0$, so the only real solution is $x = 3$.

5. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

Solution: 216

There are $4! = 24$ ways to place the four purple balls into the grid. Choose any purple ball, and place two green balls, one in its row and the other in its column. There are four boxes that do not yet lie in the same row or column as a green ball, and at least one of these contains a purple ball (otherwise the two rows containing green balls would contain the original purple ball as well as the two in the columns not containing green balls). It is then easy to see that there is a unique way to place the remaining green balls. Therefore, there are a total of $24 \cdot 9 = 216$ ways.

6. In an election, there are two candidates, A and B , who each have 5 supporters. Each supporter, independent of other supporters, has a $\frac{1}{2}$ probability of voting for his or her candidate and a $\frac{1}{2}$ probability of being lazy and not voting. What is the probability of a tie (which includes the case in which no one votes)?

Solution: 63/256

The probability that exactly k supporters of A vote and exactly k supporters of B vote is $\binom{5}{k}^2 \cdot \frac{1}{2^{10}}$. Summing over k from 0 to 5 gives

$$\left(\frac{1}{2^{10}}\right) (1 + 25 + 100 + 100 + 25 + 1) = \frac{252}{1024} = \frac{63}{256}.$$

7. If $a, b, c > 0$, what is the smallest possible value of $\lfloor \frac{a+b}{c} \rfloor + \lfloor \frac{b+c}{a} \rfloor + \lfloor \frac{c+a}{b} \rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x .)

Solution: 4

Since $\lfloor x \rfloor > x - 1$ for all x , we have that

$$\begin{aligned} \left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor &> \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} - 3 \\ &= \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) - 3. \end{aligned}$$

But by the AM-GM inequality, each of the first three terms in the last line is at least 2. Therefore, the lefthand side is greater than $2 + 2 + 2 - 3 = 3$. Since it is an integer, the smallest value it can be is therefore 4. This is in fact attainable by letting $(a, b, c) = (6, 8, 9)$.

8. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

Solution: 28

First note that all the integers must be at least 2, because the greatest common divisor of any two positive integers is at least 1. Let n be the largest integer in the circle. The greatest common divisor of its two neighbors is $n - 1$. Therefore, each of the two

neighbors is at least $n - 1$ but at most n , so since $n - 1 \nmid n$ for $n - 1 \geq 2$, they must both be equal to $n - 1$. Let m be one of the numbers on the other side of $n - 1$ from n . Then $\gcd(n, m) = n - 2$. Since $n - 2 \geq 0$, $n - 2 \mid n$ only for $n = 3$ or 4 . If $n = 3$, each number must be 2 or 3, and it is easy to check that there is no solution. If $n = 4$, then it is again not hard to find that there is a unique solution up to rotation, namely 4322343223. The only possible sum is therefore 28.

9. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

Solution: $\boxed{2/3}$

Let the triangle be denoted ABC , and suppose we fold parallel to BC . Let the distance from A to BC be h , and suppose we fold along a line at a distance of ch from A . We will assume that neither angle B nor C is obtuse, for the area of overlap will only be smaller if either is obtuse. If $c \leq \frac{1}{2}$, then A does not fold past the edge BC , so the overlap is a triangle similar to the original with height ch ; the area of the figure is then $1 - c^2 \geq \frac{3}{4}$. Suppose $c > \frac{1}{2}$, so that A does fold past BC . Then the overlap is a trapezoid formed by taking a triangle of height ch similar to the original and removing a triangle of height $(2c - 1)h$ similar to the original. The area of the resulting figure is thus $1 - c^2 + (2c - 1)^2 = 3c^2 - 4c + 2$. This is minimized when $c = \frac{2}{3}$, when the area is $\frac{2}{3} < \frac{3}{4}$; the minimum possible area is therefore $\frac{2}{3}$.

10. What is the smallest integer x larger than 1 such that x^2 ends in the same three digits as x does?

Solution: $\boxed{376}$

The condition is that $1000 \mid x^2 - x = x(x - 1)$. Since $1000 = 2^3 \cdot 5^3$, and 2 cannot divide both x and $x - 1$, $2^3 = 8$ must divide one of them. Similarly, $5^3 = 125$ must divide either x or $x - 1$. We try successive values of x that are congruent to 0 or 1 modulo 125 and see which ones have the property that x or $x - 1$ is divisible by 8. It is easy to check that 125, 126, 250, 251, and 375 do not work, but the next value, 376, does, so this is the answer.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 2 — Solutions

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

Solution: $\boxed{6\sqrt{3}}$

If the side length of the cube is s inches, then the condition implies $s^3 + 3 \cdot 12s = 2 \cdot 6s^2$, or $s(s^2 - 12s + 36) = s(s - 6)^2 = 0$. Therefore $s = 6$, and the long diagonal has length $s\sqrt{3} = 6\sqrt{3}$.

2. Find three real numbers $a < b < c$ satisfying:

$$\begin{aligned}a + b + c &= 21/4 \\ 1/a + 1/b + 1/c &= 21/4 \\ abc &= 1.\end{aligned}$$

Solution: $\boxed{1/4, 1, 4}$

By inspection, one notices that if b is a number such that $b + 1/b = 17/4$, then $a = 1$, $c = 1/b$ will work. Again by inspection (or by solving the quadratic $b^2 - 17b/4 + 1 = 0$), one finds $b = 1/4$ or 4 , so the numbers are $1/4$, 1 , and 4 .

Alternative Solution: Note that $ab + bc + ca = abc(1/a + 1/b + 1/c) = 21/4$, so a , b , and c are the roots of the polynomial $x^3 - 21x^2/4 + 21x/4 - 1$, which factors as $\frac{1}{4}(x - 1)(x - 4)(4x - 1)$, giving the same answer.

3. Working together, Jack and Jill can paint a house in 3 days; Jill and Joe can paint the same house in 4 days; or Joe and Jack can paint the house in 6 days. If Jill, Joe, and Jack all work together, how many days will it take them?

Solution: $\boxed{8/3}$

Suppose that Jack paints x houses per day, Jill paints y houses per day, and Joe paints z houses per day. Together, Jack and Jill paint $1/3$ of a house in a day — that is,

$$x + y = 1/3.$$

Similarly,

$$y + z = 1/4,$$

and

$$z + x = 1/6.$$

Adding all three equations and dividing by 2 gives

$$x + y + z = 3/8.$$

So, working together, the three folks can paint $3/8$ houses in a day, or $8/3$ days per house.

4. In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?

Solution: $\boxed{5040}$

Let us place Alice and Bob as a single person; there are then $7! = 5040$ different arrangements. Alice can be in front of Bob or vice versa, multiplying the number of possibilities by 2, but Carol is behind Dan in exactly half of those, so that the answer is just 5040.

5. You and I play the following game on an 8×8 square grid of boxes: Initially, every box is empty. On your turn, you choose an empty box and draw an X in it; if any of the four adjacent boxes are empty, you mark them with an X as well. (Two boxes are adjacent if they share an edge.) We alternate turns, with you moving first, and whoever draws the last X wins. How many choices do you have for a first move that will enable you to guarantee a win no matter how I play?

Solution: $\boxed{0}$

I can follow a symmetry strategy: whenever you play in the box S , I play in the image of S under the 180° rotation about the center of the board. This ensures that the board will always be centrally symmetric at the beginning of your turn. Thus, if you play in an empty box S , its symmetric image S' is also empty at the beginning of your turn, and it remains so after your turn, since the even size of the board ensures that S can be neither equal to nor adjacent to S' . In particular, I always have a move available. Since the first person without an available move loses, you are guaranteed to lose. So the answer is that you have 0 choices for a first move that will guarantee your win.

6. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

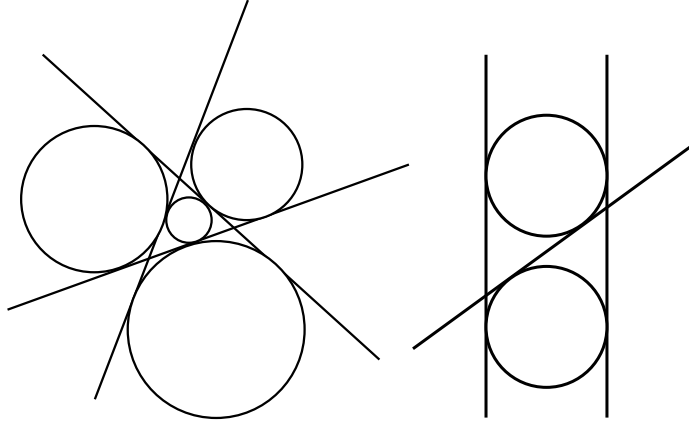
Solution: $\boxed{2/3}$

First note that the long diagonal of the cube has length $2\sqrt{3}$, so the radius of the sphere is $\sqrt{3}$. Let x be the side length of the smaller cube. Then the distance from the center of the sphere to the far face of the smaller cube is $1 + x$, while the distance from the center of the far face to a vertex lying on the sphere is $\frac{x\sqrt{2}}{2}$. Therefore, the square of the radius is $3 = (1 + x)^2 + \frac{x^2}{2}$, or $3x^2 + 4x - 4 = (3x - 2)(x + 2) = 0$, so $x = \frac{2}{3}$.

7. Three distinct lines are drawn in the plane. Suppose there exist exactly n circles in the plane tangent to all three lines. Find all possible values of n .

Solution: $\boxed{0, 2, 4}$

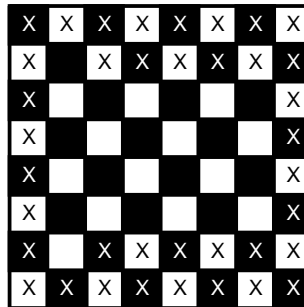
If the three lines form a triangle, then there are 4 circles, namely the incircle and the three excircles. If the three lines concur or are all parallel, then there are 0 circles. If two lines are parallel and the third is not, then there are 2 circles lying between the two parallel lines, one on each side of the transverse line. These are the only possible configurations, so the answers are 0, 2, and 4.



8. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

Solution: 38

If the chessboard is colored black and white as usual, then any diagonal is a solid color, so we may consider bishops on black and white squares separately. In one direction, the lengths of the black diagonals are 2, 4, 6, 8, 6, 4, and 2. Each of these can have at most three bishops, except the first and last which can have at most two, giving a total of at most $2 + 3 + 3 + 3 + 3 + 3 + 2 = 19$ bishops on black squares. Likewise there can be at most 19 bishops on white squares for a total of at most 38 bishops. This is indeed attainable as in the diagram below.



9. In how many ways can the cells of a 4×4 table be filled in with the digits $1, 2, \dots, 9$ so that each of the 4-digit numbers formed by the columns is divisible by each of the 4-digit numbers formed by the rows?

Solution: 9

If a and b are 4-digit numbers with the same first digit, and a divides b , then since $b < a + 1000 \leq 2a$, b must equal a . In particular, since the number formed by the first row of the table divides the number in the first column (and both have the same first digit), these numbers must be equal; call their common value n . Then, for $k = 2, 3$, or 4 , we find that the number in the k th column and the number in the k th row have the same first digit (namely the k th digit of n), so by the same reasoning, they are equal. Also, the smallest number b formed by any column is divisible by the largest number a formed by any row, but by the symmetry just proven, a is also the largest number formed by any column, so $a \geq b$. Since b is divisible by a , we must have equality. Then all columns contain the same number — and hence all rows also contain the same

number — which is only possible if all 16 cells contain the same digit. Conversely, for each $d = 1, \dots, 9$, filling in all 16 cells with the digit d clearly gives a table meeting the required condition, so we have exactly 9 such tables, one for each digit.

10. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x . How many positive integers less than 2005 can be expressed in the form $\lfloor x \lfloor x \rfloor \rfloor$ for some positive real x ?

Solution: 990

Let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of x . Note that

$$\lfloor x \lfloor x \rfloor \rfloor = \lfloor (\lfloor x \rfloor + \{x\}) \lfloor x \rfloor \rfloor = \lfloor x \rfloor^2 + \lfloor \{x\} \lfloor x \rfloor \rfloor.$$

Because $\{x\}$ may take on any value in the half-open interval $[0, 1)$, the quantity $\lfloor \{x\} \lfloor x \rfloor \rfloor$ can take on any integer value between 0 and $\lfloor x \rfloor - 1$, inclusive.

If $\lfloor x \rfloor = n$, then $\lfloor x \lfloor x \rfloor \rfloor$ can be any of the numbers $n^2, n^2 + 1, \dots, n^2 + n - 1$. In other words, there are precisely n possible values that $\lfloor x \lfloor x \rfloor \rfloor$ can take, and moreover, all of them are less than $(n + 1)^2$. Because $44^2 + 43 = 1979 < 2005$ and $45^2 = 2025 > 2005$, n can range between 1 and 44, inclusive. Therefore, the answer is

$$\sum_{n=1}^{44} n = \frac{44 \cdot 45}{2} = 990.$$

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Geometry Subject Test — Solutions

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

Solution: $\boxed{6\sqrt{3}}$

If the side length of the cube is s inches, then the condition implies $s^3 + 3 \cdot 12s = 2 \cdot 6s^2$, or $s(s^2 - 12s + 36) = s(s - 6)^2 = 0$. Therefore $s = 6$, and the long diagonal has length $s\sqrt{3} = 6\sqrt{3}$.

2. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

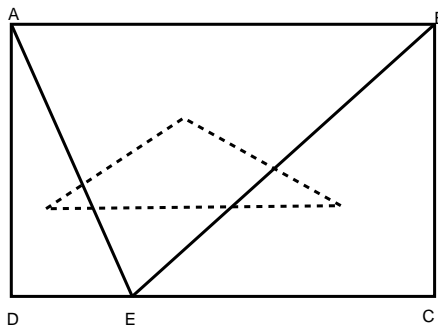
Solution: $\boxed{1 + 2\sqrt{3}}$

The plane intersects each face of the tetrahedron in a midline of the face; by symmetry it follows that the intersection of the plane with the tetrahedron is a square of side length 1. The surface area of each piece is half the total surface area of the tetrahedron plus the area of the square, that is, $\frac{1}{2} \cdot 4 \cdot \frac{2^2\sqrt{3}}{4} + 1 = 1 + 2\sqrt{3}$.

3. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD . What is the area of the triangle formed by the centroids of triangles ABE , BCE , and ADE ?

Solution: $\boxed{1/9}$

Let the centroids of ABE , BCE , and ADE be denoted by X , Y , and Z , respectively. Let $d(P, QR)$ denote the distance from P to line QR . Since the centroid lies two-thirds of the distance from each vertex to the midpoint of the opposite edge, $d(X, AB) = d(Y, CD) = d(Z, CD) = \frac{1}{3}BC$, so YZ is parallel to CD and $d(X, YZ) = BC - \frac{2}{3}BC = \frac{1}{3}BC$. Likewise, $d(Z, AD) = \frac{1}{3}DE$ and $d(Y, BC) = \frac{1}{3}CE$, so that since YZ is perpendicular to AD and BC , we have that $YZ = CD - \frac{1}{3}(DE + CE) = \frac{2}{3}CD$. Therefore, the area of XYZ is $\frac{1}{2}(\frac{1}{3}BC)(\frac{2}{3}CD) = \frac{1}{9}BC \cdot CD = \frac{1}{9}$.



4. Let XYZ be a triangle with $\angle X = 60^\circ$ and $\angle Y = 45^\circ$. A circle with center P passes through points A and B on side XY , C and D on side YZ , and E and F on side ZX . Suppose $AB = CD = EF$. Find $\angle XPY$ in degrees.

Solution: $\boxed{255/2}$

Since PAB , PCD , and PEF are all isosceles triangles with equal legs and equal bases, they are congruent. It follows that the heights of each are the same, so that P is equidistant from the sides of XYZ . Therefore, P is the incenter and therefore lies on the angle bisectors of XYZ . Thus $\angle YXP = \frac{1}{2}\angle YXZ = 30^\circ$ and $\angle PYX = \frac{1}{2}\angle ZYX = \frac{45^\circ}{2}$. Therefore $\angle XPY = 180^\circ - 30^\circ - \frac{45^\circ}{2} = \frac{255^\circ}{2}$.

5. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

Solution: $\boxed{2/3}$

First note that the long diagonal of the cube has length $2\sqrt{3}$, so the radius of the sphere is $\sqrt{3}$. Let x be the side length of the smaller cube. Then the distance from the center of the sphere to the far face of the smaller cube is $1 + x$, while the distance from the center of the far face to a vertex lying on the sphere is $\frac{x\sqrt{2}}{2}$. Therefore, the square of the radius is $3 = (1 + x)^2 + \frac{x^2}{2}$, or $3x^2 + 4x - 4 = (3x - 2)(x + 2) = 0$, so $x = \frac{2}{3}$.

6. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

Solution: $\boxed{2/3}$

Let the triangle be denoted ABC , and suppose we fold parallel to BC . Let the distance from A to BC be h , and suppose we fold along a line at a distance of ch from A . We will assume that neither angle B nor C is obtuse, for the area of overlap will only be smaller if either is obtuse. If $c \leq \frac{1}{2}$, then A does not fold past the edge BC , so the overlap is a triangle similar to the original with height ch ; the area of the figure is then $1 - c^2 \geq \frac{3}{4}$. Suppose $c > \frac{1}{2}$, so that A does fold past BC . Then the overlap is a trapezoid formed by taking a triangle of height ch similar to the original and removing a triangle of height $(2c - 1)h$ similar to the original. The area of the resulting figure is thus $1 - c^2 + (2c - 1)^2 = 3c^2 - 4c + 2$. This is minimized when $c = \frac{2}{3}$, when the area is $\frac{2}{3} < \frac{3}{4}$; the minimum possible area is therefore $\frac{2}{3}$.

7. Let $ABCD$ be a tetrahedron such that edges AB , AC , and AD are mutually perpendicular. Let the areas of triangles ABC , ACD , and ADB be denoted by x , y , and z , respectively. In terms of x , y , and z , find the area of triangle BCD .

Solution: $\boxed{\sqrt{x^2 + y^2 + z^2}}$

Place A , B , C , and D at $(0, 0, 0)$, $(b, 0, 0)$, $(0, c, 0)$, and $(0, 0, d)$ in Cartesian coordinate space, with b , c , and d positive. Then the plane through B , C , and D is given by the equation $\frac{x}{b} + \frac{y}{c} + \frac{z}{d} = 1$. The distance from the origin to this plane is then

$$\frac{1}{\sqrt{\frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2}}} = \frac{bcd}{\sqrt{b^2c^2 + c^2d^2 + d^2b^2}} = \frac{bcd}{2\sqrt{x^2 + y^2 + z^2}}.$$

Then if the area of BCD is K , the volume of the tetrahedron is

$$\frac{bcd}{6} = \frac{bcdK}{6\sqrt{x^2 + y^2 + z^2}},$$

implying $K = \sqrt{x^2 + y^2 + z^2}$.

Alternative Solution: The area of BCD is also half the length of the cross product of the vectors $\overrightarrow{BC} = (0, -c, d)$ and $\overrightarrow{BD} = (-b, 0, d)$. This cross product is $(-cd, -db, -bc) = -2(y, z, x)$, which has length $2\sqrt{x^2 + y^2 + z^2}$. Thus the area of BCD is $\sqrt{x^2 + y^2 + z^2}$.

8. Let T be a triangle with side lengths 26, 51, and 73. Let S be the set of points inside T which do not lie within a distance of 5 of any side of T . Find the area of S .

Solution: 135/28

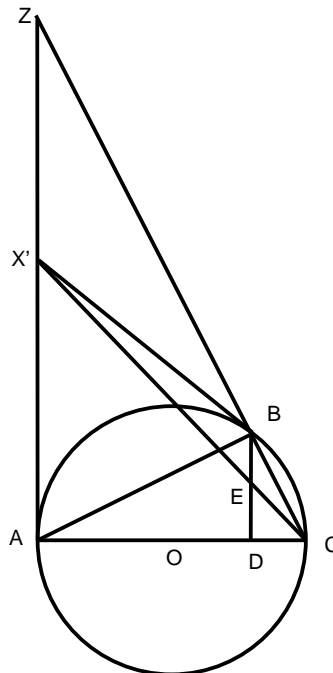
Note that the sides of S are parallel to the sides of T , so S is a triangle similar to T . The semiperimeter of T is $s = \frac{1}{2}(26 + 51 + 73) = 75$. By Heron's formula, the area of T is $\sqrt{75 \cdot 49 \cdot 24 \cdot 2} = 420$. If r is the inradius of T , then the area of T is rs , so $r = 420/75 = 28/5$. It follows that the inradius of S is $r - 5 = 3/5$, and the ratio of similitude between S and T is $3/28$. Therefore, the area of S is $420 \cdot (3/28)^2 = 135/28$.

9. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = 1/5$. Let B be the point on ω such that DB is perpendicular to AC , and let E be the midpoint of DB . The line tangent to ω at B intersects line CE at the point X . Compute AX .

Solution: 3

We first show that AX is perpendicular to AC . Let the tangent to ω at A intersect CB at Z and CE at X' . Since ZA is parallel to BD and $BE = ED$, $ZX' = X'A$. Therefore, X' is the midpoint of the hypotenuse of the right triangle ABZ , so it is also its circumcenter. Thus $X'A = X'B$, and since $X'A$ is tangent to ω and B lies on ω , we must have that $X'B$ is tangent to ω , so $X = X'$.

Let O be the center of ω . Then $OD = \frac{4}{5}$, so $BD = \frac{3}{5}$ and $DE = \frac{3}{10}$. Then $AX = DE \cdot \frac{AC}{DC} = \frac{3}{10} \cdot \frac{2}{1/5} = 3$.



10. Let AB be the diameter of a semicircle Γ . Two circles, ω_1 and ω_2 , externally tangent to each other and internally tangent to Γ , are tangent to the line AB at P and Q , respectively, and to semicircular arc AB at C and D , respectively, with $AP < AQ$. Suppose F lies on Γ such that $\angle FQB = \angle CQA$ and that $\angle ABF = 80^\circ$. Find $\angle PDQ$ in degrees.

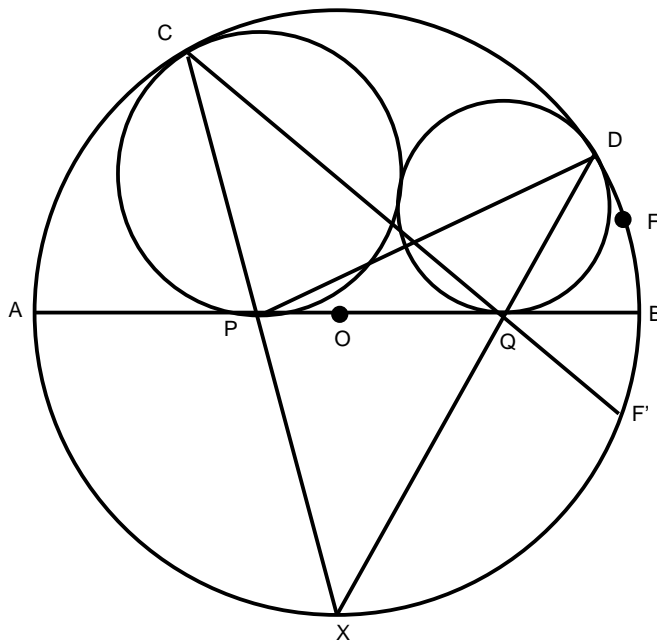
Solution: 35

Extend the semicircle centered at O to an entire circle ω , and let the reflection of F over AB be F' . Then CQF' is a straight line. Also, the homothety centered at C taking ω_1 into ω takes P to a point X on ω and AB to the parallel line tangent to ω at X . Therefore, X is the midpoint of semicircle AXB , and C, P , and X lie on a line. Similarly, D, Q , and X lie on a line. So,

$$45^\circ = \angle XCB = \angle PCB = \angle PCQ + \angle QCB = \angle PCQ + 10^\circ,$$

since $\angle QCB = \angle F'CB = \angle F'AB = \angle FAB = 90^\circ - \angle ABF = 10^\circ$. Thus $\angle PCQ = 35^\circ$. We will show that $\angle PCQ = \angle PDQ$ to get that $\angle PDQ = 35^\circ$.

Note that $\angle XPQ$ subtends the sum of arcs AC and BX , which is equal to arc XC . Therefore $\angle XPQ = \angle CDX$, so $CDQP$ is cyclic and $\angle PCQ = \angle PDQ$. The conclusion follows.



Harvard-MIT Mathematics Tournament

February 19, 2005

Guts Round — Solutions

1. Find the largest positive integer n such that $1 + 2 + 3 + \cdots + n^2$ is divisible by $1 + 2 + 3 + \cdots + n$.

Solution: $\boxed{1}$

The statement is

$$\frac{n(n+1)}{2} \mid \frac{n^2(n^2+1)}{2} \Leftrightarrow n+1 \mid n(n^2+1) = n^3+n.$$

But $n+1$ also divides $(n+1)(n^2-n+2) = n^3+n+2$, so $n+1$ must divide 2. Hence, n cannot be greater than 1. And $n=1$ clearly works, so that is the answer.

2. Let x, y , and z be positive real numbers such that $(x \cdot y) + z = (x + z) \cdot (y + z)$. What is the maximum possible value of xyz ?

Solution: $\boxed{1/27}$

The condition is equivalent to $z^2 + (x+y-1)z = 0$. Since z is positive, $z = 1 - x - y$, so $x + y + z = 1$. By the AM-GM inequality,

$$xyz \leq \left(\frac{x+y+z}{3} \right)^3 = \frac{1}{27},$$

with equality when $x = y = z = \frac{1}{3}$.

3. Find the sum

$$\frac{2^1}{4^1-1} + \frac{2^2}{4^2-1} + \frac{2^4}{4^4-1} + \frac{2^8}{4^8-1} + \cdots.$$

Solution: $\boxed{1}$

Notice that

$$\frac{2^{2^k}}{4^{2^k}-1} = \frac{2^{2^k}+1}{4^{2^k}-1} - \frac{1}{4^{2^k}-1} = \frac{1}{2^{2^k}-1} - \frac{1}{4^{2^k}-1} = \frac{1}{4^{2^{k-1}}-1} - \frac{1}{4^{2^k}-1}.$$

Therefore, the sum telescopes as

$$\left(\frac{1}{4^{2^{-1}}-1} - \frac{1}{4^{2^0}-1} \right) + \left(\frac{1}{4^{2^0}-1} - \frac{1}{4^{2^1}-1} \right) + \left(\frac{1}{4^{2^1}-1} - \frac{1}{4^{2^2}-1} \right) + \cdots$$

and evaluates to $1/(4^{2^{-1}}-1) = 1$.

4. What is the probability that in a randomly chosen arrangement of the numbers and letters in “HMMT2005,” one can read either “HMMT” or “2005” from left to right? (For example, in “5HM0M20T,” one can read “HMMT.”)

Solution: $\boxed{23/144}$

To read "HMMT," there are $\binom{8}{4}$ ways to place the letters, and $\frac{4!}{2}$ ways to place the numbers. Similarly, there are $\binom{8}{4}\frac{4!}{2}$ arrangements where one can read "2005." The number of arrangements in which one can read both is just $\binom{8}{4}$. The total number of arrangements is $\frac{8!}{4}$, thus the answer is

$$\frac{\binom{8}{4}\frac{4!}{2} + \binom{8}{4}\frac{4!}{2} - \binom{8}{4}}{\frac{8!}{4}} = \binom{8}{4}\frac{4}{8!} \cdot 23 = \frac{23}{144}.$$

5. For how many integers n between 1 and 2005, inclusive, is $2 \cdot 6 \cdot 10 \cdots (4n - 2)$ divisible by $n!$?

Solution: 2005

Note that

$$\begin{aligned} 2 \cdot 6 \cdot 10 \cdots (4n - 2) &= 2^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n - 1) \\ &= 2^n \cdot \frac{1 \cdot 2 \cdot 3 \cdots 2n}{2 \cdot 4 \cdot 6 \cdots 2n} \\ &= \frac{1 \cdot 2 \cdot 3 \cdots 2n}{1 \cdot 2 \cdot 3 \cdots n}, \end{aligned}$$

that is, it is just $(2n)!/n!$. Therefore, since $(2n)!/(n!)^2 = \binom{2n}{n}$ is always an integer, the answer is 2005.

6. Let $m \circ n = (m + n)/(mn + 4)$. Compute $((\cdots((2005 \circ 2004) \circ 2003) \circ \cdots \circ 1) \circ 0)$.

Solution: 1/12

Note that $m \circ 2 = (m + 2)/(2m + 4) = \frac{1}{2}$, so the quantity we wish to find is just $(\frac{1}{2} \circ 1) \circ 0 = \frac{1}{3} \circ 0 = 1/12$.

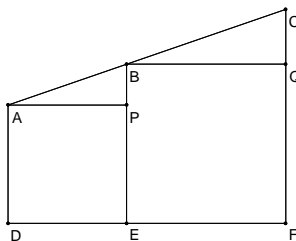
7. Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

Solution: $\sqrt{21}$

If A , B , and C are the tops of the heads of three successive people and D , E , and F are their respective feet, let P be the foot of the perpendicular from A to BE and let Q be the foot of the perpendicular from B to CF . Then, by equal angles, $\triangle ABP \sim \triangle BCQ$, so

$$\frac{CF}{BE} = \frac{CF}{BQ} = \frac{CQ}{BQ} + 1 = \frac{BP}{AP} + 1 = \frac{BE}{AP} = \frac{BE}{AD}.$$

Therefore the heights of successive people are in geometric progression. Hence, the heights of all five people are in geometric progression, so the middle height is $\sqrt{3 \cdot 7} = \sqrt{21}$ feet.



8. Let $ABCD$ be a convex quadrilateral inscribed in a circle with shortest side AB . The ratio $[BCD]/[ABD]$ is an integer (where $[XYZ]$ denotes the area of triangle XYZ .) If the lengths of AB , BC , CD , and DA are distinct integers no greater than 10, find the largest possible value of AB .

Solution: 5

Note that

$$\frac{[BCD]}{[ABD]} = \frac{\frac{1}{2}BC \cdot CD \cdot \sin C}{\frac{1}{2}DA \cdot AB \cdot \sin A} = \frac{BC \cdot CD}{DA \cdot AB}$$

since $\angle A$ and $\angle C$ are supplementary. If $AB \geq 6$, it is easy to check that no assignment of lengths to the four sides yields an integer ratio, but if $AB = 5$, we can let $BC = 10$, $CD = 9$, and $DA = 6$ for a ratio of 3. The maximum value for AB is therefore 5.

9. Farmer Bill's 1000 animals — ducks, cows, and rabbits — are standing in a circle. In order to feel safe, every duck must either be standing next to at least one cow or between two rabbits. If there are 600 ducks, what is the least number of cows there can be for this to be possible?

Solution: 201

Suppose Bill has r rabbits and c cows. At most $r - 1$ ducks can be between two rabbits: each rabbit can serve up to two such ducks, so at most $2r/2 = r$ ducks will each be served by two rabbits, but we cannot have equality, since this would require alternating between rabbits and ducks all the way around the circle, contradicting the fact that more than half the animals are ducks. Also, at most $2c$ ducks can each be adjacent to a cow. So we need $600 \leq r - 1 + 2c = (400 - c) - 1 + 2c$, giving $c \geq 201$. Conversely, an arrangement with 201 cows is possible:

$$\underbrace{RDRDR \cdots DR}_{199 R, 198 D} \underbrace{DCD DCD DCD \cdots DCD}_{201 C, 402 D}.$$

So 201 is the answer.

10. You are given a set of cards labeled from 1 to 100. You wish to make piles of three cards such that in any pile, the number on one of the cards is the product of the numbers on the other two cards. However, no card can be in more than one pile. What is the maximum number of piles you can form at once?

Solution: 8

Certainly, the two factors in any pile cannot both be at least 10, since then the product would be at least $10 \times 11 > 100$. Also, the number 1 can not appear in any pile, since then the other two cards in the pile would have to be the same. So each pile must use

one of the numbers $2, 3, \dots, 9$ as one of the factors, meaning we have at most 8 piles. Conversely, it is easy to construct a set of 8 such piles, for example:

$$\begin{aligned} &\{9, 11, 99\} \quad \{8, 12, 96\} \quad \{7, 13, 91\} \quad \{6, 14, 84\} \\ &\{5, 15, 75\} \quad \{4, 16, 64\} \quad \{3, 17, 51\} \quad \{2, 18, 36\} \end{aligned}$$

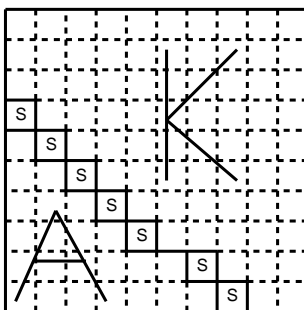
11. The Dingoberry Farm is a 10 mile by 10 mile square, broken up into 1 mile by 1 mile patches. Each patch is farmed either by Farmer Keith or by Farmer Ann. Whenever Ann farms a patch, she also farms all the patches due west of it and all the patches due south of it. Ann puts up a scarecrow on each of her patches that is adjacent to exactly two of Keith's patches (and nowhere else). If Ann farms a total of 30 patches, what is the largest number of scarecrows she could put up?

Solution: 7

Whenever Ann farms a patch P , she also farms all the patches due west of P and due south of P . So, the only way she can put a scarecrow on P is if Keith farms the patch immediately north of P and the patch immediately east of P , in which case Ann cannot farm any of the patches due north of P or due east of P . That is, Ann can only put a scarecrow on P if it is the easternmost patch she farms in its east-west row, and the northernmost in its north-south column. In particular, all of her scarecrow patches are in different rows and columns. Suppose that she puts up n scarecrows. The farthest south of these must be in the 10th row or above, so she farms at least 1 patch in that column; the second-farthest south must be in the 9th row above, so she farms at least 2 patches in that column; the third-farthest south must be in the 8th row or above, so she farms at least 3 patches in that column, and so forth, for a total of at least

$$1 + 2 + \dots + n = n(n + 1)/2$$

patches. If Ann farms a total of $30 < 8 \cdot 9/2$ patches, then we have $n < 8$. On the other hand, $n = 7$ scarecrows are possible, as shown:



12. Two vertices of a cube are given in space. The locus of points that could be a third vertex of the cube is the union of n circles. Find n .

Solution: 10

Let the distance between the two given vertices be 1. If the two given vertices are adjacent, then the other vertices lie on four circles, two of radius 1 and two of radius $\sqrt{2}$. If the two vertices are separated by a diagonal of a face of the cube, then the locus of possible vertices adjacent to both of them is a circle of radius $\frac{1}{2}$, the locus of

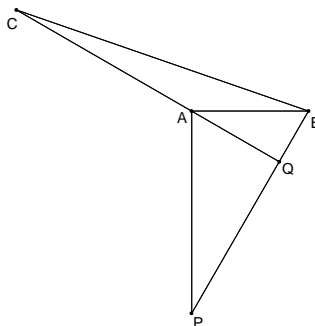
possible vertices adjacent to exactly one of them is two circles of radius $\frac{\sqrt{2}}{2}$, and the locus of possible vertices adjacent to neither of them is a circle of radius $\frac{\sqrt{3}}{2}$. If the two given vertices are separated by a long diagonal, then each of the other vertices lie on one of two circles of radius $\frac{\sqrt{2}}{3}$, for a total of 10 circles.

13. Triangle ABC has $AB = 1$, $BC = \sqrt{7}$, and $CA = \sqrt{3}$. Let ℓ_1 be the line through A perpendicular to AB , ℓ_2 the line through B perpendicular to AC , and P the point of intersection of ℓ_1 and ℓ_2 . Find PC .

Solution: 3

By the Law of Cosines, $\angle BAC = \cos^{-1} \frac{3+1-7}{2\sqrt{3}} = \cos^{-1}(-\frac{\sqrt{3}}{2}) = 150^\circ$. If we let Q be the intersection of ℓ_2 and AC , we notice that $\angle QBA = 90^\circ - \angle QAB = 90^\circ - 30^\circ = 60^\circ$. It follows that triangle ABP is a 30-60-90 triangle and thus $PB = 2$ and $PA = \sqrt{3}$. Finally, we have $\angle PAC = 360^\circ - (90^\circ + 150^\circ) = 120^\circ$, and

$$PC = (PA^2 + AC^2 - 2PA \cdot AC \cos 120^\circ)^{1/2} = (3 + 3 + 3)^{1/2} = 3.$$



14. Three noncollinear points and a line ℓ are given in the plane. Suppose no two of the points lie on a line parallel to ℓ (or ℓ itself). There are exactly n lines perpendicular to ℓ with the following property: the three circles with centers at the given points and tangent to the line all concur at some point. Find all possible values of n .

Solution: 1

The condition for the line is that each of the three points lies at an equal distance from the line as from some fixed point; in other words, the line is the directrix of a parabola containing the three points. Three noncollinear points in the coordinate plane determine a quadratic polynomial in x unless two of the points have the same x -coordinate. Therefore, given the direction of the directrix, three noncollinear points determine a parabola, unless two of the points lie on a line perpendicular to the directrix. This case is ruled out by the given condition, so the answer is 1.

15. Let S be the set of lattice points inside the circle $x^2 + y^2 = 11$. Let M be the greatest area of any triangle with vertices in S . How many triangles with vertices in S have area M ?

Solution: 16

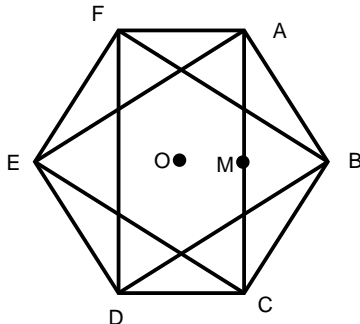
The boundary of the convex hull of S consists of points with (x, y) or $(y, x) = (0, \pm 3)$, $(\pm 1, \pm 3)$, and $(\pm 2, \pm 2)$. For any triangle T with vertices in S , we can increase its area by moving a vertex not on the boundary to some point on the boundary. Thus,

if T has area M , its vertices are all on the boundary of S . The next step is to see (either by inspection or by noting that T has area no larger than that of an equilateral triangle inscribed in a circle of radius $\sqrt{10}$, which has area less than 13) that $M = 12$. There are 16 triangles with area 12, all congruent to one of the following three: vertices $(2, 2)$, $(1, -3)$, and $(-3, 1)$; vertices $(3, -1)$, $(-3, -1)$, and $(1, 3)$; or vertices $(3, -1)$, $(-3, -1)$, and $(0, 3)$.

16. A regular octahedron has a side length of 1. What is the distance between two opposite faces?

Solution: $\boxed{\sqrt{6}/3}$

Imagine orienting the octahedron so that the two opposite faces are horizontal. Project onto a horizontal plane; these two faces are congruent equilateral triangles which (when projected) have the same center and opposite orientations. Hence, the vertices of the octahedron project to the vertices of a regular hexagon $ABCDEF$.



Let O be the center of the hexagon and M the midpoint of AC . Now ABM is a 30-60-90 triangle, so $AB = AM/(\sqrt{3}/2) = (1/2)/(\sqrt{3}/2) = \sqrt{3}/3$. If we let d denote the desired vertical distance between the opposite faces (which project to ACE and BDF), then by the Pythagorean Theorem, $AB^2 + d^2 = 1^2$, so $d = \sqrt{1 - AB^2} = \sqrt{6}/3$.

17. Compute

$$2\sqrt{2\sqrt{2^3\sqrt{2^4\sqrt{2^5\sqrt{2\cdots}}}}}$$

Solution: $\boxed{2^{e-1}}$

Taking the base 2 logarithm of the expression gives

$$1 + \frac{1}{2} \left(1 + \frac{1}{3} \left(1 + \frac{1}{4} (1 + \cdots) \right) \right) = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots = e - 1.$$

Therefore the expression is just 2^{e-1} .

18. If a , b , and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a , b , and c ?

Solution: $\boxed{1/4}$

Let d be a fourth random variable, also chosen uniformly from $[0, 1]$. For fixed a , b , and c , the probability that $d < \min\{a, b, c\}$ is evidently equal to $\min\{a, b, c\}$. Hence,

if we average over all choices of a, b, c , the average value of $\min\{a, b, c\}$ is equal to the probability that, when a, b, c , and d are independently randomly chosen, $d < \min\{a, b, c\}$, i.e., that d is the smallest of the four variables. On the other hand, by symmetry, the probability that d is the smallest of the four is simply equal to $1/4$, so that is our answer.

19. Regular tetrahedron $ABCD$ is projected onto a plane sending A, B, C , and D to A', B', C' , and D' respectively. Suppose $A'B'C'D'$ is a convex quadrilateral with $A'B' = A'D'$ and $C'B' = C'D'$, and suppose that the area of $A'B'C'D' = 4$. Given these conditions, the set of possible lengths of AB consists of all real numbers in the interval $[a, b)$. Compute b .

Solution: $\boxed{2\sqrt[4]{6}}$

The value of b occurs when the quadrilateral $A'B'C'D'$ degenerates to an isosceles triangle. This occurs when the altitude from A to BCD is parallel to the plane. Let $s = AB$. Then the altitude from A intersects the center E of face BCD . Since $EB = \frac{s}{\sqrt{3}}$, it follows that $A'C' = AE = \sqrt{s^2 - \frac{s^2}{3}} = \frac{s\sqrt{6}}{3}$. Then since BD is parallel to the plane, $B'D' = s$. Then the area of $A'B'C'D'$ is $4 = \frac{1}{2} \cdot \frac{s^2\sqrt{6}}{3}$, implying $s^2 = 4\sqrt{6}$, or $s = 2\sqrt[4]{6}$.

20. If n is a positive integer, let $s(n)$ denote the sum of the digits of n . We say that n is *zesty* if there exist positive integers x and y greater than 1 such that $xy = n$ and $s(x)s(y) = s(n)$. How many zesty two-digit numbers are there?

Solution: $\boxed{34}$

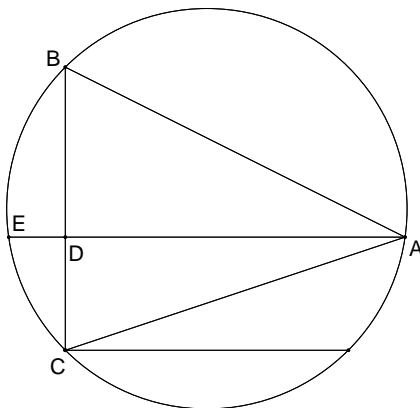
Let n be a zesty two-digit number, and let x and y be as in the problem statement. Clearly if both x and y are one-digit numbers, then $s(x)s(y) = n \neq s(n)$. Thus either x is a two-digit number or y is. Assume without loss of generality that it is x . If $x = 10a + b$, $1 \leq a \leq 9$ and $0 \leq b \leq 9$, then $n = 10ay + by$. If both ay and by are less than 10, then $s(n) = ay + by$, but if either is at least 10, then $s(n) < ay + by$. It follows that the two digits of n share a common factor greater than 1, namely y . It is now easy to count the zesty two-digit numbers by first digit starting with 2; there are a total of $5 + 4 + 5 + 2 + 7 + 2 + 5 + 4 = 34$.

21. In triangle ABC with altitude AD , $\angle BAC = 45^\circ$, $DB = 3$, and $CD = 2$. Find the area of triangle ABC .

Solution: $\boxed{15}$

Suppose first that D lies between B and C . Let ABC be inscribed in circle ω , and extend AD to intersect ω again at E . Note that A subtends a quarter of the circle, so in particular, the chord through C perpendicular to BC and parallel to AD has length $BC = 5$. Therefore, $AD = 5 + DE$. By power of a point, $6 = BD \cdot DC = AD \cdot DE = AD^2 - 5AD$, implying $AD = 6$, so the area of ABC is $\frac{1}{2}BC \cdot AD = 15$.

If D does not lie between B and C , then $BC = 1$, so A lies on a circle of radius $\sqrt{2}/2$ through B and C . But then it is easy to check that the perpendicular to BC through D cannot intersect the circle, a contradiction.



22. Find

$$\{\ln(1 + e)\} + \{\ln(1 + e^2)\} + \{\ln(1 + e^4)\} + \{\ln(1 + e^8)\} + \cdots,$$

where $\{x\} = x - [x]$ denotes the fractional part of x .

Solution: $\boxed{1 - \ln(e - 1)}$

Since $\ln(1 + e^{2^k})$ is just larger than 2^k , its fractional part is $\ln(1 + e^{2^k}) - \ln e^{2^k} = \ln(1 + e^{-2^k})$. But now notice that

$$\prod_{k=0}^n (1 + x^{2^k}) = 1 + x + x^2 + \cdots + x^{2^{n+1}-1}.$$

(This is easily proven by induction or by noting that every nonnegative integer less than 2^{n+1} has a unique $(n+1)$ -bit binary expansion.) If $|x| < 1$, this product converges to $\frac{1}{1-x}$ as n goes to infinity. Therefore,

$$\sum_{k=0}^{\infty} \ln(1 + e^{-2^k}) = \ln \prod_{k=0}^{\infty} (1 + (e^{-1})^{2^k}) = \ln \frac{1}{1 - e^{-1}} = \ln \frac{e}{e - 1} = 1 - \ln(e - 1).$$

23. The sides of a regular hexagon are trisected, resulting in 18 points, including vertices. These points, starting with a vertex, are numbered clockwise as A_1, A_2, \dots, A_{18} . The line segment $A_k A_{k+4}$ is drawn for $k = 1, 4, 7, 10, 13, 16$, where indices are taken modulo 18. These segments define a region containing the center of the hexagon. Find the ratio of the area of this region to the area of the large hexagon.

Solution: $\boxed{9/13}$

Let us assume all sides are of side length 3. Consider the triangle $A_1 A_4 A_5$. Let P be the point of intersection of $A_1 A_5$ with $A_4 A_8$. This is a vertex of the inner hexagon. Then $\angle A_4 A_1 A_5 = \angle A_5 A_4 P$, by symmetry. It follows that $A_1 A_4 A_5 \sim A_4 P A_5$. Also, $\angle A_1 A_4 A_5 = 120^\circ$, so by the Law of Cosines $A_1 A_5 = \sqrt{13}$. It follows that $PA_5 = (A_4 A_5) \cdot (A_4 A_5) / (A_1 A_5) = 1/\sqrt{13}$. Let Q be the intersection of $A_1 A_5$ and $A_{16} A_2$. By similar reasoning, $A_1 Q = 3/\sqrt{13}$, so $PQ = A_1 A_5 - A_1 Q - PA_5 = 9/\sqrt{13}$. By symmetry, the inner region is a regular hexagon with side length $9/\sqrt{13}$. Hence the ratio of the area of the smaller to larger hexagon is $(3/\sqrt{13})^2 = 9/13$.

24. In the base 10 arithmetic problem $HMMT + GUTS = ROUND$, each distinct letter represents a different digit, and leading zeroes are not allowed. What is the maximum possible value of $ROUND$?

Solution: $\boxed{16352}$

Clearly $R = 1$, and from the hundreds column, $M = 0$ or 9 . Since $H + G = 9 + O$ or $10 + O$, it is easy to see that O can be at most 7 , in which case H and G must be 8 and 9 , so $M = 0$. But because of the tens column, we must have $S + T \geq 10$, and in fact since D cannot be 0 or 1 , $S + T \geq 12$, which is impossible given the remaining choices. Therefore, O is at most 6 .

Suppose $O = 6$ and $M = 9$. Then we must have H and G be 7 and 8 . With the remaining digits $0, 2, 3, 4,$ and 5 , we must have in the ones column that T and S are 2 and 3 , which leaves no possibility for N . If instead $M = 0$, then H and G are 7 and 9 . Since again $S + T \geq 12$ and $N = T + 1$, the only possibility is $S = 8, T = 4,$ and $N = 5$, giving $ROUND = 16352 = 7004 + 9348 = 9004 + 7348$.

25. An ant starts at one vertex of a tetrahedron. Each minute it walks along a random edge to an adjacent vertex. What is the probability that after one hour the ant winds up at the same vertex it started at?

Solution: $\boxed{(3^{59} + 1)/(4 \cdot 3^{59})}$

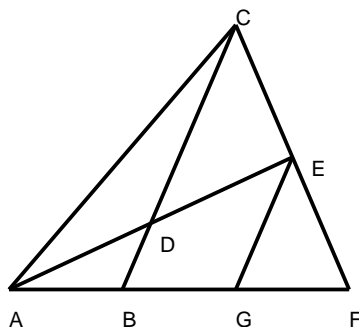
Let p_n be the probability that the ant is at the original vertex after n minutes; then $p_0 = 1$. The chance that the ant is at each of the other three vertices after n minutes is $\frac{1}{3}(1 - p_n)$. Since the ant can only walk to the original vertex from one of the three others, and at each there is a $\frac{1}{3}$ probability of doing so, we have that $p_{n+1} = \frac{1}{3}(1 - p_n)$. Let $q_n = p_n - \frac{1}{4}$. Substituting this into the recurrence, we find that $q_{n+1} = \frac{1}{4} + \frac{1}{3}(-q_n - \frac{3}{4}) = -\frac{1}{3}q_n$. Since $q_0 = \frac{3}{4}$, $q_n = \frac{3}{4} \cdot (-\frac{1}{3})^n$. In particular, this implies that

$$p_{60} = \frac{1}{4} + q_{60} = \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{3^{60}} = \frac{3^{59} + 1}{4 \cdot 3^{59}}.$$

26. In triangle ABC , $AC = 3AB$. Let AD bisect angle A with D lying on BC , and let E be the foot of the perpendicular from C to AD . Find $[ABD]/[CDE]$. (Here, $[XYZ]$ denotes the area of triangle XYZ).

Solution: $\boxed{1/3}$

By the Angle Bisector Theorem, $DC/DB = AC/AB = 3$. We will show that $AD = DE$. Let CE intersect AB at F . Then since AE bisects angle A , $AF = AC = 3AB$, and $EF = EC$. Let G be the midpoint of BF . Then $BG = GF$, so $GE \parallel BC$. But then since B is the midpoint of AG , D must be the midpoint of AE , as desired. Then $[ABD]/[CDE] = (AD \cdot BD)/(ED \cdot CD) = 1/3$.



27. In a chess-playing club, some of the players take lessons from other players. It is possible (but not necessary) for two players both to take lessons from each other. It so happens that for any three distinct members of the club, A , B , and C , exactly one of the following three statements is true: A takes lessons from B ; B takes lessons from C ; C takes lessons from A . What is the largest number of players there can be?

Solution: 4

If P , Q , R , S , and T are any five distinct players, then consider all pairs $A, B \in \{P, Q, R, S, T\}$ such that A takes lessons from B . Each pair contributes to exactly three triples (A, B, C) (one for each of the choices of C distinct from A and B); three triples (C, A, B) ; and three triples (B, C, A) . On the other hand, there are $5 \times 4 \times 3 = 60$ ordered triples of distinct players among these five, and each includes exactly one of our lesson-taking pairs. That means that there are $60/9$ such pairs. But this number isn't an integer, so there cannot be five distinct people in the club.

On the other hand, there can be four people, P , Q , R , and S : let P and Q both take lessons from each other, and let R and S both take lessons from each other; it is easy to check that this meets the conditions. Thus the maximum number of players is 4.

28. There are three pairs of real numbers (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) that satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$. Compute $\left(1 - \frac{x_1}{y_1}\right) \left(1 - \frac{x_2}{y_2}\right) \left(1 - \frac{x_3}{y_3}\right)$.

Solution: 1/1002

By the given, $2004(x^3 - 3xy^2) - 2005(y^3 - 3x^2y) = 0$. Dividing both sides by y^3 and setting $t = \frac{x}{y}$ yields $2004(t^3 - 3t) - 2005(1 - 3t^2) = 0$. A quick check shows that this cubic has three real roots. Since the three roots are precisely $\frac{x_1}{y_1}$, $\frac{x_2}{y_2}$, and $\frac{x_3}{y_3}$, we must have $2004(t^3 - 3t) - 2005(1 - 3t^2) = 2004 \left(t - \frac{x_1}{y_1}\right) \left(t - \frac{x_2}{y_2}\right) \left(t - \frac{x_3}{y_3}\right)$. Therefore,

$$\left(1 - \frac{x_1}{y_1}\right) \left(1 - \frac{x_2}{y_2}\right) \left(1 - \frac{x_3}{y_3}\right) = \frac{2004(1^3 - 3(1)) - 2005(1 - 3(1)^2)}{2004} = \frac{1}{1002}.$$

29. Let $n > 0$ be an integer. Each face of a regular tetrahedron is painted in one of n colors (the faces are not necessarily painted different colors.) Suppose there are n^3 possible colorings, where rotations, but not reflections, of the same coloring are considered the same. Find all possible values of n .

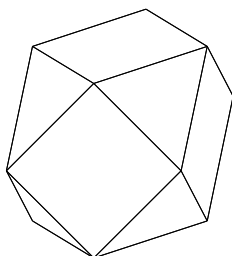
Solution: 1, 11

We count the possible number of colorings. If four colors are used, there are two different colorings that are mirror images of each other, for a total of $2\binom{n}{4}$ colorings. If three colors are used, we choose one color to use twice (which determines the coloring), for a total of $3\binom{n}{3}$ colorings. If two colors are used, we can either choose one of those colors and color three faces with it, or we can color two faces each color, for a total of $3\binom{n}{2}$ colorings. Finally, we can also use only one color, for $\binom{n}{1}$ colorings. This gives a total of

$$2\binom{n}{4} + 3\binom{n}{3} + 3\binom{n}{2} + \binom{n}{1} = \frac{1}{12}n^2(n^2 + 11)$$

colorings. Setting this equal to n^3 , we get the equation $n^2(n^2 + 11) = 12n^3$, or equivalently $n^2(n - 1)(n - 11) = 0$, giving the answers 1 and 11.

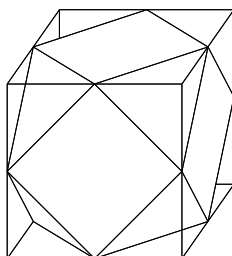
30. A cuboctahedron is a polyhedron whose faces are squares and equilateral triangles such that two squares and two triangles alternate around each vertex, as shown.



What is the volume of a cuboctahedron of side length 1?

Solution: $\boxed{5\sqrt{2}/3}$

We can construct a cube such that the vertices of the cuboctahedron are the midpoints of the edges of the cube.

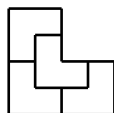


Let s be the side length of this cube. Now, the cuboctahedron is obtained from the cube by cutting a tetrahedron from each corner. Each such tetrahedron has a base in the form of an isosceles right triangle of area $(s/2)^2/2$ and height $s/2$ for a volume of $(s/2)^3/6$. The total volume of the cuboctahedron is therefore

$$s^3 - 8 \cdot (s/2)^3/6 = 5s^3/6.$$

Now, the side of the cuboctahedron is the hypotenuse of an isosceles right triangle of leg $s/2$; thus $1 = (s/2)\sqrt{2}$, giving $s = \sqrt{2}$, so the volume of the cuboctahedron is $5\sqrt{2}/3$.

31. The L shape made by adjoining three congruent squares can be subdivided into four smaller L shapes.



Each of these can in turn be subdivided, and so forth. If we perform 2005 successive subdivisions, how many of the 4^{2005} L's left at the end will be in the same orientation as the original one?

Solution: $\boxed{4^{2004} + 2^{2004}}$

After n successive subdivisions, let a_n be the number of small L's in the same orientation as the original one; let b_n be the number of small L's that have this orientation rotated counterclockwise 90° ; let c_n be the number of small L's that are rotated 180° ; and let d_n be the number of small L's that are rotated 270° . When an L is subdivided, it produces two smaller L's of the same orientation, one of each of the neighboring orientations, and none of the opposite orientation. Therefore,

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (d_n + 2a_n + b_n, a_n + 2b_n + c_n, b_n + 2c_n + d_n, c_n + 2d_n + a_n).$$

It is now straightforward to show by induction that

$$(a_n, b_n, c_n, d_n) = (4^{n-1} + 2^{n-1}, 4^{n-1}, 4^{n-1} - 2^{n-1}, 4^{n-1})$$

for each $n \geq 1$. In particular, our desired answer is $a_{2005} = 4^{2004} + 2^{2004}$.

32. Let $a_1 = 3$, and for $n \geq 1$, let $a_{n+1} = (n+1)a_n - n$. Find the smallest $m \geq 2005$ such that $a_{m+1} - 1 \mid a_m^2 - 1$.

Solution: $\boxed{2010}$

We will show that $a_n = 2 \cdot n! + 1$ by induction. Indeed, the claim is obvious for $n = 1$, and $(n+1)(2 \cdot n! + 1) - n = 2 \cdot (n+1)! + 1$. Then we wish to find $m \geq 2005$ such that $2(m+1)! \mid 4(m!)^2 + 4m!$, or dividing by $2 \cdot m!$, we want $m+1 \mid 2(m! + 1)$. Suppose $m+1$ is composite. Then it has a proper divisor $d > 2$, and since $d \mid m!$, we must have $d \mid 2$, which is impossible. Therefore, $m+1$ must be prime, and if this is the case, then $m+1 \mid m! + 1$ by Wilson's Theorem. Therefore, since the smallest prime greater than 2005 is 2011, the smallest possible value of m is 2010.

33. Triangle ABC has incircle ω which touches AB at C_1 , BC at A_1 , and CA at B_1 . Let A_2 be the reflection of A_1 over the midpoint of BC , and define B_2 and C_2 similarly. Let A_3 be the intersection of AA_2 with ω that is closer to A , and define B_3 and C_3 similarly. If $AB = 9$, $BC = 10$, and $CA = 13$, find $[A_3B_3C_3]/[ABC]$. (Here $[XYZ]$ denotes the area of triangle XYZ .)

Solution: $\boxed{14/65}$

Notice that A_2 is the point of tangency of the excircle opposite A to BC . Therefore, by considering the homothety centered at A taking the excircle to the incircle, we notice that A_3 is the intersection of ω and the tangent line parallel to BC . It follows that

$A_1B_1C_1$ is congruent to $A_3B_3C_3$ by reflecting through the center of ω . We therefore need only find $[A_1B_1C_1]/[ABC]$. Since

$$\frac{[A_1B_1C_1]}{[ABC]} = \frac{A_1B \cdot BC_1}{AB \cdot BC} = \frac{((9 + 10 - 13)/2)^2}{9 \cdot 10} = \frac{1}{10},$$

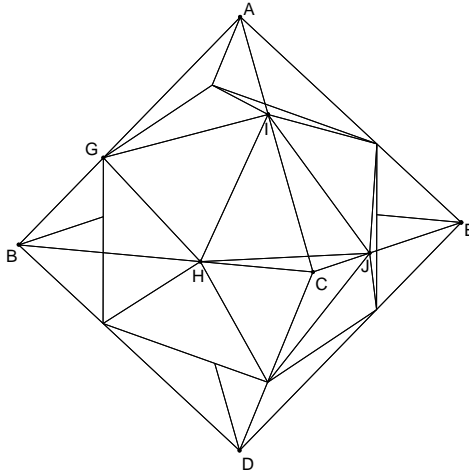
and likewise $[A_1B_1C]/[ABC] = 49/130$ and $[AB_1C_1]/[ABC] = 4/13$, we get that

$$\frac{[A_3B_3C_3]}{[ABC]} = 1 - \frac{1}{10} - \frac{49}{130} - \frac{4}{13} = \frac{14}{65}.$$

34. A regular octahedron $ABCDEF$ is given such that AD , BE , and CF are perpendicular. Let G , H , and I lie on edges AB , BC , and CA respectively such that $\frac{AG}{GB} = \frac{BH}{HC} = \frac{CI}{IA} = \rho$. For some choice of $\rho > 1$, GH , HI , and IG are three edges of a regular icosahedron, eight of whose faces are inscribed in the faces of $ABCDEF$. Find ρ .

Solution: $\boxed{(1 + \sqrt{5})/2}$

Let J lie on edge CE such that $\frac{EJ}{JC} = \rho$. Then we must have that HIJ is another face of the icosahedron, so in particular, $HI = HJ$. But since BC and CE are perpendicular, $HJ = HC\sqrt{2}$. By the Law of Cosines, $HI^2 = HC^2 + CI^2 - 2HC \cdot CI \cos 60^\circ = HC^2(1 + \rho^2 - \rho)$. Therefore, $2 = 1 + \rho^2 - \rho$, or $\rho^2 - \rho - 1 = 0$, giving $\rho = \frac{1+\sqrt{5}}{2}$.



35. Let $p = 2^{24036583} - 1$, the largest prime currently known. For how many positive integers c do the quadratics $\pm x^2 \pm px \pm c$ all have rational roots?

Solution: $\boxed{0}$

This is equivalent to both discriminants $p^2 \pm 4c$ being squares. In other words, p^2 must be the average of two squares a^2 and b^2 . Note that a and b must have the same parity, and that $(\frac{a+b}{2})^2 + (\frac{a-b}{2})^2 = \frac{a^2+b^2}{2} = p^2$. Therefore, p must be the hypotenuse in a Pythagorean triple. Such triples are parametrized by $k(m^2 - n^2, 2mn, m^2 + n^2)$. But $p \equiv 3 \pmod{4}$ and is therefore not the sum of two squares. This implies that p is not the hypotenuse of any Pythagorean triple, so the answer is 0.

36. One hundred people are in line to see a movie. Each person wants to sit in the front row, which contains one hundred seats, and each has a favorite seat, chosen randomly and independently. They enter the row one at a time from the far right. As they walk, if they reach their favorite seat, they sit, but to avoid stepping over people, if they encounter a person already seated, they sit to that person's right. If the seat furthest to the right is already taken, they sit in a different row. What is the most likely number of people that will get to sit in the first row?

Solution: 10

Let $S(i)$ be the favorite seat of the i th person, counting from the right. Let $P(n)$ be the probability that *at least* n people get to sit. At least n people sit if and only if $S(1) \geq n$, $S(2) \geq n - 1$, \dots , $S(n) \geq 1$. This has probability:

$$P(n) = \frac{100 - (n - 1)}{100} \cdot \frac{100 - (n - 2)}{100} \cdots \frac{100}{100} = \frac{100!}{(100 - n)! \cdot 100^n}.$$

The probability, $Q(n)$, that exactly n people sit is

$$P(n) - P(n + 1) = \frac{100!}{(100 - n)! \cdot 100^n} - \frac{100!}{(99 - n)! \cdot 100^{n+1}} = \frac{100! \cdot n}{(100 - n)! \cdot 100^{n+1}}.$$

Now,

$$\frac{Q(n)}{Q(n - 1)} = \frac{100! \cdot n}{(100 - n)! \cdot 100^{n+1}} \cdot \frac{(101 - n)! \cdot 100^n}{100! \cdot (n - 1)} = \frac{n(101 - n)}{100(n - 1)} = \frac{101n - n^2}{100n - 100},$$

which is greater than 1 exactly when $n^2 - n - 100 < 0$, that is, for $n \leq 10$. Therefore, the maximum value of $Q(n)$ occurs for $n = 10$.

37. Let $a_1, a_2, \dots, a_{2005}$ be real numbers such that

$$\begin{array}{cccccccc} a_1 \cdot 1 & + & a_2 \cdot 2 & + & a_3 \cdot 3 & + & \cdots & + & a_{2005} \cdot 2005 & = & 0 \\ a_1 \cdot 1^2 & + & a_2 \cdot 2^2 & + & a_3 \cdot 3^2 & + & \cdots & + & a_{2005} \cdot 2005^2 & = & 0 \\ a_1 \cdot 1^3 & + & a_2 \cdot 2^3 & + & a_3 \cdot 3^3 & + & \cdots & + & a_{2005} \cdot 2005^3 & = & 0 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_1 \cdot 1^{2004} & + & a_2 \cdot 2^{2004} & + & a_3 \cdot 3^{2004} & + & \cdots & + & a_{2005} \cdot 2005^{2004} & = & 0 \end{array}$$

and

$$a_1 \cdot 1^{2005} + a_2 \cdot 2^{2005} + a_3 \cdot 3^{2005} + \cdots + a_{2005} \cdot 2005^{2005} = 1.$$

What is the value of a_1 ?

Solution: 1/2004!

The polynomial $p(x) = x(x - 2)(x - 3) \cdots (x - 2005)/2004!$ has zero constant term, has the numbers $2, 3, \dots, 2005$ as roots, and satisfies $p(1) = 1$. Multiplying the n th equation by the coefficient of x^n in the polynomial $p(x)$ and summing over all n gives

$$a_1 p(1) + a_2 p(2) + a_3 p(3) + \cdots + a_{2005} p(2005) = 1/2004!$$

(since the leading coefficient is $1/2004!$). The left side just reduces to a_1 , so $1/2004!$ is the answer.

38. In how many ways can the set of ordered pairs of integers be colored red and blue such that for all a and b , the points (a, b) , $(-1 - b, a + 1)$, and $(1 - b, a - 1)$ are all the same color?

Solution: 16

Let φ_1 and φ_2 be 90° counterclockwise rotations about $(-1, 0)$ and $(1, 0)$, respectively. Then $\varphi_1(a, b) = (-1 - b, a + 1)$, and $\varphi_2(a, b) = (1 - b, a - 1)$. Therefore, the possible colorings are precisely those preserved under these rotations. Since $\varphi_1(1, 0) = (-1, 2)$, the colorings must also be preserved under 90° rotations about $(-1, 2)$. Similarly, one can show that they must be preserved under rotations about any point (x, y) , where x is odd and y is even. Decompose the lattice points as follows:

$$\begin{aligned} L_1 &= \{(x, y) \mid x + y \equiv 0 \pmod{2}\} \\ L_2 &= \{(x, y) \mid x \equiv y - 1 \equiv 0 \pmod{2}\} \\ L_3 &= \{(x, y) \mid x + y - 1 \equiv y - x + 1 \equiv 0 \pmod{4}\} \\ L_4 &= \{(x, y) \mid x + y + 1 \equiv y - x - 1 \equiv 0 \pmod{4}\} \end{aligned}$$

Within any of these sublattices, any point can be brought to any other through appropriate rotations, but no point can be brought to any point in a different sublattice. It follows that every sublattice must be colored in one color, but that different sublattices can be colored differently. Since each of these sublattices can be colored in one of two colors, there are $2^4 = 16$ possible colorings.

$$\begin{array}{cccc|cccc} 1 & 2 & 1 & & 2 & 1 & 2 & 1 \\ 4 & 1 & 3 & & 1 & 4 & 1 & 3 \\ 1 & 2 & 1 & & 2 & 1 & 2 & 1 \\ \hline 3 & 1 & 4 & & 1 & 3 & 1 & 4 \\ \hline 1 & 2 & 1 & & 2 & 1 & 2 & 1 \\ 4 & 1 & 3 & & 1 & 4 & 1 & 3 \\ 1 & 2 & 1 & & 2 & 1 & 2 & 1 \end{array}$$

39. How many regions of the plane are bounded by the graph of

$$x^6 - x^5 + 3x^4y^2 + 10x^3y^2 + 3x^2y^4 - 5xy^4 + y^6 = 0?$$

Solution: 5

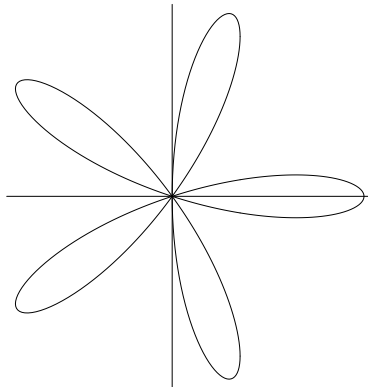
The left-hand side decomposes as

$$(x^6 + 3x^4y^2 + 3x^2y^4 + y^6) - (x^5 - 10x^3y^2 + 5xy^4) = (x^2 + y^2)^3 - (x^5 - 10x^3y^2 + 5xy^4).$$

Now, note that

$$(x + iy)^5 = x^5 + 5ix^4y - 10x^3y^2 - 10ix^2y^3 + 5xy^4 + iy^5,$$

so that our function is just $(x^2 + y^2)^3 - \Re((x + iy)^5)$. Switching to polar coordinates, this is $r^6 - \Re(r^5(\cos \theta + i \sin \theta)^5) = r^6 - r^5 \cos 5\theta$ by de Moivre's rule. The graph of our function is then the graph of $r^6 - r^5 \cos 5\theta = 0$, or, more suitably, of $r = \cos 5\theta$. This is a five-petal rose, so the answer is 5.



40. In a town of n people, a governing council is elected as follows: each person casts one vote for some person in the town, and anyone that receives at least five votes is elected to council. Let $c(n)$ denote the average number of people elected to council if everyone votes randomly. Find $\lim_{n \rightarrow \infty} c(n)/n$.

Solution: $\boxed{1 - 65/24e}$

Let $c_k(n)$ denote the expected number of people that will receive exactly k votes. We will show that $\lim_{n \rightarrow \infty} c_k(n)/n = 1/(e \cdot k!)$. The probability that any given person receives exactly k votes, which is the same as the average proportion of people that receive exactly k votes, is

$$\binom{n}{k} \cdot \left(\frac{1}{n}\right)^k \cdot \left(\frac{n-1}{n}\right)^{n-k} = \left(\frac{n-1}{n}\right)^n \cdot \frac{n(n-1) \cdots (n-k+1)}{k! \cdot (n-1)^k}.$$

Taking the limit as $n \rightarrow \infty$ and noting that $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}$ gives that the limit is $1/(e \cdot k!)$, as desired. Therefore, the limit of the average proportion of the town that receives at least five votes is

$$1 - \frac{1}{e} \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \right) = 1 - \frac{65}{24e}.$$

41. There are 42 stepping stones in a pond, arranged along a circle. You are standing on one of the stones. You would like to jump among the stones so that you move counterclockwise by either 1 stone or 7 stones at each jump. Moreover, you would like to do this in such a way that you visit each stone (except for the starting spot) exactly once before returning to your initial stone for the first time. In how many ways can you do this?

Solution: $\boxed{63}$

Number the stones $0, 1, \dots, 41$, treating the numbers as values modulo 42, and let r_n be the length of your jump from stone n . If you jump from stone n to $n+7$, then you cannot jump from stone $n+6$ to $n+7$ and so must jump from $n+6$ to $n+13$. That is, if $r_n = 7$, then $r_{n+6} = 7$ also. It follows that the 7 values $r_n, r_{n+6}, r_{n+12}, \dots, r_{n+36}$ are all equal: if one of them is 7, then by the preceding argument applied repeatedly, all of them must be 7, and otherwise all of them are 1. Now, for $n = 0, 1, 2, \dots, 42$, let s_n be the stone you are on after n jumps. Then $s_{n+1} = s_n + r_{s_n}$, and we have $s_{n+1} = s_n + r_{s_n} \equiv s_n + 1 \pmod{6}$. By induction, $s_{n+i} \equiv s_n + i \pmod{6}$; in particular

$s_{n+6} \equiv s_n$, so $r_{s_{n+6}} = r_{s_n}$. That is, the sequence of jump lengths is periodic with period 6 and so is uniquely determined by the first 6 jumps. So this gives us at most $2^6 = 64$ possible sequences of jumps $r_{s_0}, r_{s_1}, \dots, r_{s_{41}}$.

Now, the condition that you visit each stone exactly once before returning to the original stone just means that s_0, s_1, \dots, s_{41} are distinct and $s_{42} = s_0$. If all jumps are length 7, then $s_6 = s_0$, so this cannot happen. On the other hand, if the jumps are not all of length 7, then we claim s_0, \dots, s_{41} are indeed all distinct. Indeed, suppose $s_i = s_j$ for some $0 \leq i < j < 42$. Since $s_j \equiv s_i + (j - i) \pmod{6}$, we have $j \equiv i \pmod{6}$, so $j - i = 6k$ for some k . Moreover, since the sequence of jump lengths has period 6, we have

$$s_{i+6} - s_i = s_{i+12} - s_{i+6} = \dots = s_{i+6k} - s_{i+6(k-1)}.$$

Calling this common value l , we have $kl \equiv 0 \pmod{42}$. But l is divisible by 6, and $j - i < 42 \Rightarrow k < 7$ means that k is not divisible by 7, so l must be. So l , the sum of six successive jump lengths, is divisible by 42. Hence the jumps must all be of length 7, as claimed.

This shows that, for the $64 - 1 = 63$ sequences of jumps that have period 6 and are not all of length 7, you do indeed reach every stone once before returning to the starting point.

42. In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid of boxes such that every row and column contains two balls of one color and one ball of the other color? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

Solution: 5184

In each row or column, exactly one box is left empty. There are $4! = 24$ ways to choose the empty spots. Once that has been done, there are 6 ways to choose which two rows have 2 purple balls each. Now, assume without loss of generality that boxes $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$ are the empty ones, and that rows 1 and 2 have two purple balls each. Let A , B , C , and D denote the 2×2 squares in the top left, top right, bottom left, and bottom right corners, respectively (so A is formed by the first two rows and first two columns, etc.). Let a , b , c , and d denote the number of purple balls in A , B , C , and D , respectively. Then $0 \leq a, d \leq 2$, $a + b = 4$, and $b + d \leq 4$, so $a \geq d$.

Now suppose we are given the numbers a and d , satisfying $0 \leq d \leq a \leq 2$. Fortunately, the numbers of ways to color the balls in A , B , C , and D are independent of each other. For example, given $a = 1$ and $d = 0$, there are 2 ways to color A and 1 way to color D and, no matter how the coloring of A is done, there are always 2 ways to color B and 3 ways to color C . The numbers of ways to choose the colors of all the balls is as follows:

$a \backslash d$	0	1	2
0	$1 \cdot (1 \cdot 2) \cdot 1 = 2$	0	0
1	$2 \cdot (2 \cdot 3) \cdot 1 = 12$	$2 \cdot (1 \cdot 1) \cdot 2 = 4$	0
2	$1 \cdot (2 \cdot 2) \cdot 1 = 4$	$1 \cdot (3 \cdot 2) \cdot 2 = 12$	$1 \cdot (2 \cdot 1) \cdot 1 = 2$

In each square above, the four factors are the number of ways of arranging the balls in A , B , C , and D , respectively. Summing this over all pairs (a, d) satisfying $0 \leq d \leq a \leq 2$ gives a total of 36. The answer is therefore $24 \cdot 6 \cdot 36 = 5184$.

43. Write down an integer N between 0 and 20 inclusive. If more than N teams write down N , your score is N ; otherwise it is 0.

Remark: Well, maybe you can get 1 point...

44. Write down a set S of positive integers, all greater than 1, such that for each $x \in S$, x is a proper divisor of $(P/x) + 1$, where P is the product of all the elements of S . Your score is $2n$, where $n = |S|$.

Remark: This question was posed by Stefan Znám in 1972. Solutions exist for $|S| \geq 5$, but an explicit solution has been found only for $|S| \leq 13$. The simplest solution is $S = \{2, 3, 11, 23, 31\}$. A solution for $|S| = 7$ is given by $S = \{2, 3, 11, 17, 101, 149, 3109\}$. For more information, see <http://mathworld.wolfram.com/ZnamsProblem.html>.

45. A *binary word* is a finite sequence of 0's and 1's. A *square subword* is a subsequence consisting of two identical chunks next to each other. For example, the word 100101011 contains the square subwords 00, 0101 (twice), 1010, and 11.

Find a long binary word containing a small number of square subwords. Specifically, write down a binary word of any length $n \leq 50$. Your score will be $\max\{0, n - s\}$, where s is the number of occurrences of square subwords. (That is, each different square subword will be counted according to the number of times it appears.)

Remark: See http://www.combinatorics.org/Volume_10/PDF/v10i1r12.pdf for analysis of this problem. The maximum possible score is on the order of 25. In general, the minimum number of square subwords of a word of length n tends to roughly $0.551n$ as $n \rightarrow \infty$.

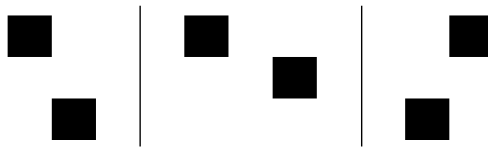
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round A — Solutions

Disconnected Domino Rally [175]

On an infinite checkerboard, the union of any two distinct unit squares is called a (*disconnected*) *domino*. A domino is said to be of *type* (a, b) , with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)

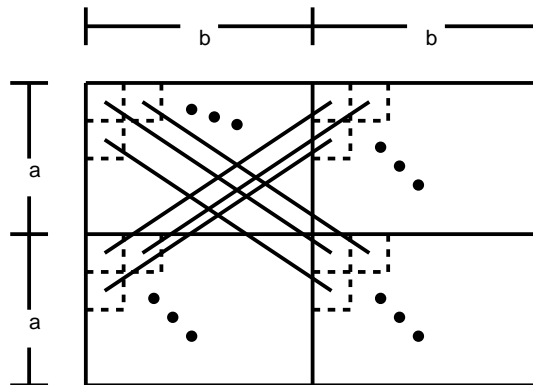


Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b) -*tileable* if it can be partitioned into dominoes of type (a, b) .

- [15] Prove that for any two types of dominoes, there exists a rectangle that can be tiled by dominoes of either type.

Solution: Note that a type (a, b) domino tiles a $\max\{1, 2a\} \times 2b$ rectangle (see diagram for $a > 0$). Then both type (a, b) and type (a', b') dominoes tile a $(\max\{1, 2a\} \cdot \max\{1, 2a'\}) \times (2b \cdot 2b')$ rectangle.



- [25] Suppose $0 < a \leq b$ and $4 \nmid mn$. Prove that the number of ways in which an $m \times n$ rectangle can be partitioned into dominoes of type (a, b) is even.

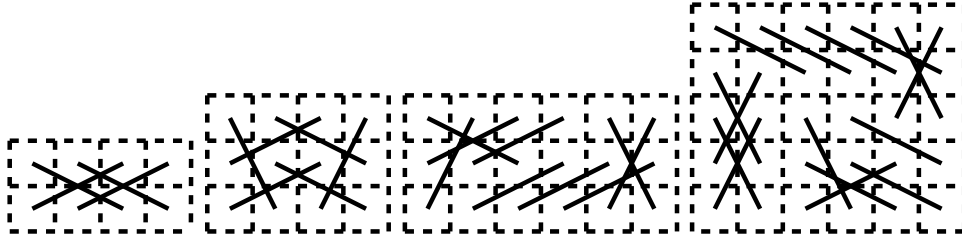
Solution: If the rectangle is tileable, it can be partitioned into an odd number of dominoes. Consider the reflection of the partitioned rectangle over one axis. This gives another partition of the rectangle. In fact, it cannot be the same partition, for suppose it were. Then we can pair each domino with its reflected image, but since there are an odd number of dominoes, one must reflect into itself. Since $a > 0$, this is not possible. Therefore, we can pair off partitions and their reflections, and it follows that the total number of partitions is even.

3. [10] Show that no rectangle of the form $1 \times k$ or $2 \times n$, where $4 \nmid n$, is $(1, 2)$ -tileable.

Solution: The claim is obvious for $1 \times k$ rectangles. For the others, color the first two columns black, the next two white, the next two black, etc. Each $(1, 2)$ domino will contain one square of each color, so in order to be tileable, the rectangle must contain the same number of black and white squares. This is the case only when $4 \mid n$.

4. [35] Show that all other rectangles of even area are $(1, 2)$ -tileable.

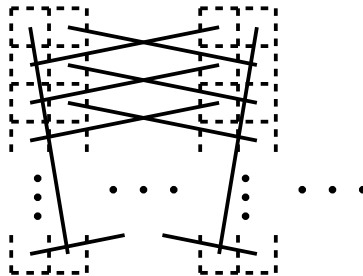
Solution: First, we demonstrate that there exist $(1, 2)$ -tilings of 2×4 , 3×4 , 3×6 , and 5×6 rectangles.



Now, notice that by combining these rectangles, we can form any rectangle of even area other than those described in the previous problem: using the first rectangle, we can form any $2 \times n$ rectangle with $4 \mid n$. By combining the first two, we can form any $m \times 4$ rectangle with $m \geq 2$, and by combining the last three, we can form any $m \times 6$ rectangle with $m \geq 3$. From these, we can form any $m \times n$ rectangle with $m \geq 3$ and n even and greater than 2, completing the proof.

5. [25] Show that for b even, there exists some M such that for every $n > M$, a $2b \times n$ rectangle is $(1, b)$ -tileable.

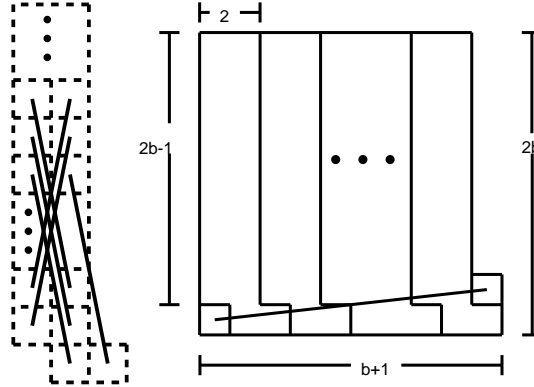
Solution: Recall from above that we can tile a $2 \times 2b$ rectangle. Four columns of a $(b + 1) \times 2b$ rectangle can be tiled as shown below, and repeating this $\frac{b}{2}$ times tiles the entire rectangle. Since any integer at least b can be written as a positive linear combination of 2 and $b + 1$, we can tile any $2b \times n$ rectangle for $n \geq b$.



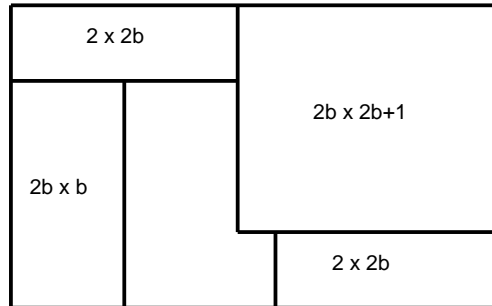
6. [40] Show that for b even, there exists some M such that for every $m, n > M$ with mn even, an $m \times n$ rectangle is $(1, b)$ -tileable.

Solution: By the diagram below, it is possible to tile a $(2b + 2) \times (4b + 1)$ rectangle. Since we can already tile a $(2b + 2) \times 2b$ rectangle by above, and $2b$ is relatively prime to $4b + 1$, this will allow us to tile any $(2b + 2) \times n$ rectangle for n sufficiently large. Combining this with the previous problem, this will allow us to tile any $m \times n$ rectangle for m and n sufficiently large and m even, completing the proof.

To tile the $(2b + 2) \times (4b + 1)$ rectangle, we first tile the following piece:



This is then combined with two $2 \times 2b$ rectangles, a $2b \times b$ rectangle, and a $2b \times (2b + 1)$ rectangle as follows:



7. [25] Prove that neither of the previous two problems holds if b is odd.

Solution: Color the grid black and white in checkerboard fashion. Then if b is odd, the two squares that make up a $(1, b)$ domino always have the same color. Therefore, for an $m \times n$ rectangle to be $(1, b)$ -tileable, it must have an even number of squares of each color. Then for any M , we can choose m and n larger than M such that n is odd and $4 \nmid m$. A $2b \times n$ rectangle and an $m \times n$ rectangle then contain bn and $mn/2$ squares of each color, respectively. Since both bn and $mn/2$ are odd, neither of these rectangles is $(1, b)$ -tileable.

An Interlude — Discovering One's Roots [100]

A k th root of unity is any complex number ω such that $\omega^k = 1$. You may use the following facts: if $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{k-1} = 0,$$

and if $1, \omega, \dots, \omega^{k-1}$ are distinct, then

$$(x^k - 1) = (x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{k-1}).$$

8. [25] Suppose x is a fifth root of unity. Find, in radical form, all possible values of

$$2x + \frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4}.$$

Solution: Note that

$$\frac{x}{1+x^2} + \frac{x^2}{1+x^3} = \frac{x^6}{1+x^2} + \frac{x^4}{x^2+x^5} = \frac{x^4+x^6}{1+x^2} = x^4 = \frac{1}{x}, \text{ and}$$

$$\frac{1}{1+x} + \frac{x^3}{1+x^4} = \frac{x^5}{1+x} + \frac{x^4}{x+x^5} = \frac{x^4+x^5}{1+x} = x^4 = \frac{1}{x}.$$

Therefore, the sum is just $2x + \frac{2}{x}$. If $x = 1$, this is 4. Otherwise, let $y = x + \frac{1}{x}$. Then x satisfies

$$0 = 1 + x + x^2 + x^3 + x^4 = \left(x^2 + \frac{1}{x^2} + 2\right) + \left(x + \frac{1}{x}\right) - 1 = y^2 + y - 1,$$

so solving this quadratic yields $y = \frac{-1 \pm \sqrt{5}}{2}$, or $2y = -1 \pm \sqrt{5}$. Since each value of y can correspond to only 2 possible values of x , and there are 4 possible values of x besides 1, both of these values for y are possible, which yields the answers, 4 and $-1 \pm \sqrt{5}$.

9. [25] Let $A_1 A_2 \dots A_k$ be a regular k -gon inscribed in a circle of radius 1, and let P be a point lying on or inside the circumcircle. Find the maximum possible value of $(PA_1)(PA_2) \dots (PA_k)$.

Solution: Place the vertices at the k th roots of unity, $1, \omega, \dots, \omega^{k-1}$, and place P at some complex number p . Then

$$\begin{aligned} ((PA_1)(PA_2) \dots (PA_k))^2 &= \prod_{i=0}^{k-1} |p - \omega^i|^2 \\ &= |p^k - 1|^2, \end{aligned}$$

since $x^k - 1 = (x - 1)(x - \omega) \dots (x - \omega^{k-1})$. This is maximized when p^k is as far as possible from 1, which occurs when $p^k = -1$. Therefore, the maximum possible value of $(PA_1)(PA_2) \dots (PA_k)$ is 2.

10. [25] Let P be a regular k -gon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of P .

Solution: Place the vertices of P at the k th roots of unity, $1, \omega, \omega^2, \dots, \omega^{k-1}$. We will first calculate the sum of the squares of the lengths of the sides and diagonals that contain the vertex 1. This is

$$\begin{aligned} \sum_{i=0}^{k-1} |1 - \omega^i|^2 &= \sum_{i=0}^{k-1} (1 - \omega^i)(1 - \bar{\omega}^i) \\ &= \sum_{i=0}^{k-1} (2 - \omega^i - \bar{\omega}^i) \\ &= 2k - 2 \sum_{i=0}^{k-1} \omega^i \\ &= 2k, \end{aligned}$$

using the fact that $1 + \omega + \dots + \omega^{k-1} = 0$. Now, by symmetry, this is the sum of the squares of the lengths of the sides and diagonals emanating from any vertex. Since there are k vertices and each segment has two endpoints, the total sum is $2k \cdot k/2 = k^2$.

11. [25] Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients, $a_n \neq 0$. Suppose every root of P is a root of unity, but $P(1) \neq 0$. Show that the coefficients of P are symmetric; that is, show that $a_n = a_0, a_{n-1} = a_1, \dots$

Solution: Since the coefficients of P are real, the complex conjugates of the roots of P are also roots of P . Now, if x is a root of unity, then $x^{-1} = \bar{x}$. But the roots of

$$x^n P(x^{-1}) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$$

are then just the complex conjugates of the roots of P , so they are the roots of P . Therefore, $P(x)$ and $x^n P(x^{-1})$ differ by a constant multiple c . Since $a_n = ca_0$ and $a_0 = ca_n$, c is either 1 or -1 . But if it were -1 , then

$$P(1) = a_n + a_{n-1} + \cdots + a_0 = \frac{1}{2}((a_n + a_0) + (a_{n-1} + a_1) + \cdots + (a_0 + a_n)) = 0,$$

a contradiction. Therefore $c = 1$, giving the result.

Early Re-tile-ment [125]

Let $S = \{s_0, \dots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \dots, s_n + k\}$. We say that two sets S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k . Given a (possibly infinite) set of integers A , we say that S *tiles* A if A can be partitioned into subsets equivalent to S . Such a partition is called a *tiling* of A by S .

12. [20] Suppose the elements of A are either bounded below or bounded above. Show that if S tiles A , then it does so uniquely, i.e., there is a unique tiling of A by S .

Solution: Assume A is bounded below; the other case is analogous. In choosing the tiling of A , note that there is a unique choice for the set S_0 that contains the minimum element of A . But then there is a unique choice for the set S_1 that contains the minimum element of $A \setminus S_0$. Continuing in this manner, there is a unique choice for the set containing the minimum element not yet covered, so we see that the tiling is uniquely determined.

13. [35] Let B be a set of integers either bounded below or bounded above. Then show that if S tiles all other integers $\mathbf{Z} \setminus B$, then S tiles all integers \mathbf{Z} .

Solution: Assume B is bounded above; the other case is analogous. Let a be the difference between the largest and smallest element of S . Denote the sets in the partition of $\mathbf{Z} \setminus B$ by S_k , $k \in \mathbf{Z}$, such that the minimum element of S_k , which we will denote c_k , is strictly increasing as k increases. Since B is bounded above, there exists some k_0 such that c_{k_0} is larger than all the elements of B . Let

$$T_l = \bigcup_{k=l}^{\infty} S_k.$$

Suppose $l \geq k_0$. Note that any element in S_k , $k < l$, is at most $c_l - 1 + a$. Therefore, T_l contains all integers that are at least $c_l + a$. Since the minimum element of T_l is c_l , T_l is completely determined by which of the integers $c_l + 1, c_l + 2, \dots, c_l + a - 1$ it contains. This implies that there are at most 2^{a-1} possible nonequivalent sets T_l when $l \geq k_0$ (here we extend the notion of equivalence to infinite sets in the natural way.) By the Pigeonhole Principle, there must then be some $l_2 > l_1 \geq k_0$ such that $T_{l_1} \sim T_{l_2}$. But then it is easy to see that the set $S_{l_1} \cup S_{l_1+1} \cup \cdots \cup S_{l_2-1}$ tiles \mathbf{Z} , so S tiles \mathbf{Z} .

14. [35] Suppose S tiles the natural numbers \mathbf{N} . Show that S tiles the set $\{1, 2, \dots, k\}$ for some positive integer k .

Solution: Using the notation from above, we can find $l_1 < l_2$ such that $T_{l_1} \sim T_{l_2}$. By the same argument as in problem 12, as long as $T_{l_1} \neq \mathbf{N}$, there is a unique choice for S_{l_1-1} that contains the largest integer not in T_{l_1} . Since the same can be said for T_{l_2} , we must have that $T_{l_1-1} \sim T_{l_2-1}$. Continuing in this manner, we find that there must exist some l for which $\mathbf{N} \sim T_l$; then S tiles $\mathbf{N} \setminus T_l = \{1, 2, \dots, c_l - 1\}$.

15. [35] Suppose S tiles \mathbf{N} . Show that S is symmetric; that is, if $-S = \{-s_n, \dots, -s_0\}$, show that $S \sim -S$.

Solution: Assume without loss of generality that the minimum element of S is 0. By the previous problem, S tiles the set $\{1, 2, \dots, k\}$ for some positive integer k . Then let $P(x)$ be the polynomial $\sum_{i=0}^n x^{s_i}$. To say that the set $\{1, 2, \dots, k\}$, or equivalently the set $\{0, 1, \dots, k-1\}$, is tiled by S is to say that there is some polynomial $Q(x)$ with coefficients 0 or 1 such that $P(x)Q(x) = 1 + x + \dots + x^{k-1} = (x^k - 1)/(x - 1)$. It follows that all the roots of $P(x)$ are roots of unity, but $P(1) \neq 0$. By question 11 above, this implies that $P(x)$ is symmetric. Therefore, $s_0 + s_n = s_1 + s_{n-1} = \dots = s_n + s_0$, so S is symmetric.

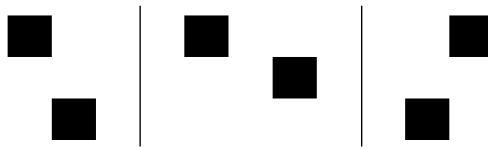
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round B — Solutions

Disconnected Domino Rally [150]

On an infinite checkerboard, the union of any two distinct unit squares is called a (*disconnected*) *domino*. A domino is said to be of *type* (a, b) , with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b) -*tileable* if it can be partitioned into dominoes of type (a, b) .

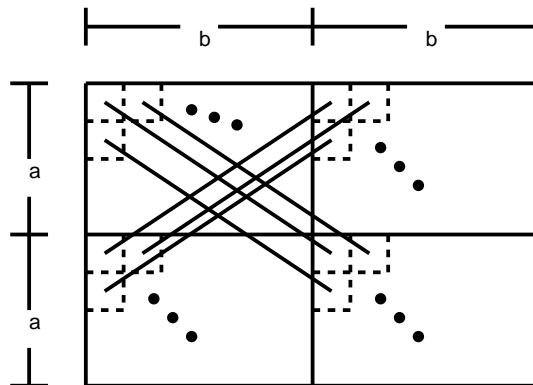
- [15] Let $0 < m \leq n$ be integers. How many different (i.e., noncongruent) dominoes can be formed by choosing two squares of an $m \times n$ array?

Solution: We must have $0 \leq a < m$, $0 \leq b < n$, $a \leq b$, and a and b not both 0. The number of pairs (a, b) with $b < a < m$ is $m(m-1)/2$, so the answer is

$$mn - \frac{m(m-1)}{2} - 1 = mn - \frac{m^2 - m + 2}{2}.$$

- [10] What are the dimensions of the rectangle of smallest area that is (a, b) -tileable?

Solution: If $a = 0$, then a $1 \times 2b$ rectangle is tileable in an obvious way. If $a > 0$, then a $2a \times 2b$ rectangle is tileable by dividing the rectangle into quarters and pairing each square with a square from the diagonally opposite quarter. The answer is therefore $\max\{1, 2a\} \times 2b$. Minimality of area follows from the next question.



3. [20] Prove that every (a, b) -tileable rectangle contains a rectangle of these dimensions.

Solution: An $m \times n$ rectangle, $m \leq n$, does not contain a $\max\{1, 2a\} \times 2b$ rectangle if and only if $m < 2a$ or $n < 2b$. But if either of these is the case, then the square closest to the center of the rectangle cannot be paired with any other square of the rectangle to form a domino of type (a, b) , so the rectangle cannot be (a, b) -tileable.

4. [30] Prove that an $m \times n$ rectangle is (b, b) -tileable if and only if $2b \mid m$ and $2b \mid n$.

Solution: Color the first b rows of an $m \times n$ rectangle black, the next b white, the next b black, etc. Any (b, b) domino covers one square of each color, so for the rectangle to be (b, b) -tileable, there must be the same number of black squares as white squares. This is possible only when $2b \mid m$. Similarly, we must have $2b \mid n$. It is easy to exhibit a tiling of all such rectangles, proving the claim. (It is also possible to prove this using the lemma described below.)

5. [35] Prove that an $m \times n$ rectangle is $(0, b)$ -tileable if and only if $2b \mid m$ or $2b \mid n$.

Solution: It is easy to exhibit a tiling of such a rectangle. The other direction follows from below. (It is also possible to prove this using a coloring argument as above: starting in one corner, divide the board into $b \times b$ blocks and color them checkerboard fashion. The details are left to the reader.)

6. [40] Let k be an integer such that $k \mid a$ and $k \mid b$. Prove that if an $m \times n$ rectangle is (a, b) -tileable, then $2k \mid m$ or $2k \mid n$.

Solution: We prove the following lemma.

Lemma. *Let k be a positive integer such that $k \mid a$ and $k \mid b$. Then an $m \times n$ rectangle is (a, b) -tileable if and only if an $m' \times n'$ rectangle is $(\frac{a}{k}, \frac{b}{k})$ -tileable for $\lfloor \frac{m}{k} \rfloor \leq m' \leq \lceil \frac{m}{k} \rceil$ and $\lfloor \frac{n}{k} \rfloor \leq n' \leq \lceil \frac{n}{k} \rceil$. (Here, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x , while $\lceil x \rceil$ denotes the least integer greater than or equal to x .)*

Proof. Number the rows and columns in order. For each pair $0 \leq i, j < k$, consider the set of squares in a row congruent to i modulo k and in a column congruent to j modulo k . If one square of a type (a, b) domino lies in this set, then so does the other. We can therefore partition the rectangle into these sets and then tile these sets instead. Each such set is a rectangular array of dimensions $m' \times n'$, with $\lfloor \frac{m}{k} \rfloor \leq m' \leq \lceil \frac{m}{k} \rceil$ and $\lfloor \frac{n}{k} \rfloor \leq n' \leq \lceil \frac{n}{k} \rceil$, and a type (a, b) domino on the original rectangle is a type $(\frac{a}{k}, \frac{b}{k})$ domino on this new array. Since all possible pairs (m', n') occur, the result follows. \square

Suppose $2k \nmid m$ and $2k \nmid n$. Then at least one of $\lfloor \frac{m}{k} \rfloor$ and $\lceil \frac{m}{k} \rceil$ is odd, so we can choose m' odd. Likewise we can choose n' odd. But then an $m' \times n'$ rectangle has odd area and so cannot be tileable, implying that the $m \times n$ rectangle is not tileable.

An Interlude — Discovering One's Roots [100]

A k th root of unity is any complex number ω such that $\omega^k = 1$.

7. [15] Find a real, irreducible quartic polynomial with leading coefficient 1 whose roots are all twelfth roots of unity.

Solution: All twelfth roots of unity are roots of

$$\begin{aligned} x^{12} - 1 &= (x^6 - 1)(x^6 + 1) \\ &= (x^3 - 1)(x^3 + 1)(x^6 + 1) \\ &= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)(x^2 + 1)(x^4 - x^2 + 1), \end{aligned}$$

so the answer is $x^4 - x^2 + 1$.

8. [25] Let x and y be two k th roots of unity. Prove that $(x + y)^k$ is real.

Solution: Note that

$$\begin{aligned} (x + y)^k &= \sum_{i=0}^k \binom{k}{i} x^i y^{k-i} \\ &= \frac{1}{2} \sum_{i=0}^k \binom{k}{i} (x^i y^{k-i} + x^{k-i} y^i) \end{aligned}$$

by pairing the i th and $(k - i)$ th terms. But $x^{k-i} y^i = (x^i y^{k-i})^{-1}$ since x and y are k th roots of unity. Moreover, since x and y have absolute value 1, so does $x^i y^{k-i}$, so $x^{k-i} y^i$ is in fact its complex conjugate. It follows that their sum is real, thus so is $(x + y)^k$.

This can also be shown geometrically. The argument of x (the angle between the vector x and the positive x -axis) is an integer multiple of $\frac{2\pi}{k}$, as is the argument of y . Since $x + y$ bisects the angle between x and y , its argument is an integer multiple of $\frac{\pi}{k}$. Multiplying this angle by k gives a multiple of π , so $(x + y)^k$ is real.

9. [30] Let x and y be two distinct roots of unity. Prove that $x + y$ is also a root of unity if and only if $\frac{y}{x}$ is a cube root of unity.

Solution: This is easiest to see geometrically. The vectors corresponding to x , y , and $-x - y$ sum to 0, so they form a triangle. In order for them all to be roots of unity, they must all have length one, so the triangle must be equilateral. Therefore the angle between x and y is $\pm \frac{2\pi}{3}$, that is, $\frac{y}{x}$ is a cube root of unity.

10. [30] Let x , y , and z be three roots of unity. Prove that $x + y + z$ is also a root of unity if and only if $x + y = 0$, $y + z = 0$, or $z + x = 0$.

Solution: Again, we consider the geometric picture. Arrange the vectors x , y , z , and $-x - y - z$ so as to form a quadrilateral. If they are all roots of unity, they form a quadrilateral all of whose side lengths are 1. If the quadrilateral is degenerate, then two of the vectors sum to 0, which implies the result. But even if it is not degenerate, the quadrilateral must be a rhombus, and since opposite sides of a rhombus are parallel, this again implies that two of the four roots of unity sum to 0.

Early Re-tiling [150]

Let $S = \{s_0, \dots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \dots, s_n + k\}$. We say that S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k . Given a (possibly infinite) set of integers A , we say that S *tiles* A if A can be partitioned into subsets equivalent to S . Such a partition is called a *tiling* of A by S .

11. [20] Find all sets S with minimum element 1 that tile $A = \{1, \dots, 12\}$.

Solution: This can be done by brute force. Alternatively, note that if $P(x)$ and $Q(x)$ are polynomials with coefficients either 0 or 1 with $P(x)Q(x) = x + x^2 + \dots + x^{12}$, then the set consisting of the exponents of nonzero terms in P tiles A . Either way, we find that S is one of the following: $\{1\}$, $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 7\}$, $\{1, 2, 3\}$, $\{1, 3, 5\}$, $\{1, 5, 9\}$, $\{1, 2, 3, 4\}$, $\{1, 2, 7, 8\}$, $\{1, 4, 7, 10\}$, $\{1, 2, 3, 4, 5, 6\}$, $\{1, 2, 3, 7, 8, 9\}$, $\{1, 2, 5, 6, 9, 10\}$, $\{1, 3, 5, 7, 9, 11\}$, or A itself.

12. [35] Let A be a finite set with more than one element. Prove that the number of nonequivalent sets S which tile A is always even.

Solution: Suppose A can be partitioned into sets S_0, \dots, S_m , each equivalent to S . (This partition is unique, simply by choosing S_0 to contain the smallest element of A , S_1 the smallest element of A not in S_0 , etc.) Then if $S_j = S + t_j$, each element of A can be written uniquely as $s_i + t_j$ for some i and j . But then the set T containing all t_j also tiles A by translation by the s_i . We cannot have S and T equivalent, for if so, since A has more than one element, both S and T would as well. This would imply that $s_0 + t_1 = s_1 + t_0$, an overlap in the tiling of A . We can thus pair together S and T , each of which tile A , so that the total number of sets tiling A must be even.

13. [25] Exhibit a set S which tiles the integers \mathbf{Z} but not the natural numbers \mathbf{N} .

Solution: One example is $\{1, 3, 4, 6\}$. Since its elements are all distinct modulo 4, it tiles \mathbf{Z} by translation by multiples of 4. On the other hand, it is easy to see that it cannot tile \mathbf{N} : 1 is contained in $\{1, 3, 4, 6\}$, but then there is no possible set for 2 to be contained in that does not overlap. Another example is $\{1, 2, 6\}$.

14. [30] Suppose that S tiles the set of all integer cubes. Prove that S has only one element.

Solution: Let the difference between the smallest and largest element of S be a . Then the set equivalent to S that contains b^3 can only contain integers between $b^3 - a$ and $b^3 + a$, inclusive. But for sufficiently large b , b^3 is the only cube in this range, so S can only have one element.

15. [40] Suppose that S tiles the set of odd prime numbers. Prove that S has only one element.

Solution: Consider the set S_0 equivalent to S that contains 3. If it contains 5 but not 7, then the set S_1 equivalent to S containing 7 must contain 9, which is not prime. Likewise, S_0 cannot contain 7 but not 5, because then the set S_1 containing 5 must contain 9. Suppose S_0 contains 3, 5, and 7. Then any other set S_1 of the tiling contains elements p , $p + 2$, and $p + 4$. But not all of these can be prime, because one of them is divisible by 3. Finally, suppose S_0 contains 3 and has second-smallest element $p > 7$. Then the set S_1 containing 5 does not contain 7 but does contain $p + 2$, and the set S_2 containing 7 contains $p + 4$. But as before, not all of p , $p + 2$, and $p + 4$ can be prime. Therefore S has no second-smallest element, so it has only one element.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Algebra Subject Test — Solutions

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

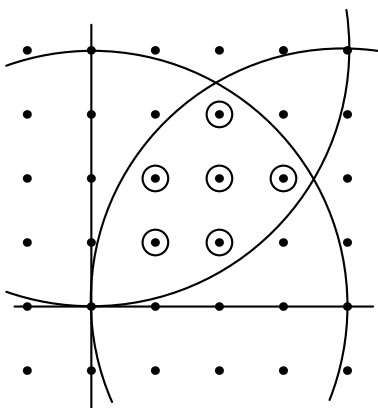
$$a^2 + b^2 < 16$$

$$a^2 + b^2 < 8a$$

$$a^2 + b^2 < 8b$$

Solution: $\boxed{6}$

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0,0)$, $(4,0)$, $(0,4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



2. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

Solution: $\boxed{6}$

For positive integers a, b , we have

$$a! \mid b! \Leftrightarrow a! \leq b! \Leftrightarrow a \leq b.$$

Thus,

$$((n!)!) \mid (2004!) \Leftrightarrow (n!) \leq 2004 \Leftrightarrow n! \leq 2004 \Leftrightarrow n \leq 6.$$

3. Compute:

$$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor.$$

Solution: $\boxed{8}$

Let $x = 2004$. Then the expression inside the floor brackets is

$$\frac{(x+1)^3}{(x-1)x} - \frac{(x-1)^3}{x(x+1)} = \frac{(x+1)^4 - (x-1)^4}{(x-1)x(x+1)} = \frac{8x^3 + 8x}{x^3 - x} = 8 + \frac{16x}{x^3 - x}.$$

Since x is certainly large enough that $0 < 16x/(x^3 - x) < 1$, the answer is 8.

4. Evaluate the sum

$$\frac{1}{2[\sqrt{1}] + 1} + \frac{1}{2[\sqrt{2}] + 1} + \frac{1}{2[\sqrt{3}] + 1} + \cdots + \frac{1}{2[\sqrt{100}] + 1}.$$

Solution: $\boxed{190/21}$

The first three terms all equal $1/3$, then the next five all equal $1/5$; more generally, for each $a = 1, 2, \dots, 9$, the terms $1/(2[\sqrt{a^2}] + 1)$ to $1/(2[\sqrt{a^2 + 2a}] + 1)$ all equal $1/(2a + 1)$, and there are $2a + 1$ such terms. Thus our terms can be arranged into 9 groups, each with sum 1, and only the last term $1/(2[\sqrt{100}] + 1)$ remains, so the answer is $9 + 1/21 = 190/21$.

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2 .

Solution: $\boxed{(-1 + \sqrt{5})/2}$

Draw a right triangle with legs $1, x$; then the angle θ opposite x is $\tan^{-1} x$, and we can compute $\cos(\theta) = 1/\sqrt{x^2 + 1}$. Thus, we only need to solve $x = 1/\sqrt{x^2 + 1}$. This is equivalent to $x\sqrt{x^2 + 1} = 1$. Square both sides to get $x^4 + x^2 = 1 \Rightarrow x^4 + x^2 - 1 = 0$. Use the quadratic formula to get the solution $x^2 = (-1 + \sqrt{5})/2$ (unique since x^2 must be positive).

6. Find all real solutions to $x^4 + (2 - x)^4 = 34$.

Solution: $\boxed{1 \pm \sqrt{2}}$

Let $y = 2 - x$, so $x + y = 2$ and $x^4 + y^4 = 34$. We know

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = x^4 + y^4 + 2xy(2x^2 + 2y^2 + 3xy).$$

Moreover, $x^2 + y^2 = (x + y)^2 - 2xy$, so the preceding equation becomes $2^4 = 34 + 2xy(2 \cdot 2^2 - xy)$, or $(xy)^2 - 8xy - 9 = 0$. Hence $xy = 9$ or -1 . Solving $xy = 9, x + y = 2$ produces complex solutions, and solving $xy = -1, x + y = 2$ produces $(x, y) = (1 + \sqrt{2}, 1 - \sqrt{2})$ or $(1 - \sqrt{2}, 1 + \sqrt{2})$. Thus, $x = 1 \pm \sqrt{2}$.

7. If x, y, k are positive reals such that

$$3 = k^2 \left(\frac{x^2}{y^2} + \frac{y^2}{x^2} \right) + k \left(\frac{x}{y} + \frac{y}{x} \right),$$

find the maximum possible value of k .

Solution: $\boxed{(-1 + \sqrt{7})/2}$

We have $3 = k^2(x^2/y^2 + y^2/x^2) + k(x/y + y/x) \geq 2k^2 + 2k$, hence $7 \geq 4k^2 + 4k + 1 = (2k + 1)^2$, hence $k \leq (\sqrt{7} - 1)/2$. Obviously k can assume this value, if we let $x = y = 1$.

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

Solution: $\boxed{128}$

For any integer $n \geq 0$, the given implies $x^{n+3} = -4x^{n+1} + 8x^n$, so we can rewrite any such power of x in terms of lower powers. Carrying out this process iteratively gives

$$\begin{aligned} x^7 &= -4x^5 + 8x^4 \\ &= 8x^4 + 16x^3 - 32x^2 \\ &= 16x^3 - 64x^2 + 64x \\ &= -64x^2 + 128. \end{aligned}$$

Thus, our answer is 128.

9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

Solution: 677

If d is the relevant greatest common divisor, then $a_{1000} = a_{999}^2 + 1 \equiv 1 = a_0 \pmod{d}$, which implies (by induction) that the sequence is periodic modulo d , with period 1000. In particular, $a_4 \equiv a_{2004} \equiv 0$. So d must divide a_4 . Conversely, we can see that $a_5 = a_4^2 + 1 \equiv 1 = a_0$ modulo a_4 , so (again by induction) the sequence is periodic modulo a_4 with period 5, and hence a_{999}, a_{2004} are indeed both divisible by a_4 . So the answer is a_4 , which we can compute directly; it is 677.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $P(1)/P(-1)$.

Solution: -2010012/2010013

Let z_1, \dots, z_5 be the roots of $Q(z) = z^5 + 2004z - 1$. We can check these are distinct (by using the fact that there's one in a small neighborhood of each root of $z^5 + 2004z$, or by noting that $Q(z)$ is relatively prime to its derivative). And certainly none of the roots of Q is the negative of another, since $z^5 + 2004z = 1$ implies $(-z)^5 + 2004(-z) = -1$, so their squares are distinct as well. Then, z_1^2, \dots, z_5^2 are the roots of P , so if we write C for the leading coefficient of P , we have

$$\begin{aligned} \frac{P(1)}{P(-1)} &= \frac{C(1 - z_1^2) \cdots (1 - z_5^2)}{C(-1 - z_1^2) \cdots (-1 - z_5^2)} \\ &= \frac{[(1 - z_1) \cdots (1 - z_5)] \cdot [(1 + z_1) \cdots (1 + z_5)]}{[(i - z_1) \cdots (i - z_5)] \cdot [(i + z_1) \cdots (i + z_5)]} \\ &= \frac{[(1 - z_1) \cdots (1 - z_5)] \cdot [(-1 - z_1) \cdots (-1 - z_5)]}{[(i - z_1) \cdots (i - z_5)] \cdot [(-i - z_1) \cdots (-i - z_5)]} \\ &= \frac{(1^5 + 2004 \cdot 1 - 1)(-1^5 + 2004 \cdot (-1) - 1)}{(i^5 + 2004 \cdot i - 1)(-i^5 + 2004 \cdot (-i) - 1)} \\ &= \frac{(2004)(-2006)}{(-1 + 2005i)(-1 - 2005i)} \\ &= -\frac{2005^2 - 1}{2005^2 + 1} \\ &= -4020024/4020026 = -2010012/2010013. \end{aligned}$$

Alternative Solution: In fact, we can construct the polynomial P explicitly (up to multiplication by a constant). We write $P(z^2)$ as a polynomial in z ; it must use only

even powers of z and be divisible by $z^5 + 2004z - 1$, so we are inspired to try a difference of squares,

$$P(z^2) = (z^5 + 2004z - 1)(z^5 + 2004z + 1) = (z^5 + 2004z)^2 - 1^2 = z^2(z^4 + 2004)^2 - 1,$$

giving

$$P(z) = z(z^2 + 2004)^2 - 1.$$

Now plugging in $z = 1$ and $z = -1$ rapidly gives $(2005^2 - 1)/(-2005^2 - 1)$ as before.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Calculus Subject Test — Solutions

1. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h \rightarrow 0} \frac{f(x+h) - f(h)}{x}$ at $x = \pi$.

Solution: $\boxed{0}$

The expression $\frac{f(x+h) - f(h)}{x}$ is continuous at $h = 0$, so the limit is just $\frac{f(x) - f(0)}{x}$. Letting $x = \pi$ yields $\frac{\sin(\sin \pi) - \sin(\sin 0)}{\pi} = 0$.

2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

Solution: $\boxed{19}$

Let $g(x) = f(x) - f(2x)$. Then we want the derivative of

$$f(x) - f(4x) = (f(x) - f(2x)) + (f(2x) - f(4x)) = g(x) + g(2x)$$

at $x = 1$. This is $g'(x) + 2g'(2x)$ at $x = 1$, or $5 + 2 \cdot 7 = 19$.

3. Find $\lim_{x \rightarrow \infty} (\sqrt[3]{x^3 + x^2} - \sqrt[3]{x^3 - x^2})$.

Solution: $\boxed{2/3}$

Observe that

$$\lim_{x \rightarrow \infty} [(x + 1/3) - \sqrt[3]{x^3 + x^2}] = \lim_{x \rightarrow \infty} \frac{x/3 + 1/27}{(\sqrt[3]{x^3 + x^2})^2 + (\sqrt[3]{x^3 + x^2})(x + 1/3) + (x + 1/3)^2},$$

by factoring the numerator as a difference of cubes. The numerator is linear in x , while the denominator is at least $3x^2$, so the limit as $x \rightarrow \infty$ is 0. By similar arguments, $\lim_{x \rightarrow \infty} [(x - 1/3) - \sqrt[3]{x^3 - x^2}] = 0$. So, the desired limit equals

$$2/3 + \lim_{x \rightarrow \infty} [(x - 1/3) - \sqrt[3]{x^3 - x^2}] - \lim_{x \rightarrow \infty} [(x + 1/3) - \sqrt[3]{x^3 + x^2}] = 2/3.$$

4. Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos x)))))))$, and suppose that the number a satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in a .

Solution: $\boxed{a^8 - 4a^6 + 6a^4 - 4a^2 + 1}$

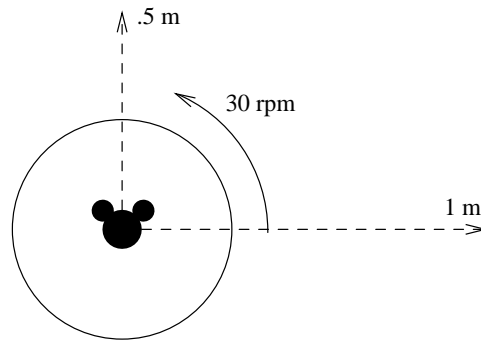
This is an exercise using the chain rule. Define $f_0(x) = x$ and $f_n(x) = \cos f_{n-1}(x)$ for $n \geq 0$. We will show by induction that $f_n(a) = a$ and $f'_n(a) = (-\sin a)^n$ for all n . The case $n = 0$ is clear. Then $f_n(a) = \cos f_{n-1}(a) = \cos a = a$, and

$$f'_n(a) = f'_{n-1}(a) \cdot (-\sin f_{n-1}(a)) = (-\sin a)^{n-1} \cdot (-\sin a) = (-\sin a)^n$$

by induction. Now, $f(x) = f_8(x)$, so $f'(a) = (-\sin a)^8 = \sin^8 a$. But $\sin^2 a = 1 - \cos^2 a = 1 - a^2$, so $f'(a) = (1 - a^2)^4 = a^8 - 4a^6 + 6a^4 - 4a^2 + 1$.

5. A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second.

Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set S of points the mouse can reach in his car within 1 second after the turntable is set in motion. (For example, the arrows in the figure below represent two possible paths the mouse can take.) What is the area of S , in square meters?



Solution: $\boxed{\pi/6}$

The mouse can wait while the table rotates through some angle θ and then spend the remainder of the time moving along that ray at 1 m/s. He can reach any point between the starting point and the furthest reachable point along the ray, $(1 - \theta/\pi)$ meters out. So the area is given by the polar integral

$$\int_0^\pi \frac{(1 - \theta/\pi)^2}{2} d\theta = \frac{1}{2} \cdot \frac{1}{\pi^2} \int_0^\pi \phi^2 d\phi = \pi/6$$

(where we have used the change of variables $\phi = \pi - \theta$).

6. For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.

Solution: $\boxed{1}$

Let $g(x) = \log f(x) = x \log x$. Then $f'(x)/f(x) = g'(x) = 1 + \log x$. Therefore $f(x) = f'(x)$ when $1 + \log x = 1$, that is, when $x = 1$.

7. Find the area of the region in the xy -plane satisfying $x^6 - x^2 + y^2 \leq 0$.

Solution: $\boxed{\pi/2}$

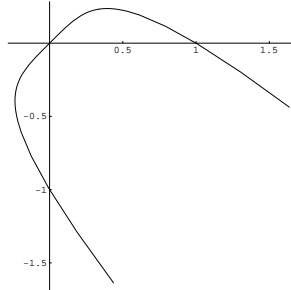
Rewrite the condition as $|y| \leq \sqrt{x^2 - x^6}$. The right side is zero when x is -1 , 0 , or 1 , and it bounds an area symmetric about the x - and y -axes. Therefore, we can calculate the area by the integral

$$2 \int_{-1}^1 \sqrt{x^2 - x^6} dx = 4 \int_0^1 x \sqrt{1 - x^4} dx = 2 \int_0^1 \sqrt{1 - u^2} du = \pi/2.$$

8. If x and y are real numbers with $(x + y)^4 = x - y$, what is the maximum possible value of y ?

Solution: $\boxed{3\sqrt[3]{2}/16}$

By drawing the graph of the curve (as shown), which is just a 135° clockwise rotation and scaling of $y = x^4$, we see that the maximum is achieved at the unique point where $dy/dx = 0$. Implicit differentiation gives $4(dy/dx + 1)(x + y)^3 = 1 - dy/dx$, so setting $dy/dx = 0$ gives $4(x + y)^3 = 1$. So $x + y = 1/\sqrt[3]{4} = \sqrt[3]{2}/2$, and $x - y = (x + y)^4 = \sqrt[3]{2}/8$. Subtracting and dividing by 2 gives $y = (\sqrt[3]{2}/2 - \sqrt[3]{2}/8)/2 = 3\sqrt[3]{2}/16$.



9. Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

converges for $c > c_0$ and diverges for $0 < c < c_0$.

Solution: $\boxed{1/e}$

The ratio test tells us that the series converges if

$$\lim_{n \rightarrow \infty} \frac{(n+1)!/(c(n+1))^{n+1}}{n!/(cn)^n} = \frac{1}{c} \cdot \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n$$

is less than one and diverges if it is greater than one. But

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \right)^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^{-n} = \frac{1}{e}.$$

Then the limit above is just $1/ce$, so the series converges for $c > 1/e$ and diverges for $0 < c < 1/e$.

10. Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \geq 1$, let $P^{[n+1]}(x) = P^{[n]}(P(x))$. Evaluate $\int_0^1 P^{[2004]}(x) dx$.

Solution: $\boxed{1/2}$

By Note that $P(1-x) = 1 - P(x)$. It follows easily by induction that $P^{[k]}(1-x) = 1 - P^{[k]}(x)$ for all positive integers k . Hence

$$\begin{aligned} \int_0^1 P^{[2004]}(x) dx &= \int_0^1 1 - P^{[2004]}(1-x) dx \\ &= 1 - \int_0^1 P^{[2004]}(1-x) dx \\ &= 1 - \int_0^1 P^{[2004]}(u) du \quad (u = 1-x). \end{aligned}$$

Hence $\int_0^1 P^{[2004]}(x) dx = 1/2$.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Combinatorics Subject Test — Solutions

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

Solution: 31

We can prove by induction on n that the following pattern holds for $0 \leq n \leq 499$: after $2n$ minutes, the first room contains $1000 - 2n$ people and the next n rooms each contain 2 people, and after $2n + 1$ minutes, the first room contains $1000 - (2n + 1)$ people, the next n rooms each contain 2 people, and the next room after that contains 1 person. So, after 60 minutes, we have one room with 940 people and 30 rooms with 2 people each.

2. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

Solution: 21600

In the top row, you can mark any of the 6 squares that is not a corner. In the bottom row, you can then mark any of the 5 squares that is not a corner and not in the same column as the square just marked. Then, in the second row, you have 6 choices for a square not in the same column as either of the two squares already marked; then there are 5 choices remaining for the third row, and so on down to 1 for the seventh row, in which you make the last mark. Thus, altogether, there are $6 \cdot 5 \cdot (6 \cdot 5 \cdots 1) = 30 \cdot 6! = 30 \cdot 720 = 21600$ possible sets of squares.

3. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

Solution: 6

Suppose the last student solved n problems, and the total number of problems on the test was p . Then the total number of correct solutions written was $7p$ (seven per problem), and also equal to $36 + n$ (the sum of the students' scores), so $p = (36 + n)/7$. The smallest $n \geq 0$ for which this is an integer is $n = 6$. But we also must have $n \leq p$, so $7n \leq 36 + n$, and solving gives $n \leq 6$. Thus $n = 6$ is the answer.

4. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH ?

Solution: 1/4

The only way that Andrea can ever flip HH is if she never flips T , in which case she must flip two heads immediately at the beginning. This happens with probability $\frac{1}{4}$.

5. A best-of-9 series is to be played between two teams; that is, the first team to win 5 games is the winner. The Mathletes have a chance of $\frac{2}{3}$ of winning any given game. What is the probability that exactly 7 games will need to be played to determine a winner?

Solution: $\boxed{20/81}$

If the Mathletes are to win, they must win exactly 5 out of the 7 games. One of the 5 games they win must be the 7th game, because otherwise they would win the tournament before 7 games are completed. Thus, in the first 6 games, the Mathletes must win 4 games and lose 2. The probability of this happening and the Mathletes winning the last game is

$$\left[\binom{6}{2} \cdot \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right)^2 \right] \cdot \left(\frac{2}{3}\right).$$

Likewise, the probability of the other team winning on the 7th game is

$$\left[\binom{6}{2} \cdot \left(\frac{1}{3}\right)^4 \cdot \left(\frac{2}{3}\right)^2 \right] \cdot \left(\frac{1}{3}\right).$$

Summing these values, we obtain $160/729 + 20/729 = 20/81$.

6. A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Bill and Karl must serve together or not at all, and Alice and Jane refuse to serve with each other?

Solution: $\boxed{41}$

If Bill and Karl are on the committee, there are $\binom{7}{3} = 35$ ways for the other group members to be chosen. However, if Alice and Jane are on the committee with Bill and Karl, there are $\binom{5}{1} = 5$ ways for the last member to be chosen, yielding 5 unacceptable committees. If Bill and Karl are not on the committee, there are $\binom{7}{5} = 21$ ways for the 5 members to be chosen, but again if Alice and Jane were to be on the committee, there would be $\binom{5}{3} = 10$ ways to choose the other three members, yielding 10 more unacceptable committees. So, we obtain $(35 - 5) + (21 - 10) = 41$ ways the committee can be chosen.

7. We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traversing every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?

Solution: $\boxed{20}$

This is obtainable by construction. Consider two tetrahedrons glued along a face; this gives us 5 vertices, 9 edges, and 6 faces, for a total of 20, and one readily checks that the required Eulerian path exists. Now, to see that we cannot do better, first notice that the number v of vertices is at least 5, since otherwise we must have a tetrahedron, which does not have an Eulerian path. Each vertex is incident to at least 3 edges, and

in fact, since there is an Eulerian path, all except possibly two vertices are incident to an even number of edges. So the number of edges is at least $(3+3+4+4+4)/2$ (since each edge meets two vertices) = 9. Finally, if $f = 4$ then each face must be a triangle, because there are only 3 other faces for it to share edges with, and we are again in the case of a tetrahedron, which is impossible; therefore $f \geq 5$. So $f+v+e \geq 5+5+9 = 19$. But since $f+v-e = 2-2g$ (where g is the number of holes in the polyhedron), $f+v+e$ must be even. This strengthens our bound to 20 as needed.

8. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

Solution: 7/15

This is a case of conditional probability; the answer is the probability that the first ball is red and the second ball is black, divided by the probability that the second ball is black.

First, we compute the numerator. If the first ball is drawn from Urn A, we have a probability of $2/6$ of getting a red ball, then a probability of $1/2$ of drawing the second ball from Urn B, and a further probability of $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, we have probability $3/6$ of getting a red ball, then $1/2$ of drawing the second ball from Urn B, and $3/5$ of getting a black ball. So our numerator is

$$\frac{1}{2} \left(\frac{2}{6} \cdot \frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{1}{2} \cdot \frac{3}{5} \right) = \frac{7}{60}.$$

We similarly compute the denominator: if the first ball is drawn from Urn A, we have a probability of $1/2$ of drawing the second ball from Urn B, and $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, then we have probability $3/6$ that it is red, in which case the second ball will be black with probability $(1/2) \cdot (3/5)$, and probability $3/6$ that the first ball is black, in which case the second is black with probability $(1/2) \cdot (2/5)$. So overall, our denominator is

$$\frac{1}{2} \left(\frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6} \left[\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5} \right] \right) = \frac{1}{4}.$$

Thus, the desired conditional probability is $(7/60) / (1/4) = 7/15$.

9. A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there?

Solution: 962

The set of empty desks must be of the form (non-full rows) \times (non-full columns): each empty desk is in a non-full column and a non-full row, and the given condition implies that each desk in such a position is empty. So if there are fewer than 25 students, then

both of these sets are nonempty; we have $2^5 - 1 = 31$ possible sets of non-full rows, and 31 sets of non-full columns, for 961 possible arrangements. Alternatively, there may be 25 students, and then only 1 arrangement is possible. Thus there are 962 possibilities altogether.

10. In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a row, and the queen starts in the center; the card configuration is thus RQR. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after the first move is either RRQ or QRR). What is the probability that, after 2004 moves, the center card is the queen?

Solution: $\boxed{1/3 + 1/(3 \cdot 2^{2003})}$

If the probability that the queen is the center card after move n is p_n , then the probability that the queen is an edge card is $1 - p_n$, and the probability that the queen is the center card after move $n + 1$ is $p_{n+1} = (1 - p_n)/2$. This recursion allows us to calculate the first few values of p_n . We might then notice in $1, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{11}{32}, \dots$, that the value of each fraction is close to $1/3$, and getting closer for larger n . In fact subtracting $1/3$ from each fraction yields $\frac{2}{3}, -\frac{1}{3}, \frac{1}{6}, -\frac{1}{12}, \frac{1}{24}, -\frac{1}{48}, \dots$. This suggests the formula $p_n = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^n$, and one can then prove that this formula is in fact correct by induction. Thus, $p(2004) = \frac{1}{3} + \frac{2}{3}(-\frac{1}{2})^{2004} = \frac{1}{3} + \frac{1}{3 \cdot 2^{2003}}$.

The recurrence can also be solved without guessing — by generating functions, for example, or by using the fundamental theorem of linear recurrences, which ensures that the solution is of the form $p_n = a + b(-1/2)^n$ for some constants a, b .

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: General Test, Part 1 — Solutions

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

Solution: 31

We can prove by induction on n that the following pattern holds for $0 \leq n \leq 499$: after $2n$ minutes, the first room contains $1000 - 2n$ people and the next n rooms each contain 2 people, and after $2n + 1$ minutes, the first room contains $1000 - (2n + 1)$ people, the next n rooms each contain 2 people, and the next room after that contains 1 person. So, after 60 minutes, we have one room with 940 people and 30 rooms with 2 people each.

2. What is the largest whole number that is equal to the product of its digits?

Solution: 9

Suppose the number n has $k + 1$ digits, the first of which is d . Then the number is at least $d \cdot 10^k$. On the other hand, each of the digits after the first is at most 9, so the product of the digits is at most $d \cdot 9^k$. Thus, if n equals the product of its digits, then

$$d \cdot 10^k \leq n \leq d \cdot 9^k$$

which forces $k = 0$, i.e., the number has only one digit. So $n = 9$ is clearly the largest possible value.

3. Suppose f is a function that assigns to each real number x a value $f(x)$, and suppose the equation

$$f(x_1 + x_2 + x_3 + x_4 + x_5) = f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - 8$$

holds for all real numbers x_1, x_2, x_3, x_4, x_5 . What is $f(0)$?

Solution: 2

Plug in $x_1 = x_2 = x_3 = x_4 = x_5 = 0$. Then the equation reads $f(0) = 5f(0) - 8$, so $4f(0) = 8$, so $f(0) = 2$.

4. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

Solution: 21600

In the top row, you can mark any of the 6 squares that is not a corner. In the bottom row, you can then mark any of the 5 squares that is not a corner and not in the same column as the square just marked. Then, in the second row, you have

6 choices for a square not in the same column as either of the two squares already marked; then there are 5 choices remaining for the third row, and so on down to 1 for the seventh row, in which you make the last mark. Thus, altogether, there are $6 \cdot 5 \cdot (6 \cdot 5 \cdots 1) = 30 \cdot 6! = 30 \cdot 720 = 21600$ possible sets of squares.

5. A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

Solution: 5

If the sides are x and y , we have $2x + 2y = 10$, so $x + y = 5$, and $\sqrt{x^2 + y^2} = \sqrt{15}$, so $x^2 + y^2 = 15$. Squaring the first equation gives $x^2 + 2xy + y^2 = 25$, and subtracting the second equation gives $2xy = 10$, so the area is $xy = 5$.

6. Find the ordered quadruple of digits (A, B, C, D) , with $A > B > C > D$, such that

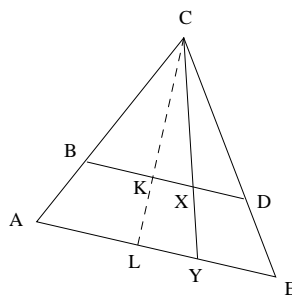
$$\begin{array}{r} ABCD \\ - DCBA \\ \hline = BDAC. \end{array}$$

Solution: (7, 6, 4, 1)

Since $D < A$, when A is subtracted from D in the ones' column, there will be a borrow from C in the tens' column. Thus, $D + 10 - A = C$. Next, consider the subtraction in the tens' column, $(C - 1) - B$. Since $C < B$, there will be a borrow from the hundreds' column, so $(C - 1 + 10) - B = A$. In the hundreds' column, $B - 1 \geq C$, so we do not need to borrow from A in the thousands' column. Thus, $(B - 1) - C = D$ and $A - D = B$. We thus have a system of four equations in four variables A, B, C, D , and solving by standard methods (e.g. substitution) produces $(A, B, C, D) = (7, 6, 4, 1)$.

7. Let ACE be a triangle with a point B on segment AC and a point D on segment CE such that BD is parallel to AE . A point Y is chosen on segment AE , and segment CY is drawn. Let X be the intersection of CY and BD . If $CX = 5$, $XY = 3$, what is the ratio of the area of trapezoid $ABDE$ to the area of triangle BCD ?

Solution: 39/25



Draw the altitude from C to AE , intersecting line BD at K and line AE at L . Then CK is the altitude of triangle BCD , so triangles CKX and CLY are similar. Since $CY/CX = 8/5$, $CL/CK = 8/5$. Also triangles CKB and CLA are similar, so that $CA/CB = 8/5$, and triangles BCD and ACE are similar, so that $AE/BD = 8/5$. The area of ACE is $(1/2)(AE)(CL)$, and the area of BCD is $(1/2)(BD)(CK)$, so the

ratio of the area of ACE to the area of BCD is $64/25$. Therefore, the ratio of the area of $ABDE$ to the area of BCD is $39/25$.

8. You have a 10×10 grid of squares. You write a number in each square as follows: you write $1, 2, 3, \dots, 10$ from left to right across the top row, then $11, 12, \dots, 20$ across the second row, and so on, ending with a 100 in the bottom right square. You then write a second number in each square, writing $1, 2, \dots, 10$ in the first column (from top to bottom), then $11, 12, \dots, 20$ in the second column, and so forth.

When this process is finished, how many squares will have the property that their two numbers sum to 101 ?

Solution: 10

The number in the i th row, j th column will receive the numbers $10(i - 1) + j$ and $10(j - 1) + i$, so the question is how many pairs (i, j) ($1 \leq i, j \leq 10$) will have

$$101 = [10(i - 1) + j] + [10(j - 1) + i] \Leftrightarrow 121 = 11i + 11j = 11(i + j).$$

Now it is clear that this is achieved by the ten pairs $(1, 10), (2, 9), (3, 8), \dots, (10, 1)$ and no others.

9. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

Solution: 7/15

This is a case of conditional probability; the answer is the probability that the first ball is red and the second ball is black, divided by the probability that the second ball is black.

First, we compute the numerator. If the first ball is drawn from Urn A, we have a probability of $2/6$ of getting a red ball, then a probability of $1/2$ of drawing the second ball from Urn B, and a further probability of $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, we have probability $3/6$ of getting a red ball, then $1/2$ of drawing the second ball from Urn B, and $3/5$ of getting a black ball. So our numerator is

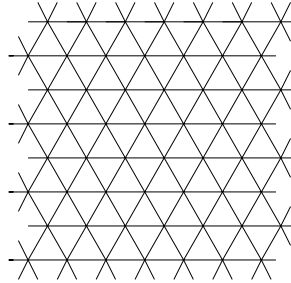
$$\frac{1}{2} \left(\frac{2}{6} \cdot \frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{1}{2} \cdot \frac{3}{5} \right) = \frac{7}{60}.$$

We similarly compute the denominator: if the first ball is drawn from Urn A, we have a probability of $1/2$ of drawing the second ball from Urn B, and $3/6$ of drawing a black ball. If the first ball is drawn from Urn B, then we have probability $3/6$ that it is red, in which case the second ball will be black with probability $(1/2) \cdot (3/5)$, and probability $3/6$ that the first ball is black, in which case the second is black with probability $(1/2) \cdot (2/5)$. So overall, our denominator is

$$\frac{1}{2} \left(\frac{1}{2} \cdot \frac{3}{6} + \frac{3}{6} \left[\frac{1}{2} \cdot \frac{3}{5} + \frac{1}{2} \cdot \frac{2}{5} \right] \right) = \frac{1}{4}.$$

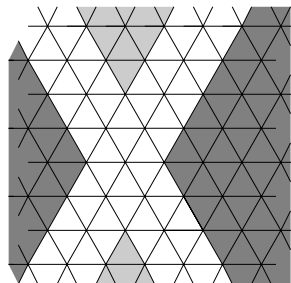
Thus, the desired conditional probability is $(7/60) / (1/4) = 7/15$.

10. A floor is tiled with equilateral triangles of side length 1, as shown. If you drop a needle of length 2 somewhere on the floor, what is the largest number of triangles it could end up intersecting? (Only count the triangles whose interiors are met by the needle — touching along edges or at corners doesn't qualify.)

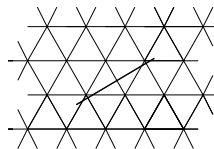


Solution: 8

Let L be the union of all the lines of the tiling. Imagine walking from one end of the needle to the other. We enter a new triangle precisely when we cross one of the lines of the tiling. Therefore, the problem is equivalent to maximizing the number of times the needle crosses L . Now, the lines of the tiling each run in one of three directions. It is clear that the needle cannot cross more than three lines in any given direction, since the lines are a distance $\sqrt{3}/2$ apart and the needle would therefore have to be of length greater than $3\sqrt{3}/2 > 2$. Moreover, it cannot cross three lines in each of two different directions. To see this, notice that its endpoints would have to lie in either the two light-shaded regions or the two dark-shaded regions shown, but the closest two points of such opposite regions are at a distance of 2 (twice the length of a side of a triangle), so the needle cannot penetrate both regions.



Therefore, the needle can cross at most three lines in one direction and two lines in each of the other two directions, making for a maximum of $3 + 2 + 2 = 7$ crossings and $7 + 1 = 8$ triangles intersected. The example shows that 8 is achievable, as long as the needle has length greater than $\sqrt{3} < 2$.



Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: General Test, Part 2 — Solutions

1. Find the largest number n such that $(2004)!$ is divisible by $((n!)!)!$.

Solution: $\boxed{6}$

For positive integers a, b , we have

$$a! \mid b! \Leftrightarrow a! \leq b! \Leftrightarrow a \leq b.$$

Thus,

$$((n!)!) \mid (2004)! \Leftrightarrow (n!) \leq 2004 \Leftrightarrow n! \leq 2004 \Leftrightarrow n \leq 6.$$

2. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH ?

Solution: $\boxed{1/4}$

The only way that Andrea can ever flip HH is if she never flips T , in which case she must flip two heads immediately at the beginning. This happens with probability $\frac{1}{4}$.

3. How many ordered pairs of integers (a, b) satisfy all of the following inequalities?

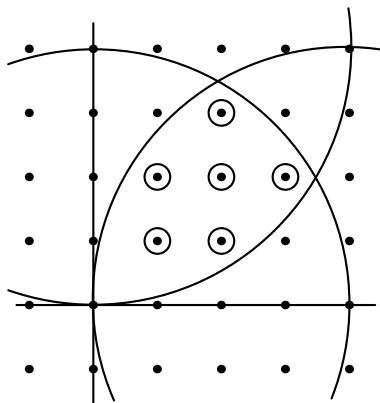
$$a^2 + b^2 < 16$$

$$a^2 + b^2 < 8a$$

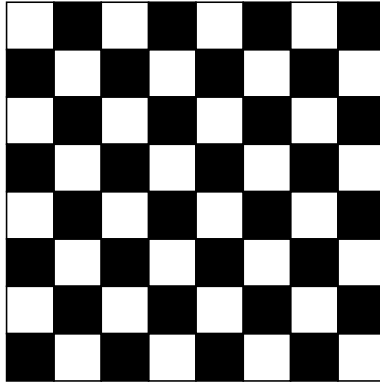
$$a^2 + b^2 < 8b$$

Solution: $\boxed{6}$

This is easiest to see by simply graphing the inequalities. They correspond to the (strict) interiors of circles of radius 4 and centers at $(0, 0)$, $(4, 0)$, $(0, 4)$, respectively. So we can see that there are 6 lattice points in their intersection (circled in the figure).



4. A horse stands at the corner of a chessboard, a white square. With each jump, the horse can move either two squares horizontally and one vertically or two vertically and one horizontally (like a knight moves). The horse earns two carrots every time it lands on a black square, but it must pay a carrot in rent to rabbit who owns the chessboard for every move it makes. When the horse reaches the square on which it began, it can leave. What is the maximum number of carrots the horse can earn without touching any square more than twice?



Solution: 0

The horse must alternate white and black squares, and it ends on the same square where it started. Thus it lands on the same number of black squares (b) as white squares (w). Thus, its net earnings will be $2b - (b + w) = b - w = 0$ carrots, regardless of its path.

5. Eight strangers are preparing to play bridge. How many ways can they be grouped into two bridge games — that is, into unordered pairs of unordered pairs of people?

Solution: 315

Putting 8 people into 4 pairs and putting those 4 pairs into 2 pairs of pairs are independent. If the people are numbered from 1 to 8, there are 7 ways to choose the person to pair with person 1. Then there are 5 ways to choose the person to pair with the person who has the lowest remaining number, 3 ways to choose the next, and 1 way to choose the last (because there are only 2 people remaining). Thus, there are $7 \cdot 5 \cdot 3 \cdot 1$ ways to assign 8 people to pairs and similarly there are $3 \cdot 1$ ways to assign 4 pairs to 2 pairs of pairs, so there are $7 \cdot 5 \cdot 3 \cdot 3 = 315$ ways.

6. a and b are positive integers. When written in binary, a has 2004 1's, and b has 2005 1's (not necessarily consecutive). What is the smallest number of 1's $a + b$ could possibly have?

Solution: 1

Consider the following addition:

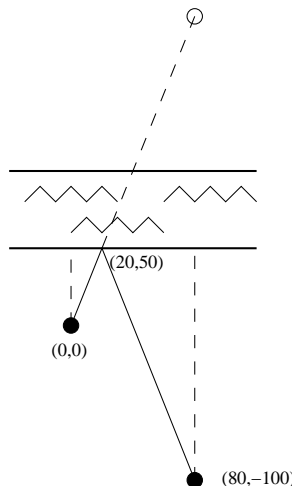
$$\begin{array}{r}
 111 \dots 100 \dots 01 \\
 + \quad \quad \quad \underline{11 \dots 11} \\
 = 1000 \dots \dots \dots 00
 \end{array}$$

By making the blocks of 1's and 0's appropriately long, we can ensure that the addends respectively contain 2004 and 2005 1's. (To be precise, we get $a = 2^{4008} - 2^{2005} + 1$ and $b = 2^{2005} - 1$.) Then the sum has only one 1. And clearly it is not possible to get any less than one 1.

7. Farmer John is grazing his cows at the origin. There is a river that runs east to west 50 feet north of the origin. The barn is 100 feet to the south and 80 feet to the east of the origin. Farmer John leads his cows to the river to take a swim, then the cows leave the river from the same place they entered and Farmer John leads them to the barn. He does this using the shortest path possible, and the total distance he travels is d feet. Find the value of d .

Solution: $\boxed{40\sqrt{29}}$

Suppose we move the barn to its reflection across the river's edge. Then paths from the origin to the river and then to the old barn location correspond to paths from the origin to the river and then to the new barn location, by reflecting the second part of the path across the river, and corresponding paths have the same length. Now the shortest path from the origin to the river and then to the new barn location is a straight line. The new barn location is 200 feet north and 80 feet east of the origin, so the value of d is $\sqrt{200^2 + 80^2} = 40\sqrt{29}$.



8. A freight train leaves the town of Jenkinsville at 1:00 PM traveling due east at constant speed. Jim, a hobo, sneaks onto the train and falls asleep. At the same time, Julie leaves Jenkinsville on her bicycle, traveling along a straight road in a northeasterly direction (but not due northeast) at 10 miles per hour. At 1:12 PM, Jim rolls over in his sleep and falls from the train onto the side of the tracks. He wakes up and immediately begins walking at 3.5 miles per hour directly towards the road on which Julie is riding. Jim reaches the road at 2:12 PM, just as Julie is riding by. What is the speed of the train in miles per hour?

Solution: $\boxed{62.5}$

Julie's distance is $(10 \text{ mph}) \cdot (6/5 \text{ hrs}) = 12$ miles. Jim's walking distance, after falling off the train, is $(3.5 \text{ mph}) \cdot (1 \text{ hr}) = 3.5$ miles at a right angle to the road. Therefore, Jim rode the train for $\sqrt{12^2 + 3.5^2} = \frac{1}{2}\sqrt{24^2 + 7^2} = 25/2$ miles, and its speed is $(25/2 \text{ mi})/(1/5 \text{ hr}) = 62.5 \text{ mph}$.

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

Solution: $\boxed{1/2}$

Imagine placing the tetrahedron $ABCD$ flat on a table with vertex A at the top. By vectors or otherwise, we see that the center is $3/4$ of the way from A to the bottom face, so the reflection of this face lies in a horizontal plane halfway between A and BCD . In particular, it cuts off the smaller tetrahedron obtained by scaling the original tetrahedron by a factor of $1/2$ about A . Similarly, the reflections of the other three faces cut off tetrahedra obtained by scaling $ABCD$ by $1/2$ about B , C , and D . On the other hand, the octahedral piece remaining after we remove these four smaller tetrahedra is in the intersection of $ABCD$ with its reflection, since the reflection sends this piece to itself. So the answer we seek is just the volume of this piece, which is

$$\begin{aligned} & (\text{volume of } ABCD) - 4 \cdot (\text{volume of } ABCD \text{ scaled by a factor of } 1/2) \\ &= 1 - 4(1/2)^3 = 1/2. \end{aligned}$$

10. A *lattice point* is a point whose coordinates are both integers. Suppose Johann walks in a line from the point $(0, 2004)$ to a random lattice point in the interior (not on the boundary) of the square with vertices $(0, 0)$, $(0, 99)$, $(99, 99)$, $(99, 0)$. What is the probability that his path, including the endpoints, contains an even number of lattice points?

Solution: $\boxed{3/4}$

If Johann picks the point (a, b) , the path will contain $\gcd(a, 2004 - b) + 1$ points. There will be an odd number of points in the path if $\gcd(a, 2004 - b)$ is even, which is true if and only if a and b are both even. Since there are 49^2 points with a, b both even and 98^2 total points, the probability that the path contains an even number of points is

$$\frac{98^2 - 49^2}{98^2} = \frac{49^2(2^2 - 1^2)}{49^2(2^2)} = \frac{3}{4}.$$

Harvard-MIT Mathematics Tournament

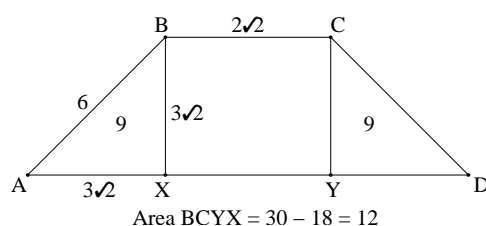
February 28, 2004

Individual Round: Geometry Subject Test — Solutions

1. In trapezoid $ABCD$, AD is parallel to BC . $\angle A = \angle D = 45^\circ$, while $\angle B = \angle C = 135^\circ$. If $AB = 6$ and the area of $ABCD$ is 30, find BC .

Solution: $\boxed{2\sqrt{2}}$

Draw altitudes from B and C to AD and label the points of intersection X and Y , respectively. Then ABX and CDY are $45^\circ-45^\circ-90^\circ$ triangles with $BX = CY = 3\sqrt{2}$. So, the area of ABX and the area of CDY are each 9, meaning that the area of rectangle $BCYX$ is 12. Since $BX = 3\sqrt{2}$, $BC = 12/(3\sqrt{2}) = 2\sqrt{2}$.



2. A parallelogram has 3 of its vertices at $(1, 2)$, $(3, 8)$, and $(4, 1)$. Compute the sum of the possible x -coordinates for the 4th vertex.

Solution: $\boxed{8}$

There are 3 possible locations for the 4th vertex. Let (a, b) be its coordinates. If it is opposite to vertex $(1, 2)$, then since the midpoints of the diagonals of a parallelogram coincide, we get $(\frac{a+1}{2}, \frac{b+2}{2}) = (\frac{3+4}{2}, \frac{8+1}{2})$. Thus $(a, b) = (6, 7)$. By similar reasoning for the other possible choices of opposite vertex, the other possible positions for the fourth vertex are $(0, 9)$ and $(2, -5)$, and all of these choices do give parallelograms. So the answer is $6 + 0 + 2 = 8$.

3. A swimming pool is in the shape of a circle with diameter 60 ft. The depth varies linearly along the east-west direction from 3 ft at the shallow end in the east to 15 ft at the diving end in the west (this is so that divers look impressive against the sunset) but does not vary at all along the north-south direction. What is the volume of the pool, in ft^3 ?

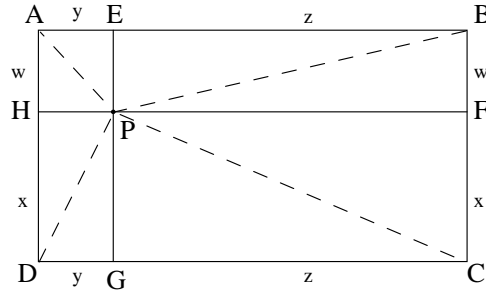
Solution: $\boxed{8100\pi}$

Take another copy of the pool, turn it upside-down, and put the two together to form a cylinder. It has height 18 ft and radius 30 ft, so the volume is $\pi(30 \text{ ft})^2 \cdot 18 \text{ ft} = 16200\pi \text{ ft}^3$; since our pool is only half of that, the answer is $8100\pi \text{ ft}^3$.

4. P is inside rectangle $ABCD$. $PA = 2$, $PB = 3$, and $PC = 10$. Find PD .

Solution: $\boxed{\sqrt{95}}$

Draw perpendiculars from P to E on AB , F on BC , G on CD , and H on DA , and let $AH = BF = w$, $HD = FC = x$, $AE = DG = y$, and $EB = GC = z$. Then $PA^2 = w^2 + y^2$, $PB^2 = w^2 + z^2$, $PC^2 = x^2 + z^2$, and $PD^2 = x^2 + y^2$. Adding and subtracting, we see that $PD^2 = PA^2 - PB^2 + PC^2 = 95$, so $PD = \sqrt{95}$.



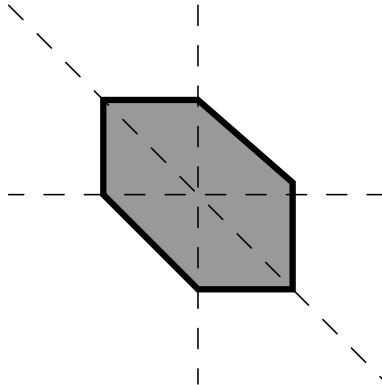
5. Find the area of the region of the xy -plane defined by the inequality $|x|+|y|+|x+y| \leq 1$.

Solution: $\boxed{3/4}$

To graph this region we divide the xy -plane into six sectors depending on which of $x, y, x + y$ are ≥ 0 , or ≤ 0 . The inequality simplifies in each case:

Sector	Inequality	Simplified inequality
$x \geq 0, y \geq 0, x + y \geq 0$	$x + y + (x + y) \leq 1$	$x + y \leq 1/2$
$x \geq 0, y \leq 0, x + y \geq 0$	$x - y + (x + y) \leq 1$	$x \leq 1/2$
$x \geq 0, y \leq 0, x + y \leq 0$	$x - y - (x + y) \leq 1$	$y \geq -1/2$
$x \leq 0, y \geq 0, x + y \geq 0$	$-x + y + (x + y) \leq 1$	$y \leq 1/2$
$x \leq 0, y \geq 0, x + y \leq 0$	$-x + y - (x + y) \leq 1$	$x \geq -1/2$
$x \leq 0, y \leq 0, x + y \leq 0$	$-x - y - (x + y) \leq 1$	$x + y \geq -1/2$

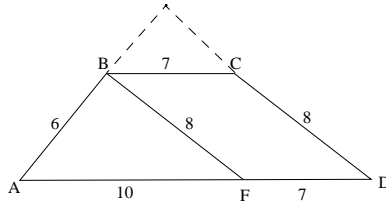
We then draw the region; we get a hexagon as shown. The hexagon intersects each region in an isosceles right triangle of area $1/8$, so the total area is $6 \cdot 1/8 = 3/4$.



6. In trapezoid $ABCD$ shown, AD is parallel to BC , and $AB = 6, BC = 7, CD = 8, AD = 17$. If sides AB and CD are extended to meet at E , find the resulting angle at E (in degrees).

Solution: $\boxed{90}$

Choose point F on AD so that $BCDF$ is a parallelogram. Then $BF = CD = 8$, and $AF = AD - DF = AD - BC = 10$, so $\triangle ABF$ is a 6-8-10 right triangle. The required angle is equal to $\angle ABF = 90^\circ$.



7. Yet another trapezoid $ABCD$ has AD parallel to BC . AC and BD intersect at P . If $[ADP]/[BCP] = 1/2$, find $[ADP]/[ABCD]$. (Here the notation $[P_1 \cdots P_n]$ denotes the area of the polygon $P_1 \cdots P_n$.)

Solution: $\boxed{3 - 2\sqrt{2}}$

A homothety (scaling) about P takes triangle ADP into BCP , since AD, BC are parallel and A, P, C ; B, P, D are collinear. The ratio of homothety is thus $\sqrt{2}$. It follows that, if we rescale to put $[ADP] = 1$, then $[ABP] = [CDP] = \sqrt{2}$, just by the ratios of lengths of bases. So $[ABCD] = 3 + 2\sqrt{2}$, so $[ADP]/[ABCD] = 1/(3 + 2\sqrt{2})$. Simplifying this, we get $3 - 2\sqrt{2}$.

8. A triangle has side lengths 18, 24, and 30. Find the area of the triangle whose vertices are the incenter, circumcenter, and centroid of the original triangle.

Solution: $\boxed{3}$

There are many solutions to this problem, which is straightforward. The given triangle is a right 3-4-5 triangle, so the circumcenter is the midpoint of the hypotenuse. Coordinatizing for convenience, put the vertex at $(0, 0)$ and the other vertices at $(0, 18)$ and $(24, 0)$. Then the circumcenter is $(12, 9)$. The centroid is at one-third the sum of the three vertices, which is $(8, 6)$. Finally, since the area equals the inradius times half the perimeter, we can see that the inradius is $(18 \cdot 24/2) / ((18 + 24 + 30)/2) = 6$. So the incenter of the triangle is $(6, 6)$. So the small triangle has a base of length 2 and a height of 3, hence its area is 3.

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

Solution: $\boxed{1/2}$

Imagine placing the tetrahedron $ABCD$ flat on a table with vertex A at the top. By vectors or otherwise, we see that the center is $3/4$ of the way from A to the bottom face, so the reflection of this face lies in a horizontal plane halfway between A and BCD . In particular, it cuts off the smaller tetrahedron obtained by scaling the original tetrahedron by a factor of $1/2$ about A . Similarly, the reflections of the other three faces cut off tetrahedra obtained by scaling $ABCD$ by $1/2$ about B , C , and D . On the other hand, the octahedral piece remaining remaining after we remove these four smaller tetrahedra is in the intersection of $ABCD$ with its reflection, since the reflection sends this piece to itself. So the answer we seek is just the volume of this piece, which is

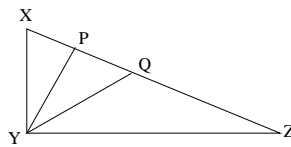
$$\begin{aligned} & (\text{volume of } ABCD) - 4 \cdot (\text{volume of } ABCD \text{ scaled by a factor of } 1/2) \\ &= 1 - 4(1/2)^3 = 1/2. \end{aligned}$$

10. Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

Solution: 1370736

The triangle's area is $(228 \cdot 2004)/2 = 228456$. All the angles at Y are 30 degrees, so by the sine area formula, the areas of the three small triangles in the diagram are $QY \cdot YZ/4$, $PY \cdot QY/4$, and $XY \cdot PY/4$, which sum to the area of the triangle. So expanding $(PY + YZ)(QY + XY)$, we see that it equals

$$4 \cdot 228456 + XY \cdot YZ = 6 \cdot 228456 = 1370736.$$



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Guts Round — Solutions

1. Find the value of

$$\binom{6}{1}2^1 + \binom{6}{2}2^2 + \binom{6}{3}2^3 + \binom{6}{4}2^4 + \binom{6}{5}2^5 + \binom{6}{6}2^6.$$

Solution: $\boxed{728}$

This sum is the binomial expansion of $(1+2)^6$, except that it is missing the first term, $\binom{6}{0}2^0 = 1$. So we get $3^6 - 1 = 728$.

2. If the three points

$$(1, a, b)$$

$$(a, 2, b)$$

$$(a, b, 3)$$

are collinear (in 3-space), what is the value of $a + b$?

Solution: $\boxed{4}$

The first two points are distinct (otherwise we would have $a = 1$ and $a = 2$ simultaneously), and they both lie on the plane $z = b$, so the whole line is in this plane and $b = 3$. Reasoning similarly with the last two points gives $a = 1$, so $a + b = 4$.

3. If the system of equations

$$|x + y| = 99$$

$$|x - y| = c$$

has exactly two real solutions (x, y) , find the value of c .

Solution: $\boxed{0}$

If $c < 0$, there are no solutions. If $c > 0$ then we have four possible systems of linear equations given by $x + y = \pm 99$, $x - y = \pm c$, giving four solutions (x, y) . So we must have $c = 0$, and then we do get two solutions ($x = y$, so they must both equal $\pm 99/2$).

4. A tree grows in a rather peculiar manner. Lateral cross-sections of the trunk, leaves, branches, twigs, and so forth are circles. The trunk is 1 meter in diameter to a height of 1 meter, at which point it splits into two sections, each with diameter .5 meter. These sections are each one meter long, at which point they each split into two sections, each with diameter .25 meter. This continues indefinitely: every section of tree is 1 meter long and splits into two smaller sections, each with half the diameter of the previous.

What is the total volume of the tree?

Solution: $\boxed{\pi/2}$

If we count the trunk as level 0, the two sections emerging from it as level 1, and so forth, then the n th level consists of 2^n sections each with diameter $1/2^n$, for a volume of $2^n(\pi/4 \cdot 2^{-2n}) = (\pi/4) \cdot 2^{-n}$. So the total volume is given by a simple infinite sum,

$$.25\pi \cdot (1 + 1/2 + 1/4 + \dots) = .25\pi \cdot 2 = \pi/2.$$

5. Augustin has six $1 \times 2 \times \pi$ bricks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation so long as it rests flat on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

Solution: $\boxed{28}$

If there are k bricks which are placed so that they contribute either 1 or 2 height, then the height of these k bricks can be any integer from k to $2k$. Furthermore, towers with different values of k cannot have the same height. Thus, for each k there are $k + 1$ possible tower heights, and since k is any integer from 0 to 6, there are $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$ possible heights.

6. Find the smallest integer n such that $\sqrt{n + 99} - \sqrt{n} < 1$.

Solution: $\boxed{2402}$

This is equivalent to

$$\begin{aligned}\sqrt{n + 99} &< \sqrt{n} + 1 \\ n + 99 &< n + 1 + 2\sqrt{n} \\ 49 &< \sqrt{n}\end{aligned}$$

So the smallest integer n with this property is $49^2 + 1 = 2402$.

7. Find the shortest distance from the line $3x + 4y = 25$ to the circle $x^2 + y^2 = 6x - 8y$.

Solution: $\boxed{7/5}$

The circle is $(x - 3)^2 + (y + 4)^2 = 5^2$. The center $(3, -4)$ is a distance of

$$\frac{|3 \cdot 3 + 4 \cdot -4 - 25|}{\sqrt{3^2 + 4^2}} = \frac{32}{5}$$

from the line, so we subtract 5 for the radius of the circle and get $7/5$.

8. I have chosen five of the numbers $\{1, 2, 3, 4, 5, 6, 7\}$. If I told you what their product was, that would not be enough information for you to figure out whether their sum was even or odd. What is their product?

Solution: $\boxed{420}$

Giving you the product of the five numbers is equivalent to telling you the product of the two numbers I didn't choose. The only possible products that are achieved by more than one pair of numbers are 12 ($\{3, 4\}$ and $\{2, 6\}$) and 6 ($\{1, 6\}$ and $\{2, 3\}$). But in the second case, you at least know that the two unchosen numbers have odd sum (and so the five chosen numbers have odd sum also). Therefore, the first case must hold, and the product of the five chosen numbers is

$$1 \cdot 2 \cdot 5 \cdot 6 \cdot 7 = 1 \cdot 3 \cdot 4 \cdot 5 \cdot 7 = 420.$$

9. A positive integer n is *picante* if $n!$ ends in the same number of zeroes whether written in base 7 or in base 8. How many of the numbers $1, 2, \dots, 2004$ are picante?

Solution: $\boxed{4}$

The number of zeroes in base 7 is the total number of factors of 7 in $1 \cdot 2 \cdots n$, which is

$$\lfloor n/7 \rfloor + \lfloor n/7^2 \rfloor + \lfloor n/7^3 \rfloor + \cdots .$$

The number of zeroes in base 8 is $\lfloor a \rfloor$, where

$$a = (\lfloor n/2 \rfloor + \lfloor n/2^2 \rfloor + \lfloor n/2^3 \rfloor + \cdots)/3$$

is one-third the number of factors of 2 in the product $n!$. Now $\lfloor n/2^k \rfloor/3 \geq \lfloor n/7^k \rfloor$ for all k , since $(n/2^k)/3 \geq n/7^k$. But n can only be picante if the two sums differ by at most $2/3$, so in particular this requires $(\lfloor n/2^2 \rfloor)/3 \leq \lfloor n/7^2 \rfloor + 2/3 \Leftrightarrow \lfloor n/4 \rfloor \leq 3 \lfloor n/49 \rfloor + 2$. This cannot happen for $n \geq 12$; checking the remaining few cases by hand, we find $n = 1, 2, 3, 7$ are picante, for a total of 4 values.

10. Let $f(x) = x^2 + x^4 + x^6 + x^8 + \cdots$, for all real x such that the sum converges. For how many real numbers x does $f(x) = x$?

Solution: 2

Clearly $x = 0$ works. Otherwise, we want $x = x^2/(1 - x^2)$, or $x^2 + x - 1 = 0$. Discard the negative root (since the sum doesn't converge there), but $(-1 + \sqrt{5})/2$ works, for a total of 2 values.

11. Find all numbers n with the following property: there is exactly one set of 8 different positive integers whose sum is n .

Solution: 36, 37

The sum of 8 different positive integers is at least $1 + 2 + 3 + \cdots + 8 = 36$, so we must have $n \geq 36$. Now $n = 36$ satisfies the desired property, since in this case we must have equality — the eight numbers must be $1, \dots, 8$. And if $n = 37$ the eight numbers must be $1, 2, \dots, 7, 9$: if the highest number is 8 then the sum is $36 < n$, while if the highest number is more than 9 the sum is $> 1 + 2 + \cdots + 7 + 9 = 37 = n$. So the highest number must be 9, and then the remaining numbers must be $1, 2, \dots, 7$. Thus $n = 37$ also has the desired property.

However, no other values of n work: if $n > 37$ then $\{1, 2, 3, \dots, 7, n - 28\}$ and $\{1, 2, \dots, 6, 8, n - 29\}$ are both sets of 8 distinct positive integers whose sum is n . So $n = 36, 37$ are the only solutions.

12. A convex quadrilateral is drawn in the coordinate plane such that each of its vertices (x, y) satisfies the equations $x^2 + y^2 = 73$ and $xy = 24$. What is the area of this quadrilateral?

Solution: 110

The vertices all satisfy $(x + y)^2 = x^2 + y^2 + 2xy = 73 + 2 \cdot 24 = 121$, so $x + y = \pm 11$. Similarly, $(x - y)^2 = x^2 + y^2 - 2xy = 73 - 2 \cdot 24 = 25$, so $x - y = \pm 5$. Thus, there are four solutions: $(x, y) = (8, 3), (3, 8), (-3, -8), (-8, -3)$. All four of these solutions satisfy the original equations. The quadrilateral is therefore a rectangle with side lengths of $5\sqrt{2}$ and $11\sqrt{2}$, so its area is 110.

13. Find all positive integer solutions (m, n) to the following equation:

$$m^2 = 1! + 2! + \cdots + n!.$$

Solution: $\boxed{(1, 1), (3, 3)}$

A square must end in the digit 0, 1, 4, 5, 6, or 9. If $n \geq 4$, then $1! + 2! + \cdots + n!$ ends in the digit 3, so cannot be a square. A simple check for the remaining cases reveals that the only solutions are (1, 1) and (3, 3).

14. If $a_1 = 1, a_2 = 0$, and $a_{n+1} = a_n + \frac{a_{n+2}}{2}$ for all $n \geq 1$, compute a_{2004} .

Solution: $\boxed{-2^{1002}}$

By writing out the first few terms, we find that $a_{n+4} = -4a_n$. Indeed,

$$a_{n+4} = 2(a_{n+3} - a_{n+2}) = 2(a_{n+2} - 2a_{n+1}) = 2(-2a_n) = -4a_n.$$

Then, by induction, we get $a_{4k} = (-4)^k$ for all positive integers k , and setting $k = 501$ gives the answer.

15. A regular decagon $A_0A_1A_2 \cdots A_9$ is given in the plane. Compute $\angle A_0A_3A_7$ in degrees.

Solution: $\boxed{54^\circ}$

Put the decagon in a circle. Each side subtends an arc of $360^\circ/10 = 36^\circ$. The inscribed angle $\angle A_0A_3A_7$ contains 3 segments, namely A_7A_8, A_8A_9, A_9A_0 , so the angle is $108^\circ/2 = 54^\circ$.

16. An n -string is a string of digits formed by writing the numbers $1, 2, \dots, n$ in some order (in base ten). For example, one possible 10-string is

35728910461

What is the smallest $n > 1$ such that there exists a palindromic n -string?

Solution: $\boxed{19}$

The following is such a string for $n = 19$:

9|18|7|16|5|14|3|12|1|10|11|2|13|4|15|6|17|8|19

where the vertical bars indicate breaks between the numbers. On the other hand, to see that $n = 19$ is the minimum, notice that only one digit can occur an odd number of times in a palindromic n -string (namely the center digit). If $n \leq 9$, then (say) the digits 1, 2 each appear once in any n -string, so we cannot have a palindrome. If $10 \leq n \leq 18$, then 0, 9 each appear once, and we again cannot have a palindrome. So 19 is the smallest possible n .

17. Kate has four red socks and four blue socks. If she randomly divides these eight socks into four pairs, what is the probability that none of the pairs will be mismatched? That is, what is the probability that each pair will consist either of two red socks or of two blue socks?

Solution: $\boxed{3/35}$

The number of ways Kate can divide the four red socks into two pairs is $\binom{4}{2}/2 = 3$. The number of ways she can divide the four blue socks into two pairs is also 3. Therefore, the number of ways she can form two pairs of red socks and two pairs of blue socks is

$3 \cdot 3 = 9$. The total number of ways she can divide the eight socks into four pairs is $[8!/(2! \cdot 2! \cdot 2! \cdot 2!)]/4! = 105$, so the probability that the socks come out paired correctly is $9/105 = 3/35$.

To see why 105 is the correct denominator, we can look at each $2!$ term as representing the double counting of pair (ab) and pair (ba) , while the $4!$ term represents the number of different orders in which we can select the same four pairs. Alternatively, we know that there are three ways to select two pairs from four socks. To select three pairs from six socks, there are five different choices for the first sock's partner and then three ways to pair up the remaining four socks, for a total of $5 \cdot 3 = 15$ pairings. To select four pairs from eight socks, there are seven different choices for the first sock's partner and then fifteen ways to pair up the remaining six socks, for a total of $7 \cdot 15 = 105$ pairings.

18. On a spherical planet with diameter 10,000 km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within 5,000 km of ground zero and leave anything beyond 5,000 km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

Solution: $\boxed{100,000,000\pi}$

The explosives have the same radius as the planet, so the surface area of the "cap" removed is the same as the new surface area revealed in the resulting "dimple." Thus the area is preserved by the explosion and remains $\pi \cdot (10,000)^2$.

19. The Fibonacci numbers are defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If the number

$$\frac{F_{2003}}{F_{2002}} - \frac{F_{2004}}{F_{2003}}$$

is written as a fraction in lowest terms, what is the numerator?

Solution: $\boxed{1}$

Before reducing, the numerator is $F_{2003}^2 - F_{2002}F_{2004}$. We claim $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$, which will immediately imply that the answer is 1 (no reducing required). This claim is straightforward to prove by induction on n : it holds for $n = 2$, and if it holds for some n , then

$$F_{n+1}^2 - F_n F_{n+2} = F_{n+1}(F_{n-1} + F_n) - F_n(F_n + F_{n+1}) = F_{n+1}F_{n-1} - F_n^2 = -(-1)^{n+1} = (-1)^{n+2}.$$

20. Two positive rational numbers x and y , when written in lowest terms, have the property that the sum of their numerators is 9 and the sum of their denominators is 10. What is the largest possible value of $x + y$?

Solution: $\boxed{73/9}$

For fixed denominators $a < b$ (with sum 10), we maximize the sum of the fractions by giving the smaller denominator as large a numerator as possible: $8/a + 1/b$. Then, if $a \geq 2$, this quantity is at most $8/2 + 1/1 = 5$, which is clearly smaller than the sum we get by setting $a = 1$, namely $8/1 + 1/9 = 73/9$. So this is the answer.

21. Find all ordered pairs of integers (x, y) such that $3^x 4^y = 2^{x+y} + 2^{2(x+y)-1}$.

Solution: $\boxed{(0, 1), (1, 1), (2, 2)}$

The right side is $2^{x+y}(1 + 2^{x+y-1})$. If the second factor is odd, it needs to be a power of 3, so the only options are $x + y = 2$ and $x + y = 4$. This leads to two solutions, namely (1,1) and (2,2). The second factor can also be even, if $x + y - 1 = 0$. Then $x + y = 1$ and $3^x 4^y = 2 + 2$, giving (0, 1) as the only other solution.

22. I have written a strictly increasing sequence of six positive integers, such that each number (besides the first) is a multiple of the one before it, and the sum of all six numbers is 79. What is the largest number in my sequence?

Solution: 48

If the fourth number is ≥ 12 , then the last three numbers must sum to at least $12 + 2 \cdot 12 + 2^2 \cdot 12 = 84 > 79$. This is impossible, so the fourth number must be less than 12. Then the only way we can have the required divisibilities among the first four numbers is if they are 1, 2, 4, 8. So the last two numbers now sum to $79 - 15 = 64$. If we call these numbers $8a, 8ab$ ($a, b > 1$) then we get $a(1 + b) = a + ab = 8$, which forces $a = 2, b = 3$. So the last two numbers are 16, 48.

23. Find the largest integer n such that $3^{512} - 1$ is divisible by 2^n .

Solution: 11

Write

$$\begin{aligned} 3^{512} - 1 &= (3^{256} + 1)(3^{256} - 1) = (3^{256} + 1)(3^{128} + 1)(3^{128} - 1) \\ &= \dots = (3^{256} + 1)(3^{128} + 1) \cdots (3 + 1)(3 - 1). \end{aligned}$$

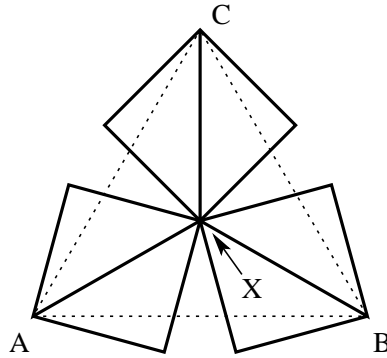
Now each factor $3^{2^k} + 1$, $k \geq 1$, is divisible by just one factor of 2, since $3^{2^k} + 1 = (3^2)^{2^{k-1}} + 1 \equiv 1^{2^{k-1}} + 1 = 2 \pmod{4}$. Thus we get 8 factors of 2 here, and the remaining terms $(3 + 1)(3 - 1) = 8$ give us 3 more factors of 2, for a total of 11.

24. We say a point is *contained* in a square if it is in its interior or on its boundary. Three unit squares are given in the plane such that there is a point contained in all three. Furthermore, three points A, B, C , are given, each contained in at least one of the squares. Find the maximum area of triangle ABC .

Solution: $3\sqrt{3}/2$

Let X be a point contained in all three squares. The distance from X to any point in any of the three squares is at most $\sqrt{2}$, the length of the diagonal of the squares. Therefore, triangle ABC is contained in a circle of radius $\sqrt{2}$, so its circumradius is at most $\sqrt{2}$. The triangle with greatest area that satisfies this property is the equilateral triangle in a circle of radius $\sqrt{2}$. (This can be proved, for example, by considering that the maximum altitude to any given side is obtained by putting the opposite vertex at the midpoint of its arc, and it follows that all the vertices are equidistant.) The equilateral triangle is also attainable, since making X the circumcenter and positioning the squares such that AX, BX , and CX are diagonals (of the three squares) and ABC is equilateral, leads to such a triangle. This triangle has area $3\sqrt{3}/2$, which may be calculated, for example, using the sine formula for area applied to ABX, ACX , and BCX , to get $3/2(\sqrt{2})^2 \sin 120^\circ$. (See diagram, next page.)

25. Suppose $x^3 - ax^2 + bx - 48$ is a polynomial with three positive roots p, q , and r such that $p < q < r$. What is the minimum possible value of $1/p + 2/q + 3/r$?



Solution: $\boxed{3/2}$

We know $pqr = 48$ since the product of the roots of a cubic is the constant term. Now,

$$\frac{1}{p} + \frac{2}{q} + \frac{3}{r} \geq 3\sqrt[3]{\frac{6}{pqr}} = \frac{3}{2}$$

by AM-GM, with equality when $1/p = 2/q = 3/r$. This occurs when $p = 2$, $q = 4$, $r = 6$, so $3/2$ is in fact the minimum possible value.

26. How many of the integers $1, 2, \dots, 2004$ can be represented as $(mn + 1)/(m + n)$ for positive integers m and n ?

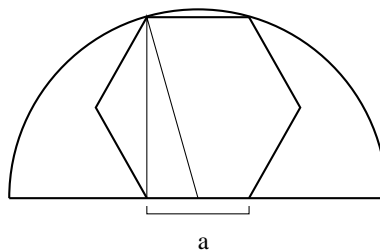
Solution: $\boxed{2004}$

For any positive integer a , we can let $m = a^2 + a - 1$, $n = a + 1$ to see that every positive integer has this property, so the answer is 2004.

27. A regular hexagon has one side along the diameter of a semicircle, and the two opposite vertices on the semicircle. Find the area of the hexagon if the diameter of the semicircle is 1.

Solution: $\boxed{3\sqrt{3}/26}$

The midpoint of the side of the hexagon on the diameter is the center of the circle. Draw the segment from this center to a vertex of the hexagon on the circle. This segment, whose length is $1/2$, is the hypotenuse of a right triangle whose legs have lengths $a/2$ and $a\sqrt{3}$, where a is a side of the hexagon. So $1/4 = a^2(1/4 + 3)$, so $a^2 = 1/13$. The hexagon consists of 6 equilateral triangles of side length a , so the area of the hexagon is $3a^2\sqrt{3}/2 = 3\sqrt{3}/26$.



28. Find the value of

$$\binom{2003}{1} + \binom{2003}{4} + \binom{2003}{7} + \cdots + \binom{2003}{2002}.$$

Solution: $\boxed{(2^{2003} - 2)/3}$

Let $\omega = -1/2 + i\sqrt{3}/2$ be a complex cube root of unity. Then, by the binomial theorem, we have

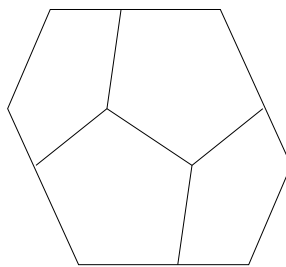
$$\begin{aligned} \omega^2(\omega + 1)^{2003} &= \binom{2003}{0}\omega^2 + \binom{2003}{1}\omega^3 + \binom{2003}{2}\omega^4 + \cdots + \binom{2003}{2003}\omega^{2005} \\ 2^{2003} &= \binom{2003}{0} + \binom{2003}{1} + \binom{2003}{2} + \cdots + \binom{2003}{2003} \\ \omega^{-2}(\omega^{-1} + 1)^{2003} &= \binom{2003}{0}\omega^{-2} + \binom{2003}{1}\omega^{-3} + \binom{2003}{2}\omega^{-4} + \cdots + \binom{2003}{2003}\omega^{-2005} \end{aligned}$$

If we add these together, then the terms $\binom{2003}{n}$ for $n \equiv 1 \pmod{3}$ appear with coefficient 3, while the remaining terms appear with coefficient $1 + \omega + \omega^2 = 0$. Thus the desired sum is just $(\omega^2(\omega + 1)^{2003} + 2^{2003} + \omega^{-2}(\omega^{-1} + 1)^{2003})/3$. Simplifying using $\omega + 1 = -\omega^2$ and $\omega^{-1} + 1 = -\omega$ gives $(-1 + 2^{2003} + -1)/3 = (2^{2003} - 2)/3$.

29. A regular dodecahedron is projected orthogonally onto a plane, and its image is an n -sided polygon. What is the smallest possible value of n ?

Solution: $\boxed{6}$

We can achieve 6 by projecting onto a plane perpendicular to an edge of the dodecahedron. Indeed, if we imagine viewing the dodecahedron in such a direction, then 4 of the faces are projected to line segments (namely, the two faces adjacent to the edge and the two opposite faces), and of the remaining 8 faces, 4 appear on the front of the dodecahedron and the other 4 are on the back. Thus, the dodecahedron appears as shown.



To see that we cannot do better, note that, by central symmetry, the number of edges of the projection must be even. So we just need to show that the answer cannot be 4. But if the projection had 4 sides, one of the vertices would give a projection forming an acute angle, which is not possible. So 6 is the answer.

30. We have an n -gon, and each of its vertices is labeled with a number from the set $\{1, \dots, 10\}$. We know that for any pair of distinct numbers from this set there is

at least one side of the polygon whose endpoints have these two numbers. Find the smallest possible value of n .

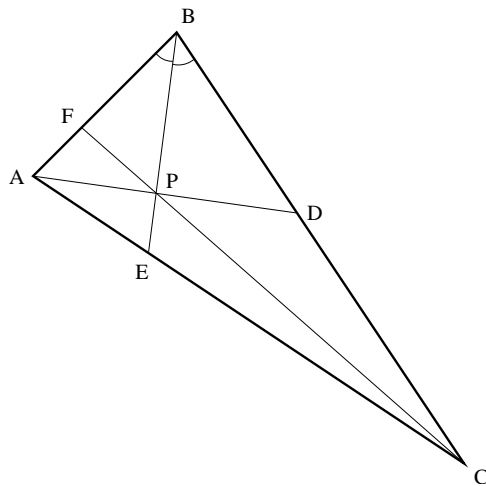
Solution: 50

Each number be paired with each of the 9 other numbers, but each vertex can be used in at most 2 different pairs, so each number must occur on at least $\lceil 9/2 \rceil = 5$ different vertices. Thus, we need at least $10 \cdot 5 = 50$ vertices, so $n \geq 50$.

To see that $n = 50$ is feasible, let the numbers $1, \dots, 10$ be the vertices of a complete graph. Then each vertex has degree 9, and there are $\binom{10}{2} = 45$ edges. If we attach extra copies of the edges 1-2, 3-4, 5-6, 7-8, and 9-10, then every vertex will have degree 10. In particular, the graph has an Eulerian tour, so we can follow this tour, successively numbering vertices of the 50-gon according to the vertices of the graph we visit. Then, for each edge of the graph, there will be a corresponding edge of the polygon with the same two vertex labels on its endpoints. It follows that every pair of distinct numbers occurs at the endpoints of some edge of the polygon, and so $n = 50$ is the answer.

31. P is a point inside triangle ABC , and lines AP, BP, CP intersect the opposite sides BC, CA, AB in points D, E, F , respectively. It is given that $\angle APB = 90^\circ$, and that $AC = BC$ and $AB = BD$. We also know that $BF = 1$, and that $BC = 999$. Find AF .

Solution: 499/500



Let $AC = BC = s$, $AB = BD = t$. Since BP is the altitude in isosceles triangle ABD , it bisects angle B . So, the Angle Bisector Theorem in triangle ABC gives $AE/EC = AB/BC = t/s$. Meanwhile, $CD/DB = (s - t)/t$. Now Ceva's theorem gives us

$$\frac{AF}{FB} = \left(\frac{AE}{EC}\right) \cdot \left(\frac{CD}{DB}\right) = \frac{s-t}{s}$$

$$\Rightarrow \frac{AB}{FB} = 1 + \frac{s-t}{s} = \frac{2s-t}{s} \Rightarrow FB = \frac{st}{2s-t}.$$

Now we know $s = 999$, but we need to find t given that $st/(2s-t) = FB = 1$. So $st = 2s - t \Rightarrow t = 2s/(s+1)$, and then

$$AF = FB \cdot \frac{AF}{FB} = 1 \cdot \frac{s-t}{s} = \frac{(s^2 - s)/(s+1)}{s} = \frac{s-1}{s+1} = \frac{499}{500}.$$

32. Define the sequence b_0, b_1, \dots, b_{59} by

$$b_i = \begin{cases} 1 & \text{if } i \text{ is a multiple of } 3 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{a_i\}$ be a sequence of elements of $\{0, 1\}$ such that

$$b_n \equiv a_{n-1} + a_n + a_{n+1} \pmod{2}$$

for $0 \leq n \leq 59$ ($a_0 = a_{60}$ and $a_{-1} = a_{59}$). Find all possible values of $4a_0 + 2a_1 + a_2$.

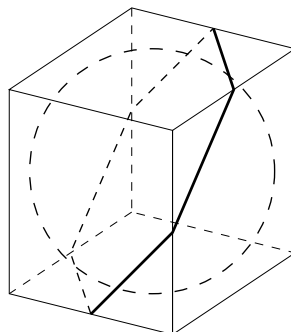
Solution: $\boxed{0, 3, 5, 6}$

Try the four possible combinations of values for a_0 and a_1 . Since we can write $a_n \equiv b_{n-1} - a_{n-2} - a_{n-1}$, these two numbers completely determine the solution $\{a_i\}$ beginning with them (if there is one). For $a_0 = a_1 = 0$, we can check that the sequence beginning $0, 0, 0, 0, 1, 1$ and repeating every 6 indices is a possible solution for $\{a_i\}$, so one possible value for $4a_0 + 2a_1 + a_2$ is 0. The other three combinations for a_0 and a_1 similarly lead to valid sequences (produced by repeating the sextuples $0, 1, 1, 1, 0, 1$; $1, 0, 1, 1, 1, 0$; $1, 1, 0, 1, 0, 1$, respectively); we thus obtain the values 3, 5, and 6.

33. A plane P slices through a cube of volume 1 with a cross-section in the shape of a regular hexagon. This cube also has an inscribed sphere, whose intersection with P is a circle. What is the area of the region inside the regular hexagon but outside the circle?

Solution: $\boxed{(3\sqrt{3} - \pi)/4}$

One can show that the hexagon must have as its vertices the midpoints of six edges of the cube, as illustrated; for example, this readily follows from the fact that opposite sides of the hexagons and the medians between them are parallel. We then conclude that the side of the hexagon is $\sqrt{2}/2$ (since it cuts off an isosceles triangle of leg $1/2$ from each face), so the area is $(3/2)(\sqrt{2}/2)^2(\sqrt{3}) = 3\sqrt{3}/4$. Also, the plane passes through the center of the sphere by symmetry, so it cuts out a cross section of radius $1/2$, whose area (which is contained entirely inside the hexagon) is then $\pi/4$. The sought area is thus $(3\sqrt{3} - \pi)/4$.



34. Find the number of 20-tuples of integers $x_1, \dots, x_{10}, y_1, \dots, y_{10}$ with the following properties:

- $1 \leq x_i \leq 10$ and $1 \leq y_i \leq 10$ for each i ;

- $x_i \leq x_{i+1}$ for $i = 1, \dots, 9$;
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$.

Solution: $\boxed{\binom{109}{10}}$

By setting $z_i = 10x_i + y_i$, we see that the problem is equivalent to choosing a nondecreasing sequence of numbers z_1, z_2, \dots, z_{10} from the values $11, 12, \dots, 110$. Making a further substitution by setting $w_i = z_i - 11 + i$, we see that the problem is equivalent to choosing a strictly increasing sequence of numbers w_1, \dots, w_{10} from among the values $1, 2, \dots, 109$. There are $\binom{109}{10}$ ways to do this.

35. There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n .

Solution: $\boxed{106}$

The sum of the angles of an n -gon is $(n - 2)180$, so the average angle measure is $(n - 2)180/n$. The common difference in this arithmetic progression is at least 1, so the difference between the largest and smallest angles is at least $n - 1$. So the largest angle is at least $(n - 1)/2 + (n - 2)180/n$. Since the polygon is convex, this quantity is no larger than 179: $(n - 1)/2 + (n - 2)180/n \leq 179$, so that $360/n - n/2 \geq 1/2$. Multiplying by $2n$ gives $720 - n^2 \geq n$. So $n(n + 1) \leq 720$, which forces $n \leq 26$. Of course, since the common difference is an integer, and the angle measures are integers, $(n - 2)180/n$ must be an integer or a half integer, so $(n - 2)360/n = 360 - 720/n$ is an integer, and then $720/n$ must be an integer. This leaves only $n = 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20, 24$ as possibilities. When n is even, $(n - 2)180/n$ is not an angle of the polygon, but the mean of the two middle angles. So the common difference is at least 2 when $(n - 2)180/n$ is an integer. For $n = 20$, the middle angle is 162, so the largest angle is at least $162 + 38/2 = 181$, since 38 is no larger than the difference between the smallest and largest angles. For $n = 24$, the middle angle is 165, again leading to a contradiction. So no solution exists for $n = 20, 24$. All of the others possess solutions:

n	angles
3	59, 60, 61
4	87, 89, 91, 93
5	106, 107, 108, 109, 110
6	115, 117, 119, 121, 123, 125
8	128, 130, 132, 134, 136, 138, 140, 142
9	136, \dots , 144
10	135, 137, 139, \dots , 153
12	139, 141, 143, \dots , 161
15	149, 150, \dots , 163
16	150, 151, \dots , 165
18	143, 145, \dots , 177

(These solutions are quite easy to construct.) The desired value is then $3 + 4 + 5 + 6 + 8 + 9 + 10 + 12 + 15 + 16 + 18 = 106$.

36. For a string of P 's and Q 's, the *value* is defined to be the product of the positions of the P 's. For example, the string $PPQPQQ$ has value $1 \cdot 2 \cdot 4 = 8$.

Also, a string is called *antipalindromic* if writing it backwards, then turning all the P 's into Q 's and vice versa, produces the original string. For example, $PPQPQQ$ is antipalindromic.

There are 2^{1002} antipalindromic strings of length 2004. Find the sum of the reciprocals of their values.

Solution: $\boxed{2005^{1002}/2004!}$

Consider the product

$$\left(\frac{1}{1} + \frac{1}{2004}\right) \left(\frac{1}{2} + \frac{1}{2003}\right) \left(\frac{1}{3} + \frac{1}{2002}\right) \cdots \left(\frac{1}{1002} + \frac{1}{1003}\right).$$

This product expands to 2^{1002} terms, and each term gives the reciprocal of the value of a corresponding antipalindromic string of P 's and Q 's as follows: if we choose the term $1/n$ for the n th factor, then our string has a P in position n and Q in position $2005 - n$; if we choose the term $1/(2005 - n)$, then we get a Q in position n and P in position $2005 - n$. Conversely, each antipalindromic string has its value represented by exactly one of our 2^{1002} terms. So the value of the product is the number we are looking for. But when we simplify this product, the n th factor becomes $1/n + 1/(2005 - n) = 2005/n(2005 - n)$. Multiplying these together, we get 1002 factors of 2005 in the numerator and each integer from 1 to 2004 exactly once in the denominator, for a total of $2005^{1002}/2004!$.

37. Simplify $\prod_{k=1}^{2004} \sin(2\pi k/4009)$.

Solution: $\boxed{\frac{\sqrt{4009}}{2^{2004}}}$

Let $\zeta = e^{2\pi i/4009}$ so that $\sin(2\pi k/4009) = \frac{\zeta^k - \zeta^{-k}}{2i}$ and $x^{4009} - 1 = \prod_{k=0}^{4008} (x - \zeta^k)$. Hence $1 + x + \cdots + x^{4008} = \prod_{k=1}^{4008} (x - \zeta^k)$. Comparing constant coefficients gives $\prod_{k=1}^{4008} \zeta^k = 1$, setting $x = 1$ gives $\prod_{k=1}^{4008} (1 - \zeta^k) = 4009$, and setting $x = -1$ gives $\prod_{k=1}^{4008} (1 + \zeta^k) = 1$. Now, note that $\sin(2\pi(4009 - k)/4009) = -\sin(2\pi k/4009)$, so

$$\begin{aligned} \left(\prod_{k=1}^{2004} \sin(2\pi k/4009)\right)^2 &= (-1)^{2004} \prod_{k=1}^{4008} \sin(2\pi k/4009) \\ &= \prod_{k=1}^{4008} \frac{\zeta^k - \zeta^{-k}}{2i} \\ &= \frac{1}{(2i)^{4008}} \prod_{k=1}^{4008} \frac{\zeta^{2k} - 1}{\zeta^k} \\ &= \frac{1}{2^{4008}} \prod_{k=1}^{4008} (\zeta^{2k} - 1) \\ &= \frac{1}{2^{4008}} \prod_{k=1}^{4008} (\zeta^k - 1)(\zeta^k + 1) \\ &= \frac{4009 \cdot 1}{2^{4008}}. \end{aligned}$$

However, $\sin(x)$ is nonnegative on the interval $[0, \pi]$, so our product is positive. Hence it is $\frac{\sqrt{4009}}{2^{2004}}$.

38. Let $S = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \dots, p_n \text{ are distinct primes and } p_1, \dots, p_n < 30\}$. Assume 1 is in S . Let a_1 be an element of S . We define, for all positive integers n :

$$a_{n+1} = a_n / (n + 1) \quad \text{if } a_n \text{ is divisible by } n + 1;$$

$$a_{n+1} = (n + 2)a_n \quad \text{if } a_n \text{ is not divisible by } n + 1.$$

How many distinct possible values of a_1 are there such that $a_j = a_1$ for infinitely many j 's?

Solution: 512

If a_1 is odd, then we can see by induction that $a_j = (j+1)a_1$ when j is even and $a_j = a_1$ when j is odd (using the fact that no even j can divide a_1). So we have infinitely many j 's for which $a_j = a_1$.

If $a_1 > 2$ is even, then a_2 is odd, since $a_2 = a_1/2$, and a_1 may have only one factor of 2. Now, in general, let $p = \min(\{p_1, \dots, p_n\} \setminus \{2\})$. Suppose $1 < j < p$. By induction, we have $a_j = (j+1)a_1/2$ when j is odd, and $a_j = a_1/2$ when j is even. So $a_i \neq a_1$ for all $1 < j < p$. It follows that $a_p = a_1/2p$. Then, again using induction, we get for all nonnegative integers k that $a_{p+k} = a_p$ if k is even, and $a_{p+k} = (p+k+1)a_p$ if k is odd. Clearly, $a_p \neq a_1$ and $p+k+1 \neq 2p$ when k is odd (the left side is odd, and the right side even). It follows that $a_j = a_1$ for no $j > 1$. Finally, when $a_1 = 2$, we can check inductively that $a_j = j+1$ for j odd and $a_j = 1$ for j even.

So our answer is just the number of odd elements in S . There are 9 odd prime numbers smaller than 30, so the answer is $2^9 = 512$.

39. You want to arrange the numbers $1, 2, 3, \dots, 25$ in a sequence with the following property: if n is divisible by m , then the n th number is divisible by the m th number. How many such sequences are there?

Solution: 24

Let the rearranged numbers be a_1, \dots, a_{25} . The number of pairs (n, m) with $n \mid m$ must equal the number of pairs with $a_n \mid a_m$, but since each pair of the former type is also of the latter type, the converse must be true as well. Thus, $n \mid m$ if and only if $a_n \mid a_m$. Now for each $n = 1, 2, \dots, 6$, the number of values divisible by n uniquely determines n , so $n = a_n$. Similarly, 7, 8 must either be kept fixed by the rearrangement or interchanged, because they are the only values that divide exactly 2 other numbers in the sequence; since 7 is prime and 8 is not, we conclude they are kept fixed. Then we can easily check by induction that $n = a_n$ for all larger composite numbers $n \leq 25$ (by using $m = a_m$ for all proper factors m of n) and $n = 11$ (because it is the only prime that divides exactly 1 other number). So we have only the primes $n = 13, 17, 19, 23$ left to rearrange, and it is easily seen that these can be permuted arbitrarily, leaving $4!$ possible orderings altogether.

40. You would like to provide airline service to the 10 cities in the nation of Schizophrenia, by instituting a certain number of two-way routes between cities. Unfortunately, the government is about to divide Schizophrenia into two warring countries of five cities

each, and you don't know which cities will be in each new country. All airplane service between the two new countries will be discontinued. However, you want to make sure that you set up your routes so that, for any two cities in the same new country, it will be possible to get from one city to the other (without leaving the country).

What is the minimum number of routes you must set up to be assured of doing this, no matter how the government divides up the country?

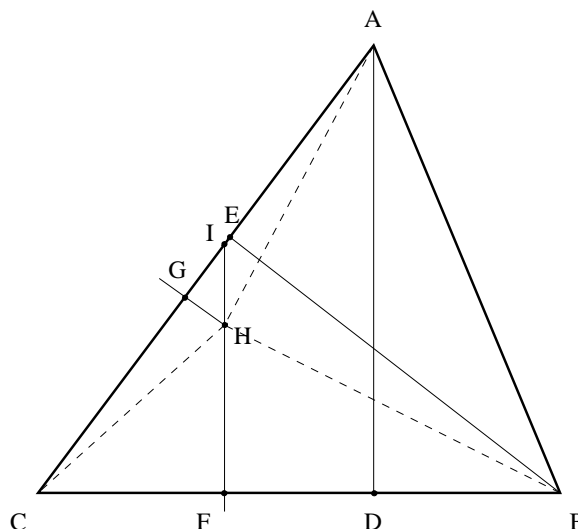
Solution: 30

Each city C must be directly connected to at least 6 other cities, since otherwise the government could put C in one country and all its connecting cities in the other country, and there would be no way out of C . This means that we have 6 routes for each of 10 cities, counted twice (since each route has two endpoints) $\Rightarrow 6 \cdot 10/2 = 30$ routes. On the other hand, this is enough: picture the cities arranged around a circle, and each city connected to its 3 closest neighbors in either direction. Then if C and D are in the same country but mutually inaccessible, this means that on each arc of the circle between C and D , there must be (at least) three consecutive cities in the other country. Then this second country would have 6 cities, which is impossible. So our arrangement achieves the goal with 30 routes.

41. A tetrahedron has all its faces triangles with sides 13, 14, 15. What is its volume?

Solution: $42\sqrt{55}$

Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let AD, BE be altitudes. Then $BD = 5, CD = 9$. (If you don't already know this, it can be deduced from the Pythagorean Theorem: $CD^2 - BD^2 = (CD^2 + AD^2) - (BD^2 + AD^2) = AC^2 - AB^2 = 56$, while $CD + BD = BC = 14$, giving $CD - BD = 56/14 = 4$, and now solve the linear system.) Also, $AD = \sqrt{AB^2 - BD^2} = 12$. Similar reasoning gives $AE = 33/5, EC = 42/5$.



Now let F be the point on BC such that $CF = BD = 5$, and let G be on AC such that $CG = AE = 33/5$. Imagine placing face ABC flat on the table, and letting X be a point in space with $CX = 13, BX = 14$. By mentally rotating triangle BCX

about line BC , we can see that X lies on the plane perpendicular to BC through F . In particular, this holds if X is the fourth vertex of our tetrahedron $ABCX$. Similarly, X lies on the plane perpendicular to AC through G . Let the mutual intersection of these two planes and plane ABC be H . Then XH is the altitude of the tetrahedron.

To find XH , extend FH to meet AC at I . Then $\triangle CFI \sim \triangle CDA$, a 3-4-5 triangle, so $FI = CF \cdot 4/3 = 20/3$, and $CI = CF \cdot 5/3 = 25/3$. Then $IG = CI - CG = 26/15$, and $HI = IG \cdot 5/4 = 13/6$. This leads to $HF = FI - HI = 9/2$, and finally $XH = \sqrt{XF^2 - HF^2} = \sqrt{AD^2 - HF^2} = 3\sqrt{55}/2$.

Now $XABC$ is a tetrahedron whose base $\triangle ABC$ has area $AD \cdot BC/2 = 12 \cdot 14/2 = 84$, and whose height XH is $3\sqrt{55}/2$, so its volume is $(84)(3\sqrt{55}/2)/3 = 42\sqrt{55}$.

42. S is a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. Suppose that the number N of elements of S with absolute value at most 1 is finite. What is the largest possible value of N ?

Solution: 13

First, if S contained some $u \neq 0$ with absolute value < 1 , then (by the first condition) every power of u would be in S , and S would contain infinitely many different numbers of absolute value < 1 . This is a contradiction. Now suppose S contains some number u of absolute value 1 and argument θ . If θ is not an integer multiple of $\pi/6$, then u has some power v whose argument lies strictly between $\theta + \pi/3$ and $\theta + \pi/2$. Then $u^2 + v^2 = u^2(1 + (v/u)^2)$ has absolute value between 0 and 1, since $(v/u)^2$ lies on the unit circle with angle strictly between $2\pi/3$ and π . But $u^2 + v^2 \in S$, so this is a contradiction.

This shows that the only possible elements of S with absolute value ≤ 1 are 0 and the points on the unit circle whose arguments are multiples of $\pi/6$, giving $N \leq 1 + 12 = 13$. To show that $N = 13$ is attainable, we need to show that there exists a possible set S containing all these points. Let T be the set of all numbers of the form $a + b\omega$, where a, b are integers and ω is a complex cube root of 1. Since $\omega^2 = -1 - \omega$, T is closed under multiplication and addition. Then, if we let S be the set of numbers u such that $u^2 \in T$, S has the required properties, and it contains the 13 complex numbers specified, so we're in business.

43. Write down an integer from 0 to 20 inclusive. This problem will be scored as follows: if N is the second-largest number from among the responses submitted, then each team that submits N gets N points, and everyone else gets zero. (If every team picks the same number then nobody gets any points.)

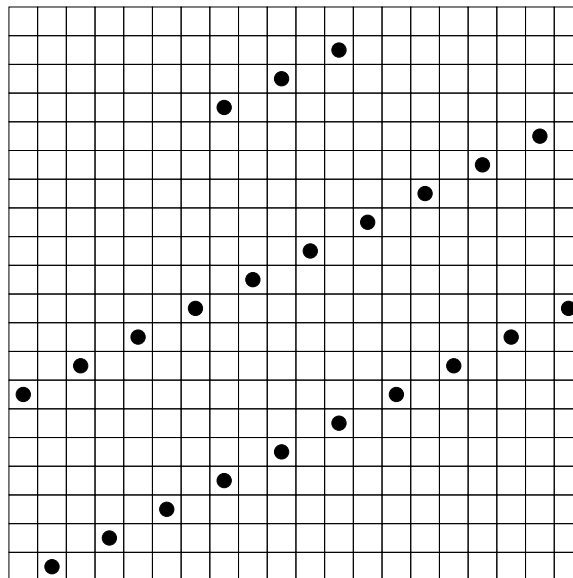
Solution: The only Nash equilibria of this game (where each team plays its best possible move given the other teams' choices) are fairly degenerate: every team but one plays 1, and the remaining team is more likely to choose 2 than any higher number. Of course, we cannot assume perfectly rational play in reality — nor are the utility functions the same, since the goal is to score higher than other teams, not to maximize one's own expected number of points. It will be interesting to see what the submissions are.

44. Shown on your answer sheet is a 20×20 grid. Place as many queens as you can so that each of them attacks at most one other queen. (A queen is a chess piece that can

move any number of squares horizontally, vertically, or diagonally.) It's not very hard to get 20 queens, so you get no points for that, but you get 5 points for each further queen beyond 20. You can mark the grid by placing a dot in each square that contains a queen.

Solution: An elementary argument shows there cannot be more than 26 queens: we cannot have more than 2 in a row or column (or else the middle queen would attack the other two), so if we had 27 queens, there would be at least 7 columns with more than one queen and thus at most 13 queens that are alone in their respective columns. Similarly, there would be at most 13 queens that are alone in their respective rows. This leaves $27 - 13 - 13 = 1$ queen who is not alone in her row or column, and she therefore attacks two other queens, contradiction.

Of course, this is not a very strong argument since it makes no use of the diagonals. The best possible number of queens is not known to us; the following construction gives 23:



45. A *binary string of length n* is a sequence of n digits, each of which is 0 or 1. The *distance* between two binary strings of the same length is the number of positions in which they disagree; for example, the distance between the strings 01101011 and 00101110 is 3 since they differ in the second, sixth, and eighth positions.

Find as many binary strings of length 8 as you can, such that the distance between any two of them is at least 3. You get one point per string.

Solution: The maximum possible number of such strings is 20. An example of a set

attaining this bound is

00000000	00110101
11001010	10011110
11100001	01101011
11010100	01100110
10111001	10010011
01111100	11001101
00111010	10101100
01010111	11110010
00001111	01011001
10100111	11111111

This example is taken from page 57 of F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error Correcting Codes* (New York: Elsevier Publishing, 1977). The proof that 20 is the best possible is elementary but too long to reproduce here; see pages 537–541 of MacWilliams and Sloane for details.

In general, a set of M strings of length n such that any two have a distance of at least d is called an (n, M, d) -code. These objects are of basic importance in coding theory, which studies how to transmit information through a channel with a known error rate. For example, since the code given above has minimum distance 3, I can transmit to you a message consisting of strings in this code, and even if there is a possible error rate of one digit in each string, you will still be able to determine the intended message uniquely.

Harvard-MIT Mathematics Tournament

February 28, 2004

Team Round — Solutions

A Build-It-Yourself Table

An infinite table of nonnegative integers is constructed as follows: in the top row, some number is 1 and all other numbers are 0's; in each subsequent row, every number is the sum of some two of the three closest numbers in the preceding row. An example of such a table is shown below.

$$\begin{array}{cccccccccc} \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & 3 & 3 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 2 & 4 & 4 & 6 & 3 & 2 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The top row (with the one 1) is called row 0; the next row is row 1; the next row is row 2, and so forth.

Note that the following problems require you to prove the statements for *every* table that can be constructed by the process described above, not just for the example shown.

1. Show that any number in row n (for $n > 0$) is at most 2^{n-1} .

Solution: We use induction on n . It is clear that any number in row 1 is at most $1 = 2^0$. Now, if every number in row n is at most 2^{n-1} , then every number in row $n+1$ is the sum of two numbers in row n and so is at most $2^{n-1} + 2^{n-1} = 2^n$. This gives the induction step, and the result follows.

2. What is the earliest row in which the number 2004 may appear?

Solution:

$$\begin{array}{cccccc} \dots & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & 0 & \dots \\ \dots & 0 & 1 & 2 & 2 & 0 & \dots \\ \dots & 0 & 3 & 4 & 4 & 0 & \dots \\ \dots & 0 & 7 & 8 & 8 & 0 & \dots \\ \dots & 0 & 15 & 16 & 16 & 0 & \dots \\ \dots & 0 & 31 & 31 & 32 & 0 & \dots \\ \dots & 0 & 62 & 63 & 63 & 0 & \dots \\ \dots & 0 & 125 & 125 & 126 & 0 & \dots \\ \dots & 0 & 250 & 251 & 251 & 0 & \dots \\ \dots & 0 & 501 & 501 & 502 & 0 & \dots \\ \dots & 0 & 1002 & 1002 & 1003 & 0 & \dots \\ \dots & 0 & 2004 & 2004 & 2005 & 0 & \dots \end{array}$$

By the previous problem, it cannot appear before row 12. By starting off the table as shown above, we see that row 12 is possible, so this is the answer.

3. Let

$$S(n, r) = \binom{n-1}{r-1} + \binom{n-1}{r} + \binom{n-1}{r+1} + \cdots + \binom{n-1}{n-1}$$

for all $n, r > 0$, and in particular $S(n, r) = 0$ if $r > n > 0$. Prove that the number in row n of the table, r columns to the left of the 1 in the top row, is at most $S(n, r)$. (**Hint:** First prove that $S(n-1, r-1) + S(n-1, r) = S(n, r)$.)

Solution: First, we prove the statement in the hint: adding the i th term of the sum for $S(n-1, r-1)$ to the i th term for $S(n-1, r)$, for each i , we get that $S(n-1, r-1) + S(n-1, r)$ equals

$$\begin{aligned} & \left(\binom{n-2}{r-2} + \binom{n-2}{r-1} \right) + \left(\binom{n-2}{r-1} + \binom{n-2}{r} \right) + \cdots \\ & \cdots + \left(\binom{n-2}{n-3} + \binom{n-2}{n-2} \right) + \binom{n-2}{n-2} \\ & = \binom{n-1}{r-1} + \binom{n-1}{r} + \cdots + \binom{n-1}{n-1} = S(n, r). \end{aligned}$$

Now we can prove the main statement by induction on n . The base case $n = 1$ is clear. If the statement holds for $n-1$, then first suppose $r > 1$. Then the number in row n , r columns to the left, is the sum of two of the three numbers above it, which, by the induction hypothesis, are at most $S(n-1, r-1)$, $S(n-1, r)$, $S(n-1, r+1)$ respectively. Since the first two of these are greater than the last (because the summation formula gives $S(n-1, r-1) = S(n-1, r) + \binom{n-1}{r-2}$ and $S(n-1, r) = S(n-1, r+1) + \binom{n-1}{r-1}$), we have an upper bound of $S(n-1, r-1) + S(n-1, r) = S(n, r)$ by the above. So the result follows by induction. Finally, in the case $r = 1$, the quantity in question is just 2^{n-1} , and the result holds by Problem 1.

4. Show that the sum of all the numbers in row n is at most $(n+2)2^{n-1}$.

Solution: The previous problem gives an upper bound on the number located r columns to the left of the initial 1; adding over all $r = 1, 2, \dots, n$ gives

$$\sum_{s=0}^{n-1} (s+1) \binom{n-1}{s}$$

since the term $\binom{n-1}{s}$ occurs for the $s+1$ values $r = 1, 2, \dots, s+1$. But this sum equals $(n+1)2^{n-2}$. For example, add the sum to itself and reverse the terms of the second sum to get

$$\begin{aligned} & \sum_{s=0}^{n-1} (s+1) \binom{n-1}{s} + \sum_{s=0}^{n-1} ([n-1-s]+1) \binom{n-1}{n-1-s} \\ & = \sum_{s=0}^{n-1} (n+1) \binom{n-1}{s} = (n+1) \sum_{s=0}^{n-1} \binom{n-1}{s} = (n+1)2^{n-1}, \end{aligned}$$

and our original sum is half of this.

So the sum of the terms in row n to the left of the central column is at most $(n+1)2^{n-2}$. Similarly, the sum of the terms to the right of the central column is at most $(n+1)2^{n-2}$. Adding these together, plus the upper bound of 2^{n-1} for the central number (Problem 1), gives our upper bound of $(n+2)2^{n-1}$ for the sum of all the numbers in the row.

A pair of successive numbers in the same row is called a *switch pair* if one number in the pair is even and the other is odd.

5. Prove that the number of switch pairs in row n is at most twice the number of odd numbers in row n .

Solution: Each switch pair contains an odd number, and each odd number can belong to at most two switch pairs (since it has only two neighbors).

6. Prove that the number of odd numbers in row n is at most twice the number of switch pairs in row $n - 1$.

Solution: Each odd number in row n is the sum of two of the three numbers above it in row $n - 1$; these three numbers cannot all have the same parity (or else any sum of two of them would be even), so somewhere among them is a switch pair. Since each switch pair in row $n - 1$ can contribute to at most two odd numbers in row n in this manner (namely, the two numbers immediately below the members of the pair), the result follows.

7. Prove that the number of switch pairs in row n is at most twice the number of switch pairs in row $n - 1$.

Solution: If we go sufficiently far to the left or right in row n , we get to zeroes. Therefore, row n consists of a finite number of “odd blocks” of the form

$$E O O O \dots O E$$

(where E represents an even number and O an odd number), which are separated by even numbers, except that one even number may simultaneously be an endpoint of two odd blocks. Each odd block contributes two switch pairs to row n , so it is enough to show that each odd block has a switch pair somewhere above it in row $n - 1$. But the odd block consists of at least one O between two E 's, making for at least three terms. If there were no switch pairs above the block, then in particular, the first three terms immediately above it would be all odd or all even, and then the second term in our block would have to be even, contradicting the assumption that it was O . This proves the result.

Written In The Stars

Suppose S is a finite set with a binary operation \star — that is, for any elements a, b of S , there is defined an element $a \star b$ of S . It is given that $(a \star b) \star (a \star b) = b \star a$ for all $a, b \in S$.

8. Prove that $a \star b = b \star a$ for all $a, b \in S$.

Solution: We have

$$\begin{aligned} a \star b &= (b \star a) \star (b \star a) \\ &= ([a \star b] \star [a \star b]) \star ([a \star b] \star [a \star b]) \\ &= [a \star b] \star [a \star b] \\ &= b \star a. \end{aligned}$$

Let T be the set of elements of the form $a \star a$ for $a \in S$.

9. If b is any element of T , prove that $b \star b = b$.

Solution: If $b \in T$, then $b = a \star a$ for some a , so $b \star b = (a \star a) \star (a \star a) = a \star a$ (by the given property) $= b$.

Now suppose further that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$. (Thus we can write an expression like $a \star b \star c \star d$ without ambiguity.)

10. Let a be an element of T . Let the *image* of a be the set of all elements of T that can be represented as $a \star b$ for some $b \in T$. Prove that if c is in the image of a , then $a \star c = c$.

Solution: Write $c = a \star b$, and then $a \star c = a \star a \star b = a \star b$ (since $a \star a = a$) $= c$.

11. Prove that there exists an element $a \in T$ such that the equation $a \star b = a$ holds for all $b \in T$.

Solution: Choose a whose image contains as few elements as possible — we know we can do this, since T , being a subset of S , is finite. We claim that this a works. Indeed, suppose c is in the image of a . Then, for any d in the image of c , $a \star (c \star d) = (a \star c) \star d = c \star d = d$, so d is also in the image of a . So the image of c is contained in the image of a . But a was chosen to have image as small as possible, so the two images must coincide. In particular, $a \star a = a$ is in the image of c . So

$$a = c \star a = a \star c = c.$$

This argument shows that a is the only element of the image of a , which gives what we wanted.

Alternative Solution: This can also be solved without using Problem 10: The product of any two elements of T is also in T , since commutativity and associativity give $(b \star b) \star (c \star c) = (b \star c) \star (b \star c)$ for $b, c \in S$. Then let a_1, a_2, \dots, a_n be all the elements of T , and put $a = a_1 \star a_2 \star \dots \star a_n$; this value does not depend on the ordering of the elements. If $b \in T$, then $a = c \star b$, where c is the \star -product of all elements of T different from b , and consequently $a \star b = (c \star b) \star b = c \star (b \star b) = c \star b = a$.

12. Prove that there exists an element $a \in S$ such that the equation $a \star b = a$ holds for all $b \in S$.

Solution: The same a as in the previous problem will do the trick. Indeed, for any $b \in S$, we have

$$a \star b = (b \star a) \star (b \star a) = (a \star a) \star (b \star b)$$

(we have used commutativity and associativity). But $a \star a = a$, and $b \star b \in T$, so this expression equals $a \star (b \star b) = a$, as required.

Sigma City

13. Let n be a positive odd integer. Prove that

$$\lfloor \log_2 n \rfloor + \lfloor \log_2(n/3) \rfloor + \lfloor \log_2(n/5) \rfloor + \lfloor \log_2(n/7) \rfloor + \dots + \lfloor \log_2(n/n) \rfloor = (n - 1)/2.$$

Solution: Note that $\lfloor \log_2 k \rfloor$ is the cardinality of the set $\{2, 4, 8, \dots, 2^{\lfloor \log_2 k \rfloor}\}$, i.e., the number of powers of 2 that are even and are at most k . Then $\lfloor \log_2(n/k) \rfloor$ is the number of even powers of 2 that are at most n/k , or equivalently (multiplying each such number by k) the number of positive even numbers $\leq n$ whose greatest odd divisor is k . Summing over all odd k , we get the number of even numbers $\leq n$, which is just $(n-1)/2$.

Let $\sigma(n)$ denote the sum of the (positive) divisors of n , including 1 and n itself.

14. Prove that

$$\sigma(1) + \sigma(2) + \sigma(3) + \cdots + \sigma(n) \leq n^2$$

for every positive integer n .

Solution: The i th term on the left is the sum of all d dividing i . If we write this sum out explicitly, then each term $d = 1, 2, \dots, n$ appears $\lfloor n/d \rfloor$ times — once for each multiple of d that is $\leq n$. Thus, the sum equals

$$\begin{aligned} \lfloor n/1 \rfloor + 2 \lfloor n/2 \rfloor + 3 \lfloor n/3 \rfloor + \cdots + n \lfloor n/n \rfloor &\leq n/1 + 2n/2 + 3n/3 + \cdots + n/n \\ &= n + n + \cdots + n \\ &= n^2. \end{aligned}$$

15. Prove that

$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + \frac{\sigma(3)}{3} + \cdots + \frac{\sigma(n)}{n} \leq 2n$$

for every positive integer n .

Solution: This is similar to the previous solution. If d is a divisor of i , then so is i/d , and $(i/d)/i = 1/d$. Summing over all d , we see that $\sigma(i)/i$ is the sum of the reciprocals of the divisors of i , for each positive integer i . So, summing over all i from 1 to n , we get the value $1/d$ appearing $\lfloor n/d \rfloor$ times, once for each multiple of d that is at most n . In particular, the sum is

$$\frac{1}{1} \lfloor \frac{n}{1} \rfloor + \frac{1}{2} \lfloor \frac{n}{2} \rfloor + \frac{1}{3} \lfloor \frac{n}{3} \rfloor + \cdots + \frac{1}{n} \lfloor \frac{n}{n} \rfloor < \frac{n}{1^2} + \frac{n}{2^2} + \cdots + \frac{n}{n^2}.$$

So now all we need is $1/1^2 + 1/2^2 + \cdots + 1/n^2 < 2$. This can be obtained from the classic formula $1/1^2 + 1/2^2 + \cdots = \pi^2/6$, or from the more elementary estimate

$$\begin{aligned} 1/2^2 + 1/3^2 + \cdots + 1/n^2 &< 1/(1 \cdot 2) + 1/(2 \cdot 3) + \cdots + 1/((n-1) \cdot n) \\ &= (1/1 - 1/2) + (1/2 - 1/3) + \cdots + (1/(n-1) - 1/n) \\ &= 1 - 1/n \\ &< 1. \end{aligned}$$

16. Now suppose again that n is odd. Prove that

$$\sigma(1) \lfloor \log_2 n \rfloor + \sigma(3) \lfloor \log_2(n/3) \rfloor + \sigma(5) \lfloor \log_2(n/5) \rfloor + \cdots + \sigma(n) \lfloor \log_2(n/n) \rfloor < n^2/8.$$

Solution: The term $\sigma(i) \lfloor \log_2(n/i) \rfloor$ is the sum of the divisors of i times the number of even numbers $\leq n$ whose greatest odd divisor is i . Thus, summing over all odd i ,

we get the sum of d over all pairs (d, j) , where $j < n$ is even and d is an odd divisor of j . Each odd number d then appears $\lfloor n/2d \rfloor$ times, since this is the number of even numbers $< n$ that have d as a divisor. So the sum equals

$$\begin{aligned} \lfloor n/2 \rfloor + 3 \lfloor n/6 \rfloor + 5 \lfloor n/10 \rfloor + \cdots + n \lfloor n/2n \rfloor \\ \leq (n-1)/2 + 3(n-1)/6 + \cdots + m(n-1)/2m, \end{aligned}$$

where m is the greatest odd integer less than $n/2$. (We can ignore the terms $d \lfloor n/2d \rfloor$ for $d > m$ because these floors are zero.) This expression equals

$$(n-1)/2 + (n-1)/2 + \cdots + (n-1)/2 = (n-1)(m+1)/4 \leq (n-1)(n+1)/8,$$

which is less than $n^2/8$, as required.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Algebra Subject Test — Solutions

1. Find the smallest value of x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a .

Solution: 49

We want to find the smallest value of x such that $x \geq 14\sqrt{a} - a$ for all a . This is just the maximum possible value of $14\sqrt{a} - a = 49 - (\sqrt{a} - 7)^2$, which is clearly 49, achieved when $a = 49$.

2. Compute $\frac{\tan^2(20^\circ) - \sin^2(20^\circ)}{\tan^2(20^\circ) \sin^2(20^\circ)}$.

Solution: 1

If we multiply top and bottom by $\cos^2(20^\circ)$, the numerator becomes $\sin^2(20^\circ) \cdot (1 - \cos^2 20^\circ) = \sin^4(20^\circ)$, while the denominator becomes $\sin^4(20^\circ)$ also. So they are equal, and the ratio is 1.

3. Find the smallest n such that $n!$ ends in 290 zeroes.

Solution: 1170

Each 0 represents a factor of $10 = 2 \cdot 5$. Thus, we wish to find the smallest factorial that contains at least 290 2's and 290 5's in its prime factorization. Let this number be $n!$, so the factorization of $n!$ contains 2 to the power p and 5 to the power q , where

$$p = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \left\lfloor \frac{n}{2^3} \right\rfloor + \cdots \quad \text{and} \quad q = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots$$

(this takes into account one factor for each single multiple of 2 or 5 that is $\leq n$, an additional factor for each multiple of 2^2 or 5^2 , and so on). Naturally, $p \geq q$ because 2 is smaller than 5. Thus, we want to bring q as low to 290 as possible. If $q = \lfloor \frac{n}{5} \rfloor + \lfloor \frac{n}{5^2} \rfloor + \lfloor \frac{n}{5^3} \rfloor + \cdots$, we form a rough geometric sequence (by taking away the floor function) whose sum is represented by $290 \approx \frac{n/5}{1-1/5}$. Hence we estimate $n = 1160$, and this gives us $q = 288$. Adding 10 to the value of n gives the necessary two additional factors of 5, and so the answer is 1170.

4. Simplify: $2\sqrt{1.5 + \sqrt{2}} - (1.5 + \sqrt{2})$.

Solution: 1/2

The given expression equals $\sqrt{6 + 4\sqrt{2}} - (1.5 + \sqrt{2}) = \sqrt{6 + 2\sqrt{8}} - (1.5 + \sqrt{2})$. But on inspection, we see that $(\sqrt{2} + \sqrt{4})^2 = 6 + 2\sqrt{8}$, so the answer is $(\sqrt{2} + \sqrt{4}) - (1.5 + \sqrt{2}) = 2 - 3/2 = 1/2$.

5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, ..., n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

Solution: 2002

The sum of all the numbers is $n_1 + 2n_2 + \cdots + 2003n_{2003}$, while the number of numbers is $n_1 + n_2 + \cdots + n_{2003}$. Hence, the desired quantity equals

$$\begin{aligned} & (n_1 + 2n_2 + \cdots + 2003n_{2003}) - (n_1 + n_2 + \cdots + n_{2003}) \\ &= (\text{sum of the numbers}) - (\text{number of numbers}) \\ &= 2003 - (\text{number of numbers}), \end{aligned}$$

which is maximized when the number of numbers is minimized. Hence, we should have just one number, equal to 2003, and then the specified sum is $2003 - 1 = 2002$.

Comment: On the day of the contest, a protest was lodged (successfully) on the grounds that the use of the words “several” and “their” in the problem statement implies there must be at least 2 numbers. Then the answer is 2001, and this maximum is achieved by any two numbers whose sum is 2003. (me way.)

6. Let $a_1 = 1$, and let $a_n = \lfloor n^3/a_{n-1} \rfloor$ for $n > 1$. Determine the value of a_{999} .

Solution: 999

We claim that for any odd n , $a_n = n$. The proof is by induction. To get the base cases $n = 1, 3$, we compute $a_1 = 1$, $a_2 = \lfloor 2^3/1 \rfloor = 8$, $a_3 = \lfloor 3^3/8 \rfloor = 3$. And if the claim holds for odd $n \geq 3$, then $a_{n+1} = \lfloor (n+1)^3/n \rfloor = n^2 + 3n + 3$, so $a_{n+2} = \lfloor (n+2)^3/(n^2+3n+3) \rfloor = \lfloor (n^3+6n^2+12n+8)/(n^2+3n+2) \rfloor = \lfloor n+2+\frac{n^2+3n+2}{n^2+3n+3} \rfloor = n+2$. So the claim holds, and in particular, $a_{999} = 999$.

7. Let a, b, c be the three roots of $p(x) = x^3 + x^2 - 333x - 1001$. Find $a^3 + b^3 + c^3$.

Solution: 2003

We know that $x^3 + x^2 - 333x - 1001 = (x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$. Also, $(a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc = a^3 + b^3 + c^3$. Thus, $a^3 + b^3 + c^3 = (-1)^3 - 3(-1)(-333) + 3 \cdot 1001 = 2003$.

8. Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \cdots$.

Solution: 13/36

Each term takes the form

$$\frac{1}{n^2 + (n-2)} = \frac{1}{(n+2) \cdot (n-1)}.$$

Using the method of partial fractions, we can write (for some constants A, B)

$$\begin{aligned} \frac{1}{(n+2) \cdot (n-1)} &= \frac{A}{n+2} + \frac{B}{n-1} \\ \Rightarrow 1 &= A \cdot (n-1) + B \cdot (n+2) \end{aligned}$$

Setting $n = 1$ we get $B = \frac{1}{3}$, and similarly with $n = -2$ we get $A = -\frac{1}{3}$. Hence the sum becomes

$$\frac{1}{3} \cdot \left[\left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{3} - \frac{1}{6} \right) + \left(\frac{1}{4} - \frac{1}{7} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \cdots \right].$$

Thus, it telescopes, and the only terms that do not cancel produce a sum of $\frac{1}{3} \cdot \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = \frac{13}{36}$.

9. For how many integers n , for $1 \leq n \leq 1000$, is the number $\frac{1}{2} \binom{2n}{n}$ even?

Solution: 990

In fact, the expression $\binom{2n}{n}$ is always even, and it is not a multiple of four if and only if n is a power of 2, and there are 10 powers of 2 between 1 and 1000.

Let $f(N)$ denote the number of factors of 2 in N . Thus,

$$f(n!) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{8} \right\rfloor + \cdots = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor.$$

Also, it is clear that $f(ab) = f(a) + f(b)$ and $f\left(\frac{a}{b}\right) = f(a) - f(b)$ for integers a, b . Now for any positive integer n , let m be the integer such that $2^m \leq n < 2^{m+1}$. Then

$$\begin{aligned} f\left(\binom{2n}{n}\right) &= f\left(\frac{(2n)!}{n!n!}\right) = \sum_{k=1}^{\infty} \left\lfloor \frac{2n}{2^k} \right\rfloor - 2 \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \right) \\ &= \sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^{k-1}} \right\rfloor - 2 \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \right) \\ &= \lfloor n \rfloor - \left(\sum_{k=1}^{\infty} \left\lfloor \frac{n}{2^k} \right\rfloor \right) \\ &= n - \left(\sum_{k=1}^m \left\lfloor \frac{n}{2^k} \right\rfloor \right) \\ &\geq n - \left(\sum_{k=1}^m \frac{n}{2^k} \right) \\ &= n - n \left(\frac{2^m - 1}{2^m} \right) = \frac{n}{2^m} \geq 1. \end{aligned}$$

Both equalities hold when $n = 2^m$, and otherwise, $f\left(\binom{2n}{n}\right) > 1$.

10. Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.

Solution: -5/21

Cross-multiplying gives $(x+7)P(2x) = 8xP(x+1)$. If P has degree n and leading coefficient c , then the leading coefficients of the two sides are $2^n c$ and $8c$, so $n = 3$. Now $x = 0$ is a root of the right-hand side, so it's a root of the left-hand side, so that $P(x) = xQ(x)$ for some polynomial $Q \Rightarrow 2x(x+7)Q(2x) = 8x(x+1)Q(x+1)$ or $(x+7)Q(2x) = 4(x+1)Q(x+1)$. Similarly, we see that $x = -1$ is a root of the left-hand side, giving $Q(x) = (x+2)R(x)$ for some polynomial $R \Rightarrow 2(x+1)(x+7)R(2x) = 4(x+1)(x+3)R(x+1)$, or $(x+7)R(2x) = 2(x+3)R(x+1)$. Now $x = -3$ is a root of the left-hand side, so $R(x) = (x+6)S(x)$ for some polynomial S .

At this point, $P(x) = x(x+2)(x+6)S(x)$, but P has degree 3, so S must be a constant. Since $P(1) = 1$, we get $S = 1/21$, and then $P(-1) = (-1)(1)(5)/21 = -5/21$.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Calculus Subject Test — Solutions

1. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

Solution: $\boxed{2/3}$

The probability of the point falling between a distance r and $r + dr$ from the center is the ratio of the area of the corresponding annulus to the area of the whole circle: $\frac{\pi[(r+dr)^2-r^2]}{\pi} \rightarrow \frac{2\pi r dr}{\pi} = 2r dr$ for small values of dr . Then the expected distance is $\int_0^1 r \cot 2r dr = \frac{2}{3}$.

2. A particle moves along the x -axis in such a way that its velocity at position x is given by the formula $v(x) = 2 + \sin x$. What is its acceleration at $x = \frac{\pi}{6}$?

Solution: $\boxed{5\sqrt{3}/4}$

Acceleration is given by $a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v = \cos x \cdot (2 + \sin x) = 5\sqrt{3}/4$.

3. What is the area of the region bounded by the curves $y = x^{2003}$ and $y = x^{1/2003}$ and lying above the x -axis?

Solution: $\boxed{1001/1002}$

The two curves intersect at $(0, 0)$ and $(1, 1)$, so the desired area is

$$\int_0^1 (x^{1/2003} - x^{2003}) dx = \left[\frac{x^{2004/2003}}{2004/2003} - \frac{x^{2004}}{2004} \right]_0^1 = \frac{1001}{1002}.$$

4. The sequence of real numbers x_1, x_2, x_3, \dots satisfies $\lim_{n \rightarrow \infty} (x_{2n} + x_{2n+1}) = 315$ and $\lim_{n \rightarrow \infty} (x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim_{n \rightarrow \infty} (x_{2n}/x_{2n+1})$.

Solution: $\boxed{-1}$

We have $\lim_{n \rightarrow \infty} (x_{2n+1} - x_{2n-1}) = \lim_{n \rightarrow \infty} [(x_{2n} + x_{2n+1}) - (x_{2n} + x_{2n-1})] = 315 - 2003 = -1688$; it follows that $x_{2n+1} \rightarrow -\infty$ as $n \rightarrow \infty$. Then

$$\lim_{n \rightarrow \infty} \frac{x_{2n}}{x_{2n+1}} = \lim_{n \rightarrow \infty} \frac{x_{2n} + x_{2n+1}}{x_{2n+1}} - 1 = -1,$$

since $x_{2n} + x_{2n+1} \rightarrow 315$ while $x_{2n+1} \rightarrow -\infty$.

5. Find the minimum distance from the point $(0, 5/2)$ to the graph of $y = x^4/8$.

Solution: $\boxed{\sqrt{17}/2}$

We want to minimize $x^2 + (x^4/8 - 5/2)^2 = x^8/64 - 5x^4/8 + x^2 + 25/4$, which is equivalent to minimizing $z^4/4 - 10z^2 + 16z$, where we have set $z = x^2$. The derivative of this expression is $z^3 - 20z + 16$, which is seen on inspection to have 4 as a root, leading to the factorization $(z - 4)(z + 2 - 2\sqrt{2})(z + 2 + 2\sqrt{2})$. Since $z = x^2$ ranges over $[0, \infty)$, the possible minima are at $z = 0$, $z = -2 + 2\sqrt{2}$, and $z = 4$. However, the derivative is positive on $(0, -2 + 2\sqrt{2})$, so this leaves only 0 and 4 to be tried. We find that the minimum is in fact achieved at $z = 4$, so the closest point on the graph is given by $x = \pm 2$, with distance $\sqrt{2^2 + (2^4/8 - 5/2)^2} = \sqrt{17}/2$.

6. For n an integer, evaluate

$$\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2 - 0^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \cdots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right).$$

Solution: $\boxed{\pi/2}$

Note that $\frac{1}{\sqrt{n^2 - i^2}} = \frac{1}{n} \cdot \frac{1}{\sqrt{1 - (\frac{i}{n})^2}}$, so that the sum we wish to evaluate is just a Riemann sum. Then,

$$\lim_{n \rightarrow \infty} \left(\frac{1}{n} \sum_{i=0}^{n-1} \frac{1}{\sqrt{1 - (\frac{i}{n})^2}} \right) = \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx = [\sin^{-1} x]_0^1 = \frac{\pi}{2}.$$

7. For what value of $a > 1$ is

$$\int_a^{a^2} \frac{1}{x} \log \frac{x-1}{32} dx$$

minimum?

Solution: $\boxed{3}$

Let $f(a) = \int_a^{a^2} \frac{1}{x} \log \frac{x-1}{32} dx$. Then we want $\frac{df}{da} = 0$; by the Fundamental Theorem of Calculus and the chain rule, this implies that

$$2a \left(\frac{1}{a^2} \log \frac{a^2-1}{32} \right) - \frac{1}{a} \log \frac{a-1}{32} = \frac{d}{da} \left(\int_c^{a^2} \frac{1}{x} \log \frac{x-1}{32} dx - \int_c^a \frac{1}{x} \log \frac{x-1}{32} dx \right) = 0,$$

where c is any constant with $1 < c < a$. Then $2 \log \frac{a^2-1}{32} = \log \frac{a-1}{32}$, so that $(\frac{a^2-1}{32})^2 = \frac{a-1}{32}$. After canceling factors of $(a-1)/32$ (since $a > 1$), this simplifies to $(a^2-1)(a+1) = 32 \Rightarrow a^3 + a^2 - a - 33 = 0$, which in turn factors as $(a-3)(a^2 + 4a + 11) = 0$. The quadratic factor has no real solutions, so this leaves only $a = 3$. However, we have that $a > 1$, and we can check that $f(1) = 0$, $\lim_{a \rightarrow \infty} f(a) > 0$, and $f(3) < 0$, so the global minimum does occur at $a = 3$.

8. A right circular cone with a height of 12 inches and a base radius of 3 inches is filled with water and held with its vertex pointing downward. Water flows out through a hole at the vertex at a rate in cubic inches per second numerically equal to the height of the water in the cone. (For example, when the height of the water in the cone is 4 inches, water flows out at a rate of 4 cubic inches per second.) Determine how many seconds it will take for all of the water to flow out of the cone.

Solution: $\boxed{9\pi/2}$

When the water in the cone is h inches high, it forms a cone similar to the original, so that its base has radius $h/4$ and its volume is hence $\pi h^3/48$. The given condition then states that

$$\frac{d}{dt} \left(\frac{\pi h^3}{48} \right) = -h \Rightarrow \frac{\pi h^2}{16} \cdot \frac{dh}{dt} = -h \Rightarrow 2h \cdot \frac{dh}{dt} = -\frac{32}{\pi}.$$

Integrating with respect to t , we get that $h^2 = -32t/\pi + C$; setting $t = 0$, $h = 12$, we get $C = 144$. The cone empties when $h = 0$, so $0 = -32t/\pi + 144 \Rightarrow t = 9\pi/2$.

9. Two differentiable real functions $f(x)$ and $g(x)$ satisfy

$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$

for all x , and $f(0) = g(2003) = 1$. Find the largest constant c such that $f(2003) > c$ for all such functions f, g .

Solution: $\boxed{1 - \ln 2}$

Rearranging the given equation gives $f'(x)e^{-f(x)} = g'(x)e^{-g(x)}$ for all x , so $\frac{d}{dx}(e^{-f(x)} - e^{-g(x)}) = -f'(x)e^{-f(x)} + g'(x)e^{-g(x)} = 0$. Thus, $e^{-f(x)} - e^{-g(x)}$ is a constant, and it must be less than $e^{-f(0)} = e^{-1}$. Thus, $e^{-f(2003)} < e^{-g(2003)} + e^{-1} = 2e^{-1} = e^{\ln 2 - 1} \Rightarrow f(2003) > 1 - \ln 2$. On the other hand, we can find positive-valued functions $e^{-f(x)}, e^{-g(x)}$ that take on the required values at 0 and 2003 and have constant difference arbitrarily close to e^{-1} . For example, for arbitrarily large t , we can set $e^{-f(x)} = e^{-(t(2003-x)+1)} + e^{-1} - e^{-(2003t+1)}$ and $e^{-g(x)} = e^{-(t(2003-x)+1)}$, and we can check that the resulting functions f, g satisfy the required conditions. Thus, we can make $f(2003)$ arbitrarily close to $1 - \ln 2$, so this is the answer.

10. Evaluate

$$\int_{-\infty}^{\infty} \frac{1-x^2}{1+x^4} dx.$$

Solution: $\boxed{0}$

Let $S = \int_0^{\infty} 1/(x^4 + 1) dx$; note that the integral converges absolutely. Substituting $x = 1/u$, so that $dx = -1/u^2 du$, we have

$$\begin{aligned} S &= \int_0^{\infty} \frac{1}{1+x^4} dx = \int_{\infty}^0 \frac{1}{1+u^{-4}} \frac{du}{-u^2} = \int_{\infty}^0 \frac{-u^2}{u^4+1} du \\ &= \int_0^{\infty} \frac{u^2}{1+u^4} du = \int_0^{\infty} \frac{x^2}{1+x^4} dx \end{aligned}$$

(the manipulations are justified by absolute convergence), from which we see that $\int_0^{\infty} (1-x^2)/(1+x^4) dx = 0$. Since the integrand is an even function, it follows that the integral from $-\infty$ to ∞ is zero as well.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Combinatorics Subject Test — Solutions

1. You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, and all configurations of switches are equally likely. You perform the following operation: for each switch S , if the two switches next to S were initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?

Solution: $\boxed{1/2^{2002}}$

There are 2^{2003} equally likely starting configurations. All switches end up ON if and only if switches 1, 3, 5, 7, \dots , 2003, 2, 4, \dots , 2002 — i.e. all 2003 of them — were initially in the same position. This initial position can be ON or OFF, so this situation occurs with probability $2/2^{2003} = 1/2^{2002}$.

2. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

Solution: $\boxed{36}$

Since each row has only two squares, it is impossible for two marked squares to be in the same row. Therefore, exactly nine of the ten rows contain marked squares. Consider two cases:

Case 1: The first or last row is empty. These two cases are symmetrical, so assume without loss of generality that the first row is empty. There are two possibilities for the second row: either the first square is marked, or the second square is marked. Since the third row must contain a marked square, and it cannot be in the same column as the marked square in the second row, the third row is determined by the second. Similarly, all the remaining rows are determined. This leaves two possibilities if the first row is empty. Thus, there are four possibilities if the first or last row is empty.

Case 2: The empty row is not the first or last. Then, there are two blocks of (one or more) consecutive rows of marked squares. As above, the configuration of the rows in each of the two blocks is determined by the position of the marked square in the first of its rows. That makes $2 \times 2 = 4$ possible configurations. There are eight possibilities for the empty row, making a total of 32 possibilities in this case.

Together, there are 36 possible configurations of marked squares.

3. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

Solution: $\boxed{9/13}$

Consider the situation after two points. Daniel has a $9/25$ chance of winning, Scott, $4/25$, and there is a $12/25$ chance that the players will be tied. In the latter case, we revert to the original situation. In particular, after every two points, either the game

returns to the original situation, or one player wins. If it is given that the game lasts $2k$ rounds, then the players must be at par after $2(k-1)$ rounds, and then Daniel wins with probability $(9/25)/(9/25 + 4/25) = 9/13$. Since this holds for any k , we conclude that Daniel wins the game with probability $9/13$.

4. In a certain country, there are 100 senators, each of whom has 4 aides. These senators and aides serve on various committees. A committee may consist either of 5 senators, of 4 senators and 4 aides, or of 2 senators and 12 aides. Every senator serves on 5 committees, and every aide serves on 3 committees. How many committees are there altogether?

Solution: 160

If each senator gets a point for every committee on which she serves, and every aide gets $1/4$ point for every committee on which he serves, then the 100 senators get 500 points altogether, and the 400 aides get 300 points altogether, for a total of 800 points. On the other hand, each committee contributes 5 points, so there must be $800/5 = 160$ committees.

5. We wish to color the integers $1, 2, 3, \dots, 10$ in red, green, and blue, so that no two numbers a and b , with $a - b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

Solution: 186

The condition is equivalent to never having an odd number and an even number in the same color. We can choose one of the three colors for the odd numbers and distribute the other two colors freely among the 5 even numbers; this can be done in $3 \cdot 2^5 = 96$ ways. We can also choose one color for the even numbers and distribute the other two colors among the 5 odd numbers, again in 96 ways. This gives a total of 192 possibilities. However, we have double-counted the $3 \cdot 2 = 6$ cases where all odd numbers are the same color and all even numbers are the same color, so there are actually $192 - 6 = 186$ possible colorings.

6. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

Solution: 0

Color the chairs red and black in checkerboard fashion, with the center chair black. Then all 18 red chairs are initially occupied. Also notice that adjacent chairs have different colors. It follows that we need 18 black chairs to accommodate the reassignment, but there are only 17 of them. Thus, the answer is 0.

7. You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, \dots$. Each ball has probability $1/2^n$ of being put into box n . The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

Solution: 5/7

Notice that the answer is the sum of the probabilities that boxes $1, 2, \dots$, respectively, contain at least 2 balls, since those events are mutually exclusive. For box n , the probability of having at least 2 balls is

$$3[(1/2^n)^2(1 - 1/2^n)] + (1/2^n)^3 = 3/2^{2n} - 2/2^{3n} = 3/4^n - 2/8^n.$$

Summing to infinity using the geometric series formula, we get the answer $(3/4)/(1 - 1/4) - (2/8)/(1 - 1/8)$, which is equal to $5/7$.

8. For any subset $S \subseteq \{1, 2, \dots, 15\}$, a number n is called an “anchor” for S if n and $n + |S|$ are both members of S , where $|S|$ denotes the number of members of S . Find the average number of anchors over all possible subsets $S \subseteq \{1, 2, \dots, 15\}$.

Solution: 13/8

We first find the sum of the numbers of anchors of all subsets S ; this is equivalent to finding, for each n , the number of sets for which n is an anchor, and then summing over all n . Suppose that n is an anchor for S , and S has k elements. Then $n, n + k \in S \Rightarrow k \geq 2$, and also $n + k \leq 15$, or $k \leq 15 - n$. The remaining $k - 2$ elements of S (other than n and $n + k$) may be freely chosen from the remaining 13 members of $\{1, 2, \dots, 15\}$, so we get $\binom{13}{k-2}$ possible sets S . Summing over all allowed values of k , we then have $\binom{13}{0} + \binom{13}{1} + \binom{13}{2} + \dots + \binom{13}{13-n}$ sets with n as an anchor. If we sum over all $n = 1, 2, \dots, 13$ (since there are no possible values of k when $n > 13$), we get a total of

$$13\binom{13}{0} + 12\binom{13}{1} + 11\binom{13}{2} + \dots + \binom{13}{12}.$$

If we call this quantity A , then, by symmetry, $2A$ equals

$$\begin{aligned} & 13\binom{13}{0} + 12\binom{13}{1} + 11\binom{13}{2} + \dots + \binom{13}{12} \\ & \quad + \binom{13}{1} + 2\binom{13}{2} + \dots + 12\binom{13}{12} + 13\binom{13}{13} \\ = & 13 \left[\binom{13}{0} + \binom{13}{1} + \binom{13}{2} + \dots + \binom{13}{12} + \binom{13}{13} \right] = 13 \cdot 2^{13}. \end{aligned}$$

So $A = 13 \cdot 2^{12}$ is the total number of anchors over all possible sets S . Finally, to find the average number of anchors, we divide by the number of sets, which is 2^{15} ; thus, the answer is $13 \cdot 2^{12}/2^{15} = 13/8$.

9. At a certain college, there are 10 clubs and some number of students. For any two different students, there is some club such that exactly one of the two belongs to that club. For any three different students, there is some club such that either exactly one or all three belong to that club. What is the largest possible number of students?

Solution: 513

Let C be the set of clubs; each student then corresponds to a subset of C (the clubs to which that student belongs). The two-student condition implies that these subsets must be all distinct. Now (assuming there is more than one student) some student belongs to a nonempty set S of clubs. For every subset $T \subseteq C$, let $f(T)$ be the subset of C consisting of those clubs that are in exactly one of S and T (so that $f(T) = (S \cup T) - (S \cap T)$). It is straightforward to check that $f(f(T)) = T$ and $f(T) \neq T$, so that the collection of all 2^{10} subsets of C is partitioned into pairs $\{T, f(T)\}$. Moreover,

as long as S is distinct from T and $f(T)$, every club is in either none or exactly two of the sets S, T , and $f(T)$, so we cannot have a student corresponding to T and another corresponding to $f(T)$. This puts an upper bound of 513 possible students (one for S , one for $\emptyset = f(S)$, and one for each of the 511 other pairs). On the other hand, if we take some club c , we can have one student belonging to no clubs and 512 other students all belonging to c and to the 512 possible subsets of the other 9 clubs, respectively. It is readily checked that this arrangement meets the conditions — for the three-student condition, either all three students are in c , or one is the student who belongs to no clubs and we reduce to the two-student condition — so 513 is achievable.

10. A calculator has a display, which shows a nonnegative integer N , and a button, which replaces N by a random integer chosen uniformly from the set $\{0, 1, \dots, N - 1\}$, provided that $N > 0$. Initially, the display holds the number $N = 2003$. If the button is pressed repeatedly until $N = 0$, what is the probability that the numbers 1, 10, 100, and 1000 will each show up on the display at some point?

Solution: 1/2224222

First, we claim that if the display starts at some N , the probability that any given number $M < N$ will appear at some point is $1/(M + 1)$. We can show this by induction on N . If $N = M + 1$ (the base case), M can only be reached if it appears after the first step, and this occurs with probability $1/N = 1/(M + 1)$. If $N > M + 1$ and the claim holds for $N - 1$, then there are two possibilities starting from N . If the first step leads to $N - 1$ (this occurs with probability $1/N$), the probability of seeing M subsequently is $1/(M + 1)$ by the induction hypothesis. If the first step leads to something less than $N - 1$ (probability $(N - 1)/N$), then it leads to any of the integers $\{0, 1, \dots, N - 2\}$ with equal probability. But this is exactly what the first step would have been if we had started from $N - 1$; hence, the probability of seeing M is again $1/(M + 1)$ by induction. Thus, the overall probability of seeing M is $\frac{1}{N} \cdot \frac{1}{M + 1} + \frac{N - 1}{N} \cdot \frac{1}{M + 1} = 1/(M + 1)$, proving the induction step and the claim.

Now let $P(N, M)$ ($M < N$) be the probability of eventually seeing the number M if we start at N ; note that this is the same as the conditional probability of seeing M given that we see N . Hence, the desired probability is

$$P(2003, 1000) \cdot P(1000, 100) \cdot P(100, 10) \cdot P(10, 1) = \frac{1}{1001} \cdot \frac{1}{101} \cdot \frac{1}{11} \cdot \frac{1}{2} = \frac{1}{2224222}.$$

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 1 — Solutions

1. 10 people are playing musical chairs with n chairs in a circle. They can be seated in $7!$ ways (assuming only one person fits on each chair, of course), where different arrangements of the same people on chairs, even rotations, are considered different. Find n .

Solution: $\boxed{4}$

The number of ways 10 people can be seated on n chairs is $n!$ multiplied by the number of ways one can choose n people out of 10. Hence we must solve $7! = n! \cdot 10! / (n! \cdot (10 - n)!)$. This is equivalent to $(10 - n)! = 10! / 7! = 8 \cdot 9 \cdot 10 = 720 = 6!$. We therefore have $n = 4$.

2. $OPEN$ is a square, and T is a point on side NO , such that triangle TOP has area 62 and triangle TEN has area 10. What is the length of a side of the square?

Solution: $\boxed{12}$

$62 = PO \cdot OT / 2$ and $10 = EN \cdot TN / 2 = PO \cdot TN / 2$, so adding gives $72 = PO \cdot (OT + TN) / 2 = PO \cdot ON / 2 = PO^2 / 2 \Rightarrow PO = 12$.

3. There are 16 members on the Height-Measurement Matching Team. Each member was asked, "How many other people on the team — not counting yourself — are exactly the same height as you?" The answers included six 1's, six 2's, and three 3's. What was the sixteenth answer? (Assume that everyone answered truthfully.)

Solution: $\boxed{3}$

For anyone to have answered 3, there must have been exactly 4 people with the same height, and then each of them would have given the answer 3. Thus, we need at least four 3's, so 3 is the remaining answer. (More generally, a similar argument shows that the number of members answering n must be divisible by $n + 1$.)

4. How many 2-digit positive integers have an even number of positive divisors?

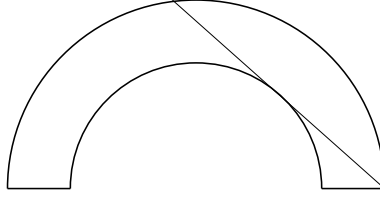
Solution: $\boxed{84}$

An integer has an odd number of divisors precisely if it is a square. So we take the 90 2-digit numbers $(10, 11, \dots, 99)$ and remove the 6 squares $(4^2, 5^2, \dots, 9^2)$, for a total of 84.

5. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?

Solution: $\boxed{18\pi}$

The maximal distance is as shown in the figure (next page). Call the radii R and r , $R > r$. Then $R^2 - r^2 = 6^2$ by the Pythagorean theorem, so the area is $(\pi/2) \cdot (R^2 - r^2) = 18\pi$.



6. In how many ways can 3 bottles of ketchup and 7 bottles of mustard be arranged in a row so that no bottle of ketchup is immediately between two bottles of mustard? (The bottles of ketchup are mutually indistinguishable, as are the bottles of mustard.)

Solution: 22

Consider the blocks of consecutive bottles of ketchup in such an arrangement. A block of just one bottle must occur at the beginning or the end of the row, or else it would be between two bottles of mustard. However, a block of two or three bottles can occur anywhere. We cannot have three blocks of one bottle each, since there are only two possible locations for such blocks. Thus, we either have a block of one bottle and a block of two, or one block of all three bottles. In the first case, if the single bottle occurs at the beginning of the row, then anywhere from 1 to 7 bottles of mustard may intervene before the block of 2 ketchup bottles, giving 7 possible arrangements. We likewise have 7 arrangements if the single bottle occurs at the end of the row. Finally, if there is just one block of three bottles, anywhere from 0 to 7 mustard bottles may precede it, giving 8 possible arrangements. So, altogether, we have $7 + 7 + 8 = 22$ configurations.

7. Find the real value of x such that $x^3 + 3x^2 + 3x + 7 = 0$.

Solution: $-1 - \sqrt[3]{6}$

Rewrite the equation as $(x + 1)^3 + 6 = 0$ to get $(x + 1)^3 = -6 \Rightarrow x + 1 = \sqrt[3]{-6} \Rightarrow x = -1 - \sqrt[3]{6}$.

8. A broken calculator has the $+$ and \times keys switched. For how many ordered pairs (a, b) of integers will it correctly calculate $a + b$ using the labelled $+$ key?

Solution: 2

We need $a + b = ab$, or $a = \frac{b}{b-1} = 1 - \frac{1}{b-1}$, so $1/(b-1)$ is an integer. Thus b must be 0 or 2, and a is 0 or 2, respectively. So there are 2.

9. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

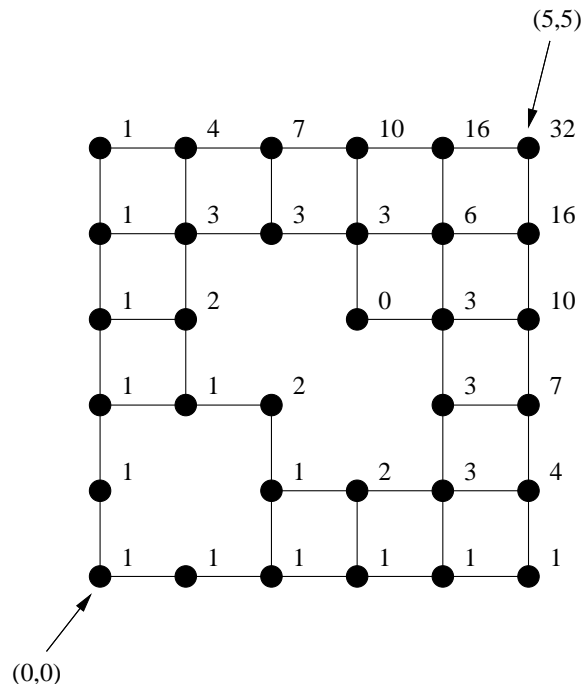
Solution: 1999

By induction, it follows easily that any triangulation of an n -gon inscribed in a circle has $n - 2$ triangles. A triangle is obtuse unless it contains the center of the circle in its interior (in which case it is acute) or on one of its edges (in which case it is right). It is then clear that there are at most 2 non-obtuse triangles, and 2 is achieved when the center of the circle is on one of the diagonals of the triangulation. So the minimum number of obtuse triangles is $2001 - 2 = 1999$.

10. Bessie the cow is trying to navigate her way through a field. She can travel only from lattice point to adjacent lattice point, can turn only at lattice points, and can travel only to the east or north. (A lattice point is a point whose coordinates are both integers.) $(0,0)$ is the southwest corner of the field. $(5,5)$ is the northeast corner of the field. Due to large rocks, Bessie is unable to walk on the points $(1,1)$, $(2,3)$, or $(3,2)$. How many ways are there for Bessie to travel from $(0,0)$ to $(5,5)$ under these constraints?

Solution: 32

In the figure, each point is labeled with the number of ways to reach that point. The numbers are successively computed as follows: The point $(0,0)$ can trivially be reached in 1 way. When Bessie reaches any subsequent point (x,y) (other than a rock), she can arrive either via a northward or an eastward step, so the number of ways she can reach that point equals the number of ways of reaching $(x-1,y)$ plus the number of ways of reaching $(x,y-1)$. By iterating this calculation, we eventually find that $(5,5)$ can be reached in 32 ways.



Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 2 — Solutions

1. A compact disc has the shape of a circle of diameter 5 inches with a 1-inch-diameter circular hole in the center. Assuming the capacity of the CD is proportional to its area, how many inches would need to be added to the outer diameter to double the capacity?

Solution: 2

Doubling the capacity is equivalent to doubling the area, which is initially $\pi[(5/2)^2 - (1/2)^2] = 6\pi$. Thus we want to achieve an area of 12π , so if the new diameter is d , we want $\pi[(d/2)^2 - (1/2)^2] = 12\pi \Rightarrow d = 7$. Thus we need to add 2 inches to the diameter.

2. You have a list of real numbers, whose sum is 40. If you replace every number x on the list by $1 - x$, the sum of the new numbers will be 20. If instead you had replaced every number x by $1 + x$, what would the sum then be?

Solution: 100

Let n be the number of numbers on the list. If each initial number is replaced by its negative, the sum will then be -40 , and adding 1 to every number on this list increases the sum by n , so $n - 40 = 20 \Rightarrow n = 60$. Then, if we had simply added 1 to each of the initial numbers (without negating first), the sum would increase to $40 + n = 40 + 60 = 100$.

3. How many positive rational numbers less than π have denominator at most 7 when written in lowest terms? (Integers have denominator 1.)

Solution: 54

We can simply list them. The table shows that there are $3+3+6+6+12+6+18 = 54$.

Denominator	Values
1	$\frac{1}{1}, \frac{2}{1}, \frac{3}{1}$
2	$\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$
3	$\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{7}{3}, \frac{8}{3}$
4	$\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}$
5	$\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{6}{5}, \frac{7}{5}, \frac{8}{5}, \frac{9}{5}, \frac{11}{5}, \frac{12}{5}, \frac{13}{5}, \frac{14}{5}$
6	$\frac{1}{6}, \frac{5}{6}, \frac{7}{6}, \frac{11}{6}, \frac{13}{6}, \frac{17}{6}$
7	$\frac{1}{7}, \frac{2}{7}, \dots, \frac{6}{7}, \frac{8}{7}, \frac{9}{7}, \dots, \frac{13}{7}, \frac{15}{7}, \frac{16}{7}, \dots, \frac{20}{7}$

4. In triangle ABC with area 51, points D and E trisect AB and points F and G trisect BC . Find the largest possible area of quadrilateral $DEFG$.

Solution: 17

Assume E is between D and B , and F is between G and B (the alternative is to switch two points, say D and E , which clearly gives a non-convex quadrilateral with smaller

area). If two triangles have their bases on the same line and the same opposite vertex, then it follows from the $\frac{1}{2}bh$ formula that their areas are in the same ratio as their bases. In particular (brackets denote areas),

$$\frac{[DBG]}{[ABC]} = \frac{[DBG]}{[ABG]} \cdot \frac{[ABG]}{[ABC]} = \frac{DB}{AB} \cdot \frac{BG}{BC} = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9},$$

and similarly $[EBF]/[ABC] = 1/9$. Subtracting gives $[DEFG]/[ABC] = 1/3$, so the answer is $[ABC]/3 = 17$.

5. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

Solution: 36

Since each row has only two squares, it is impossible for two marked squares to be in the same row. Therefore, exactly nine of the ten rows contain marked squares. Consider two cases:

Case 1: The first or last row is empty. These two cases are symmetrical, so assume without loss of generality that the first row is empty. There are two possibilities for the second row: either the first square is marked, or the second square is marked. Since the third row must contain a marked square, and it cannot be in the same column as the marked square in the second row, the third row is determined by the second. Similarly, all the remaining rows are determined. This leaves two possibilities if the first row is empty. Thus, there are four possibilities if the first or last row is empty.

Case 2: The empty row is not the first or last. Then, there are two blocks of (one or more) consecutive rows of marked squares. As above, the configuration of the rows in each of the two blocks is determined by the position of the marked square in the first of its rows. That makes $2 \times 2 = 4$ possible configurations. There are eight possibilities for the empty row, making a total of 32 possibilities in this case.

Together, there are 36 possible configurations of marked squares.

6. The numbers 112, 121, 123, 153, 243, 313, and 322 are among the rows, columns, and diagonals of a 3×3 square grid of digits (rows and diagonals read left-to-right, and columns read top-to-bottom). What 3-digit number completes the list?

Solution: 524

1	1	2
5	2	4
3	1	3

The center digit is the middle digit of 4 numbers (hence at least 3 members of the above list), so it must be 2. The top-left digit begins at least 2 members of the above list, so it must be 1 or 3. If it is 3, then after placing 313 we see that we need three more numbers starting with 3, impossible; hence, it is 1. So 243 and 313 must (in some order) be the last row and the last column, and now it is easy to complete the grid as shown; the answer is 524.

7. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

Solution: $\boxed{9/13}$

Consider the situation after two points. Daniel has a $9/25$ chance of winning, Scott, $4/25$, and there is a $12/25$ chance that the players will be tied. In the latter case, we revert to the original situation. In particular, after every two points, either the game returns to the original situation, or one player wins. If it is given that the game lasts $2k$ rounds, then the players must be at par after $2(k-1)$ rounds, and then Daniel wins with probability $(9/25)/(9/25 + 4/25) = 9/13$. Since this holds for any k , we conclude that Daniel wins the game with probability $9/13$.

8. If $x \geq 0$, $y \geq 0$ are integers, randomly chosen with the constraint $x + y \leq 10$, what is the probability that $x + y$ is even?

Solution: $\boxed{6/11}$

For each $p \leq 10$, if $x + y = p$, x can range from 0 to p , yielding $p + 1$ ordered pairs (x, y) . Thus there are a total of $1 + 2 + 3 + \dots + 11$ allowable ordered pairs (x, y) , but $1 + 3 + 5 + \dots + 11$ of these pairs have an even sum. So the desired probability is

$$\frac{1 + 3 + 5 + \dots + 11}{1 + 2 + 3 + \dots + 11} = \frac{6^2}{11 \cdot 6} = \frac{6}{11}.$$

9. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

Solution: $\boxed{0}$

Color the chairs red and black in checkerboard fashion, with the center chair black. Then all 18 red chairs are initially occupied. Also notice that adjacent chairs have different colors. It follows that we need 18 black chairs to accommodate the reassignment, but there are only 17 of them. Thus, the answer is 0.

10. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, \dots , n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \dots + 2002n_{2003}.$$

Solution: $\boxed{2002}$

The sum of all the numbers is $n_1 + 2n_2 + \dots + 2003n_{2003}$, while the number of numbers is $n_1 + n_2 + \dots + n_{2003}$. Hence, the desired quantity equals

$$\begin{aligned} & (n_1 + 2n_2 + \dots + 2003n_{2003}) - (n_1 + n_2 + \dots + n_{2003}) \\ &= (\text{sum of the numbers}) - (\text{number of numbers}) \end{aligned}$$

$$= 2003 - (\text{number of numbers}),$$

which is maximized when the number of numbers is minimized. Hence, we should have just one number, equal to 2003, and then the specified sum is $2003 - 1 = 2002$.

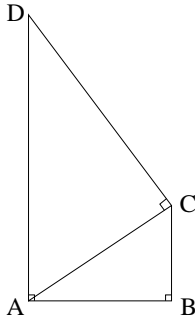
Comment: On the day of the contest, a protest was lodged (successfully) on the grounds that the use of the words “several” and “their” in the problem statement implies there must be at least 2 numbers. Then the answer is 2001, and this maximum is achieved by any two numbers whose sum is 2003.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Geometry Subject Test — Solutions

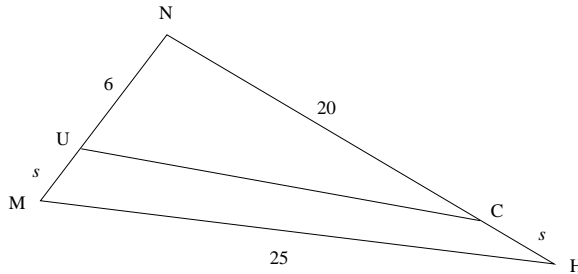
1. AD and BC are both perpendicular to AB , and CD is perpendicular to AC . If $AB = 4$ and $BC = 3$, find CD .



Solution: $20/3$

By Pythagoras in $\triangle ABC$, $AC = 5$. But $\angle CAD = 90^\circ - \angle BAC = \angle ACB$, so right triangles CAD, BCA are similar, and $CD/AC = BA/CB = 4/3 \Rightarrow CD = 20/3$.

2. As shown, U and C are points on the sides of triangle MNH such that $MU = s$, $UN = 6$, $NC = 20$, $CH = s$, $HM = 25$. If triangle UNC and quadrilateral $MUCH$ have equal areas, what is s ?



Solution: 4

Using brackets to denote areas, we have $[MCH] = [UNC] + [MUCH] = 2[UNC]$. On the other hand, triangles with equal altitudes have their areas in the same ratio as their bases, so

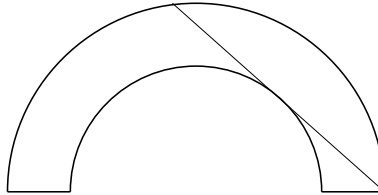
$$2 = \frac{[MNH]}{[UNC]} = \frac{[MNH]}{[MNC]} \cdot \frac{[MNC]}{[UNC]} = \frac{NH}{NC} \cdot \frac{MN}{UN} = \frac{s+20}{20} \cdot \frac{s+6}{6}.$$

Clearing the denominator gives $(s+20)(s+6) = 240$, and solving the quadratic gives $s = 4$ or -30 . Since $s > 0$, we must have $s = 4$.

3. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?

Solution: $\boxed{18\pi}$

The maximal distance is as shown in the figure. Call the radii R and r , $R > r$. Then $R^2 - r^2 = 6^2$ by the Pythagorean theorem, so the area is $(\pi/2) \cdot (R^2 - r^2) = 18\pi$.



4. Farmer John is inside of an ellipse with reflective sides, given by the equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b > 0$. He is standing at the point $(3, 0)$, and he shines a laser pointer in the y -direction. The light reflects off the ellipse and proceeds directly toward Farmer Brown, traveling a distance of 10 before reaching him. Farmer John then spins around in a circle; wherever he points the laser, the light reflects off the wall and hits Farmer Brown. What is the ordered pair (a, b) ?

Solution: $\boxed{(5, 4)}$

The points where the farmers are standing must be the foci of the ellipse, so they are $(3, 0)$ and $(-3, 0)$. If the total distance traveled is 10, then a must be half of that, or 5, since the distance traveled by a ray reflecting off the wall from when it leaves one focus to when it reaches the other focus is $2a$, the length of the major axis. If a focus is at $x = 3$, then we have $3 = \sqrt{a^2 - b^2} = \sqrt{25 - b^2}$, yielding $b = 4$.

5. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

Solution: $\boxed{1999}$

By induction, it follows easily that any triangulation of an n -gon inscribed in a circle has $n - 2$ triangles. A triangle is obtuse unless it contains the center of the circle in its interior (in which case it is acute) or on one of its edges (in which case it is right). It is then clear that there are at most 2 non-obtuse triangles, and 2 is achieved when the center of the circle is on one of the diagonals of the triangulation. So the minimum number of obtuse triangles is $2001 - 2 = 1999$.

6. Take a clay sphere of radius 13, and drill a circular hole of radius 5 through its center. Take the remaining "bead" and mold it into a new sphere. What is this sphere's radius?

Solution: $\boxed{12}$

Let r be the radius of the sphere. We take cross sections of the bead perpendicular to the line of the drill and compare them to cross sections of the sphere at the same

distance from its center. At a height h , the cross section of the sphere is a circle with radius $\sqrt{r^2 - h^2}$ and thus area $\pi(r^2 - h^2)$. At the same height, the cross section of the bead is an annulus with outer radius $\sqrt{13^2 - h^2}$ and inner radius 5, for an area of $\pi(13^2 - h^2) - \pi(5^2) = \pi(12^2 - h^2)$ (since $13^2 - 5^2 = 12^2$). Thus, if $r = 12$, the sphere and the bead will have the same cross-sectional area $\pi(12^2 - h^2)$ for $|h| \leq 12$ and 0 for $|h| > 12$. Since all the cross sections have the same area, the two clay figures then have the same volume. And certainly there is only one value of r for which the two volumes are equal, so $r = 12$ is the answer.

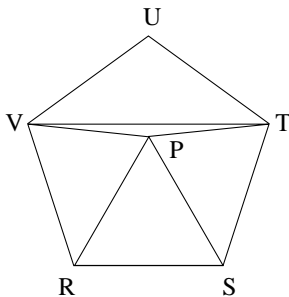
7. Let $RSTUV$ be a regular pentagon. Construct an equilateral triangle PRS with point P inside the pentagon. Find the measure (in degrees) of angle PTV .

Solution: 6

We have $\angle PRV = \angle SRV - \angle SRP = 108^\circ - 60^\circ = 48^\circ$. Since $PR = RS = RV$, triangle PRV is isosceles, so that $\angle VPR = \angle RVP = (180^\circ - \angle PRV)/2 = 66^\circ$. Likewise, we have $\angle TPS = 66^\circ$, so that

$$\angle TPV = 360^\circ - (\angle VPR + \angle RPS + \angle SPT) = 360^\circ - (66^\circ + 60^\circ + 66^\circ) = 168^\circ.$$

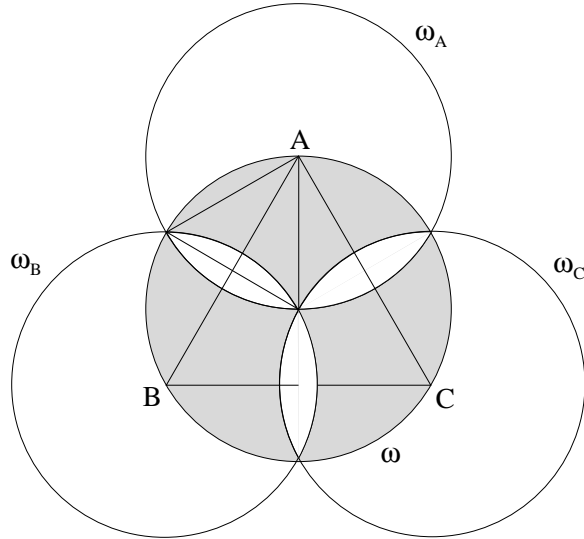
Finally, by symmetry, triangle PTV is isosceles ($PT = TV$), so $\angle PTV = \angle TVP = (180^\circ - \angle TPV)/2 = 6^\circ$. (See the figure.)



8. Let ABC be an equilateral triangle of side length 2. Let ω be its circumcircle, and let $\omega_A, \omega_B, \omega_C$ be circles congruent to ω centered at each of its vertices. Let R be the set of all points in the plane contained in exactly two of these four circles. What is the area of R ?

Solution: $2\sqrt{3}$

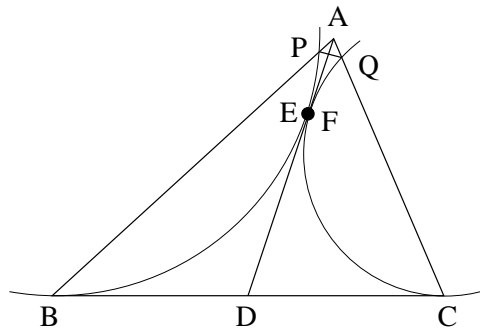
$\omega_A, \omega_B, \omega_C$ intersect at the circumcenter; thus, every point within the circumcircle, and no point outside of it, is in two or more circles. The area inside exactly two circles is shaded in the figure. The two intersection points of ω_A and ω_B , together with A , form the vertices of an equilateral triangle. As shown, this equilateral triangle cuts off a “lip” of ω (bounded by a 60° arc of ω and the corresponding chord) and another, congruent lip of ω_B that is not part of the region of interest. By rotating the first lip to the position of the second, we can reassemble the equilateral triangle. Doing this for each of the 6 such triangles, we see that the desired area equals the area of a regular hexagon inscribed in ω . The side length of this hexagon is $(2/3) \cdot (\sqrt{3}/2) \cdot 2 = 2\sqrt{3}/3$, so its area is $6 \cdot (\sqrt{3}/4) \cdot (2\sqrt{3}/3)^2 = 2\sqrt{3}$, and this is the answer.



9. In triangle ABC , $\angle ABC = 50^\circ$ and $\angle ACB = 70^\circ$. Let D be the midpoint of side BC . A circle is tangent to BC at B and is also tangent to segment AD ; this circle intersects AB again at P . Another circle is tangent to BC at C and is also tangent to segment AD ; this circle intersects AC again at Q . Find $\angle APQ$ (in degrees).

Solution: 70

Suppose the circles are tangent to AD at E, F , respectively; then, by equal tangents, $DE = DB = DC = DF \Rightarrow E = F$ (as shown). So, by the Power of a Point Theorem, $AP \cdot AB = AE^2 = AF^2 = AQ \cdot AC \Rightarrow AP/AQ = AC/AB \Rightarrow \triangle APQ \sim \triangle ACB$, giving $\angle APQ = \angle ACB = 70^\circ$.



10. Convex quadrilateral $MATH$ is given with $HM/MT = 3/4$, and $\angle ATM = \angle MAT = \angle AHM = 60^\circ$. N is the midpoint of MA , and O is a point on TH such that lines MT, AH, NO are concurrent. Find the ratio HO/OT .

Solution: 9/16

Triangle MAT is equilateral, so $HM/AT = HM/MT = 3/4$. Also, $\angle AHM = \angle ATM$, so the quadrilateral is cyclic. Now, let P be the intersection of MT, AH, NO . Extend MH and NO to intersect at point Q . Then by Menelaus's theorem, applied to triangle

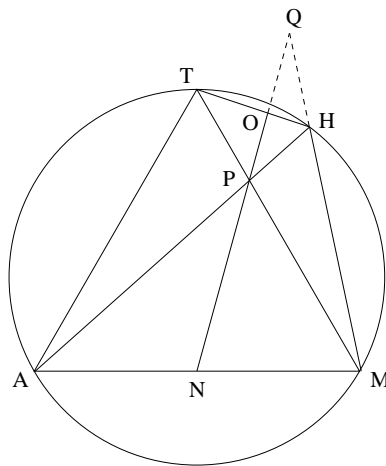
AHM and line QNP , we have

$$\frac{HQ}{QM} \cdot \frac{MN}{NA} \cdot \frac{AP}{PH} = 1,$$

while applying the same theorem to triangle THM and line QPO gives

$$\frac{HQ}{QM} \cdot \frac{MP}{PT} \cdot \frac{TO}{OH} = 1.$$

Combining gives $HO/OT = (MP/PT) \cdot (AN/NM) \cdot (HP/PA) = (MP/PA) \cdot (HP/PT)$ (because $AN/NM = 1$). But since $MATH$ is cyclic, $\triangle APT \sim \triangle MPH$, so $MP/PA = HP/PT = HM/AT = 3/4$, and the answer is $(3/4)^2 = 9/16$. (See figure.)



Harvard-MIT Mathematics Tournament

March 15, 2003

Guts Round — Solutions

1. Simplify $\sqrt[2003]{2\sqrt{11} - 3\sqrt{5}} \cdot \sqrt[4006]{89 + 12\sqrt{55}}$.

Solution: $\boxed{-1}$

Note that $(2\sqrt{11} + 3\sqrt{5})^2 = 89 + 12\sqrt{55}$. So, we have

$$\begin{aligned}\sqrt[2003]{2\sqrt{11} - 3\sqrt{5}} \cdot \sqrt[4006]{89 + 12\sqrt{55}} &= \sqrt[2003]{2\sqrt{11} - 3\sqrt{5}} \cdot \sqrt[2003]{2\sqrt{11} + 3\sqrt{5}} \\ &= \sqrt[2003]{(2\sqrt{11})^2 - (3\sqrt{5})^2} = \sqrt[2003]{-1} = -1.\end{aligned}$$

2. The graph of $x^4 = x^2y^2$ is a union of n different lines. What is the value of n ?

Solution: $\boxed{3}$

The equation $x^4 - x^2y^2 = 0$ factors as $x^2(x + y)(x - y) = 0$, so its graph is the union of the three lines $x = 0$, $x + y = 0$, and $x - y = 0$.

3. If a and b are positive integers that can each be written as a sum of two squares, then ab is also a sum of two squares. Find the smallest positive integer c such that $c = ab$, where $a = x^3 + y^3$ and $b = x^3 + y^3$ each have solutions in integers (x, y) , but $c = x^3 + y^3$ does not.

Solution: $\boxed{4}$

We can't have $c = 1 = 1^3 + 0^3$ or $c = 2 = 1^3 + 1^3$, and if $c = 3$, then a or $b = \pm 3$ which is not a sum of two cubes (otherwise, flipping signs of x and y if necessary, we would get either a sum of two nonnegative cubes to equal 3, which clearly does not happen, or a difference of two nonnegative cubes to equal 3, but the smallest difference between two successive cubes ≥ 1 is $2^3 - 1^3 = 7$). However, $c = 4$ does meet the conditions, with $a = b = 2 = 1^3 + 1^3$ (an argument similar to the above shows that there are no x, y with $4 = x^3 + y^3$), so 4 is the answer.

4. Let $z = 1 - 2i$. Find $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots$.

Solution: $\boxed{(2i - 1)/4}$

Let $x = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \dots$, so $z \cdot x = \left(1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \dots\right)$. Then $z \cdot x - x = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots = \frac{1}{1-1/z} = \frac{z}{z-1}$. Solving for x in terms of z , we obtain $x = \frac{z}{(z-1)^2}$. Plugging in the original value of z produces $x = (2i - 1)/4$.

5. Compute the surface area of a cube inscribed in a sphere of surface area π .

Solution: $\boxed{2}$

The sphere's radius r satisfies $4\pi r^2 = \pi \Rightarrow r = 1/2$, so the cube has body diagonal 1, hence side length $1/\sqrt{3}$. So, its surface area is $6(1/\sqrt{3})^2 = 2$.

6. Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For how many n , $0 \leq n \leq 100$, is F_n a multiple of 13?

Solution: 15

The sequence of remainders modulo 13 begins 0, 1, 1, 2, 3, 5, 8, 0, and then we have $F_{n+7} \equiv 8F_n$ modulo 13 by a straightforward induction. In particular, F_n is a multiple of 13 if and only if $7 \mid n$, so there are 15 such n .

7. a and b are integers such that $a + \sqrt{b} = \sqrt{15 + \sqrt{216}}$. Compute a/b .

Solution: 1/2

Squaring both sides gives $a^2 + b + 2a\sqrt{b} = 15 + \sqrt{216}$; separating rational from irrational parts, we get $a^2 + b = 15$, $4a^2b = 216$, so a^2 and b equal 6 and 9. a is an integer, so $a^2 = 9$, $b = 6 \Rightarrow a/b = 3/6 = 1/2$. (We cannot have $a = -3$, since $a + \sqrt{b}$ is positive.)

8. How many solutions in nonnegative integers (a, b, c) are there to the equation

$$2^a + 2^b = c! \quad ?$$

Solution: 5

We can check that $2^a + 2^b$ is never divisible by 7, so we must have $c < 7$. The binary representation of $2^a + 2^b$ has at most two 1's. Writing $0!, 1!, 2!, \dots, 6!$ in binary, we can check that the only possibilities are $c = 2, 3, 4$, giving solutions $(0, 0, 2), (1, 2, 3), (2, 1, 3), (3, 4, 4), (4, 3, 4)$.

9. For x a real number, let $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. How many solutions are there to the equation

$$f(f(f(f(x)))) = x?$$

Solution: 2

Certainly 0, 2 are fixed points of f and therefore solutions. On the other hand, there can be no solutions for $x < 0$, since f is nonnegative-valued; for $0 < x < 2$, we have $0 \leq f(x) < x < 2$ (and $f(0) = 0$), so iteration only produces values below x , and for $x > 2$, $f(x) > x$, and iteration produces higher values. So there are no other solutions.

10. Suppose that A, B, C, D are four points in the plane, and let Q, R, S, T, U, V be the respective midpoints of AB, AC, AD, BC, BD, CD . If $QR = 2001, SU = 2002, TV = 2003$, find the distance between the midpoints of QU and RV .

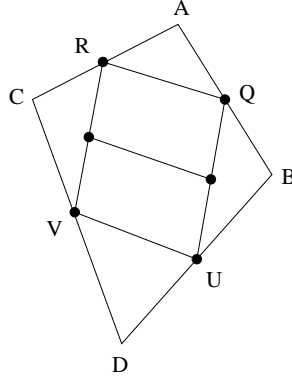
Solution: 2001

This problem has far more information than necessary: QR and UV are both parallel to BC , and QU and RV are both parallel to AD . Hence, $QUVR$ is a parallelogram, and the desired distance is simply the same as the side length QR , namely 2001. (See figure, next page.)

11. Find the smallest positive integer n such that $1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2$ is divisible by 100.

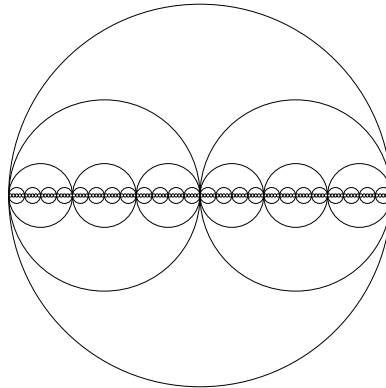
Solution: 24

The sum of the first n squares equals $n(n+1)(2n+1)/6$, so we require $n(n+1)(2n+1)$ to be divisible by $600 = 24 \cdot 25$. The three factors are pairwise relatively prime, so



one of them must be divisible by 25. The smallest n for which this happens is $n = 12$ ($2n + 1 = 25$), but then we do not have enough factors of 2. The next smallest is $n = 24$ ($n + 1 = 25$), and this works, so 24 is the answer.

12. As shown in the figure, a circle of radius 1 has two equal circles whose diameters cover a chosen diameter of the larger circle. In each of these smaller circles we similarly draw three equal circles, then four in each of those, and so on. Compute the area of the region enclosed by a positive even number of circles.



Solution: $\boxed{\pi/e}$

At the n th step, we have $n!$ circles of radius $1/n!$ each, for a total area of $n! \cdot \pi/(n!)^2 = \pi/n!$. The desired area is obtained by adding the areas of the circles at step 2, then subtracting those at step 3, then adding those at step 4, then subtracting those at step 5, and so forth. Thus, the answer is

$$\frac{\pi}{2!} - \frac{\pi}{3!} + \frac{\pi}{4!} - \dots = \pi \left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \right) = \pi e^{-1}.$$

13. If $xy = 5$ and $x^2 + y^2 = 21$, compute $x^4 + y^4$.

Solution: $\boxed{391}$

We have $441 = (x^2 + y^2)^2 = x^4 + y^4 + 2(xy)^2 = x^4 + y^4 + 50$, yielding $x^4 + y^4 = 391$.

14. A positive integer will be called “sparkly” if its smallest (positive) divisor, other than 1, equals the total number of divisors (including 1). How many of the numbers $2, 3, \dots, 2003$ are sparkly?

Solution: $\boxed{3}$

Suppose n is sparkly; then its smallest divisor other than 1 is some prime p . Hence, n has p divisors. However, if the full prime factorization of n is $p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$, the number of divisors is $(e_1 + 1)(e_2 + 1) \cdots (e_r + 1)$. For this to equal p , only one factor can be greater than 1, so n has only one prime divisor — namely p — and we get $e_1 = p - 1 \Rightarrow n = p^{p-1}$. Conversely, any number of the form p^{p-1} is sparkly. There are just three such numbers in the desired range ($2^1, 3^2, 5^4$), so the answer is 3.

15. The product of the digits of a 5-digit number is 180. How many such numbers exist?

Solution: $\boxed{360}$

Let the digits be a, b, c, d, e . Then $abcde = 180 = 2^2 \cdot 3^2 \cdot 5$. We observe that there are 6 ways to factor 180 into digits a, b, c, d, e (ignoring differences in ordering): $180 = 1 \cdot 1 \cdot 4 \cdot 5 \cdot 9 = 1 \cdot 1 \cdot 5 \cdot 6 \cdot 6 = 1 \cdot 2 \cdot 2 \cdot 5 \cdot 9 = 1 \cdot 2 \cdot 3 \cdot 5 \cdot 6 = 1 \cdot 3 \cdot 3 \cdot 4 \cdot 5 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. There are (respectively) 60, 30, 60, 120, 60, and 30 permutations of these breakdowns, for a total of 360 numbers.

16. What fraction of the area of a regular hexagon of side length 1 is within distance $\frac{1}{2}$ of at least one of the vertices?

Solution: $\boxed{\pi\sqrt{3}/9}$

The hexagon has area $6(\sqrt{3}/4)(1)^2 = 3\sqrt{3}/2$. The region we want consists of six 120° arcs of circles of radius $1/2$, which can be reassembled into two circles of radius $1/2$. So its area is $\pi/2$, and the ratio of areas is $\pi\sqrt{3}/9$.

17. There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is called a “hub” if it is directly connected to every other city. What is the largest possible number of hubs?

Solution: $\boxed{6}$

If there are h hubs, then $\binom{h}{2}$ roads connect the hubs to each other, and each hub is connected to the other $10 - h$ cities; we thus get $\binom{h}{2} + h(10 - h)$ distinct roads. So, $40 \geq \binom{h}{2} + h(10 - h) = -h^2/2 + 19h/2$, or $80 \geq h(19 - h)$. The largest $h \leq 10$ satisfying this condition is $h = 6$, and conversely, if we connect each of 6 cities to every other city and place the remaining $40 - [\binom{6}{2} + 6(10 - 6)] = 1$ road wherever we wish, we can achieve 6 hubs. So 6 is the answer.

18. Find the sum of the reciprocals of all the (positive) divisors of 144.

Solution: $\boxed{403/144}$

As d ranges over the divisors of 144, so does $144/d$, so the sum of $1/d$ is $1/144$ times the sum of the divisors of 144. Using the formula for the sum of the divisors of a number (or just counting them out by hand), we get that this sum is 403, so the answer is $403/144$.

19. Let r, s, t be the solutions to the equation $x^3 + ax^2 + bx + c = 0$. What is the value of $(rs)^2 + (st)^2 + (rt)^2$ in terms of a, b , and c ?

Solution: $\boxed{b^2 - 2ac}$

We have $(x - r)(x - s)(x - t) = x^3 + ax^2 + bx + c$, so

$$a = -(r + s + t), \quad b = rs + st + rt, \quad c = -rst.$$

So we have

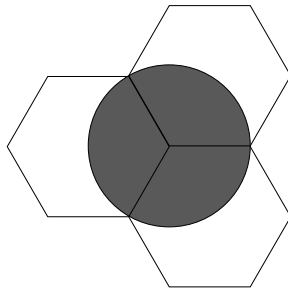
$$(rs)^2 + (st)^2 + (rt)^2 = (rs + st + rt)^2 - 2rst(r + s + t) = b^2 - 2ac.$$

20. What is the smallest number of regular hexagons of side length 1 needed to completely cover a disc of radius 1?

Solution: $\boxed{3}$

First, we show that two hexagons do not suffice. Specifically, we claim that a hexagon covers less than half of the disc's boundary. First, a hexagon of side length 1 may be inscribed in a circle, and this covers just 6 points. Translating the hexagon vertically upward (regardless of its orientation) will cause it to no longer touch any point on the lower half of the circle, so that it now covers less than half of the boundary. By rotational symmetry, the same argument applies to translation in any other direction, proving the claim. Then, two hexagons cannot possibly cover the disc.

The disc can be covered by three hexagons as follows. Let P be the center of the circle. Put three non-overlapping hexagons together at point P . This will cover the circle, since each hexagon will cover a 120° sector of the circle.



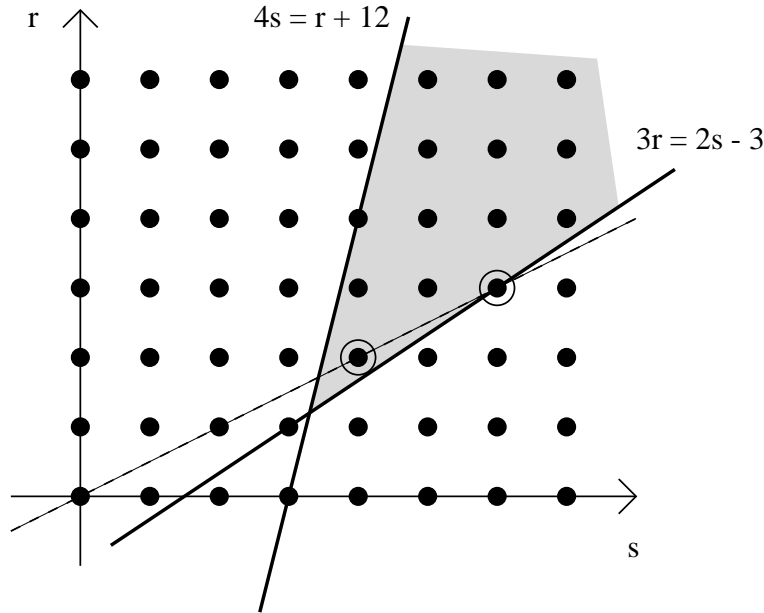
21. r and s are integers such that

$$3r \geq 2s - 3 \text{ and } 4s \geq r + 12.$$

What is the smallest possible value of r/s ?

Solution: $\boxed{1/2}$

We simply plot the two inequalities in the sr -plane and find the lattice point satisfying both inequalities such that the slope from it to the origin is as low as possible. We find that this point is $(2, 4)$ (or $(3, 6)$), as circled in the figure, so the answer is $2/4 = 1/2$.



22. There are 100 houses in a row on a street. A painter comes and paints every house red. Then, another painter comes and paints every third house (starting with house number 3) blue. Another painter comes and paints every fifth house red (even if it is already red), then another painter paints every seventh house blue, and so forth, alternating between red and blue, until 50 painters have been by. After this is finished, how many houses will be red?

Solution: 52

House n ends up red if and only if the largest odd divisor of n is of the form $4k + 1$. We have 25 values of $n = 4k + 1$; 13 values of $n = 2(4k + 1)$ (given by $k = 0, 1, 2, \dots, 12$); 7 values of $n = 4(4k + 1)$ ($k = 0, 1, \dots, 6$); 3 values of $n = 8(4k + 1)$ ($k = 0, 1, 2$); 2 of the form $n = 16(4k + 1)$ (for $k = 0, 1$); 1 of the form $n = 32(4k + 1)$; and 1 of the form $n = 64(4k + 1)$. Thus we have a total of $25 + 13 + 7 + 3 + 2 + 1 + 1 = 52$ red houses.

23. How many lattice points are enclosed by the triangle with vertices $(0, 99)$, $(5, 100)$, and $(2003, 500)$? Don't count boundary points.

Solution: 0

Using the determinant formula, we get that the area of the triangle is

$$\left| \begin{array}{cc} 5 & 1 \\ 2003 & 401 \end{array} \right| / 2 = 1.$$

There are 4 lattice points on the boundary of the triangle (the three vertices and $(1004, 300)$), so it follows from Pick's Theorem that there are 0 in the interior.

24. Compute the radius of the inscribed circle of a triangle with sides 15, 16, and 17.

Solution: $\sqrt{21}$

Hero's formula gives that the area is $\sqrt{24 \cdot 9 \cdot 8 \cdot 7} = 24\sqrt{21}$. Then, using the result that the area of a triangle equals the inradius times half the perimeter, we see that the radius is $\sqrt{21}$.

25. Let ABC be an isosceles triangle with apex A . Let I be the incenter. If $AI = 3$ and the distance from I to BC is 2, then what is the length of BC ?

Solution: $\boxed{4\sqrt{5}}$

Let X and Y be the points where the incircle touches AB and BC , respectively. Then AXI and AYB are similar right triangles. Since I is the incenter, we have $IX = IY = 2$. Using the Pythagorean theorem on triangle AXI , we find $AX = \sqrt{5}$. By similarity, $AY/AX = BY/IX$. Plugging in the numbers given, $5/\sqrt{5} = BY/2$, so $BY = 2\sqrt{5}$. Y is the midpoint of BC , so $BC = 4\sqrt{5}$.

26. Find all integers m such that $m^2 + 6m + 28$ is a perfect square.

Solution: $\boxed{6, -12}$

We must have $m^2 + 6m + 28 = n^2$, where n is an integer. Rewrite this as $(m+3)^2 + 19 = n^2 \Rightarrow n^2 - (m+3)^2 = 19 \Rightarrow (n-m-3)(n+m+3) = 19$. Let $a = n-m-3$ and $b = n+m+3$, so we want $ab = 19$. This leaves only 4 cases:

- $a = 1, b = 19$. Solve the system $n - m - 3 = 1$ and $n + m + 3 = 19$ to get $n = 10$ and $m = 6$, giving one possible solution.
- $a = 19, b = 1$. Solve the system, as above, to get $n = 10$ and $m = -12$.
- $a = -1, b = -19$. We get $n = -10$ and $m = -12$.
- $a = -19, b = -1$. We get $n = -10$ and $m = 6$.

Thus the only m are 6 and -12 .

27. The rational numbers x and y , when written in lowest terms, have denominators 60 and 70, respectively. What is the smallest possible denominator of $x + y$?

Solution: $\boxed{84}$

Write $x + y = a/60 + b/70 = (7a + 6b)/420$. Since a is relatively prime to 60 and b is relatively prime to 70, it follows that none of the primes 2, 3, 7 can divide $7a + 6b$, so we won't be able to cancel any of these factors in the denominator. Thus, after reducing to lowest terms, the denominator will still be at least $2^2 \cdot 3 \cdot 7 = 84$ (the product of the powers of 2, 3, and 7 dividing 420). On the other hand, 84 is achievable, by taking (e.g.) $1/60 + 3/70 = 25/420 = 5/84$. So 84 is the answer.

28. A point in three-space has distances 2, 6, 7, 8, 9 from five of the vertices of a regular octahedron. What is its distance from the sixth vertex?

Solution: $\boxed{\sqrt{21}}$

By a simple variant of the British Flag Theorem, if $ABCD$ is a square and P any point in space, $AP^2 + CP^2 = BP^2 + DP^2$. Four of the five given vertices must form a square $ABCD$, and by experimentation we find their distances to the given point P must be $AP = 2, BP = 6, CP = 9, DP = 7$. Then A, C , and the other two vertices E, F also form a square $AECF$, so $85 = AP^2 + CP^2 = EP^2 + FP^2 = 8^2 + FP^2 \Rightarrow FP = \sqrt{21}$.

29. A *palindrome* is a positive integer that reads the same backwards as forwards, such as 82328. What is the smallest 5-digit palindrome that is a multiple of 99?

Solution: $\boxed{54945}$

Write the number as \underline{XYZYX} . This is the same as $10000X + 1000Y + 100Z + 10Y + X = 99(101X + 10Y + Z) + 20Y + 2X + Z$. We thus want $20Y + 2X + Z$ to be a multiple of 99, with X as small as possible. This expression cannot be larger than $20 \cdot 9 + 2 \cdot 9 + 9 = 207$, and it is greater than 0 (since $X \neq 0$), so for this to be a multiple of 99, it must equal 99 or 198. Consider these two cases.

To get 198, we must have $Y = 9$, which then leaves $2X + Z = 18$. The smallest possible X is 5, and then Z becomes 8 and we have the number 59895.

To get 99, we must have $Y = 4$. Then, $2X + Z = 19$, and, as above, we find the minimal X is 5 and then $Z = 9$. This gives us the number 54945. This is smaller than the other number, so it is the smallest number satisfying the conditions of the problem.

30. The sequence a_1, a_2, a_3, \dots of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5 .

Solution: 3

Let $b_n = a_n + 1$. Then the recurrence becomes $b_{n+1} - 1 = (b_n^2 - b_{n-1})/b_{n-1} = b_n^2/b_{n-1} - 1$, so $b_{n+1} = b_n^2/b_{n-1}$. It follows that the sequence (b_n) is a geometric progression, from which $b_5^2 = b_1 b_9 = 2 \cdot 8 = 16 \Rightarrow b_5 = \pm 4$. However, since all b_n are real, they either alternate in sign or all have the same sign (depending on the sign of the progression's common ratio); either way, b_5 has the same sign as b_1 , so $b_5 = 4 \Rightarrow a_5 = 3$.

31. A cylinder of base radius 1 is cut into two equal parts along a plane passing through the center of the cylinder and tangent to the two base circles. Suppose that each piece's surface area is m times its volume. Find the greatest lower bound for all possible values of m as the height of the cylinder varies.

Solution: 3

Let h be the height of the cylinder. Then the volume of each piece is half the volume of the cylinder, so it is $\frac{1}{2}\pi h$. The base of the piece has area π , and the ellipse formed by the cut has area $\pi \cdot 1 \cdot \sqrt{1 + \frac{h^2}{4}}$ because its area is the product of the semiaxes times π . The rest of the area of the piece is half the lateral area of the cylinder, so it is πh . Thus, the value of m is

$$\begin{aligned} \frac{\pi + \pi\sqrt{1 + h^2/4} + \pi h}{\pi h/2} &= \frac{2 + 2h + \sqrt{4 + h^2}}{h} \\ &= \frac{2}{h} + 2 + \sqrt{\frac{4}{h^2} + 1}, \end{aligned}$$

a decreasing function of h whose limit as $h \rightarrow \infty$ is 3. Therefore the greatest lower bound of m is 3.

32. If x , y , and z are real numbers such that $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, find the maximum possible value of $x - y + z$.

Solution: 4

The equation rearranges as $(x - 1)^2 + (y + 2)^2 + (x - z)^2 = 0$, so we must have $x = 1$, $y = -2$, $z = 1$, giving us 4.

33. We are given triangle ABC , with $AB = 9$, $AC = 10$, and $BC = 12$, and a point D on BC . B and C are reflected in AD to B' and C' , respectively. Suppose that lines BC' and $B'C$ never meet (i.e., are parallel and distinct). Find BD .

Solution: 6

The lengths of AB and AC are irrelevant. Because the figure is symmetric about AD , lines BC' and $B'C$ meet if and only if they meet at a point on line AD . So, if they never meet, they must be parallel to AD . Because AD and BC' are parallel, triangles ABD and ADC' have the same area. Then ABD and ADC also have the same area. Hence, BD and CD must have the same length, so $BD = \frac{1}{2}BC = 6$.

34. $OKRA$ is a trapezoid with OK parallel to RA . If $OK = 12$ and RA is a positive integer, how many integer values can be taken on by the length of the segment in the trapezoid, parallel to OK , through the intersection of the diagonals?

Solution: 10

Let $RA = x$. If the diagonals intersect at X , and the segment is PQ with P on KR , then $\triangle PKX \sim \triangle RKA$ and $\triangle OKX \sim \triangle RAX$ (by equal angles), giving $RA/PX = AK/XK = 1 + AX/XK = 1 + AR/OK = (x + 12)/12$, so $PX = 12x/(12 + x)$. Similarly $XQ = 12x/(12 + x)$ also, so $PQ = 24x/(12 + x) = 24 - \frac{288}{12+x}$. This has to be an integer. $288 = 2^5 3^2$, so it has $(5 + 1)(3 + 1) = 18$ divisors. $12 + x$ must be one of these. We also exclude the 8 divisors that don't exceed 12, so our final answer is 10.

35. A certain lottery has tickets labeled with the numbers $1, 2, 3, \dots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is randomly drawn (again without replacement), and so forth, until an odd number is drawn. Then, every person whose ticket number was drawn (at any point in the process) wins a prize.

You have ticket number 1000. What is the probability that you get a prize?

Solution: 1/501

Notice that the outcome is the same as if the lottery instead draws *all* the tickets, in random order, and awards a prize to the holder of the odd ticket drawn earliest and each even ticket drawn before it. Thus, the probability of your winning is the probability that, in a random ordering of the tickets, your ticket precedes all the odd tickets. This, in turn, equals the probability that your ticket is the first in the set consisting of your own and all odd-numbered tickets, irrespective of the other even-numbered tickets. Since all $501!$ orderings of these tickets are equally likely, the desired probability is $1/501$.

36. A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples? (A whole uncut apple counts as one piece.)

Solution: 611

Consider a bipartite graph, with 221 vertices representing the apples and 403 vertices representing the students; each student is connected to each apple that she gets a

piece of. The number of pieces then equals the number of edges in the graph. Each student gets a total of $221/403 = 17/31$ apple, but each component of the graph represents a complete distribution of an integer number of apples to an integer number of students and therefore uses at least 17 apple vertices and 31 student vertices. Then we have at most $221/17 = 403/31 = 13$ components in the graph, so there are at least $221 + 403 - 13 = 611$ edges. On the other hand, if we simply distribute in the straightforward manner — proceeding through the students, cutting up a new apple whenever necessary but never returning to a previous apple or student — we can create a graph without cycles, and each component does involve 17 apples and 31 students. Thus, we get 13 trees, and 611 edges is attainable.

37. A *quagga* is an extinct chess piece whose move is like a knight's, but much longer: it can move 6 squares in any direction (up, down, left, or right) and then 5 squares in a perpendicular direction. Find the number of ways to place 51 quaggas on an 8×8 chessboard in such a way that no quagga attacks another. (Since quaggas are naturally belligerent creatures, a quagga is considered to attack quaggas on any squares it can move to, as well as any other quaggas on the same square.)

Solution: 68

Represent the 64 squares of the board as vertices of a graph, and connect two vertices by an edge if a quagga can move from one to the other. The resulting graph consists of 4 paths of length 5 and 4 paths of length 3 (given by the four rotations of the two paths shown, next page), and 32 isolated vertices. Each path of length 5 can accommodate at most 3 nonattacking quaggas in a unique way (the first, middle, and last vertices), and each path of length 3 can accommodate at most 2 nonattacking quaggas in a unique way; thus, the maximum total number of nonattacking quaggas we can have is $4 \cdot 3 + 4 \cdot 2 + 32 = 52$. For 51 quaggas to fit, then, just one component of the graph must contain one less quagga than its maximum.

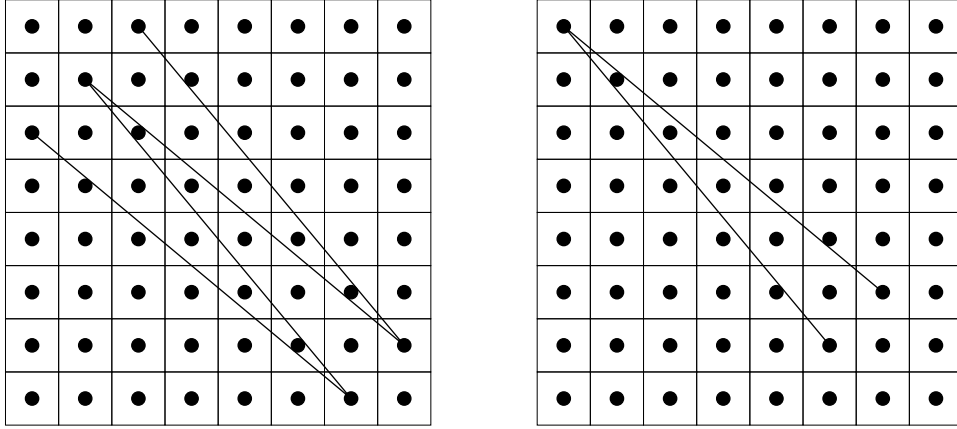
If this component is a path of length 5, there are $\binom{5}{2} - 4 = 6$ ways to place the two quaggas on nonadjacent vertices, and then all the other locations are forced; the 4 such paths then give us $4 \cdot 6 = 24$ possibilities this way. If it is a path of length 3, there are 3 ways to place one quagga, and the rest of the board is forced, so we have $4 \cdot 3 = 12$ possibilities here. Finally, if it is one of the 32 isolated vertices, we simply leave this square empty, and the rest of the board is forced, so we have 32 possibilities here. So the total is $24 + 12 + 32 = 68$ different arrangements.

38. Given are real numbers x, y . For any pair of real numbers a_0, a_1 , define a sequence by $a_{n+2} = xa_{n+1} + ya_n$ for $n \geq 0$. Suppose that there exists a fixed nonnegative integer m such that, for every choice of a_0 and a_1 , the numbers a_m, a_{m+1}, a_{m+3} , in this order, form an arithmetic progression. Find all possible values of y .

Solution: $0, 1, (1 \pm \sqrt{5})/2$

Note that $x = 1$ (or $x = 0$), $y = 0$ gives a constant sequence, so it will always have the desired property. Thus, $y = 0$ is one possibility. For the rest of the proof, assume $y \neq 0$.

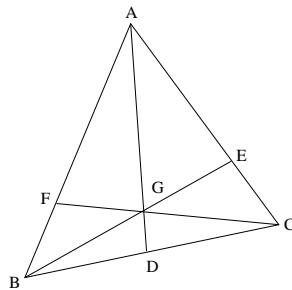
We will prove that a_m and a_{m+1} may take on any pair of values, for an appropriate choice of a_0 and a_1 . Use induction on m . The case $m = 0$ is trivial. Suppose that a_m and a_{m+1} can take on any value. Let p and q be any real numbers. By setting



$a_m = \frac{q-xp}{y}$ (remembering that $y \neq 0$) and $a_{m+1} = p$, we get $a_{m+1} = p$ and $a_{m+2} = q$. Therefore, a_{m+1} and a_{m+2} can have any values if a_m and a_{m+1} can. That completes the induction.

Now we determine the nonzero y such that a_m, a_{m+1}, a_{m+3} form an arithmetic sequence; that is, such that $a_{m+3} - a_{m+1} = a_{m+1} - a_m$. But because $a_{m+3} = (x^2 + y)a_{m+1} + xy a_m$ by the recursion formula, we can eliminate a_{m+3} from the equation, obtaining the equivalent condition $(x^2 + y - 2)a_{m+1} + (xy + 1)a_m = 0$. Because the pair a_m, a_{m+1} can take on any values, this condition means exactly that $x^2 + y - 2 = xy + 1 = 0$. Then $x = -1/y$, and $1/y^2 + y - 2 = 0$, or $y^3 - 2y^2 + 1 = 0$. One root of this cubic is $y = 1$, and the remaining quadratic factor $y^2 - y - 1$ has the roots $(1 \pm \sqrt{5})/2$. Since each such y gives an x for which the condition holds, we conclude that the answer to the problem is $y = 0, 1$, or $(1 \pm \sqrt{5})/2$.

39. In the figure, if $AE = 3$, $CE = 1$, $BD = CD = 2$, and $AB = 5$, find AG .



Solution: $\boxed{3\sqrt{66}/7}$

By Stewart's Theorem, $AD^2 \cdot BC + CD \cdot BD \cdot BC = AB^2 \cdot CD + AC^2 \cdot BD$, so $AD^2 = (5^2 \cdot 2 + 4^2 \cdot 2 - 2 \cdot 2 \cdot 4)/4 = (50 + 32 - 16)/4 = 33/2$. By Menelaus's Theorem applied to line BGE and triangle ACD , $DG/GA \cdot AE/EC \cdot CB/BD = 1$, so $DG/GA = 1/6 \Rightarrow AD/AG = 7/6$. Thus $AG = 6 \cdot AD/7 = 3\sqrt{66}/7$.

40. All the sequences consisting of five letters from the set $\{T, U, R, N, I, P\}$ (with repetitions allowed) are arranged in alphabetical order in a dictionary. Two sequences are called "anagrams" of each other if one can be obtained by rearranging the letters of the

other. How many pairs of anagrams are there that have exactly 100 other sequences between them in the dictionary?

Solution: $\boxed{0}$

Convert each letter to a digit in base 6: $I \mapsto 0, N \mapsto 1, P \mapsto 2, R \mapsto 3, T \mapsto 4, U \mapsto 5$. Then the dictionary simply consists of all base-6 integers from 00000_6 to 55555_6 in numerical order. If one number can be obtained from another by a rearrangement of digits, then the numbers are congruent modulo 5 (this holds because a number $abcde_6 = 6^4 \cdot a + 6^3 \cdot b + 6^2 \cdot c + 6 \cdot d + e$ is congruent modulo 5 to $a + b + c + d + e$), but if there are 100 other numbers between them, then their difference is 101, which is not divisible by 5. So there are no such pairs.

41. A hotel consists of a 2×8 square grid of rooms, each occupied by one guest. All the guests are uncomfortable, so each guest would like to move to one of the adjoining rooms (horizontally or vertically). Of course, they should do this simultaneously, in such a way that each room will again have one guest. In how many different ways can they collectively move?

Solution: $\boxed{1156}$

Imagine that the rooms are colored black and white, checkerboard-style. Each guest in a black room moves to an adjacent white room (and vice versa). If, for each such guest, we place a domino over the original room and the new room, we obtain a covering of the $2 \times n$ grid by n dominoes, since each black square is used once and each white square is used once. Applying a similar procedure to each guest who begins in a white room and moves to a black room, we obtain a second domino tiling. Conversely, it is readily verified that any pair of such tilings uniquely determines a movement pattern. Also, it is easy to prove by induction that the number of domino tilings of a $2 \times n$ grid is the $(n + 1)$ th Fibonacci number (this holds for the base cases $n = 1, 2$, and for a $2 \times n$ rectangle, the two rightmost squares either belong to one vertical domino, leaving a $2 \times (n - 1)$ rectangle to be covered arbitrarily, or to two horizontal dominoes which also occupy the adjoining squares, leaving a $2 \times (n - 2)$ rectangle to be covered freely; hence, the numbers of tilings satisfy the Fibonacci recurrence). So the number of domino tilings of a 2×8 grid is 34, and the number of pairs of such tilings is $34^2 = 1156$, the answer.

42. A tightrope walker stands in the center of a rope of length 32 meters. Every minute she walks forward one meter with probability $3/4$ and backward one meter with probability $1/4$. What is the probability that she reaches the end in front of her before the end behind her?

Solution: $\boxed{3^{16}/(3^{16} + 1)}$

After one minute, she is three times as likely to be one meter forward as one meter back. After two minutes she is either in the same place, two meters forward, or two meters back. The chance of being two meters forward is clearly $(3/4)^2$ and thus $3^2 = 9$ times greater than the chance of being two back $((1/4)^2)$. We can group the minutes into two-minute periods and ignore the periods of no net movement, so we can consider her to be moving 2 meters forward or backward each period, where forward movement is 3^2 times as likely as backward movement. Repeating the argument inductively, we eventually find that she is 3^{16} times more likely to move 16 meters forward than 16

meters backward, and thus the probability is $3^{16}/(3^{16} + 1)$ that she will meet the front end of the rope first.

43. Write down an integer N between 0 and 10, inclusive. You will receive N points — unless some other team writes down the same N , in which case you receive nothing.

Comments: Somewhat surprisingly, this game does not appear to be well-documented in the literature. A better-known relative is Undercut, described by Douglas Hofstadter in his book *Metamagical Themas: Questing for the Essence of Mind and Pattern*. In this game, two players each choose an integer in a specified interval and receive that many points — unless the two numbers differ by 1, in which case the lower player receives a number of points equal to the sum of the two numbers. See, for example, <http://mathforum.org/mam/96/undercut/> .

44. A *partition* of a number n is a sequence of positive integers, arranged in descending order, whose sum is n . For example, $n = 4$ has 5 partitions: $1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4$. Given two different partitions of the same number, $n = a_1 + a_2 + \dots + a_k = b_1 + b_2 + \dots + b_l$, where $k \leq l$, the first partition is said to *dominate* the second if all of the following inequalities hold:

$$\begin{aligned} a_1 &\geq b_1; \\ a_1 + a_2 &\geq b_1 + b_2; \\ a_1 + a_2 + a_3 &\geq b_1 + b_2 + b_3; \\ &\vdots \\ a_1 + a_2 + \dots + a_k &\geq b_1 + b_2 + \dots + b_k. \end{aligned}$$

Find as many partitions of the number $n = 20$ as possible such that none of the partitions dominates any other. Your score will be the number of partitions you find. If you make a mistake and one of your partitions does dominate another, your score is the largest m such that the first m partitions you list constitute a valid answer.

Comments: Computer searches have found that the maximum possible number of partitions is 20, achieved e.g. by the following:

9 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	6 + 3 + 3 + 2 + 2 + 2 + 2
8 + 2 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	5 + 5 + 4 + 1 + 1 + 1 + 1 + 1 + 1
7 + 4 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1	5 + 5 + 3 + 2 + 2 + 1 + 1 + 1
7 + 3 + 3 + 1 + 1 + 1 + 1 + 1 + 1 + 1	5 + 5 + 2 + 2 + 2 + 2 + 2
7 + 3 + 2 + 2 + 2 + 1 + 1 + 1 + 1	5 + 4 + 4 + 3 + 1 + 1 + 1 + 1
7 + 2 + 2 + 2 + 2 + 2 + 2 + 1	5 + 4 + 4 + 2 + 2 + 2 + 1
6 + 5 + 2 + 1 + 1 + 1 + 1 + 1 + 1 + 1	5 + 4 + 3 + 3 + 3 + 1 + 1
6 + 4 + 3 + 2 + 1 + 1 + 1 + 1 + 1	5 + 3 + 3 + 3 + 3 + 3
6 + 4 + 2 + 2 + 2 + 2 + 1 + 1	4 + 4 + 4 + 4 + 2 + 1 + 1
6 + 3 + 3 + 3 + 2 + 1 + 1 + 1	4 + 4 + 4 + 3 + 3 + 2

In general, as shown in

<http://www-math.mit.edu/~efedula/chains.ps>
 “Chain Lengths in the Dominance Lattice,” by Edward Early, 2002

the maximum number of mutually nondominating partitions of n is roughly on the order of $e^{\pi\sqrt{2n/3}}$ (in fact, the ratio of the total number of partitions to n to the maximum number of nondominating partitions has polynomial order of magnitude), although a proof by explicit construction is not known.

45. Find a set S of positive integers such that no two distinct subsets of S have the same sum. Your score will be $\lfloor 20(2^n/r - 2) \rfloor$, where n is the number of elements in the set S , and r is the largest element of S (assuming, of course, that this number is nonnegative).

Comments: The obvious such sets are those of the form $\{1, 2, 4, \dots, 2^{n-1}\}$, but these achieve a score of 0. A bit of experimentation finds that the set $\{3, 5, 6, 7\}$ also works; this answer achieves 5 points. The set $\{6, 9, 11, 12, 13\}$ achieves 9 points. Doing significantly better than this by hand computation becomes difficult. The reference

http://www.combinatorics.org/Volume_5/PDF/v5i1r3.pdf
“A Construction for Sets of Integers with Distinct Subset Sums,”
by Tom Bohman, 1997

(from whence the above examples) provides a construction that achieves a score of 50. Erdős conjectured in effect that a maximum possible score exists, but this conjecture remains open. The strongest known result in this direction is that, if S has n elements, the largest element is at least a constant times $2^n/\sqrt{n}$ (proven by Erdős and Moser in 1955, with an improvement on the constant by Elkies in 1986).

Hej då!

Harvard-MIT Mathematics Tournament

March 15, 2003

Team Round — Solutions

Completions and Configurations

Given a set A and a nonnegative integer k , the k -completion of A is the collection of all k -element subsets of A , and a k -configuration of A is any subset of the k -completion of A (including the empty set and the entire k -completion). For instance, the 2-completion of $A = \{1, 2, 3\}$ is $\{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, and the 2-configurations of A are

$$\begin{array}{ll}
 \{\} & \{\{1, 2\}\} \\
 \{\{1, 3\}\} & \{\{2, 3\}\} \\
 \{\{1, 2\}, \{1, 3\}\} & \{\{1, 2\}, \{2, 3\}\} \\
 \{\{1, 3\}, \{2, 3\}\} & \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}
 \end{array}$$

The *order* of an element a of A with respect to a given k -configuration of A is the number of subsets in the k -configuration that contain a . A k -configuration of a set A is *consistent* if the order of every element of A is the same, and the *order* of a consistent k -configuration is this common value.

1. (a) How many k -configurations are there of a set that has n elements?

Solution: An n -element set has $\binom{n}{k}$ subsets of size k , and we can construct a k -configuration by independently choosing, for each subset, whether or not to include it, so there are $2^{\binom{n}{k}}$ k -configurations.

- (b) How many k -configurations that have m elements are there of a set that has n elements?

Solution: Again, an n -element set has $\binom{n}{k}$ subsets of size k , so there are $\binom{\binom{n}{k}}{m}$ k -configurations with m elements.

2. Suppose A is a set with n elements, and k is a divisor of n . Find the number of consistent k -configurations of A of order 1.

Solution: Given such a k -configuration, we can write out all the elements of one of the k -element subsets, then all the elements of another subset, and so forth, eventually obtaining an ordering of all n elements of A . Conversely, given any ordering of the elements of A , we can construct a consistent k -configuration of order 1 from it by grouping together the first k elements, then the next k elements, and so forth. In fact, each consistent k -configuration of order 1 corresponds to $(n/k)!(k!)^{n/k}$ different such orderings, since the elements of A within each of the n/k k -element subsets can be ordered in $k!$ ways, and the various subsets can also be ordered with respect to each other in $(n/k)!$ different ways. Thus, since there are $n!$ orderings of the elements of A , we get $\frac{n!}{(n/k)!(k!)^{n/k}}$ different consistent k -configurations of order 1.

3. (a) Let $A_n = \{a_1, a_2, a_3, \dots, a_n, b\}$, for $n \geq 3$, and let C_n be the 2-configuration consisting of $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, $\{a_1, a_n\}$, and $\{a_i, b\}$ for $1 \leq i \leq n$. Let $S_e(n)$ be the number of subsets of C_n that are consistent of order e . Find $S_e(101)$ for $e = 1, 2$, and 3.

Solution: For convenience, we assume the a_i are indexed modulo 101, so that $a_{i+1} = a_1$ when $a_i = a_{101}$.

In any consistent subset of C_{101} of order 1, b must be paired with exactly one a_i , say a_1 . Then, a_2 cannot be paired with a_1 , so it must be paired with a_3 , and likewise we find we use the pairs $\{a_4, a_5\}, \{a_6, a_7\}, \dots, \{a_{100}, a_{101}\}$ — and this does give us a consistent subset of order 1. Similarly, pairing b with any other a_i would give us a unique extension to a consistent configuration of order 1. Thus, we have one such 2-configuration for each i , giving $S_1(101) = 101$ altogether.

In a consistent subset of order 2, b must be paired with two other elements. Suppose one of them is a_i . Then a_i is also paired with either a_{i-1} or a_{i+1} , say a_{i+1} . But then a_{i-1} needs to be paired up with two other elements, and a_i is not available, so it must be paired with a_{i-2} and b . Now b has its two pairs determined, so nothing else can be paired with b . Thus, for $j \neq i - 1, i$, we have that a_j must be paired with a_{j-1} and a_{j+1} . So our subset must be of the form

$$\{\{b, a_i\}, \{a_i, a_{i+1}\}, \{a_{i+1}, a_{i+2}\}, \dots, \{a_{101}, a_1\}, \dots, \{a_{i-2}, a_{i-1}\}, \{a_{i-1}, b\}\}$$

for some i . On the other hand, for any $i = 1, \dots, 101$, this gives a subset meeting our requirements. So, we have 101 possibilities, and $S_2(101) = 101$.

Finally, in a consistent subset of order 3, each a_i must be paired with a_{i-1} , a_{i+1} , and b . But then b occurs in 101 pairs, not just 3, so we have a contradiction. Thus, no such subset exists, so $S_3(101) = 0$.

- (b) Let $A = \{V, W, X, Y, Z, v, w, x, y, z\}$. Find the number of subsets of the 2-configuration

$$\begin{aligned} & \{ \{V, W\}, \{W, X\}, \{X, Y\}, \{Y, Z\}, \{Z, V\}, \{v, x\}, \{v, y\}, \{w, y\}, \{w, z\}, \{x, z\}, \\ & \{V, v\}, \{W, w\}, \{X, x\}, \{Y, y\}, \{Z, z\} \} \end{aligned}$$

that are consistent of order 1.

Solution: No more than two of the pairs $\{v, x\}, \{v, y\}, \{w, y\}, \{w, z\}, \{x, z\}$ may be included in a 2-configuration of order 1, since otherwise at least one of v, w, x, y, z would occur more than once. If exactly one is included, say $\{v, x\}$, then w, y, z must be paired with W, Y, Z , respectively, and then V and X cannot be paired. So either none or exactly two of the five pairs above must be used. If none, then v, w, x, y, z must be paired with V, W, X, Y, Z , respectively, and we have 1 2-configuration arising in this manner. If exactly two are used, we can check that there are 5 ways to do this without duplicating an element:

$$\{v, x\}, \{w, y\} \quad \{v, x\}, \{w, z\} \quad \{v, y\}, \{w, z\} \quad \{v, y\}, \{x, z\} \quad \{w, y\}, \{x, z\}$$

In each case, it is straightforward to check that there is a unique way of pairing up the remaining elements of A . So we get 5 2-configurations in this way, and the total is 6.

- (c) Let $A = \{a_1, b_1, a_2, b_2, \dots, a_{10}, b_{10}\}$, and consider the 2-configuration C consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq 10$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq 9$, and $\{b_i, b_{i+1}\}$ for all $1 \leq i \leq 9$. Find the number of subsets of C that are consistent of order 1.

Solution: Let $A_n = \{a_1, b_1, a_2, b_2, \dots, a_n, b_n\}$ for $n \geq 1$, and consider the 2-configuration C_n consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq n$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n - 1$, and $\{b_i, b_{i+1}\}$

for all $1 \leq i \leq n-1$. Let N_n be the number of subsets of C_n that are consistent of order 1 (call these “matchings” of C_n). Consider any matching of C_{n+2} . Either a_{n+2} is paired with b_{n+2} , in which case the remaining elements of our matching form a matching of C_{n+1} ; or a_{n+2} is paired with a_{n+1} , in which case b_{n+2} must be paired with b_{n+1} , and the remaining elements form a matching of C_n . It follows that $N_{n+2} = N_{n+1} + N_n$. By direct calculation, $N_1 = 1$ and $N_2 = 2$, and now computing successive values of N_n using the recurrence yields $N_{10} = 89$.

Define a k -configuration of A to be m -separable if we can label each element of A with an integer from 1 to m (inclusive) so that there is no element E of the k -configuration all of whose elements are assigned the same integer. If C is any subset of A , then C is m -separable if we can assign an integer from 1 to m to each element of C so that there is no element E of the k -configuration such that $E \subseteq C$ and all elements of E are assigned the same integer.

4. (a) Suppose A has n elements, where $n \geq 2$, and C is a 2-configuration of A that is not m -separable for any $m < n$. What is (in terms of n) the smallest number of elements that C can have?

Solution: We claim that every pair of elements of A must belong to C , so that the answer is $\binom{n}{2}$. Indeed, if $a, b \in A$ and $\{a, b\}$ is not in the 2-configuration, then we can assign the other elements of A the numbers $1, 2, \dots, n-2$ and assign a and b both the number $n-1$, so that C is $(n-1)$ -separable. On the other hand, if every pair of elements of A is in the configuration, then A cannot be m -separable for $m < n$, since this would require assigning the same number to at least two elements, and then we would have a pair whose elements have the same number.

- (b) Show that every 3-configuration of an n -element set A is m -separable for every integer $m \geq n/2$.

Solution: We can successively label the elements of A with the numbers $1, 1, 2, 2, 3, 3, \dots, \lceil n/2 \rceil$. Then surely no 3-element subset can have all its elements labeled with the same number, since no label is assigned to more than two elements. Thus, when $m \geq n/2 \Rightarrow m \geq \lceil n/2 \rceil$, this labeling shows that any 3-configuration is m -separable.

- (c) Fix $k \geq 2$, and suppose A has k^2 elements. Show that any k -configuration of A with fewer than $\binom{k^2-1}{k-1}$ elements is k -separable.

Solution: The argument is similar to that used in problem 2. Suppose the configuration is not k -separable. Consider all possible orderings of the k^2 elements of A . For each ordering, assign the first k elements the number 1, the next k elements the number 2, and so forth. By assumption, for each such assignment, there exists some element of the k -configuration whose elements are all assigned the same number. Now consider any given element E of the k -configuration. For each i , we count the orderings in which all k elements of E receive the number i : there are $k!$ possible orderings for the elements of E , and there are $(k^2 - k)!$ possible orderings for the remaining elements of A . Altogether, this gives $k \cdot k! \cdot (k^2 - k)!$ orderings in which the elements of E all receive the same label. So if, in each of the $(k^2)!$ orderings of the elements of A , there is some E all of whose members receive the same label, then there must be at least

$$\frac{(k^2)!}{k \cdot k! \cdot (k^2 - k)!} = \frac{(k^2 - 1)!}{(k - 1)!(k^2 - k)!} = \binom{k^2 - 1}{k - 1}$$

elements E of the k -configuration. Hence, if there are fewer elements, the k -configuration is k -separable, as desired.

5. Let $B_k(n)$ be the largest possible number of elements in a 2-separable k -configuration of a set with $2n$ elements ($2 \leq k \leq n$). Find a closed-form expression (i.e. an expression not involving any sums or products with a variable number of terms) for $B_k(n)$.

Solution: First, a lemma: For any a with $0 \leq a \leq 2n$, $\binom{a}{k} + \binom{2n-a}{k} \geq 2\binom{n}{k}$. (By convention, we set $\binom{a}{k} = 0$ when $a < k$.) Proof: We may assume $a \geq n$, since otherwise we can replace a with $2n - a$. Now we prove the result by induction on a . For the base case, if $a = n$, then the lemma states that $2\binom{n}{k} \geq 2\binom{n}{k}$, which is trivial. If the lemma holds for some $a > 0$, then by the familiar identity,

$$\begin{aligned} & \left[\binom{a+1}{k} + \binom{2n-a-1}{k} \right] - \left[\binom{a}{k} + \binom{2n-a}{k} \right] \\ &= \left[\binom{a+1}{k} - \binom{a}{k} \right] - \left[\binom{2n-a}{k} + \binom{2n-a-1}{k} \right] \\ &= \binom{a}{k-1} - \binom{2n-a-1}{k-1} > 0 \end{aligned}$$

(since $a > 2n - a - 1$), so $\binom{a+1}{k} + \binom{2n-a-1}{k} > \binom{a}{k} + \binom{2n-a}{k} \geq 2\binom{n}{k}$, giving the induction step. The lemma follows.

Now suppose that the elements of A are labeled such that a elements of the set A receive the number 1 and $2n - a$ elements receive the number 2. Then the k -configuration can include all k -element subsets of A except those contained among the a elements numbered 1 or the $2n - a$ elements numbered 2. Thus, we have at most $\binom{2n}{k} - \binom{a}{k} - \binom{2n-a}{k}$ elements in the k -configuration, and by the lemma, this is at most

$$\binom{2n}{k} - 2\binom{n}{k}.$$

On the other hand, we can achieve $\binom{2n}{k} - 2\binom{n}{k}$ via the recipe above — take all the k -element subsets of A , except those contained entirely within the first n elements or entirely within the last n elements. Then, labeling the first n elements with the number 1 and the last n elements with the number 2 shows that the configuration is 2-separable. So, $B_k(n) = \binom{2n}{k} - 2\binom{n}{k}$.

6. Prove that any 2-configuration containing e elements is m -separable for some $m \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$.

Solution: Suppose m is the minimum integer for which the given configuration C on set A is m -separable, and fix a corresponding labeling of the elements of A . Let A_i be the set of all elements with the label i . Then, for any i, j with $1 \leq i < j \leq m$, there must exist $a_i \in A_i, a_j \in A_j$ with $\{a_i, a_j\} \in C$, since otherwise the elements of A_j could have been reassigned the label i , decreasing the number of distinct labels necessary and thus contradicting the minimality of m . We thus get at least $\binom{m}{2}$ different elements of C . Therefore, $e \geq \binom{m}{2} = m(m-1)/2 = [(m - \frac{1}{2})^2 - \frac{1}{4}]/2$, and solving for m gives the desired result.

A *cell* of a 2-configuration of a set A is a nonempty subset C of A such that

- i. for any two distinct elements a, b of C , there exists a sequence c_0, c_1, \dots, c_n of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \dots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration, and
- ii. if a is an element of C and b is an element of A but not of C , there does NOT exist a sequence c_0, c_1, \dots, c_n of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \dots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration.

Also, we define a 2-configuration of A to be *barren* if there is no subset $\{a_0, a_1, \dots, a_n\}$ of A , with $n \geq 2$, such that $\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}$ and $\{a_n, a_0\}$ are all elements of the 2-configuration.

7. Show that, given any 2-configuration of a set A , every element of A belongs to exactly one cell.

Solution: First, given a , let C_a be the set of all $b \in A$ for which there exists a sequence $a = c_0, c_1, \dots, c_n = b$ as in the definition of a cell. Certainly $a \in C_a$ (take $n = 0$); we claim that C_a is a cell. If $b, b' \in C_a$, then there exist sequences $a = c_0, c_1, \dots, c_n = b$ and $a = c'_0, c'_1, \dots, c'_m = b'$, so the sequence $b = c_n, c_{n-1}, \dots, c_1, c_0, c'_1, \dots, c'_m = b'$ shows that the first condition is met. For the second, suppose that there does exist a sequence $b = c_0, c_1, \dots, c_n = b'$ with $b \in C_a, b' \notin C_a$. Then, concatenating with our sequence from a to b , we get a sequence from a to b' , contradicting the assumption $b' \notin C_a$. Thus, the second condition holds, and C_a is a cell. So a lies in at least one cell.

But now, note that if C is a cell containing a , then all b for which such a sequence from a to b exists must lie in C (or the second condition is violated), and if no such sequence exists, then b cannot lie in C (or the first condition is violated). Thus, the elements of C are uniquely determined, so there is exactly one cell containing a , and the proof is complete.

8. (a) Given a set A with $n \geq 1$ elements, find the number of consistent 2-configurations of A of order 1 with exactly 1 cell.

Solution: There must be some pair $\{a, b\}$ in the 2-configuration, since each element $a \in A$ must belong to one pair. Since neither a nor b can now belong to any other pair, this must be the entire cell. Thus, there is 1 such 2-configuration when $n = 2$, and there are none when $n \neq 2$.

- (b) Given a set A with 10 elements, find the number of consistent 2-configurations of A of order 2 with exactly 1 cell.

Solution: Consider such a configuration; let $\{a_1, a_2\}$ be an element of it. Then a_2 belongs to exactly one other pair; call it $\{a_2, a_3\}$. Likewise, a_3 belongs to exactly one other pair $\{a_3, a_4\}$, and so forth; since we have finitely many elements, we must eventually reach some pair $\{a_m, a_k\}$ that revisits a previously used element ($m > k$). But this is only possible if $k = 1$, since each other a_k with $k < m$ is already used in two pairs. Now, $\{a_1, \dots, a_m\}$ constitutes a complete cell, because none of these elements can be used in any more pairs, so $m = 10$. Thus, every consistent 2-configuration of order 2 with exactly 1 cell gives rise to a permutation a_1, \dots, a_{10} of the elements of A , and conversely, each such permutation gives us a 2-configuration $\{\{a_1, a_2\}, \dots, \{a_9, a_{10}\}, \{a_{10}, a_1\}\}$. In fact, each configuration corresponds to exactly 20 permutations, depending which of the

10 elements of the 2-configuration we choose as $\{a_1, a_2\}$ and which of these two elements of A we in turn choose to designate as a_1 (as opposed to a_2). Therefore, the number of such 2-configurations is $10!/20 = 9!/2 = 181440$. (More generally, the same argument shows that, when A has $n \geq 3$ elements, there are $(n-1)!/2$ such 2-configurations.)

- (c) Given a set A with 10 elements, find the number of consistent 2-configurations of order 2 with exactly 2 cells.

Solution: Notice that if we look only at the pairs contained within any fixed cell, each element of that cell still lies in 2 such pairs, since all the pairs it belongs to are contained within that cell. Thus we have an induced consistent 2-configuration of order 2 of each cell.

Now, each cell must have at least 3 elements for the configuration to be 2-consistent. So we can have either two 5-element cells, a 4-element cell and a 6-element cell, or a 3-element cell and a 7-element cell. If there are two 5-element cells, we can choose the members of the first cell in $\binom{10}{5}$ ways, and then (by the reasoning in the previous problem) we have $4!/2$ ways to build a consistent 2-configuration of order 2 of each cell. However, choosing 5 elements for the first cell is equivalent to choosing the other 5 elements for the first cell, since the two cells are indistinguishable; thus, we have overcounted by a factor of 2. So we have $\binom{10}{5} \cdot (4!/2)^2/2 = 252 \cdot 144/2 = 18144$ ways to form our configuration if we require it to have two cells of 5 elements each.

If we have one 4-element cell and one 6-element cell, then there are $\binom{10}{4}$ ways to determine which 4 elements go in the smaller cell, and then $3!/2$ ways and $5!/2$ ways, respectively, to construct the 2-configurations of the two cells, for a total of $\binom{10}{4} \cdot (3!/2) \cdot (5!/2) = 210 \cdot 3 \cdot 60 = 37800$ configurations (no overcounting here), and by similar reasoning, we have $\binom{10}{3} \cdot (2!/2) \cdot (6!/2) = 120 \cdot 1 \cdot 360 = 43200$ configurations with one 3-element cell and one 7-element cell. Thus, altogether, we have a total of $18144 + 37800 + 43200 = 99144$ consistent 2-configurations of order 2 with exactly 2 cells.

9. (a) Show that if every cell of a 2-configuration of a finite set A is m -separable, then the whole 2-configuration is m -separable.

Solution: Let C be a 2-configuration of A with cells A_1, \dots, A_n , so that there is no element of C with one element in A_i and another in A_j for $i \neq j$. Suppose that each cell is m -separable, so that for each i , $1 \leq i \leq n$, there is a labeling function $f_i : A_i \rightarrow \{1, \dots, m\}$ such that no two elements in the same pair of C are assigned the same number. Then, by combining, we get a function f on all of A whose restriction to A_i is f_i for each i . By the definition of f_i , within each A_i there is no element of C both of whose elements are mapped to the same integer; and as above, we know that there are no elements of C not contained inside any A_i . Thus, C is m -separable, by the existence of f .

- (b) Show that any barren 2-configuration of a finite set A is 2-separable.

Solution: It is sufficient to show each cell is 2-separable, by part (a). A barren 2-configuration by definition cannot have any cycles (i.e. subsets $\{a_0, \dots, a_n\}$, $n \geq 2$, where each $\{a_i, a_{i+1}\}$ and $\{a_n, a_0\}$ all belong to the 2-configuration). For any two distinct elements a, b of A in the same cell of a 2-configuration C , define the *distance* between them to be the smallest n such that there exists a sequence $a = a_0, a_1, \dots, a_n =$

b with $\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}$ all belonging to the 2-configuration. Notice that the terms of this sequence are all distinct: if $a_i = a_j$ for $i < j$, then we have the shorter sequence $a_0, a_1, \dots, a_i, a_{j+1}, \dots, a_n$, contradicting minimality.

Now let C be a barren 2-configuration of A . Pick any element a of A ; label it and all elements at even distance from it with the integer 1, and label all elements at odd distance from it with the integer 2. We claim no two different elements with the same label appear in the same element of C . Otherwise, let b and c be such elements, and let $a = a_0, a_1, \dots, a_n = b$ and $a = a'_0, a'_1, \dots, a'_m = c$ be the corresponding minimal sequences. Consider the largest p such that $a_p \in \{a'_0, \dots, a'_m\}$; write $a_p = a'_q$. We claim the set $\{a_n, a_{n-1}, \dots, a_p, a'_{q+1}, \dots, a'_m\}$ is then a cycle. It is straightforward to check that all its elements are distinct; the only issue is whether it has at least 3 elements. If not, we would have $a_p = a_n$ or a'_m . Assume that $a_p = a_n \Rightarrow p = n \Rightarrow q = m - 1$ (by minimality of our sequence (a'_i)), but this means that $m = n + 1$, so the distances of b and c from a have opposite parities, contradicting the assumption that they have the same label. The case $a'_q = a'_m$ is similar. Thus, our set really does have at least three elements, and it is a cycle. But since A is barren, it contains no cycles, and we have a contradiction.

Thus, after all, no two elements with the same label appear in the same pair of C , so the cell containing a is 2-separable, and we are done.

10. Show that every consistent 2-configuration of order 4 on a finite set A has a subset that is a consistent 2-configuration of order 2.

Solution: First, assume the 2-configuration has just one cell. We claim there exists a sequence a_0, a_1, \dots, a_n of elements of A (not necessarily all distinct) such that the list

$$\{a_0, a_1\}, \{a_1, a_2\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}$$

contains each element of the 2-configuration exactly once. To see this, consider the longest sequence such that $\{a_0, a_1\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}$ are all distinct elements of the 2-configuration. (We may take $n = 0$ if necessary. Note that the finiteness condition ensures such a maximal sequence exists.) Each element of A occurs an even number of times among these pairs (since each occurrence in the sequence contributes to two pairs). If every element occurs 4 times or 0 times, then the elements occurring in the sequence form a cell, since they cannot occur in any other pairs in the 2-configuration. Hence, they are all of A , and our sequence uses all the pairs in the 2-configuration, so the claim follows. Otherwise, there is some element a_i occurring exactly twice. Choose b_1 so that $\{a_i, b_1\}$ is one of the two pairs in the 2-configuration not used by our sequence. Then choose b_2 so that $\{b_1, b_2\}$ be another pair not used thus far. Continue in this manner, choosing new elements b_k with $\{b_k, b_{k+1}\}$ a pair not already used, until we reach a point where finding another unused pair is impossible. Now, our pairs so far are

$$\begin{aligned} &\{a_0, a_1\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}, \\ &\{a_i, b_1\}, \{b_1, b_2\}, \dots, \{b_{k-1}, b_k\}. \end{aligned}$$

Every element is used in an even number of these pairs, except possibly a_i , which is used in three pairs, and b_k , which is used in an odd number of pairs (so one or three) — unless $a_i = b_k$, in which case this element occurs four times. But since it is impossible to continue the sequence, b_k must indeed have been used four times, so $b_k = a_i$.

But now we can construct the following sequence of distinct elements of the 2-configuration:

$$\{a_0, a_1\}, \dots, \{a_{i-1}, a_i\}, \{a_i, b_1\}, \{b_1, b_2\}, \dots, \{b_{k-1}, a_i\},$$

$$\{a_i, a_{i+1}\}, \dots, \{a_{n-1}, a_n\}, \{a_n, a_0\}.$$

This contradicts the maximality of our original sequence. This contradiction means that our original sequence must have used all the pairs in the 2-configuration, after all.

So we can express the 2-configuration via such a sequence of pairs, where each pair's second element equals the first element of the next pair. If A has n elements, then (since each element appears in four pairs) we have $2n$ pairs. So we can choose the 1st, 3rd, 5th, \dots , $(2n - 1)$ th pairs, and then each element of A belongs to just two of these pairs, because each occurrence of the element as an a_i contributes to two consecutive pairs from our original sequence (or the first and last such pairs). Thus, we have our consistent 2-configuration of order 2, as desired.

Finally, if A consists of more than one cell, then the pairs within any given cell form a consistent 2-configuration of order 4 on that cell. So we simply apply the above procedure to obtain a consistent 2-configuration of order 2 on each cell, and then combining these gives a consistent 2-configuration of order 2 on A , as desired.

Comments: A note for those who might have found these problems rather foreign — the objects described here are actually of considerable importance; they constitute the elements of graph theory, one of the major research areas of modern mathematics. What we have called a “2-configuration” is generally called a *graph*, and what we have called a “ k -configuration” ($k > 2$) is generally called a *hypergraph*. The graph in problem 3b is the *Petersen graph*, a ubiquitous counterexample in graph theory. A consistent 2-configuration of order n is an *n -regular graph*; a cell is a *component*; a barren 2-configuration is a *forest* (and a forest with one component is a *tree*); and an m -separable configuration is *m -colorable* (and the minimum m for which a graph is m -colorable is its *chromatic number*).

Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Advanced Topics**

1. Eight knights are randomly placed on a chessboard (not necessarily on distinct squares). A knight on a given square attacks all the squares that can be reached by moving either (1) two squares up or down followed by one squares left or right, or (2) two squares left or right followed by one square up or down. Find the probability that every square, occupied or not, is attacked by some knight.

Solution: $\boxed{0}$. Since every knight attacks at most eight squares, the event can only occur if every knight attacks exactly eight squares. However, each corner square must be attacked, and some experimentation readily finds that it is impossible to place a knight so as to attack a corner and seven other squares as well.

2. A certain cafeteria serves ham and cheese sandwiches, ham and tomato sandwiches, and tomato and cheese sandwiches. It is common for one meal to include multiple types of sandwiches. On a certain day, it was found that 80 customers had meals which contained both ham and cheese; 90 had meals containing both ham and tomatoes; 100 had meals containing both tomatoes and cheese. 20 customers' meals included all three ingredients. How many customers were there?

Solution: $\boxed{230}$. Everyone who ate just one sandwich is included in exactly one of the first three counts, while everyone who ate more than one sandwich is included in all four counts. Thus, to count each customer exactly once, we must add the first three figures and subtract the fourth twice: $80 + 90 + 100 - 2 \cdot 20 = 230$.

3. How many four-digit numbers are there in which at least one digit occurs more than once?

Solution: $\boxed{4464}$. There are 9000 four-digit numbers altogether. If we consider how many four-digit numbers have all their digits distinct, there are 9 choices for the first digit (since we exclude leading zeroes), and then 9 remaining choices for the second digit, then 8 for the third, and 7 for the fourth, for a total of $9 \cdot 9 \cdot 8 \cdot 7 = 4536$. Thus the remaining $9000 - 4536 = 4464$ numbers have a repeated digit.

4. Two fair coins are simultaneously flipped. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that the other coin also came up heads on this last flip?

Solution: $\boxed{1/3}$. Let the desired probability be p . There is a $1/4$ chance that both coins will come up heads on the first toss. Otherwise, both can come up heads simultaneously only if both are tails on the first toss, and then the process restarts as if from the beginning; thus this situation occurs with probability $p/4$. Thus $p = 1/4 + p/4$; solving, $p = 1/3$.

Alternate Solution: The desired event is equivalent to both coins coming up tails for n successive turns (for some $n \geq 0$), then both coins coming up heads. For any fixed value of

n , the probability of this occurring is $1/4^{n+1}$. Since all these events are disjoint, the total probability is $1/4 + 1/4^2 + 1/4^3 + \dots = 1/3$.

5. Determine the number of subsets S of $\{1, 2, 3, \dots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S, b \notin S, c \in S$.

Solution: 968 There are $2^{10} = 1024$ subsets of $\{1, 2, \dots, 10\}$ altogether. Any subset without the specified property must be either the empty set or a block of consecutive integers. To specify a block of consecutive integers, we either have just one element (10 choices) or a pair of distinct endpoints ($\binom{10}{2} = 45$ choices). So the number of sets with our property is $1024 - (1 + 10 + 45) = 968$.

6. In how many ways can the numbers $1, 2, \dots, 2002$ be placed at the vertices of a regular 2002-gon so that no two adjacent numbers differ by more than 2? (Rotations and reflections are considered distinct.)

Solution: 4004. There are 2002 possible positions for the 1. The two numbers adjacent to the 1 must be 2 and 3; there are two possible ways of placing these. The positions of these numbers uniquely determine the rest: for example, if 3 lies clockwise from 1, then the number lying counterclockwise from 2 must be 4; the number lying clockwise from 3 must be 5; the number lying counterclockwise from 4 must now be 6; and so forth. Eventually, 2002 is placed adjacent to 2000 and 2001, so we do get a valid configuration. Thus there are $2002 \cdot 2$ possible arrangements.

7. A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given flight. There are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the same probability of crashing?

Solution: They have the same probability of failing if $\binom{5}{2}p^3(1-p)^2 + \binom{5}{1}p^4(1-p) + p^5 = \binom{3}{1}p^2(1-p) + p^3$, which is true iff $p^2(6p^3 - 15p^2 + 12p - 3) = 0$. This is clearly true for $p = 0$. We know it is true for $p = 1$, since both probabilities would be 1 in this case, so we know $p - 1$ is a factor of $6p^3 - 15p^2 + 12p - 3$. Thus, factoring gives that the engines have the same probability of failing if $p^2(p - 1)(6p^2 - 9p + 3) = 0$. By the quadratic formula (or by factoring), the quadratic has roots $p = \frac{1}{2}, 1$, so the answer is $0, \frac{1}{2}, 1$.

8. Given a 9×9 chess board, we consider all the rectangles whose edges lie along grid lines (the board consists of 81 unit squares, and the grid lines lie on the borders of the unit squares). For each such rectangle, we put a mark in every one of the unit squares inside it. When this process is completed, how many unit squares will contain an even number of marks?

Solution: 56. Consider the rectangles which contain the square in the i th row and j th column. There are i possible positions for the upper edge of such a rectangle, $10 - i$ for the lower edge, j for the left edge, and $10 - j$ for the right edge; thus we have $i(10 - i)j(10 - j)$ rectangles altogether, which is odd iff i, j are both odd, i.e. iff $i, j \in \{1, 3, 5, 7, 9\}$. There are thus 25 unit squares which lie in an odd number of rectangles, so the answer is $81 - 25 = 56$.

9. Given that a, b, c are positive real numbers and $\log_a b + \log_b c + \log_c a = 0$, find the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$.

Solution: $\boxed{3}$. Let $x = \log_a b$ and $y = \log_b c$; then $\log_c a = -(x + y)$. Thus we want to compute the value of $x^3 + y^3 - (x + y)^3 = -3x^2y - 3xy^2 = -3xy(x + y)$. On the other hand, $-xy(x + y) = (\log_a b)(\log_b c)(\log_c a) = 1$, so the answer is 3.

10. One fair die is rolled; let a denote the number that comes up. We then roll a dice; let the sum of the resulting a numbers be b . Finally, we roll b dice, and let c be the sum of the resulting b numbers. Find the expected (average) value of c .

Solution: $\boxed{343/8}$ The expected result of an individual die roll is $(1+2+3+4+5+6)/6 = 7/2$. For any particular value of b , if b dice are rolled independently, then the expected sum is $(7/2)b$. Likewise, when we roll a dice, the expected value of their sum b is $(7/2)a$, so the expected value of c is $(7/2)^2a$. Similar reasoning again shows us that the expected value of a is $7/2$ and so the expected value of c overall is $(7/2)^3 = 343/8$.

Harvard-MIT Math Tournament

March 17, 2002

Individual Subject Test: Algebra

1. Nine nonnegative numbers have average 10. What is the greatest possible value for their median?

Solution: $\boxed{18}$ If the median is m , then the five highest numbers are all at least m , so the sum of all the numbers is at least $5m$. Thus $90 \geq 5m \Rightarrow m \leq 18$. Conversely, we can achieve $m = 18$ by taking four 0's and five 18's.

2. p and q are primes such that the numbers $p + q$ and $p + 7q$ are both squares. Find the value of p .

Solution: $\boxed{2}$. Writing $x^2 = p + q, y^2 = p + 7q$, we have $6q = y^2 - x^2 = (y - x)(y + x)$. Since $6q$ is even, one of the factors $y - x, y + x$ is even, and then the other is as well; thus $6q$ is divisible by 4 $\Rightarrow q$ is even $\Rightarrow q = 2$ and $6q = 12$. We may assume x, y are both taken to be positive; then we must have $y - x = 2, y + x = 6 \Rightarrow x = 2$, so $p + 2 = 2^2 = 4 \Rightarrow p = 2$ also.

3. Real numbers a, b, c satisfy the equations $a + b + c = 26, 1/a + 1/b + 1/c = 28$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

Solution: $\boxed{725}$ Multiplying the two given equations gives

$$\frac{a}{a} + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{b} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + \frac{c}{c} = 26 \cdot 28 = 728,$$

and subtracting 3 from both sides gives the answer, 725.

4. If a positive integer multiple of 864 is picked randomly, with each multiple having the same probability of being picked, what is the probability that it is divisible by 1944?

Solution: The probability that a multiple of $864 = 2^5 3^3$ is divisible by $2744 = 2^3 3^5$ is the same as the probability that a multiple of 2^2 is divisible by 3^2 , which since 4 and 9 are relatively prime is $\boxed{\frac{1}{9}}$.

5. Find the greatest common divisor of the numbers $2002 + 2, 2002^2 + 2, 2002^3 + 2, \dots$

Solution: $\boxed{6}$. Notice that $2002 + 2$ divides $2002^2 - 2^2$, so any common divisor of $2002 + 2$ and $2002^2 + 2$ must divide $(2002^2 + 2) - (2002^2 - 2^2) = 6$. On the other hand, every number in the sequence is even, and the n th number is always congruent to $1^n + 2 \equiv 0$ modulo 3. Thus, 6 divides every number in the sequence.

6. Find the sum of the even positive divisors of 1000.

Solution: $\boxed{2184}$. Notice that $2k$ is a divisor of 1000 iff k is a divisor of 500, so we need only find the sum of the divisors of 500 and multiply by 2. This can be done by enumerating the divisors individually, or simply by using the formula: $\sigma(2^2 \cdot 5^3) = (1 + 2 + 2^2)(1 + 5 + 5^2 + 5^3) = 1092$, and then doubling gives 2184. **Alternate Solution:** The sum of all the divisors of 1000 is $(1 + 2 + 2^2 + 2^3)(1 + 5 + 5^2 + 5^3) = 2340$. The odd divisors of 1000 are simply the divisors of 125, whose sum is $1 + 5 + 5^2 + 5^3 = 156$; subtracting this from 2340, we are left with the sum of the even divisors of 1000, which is 2184.

7. The real numbers x, y, z, w satisfy

$$\begin{aligned} 2x + y + z + w &= 1 \\ x + 3y + z + w &= 2 \\ x + y + 4z + w &= 3 \\ x + y + z + 5w &= 25. \end{aligned}$$

Find the value of w .

Solution: $\boxed{11/2}$. Multiplying the four equations by 12, 6, 4, 3 respectively, we get

$$\begin{aligned} 24x + 12y + 12z + 12w &= 12 \\ 6x + 18y + 6z + 6w &= 12 \\ 4x + 4y + 16z + 4w &= 12 \\ 3x + 3y + 3z + 15w &= 75. \end{aligned}$$

Adding these yields $37x + 37y + 37z + 37w = 111$, or $x + y + z + w = 3$. Subtract this from the fourth given equation to obtain $4w = 22$, or $w = 11/2$.

8. Determine the value of the sum

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \cdots + \frac{29}{14^2 \cdot 15^2}.$$

Solution: $\boxed{224/225}$ The sum telescopes as

$$\left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{1}{14^2} - \frac{1}{15^2}\right) = \frac{1}{1^2} - \frac{1}{15^2} = \frac{224}{225}.$$

9. For any positive integer n , let $f(n)$ denote the number of 1's in the base-2 representation of n . For how many values of n with $1 \leq n \leq 2002$ do we have $f(n) = f(n+1)$?

Solution: $\boxed{501}$. If n is even, then $n+1$ is obtained from n in binary by changing the final 0 to a 1; thus $f(n+1) = f(n) + 1$. If n is odd, then $n+1$ is obtained by changing the last 0 to a 1, the ensuing string of 1s to 0s, and then changing the next rightmost 0 to a 1. This produces no net change in the number of 1's iff n ends in 01 in base 2. Thus,

$f(n+1) = f(n)$ if and only if n is congruent to 1 mod 4, and there are 501 such numbers in the specified range.

10. Determine the value of

$$2002 + \frac{1}{2}(2001 + \frac{1}{2}(2000 + \cdots + \frac{1}{2}(3 + \frac{1}{2} \cdot 2)) \cdots).$$

Solution: $\boxed{4002}$. We can show by induction that $n + \frac{1}{2}([n-1] + \frac{1}{2}(\cdots + \frac{1}{2} \cdot 2) \cdots) = 2(n-1)$. For $n = 3$ we have $3 + \frac{1}{2} \cdot 2 = 4$, giving the base case, and if the result holds for n , then $(n+1) + \frac{1}{2}2(n-1) = 2n = 2(n+1) - 2$. Thus the claim holds, and now plug in $n = 2002$.

Alternate Solution: Expand the given expression as $2002 + 2001/2 + 2000/2^2 + \cdots + 2/2^{2000}$. Letting S denote this sum, we have $S/2 = 2002/2 + 2001/2^2 + \cdots + 2/2^{2001}$, so $S - S/2 = 2002 - (1/2 + 1/4 + \cdots + 1/2^{2000}) - 2/2^{2001} = 2002 - (1 - 1/2^{2000}) - 1/2^{2000} = 2001$, so $S = 4002$.

Harvard-MIT Math Tournament

March 17, 2002

Individual Subject Test: Calculus

1. Two circles have centers that are d units apart, and each has diameter \sqrt{d} . For any d , let $A(d)$ be the area of the smallest circle that contains both of these circles. Find $\lim_{d \rightarrow \infty} \frac{A(d)}{d^2}$.

Solution: This equals $\lim_{d \rightarrow \infty} \frac{\pi \left(\frac{d+\sqrt{d}}{2}\right)^2}{d^2} = \boxed{\frac{\pi}{4}}$.

2. Find $\lim_{h \rightarrow 0} \frac{x^2 - (x+h)^2}{h}$.

Solution: This equals $\lim_{h \rightarrow 0} \frac{x^2 - x^2 - h^2 - 2hx}{h} = \lim_{h \rightarrow 0} -h - 2x = \boxed{-2x}$. Alternate Solution: This is the definition of the derivative of $-x^2$ with respect to x , which is $-2x$.

3. We are given the values of the differentiable real functions f, g, h , as well as the derivatives of their pairwise products, at $x = 0$:

$$f(0) = 1; g(0) = 2; h(0) = 3; (gh)'(0) = 4; (hf)'(0) = 5; (fg)'(0) = 6.$$

Find the value of $(fgh)'(0)$.

Solution: $\boxed{16}$ By the product rule, $(fgh)' = f'gh + fg'h + fgh' = ((fg)'h + (gh)'f + (hf)'g)/2$. Evaluated at 0, this gives 16.

4. Find the area of the region in the first quadrant $x > 0, y > 0$ bounded above the graph of $y = \arcsin(x)$ and below the graph of the $y = \arccos(x)$.

Solution: We can integrate over y rather than x . In particular, the solution is $\int_0^{\pi/4} \sin y \, dy + \int_{\pi/4}^{\pi/2} \cos y \, dy = \left(1 - \frac{1}{\sqrt{2}}\right) 2 = \boxed{2 - \sqrt{2}}$.

5. What is the minimum vertical distance between the graphs of $2 + \sin(x)$ and $\cos(x)$?

Solution: The derivative of $2 + \sin(x) - \cos(x)$ is $\cos x + \sin x$, which in the interval $0 \leq x < 2\pi$ is zero at $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. At $\frac{7\pi}{4}$, when $\sin(x)$ is negative and $\cos(x)$ is positive, the distance reaches its minimal value of $\boxed{2 - \sqrt{2}}$.

6. Determine the positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0), (a, 0), (0, a^2 + 1),$ and $(a, a^2 + 1)$.

Solution: $\boxed{\sqrt{3}}$ The area of the rectangle is $a^3 + a$. The portion under the parabola has area $\int_0^a x^2 + 1 \, dx = a^3/3 + a$. Thus we wish to solve the equation $a^3 + a = 2(a^3/3 + a)$; dividing by a and rearranging gives $a^2/3 = 1$, so $a = \sqrt{3}$.

7. Denote by $\langle x \rangle$ the fractional part of the real number x (for instance, $\langle 3.2 \rangle = 0.2$). A positive integer N is selected randomly from the set $\{1, 2, 3, \dots, M\}$, with each integer having the same probability of being picked, and $\langle \frac{87}{303}N \rangle$ is calculated. This procedure is repeated M times and the average value $A(M)$ is obtained. What is $\lim_{M \rightarrow \infty} A(M)$?

Solution: This method of picking N is equivalent to uniformly randomly selecting a positive integer. Call this the average value of $\langle \frac{87}{303}N \rangle$ for N a positive integer. In lowest terms, $\frac{87}{303} = \frac{29}{101}$, so the answer is the same as the average value of $\frac{0}{101}, \frac{1}{101}, \dots, \frac{100}{101}$, which is $\frac{1+2+\dots+100}{101 \cdot 101} = \frac{100 \cdot 101 / 2}{101 \cdot 101} = \boxed{\frac{50}{101}}$.

8. Evaluate $\int_0^{(\sqrt{2}-1)/2} \frac{dx}{(2x+1)\sqrt{x^2+x}}$.

Solution: Let $u = \sqrt{x^2 + x}$. Then $du = \frac{2x+1}{2\sqrt{x^2+x}} dx$. So the integral becomes $2 \int \frac{du}{(4x^2+4x+1)}$, or $2 \int \frac{du}{4u^2+1}$. This is $\tan^{-1}(2u)$, yielding a final answer of $\tan^{-1}(2\sqrt{x^2+x}) + C$ for the indefinite integral. The definite integral becomes $\tan^{-1}(1) - \tan^{-1}(0) = \boxed{\frac{\pi}{4}}$.

9. Suppose f is a differentiable real function such that $f(x) + f'(x) \leq 1$ for all x , and $f(0) = 0$. What is the largest possible value of $f(1)$? (Hint: consider the function $e^x f(x)$.)

Solution: $\boxed{1 - 1/e}$ Let $g(x) = e^x f(x)$; then $g'(x) = e^x(f(x) + f'(x)) \leq e^x$. Integrating from 0 to 1, we have $g(1) - g(0) = \int_0^1 g'(x) dx \leq \int_0^1 e^x dx = e - 1$. But $g(1) - g(0) = e \cdot f(1)$, so we get $f(1) \leq (e - 1)/e$. This maximum is attained if we actually have $g'(x) = e^x$ everywhere; this entails the requirement $f(x) + f'(x) = 1$, which is met by $f(x) = 1 - e^{-x}$.

10. A continuous real function f satisfies the identity $f(2x) = 3f(x)$ for all x . If $\int_0^1 f(x) dx = 1$, what is $\int_1^2 f(x) dx$?

Solution: $\boxed{5}$ Let $S = \int_1^2 f(x) dx$. By setting $u = 2x$, we see that $\int_{1/2}^1 f(x) dx = \int_{1/2}^1 f(2x)/3 dx = \int_1^2 f(u)/6 du = S/6$. Similarly, $\int_{1/4}^{1/2} f(x) dx = S/36$, and in general $\int_{1/2^{n-1}}^{1/2^n} f(x) dx = S/6^n$. Adding finitely many of these, we have $\int_{1/2^n}^1 f(x) dx = S/6 + S/36 + \dots + S/6^n = S \cdot (1 - 1/6^n)/5$. Taking the limit as $n \rightarrow \infty$, we have $\int_0^1 f(x) dx = S/5$. Thus $S = 5$, the answer.

Harvard-MIT Math Tournament

March 17, 2002

Individual General Test: **Part 1**

1. What is the maximum number of lattice points (i.e. points with integer coordinates) in the plane that can be contained strictly inside a circle of radius 1?

Solution: $\boxed{4}$. The circle centered at $(1/2, 1/2)$ shows that 4 is achievable. On the other hand, no two points within the circle can be at a mutual distance of 2 or greater. If there are more than four lattice points, classify all such points by the parity of their coordinates: (even, even), (even, odd), (odd, even), or (odd, odd). Then some two points lie in the same class. Since they are distinct, this means either their first or second coordinates must differ by at least 2, so their distance is at least 2, a contradiction.

2. Eight knights are randomly placed on a chessboard (not necessarily on distinct squares). A knight on a given square attacks all the squares that can be reached by moving either (1) two squares up or down followed by one square left or right, or (2) two squares left or right followed by one square up or down. Find the probability that every square, occupied or not, is attacked by some knight.

Solution: $\boxed{0}$. Since every knight attacks at most eight squares, the event can only occur if every knight attacks exactly eight squares. However, each corner square must be attacked, and some experimentation readily finds that it is impossible to place a knight so as to attack a corner and seven other squares as well.

3. How many triples (A, B, C) of positive integers (positive integers are the numbers $1, 2, 3, 4, \dots$) are there such that $A + B + C = 10$, where order does not matter (for instance the triples $(2, 3, 5)$ and $(3, 2, 5)$ are considered to be the same triple) and where two of the integers in a triple could be the same (for instance $(3, 3, 4)$ is a valid triple).

Solution: The triples can merely be enumerated: $(1, 1, 8)$, $(1, 2, 7)$, $(1, 3, 6)$, $(1, 4, 5)$, $(2, 2, 6)$, $(2, 3, 5)$, $(2, 4, 4)$, and $(3, 3, 4)$. There are $\boxed{8}$ elements.

4. We call a set of professors and committees on which they serve a *university* if

(1) given two distinct professors there is one and only one committee on which they both serve,

(2) given any committee, C , and any professor, P , not on that committee, there is exactly one committee on which P serves and no professors on committee C serve, and

(3) there are at least two professors on each committee; there are at least two committees.

What is the smallest number of committees a university can have?

Solution: Let C be any committee. Then there exists a professor P not on C (or else there would be no other committees). By axiom 2, P serves on a committee D having no common members with C . Each of these committees has at least two members, and for each $Q \in C, R \in D$, there exists (by axiom 1) a committee containing Q and R , which (again by axiom 1) has no other common members with C or D . Thus we have at least

$2 + 2 \cdot 2 = \boxed{6}$ committees. This minimum is attainable - just take four professors and let any two professors form a committee.

5. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

Solution: The hexagon is composed of six equilateral triangles each of side length $\sqrt[4]{3}$ (with base $b = \sqrt[4]{3}$ and height $\frac{\sqrt{3}}{2}b$), so the total area is $\boxed{\frac{9}{2}}$.

6. A man, standing on a lawn, is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result?

Solution: $\boxed{60\pi \text{ ft}^2}$ Let O be the center of the man's circular trajectory. The sombrero kills all the grass that is within 3 feet of any point that is 5 feet away from O — i.e. all the grass at points P with $2 \leq OP \leq 8$. The area of this annulus is then $\pi(8^2 - 2^2) = 60\pi$ square feet.

7. A circle is inscribed in a square dartboard. If a dart is thrown at the dartboard and hits the dartboard in a random location, with all locations having the same probability of being hit, what is the probability that it lands within the circle?

Solution: The answer is the area of the circle over the area of the square, which is $\boxed{\frac{\pi}{4}}$.

8. Count the number of triangles with positive area whose vertices are points whose (x, y) -coordinates lie in the set $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.

Solution: There are $\binom{9}{3} = 84$ triples of points. 8 of them form degenerate triangles (the ones that lie on a line), so there are $84 - 8 = \boxed{76}$ nondegenerate triangles.

9. Real numbers a, b, c satisfy the equations $a + b + c = 26, 1/a + 1/b + 1/c = 28$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

Solution: $\boxed{725}$ Multiplying the two given equations gives

$$\frac{a}{a} + \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{b} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} + \frac{c}{c} = 26 \cdot 28 = 728,$$

and subtracting 3 from both sides gives the answer, 725.

10. A certain cafeteria serves ham and cheese sandwiches, ham and tomato sandwiches, and tomato and cheese sandwiches. It is common for one meal to include multiple types of sandwiches. On a certain day, it was found that 80 customers had meals which contained both ham and cheese; 90 had meals containing both ham and tomatoes; 100 had meals containing both tomatoes and cheese. 20 customers' meals included all three ingredients. How many customers were there?

Solution: $\boxed{230}$. Everyone who ate just one sandwich is included in exactly one of the first three counts, while everyone who ate more than one sandwich is included in all four counts. Thus, to count each customer exactly once, we must add the first three figures and subtract the fourth twice: $80 + 90 + 100 - 2 \cdot 20 = 230$.

Harvard-MIT Math Tournament

March 17, 2002

Individual General Test: **Part 2**

1. The squares of a chessboard are numbered from left to right and top to bottom (so that the first row reads $1, 2, \dots, 8$, the second reads $9, 10, \dots, 16$, and so forth). The number 1 is on a black square. How many black squares contain odd numbers?

Solution: $\boxed{16}$. The black squares in the n th row contain odd numbers when n is odd and even numbers when n is even; thus there are four rows where the black squares contain odd numbers, and each such row contributes four black squares.

2. You are in a completely dark room with a drawer containing 10 red, 20 blue, 30 green, and 40 khaki socks. What is the smallest number of socks you must randomly pull out in order to be sure of having at least one of each color?

Solution: $\boxed{91}$. The maximum number of socks that can be pulled out *without* representing every color is $20 \text{ blue} + 30 \text{ green} + 40 \text{ khaki} = 90$, so 91 is the minimum needed to ensure that this doesn't happen.

3. Solve for x in $3 = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$.

Solution: Squaring both sides and subtracting x from both sides, we get $9 - x = 3$, or $x = \boxed{6}$.

4. Dan is holding one end of a 26 inch long piece of light string that has a heavy bead on it with each hand (so that the string lies along two straight lines). If he starts with his hands together at the start and leaves his hands at the same height, how far does he need to pull his hands apart so that the bead moves upward by 8 inches?

Solution: After he pulls the bead is 5 inches below his hands, and it is 13 inches from each hand. Using the Pythagorean theorem, his hands must be $2 \cdot 12 = \boxed{24}$ inches apart.

5. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

Solution: The hexagon is composed of six equilateral triangles each of side length $\sqrt[4]{3}$ (with base $b = \sqrt[4]{3}$ and height $\frac{\sqrt{3}}{2}b$), so the total area is $\boxed{\frac{9}{2}}$.

6. Nine nonnegative numbers have average 10. What is the greatest possible value for their median?

Solution: $\boxed{18}$ If the median is m , then the five highest numbers are all at least m , so the sum of all the numbers is at least $5m$. Thus $90 \geq 5m \Rightarrow m \leq 18$. Conversely, we can achieve $m = 18$ by taking four 0's and five 18's.

7. p and q are primes such that the numbers $p + q$ and $p + 7q$ are both squares. Find the value of p .

Solution: $\boxed{2}$. Writing $x^2 = p + q, y^2 = p + 7q$, we have $6q = y^2 - x^2 = (y - x)(y + x)$. Since $6q$ is even, one of the factors $y - x, y + x$ is even, and then the other is as well; thus $6q$ is divisible by 4 $\Rightarrow q$ is even $\Rightarrow q = 2$ and $6q = 12$. We may assume x, y are both taken to be positive; then we must have $y - x = 2, y + x = 6 \Rightarrow x = 2$, so $p + 2 = 2^2 = 4 \Rightarrow p = 2$ also.

8. Two fair coins are simultaneously flipped. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that the other coin also came up heads on this last flip?

Solution: $\boxed{1/3}$. Let the desired probability be p . There is a $1/4$ chance that both coins will come up heads on the first toss. Otherwise, both can come up heads simultaneously only if both are tails on the first toss, and then the process restarts as if from the beginning; thus this situation occurs with probability $p/4$. Thus $p = 1/4 + p/4$; solving, $p = 1/3$.

Alternate Solution: The desired event is equivalent to both coins coming up tails for n successive turns (for some $n \geq 0$), then both coins coming up heads. For any fixed value of n , the probability of this occurring is $1/4^{n+1}$. Since all these events are disjoint, the total probability is $1/4 + 1/4^2 + 1/4^3 + \dots = 1/3$.

9. A and B are two points on a circle with center O , and C lies outside the circle, on ray AB . Given that $AB = 24, BC = 28, OA = 15$, find OC .

Solution: $\boxed{41}$. Let M be the midpoint of AB ; then $\triangle OMB$ is a right triangle with $OB = 15, MB = 12$, so $OM = 9$. Now $\triangle OMC$ is a right triangle with $OM = 9, MC = 40$, so $OC = 41$.

10. How many four-digit numbers are there in which at least one digit occurs more than once?

Solution: $\boxed{4464}$. There are 9000 four-digit numbers altogether. If we consider how many four-digit numbers have all their digits distinct, there are 9 choices for the first digit (since we exclude leading zeroes), and then 9 remaining choices for the second digit, then 8 for the third, and 7 for the fourth, for a total of $9 \cdot 9 \cdot 8 \cdot 7 = 4536$. Thus the remaining $9000 - 4536 = 4464$ numbers have a repeated digit.

Harvard-MIT Math Tournament

March 17, 2002

Individual Subject Test: **Geometry**

1. A man, standing on a lawn, is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result?

Solution: $\boxed{60\pi \text{ ft}^2}$ Let O be the center of the man's circular trajectory. The sombrero kills all the grass that is within 3 feet of any point that is 5 feet away from O — i.e. all the grass at points P with $2 \leq OP \leq 8$. The area of this annulus is then $\pi(8^2 - 2^2) = 60\pi$ square feet.

2. Dan is holding one end of a 26 inch long piece of light string that has a heavy bead on it with each hand (so that the string lies along straight lines). If he starts with his hands together at the start and leaves his hands at the same height, how far does he need to pull his hands apart so that the bead moves upward by 8 inches?

Solution: After he pulls the bead is 5 inches below his hands, and it is 13 inches from each hand. Using the Pythagorean theorem, his hands must be $2 \cdot 12 = \boxed{24}$ inches apart.

3. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

Solution: The hexagon is composed of six equilateral triangles each of side length $\sqrt[4]{3}$ (with base $b = \sqrt[4]{3}$ and height $\frac{\sqrt{3}}{2}b$), so the total area is $\boxed{\frac{9}{2}}$.

4. We call a set of professors and committees on which they serve a *university* if

(1) given two distinct professors there is one and only one committee on which they both serve,

(2) given any committee, C , and any professor, P , not on that committee, there is exactly one committee on which P serves and no professors on committee C serve, and

(3) there are at least two professors on each committee; there are at least two committees.

What is the smallest number of committees a university can have?

Solution: Let C be any committee. Then there exists a professor P not on C (or else there would be no other committees). By axiom 2, P serves on a committee D having no common members with C . Each of these committees has at least two members, and for each $Q \in C, R \in D$, there exists (by axiom 1) a committee containing Q and R , which (again by axiom 1) has no other common members with C or D . Thus we have at least $2 + 2 \cdot 2 = \boxed{6}$ committees. This minimum is attainable - just take four professors and let any two professors form a committee.

5. Consider a square of side length 1. Draw four lines that each connect a midpoint of a side with a corner not on that side, such that each midpoint and each corner is touched by only one line. Find the area of the region completely bounded by these lines.

Solution: In unit square $ABCD$, denote by E, F, G, H the respective midpoints of sides AB, BC, CD, DA . Let I be the intersection of AF and DE , let J be the intersection of BG and AF , let K be the intersection of CH and BG , and let L be the intersection of DE and CH . We want to find the area of square $IJKL$. The area of ABF is $\frac{1}{4}$, which is equal to $\frac{1}{2}AF \cdot BJ = \frac{\sqrt{5}}{4}BJ$, so $BJ = \frac{1}{\sqrt{5}}$. Using similar triangles, $GK = JF = \frac{1}{2}BJ$. Thus the length of a side of $IJKL$ is $JK = \frac{\sqrt{5}}{2} - \frac{1}{\sqrt{5}} - \frac{1}{2} \cdot \frac{1}{\sqrt{5}} = \frac{1}{\sqrt{5}}$, and the area of $IJKL$ is $\boxed{\frac{1}{5}}$.

6. If we pick (uniformly) a random square of area 1 with sides parallel to the x - and y -axes that lies entirely within the 5-by-5 square bounded by the lines $x = 0, x = 5, y = 0, y = 5$ (the corners of the square need not have integer coordinates), what is the probability that the point $(x, y) = (4.5, 0.5)$ lies within the square of area 1?

Solution: The upper-left corner of the unit square is picked uniformly from the square $0 \leq x \leq 4; 1 \leq y \leq 5$, and for it to contain the desired point it must lie in the square $3.5 \leq x \leq 4; 1 \leq y \leq 1.5$. The answer is the ratio of the squares' areas, $\frac{1}{4}/16 = \boxed{\frac{1}{64}}$.

7. Equilateral triangle ABC of side length 2 is drawn. Three squares external to the triangle, $ABDE, BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that contains these squares?

Solution: The equilateral triangle with sides lying on lines DG, EH , and FI has minimal area. (The only other reasonable candidate is the triangle with sides along DE, FG, HI , but a quick sketch shows that it is larger.) Let J, K , and L be the vertices of this triangle closest to D, H , and F , respectively. Clearly, $KI = FL = 2$. Triangle FCI is a $30^\circ - 30^\circ - 120^\circ$ triangle, so we can calculate the length of FI as $2\sqrt{3}$, making the side length of $\triangle JKL$ $4 + 2\sqrt{3}$, and its area $\boxed{12 + 7\sqrt{3}}$.

8. Equilateral triangle ABC of side length 2 is drawn. Three squares containing the triangle, $ABDE, BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that contains these squares?

Solution: The equilateral triangle with sides lying on lines DE, FG , and HI has minimal area. Let J, K , and L be the vertices of this triangle closest to D, H , and F , respectively. Clearly, $DE = 2$. Denote by M the intersection of CI and BD . Using the $30^\circ - 30^\circ - 120^\circ$ triangle BCM , we get $BM = 2/\sqrt{3}$, and thus $MD = 2 - 2/\sqrt{3}$. By symmetry, MJ bisects angle DMI , from which we see that $\triangle JMD$ is a $30^\circ - 60^\circ - 90^\circ$. We then get $JD = 2\sqrt{3} - 2$, making the side length of JKL $4\sqrt{3} - 2$, and its area $\boxed{13\sqrt{3} - 12}$.

9. A and B are two points on a circle with center O , and C lies outside the circle, on ray AB . Given that $AB = 24, BC = 28, OA = 15$, find OC .

Solution: $\boxed{41}$. Let M be the midpoint of AB ; then $\triangle OMB$ is a right triangle with $OB = 15, MB = 12$, so $OM = 9$. Now $\triangle OMC$ is a right triangle with $OM = 9, MC = 40$, so $OC = 41$.

10. Let $\triangle ABC$ be equilateral, and let D, E, F be points on sides BC, CA, AB respectively, with $FA = 9, AE = EC = 6, CD = 4$. Determine the measure (in degrees) of $\angle DEF$.

Solution: 60. Let H, I be the respective midpoints of sides BC, AB , and also extend CB and EF to intersect at J . By equal angles, $\triangle EIF \sim \triangle JBF$. However, $BF = 12 - 9 = 3 = 9 - 6 = IF$, so in fact $\triangle EIF \cong \triangle JBF$, and then $JB = 6$. Now let HI intersect EF at K , and notice that $\triangle EIK \sim \triangle JHK \Rightarrow IK/HK = EI/JH = 6/12 = 1/2 \Rightarrow HK = 4$, since $IK + HK = HI = 6$. Now consider the 60° rotation about E carrying triangle CHE to triangle HIE ; we see that it also takes D to K , and thus $\angle DEF = \angle DEK = 60^\circ$.

Harvard-MIT Math Tournament

March 17, 2002

Guts Round

1. An (l, a) -*design* of a set is a collection of subsets of that set such that each subset contains exactly l elements and that no two of the subsets share more than a elements. How many $(2,1)$ -designs are there of a set containing 8 elements?

Solution: There are $\binom{8}{2} = 28$ 2-element subsets. Any two distinct such subsets have at most 1 common element; hence, for each subset, we can decide independently whether or not it belongs to the design, and we thus obtain $\boxed{2^{28}}$ designs.

2. A *lattice point* in the plane is a point of the form (n, m) , where n and m are integers. Consider a set S of lattice points. We construct the *transform* of S , denoted by S' , by the following rule: the pair (n, m) is in S' if and only if any of $(n, m - 1)$, $(n, m + 1)$, $(n - 1, m)$, $(n + 1, m)$, and (n, m) is in S . How many elements are in the set obtained by successively transforming $\{(0, 0)\}$ 14 times?

Solution: Transforming it $k \geq 1$ times yields the “diamond” of points (n, m) such that $|n| + |m| \leq k$. The diamond contains $(k + 1)^2 + k^2$ lattice points (this can be seen by rotating the plane 45 degrees and noticing the lattice points in the transforms form two squares, one of which is contained in the other), so the answer is $\boxed{421}$.

3. How many elements are in the set obtained by transforming $\{(0, 0), (2, 0)\}$ 14 times?

Solution: Transforming it $k \geq 1$ times yields the diamond $\{(n, m) : |n - 1| + |m| \leq k + 1\}$ with the points $(1, k)$, $(1, k + 1)$, $(1, -k)$, $(1, -k - 1)$ removed (this can be seen inductively). So we get $(k + 1)^2 + k^2 - 4$ lattice points, making the answer $\boxed{477}$.

4. How many ways are there of using diagonals to divide a regular 6-sided polygon into triangles such that at least one side of each triangle is a side of the original polygon and that each vertex of each triangle is a vertex of the original polygon?

Solution: The number of ways of triangulating a convex $(n + 2)$ -sided polygon is $\binom{2n}{n} \frac{1}{n + 1}$, which is 14 in this case. However, there are two triangulations of a hexagon which produce one triangle sharing no sides with the original polygon, so the answer is $14 - 2 = \boxed{12}$.

5. Two 4×4 squares are randomly placed on an 8×8 chessboard so that their sides lie along the grid lines of the board. What is the probability that the two squares overlap?

Solution: $\boxed{529/625}$. Each square has 5 horizontal \cdot 5 vertical = 25 possible positions, so there are 625 possible placements of the squares. If they do not overlap, then either one square lies in the top four rows and the other square lies in the bottom four rows, or one square lies in the left four columns and the other lies in the right four columns. The first possibility can happen in $2 \cdot 5 \cdot 5 = 50$ ways (two choices of which square goes on top, and five horizontal positions for each square); likewise, so can the second. However,

this double-counts the 4 cases in which the two squares are in opposite corners, so we have $50 + 50 - 4 = 96$ possible non-overlapping arrangements $\Rightarrow 25^2 - 96 = 529$ overlapping arrangements.

6. Find all values of x that satisfy $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots$ (be careful; this is tricky).

Solution: Multiplying both sides by $1 + x$ gives $(1 + x)x = 1$, or $x = \frac{-1 \pm \sqrt{5}}{2}$. However, the series only converges for $|x| < 1$, so only the answer $x = \boxed{\frac{-1 + \sqrt{5}}{2}}$ makes sense.

7. A rubber band is 4 inches long. An ant begins at the left end. Every minute, the ant walks one inch along rightwards along the rubber band, but then the band is stretched (uniformly) by one inch. For what value of n will the ant reach the right end during the n th minute?

Solution: $\boxed{7}$ The ant traverses $1/4$ of the band's length in the first minute, $1/5$ of the length in the second minute (the stretching does not affect its position as a fraction of the band's length), $1/6$ of the length in the third minute, and so on. Since

$$1/4 + 1/5 + \dots + 1/9 < 0.25 + 0.20 + 0.167 + 0.143 + 0.125 + 0.112 = 0.997 < 1,$$

the ant does not cover the entire band in six minutes. However,

$$1/4 + \dots + 1/10 > 0.25 + 0.20 + 0.16 + 0.14 + 0.12 + 0.11 + 0.10 = 1.08 > 1,$$

so seven minutes suffice.

8. Draw a square of side length 1. Connect its sides' midpoints to form a second square. Connect the midpoints of the sides of the second square to form a third square. Connect the midpoints of the sides of the third square to form a fourth square. And so forth. What is the sum of the areas of all the squares in this infinite series?

Solution: The area of the first square is 1, the area of the second is $\frac{1}{2}$, the area of the third is $\frac{1}{4}$, etc., so the answer is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = \boxed{2}$.

9. Find all values of x with $0 \leq x < 2\pi$ that satisfy $\sin x + \cos x = \sqrt{2}$.

Solution: Squaring both sides gives $\sin^2 x + \cos^2 x + 2 \sin x \cos x = 1 + \sin 2x = 2$, so $x = \boxed{\frac{\pi}{4}, \frac{5\pi}{4}}$.

10. The mathematician John is having trouble remembering his girlfriend Alicia's 7-digit phone number. He remembers that the first four digits consist of one 1, one 2, and two 3s. He also remembers that the fifth digit is either a 4 or 5. While he has no memory of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. If this is all the information he has, how many phone numbers does he have to try if he is to make sure he dials the correct number?

Solution: There are $\frac{4!}{2!} = 12$ possibilities for the first four digits. There are two possibilities for the fifth digit. There are 10 possibilities for the sixth digit, and this uniquely determines the seventh digit. So he has to dial $12 \cdot 2 \cdot 10 = \boxed{240}$ numbers.

11. How many real solutions are there to the equation

$$|||x| - 2| - 2| - 2| = |||x| - 3| - 3| - 3| ?$$

Solution: $\boxed{6}$. The graphs of the two sides of the equation can be graphed on the same plot to reveal six intersection points.

12. This question forms a three question multiple choice test. After each question, there are 4 choices, each preceded by a letter. Please write down your answer as the ordered triple (letter of the answer of Question #1, letter of the answer of Question #2, letter of the answer of Question #3). If you find that all such ordered triples are logically impossible, then write “no answer” as your answer. If you find more than one possible set of answers, then provide all ordered triples as your answer.

When we refer to “the correct answer to Question X ” it is the actual answer, not the letter, to which we refer. When we refer to “the letter of the correct answer to question X ” it is the letter contained in parentheses that precedes the answer to which we refer.

You are given the following condition: No two correct answers to questions on the test may have the same letter.

Question 1. If a fourth question were added to this test, and if the letter of its correct answer were (C), then:

- (A) This test would have no logically possible set of answers.
- (B) This test would have one logically possible set of answers.
- (C) This test would have more than one logically possible set of answers.
- (D) This test would have more than one logically possible set of answers.

Question 2. If the answer to Question 2 were “Letter (D)” and if Question 1 were not on this multiple-choice test (still keeping Questions 2 and 3 on the test), then the letter of the answer to Question 3 would be:

- (A) Letter (B)
- (B) Letter (C)
- (C) Letter (D)
- (D) Letter (A)

Question 3. Let $P_1 = 1$. Let $P_2 = 3$. For all $i > 2$, define $P_i = P_{i-1}P_{i-2} - P_{i-2}$. Which is a factor of P_{2002} ?

- (A) 3
- (B) 4
- (C) 7
- (D) 9

Solution: $\boxed{(A, C, D)}$. Question 2: In order for the answer to be consistent with the condition, “If the answer to Question 2 were Letter (D),” the answer to this question actually must be “Letter (D).” The letter of this answer is (C).

Question 1: If a fourth question had an answer with letter (C), then at least two answers would have letter (C) (the answers to Questions 2 and 4). This is impossible. So, (A) must be the letter of the answer to Question 1.

Question 3: If we inspect the sequence P_i modulo 3, 4, 7, and 9 (the sequences quickly become periodic), we find that 3, 7, and 9 are each factors of P_{2002} . We know that letters (A) and (C) cannot be repeated, so the letter of this answer must be (D).

13. A *domino* is a 1-by-2 or 2-by-1 rectangle. A *domino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping dominoes. For instance, there is one domino tiling of a 2-by-1 rectangle and there are 2 tilings of a 2-by-2 rectangle (one consisting of two horizontal dominoes and one consisting of two vertical dominoes). How many domino tilings are there of a 2-by-10 rectangle?

Solution: The number of tilings of a 2-by- n , rectangle is the n th Fibonacci number F_n , where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. (This is not hard to show by induction.) The answer is $\boxed{89}$.

14. An *omino* is a 1-by-1 square or a 1-by-2 horizontal rectangle. An *omino tiling* of a region of the plane is a way of covering it (and only it) by ominoes. How many omino tilings are there of a 2-by-10 horizontal rectangle?

Solution: There are exactly as many omino tilings of a 1-by- n rectangle as there are domino tilings of a 2-by- n rectangle. Since the rows don't interact at all, the number of omino tilings of an m -by- n rectangle is the number of omino tilings of a 1-by- n rectangle raised to the m th power, F_n^m . The answer is $89^2 = \boxed{7921}$.

15. How many sequences of 0s and 1s are there of length 10 such that there are no three 0s or 1s consecutively anywhere in the sequence?

Solution: We can have blocks of either 1 or 2 0s and 1s, and these blocks must be alternating between 0s and 1s. The number of ways of arranging blocks to form a sequence of length n is the same as the number of omino tilings of a 1-by- n rectangle, and we may start each sequence with a 0 or a 1, making $2F_n$ or, in this case, $\boxed{178}$ sequences.

16. Divide an m -by- n rectangle into mn nonoverlapping 1-by-1 squares. A *polyomino* of this rectangle is a subset of these unit squares such that for any two unit squares S, T in the polyomino, either

- (1) S and T share an edge or
- (2) there exists a positive integer n such that the polyomino contains unit squares $S_1, S_2, S_3, \dots, S_n$ such that S and S_1 share an edge, S_n and T share an edge, and for all positive integers $k < n$, S_k and S_{k+1} share an edge.

We say a polyomino of a given rectangle *spans* the rectangle if for each of the four edges of the rectangle the polyomino contains a square whose edge lies on it.

What is the minimum number of unit squares a polyomino can have if it spans a 128-by-343 rectangle?

Solution: To span an $a \times b$ rectangle, we need at least $a + b - 1$ squares. Indeed, consider a square of the polyomino bordering the left edge of the rectangle and one bordering the right edge. There exists a path connecting these squares; suppose it runs through c different rows. Then the path requires at least $b - 1$ horizontal and $c - 1$ vertical steps, so it uses at least $b + c - 1$ different squares. However, since the polyomino also hits the top and bottom edges of the rectangle, it must run into the remaining $a - c$ rows as well, so altogether we need at least $a + b - 1$ squares. On the other hand, this many squares suffice — just consider all the squares bordering the lower or right edges of the rectangle. So, in our case, the answer is $128 + 343 - 1 = \boxed{470}$.

17. Find the number of *pentominoes* (5-square polyominoes) that span a 3-by-3 rectangle, where polyominoes that are flips or rotations of each other are considered the same polyomino.

Solution: By enumeration, the answer is $\boxed{6}$.

18. Call the pentominoes found in the last problem *square pentominoes*. Just like dominos and ominoes can be used to tile regions of the plane, so can square pentominoes. In particular, a *square pentomino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping square pentominoes. How many square pentomino tilings are there of a 12-by-12 rectangle?

Solution: Since 5 does not divide 144, there are $\boxed{0}$.

19. For how many integers a ($1 \leq a \leq 200$) is the number a^a a square?

Solution: $\boxed{107}$ If a is even, we have $a^a = (a^{a/2})^2$. If a is odd, $a^a = (a^{(a-1)/2})^2 \cdot a$, which is a square precisely when a is. Thus we have 100 even values of a and 7 odd square values ($1^2, 3^2, \dots, 13^2$) for a total of 107.

20. The Antartic language has an alphabet of just 16 letters. Interestingly, every word in the language has exactly 3 letters, and it is known that no word's first letter equals any word's last letter (for instance, if the alphabet were $\{a, b\}$ then aab and aaa could not both be words in the language because a is the first letter of a word and the last letter of a word; in fact, just aaa alone couldn't be in the language). Given this, determine the maximum possible number of words in the language.

Solution: $\boxed{1024}$ Every letter can be the first letter of a word, or the last letter of a word, or possibly neither, but not both. If there are a different first letters and b different last letters, then we can form $a \cdot 16 \cdot b$ different words (and the desired conditions will be met). Given the constraints $0 \leq a, b$; $a + b \leq 16$, this product is maximized when $a = b = 8$, giving the answer.

21. The Dyslexian alphabet consists of consonants and vowels. It so happens that a finite sequence of letters is a word in Dyslexian precisely if it alternates between consonants and vowels (it may begin with either). There are 4800 five-letter words in Dyslexian. How many letters are in the alphabet?

Solution: $\boxed{12}$ Suppose there are c consonants, v vowels. Then there are $c \cdot v \cdot c \cdot v \cdot c + v \cdot c \cdot v \cdot c \cdot v = (cv)^2(c+v)$ five-letter words. Thus, $c+v = 4800/(cv)^2 = 3 \cdot (40/cv)^2$, so cv is a divisor of 40. If $cv \leq 10$, we have $c+v \geq 48$, impossible for c, v integers; if $cv = 40$, then $c+v = 3$ which is again impossible. So $cv = 20$, giving $c+v = 12$, the answer. As a check, this does have integer solutions: $(c, v) = (2, 10)$ or $(10, 2)$.

22. A *path* of length n is a sequence of points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with integer coordinates such that for all i between 1 and $n-1$ inclusive, either

- (1) $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$ (in which case we say the i th step is *rightward*) or
- (2) $x_{i+1} = x_i$ and $y_{i+1} = y_i + 1$ (in which case we say that the i th step is *upward*).

This path is said to *start* at (x_1, y_1) and *end* at (x_n, y_n) . Let $P(a, b)$, for a and b nonnegative integers, be the number of paths that start at $(0, 0)$ and end at (a, b) .

Find $\sum_{i=0}^{10} P(i, 10-i)$.

Solution: This is just the number of paths of length 10. The i th step can be either upward or rightward, so there are $2^{10} = \boxed{1024}$ such paths.

23. Find $P(7, 3)$.

Solution: The number of paths that start at $(0, 0)$ and end at (n, m) is $\binom{n+m}{n}$, since we must choose n of our $n+m$ steps to be rightward steps. In this case, the answer is $\boxed{120}$.

24. A *restricted path* of length n is a path of length n such that for all i between 1 and $n-2$ inclusive, if the i th step is upward, the $i+1$ st step must be rightward.

Find the number of restricted paths that start at $(0, 0)$ and end at $(7, 3)$.

Solution: This is equal to the number of lattice paths from $(0, 0)$ to $(7, 3)$ that use only rightward and diagonal (upward+rightward) steps plus the number of lattice paths from $(0, 0)$ to $(7, 2)$ that use only rightward and diagonal steps, which is equal to the number of paths (as defined above) from $(0, 0)$ to $(4, 3)$ plus the number of paths from $(0, 0)$ to $(5, 2)$, or $\binom{4+3}{3} + \binom{5+2}{2} = \boxed{56}$.

25. A math professor stands up in front of a room containing 100 very smart math students and says, "Each of you has to write down an integer between 0 and 100, inclusive, to guess 'two-thirds of the average of all the responses.' Each student who guesses the highest integer that is not higher than two-thirds of the average of all responses will receive a prize." If among all the students it is common knowledge that everyone will write down the best response, and there is no communication between students, what single integer should each of the 100 students write down?

Solution: Since the average cannot be greater than 100, no student will write down a number greater than $\frac{2}{3} \cdot 100$. But then the average cannot be greater than $\frac{2}{3} \cdot 100$, and, realizing this, each student will write down a number no greater than $(\frac{2}{3})^2 \cdot 100$. Continuing in this manner, we eventually see that no student will write down an integer greater than $\boxed{0}$, so this is the answer.

26. Another professor enters the same room and says, “Each of you has to write down an integer between 0 and 200. I will then compute X , the number that is 3 greater than half the average of all the numbers that you will have written down. Each student who writes down the number closest to X (either above or below X) will receive a prize.” One student, who misunderstood the question, announces to the class that he will write the number 107. If among the other 99 students it is common knowledge that all 99 of them will write down the best response, and there is no further communication between students, what single integer should each of the 99 students write down?

Solution: Use the same logic to get $\boxed{7}$. Note 6 and 8 do not work.

27. Consider the two hands of an analog clock, each of which moves with constant angular velocity. Certain positions of these hands are possible (e.g. the hour hand halfway between the 5 and 6 and the minute hand exactly at the 6), while others are impossible (e.g. the hour hand exactly at the 5 and the minute hand exactly at the 6). How many different positions are there that would remain possible if the hour and minute hands were switched?

Solution: $\boxed{143}$ We can look at the twelve-hour cycle beginning at midnight and ending just before noon, since during this time, the clock goes through each possible position exactly once. The minute hand has twelve times the angular velocity of the hour hand, so if the hour hand has made t revolutions from its initial position ($0 \leq t < 1$), the minute hand has made $12t$ revolutions. If the hour hand were to have made $12t$ revolutions, the minute hand would have made $144t$. So we get a valid configuration by reversing the hands precisely when $144t$ revolutions land the hour hand in the same place as t revolutions — i.e. when $143t = 144t - t$ is an integer, which clearly occurs for exactly 143 values of t corresponding to distinct positions on the clock ($144 - 1 = 143$).

28. Count how many 8-digit numbers there are that contain exactly four nines as digits.

Solution: There are $\binom{8}{4} \cdot 9^4$ sequences of 8 numbers with exactly four nines. A sequence of digits of length 8 is not an 8-digit number, however, if and only if the first digit is zero. There are $\binom{7}{4} 9^3$ 8-digit sequences that are not 8-digit numbers. The answer is thus $\binom{8}{4} \cdot 9^4 - \binom{7}{4} 9^3 = \boxed{433755}$.

29. A sequence $s_0, s_1, s_2, s_3, \dots$ is defined by $s_0 = s_1 = 1$ and, for every positive integer n , $s_{2n} = s_n, s_{4n+1} = s_{2n+1}, s_{4n-1} = s_{2n-1} + s_{2n-1}^2/s_{n-1}$. What is the value of s_{1000} ?

Solution: $\boxed{720}$ Some experimentation with small values may suggest that $s_n = k!$, where k is the number of ones in the binary representation of n , and this formula is in fact provable by a straightforward induction. Since $1000_{10} = 1111101000_2$, with six ones, $s_{1000} = 6! = 720$.

30. A conical flask contains some water. When the flask is oriented so that its base is horizontal and lies at the bottom (so that the vertex is at the top), the water is 1 inch deep. When the flask is turned upside-down, so that the vertex is at the bottom, the water is 2 inches deep. What is the height of the cone?

Solution: $\boxed{\frac{1}{2} + \frac{\sqrt{93}}{6}}$. Let h be the height, and let V be such that Vh^3 equals the volume of the flask. When the base is at the bottom, the portion of the flask not occupied by water forms a cone similar to the entire flask, with a height of $h - 1$; thus its volume is $V(h - 1)^3$. When the base is at the top, the water occupies a cone with a height of 2, so its volume is $V \cdot 2^3$. Since the water's volume does not change,

$$\begin{aligned} Vh^3 - V(h - 1)^3 &= 8V \\ \Rightarrow 3h^2 - 3h + 1 &= h^3 - (h - 1)^3 = 8 \\ \Rightarrow 3h^2 - 3h - 7 &= 0. \end{aligned}$$

Solving via the quadratic formula and taking the positive root gives $h = \frac{1}{2} + \frac{\sqrt{93}}{6}$.

31. Express, as concisely as possible, the value of the product

$$(0^3 - 350)(1^3 - 349)(2^3 - 348)(3^3 - 347) \cdots (349^3 - 1)(350^3 - 0).$$

Solution: $\boxed{0}$. One of the factors is $7^3 - 343 = 0$, so the whole product is zero.

32. Two circles have radii 13 and 30, and their centers are 41 units apart. The line through the centers of the two circles intersects the smaller circle at two points; let A be the one outside the larger circle. Suppose B is a point on the smaller circle and C a point on the larger circle such that B is the midpoint of AC . Compute the distance AC .

Solution: $\boxed{12\sqrt{13}}$ Call the large circle's center O_1 . Scale the small circle by a factor of 2 about A ; we obtain a new circle whose center O_2 is at a distance of $41 - 13 = 28$ from O_1 , and whose radius is 26. Also, the dilation sends B to C , which thus lies on circles O_1 and O_2 . So points O_1, O_2, C form a 26-28-30 triangle. Let H be the foot of the altitude from C to O_1O_2 ; we have $CH = 24$ and $HO_2 = 10$. Thus, $HA = 36$, and $AC = \sqrt{24^2 + 36^2} = 12\sqrt{13}$.

33. The expression $[x]$ denotes the greatest integer less than or equal to x . Find the value of

$$\left\lfloor \frac{2002!}{2001! + 2000! + 1999! + \cdots + 1!} \right\rfloor.$$

Solution: $\boxed{2000}$ We break up $2002! = 2002(2001)!$ as

$$\begin{aligned} 2000(2001!) + 2 \cdot 2001(2000!) &= 2000(2001!) + 2000(2000!) + 2002 \cdot 2000(1999!) \\ &> 2000(2001! + 2000! + 1999! + \cdots + 1!). \end{aligned}$$

On the other hand,

$$2001(2001! + 2000! + \cdots + 1!) > 2001(2001! + 2000!) = 2001(2001!) + 2001! = 2002!.$$

Thus we have $2000 < 2002!/(2001! + \cdots + 1!) < 2001$, so the answer is 2000.

34. Points P and Q are 3 units apart. A circle centered at P with a radius of $\sqrt{3}$ units intersects a circle centered at Q with a radius of 3 units at points A and B . Find the area of quadrilateral $APBQ$.

Solution: The area is twice the area of triangle APQ , which is isosceles with side lengths 3, 3, $\sqrt{3}$. By Pythagoras, the altitude to the base has length $\sqrt{3^2 - (\sqrt{3}/2)^2} = \sqrt{33}/2$, so the triangle has area $\frac{\sqrt{99}}{4}$. Double this to get $\boxed{\frac{3\sqrt{11}}{2}}$.

35. Suppose a, b, c, d are real numbers such that

$$|a - b| + |c - d| = 99; \quad |a - c| + |b - d| = 1.$$

Determine all possible values of $|a - d| + |b - c|$.

Solution: $\boxed{99}$ If $w \geq x \geq y \geq z$ are four arbitrary real numbers, then $|w - z| + |x - y| = |w - y| + |x - z| = w + x - y - z \geq w - x + y - z = |w - x| + |y - z|$. Thus, in our case, two of the three numbers $|a - b| + |c - d|$, $|a - c| + |b - d|$, $|a - d| + |b - c|$ are equal, and the third one is less than or equal to these two. Since we have a 99 and a 1, the third number must be 99.

36. Find the set consisting of all real values of x such that the three numbers $2^x, 2^{x^2}, 2^{x^3}$ form a non-constant arithmetic progression (in that order).

Solution: The empty set, $\boxed{\emptyset}$. Trivially, $x = 0, 1$ yield constant arithmetic progressions; we show that there are no other possibilities. If these numbers do form a progression, then, by the AM-GM (arithmetic mean-geometric mean) inequality,

$$\begin{aligned} 2 \cdot 2^{x^2} &= 2^x + 2^{x^3} \geq 2\sqrt{2^x \cdot 2^{x^3}} \\ \Rightarrow 2^{x^2} &\geq 2^{(x+x^3)/2} \Rightarrow x^2 \geq (x+x^3)/2 \\ \Rightarrow x(x-1)^2 &= x^3 - 2x^2 + x \leq 0. \end{aligned}$$

Assuming $x \neq 0, 1$, we can divide by $(x-1)^2 > 0$ and obtain $x < 0$. However, then $2^x, 2^{x^3}$ are less than 1, while 2^{x^2} is more than 1, so the given sequence cannot possibly be an arithmetic progression.

37. Call a positive integer “mild” if its base-3 representation never contains the digit 2. How many values of n ($1 \leq n \leq 1000$) have the property that n and n^2 are both mild?

Solution: $\boxed{7}$ Such a number, which must consist entirely of 0’s and 1’s in base 3, can never have more than one 1. Indeed, if $n = 3^a + 3^b + \text{higher powers}$ where $b > a$, then $n^2 = 3^{2a} + 2 \cdot 3^{a+b} + \text{higher powers}$ which will not be mild. On the other hand, if n does just have one 1 in base 3, then clearly n and n^2 are mild. So the values of $n \leq 1000$ that work are $3^0, 3^1, \dots, 3^6$; there are 7 of them.

38. Massachusetts Avenue is ten blocks long. One boy and one girl live on each block. They want to form friendships such that each boy is friends with exactly one girl and vice-versa. Nobody wants a friend living more than one block away (but they may be on the same block). How many pairings are possible?

Solution: 89 Let a_n be the number of pairings if there are n blocks; we have $a_1 = 1, a_2 = 2$, and we claim the Fibonacci recurrence is satisfied. Indeed, if there are n blocks, either the boy on block 1 is friends with the girl on block 1, leaving a_{n-1} possible pairings for the people on the remaining $n - 1$ blocks, or he is friends with the girl on block 2, in which case the girl on block 1 must be friends with the boy on block 2, and then there are a_{n-2} possibilities for the friendships among the remaining $n - 2$ blocks. So $a_n = a_{n-1} + a_{n-2}$, and we compute: $a_3 = 3, a_4 = 5, \dots, a_{10} = 89$.

39. In the x - y plane, draw a circle of radius 2 centered at $(0, 0)$. Color the circle red above the line $y = 1$, color the circle blue below the line $y = -1$, and color the rest of the circle white. Now consider an arbitrary straight line at distance 1 from the circle. We color each point P of the line with the color of the closest point to P on the circle. If we pick such an arbitrary line, randomly oriented, what is the probability that it contains red, white, and blue points?

Solution: Let $O = (0, 0), P = (1, 0)$, and H the foot of the perpendicular from O to the line. If $\angle POH$ (as measured counterclockwise) lies between $\pi/3$ and $2\pi/3$, the line will fail to contain blue points; if it lies between $4\pi/3$ and $5\pi/3$, the line will fail to contain red points. Otherwise, it has points of every color. Thus, the answer is $1 - \frac{2\pi}{3}/2\pi = \boxed{\frac{2}{3}}$.

40. Find the volume of the three-dimensional solid given by the inequality $\sqrt{x^2 + y^2} + |z| \leq 1$.

Solution: $2\pi/3$. The solid consists of two cones, one whose base is the circle $x^2 + y^2 = 1$ in the xy -plane and whose vertex is $(0, 0, 1)$, and the other with the same base but vertex $(0, 0, -1)$. Each cone has a base area of π and a height of 1, for a volume of $\pi/3$, so the answer is $2\pi/3$.

41. For any integer n , define $\lfloor n \rfloor$ as the greatest integer less than or equal to n . For any positive integer n , let

$$f(n) = \lfloor n \rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \dots + \left\lfloor \frac{n}{n} \right\rfloor.$$

For how many values of $n, 1 \leq n \leq 100$, is $f(n)$ odd?

Solution: 55 Notice that, for fixed a , $\lfloor n/a \rfloor$ counts the number of integers $b \in \{1, 2, \dots, n\}$ which are divisible by a ; hence, $f(n)$ counts the number of pairs $(a, b), a, b \in \{1, 2, \dots, n\}$ with b divisible by a . For any fixed b , the number of such pairs is $d(b)$ (the number of divisors of b), so the total number of pairs $f(n)$ equals $d(1) + d(2) + \dots + d(n)$. But $d(b)$ is odd precisely when b is a square, so $f(n)$ is odd precisely when there are an odd number of squares in $\{1, 2, \dots, n\}$. This happens for $1 \leq n < 4; 9 \leq n < 16; \dots; 81 \leq n < 100$. Adding these up gives 55 values of n .

42. Find all the integers $n > 1$ with the following property: the numbers $1, 2, \dots, n$ can be arranged in a line so that, of any two adjacent numbers, one is divisible by the other.

Solution: $\boxed{2, 3, 4, 6}$ The values $n = 2, 3, 4, 6$ work, as shown by respective examples $1, 2$; $2, 1, 3$; $2, 4, 1, 3$; $3, 6, 2, 4, 1, 5$. We shall show that there are no other possibilities. If $n = 2k + 1$ is odd, then none of the numbers $k + 1, k + 2, \dots, 2k + 1$ can divide any other, so no two of these numbers are adjacent. This is only possible if they occupy the 1st, 3rd, $\dots, (2k + 1)$ th positions in the line, which means every number $\leq k$ is adjacent to two of these and hence divides two of them. But k only divides one of these numbers when $k \geq 2$. Thus no odd $n \geq 5$ works. If $n = 2k$ is even, the numbers $k + 1, k + 2, \dots, 2k$ again must be mutually nonadjacent, but now this means we can have up to two numbers $\leq k$ each of which is adjacent to only one number $> k$, and if there are two such numbers, they must be adjacent. If $k \geq 4$, then each of $k - 1, k$ divides only one of the numbers $k + 1, \dots, 2k$, so $k - 1, k$ must be adjacent, but this is impossible. Thus no even $k \geq 8$ works, and we are done.

43. Given that a, b, c are positive integers satisfying

$$a + b + c = \gcd(a, b) + \gcd(b, c) + \gcd(c, a) + 120,$$

determine the maximum possible value of a .

Solution: $\boxed{240}$. Notice that $(a, b, c) = (240, 120, 120)$ achieves a value of 240. To see that this is maximal, first suppose that $a > b$. Notice that $a + b + c = \gcd(a, b) + \gcd(b, c) + \gcd(c, a) + 120 \leq \gcd(a, b) + b + c + 120$, or $a \leq \gcd(a, b) + 120$. However, $\gcd(a, b)$ is a proper divisor of a , so $a \geq 2 \cdot \gcd(a, b)$. Thus, $a - 120 \leq \gcd(a, b) \leq a/2$, yielding $a \leq 240$. Now, if instead $a \leq b$, then either $b > c$ and the same logic shows that $b \leq 240 \Rightarrow a \leq 240$, or $b \leq c, c > a$ (since a, b, c cannot all be equal) and then $c \leq 240 \Rightarrow a \leq b \leq c \leq 240$.

44. The unknown real numbers x, y, z satisfy the equations

$$\frac{x + y}{1 + z} = \frac{1 - z + z^2}{x^2 - xy + y^2}; \quad \frac{x - y}{3 - z} = \frac{9 + 3z + z^2}{x^2 + xy + y^2}.$$

Find x .

Solution: $\boxed{\sqrt[3]{14}}$ Cross-multiplying in both equations, we get, respectively, $x^3 + y^3 = 1 + z^3, x^3 - y^3 = 27 - z^3$. Now adding gives $2x^3 = 28$, or $x = \sqrt[3]{14}$.

45. Find the number of sequences a_1, a_2, \dots, a_{10} of positive integers with the property that $a_{n+2} = a_{n+1} + a_n$ for $n = 1, 2, \dots, 8$, and $a_{10} = 2002$.

Solution: $\boxed{3}$ Let $a_1 = a, a_2 = b$; we successively compute $a_3 = a + b$; $a_4 = a + 2b$; \dots ; $a_{10} = 21a + 34b$. The equation $2002 = 21a + 34b$ has three positive integer solutions, namely $(84, 7), (50, 28), (16, 49)$, and each of these gives a unique sequence.

46. Points A, B, C in the plane satisfy $\overline{AB} = 2002, \overline{AC} = 9999$. The circles with diameters AB and AC intersect at A and D . If $\overline{AD} = 37$, what is the shortest distance from point A to line BC ?

Solution: $\angle ADB = \angle ADC = \pi/2$ since D lies on the circles with AB and AC as diameters, so D is the foot of the perpendicular from A to line BC , and the answer is the given $\boxed{37}$.

47. The real function f has the property that, whenever a, b, n are positive integers such that $a + b = 2^n$, the equation $f(a) + f(b) = n^2$ holds. What is $f(2002)$?

Solution: We know $f(a) = n^2 - f(2^n - a)$ for any a, n with $2^n > a$; repeated application gives

$$\begin{aligned} f(2002) &= 11^2 - f(46) = 11^2 - (6^2 - f(18)) = 11^2 - (6^2 - (5^2 - f(14))) \\ &= 11^2 - (6^2 - (5^2 - (4^2 - f(2))))). \end{aligned}$$

But $f(2) = 2^2 - f(2)$, giving $f(2) = 2$, so the above simplifies to $11^2 - (6^2 - (5^2 - (4^2 - 2))) = \boxed{96}$.

48. A *permutation* of a finite set is a one-to-one function from the set to itself; for instance, one permutation of $\{1, 2, 3, 4\}$ is the function π defined such that $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 4$, and $\pi(4) = 2$. How many permutations π of the set $\{1, 2, \dots, 10\}$ have the property that $\pi(i) \neq i$ for each $i = 1, 2, \dots, 10$, but $\pi(\pi(i)) = i$ for each i ?

Solution: For each such π , the elements of $\{1, 2, \dots, 10\}$ can be arranged into pairs $\{i, j\}$ such that $\pi(i) = j$; $\pi(j) = i$. Choosing a permutation π is thus tantamount to choosing a partition of $\{1, 2, \dots, 10\}$ into five disjoint pairs. There are 9 ways to pair off the number 1, then 7 ways to pair off the smallest number not yet paired, and so forth, so we have $9 \cdot 7 \cdot 5 \cdot 3 \cdot 1 = \boxed{945}$ partitions into pairs.

49. Two integers are *relatively prime* if they don't share any common factors, i.e. if their greatest common divisor is 1. Define $\varphi(n)$ as the number of positive integers that are less than n and relatively prime to n . Define $\varphi_d(n)$ as the number of positive integers that are less than dn and relatively prime to n .

What is the least n such that $\varphi_x(n) = 64000$, where $x = \varphi_y(n)$, where $y = \varphi(n)$?

Solution: For fixed n , the pattern of integers relatively prime to n repeats every n integers, so $\varphi_d(n) = d\varphi(n)$. Therefore the expression in the problem equals $\varphi(n)^3$. The cube root of 64000 is 40. $\varphi(p) = p - 1$ for any prime p . Since 40 is one less than a prime, the least n such that $\varphi(n) = 40$ is $\boxed{41}$.

50. Give the set of all positive integers n such that $\varphi(n) = 2002^2 - 1$.

Solution: The empty set, $\boxed{\emptyset}$. If m is relatively prime to n and $m < n$, then $n - m$ must likewise be relatively prime to n , and these are distinct for $n > 2$ since $n/2, n$ are not relatively prime. Therefore, for all $n > 2$, $\varphi(n)$ must be even. $2002^2 - 1$ is odd, and $\varphi(2) = 1 \neq 2002^2 - 1$, so no numbers n fulfill the equation.

51. Define $\varphi^k(n)$ as the number of positive integers that are less than or equal to n/k and relatively prime to n . Find $\phi^{2001}(2002^2 - 1)$. (Hint: $\phi(2003) = 2002$.)

Solution: $\varphi^{2001}(2002^2 - 1) = \varphi^{2001}(2001 \cdot 2003) =$ the number of m that are relatively prime to both 2001 and 2003, where $m \leq 2003$. Since $\phi(n) = n - 1$ implies that n is prime, we must only check for those m relatively prime to 2001, except for 2002, which is relatively prime to $2002^2 - 1$. So $\varphi^{2001}(2002^2 - 1) = \varphi(2001) + 1 = \varphi(3 \cdot 23 \cdot 29) + 1 = (3 - 1)(23 - 1)(29 - 1) + 1 = \boxed{1233}$.

52. Let $ABCD$ be a quadrilateral, and let E, F, G, H be the respective midpoints of AB, BC, CD, DA . If $EG = 12$ and $FH = 15$, what is the maximum possible area of $ABCD$?

Solution: The area of $EFGH$ is $EG \cdot FH \sin \theta / 2$, where θ is the angle between EG and FH . This is at most 90. However, we claim the area of $ABCD$ is twice that of $EFGH$. To see this, notice that $EF = AC/2 = GH$, $FG = BD/2 = HE$, so $EFGH$ is a parallelogram. The half of this parallelogram lying inside triangle DAB has area $(BD/2)(h/2)$, where h is the height from A to BD , and triangle DAB itself has area $BD \cdot h/2 = 2 \cdot (BD/2)(h/2)$. A similar computation holds in triangle BCD , proving the claim. Thus, the area of $ABCD$ is at most $\boxed{180}$. And this maximum is attainable — just take a rectangle with $AB = CD = 15, BC = DA = 12$.

53. ABC is a triangle with points E, F on sides AC, AB , respectively. Suppose that BE, CF intersect at X . It is given that $AF/FB = (AE/EC)^2$ and that X is the midpoint of BE . Find the ratio CX/XF .

Solution: Let $x = AE/EC$. By Menelaus's theorem applied to triangle ABE and line CXF ,

$$1 = \frac{AF}{FB} \cdot \frac{BX}{XE} \cdot \frac{EC}{CA} = \frac{x^2}{x+1}.$$

Thus, $x^2 = x + 1$, and x must be positive, so $x = (1 + \sqrt{5})/2$. Now apply Menelaus to triangle ACF and line BXE , obtaining

$$1 = \frac{AE}{EC} \cdot \frac{CX}{XF} \cdot \frac{FB}{BA} = \frac{CX}{XF} \cdot \frac{x}{x^2 + 1},$$

so $CX/XF = (x^2 + 1)/x = (2x^2 - x)/x = 2x - 1 = \boxed{\sqrt{5}}$.

54. How many pairs of integers (a, b) , with $1 \leq a \leq b \leq 60$, have the property that b is divisible by a and $b + 1$ is divisible by $a + 1$?

Solution: The divisibility condition is equivalent to $b - a$ being divisible by both a and $a + 1$, or, equivalently (since these are relatively prime), by $a(a + 1)$. Any b satisfying the condition is automatically $\geq a$, so it suffices to count the number of values $b - a \in \{1 - a, 2 - a, \dots, 60 - a\}$ that are divisible by $a(a + 1)$ and sum over all a . The number of such values will be precisely $60/[a(a + 1)]$ whenever this quantity is an integer, which fortunately happens for every $a \leq 5$; we count:
 $a = 1$ gives 30 values of b ;

$a = 2$ gives 10 values of b ;
 $a = 3$ gives 5 values of b ;
 $a = 4$ gives 3 values of b ;
 $a = 5$ gives 2 values of b ;
 $a = 6$ gives 2 values ($b = 6$ or 48);
any $a \geq 7$ gives only one value, namely $b = a$, since $b > a$ implies $b \geq a + a(a + 1) > 60$.

Adding these up, we get a total of $\boxed{106}$ pairs.

55. A sequence of positive integers is given by $a_1 = 1$ and $a_n = \gcd(a_{n-1}, n) + 1$ for $n > 1$. Calculate a_{2002} .

Solution: $\boxed{3}$. It is readily seen by induction that $a_n \leq n$ for all n . On the other hand, a_{1999} is one greater than a divisor of 1999. Since 1999 is prime, we have $a_{1999} = 2$ or 2000; the latter is not possible since $2000 > 1999$, so we have $a_{1999} = 2$. Now we straightforwardly compute $a_{2000} = 3$, $a_{2001} = 4$, and $a_{2002} = 3$.

56. x, y are positive real numbers such that $x + y^2 = xy$. What is the smallest possible value of x ?

Solution: $\boxed{4}$ Notice that $x = y^2/(y-1) = 2 + (y-1) + 1/(y-1) \geq 2 + 2 = 4$. Conversely, $x = 4$ is achievable, by taking $y = 2$.

57. How many ways, without taking order into consideration, can 2002 be expressed as the sum of 3 positive integers (for instance, $1000 + 1000 + 2$ and $1000 + 2 + 1000$ are considered to be the same way)?

Solution: Call the three numbers that sum to 2002 A, B , and C . In order to prevent redundancy, we will consider only cases where $A \leq B \leq C$. Then A can range from 1 to 667, inclusive. For odd A , there are $1000 - \frac{3(A-1)}{2}$ possible values for B . For each choice of A and B , there can only be one possible C , since the three numbers must add up to a fixed value. We can add up this arithmetic progression to find that there are 167167 possible combinations of A, B, C , for odd A . For each even A , there are $1002 - \frac{3A}{2}$ possible values for B . Therefore, there are 166833 possible combinations for even A . In total, this makes $\boxed{334000}$ possibilities.

58. A sequence is defined by $a_0 = 1$ and $a_n = 2^{a_{n-1}}$ for $n \geq 1$. What is the last digit (in base 10) of a_{15} ?

Solution: $\boxed{6}$. Certainly $a_{13} \geq 2$, so a_{14} is divisible by $2^2 = 4$. Writing $a_{14} = 4k$, we have $a_{15} = 2^{4k} = 16^k$. But every power of 16 ends in 6, so this is the answer.

59. Determine the value of

$$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots + 2001 \cdot 2002.$$

Solution: $\boxed{2004002}$. Rewrite the expression as

$$2 + 3 \cdot (4 - 2) + 5 \cdot (6 - 4) + \cdots + 2001 \cdot (2002 - 2000)$$

$$= 2 + 6 + 10 + \cdots + 4002.$$

This is an arithmetic progression with $(4002 - 2)/4 + 1 = 1001$ terms and average 2002, so its sum is $1001 \cdot 2002 = 2004002$.

60. A 5×5 square grid has the number -3 written in the upper-left square and the number 3 written in the lower-right square. In how many ways can the remaining squares be filled in with integers so that any two adjacent numbers differ by 1, where two squares are adjacent if they share a common edge (but not if they share only a corner)?

Solution: $\boxed{250}$ If the square in row i , column j contains the number k , let its “index” be $i + j - k$. The constraint on adjacent squares now says that if a square has index r , the squares to its right and below it each have index r or $r + 2$. The upper-left square has index 5, and the lower-right square has index 7, so every square must have index 5 or 7. The boundary separating the two types of squares is a path consisting of upward and rightward steps; it can be extended along the grid’s border so as to obtain a path between the lower-left and upper-right corners. Conversely, any such path uniquely determines each square’s index and hence the entire array of numbers — except that the two paths lying entirely along the border of the grid fail to separate the upper-left from the lower-right square and thus do not create valid arrays (since these two squares should have different indices). Each path consists of 5 upward and 5 rightward steps, so there are $\binom{10}{5} = 252$ paths, but two are impossible, so the answer is 250.

61. Bob Barker went back to school for a PhD in math, and decided to raise the intellectual level of *The Price is Right* by having contestants guess how many objects exist of a certain type, without going over. The number of points you will get is the percentage of the correct answer, divided by 10, with no points for going over (i.e. a maximum of 10 points).

Let’s see the first object for our contestants...a *table* of shape $(5, 4, 3, 2, 1)$ is an arrangement of the integers 1 through 15 with five numbers in the top row, four in the next, three in the next, two in the next, and one in the last, such that each row and each column is increasing (from left to right, and top to bottom, respectively). For instance:

1	2	3	4	5
6	7	8	9	
10	11	12		
13	14			
15				

is one table. How many tables are there?

Solution: $15!/(3^4 \cdot 5^3 \cdot 7^2 \cdot 9) = \boxed{292864}$. These are Standard Young Tableaux.

62. Our next object up for bid is an arithmetic progression of primes. For example, the primes 3, 5, and 7 form an arithmetic progression of length 3. What is the largest possible length of an arithmetic progression formed of positive primes less than 1,000,000? Be prepared to justify your answer.

Solution: $\boxed{12}$. We can get 12 with 110437 and difference 13860.

63. Our third and final item comes to us from Germany, I mean Geometry. It is known that a regular n -gon can be constructed with straightedge and compass if n is a prime that is 1 plus a power of 2. It is also possible to construct a $2n$ -gon whenever an n -gon is constructible, or a $p_1 p_2 \cdots p_m$ -gon where the p_i 's are distinct primes of the above form. What is really interesting is that these conditions, together with the fact that we can construct a square, is that they give us all constructible regular n -gons. What is the largest n less than 4,300,000,000 such that a regular n -gon is constructible?

Solution: The known primes of this form (Fermat primes) are 3, 5, 17, 257, and 65537, and the result is due to Gauss (German). If there are other such primes (unknown), then they are much bigger than 10^{10} . So for each product of these primes, we can divide $4.3 \cdot 10^9$ by that number and take \log_2 to find the largest power of 2 to multiply by, then compare the resulting numbers. There are 32 cases to check, or just observe that $2^{32} = 4,294,967,296$ is so close that there's likely a shortcut. Note that $2^{32} + 1$ is divisible by 641, and hence not prime. $3 \cdot 5 \cdot 17 \cdot 257 \cdot 65537 = 2^{32} - 1$ is smaller; replacing any of the factors by the closest power of 2 only decreases the product, and there's not enough room to squeeze in an extra factor of 2 without replacing all of them, and that gives us $\boxed{2^{32}}$, so indeed that it is the answer.

Help control the pet population. Have your pets spayed or neutered. Bye-bye.

Team Event Solutions

HMMT 2002

Palindromes. A *palindrome* is a positive integer n not divisible by 10 such that if you write the decimal digits of n in reverse order, the number you get is n itself. For instance, the numbers 4 and 25752 are palindromes.

1. [15] Determine the number of palindromes that are less than 1000.

Solution. Every one-digit number (there are nine) is a palindrome. The two-digit palindromes have the form $\underline{a}a$ for a nonzero digit a , so there are nine of them. A three-digit palindrome is $\underline{a}b\underline{a}$ with a a nonzero digit and b any digit, so there are $9 \times 10 = 90$ of these. Thus the number of palindromes less than 1000 is $9 + 9 + 90 = \mathbf{108}$.

2. [30] Determine the number of four-digit integers n such that n and $2n$ are both palindromes.

Solution. Let $n = \underline{a}b\underline{b}a$. If $a, b \leq 4$ then there are no carries in the multiplication $n \times 2$, and $2n = \underline{(2a)}(2b)\underline{(2b)}(2a)$ is a palindrome. We shall show conversely that if n and $2n$ are palindromes, then necessarily $\underline{a}, b \leq 4$. Hence the answer to the problem is $4 \times 5 = \mathbf{20}$ (because a cannot be zero).

If $a \geq 5$ then $2n$ is a five-digit number whose most significant digit is 1, but because $2n$ is even, its least significant digit is even, contradicting the assumption that $2n$ is a palindrome. Therefore $a \leq 4$. Consequently $2n$ is a four-digit number, and its tens and hundreds digits must be equal. Because $a \leq 4$, there is no carry out of the ones place in the multiplication $n \times 2$, and therefore the tens digit of $2n$ is the ones digit of $2b$. In particular, the tens digit of $2n$ is even. But if $b \geq 5$, the carry out of the tens place makes the hundreds digit of $2n$ *odd*, which is impossible. Hence $b \leq 4$ as well.

3. [40] Suppose that a positive integer n has the property that $n, 2n, 3n, \dots, 9n$ are all palindromes. Prove that the decimal digits of n are all zeros or ones.

Solution. First consider the ones digit a of n ; we claim that $a = 1$. Certainly a cannot be even, for then $5n$ would be divisible by 10. If a is 5, 7, or 9, then $2n$ has an even ones digit, while its most significant digit is 1. If a is 3, then $4n$ has an even ones digit but most significant digit 1. Thus $a = 1$ is the only possibility. Moreover $9n$ has the same number of digits as n , for otherwise $9n$ would have most significant digit 1 but least significant digit 9, which is forbidden.

Now suppose n has at least one digit that is neither a zero nor a one. Let b be the leftmost (i.e., most significant) such digit, so that the left end of the decimal representation of n looks like

$$\underline{a_1} \dots \underline{a_r} \underline{b} \dots$$

for some $r \geq 1$ and digits $a_i \in \{0, 1\}$. When n is multiplied by 9, there will be a carry out of the column containing b . In particular, the r^{th} digit from the left in $9n$ will not be $9a_r$. But the right end of the decimal representation of n is

$$\dots \underline{a_r} \dots \underline{a_1};$$

because each a_i is 0 or 1, there are no carries out of the first $r - 1$ columns, so the r^{th} digit from the right in $9n$ will be $9a_r$. Thus $9n$ is not a palindrome, a contradiction. This completes the proof.

Floor functions. The notation $\lfloor x \rfloor$ stands for the largest integer less than or equal to x .

4. [15] Let n be an integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor = n.$$

Solution. Suppose $n = 2m$ is even. Then $\lfloor n/2 \rfloor = \lfloor m \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1/2 \rfloor = m$, whose sum is $m + m = 2m = n$. Otherwise $n = 2m + 1$ is odd. In this case $\lfloor n/2 \rfloor = \lfloor m + 1/2 \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1 \rfloor = m + 1$, whose sum is $m + (m + 1) = 2m + 1 = n$, as desired.

5. [20] Prove for integers n that

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Solution. Suppose $n = 2m$ is even; then $\lfloor n/2 \rfloor = \lfloor m \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1/2 \rfloor = m$, whose product is $m^2 = \lfloor m^2 \rfloor = \lfloor (2m)^2/4 \rfloor$. Otherwise $n = 2m + 1$ is odd, so that $\lfloor n/2 \rfloor = \lfloor m + 1/2 \rfloor = m$ and $\lfloor (n+1)/2 \rfloor = \lfloor m + 1 \rfloor = m + 1$, whose product is $m^2 + m$. On the other side, we find that

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{4m^2 + 4m + 1}{4} \right\rfloor = \left\lfloor m^2 + m + \frac{1}{4} \right\rfloor = m^2 + m,$$

as desired.

In problems 6–7 you may use without proof the known summations

$$\sum_{n=1}^L n = n(n+1)/2 \quad \text{and} \quad \sum_{n=1}^L n^3 = n^2(n+1)^2/4 \quad \text{for positive integers } L.$$

6. [20] For positive integers L , let $S_L = \sum_{n=1}^L \lfloor n/2 \rfloor$. Determine all L for which S_L is a square number.

Solution. We distinguish two cases depending on the parity of L . Suppose first that $L = 2k - 1$ is odd, where $k \geq 1$. Then

$$S_L = \sum_{1 \leq n \leq 2k-1} \left\lfloor \frac{n}{2} \right\rfloor = 2 \sum_{0 \leq m < k} m = 2 \cdot \frac{k(k-1)}{2} = k(k-1).$$

If $k = 1$, this is the square number 0. If $k > 1$ then $(k-1)^2 < k(k-1) < k^2$, so $k(k-1)$ is not square. Now suppose $L = 2k$ is even, where $k \geq 1$. Then $S_L = S_{L-1} + k = k^2$ is always square. Hence S_L is square exactly when $L = 1$ or L is even.

7. [45] Let $T_L = \sum_{n=1}^L \lfloor n^3/9 \rfloor$ for positive integers L . Determine all L for which T_L is a square number.

Solution. Since T_L is square if and only if $9T_L$ is square, we may consider $9T_L$ instead of T_L .

It is well known that n^3 is congruent to 0, 1, or 8 modulo 9 according as n is congruent to 0, 1, or 2 modulo 3. (Proof: $(3m+k)^3 = 27m^3 + 3(9m^2)k + 3(3m)k^2 + k^3 \equiv k^3 \pmod{9}$.) Therefore

$n^3 - 9 \lfloor n^3/9 \rfloor$ is 0, 1, or 8 according as n is congruent to 0, 1, or 2 modulo 3. We find therefore that

$$\begin{aligned} 9T_L &= \sum_{1 \leq n \leq L} 9 \left\lfloor \frac{n^3}{9} \right\rfloor \\ &= \sum_{1 \leq n \leq L} n^3 - \#\{1 \leq n \leq L : n \equiv 1 \pmod{3}\} - 8\#\{1 \leq n \leq L : n \equiv 2 \pmod{3}\} \\ &= \left(\frac{1}{2}L(L+1) \right)^2 - \left\lfloor \frac{L+2}{3} \right\rfloor - 8 \left\lfloor \frac{L+1}{3} \right\rfloor. \end{aligned}$$

Clearly $9T_L < (L(L+1)/2)^2$ for $L \geq 1$. We shall prove that $9T_L > (L(L+1)/2 - 1)^2$ for $L \geq 4$, whence $9T_L$ is not square for $L \geq 4$. Because

$$(L(L+1)/2 - 1)^2 = (L(L+1)/2)^2 - L(L+1) + 1,$$

we need only show that

$$\left\lfloor \frac{L+2}{3} \right\rfloor + 8 \left\lfloor \frac{L+1}{3} \right\rfloor \leq L^2 + L - 2.$$

But the left-hand side of this is bounded above by $3L+10/3$, and the inequality $3L+10/3 \leq L^2+L-2$ means exactly $L^2 - 2L - 16/3 \geq 0$ or $(L-1)^2 \geq 19/3$, which is true for $L \geq 4$, as desired.

Hence T_L is not square for $L \geq 4$. By direct computation we find $T_1 = T_2 = 0$ and $T_3 = 3$, so T_L is square only for $L \in \{1, 2\}$.

Luck of the dice. Problems 8–12 concern a two-player game played on a board consisting of fourteen spaces in a row. The leftmost space is labeled *START*, and the rightmost space is labeled *END*. Each of the twelve other squares, which we number 1 through 12 from left to right, may be blank or may be labeled with an arrow pointing to the right. The term *blank square* will refer to one of these twelve squares that is not labeled with an arrow. The set of blank squares on the board will be called a *board configuration*; the board below uses the configuration $\{1, 2, 3, 4, 7, 8, 10, 11, 12\}$.

<i>START</i>					⇒	⇒			⇒				<i>END</i>
	1	2	3	4	5	6	7	8	9	10	11	12	

For $i \in \{1, 2\}$, player i has a die that produces each integer from 1 to s_i with probability $1/s_i$. Here s_1 and s_2 are positive integers fixed before the game begins. The game rules are as follows:

1. The players take turns alternately, and player 1 takes the first turn.
2. On each of his turns, player i rolls his die and moves his piece to the right by the number of squares that he rolled. If his move ends on a square marked with an arrow, he moves his piece forward another s_i squares. If that move ends on an arrow, he moves another s_i squares, repeating until his piece comes to rest on a square without an arrow.
3. If a player's move would take him past the *END* square, instead he lands on the *END* square.
4. Whichever player reaches the *END* square first wins.

As an example, suppose that $s_1 = 3$ and the first player is on square 4 in the sample board shown above. If the first player rolls a 2, he moves to square 6, then to square 9, finally coming to rest on

square 12. If the second player does not reach the *END* square on her next turn, the first player will necessarily win on his next turn, as he must roll at least a 1.

8. [35] In this problem only, assume that $s_1 = 4$ and that exactly one board square, say square number n , is marked with an arrow. Determine all choices of n that maximize the average distance in squares the first player will travel in his first two turns.

Solution. Because expectation is linear, the average distance the first player travels in his first two turns is the average sum of two rolls of his die (which does not depend on the board configuration) plus four times the probability that he lands on the arrow on one of his first two turns. Thus we just need to maximize the probability that player 1 lands on the arrow in his first two turns. If $n \geq 5$, player 1 cannot land on the arrow in his first turn, so he encounters the arrow with probability at most $1/4$. If instead $n \leq 4$, player 1 has a $1/4$ chance of landing on the arrow on his first turn. If he misses, then he has a $1/4$ chance of hitting the arrow on his second turn provided that he is not beyond square n already. The chance that player 1's first roll left him on square $n - 1$ or farther left is $(n - 1)/4$. Hence his probability of benefiting from the arrow in his first two turns is $1/4 + (1/4)(n - 1)/4$, which is maximized for $n = 4$, where it is greater than the value of $1/4$ that we get from $n \geq 5$. Hence the answer is $n = 4$.

9. [30] In this problem suppose that $s_1 = s_2$. Prove that for each board configuration, the first player wins with probability strictly greater than $\frac{1}{2}$.

Solution. Let σ_1 and σ_2 denote the sequence of the next twelve die rolls that players 1 and 2 respectively will make. The outcome of the game is completely determined by the σ_i . Now player 1 wins in all cases in which $\sigma_1 = \sigma_2$, for then each of player 2's moves bring her piece to a square already occupied by player 1's piece. It is sufficient, therefore, to show that player 1 wins at least half the cases in which $\sigma_1 \neq \sigma_2$. But all these cases can be partitioned into disjoint pairs

$$\{(\sigma_1, \sigma_2), (\sigma_2, \sigma_1)\},$$

and player 1 wins in at least one case in each pair. For if player 2 wins in the case (σ_1, σ_2) , say on her n^{th} turn, the first n elements of σ_1 do not take player 1 beyond space 12, while the first n elements of σ_2 must take player 2 beyond space 12. Clearly, then, player 1 wins (σ_2, σ_1) on his n^{th} turn.

10. [30] Exhibit a configuration of the board and a choice of s_1 and s_2 so that $s_1 > s_2$, yet the *second* player wins with probability strictly greater than $\frac{1}{2}$.

Solution. Let $s_1 = 3$ and $s_2 = 2$ and place an arrow on all the even-numbered squares. In this configuration, player 1 can move at most six squares in a turn: up to three from his roll and an additional three if his roll landed him on an arrow. Hence player 1 cannot win on his first or second turn. Player 2, however, wins immediately if she ever lands on an arrow. Thus player 2 has probability $1/2$ of winning on her first turn, and failing that, she has probability $1/2$ of winning on her second turn. Hence player 2 wins with probability at least $1/2 + (1/2)(1/2) = 3/4$.

11. [55] In this problem assume $s_1 = 3$ and $s_2 = 2$. Determine, with proof, the nonnegative integer k with the following property:

1. For every board configuration with strictly fewer than k blank squares, the first player wins with probability strictly greater than $\frac{1}{2}$; but

2. there exists a board configuration with exactly k blank squares for which the second player wins with probability strictly greater than $\frac{1}{2}$.

Solution. The answer is $k = 3$. Consider the configuration whose blank squares are 2, 6, and 10. Because these numbers represent all congruence classes modulo 3, player 1 cannot win on his first turn: he will come to rest on one of the blank squares. But player 2 will win on her first turn if she rolls a 1, for 2, 6, and 10 are all even. Thus player 2 wins on her first turn with probability $1/2$. Failing this, player 1 may fail to win on his second turn, for instance, if he rolled a 2 previously and now rolls a 1, ending up on square 6. Then player 2 will again have probability $1/2$ of winning on her next turn. Thus player 2 wins the game with probability exceeding $1/2$.

We must now prove that all configurations with fewer than three blanks favor player 1. If the numbers of the blank squares represent at most one residue class modulo 3, then clearly player 1 wins on his first turn with probability at least $2/3$. This disposes of the cases of no blanks, just one blank, and two blanks that are congruent modulo 3. In the remaining case, there are two blank squares whose indices are incongruent modulo 3. Then player 1 wins on his first turn with probability only $1/3$. If he does not win immediately, player 2 wins on her first turn with probability at most $1/2$, for there is a blank in at least one congruence class modulo 2. If player 2 does not win on her first turn, then player 1 wins on his second turn with probability at least $2/3$, for there is only one blank square in front of him now. Thus player 1 wins the game with probability at least $1/3 + (2/3)(1/2)(2/3) = 5/9 > 1/2$, as desired.

12. [65] Now suppose that before the game begins, the players choose the initial game state as follows:

1. The first player chooses s_1 subject to the constraint that $2 \leq s_1 \leq 5$; then
2. the second player chooses s_2 subject to the constraint that $2 \leq s_2 \leq 5$ and then specifies the board configuration.

Prove that the second player can always make her decisions so that she will win the game with probability strictly greater than $\frac{1}{2}$.

Solution. If $s_1 \in \{3, 5\}$, take $s_2 = 2$ and put arrows on the even-numbered squares. Player 1 cannot win on his first turn because he can move at most $2s_1$ spaces in a turn. Player 2 wins on her first turn with probability $1/2$. Failing that, player 1 might fail to win on his second turn, and player 2 will again have probability $1/2$ of winning on her second turn, so her probability of winning the game is certainly greater than $1/2$.

If $s_1 = 4$, take $s_2 = 3$ and leave blank only squares 1, 4, 7, and 10. These occupy all congruence classes modulo 4, so player 1 cannot win on his first turn. But the blank squares lie in the same congruence class modulo 3, so player 2 then wins on her first turn with probability $2/3$.

Finally, if $s_1 = 2$, take $s_2 = 5$ and leave all squares blank. Then player 2 moves 3 squares in a turn on average, hence covers 15 squares on average in her first five turns. Moreover, the distribution of player 2's distance traveled in five turns is symmetric about 15. Thus player 2 has probability greater than $1/2$ of reaching *END* by the end of her fifth turn. Player 1, on the other hand, cannot win in five turns because he can move at most 10 squares in those turns.

Advanced Topics Test Solutions

Harvard-MIT Math Tournament

March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

Solution: $\frac{24}{6 \cdot 3} = \frac{x^4 - y^4}{(x^2 + y^2)(x + y)} = \frac{(x^2 + y^2)(x + y)(x - y)}{(x^2 + y^2)(x + y)} = x - y = \boxed{\frac{4}{3}}$.

2. Find $\log_n \left(\frac{1}{2}\right) \log_{n-1} \left(\frac{1}{3}\right) \cdots \log_2 \left(\frac{1}{n}\right)$ in terms of n .

Solution: Using $\log \frac{1}{x} = -\log x$ and $\log_b a = \frac{\log a}{\log b}$, we get that the product equals $\frac{(-\log 2)(-\log 3)\cdots(-\log n)}{\log n \cdots \log 3 \log 2} = \boxed{(-1)^{n-1}}$.

3. Calculate the sum of the coefficients of $P(x)$ if $(20x^{27} + 2x^2 + 1)P(x) = 2001x^{2001}$.

Solution: The sum of coefficients of $f(x)$ is the value of $f(1)$ for any polynomial f . Plugging in 1 to the above equation, $P(1) = \frac{2001}{23} = \boxed{87}$.

4. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n . (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

Solution: There are 9^7 total shapes possible, since each of the 7 columns can contain anywhere from 0 to 8 checkers. The number of shapes symmetric with respect to a horizontal flip is the number of shapes of the leftmost four columns, since the configuration of these four columns uniquely determines the configuration of the remaining columns if it is known the shape is symmetric: 9^4 . Now we know there are $9^7 - 9^4$ non-symmetric shapes, so there are $\frac{9^7 - 9^4}{2}$ non-symmetric shapes modulo flips. Thus the total number of shapes modulo flips is $9^4 + \frac{9^7 - 9^4}{2} = 9^4 \left(1 + \frac{9^3 - 1}{2}\right) = 9^4 \left(\frac{9^3 + 1}{2}\right) = \left(\frac{3^8(3^6 - 1)}{2}\right) = 9 \frac{3^6(3^6 + 1)}{2} = 9(1 + 2 + \cdots + 3^6)$, so $n = 3^6 = \boxed{729}$.

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

Solution: Because the last digit of the product is 2, none of the three consecutive even integers end in 0. Thus they must end in 2, 4, 6 or 4, 6, 8, so they must end in 4, 6, 8 since $2 \cdot 4 \cdot 6$

does not end in 2. Call the middle integer n . Then the product is $(n-2)n(n+2) = n^3 - 4n$, so $n > \sqrt[3]{200000} = \sqrt[3]{200 \cdot 10^3} \approx 60$, but clearly $n < \sqrt[3]{300000} = \sqrt[3]{300 \cdot 10^3} < 70$. Thus $n = 66$, and the product is $66^3 - 4 \cdot 66 = \boxed{287232}$.

6. There are two red, two black, two white, and a positive but unknown number of blue socks in a drawer. It is empirically determined that if two socks are taken from the drawer without replacement, the probability they are of the same color is $\frac{1}{5}$. How many blue socks are there in the drawer?

Solution: Let the number of blue socks be $x > 0$. Then the probability of drawing a red sock from the drawer is $\frac{2}{6+x}$ and the probability of drawing a second red sock from the drawer is $\frac{1}{6+x-1} = \frac{1}{5+x}$, so the probability of drawing two red socks from the drawer without replacement is $\frac{2}{(6+x)(5+x)}$. This is the same as the probability of drawing two black socks from the drawer and the same as the probability of drawing two white socks from the drawer. The probability of drawing two blue socks from the drawer, similarly, is $\frac{x(x-1)}{(6+x)(5+x)}$. Thus the probability of drawing two socks of the same color is the sum of the probabilities of drawing two red, two black, two white, and two blue socks from the drawer: $3 \frac{2}{(6+x)(5+x)} + \frac{x(x-1)}{(6+x)(5+x)} = \frac{x^2-x+6}{(6+x)(5+x)} = \frac{1}{5}$. Cross-multiplying and distributing gives $5x^2 - 5x + 30 = x^2 + 11x + 30$, so $4x^2 - 16x = 0$, and $x = 0$ or 4 . But since $x > 0$, there are $\boxed{4}$ blue socks.

7. Order these four numbers from least to greatest: $5^{56}, 10^{51}, 17^{35}, 31^{28}$.

Solution: $10^{51} > 9^{51} = 3^{102} = 27^{34} > 17^{35} > 16^{35} = 32^{28} > 31^{28} > 25^{28} = 5^{56}$, so the ordering is $\boxed{5^{56}, 31^{28}, 17^{35}, 10^{51}}$.

8. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

Solution: If $n = 1$, the relation can not hold, so assume otherwise. If $x > y$, the left hand side factors as $n^y(n^{x-y} + 1)$ so $n^{x-y} + 1$ is a power of n . But it leaves a remainder of 1 when divided by n and is greater than 1, a contradiction. We reach a similar contradiction if $y > x$. So $y = x$ and $2n^x = n^z$, so 2 is a power of n and $n = 2$. So all solutions are of the form $2^x + 2^x = 2^{x+1}$, which holds for all x . $2^{x+1} < 2001$ implies $x < 11$, so there are $\boxed{10}$ solutions.

9. Find the real solutions of $(2x+1)(3x+1)(5x+1)(30x+1) = 10$.

Solution: $(2x+1)(3x+1)(5x+1)(30x+1) = [(2x+1)(30x+1)][(3x+1)(5x+1)] = (60x^2 + 32x + 1)(15x^2 + 8x + 1) = (4y+1)(y+1) = 10$, where $y = 15x^2 + 8x$. The quadratic equation in y yields $y = 1$ and $y = -\frac{9}{4}$. For $y = 1$, we have $15x^2 + 8x - 1 = 0$, so $x = \frac{-4 \pm \sqrt{31}}{15}$. For $y = -\frac{9}{4}$, we have $15x^2 + 8x + \frac{9}{4} = 0$, which yields only complex solutions for x . Thus the real solutions are $\boxed{\frac{-4 \pm \sqrt{31}}{15}}$.

10. Alex picks his favorite point (x, y) in the first quadrant on the unit circle $x^2 + y^2 = 1$, such that a ray from the origin through (x, y) is θ radians counterclockwise from the positive x -axis. He then computes $\cos^{-1}\left(\frac{4x+3y}{5}\right)$ and is surprised to get θ . What is $\tan(\theta)$?

Solution: $x = \cos(\theta)$, $y = \sin(\theta)$. By the trig identity you never thought you'd need, $\frac{4x+3y}{5} = \cos(\theta - \phi)$, where ϕ has sine $3/5$ and cosine $4/5$. Now $\theta - \phi = \theta$ is impossible, since $\phi \neq 0$, so we must have $\theta - \phi = -\theta$, hence $\theta = \phi/2$. Now use the trusty half-angle identities to get $\tan(\theta) = \boxed{\frac{1}{3}}$.

Algebra Test Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

Solution: $\frac{24}{6 \cdot 3} = \frac{x^4 - y^4}{(x^2 + y^2)(x + y)} = \frac{(x^2 + y^2)(x + y)(x - y)}{(x^2 + y^2)(x + y)} = x - y = \boxed{\frac{4}{3}}$.

2. Find $(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots$, where $|x| < 1$.

Solution: Let $S = (x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots = 1 + x + x^2 + x^3 + \cdots$. Since $xS = x + x^2 + x^3 + x^4 + \cdots$, we have $(1 - x)S = 1$, so $S = \boxed{\frac{1}{1 - x}}$.

3. How many times does 24 divide into 100! (factorial)?

Solution: We first determine the number of times 2 and 3 divide into $100! = 1 \cdot 2 \cdot 3 \cdots 100$. Let $\langle N \rangle_n$ be the number of times n divides into N (i.e. we want to find $\langle 100! \rangle_{24}$). Since 2 only divides into even integers, $\langle 100! \rangle_2 = \langle 2 \cdot 4 \cdot 6 \cdots 100 \rangle$. Factoring out 2 once from each of these multiples, we get that $\langle 100! \rangle_2 = \langle 2^{50} \cdot 1 \cdot 2 \cdot 3 \cdots 50 \rangle_2$. Repeating this process, we find that $\langle 100! \rangle_2 = \langle 2^{50+25+12+6+3+1} \cdot 1 \rangle_2 = 97$. Similarly, $\langle 100! \rangle_3 = \langle 3^{33+11+3+1} \rangle_3 = 48$. Now $24 = 2^3 \cdot 3$, so for each factor of 24 in 100! there needs to be three multiples of 2 and one multiple of 3 in 100!. Thus $\langle 100! \rangle_{24} = (\lfloor \langle 100! \rangle_2 / 3 \rfloor + \langle 100! \rangle_3) = \boxed{32}$, where $[N]$ is the greatest integer less than or equal to N .

4. Given that 7,999,999,999 has at most two prime factors, find its largest prime factor.

Solution: $7,999,999,999 = 8 \cdot 10^9 - 1 = 2000^3 - 1 = (2000 - 1)(2000^2 + 2000 + 1)$, so $(2000^2 + 2000 + 1) = \boxed{4,002,001}$ is its largest prime factor.

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

Solution: Because the last digit of the product is 2, none of the three consecutive even integers end in 0. Thus they must end in 2, 4, 6 or 4, 6, 8, so they must end in 4, 6, 8 since $2 \cdot 4 \cdot 6$ does not end in 2. Call the middle integer n . Then the product is $(n - 2)n(n + 2) = n^3 - 4n$, so $n > \sqrt[3]{200000} = \sqrt[3]{200 \cdot 10^3} \approx 60$, but clearly $n < \sqrt[3]{300000} = \sqrt[3]{300 \cdot 10^3} < 70$. Thus $n = 66$, and the product is $66^3 - 4 \cdot 66 = \boxed{287232}$.

6. What is the last digit of $1^1 + 2^2 + 3^3 + \cdots + 100^{100}$?

Solution: Let $L(d, n)$ be the last digit of a number ending in d to the n th power. For $n \geq 1$, we know that $L(0, n) = 0$, $L(1, n) = 1$, $L(5, n) = 5$, $L(6, n) = 6$. All numbers ending in odd digits in this series are raised to odd powers; for odd n , $L(3, n) = 3$ or 7, $L(7, n) = 3$ or 7, $L(9, n) = 9$. All numbers ending in even digits are raised to even powers; for even n , $L(2, n) = 4$ or 6, $L(4, n) = L(6, n) = 6$, $L(8, n) = 6$ or 4. Further, for each

last digit that has two possible values, the possible values will be present equally as often. Now define $S(d)$ such that $S(0)=0$ and for $1 \leq d \leq 9$, $S(d) = L(d, d) + L(d, d + 10) + L(d, d + 20) + L(d, d + 30) + \dots + L(d, d + 90)$, so that the sum we want to calculate becomes $S(0) + S(1) + S(2) + \dots + S(9)$. But by the above calculations all $S(d)$ are divisible by 10, so their sum is divisible by 10, which means its last digit is $\boxed{0}$.

7. A polynomial P has four roots, $\frac{1}{4}, \frac{1}{2}, 2, 4$. The product of the roots is 1, and $P(1) = 1$. Find $P(0)$.

Solution: A polynomial Q with n roots, x_1, \dots, x_n , and $Q(x_0) = 1$ is given by $Q(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}$, so $P(0) = \frac{1}{\frac{3}{4} \cdot \frac{1}{2} \cdot (-1) \cdot (-3)} = \boxed{\frac{8}{9}}$.

8. How many integers between 1 and 2000 inclusive share no common factors with 2001?

Solution: Two integers are said to be *relatively prime* if they share no common factors, that is if there is no integer greater than 1 that divides evenly into both of them. Note that 1 is relatively prime to all integers. Let $\varphi(n)$ be the number of integers less than n that are relatively prime to n . Since $\varphi(mn) = \varphi(m)\varphi(n)$ for m and n relatively prime, we have $\varphi(2001) = \varphi(3 \cdot 23 \cdot 29) = (3 - 1)(23 - 1)(29 - 1) = \boxed{1232}$.

9. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

Solution: If $n = 1$, the relation cannot hold, so assume otherwise. If $x > y$, the left hand side factors as $n^y(n^{x-y} + 1)$ so $n^{x-y} + 1$ is a power of n . But it leaves a remainder of 1 when divided by n and is greater than 1, a contradiction. We reach a similar contradiction if $y > x$. So $y = x$ and $2n^x = n^z$, so 2 is a power of n and $n = 2$. So all solutions are of the form $2^x + 2^x = 2^{x+1}$, which holds for all x . $2^{x+1} < 2001$ implies $x < 11$, so there are $\boxed{10}$ solutions.

10. Find the real solutions of $(2x + 1)(3x + 1)(5x + 1)(30x + 1) = 10$.

Solution: $(2x + 1)(3x + 1)(5x + 1)(30x + 1) = [(2x + 1)(30x + 1)][(3x + 1)(5x + 1)] = (60x^2 + 32x + 1)(15x^2 + 8x + 1) = (4y + 1)(y + 1) = 10$, where $y = 15x^2 + 8x$. The quadratic equation in y yields $y = 1$ and $y = -\frac{9}{4}$. For $y = 1$, we have $15x^2 + 8x - 1 = 0$, so $x = \frac{-4 \pm \sqrt{31}}{15}$. For $y = -\frac{9}{4}$, we have $15x^2 + 8x + \frac{9}{4}$, which yields only complex solutions for x . Thus the real solutions are $\boxed{\frac{-4 \pm \sqrt{31}}{15}}$.

Calculus Test Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. A sequence of ants walk from $(0, 0)$ to $(1, 0)$ in the plane. The n th ant walks along n semicircles of radius $\frac{1}{n}$ with diameters lying along the line from $(0, 0)$ to $(1, 0)$. Let L_n be the length of the path walked by the n th ant. Compute $\lim_{n \rightarrow \infty} L_n$.

Solution: A semicircle of radius $\frac{1}{n}$ has length $\frac{1}{2}\pi\left(\frac{2}{n}\right) = \frac{\pi}{n}$, so n such semicircles have total length $\boxed{\pi}$.

2. The polynomial $3x^5 - 250x^3 + 735x$ is interesting because it has the maximum possible number of relative extrema and points of inflection at integer lattice points for a quintic polynomial. What is the sum of the x -coordinates of these points?

Solution: The first derivative is $15x^4 - 750x^2 + 735$, whose roots (which give the relative extrema) sum to $750/15 = 50$. The second derivative is $60x^3 - 1500x$, whose roots (which give the points of inflection) sum to $1500/60 = 25$, for a grand total of $\boxed{75}$.

3. A balloon that blows up in the shape of a perfect cube is being blown up at a rate such that at time t fortnights, it has surface area $6t$ square furlongs. At how many cubic furlongs per fortnight is the air being pumped in when the surface area is 144 square furlongs?

Solution: The surface area at time t is $6t$, so the volume is $t^{3/2}$. Hence the air is being pumped in at a rate of $\frac{3}{2}\sqrt{t}$. When the surface area is 144, $t = 24$, so the answer is $\boxed{3\sqrt{6}}$.

4. What is the size of the largest rectangle that can be drawn inside of a 3-4-5 right triangle with one of the rectangle's sides along one of the legs of the triangle?

Solution: Clearly one vertex of the rectangle will be at the right angle. Position the triangle with the leg of length 4 along the x -axis and the leg of length 3 along the y -axis. Then the hypotenuse is along the line $y = 3 - (3/4)x$.

Suppose the rectangle has a side of length y along the leg of length 3. Then the area is $y(4/3)(3 - y) = 4y - (4/3)y^2$. The derivative of this is 0 when $4 - (8/3)y = 0$, or $y = 3/2$, giving an area of 3.

Or, if you prefer, suppose the rectangle has a side of length x along the leg of length 4. Then the area is $x(3 - (3/4)x) = 3x - (3/4)x^2$. The derivative of this is 0 when $3 - (3/2)x = 0$, or $x = 2$, again giving an area of $\boxed{3}$.

5. Same as question 4, but now we want one of the rectangle's sides to be along the hypotenuse.

Solution: Put the hypotenuse along the x -axis, with the short leg starting at the origin so that the right angle is at the point $(9/5, 12/5)$. For notational convenience, let's just scale everything by a factor of 5 and then remember to divide the final area by 25, so now the top point is at $(9, 12)$.

Let $(a, 0)$ be the point where the edge of the rectangle along the hypotenuse starts. Then the height is $h = (4/3)a$ since the leg of length 3 is along the line $y = (4/3)x$. The leg of length 4 is along the line $x = 25 - (4/3)y$, so the horizontal edge of the rectangle ends at $b = 25 - (4/3)h = 25 - (16/9)a$. The area of the rectangle is $(b - a)h = (25 - (16/9)a - a)(4/3)a = \frac{100}{3}a - \frac{100}{27}a^2$. The derivative of this is 0 when $\frac{100}{3} = \frac{200}{27}a$, or $a = 9/2$. Thus the maximum area is $\frac{(100/3)(9/2) - (100/27)(81/4)}{25} = \boxed{3}$.

6. The graph of $x^2 - (y - 1)^2 = 1$ has one tangent line with positive slope that passes through $(x, y) = (0, 0)$. If the point of tangency is (a, b) , find $\sin^{-1}(\frac{a}{b})$ in radians.

Solution: Differentiating both sides of the equation, we find that $2x - 2(y - 1)\frac{dy}{dx} = 0$, and so $\frac{dy}{dx} = \frac{x}{y-1} = \frac{a}{b-1}$. The line passing through $(0, 0)$ and (a, b) has slope $\frac{b}{a}$, so $\frac{b}{a} = \frac{a}{b-1}$. Solving simultaneously with $a^2 - (b - 1)^2 = 1$, we get $b^2 - b - [(b - 1)^2 + 1] = 0$, and so $b = 2$, $a = \sqrt{2}$. Finally, $\sin^{-1}(\frac{a}{b}) = \boxed{\frac{\pi}{4}}$.

7. Find the coefficient of x^{12} in the Maclaurin series (i.e. Taylor series around $x = 0$) for $\frac{1}{1-3x+2x^2}$.

Solution: If you know formal power series, then this is not such a hard question, but since this is a calculus test... Use partial fractions to get $\frac{1}{1-3x+2x^2} = \frac{1/2}{1-2x} - \frac{1}{1-x}$. Now each of these can be expanded as a geometric series (or take derivatives and get the same result) to get $\frac{1}{2}(1 + 2x + 4x^2 + 8x^3 + \dots) - (1 + x + x^2 + x^3 + \dots)$, so the coefficient of x^n is $2^{n-1} - 1$. When $n = 12$, that's $\boxed{2047}$.

8. Evaluate $\sum_{n=0}^{\infty} \cot^{-1}(n^2 + n + 1)$.

Solution: $\sum_{n=0}^{\infty} \cot^{-1}(n^2 + n + 1) = \sum_{n=0}^{\infty} \arctan(\frac{1}{n^2+n+1}) = \sum_{n=0}^{\infty} \arctan(n + 1) - \arctan(n)$ by the sum/difference formula for tangent. This sum, taken out to $n = N$, telescopes to $-\arctan(0) + \arctan(N + 1)$. So as N goes to infinity, the sum goes to $\boxed{\pi/2}$.

9. On the planet Lemniscate, the people use the elliptic table of elements, a far more advanced version of our periodic table. They're not very good at calculus, though, so they've asked for your help. They know that Kr is somewhat radioactive and deteriorates into Pl, a very unstable element that deteriorates to form the stable element As. They started with a block of Kr of size 10 and nothing else. (Their units don't translate into English, sorry.) and nothing else. At time t , they let $x(t)$ be the amount of Kr, $y(t)$ the amount of Pl, and $z(t)$

the amount of As. They know that $x'(t) = -x$, and that, in the absence of Kr, $y'(t) = -2y$. Your job is to find at what time t the quantity of Pl will be largest. You should assume that the entire amount of Kr that deteriorates has turned into Pl.

Solution: This problem is long-winded since it's giving an autonomous linear system of differential equations without using any such language (and it includes a number of subtle references). The system we have is $x' = -x$, $y' = x - 2y$. It's not hard to see that $x = 10e^{-t}$ satisfies the first equation and the initial condition. Plugging this into the second equation and using the integrating factor e^{2t} (or using eigenvalues and eigenvectors to solve the system directly, though I don't want to begin to explain what that means) lets us solve for y . More precisely, we want to solve $y' + 2y = 10e^{-t}$. Multiply by e^{2t} and simplify the left hand side to get $(ye^{2t})' = 10e^t$. Integrating both sides with respect to t then yields $ye^{2t} = 10e^t + C$, or $y = 10e^{-t} + Ce^{-2t}$. Since $y(0) = 0$, we find $C = -10$. Now to maximize y , we solve $y'(t) = 0$, or $-10e^{-t} + 20e^{-2t} = 0$, or $t = \boxed{\ln 2}$.

10. Evaluate the definite integral $\int_{-1}^{+1} \frac{2u^{332} + u^{998} + 4u^{1664} \sin u^{691}}{1 + u^{666}} du$.

Solution: The term $\frac{4u^{1664} \sin u^{691}}{1 + u^{666}}$ is odd in u , so its integral is 0. Now make the substitution $u = v^{1/333} \Rightarrow du = \frac{1}{333} v^{-332/333} dv$ to find that $\int_{-1}^{+1} \frac{2u^{332} + u^{998}}{1 + u^{666}} du = \frac{1}{333} \int_{-1}^{+1} \frac{2 + v^2}{1 + v^2} dv = \frac{1}{333} \int_{-1}^{+1} \left(1 + \frac{1}{1 + v^2}\right) dv = \frac{2}{333} \int_0^1 \left(1 + \frac{1}{1 + v^2}\right) dv = \frac{2}{333} \left(1 + \int_0^1 \frac{1}{1 + v^2} dv\right) = \frac{2}{333} (1 + \tan^{-1} 1) = \boxed{\frac{2}{333} \left(1 + \frac{\pi}{4}\right)}$.

General Test Solutions (First Half)

Harvard-MIT Math Tournament

March 3, 2001

1. What is the last digit of $17^{103} + 5$?

Solution: Let $\langle N \rangle$ be the last digit of N . $\langle 17^2 \rangle = 9$, $\langle 17^3 \rangle = 3$, $\langle 17^4 \rangle = 1$, and $\langle 17^5 \rangle = 7$. Since this pattern keeps on repeating itself, $\langle 17^{4N} \rangle = 1$ for any integer N . Thus $\langle 17^{2 \cdot 25} \rangle = \langle 17^{100} \rangle = 1$, so $\langle 17^{103} \rangle = 3$, and $\langle 17^{103} + 5 \rangle = \langle 3 + 5 \rangle = \boxed{8}$.

2. Find $x + y$, given that $x^2 - y^2 = 10$ and $x - y = 2$.

Solution: $x^2 - y^2 = (x - y)(x + y) = 2(x + y) = 10$, so $x + y = \boxed{5}$.

3. There are some red and blue marbles in a box. We are told that there are twelve more red marbles than blue marbles, and we experimentally determine that when we pick a marble randomly we get a blue marble one quarter of the time. How many marbles are there in the box?

Solution: Call the number of blue marbles x , so the number of red marbles is $x + 12$ and the total number of marbles is $2x + 12$. The probability of picking a red marble is $\frac{x}{2x+12} = \frac{1}{4} \Rightarrow x = 6$, so $2x + 12 = \boxed{24}$.

4. Find $a + b + c + d + e$ if

$$\begin{aligned}3a + 2b + 4d &= 10, \\6a + 5b + 4c + 3d + 2e &= 8, \\a + b + 2c + 5e &= 3, \\2c + 3d + 3e &= 4, \text{ and} \\a + 2b + 3c + d &= 7.\end{aligned}$$

Solution: Adding the first, third, and fifth equations, we get $5a + 5b + 5c + 5d + 5e = 10 + 3 + 7 \Rightarrow a + b + c + d + e = \boxed{4}$.

5. What is the sum of the coefficients of the expansion $(x + 2y - 1)^6$?

Solution: The sum of the coefficients of a polynomial is that polynomial evaluated at 1, which for the question at hand is $(1 + 2 \cdot 1 - 1)^6 = 2^6 = \boxed{64}$.

6. A right triangle has a hypotenuse of length 2, and one of its legs has length 1. The altitude to its hypotenuse is drawn. What is the area of the rectangle whose diagonal is this altitude?

Solution: Call the triangle ABC , with $AC = 2$ and $BC = 1$. By the Pythagorean theorem, $AB = \sqrt{3}$. Call the point at which the altitude intersects the hypotenuse D . Let

$E \neq B$ be the vertex of the rectangle on AB and $F \neq B$ be the vertex of the rectangle on BC . Triangle BDC is similar to triangle ABC , so $BD = \frac{\sqrt{3}}{2}$. Triangle DBF is similar to triangle ABC , so $DF = \frac{\sqrt{3}}{4}$ and $BF = \frac{3}{4}$. The area of the rectangle is thus $\frac{\sqrt{3}}{4} \cdot \frac{3}{4} = \boxed{\frac{3\sqrt{3}}{16}}$.

7. Find $(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots$, where $|x| < 1$.

Solution: Let $S = (x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots = 1 + x + x^2 + x^3 + \cdots$. Since $xS = x + x^2 + x^3 + x^4 + \cdots$, we have $(1 - x)S = 1$, so $S = \boxed{\frac{1}{1-x}}$.

8. How many times does 24 divide into $100!$?

Solution: We first determine the number of times 2 and 3 divide into $100! = 1 \cdot 2 \cdot 3 \cdots 100$. Let $\langle N \rangle_n$ be the number of times n divides into N (i.e. we want to find $\langle 100! \rangle_{24}$). Since 2 only divides into even integers, $\langle 100! \rangle_2 = \langle 2 \cdot 4 \cdot 6 \cdots 100 \rangle$. Factoring out 2 once from each of these multiples, we get that $\langle 100! \rangle_2 = \langle 2^{50} \cdot 1 \cdot 2 \cdot 3 \cdots 50 \rangle_2$. Repeating this process, we find that $\langle 100! \rangle_2 = \langle 2^{50+25+12+6+3+1} \cdot 1 \rangle_2 = 97$. Similarly, $\langle 100! \rangle_3 = \langle 3^{33+11+3+1} \rangle_3 = 48$. Now $24 = 2^3 \cdot 3$, so for each factor of 24 in $100!$ there needs to be three multiples of 2 and one multiple of 3 in $100!$. Thus $\langle 100! \rangle_{24} = (\lfloor \langle 100! \rangle_2 / 3 \rfloor + \langle 100! \rangle_3) = \boxed{32}$, where $\lfloor N \rfloor$ is the greatest integer less than or equal to N .

9. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n . (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

Solution: There are 9^7 total shapes possible, since each of the 7 columns can contain anywhere from 0 to 8 checkers. The number of shapes symmetric with respect to a horizontal flip is the number of shapes of the leftmost four columns, since the configuration of these four columns uniquely determines the configuration of the remaining columns if it is known the shape is symmetric: 9^4 . Now we know there are $9^7 - 9^4$ non-symmetric shapes, so there are $\frac{9^7 - 9^4}{2}$ non-symmetric shapes modulo flips. Thus the total number of shapes modulo flips is $9^4 + \frac{9^7 - 9^4}{2} = 9^4 \left(1 + \frac{9^3 - 1}{2} \right) = 9^4 \left(\frac{9^3 + 1}{2} \right) = \left(\frac{3^8(3^6 - 1)}{2} \right) = 9 \frac{3^6(3^6 + 1)}{2} = 9(1 + 2 + \cdots + 3^6)$, so $n = 3^6 = \boxed{729}$.

10. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

Solution: Because the last digit of the product is 2, none of the three consecutive even integers end in 0. Thus they must end in 2, 4, 6 or 4, 6, 8, so they must end in 4, 6, 8 since $2 \cdot 4 \cdot 6$ does not end in 2. Call the middle integer n . Then the product is $(n-2)n(n+2) = n^3 - 4n$, so $n > \sqrt[3]{200000} = \sqrt[3]{200 \cdot 10^3} \approx 60$, but clearly $n < \sqrt[3]{300000} = \sqrt[3]{300 \cdot 10^3} < 70$. Thus $n = 66$, and the product is $66^3 - 4 \cdot 66 = \boxed{287232}$.

General Test Solutions (Second Half)

Harvard-MIT Math Tournament

March 3, 2001

1. A circle of radius 3 crosses the center of a square of side length 2. Find the difference between the areas of the nonoverlapping portions of the figures.

Solution: Call the area of the square s , the area of the circle c , and the area of the overlapping portion x . The area of the circle not overlapped by the square is $c - x$ and the area of the square not overlapped by the circle is $s - x$, so the difference between these two is $(c - x) - (s - x) = c - s = \boxed{9\pi^2 - 4}$.

2. Call three sides of an opaque cube adjacent if someone can see them all at once. Draw a plane through the centers of each triple of adjacent sides of a cube with edge length 1. Find the volume of the closed figure bounded by the resulting planes.

Solution: The volume of the figure is half the volume of the cube (which can be seen by cutting the cube into 8 equal cubes and realizing that the planes cut each of these cubes in half), namely $\boxed{\frac{1}{2}}$.

3. Find x if $x^{x^{x^{\dots}}} = 2$.

Solution: $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}} = 2^{\frac{1}{2}2^{\frac{1}{2}2^{\dots}}} = 2^{1^{\dots}} = 2$, so $x = \boxed{\sqrt{2}}$.

4. Some students are taking a math contest, in which each student takes one of four tests. One third of the students take one test, one fourth take another test, one fifth take the next test, and 26 students take the last test. How many students are taking the contest in total?

Solution: Call the total number of students n . We know $n = \frac{n}{3} + \frac{n}{4} + \frac{n}{5} + 26$, so $n = \boxed{120}$.

5. What is the area of a square inscribed in a semicircle of radius 1, with one of its sides flush with the diameter of the semicircle?

Solution: Call the center of the semicircle O , a point of contact of the square and the circular part of the semicircle A , the closer vertex of the square on the diameter B , and the side length of the square x . We know $OA = 1$, $AB = x$, $OB = \frac{x}{2}$, and $\angle ABO$ is right. By the Pythagorean theorem, $x^2 = \boxed{\frac{4}{5}}$.

6. You take a wrong turn on the way to MIT and end up in Transylvania, where 99% of the inhabitants are vampires and the rest are regular humans. For obvious reasons, you want

to be able to figure out who's who. On average, nine-tenths of the vampires are correctly identified as vampires and nine-tenths of humans are correctly identified as humans. What is the probability that someone identified as a human is actually a human?

Solution: Consider a sample of 1000 inhabitants. On average, 990 are vampires and 10 are people. 99 vampires are identified as human and 9 humans are identified as human. So out of the 108 who pass, only $\frac{1}{12}$ are human.

7. A real number x is randomly chosen in the interval $[-15\frac{1}{2}, 15\frac{1}{2}]$. Find the probability that the closest integer to x is odd.

Solution: By using a graphical method, we can see that, for real x on $[-n - \frac{1}{2}, n + \frac{1}{2}]$, n an even integer, the probability that the closest integer to x is odd is $\frac{n}{2n+1}$. The desired probability is $\frac{15}{31}$.

8. A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square.

Solution: Call the point in question A , the center of the circle O , and its radius r . Consider a right triangle BOA with hypotenuse OA : OA has length r , and BO and BA have lengths $r - 1$ and $r - 2$. By the Pythagorean theorem, $(r - 1)^2 + (r - 2)^2 = r^2 \Rightarrow r^2 - 6r + 5 = 0 \Rightarrow r = 5$ since $r > 4$. The area of the square is $(2r)^2 = 100$.

9. Two circles are concentric. The area of the ring between them is A . In terms of A , find the length of the longest chord contained entirely within the ring.

Solution: Let the radii of the circles be r and $R > r$, so $A = \pi(R^2 - r^2)$. By the Pythagorean theorem, the length of the chord is $2\sqrt{R^2 - r^2} = 2\sqrt{\frac{A}{\pi}}$.

10. Find the volume of the tetrahedron with vertices $(5, 8, 10)$, $(10, 10, 17)$, $(4, 45, 46)$, $(2, 5, 4)$.

Solution: Each vertex (x, y, z) obeys $x + y = z + 3$, so all the vertices are coplanar and the volume of the tetrahedron is 0.

Geometry Test Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. A circle of radius 3 crosses the center of a square of side length 2. Find the positive difference between the areas of the nonoverlapping portions of the figures.

Solution: Call the area of the square s , the area of the circle c , and the area of the overlapping portion x . The area of the circle not overlapped by the square is $c - x$ and the area of the square not overlapped by the circle is $s - x$, so the difference between these two is $(c - x) - (s - x) = c - s = \boxed{9\pi^2 - 4}$.

2. Call three sides of an opaque cube adjacent if someone can see them all at once. Draw a plane through the centers of each triple of adjacent sides of a cube with edge length 1. Find the volume of the closed figure bounded by the resulting planes.

Solution: The volume of the figure is half the volume of the cube (which can be seen by cutting the cube into 8 equal cubes and realizing that the planes cut each of these cubes in half), namely $\boxed{\frac{1}{2}}$.

3. Square $ABCD$ is drawn. Isosceles Triangle CDE is drawn with E a right angle. Square $DEFG$ is drawn. Isosceles triangle FGH is drawn with H a right angle. This process is repeated infinitely so that no two figures overlap each other. If square $ABCD$ has area 1, compute the area of the entire figure.

Solution: Let the area of the n th square drawn be S_n and the area of the n th triangle be T_n . Since the hypotenuse of the n th triangle is of length $\sqrt{S_n}$, its legs are of length $l = \sqrt{\frac{S_n}{2}}$, so $S_{n+1} = l^2 = \frac{S_n}{2}$ and $T_n = \frac{l^2}{2} = \frac{S_n}{4}$. Using the recursion relations, $S_n = \frac{1}{2^{n-1}}$ and $T_n = \frac{1}{2^{n+1}}$, so $S_n + T_n = \frac{1}{2^{n-1}} + \frac{1}{2^{n+1}} = \left(\frac{1}{2} + 2\right) \frac{1}{2^n} = \frac{5}{2} \frac{1}{2^n}$. Thus the total area of the figure is $\sum_{n=1}^{\infty} S_n + T_n = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \boxed{\frac{5}{2}}$.

4. A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2 + \pi$, find x .

Solution: Let C be the area of the circle, S be the area of the square two of whose edges are the chords, and A be the area of the part of the circle included between the chords. The radius of the circle is $\frac{\sqrt{2}}{2}x$, so $C = \frac{\pi}{2}x^2$, and $S = x^2$. Then the area A is the area of the square plus one half of the difference between the areas of the circle and square: $A = \frac{C-S}{2} + S = \frac{C+S}{2} = \frac{1+\pi}{2}x^2$, so $x = \sqrt{\frac{2A}{1+\pi}} = \boxed{2}$.

5. Find the volume of the tetrahedron with vertices $(5, 8, 10)$, $(10, 10, 17)$, $(4, 45, 46)$, $(2, 5, 4)$.

Solution: Each vertex (x, y, z) obeys $x + y = z + 3$, so all the vertices are coplanar and the volume of the tetrahedron is $\boxed{0}$.

6. A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square.

Solution: Call the point in question A , the center of the circle O , and its radius r . Consider a right triangle BOA with hypotenuse OA : OA has length r , and BO and BA have lengths $r - 1$ and $r - 2$. By the Pythagorean theorem, $(r - 1)^2 + (r - 2)^2 = r^2 \Rightarrow r^2 - 6r + 5 = 0 \Rightarrow r = 5$ since $r > 4$. The area of the square is $(2r)^2 = \boxed{100}$.

7. Equilateral triangle ABC with side length 1 is drawn. A square is drawn such that its vertex at A is opposite to its vertex at the midpoint of BC . Find the area enclosed within the intersection of the insides of the triangle and square. Hint: $\sin 75 = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$.

Solution: Let D be the midpoint of BC , $F \neq A$ be the point of intersection of the square and triangle lying on AC , b be the length of FC , x be the side length of the triangle, and y be the length of AD . By the law of sines on triangle CDF , we have $\frac{2\sin 75}{x} = \frac{\sin 45}{b}$, so $b = \frac{x \sin 45}{2 \sin 75} = \frac{\sqrt{2}x}{4 \sin 75}$. The area of the desired figure can easily be seen to be $\frac{1}{2}(x - b)y$ since it can be seen as two triangles of width y and height $\frac{x-b}{2}$. This reduces to $\frac{1}{2} \left(1 - \frac{\sqrt{2}}{4 \sin 75}\right) xy$. Then by the Pythagorean theorem on triangle ABD , $x^2 = \left(\frac{x}{2}\right)^2 + y^2$, so $y = \frac{\sqrt{3}}{2}x$, and the area becomes $\frac{\sqrt{3}}{4} \left(1 - \frac{\sqrt{2}}{4 \sin 75}\right) x^2 = \frac{\sqrt{3}}{4} \left(1 - \frac{\sqrt{2}}{4 \sin 75}\right) = \boxed{\frac{3}{4(\sqrt{3}+1)}}$.

8. Point D is drawn on side BC of equilateral triangle ABC , and AD is extended past D to E such that angles EAC and EBC are equal. If $BE = 5$ and $CE = 12$, determine the length of AE .

Solution: By construction, $ABEC$ is a cyclic quadrilateral. Ptolemy's theorem says that for cyclic quadrilaterals, the sum of the products of the lengths of the opposite sides equals the product of the lengths of the diagonals. This yields $(BC)(AE) = (BA)(CE) + (BE)(AC)$. Since ABC is equilateral, $BC = AC = AB$, so dividing out by this common value we get $AE = CE + BE = \boxed{17}$.

9. Parallelogram $AECF$ is inscribed in square $ABCD$. It is reflected across diagonal AC to form another parallelogram $AE'CF'$. The region common to both parallelograms has area m and perimeter n . Compute the value of $\frac{m}{n^2}$ if $AF : AD = 1 : 4$.

Solution: By symmetry, the region is a rhombus, $AXCY$, centered at the center of the square, O . Consider isosceles right triangle ACD . By the technique of mass points, we find that $DO : YO = 7 : 1$. Therefore, the rhombus is composed of four triangles, whose sides are in the ratio $1 : 7 : 5\sqrt{2}$. The perimeter of the rhombus is $20\sqrt{2}N$, and the area is $14N^2$. The required ratio is thus $\boxed{\frac{7}{400}}$.

10. A is the center of a semicircle, with radius AD lying on the base. B lies on the base between A and D , and E is on the circular portion of the semicircle such that EBA is a right angle. Extend EA through A to C , and put F on line CD such that EBF is a line. Now $EA = 1$, $AC = \sqrt{2}$, $BF = \frac{2-\sqrt{2}}{4}$, $CF = \frac{2\sqrt{5}+\sqrt{10}}{4}$, and $DF = \frac{2\sqrt{5}-\sqrt{10}}{4}$. Find DE .

Solution: Let $\theta = \angle AED$ and $x = DE$. By the law of cosines on triangle ADE , we have $1 = 1 + x^2 - 2x \cos \theta \Rightarrow 2x \cos \theta = x^2$. Then by the law of cosines on triangle CDE (note that $CD = \sqrt{5}$), we have $5 = (1 + \sqrt{2})^2 + x^2 - 2(1 + \sqrt{2})x \cos \theta = (1 + \sqrt{2})^2 + x^2 - (1 + \sqrt{2})x^2$.

Solving the quadratic equation gives $x = \boxed{\sqrt{2 - \sqrt{2}}}$.

1. [5] January 3, 1911 was an *odd date* as its abbreviated representation, 1/3/1911, can be written using only odd digits (note all four digits are written for the year). To the nearest month, how many months will have elapsed between the most recent odd date and the next odd date (today is 3/3/2001, an even date).

Solution: The most recent odd date was 11/19/1999 (November has 30 days, but the assumption that it has 31 days does not change the answer), and the next odd date will be 1/1/3111. From 11/19/1999 to 1/1/2000 is about 1 month. From 2000 to 3111 is 1111 years, or $12 \cdot 1111 = 13332$ months, so the total number of months is $\boxed{13333}$.

2. [4] Ken is the best sugar cube retailer in the nation. Trevor, who loves sugar, is coming over to make an order. Ken knows Trevor cannot afford more than 127 sugar cubes, but might ask for any number of cubes less than or equal to that. Ken prepares seven cups of cubes, with which he can satisfy any order Trevor might make. How many cubes are in the cup with the most sugar?

Solution: The only way to fill seven cups to satisfy the above condition is to use a binary scheme, so the cups must contain 1, 2, 4, 8, 16, 32, and $\boxed{64}$ cubes of sugar.

3. [7] Find the number of triangulations of a general convex 7-gon into 5 triangles by 4 diagonals that do not intersect in their interiors.

Solution: Define the Catalan numbers by $C(n) = \frac{1}{n+1} \binom{2n}{n}$. The current solution is the $C(\text{number of triangles}) = C(5) = \boxed{42}$.

4. [7] Find $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$.

Solution: $\prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty} \frac{n^2-1}{n^2} = \prod_{n=2}^{\infty} \frac{(n-1)(n+1)}{n \cdot n} = \frac{1 \cdot 3}{2 \cdot 2} \frac{2 \cdot 4}{3 \cdot 3} \frac{3 \cdot 5}{4 \cdot 4} \frac{4 \cdot 6}{5 \cdot 5} \frac{5 \cdot 7}{6 \cdot 6} \dots = \frac{1 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \dots}{2 \cdot 2 \cdot 3 \cdot 3 \cdot 4 \cdot 4 \cdot 5 \cdot 5 \dots} =$

$\boxed{\frac{1}{2}}$.

5. [± 6] Let ABC be a triangle with incenter I and circumcenter O . Let the circumradius be R . What is the least upper bound of all possible values of IO ?

Solution: I always lies inside the convex hull of ABC , which in turn always lies in the circumcircle of ABC , so $IO < R$. On the other hand, if we first draw the circle Ω of radius R about O and then pick A , B , and C very close together on it, we can force the convex hull of ABC to lie outside the circle of radius $R - \epsilon$ about O for any ϵ . Thus the answer is \boxed{R} .

6. [8] Six students taking a test sit in a row of seats with aisles only on the two sides of the row. If they finish the test at random times, what is the probability that some student will have to pass by another student to get to an aisle?

Solution: The probability p that *no* student will have to pass by another student to get to an aisle is the probability that the first student to leave is one of the students on the end, the next student to leave is on one of the ends of the remaining students, etc.: $p = \frac{2}{6} \cdot \frac{2}{5} \cdot \frac{2}{4} \cdot \frac{2}{3}$, so the desired probability is $1 - p = \boxed{\frac{43}{45}}$.

7. [5] Suppose $a, b, c, d,$ and e are objects that we can multiply together, but the multiplication doesn't necessarily satisfy the associative law, i.e. $(xy)z$ does not necessarily equal $x(yz)$. How many different ways are there to interpret the product $abcde$?

Solution: $C(\text{number of letters} - 1) = C(4) = \boxed{14}$.

8. [10] Compute $1 \cdot 2 + 2 \cdot 3 + \dots + (n - 1)n$.

Solution: Let $S = 1 \cdot 2 + 2 \cdot 3 + \dots + (n - 1)n$. We know $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$. So $S = 1(1 + 1) + 2(2 + 1) + \dots + (n - 1)n = (1^2 + 2^2 + \dots + (n - 1)^2) + (1 + 2 + \dots + (n - 1)) = \frac{(n-1)(n)(2n-1)}{6} + \frac{(n-1)(n)}{2} = \boxed{\frac{(n-1)n(n+1)}{3}}$.

We can also arrive at the solution by realizing that $\sum_{i=1}^n i^2 = \sum_{i=1}^n i^2 + \sum_{i=2}^n i^2 + \sum_{i=3}^n i^2 + \dots + \sum_{i=n}^n i^2 = n \sum_{i=1}^n i^2 - \left(\sum_{i=1}^1 i^2 + \sum_{i=1}^2 i^2 + \sum_{i=1}^3 i^2 + \dots + \sum_{i=1}^{n-1} i^2 \right) = n \frac{n(n+1)}{2} - \left(\frac{1 \cdot 2}{2} + \frac{2 \cdot 3}{2} + \dots + \frac{(n-1)n}{2} \right) = n \frac{n(n+1)}{2} - \frac{1}{2}S = \frac{n(n+1)(2n+1)}{6}$, so $S = \frac{(n-1)n(n+1)}{3}$.

9. [5] Suppose x satisfies $x^3 + x^2 + x + 1 = 0$. What are all possible values of $x^4 + 2x^3 + 2x^2 + 2x + 1$?

Solution: $x^4 + 2x^3 + 2x^2 + 2x + 1 = (x + 1)(x^3 + x^2 + x + 1) = \boxed{0}$ is the only possible solution.

10. [8] Two concentric circles have radii r and $R > r$. Three new circles are drawn so that they are each tangent to the big two circles and tangent to the other two new circles. Find $\frac{R}{r}$.

Solution: The centers of the three new circles form a triangle. The diameter of the new circles is $R - r$, so the side length of the triangle is $R - r$. Call the center of the concentric circle O , two vertices of the triangle A and B , and AB 's midpoint D . OA is the average R and r , namely $\frac{R+r}{2}$. Using the law of sines on triangle DAO , we get $\frac{\sin(30)}{AD} = \frac{\sin(90)}{AO} \Rightarrow R = 3r$, so $\frac{R}{r} = \boxed{3}$.

11. [8] 12 points are placed around the circumference of a circle. How many ways are there to draw 6 non-intersecting chords joining these points in pairs?

Solution: $C(\text{number of chords}) = C(6) = \boxed{132}$.

12. [± 6] How many distinct sets of 8 positive odd integers sum to 20?

Solution: This is the same as the number of ways 8 nonnegative even integers sum to 12 (we subtract 1 from each integer in the above sum). All $\boxed{11}$ possibilities are (leaving out 0s): 12, 10 + 2, 8 + 4, 8 + 2 + 2, 6 + 6, 6 + 4 + 2, 6 + 2 + 2 + 2 + 2, 4 + 4 + 4, 4 + 4 + 2 + 2, 4 + 2 + 2 + 2 + 2, 2 + 2 + 2 + 2 + 2 + 2.

13. [5] Find the number of real zeros of $x^3 - x^2 - x + 2$.

Solution: Let $f(x) = x^3 - x^2 - x + 2$, so $f'(x) = 3x^2 - 2x - 1$. The slope is zero when $3x^2 - 2x - 1 = 0$, where $x = -\frac{1}{3}$ and $x = 1$. Now $f(\frac{1}{3}) > 0$ and $f(1) > 0$, so there are no zeros between $x = -\frac{1}{3}$ and $x = 1$. Since $\lim_{x \rightarrow +\infty} f(x) > 0$, there are no zeros for $x > 1$. Since $\lim_{x \rightarrow -\infty} f(x) < 0$, there is one zero for $x < -\frac{1}{3}$, for a total of $\boxed{1}$ zero.

14. [8] Find the exact value of $1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}$.

Solution: Let x be what we are trying to find. $x - 1 = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \Rightarrow \frac{1}{x-1} - 1 = \frac{2}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}} \Rightarrow \frac{2}{\frac{1}{x-1} - 1} = x \Rightarrow x^2 - 2 = 0$, so $x = \boxed{\sqrt{2}}$ since $x > 0$.

15. [6] A beaver walks from (0, 0) to (4, 4) in the plane, walking one unit in the positive x direction or one unit in the positive y direction at each step. Moreover, he never goes to a point (x, y) with $y > x$. How many different paths can he walk?

Solution: $C(4) = \boxed{14}$.

16. [6] After walking so much that his feet get really tired, the beaver staggers so that, at each step, his coordinates change by either $(+1, +1)$ or $(+1, -1)$. Now he walks from (0, 0) to (8, 0) without ever going below the x -axis. How many such paths are there?

Solution: $C(4) = \boxed{14}$.

17. [4] Frank and Joe are playing ping pong. For each game, there is a 30% chance that Frank wins and a 70% chance Joe wins. During a match, they play games until someone wins a total of 21 games. What is the expected value of number of games played per match?

Solution: The expected value of the ratio of Frank's to Joe's score is 3:7, so Frank is expected to win 9 games for each of Frank's 21. Thus the expected number of games in a match is $\boxed{30}$.

18. [5] Find the largest prime factor of $-x^{10} - x^8 - x^6 - x^4 - x^2 - 1$, where $x = 2i$, $i = \sqrt{-1}$.

Solution: $\boxed{13}$.

19. [9] Calculate $\sum_{n=1}^{2001} n^3$.

Solution: $\sum_{n=1}^{2001} n^3 = \left(\sum_{n=1}^{2001} n\right)^2 = \left(\frac{2001 \cdot 2002}{2}\right)^2 = \boxed{4012013006001}$.

20. [± 4] Karen has seven envelopes and seven letters of congratulations to various HMMT coaches. If she places the letters in the envelopes at random with each possible configuration having an equal probability, what is the probability that exactly six of the letters are in the correct envelopes?

Solution: $\boxed{0}$, since if six letters are in their correct envelopes the seventh is as well.

21. [10] Evaluate $\sum_{i=1}^{\infty} \frac{(i+1)(i+2)(i+3)}{(-2)^i}$.

Solution: This is the power series of $\frac{6}{(1+x)^4}$ expanded about $x = 0$ and evaluated at $x = -\frac{1}{2}$, so the solution is $\boxed{96}$.

22. [6] A man is standing on a platform and sees his train move such that after t seconds it is $2t^2 + d_0$ feet from his original position, where d_0 is some number. Call the smallest (constant) speed at which the man have to run so that he catches the train v . In terms of n , find the n th smallest value of d_0 that makes v a perfect square.

Solution: The train's distance from the man's original position is $t^2 + d_0$, and the man's distance from his original position if he runs at speed v is vt at time t . We need to find where $t^2 + d_0 = vt$ has a solution. Note that this is a quadratic equation with discriminant $D = \sqrt{v^2 - 4d_0}$, so it has solutions for real D , i.e. where $v \geq \sqrt{4d_0}$, so $4d_0$ must be a perfect square. This happens when $4d_0$ is an even power of 2: the smallest value is 2^0 , the second smallest is 2^2 , the third smallest is 2^4 , and in general the n th smallest is $\boxed{2^{2(n-1)}}$, or $\boxed{4^{n-1}}$.

23. [5] Alice, Bob, and Charlie each pick a 2-digit number at random. What is the probability that all of their numbers' tens' digits are different from each others' tens' digits and all of their numbers' ones digits are different from each others' ones' digits?

Solution: $\frac{9}{10} \cdot \frac{8}{10} \cdot \frac{8}{9} \cdot \frac{7}{9} = \boxed{\frac{112}{225}}$.

24. [6] Square $ABCD$ has side length 1. A dilation is performed about point A , creating square $AB'C'D'$. If $BC' = 29$, determine the area of triangle BDC' .

Solution: $29^2 - 2 \cdot \frac{1}{2}(29) \left(\frac{29}{2}\right) - \frac{1}{2} = \boxed{420}$.

25. [± 10] What is the remainder when $100!$ is divided by 101 ?

Solution: Wilson's theorem says that for p a prime, $(p - 1)! \equiv -1 \pmod{p}$, so the remainder is $\boxed{100}$.

26. [6] A circle with center at O has radius 1. Points P and Q outside the circle are placed such that PQ passes through O . Tangent lines to the circle through P hit the circle at P_1 and P_2 , and tangent lines to the circle through Q hit the circle at Q_1 and Q_2 . If $\angle P_1PP_2 = 45^\circ$ and $\angle Q_1QQ_2 = 30^\circ$, find the minimum possible length of arc P_2Q_2 .

Solution: $(45 - 30)^\circ = \boxed{\frac{\pi}{12}}$.

27. [5] Mona has 12 match sticks of length 1, and she has to use them to make regular polygons, with each match being a side or a fraction of a side of a polygon, and no two matches overlapping or crossing each other. What is the smallest total area of the polygons Mona can make?

Solution: $4\sqrt{\frac{3}{4}} = \boxed{\sqrt{3}}$.

28. [4] How many different combinations of 4 marbles can be made from 5 indistinguishable red marbles, 4 indistinguishable blue marbles, and 2 indistinguishable black marbles?

Solution: $5 + 4 + 3 = \boxed{12}$.

29. [10] Count the number of sequences a_1, a_2, a_3, a_4, a_5 of integers such that $a_i \leq 1$ for all i and all partial sums ($a_1, a_1 + a_2$, etc.) are non-negative.

Solution: $C(\text{length} + 1) = C(6) = \boxed{132}$.

30. [± 4] How many roots does $\arctan x = x^2 - 1.6$ have, where the arctan function is defined in the range $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$?

Solution: $\boxed{2}$.

31. [5] If two fair dice are tossed, what is the probability that their sum is divisible by 5?

Solution: $\boxed{\frac{1}{4}}$.

32. [10] Count the number of permutations $a_1a_2 \dots a_7$ of 1234567 with longest decreasing subsequence of length at most two (i.e. there does not exist $i < j < k$ such that $a_i > a_j > a_k$).

Solution: $C(7) = \boxed{429}$.

33. [± 5] A line of soldiers 1 mile long is jogging. The drill sergeant, in a car, moving at twice their speed, repeatedly drives from the back of the line to the front of the line and

back again. When each soldier has marched 15 miles, how much mileage has been added to the car, to the nearest mile?

Solution: $\boxed{30}$.

34. [8] Find all the values of m for which the zeros of $2x^2 - mx - 8$ differ by $m - 1$.

Solution: $\boxed{6, -\frac{10}{3}}$.

35. [7] Find the largest integer that divides $m^5 - 5m^3 + 4m$ for all $m \geq 5$.

Solution: $\boxed{120}$.

36. [4] Count the number of sequences $1 \leq a_1 \leq a_2 \leq \dots \leq a_5$ of integers with $a_i \leq i$ for all i .

Solution: $C(\text{number of terms}) = C(5) = \boxed{42}$.

37. [5] Alex and Bob have 30 matches. Alex picks up somewhere between one and six matches (inclusive), then Bob picks up somewhere between one and six matches, and so on. The player who picks up the last match wins. How many matches should Alex pick up at the beginning to guarantee that he will be able to win?

Solution: $\boxed{2}$.

38. [9] The cafeteria in a certain laboratory is open from noon until 2 in the afternoon every Monday for lunch. Two professors eat 15 minute lunches sometime between noon and 2. What is the probability that they are in the cafeteria simultaneously on any given Monday?

Solution: $\boxed{\frac{15}{64}}$.

39. [9] What is the remainder when 2^{2001} is divided by $2^7 - 1$?

Solution: $2^{2001 \pmod{7}} = 2^6 = \boxed{64}$.

40. [5] A product of five primes is of the form ABC, ABC , where A, B , and C represent digits. If one of the primes is 491, find the product ABC, ABC .

Solution: $491 \cdot 1001 \cdot 2 = \boxed{982,982}$.

41. [4] If $(a + \frac{1}{a})^2 = 3$, find $(a + \frac{1}{a})^3$ in terms of a .

Solution: $\boxed{0}$.

42. [10] Solve $x = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$ for x .

Solution: $\boxed{\frac{1+\sqrt{5}}{2}}$.

43. [4] When a single number is added to each member of the sequence 20, 50, 100, the sequence becomes expressible as x, ax, a^2x . Find a .

Solution: $\boxed{\frac{5}{3}}$.

44. [7] Through a point in the interior of a triangle ABC , three lines are drawn, one parallel to each side. These lines divide the sides of the triangle into three regions each. Let a, b , and c be the lengths of the sides opposite $\angle A, \angle B$, and $\angle C$, respectively, and let a', b' , and c' be the lengths of the middle regions of the sides opposite $\angle A, \angle B$, and $\angle C$, respectively. Find the numerical value of $a'/a + b'/b + c'/c$.

Solution: $\boxed{1}$.

45. [4] A stacking of circles in the plane consists of a base, or some number of unit circles centered on the x -axis in a row without overlap or gaps, and circles above the x -axis that must be tangent to two circles below them (so that if the ends of the base were secured and gravity were applied from below, then nothing would move). How many stackings of circles in the plane have 4 circles in the base?

Solution: $C(4) = \boxed{14}$.

46. [± 5] Draw a rectangle. Connect the midpoints of the opposite sides to get 4 congruent rectangles. Connect the midpoints of the lower right rectangle for a total of 7 rectangles. Repeat this process infinitely. Let n be the minimum number of colors we can assign to the rectangles so that no two rectangles sharing an edge have the same color and m be the minimum number of colors we can assign to the rectangles so that no two rectangles sharing a corner have the same color. Find the ordered pair (n, m) .

Solution: $\boxed{(3, 4)}$.

47. [7] For the sequence of numbers n_1, n_2, n_3, \dots , the relation $n_i = 2n_{i-1} + a$ holds for all $i > 1$. If $n_2 = 5$ and $n_8 = 257$, what is n_5 ?

Solution: $\boxed{33}$.

48. [8] What is the smallest positive integer x for which $x^2 + x + 41$ is not a prime?

Solution: $\boxed{40}$.

49. [5] If $\frac{1}{9}$ of 60 is 5, what is $\frac{1}{20}$ of 80?

Solution: In base 15, $\boxed{6}$.

50. [9] The Fibonacci sequence F_1, F_2, F_3, \dots is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Find the least positive integer t such that for all $n > 0$, $F_n = F_{n+t}$.

Solution: $\boxed{60}$.

51. [5] Some people like to write with larger pencils than others. Ed, for instance, likes to write with the longest pencils he can find. However, the halls of MIT are of limited height L and width L . What is the longest pencil Ed can bring through the halls so that he can negotiate a square turn?

Solution: $\boxed{3L}$.

52. [6] Find all ordered pairs (m, n) of integers such that $231m^2 = 130n^2$.

Solution: The unique solution is $\boxed{(0, 0)}$.

53. [7] Find the sum of the infinite series $\frac{1}{3^2-1^2} \left(\frac{1}{1^2} - \frac{1}{3^2}\right) + \frac{1}{5^2-3^2} \left(\frac{1}{3^2} - \frac{1}{5^2}\right) + \frac{1}{7^2-5^2} \left(\frac{1}{5^2} - \frac{1}{7^2}\right) + \dots$.

Solution: $1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \dots = \boxed{1}$.

54. [10] The set of points (x_1, x_2, x_3, x_4) in \mathbf{R}^4 such that $x_1 \geq x_2 \geq x_3 \geq x_4$ is a cone (or hypercone, if you insist). Into how many regions is this cone sliced by the hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$?

Solution: $C(4) = \boxed{14}$.

55. [7] How many multiples of 7 between 10^6 and 10^9 are perfect squares?

Solution: $\left\lfloor \sqrt{\frac{10^9}{7^2}} \right\rfloor - \left\lfloor \sqrt{\frac{10^6}{7^2}} \right\rfloor = 4517 - 142 = \boxed{4375}$.

56. [6] A triangle has sides of length 888, 925, and $x > 0$. Find the value of x that minimizes the area of the circle circumscribed about the triangle.

Solution: $\boxed{259}$.

57. [5] Let $x = 2001^{1002} - 2001^{-1002}$ and $y = 2001^{1002} + 2001^{-1002}$. Find $x^2 - y^2$.

Solution: $\boxed{-4}$.

58. [9] Let (x, y) be a point in the cartesian plane, $x, y > 0$. Find a formula in terms of x and y for the minimal area of a right triangle with hypotenuse passing through (x, y) and legs contained in the x and y axes.

Solution: $\boxed{2xy}$.

59. [10] Trevor and Edward play a game in which they take turns adding or removing beans from a pile. On each turn, a player must either add or remove the largest perfect square number of beans that is in the heap. The player who empties the pile wins. For example, if Trevor goes first with a pile of 5 beans, he can either add 4 to make the total 9, or remove 4 to make the total 1, and either way Edward wins by removing all the beans. There is no limit to how large the pile can grow; it just starts with some finite number of beans in it, say fewer than 1000.

Before the game begins, Edward dispatches a spy to find out how many beans will be in the opening pile, call this n , then “graciously” offers to let Trevor go first. Knowing that the first player is more likely to win, but not knowing n , Trevor logically but unwisely accepts, and Edward goes on to win the game. Find a number n less than 1000 that would prompt this scenario, assuming both players are perfect logicians. A correct answer is worth the nearest integer to $\log_2(n - 4)$ points.

Solution: The correct answers are 0 (worth imaginary points), 5 (worth 0 points), 20 (4 points), 29, 45 (5 points), 80 (6 points), 101, 116, 135, 145, 165, 173 (7 points), 236, 257 (8 points), 397, 404, 445, 477, 540, 565, 580, 629, 666 (9 points), 836, 845, 885, 909, 944, 949, 954, 975 (10 points). This game is called Epstein’s Put-or-Take-a-Square game. It is unknown whether or not these numbers (or the first player’s win positions) have positive density.

60. [∞] Find an n such that $n! - (n - 1)! + (n - 2)! - (n - 3)! + \cdots \pm 1!$ is prime. Be prepared to justify your answer for $\begin{cases} n, & n \leq 25 \\ \lfloor \frac{n+225}{10} \rfloor, & n > 25 \end{cases}$ points, where $[N]$ is the greatest integer less than N .

Solution: 3, 4, 5, 6, 7, 8, 10, 15, 19, 41 (26 points), 59, 61 (28 points), 105 (33 points), 160 (38 points) are the only ones less than or equal to 335. If anyone produces an answer larger than 335, then we ask for justification to call their bluff. It is not known whether or not there are infinitely many such n .

Team Round Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. How many digits are in the base two representation of $10!$ (factorial)?

Solution: We write $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$. The number of digits (base 2) of $10!$ is equal to $\lceil \log_2 10! \rceil = 8 + \log_2(3^4 \cdot 5^2 \cdot 7)$. Since $2^13 < 3^4 \cdot 5^2 \cdot 7 < 2^{14}$, the number of digits is $8 + 13 = \boxed{21}$.

2. On a certain unidirectional highway, trucks move steadily at 60 miles per hour spaced $1/4$ of a mile apart. Cars move steadily at 75 miles per hour spaced 3 seconds apart. A lone sports car weaving through traffic at a steady forward speed passes two cars between each truck it passes. How quickly is it moving in miles per hour?

Solution: The cars are $1/8$ of a mile apart. Consider the reference frame in which the trucks move at 0 velocity (and the cars move at 15). Call the speed of the sports car in this reference frame v . The amount of time for the sports car to move from one truck to the next is $\frac{1/4 \text{ miles}}{v}$, and the amount of time for two regular cars to pass the truck is $\frac{1/8 \text{ miles}}{15 \text{ mph}}$. Equating these, we get $v = 30$, and $v + 60 = \boxed{90}$ mph.

3. What is the 18th digit after the decimal point of $\frac{10000}{9899}$?

Solution: $\frac{10000}{9899}$ satisfies $100(x - 1) = 1.01x$, so each pair of adjacent digits is generated by adding the previous two pairs of digits. So the decimal is $1.01020305081321345590\dots$, and the 18th digit is $\boxed{5}$.

4. P is a polynomial. When P is divided by $x - 1$, the remainder is -4 . When P is divided by $x - 2$, the remainder is -1 . When P is divided by $x - 3$, the remainder is 4 . Determine the remainder when P is divided by $x^3 - 6x^2 + 11x - 6$.

Solution: The remainder polynomial is simply the order two polynomial that goes through the points $(1, -4)$, $(2, -1)$, and $(3, 4)$: $\boxed{x^2 - 5}$.

5. Find all x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $1 - \sin^4 x - \cos^2 x = \frac{1}{16}$.

Solution: $1 - \sin^4 x - \cos^2 x = \frac{1}{16} \Rightarrow (16 - 16\cos^2 x) - \sin^4 x - 1 = 0 \Rightarrow 16\sin^4 x - 16\sin^2 x + 1 = 0$. Use the quadratic formula in $\sin x$ to obtain $\sin^2 x = \frac{1}{2} \pm \frac{\sqrt{3}}{4}$. Since $\cos 2x = 1 - 2\sin^2 x = \pm \frac{\sqrt{3}}{2}$, we get $x = \boxed{\pm \frac{\pi}{12}, \pm \frac{5\pi}{12}}$.

6. What is the radius of the smallest sphere in which 4 spheres of radius 1 will fit?

Solution: The centers of the smaller spheres lie on a tetrahedron. Let the points of the tetrahedron be $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$, and $(1, -1, -1)$. These points have distance $\sqrt{3}$ from the center, and $\sqrt{2}$ from each other, so the radius of the smallest sphere in which 4 spheres of radius $\sqrt{2}$ will fit is $\sqrt{2} + \sqrt{3}$. Scale this to the correct answer by dividing by $\sqrt{2}$: $\boxed{\frac{2+\sqrt{6}}{2}}$.

7. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. The Lucas numbers are defined by $L_1 = 1$, $L_2 = 2$, and $L_{n+2} = L_{n+1} + L_n$ for $n \geq 1$.

Calculate $\frac{\prod_{n=1}^{15} \frac{F_{2n}}{F_n}}{\prod_{n=1}^{13} L_n}$.

Solution: It is easy to show that $L_n = \frac{F_{2n}}{F_n}$, so the product above is $L_1 L_4 L_9 = 843 \cdot 1364 = \boxed{1149852}$.

8. Express $\frac{\sin 10 + \sin 20 + \sin 30 + \sin 40 + \sin 50 + \sin 60 + \sin 70 + \sin 80}{\cos 5 \cos 10 \cos 20}$ without using trigonometric functions.

Solution: We will use the identities $\cos a + \cos b = 2 \cos \frac{a+b}{2} \cos \frac{a-b}{2}$ and $\sin a + \sin b = 2 \sin \frac{a+b}{2} \cos \frac{a-b}{2}$. The numerator is $(\sin 10 + \sin 80) + (\sin 20 + \sin 70) + (\sin 30 + \sin 60) + (\sin 40 + \sin 50) = 2 \sin 45 (\cos 35 + \cos 25 + \cos 15 + \cos 35) = 2 \sin 45 ((\cos 35 + \cos 5) + (\cos 25 + \cos 15)) = 4 \sin 45 \cos 20 (\cos 15 + \cos 5) = 8 \sin 45 \cos 20 \cos 10 \cos 5$, so the fraction equals $8 \sin 45 = \boxed{4\sqrt{2}}$.

9. Compute $\sum_{i=1}^{\infty} \frac{a^i}{a^i}$ for $a > 1$.

Solution: The sum $S = a + ax + ax^2 + ax^3 + \dots$ for $x < 1$ can be determined by realizing that $xS = ax + ax^2 + ax^3 + \dots$ and $(1-x)S = a$, so $S = \frac{a}{1-x}$. Using this, we have $\sum_{i=1}^{\infty} \frac{a^i}{a^i} = a \sum_{i=1}^{\infty} \frac{i}{a^i} = a \left[\frac{1}{a} + \frac{2}{a^2} + \frac{3}{a^3} + \dots \right] = a \left[\left(\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \dots \right) + \left(\frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \dots \right) + \dots \right] = a \left[\frac{1}{1-a} + \frac{1}{a} \frac{1}{1-a} + \frac{1}{a^2} \frac{1}{1-a} + \dots \right] = \frac{a}{1-a} \left[1 + \frac{1}{a} + \frac{1}{a^2} + \dots \right] = \boxed{\left(\frac{a}{1-a} \right)^2}$.

10. Define a monic irreducible polynomial with integral coefficients to be a polynomial with leading coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 as the factorization into monic irreducible polynomials. How many not necessarily distinct monic irreducible polynomials are there in the prime factorization of $(x^8 + x^4 + 1)(x^8 + x + 1)$ (for instance, $(x+1)^2$ has two prime factors)?

Solution: $x^8 + x^4 + 1 = (x^8 + 2x^4 + 1) - x^4 = (x^4 + 1)^2 - (x^2)^2 = (x^4 - x^2 + 1)(x^4 + x^2 + 1) = (x^4 - x^2 + 1)(x^2 + x + 1)(x^2 - x + 1)$, and $x^8 + x + 1 = (x^2 + x + 1)(x^6 - x^5 + x^3 - x^2 + 1)$. If an integer polynomial $f(x) = a_n x^n + \dots + a_0 \pmod{p}$, where p does not divide a_n , has no zeros, then f

has no rational roots. Taking $p = 2$, we find $x^6 - x^5 + x^3 - x^2 + 1$ is irreducible. The prime factorization of our polynomial is thus $(x^4 - x^2 + 1)(x^2 - x + 1)(x^2 + x + 1)^2(x^6 - x^5 + x^3 - x^2 + 1)$, so the answer is $\boxed{5}$.

11. Define $a? = (a - 1)/(a + 1)$ for $a \neq -1$. Determine all real values N for which $(N?)? = \tan 15$.

Solution: Let $x = N?$. Then $(x - 1) \cos 15 = (x + 1) \sin 15$. Squaring and rearranging terms, and using the fact that $\cos^2 15 - \sin^2 15 = \cos 30 = \frac{\sqrt{3}}{2}$, we have $3x^2 - 4\sqrt{3}x + 3 = 0$. Solving, we find that $x = \sqrt{3}$ or $\frac{\sqrt{3}}{3}$. However, we may reject the second root because it yields a negative value for $(N?)?$. Therefore $x = \sqrt{3}$ and $N = \frac{1+x}{1-x} = \frac{1+\sqrt{3}}{1-\sqrt{3}} = \boxed{-2 - \sqrt{3}}$.

12. All subscripts in this problem are to be considered modulo 6, that means for example that ω_7 is the same as ω_1 . Let $\omega_1, \dots, \omega_6$ be circles of radius r , whose centers lie on a regular hexagon of side length 1. Let P_i be the intersection of ω_i and ω_{i+1} that lies further from the center of the hexagon, for $i = 1, \dots, 6$. Let $Q_i, i = 1 \dots 6$, lie on ω_i such that Q_i, P_i, Q_{i+1} are colinear. Find the number of possible values of r .

Solution: Consider two consecutive circles ω_i and ω_{i+1} . Let Q_i, Q'_i be two points on ω_i and Q_{i+1}, Q'_{i+1} on ω_{i+1} such that Q_i, P_i and Q_{i+1} are colinear and also Q'_i, P_i and Q'_{i+1} . Then $\angle Q_i Q'_i = 2\angle Q_i P_i Q'_i = 2\angle Q_{i+1} P_i Q'_{i+1} = \angle Q_{i+1} Q'_{i+1}$. Refer to the center of ω_i as O_i . The previous result shows that the lines $O_i Q_i$ and $O_{i+1} Q_{i+1}$ meet at the same angle as the lines $O_i Q'_i$ and $O_{i+1} Q'_{i+1}$, call this angle ψ_i . ψ_i is a function solely of the circles ω_i and ω_{i+1} and the distance between them (we have just showed that any two points Q_i and Q'_i on ω_i give the same value of ψ_i , so ψ_i can't depend on this.) Now, the geometry of ω_i and ω_{i+1} is the same for every i , so ψ_i is simply a constant ψ which depends only on r . We know $6\psi = 0 \pmod{2\pi}$ because $Q_7 = Q_1$.

We now compute ψ . It suffices to do the computation for some specific choice of Q_i . Take Q_i to be the intersection of $O_i O_{i+1}$ and ω_i which is further from O_{i+1} . We are to compute the angle between $O_i Q_i$ and $O_{i+1} Q_{i+1}$ which is the same as $\angle O_i O_{i+1} Q_{i+1}$. Note the triangle $\triangle O_i P_i O_{i+1}$ is isosceles, call the base angle ξ . We have $\angle O_i O_{i+1} Q_{i+1} = \angle O_i O_{i+1} P_i + \angle P_i O_{i+1} Q_{i+1} = \xi + (\pi - 2\angle O_{i+1} P_i Q_{i+1}) = \xi + (\pi - 2(\pi - \angle Q_i O_{i+1} P_i - \angle P_i Q_i O_{i+1})) = \xi - \pi + 2(\xi + (1/2)\angle P_i O_i O_{i+1}) = \xi - \pi + 2(\xi + (1/2)\xi) = 4\xi - \pi$.

So we get $6(4\xi - \pi) = 0 \pmod{2\pi}$. Noting that ξ must be acute, $\xi = \pi/12, \pi/6, \pi/4, \pi/3$ or $5\pi/12$. r is uniquely determined as $(1/2) \sec \xi$ so there are $\boxed{5}$ possible values of r .

Advanced Topics Solutions

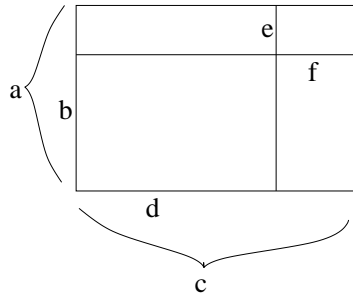
Rice Mathematics Tournament 2000

1. Assume we have 4 colors - 1, 2, 3, and 4. Fix the bottom as color 1. On the remaining sides you can have colors 2, 3, 4 (in that order), or 2, 4, 3, which are not rotationally identical. So, there are **2** ways to color it.
2. Since $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, we can write the first term as $(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3})^6$. Since $\cos \frac{4\pi}{3} = -\frac{1}{2}$ and $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, we can write the second term as $(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3})^6$. Now, we apply DeMoivre's Theorem to simplify the first expression to $(\cos 6 \cdot \frac{2\pi}{3} + i \sin 6 \cdot \frac{2\pi}{3}) = (\cos 4\pi + i \sin 4\pi) = 1 + 0 = 1$. Similarly, we simplify the second expression to $(\cos 6 \cdot \frac{4\pi}{3} + i \sin 6 \cdot \frac{4\pi}{3}) = (\cos 8\pi + i \sin 8\pi) = 1 + 0 = 1$. Thus, the total sum is $1 + 1 = \mathbf{2}$.
3. We know that $\frac{1}{n^2+2n} = \frac{1}{n(n+2)} = \frac{\frac{1}{n} - \frac{1}{n+2}}{2}$. So, if we sum this from 1 to ∞ , all terms except for $\frac{1}{2} + \frac{1}{2}$ will cancel out (a "telescoping" series). Therefore, the sum will be $\frac{3}{4}$.
4. The possibilities for the numbers are:
 - all five are divisible by 3
 - three are divisible by 3, one is $\equiv 1 \pmod{3}$ and one is $\equiv 2 \pmod{3}$
 - two are divisible by 3, and the other three are either $\equiv 1 \pmod{3}$ or $\equiv 2 \pmod{3}$
 - one is divisible by 3, two are $\equiv 1 \pmod{3}$ and two are $\equiv 2 \pmod{3}$
 - four are $\equiv 1 \pmod{3}$ and one is $\equiv 2 \pmod{3}$
 - four are $\equiv 2 \pmod{3}$ and one is $\equiv 1 \pmod{3}$

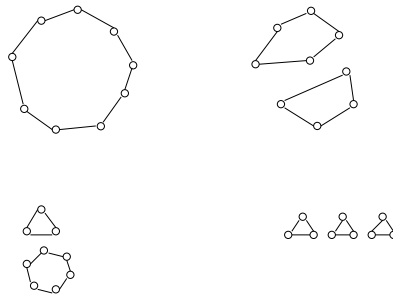
This gives us 1001 possible combinations out of $\binom{15}{5}$ or 3003. So, the probability is $\frac{1001}{3003} = \frac{1}{3}$.

5. **153,370,371,407**

6. There are 6 people that could get their hat back, so we must multiply 6 by the number of ways that the other 5 people can arrange their hats such that no one gets his/her hat back. So, the number of ways this will happen is $(6 \cdot \text{derangement of } 5)$, or $6 * 44 = 264$. Since there are $6! = 720$ possible arrangements of hats, the probability of exactly one person getting their hat back is $\frac{264}{720} = \frac{11}{30}$.
7. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2 + f^2} = 15$ (since this is $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2}$). Also, note that $ac + bd - ad - bc = (a - b)(c - d) = ef$. So, solving for e and f , we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2 + 4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4} \cdot 12 = 9$. Therefore, $ef = 9 * 12 = \mathbf{108}$.



8. It suffices to consider the complements of the graphs, so we are looking for graphs with 9 vertices, where each vertex is connected to 2 others. There are 4 different graphs - see below.



9. The probability of the Reals hitting 0 singles is $(\frac{2}{3})^3$. The probability of the Reals hitting exactly 1 single is $\binom{3}{2} \cdot (\frac{2}{3})^3 \cdot \frac{1}{3}$, since there are 3 spots to put the two outs (the last spot must be an out, since the inning has to end on an out). The probability of the Reals hitting exactly 2 singles is $\binom{4}{2} \cdot (\frac{2}{3})^3 \cdot (\frac{1}{3})^3$. The probability of the Reals hitting exactly 3 singles is $\binom{5}{2} \cdot (\frac{2}{3})^3 \cdot (\frac{1}{3})^3$. If any of these happen, the Alphas win right away. Adding these gives us a $\frac{656}{729}$ chance of this happening. If exactly 4 singles occur (with probability $\binom{6}{2} \cdot (\frac{2}{3})^3 \cdot (\frac{1}{3})^4$), then there is a $\frac{2}{5}$ chance that the Alphas win. The probability of this happening is $\frac{2}{5} \cdot \frac{40}{729}$. Thus, the total probability of the Alphas winning is the sum of these two probabilities, or $\frac{656}{729} + \frac{16}{729} = \frac{224}{243}$.
10. A will say yes when B says no to $n - 1$ or n , as A will then know B's number is one greater than A's number. Thus, A responds first, after $\frac{n-1}{2}$ "no" responses if n is odd, after $\frac{n}{2}$ "no" responses if n is even.

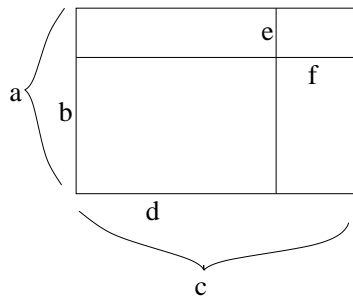
Algebra Solutions

Rice Mathematics Tournament 2000

1. The only integers that satisfy $|x| + 5 < 7$ are the ones that satisfy $|x| < 2$ - namely, $-1, 0, 1$. The integers that satisfy $|x - 3| > 2$ are $6, 7, 8, \dots$ and $0, -1, -2, \dots$. So, the integers that satisfy both are $0, -1$, and there are **2** of them.
2. $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$ can be factored into $(2000 - 1999)2000^2 + 1999^2(-2000 + 1999)$, which reduces to $2000^2 - 1999^2$. This factors into $(2000 + 1999)(2000 - 1999)$ which is equal to **3999**.
3. Let the scores be a, b, c, d, e , where $0 \leq a \leq b \leq c \leq d \leq e \leq 100$. So, the mean is $\frac{1}{5}(a + b + c + d + e)$, and the median is c . So, we want to maximize $\frac{1}{5}(a + b + c + d + e) - c$. To do this, we must maximize d and e and minimize or maximize c . One way to do this is to let $a = b = c = 0$ and $d = e = 100$, so the difference between the mean and the median is $\frac{1}{5}(0 + 0 + 0 + 100 + 100) - 0 = \frac{200}{5} = \mathbf{40}$. If we maximize c , then $c = d = e = 100$, and then the mean is $\frac{1}{5}(0 + 0 + 100 + 100 + 100) = 60$, and the median is 60, with a difference of 40 as well.
4. The price starts at \$100. Clearly, the order of price changes does not matter. It is reduced by 10% three times ($\$100 \rightarrow \$90 \rightarrow \$81 \rightarrow \72.90), and the new price is \$72.90. It is increased by 10% four times ($\$72.90 \rightarrow \$80.19 \rightarrow \$88.209 \rightarrow \$97.0299 \rightarrow \$106.73289$), and the new price is \$106.73289. Rounded to the nearest cent, this is **\$106.73**.
5. Every day Edward works, he gets $\frac{1}{9}$ of the test done. Similarly, every day Barbara works, she gets $\frac{1}{10}$ of the test done, every day Abhinav works, he gets $\frac{1}{11}$ of the test done, and every day Alex works, he gets $\frac{1}{12}$ of the test done. So, after 4 days (after everyone has worked on the test one day, they have completed $\frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} = 38.535\%$ of the test. After 8 days, they have completed twice that, or 77.0707% of the test. After Edward, Barbara, and Abhinav each work one more day, the test will be complete in the minimum amount of time, so the test will take **11 days** to complete. If the least efficient workers work after the 8th day, the test still takes 11 days to complete.
6. A shortest path is $x \rightarrow x^2 \rightarrow x^4 \rightarrow x^8 \rightarrow x^{12} \rightarrow x^{24} \rightarrow x^{25} \rightarrow x^{50} \rightarrow x^{100} \rightarrow x^{200} \rightarrow x^{400} \rightarrow x^{800} \rightarrow x^{1600} \rightarrow x^{2000}$, using **13 multiplications**.
7. All multiplicatively perfect numbers have exactly 4 distinct positive divisors, or 1. So, we must look for numbers that are either
 - 1
 - a product of two distinct primes
 - a cube of a prime

Numbers satisfying one of these conditions less than 100 are: 1, 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 39, 46, 51, 55, 57, 58, 62, 65, 69, 74, 77, 82, 85, 86, 87, 91, 93, 94, 95. There are **33** of these.

8. $168 = 2^3 \cdot 3 \cdot 7$. There are only 2 combinations of these whose sums allow indistinguishability of the ages. If there are 27 trees, 2, 4, 21 and 1, 12, 14 years are possible. If there are 21 trees, 2, 7, 12 and 3, 4, 14 are possible. So, the possible ages of the oldest daughter are **12, 14, 21**.
9. We can view these conditions as a geometry diagram as seen below. So, we know that $\frac{e}{f} = \frac{3}{4}$ (since $e = a - b = \frac{3}{4}c - \frac{3}{4}d = \frac{3}{4}f$ and we know that $\sqrt{e^2 + f^2} = 15$ (since this is $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2}$). Also, note that $ac + bd - ad - bc = (a - b)(c - d) = ef$. So, solving for e and f , we find that $e^2 + f^2 = 225$, so $16e^2 + 16f^2 = 3600$, so $(4e)^2 + (4f)^2 = 3600$, so $(3f)^2 + (4f)^2 = 3600$, so $f^2(3^2 + 4^2) = 3600$, so $25f^2 = 3600$, so $f^2 = 144$ and $f = 12$. Thus, $e = \frac{3}{4}12 = 9$. Therefore, **$ef = 9 * 12 = 108$** .



10. $a = 1$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 2$, which is prime. $a = 2$ clearly does not work, since if $x = 1$, then $x^4 + a^2 = 5$, which is also prime. Here is a table for a 's and values of x that show they do not work:

a	x	$a^4 + x^2$
3	10	10009
4	1	17
5	2	41
6	1	37
7	20	160049

So, let us consider $a = 8$ - i.e. the sum $x^4 + 64$. This is the same as $(x^2 + 8)^2 - 16x^2 = (x^2 + 4x + 8)(x^2 - 4x + 8)$ by the difference of squares. This is clearly not prime for any integer x . So, the answer is **$a = 8$** .

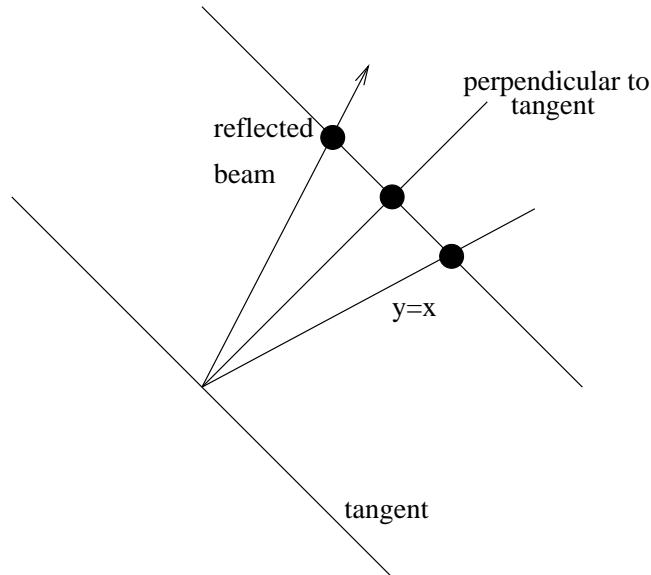
Calculus Solutions

Rice Mathematics Tournament 2000

1. $y = x^3 - 3x^2 + 6x + 2000$, so $y' = 3x^2 - 6x + 6$ and $y'' = 6x - 6$, so the point of inflection is the solution to $6x - 6 = 0$, or $x = 1$. At $x = 1$, the slope is $f'|_{x=1} = 3(1)^2 - 6(1) + 6 = \mathbf{3}$.
2. The change in Karen's position is $x - x^3$. The optimal length to climb is at a critical point. The only realistic critical point is at the solution to $1 - 3x^2 = 0$ or $\mathbf{x = \frac{\sqrt{3}}{3}}$.
3. $\int_0^{\frac{1}{4}\pi} \sin x + C = 0$, from the statement of the problem. So, $[-\cos x + Cx]|_0^{\frac{\pi}{4}} = 0$. Thus, $\cos \frac{\pi}{4} + \frac{\pi}{4}C + 1 = 0$. So, $-\frac{\sqrt{2}}{2} + \frac{\pi}{4}C + 1 = 0$, and solving for C , we find that $\mathbf{C = \frac{2\sqrt{2}-4}{\pi}}$.
4. Let $y = \tan x$. So, we want to find the minimum of $y + \frac{1}{y}$, where $0 \leq y \leq \infty$. Taking the derivative and minimizing, we find that the minimum occurs at $y = 1$, so the minimum of the given function occurs at $\arctan 1 = \frac{\pi}{4}$.
5. $f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i}$. So, $f'(x) = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$. Thus, $\mathbf{f(x) = -\ln(1-x)}$.
6. Assume the pipe barely fits around the corner (i.e. it is in contact with the corner). The lower corner is at $(0, 0)$ and the upper corner is at $(6, 6\sqrt{5})$. Call x_0 the point on the lower wall it hits at the tightest spot. Given an x_0 , the longest a pipe could be with one end at x_0 and leaning against the $(6, 6\sqrt{5})$ corner is $\sqrt{x_0^2 + (6\sqrt{5} + \frac{36\sqrt{5}}{x_0-6})^2}$. We want the minimum of all of these "longest pipes", because the pipe needs to fit at all angles around the corner. Taking the derivative (without the square root for simplicity) and setting it equal to 0, we need to solve $x_0^3 - 6x_0^2 + 36x_0 - 1296 = 0$. We can quickly find that $x_0 = 12$ is the only good solution, so the maximum length is $\mathbf{12\sqrt{6}}$.
7. Using the hint, take $\frac{dy}{dx}$. Set this equal to 0 and solve for x relative to x_0 . Plug this in for x_0 in the given family of lines to obtain the envelope $\mathbf{y = x + \frac{1}{x}, x > 0}$.
8. Let I denote the given integral. Under the transformation $\theta \rightarrow \frac{\pi}{2} - \theta$, I transforms to $\int_0^{\frac{\pi}{2}} \ln(\cos(\theta))d\theta$. So,

$$\begin{aligned}
 2I &= \int_0^{\frac{\pi}{2}} \ln(\sin \theta \cos \theta)d\theta \\
 &= \int_0^{\pi} (\ln(\sin 2\theta) - \ln 2)d(2\theta)/2 \\
 &= -\frac{\pi}{2\ln 2} + \frac{1}{2} \int_0^{\pi} \sin(\alpha)d\alpha \quad \text{giving } \mathbf{I = -\frac{\pi}{2} \ln 2}. \\
 &= -\frac{\pi}{2} \cdot \ln 2 + \int_0^{\frac{\pi}{2}} \sin(\alpha)d\alpha \\
 &= -\frac{\pi}{2} \cdot \ln 2 + I
 \end{aligned}$$
9. Note first that $([f(x)]^2 - x)^2 = f(x)$, so if $f(a) = 4$, then $(16 - a)^2 = 4$, so $a = 14$. Now, $f'(x) = \frac{1 + \frac{f'(x)}{2([f(x)]^2 - x)}}{2f(x)}$, so $f'(14) = \frac{4}{31}$.
10. Solution: Throughout this solution we will use the fact that when light bounces off a mirror, the angle of incidence is equal to the angle of reflection. First the beam hits the point $(8, -1)$, then $(6, 1)$, $(4, -1)$, $(2, 1)$, and then is travelling along the line $y = x - 1$. Thus the beam hits the parabola at the point $(1 + \frac{1-\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2})$. To estimate $\sqrt{5}$, notice that $22^2 = 484$ and $23^2 = 529$, so $\sqrt{5} = \frac{\sqrt{500}}{10} = 2.2\dots$. Thus $\frac{1-\sqrt{5}}{2} = -.6\dots$, so the light

hits the parabola at approximately $(.4, -.6)$. The slope of the tangent to the parabola at this point is $\frac{-1}{2}(.4)^{-1/2}$, which is about $-.8$, so we need to find the slope of the beam after it reflects off of this tangent. For purposes of finding this slope, change coordinates so that the point of intersection is the origin. The beam is coming in along $y = x$, and $y = 1.2x$ is perpendicular to the tangent. The diagram below should clarify the setup.



We will find the new path of the light by finding the reflection about the line $y = 1.2x$ of a point on its incoming path. We know the point $(1, 1.2)$ is on the line $y = 1.2x$, so a perpendicular through this point is $y - 1.2 = -.8(x - 1)$, which intersects $y = x$ at the point $(1.1, 1.1)$. Thus the new path goes through the point $(.9, 1.3)$, so it has slope 1.4 (all values rounded to one decimal place). Going back to our original coordinate system, the light is now travelling along the line $y + .6 = 1.4(x - .4)$, so it next hits the mirror at $(1.5, 1)$. After that the x coordinate increases by $2/1.4 = 1.4$ between bounces, so it hits $(2.9, -1)$, $(4.3, 1)$, $(5.7, -1)$, $(7.1, 1)$, $(8.5, -1)$, and finally $(9.9, 1)$. A closer examination of the approximations made (e.g. by refining them to two decimal places) reveals that the last bounce is actually further to the left (at $(9.21, 1)$, to be more precise), so indeed the light does bounce **12** times.

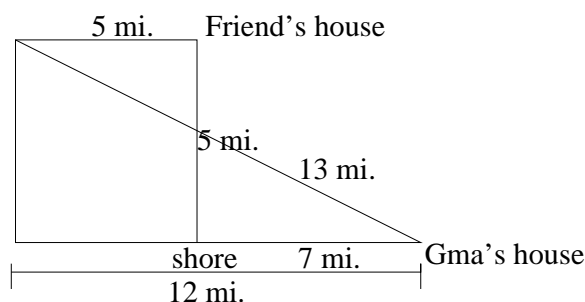
General Solutions

Rice Mathematics Tournament 2000

1. Since $d = a - c$, substitute into the equation $b = 2c + d$ to get $b = 2c + a - c = a + c$. Also, substitute into $2c = d + a - 1$ to get $2c = a - c + a - 1$, or $3c = 2a - 1$. Now, since $b = a + c$, we can substitute into $a = 2b + c$ to get $a = 2a + 2c + c$, or $a = -3c$. Since we know $3c = 2a - 1$ from above, we substitute in to get $3c = -6c - 1$, or $c = -\frac{1}{9}$. Thus, we find that $a = \frac{1}{3}$, $d = \frac{4}{9}$, and $b = \frac{2}{9}$.
2. Let x be the temperature we are looking for, so $x = \frac{9}{5} + 32$. So, $-\frac{4}{5}x = 32$, so $x = -\frac{5}{4} \cdot 32 = -40$.
3. Let x be the length of Henry's shadow in feet. Using similar triangles, we find that $\frac{5.5}{12} = \frac{x}{5}$, so $x = \frac{5.5}{12} \cdot 5 = \frac{11}{24} \cdot 5 = \frac{55}{24}$.
4. Let x be the number of students and y be the number of non-students. We then have the equations $x + y = 3000$ and $10x + 15y = 36250$. Substituting, we find that $10(3000 - y) + 15y = 36250$, or $30000 - 10y + 15y = 36250$, so $30000 + 5y = 36250$. So, $5y = 6250$, and $y = 1250$, so $x = 1750$.
5. The total number of degrees in an octagon is $(8 - 2) \cdot 180 = 1080$. Since the degrees are evenly distributed among the angles, the measure of one interior angle is $\frac{1080}{8} = 135^\circ$.
6. Pick any card first, then pick the other face-down card.
 - (a) $\frac{1}{3}$
 - (b) $\frac{2}{3}$
7. $\sqrt{19992000} \approx 4471.241$, and $[4471.241] = 4471$.
8. The time that Bobo can juggle is the number of cows times seconds. So, we get the following table:

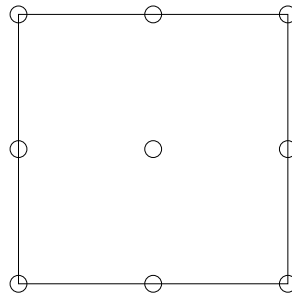
cows started juggling	1	2	3	4	5	6	7	8	9	10	11
total time	64	110	141	160	165	162	154	144	126	130	132
cows started juggling	12	13	14	15	16	17	18	19	20	21	22
total time	132	130	126	120	112	102	90	76	60	42	22
- Thus, we see that the maximum occurs with **5 cows**, and the total time is 165 seconds = **$2\frac{3}{4}$ minutes**.
9. Let p be the price of HMMT. So $p = k \cdot \frac{x}{y}$, where k is a constant to be determined. We know that when $x = 8$ and $y = 4$ that $p = 12$, so, solving for k , we find that $k = 6$. So, when $x = 4$ and $y = 8$, we find that $p = 6 \cdot \frac{4}{8} = 3$.
10. On the 12 foot sides, he needs 7 posts, and on the 20 foot sides, he needs 9 posts, so he needs $7 + 9 + 7 + 9 = 32$ total posts. Let x be the number of normal fenceposts and y be the number of strong fenceposts, so $x + y = 32$. To spend \$70, we have the equation $2x + 3y = 70$. Substituting, we find that $2(32 - y) + 3y = 70$, so $64 - 2y + 3y = 70$, and $64 + y = 70$, so $y = 6$.
11. Substituting, we find that $\frac{3+n}{3-n} = 3$, so $3 + n = 9 - 3n$, thus $4n = 6$. So, $n = \frac{3}{2}$.

12. Yes - one possible path is $4 \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 5$, so the difference between the start island and end island is **1**.
13. The number of rearrangements keeping 1 number in its spot and rearranging the other 5 such that none are in the right spot is 44. There are 6 numbers to fix, this gives us an answer of $44 \cdot 6 = \mathbf{264}$.
14. February 26, 2000 is a Saturday. April 24, 2000 is $23 \cdot 365 + 1 + 1 + 1 + 1 + 1 + 1 = 8401$ days away from April 24, 1977 (including the leap years). So, February 26, 2000 is $8401 - 3 - 31 - 24 = 8343$ days after April 24, 1977. Now, $\frac{8343}{7} = 1191$ with a remainder of 6. So, 6 days before Saturday is **Sunday**.
15. Notice that $3^6 = 729$ while $5^4 = 625$, and since $\ln 5 > \ln 3$, it follows that $5^4 \ln 3 < 3^6 \ln 5$, so $5^3 * 5 \ln 3 < 3^5 * 3 \ln 5$, and so by laws of logarithms, $5^3 \ln 3^5 < 3^5 \ln 5^3$. Again applying laws of logarithms, it follows that $\ln (3^5)^{(5^3)} < \ln (5^3)^{(3^5)}$. So, since $\ln x$ is an increasing function, it follows that $(3^5)^{(5^3)} < (5^3)^{(3^5)}$, and so it follows that **$(5^3)^{(3^5)}$ is greater**. One can also use the laws of exponents to reduce the values to $3^{(5^4)}$ and $5^{(3^6)}$. The second is clearly larger.
16. Expressing the total time Joe has biked in hours leads to the equation $\frac{x}{20} + \frac{x}{20} + \frac{x+2}{14} = 1$. So, $x = 5$. Thus, we can construct the diagram below, and find the total time that it takes to get back: $13 * \frac{1}{78} = \frac{1}{6}$ hours, or **10 minutes**.



17. There are $7!$ ways to arrange those letters. However, for every distinct arrangement, there are $2! * 2! = 4$ total arrangements of the 2 T's and 2 N's. Therefore, the total number of distinct ways to arrange the letters is $\frac{7!}{2!2!} = \mathbf{1260}$.
18. The only digits possible are 4, 6, 8, and 9. The only groups of numbers allowed keeping the average at 5 are 8444 and 6464. There are 4 ways to arrange 8444 and 6 ways to arrange 6464 so there are only **10 combinations** to try.
19. Let y be the number of coins in the chest. From the problem, we know that $y \equiv 5 \pmod{11}$, $y \equiv 3 \pmod{10}$, and $y \equiv 0 \pmod{9}$. Combining these gives us that $y \equiv 423 \pmod{990}$, so the answer is **423**.
20. The area of the big circle is $(\frac{5}{2})^2 \pi = \frac{25}{4} \pi$. The area of the circle with diameter AB is π , and the area of the circle with diameter BC is $\frac{9}{4} \pi$. Thus, the percentage of the big circle that is shaded is $\frac{\frac{25}{4} \pi - \pi - \frac{9}{4} \pi}{\frac{25}{4} \pi} = \frac{25 - 4 - 9}{25} = \frac{12}{25} = \mathbf{48\%}$.
21. It is clear that we can split the figure into 12 equilateral triangles, all of which have side length s . So, since the area of one of these triangles is $\frac{s^2 \sqrt{3}}{4}$, the total area is **$3s^2 \sqrt{3}$** .

22. $A_{iso} = \frac{A_{eq}}{2}$ so $\Delta A = A_{eq} - A_{iso} = \frac{1}{2}A_{eq}$. Also, $A_{eq} = \frac{4^2 \cdot \sqrt{3}}{4} = 4\sqrt{3}$, so $\Delta A = 2\sqrt{3}$. Thus, the area is $2\sqrt{3}$.
23. Notice that $7^{2k+1} \equiv 3 \pmod{4}$ for $k \in \mathbf{Z}$ (k is an integer). Also, $7^1 \equiv 7 \pmod{100}$, $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, $7^4 \equiv 1 \pmod{100}$, with the cycle repeating afterwards. So, clearly $7^{7^{7^7}} \equiv 3 \pmod{4}$, and since the cycle has period 4 $\pmod{100}$, we can conclude that $7^{7^{7^7}} \equiv \mathbf{43} \pmod{100}$.
24. If there are no empty boxes, there are 126 ways of distributing the identical candy. If there is one empty box, there are 84 ways of distributing the candy and 5 ways of choosing the empty box, so there are $84 \cdot 5 = 420$ ways. If there are two empty boxes, there are 36 ways of distributing the candy and 6 combinations of assigning the empty boxes so that they are not adjacent, so there are $36 \cdot 6 = 216$ ways. If there are three empty boxes, there are 9 ways of distributing the candy, and only 1 way of arranging the empty boxes. Having four or five empty boxes is impossible, since some two would have to be adjacent. So, the total number of ways is $126 + 420 + 216 + 9 = \mathbf{771}$.
25. We can place 9 points as shown, all at least $\frac{1}{2}$ unit apart, but the next point must be less than $\frac{1}{2}$, so **10** points must be placed. There is no arrangement of 10 points with distance at least $\frac{1}{2}$. The proof of this is a simple application of the Pigeonhole Principle.

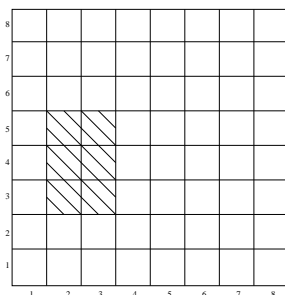


26. Janet is at $(5, -5)$ and Tim is at $(7, 4)$. They are $\sqrt{85}$ miles apart.

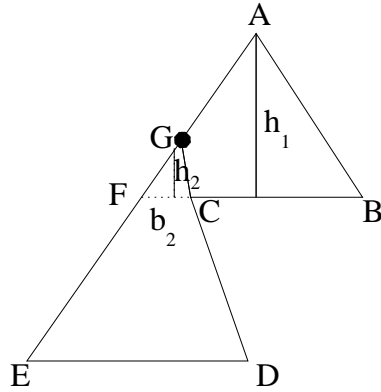
Geometry Solutions

Rice Mathematics Tournament 2000

1. Consider the board labeled as below, with labels for columns and rows. To choose any rectangle on the board, it is sufficient to choose some number (1-8) of adjacent columns, and some number (1-8) of adjacent rows, since the rectangle can be created by forming the intersection of the columns and rows. For instance, the intersection of columns 2,3 and rows 3,4,5 is the rectangle shaded below. So, there are 8 ways to choose 1 adjacent column, 7 ways to choose 2 adjacent columns, . . . , 1 way to choose 8 adjacent columns, so there are $8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 36$ total ways to choose the columns, and 36 ways to choose the rows. Thus, the total number of ways to choose a rectangle (i.e. the total number of rectangles) is $36^2 = 1296$.

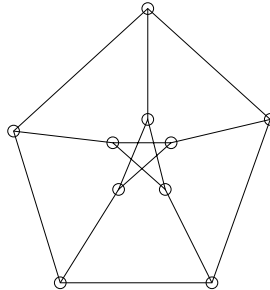


2. The maximum occurs in an equilateral triangle, in which case the sides $a = b = c$ are given by $a^2 + b^2 + c^2 = 3a^2 = 96$, so $a = 4\sqrt{2}$. Thus, the medians are $3 \cdot (\frac{a\sqrt{3}}{2}) = 6\sqrt{6}$.
3. Let s be the side length of $ABCD$. Since $ABCD$ is a square, we can write $2s^2 = (PA)^2 + (PC)^2 = (PB)^2 + (PD)^2$. So, we can substitute in for PA, PC , and PD to get that $PB = 24$.
4. Notice that in general, when there is a rectangle of side length x and y , the area of the non-triangle regions (created by drawing a line connecting the midpoint of two opposite lines and a line connecting two opposite corners - see diagram for examples) is simply $\frac{3}{4}$ of the original area of the box, since the area of the excluded triangles are $\frac{1}{2} \cdot \frac{y}{2} \cdot \frac{x}{2} + \frac{1}{2} \cdot \frac{y}{2} \cdot \frac{x}{2} = 2 \cdot \frac{xy}{4} = \frac{xy}{2}$, so the desired area is simply $xy - \frac{xy}{2} = \frac{xy}{2}$, as desired. So, if we consider the four rectangles making up the large rectangle that are divided this way (the one with dimensions $\frac{a}{4} \times \frac{b}{3}$, etc.), we can say that the total shaded area is $\frac{3}{4}$ of the total area of the rectangle - that is, $\frac{3}{4}ab$.
5. The area of $\triangle ABF = \frac{1}{2}bh = \frac{1}{2} \cdot 3h = \frac{3}{2}(3 \sin 60^\circ) = \frac{9}{2} \sin 60^\circ$. The area of $\triangle FCDE =$ area of $\triangle ABF$ - area of $\triangle FGC = \frac{9}{2} \sin 60^\circ - \frac{1}{2} \sin 60^\circ = 4 \sin 60^\circ$. Therefore, area of $ABCDE =$ area of $\triangle ABF +$ area of $FCDE = \frac{9}{2} \sin 60^\circ + 4 \sin 60^\circ = \frac{17}{2} \sin 60^\circ = \frac{17\sqrt{3}}{4}$.
6. The area of an equilateral triangle with side length s is $\frac{\sqrt{3}}{4}s^2$. Therefore, the side length is $4\sqrt{2}$ and height $h = 2\sqrt{6}$. Now, if r is the radius of the inscribed circle, then $r = \frac{h}{3}$, since we have an equilateral triangle. Thus the area is $\pi r^2 = \frac{8\pi}{3}$.
7. Let's "resize" the coordinates to be $x' = x, y' = \frac{2y}{3}$. This keeps the origin at $(0,0)$, but turns our ellipse into a circle of radius 2. Thus now, the triangle is equilateral, and we



can see it now has area $3\sqrt{3}$. Once we expand back, we can see we are just multiplying the area by $\frac{3}{2}$ and so the answer is $\frac{9\sqrt{3}}{2}$.

8. By standard formula, we have that the radius of the inscribed circle, r , is $r = \frac{ab \sin A}{2a+b}$ (isoceles triangle that is formed by cutting the pyramid vertically in half (cuts the base into 2 equal rectangles)). $h^2 + (\frac{b}{2})^2 = a^2$ gives $a = \sqrt{h^2 + \frac{b^2}{4}}$. Also $\sin A = \frac{h}{a}$. Therefore, $r = \frac{bh}{2\sqrt{h^2 + \frac{b^2}{4}} + b}$. Note that the diameter of the cube is the diameter of the sphere. Let l be the length of the side of the cube, so the diameter of the cube is $l\sqrt{3} = 2r$, so $l = \frac{2r}{\sqrt{3}}$. So, the volume of the pyramid is $\frac{1}{3}b^2h$ and the cube volume if l^3 . So, the ratio is $\frac{25\sqrt{3}}{6}$.
9. A hexagon can be formed by removing any vertex, removing all vertices connected to that vertex, and then removing any edges connected to any of the removed vertices, and these are the only hexagons in the diagram. Thus, since there are 10 vertices, there are **10 hexagons** in the figure below.



10. Consider the flattened version of the situation. Then let O be the center fo the spheres, A be the first reflection point, B be the point of C_1 such that OB is perpendicular to OP . Then since $OB = 1, OP = \sqrt{3}$, $\angle PBO = 60^\circ$ and P, B, A are collinear, $\angle BAO = 60^\circ$ implies that $\angle POA = 30^\circ$. Therefore, each reflection takes the ray $\frac{1}{12}$ around the circle, so there are **11 reflections**.

Guts Solutions
HMMT 2000

1. $\pi^e/3$
2. $\frac{XY}{X+Y-1}$
3. 57
4. (1,2)
5. 1, 512, 4913, 5832
6. 3
7. $(\frac{x^2+1}{2})^m$.
8. 14
9. 15
10. $\pi R\sqrt{R^2 + H^2} + 2\pi r^2(1 + \sqrt{1 - \frac{R^2}{r^2}})$
11. 58
12. 274
13. 3
14. $\sqrt{51}$.
15. 2^{49} .
16. (-1,3) and (3,-1)
17. $1 = 3^0$
18. $-\pi/4$
19. 1
20. $3 + 2\sqrt{2} + \sqrt{3} + \sqrt{6}$
21. 546
22. 2008
23. 224640

24. 6

25. 1113122113111221

26. 313

27. 325

28. $1/6$

29. 3

30. 1

31. D is harmonic conjugate of C, standard construction.

32.

33. Of the form $4n + 2$.

34.

35. $2/3$

36. 90 degrees

37.

38. $3^{3^{3^{888}}}$

39. 254.50000... (more than 100 zeros).

40. n any odd number, $p = 2$

41. $h = \frac{1}{4b} \sqrt{(a+b)(3b-a)}$

42. $n(n^2 + 1)/2$

43. 120/1147

44. $f(0) = -1, f(1) = 1$

45. 1999000, 2000

46. 1,2

47.

Oral Round Solutions

HMMT 2000

1. The equation involves only even powers, so we may assume m, n are positive (negative solutions may be generated by switching signs). $n^6 \leq n^6 + 1 \leq n^6 + 2n^3 + 1$, so $n^6 + 1$ lies between two squares, with strict inequality unless $n = 0$. So the only solutions are $m = \pm 1, n = 0$.
2. Dividing by x^6 , we get the equation $x^4 + 7x^3 + 14x^2 + 1729x - 1379 = 0$. This is a monotonically increasing function of x on $[0, \infty)$. From $f(0) < 0$ and $f(x) \rightarrow \infty$ as $x \rightarrow \infty$ we see that there's only one positive solution. In fact, $f(1) > 0$, so there's no positive integer solution.
3. Assume w.l.o.g that $c > b \geq a$. Then $c^n - b^n = (c - b)(c^{n-1} + \dots + b^{n-1}) > 1(nb^{n-1}) \geq na^{n-1}$. So $a^n > na^{n-1}$, hence $a > n$.
4. Let m be the smallest number, written on some square S . Then clearly all of the adjacent squares to S must have m on them (they can't have anything smaller, since m is the smallest, and can't have anything larger since m is the average of those adjacent numbers). Since this applies to every square on which m is written, all of the squares must have m on them. Otherwise there will be some "boundary" square with m on it, which will not have m written on one of its neighbours, a contradiction.
5. Suppose there is such a triangle with vertices at $(m_1, n_1), (m_2, n_2), (m_3, n_3)$. Then we are given $\sqrt{(m_1 - m_2)^2 + (n_1 - n_2)^2}$ is an odd integer, so that $(m_1 - m_2)^2 + (n_1 - n_2)^2$ is 1 (mod 4). Hence exactly one of $m_1 - m_2$ and $n_1 - n_2$ must be odd and the other must be even. Suppose w.l.o.g $m_1 - m_2$ is odd and $n_1 - n_2$ is even. Similarly, exactly one of $m_1 - m_3$ and $n_1 - n_3$ must be odd and the other must be even. If $m_1 - m_3$ is odd and $n_1 - n_3$ even, then $m_2 - m_3$ and $n_2 - n_3$ are both even, which is a contradiction since we want $(m_2 - m_3)^2 + (n_2 - n_3)^2$ to be 1 (mod 4). Similarly if $m_1 - m_3$ is even and $n_1 - n_3$ is odd then $m_2 - m_3$ and $n_2 - n_3$ are both odd, and $(m_2 - m_3)^2 + (n_2 - n_3)^2$ is 2 (mod 4), a contradiction.
Alternate solution: By Pick's theorem ($A = I + \frac{1}{2}B - 1$), the area is an integer or a half-integer. But by Hero's formula, the area is $\frac{1}{4}\sqrt{\text{product of 4 odd integers}}$. this is a contradiction.
6. Any multiple of 3 can be written as $6k$ or $6k + 15$. Now, $6k = (k + 1)^3 + (k - 1)^3 + (-k)^3 + (-k)^3$. $6k + 15 = (k + 2)^3 + (-k + 2)^3 + (2k - 1)^3 + (-2k)^3$.

7. If the tetrahedron is oriented so that two opposite edges of the tetrahedron are parallel to the ground, it is clear from symmetry that the plane parallel to the ground and passing through the center of the tetrahedron splits it into equal spaces of volume $1/2$. Let's call this position 1. If, instead, it is oriented with the point down and the top face parallel to the ground, the plane through the center splits it into regions of volume $27/64$ and $37/64$. To see this, note that the center of a regular tetrahedron is $3/4$ of the way down the altitude from each vertex. Hence the tetrahedral section has volume $(3/4)^3 = 27/64$. Let's call this position position 2. If the tetrahedron is turned continuously from position 1 to position 2, because $27/64 < 7/16 < 1/2$, it must pass through an orientation s.t. the fraction of volume below the center is $7/16$.

The cube has 180° rotational symmetry, so any plane through the center splits it into sections of equal volume. Hence it is impossible to get a volume of $7/16$.

8. Let f be one such polynomial, for example, the obvious one: $f(x) = x^2 + 1$. Then all such polynomials must be of the form $f(x) + g(x)$ where $g(x) = 0$ at $x = 1, 2, \dots, n$. $g(x)$ must be of the form $C(x-1)(x-2)\dots(x-n)$ and since $f + g$ has to have integer coefficients, C must be an integer. At 0 this gives us $(f + g)(0) = f(0) + g(0) = 1 + C(-1)^n n!$. Since C can be any positive or negative integer, the answer is that $f(0)$ can be any integer congruent to $1 \pmod{n!}$, i.e. of the form $1 + Cn!$, C an integer.
9. This is a special case of the problem with $n + 2$ vectors in n dimensions. First it is clear that we can take all the vectors to be of length 1. Then we induct on n . The first case is $n=1$. Here the statement is that given a, b, c real numbers, at least one of ab, bc , and ac is nonnegative. Without loss of generality, we can assume that a and b are of the same sign, but then $ab \geq 0$. Now assume the statement is false for the n -dimensional case. Choose some vector, say \vec{v}_{n+2} , and project the other vectors onto the space perpendicular to \vec{v}_{n+2} to get $\vec{v}_i' = \vec{v}_i - (\vec{v}_i \cdot \vec{v}_{n+2})\vec{v}_{n+2}$. This is essentially taking out the space parallel to \vec{v}_{n+2} and reducing the problem by one dimension. The only thing left to check is that if $\vec{v}_i \cdot \vec{v}_j < 0$ for all i, j then $\vec{v}_i' \cdot \vec{v}_j' < 0$ for all i, j . This is just a calculation: $\vec{v}_i' \cdot \vec{v}_j' = \vec{v}_i \cdot \vec{v}_j - 2(\vec{v}_j \cdot \vec{v}_{n+2})(\vec{v}_i \cdot \vec{v}_{n+2}) + (\vec{v}_{n+2} \cdot \vec{v}_{n+2})(\vec{v}_j \cdot \vec{v}_{n+2})(\vec{v}_i \cdot \vec{v}_{n+2})$ and since $(\vec{v}_{n+2} \cdot \vec{v}_{n+2}) = 1$, this is just $\vec{v}_i \cdot \vec{v}_j - (\vec{v}_j \cdot \vec{v}_{n+2})(\vec{v}_i \cdot \vec{v}_{n+2})$. $(\vec{v}_j \cdot \vec{v}_{n+2})$ and $(\vec{v}_i \cdot \vec{v}_{n+2})$ are both negative by assumption so their product is positive and $\vec{v}_i' \cdot \vec{v}_j' < \vec{v}_i \cdot \vec{v}_j < 0$.
10. Let ω be a 23rd root of unity, a solution to the equation $x^{23} = 1$. We can encode the state of the game as 23 rational numbers a_0, \dots, a_{22} representing the amount of water each person has, with Alex having the fraction a_0 of the water and continuing to his right. Then we can look at the polynomial $f(x) = a_{22}x^{22} + \dots + a_0$ which still encodes the state of the game. In this schema the original state is the constant 1, and we have at all times the condition $f(1) = 1$ (conservation of water). However, we want a cyclic representation in order to encode the play of the game, so we let the state of the game be $f(\omega) = a_{22}\omega^{22} + \dots + a_0$. In this representation, a step in the game can be encoded as multiplication by another polynomial in ω , $g(\omega) = b_{22}\omega^{22} + \dots + b_0$ where the b_i are the fraction of water that Alex initially gives to the i -th person on his right. The example in the problem is represented by $\frac{1}{6}\omega + \frac{1}{3} + \frac{1}{2}\omega^{22}$. So we would like to know which g are solutions to $g(\omega)^{23} = 1$. We can take 23rd roots of both sides and find that $g(\omega) = b_{22}\omega^{22} + \dots + b_0 = \omega^k$ for some integer k . To simplify this we need a small lemma.

Lemma: If $a_{22}\omega^{22} + \dots + a_0$ is some rational linear combination of the powers of ω and is equal to 0, then $a_{22} = a_{21} = \dots = a_0$.

Proof: Assume this is false. So there exists some polynomial $A(x) = a_{22}x^{22} + \dots + a_0$ with ω as a root that is not a constant multiple of $B(x) = x^{22} + x^{21} + \dots + 1$. Now apply Euclid's algorithm to find the G.C.D. of $A(x)$ and $B(x)$ and call it $D(x)$. But $D(x)$ is a rational polynomial that divides $B(x)$, which has no non-trivial rational divisors. Hence $D(x)$ must be either a constant or of degree 22. Since $A(\omega) = B(\omega) = 0$ all the terms in the algorithm have a root at ω and so does $D(x)$. So since 0 is not an acceptable value for a G.C.D., $D(x)$ must have degree 22. Since the degree of each successive term in Euclid's algorithm decreases by at least one, this means that $D(x)$ must be $A(x)$ and hence that $B(x)$ is a constant multiple of $A(x)$. This contradicts the original assumption and the Lemma is proved.

So the Lemma tells us that $b_{22} = b_{21} = \dots = b_k - 1 = \dots = b_0$. Since we already know that $g(1) = 1$ and hence $b_{22} + b_{21} + \dots + b_0 = 1$, and that all the b_i are nonnegative, we get that $b_i = 0$ for $i \neq k$ and $b_k = 1$. Since this is a solution for all k , the possible schemes for Alex to use are just those involving giving all his water to any one other person.

Power Test Solutions

Rice Mathematics Tournament 2000

1. (a) $D_4 = 9, D_5 = 44$
 (b) This is $D_7 = 1854$
 (c) $D_n = (n - 1)(D_{n-1} + D_{n-2})$
 (d) $D_n = n \cdot D_{n-1} + (-1)^n$
2. $10 \cdot 9 \cdot 8$ give the number of all different 3 letter words. One sixth of these are in alphabetical order. $\frac{10 \cdot 9 \cdot 8}{6} = 120$.
3. $A < B < C$. More specifically, $A = .26; B = .37; C = .56$.
4. This is just derangements of 23541. $D_5 = 44$.
5. We want $1 - 365 \cdot 364 \cdot 363 \dots (365 - n + 1) / 365^n > 50\%$. Just guesstimating gives the answer of **23** people.
6. Since we want to choose k times from n distinct elements, this is equivalent to choosing where to put $n - 1$ "dividers" that separate the choices. For example, if we wanted to choose 3 scoops of ice cream from the flavors chocolate, vanilla, strawberry, and coffee, we can represent the choice of 2 vanilla and 1 coffee by (divider),choice,choice,(divider),(divider),cho. Notice that the choices of positions for the dividers completely determines which elements we choose. Therefore, we have $n + k - 1$ spaces to fill with $n - 1$ dividers, so the number of ways of doing this is $\binom{n+k-1}{n-1}$ or $\binom{n+k-1}{k}$
7. We can use the same method as the above argument, except that we know every element occurs at least once. So, this is the same as choosing $k - n$ times from the n distinct objects, so the answer is $\binom{k-n-n+1}{n-1}$, or $\binom{k-1}{n-1}$ or $\binom{k-1}{k-n}$.
8. FIX????????? $1 - \frac{971}{1033} = \frac{62}{1033}$
9. Let S be the sum desired. Then $101 \cdot S = \frac{101}{1} \binom{100}{0} + \frac{101}{2} \binom{100}{1} + \frac{101}{3} \binom{100}{2} + \dots + \frac{101}{101} \binom{100}{100}$. Now, consider a general term in this expression - i.e. $\frac{101}{i+1} \binom{100}{i}$. This is equal to $\frac{101}{i+1} \cdot \frac{100!}{(100-i)!i!} = \frac{101 \cdot 100!}{(101-(i+1))!i!(i+1)} = \frac{101!}{(101-(i+1))!i+1!} = \binom{101}{i+1}$. So, we can simplify the terms to get $101 \cdot S = \binom{101}{1} + \binom{101}{2} + \binom{101}{3} + \dots + \binom{101}{100} + \binom{101}{101}$. Thus, $101 \cdot S = 2^{101} - 1$, so $S = \frac{2^{101}-1}{101}$.
10. $\binom{n+m-1}{m}$ or $\binom{n+m-1}{n-1}$. This can be seen from the fact that each distribution can be described by a combination of letters, where each letter represents a different box. The number of times each letter occurs in the combination is determined by the number of balls in the corresponding box.
11. $\frac{m!}{[m_1!m_2!m_3!\dots m_n!]}$. This is also equal to $\binom{m}{m_1} \binom{m-m_1}{m_2} \binom{m-m_1-m_2}{m_3} \dots$
12. This is very similiar to 11. A) $\frac{7!}{2!3!2!} = 210$ B) $-7!/(3!3!) = -140$
13. This is the sum of all the distribution numbers in which the numbers $m_1 \dots m_n$ run through all possible sequences of n positive integers adding up to m : $\sum_{m_i > 1} \frac{m!}{m_1!m_2!\dots m_n!}$

14. The boxes can be in any order, so we have a factor of 6. Seven can be partitioned into three distinct parts as 0,1,6; 0,2,5; 0,3,4; or 1,2,4. So, the answer is $6 \left[\frac{7!}{0!1!6!} + \frac{7!}{0!2!5!} + \frac{7!}{0!3!4!} + \frac{7!}{1!2!4!} \right] = \mathbf{1008}$.
15. This is partitioning 10 into 3 partitions, 2 of 1 type and one of the other. Thus, the answer is $\frac{10!}{3!3!4!2!} = \mathbf{2100}$.
16. This is equivalent to finding the number of sequences of length 10 composed of 0's and 1's. (0 in a spot corresponds to that spot's number (0-9) not being in the subset.) However, we can't have two consecutive 1's. If we try to generalize, let $0 = A$, $1 = B$ and we are doing an n -letter "word" instead of ten. Set $w_n =$ number of n -letter words (satisfying the conditions); set $a_n =$ number of words counted by w_n that begin with A; set $b_n =$ number of words counted by w_n that begin with B. $w_n = a_n + b_n$. $a_n = b_{n-1}$. $b_n = w_{n-1}$. Combining these we get the recursive relationship $w_n = w_{n-1} + w_{n-2}$. Then we can build up to find that $w_{10} = \mathbf{144}$.
17. $T(m, n) = n(T(m-1, n-1) + T(m-1, n))$ for $1 < n < m$. To prove this, look at $T(5, 3)$ and think of it as the number of 5-letter words from $\{A, B, C\}$ with no missing letters. There are 3 choices for the first letter. After this, the remaining four letters must be filled in, and the first letter (call it X) does not have to be used again. There are two cases:
- If X does not occur again, then the word can be completed in $T(4, 2)$ ways.
 - If X does occur again, then the number of ways to complete the word is $T(4, 3)$.
- As we have n choices for the letter X (first letter), we get that $T(5, 3) = 3 \cdot (T(4, 2) + T(4, 3))$, or the above, in general.
18. $a_n = a_{n-1} + a_{n-2} + a_{n-5}$. Every way to make n cents ends in either a 1 cent, 2 cents, or 5 cents (since order matters, there is a distinct last stamp). There are a_{n-1} ways to make n cents ending in a 1 cent stamp, since this is the number of ways to make $n-1$ cents. Similarly, there are a_{n-2} ways to make n cents ending in a 2 cents stamp, and a_{n-5} ways to make n cents ending in a 5 cents stamp. Since every way to make n cents ends in one of these stamps, and there is no overlap, the total number of ways to make n cents is the sum of these, or $a_{n-1} + a_{n-2} + a_{n-5}$.
19. (a) $r_n = r_{n-1} + n$
 (b) $\binom{n+1}{2} + 1$
20. First, we will show that a positive integer x is not a difference of squares if and only if $x \equiv 2 \pmod{4}$.
- First, suppose $x = a^2 - b^2$ for some integers a, b , so $x = (a-b)(a+b)$. Now, if $a-b = 1$, then $a+b = 2n+1$ for some integer n , so $x = (a-b)(a+b) = 2n+1$, and x is odd.
- Conversely, if x is odd, then $x = 2n+1$ for some integer n , so $x = 1(2n+1) = (n+1-n)(n+1+n) = (n+1)^2 - n^2$.
- Now, if $a-b = 2$, then $a+b = 2n+2$ for some integer n , so $x = (a-b)(a+b) = 2(2n+2) = 4n+4$
- Conversely, if $x \equiv 0 \pmod{4}$, then $x = 4n+4$ for some integer n , so $x = (n+2)^2 - n^2$.
- So, for all other cases, $a-b \geq 2$. If $a-b$ is even, then we know $a-b = 2n$ for some integer n , and $a+b = 2n+2m$ for some integer m , so $x = (a-b)(a+b) = 2n \cdot (2n+2m) = 4nm + 4n^2$, so $x \equiv 0 \pmod{4}$, and we have already covered this case.

If $a - b$ is odd, then we know $a - b = 2n + 1$ for some integer n , and $a + b = 2n + 2m + 1$, where m is some integer, so $x = (a - b)(a + b) = (2n + 1) \cdot (2n + 2m + 1) = 4(n^2 + nm + n) + 2n + 2m + 1 \equiv 1 \pmod{4}$, which is a case that we have already covered.

Therefore, x is not a difference of two squares if and only if $x \equiv 2 \pmod{4}$. So, the 2000th number is $4 \cdot 1999 + 2 = \mathbf{7998}$.

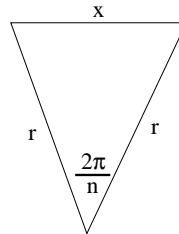
Team Test Solutions

Rice Mathematics Tournament 2000

1. .6445 rounds to .645 to .65 to .7. Otherwise .6444... rounds to .644. So the smallest number is **.6445**.
2. Let c =price, p =purity, d =diameter, h =depth of gold mine, k_i =constant. We are given $c = k_1 p^2 d^3$, $p = k_2 \frac{c^2}{h^2}$, and $d = k_3 \sqrt[3]{ch}$. So, $c = k_1 k_2^2 \frac{c^4}{h^4} k_3^3 c h^3 = k_4 c^5 \frac{1}{h}$. Thus, $k_4 c^4 = h$, and $c = k_5 h^{\frac{1}{4}}$. Thus, p varies as $h^{\frac{1}{4}}$.
3. The sum of the numbers from 700 to 799 is $\frac{799 \cdot 800}{2} - \frac{699 \cdot 700}{2} = 74950$. The sum of the numbers from 70 to 79 is $\frac{79 \cdot 80}{2} - \frac{69 \cdot 70}{2} = 745$. So, all numbers that end from 70 to 79 (excluding those starting with 7, since we counted those from 700 to 799) is $745 \cdot 9 + 10(100 + 200 + \dots + 600 + 800 + 900) = 44705$. The sum of all numbers ending in 7 is $9(7+17+27+37+47+57+67+87+97) + 9(100+200+\dots+600+800+900) = 38187$. So, the total sum of numbers containing a 7 is $74950 + 44705 + 38186 = \mathbf{157842}$.
4. **738,826**. This can be arrived at by stepping down, starting with finding how many combinations are there that begin with a letter other than V or W, and so forth. The answer is $\frac{8 \cdot 9!}{2} + \frac{4 \cdot 7!}{2} + 4 \cdot 6! + 4 \cdot 4! + 3! + 2! + 2! = \mathbf{738826}$.
5. $0 = \cos(\alpha + \beta) + \sin(\alpha - \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta + \sin \alpha \cdot \cos \beta - \sin \beta \cos \alpha = (\cos \alpha + \sin \alpha) \cdot (\cos \beta - \sin \beta)$. So $\cos \alpha + \sin \alpha = 0$ or $\cos \beta - \sin \beta = 0$. Then $\tan \alpha = -1$ or $\tan \beta = 1$. Since $\tan \beta$ is given as $\frac{1}{2000}$, $\tan \alpha = -\mathbf{1}$.
6. Since $\alpha^3 - \alpha - 1 = 0$, then $\alpha^{10} = \alpha^8 + \alpha^7$. So, we can reduce our expression to $3\alpha^8 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $3\alpha^8 - 3\alpha^6 - 3\alpha^5 = 0$, so our expression reduces to $4\alpha^4 + 2\alpha^3 - 4\alpha^2 - 6\alpha - 17$. Also, $4\alpha^4 - 4\alpha^2 - 4\alpha^0$, so our expression reduces to $2\alpha^3 - 2\alpha - 17$. Now, $2\alpha^3 - 2\alpha - 2 = 0$, so our expression reduces to **-15**, which is our answer.
7. Another 4-digit number that satisfies this property is **9801**, since $9801 = 9 \cdot 1089$.
8. If she has a silver dollar, then she would have too many other coins, as 0 half dollars, 2 quarters, 3 dimes, etc. would be greater than the total. So she has no silver dollars, and at least one of every other denomination. Continuing, it turns out the only feasible solution is 0 silver dollars, 1 half dollar, 2 quarters, 3 dimes, 4 nickels, 8 pennies, for a total of **18** coins.
9. It suffices to consider $x \geq 1$, since $4(-x)^4 + 1 = 4(x)^4 + 1$, and $4(0) + 1 = 1$ is not prime. So, $4x^4 + 1 = (4x^4 + 4x^2 + 1) - 4x^2 = (2x^2 + 1)^2 - (2x)^2 = (2x^2 + 1 - 2x)(2x^2 + 1 + 2x)$. For integers x , both $2x^2 - 2x + 1$ and $2x^2 + 2x + 1$ are integers, so this factors $4x^4 + 1$ unless $2x^2 - 2x + 1 = \pm 1$ or $2x^2 + 2x + 1 = \pm 1$. Since $x > 0$, then $2x^2 + 2x + 1 > 1$, so we must have $2x^2 - 2x + 1 = \pm 1$. $2x^2 - 2x + 1 = -1$ is absurd ($4x^4 + 1, 2x^2 + 2x + 1 > 0$, so $2x^2 - 2x + 1 = \frac{4x^4 + 1}{2x^2 + 2x + 1} > 0$), so we solve $2x^2 - 2x + 1 = 1$, or $2x^2 - 2x = 0$, so $x(x - 1) = 0$, and $x = 0$ or $x = 1$. We have already rejected $x = 0$, so the only case left is $x = 1$, or $4(1)^4 + 1 = \mathbf{5}$.
10. The second hand crosses the minute hand 59 times an hour. The second hand crosses the hour hand 60 times an hour, except for 2 of the hours, due to the movement of the hour hand. The minute hand and the hour hand cross 22 times total, because the hour hand

completes 2 rotations in a day, and the minute hand completes 24. The second, hour, and minute hand all coincide only at noon and midnight, but we've counted each of these 12:00's 3 times instead of once. Therefore, the answer is $59 \cdot 60 + 60 \cdot 24 - 2 + 22 - 2 \cdot 2$, giving us **2872** crossings.

11. $f(x)$ is either **0** or something of the form $\pm x^m$, where $m \geq 0$.
12. A 's position is $(a - V_a t, 0)$ and P 's position is $(0, b - V_b t)$. So, at time t , the distance between them is $\sqrt{(a - V_a t)^2 + (b - V_b t)^2}$. Notice that this distance is the same as the distance between the point (a, b) and the line $(V_a t, V_b t)$, which is the same as the line $V_b x - V_a y = 0$. The distance from a line $Ax + By + C = 0$ and (x_0, y_0) is $\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$, so the answer is $\frac{|bV_a - aV_b|}{\sqrt{V_a^2 + V_b^2}}$.
13. Given any 4 vertices, there is exactly one intersection of all the diagonals connecting them. So, the answer is $\binom{n}{4}$.
14. x_0 if $n \equiv 0 \pmod{4}$, $\frac{1+x_0}{1-x_0}$ if $n \equiv 1 \pmod{4}$, $-\frac{1}{x_0}$ if $n \equiv 2 \pmod{4}$, $\frac{x_0-1}{x_0+1}$ if $n \equiv 3 \pmod{4}$.
15. Consider a regular n -gon with radius r . Let x be the side length of the n -gon. So, since the central angle is $\frac{2\pi}{n}$ (see diagram below), use the Law of Cosines to find that $x^2 = r^2 + r^2 - 2r * r \cos \frac{2\pi}{n}$, so $x^2 = 2r^2(1 - \cos \frac{2\pi}{n})$. Thus, $x = r\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. So, the total perimeter of the n -gon is $nx = nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}}$. Now, if we take $\lim_{n \rightarrow \infty}$ of the perimeter, the result will be $2\pi n$, since the n -gon approaches a circle, so $\lim_{n \rightarrow \infty} nr\sqrt{2}\sqrt{1 - \cos \frac{2\pi}{n}} = 2\pi r$, and so $\lim_{n \rightarrow \infty} nr\sqrt{1 - \cos \frac{2\pi}{n}} = \pi r\sqrt{2}$.



Advanced Topics Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem AT1 [3 points]

One of the receipts for a math tournament showed that 72 identical trophies were purchased for \$-99.9-, where the first and last digits were illegible. How much did each trophy cost?

Solution: The price must be divisible by 8 and 9. Thus the last 3 digits must be divisible by 8, so the price ends with 992, and the first digit must be 7 to make the total divisible by 9. $\$799.92/72 = \11.11 .

Problem AT2 [3 points]

Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending \$1024. When she comes out of the last shop she has no money left. What is the minimum possible value of d ?

Solution: Work backwards. Before going into the last shop she had \$1024, before the lottery she had \$512, then \$1536, \$768, \dots . We can easily prove by induction that if she ran out of money after n shops, $0 \leq n \leq 10$, she must have started with $1024 - 2^{10-n}$ dollars. Therefore d is **1023**.

Problem AT3 [4 points]

An unfair coin has the property that when flipped four times, it has the same probability of turning up 2 heads and 2 tails (in any order) as 3 heads and 1 tail (in any order). What is the probability of getting a head in any one flip?

Solution: Let p be the probability of getting a head in one flip. There are 6 ways to get 2 heads and 2 tails, each with probability $p^2(1-p)^2$, and 4 ways to get 3 heads and 1 tail, each with probability $p^3(1-p)$. We are given that $6p^2(1-p)^2 = 4p^3(1-p)$. Clearly p is not 0 or 1, so we can divide by $p^2(1-p)$ to get $6(1-p) = 4p$. Therefore p is $\frac{3}{5}$.

Problem AT4 [4 points]

You are given 16 pieces of paper numbered 16, 15, \dots , 2, 1 in that order. You want to put them in the order 1, 2, \dots , 15, 16 switching only two adjacent pieces of paper at a time. What is the minimum number of switches necessary?

Solution: Piece 16 has to move to the back 15 times, piece 15 has to move to the back 14 times, \dots , piece 2 has to move to the back 1 time, piece 1 has to move to the back 0 times. Since only one piece can move back in each switch, we must have at least $15 + 14 + \dots + 1 = \mathbf{120}$ switches.

Problem AT5 [5 points]

For any finite set S , let $f(S)$ be the sum of the elements of S (if S is empty then $f(S) = 0$). Find the sum over all subsets E of S of $\frac{f(E)}{f(S)}$ for $S = \{1, 2, \dots, 1999\}$.

Solution: An n element set has 2^n subsets, so each element of S appears in 2^{1998} subsets E , so our sum is $2^{1998} \cdot \frac{1+2+\dots+1999}{1+2+\dots+1999} = \mathbf{2^{1998}}$.

Problem AT6 [5 points]

Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?

Solution: The number of sheets will leave a remainder of 1 when divided by the least common multiple of 2, 3, 4, 5, 6, 7, and 8, which is $8 \cdot 3 \cdot 5 \cdot 7 = 840$. Since the number of sheets is between 1000 and 2000, the only possibility is 1681. The number of piles must be a divisor of $1681 = 41^2$, hence it must be **41**.

Problem AT7 [5 points]

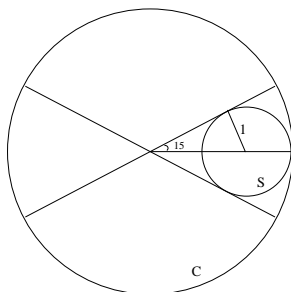
Find an ordered pair (a, b) of real numbers for which $x^2 + ax + b$ has a non-real root whose cube is 343.

Solution: The cube roots of 343 are the roots of $x^3 - 343$, which is $(x - 7)(x^2 + 7x + 49)$. Therefore the ordered pair we want is **(7,49)**.

Problem AT8 [6 points]

Let C be a circle with two diameters intersecting at an angle of 30 degrees. A circle S is tangent to both diameters and to C , and has radius 1. Find the largest possible radius of C .

Solution: For C to be as large as possible we want S to be as small as possible. It is not hard to see that this happens in the situation shown below. Then the radius of C is $1 + \csc 15 = \mathbf{1 + \sqrt{2} + \sqrt{6}}$. The computation of $\sin 15$ can be done via the half angle formula.



Problem AT9 [7 points]

As part of his effort to take over the world, Edward starts producing his own currency. As part of an effort to stop Edward, Alex works in the mint and produces 1 counterfeit coin for every 99 real ones. Alex isn't very good at this, so none of the counterfeit coins are the right weight. Since the mint is not perfect, each coin is weighed before leaving. If the coin is not the right weight, then it is sent to a lab for testing. The scale is accurate 95% of the time, 5% of all the coins minted are sent to the lab, and the lab's test is accurate 90% of the time. If the lab says a coin is counterfeit, what is the probability that it really is?

Solution: 5% of the coins are sent to the lab, and only .95% of the coins are sent to the lab and counterfeit, so there is a 19% chance that a coin sent to the lab is counterfeit and an 81% chance that it is real. The lab could correctly detect a counterfeit coin or falsely accuse a real one of being counterfeit, so the probability that a coin the lab says is counterfeit really is counterfeit is $\frac{19/100 \cdot 9/10}{19/100 \cdot 9/10 + 81/100 \cdot 1/10} = \frac{19}{28}$.

Problem AT10 [8 points]

Find the minimum possible value of the largest of xy , $1-x-y+xy$, and $x+y-2xy$ if $0 \leq x \leq y \leq 1$.

Solution: I claim the answer is $4/9$. Let $s = x + y$, $p = xy$, so x and y are $\frac{s \pm \sqrt{s^2 - 4p}}{2}$. Since x and y are real, $s^2 - 4p \geq 0$. If one of the three quantities is less than or equal to $1/9$, then at least one of the others is at least $4/9$ by the pigeonhole principle since they add up to 1. Assume that $s - 2p < 4/9$, then $s^2 - 4p < (4/9 + 2p)^2 - 4p$, and since the left side is non-negative we get $0 \leq p^2 - \frac{5}{9}p + \frac{4}{81} = (p - \frac{1}{9})(p - \frac{4}{9})$. This implies that either $p \leq \frac{1}{9}$ or $p \geq \frac{4}{9}$, and either way we're done. This minimum is achieved if x and y are both $1/3$, so the answer is $\frac{4}{9}$, as claimed.

Algebra Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem A1 [3 points]

If $a@b = \frac{a^3-b^3}{a-b}$, for how many real values of a does $a@1 = 0$?

Solution: If $\frac{a^3-1}{a-1} = 0$, then $a^3 - 1 = 0$, or $(a - 1)(a^2 + a + 1) = 0$. Thus $a = 1$, which is an extraneous solution since that makes the denominator of the original expression 0, or a is a root of $a^2 + a + 1$. But this quadratic has no real roots, in particular its roots are $\frac{-1 \pm \sqrt{-3}}{2}$. Therefore there are no such real values of a , so the answer is **0**.

Problem A2 [3 points]

For what single digit n does 91 divide the 9-digit number $12345n789$?

Solution 1: 123450789 leaves a remainder of 7 when divided by 91, and 1000 leaves a remainder of 90, or -1, so adding **7** multiples of 1000 will give us a multiple of 91.

Solution 2: For those who don't like long division, there is a quicker way. First notice that $91 = 7 \cdot 13$, and $7 \cdot 11 \cdot 13 = 1001$. Observe that $12345n789 = 123 \cdot 1001000 + 45n \cdot 1001 - 123 \cdot 1001 + 123 - 45n + 789$. It follows that 91 will divide $12345n789$ iff 91 divides $123 - 45n + 789 = 462 - n$. The number 462 is divisible by 7 and leaves a remainder of **7** when divided by 13.

Problem A3 [4 points]

Alex is stuck on a platform floating over an abyss at 1 ft/s. An evil physicist has arranged for the platform to fall in (taking Alex with it) after traveling 100ft. One minute after the platform was launched, Edward arrives with a second platform capable of floating all the way across the abyss. He calculates for 5 seconds, then launches the second platform in such a way as to maximize the time that one end of Alex's platform is between the two ends of the new platform, thus giving Alex as much time as possible to switch. If both platforms are 5 ft long and move with constant velocity once launched, what is the speed of the second platform (in ft/s)?

Solution: The slower the second platform is moving, the longer it will stay next to the first platform. However, it needs to be moving fast enough to reach the first platform before it's too late. Let v be the velocity of the second platform. It starts 65 feet behind the first platform, so it reaches the back of the first platform at $\frac{60}{v-1}$ seconds, and passes the front at $\frac{70}{v-1}$ seconds, so the time to switch is $\frac{10}{v-1}$. Hence we want v to be as small as possible while still allowing the switch before the first platform falls. Therefore the time to switch will be maximized if the back of the second platform lines up with the front of the first platform at the instant that the first platform has travelled 100ft, which occurs after 100 seconds. Since the second platform is launched 65 seconds later and has to travel 105 feet, its speed is $105/35 = \mathbf{3ft/s}$.

Problem A4 [4 points]

Find all possible values of $\frac{d}{a}$ where $a^2 - 6ad + 8d^2 = 0$, $a \neq 0$.

Solution: Dividing $a^2 - 6ad + 8d^2 = 0$ by a^2 , we get $1 - 6\frac{d}{a} + 8(\frac{d}{a})^2 = 0$. The roots of this quadratic are $\frac{1}{2}, \frac{1}{4}$.

Problem A5 [5 points]

You are trapped in a room with only one exit, a long hallway with a series of doors and land mines. To get out you must open all the doors and disarm all the mines. In the room is a panel with 3 buttons, which conveniently contains an instruction manual. The red button arms a mine, the yellow button disarms two mines and closes a door, and the green button opens two doors. Initially 3 doors are closed and 3 mines are armed. The manual warns that attempting to disarm two mines or open two doors when only one is armed/closed will reset the system to its initial state. What is the minimum number of buttons you must push to get out?

Solution: Clearly we do not want to reset the system at any time. After pressing the red button r times, the yellow button y times, and the green button g times, there will be $3 + r - 2y$ armed mines and $3 + y - 2g$ closed doors, so we want the values of r , y , and g that make both of these quantities 0 while minimizing $r + y + g$. From the number of doors we see that y must be odd, from the number of mines we see $y = (3 + r)/2 \geq 3/2$, so $y \geq 3$. Then $g = (3 + y)/2 \geq 3$, and $r = 2y - 3 \geq 3$, so $r + y + g \geq 9$. Call the red, yellow, and green buttons 1, 2, and 3 respectively for notational convenience, then a sequence of buttons that would get you out is 123123123. Another possibility is 111222333, and of course there are others. Therefore the answer is **9**.

Problem A6 [5 points]

Carl and Bob can demolish a building in 6 days, Anne and Bob can do it in 3, Anne and Carl in 5. How many days does it take all of them working together if Carl gets injured at the end of the first day and can't come back? Express your answer as a fraction in lowest terms.

Solution: Let a be the portion of the work that Anne does in one day, similarly b for Bob and c for Carl. Then what we are given is the system of equations $b + c = 1/6$, $a + b = 1/3$, and $a + c = 1/5$. Thus in the first day they complete $a + b + c = \frac{1}{2}(1/6 + 1/3 + 1/5) = 7/20$, leaving $13/20$ for Anne and Bob to complete. This takes $\frac{13/20}{1/3} = 39/20$ days, for a total of $\frac{59}{20}$.

Problem A7 [5 points]

Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?

Solution: The number of sheets will leave a remainder of 1 when divided by the least common multiple of 2, 3, 4, 5, 6, 7, and 8, which is $8 \cdot 3 \cdot 5 \cdot 7 = 840$. Since the number of sheets is between 1000 and 2000, the only possibility is 1681. The number of piles must be a divisor of $1681 = 41^2$, hence it must be **41**.

Problem A8 [6 points]

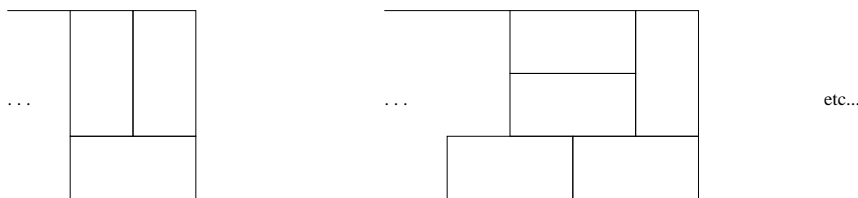
If $f(x)$ is a monic quartic polynomial such that $f(-1) = -1$, $f(2) = -4$, $f(-3) = -9$, and $f(4) = -16$, find $f(1)$.

Solution: The given data tells us that the roots of $f(x) + x^2$ are $-1, 2, -3$, and 4 . Combining with the fact that f is monic and quartic we get $f(x) + x^2 = (x + 1)(x - 2)(x + 3)(x - 4)$. Hence $f(1) = (2)(-1)(4)(-3) - 1 = \mathbf{23}$.

Problem A9 [7 points]

How many ways are there to cover a 3×8 rectangle with 12 identical dominoes?

Solution: Trivially there is 1 way to tile a 3×0 rectangle, and it is not hard to see there are 3 ways to tile a 3×2 . Let T_n be the number of tilings of a $3 \times n$ rectangle, where n is even. From the diagram below we see the recursion $T_n = 3T_{n-2} + 2(T_{n-4} + T_{n-6} + \dots + T_2 + T_0)$. Given that, we can just calculate $T_4 = 11$, $T_6 = 41$, and T_8 is **153**.



Problem A10 [8 points]

Pyramid *EARLY* is placed in (x, y, z) coordinates so that $E = (10, 10, 0)$, $A = (10, -10, 0)$, $R = (-10, -10, 0)$, $L = (-10, 10, 0)$, and $Y = (0, 0, 10)$. Tunnels are drilled through the pyramid in such a way that one can move from (x, y, z) to any of the 9 points $(x, y, z - 1)$, $(x \pm 1, y, z - 1)$, $(x, y \pm 1, z - 1)$, $(x \pm 1, y \pm 1, z - 1)$. Sean starts at Y and moves randomly down to the base of the pyramid, choosing each of the possible paths with probability $\frac{1}{9}$ each time. What is the probability that he ends up at the point $(8, 9, 0)$?

Solution 1: Start by figuring out the probabilities of ending up at each point on the way down the pyramid. Obviously we start at the top vertex with probability 1, and each point on the next level down with probability $1/9$. Since each probability after n steps will be some integer over 9^n , we will look only at those numerators. The third level down has probabilities as shown below. Think of this as what you would see if you looked at the pyramid from above, and peeled off the top two layers.

1 2 3 2 1
 2 4 6 4 2
 3 6 9 6 3
 2 4 6 4 2
 1 2 3 2 1

What we can observe here is not only the symmetry along vertical, horizontal, and diagonal axes, but also that each number is the product of the numbers at the ends of its row and column (e.g. $6 = 2 \cdot 3$). This comes from the notion of *independence* of events, i.e. that if we east and then south, we end up in the same place as if we had moved south and then east. Since we are only looking for the probability of ending up at $(8, 9, 0)$, we need only know that this is true for the top two rows of the square of probabilities, which depend only on the top two rows of the previous layer. This will follow from the calculation of the top row of each square, which we can do via an algorithm similar to Pascal's triangle. In the diagram below, each element is the sum of the 3 above it.

```

      1
    1 1 1
  1 2 3 2 1
1 3 6 7 6 3 1
1 4 10 16 19 16 10 4 1
1 5 15 30 45 51 45 30 15 5 1

```

Now observe that the first 3 numbers in row n , where the top is row 0, are $1, n, \frac{n(n+1)}{2}$. This fact is easily proved by induction on n , so the details are left to the reader. Now we can calculate the top two rows of each square via another induction argument, or by independence, to establish that the second row is always n times the first row. Therefore the probability of ending up at the point $(8,9,0)$ is $\frac{550}{9^{10}}$.

Solution 2: At each move, the x and y coordinates can each increase by 1, decrease by 1, or stay the same. The y coordinate must increase 9 times and stay the same 1 times, the x coordinate can either increase 8 times and stay the same 1 time or decrease 1 time and increase 9 times. Now we consider every possible case. First consider the cases where the x coordinate decreases once. If the x coordinate decreases while the y coordinate increases, then we have 8 moves that are the same and 2 that are different, which can be done in $\frac{10!}{8!} = 90$ ways. If the x coordinate decreases while the y coordinate stays the same, then we have 9 moves that are the same and 1 other, which can be done in $\frac{10!}{9!} = 10$ ways. Now consider the cases where the x coordinate stays the same twice. If the y coordinate stays the same while the x coordinate increases, then we have 7 moves that are the same, 2 that are the same, and 1 other, which can be done in $\frac{10!}{7!2!} = 360$ ways. If the y coordinate stays the same while the x coordinate stays the same, then we have 8 moves that are the same and 2 that are different, which can be done in $\frac{10!}{8!} = 90$ ways. Therefore there are $360 + 90 + 90 + 10 = 550$ paths to $(8,9,0)$, out of 9^{10} possible paths to the bottom, so the probability of ending up at the point $(8,9,0)$ is $\frac{550}{9^{10}}$.

Calculus Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem C1 [3 points]

Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

Solution: Since $f''(x) = 0$ we must have $f(x) = ax + b$ for some real numbers a, b . Thus $f(0) = b = 19$ and $f(1) = a + 19 = 99$, so $a = 80$. Therefore $f(x) = \mathbf{80x + 19}$.

Problem C2 [3 points]

A rectangle has sides of length $\sin x$ and $\cos x$ for some x . What is the largest possible area of such a rectangle?

Solution: We wish to maximize $\sin x \cdot \cos x = \frac{1}{2} \sin 2x$. But $\sin 2x \leq 1$, with equality holding for $x = \pi/4$, so the maximum is $\frac{1}{2}$.

Problem C3 [4 points]

Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} \left(\frac{\sin x}{1+x^4} + 1 \right) dx.$$

Solution: The function $\frac{\sin x}{1+x^4}$ is odd, so its integral over this interval is 0. Thus we get the same answer if we just integrate dx , namely, $\mathbf{8\pi\sqrt{2}}$.

Problem C4 [4 points]

f is a continuous real-valued function such that $f(x+y) = f(x)f(y)$ for all real x, y . If $f(2) = 5$, find $f(5)$.

Solution 1: Since $f(nx) = f(x)^n$ for all integers n , $f(5) = f(1)^5$ and $f(2) = f(1)^2$, so $f(5) = f(2)^{5/2} = \mathbf{25\sqrt{5}}$.

Solution 2: More generally, since $f(nx) = f(x)^n$ for all integers n , $f(1) = c = f(1/n)^n$ for some constant c and all integers n . Thus $f(k/n) = f(1/n)^k = f(1)^{k/n} = c^{k/n}$ for all rational numbers k/n . By continuity, it follows that $f(x) = c^x$ for all real numbers x . Since $f(2) = 5$, $c = \sqrt{5}$, so $f(5) = \mathbf{25\sqrt{5}}$.

Problem C5 [5 points]

Let $f(x) = x + \frac{1}{2x + \frac{1}{2x + \frac{1}{2x + \dots}}}$ for $x > 0$. Find $f(99)f'(99)$.

Solution: Assume that the continued fraction converges (it does) so that $f(x)$ is well defined. Notice that $f(x) - x = \frac{1}{x + f(x)}$, so $f(x)^2 - x^2 = 1$, or $f(x) = \sqrt{1 + x^2}$ (we need the positive square root since $x > 0$). Thus $f'(x) = \frac{x}{\sqrt{1+x^2}}$, so $f(x)f'(x) = x$. In particular, $f(99)f'(99) = \mathbf{99}$.

Problem C6 [5 points]

Evaluate $\frac{d}{dx}(\sin x - \frac{4}{3} \sin^3 x)$ when $x = 15$.

Solution: Of course this problem can be done by brute force, differentiating and then using the half angle formula to find \sin and \cos of 15, but there is a quicker way. $e^{ix} = \cos x + i \sin x$, so $\sin(3x)$ is the imaginary part of $(\cos x + i \sin x)^3$, which is $3 \cos^2 x \sin x - \sin^3 x = 3 \sin x - 4 \sin^3 x$, so the expression we are differentiating is just $\frac{1}{3} \sin(3x)$. Hence the derivative is $\cos(3x)$, and $\cos 45 = \frac{\sqrt{2}}{2}$.

Problem C7 [5 points]

If a right triangle is drawn in a semicircle of radius 1/2 with one leg (not the hypotenuse) along the diameter, what is the triangle's maximum possible area?

Solution: It is easy to see that we will want one vertex of the triangle to be where the diameter meets the semicircle, so the diameter is divided into segments of length x and $1 - x$, where x is the length of the leg on the diameter. The other leg of the triangle will be the geometric mean of these two numbers, $\sqrt{x(1-x)}$. Therefore the area of the triangle is $\frac{x\sqrt{x(1-x)}}{2}$, so it will be maximized when $\frac{d}{dx}(x^3 - x^4) = 3x^2 - 4x^3 = 0$, or $x = 3/4$. Therefore the maximum area is $\frac{3\sqrt{3}}{32}$.

Problem C8 [6 points]

A circle is randomly chosen in a circle of radius 1 in the sense that a point is randomly chosen for its center, then a radius is chosen at random so that the new circle is contained in the original circle. What is the probability that the new circle contains the center of the original circle?

Solution: If the center of the new circle is more than 1/2 away from the center of the original circle then the new circle cannot possibly contain the center of the original one. Let x be the distance between the centers (by symmetry this is all we need to consider), then for $0 \leq x \leq 1/2$ the probability of the new circle containing the center of the original one is $1 - \frac{x}{1-x}$. Hence we need to compute $\int_0^{1/2} (1 - \frac{x}{1-x}) dx = \frac{1}{2} - \int_0^{1/2} \frac{x}{1-x} dx$. To evaluate the integral, we can integrate by parts to get

$$-x \ln(1-x) \Big|_0^{1/2} - \int_0^{1/2} -\ln(1-x) dx = -\frac{1}{2} \ln\left(\frac{1}{2}\right) - [(1-x) \ln(1-x) - (1-x)] \Big|_0^{1/2} = \ln 2 - \frac{1}{2}.$$

Alternatively, we can use polynomial division to find that $\frac{x}{1-x} = -1 + \frac{1}{1-x}$, so $\int_0^{1/2} \frac{x}{1-x} dx = \int_0^{1/2} (-1 + \frac{1}{1-x}) dx = \ln 2 - \frac{1}{2}$. Therefore the probability is $\frac{1}{2} - (\ln 2 - \frac{1}{2}) = \mathbf{1 - \ln 2}$.

Problem C9 [7 points]

What fraction of the Earth's volume lies above the 45 degrees north parallel? You may assume the Earth is a perfect sphere. The volume in question is the smaller piece that we would get if the sphere were sliced into two pieces by a plane.

Solution 1: Without loss of generality, look at a sphere of radius 1 centered at the origin. If you like cartesian coordinates, then you can slice the sphere into discs with the same z coordinate, which have radius $\sqrt{1-z^2}$, so the region we are considering has volume $\int_{\sqrt{2}/2}^1 \pi(1-z^2) dz = \pi(\frac{2}{3} - \frac{5\sqrt{2}}{12})$, and dividing by $4\pi/3$ we get $\frac{8-5\sqrt{2}}{16}$.

Solution 2: For those who prefer spherical coordinates, we can find the volume of the spherical cap plus a cone whose vertex is the center of the sphere. This region is where r ranges from 0 to 1, θ ranges from 0 to 2π , and ϕ ranges from 0 to $\pi/4$. Remembering we need to subtract off the volume of the cone, which has height $\frac{1}{\sqrt{2}}$ and a circular base of radius $\frac{1}{\sqrt{2}}$, then divide by $\frac{4}{3}\pi$ to get the fraction of the volume of the sphere, we find that we need to evaluate $\frac{\int_0^1 \int_0^{2\pi} \int_0^{\pi/4} r^2 \sin \phi d\phi d\theta dr - \frac{1}{3}\pi(\frac{1}{\sqrt{2}})^2 \frac{1}{\sqrt{2}}}{4\pi/3}$. The integral is just $2\pi \frac{1}{3}(-\cos \pi/4 + \cos 0) = \frac{4\pi-2\pi\sqrt{2}}{6}$. Putting this back in the answer and simplifying yields $\frac{8-5\sqrt{2}}{16}$.

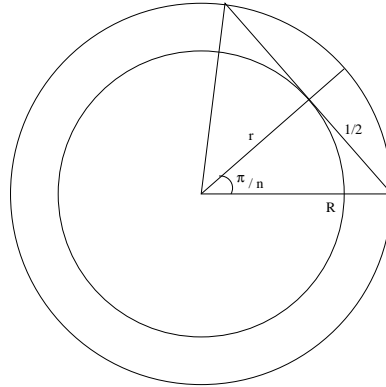
Solution 3: Cavalieri's Principle states that if two solids have the same cross-sectional areas at every height, then they have the same volume. This principle is very familiar in the plane, where we know that the area of a triangle depends only on the base and height, not the precise position of the apex. To apply it to a sphere, consider a cylinder with radius 1 and height 1. Now cut out a cone whose base is the upper circle of the cylinder and whose apex is the center of the lower circle. Then at a height z the area is $\pi(1^2 - z^2)$, exactly the same as for the upper hemisphere! The portion lying above the the 45 degrees north parallel is that which ranges from height $\frac{1}{\sqrt{2}}$ to 1. The volume of the cylinder in this range is $\pi \cdot 1^2(1 - \frac{1}{\sqrt{2}})$. The volume of the cone in this range is the volume of the entire cone minus the portion from height 0 to $\frac{1}{\sqrt{2}}$, i.e., $\frac{1}{3}\pi(1^2 \cdot 1 - (\frac{1}{\sqrt{2}})^2 \frac{1}{\sqrt{2}})$. Therefore the fraction of the Earth's volume that lies above the 45 degrees north parallel is $\frac{\pi(1-\frac{1}{\sqrt{2}}) - \frac{1}{3}\pi(1-\frac{1}{2\sqrt{2}})}{4\pi/3} = \frac{8-5\sqrt{2}}{16}$.

Solution 4: Another way to approach this problem is to integrate the function $\sqrt{1-x^2-y^2}$ over the region $x^2+y^2 \leq \frac{1}{\sqrt{2}}$, subtract off a cylinder of radius and height $\frac{1}{\sqrt{2}}$, then divide by the volume of the sphere. One could also use the nontrivial fact that the surface area of a portion of a sphere of radius r between two parallel planes separated by a distance z is $2\pi r^2 z$, so in particular the surface area of this cap is $2\pi(1 - \frac{1}{\sqrt{2}})$. Now, the ratio of the surface area of the cap to the surface area of the sphere is the same as the ratio of the volume of the cap plus the cone considered in Solution 2 to the volume of the whole sphere, so this allows us to avoid integration entirely.

Problem C10 [8 points]

Let A_n be the area outside a regular n -gon of side length 1 but inside its circumscribed circle, let B_n be the area inside the n -gon but outside its inscribed circle. Find the limit as n tends to infinity of $\frac{A_n}{B_n}$.

Solution: The radius of the inscribed circle is $\frac{1}{2} \cot \frac{\pi}{n}$, the radius of the circumscribed circle is $\frac{1}{2} \csc \frac{\pi}{n}$, and the area of the n -gon is $\frac{n}{4} \cot \frac{\pi}{n}$. The diagram below should help you verify that these are correct.



Then $A_n = \pi(\frac{1}{2} \csc \frac{\pi}{n})^2 - \frac{n}{4} \cot \frac{\pi}{n}$, and $B_n = \frac{n}{4} \cot \frac{\pi}{n} - \pi(\frac{1}{2} \cot \frac{\pi}{n})^2$, so $\frac{A_n}{B_n} = \frac{\pi(\csc \frac{\pi}{n})^2 - n \cot \frac{\pi}{n}}{n \cot \frac{\pi}{n} - \pi(\cot \frac{\pi}{n})^2}$. Let s denote $\sin \frac{\pi}{n}$ and c denote $\cos \frac{\pi}{n}$. Multiply numerator and denominator by s^2 to get $\frac{A_n}{B_n} = \frac{\pi - ncs}{ncs - \pi c^2}$.

Now use Taylor series to replace s by $\frac{\pi}{n} - \frac{(\frac{\pi}{n})^3}{6} + \dots$ and c by $1 - \frac{(\frac{\pi}{n})^2}{2} + \dots$. By l'Hôpital's rule it will suffice to take just enough terms so that the highest power of n in the numerator and denominator is determined, and that turns out to be n^{-2} in each case. In particular, we get the

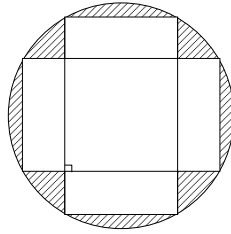
$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \frac{\pi - n \frac{\pi}{n} + n \frac{2}{3} (\frac{\pi}{n})^3 + \dots}{n \frac{\pi}{n} - n \frac{2}{3} (\frac{\pi}{n})^3 - \pi + \pi (\frac{\pi}{n})^2 + \dots} = \frac{\frac{2}{3} \frac{\pi^3}{n^2} + \dots}{\frac{1}{3} \frac{\pi^3}{n^2} + \dots} \rightarrow \mathbf{2}.$$

Geometry Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem G1 [3]

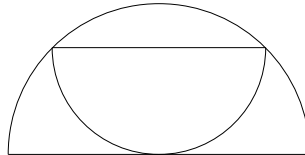
Two 10×24 rectangles are inscribed in a circle as shown. Find the shaded area.



Solution: The rectangles are 10×24 , so their diagonals, which are diameters of the circle, have length 26. Therefore the area of the circle is $\pi 13^2$, and the overlap is a 10×10 square, so the shaded area is $\pi 13^2 - 2 \cdot 10 \cdot 24 + 10^2 = \mathbf{169\pi - 380}$.

Problem G2 [3]

A semicircle is inscribed in a semicircle of radius 2 as shown. Find the radius of the smaller semicircle.



Solution: Draw a line from the center of the smaller semicircle to the center of the larger one, and a line from the center of the larger semicircle to one of the other points of intersection of the two semicircles. We now have a right triangle whose legs are both the radius of the smaller semicircle and whose hypotenuse is 2, therefore the radius of the smaller semicircle is $\sqrt{2}$.

Problem G3 [4]

In a cube with side length 6, what is the volume of the tetrahedron formed by any vertex and the three vertices connected to that vertex by edges of the cube?

Solution: We have a tetrahedron whose base is half a face of the cube and whose height is the side length of the cube, so its volume is $\frac{1}{3} \cdot (\frac{1}{2} \cdot 6^2) \cdot 6 = \mathbf{36}$.

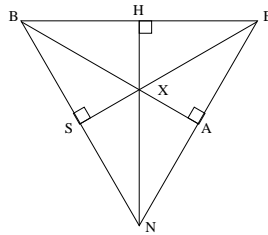
Problem G4 [4]

A cross-section of a river is a trapezoid with bases 10 and 16 and slanted sides of length 5. At this section the water is flowing at π mph. A little ways downstream is a dam where the water flows through 4 identical circular holes at 16 mph. What is the radius of the holes?

Solution: The volume of water going through any cross-section of the river in an hour (assuming the cross-sections are parallel) is the area times the velocity. The trapezoid has height 4, hence area 52, so the volume of water going through at any hour is 52π . Let r be the radius of the holes, then the total area is $4\pi r^2$, so the volume of water is $64\pi r^2$. Therefore $64\pi r^2 = 52\pi$, so $r = \frac{\sqrt{13}}{4}$.

Problem G5 [5]

In triangle BEN shown below with its altitudes intersecting at X , $NA = 7$, $EA = 3$, $AX = 4$, and $NS = 8$. Find the area of BEN .



Solution: The idea is to try to find a base and height for the triangle so that we can find the area. By the Pythagorean theorem, $EX = 5$, $NX = \sqrt{65}$, and $SX = 1$. Triangles AXE and BXS are similar since they have the same angles. The ratio of their side lengths is 4:1, so $BS = 3/4$ and $BX = 5/4$. Now using either NE or NB as a base, we get that the area of BEN is $\frac{1}{2} \cdot (8 + \frac{3}{4}) \cdot 6$ or $\frac{1}{2} \cdot (4 + \frac{5}{4}) \cdot 10$, both of which simplify to $\frac{105}{4}$.

Problem G6 [5]

A sphere of radius 1 is covered in ink and rolling around between concentric spheres of radii 3 and 5. If this process traces a region of area 1 on the larger sphere, what is the area of the region traced on the smaller sphere?

Solution: The figure drawn on the smaller sphere is just a scaled down version of what was drawn on the larger sphere, so the ratio of the areas is the ratio of the surface area of the spheres. This is the same as the ratio of the squares of the radii, which is $\frac{9}{25}$.

Problem G7 [5]

A dart is thrown at a square dartboard of side length 2 so that it hits completely randomly. What is the probability that it hits closer to the center than any corner, but within a distance 1 of a corner?

Solution: By symmetry it will suffice to consider one quarter of the dartboard, which is a square of side length 1. Therefore the probability is the area of the desired region in this square. The desired region is the part of the circle of radius 1 centered at a corner that is closer to the opposite corner. The points closer to the opposite corner are those that are on the other side of the diagonal through the other two corners, so the desired region is a quarter of a circle of radius 1 minus a right triangle with legs of length 1. Therefore the area (and hence the probability) is $\frac{\pi-2}{4}$.

Problem G8 [6]

Squares $ABKL, BCMN, CAOP$ are drawn externally on the sides of a triangle ABC . The line segments KL, MN, OP , when extended, form a triangle $A'B'C'$. Find the area of $A'B'C'$ if ABC is an equilateral triangle of side length 2.

Solution: Triangle ABC has area $\sqrt{3}$, and each of the three squares has area 4. The three remaining regions are congruent, so just consider the one that includes vertex B . Triangle KBN has two sides of length 2 and an angle of 120° between them, so bisecting that angle we get two halves of an equilateral triangle of side length 2, so the area is again $\sqrt{3}$. The remaining region is an equilateral triangle of side length $2\sqrt{3}$, so its area is $(2\sqrt{3})^2\sqrt{3}/4 = 3\sqrt{3}$. Therefore the area of $A'B'C'$ is $\sqrt{3} + 3 \cdot 4 + 3 \cdot \sqrt{3} + 3 \cdot 3\sqrt{3} = \mathbf{12 + 13\sqrt{3}}$.

Note that this problem is still solvable, but much harder, if the first triangle is not equilateral.

Problem G9 [7]

A regular tetrahedron has two vertices on the body diagonal of a cube with side length 12. The other two vertices lie on one of the face diagonals not intersecting that body diagonal. Find the side length of the tetrahedron.

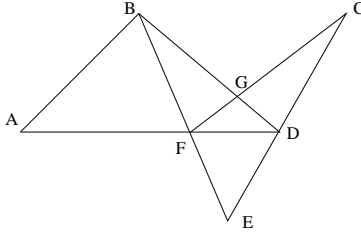
Solution: Let $ABCD$ be a tetrahedron of side s . We want to find the distance between two of its opposite sides. Let E be the midpoint of AD , F the midpoint of BC . Then $AE = s/2$, $AF = s\sqrt{3}/2$, and angle $AEF = 90^\circ$. So the distance between the two opposite sides is $EF = \sqrt{AF^2 - AE^2} = \sqrt{3s^2/4 - s^2/4} = s/\sqrt{2}$.

Now we find the distance between a body diagonal and a face diagonal of a cube of side a . Let O be the center of the cube and P be the midpoint of the face diagonal. Then the plane containing P and the body diagonal is perpendicular to the face diagonal. So the distance between the body and face diagonals is the distance between P and the body diagonal, which is $\frac{a}{2}\sqrt{\frac{2}{3}}$ (the altitude from P of right triangle OPQ , where Q is the appropriate vertex of the cube). So now $\frac{s}{\sqrt{2}} = \frac{a}{2}\sqrt{\frac{2}{3}}$, thus $s = a/\sqrt{3} = 12/\sqrt{3} = \mathbf{4\sqrt{3}}$.

Problem G10 [8]

In the figure below, $AB = 15, BD = 18, AF = 15, DF = 12, BE = 24$, and $CF = 17$. Find $BG : FG$.

Solution: Our goal is to find the lengths BG and FG . There are several ways to go about doing this, but we will show only one here. We will make several uses of Stewart's theorem, which can



be proved using the law of cosines twice. By Stewart's theorem on triangle ABD and line BF , $15^2 \cdot 12 + 18^2 \cdot 15 = BF^2 \cdot 27 + 15 \cdot 12 \cdot 27$, so $BF = 10$ and $EF = 14$. By Stewart's theorem on triangle ABE and line AF , $AE^2 \cdot 10 + 15^2 \cdot 14 = 15^2 \cdot 24 + 14 \cdot 10 \cdot 24$, so $AE = \sqrt{561}$. By Stewart's theorem on triangle AED and line EF , $ED^2 \cdot 15 + 561 \cdot 12 = 14^2 \cdot 27 + 12 \cdot 15 \cdot 27$, so $ED = 2\sqrt{57}$. By Stewart's theorem on triangle CFE and line FD , $14^2 \cdot CD + 17^2 \cdot 2\sqrt{57} = 12^2 \cdot (CD + 2\sqrt{57}) + 2\sqrt{57} \cdot CD \cdot (CD + 2\sqrt{57})$, so $CD = \sqrt{57}$ and $CE = 3\sqrt{57}$. Note that $DG = 18 - BG$ and apply Menelaus' theorem to triangle BED and the line through C , G , and F to get $3 \cdot \frac{18-BG}{BG} \cdot \frac{10}{14} = 1$, so $BG = \frac{135}{11}$. Similarly $CG = 17 - FG$, so applying Menelaus' theorem to triangle CFE and the line through B , G , and D we get $\frac{24}{10} \cdot \frac{FG}{17-FG} \cdot \frac{1}{2} = 1$, so $FG = \frac{85}{11}$. Therefore $BG : FG = 27 : 17$.

Oral Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem O1 [25 points]

Start with an angle of 60° and bisect it, then bisect the lower 30° angle, then the upper 15° angle, and so on, always alternating between the upper and lower of the previous two angles constructed. This process approaches a limiting line that divides the original 60° angle into two angles. Find the measure (degrees) of the smaller angle.

Solution: The fraction of the original angle is $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - + \dots$. This is just a geometric series with first term $1/2$ and ratio $-1/2$, so the sum is $1/3$. Therefore the smaller angle is 20° .

Score 20 points for the correct answer, 5 points for a correct justification. Just setting up the sum is worth 6 points, getting $1/3$ is worth 9 more.

Problem O2 [25 points]

Alex, Pei-Hsin, and Edward got together before the contest to send a mailing to all the invited schools. Pei-Hsin usually just stuffs the envelopes, but if Alex leaves the room she has to lick them as well and has a 25% chance of dying from an allergic reaction before he gets back. Licking the glue makes Edward a bit psychotic, so if Alex leaves the room there is a 20% chance that Edward will kill Pei-Hsin before she can start licking envelopes. Alex leaves the room and comes back to find Pei-Hsin dead. What is the probability that Edward was responsible?

Solution: There are two possibilities: either Edward killed Pei-Hsin or the envelopes did. The envelope could only be responsible if Edward was not, so the chances of that would be $4/5 \cdot 1/4 = 1/5$. This is the same as the probability that Edward killed her, so the events are equally likely and the answer is **50%**, or **1/2**.

Score 20 points for the correct answer, 5 points for a correct justification. Not many places to give partial credit.

Problem O3 [30 points]

If x , y , and z are distinct positive integers such that $x^2 + y^2 = z^3$, what is the smallest possible value of $x + y + z$.

Solution 1: Without loss of generality let $x > y$. We must have z^3 expressible as the sum of two squares, and this first happens when $z = 5$. Then x and y can be 10 and 5 or 11 and 2. If $z > 5$ then $z \geq 10$ for z^3 to be a sum of two distinct squares, so $x^2 > 500$, $x > 22$, so $x + y + z > 32$. Thus the smallest possible value of $x + y + z$ is $11 + 2 + 5 = \mathbf{18}$.

Solution 2: If $z > 5$, then $z \geq 6$, so $z^3 \geq 216$. Now $x^2 + y^2 \geq 216$, so $x \geq 11$ and $y \geq 1$, thus $x + y + z \geq 18$. Since $x = 11, y = 1, z = 6$ does not work, we must have $x + y + z > 18$, and the solution given is the best possible.

Score 20 points for the correct answer, 5 points for justifying that we can't do better with $z \leq 5$, 5 points for justifying that we can't do better with $z > 5$.

Problem O4 [35 points]

Evaluate $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$, where $\cos \theta = \frac{1}{5}$.

Solution: $\cos n\theta$ is the real part of $e^{in\theta}$, so the sum is the real part of $\sum_{n=0}^{\infty} \frac{e^{in\theta}}{2^n}$. This is a geometric series with initial term 1 and ratio $\frac{e^{i\theta}}{2}$, so its sum is $\frac{1}{1-e^{i\theta}/2}$. We are given $\cos \theta = \frac{1}{5}$, so $\sin \theta = \pm \frac{2\sqrt{6}}{5}$. Thus the sum is $\frac{10}{10-1\mp 2i\sqrt{6}} = \frac{90\pm 20i\sqrt{6}}{105}$, and the real part is $\frac{6}{7}$.

Score 20 points for the correct answer, 15 points for a correct justification.

Problem O5 [45 points]

Let r be the radius of the inscribed circle of triangle ABC . Take a point D on side BC , and let r_1 and r_2 be the inradii of triangles ABD and ACD . Prove that r , r_1 , and r_2 can always be the side lengths of a triangle.

Solution: We must show that r , r_1 , and r_2 satisfy the triangle inequality, i.e. that the sum of any two of them exceeds the third. Clearly r is the largest of the three, so we need only verify that $r_1 + r_2 > r$. Let K and s be the area and semiperimeter of triangle ABC . Similarly define K_1 , K_2 , s_1 , and s_2 . Observe that s is larger than s_1 or s_2 and that $K_1 + K_2 = K$. While these facts are almost trivial to verify, they must be stated. Then $r = K/s$, $r_1 = K_1/s_1$, and $r_2 = K_2/s_2$, so $r_1 + r_2 = K_1/s_1 + K_2/s_2 > K_1/s + K_2/s = K/s = r$.

The correct use of areas and semiperimeters is worth 25 points, each of the critical facts is worth 10 points. I don't know of any other way to do this problem, so attempts at alternate proofs should get at most 15 points for effort unless they really are on the right track to another solution.

Problem O6 [45 points]

You want to sort the numbers 5 4 3 2 1 using block moves. In other words, you can take any set of numbers that appear consecutively and put them back in at any spot as a block. For example, 6 5 3 4 2 1 \rightarrow 4 2 6 5 3 1 is a valid block move for 6 numbers. What is the minimum number of block moves necessary to get 1 2 3 4 5?

Solution 1: Here is a sequence of 3 moves that works: 54321 \rightarrow 32541 \rightarrow 34125 \rightarrow 12345. But how do we know we can't do it in 2 moves? From any position there are 20 possible permutations via block moves, 16 from moving a block of size 1 and 4 from moving a block of size 2. One could simply write the 20 permutations of 54321 and the 20 permutations of 12345 and try to see that they have nothing in common, which would suffice since the inverse of a block move is also a block move. A more clever method is to notice that if we could sort 54321 in 2 moves then we could sort 4321 in 2 moves as well by simply deleting the 5 from each step. But 4321 has only 10 permutations from block moves, namely 3421, 3241, 3214, 4231, 4213, 2431, 4312, 1432, 4132, and 2143. The 10

permutations of 1234 are 2134, 2314, 2341, 1324, 1342, 3124, 1243, 4123, 1423, and 3412. These two sets of permutations have nothing in common, thus it takes at least 3 moves to sort 4321, and hence at least **3** moves to sort 54321.

Solution 2: There is a more elegant way to show we need at least 3 moves. Given a permutation of $\{1, 2, 3, 4, 5\}$ (or any ordered set), define a *descent* to be an adjacent pair of numbers in the permutation such that the left number is greater than the right one. For example, 12345, 34215, and 54321 have 0, 2, and 4 descents, respectively. Any permutation obtained from 12345 by one block move has (at most) one descent, at the left edge of the moved block. Similarly, any permutation obtained from 54321 by one block move has (at least) three descents, so that we can't get from 54321 to 12345 by two block moves.

Score 20 points for the correct answer, 10 points for a numerical example proving that is attainable, and 15 points for proving it can't be done in 2 or fewer moves.

Problem O7 [55 points]

Evaluate $\sum_{n=1}^{\infty} \frac{n^5}{n!}$.

Solution: We start by noticing that $\sum_{n=1}^{\infty} \frac{n}{n!} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} = e$. Next we see that $\sum_{n=1}^{\infty} \frac{n^2}{n!} = \sum_{n=1}^{\infty} \frac{n}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1+n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{n}{n!} = e + e = 2e$. Let $f(k) = \sum_{n=1}^{\infty} \frac{n^k}{n!}$, then $\sum_{n=1}^{\infty} \frac{n^k}{n!} = \sum_{n=1}^{\infty} \frac{n^{k-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{(1+n)^{k-1}}{n!}$, so by the binomial theorem $f(k) = \sum_{j=0}^{k-1} \binom{k-1}{j} \cdot f(j)$. Armed with this formula, we can easily compute $f(3) = f(0) + 2f(1) + f(2) = e + 2e + 2e = 5e$, $f(4) = 1 \cdot e + 3 \cdot e + 3 \cdot 2e + 1 \cdot 5e = 15e$, and $f(5) = 1 \cdot e + 4 \cdot e + 6 \cdot 2e + 4 \cdot 5e + 1 \cdot 15e = \mathbf{52e}$.

Score 30 points for the correct answer, 25 points for a correct justification. If on the right track with a good justification, but an arithmetic error is made along the way, score 5 points for each $f(j), j = 0, 1, 2, 3, 4$, correctly computed.

Problem O8 [55 points]

What is the smallest square-free composite number that can divide a number of the form $4242 \dots 42 \pm 1$?

Solution: It is easy to see that such a number can never be divisible by 2, 3, 5, or 7. They can be divisible by 11, the smallest example being $4242424241 = 11 \cdot 547 \cdot 705073$. What makes this problem hard is finding the next prime that can divide such a number. Let $T_n = \sum_{i=0}^n 42 \cdot 10^{2i}$. Then the numbers T_n modulo a prime p will always be periodic, since $T_n = 100T_{n-1} + 42$, so we just need to compute one period and see if it contains ± 1 . Thus we find that modulo 13 we get 3, 4, 0, 3, \dots , modulo 17 we get 8, 9, 7, 11, 3, 2, 4, 0, 8, \dots , modulo 19 we get 4, 5, 10, 16, 8, 6, 15, 3, 0, 4, \dots , and modulo 23 we get 19, 10, 7, 6, 21, 3, 20, 18, 2, 12, 0, 19, \dots , so none of these primes can ever divide $T_n \pm 1$. But $424241 = 29 \cdot 14629$, so 29 can also divide numbers of this form. Therefore the smallest composite number that can divide $T_n \pm 1$ for some n is **319**, and the smallest such n is 83.

Score 30 points for the correct answer, 15 points for showing it is the smallest, 10 points for showing it does work. Just seeing that 11 is the smallest prime divisor is worth 5 points, finding for exactly which n is worth 5 more.

Problem O9 [60 points]

You are somewhere on a ladder with 5 rungs. You have a fair coin and an envelope that contains either a double-headed coin or a double-tailed coin, each with probability $1/2$. Every minute you flip a coin. If it lands heads you go up a rung, if it lands tails you go down a rung. If you move up from the top rung you win, if you move down from the bottom rung you lose. You can open the envelope at any time, but if you do then you must immediately flip that coin once, after which you can use it or the fair coin whenever you want. What is the best strategy (i.e. on what rung(s) should you open the envelope)?

Solution: First consider the probability of winning if you never open the envelope. Let $q(n)$ be the probability of winning from the n th rung with just the fair coin, then $q(n) = \frac{q(n-1)+q(n+1)}{2}$, so it is not hard to calculate that $q(n) = n/6$. If we open the envelope, then there's a $1/2$ chance that it is heads and we win, and a $1/2$ chance that it is tails and we end up one rung down with just the fair coin (obviously we keep using the double sided coin iff it is double headed). Let us start by analyzing rung 1. If we don't open the envelope, then we have a $1/2$ chance of losing and a $1/2$ chance of ending up on rung 2 with the envelope. If we do open the envelope, then we have a $1/2$ chance of losing and a $1/2$ chance of winning, which is a better outcome, so we should **open** the envelope on rung 1. Next we look at rung 5. If we don't open the envelope, then we have a $1/2$ chance of winning and a $1/2$ chance of moving down to rung 4 with the envelope. If we do open the envelope, then we we have a $1/2$ chance of winning and a $1/2$ chance of moving down to rung 4 without the envelope. Let $p(n)$ be the probability of winning from rung n if we are there with the envelope still unopened. Then clearly $p(n) \geq q(n)$ for all n if we're using optimal strategy, so we should **not open** the envelope on rung 5. Next we look at rung 4. If we open the envelope, then our chance of winning is $1/2 + q(3)/2 = 3/4$. If we don't, then our chance of winning is $p(5)/2 + p(3)/2$. We do know that $p(5) = 1/2 + p(4)/2$, but this is not enough to tell us what to do on rung 4. Looking at rung 3, we can open the envelope for a probability $1/2 + q(2)/2 = 2/3$ of winning, and we can not open the envelope for a probability $p(4)/2 + p(2)/2$ of winning. On rung 2, we can open the envelope for a probability $1/2 + q(1)/2 = 7/12$ of winning, and we can not open the envelope for a probability $p(3)/2 + p(1)/2 = p(3)/2 + 1/4$ of winning.

Now we can use all this information together for the complete answer. We know $p(2) \geq 7/12$, therefore $p(3) \geq p(4)/2 + 7/24$, and we know $p(4) \geq p(5)/2 + p(3)/2 \geq \frac{1/2+p(4)/2}{2} + p(3)/2 \geq \frac{1/2+p(4)/2}{2} + \frac{p(4)/2+7/24}{2}$. Isolating $p(4)$ in this inequality, we get $p(4) \geq 1/2+7/24 > 3/4$, therefore we should **not open** the envelope on rung 4. Now from $p(4) = p(5)/2+p(3)/2$ and $p(5) = 1/2+p(4)/2$ we have $p(4) = \frac{1/2+p(4)/2}{2} + p(3)/2$, so $p(3) = 3p(4)/2 - 1/2 \geq 11/16 > 2/3$, so we should **not open** the envelope on rung 3. Now $p(2) \geq p(3)/2 + 1/4 \geq 19/32 > 7/12$, so we should **not open** the envelope on rung 2. Therefore the best strategy is to open the envelope iff we are on the bottom rung.

For each rung, score 4 points for the correct answer and 8 more for a correct justification.

Problem O10 [75 points]

A, B, C, D , and E are relatively prime integers (i.e., have no single common factor) such that the polynomials $5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$ and $10Ax^3 + 6Bx^2 + 3Cx + D$ together have 7 distinct integer roots. What are all possible values of A ? Your team has been given a sealed envelope that contains a hint for this problem. If you open the envelope, the value of this problem decreases by 20 points. To get full credit, give the sealed envelope to the judge before presenting your solution.

Hint: Consider $A = 1, B = D = 0, C = 750$, and $E = 19845$.

Solution: Call the negatives of the roots of the first polynomial a, b, c, d , and the negatives of the roots of the second polynomial e, f, g (using the negatives avoids negative signs for the rest of the proof, thus preventing the possibility of dropping a sign). Then $5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E = 5A(x+a)(x+b)(x+c)(x+d)$ and $10Ax^3 + 6Bx^2 + 3Cx + D = 10A(x+e)(x+f)(x+g)$. Thus $B = \frac{5}{4}A(a+b+c+d) = \frac{5}{3}A(e+f+g)$, $C = \frac{5}{3}A(ab+ac+ad+bc+bd+cd) = \frac{10}{3}A(ef+eg+fg)$, $D = \frac{5}{2}A(abc+abd+acd+bcd) = 10A(efg)$, and $E = 5A(abcd)$. From these equations we see that since all the variables are integers, it must be the case that $A|B$, $A|3C$, $A|D$, and $A|E$, therefore A, B, C, D , and E can only be relatively prime if A is ± 1 or ± 3 . Now we need numerical examples to show that both of these are possible. Without loss of generality let $g = 0$, so $D = 0$. Then e and f are $\frac{-3B \pm \sqrt{9B^2 - 30AC}}{10A}$, so $9B^2 - 30AC$ must be a perfect square. Let $B = 0$ in the hope that solutions will still exist to this simplified problem. First let us try to find an example with $A = -1$, so we need $30C$ to be a perfect square. This first happens for $C = 30$, and in that case e and f are ± 3 . We need $a + b + c + d = 0$, so let's try to look for $a = -b, c = -d$. This doesn't work for $C = 30$ since $3C/5 = 18$ is not the sum of two distinct squares. For that we will need to try $C = 5^2 \cdot 30 = 750$, for which we get $e, f = \pm 15, a, b = \pm 3$, and $c, d = \pm 21$. Thus for $A = \pm 1$ we can use $B = D = 0, C = \mp 750$, and $E = \pm 19845$. Similarly we can find $A = \pm 3, B = D = 0, C = \mp 250$, and $E = \pm 735$, for which $(a, b, c, d, e, f, g) = (1, -1, 7, -7, 5, -5, 0)$.

Note that these give us quintic polynomials with integer coefficients possessing 4 relative extrema and 3 points of inflection at lattice points, such as $3x^5 - 250x^3 + 735x$.

Score 20 points for the correct answer, 15 points for a correct justification, 20 points for a numerical example for ± 1 , 20 points for a numerical example for ± 3 . If the sealed envelope isn't presented at the beginning of the solution, no credit is given for the ± 1 example. If the \pm is forgotten, give 10 points for the answer, 10 for the justification, and 10 for each numerical example. There are infinitely many possible examples, so anything other than the two given above must be checked for accuracy.

Team Solutions

Harvard-MIT Math Tournament
February 27, 1999

Problem T1 [15]

A combination lock has a 3 number combination, with each number an integer between 0 and 39 inclusive. Call the numbers n_1 , n_2 , and n_3 . If you know that n_1 and n_3 leave the same remainder when divided by 4, and n_2 and $n_1 + 2$ leave the same remainder when divided by 4, how many possible combinations are there?

Solution: There are 40 choices for the last number, and for each of these we have 10 choices for each of the first two numbers, thus giving us a total of **4000** possible combinations. It is interesting to note that these restrictions are actually true for Master locks.

Problem T2 [15]

A ladder is leaning against a house with its lower end 15 feet from the house. When the lower end is pulled 9 feet farther from the house, the upper end slides 13 feet down. How long is the ladder (in feet)?

Solution: Of course the house makes a right angle with the ground, so we can use the Pythagorean theorem. Let x be the length of the ladder and y be the original height at which it touched the house. Then we are given $x^2 = 15^2 + y^2 = 24^2 + (y - 13)^2$. Isolating y in the second equation we get $y = 20$, thus x is **25**.

Problem T3 [20]

How many non-empty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have exactly k elements and do not contain the element k for some $k = 1, 2, \dots, 8$.

Solution: Probably the easiest way to do this problem is to count how many non-empty subsets of $\{1, 2, \dots, n\}$ have k elements and do contain the element k for some k . The element k must have $k - 1$ other elements with it to be in a subset of k elements, so there are $\binom{n-1}{k-1}$ such subsets. Now $\sum_{k=1}^n \binom{n-1}{k-1} = (1 + 1)^{n-1} = 2^{n-1}$, so that is how many non-empty sets contain some k and have k elements. The set $\{1, 2, \dots, n\}$ has 2^n subsets (each element either is or is not in a particular subset), one of which is the empty set, so the number of non-empty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have exactly k elements and do not contain the element k for some k is $2^n - 2^{n-1} - 1 = 2^{n-1} - 1$. In the case $n = 8$, this yields **127**.

Problem T4 [20]

Consider the equation $FORTY + TEN + TEN = SIXTY$, where each of the ten letters represents a distinct digit from 0 to 9. Find all possible values of $SIXTY$.

Solution: Since $Y + N + N$ ends in Y , N must be 0 or 5. But if $N = 5$ then $T + E + E + 1$ ends in T , which is impossible, so $N = 0$ and $E = 5$. Since $F \neq S$ we must have $O = 9$, $R + T + T + 1 > 10$, and $S = F + 1$. Now $I \neq 0$, so it must be that $I = 1$ and $R + T + T + 1 > 20$. Thus R and T are 6 and 7, 6 and 8, or 7 and 8 in some order. But X can't be 0 or 1 since those are taken, and X cannot be 3 since F and S have to be consecutive, so it must be that $R + T + T + 1$ is 21 or 23. This is satisfied only for $R = 7, T = 8$, so $F = 2, S = 3$, and $Y = 6$. This *SIXTY* = **31486**.

Problem T5 [30]

If a and b are randomly selected real numbers between 0 and 1, find the probability that the nearest integer to $\frac{a-b}{a+b}$ is odd.

Solution: The only reasonable way I know of to do this problem is geometrically (yes, you can use integrals to find the areas of the triangles involved, but I don't consider that reasonable). First let us find the points (a, b) in the plane for which the nearest integer to $\frac{a-b}{a+b}$ is 0, i.e. $-\frac{1}{2} \leq \frac{a-b}{a+b} \leq \frac{1}{2}$. Taking the inequalities one at a time, $-\frac{1}{2} \leq \frac{a-b}{a+b}$ implies that $a + b \geq 2(b - a)$, or $b \leq 3a$, so these points must lie below the line $y = 3x$. Similarly, $\frac{a-b}{a+b} \leq \frac{1}{2}$ implies that (a, b) must lie above the line $y = \frac{1}{3}x$. Now we can look for the points (a, b) for which the nearest integer to $\frac{a-b}{a+b}$ is 1, i.e. $\frac{1}{2} \leq \frac{a-b}{a+b} \leq \frac{3}{2}$, and we find that all points in the first quadrant that lie above the line $y = 3x$ satisfy this inequality. Similarly, the closest integer to $\frac{a-b}{a+b}$ is -1 for all points in the first quadrant below the line $y = \frac{1}{3}x$. For a and b between 0 and 1, the locus of points (a, b) for which the nearest integer to $\frac{a-b}{a+b}$ is odd is two right triangles with legs of length 1 and $\frac{1}{3}$, so together they have area $\frac{1}{3}$. The locus of all points (a, b) with a and b between 0 and 1 is a square of side length 1, and thus has area 1. Therefore the probability that the nearest integer to $\frac{a-b}{a+b}$ is odd is $\frac{1}{3}$.

Problem T6 [30]

Reduce the number $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$.

Solution: Observe that $(\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}})^3 = (2 + \sqrt{5}) - 3(\sqrt[3]{2 + \sqrt{5}}) - 3(\sqrt[3]{2 - \sqrt{5}}) + (2 - \sqrt{5}) = 4 - 3(\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}})$. Hence $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}}$ is a root of the cubic $x^3 + 3x - 4 = (x - 1)(x^2 + x + 4)$. The roots of $x^2 + x + 4$ are imaginary, so $\sqrt[3]{2 + \sqrt{5}} + \sqrt[3]{2 - \sqrt{5}} = \mathbf{1}$.

Problem T7 [30]

Let $\frac{1}{1-x-x^2-x^3} = \sum_{i=0}^{\infty} a_n x^n$, for what positive integers n does $a_{n-1} = n^2$?

Solution: Multiplying both sides by $1 - x - x^2 - x^3$ the right hand side becomes $a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + \dots$, and setting coefficients of x^n equal to each other we find that $a_0 = 1, a_1 = 1, a_2 = 2$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$. Thus the sequence of a_n 's starts 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, \dots . So we now see that $a_0 = 1^2$ and $a_8 = 9^2$. What makes it impossible for this to happen again is that the sequence is growing exponentially. It will suffice to show that $a_n > 1.5^n$ for $n > 2$, since $n^2 / (n - 1)^2 < 1.5$ for $n \geq 6$, thus when a_{n-1} exceeds n^2 at $n = 10$ there can be no more solutions to $a_{n-1} = n^2$. Observe that $a_n > 1.5a_{n-1}$ for $n = 3, 4, 5$. By way of induction, assume it for $n - 2$,

$n-1$, and n , then $a_{n+1} = a_n + a_{n-1} + a_{n-2} > 1.5^n + 1.5^{n-1} + 1.5^{n-2} = 1.5^{n-2}(1 + 1.5 + 1.5^2) > 1.5^{n+1}$. Thus, by induction, $a_n > 1.5^n$ for $n > 2$, so the only solutions are **1, 9**.

Problem T8 [35]

Find all the roots of $(x^2 + 3x + 2)(x^2 - 7x + 12)(x^2 - 2x - 1) + 24 = 0$.

Solution: We re-factor as $(x + 1)(x - 3)(x + 2)(x - 4)(x^2 - 2x - 1) + 24$, or $(x^2 - 2x - 3)(x^2 - 2x - 8)(x^2 - 2x - 1) + 24$, and this becomes $(y - 4)(y - 9)(y - 2) + 24$ where $y = (x - 1)^2$. Now, $(y - 4)(y - 9)(y - 2) + 24 = (y - 8)(y - 6)(y - 1)$, so y is 1, 6, or 8. Thus the roots of the original polynomial are **0, 2, $1 \pm \sqrt{6}$, $1 \pm 2\sqrt{2}$** .

Problem T9 [40]

Evaluate $\sum_{n=2}^{17} \frac{n^2+n+1}{n^4+2n^3-n^2-2n}$.

Solution: Observe that the denominator $n^4 + 2n^3 - n^2 - 2n = n(n-1)(n+1)(n+2)$. Thus we can rewrite the fraction as $\frac{n^2-n+1}{n^4+2n^3-n^2-2n} = \frac{a}{n-1} + \frac{b}{n} + \frac{c}{n+1} + \frac{d}{n+2}$ for some real numbers a, b, c , and d . This method is called *partial fractions*. Condensing the right hand side as a fraction over $n^4+2n^3-n^2-2n$ we get $n^2 - n + 1 = a(n^3 + 3n^2 + 2n) + b(n^3 + 2n^2 - n - 2) + c(n^3 + n^2 - 2n) + d(n^3 - n)$. Comparing coefficients of each power of n we get $a+b+c+d = 0$, $3a+2b+c = 2$, $2a-b-2c-d = 2$, and $-2b = 2$. This is a system of 4 equations in 4 variables, and its solution is $a = 1/2$, $b = -1/2$, $c = 1/2$, and $d = -1/2$. Thus the summation becomes $\frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots + \frac{1}{16} - \frac{1}{17} + \frac{1}{18} - \frac{1}{19})$. Notice that almost everything cancels to leave us with $\frac{1}{2}(1 + \frac{1}{3} - \frac{1}{17} - \frac{1}{19}) = \frac{592}{969}$.

Problem T10 [45]

If 5 points are placed in the plane at lattice points (i.e. points (x, y) where x and y are both integers) such that no three are collinear, then there are 10 triangles whose vertices are among these points. What is the minimum possible number of these triangles that have area greater than $1/2$?

Solution: By the pigeonhole principle, the 5 points cannot all be distinct modulo 2, so two of them must have a midpoint that is also a lattice point. This midpoint is not one of the 5 since no 3 are collinear. Pick's theorem states that the area of a polygon whose vertices are lattice points is $B/2 + I - 1$ where B is the number of lattice points on the boundary and I is the number in the interior. Thus those two points form the base of 3 triangles whose area will be greater than $1/2$ by Pick's theorem since there are 4 lattice points on the boundary. Now it also turns out that at least one of the triangles must contain a lattice point, thus giving us a fourth triangle with area greater than $1/2$. This is actually pretty easy to show with the aid of a picture or some visualization. Suppose we have 4 points and we're trying to find a 5th one so that no triangle will contain an interior lattice point. The 4 lattice points must form a quadrilateral of area 1, so in fact it is a parallelogram (think deeply about it). Draw the four sides, extending them throughout the plain. Each vertex is now the tip of an infinite triangular region of the plane, and if the 5th lattice point is chosen in that region then the triangle formed by the 5th point and the two vertices of the parallelogram adjacent to the one we are considering will form a triangle containing the vertex we

are considering. But the part of the plane that isn't in one of these 4 regions contains no lattice points or else we could draw a parallelogram congruent to the first one with lattice point vertices and containing that lattice point, but that would violate Pick's theorem since the parallelogram has area 1. Therefore we must have a fourth triangle with area greater than $1/2$ (one must justify that this really is in addition to the 3 triangles we already knew we'd get). An example that achieves this minimum is the points $(0,0)$, $(1,0)$, $(1,1)$, $(2,1)$, and $(2,-1)$. Therefore the minimum possible number of these triangles that have area greater than $1/2$ is **4**.

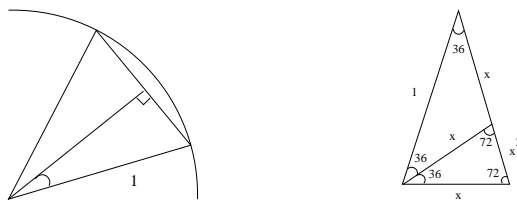
A less trivial example that achieves the minimum is $(0,0)$, $(1,1)$, $(2,1)$, $(3,2)$, and $(7,5)$.

Problem T11 [55]

Circles C_1, C_2, C_3 have radius 1 and centers O, P, Q respectively. C_1 and C_2 intersect at A , C_2 and C_3 intersect at B , C_3 and C_1 intersect at C , in such a way that $\angle APB = 60^\circ$, $\angle BQC = 36^\circ$, and $\angle COA = 72^\circ$. Find angle ABC (degrees).

Solution: Using a little trig, we have $BC = 2 \sin 18$, $AC = 2 \sin 36$, and $AB = 2 \sin 30$ (see left diagram). Call these a, b , and c , respectively. By the law of cosines, $b^2 = a^2 + c^2 - 2ac \cos ABC$, therefore $\cos ABC = \frac{\sin^2 18 + \sin^2 30 - \sin^2 36}{2 \sin 18 \sin 30}$. In the right diagram below we let $x = 2 \sin 18$ and see that $x + x^2 = 1$, hence $\sin 18 = \frac{-1 + \sqrt{5}}{4}$. Using whatever trig identities you prefer you can find that $\sin^2 36 = \frac{5 - \sqrt{5}}{4}$, and of course $\sin 30 = \frac{1}{2}$. Now simplification yields $\sin^2 18 + \sin^2 30 - \sin^2 36 = 0$, so $\angle ABC = 90^\circ$.

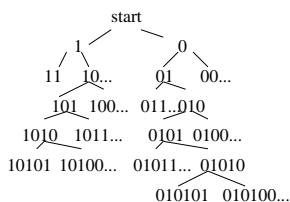
Note that this means that if a regular pentagon, hexagon, and decagon are inscribed in a circle, then we can take one side from each and form a right triangle.



Problem T12 [65]

A fair coin is flipped every second and the results are recorded with 1 meaning heads and 0 meaning tails. What is the probability that the sequence 10101 occurs before the first occurrence of the sequence 010101?

Solution: Call it a win if we reach 10101, a loss if we reach 010101. Let x be the probability of winning if the first flip is a 1, let y be the probability of winning if the first flip is a 0. Then the probability of winning is $(x + y)/2$ since the first flip is 1 or 0, each with probability $1/2$. If we ever get two 1's in a row, that is the same as starting with a 1 as far as the probability is concerned. Similarly, if we get two 0's in a row, then we might as well have started with a single 0.



From the tree of all possible sequences, shown above, in which the probability of moving along any particular line is $1/2$, we see that $x = x(1/2 + 1/8) + y(1/4 + 1/16) + 1/16$, and $y = x(1/4 + 1/16) + y(1/2 + 1/8 + 1/32)$. Solving these two equations in two unknowns we get $x = 11/16$ and $y = 5/8$. Therefore the probability that the sequence 10101 occurs before the first occurrence of the sequence 010101 is **21/32**.

Advanced Topics

1. Evaluate

$$\sin(1998^\circ + 237^\circ) \sin(1998^\circ - 1653^\circ).$$

Answer: $\boxed{-\frac{1}{4}}$. We have $\sin(1998^\circ + 237^\circ) \sin(1998^\circ - 1653^\circ) = \sin(2235^\circ) \sin(345^\circ) = \sin(75^\circ) \sin(-15^\circ) = -\sin(75^\circ) \sin(15^\circ) = -\sin(15^\circ) \cos(15^\circ) = -\frac{\sin(30^\circ)}{2} = -\frac{1}{4}$.

2. How many values of x , $-19 < x < 98$, satisfy

$$\cos^2 x + 2 \sin^2 x = 1?$$

Answer: $\boxed{38}$. For any x , $\sin^2 x + \cos^2 x = 1$. Subtracting this from the given equation gives $\sin^2 x = 0$, or $\sin x = 0$. Thus x must be a multiple of π , so $-19 < k\pi < 98$ for some integer k , or approximately $-6.1 < k < 31.2$. There are 32 values of k that satisfy this, so there are 32 values of x that satisfy $\cos^2 x + 2 \sin^2 x = 1$.

3. Find the sum of the infinite series

$$1 + 2\left(\frac{1}{1998}\right) + 3\left(\frac{1}{1998}\right)^2 + 4\left(\frac{1}{1998}\right)^3 + \dots$$

Answer: $\boxed{\left(\frac{1998}{1997}\right)^2}$ or $\frac{3992004}{3988009}$. We can rewrite the sum as

$$\left(1 + \frac{1}{1998} + \left(\frac{1}{1998}\right)^2 + \dots\right) + \left(\frac{1}{1998} + \left(\frac{1}{1998}\right)^2 + \left(\frac{1}{1998}\right)^3 + \dots\right) + \left(\left(\frac{1}{1998}\right)^2 + \left(\frac{1}{1998}\right)^3 + \dots\right) + \dots$$

Evaluating each of the infinite sums gives

$$\frac{1}{1 - \frac{1}{1998}} + \frac{\frac{1}{1998}}{1 - \frac{1}{1998}} + \frac{\left(\frac{1}{1998}\right)^2}{1 - \frac{1}{1998}} + \dots = \frac{1998}{1997} \cdot \left(1 + \frac{1}{1998} + \left(\frac{1}{1998}\right)^2 + \dots\right) = \frac{1998}{1997} \cdot \left(1 + \frac{1}{1998} + \left(\frac{1}{1998}\right)^2 + \dots\right),$$

which is equal to $\left(\frac{1998}{1997}\right)^2$, or $\frac{3992004}{3988009}$, as desired.

4. Find the range of

$$f(A) = \frac{(\sin A)(3 \cos^2 A + \cos^4 A + 3 \sin^2 A + (\sin^2 A)(\cos^2 A))}{(\tan A)(\sec A - (\sin A)(\tan A))}$$

if $A \neq \frac{n\pi}{2}$.

Answer: $\boxed{(3, 4)}$. We factor the numerator and write the denominator in term of fractions to get

$$\frac{(\sin A)(3 + \cos^2 A)(\sin^2 A + \cos^2 A)}{\left(\frac{\sin A}{\cos A}\right)\left(\frac{1}{\cos A} - \frac{\sin^2 A}{\cos A}\right)} = \frac{(\sin A)(3 + \cos^2 A)(\sin^2 A + \cos^2 A)}{\frac{(\sin A)(1 - \sin^2 A)}{\cos^2 A}}.$$

Because $\sin^2 A + \cos^2 A = 1$, $1 - \sin^2 A = \cos^2 A$, so the expression is simply equal to $3 + \cos^2 A$. The range of $\cos^2 A$ is $(0, 1)$ (0 and 1 are not included because $A \neq \frac{n\pi}{2}$, so the range of $3 + \cos^2 A$ is $(3, 4)$.

5. How many positive integers less than 1998 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

Answer: $\boxed{1487}$. The factorization of 1547 is $7 \cdot 13 \cdot 17$, so we wish to find the number of positive integers less than 1998 that are not divisible by 7, 13, or 17. By the Principle of Inclusion-Exclusion, we first subtract the numbers that are divisible by one of 7, 13, and 17, add back those that are divisible by two of 7, 13, and 17, then subtract those divisible by three of them. That is,

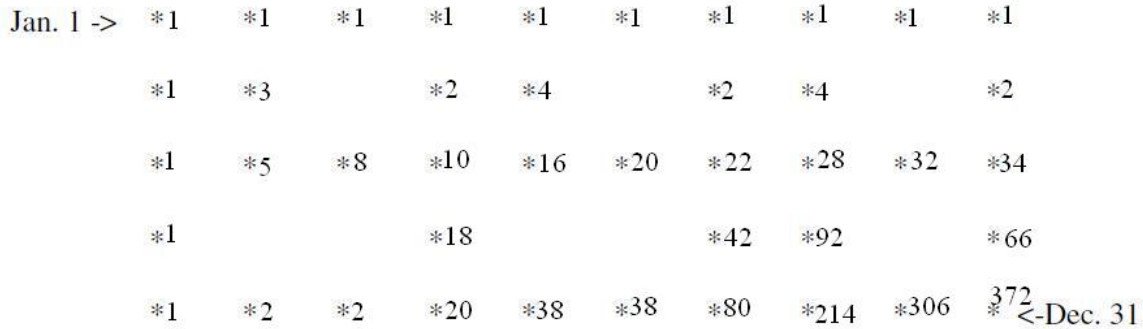
$$1997 - \left\lfloor \frac{1997}{7} \right\rfloor - \left\lfloor \frac{1997}{13} \right\rfloor - \left\lfloor \frac{1997}{17} \right\rfloor + \left\lfloor \frac{1997}{7 \cdot 13} \right\rfloor + \left\lfloor \frac{1997}{7 \cdot 17} \right\rfloor + \left\lfloor \frac{1997}{13 \cdot 17} \right\rfloor - \left\lfloor \frac{1997}{7 \cdot 13 \cdot 17} \right\rfloor,$$

or 1487.

6. In the diagram below, how many distinct paths are there from January 1 to December 31, moving from one adjacent dot to the next either to the right, down, or diagonally down to the right?



Answer: 372. For each dot in the diagram, we can count the number of paths from January 1 to it by adding the number of ways to get to the dots to the left of it, above it, and above and to the left of it, starting from the topmost leftmost dot. This yields the following numbers of paths:



So the number of paths from January 1 to December 31 is 372.

7. The Houston Association of Mathematics Educators decides to hold a grand forum on mathematics education and invites a number of politicians from the United States to participate. Around lunch time the politicians decide to play a game. In this game, players can score 19 points for pegging the coordinator of the gathering with a spit ball, 9 points for downing an entire cup of the forum's interpretation of coffee, or 8 points for quoting more than three consecutive words from the speech Senator Bobbo delivered before lunch. What is the product of the two greatest scores that a player **cannot** score in this game?

Answer: 1209. Attainable scores are positive integers that can be written in the form $8a + 9b + 19c$, where a , b , and c are nonnegative integers. Consider attainable number of points modulo 8.

Scores that are $0 \pmod{8}$ can be obtained with $8a$ for positive a .

Scores that are $1 \pmod{8}$ greater than or equal to 9 can be obtained with $9 + 8a$ for nonnegative a .

Scores that are $2 \pmod{8}$ greater than or equal to 18 can be obtained with $9 \cdot 2 + 8a$.

Scores that are $3 \pmod{8}$ greater than or equal to 19 can be obtained with $19 + 8a$.

Scores that are $4 \pmod{8}$ greater than or equal to $19 + 9 = 28$ can be obtained with $19 + 9 + 8a$.

Scores that are $5 \pmod{8}$ greater than or equal to $19 + 9 \cdot 2 = 37$ can be obtained with $19 + 9 \cdot 2 + 8a$.

Scores that are $6 \pmod{8}$ greater than or equal to $19 \cdot 2 = 38$ can be obtained with $19 \cdot 2 + 8a$.

Scores that are $7 \pmod{8}$ greater than or equal to $19 \cdot 2 + 9 = 47$ can be obtained with $19 \cdot 2 + 9 + 8a$.

So the largest two unachievable values are 39 and 31. Multiplying them gives 1209.

8. Given any two positive real numbers x and y , then $x \diamond y$ is a positive real number defined in terms of x and y by some fixed rule. Suppose the operation $x \diamond y$ satisfies the equations $(x \cdot y) \diamond y = x(y \diamond y)$ and $(x \diamond 1) \diamond x = x \diamond 1$ for all $x, y > 0$. Given that $1 \diamond 1 = 1$, find $19 \diamond 98$.

Answer: $\boxed{19}$. Note first that $x \diamond 1 = (x \cdot 1) \diamond 1 = x \cdot (1 \diamond 1) = x \cdot 1 = x$. Also, $x \diamond x = (x \diamond 1) \diamond x = x \diamond 1 = x$. Now, we have $(x \cdot y) \diamond y = x \cdot (y \diamond y) = x \cdot y$. So $19 \diamond 98 = (\frac{19}{98} \cdot 98) \diamond 98 = \frac{19}{98} \cdot (98 \diamond 98) = \frac{19}{98} \cdot 98 = 19$.

9. Bob's Rice ID number has six digits, each a number from 1 to 9, and any digit can be used any number of times. The ID number satisfies the following property: the first two digits is a number divisible by 2, the first three digits is a number divisible by 3, etc. so that the ID number itself is divisible by 6. One ID number that satisfies this condition is 123252. How many different possibilities are there for Bob's ID number?

Answer: $\boxed{324}$. We will count the number of possibilities for each digit in Bob's ID number, then multiply them to find the total number of possibilities for Bob's ID number. There are 3 possibilities for the first digit given any last 5 digits, because the entire number must be divisible by 3, so the sum of the digits must be divisible by 3. Because the first two digits are a number divisible by 2, the second digit must be 2, 4, 6, or 8, which is 4 possibilities. Because the first five digits are a number divisible by 5, the fifth digit must be a 5. Now, if the fourth digit is a 2, then the last digit has two choices, 2, 8, and the third digit has 5 choices, 1, 3, 5, 7, 9. If the fourth digit is a 4, then the last digit must be a 6, and the third digit has 4 choices, 2, 4, 6, 8. If the fourth digit is a 6, then the last digit must be a 4, and the third digit has 5 choices, 1, 3, 5, 7, 9. If the fourth digit is an 8, then the last digit has two choices, 2, 8, and the third digit has 4 choices, 2, 4, 6, 8. So there are a total of $3 \cdot 4 (2 \cdot 5 + 4 + 5 + 2 \cdot 4) = 3 \cdot 4 \cdot 27 = 324$ possibilities for Bob's ID number.

10. In the fourth annual Swirled Series, the Oakland Alphas are playing the San Francisco Gammas. The first game is played in San Francisco and succeeding games alternate in location. San Francisco has a 50% chance of winning their home games, while Oakland has a probability of 60% of winning at home. Normally, the series will stretch on forever until one team gets a three-game lead, in which case they are declared the winners. However, after each game in San Francisco there is a 50% chance of an earthquake, which will cause the series to end with the team that has won more games declared the winner. What is the probability that the Gammas will win?

Answer: $\boxed{\frac{34}{73}}$. Let $F(x)$ be the probability that the Gammas will win the series if they are ahead by x games and are about to play in San Francisco, and let $A(x)$ be the probability that the Gammas will win the series if they are ahead by x games and are about to play in Oakland. Then we have

$$\begin{aligned} F(2) &= \frac{3}{4} + \frac{A(1)}{4} \\ A(1) &= \frac{6F(0)}{10} + \frac{4F(2)}{10} \\ F(0) &= \frac{1}{4} + \frac{A(1)}{4} + \frac{A(-1)}{4} \\ A(-1) &= \frac{6F(-2)}{10} + \frac{4F(0)}{10} \\ F(-2) &= \frac{A(-1)}{4} \end{aligned}$$

Plugging $A(1) = \frac{6F(0)}{10} + \frac{4F(2)}{10}$ into $F(2) = \frac{3}{4} + \frac{A(1)}{4}$, we get

$$\begin{aligned} F(2) &= \frac{3}{4} + \frac{1}{4} \left(\frac{6F(0)}{10} + \frac{4F(2)}{10} \right) \\ \frac{9F(2)}{10} &= \frac{3}{4} + \frac{6F(0)}{40} \Leftrightarrow F(2) = \frac{5}{6} + \frac{F(0)}{6} \end{aligned}$$

Plugging $A(-1) = \frac{6F(-2)}{10} + \frac{4F(0)}{10}$ into $F(-2) = \frac{A(-1)}{4}$, we get

$$\frac{34A(-1)}{40} = \frac{4F(0)}{10} \Leftrightarrow F(-2) = \frac{2F(0)}{17}$$

Now,

$$F(0) = \frac{1}{4} + \frac{1}{4} \left(\frac{6F(0)}{10} + \frac{4F(2)}{10} \right) + \frac{1}{4} \left(\frac{6F(-2)}{10} + \frac{4F(0)}{10} \right)$$

This simplifies to $F(0) = \frac{1}{4} + \frac{F(0)}{4} + \frac{F(2)}{10} + \frac{6F(-2)}{40}$. Then, plugging our formulas in, we get

$$F(0) = \frac{1}{4} + \frac{F(0)}{4} + \frac{1}{10} \left(\frac{5}{6} + \frac{F(0)}{6} \right) + \frac{3F(0)}{170}.$$

$$\frac{73F(0)}{102} = \frac{1}{3} \Leftrightarrow F(0) = \frac{34}{73}.$$

Since $F(0)$ is the situation before the Series has started, the probability that the Gammas will win is $\frac{34}{73}$.

HMMT 1998: Algebra Solutions

1. *Problem:* The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is \$23.50. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is \$39.50. What is the cost of 2 hamburgers, 2 milk shakes, and 2 orders of fries at this restaurant?

Solution: Let H = hamburger, M = milk shake, and F = order of fries. Then $3H + 5M + F = \$23.50$. Multiplying the equation by 2 yields $6H + 10M + 2F = \$47$. Also, it is given that $5H + 9M + F = \$39.50$. Then subtracting the following equations

$$\begin{array}{r} 6H + 10M + 2F = \$47.00 \\ 5H + 9M + F = \$39.50 \end{array}$$

yields $H + M + F = \$7.50$. Multiplying the equation by 2 yields $2H + 2M + 2F = \boxed{\$15}$.

2. *Problem:* Bobbo starts swimming at 2 feet/s across a 100 foot wide river with a current of 5 feet/s. Bobbo doesn't know that there is a waterfall 175 feet from where he entered the river. He realizes his predicament midway across the river. What is the minimum speed that Bobbo must increase to make it to the other side of the river safely?

Solution: When Bobbo is midway across the river, he has travelled 50 feet. Going at a speed of 2 feet/s, this means that Bobbo has already been in the river for $\frac{50 \text{ feet}}{20 \text{ feet/s}} = 25$ s. Then he has traveled $5 \text{ feet/s} \cdot 25 \text{ s} = 125$ feet down the river. Then he has $175 \text{ feet} - 125 \text{ feet} = 50$ feet left to travel downstream before he hits the waterfall.

Bobbo travels at a rate of 5 feet/s downstream. Thus there are $\frac{50 \text{ feet}}{5 \text{ feet/s}} = 10$ s before he hits the waterfall. He still has to travel 50 feet horizontally across the river. Thus he must travel at a speed of $\frac{50 \text{ feet}}{10 \text{ s}} = 5$ feet/s. This is a $\boxed{3 \text{ feet/s}}$ difference from Bobbo's original speed of 2 feet/s.

3. *Problem:* Find the sum of every even positive integer less than 233 not divisible by 10.

Solution: We find the sum of all positive even integers less than 233 and then subtract all the positive integers less than 233 that are divisible by 10.

$2+4+\dots+232 = 2(1+2+\dots+116) = 116 \cdot 117 = 13572$. The sum of all positive integers less than 233 that are divisible by 10 is $10+20+\dots+230 = 10(1+2+\dots+23) = 2760$. Then our answer is $13572-2760 = \boxed{10812}$.

4. *Problem:* Given that r and s are relatively prime positive integers such that $\frac{r}{s} = \frac{2(\sqrt{2}+\sqrt{10})}{5(\sqrt{3}+\sqrt{5})}$, find r and s .

Solution: Squaring both sides of the given equation yields $\frac{r^2}{s^2} = \frac{4(12+4\sqrt{5})}{25(3+\sqrt{5})} = \frac{16(3+\sqrt{5})}{25(3+\sqrt{5})} = \frac{16}{25}$. Because r and s are positive and relatively prime, then by inspection, $\boxed{r = 4 \text{ and } s = 5}$.

5. *Problem:* A man named Juan has three rectangular solids, each having volume 128. Two of the faces of one solid have areas 4 and 32. Two faces of another solid have areas 64 and 16. Finally, two faces of the last solid have areas 8 and 32. What is the minimum possible exposed surface area of the tallest tower Juan can construct by stacking his solids one on top of the other, face to face? (Assume that the base of the tower is not exposed).

Solution: Suppose that x, y, z are the sides of the following solids. Then Volume = $xyz = 128$. For the first solid, without loss of generality (with respect to assigning lengths to x, y, z), $xy = 4$ and $. Then $xy^2z = 128$. Then $y = 1$. Solving the remaining equations yields $x = 4$ and $z = 32$. Then the first solid has dimensions $4 \times 1 \times 32$.$

For the second solid, without loss of generality, $xy = 64$ and $yz = 16$. Then $xy^2z = 1024$. Then $y = 8$. Solving the remaining equations yields $x = 8$ and $z = 2$. Then the second solid has dimensions $8 \times 8 \times 2$.

For the third solid, without loss of generality, $xy = 8$ and $yz = 32$. Then $y = 2$. Solving the remaining equations yields $x = 4$ and $z = 16$. Then the third solid has dimensions $4 \times 2 \times 16$.

To obtain the tallest structure, Juan must stack the boxes such that the longest side of each solid is oriented vertically. Then for the first solid, the base must be 1×4 , so that the side of length 32 can

contribute to the height of the structure. Similarly, for the second solid, the base must be 8×2 , so that the side of length 8 can contribute to the height. Finally, for the third solid, the base must be 4×2 . Thus the structure is stacked, from bottom to top: second solid, third solid, first solid. This order is necessary, so that the base of each solid will fit entirely on the top of the solid directly beneath it.

All the side faces of the solids contribute to the surface area of the final solid. The side faces of the bottom solid have area $8 \cdot (8 + 2 + 8 + 2) = 160$. The side faces of the middle solid have area $16 \cdot (4 + 2 + 4 + 2) = 192$. The side faces of the top solid have area $32 \cdot (4 + 1 + 4 + 1) = 320$.

Furthermore, the top faces of each of the solids are exposed. The top face of the bottom solid is partially obscured by the middle solid. Thus the total exposed area of the top face of the bottom solid is $8 \cdot 2 - 4 \cdot 2 = 8$. The top face of the middle solid is partially obscured by the top solid. Thus the total exposed area of the top face of the middle solid is $4 \cdot 2 - 4 \cdot 1 = 4$. The top face of the top solid is fully exposed. Thus the total exposed area of the top face of the top solid is $4 \cdot 1 = 4$.

Then the total surface area of the entire structure is $160 + 192 + 320 + 8 + 4 + 4 = \boxed{688}$.

6. *Problem:* How many pairs of positive integers (a, b) with $a \leq b$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

Solution: $\frac{1}{a} + \frac{1}{b} = \frac{1}{6} \Rightarrow \frac{a+b}{ab} = \frac{1}{6} \Rightarrow ab = 6a+6b \Rightarrow ab-6a-6b = 0$. Factoring yields $(a-6)(b-6) - 36 = 0$. Then $(a-6)(b-6) = 36$. Because a and b are positive integers, only the factor pairs of 36 are possible values of $a-6$ and $b-6$. The possible pairs are:

$$\begin{aligned} a-6 &= 1, b-6 = 36 \\ a-6 &= 2, b-6 = 18 \\ a-6 &= 3, b-6 = 12 \\ a-6 &= 4, b-6 = 9 \\ a-6 &= 6, b-6 = 6 \end{aligned}$$

Because $a \leq b$, the symmetric cases, such as $a-6 = 12, b-6 = 3$ are not applicable. Then there are $\boxed{5}$ possible pairs.

7. *Problem:* Given that three roots of $f(x) = x^4 + ax^2 + bx + c$ are 2, -3, and 5, what is the value of $a+b+c$?

Solution: By definition, the coefficient of x^3 is negative the sum of the roots. In $f(x)$, the coefficient of x^3 is 0. Thus the sum of the roots of $f(x)$ is 0. Then the fourth root is -4. Then $f(x) = (x-2)(x+3)(x-5)(x+4)$. Notice that $f(1)$ is $1 + a + b + c$. Thus our answer is $f(1) - 1 = (1-2)(1+3)(1-5)(1+4) - 1 = \boxed{79}$.

8. *Problem:* Find the set of solutions for x in the inequality $\frac{x+1}{x+2} > \frac{3x+4}{2x+9}$ when $x \neq -2, x \neq \frac{9}{2}$.

Solution: There are 3 possible cases of x : 1) $-\frac{9}{2} < x$, 2) $\frac{9}{2} \leq x \leq -2$, 3) $-2 < x$. For the cases (1) and (3), $x+2$ and $2x+9$ are both positive or negative, so the following operation can be carried out without changing the inequality sign:

$$\begin{aligned} \frac{x+1}{x+2} &> \frac{3x+4}{2x+9} \\ \Rightarrow 2x^2 + 11x + 9 &> 3x^2 + 10x + 8 \\ \Rightarrow 0 &> x^2 - x - 1 \end{aligned}$$

The inequality holds for all $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$. The initial conditions were $-\frac{9}{2} < x$ or $-2 < x$. The intersection of these three conditions occurs when $\frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}$.

Case (2) is $\frac{9}{2} \leq x \leq -2$. For all x satisfying these conditions, $x+2 < 0$ and $2x+9 > 0$. Then the following operations will change the direction of the inequality:

$$\begin{aligned}\frac{x+1}{x+2} &> \frac{3x+4}{2x+9} \\ \Rightarrow 2x^2 + 11x + 9 &< 3x^2 + 10x + 8 \\ \Rightarrow 0 &< x^2 - x - 1\end{aligned}$$

The inequality holds for all $x < \frac{1-\sqrt{5}}{2}$ and $\frac{1+\sqrt{5}}{2} < x$. The initial condition was $-\frac{9}{2} \leq x \leq -2$. Hence the intersection of these conditions yields all x such that $-\frac{9}{2} \leq x \leq -2$. Then all possible cases of x are

$$\boxed{-\frac{9}{2} \leq x \leq -2 \cup \frac{1-\sqrt{5}}{2} < x < \frac{1+\sqrt{5}}{2}}.$$

9. *Problem:* Suppose $f(x)$ is a rational function such that $3f(\frac{1}{x}) + \frac{2f(x)}{x} = x^2$ for $x \neq 0$. Find $f(-2)$.

Solution: Let $x = \frac{-1}{2}$. Then

$$\begin{aligned}3f(-2) + \frac{2f(\frac{-1}{2})}{\frac{-1}{2}} &= \frac{1}{4} \\ \Rightarrow 3f(-2) - 4f(\frac{-1}{2}) &= \frac{1}{4}\end{aligned}\quad (1)$$

Let $x = -2$. Then

$$\begin{aligned}3f(\frac{-1}{2}) + \frac{2f(-2)}{-2} &= 4 \\ \Rightarrow 3f(\frac{-1}{2}) - f(-2) &= 4\end{aligned}\quad (2)$$

Solving this system of equations $\{(1), (2)\}$ for $f(-2)$ yields $f(-2) = \boxed{\frac{67}{20}}$.

10. *Problem:* G.H. Hardy once went to visit Srinivasa Ramanujan in the hospital, and he started the conversation with: "I came here in taxi-cab number 1729. That number seems dull to me, which I hope isn't a bad omen." "Nonsense," said Ramanujan. "The number isn't dull at all. It's quite interesting. It's the smallest number that can be expressed as the sum of two cubes in two different ways." Ramanujan had immediately seen that $1729 = 12^3 + 1^3 = 10^3 + 9^3$. What is the smallest positive integer representable as the sum of the cubes of *three* positive integers in two different ways?

Solution: Let this smallest positive integer be represented as $a^3 + b^3 + c^3 = d^3 + e^3 + f^3$. By inspection, a solution is not possible with the first 4 cubes. We prove that it is impossible to write the same number as two different sums of the first 5 cubes. Because we necessarily need to use the 5th cube (otherwise, this proof would be for the first 4 cubes), we have $5^3 + b^3 + c^3 = d^3 + e^3 + f^3$. Without loss of generality, suppose $d = 5$. By inspection, there is no solution to $b^3 + c^3 = e^3 + f^3$, such that $b, c, e, f \leq 5$ and b, c and e, f are unique.

Then none of d, e, f are 5. Then at least two must be 4, otherwise the RHS would be too small. Without loss of generality, suppose $d = e = 4$. Then $b^3 + c^3 = 3 + f^3$. By inspection, there are no possible solutions if $b, c, f \leq 4$.

Thus if $a = 5$, there are no solutions.

Suppose that there is a solution within the first 6 cubes. Then $a = 6$. By the same analysis as above, $d = e = 5$, otherwise the RHS would be too small. Then $b^3 + c^3 = 34 + f^3$. By inspection, we see that a possible solution is $b = 3, c = 2, f = 1$. Then the desired integer is $6^3 + 3^3 + 2^3 = \boxed{251}$.

HMMT 1998: Calculus Solutions

1. *Problem:* Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after t minutes he is at position $(t, \sin t)$.

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for m minutes, his position is $(m, \cos t)$.

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

Solution: At arbitrary time t , Farmer Tim is at position $(t, \sin t)$ and Alex is at position $(t - 5, \cos t)$. Hence at time t , the distance, d , between Tim and Alex is $d = \sqrt{(\sin t - \cos t)^2 + 25}$. To find the maximum value of d , we solve for t such that $\frac{dd}{dt} = 0$.

$\frac{dd}{dt} = \frac{(\sin t - \cos t)(\cos t + \sin t)}{\sqrt{(\sin t - \cos t)^2 + 25}}$. Then $\frac{dd}{dt} = 0 \Rightarrow \sin^2 t - \cos^2 t = 0 \Rightarrow \sin^2 t = \cos^2 t$. Equality happens if t is any constant multiple of $\frac{\pi}{4}$.

Notice that to maximize d , we need to maximize $(\sin t - \cos t)^2$. This is achieved when $\cos t = -\sin t$. Because we determined earlier that t is a constant multiple of $\frac{\pi}{4}$, then with this new condition, we see that t must be a constant multiple of $\frac{3\pi}{4}$.

Then $(\sin t - \cos t)^2 = 2 \Rightarrow d = \boxed{\sqrt{29}}$.

2. *Problem:* A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1m. If the water level in the cube is falling at a rate of 1 cm/s, at what rate is the water level in the cylinder rising?

Solution: The magnitude of the change in volume per unit time of the two solids is the same. The change in volume per unit time of the cube is 1 cm·m²/s. The change in volume per unit time of the cylinder is $\pi \cdot \frac{dh}{dt} \cdot m^2$, where $\frac{dh}{dt}$ is the rate at which the water level in the cylinder is rising.

Solving the equation $\pi \cdot \frac{dh}{dt} \cdot m^2 = 1 \text{ cm} \cdot m^2/\text{s}$ yields $\boxed{\frac{1}{\pi} \text{ cm/s}}$.

3. *Problem:* Find the area of the region bounded by the graphs of $y = x^2$, $y = x$, and $x = 2$.

Solution: There are two regions to consider. First, there is the region bounded by $y = x^2$ and $y = x$, in the interval $[0, 1]$. In this interval, the values of $y = x$ are greater than the values of $y = x^2$, thus the area is calculated by $\int_0^1 (x - x^2) dx$.

Second, there is the region bounded by $y = x^2$ and $y = x$ and $x = 2$, in the interval $[1, 2]$. In this interval, the values of $y = x^2$ are greater than the values of $y = x$, thus the area is calculated by $\int_1^2 (x^2 - x) dx$.

Then the total area of the region bounded by the three graphs is $\int_0^1 (x - x^2) dx + \int_1^2 (x^2 - x) dx = \boxed{1}$.

4. *Problem:* Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \dots$, for $-1 \leq x \leq 1$. Find $\sqrt{e \int_0^1 f(x) dx}$.

Solution: Observe that $f(x)$ is merely an infinite geometric series. Thus $f(x) = \frac{1}{1 - \frac{x}{2}} = \frac{2}{2-x}$. Then $\int_0^1 \frac{2}{2-x} = 2 \ln 2$. Then $\sqrt{e^{2 \ln 2}} = \sqrt{2^2} = \boxed{2}$.

5. *Problem:* Evaluate $\lim_{x \rightarrow 1} x^{\frac{x}{\sin(1-x)}}$.

Solution: Rewrite the expression to evaluate as $e^{\ln x^{\frac{x}{\sin(1-x)}}}$. Then we must evaluate $\lim_{x \rightarrow 1} e^{\ln x^{\frac{x}{\sin(1-x)}}}$.

$\lim_{x \rightarrow 1} \ln x^{\frac{x}{\sin(1-x)}} = \lim_{x \rightarrow 1} \left(\frac{x}{\sin(1-x)} \ln x \right)$. Because direct calculation of the limit results in indeterminate form $(\frac{1}{0} \cdot 0)$, we can use L'Hopital's rule to evaluate the limit. By L'Hopital's rule, $\lim_{x \rightarrow 1} \left(\frac{x}{\sin(1-x)} \ln x \right) =$

$\lim_{x \rightarrow 1} \frac{\ln x + 1}{-\cos(1-x)}$. This limit is simply -1.

Hence $\lim_{x \rightarrow 1} e^{\ln x \frac{x}{\sin(1-x)}} = e^{-1} = \boxed{\frac{1}{e}}$.

6. *Problem:* Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place 1,875 feet from the prison and was spotted by a guard with a crossbow. The guard fired an arrow with an initial velocity of 100 ft/s. At the same time, Edward started running away with an acceleration of 1 ft/s². Assuming that air resistance causes the arrow to decelerate at 1 ft/s² and that it does hit Edward, how fast was the arrow moving at the moment of impact (in ft/s)?

Solution: We use the formula for distance, $d = \frac{1}{2}at^2 + vt + d_0$. Then after t seconds, Edward is at location $1875 + \frac{1}{2}(1)(t^2)$ from the prison. After t seconds, the arrow is at location $\frac{1}{2}(-1)(t^2) + 100t$ from the prison. When the arrow hits Edward, both objects are at the same distance away from the tower. Hence $1875 + \frac{1}{2}(1)(t^2) = \frac{1}{2}(-1)(t^2) + 100t$. Solving for t yields $t^2 - 100t + 1875 = 0 \Rightarrow t = 25$ or $t = 75$. Then it must be $t = 25$, because after the arrow hits Edward, he will stop running.

After 25 seconds, the arrow is moving at a velocity of $100 - 25(1) = \boxed{75 \text{ ft/s}}$.

7. *Problem:* A parabola is inscribed in equilateral triangle ABC of side length 1 in the sense that AC and BC are tangent to the parabola at A and B , respectively. Find the area between AB and the parabola.

Solution: Suppose $A = (0, 0)$, $B = (1, 0)$, and $C = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Then the parabola in question goes through $(0, 0)$ and $(1, 0)$ and has tangents with slopes of $\sqrt{3}$ and $-\sqrt{3}$, respectively, at these points. Suppose the parabola has equation $y = ax^2 + bx + c$. Then $\frac{dy}{dx} = 2ax + b$.

At point $(0, 0)$, $\frac{dy}{dx} = b$. Also the slope at $(0, 0)$, as we determined earlier, is $\sqrt{3}$. Hence $b = \sqrt{3}$. Similarly, at point $(1, 0)$, $\frac{dy}{dx} = 2a + b$. The slope at $(1, 0)$, as we determined earlier, is $-\sqrt{3}$. Then $a = -\sqrt{3}$.

Since the parabola goes through $(0, 0)$, $c = 0$. Hence the equation of the parabola is $y = -\sqrt{3}x^2 + \sqrt{3}x$. The desired area is simply the area under the parabolic curve in the interval $[0, 1]$.

Hence $\int_0^1 (-\sqrt{3}x^2 + \sqrt{3}x) dx = \boxed{\frac{\sqrt{3}}{6}}$.

8. *Problem:* Find the slopes of all lines passing through the origin and tangent to the curve $y^2 = x^3 + 39x - 35$.

Solution: Any line passing through the origin has equation $y = mx$, where m is the slope of the line. If a line is tangent to the given curve, then at the point of tangency, (x, y) , $\frac{dy}{dx} = m$.

First, we calculate $\frac{dy}{dx}$ of the curve: $2ydy = 3x^2dx + 39dx \Rightarrow \frac{dy}{dx} = \frac{3x^2 + 39}{2y}$. Substituting mx for y , we get the following system of equations:

$$\begin{aligned} m^2x^2 &= x^3 + 39x - 35 \\ m &= \frac{3x^2 + 39}{2mx} \end{aligned}$$

Solving for x yields the equation $x^3 - 39x + 70 = 0 \Rightarrow (x - 2)(x + 7)(x - 5) = 0 \Rightarrow x = 2$ or $x = -7$ or $x = 5$. These solutions indicate the x -coordinate of the points at which the desired lines are tangent to the curve. Solving for the slopes of these lines, we get $m = \pm \frac{\sqrt{51}}{2}$ for $x = 2$, no real solutions for $x = -7$, and $m = \pm \frac{\sqrt{285}}{5}$ for $x = 5$. Thus $\boxed{m = \pm \frac{\sqrt{51}}{2}, \pm \frac{\sqrt{285}}{5}}$.

9. *Problem:* Evaluate $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$.

Solution: Note that if we take the integral of $f(x)$ in problem 4, we get the function $F(x) = x + \frac{x^2}{2 \cdot 2} + \frac{x^3}{3 \cdot 2^2} + \dots$. Evaluating this integral in the interval $[0, 1]$, we get $1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2^2} + \dots$, which is the desired sum.

Hence $\int_0^1 \frac{2}{2-x} dx = 2 \ln 2$.

10. *Problem:* Let S be the locus of all points (x, y) in the first quadrant such that $\frac{x}{t} + \frac{y}{1-t} = 1$ for some t with $0 < t < 1$. Find the area of S .

Solution: Solving for t in the given equation, we get $t^2 + (y - x - 1)t + x = 0$. Using the quadratic equation, we get $t = \frac{(x+1-y) \pm \sqrt{(y-x-1)^2 - 4x}}{2}$. For all valid combinations of (x, y) , t is positive and less than 1 (this is easy to see by inspection). All valid combinations of (x, y) are those that make $(y - x - 1)^2 - 4x \geq 0$.

Solving for y in the equation $(y - x - 1)^2 - 4x = 0$ yields $y^2 - (2x+2)y + (x-1)^2 \geq 0 \Rightarrow y = (x+1) \pm 2\sqrt{x}$. In the original equation, it is given that $\frac{x}{t} + \frac{y}{1-t} = 1$, and $0 < t < 1$. This implies that $x, y < 1$. Then the only possible $y < 1$ that satisfies $(y - x - 1)^2 - 4x = 0$ is $y = x + 1 - 2\sqrt{x}$.

Then to satisfy the inequality $(y - x - 1)^2 - 4x \geq 0$, we must have $y \leq x + 1 - 2\sqrt{x}$. Recall that this is when $0 < y < 1$. Hence we integrate in the interval $[0, 1]$: $\int_0^1 x + 1 - 2\sqrt{x} = \boxed{\frac{1}{6}}$.

Geometry Solutions

Harvard-MIT Math Tournament

Written by Anders Kaseorg

1. $m\angle EDX = 180^\circ - m\angle LAX - m\angle ELA = 180^\circ - m\angle LAX - (180^\circ - m\angle AXE) = \boxed{80^\circ}$.
2. If the angle in radians is θ , then Anne travels $R\theta$ and Lisa travels $(R - r) + r\theta + (R - r)$. Setting these equal yields $R(\theta - 2) = r(\theta - 2)$, so $\theta = \boxed{2 \text{ radians}}$.
3. $m\angle MLD = \frac{1}{2}\widehat{AB} = \boxed{45^\circ}$.
4. The cylinder has a cross-sectional area π times greater than the cube, so the water raises $\frac{1}{\pi}$ times as quickly in the cylinder as it lowers in the cube; that is, at $\boxed{\frac{1}{\pi} \frac{\text{cm}}{\text{s}}}$.
5. If the new circle has radius r , then the distance from its center to E can be computed either as $1 + r$ or $(2 - r)\sqrt{2}$. Setting these equal yields $r = \frac{2\sqrt{2}-1}{\sqrt{2}+1} = \boxed{5 - 3\sqrt{2}}$.
6. The central circle has area $\pi\left(\frac{1}{2\sqrt{3}}\right)^2 = \frac{\pi}{12}$, and each of the three small triangles are copies of the entire figure dilated by $\frac{1}{3}$. Therefore, the total area is given by $K = \frac{\pi}{12} + 3 \cdot \left(\frac{1}{3}\right)^2 K \iff \frac{2}{3}K = \frac{\pi}{12} \iff K = \boxed{\frac{\pi}{8}}$.
7. Let $E = (a, b, 0)$, $A = (-c, b, 0)$, $R = (-c, -d, 0)$, $L = (a, -d, 0)$, $Y = (0, 0, h)$, and observe that $EY^2 + RY^2 = a^2 + b^2 + c^2 + d^2 + 2h^2 = AY^2 + LY^2$, which can only be satisfied by $EY = 1$, $AY = 4$, $RY = 8$, $LY = 7$ (or the symmetric configurations). Since EA is an integral side of a triangle whose other sides are 1 and 4, we must have $EA = 4$; similarly, $EL = 7$. Therefore, the area of rectangle $EARL$ is $\boxed{28}$. (Such a pyramid may be constructed by taking $a = \frac{1}{8}$, $b = \frac{1}{14}$, $c = \frac{31}{8}$, $d = \frac{97}{14}$, $h = \frac{\sqrt{3071}}{56}$.)
8. Since $OD \perp AC$ and $\triangle AOC$ is equilateral, we have $\angle AOD = 30^\circ$. So $AE = \frac{1}{\sqrt{3}}$, and $BH = \sqrt{AB^2 + AH^2} = \sqrt{2^2 + \left(3 - \frac{1}{\sqrt{3}}\right)^2} = \sqrt{\frac{40}{3} - 2\sqrt{3}} \approx 3.141533339$.
9. The dilation of ratio $-\frac{2}{3}$ about T sends C_2 to C_1 , O_2 to O_1 , and S to the other intersection of s with C_1 , which we shall call U . We can now compute $TR \cdot TS = \frac{3}{2}TR \cdot TU = \frac{3}{2}TP^2 = \frac{3}{2}(O_1T^2 - O_1P^2) = \frac{3}{2}\left(\left(\frac{O_1O_2}{1+\frac{3}{2}}\right)^2 - O_1P^2\right) = \frac{3}{2}\left(\left(\frac{20}{5/2}\right)^2 - 4^2\right) = \boxed{72}$.
10. Suppose that when the ball hits a side of the table, instead of reflecting the ball's path, we reflect the entire table over this side so that the path remains straight. If we repeatedly reflect the table over its sides in all possible ways, we get a triangular grid that tiles the plane. Whenever the path crosses n lines in this grid parallel to CT , it will cross $\frac{7}{8}n$ lines parallel to CH and $\frac{15}{8}n$ lines parallel to HT . After crossing $8 + 7 + 15 = 30$ grid lines it will have crossed three lines simultaneously again, which means that the ball will have landed in a pocket after bouncing $\boxed{27}$ times. By picturing the grid it is easy to see that the pocket in question is \boxed{H} . The distance the ball travels during the $\frac{1}{8}$ of its trip described in the problem is the third side of a triangle with an 120° angle between two sides 16 and 14, which is $\sqrt{16^2 + 14^2 - 2 \cdot 16 \cdot 14 \cos 120^\circ} = 26$, so length of the entire trip is $\boxed{208}$.

1998 Power Question Solutions

I. Graphs, total of 20 points

a. completely correct gets 1 point, total of 6 points

- i. yes. vertices A,B,C,D, edges AB,AC,AD,BD
- ii. no. A and B are connected twice
- iii. yes. vertices A,B,C, edges ABC
- iv. yes. vertices A,B,C,D,E, edges AB,AD,AE,BE,CD,CE
- v. no. the edge from B does not connect to another vertex
- vi. no. E is connected to itself

b. completely correct gets 1 point, total of 2 points

- i. BL,BE,BU,EL,EU
- ii. $A \leftrightarrow B$, $B \leftrightarrow L$, $C \leftrightarrow U$, $D \leftrightarrow E$

c. 1 point each for i-iii, 3 for iv-vi, total of 12 points

- i. no
- ii. no
- iii. no
- iv. yes
- v. no
- vi. yes

note that the second graph in v has more vertices of degree 3 than the first

II. Planar Graphs, total of 60 points

a. 1 point each for v, e, f, total of 6 points

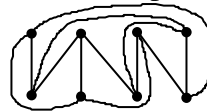
- i. $v=9$, $e=16$, $f=9$
- ii. $v=7$, $e=14$, $f=9$

note that $v-e+f=2$ by Euler's formula

b. 3 points each, total of 12 points

- i. yes
- ii. yes
- iii. no
- iv. yes

note that this is a planar drawing of ii:



for iii, note that if we remove the two vertices of degree two and make the vertices they were adjacent to adjacent to each other then we get a graph isomorphic to the one in d.ii.

c. total of 15 points

partial credit: up to 5 for effort, 5 for insight, otherwise as described below

[1 point] If G has 3 vertices and two edges then the assertion is easy since $f=1$.

[6 points] Otherwise each face is bounded by 3 or more edges, hence counting the number of edges bounding each faces and summing, then using the fact that we at most double counted [-2 for asserting we exactly double counted] each face we get $3f \leq 2e$, thus proving the left inequality.

[8 points] Now $f \leq (2/3)e$ and we want to eliminate f , so we can add $v-e$ to both sides to get $v-e+f \leq v-e+(2/3)e$, thus by Euler's formula $2 \leq v-e/3$, and rearranging yields the right inequality.

d. 5 points for i, 10 for ii, total of 15 points

partial credit: if Jordan curve thm is used then 3 points for i, 5 for ii.

on part ii: 3 for effort, 3 for insight, otherwise as described

i. Assume it is planar. $v=5, e=10, 3v-6=9 < e$, contradicting c, thus nonplanar

ii. [1 point] Note that $v=6, e=9$, so $3v-6=12 > e$ is not a contradiction

[2 points] There are no triangles in this graph, thus every face is bordered by at least 4 edges. This will allow us to prove a better inequality as in c with the assumption the graph is planar, and that will provide the contradiction.

[2 points] Thus as in the proof of c we get $4f \leq 2e$.

[4 points] Hence $f \leq e/2$, and again we want to eliminate f so adding $v-e$ to both sides and applying Euler's formula yields $2 \leq v-e/2$. (or $e \leq 2v-4$)

[1 point] $6-9/2 < 2$, contradicting the inequality just proved, thus the graph is not planar.

e. total of 12 points

partial credit: 4 for effort, 4 for insight, otherwise as described

[2 points] Without loss of generality we may assume the graph is connected.

[2 points] If the graph has fewer than 3 vertices then the assertion is obvious.

[5 points] Otherwise we can apply the inequality in c. First assume every vertex has degree at least 6, then adding the degrees of each vertex double counts the edges so $6v \leq 2e$.

[3 points] Now by c. $e \leq 3v-6$, and combining these inequalities we get $6v \leq 6v-12$, a contradiction, thus the graph must have at least one vertex of degree 5 or less.

III. Coloring, total of 56 points

a. 3 points each, total of 12 points

i. 4

ii. 5

iii. 4

iv. 3

b. total of 14 points

partial credit: 5 for effort, 5 for insight, otherwise as described

[8 points] Number the colors used in G $1, 2, \dots, \chi$. Let v_1 be the number of vertices colored with color 1. Then since no pair of them are adjacent, they are all adjacent in the new graph and thus $\bar{\chi} \geq v_1$. Repeating this procedure for v_2, \dots, v_χ and adding we get $\chi \bar{\chi} \geq v$.

[6 points] By the AM-GM inequality $\chi + \bar{\chi} \geq 2\sqrt{\chi \bar{\chi}} \geq 2\sqrt{v}$.

c. 5 points for i and iii, 20 for ii, total of 30 points

partial credit: as described below

i. By II.e. G must have a vertex of degree 5 or less

ii. [2 points] We want to remove the vertex V without decreasing the number of colors needed. In the cases where V has degree < 5 this is obviously possible.

[10 points] In the case where V has degree 5, color the adjacent vertices with colors 1, 2, 3, 4, 5 in clockwise order. If the vertices colored 1 and 3 are not connected by a walk with all vertices colored 1 or 3 then change the vertex colored 3 to 1, change any 1 vertices adjacent to it to 3, then change any 3 vertices adjacent to those to 1, and so on. This recoloring does not affect any of the other vertices adjacent to V and thus allows us to make V color 3, i.e. if a sixth color is needed for G then it is still needed after we remove V .

[8 points] If the vertices colored 1 and 3 are connected by a walk of vertices colored 1 or 3 then look at the vertices colored 2 and 4. If they are connected by a walk then by the Jordan Curve Theorem [-4 for not using this] this walk must cross the closed curve formed by the 1-3 walk and the edges from V to the vertices colored 1 and 3. The graph is planar, so we can assume it is drawn without edge crossings, thus the walk from the vertex colored 2 to the one colored 4 must cross this curve at a vertex, hence they are not connected by a walk with all vertices colored 2 or 4, and by the previous argument we can change the vertex colored 2 to 4 and make V color 2, so it can be removed without removing the need for a sixth color.

iii. [4 points] We have shown that for any graph with $\chi > 5$ we can find one with fewer vertices, and since the proof did not depend on the number of vertices we can keep doing this until we get a graph with 5 vertices, which can clearly be colored with only 5 colors, thus G could not require more than 5 colors.

[1 points] To make this rigorous some mention should be made of the well-ordering principal or infinite descent.

Historical notes and inspiration for further study of the subject:

The Four Color theorem was first conjectured in 1852. Many “proofs” were given, including one in 1879 by A. B. Kempe that was believed to be correct until P. J. Heawood found a flaw in 1890. The proof of the Four Color theorem, given in 1976 by Haken and Appel, uses a similar technique to what we just used, but instead of using 6 possible subgraphs it uses 1482. The way these theorems relate to coloring maps is that we can consider the *dual graph*, formed by putting a vertex in every face and connecting the vertices whose faces share an edge. Coloring the dual graph is the same as coloring the faces of the original graph in such a way that no two faces sharing an edge have the same color, which is what one does when coloring a map.

While it may seem that we used many definitions in this power question, there are many more that had to be omitted. Some of the other interesting definitions used in graph theory are *expansion*, which is what you get if you add vertices along an edge of a graph, and *supergraph*, which is what you get if you add new vertices anywhere and connect them arbitrarily to the other vertices. Kuratowski’s Theorem states that every nonplanar graph is a supergraph of an expansion of one of the two graphs in II.d. A corollary of this is that a graph is nonplanar iff it is a supergraph of an expansion of one of those two graphs, thus we can classify planar and nonplanar graphs in terms of just two particular graphs.

A planar graph can also be described as one that can be drawn on a sphere with no edge crossings. A torus is essentially the surface of a solid sphere with a hole drilled all the way through it. The *genus* g of a graph is the minimum number of holes that must be drilled in a sphere in order to be able to draw it without edge crossings. In other words a graph of genus g can be drawn on a g holed torus without edge crossings, but not on a $g-1$ holed torus. The Heawood Coloring Theorem states that if G has genus $g > 0$ then G can be colored with $\left\lceil \frac{7 + \sqrt{1 + 48g}}{2} \right\rceil$ colors. Note that plugging in $g=0$ yields 4, but no proof

of this theorem is known that does not depend on the condition $g > 0$ (of course we can combine the proof for $g>0$ with the four color theorem to conclude that the result holds for all g , but it would be nice to have just one proof that works equally well for all g).

Another thing we can study about graphs is what kind of walks exist in them. An *eulerian walk*, named after the great Swiss mathematician Leonhard Euler (1707-1783, pronounced Oiler), is one that uses every edge exactly once. A *hamiltonian walk*, named after the Irish mathematician Sir William Rowan Hamilton (1805-1865), is one that uses every vertex exactly once. Try to classify which graphs have such walks, with or without the condition that the starting and ending points must be the same (in the case of a hamiltonian walk the end vertex is thus counted twice and it is called a *closed hamiltonian walk* to distinguish from the *open walk*).

This power question was barely an introduction to graph theory. It is a very broad field of mathematics, closely related to topology and knot theory. I hope you enjoyed this test, learned something from it, and that you will continue your studies of mathematics for many years.

Edward Early
2/14/98