HMMT February 2020

February 15, 2020

Algebra and Number Theory

1. Let $P(x) = x^3 + x^2 - r^2 x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

2. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

3. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

4. For positive integers n and k, let $\mho(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mho(n, k)}{3^{n+k-7}}.$$

5. A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1 + n_2 + \cdots + n_{10} = N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim_{M \to \infty} p_M$.

6. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$, where each a_i is an integer between 0 and $n - 1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

7. Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n - 3}$$

is an integer.

8. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

9. Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

10. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each nonnegative integer k. For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

    We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

    in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.

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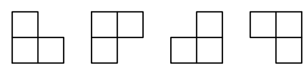
Combinatorics

1. How many ways can the vertices of a cube be colored red or blue so that the color of each vertex is the color of the majority of the three vertices adjacent to it?

2. How many positive integers at most 420 leave different remainders when divided by each of 5, 6, and 7?

3. Each unit square of a 4×4 square grid is colored either red, green, or blue. Over all possible colorings of the grid, what is the maximum possible number of L-trominos that contain exactly one square of each color? (L-trominos are made up of three unit squares sharing a corner, as shown below.)



4. Given an 8×8 checkerboard with alternating white and black squares, how many ways are there to choose four black squares and four white squares so that no two of the eight chosen squares are in the same row or column?

5. Let S be a set of intervals defined recursively as follows:

   - Initially, $[1, 1000]$ is the only interval in S.
   - If $l \neq r$ and $[l, r] \in S$, then both $\left[l, \left\lfloor \frac{l+r}{2} \right\rfloor\right], \left[\left\lfloor \frac{l+r}{2} \right\rfloor + 1, r\right] \in S$.

   (Note that S can contain intervals such as $[1, 1]$, which contain a single integer.) An integer i is chosen uniformly at random from the range $[1, 1000]$. What is the expected number of intervals in S which contain i?

6. Alice writes 1001 letters on a blackboard, each one chosen independently and uniformly at random from the set $S = \{a, b, c\}$. A move consists of erasing two distinct letters from the board and replacing them with the third letter in S. What is the probability that Alice can perform a sequence of moves which results in one letter remaining on the blackboard?

7. Anne-Marie has a deck of 16 cards, each with a distinct positive factor of 2002 written on it. She shuffles the deck and begins to draw cards from the deck without replacement. She stops when there exists a nonempty subset of the cards in her hand whose numbers multiply to a perfect square. What is the expected number of cards in her hand when she stops?

8. Let Γ_1 and Γ_2 be concentric circles with radii 1 and 2, respectively. Four points are chosen on the circumference of Γ_2 independently and uniformly at random, and are then connected to form a convex quadrilateral. What is the probability that the perimeter of this quadrilateral intersects Γ_1?

9. Farmer James wishes to cover a circle with circumference 10π with six different types of colored arcs. Each type of arc has radius 5, has length either π or 2π, and is colored either red, green, or blue. He has an unlimited number of each of the six arc types. He wishes to completely cover his circle without overlap, subject to the following conditions:

   - Any two adjacent arcs are of different colors.
   - Any three adjacent arcs where the middle arc has length π are of three different colors.

   Find the number of distinct ways Farmer James can cover his circle. Here, two coverings are equivalent if and only if they are rotations of one another. In particular, two colorings are considered distinct if they are reflections of one another, but not rotations of one another.

10. Max repeatedly throws a fair coin in a hurricane. For each throw, there is a 4% chance that the coin gets blown away. He records the number of heads H and the number of tails T before the coin is lost. (If the coin is blown away on a toss, no result is recorded for that toss.) What is the expected value of $|H - T|$?

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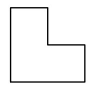
Geometry

1. Let $DIAL$, FOR, and $FRIEND$ be regular polygons in the plane. If $ID = 1$, find the product of all possible areas of OLA.

2. Let ABC be a triangle with $AB = 5$, $AC = 8$, and $\angle BAC = 60^\circ$. Let $UVWXYZ$ be a regular hexagon that is inscribed inside ABC such that U and V lie on side BA, W and X lie on side AC, and Z lies on side CB. What is the side length of hexagon $UVWXYZ$?

3. Consider the L-shaped tromino below with 3 attached unit squares. It is cut into exactly two pieces of equal area by a line segment whose endpoints lie on the perimeter of the tromino. What is the longest possible length of the line segment?



4. Let $ABCD$ be a rectangle and E be a point on segment AD. We are given that quadrilateral $BCDE$ has an inscribed circle ω_1 that is tangent to BE at T. If the incircle ω_2 of ABE is also tangent to BE at T, then find the ratio of the radius of ω_1 to the radius of ω_2.

5. Let $ABCDEF$ be a regular hexagon with side length 2. A circle with radius 3 and center at A is drawn. Find the area inside quadrilateral $BCDE$ but outside the circle.

6. Let ABC be a triangle with $AB = 5$, $BC = 6$, $CA = 7$. Let D be a point on ray AB beyond B such that $BD = 7$, E be a point on ray BC beyond C such that $CE = 5$, and F be a point on ray CA beyond A such that $AF = 6$. Compute the area of the circumcircle of DEF.

7. Let Γ be a circle, and ω_1 and ω_2 be two non-intersecting circles inside Γ that are internally tangent to Γ at X_1 and X_2, respectively. Let one of the common internal tangents of ω_1 and ω_2 touch ω_1 and ω_2 at T_1 and T_2, respectively, while intersecting Γ at two points A and B. Given that $2X_1T_1 = X_2T_2$ and that ω_1, ω_2, and Γ have radii 2, 3, and 12, respectively, compute the length of AB.

8. Let ABC be an acute triangle with circumcircle Γ. Let the internal angle bisector of $\angle BAC$ intersect BC and Γ at E and N, respectively. Let A' be the antipode of A on Γ and let V be the point where AA' intersects BC. Given that $EV = 6$, $VA' = 7$, and $A'N = 9$, compute the radius of Γ.

9. Circles $\omega_a, \omega_b, \omega_c$ have centers A, B, C, respectively and are pairwise externally tangent at points D, E, F (with $D \in BC, E \in CA, F \in AB$). Lines BE and CF meet at T. Given that ω_a has radius 341, there exists a line ℓ tangent to all three circles, and there exists a circle of radius 49 tangent to all three circles, compute the distance from T to ℓ.

10. Let Γ be a circle of radius 1 centered at O. A circle Ω is said to be *friendly* if there exist distinct circles $\omega_1, \omega_2, \ldots, \omega_{2020}$, such that for all $1 \le i \le 2020$, ω_i is tangent to Γ, Ω, and ω_{i+1}. (Here, $\omega_{2021} = \omega_1$.) For each point P in the plane, let $f(P)$ denote the sum of the areas of all friendly circles centered at P. If A and B are points such that $OA = \frac{1}{2}$ and $OB = \frac{1}{3}$, determine $f(A) - f(B)$.

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1. **[4]** Submit an integer x as your answer to this problem. The number of points you receive will be $\max(0, 8 - |8x - 100|)$. (Non-integer answers will be given 0 points.)

2. **[4]** Let ABC be a triangle and ω be its circumcircle. The point M is the midpoint of arc BC not containing A on ω and D is chosen so that DM is tangent to ω and is on the same side of AM as C. It is given that $AM = AC$ and $\angle DMC = 38^\circ$. Find the measure of angle $\angle ACB$.

3. **[4]** Let ABC be a triangle and D, E, and F be the midpoints of sides BC, CA, and AB respectively. What is the maximum number of circles which pass through at least 3 of these 6 points?

4. **[4]** Compute the value of $\sqrt{105^3 - 104^3}$, given that it is a positive integer.

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5. **[5]** Alice, Bob, and Charlie roll a 4, 5, and 6-sided die, respectively. What is the probability that a number comes up exactly twice out of the three rolls?

6. **[5]** Two sides of a regular n-gon are extended to meet at a 28° angle. What is the smallest possible value for n?

7. **[5]** Ana and Banana are rolling a standard six-sided die. Ana rolls the die twice, obtaining a_1 and a_2, then Banana rolls the die twice, obtaining b_1 and b_2. After Ana's two rolls but before Banana's two rolls, they compute the probability p that $a_1b_1 + a_2b_2$ will be a multiple of 6. What is the probability that $p = \frac{1}{6}$?

8. **[5]** Tessa picks three real numbers x, y, z and computes the values of the eight expressions of the form $\pm x \pm y \pm z$. She notices that the eight values are all distinct, so she writes the expressions down in increasing order. For example, if $x = 2$, $y = 3$, $z = 4$, then the order she writes them down is

$$-x-y-z,\ +x-y-z,\ -x+y-z,\ -x-y+z,\ +x+y-z,\ +x-y+z,\ -x+y+z,\ +x+y+z.$$

How many possible orders are there?

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9. **[6]** Let $P(x)$ be the monic polynomial with rational coefficients of minimal degree such that $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{4}}, \dots, \frac{1}{\sqrt{1000}}$ are roots of P. What is the sum of the coefficients of P?

10. **[6]** Jarris is a weighted tetrahedral die with faces F_1, F_2, F_3, F_4. He tosses himself onto a table, so that the probability he lands on a given face is proportional to the area of that face (i.e. the probability he lands on face F_i is $\frac{[F_i]}{[F_1]+[F_2]+[F_3]+[F_4]}$ where $[K]$ is the area of K). Let k be the maximum distance any part of Jarris is from the table after he rolls himself. Given that Jarris has an inscribed sphere of radius 3 and circumscribed sphere of radius 10, find the minimum possible value of the expected value of k.

11. **[6]** Find the number of ordered pairs of positive integers (x, y) with $x, y \leq 2020$ such that $3x^2 + 10xy + 3y^2$ is the power of some prime.

12. **[6]** An 11×11 grid is labeled with consecutive rows $0, 1, 2, \dots, 10$ and columns $0, 1, 2, \dots, 10$ so that it is filled with integers from 1 to 2^{10}, inclusive, and the sum of all of the numbers in row n and in column n are both divisible by 2^n. Find the number of possible distinct grids.

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13. **[8]** Let $\triangle ABC$ be a triangle with $AB = 7$, $BC = 1$, and $CA = 4\sqrt{3}$. The angle trisectors of C intersect $\overline{AB}$ at D and E, and lines $\overline{AC}$ and $\overline{BC}$ intersect the circumcircle of $\triangle CDE$ again at X and Y, respectively. Find the length of XY.

14. **[8]** Let $\varphi(n)$ denote the number of positive integers less than or equal to n which are relatively prime to n. Let S be the set of positive integers n such that $\frac{2n}{\varphi(n)}$ is an integer. Compute the sum

$$\sum_{n \in S} \frac{1}{n}.$$

15. **[8]** You have six blocks in a row, labeled 1 through 6, each with weight 1. Call two blocks $x \leq y$ connected when, for all $x \leq z \leq y$, block z has not been removed. While there is still at least one block remaining, you choose a remaining block uniformly at random and remove it. The cost of this operation is the sum of the weights of the blocks that are connected to the block being removed, including itself. Compute the expected total cost of removing all the blocks.

16. **[8]** Determine all triplets of real numbers (x, y, z) satisfying the system of equations

$$\begin{aligned} x^2y + y^2z &= 1040 \\ x^2z + z^2y &= 260 \\ (x-y)(y-z)(z-x) &= -540. \end{aligned}$$

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17. **[10]** Let ABC be a triangle with incircle tangent to the perpendicular bisector of BC. If $BC = AE = 20$, where E is the point where the A-excircle touches BC, then compute the area of $\triangle ABC$.

18. **[10]** A *vertex-induced* subgraph is a subset of the vertices of a graph together with any edges whose endpoints are both in this subset.

    An undirected graph contains 10 nodes and m edges, with no loops or multiple edges. What is the minimum possible value of m such that this graph must contain a nonempty vertex-induced subgraph where all vertices have degree at least 5?

19. **[10]** The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. There exist unique positive integers $n_1, n_2, n_3, n_4, n_5, n_6$ such that

$$\sum_{i_1=0}^{100}\sum_{i_2=0}^{100}\sum_{i_3=0}^{100}\sum_{i_4=0}^{100}\sum_{i_5=0}^{100} F_{i_1+i_2+i_3+i_4+i_5} = F_{n_1} - 5F_{n_2} + 10F_{n_3} - 10F_{n_4} + 5F_{n_5} - F_{n_6}.$$

    Find $n_1 + n_2 + n_3 + n_4 + n_5 + n_6$.

20. **[10]** There exist several solutions to the equation

$$1 + \frac{\sin x}{\sin 4x} = \frac{\sin 3x}{\sin 2x},$$

    where x is expressed in degrees and $0^\circ < x < 180^\circ$. Find the sum of all such solutions.

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21. **[12]** We call a positive integer t *good* if there is a sequence $a_0, a_1, \ldots$ of positive integers satisfying $a_0 = 15, a_1 = t$, and

$$a_{n-1}a_{n+1} = (a_n - 1)(a_n + 1)$$

    for all positive integers n. Find the sum of all good numbers.

22. **[12]** Let A be a set of integers such that for each integer m, there exists an integer $a \in A$ and positive integer n such that $a^n \equiv m \pmod{100}$. What is the smallest possible value of $|A|$?

23. **[12]** A function $f\colon A \to A$ is called *idempotent* if $f(f(x)) = f(x)$ for all $x \in A$. Let I_n be the number of idempotent functions from $\{1, 2, \ldots, n\}$ to itself. Compute

$$\sum_{n=1}^{\infty} \frac{I_n}{n!}.$$

24. **[12]** In $\triangle ABC$, ω is the circumcircle, I is the incenter and I_A is the A-excenter. Let M be the midpoint of arc $\widehat{BAC}$ on ω, and suppose that X, Y are the projections of I onto MI_A and I_A onto MI, respectively. If $\triangle XYI_A$ is an equilateral triangle with side length 1, compute the area of $\triangle ABC$.

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25. **[15]** Let S be the set of 3^4 points in four-dimensional space where each coordinate is in $\{-1, 0, 1\}$. Let N be the number of sequences of points $P_1, P_2, \ldots, P_{2020}$ in S such that $P_iP_{i+1} = 2$ for all $1 \leq i \leq 2020$ and $P_1 = (0, 0, 0, 0)$. (Here $P_{2021} = P_1$.) Find the largest integer n such that 2^n divides N.

26. **[15]** Let $ABCD$ be a cyclic quadrilateral, and let segments AC and BD intersect at E. Let W and Y be the feet of the altitudes from E to sides DA and BC, respectively, and let X and Z be the midpoints of sides AB and CD, respectively. Given that the area of AED is 9, the area of BEC is 25, and $\angle EBC - \angle ECB = 30^\circ$, then compute the area of $WXYZ$.

27. **[15]** Let $\{a_i\}_{i\geq 0}$ be a sequence of real numbers defined by

$$a_{n+1} = a_n^2 - \frac{1}{2^{2020 \cdot 2^n - 1}}$$

for $n \geq 0$. Determine the largest value for a_0 such that $\{a_i\}_{i\geq 0}$ is bounded.

28. **[15]** Let $\triangle ABC$ be a triangle inscribed in a unit circle with center O. Let I be the incenter of $\triangle ABC$, and let D be the intersection of BC and the angle bisector of $\angle BAC$. Suppose that the circumcircle of $\triangle ADO$ intersects BC again at a point E such that E lies on IO. If $\cos A = \frac{12}{13}$, find the area of $\triangle ABC$.

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29. **[18]** Let $ABCD$ be a tetrahedron such that its circumscribed sphere of radius R and its inscribed sphere of radius r are concentric. Given that $AB = AC = 1 \leq BC$ and $R = 4r$, find BC^2.

30. **[18]** Let $S = \{(x, y) \mid x > 0, y > 0, x + y < 200, \text{ and } x, y \in \mathbb{Z}\}$. Find the number of parabolas $\mathcal{P}$ with vertex V that satisfy the following conditions:

- $\mathcal{P}$ goes through both $(100, 100)$ and at least one point in S,
- V has integer coordinates, and
- $\mathcal{P}$ is tangent to the line $x + y = 0$ at V.

31. **[18]** Anastasia is taking a walk in the plane, starting from $(1, 0)$. Each second, if she is at (x, y), she moves to one of the points $(x - 1, y)$, $(x + 1, y)$, $(x, y - 1)$, and $(x, y + 1)$, each with $\frac{1}{4}$ probability. She stops as soon as she hits a point of the form (k, k). What is the probability that k is divisible by 3 when she stops?

32. **[18]** Find the smallest real constant α such that for all positive integers n and real numbers $0 = y_0 < y_1 < \cdots < y_n$, the following inequality holds:

$$\alpha \sum_{k=1}^{n} \frac{(k+1)^{3/2}}{\sqrt{y_k^2 - y_{k-1}^2}} \geq \sum_{k=1}^{n} \frac{k^2 + 3k + 3}{y_k}.$$

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33. **[22]** Estimate

$$N = \prod_{n=1}^{\infty} n^{n^{-1.25}}.$$

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N) \rfloor$ points.

34. **[22]** For odd primes p, let $f(p)$ denote the smallest positive integer a for which there does not exist an integer n satisfying $p \mid n^2 - a$. Estimate N, the sum of $f(p)^2$ over the first 10^5 odd primes p.

An estimate of $E > 0$ will receive $\lfloor 22 \min(N/E, E/N)^3 \rfloor$ points.

35. **[22]** A collection $\mathcal{S}$ of 10000 points is formed by picking each point uniformly at random inside a circle of radius 1. Let N be the expected number of points of $\mathcal{S}$ which are vertices of the convex hull of the $\mathcal{S}$. (The convex hull is the smallest convex polygon containing every point of $\mathcal{S}$.) Estimate N.

An estimate of $E > 0$ will earn $\max(\lfloor 22 - |E - N| \rfloor, 0)$ points.

36. **[22]** A *snake of length k* is an animal which occupies an ordered k-tuple $(s_1, \ldots, s_k)$ of cells in a $n \times n$ grid of square unit cells. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \ldots, k-1$. If the snake is currently occupying $(s_1, \ldots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can move to occupy $(s, s_1, \ldots, s_{k-1})$ instead.

Initially, a snake of length 4 is in the grid $\{1, 2, \ldots, 30\}^2$ occupying the positions $(1,1), (1,2), (1,3), (1,4)$ with $(1,1)$ as its head. The snake repeatedly makes a move uniformly at random among moves it can legally make. Estimate N, the expected number of moves the snake makes before it has no legal moves remaining.

An estimate of $E > 0$ will earn $\lfloor 22 \min(N/E, E/N)^4 \rfloor$ points.

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Team Round

1. **[20]** Let n be a positive integer. Define a sequence by $a_0 = 1$, $a_{2i+1} = a_i$, and $a_{2i+2} = a_i + a_{i+1}$ for each $i \geq 0$. Determine, with proof, the value of $a_0 + a_1 + a_2 + \cdots + a_{2^n-1}$.

2. **[25]** Let n be a fixed positive integer. An n-staircase is a polyomino with $\frac{n(n+1)}{2}$ cells arranged in the shape of a staircase, with arbitrary size. Here are two examples of 5-staircases:

Prove that an n-staircase can be dissected into strictly smaller n-staircases.

3. **[25]** Let ABC be a triangle inscribed in a circle ω and ℓ be the tangent to ω at A. The line through B parallel to AC meets ℓ at P, and the line through C parallel to AB meets ℓ at Q. The circumcircles of ABP and ACQ meet at $S \neq A$. Show that AS bisects BC.

4. **[35]** Alan draws a convex 2020-gon $\mathcal{A} = A_1A_2\cdots A_{2020}$ with vertices in clockwise order and chooses 2020 angles $\theta_1, \theta_2, \ldots, \theta_{2020} \in (0, \pi)$ in radians with sum 1010π. He then constructs isosceles triangles $\triangle A_iB_iA_{i+1}$ on the exterior of $\mathcal{A}$ with $B_iA_i = B_iA_{i+1}$ and $\angle A_iB_iA_{i+1} = \theta_i$. (Here, $A_{2021} = A_1$.) Finally, he erases $\mathcal{A}$ and the point B_1. He then tells Jason the angles $\theta_1, \theta_2, \ldots, \theta_{2020}$ he chose. Show that Jason can determine where B_1 was from the remaining 2019 points, i.e. show that B_1 is uniquely determined by the information Jason has.

5. **[40]** Let a_0, b_0, c_0, a, b, c be integers such that $\gcd(a_0, b_0, c_0) = \gcd(a, b, c) = 1$. Prove that there exists a positive integer n and integers $a_1, a_2, \ldots, a_n = a, b_1, b_2, \ldots, b_n = b, c_1, c_2, \ldots, c_n = c$ such that for all $1 \leq i \leq n$, $a_{i-1}a_i + b_{i-1}b_i + c_{i-1}c_i = 1$.

6. **[40]** Let $n > 1$ be a positive integer and S be a collection of $\frac{1}{2}\binom{2n}{n}$ distinct n-element subsets of $\{1, 2, \ldots, 2n\}$. Show that there exists $A, B \in S$ such that $|A \cap B| \leq 1$.

7. **[50]** Positive real numbers x and y satisfy

$$\left|\left|\cdots\left|\left||x| - y\right| - x\right|\cdots - y\right| - x\right| = \left|\left|\cdots\left|\left||y| - x\right| - y\right|\cdots - x\right| - y\right|$$

where there are 2019 absolute value signs $|\cdot|$ on each side. Determine, with proof, all possible values of $\frac{x}{y}$.

8. **[50]** Let ABC be a scalene triangle with angle bisectors AD, BE, and CF so that D, E, and F lie on segments BC, CA, and AB respectively. Let M and N be the midpoints of BC and EF respectively. Prove that line AN and the line through M parallel to AD intersect on the circumcircle of ABC if and only if $DE = DF$.

9. **[55]** Let $p > 5$ be a prime number. Show that there exists a prime number $q < p$ and a positive integer n such that p divides $n^2 - q$.

10. **[60]** Let n be a fixed positive integer, and choose n positive integers $a_1, \ldots, a_n$. Given a permutation π on the first n positive integers, let $S_\pi = \{i \mid \frac{a_i}{\pi(i)} \text{ is an integer}\}$. Let N denote the number of distinct sets S_π as π ranges over all such permutations. Determine, in terms of n, the maximum value of N over all possible values of $a_1, \ldots, a_n$.

HMIC 2020

March 28, 2020 – April 3, 2020

1. **[6]** Sir Alex is coaching a soccer team of n players of distinct heights. He wants to line them up so that for each player P, the total number of players that are either to the left of P and taller than P or to the right of P and shorter than P is even. In terms of n, how many possible orders are there?

2. **[7]** Some bishops and knights are placed on an infinite chessboard, where each square has side length 1 unit. Suppose that the following conditions hold:

   - For each bishop, there exists a knight on the same diagonal as that bishop (there may be another piece between the bishop and the knight).
   - For each knight, there exists a bishop that is exactly $\sqrt{5}$ units away from it.
   - If any piece is removed from the board, then at least one of the above conditions is no longer satisfied.

   If n is the total number of pieces on the board, find all possible values of n.

3. **[9]** Let $P_1P_2P_3P_4$ be a tetrahedron in $\mathbb{R}^3$ and let O be a point equidistant from each of its vertices. Suppose there exists a point H such that for each i, the line P_iH is perpendicular to the plane through the other three vertices. Line P_1H intersects the plane through P_2, P_3, P_4 at A, and contains a point $B \neq P_1$ such that $OP_1 = OB$. Show that $HB = 3HA$.

4. **[9]** Let $C_k = \frac{1}{k+1}\binom{2k}{k}$ denote the k^{th} Catalan number and p be an odd prime. Prove that exactly half of the numbers in the set

   $$\left\{ \sum_{k=1}^{p-1} C_k n^k \;\middle|\; n \in \{1, 2, \ldots, p-1\} \right\}$$

   are divisible by p.

5. **[11]** A triangle and a circle are in the same plane. Show that the area of the intersection of the triangle and the circle is at most one third of the area of the triangle plus one half of the area of the circle.

Time limit: 4 hours

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Algebra and Number Theory

1. What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is *palindromic* if the sequence of decimal digits are the same when read backwards.)

2. Let $N = 2^{\left(2^2\right)}$ and x be a real number such that $N^{\left(N^N\right)} = 2^{(2^x)}$. Find x.

3. Let x and y be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

4. Let $\mathbb{N}$ be the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

   - $f(1) = 1$;
   - for $n \in \mathbb{N}$, $f(2n) = 2f(n)$ and $f(2n+1) = 2f(n) - 1$.

   Determine the sum of all positive integer solutions to $f(x) = 19$ that do not exceed 2019.

5. Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1 b_1 = 20$, $a_2 b_2 = 19$, and $a_3 b_3 = 14$. Find the greatest possible value of $a_4 b_4$.

6. For positive reals p and q, define the *remainder* when p is divided by q as the smallest nonnegative real r such that $\frac{p-r}{q}$ is an integer. For an ordered pair (a, b) of positive integers, let r_1 and r_2 be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs (a, b) such that $a, b \leq 20$ and $r_1 + r_2 = \sqrt{2}$.

7. Find the value of
$$\sum_{a=1}^{\infty} \sum_{b=1}^{\infty} \sum_{c=1}^{\infty} \frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

8. There is a unique function $f : \mathbb{N} \to \mathbb{R}$ such that $f(1) > 0$ and such that
$$\sum_{d|n} f(d) f\left(\frac{n}{d}\right) = 1$$
for all $n \geq 1$. What is $f(2018^{2019})$?

9. Tessa the hyper-ant has a 2019-dimensional hypercube. For a real number k, she calls a placement of nonzero real numbers on the 2^{2019} vertices of the hypercube *k-harmonic* if for any vertex, the sum of all 2019 numbers that are edge-adjacent to this vertex is equal to k times the number on this vertex. Let S be the set of all possible values of k such that there exists a k-harmonic placement. Find $\sum_{k \in S} |k|$.

10. The sequence of integers $\{a_i\}_{i=0}^{\infty}$ satisfies $a_0 = 3, a_1 = 4$, and
$$a_{n+2} = a_{n+1} a_n + \left\lceil \sqrt{a_{n+1}^2 - 1} \sqrt{a_n^2 - 1} \right\rceil$$
for $n \geq 0$. Evaluate the sum
$$\sum_{n=0}^{\infty} \left(\frac{a_{n+3}}{a_{n+2}} - \frac{a_{n+2}}{a_n} + \frac{a_{n+1}}{a_{n+3}} - \frac{a_n}{a_{n+1}} \right).$$

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Combinatorics

1. How many distinct permutations of the letters of the word REDDER are there that do not contain a palindromic substring of length at least two? (A *substring* is a contiguous block of letters that is part of the string. A string is *palindromic* if it is the same when read backwards.)

2. Your math friend Steven rolls five fair icosahedral dice (each of which is labelled $1, 2, \dots, 20$ on its sides). He conceals the results but tells you that at least half of the rolls are 20. Suspicious, you examine the first two dice and find that they show 20 and 19 in that order. Assuming that Steven is truthful, what is the probability that all three remaining concealed dice show 20?

3. Reimu and Sanae play a game using 4 fair coins. Initially both sides of each coin are white. Starting with Reimu, they take turns to color one of the white sides either red or green. After all sides are colored, the 4 coins are tossed. If there are more red sides showing up, then Reimu wins, and if there are more green sides showing up, then Sanae wins. However, if there is an equal number of red sides and green sides, then *neither* of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that Reimu wins?

4. Yannick is playing a game with 100 rounds, starting with 1 coin. During each round, there is a $n\%$ chance that he gains an extra coin, where n is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

5. Contessa is taking a random lattice walk in the plane, starting at $(1, 1)$. (In a random lattice walk, one moves up, down, left, or right 1 unit with equal probability at each step.) If she lands on a point of the form $(6m, 6n)$ for $m, n \in \mathbb{Z}$, she ascends to heaven, but if she lands on a point of the form $(6m+3, 6n+3)$ for $m, n \in \mathbb{Z}$, she descends to hell. What is the probability that she ascends to heaven?

6. A point P lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0 = P$, and given point P_i, point P_{i+1} is obtained by reflecting P_i over one of the four lines AB, BC, CD, DA, chosen uniformly at random and independently for each i. What is the probability that $P_8 = P$?

7. In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

   What is the probability that Alice wins the election (by having more votes than Celia)?

8. For a positive integer N, we color the positive divisors of N (including 1 and N) with four colors. A coloring is called *multichromatic* if whenever a, b and $\gcd(a, b)$ are pairwise distinct divisors of N, then they have pairwise distinct colors. What is the maximum possible number of multichromatic colorings a positive integer can have if it is not the power of any prime?

9. How many ways can one fill a 3×3 square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

10. Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position (a_1, a_2, a_3, a_4) to some position (x_1, x_2, x_3, x_4) with integer coordinates satisfying

    $$(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2+(x_4-a_4)^2 = 4 \quad \text{and} \quad |(x_1+x_2+x_3+x_4)-(a_1+a_2+a_3+a_4)| = 2.$$

    In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

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Geometry

1. Let d be a real number such that every non-degenerate quadrilateral has at least two interior angles with measure less than d degrees. What is the minimum possible value for d?

2. In rectangle $ABCD$, points E and F lie on sides AB and CD respectively such that both AF and CE are perpendicular to diagonal BD. Given that BF and DE separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of BD.

3. Let AB be a line segment with length 2, and S be the set of points P on the plane such that there exists point X on segment AB with $AX = 2PX$. Find the area of S.

4. Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. AC and BD intersect at G. Given that the area of AGB is 10 more than the area of CGB, find the smallest possible area of hexagon $ABCDEF$.

5. Isosceles triangle ABC with $AB = AC$ is inscribed in a unit circle Ω with center O. Point D is the reflection of C across AB. Given that $DO = \sqrt{3}$, find the area of triangle ABC.

6. Six unit disks $C_1, C_2, C_3, C_4, C_5, C_6$ are in the plane such that they don't intersect each other and C_i is tangent to C_{i+1} for $1 \leq i \leq 6$ (where $C_7 = C_1$). Let C be the smallest circle that contains all six disks. Let r be the smallest possible radius of C, and R the largest possible radius. Find $R - r$.

7. Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let H be the orthocenter of ABC. Find the radius of the circle with nonzero radius tangent to the circumcircles of AHB, BHC, CHA.

8. In triangle ABC with $AB < AC$, let H be the orthocenter and O be the circumcenter. Given that the midpoint of OH lies on BC, $BC = 1$, and the perimeter of ABC is 6, find the area of ABC.

9. In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point A and intersects edges BF, FG, GH, HD at points P, Q, R, S respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

10. In triangle ABC, $AB = 13, BC = 14, CA = 15$. Squares ABB_1A_2, BCC_1B_2, CAA_1C_2 are constructed outside the triangle. Squares $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ are constructed outside the hexagon $A_1A_2B_1B_2C_1C_2$. Squares $A_3B_4B_5A_6$, $B_3C_4C_5B_6$, $C_3A_4A_5C_6$ are constructed outside the hexagon $A_4A_3B_4B_3C_4C_3$. Find the area of the hexagon $A_5A_6B_5B_6C_5C_6$.

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There are nine sets of four problems each. The point values of the problems in each set, in order, are 3, 4, 5, 7, 9, 12, 15, 20, 25. Good luck!

Also, a heads up: the problems in the last set (problems 33 to 36) of this guts round will **NOT** give possible partial credits as per tradition, but the set is nevertheless different from a normal guts set. **In light of this change, you may want to reserve more time for the last set than usual.**

1. [**3**] Find the sum of all real solutions to $x^2 + \cos x = 2019$.

2. [**3**] There are 100 people in a room with ages $1, 2, \ldots, 100$. A pair of people is called *cute* if each of them is at least seven years older than half the age of the other person in the pair. At most how many pairwise disjoint cute pairs can be formed in this room?

3. [**3**] Let $S(x)$ denote the sum of the digits of a positive integer x. Find the maximum possible value of $S(x + 2019) - S(x)$.

4. [**3**] Tessa has a figure created by adding a semicircle of radius 1 on each side of an equilateral triangle with side length 2, with semicircles oriented outwards. She then marks two points on the boundary of the figure. What is the greatest possible distance between the two points?

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5. [**4**] Call a positive integer n *weird* if n does not divide $(n-2)!$. Determine the number of weird numbers between 2 and 100 inclusive.

6. [**4**] The pairwise products ab, bc, cd, and da of positive integers a, b, c, and d are $64, 88, 120$, and 165 in some order. Find $a + b + c + d$.

7. [**4**] For any real number α, define

$$\operatorname{sign}(\alpha) = \begin{cases} +1 & \text{if } \alpha > 0, \\ 0 & \text{if } \alpha = 0, \\ -1 & \text{if } \alpha < 0. \end{cases}$$

How many triples $(x, y, z) \in \mathbb{R}^3$ satisfy the following system of equations

$$\begin{aligned} x &= 2018 - 2019 \cdot \operatorname{sign}(y + z), \\ y &= 2018 - 2019 \cdot \operatorname{sign}(z + x), \\ z &= 2018 - 2019 \cdot \operatorname{sign}(x + y)? \end{aligned}$$

8. [**4**] A regular hexagon $PROFIT$ has area 1. Every minute, greedy George places the largest possible equilateral triangle that does not overlap with other already-placed triangles in the hexagon, with ties broken arbitrarily. How many triangles would George need to cover at least 90% of the hexagon's area?

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9. [**5**] Define $P = \{\mathrm{S}, \mathrm{T}\}$ and let $\mathcal{P}$ be the set of all proper subsets of P. (A *proper subset* is a subset that is not the set itself.) How many ordered pairs $(\mathcal{S}, \mathcal{T})$ of proper subsets of $\mathcal{P}$ are there such that

   (a) $\mathcal{S}$ is not a proper subset of $\mathcal{T}$ and $\mathcal{T}$ is not a proper subset of $\mathcal{S}$; and

   (b) for any sets $S \in \mathcal{S}$ and $T \in \mathcal{T}$, S is not a proper subset of T and T is not a proper subset of S?

10. [**5**] Let

$$A = (1 + 2\sqrt{2} + 3\sqrt{3} + 6\sqrt{6})(2 + 6\sqrt{2} + \sqrt{3} + 3\sqrt{6})(3 + \sqrt{2} + 6\sqrt{3} + 2\sqrt{6})(6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}),$$

$$B = (1 + 3\sqrt{2} + 2\sqrt{3} + 6\sqrt{6})(2 + \sqrt{2} + 6\sqrt{3} + 3\sqrt{6})(3 + 6\sqrt{2} + \sqrt{3} + 2\sqrt{6})(6 + 2\sqrt{2} + 3\sqrt{3} + \sqrt{6}).$$

   Compute the value of A/B.

11. [**5**] In the Year 0 of Cambridge there is one squirrel and one rabbit. Both animals multiply in numbers quickly. In particular, if there are m squirrels and n rabbits in Year k, then there will be $2m + 2019$ squirrels and $4n - 2$ rabbits in Year $k + 1$. What is the first year in which there will be strictly more rabbits than squirrels?

12. [**5**] Bob is coloring lattice points in the coordinate plane. Find the number of ways Bob can color five points in $\{(x, y) \mid 1 \leq x, y \leq 5\}$ blue such that the distance between any two blue points is *not* an integer.

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13. [**7**] Reimu has 2019 coins $C_0, C_1, \ldots, C_{2018}$, one of which is fake, though they look identical to each other (so each of them is equally likely to be fake). She has a machine that takes any two coins and picks one that is not fake. If both coins are not fake, the machine picks one uniformly at random. For each $i = 1, 2, \ldots, 1009$, she puts C_0 and C_i into the machine once, and machine picks C_i. What is the probability that C_0 is fake?

14. [**7**] Let ABC be a triangle where $AB = 9, BC = 10, CA = 17$. Let Ω be its circumcircle, and let A_1, B_1, C_1 be the diametrically opposite points from A, B, C, respectively, on Ω. Find the area of the convex hexagon with the vertices A, B, C, A_1, B_1, C_1.

15. [**7**] Five people are at a party. Each pair of them are *friends*, *enemies*, or *frenemies* (which is equivalent to being both *friends* and *enemies*). It is known that given any three people A, B, C:

   - If A and B are friends and B and C are friends, then A and C are friends;
   - If A and B are enemies and B and C are enemies, then A and C are friends;
   - If A and B are friends and B and C are enemies, then A and C are enemies.

   How many possible relationship configurations are there among the five people?

16. [**7**] Let $\mathbb{R}$ be the set of real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y, we have

$$f(x^2) + f(y^2) = f(x + y)^2 - 2xy.$$

   Let $S = \sum_{n=-2019}^{2019} f(n)$. Determine the number of possible values of S.

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17. [**9**] Let ABC be a triangle with $AB = 3$, $BC = 4$, and $CA = 5$. Let A_1, A_2 be points on side BC, B_1, B_2 be points on side CA, and C_1, C_2 be points on side AB. Suppose that there exists a point P such that PA_1A_2, PB_1B_2, and PC_1C_2 are congruent equilateral triangles. Find the area of convex hexagon $A_1A_2B_1B_2C_1C_2$.

18. [**9**] 2019 points are chosen independently and uniformly at random on the interval $[0, 1]$. Tairitsu picks 1000 of them randomly and colors them black, leaving the remaining ones white. Hikari then computes the sum of the positions of the leftmost white point and the rightmost black point. What is the probability that this sum is at most 1?

19. [**9**] Complex numbers a, b, c form an equilateral triangle with side length 18 in the complex plane. If $|a + b + c| = 36$, find $|bc + ca + ab|$.

20. [**9**] On floor 0 of a weird-looking building, you enter an elevator that only has one button. You press the button twice and end up on floor 1. Thereafter, every time you press the button, you go up by one floor with probability $\frac{X}{Y}$, where X is your current floor, and Y is the total number of times you have pressed the button thus far (not including the current one); otherwise, the elevator does nothing. Between the third and the 100^{th} press inclusive, what is the expected number of pairs of consecutive presses that both take you up a floor?

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21. [**12**] A regular hexagon $ABCDEF$ has side length 1 and center O. Parabolas $P_1, P_2, \ldots, P_6$ are constructed with common focus O and directrices AB, BC, CD, DE, EF, FA respectively. Let χ be the set of all distinct points on the plane that lie on at least two of the six parabolas. Compute

$$\sum_{X \in \chi} |OX|.$$

(Recall that the focus is the point and the directrix is the line such that the parabola is the locus of points that are equidistant from the focus and the directrix.)

22. [**12**] Determine the number of subsets S of $\{1, 2, \ldots, 1000\}$ that satisfy the following conditions:
    - S has 19 elements, and
    - the sum of the elements in any non-empty subset of S is not divisible by 20.

23. [**12**] Find the smallest positive integer n such that

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{n\text{ 2's}} > \underbrace{((\cdots((100!)!)!\cdots)!)!}_{100\text{ factorials}}.$$

24. [**12**] Let S be the set of all positive factors of 6000. What is the probability of a random quadruple $(a, b, c, d) \in S^4$ satisfies

$$\operatorname{lcm}(\gcd(a, b), \gcd(c, d)) = \gcd(\operatorname{lcm}(a, b), \operatorname{lcm}(c, d))?$$

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25. [**15**] A 5 by 5 grid of unit squares is partitioned into 5 pairwise incongruent rectangles with sides lying on the gridlines. Find the maximum possible value of the product of their areas.

26. [**15**] Let ABC be a triangle with $AB = 13, BC = 14, CA = 15$. Let I_A, I_B, I_C be the A, B, C excenters of this triangle, and let O be the circumcenter of the triangle. Let $\gamma_A, \gamma_B, \gamma_C$ be the corresponding excircles and ω be the circumcircle. X is one of the intersections between γ_A and ω. Likewise, Y is an intersection of γ_B and ω, and Z is an intersection of γ_C and ω. Compute
$$\cos \angle OXI_A + \cos \angle OYI_B + \cos \angle OZI_C.$$

27. [**15**] Consider the eighth-sphere $\{(x, y, z) \mid x, y, z \geq 0,\ x^2 + y^2 + z^2 = 1\}$. What is the area of its projection onto the plane $x + y + z = 1$?

28. [**15**] How many positive integers $2 \leq a \leq 101$ have the property that there exists a positive integer N for which the last two digits in the decimal representation of a^{2^n} is the same for all $n \geq N$?

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29. [**20**] Yannick picks a number N randomly from the set of positive integers such that the probability that n is selected is 2^{-n} for each positive integer n. He then puts N identical slips of paper numbered 1 through N into a hat and gives the hat to Annie. Annie does not know the value of N, but she draws one of the slips uniformly at random and discovers that it is the number 2. What is the expected value of N given Annie's information?

30. [**20**] Three points are chosen inside a unit cube uniformly and independently at random. What is the probability that there exists a cube with side length $\frac{1}{2}$ and edges parallel to those of the unit cube that contains all three points?

31. [**20**] Let ABC be a triangle with $AB = 6$, $AC = 7$, $BC = 8$. Let I be the incenter of ABC. Points Z and Y lie on the interior of segments AB and AC respectively such that YZ is tangent to the incircle. Given point P such that
$$\angle ZPC = \angle YPB = 90^\circ,$$
find the length of IP.

32. [**20**] For positive integers a and b such that a is coprime to b, define $\text{ord}_b(a)$ as the least positive integer k such that $b \mid a^k - 1$, and define $\varphi(a)$ to be the number of positive integers less than or equal to a which are coprime to a. Find the least positive integer n such that
$$\text{ord}_n(m) < \frac{\varphi(n)}{10}$$
for all positive integers m coprime to n.

Warning: The next (and final) set contains problems in different formats.

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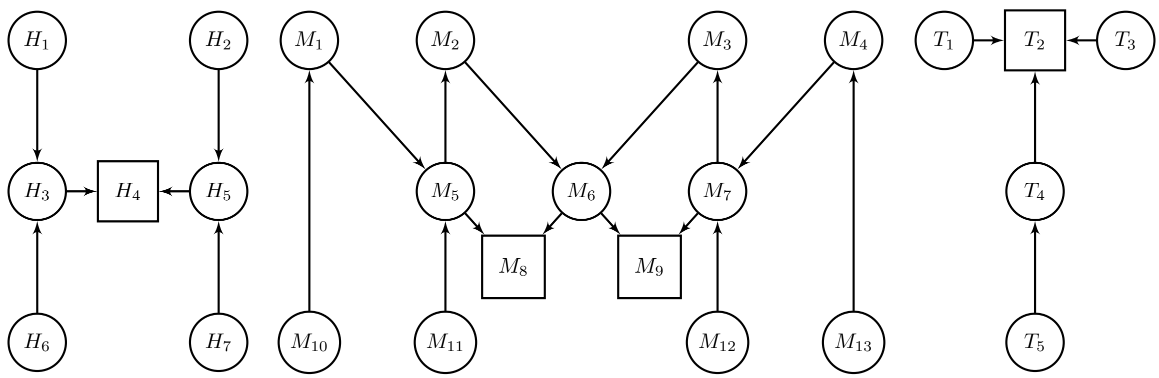
Welcome to **Ultra Relay**! This is an event where teams of twenty-five students race to solve a set of twenty-five problems and ultimately find the answers to the four "anchor" problems. Many of the problems will depend on answers to other problems, hence the team members will need to relay the answers from one person/problem to the next, along the network shown below. (The anchor problems are indicated by a box.) **Only the answers to the four anchor problems will be graded.**



What? You don't have twenty-five people on your team, you say? That's unfortunate. Anyway, here are all the problems. (You may work on them together.) Have fun!

For ease of reference, the label of each problem also denotes the answer to that problem.

H_1. Let $r = H_1$ be the answer to this problem. Given that r is a nonzero real number, what is the value of $r^4 + 4r^3 + 6r^2 + 4r$?

H_2. Given two distinct points A, B and line ℓ that is not perpendicular to AB, what is the maximum possible number of points P on ℓ such that ABP is an isosceles triangle?

H_3. Let $A = H_1, B = H_6 + 1$. A real number x is chosen randomly and uniformly in the interval $[A, B]$. Find the probability that $x^2 > x^3 > x$.

H_4. Let $A = \lceil 1/H_3 \rceil, B = \lceil H_5/2 \rceil$. How many ways are there to partition the set $\{1, 2, \ldots, A + B\}$ into two sets U and V with size A and B respectively such that the probability that a number chosen from U uniformly at random is greater than a number chosen from V uniformly at random is exactly $\frac{1}{2}$?

H_5. Let $A = H_2, B = H_7$. Two circles with radii A and B respectively are given in the plane. If the length of their common external tangent is twice the length of their common internal tangent (where both tangents are considered as segments with endpoints being the points of tangency), find the distance between the two centers.

H_6. How many ways are there to arrange the numbers $21, 22, 33, 35$ in a row such that any two adjacent numbers are relatively prime?

H_7. How many pairs of integers (x, y) are there such that $|x^2 - 2y^2| \leq 1$ and $|3x - 4y| \leq 1$?

M_1. Let $S = M_{10}$. Determine the number of ordered triples (a, b, c) of nonnegative integers such that $a + 2b + 4c = S$.

M_2. Let $S = \lfloor M_5 \rfloor$. Two integers m and n are chosen between 1 and S inclusive uniformly and independently at random. What is the probability that $m^n = n^m$?

M_3. Let $S = \lceil M_7 \rceil$. In right triangle ABC, $\angle C = 90°$, $AC = 27, BC = 36$. A circle with radius S is tangent to both AC and BC and intersects AB at X and Y. Find the length of XY.

M_4. Let $S = M_{13} + 5$. Compute the product of all positive divisors of S.

M_5. Let $A = \sqrt{M_1}, B = \lceil M_{11} \rceil$. Given complex numbers x and y such that $x + \frac{1}{y} = A$, $\frac{1}{x} + y = B$, compute the value of $xy + \frac{1}{xy}$.

M_6. Let $A = \lfloor 1/M_2 \rfloor, B = \lfloor M_3^2/100 \rfloor$. Let P and Q both be quadratic polynomials. Given that the real roots of $P(Q(x)) = 0$ are $0, A, B, C$ in some order, find the sum of all possible values of C.

M_7. Let $A = \lceil \log_2 M_4 \rceil, B = M_{12} + 1$. A 5-term sequence of positive reals satisfy that the first three terms and the last three terms both form an arithmetic sequence and the middle three terms form a geometric sequence. If the first term is A and the fifth term is B, determine the third term of the sequence.

M_8. Let $A = \lfloor M_5^2 \rfloor, B = \lfloor M_6^2 \rfloor$. A regular A-gon, a regular B-gon, and a circle are given in the plane. What is the greatest possible number of regions that these shapes divide the plane into?

M_9. Let A and B be the unit digits of $\lceil 7M_6 \rceil$ and $\lfloor 6M_7 \rfloor$ respectively. When all the positive integers not containing digit A or B are written in increasing order, what is the 2019^{th} number in the list?

M_{10}. What is the smallest positive integer with remainder $2, 3, 4$ when divided by $3, 5, 7$ respectively?

M_{11}. An equiangular hexagon has side lengths $1, 2, 3, 4, 5, 6$ in some order. Find the nonnegative difference between the largest and the smallest possible area of this hexagon.

M_{12}. Determine the second smallest positive integer n such that $n^3 + n^2 + n + 1$ is a perfect square.

M_{13}. Given that A, B are nonzero base-10 digits such that $A \cdot \overline{AB} + B = \overline{BB}$, find $\overline{AB}$.

T_1. Let S, P, A, C, E be (not necessarily distinct) decimal digits where $E \neq 0$. Given that $N = \sqrt{\overline{ESCAPE}}$ is a positive integer, find the minimum possible value of N.

T_2. Let $X = \lfloor T_1/8 \rfloor, Y = T_3 - 1, Z = T_4 - 2$. A point P lies inside the triangle ABC such that $PA = X, PB = Y, PC = Z$. Find the largest possible area of the triangle.

T_3. How many ways can one tile a 2×8 board with 1×1 and 2×2 tiles? Rotations and reflections of the same configuration are considered distinct.

T_4. Let $S = T_5$. Given real numbers a, b, c such that $a^2 + b^2 + c^2 + (a + b + c)^2 = S$, find the maximum possible value of $(a + b)(b + c)(c + a)$.

T_5. A regular tetrahedron has volume 8. What is the volume of the set of all the points in the space (*not necessarily inside the tetrahedron*) that are closer to the center of the tetrahedron than any of the four vertices?

33. [**25**] Determine the value of H_4.

34. [**25**] Determine the value of M_8.

35. [**25**] Determine the value of M_9.

36. [**25**] Determine the value of T_2.

HMMT February 2019

February 16, 2019

Team Round

1. [**20**] Let $ABCD$ be a parallelogram. Points X and Y lie on segments AB and AD respectively, and AC intersects XY at point Z. Prove that

$$\frac{AB}{AX} + \frac{AD}{AY} = \frac{AC}{AZ}.$$

2. [**20**] Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers, and let f be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Must there exist some positive integer n such that $(f(1), f(2), \ldots, f(n))$ is a permutation of $(1, 2, \ldots, n)$?

3. [**25**] For any angle $0 < \theta < \pi/2$, show that

$$0 < \sin\theta + \cos\theta + \tan\theta + \cot\theta - \sec\theta - \csc\theta < 1.$$

4. [**35**] Find all positive integers n for which there do not exist n consecutive composite positive integers less than $n!$.

5. [**40**] Find all positive integers n such that the unit segments of an $n \times n$ grid of unit squares can be partitioned into groups of three such that the segments of each group share a common vertex.

6. [**45**] Scalene triangle ABC satisfies $\angle A = 60°$. Let the circumcenter of ABC be O, the orthocenter be H, and the incenter be I. Let D, T be the points where line BC intersects the internal and external angle bisectors of $\angle A$, respectively. Choose point X on the circumcircle of $\triangle IHO$ such that $HX \parallel AI$. Prove that $OD \perp TX$.

7. [**50**] A convex polygon on the plane is called *wide* if the projection of the polygon onto any line in the same plane is a segment with length at least 1. Prove that a circle of radius $\frac{1}{3}$ can be placed completely inside any wide polygon.

8. [**50**] Can the set of lattice points $\{(x, y) | x, y \in \mathbb{Z}, 1 \leq x, y \leq 252, x \neq y\}$ be colored using 10 distinct colors such that for all $a \neq b, b \neq c$, the colors of (a, b) and (b, c) are distinct?

9. [**55**] Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of all polynomials in x with coefficients in $\mathbb{F}_p$ (the integers modulo p with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_p$ for each nonnegative integer k. For example, $(x + 2)(2x + 3) = 2x^2 + 2x + 1$ in $\mathbb{F}_5[x]$ because the corresponding coefficients are equal modulo 5.

   Let $f, g \in \mathbb{F}_p[x]$. The pair (f, g) is called *compositional* if

$$f(g(x)) \equiv x^{p^2} - x$$

   in $\mathbb{F}_p[x]$. Find, with proof, the number of compositional pairs (in terms of p).

10. [**60**] Prove that for all positive integers n, all complex roots r of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

   lie on the unit circle (i.e. $|r| = 1$).

HMMT November 2019

November 9, 2019

General Round

1. Dylan has a 100×100 square, and wants to cut it into pieces of area at least 1. Each cut must be a straight line (not a line segment) and must intersect the interior of the square. What is the largest number of cuts he can make?

2. Meghana writes two (not necessarily distinct) primes q and r in base 10 next to each other on a blackboard, resulting in the concatenation of q and r (for example, if $q = 13$ and $r = 5$, the number on the blackboard is now 135). She notices that three more than the resulting number is the square of a prime p. Find all possible values of p.

3. Katie has a fair 2019-sided die with sides labeled $1, 2, \ldots, 2019$. After each roll, she replaces her n-sided die with an $(n+1)$-sided die having the n sides of her previous die and an additional side with the number she just rolled. What is the probability that Katie's 2019^{th} roll is a 2019?

4. In $\triangle ABC$, $AB = 2019$, $BC = 2020$, and $CA = 2021$. Yannick draws three regular n-gons in the plane of $\triangle ABC$ so that each n-gon shares a side with a distinct side of $\triangle ABC$ and no two of the n-gons overlap. What is the maximum possible value of n?

5. Let a, b, c be positive real numbers such that $a \le b \le c \le 2a$. Find the maximum possible value of

$$\frac{b}{a} + \frac{c}{b} + \frac{a}{c}.$$

6. Find all ordered pairs (a, b) of positive integers such that $2a + 1$ divides $3b - 1$ and $2b + 1$ divides $3a - 1$.

7. In Middle-Earth, nine cities form a 3 by 3 grid. The top left city is the capital of Gondor and the bottom right city is the capital of Mordor. How many ways can the remaining cities be divided among the two nations such that all cities in a country can be reached from its capital via the grid-lines without passing through a city of the other country?

8. Compute the number of ordered pairs of integers (x, y) such that $x^2 + y^2 < 2019$ and

$$x^2 + \min(x, y) = y^2 + \max(x, y).$$

9. Let $ABCD$ be an isosceles trapezoid with $AD = BC = 255$ and $AB = 128$. Let M be the midpoint of CD and let N be the foot of the perpendicular from A to CD. If $\angle MBC = 90^\circ$, compute $\tan \angle NBM$.

10. An *up-right path* between two lattice points P and Q is a path from P to Q that takes steps of 1 unit either up or to the right. A lattice point (x, y) with $0 \le x, y \le 5$ is chosen uniformly at random. Compute the expected number of up-right paths from $(0, 0)$ to $(5, 5)$ not passing through (x, y).

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HMMT November 2019, November 9, 2019 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# ____________

1. **[5]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

   A polynomial is called *teeny* if it has degree at most 1 and integer coefficients between -7 and 7, inclusive.

   How many nonzero tricky teeny polynomials are there?

2. **[5]** You are trying to cross a 6 foot wide river. You can jump at most 4 feet, but you have one stone you can throw into the river; after it is placed, you may jump to that stone and, if possible, from there to the other side of the river. However, you are not very accurate and the stone ends up landing uniformly at random in the river. What is the probability that you can get across?

3. **[5]** For how many positive integers a does the polynomial

$$x^2 - ax + a$$

   have an integer root?

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4. **[6]** In 2019, a team, including professor Andrew Sutherland of MIT, found three cubes of integers which sum to 42:

$$42 = (-8053873881207597\_)^3 + (80435758145817515)^3 + (12602123297335631)^3$$

   One of the digits, labeled by an underscore, is missing. What is that digit?

5. **[6]** A point P is chosen uniformly at random inside a square of side length 2. If P_1, P_2, P_3, and P_4 are the reflections of P over each of the four sides of the square, find the expected value of the area of quadrilateral $P_1P_2P_3P_4$.

6. **[6]** Compute the sum of all positive integers $n < 2048$ such that n has an even number of 1's in its binary representation.

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7. **[7]** Let S be the set of all nondegenerate triangles formed from the vertices of a regular octagon with side length 1. Find the ratio of the largest area of any triangle in S to the smallest area of any triangle in S.

8. **[7]** There are 36 students at the Multiples Obfuscation Program, including a singleton, a pair of identical twins, a set of identical triplets, a set of identical quadruplets, and so on, up to a set of identical octuplets. Two students look the same if and only if they are from the same identical multiple. Nithya the teaching assistant encounters a random student in the morning and a random student in the afternoon (both chosen uniformly and independently), and the two look the same. What is the probability that they are actually the same person?

9. **[7]** Let p be a real number between 0 and 1. Jocelin has a coin that lands heads with probability p and tails with probability $1-p$; she also has a number written on a blackboard. Each minute, she flips the coin, and if it lands heads, she replaces the number x on the blackboard with $3x+1$; if it lands tails she replaces it with $x/2$. Given that there are constants a, b such that the expected value of the value written on the blackboard after t minutes can be written as $at+b$ for all positive integers t, compute p.

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10. **[8]** Let $ABCD$ be a square of side length 5, and let E be the midpoint of side AB. Let P and Q be the feet of perpendiculars from B and D to CE, respectively, and let R be the foot of the perpendicular from A to DQ. The segments CE, BP, DQ, and AR partition $ABCD$ into five regions. What is the median of the areas of these five regions?

11. **[8]** Let a, b, c, d be real numbers such that

$$\min(20x+19, 19x+20) = (ax+b) - |cx+d|$$

for all real numbers x. Find $ab+cd$.

12. **[8]** Four players stand at distinct vertices of a square. They each independently choose a vertex of the square (which might be the vertex they are standing on). Then, they each, at the same time, begin running in a straight line to their chosen vertex at 10mph, stopping when they reach the vertex. If at any time two players, whether moving or not, occupy the same space (whether a vertex or a point inside the square), they collide and fall over. How many different ways are there for the players to choose vertices to go to so that none of them fall over?

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13. **[9]** In $\triangle ABC$, the incircle centered at I touches sides AB and BC at X and Y, respectively. Additionally, the area of quadrilateral $BXIY$ is $\frac{2}{5}$ of the area of ABC. Let p be the smallest possible perimeter of a $\triangle ABC$ that meets these conditions and has integer side lengths. Find the smallest possible area of such a triangle with perimeter p.

14. **[9]** Compute the sum of all positive integers n for which

$$9\sqrt{n} + 4\sqrt{n+2} - 3\sqrt{n+16}$$

is an integer.

15. **[9]** Let a, b, c be positive integers such that

$$\frac{a}{77} + \frac{b}{91} + \frac{c}{143} = 1.$$

What is the smallest possible value of $a + b + c$?

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16. **[10]** Equilateral $\triangle ABC$ has side length 6. Let ω be the circle through A and B such that CA and CB are both tangent to ω. A point D on ω satisfies $CD = 4$. Let E be the intersection of line CD with segment AB. What is the length of segment DE?

17. **[10]** Kelvin the frog lives in a pond with an infinite number of lily pads, numbered 0, 1, 2, 3, and so forth. Kelvin starts on lily pad 0 and jumps from pad to pad in the following manner: when on lily pad i, he will jump to lily pad $(i + k)$ with probability $\frac{1}{2^k}$ for $k > 0$. What is the probability that Kelvin lands on lily pad 2019 at some point in his journey?

18. **[10]** The polynomial $x^3 - 3x^2 + 1$ has three real roots r_1, r_2, and r_3. Compute

$$\sqrt[3]{3r_1 - 2} + \sqrt[3]{3r_2 - 2} + \sqrt[3]{3r_3 - 2}.$$

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19. **[11]** Let ABC be a triangle with $AB = 5$, $BC = 8$, $CA = 11$. The incircle ω and A-excircle[1] Γ are centered at I_1 and I_2, respectively, and are tangent to BC at D_1 and D_2, respectively. Find the ratio of the area of $\triangle AI_1D_1$ to the area of $\triangle AI_2D_2$.

20. **[11]** Consider an equilateral triangle T of side length 12. Matthew cuts T into N smaller equilateral triangles, each of which has side length 1, 3, or 8. Compute the minimum possible value of N.

21. **[11]** A positive integer n is *infallible* if it is possible to select n vertices of a regular 100-gon so that they form a convex, non-self-intersecting n-gon having all equal angles. Find the sum of all infallible integers n between 3 and 100, inclusive.

[1] The *A-excircle* of triangle ABC is the unique circle lying outside the triangle that is tangent to segment BC and the extensions of sides AB and AC.

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22. **[12]** Let $f(n)$ be the number of distinct digits of n when written in base 10. Compute the sum of $f(n)$ as n ranges over all positive 2019-digit integers.

23. **[12]** For a positive integer n, let, $\tau(n)$ be the number of positive integer divisors of n. How many integers $1 \leq n \leq 50$ are there such that $\tau(\tau(n))$ is odd?

24. **[12]** Let P be a point inside regular pentagon $ABCDE$ such that $\angle PAB = 48^\circ$ and $\angle PDC = 42^\circ$. Find $\angle BPC$, in degrees.

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25. **[13]** In acute $\triangle ABC$ with centroid G, $AB = 22$ and $AC = 19$. Let E and F be the feet of the altitudes from B and C to AC and AB respectively. Let G' be the reflection of G over BC. If E, F, G, and G' lie on a circle, compute BC.

26. **[13]** Dan is walking down the left side of a street in New York City and must cross to the right side at one of 10 crosswalks he will pass. Each time he arrives at a crosswalk, however, he must wait t seconds, where t is selected uniformly at random from the real interval $[0, 60]$ (t can be different at different crosswalks). Because the wait time is conveniently displayed on the signal across the street, Dan employs the following strategy: if the wait time when he arrives at the crosswalk is no more than k seconds, he crosses. Otherwise, he immediately moves on to the next crosswalk. If he arrives at the last crosswalk and has not crossed yet, then he crosses regardless of the wait time. Find the value of k which minimizes his expected wait time.

27. **[13]** For a given positive integer n, we define $\varphi(n)$ to be the number of positive integers less than or equal to n which share no common prime factors with n. Find all positive integers n for which

$$\varphi(2019n) = \varphi(n^2).$$

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28. **[15]** A palindrome is a string that does not change when its characters are written in reverse order. Let S be a 40-digit string consisting only of 0's and 1's, chosen uniformly at random out of all such strings. Let E be the expected number of nonempty contiguous substrings of S which are palindromes. Compute the value of $\lfloor E \rfloor$.

29. **[15]** In isosceles $\triangle ABC$, $AB = AC$ and P is a point on side BC. If $\angle BAP = 2\angle CAP$, $BP = \sqrt{3}$, and $CP = 1$, compute AP.

30. **[15]** A function $f : \mathbb{Z} \to \mathbb{Z}$ satisfies: $f(0) = 0$ and

$$\left| f((n+1)2^k) - f(n2^k) \right| \leq 1$$

for all integers $k \geq 0$ and n. What is the maximum possible value of $f(2019)$?

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31. **[17]** James is standing at the point $(0, 1)$ on the coordinate plane and wants to eat a hamburger. For each integer $n \geq 0$, the point $(n, 0)$ has a hamburger with n patties. There is also a wall at $y = 2.1$ which James cannot cross. In each move, James can go either up, right, or down 1 unit as long as he does not cross the wall or visit a point he has already visited.

    Every second, James chooses a valid move uniformly at random, until he reaches a point with a hamburger. Then he eats the hamburger and stops moving. Find the expected number of patties that James eats on his burger.

32. **[17]** A sequence of real numbers $a_0, a_1, \dots, a_9$ with $a_0 = 0$, $a_1 = 1$, and $a_2 > 0$ satisfies

    $$a_{n+2}a_na_{n-1} = a_{n+2} + a_n + a_{n-1}$$

    for all $1 \leq n \leq 7$, but cannot be extended to a_{10}. In other words, no values of $a_{10} \in \mathbb{R}$ satisfy

    $$a_{10}a_8a_7 = a_{10} + a_8 + a_7.$$

    Compute the smallest possible value of a_2.

33. **[17]** A circle Γ with center O has radius 1. Consider pairs (A, B) of points so that A is inside the circle and B is on its boundary. The circumcircle Ω of OAB intersects Γ again at $C \neq B$, and line AC intersects Γ again at $X \neq C$. The pair (A, B) is called *techy* if line OX is tangent to Ω. Find the area of the region of points A so that there exists a B for which (A, B) is techy.

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34. **[20]** A polynomial P with integer coefficients is called *tricky* if it has 4 as a root.

    A polynomial is called k-*tiny* if it has degree at most 7 and integer coefficients between $-k$ and k, inclusive.

    A polynomial is called *nearly tricky* if it is the sum of a tricky polynomial and a 1-tiny polynomial.

    Let N be the number of nearly tricky 7-tiny polynomials. Estimate N.

    An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^4 \right\rfloor$ points.

35. **[20]** You are trying to cross a 400 foot wide river. You can jump at most 4 feet, but you have many stones you can throw into the river. You will stop throwing stones and cross the river once you have placed enough stones to be able to do so. You can throw straight, but you can't judge distance very well, so each stone ends up being placed uniformly at random along the width of the river. Estimate the expected number N of stones you must throw before you can get across the river.

    An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^3 \right\rfloor$ points.

36. **[20]** Let N be the number of sequences of positive integers $(a_1, a_2, a_3, \dots, a_{15})$ for which the polynomials

    $$x^2 - a_ix + a_{i+1}$$

    each have an integer root for every $1 \leq i \leq 15$, setting $a_{16} = a_1$. Estimate N.

    An estimate of E will earn $\left\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \right\rfloor$ points.

HMMT November 2019

November 9, 2019

Team Round

1. **[20]** Each person in Cambridge drinks a (possibly different) 12 ounce mixture of water and apple juice, where each drink has a positive amount of both liquids. Marc McGovern, the mayor of Cambridge, drinks $\frac{1}{6}$ of the total amount of water drunk and $\frac{1}{8}$ of the total amount of apple juice drunk. How many people are in Cambridge?

2. **[20]** 2019 students are voting on the distribution of N items. For each item, each student submits a vote on who should receive that item, and the person with the most votes receives the item (in case of a tie, no one gets the item). Suppose that no student votes for the same person twice. Compute the maximum possible number of items one student can receive, over all possible values of N and all possible ways of voting.

3. **[30]** The coefficients of the polynomial $P(x)$ are nonnegative integers, each less than 100. Given that $P(10) = 331633$ and $P(-10) = 273373$, compute $P(1)$.

4. **[35]** Two players play a game, starting with a pile of N tokens. On each player's turn, they must remove 2^n tokens from the pile for some nonnegative integer n. If a player cannot make a move, they lose. For how many N between 1 and 2019 (inclusive) does the first player have a winning strategy?

5. **[40]** Compute the sum of all positive real numbers $x \le 5$ satisfying

$$x = \frac{\lceil x^2 \rceil + \lceil x \rceil \cdot \lfloor x \rfloor}{\lceil x \rceil + \lfloor x \rfloor}.$$

6. **[45]** Let $ABCD$ be an isosceles trapezoid with $AB = 1$, $BC = DA = 5$, $CD = 7$. Let P be the intersection of diagonals AC and BD, and let Q be the foot of the altitude from D to BC. Let PQ intersect AB at R. Compute $\sin \angle RPD$.

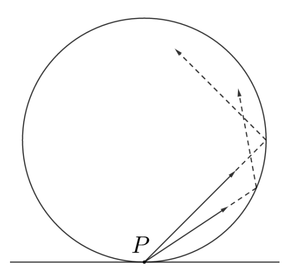
7. **[55]** Consider sequences a of the form $a = (a_1, a_2, \ldots, a_{20})$ such that each term a_i is either 0 or 1. For each such sequence a, we can produce a sequence $b = (b_1, b_2, \ldots, b_{20})$, where

$$b_i = \begin{cases} a_i + a_{i+1} & i = 1 \\ a_{i-1} + a_i + a_{i+1} & 1 < i < 20 \\ a_{i-1} + a_i & i = 20. \end{cases}$$

How many sequences b are there that can be produced by more than one distinct sequence a?

8. **[60]** In $\triangle ABC$, the external angle bisector of $\angle BAC$ intersects line BC at D. E is a point on ray $\overrightarrow{AC}$ such that $\angle BDE = 2\angle ADB$. If $AB = 10$, $AC = 12$, and $CE = 33$, compute $\frac{DB}{DE}$.

9. **[65]** Will stands at a point P on the edge of a circular room with perfectly reflective walls. He shines two laser pointers into the room, forming angles of n° and $(n+1)^\circ$ with the tangent at P, where n is a positive integer less than 90. The lasers reflect off of the walls, illuminating the points they hit on the walls, until they reach P again. (P is also illuminated at the end.) What is the minimum possible number of illuminated points on the walls of the room?



10. **[70]** A convex 2019-gon $A_1A_2\ldots A_{2019}$ is cut into smaller pieces along its 2019 diagonals of the form A_iA_{i+3} for $1 \le i \le 2019$, where $A_{2020} = A_1$, $A_{2021} = A_2$, and $A_{2022} = A_3$. What is the least possible number of resulting pieces?

HMMT November 2019

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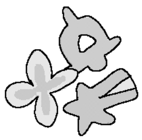
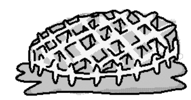
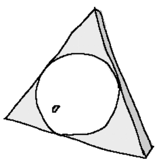
Theme Round



1. For breakfast, Mihir always eats a bowl of Lucky Charms cereal, which consists of oat pieces and marshmallow pieces. He defines the *luckiness* of a bowl of cereal to be the ratio of the number of marshmallow pieces to the total number of pieces. One day, Mihir notices that his breakfast cereal has exactly 90 oat pieces and 9 marshmallow pieces, and exclaims, "This is such an unlucky bowl!" How many marshmallow pieces does Mihir need to add to his bowl to double its luckiness?

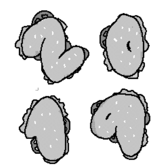
2. Sandy likes to eat waffles for breakfast. To make them, she centers a circle of waffle batter of radius 3cm at the origin of the coordinate plane and her waffle iron imprints non-overlapping unit-square holes **centered** at each lattice point. How many of these holes are contained entirely within the area of the waffle?

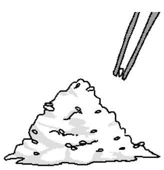




3. For breakfast, Milan is eating a piece of toast shaped like an equilateral triangle. On the piece of toast rests a single sesame seed that is one inch away from one side, two inches away from another side, and four inches away from the third side. He places a circular piece of cheese on top of the toast that is tangent to each side of the triangle. What is the area of this piece of cheese?

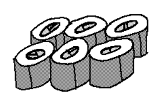
4. To celebrate 2019, Faraz gets four sandwiches shaped in the digits 2, 0, 1, and 9 at lunch. However, the four digits get reordered (but not flipped or rotated) on his plate and he notices that they form a 4-digit multiple of 7. What is the greatest possible number that could have been formed?

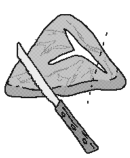




5. Alison is eating 2401 grains of rice for lunch. She eats the rice in a very peculiar manner: every step, if she has only one grain of rice remaining, she eats it. Otherwise, she finds the smallest positive integer $d > 1$ for which she can group the rice into equal groups of size d with none left over. She then groups the rice into groups of size d, eats one grain from each group, and puts the rice back into a single pile. How many steps does it take her to finish all her rice?

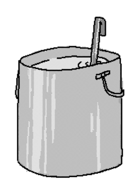
6. Wendy eats sushi for lunch. She wants to eat six pieces of sushi arranged in a 2×3 rectangular grid, but sushi is sticky, and Wendy can only eat a piece if it is adjacent to (not counting diagonally) at most two other pieces. In how many orders can Wendy eat the six pieces of sushi, assuming that the pieces of sushi are distinguishable?

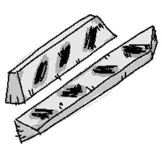




7. Carl only eats food in the shape of equilateral pentagons. Unfortunately, for dinner he receives a piece of steak in the shape of an equilateral triangle. So that he can eat it, he cuts off two corners with straight cuts to form an equilateral pentagon. The set of possible perimeters of the pentagon he obtains is exactly the interval $[a, b)$, where a and b are positive real numbers. Compute $\frac{a}{b}$.

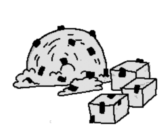
8. Omkar, Krit_1, Krit_2, and Krit_3 are sharing $x > 0$ pints of soup for dinner. Omkar always takes 1 pint of soup (unless the amount left is less than one pint, in which case he simply takes all the remaining soup). Krit_1 always takes $\frac{1}{6}$ of what is left, Krit_2 always takes $\frac{1}{5}$ of what is left, and Krit_3 always takes $\frac{1}{4}$ of what is left. They take soup in the order of Omkar, Krit_1, Krit_2, Krit_3, and then cycle through this order until no soup remains. Find all x for which everyone gets the same amount of soup.





9. For dinner, Priya is eating grilled pineapple spears. Each spear is in the shape of the quadrilateral $PINE$, with $PI = 6\text{cm}$, $IN = 15\text{cm}$, $NE = 6\text{cm}$, $EP = 25\text{cm}$, and $\angle NEP + \angle EPI = 60°$. What is the area of each spear, in cm^2?

10. For dessert, Melinda eats a spherical scoop of ice cream with diameter 2 inches. She prefers to eat her ice cream in cube-like shapes, however. She has a special machine which, given a sphere placed in space, cuts it through the planes $x = n$, $y = n$, and $z = n$ for every integer n (not necessarily positive). Melinda centers the scoop of ice cream uniformly at random inside the cube $0 \le x, y, z \le 1$, and then cuts it into pieces using her machine. What is the expected number of pieces she cuts the ice cream into?



HMIC 2019

March 30, 2019 – April 5, 2019

1. [**5**] Let ABC be an acute scalene triangle with incenter I. Show that the circumcircle of BIC intersects the Euler line of ABC in two distinct points.

   (Recall that the *Euler line* of a scalene triangle is the line that passes through its circumcenter, centroid, orthocenter, and the nine-point center.)

2. [**7**] Annie has a permutation $(a_1, a_2, \ldots, a_{2019})$ of $S = \{1, 2, \ldots, 2019\}$, and Yannick wants to guess her permutation. With each guess Yannick gives Annie an n-tuple $(y_1, y_2, \ldots, y_{2019})$ of integers in S, and then Annie gives the number of indices $i \in S$ such that $a_i = y_i$.

   (a) Show that Yannick can always guess Annie's permutation with at most 1200000 guesses.

   (b) Show that Yannick can always guess Annie's permutation with at most 24000 guesses.

3. [**8**] Do there exist four points $P_i = (x_i, y_i) \in \mathbb{R}^2$ $(1 \le i \le 4)$ on the plane such that:

   - for all $i = 1, 2, 3, 4$, the inequality $x_i^4 + y_i^4 \le x_i^3 + y_i^3$ holds, and
   - for all $i \ne j$, the distance between P_i and P_j is greater than 1?

4. [**10**] A *cactus* is a finite simple connected graph where no two cycles share an edge. Show that in a nonempty cactus, there must exist a vertex which is part of at most one cycle.

5. [**12**] Let $p = 2017$ be a prime and $\mathbb{F}_p$ be the integers modulo p. A function $f : \mathbb{Z} \to \mathbb{F}_p$ is called *good* if there is $\alpha \in \mathbb{F}_p$ with $\alpha \not\equiv 0 \pmod{p}$ such that

   $$f(x)f(y) = f(x+y) + \alpha^y f(x-y) \pmod{p}$$

   for all $x, y \in \mathbb{Z}$. How many good functions are there that are periodic with minimal period 2016?

HMMT February 2018

February 10, 2018

Algebra and Number Theory

1. For some real number c, the graphs of the equation $y = |x - 20| + |x + 18|$ and the line $y = x + c$ intersect at exactly one point. What is c?

2. Compute the positive real number x satisfying
$$x^{(2x^6)} = 3.$$

3. There are two prime numbers p so that $5p$ can be expressed in the form $\left\lfloor \frac{n^2}{5} \right\rfloor$ for some positive integer n. What is the sum of these two prime numbers?

4. Distinct prime numbers p, q, r satisfy the equation
$$2pqr + 50pq = 7pqr + 55pr = 8pqr + 12qr = A$$
for some positive integer A. What is A?

5. Let $\omega_1, \omega_2, \ldots, \omega_{100}$ be the roots of $\frac{x^{101}-1}{x-1}$ (in some order). Consider the set
$$S = \{\omega_1^1, \omega_2^2, \omega_3^3, \ldots, \omega_{100}^{100}\}.$$
Let M be the maximum possible number of unique values in S, and let N be the minimum possible number of unique values in S. Find $M - N$.

6. Let α, β, and γ be three real numbers. Suppose that
$$\cos\alpha + \cos\beta + \cos\gamma = 1$$
$$\sin\alpha + \sin\beta + \sin\gamma = 1.$$
Find the smallest possible value of $\cos\alpha$.

7. Rachel has the number 1000 in her hands. When she puts the number x in her left pocket, the number changes to $x + 1$. When she puts the number x in her right pocket, the number changes to x^{-1}. Each minute, she flips a fair coin. If it lands heads, she puts the number into her left pocket, and if it lands tails, she puts it into her right pocket. She then takes the new number out of her pocket. If the expected value of the number in Rachel's hands after eight minutes is E, then compute $\lfloor \frac{E}{10} \rfloor$.

8. For how many pairs of sequences of nonnegative integers $(b_1, b_2, \ldots, b_{2018})$ and $(c_1, c_2, \ldots, c_{2018})$ does there exist a sequence of nonnegative integers $(a_0, \ldots, a_{2018})$ with the following properties:
   - For $0 \le i \le 2018$, $a_i < 2^{2018}$;
   - For $1 \le i \le 2018$, $b_i = a_{i-1} + a_i$ and $c_i = a_{i-1} | a_i$;

   where $|$ denotes the bitwise or operation?

   (The *bitwise or* of two nonnegative integers $x = \cdots x_3 x_2 x_1 x_0$ and $y = \cdots y_3 y_2 y_1 y_0$ expressed in binary is defined as $x|y = \cdots z_3 z_2 z_1 z_0$, where $z_i = 1$ if at least one of x_i and y_i is 1, and 0 otherwise.)

9. Assume the quartic $x^4 - ax^3 + bx^2 - ax + d = 0$ has four real roots $\frac{1}{2} \le x_1, x_2, x_3, x_4 \le 2$. Find the maximum possible value of $\frac{(x_1+x_2)(x_1+x_3)x_4}{(x_4+x_2)(x_4+x_3)x_1}$ (over all valid choices of a, b, d).

10. Let S be a randomly chosen 6-element subset of the set $\{0, 1, 2, \ldots, n\}$. Consider the polynomial $P(x) = \sum_{i \in S} x^i$. Let X_n be the probability that $P(x)$ is divisible by some nonconstant polynomial $Q(x)$ of degree at most 3 with integer coefficients satisfying $Q(0) \ne 0$. Find the limit of X_n as n goes to infinity.

HMMT February 2018

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Combinatorics

1. Consider a 2×3 grid where each entry is one of 0, 1, and 2. For how many such grids is the sum of the numbers in every row and in every column a multiple of 3? One valid grid is shown below.

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

2. Let a and b be five-digit palindromes (without leading zeroes) such that $a < b$ and there are no other five-digit palindromes strictly between a and b. What are all possible values of $b - a$? (A number is a palindrome if it reads the same forwards and backwards in base 10.)

3. A 4×4 window is made out of 16 square windowpanes. How many ways are there to stain each of the windowpanes, red, pink, or magenta, such that each windowpane is the same color as exactly two of its neighbors? Two different windowpanes are neighbors if they share a side.

4. How many ways are there for Nick to travel from $(0, 0)$ to $(16, 16)$ in the coordinate plane by moving one unit in the positive x or y direction at a time, such that Nick changes direction an odd number of times?

5. A bag contains nine blue marbles, ten ugly marbles, and one special marble. Ryan picks marbles randomly from this bag with replacement until he draws the special marble. He notices that none of the marbles he drew were ugly. Given this information, what is the expected value of the number of total marbles he drew?

6. Sarah stands at $(0, 0)$ and Rachel stands at $(6, 8)$ in the Euclidean plane. Sarah can only move 1 unit in the positive x or y direction, and Rachel can only move 1 unit in the negative x or y direction. Each second, Sarah and Rachel see each other, independently pick a direction to move at the same time, and move to their new position. Sarah catches Rachel if Sarah and Rachel are ever at the same point. Rachel wins if she is able to get to $(0, 0)$ without being caught; otherwise, Sarah wins. Given that both of them play optimally to maximize their probability of winning, what is the probability that Rachel wins?

7. A tourist is learning an incorrect way to sort a permutation $(p_1, \ldots, p_n)$ of the integers $(1, \ldots, n)$. We define a *fix* on two adjacent elements p_i and p_{i+1}, to be an operation which swaps the two elements if $p_i > p_{i+1}$, and does nothing otherwise. The tourist performs $n - 1$ rounds of fixes, numbered $a = 1, 2, \ldots, n - 1$. In round a of fixes, the tourist fixes p_a and p_{a+1}, then p_{a+1} and p_{a+2}, and so on, up to p_{n-1} and p_n. In this process, there are $(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2}$ total fixes performed.

   How many permutations of $(1, \ldots, 2018)$ can the tourist start with to obtain $(1, \ldots, 2018)$ after performing these steps?

8. A permutation of $\{1, 2, \ldots, 7\}$ is chosen uniformly at random. A partition of the permutation into contiguous blocks is correct if, when each block is sorted independently, the entire permutation becomes sorted. For example, the permutation $(3, 4, 2, 1, 6, 5, 7)$ can be partitioned correctly into the blocks $[3, 4, 2, 1]$ and $[6, 5, 7]$, since when these blocks are sorted, the permutation becomes $(1, 2, 3, 4, 5, 6, 7)$.

   Find the expected value of the maximum number of blocks into which the permutation can be partitioned correctly.

9. How many ordered sequences of 36 digits have the property that summing the digits to get a number and taking the last digit of the sum results in a digit which is not in our original sequence? (Digits range from 0 to 9.)

10. Lily has a 300×300 grid of squares. She now removes 100×100 squares from each of the four corners and colors each of the remaining 50000 squares black and white. Given that no 2×2 square is colored in a checkerboard pattern, find the maximum possible number of (unordered) pairs of squares such that one is black, one is white and the squares share an edge.

HMMT February 2018

February 10, 2018

Geometry

1. Triangle GRT has $GR = 5$, $RT = 12$, and $GT = 13$. The perpendicular bisector of GT intersects the extension of GR at O. Find TO.

2. Points A, B, C, D are chosen in the plane such that segments AB, BC, CD, DA have lengths 2, 7, 5, 12, respectively. Let m be the minimum possible value of the length of segment AC and let M be the maximum possible value of the length of segment AC. What is the ordered pair (m, M)?

3. How many noncongruent triangles are there with one side of length 20, one side of length 17, and one 60° angle?

4. A paper equilateral triangle of side length 2 on a table has vertices labeled A, B, C. Let M be the point on the sheet of paper halfway between A and C. Over time, point M is lifted upwards, folding the triangle along segment BM, while A, B, and C remain on the table. This continues until A and C touch. Find the maximum volume of tetrahedron $ABCM$ at any time during this process.

5. In the quadrilateral $MARE$ inscribed in a unit circle ω, AM is a diameter of ω, and E lies on the angle bisector of $\angle RAM$. Given that triangles RAM and REM have the same area, find the area of quadrilateral $MARE$.

6. Let ABC be an equilateral triangle of side length 1. For a real number $0 < x < 0.5$, let A_1 and A_2 be the points on side BC such that $A_1B = A_2C = x$, and let $T_A = \triangle AA_1A_2$. Construct triangles $T_B = \triangle BB_1B_2$ and $T_C = \triangle CC_1C_2$ similarly.

   There exist positive rational numbers b, c such that the region of points inside all three triangles T_A, T_B, T_C is a hexagon with area
   $$\frac{8x^2 - bx + c}{(2-x)(x+1)} \cdot \frac{\sqrt{3}}{4}.$$
   Find (b, c).

7. Triangle ABC has sidelengths $AB = 14$, $AC = 13$, and $BC = 15$. Point D is chosen in the interior of $\overline{AB}$ and point E is selected uniformly at random from $\overline{AD}$. Point F is then defined to be the intersection point of the perpendicular to $\overline{AB}$ at E and the union of segments $\overline{AC}$ and $\overline{BC}$. Suppose that D is chosen such that the expected value of the length of $\overline{EF}$ is maximized. Find AD.

8. Let ABC be an equilateral triangle with side length 8. Let X be on side AB so that $AX = 5$ and Y be on side AC so that $AY = 3$. Let Z be on side BC so that AZ, BY, CX are concurrent. Let ZX, ZY intersect the circumcircle of AXY again at P, Q respectively. Let XQ and YP intersect at K. Compute $KX \cdot KQ$.

9. Po picks 100 points $P_1, P_2, \ldots, P_{100}$ on a circle independently and uniformly at random. He then draws the line segments connecting P_1P_2, $P_2P_3, \ldots, P_{100}P_1$. When all of the line segments are drawn, the circle is divided into a number of regions. Find the expected number of regions that have all sides bounded by straight lines.

10. Let ABC be a triangle such that $AB = 6, BC = 5, AC = 7$. Let the tangents to the circumcircle of ABC at B and C meet at X. Let Z be a point on the circumcircle of ABC. Let Y be the foot of the perpendicular from X to CZ. Let K be the intersection of the circumcircle of BCY with line AB. Given that Y is on the interior of segment CZ and $YZ = 3CY$, compute AK.

HMMT February 2018, February 10, 2018 — GUTS ROUND

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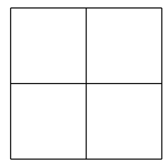
1. [**4**] A square can be divided into four congruent figures as shown:



If each of the congruent figures has area 1, what is the area of the square?

2. [**4**] John has a 1 liter bottle of pure orange juice. He pours half of the contents of the bottle into a vat, fills the bottle with water, and mixes thoroughly. He then repeats this process 9 more times. Afterwards, he pours the remaining contents of the bottle into the vat. What fraction of the liquid in the vat is now water?

3. [**4**] Allen and Yang want to share the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$. How many ways are there to split all ten numbers among Allen and Yang so that each person gets at least one number, and either Allen's numbers or Yang's numbers sum to an even number?

4. [**4**] Find the sum of the digits of $11 \cdot 101 \cdot 111 \cdot 110011$.

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5. [**6**] Randall proposes a new temperature system called *Felsius* temperature with the following conversion between Felsius $^\circ E$, Celsius $^\circ C$, and Fahrenheit $^\circ F$:

$$^\circ E = \frac{7 \times {}^\circ C}{5} + 16 = \frac{7 \times {}^\circ F - 80}{9}.$$

For example, $0\,^\circ C = 16\,^\circ E$. Let x, y, z be real numbers such that $x\,^\circ C = x\,^\circ E$, $y\,^\circ E = y\,^\circ F$, $z\,^\circ C = z\,^\circ F$. Find $x + y + z$.

6. [**6**] A bug is on a corner of a cube. A *healthy* path for the bug is a path along the edges of the cube that starts and ends where the bug is located, uses no edge multiple times, and uses at most two of the edges adjacent to any particular face. Find the number of healthy paths.

7. [**6**] A triple of integers (a, b, c) satisfies $a + bc = 2017$ and $b + ca = 8$. Find all possible values of c.

8. [**6**] Suppose a real number $x > 1$ satisfies

$$\log_2(\log_4 x) + \log_4(\log_{16} x) + \log_{16}(\log_2 x) = 0.$$

Compute

$$\log_2(\log_{16} x) + \log_{16}(\log_4 x) + \log_4(\log_2 x).$$

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9. [**7**] In a game, N people are in a room. Each of them simultaneously writes down an integer between 0 and 100 inclusive. A person wins the game if their number is exactly two-thirds of the average of all the numbers written down. There can be multiple winners or no winners in this game. Let m be the maximum possible number such that it is possible to win the game by writing down m. Find the smallest possible value of N for which it is possible to win the game by writing down m in a room of N people.

10. [**7**] Let a positive integer n be called a *cubic square* if there exist positive integers a, b with $n = \gcd(a^2, b^3)$. Count the number of cubic squares between 1 and 100 inclusive.

11. [**7**] FInd the value of

$$\sum_{k=1}^{60} \sum_{n=1}^{k} \frac{n^2}{61 - 2n}.$$

12. [**7**] $\triangle PNR$ has side lengths $PN = 20$, $NR = 18$, and $PR = 19$. Consider a point A on PN. $\triangle NRA$ is rotated about R to $\triangle N'RA'$ so that R, N', and P lie on the same line and AA' is perpendicular to PR. Find $\frac{PA}{AN}$.

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13. [**9**] Suppose $\triangle ABC$ has lengths $AB = 5$, $BC = 8$, and $CA = 7$, and let ω be the circumcircle of $\triangle ABC$. Let X be the second intersection of the external angle bisector of $\angle B$ with ω, and let Y be the foot of the perpendicular from X to BC. Find the length of YC.

14. [**9**] Given that x is a positive real, find the maximum possible value of

$$\sin\left(\tan^{-1}\left(\frac{x}{9}\right) - \tan^{-1}\left(\frac{x}{16}\right)\right).$$

15. [**9**] Michael picks a random subset of the complex numbers $\{1, \omega, \omega^2, \ldots, \omega^{2017}\}$ where ω is a primitive 2018^{th} root of unity and all subsets are equally likely to be chosen. If the sum of the elements in his subset is S, what is the expected value of $|S|^2$? (The sum of the elements of the empty set is 0.)

16. [**9**] Solve for x:

$$x\lfloor x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor\rfloor = 122.$$

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17. [**10**] Compute the value of
$$\frac{\cos 30.5^\circ + \cos 31.5^\circ + ... + \cos 44.5^\circ}{\sin 30.5^\circ + \sin 31.5^\circ + ... + \sin 44.5^\circ}.$$

18. [**10**] Compute the number of integers $n \in \{1, 2, \ldots, 300\}$ such that n is the product of two distinct primes, and is also the length of the longest leg of some nondegenerate right triangle with integer side lengths.

19. [**10**] Suppose there are 100 cookies arranged in a circle, and 53 of them are chocolate chip, with the remainder being oatmeal. Pearl wants to choose a contiguous subsegment of exactly 67 cookies and wants this subsegment to have exactly k chocolate chip cookies. Find the sum of the k for which Pearl is guaranteed to succeed regardless of how the cookies are arranged.

20. [**10**] Triangle $\triangle ABC$ has $AB = 21$, $BC = 55$, and $CA = 56$. There are two points P in the plane of $\triangle ABC$ for which $\angle BAP = \angle CAP$ and $\angle BPC = 90^\circ$. Find the distance between them.

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21. [**12**] You are the first lucky player to play in a slightly modified episode of Deal or No Deal! Initially, there are sixteen cases marked 1 through 16. The dollar amounts in the cases are the powers of 2 from $2^1 = 2$ to $2^{16} = 65536$, in some random order. The game has eight turns. In each turn, you choose a case and claim it, without opening it. Afterwards, a random remaining case is opened and revealed to you, then removed from the game.

    At the end of the game, all eight of your cases are revealed and you win all of the money inside them.

    However, the hosts do not realize you have X-ray vision and can see the amount of money inside each case! What is the expected amount of money you will make, given that you play optimally?

22. [**12**] How many graphs are there on 10 vertices labeled $1, 2, \ldots, 10$ such that there are exactly 23 edges and no triangles?

23. [**12**] Kevin starts with the vectors $(1, 0)$ and $(0, 1)$ and at each time step, he replaces one of the vectors with their sum. Find the cotangent of the minimum possible angle between the vectors after 8 time steps.

24. [**12**] Find the largest positive integer n for which there exist n finite sets $X_1, X_2, \ldots, X_n$ with the property that for every $1 \leq a < b < c \leq n$, the equation
$$|X_a \cup X_b \cup X_c| = \left\lceil \sqrt{abc} \right\rceil$$
holds.

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25. [**15**] Fran writes the numbers $1, 2, 3, \ldots, 20$ on a chalkboard. Then she erases all the numbers by making a series of moves; in each move, she chooses a number n uniformly at random from the set of all numbers still on the chalkboard, and then erases all of the divisors of n that are still on the chalkboard (including n itself). What is the expected number of moves that Fran must make to erase all the numbers?

26. [**15**] Let ABC be a triangle with $\angle A = 18°, \angle B = 36°$. Let M be the midpoint of AB, D a point on ray CM such that $AB = AD$; E a point on ray BC such that $AB = BE$, and F a point on ray AC such that $AB = AF$. Find $\angle FDE$.

27. [**15**] There are 2018 frogs in a pool and there is 1 frog on the shore. In each time-step thereafter, one random frog moves position. If it was in the pool, it jumps to the shore, and vice versa. Find the expected number of time-steps before all frogs are in the pool for the first time.

28. [**15**] Arnold and Kevin are playing a game in which Kevin picks an integer $1 \leq m \leq 1001$, and Arnold is trying to guess it. On each turn, Arnold first pays Kevin 1 dollar in order to guess a number k of Arnold's choice. If $m \geq k$, the game ends and he pays Kevin an additional $m - k$ dollars (possibly zero). Otherwise, Arnold pays Kevin an additional 10 dollars and continues guessing.

    Which number should Arnold guess first to ensure that his worst-case payment is minimized?

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29. [**17**] Let a, b, c be positive integers. All the roots of each of the quadratics

$$ax^2 + bx + c, ax^2 + bx - c, ax^2 - bx + c, ax^2 - bx - c$$

    are integers. Over all triples (a, b, c), find the triple with the third smallest value of $a + b + c$.

30. [**17**] Find the number of *unordered* pairs $\{a, b\}$, where $a, b \in \{0, 1, 2, \ldots, 108\}$ such that 109 divides $a^3 + b^3 - ab$.

31. [**17**] In triangle ABC, $AB = 6$, $BC = 7$ and $CA = 8$. Let D, E, F be the midpoints of sides BC, AC, AB, respectively. Also let O_A, O_B, O_C be the circumcenters of triangles AFD, BDE, and CEF, respectively. Find the area of triangle $O_A O_B O_C$.

32. [**17**] How many 48-tuples of positive integers $(a_1, a_2, \ldots, a_{48})$ between 0 and 100 inclusive have the property that for all $1 \leq i < j \leq 48$, $a_i \notin \{a_j, a_j + 1\}$?

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33. [**20**] 679 contestants participated in HMMT February 2017. Let N be the number of these contestants who performed at or above the median score in at least one of the three individual tests. Estimate N.

    An estimate of E earns $\left\lfloor 20 - \frac{|E-N|}{2} \right\rfloor$ or 0 points, whichever is greater.

34. [**20**] The integers $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ are written on a blackboard. Each day, a teacher chooses one of the integers uniformly at random and decreases it by 1. Let X be the expected value of the number of days which elapse before there are no longer positive integers on the board. Estimate X.

    An estimate of E earns $\lfloor 20 \cdot 2^{-|X-E|/8} \rfloor$ points.

35. [**20**] In a wooden block shaped like a cube, all the vertices and edge midpoints are marked. The cube is cut along *all* possible planes that pass through at least four marked points. Let N be the number of pieces the cube is cut into. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min(N/E, E/N) \rfloor$ points.

36. [**20**] In the game of Connect Four, there are seven vertical columns which have spaces for six tokens. These form a 7×6 grid of spaces. Two players White and Black move alternately. A player takes a turn by picking a column which is not already full and dropping a token of their color into the lowest unoccupied space in that column. The game ends when there are four consecutive tokens of the same color in a line, either horizontally, vertically, or diagonally. The player who has four tokens in a row of their color wins.

    Assume two players play this game randomly. Each player, on their turn, picks a random column which is not full and drops a token of their color into that column. This happens until one player wins or all of the columns are filled. Let P be the probability that all of the columns are filled without any player obtaining four tokens in a row of their color. Estimate P.

    An estimate of $E > 0$ earns $\lfloor 20 \min(P/E, E/P) \rfloor$ points.

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Team Round

1. [**20**] In an $n \times n$ square array of 1×1 cells, at least one cell is colored pink. Show that you can always divide the square into rectangles along cell borders such that each rectangle contains exactly one pink cell.

2. [**25**] Is the number
$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\left(1+\frac{1}{6}\right)\cdots\left(1+\frac{1}{2018}\right)$$
greater than, less than, or equal to 50?

3. [**30**] Michelle has a word with 2^n letters, where a word can consist of letters from any alphabet. Michelle performs a *switcheroo* on the word as follows: for each $k = 0, 1, \ldots, n-1$, she switches the first 2^k letters of the word with the next 2^k letters of the word. For example, for $n = 3$, Michelle changes
$$ABCDEFGH \to BACDEFGH \to CDBAEFGH \to EFGHCDBA$$
in one switcheroo.

   In terms of n, what is the minimum positive integer m such that after Michelle performs the switcheroo operation m times on any word of length 2^n, she will receive her original word?

4. [**30**] In acute triangle ABC, let D, E, and F be the feet of the altitudes from A, B, and C respectively, and let L, M, and N be the midpoints of BC, CA, and AB, respectively. Lines DE and NL intersect at X, lines DF and LM intersect at Y, and lines XY and BC intersect at Z. Find $\frac{ZB}{ZC}$ in terms of AB, AC, and BC.

5. [**30**] Is it possible for the projection of the set of points (x, y, z) with $0 \le x, y, z \le 1$ onto some two-dimensional plane to be a simple convex pentagon?

6. [**35**] Let $n \ge 2$ be a positive integer. A subset of positive integers S is said to be *comprehensive* if for every integer $0 \le x < n$, there is a subset of S whose sum has remainder x when divided by n. Note that the empty set has sum 0. Show that if a set S is comprehensive, then there is some (not necessarily proper) subset of S with at most $n-1$ elements which is also comprehensive.

7. [**50**] Let $[n]$ denote the set of integers $\{1, 2, \ldots, n\}$. We randomly choose a function $f : [n] \to [n]$, out of the n^n possible functions. We also choose an integer a uniformly at random from $[n]$. Find the probability that there exist positive integers $b, c \ge 1$ such that $f^b(1) = a$ and $f^c(a) = 1$. ($f^k(x)$ denotes the result of applying f to x k times).

8. [**60**] Allen plays a game on a tree with $2n$ vertices, each of whose vertices can be red or blue. Initially, all of the vertices of the tree are colored red. In one move, Allen is allowed to take two vertices of the same color which are connected by an edge and change both of them to the opposite color. He wins if at any time, all of the vertices of the tree are colored blue.

   (a) (**20**) Show that Allen can win if and only if the vertices can be split up into two groups V_1 and V_2 of size n, such that each edge in the tree has one endpoint in V_1 and one endpoint in V_2.

   (b) (**40**) Let $V_1 = \{a_1, \ldots, a_n\}$ and $V_2 = \{b_1, \ldots, b_n\}$ from part (a). Let M be the minimum over all permutations σ of $\{1, \ldots, n\}$ of the quantity
$$\sum_{i=1}^{n} d(a_i, b_{\sigma(i)}),$$
   where $d(v, w)$ denotes the number of edges along the shortest path between vertices v and w in the tree.

   Show that if Allen can win, then the minimum number of moves that it can take for Allen to win is equal to M.

(A *graph* consists of a set of vertices and some edges between distinct pairs of vertices. It is *connected* if every pair of vertices are connected by some path of one or more edges. A *tree* is a graph which is connected, in which the number of edges is one less than the number of vertices.)

9. [**60**] Evan has a simple graph with v vertices and e edges. Show that he can delete at least $\frac{e-v+1}{2}$ edges so that each vertex still has at least half of its original degree.

10. [**60**] Let n and m be positive integers which are at most 10^{10}. Let R be the rectangle with corners at $(0,0),(n,0),(n,m),(0,m)$ in the coordinate plane. A simple non-self-intersecting quadrilateral with vertices at integer coordinates is called *far-reaching* if each of its vertices lie on or inside R, but each side of R contains at least one vertex of the quadrilateral. Show that there is a far-reaching quadrilateral with area at most 10^6.

    (A side of a rectangle includes the two endpoints.)

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November 10, 2018

General Round

1. What is the largest factor of 130000 that does not contain the digit 0 or 5?

2. Twenty-seven players are randomly split into three teams of nine. Given that Zack is on a different team from Mihir and Mihir is on a different team from Andrew, what is the probability that Zack and Andrew are on the same team?

3. A square in the xy-plane has area A, and three of its vertices have x-coordinates $2, 0$, and 18 in some order. Find the sum of all possible values of A.

4. Find the number of eight-digit positive integers that are multiples of 9 and have all distinct digits.

5. Compute the smallest positive integer n for which
$$\sqrt{100+\sqrt{n}}+\sqrt{100-\sqrt{n}}$$
is an integer.

6. Call a polygon *normal* if it can be inscribed in a unit circle. How many non-congruent normal polygons are there such that the square of each side length is a positive integer?

7. Anders is solving a math problem, and he encounters the expression $\sqrt{15!}$. He attempts to simplify this radical by expressing it as $a\sqrt{b}$ where a and b are positive integers. The sum of all possible distinct values of ab can be expressed in the form $q \cdot 15!$ for some rational number q. Find q.

8. Equilateral triangle ABC has circumcircle Ω. Points D and E are chosen on minor arcs AB and AC of Ω respectively such that $BC = DE$. Given that triangle ABE has area 3 and triangle ACD has area 4, find the area of triangle ABC.

9. 20 players are playing in a Super Smash Bros. Melee tournament. They are ranked $1-20$, and player n will always beat player m if $n < m$. Out of all possible tournaments where each player plays 18 distinct other players exactly once, one is chosen uniformly at random. Find the expected number of pairs of players that win the same number of games.

10. Real numbers x, y, and z are chosen from the interval $[-1, 1]$ independently and uniformly at random. What is the probability that
$$|x|+|y|+|z|+|x+y+z| = |x+y|+|y+z|+|z+x|?$$

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1. [**5**] A positive integer is called *primer* if it has a prime number of distinct prime factors. Find the smallest primer number.

2. [**5**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \leq k \leq n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of all numbers in the 2018th row?

3. [**5**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×2 mazes.

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4. [**6**] Let a, b, c, n be positive real numbers such that $\frac{a+b}{a} = 3$, $\frac{b+c}{b} = 4$, and $\frac{c+a}{c} = n$. Find n.

5. [**6**] Jerry has ten distinguishable coins, each of which currently has heads facing up. He chooses one coin and flips it over, so it now has tails facing up. Then he picks another coin (possibly the same one as before) and flips it over. How many configurations of heads and tails are possible after these two flips?

6. [**6**] An equilateral hexagon with side length 1 has interior angles $90°, 120°, 150°, 90°, 120°, 150°$ in that order. Find its area.

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7. [**7**] At Easter-Egg Academy, each student has two eyes, each of which can be eggshell, cream, or cornsilk. It is known that 30% of the students have at least one eggshell eye, 40% of the students have at least one cream eye, and 50% of the students have at least one cornsilk eye. What percentage of the students at Easter-Egg Academy have two eyes of the same color?

8. [**7**] Pentagon $JAMES$ is such that $AM = SJ$ and the internal angles satisfy $\angle J = \angle A = \angle E = 90^\circ$, and $\angle M = \angle S$. Given that there exists a diagonal of $JAMES$ that bisects its area, find the ratio of the shortest side of $JAMES$ to the longest side of $JAMES$.

9. [**7**] Farmer James has some strange animals. His hens have 2 heads and 8 legs, his peacocks have 3 heads and 9 legs, and his zombie hens have 6 heads and 12 legs. Farmer James counts 800 heads and 2018 legs on his farm. What is the number of animals that Farmer James has on his farm?

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10. [**8**] Abbot writes the letter A on the board. Every minute, he replaces every occurrence of A with AB and every occurrence of B with BA, hence creating a string that is twice as long. After 10 minutes, there are $2^{10} = 1024$ letters on the board. How many adjacent pairs are the same letter?

11. [**8**] Let $\triangle ABC$ be an acute triangle, with M being the midpoint of $\overline{BC}$, such that $AM = BC$. Let D and E be the intersection of the internal angle bisectors of $\angle AMB$ and $\angle AMC$ with AB and AC, respectively. Find the ratio of the area of $\triangle DME$ to the area of $\triangle ABC$.

12. [**8**] Consider an unusual biased coin, with probability p of landing heads, probability $q \leq p$ of landing tails, and probability $\frac{1}{6}$ of landing on its side (i.e. on neither face). It is known that if this coin is flipped twice, the likelihood that both flips will have the same result is $\frac{1}{2}$. Find p.

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13. [**9**] Find the smallest positive integer n for which

$$1!2!\cdots(n-1)! > n!^2.$$

14. [**9**] Call a triangle *nice* if the plane can be tiled using congruent copies of this triangle so that any two triangles that share an edge (or part of an edge) are reflections of each other via the shared edge. How many dissimilar nice triangles are there?

15. [**9**] On a computer screen is the single character `a`. The computer has two keys: `c` (copy) and `p` (paste), which may be pressed in any sequence.

    Pressing `p` increases the number of `a`'s on screen by the number that were there the last time `c` was pressed. `c` doesn't change the number of `a`'s on screen. Determine the fewest number of keystrokes required to attain at least 2018 `a`'s on screen. (**Note**: pressing `p` before the first press of `c` does nothing).

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16. [**10**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. Find the smallest primest number.

17. [**10**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \le k \le n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. What is the sum of the absolute values of all numbers in the 2018th row?

18. [**10**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Determine the number of solvable 2×5 mazes.

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19. [**11**] Let A be the number of unordered pairs of ordered pairs of integers between 1 and 6 inclusive, and let B be the number of ordered pairs of unordered pairs of integers between 1 and 6 inclusive. (Repetitions are allowed in both ordered and unordered pairs.) Find $A - B$.

20. [**11**] Let z be a complex number. In the complex plane, the distance from z to 1 is 2, and the distance from z^2 to 1 is 6. What is the real part of z?

21. [**11**] A function $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ is said to be *nasty* if there do not exist distinct $a, b \in \{1, 2, 3, 4, 5\}$ satisfying $f(a) = b$ and $f(b) = a$. How many nasty functions are there?

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22. [**12**] In a square of side length 4, a point on the interior of the square is randomly chosen and a circle of radius 1 is drawn centered at the point. What is the probability that the circle intersects the square exactly twice?

23. [**12**] Let S be a subset with four elements chosen from $\{1, 2, \ldots, 10\}$. Michael notes that there is a way to label the vertices of a square with elements from S such that no two vertices have the same label, and the labels adjacent to any side of the square differ by at least 4. How many possibilities are there for the subset S?

24. [**12**] Let $ABCD$ be a convex quadrilateral so that all of its sides and diagonals have integer lengths. Given that $\angle ABC = \angle ADC = 90°$, $AB = BD$, and $CD = 41$, find the length of BC.

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25. [**13**] Let $a_0, a_1, \ldots$ and $b_0, b_1, \ldots$ be geometric sequences with common ratios r_a and r_b, respectively, such that

$$\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i = 1 \quad \text{and} \quad \left(\sum_{i=0}^{\infty} a_i^2\right)\left(\sum_{i=0}^{\infty} b_i^2\right) = \sum_{i=0}^{\infty} a_i b_i.$$

Find the smallest real number c such that $a_0 < c$ must be true.

26. [**13**] Points E, F, G, H are chosen on segments AB, BC, CD, DA, respectively, of square $ABCD$. Given that segment EG has length 7, segment FH has length 8, and that EG and FH intersect inside $ABCD$ at an acute angle of 30°, then compute the area of square $ABCD$.

27. [**13**] At lunch, Abby, Bart, Carl, Dana, and Evan share a pizza divided radially into 16 slices. Each one takes takes one slice of pizza uniformly at random, leaving 11 slices. The remaining slices of pizza form "sectors" broken up by the taken slices, e.g. if they take five consecutive slices then there is one sector, but if none of them take adjacent slices then there will be five sectors. What is the expected number of sectors formed?

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28. [**15**] What is the 3-digit number formed by the 9998^{th} through 10000^{th} digits after the decimal point in the decimal expansion of $\frac{1}{998}$?

Note: Make sure your answer has exactly three digits, so please include any leading zeroes if necessary.

29. [**15**] An isosceles right triangle ABC has area 1. Points D, E, F are chosen on BC, CA, AB respectively such that DEF is also an isosceles right triangle. Find the smallest possible area of DEF.

30. [**15**] Let n be a positive integer. Let there be P_n ways for Pretty Penny to make exactly n dollars out of quarters, dimes, nickels, and pennies. Also, let there be B_n ways for Beautiful Bill to make exactly n dollars out of one dollar bills, quarters, dimes, and nickels. As n goes to infinity, the sequence of fractions $\frac{P_n}{B_n}$ approaches a real number c. Find c.

Note: Assume both Pretty Penny and Beautiful Bill each have an unlimited number of each type of coin. Pennies, nickels, dimes, quarters, and dollar bills are worth 1, 5, 10, 25, 100 cents respectively.

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31. [**17**] David and Evan each repeatedly flip a fair coin. David will stop when he flips a tail, and Evan will stop once he flips 2 consecutive tails. Find the probability that David flips more total heads than Evan.

32. [**17**] Over all real numbers x and y, find the minimum possible value of

$$(xy)^2 + (x+7)^2 + (2y+7)^2.$$

33. [**17**] Let ABC be a triangle with $AB = 20, BC = 10, CA = 15$. Let I be the incenter of ABC, and let BI meet AC at E and CI meet AB at F. Suppose that the circumcircles of BIF and CIE meet at a point D different from I. Find the length of the tangent from A to the circumcircle of DEF.

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34. [**20**] A positive integer is called *primer* if it has a prime number of distinct prime factors. A positive integer is called *primest* if it has a primer number of distinct primer factors. A positive integer is called *prime-minister* if it has a primest number of distinct primest factors. Let N be the smallest prime-minister number. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right) \rfloor$ points.

35. [**20**] Pascal has a triangle. In the nth row, there are $n+1$ numbers $a_{n,0}, a_{n,1}, a_{n,2}, \ldots, a_{n,n}$ where $a_{n,0} = a_{n,n} = 1$. For all $1 \le k \le n-1$, $a_{n,k} = a_{n-1,k} - a_{n-1,k-1}$. Let N be the value of the sum

$$\sum_{k=0}^{2018} \frac{|a_{2018,k}|}{\binom{2018}{k}}.$$

    Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \cdot 2^{-|N-E|/70} \rfloor$ points.

36. [**20**] An $n \times m$ maze is an $n \times m$ grid in which each cell is one of two things: a wall, or a blank. A maze is *solvable* if there exists a sequence of adjacent blank cells from the top left cell to the bottom right cell going through no walls. (In particular, the top left and bottom right cells must both be blank.) Let N be the number of solvable 5×5 mazes. Estimate N.

    An estimate of $E > 0$ earns $\lfloor 20 \min\left(\frac{N}{E}, \frac{E}{N}\right)^2 \rfloor$ points.

HMMT November 2018

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Team Round

1. [**15**] Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

2. [**20**] Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

3. [**25**] For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

4. [**30**] Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.

5. [**35**] Find the sum of all positive integers n such that $1 + 2 + \cdots + n$ divides

$$15[(n+1)^2 + (n+2)^2 + \cdots + (2n)^2].$$

6. [**45**] Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.

7. [**50**] A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

   Note: A rectangle must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.

8. [**55**] Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and

   - if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from this vertex;
   - if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from this vertex.

   When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

9. [**60**] Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \parallel BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

10. [**65**] David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

    Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.

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November 10, 2018

Theme Round

1. Square $CASH$ and regular pentagon $MONEY$ are both inscribed in a circle. Given that they do not share a vertex, how many intersections do these two polygons have?

2. Consider the addition problem:

$$\begin{array}{cccccc} & & C & A & S & H \\ + & & & & M & E \\ \hline & O & S & I & D & E \end{array}$$

   where each letter represents a base-ten digit, and $C, M, O \neq 0$. (Distinct letters are allowed to represent the same digit) How many ways are there to assign values to the letters so that the addition problem is true?

3. HOW, BOW, and DAH are equilateral triangles in a plane such that $WO = 7$ and $AH = 2$. Given that D, A, B are collinear in that order, find the length of BA.

4. I have two cents and Bill has n cents. Bill wants to buy some pencils, which come in two different packages. One package of pencils costs 6 cents for 7 pencils, and the other package of pencils costs a *dime for a dozen* pencils (i.e. 10 cents for 12 pencils). Bill notes that he can spend **all** n of his cents on some combination of pencil packages to get P pencils. However, if I *give my two cents* to Bill, he then notes that he can instead spend **all** $n + 2$ of his cents on some combination of pencil packages to get fewer than P pencils. What is the smallest value of n for which this is possible?

   Note: Both times Bill must spend **all** of his cents on pencil packages, i.e. have zero cents after either purchase.

5. Lil Wayne, the rain god, determines the weather. If Lil Wayne makes it rain on any given day, the probability that he makes it rain the next day is 75%. If Lil Wayne doesn't make it rain on one day, the probability that he makes it rain the next day is 25%. He decides not to make it rain today. Find the smallest positive integer n such that the probability that Lil Wayne *makes it rain* n days from today is greater than 49.9%.

6. Farmer James invents a new currency, such that for every positive integer $n \leq 6$, there exists an n-coin worth $n!$ cents. Furthermore, he has exactly n copies of each n-coin. An integer k is said to be *nice* if Farmer James can make k cents using at least one copy of each type of coin. How many positive integers less than 2018 are nice?

7. Ben "One Hunna Dolla" Franklin is flying a kite $KITE$ such that IE is the perpendicular bisector of KT. Let IE meet KT at R. The midpoints of KI, IT, TE, EK are A, N, M, D, respectively. Given that $[MAKE] = 18, IT = 10, [RAIN] = 4$, find $[DIME]$.

   Note: $[X]$ denotes the area of the figure X.

8. Crisp All, a basketball player, is *dropping dimes* and nickels on a number line. Crisp drops a dime on every positive multiple of 10, and a nickel on every multiple of 5 that is not a multiple of 10. Crisp then starts at 0. Every second, he has a $\frac{2}{3}$ chance of jumping from his current location x to $x + 3$, and a $\frac{1}{3}$ chance of jumping from his current location x to $x + 7$. When Crisp jumps on either a dime or a nickel, he stops jumping. What is the probability that Crisp *stops on a dime*?

9. Circle ω_1 of radius 1 and circle ω_2 of radius 2 are concentric. Godzilla inscribes square $CASH$ in ω_1 and regular pentagon $MONEY$ in ω_2. It then writes down all 20 (not necessarily distinct) distances between a vertex of $CASH$ and a vertex of $MONEY$ and multiplies them all together. What is the maximum possible value of his result?

10. *One million bucks* (i.e. one million male deer) are in different cells of a 1000×1000 grid. The left and right edges of the grid are then glued together, and the top and bottom edges of the grid are glued together, so that the grid forms a doughnut-shaped torus. Furthermore, some of the bucks are *honest bucks*, who always tell the truth, and the remaining bucks are *dishonest bucks*, who never tell the truth. Each of the million bucks claims that "at most one of my neighboring bucks is an *honest buck*." A pair of neighboring bucks is said to be *buckaroo* if exactly one of them is an *honest buck*. What is the minimum possible number of *buckaroo* pairs in the grid?

    Note: Two bucks are considered to be *neighboring* if their cells (x_1, y_1) and (x_2, y_2) satisfy either: $x_1 = x_2$ and $y_1 - y_2 \equiv \pm 1 \pmod{1000}$, or $x_1 - x_2 \equiv \pm 1 \pmod{1000}$ and $y_1 = y_2$.

HMIC 2018

April 09, 2018

HMIC 2018

1. [**6**] Let $m > 1$ be a fixed positive integer. For a nonempty string of base-ten digits S, let $c(S)$ be the number of ways to split S into contiguous nonempty strings of digits such that the base-ten number represented by each string is divisible by m. These strings are allowed to have leading zeroes.

   In terms of m, what are the possible values that $c(S)$ can take?

   For example, if $m = 2$, then $c(1234) = 2$ as the splits 1234 and $12|34$ are valid, while the other six splits are invalid.

2. [**7**] Consider a finite set of points $T \in \mathbb{R}^n$ contained in the n-dimensional unit ball centered at the origin, and let X be the convex hull of T. Prove that for all positive integers k and all points $x \in X$, there exist points $t_1, t_2, \ldots, t_k \in T$, not necessarily distinct, such that their centroid

$$\frac{t_1 + t_2 + \cdots + t_k}{k}$$

   has Euclidean distance at most $\frac{1}{\sqrt{k}}$ from x.

   (The n-dimensional unit ball centered at the origin is the set of points in $\mathbb{R}^n$ with Euclidean distance at most 1 from the origin. The convex hull of a set of points $T \in \mathbb{R}^n$ is the smallest set of points X containing T such that each line segment between two points in X lies completely inside X.)

3. [**8**] A polygon in the plane (with no self-intersections) is called *equitable* if every line passing through the origin divides the polygon into two (possibly disconnected) regions of equal area.

   Does there exist an equitable polygon which is not centrally symmetric about the origin?

   (A polygon is centrally symmetric about the origin if a 180-degree rotation about the origin sends the polygon to itself.)

4. [**10**] Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y + xy)) = (y + 1)f(x + 1) - 1$$

   for all $x, y \in \mathbb{R}^+$.

   ($\mathbb{R}^+$ denotes the set of positive real numbers.)

5. [**11**] Let G be an undirected simple graph. Let $f(G)$ be the number of ways to orient all of the edges of G in one of the two possible directions so that the resulting directed graph has no directed cycles. Show that $f(G)$ is a multiple of 3 if and only if G has a cycle of odd length.

February 2017

February 18, 2017

Algebra and Number Theory

1. Let $Q(x) = a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with integer coefficients, and $0 \le a_i < 3$ for all $0 \le i \le n$.

   Given that $Q(\sqrt{3}) = 20 + 17\sqrt{3}$, compute $Q(2)$.

2. Find the value of

$$\sum_{1 \le a < b < c} \frac{1}{2^a 3^b 5^c}$$

   (i.e. the sum of $\frac{1}{2^a 3^b 5^c}$ over all triples of positive integers (a, b, c) satisfying $a < b < c$)

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x)f(y) = f(x - y)$. Find all possible values of $f(2017)$.

4. Find all pairs (a, b) of positive integers such that $a^{2017} + b$ is a multiple of ab.

5. Kelvin the Frog was bored in math class one day, so he wrote all ordered triples (a, b, c) of positive integers such that $abc = 2310$ on a sheet of paper. Find the sum of all the integers he wrote down. In other words, compute

$$\sum_{\substack{abc=2310 \\ a,b,c \in \mathbb{N}}} (a + b + c),$$

   where $\mathbb{N}$ denotes the positive integers.

6. A polynomial P of degree 2015 satisfies the equation $P(n) = \frac{1}{n^2}$ for $n = 1, 2, \ldots, 2016$. Find $\lfloor 2017P(2017) \rfloor$.

7. Determine the largest real number c such that for any 2017 real numbers $x_1, x_2, \ldots, x_{2017}$, the inequality

$$\sum_{i=1}^{2016} x_i(x_i + x_{i+1}) \ge c \cdot x_{2017}^2$$

   holds.

8. Consider all ordered pairs of integers (a, b) such that $1 \le a \le b \le 100$ and

$$\frac{(a + b)(a + b + 1)}{ab}$$

   is an integer.

   Among these pairs, find the one with largest value of b. If multiple pairs have this maximal value of b, choose the one with largest a. For example choose $(3, 85)$ over $(2, 85)$ over $(4, 84)$. Note that your answer should be an ordered pair.

9. The Fibonacci sequence is defined as follows: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all integers $n \ge 2$. Find the smallest positive integer m such that $F_m \equiv 0 \pmod{127}$ and $F_{m+1} \equiv 1 \pmod{127}$.

10. Let $\mathbb{N}$ denote the natural numbers. Compute the number of functions $f : \mathbb{N} \to \{0, 1, \ldots, 16\}$ such that

$$f(x + 17) = f(x) \quad \text{and} \quad f(x^2) \equiv f(x)^2 + 15 \pmod{17}$$

   for all integers $x \ge 1$.

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Combinatorics

1. Kelvin the Frog is going to roll three fair ten-sided dice with faces labelled $0, 1, 2, \ldots, 9$. First he rolls two dice, and finds the sum of the two rolls. Then he rolls the third die. What is the probability that the sum of the first two rolls equals the third roll?

2. How many ways are there to insert +'s between the digits of 111111111111111 (fifteen 1's) so that the result will be a multiple of 30?

3. There are 2017 jars in a row on a table, initially empty. Each day, a nice man picks ten consecutive jars and deposits one coin in each of the ten jars. Later, Kelvin the Frog comes back to see that N of the jars all contain the same positive integer number of coins (i.e. there is an integer $d > 0$ such that N of the jars have exactly d coins). What is the maximum possible value of N?

4. Sam spends his days walking around the following 2×2 grid of squares.

1	2
4	3

   Say that two squares are adjacent if they share a side. He starts at the square labeled 1 and every second walks to an adjacent square. How many paths can Sam take so that the sum of the numbers on every square he visits in his path is equal to 20 (not counting the square he started on)?

5. Kelvin the Frog likes numbers whose digits strictly decrease, but numbers that violate this condition in at most one place are good enough. In other words, if d_i denotes the ith digit, then $d_i \le d_{i+1}$ for at most one value of i. For example, Kelvin likes the numbers 43210, 132, and 3, but not the numbers 1337 and 123. How many 5-digit numbers does Kelvin like?

6. Emily starts with an empty bucket. Every second, she either adds a stone to the bucket or removes a stone from the bucket, each with probability $\frac{1}{2}$. If she wants to remove a stone from the bucket and the bucket is currently empty, she merely does nothing for that second (still with probability $\frac{1}{2}$). What is the probability that after 2017 seconds her bucket contains exactly 1337 stones?

7. There are 2017 frogs and 2017 toads in a room. Each frog is friends with exactly 2 distinct toads. Let N be the number of ways to pair every frog with a toad who is its friend, so that no toad is paired with more than one frog. Let D be the number of distinct possible values of N, and let S be the sum of all possible values of N. Find the ordered pair (D, S).

8. Kelvin and 15 other frogs are in a meeting, for a total of 16 frogs. During the meeting, each pair of distinct frogs becomes friends with probability $\frac{1}{2}$. Kelvin thinks the situation after the meeting is *cool* if for each of the 16 frogs, the number of friends they made during the meeting is a multiple of 4. Say that the probability of the situation being cool can be expressed in the form $\frac{a}{b}$, where a and b are relatively prime. Find a.

9. Let m be a positive integer, and let T denote the set of all subsets of $\{1, 2, \ldots, m\}$. Call a subset S of T δ-*good* if for all $s_1, s_2 \in S$, $s_1 \neq s_2$, $|\Delta(s_1, s_2)| \ge \delta m$, where Δ denotes symmetric difference (the symmetric difference of two sets is the set of elements that is in exactly one of the two sets). Find the largest possible integer s such that there exists an integer m and a $\frac{1024}{2047}$-good set of size s.

10. Compute the number of possible words $w = w_1 w_2 \ldots w_{100}$ satisfying:

    - w has exactly 50 A's and 50 B's (and no other letters).
    - For $i = 1, 2, \ldots, 100$, the number of A's among $w_1, w_2, \ldots, w_i$ is at most the number of B's among $w_1, w_2, \ldots, w_i$.
    - For all $i = 44, 45, \ldots, 57$, if w_i is an B, then w_{i+1} must be an B.

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Geometry

1. Let A, B, C, D be four points on a circle in that order. Also, $AB = 3$, $BC = 5$, $CD = 6$, and $DA = 4$. Let diagonals AC and BD intersect at P. Compute $\frac{AP}{CP}$.

2. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let ℓ be a line passing through two sides of triangle ABC. Line ℓ cuts triangle ABC into two figures, a triangle and a quadrilateral, that have equal perimeter. What is the maximum possible area of the triangle?

3. Let S be a set of 2017 distinct points in the plane. Let R be the radius of the smallest circle containing all points in S on either the interior or boundary. Also, let D be the longest distance between two of the points in S. Let a, b are real numbers such that $a \leq \frac{D}{R} \leq b$ for all possible sets S, where a is as large as possible and b is as small as possible. Find the pair (a, b).

4. Let $ABCD$ be a convex quadrilateral with $AB = 5, BC = 6, CD = 7$, and $DA = 8$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 - PQ^2$.

5. Let $ABCD$ be a quadrilateral with an inscribed circle ω and let P be the intersection of its diagonals AC and BD. Let R_1, R_2, R_3, R_4 be the circumradii of triangles APB, BPC, CPD, DPA respectively. If $R_1 = 31$ and $R_2 = 24$ and $R_3 = 12$, find R_4.

6. In convex quadrilateral $ABCD$ we have $AB = 15$, $BC = 16$, $CD = 12$, $DA = 25$, and $BD = 20$. Let M and γ denote the circumcenter and circumcircle of $\triangle ABD$. Line CB meets γ again at F, line AF meets MC at G, and line GD meets γ again at E. Determine the area of pentagon $ABCDE$.

7. Let ω and Γ by circles such that ω is internally tangent to Γ at a point P. Let AB be a chord of Γ tangent to ω at a point Q. Let $R \neq P$ be the second intersection of line PQ with Γ. If the radius of Γ is 17, the radius of ω is 7, and $\frac{AQ}{BQ} = 3$, find the circumradius of triangle AQR.

8. Let ABC be a triangle with circumradius $R = 17$ and inradius $r = 7$. Find the maximum possible value of $\sin \frac{A}{2}$.

9. Let ABC be a triangle, and let $BCDE, CAFG, ABHI$ be squares that do not overlap the triangle with centers X, Y, Z respectively. Given that $AX = 6$, $BY = 7$, and $CZ = 8$, find the area of triangle XYZ.

10. Let $ABCD$ be a quadrilateral with an inscribed circle ω. Let I be the center of ω let $IA = 12$, $IB = 16$, $IC = 14$, and $ID = 11$. Let M be the midpoint of segment AC. Compute $\frac{IM}{IN}$, where N is the midpoint of segment BD.

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1. [**4**] A random number generator will always output 7. Sam uses this random number generator once. What is the expected value of the output?

2. [**4**] Let A, B, C, D, E, F be 6 points on a circle in that order. Let X be the intersection of AD and BE, Y is the intersection of AD and CF, and Z is the intersection of CF and BE. X lies on segments BZ and AY and Y lies on segment CZ. Given that $AX = 3$, $BX = 2$, $CY = 4$, $DY = 10$, $EZ = 16$, and $FZ = 12$, find the perimeter of triangle XYZ.

3. [**4**] Find the number of pairs of integers (x, y) such that $x^2 + 2y^2 < 25$.

4. [**4**] Find the number of ordered triples of nonnegative integers (a, b, c) that satisfy

$$(ab + 1)(bc + 1)(ca + 1) = 84.$$

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5. [**6**] Find the number of ordered triples of positive integers (a, b, c) such that

$$6a + 10b + 15c = 3000.$$

6. [**6**] Let $ABCD$ be a convex quadrilateral with $AC = 7$ and $BD = 17$. Let M, P, N, Q be the midpoints of sides AB, BC, CD, DA respectively. Compute $MN^2 + PQ^2$

7. [**6**] An ordered pair of sets (A, B) is *good* if A is not a subset of B and B is not a subset of A. How many ordered pairs of subsets of $\{1, 2, \ldots, 2017\}$ are good?

8. [**6**] You have 128 teams in a single elimination tournament. The Engineers and the Crimson are two of these teams. Each of the 128 teams in the tournament is equally strong, so during each match, each team has an equal probability of winning.

   Now, the 128 teams are randomly put into the bracket.

   What is the probability that the Engineers play the Crimson sometime during the tournament?

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9. [**7**] Jeffrey writes the numbers 1 and $100000000 = 10^8$ on the blackboard. Every minute, if x, y are on the board, Jeffrey replaces them with

$$\frac{x+y}{2} \quad \text{and} \quad 2\left(\frac{1}{x}+\frac{1}{y}\right)^{-1}.$$

After 2017 minutes the two numbers are a and b. Find $\min(a, b)$ to the nearest integer.

10. [**7**] Let ABC be a triangle in the plane with $AB = 13, BC = 14, AC = 15$. Let M_n denote the smallest possible value of $(AP^n + BP^n + CP^n)^{\frac{1}{n}}$ over all points P in the plane. Find $\lim_{n\to\infty} M_n$.

11. [**7**] Consider the graph in 3-space of

$$0 = xyz(x+y)(y+z)(z+x)(x-y)(y-z)(z-x).$$

This graph divides 3-space into N connected regions. What is N?

12. [**7**] In a certain college containing 1000 students, students may choose to major in exactly one of math, computer science, finance, or English. The *diversity ratio* $d(s)$ of a student s is the defined as number of students in a different major from s divided by the number of students in the same major as s (including s). The *diversity* D of the college is the sum of all the diversity ratios $d(s)$.

Determine all possible values of D.

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13. [**9**] The game of Penta is played with teams of five players each, and there are five roles the players can play. Each of the five players chooses two of five roles they wish to play. If each player chooses their roles randomly, what is the probability that each role will have exactly two players?

14. [**9**] Mrs. Toad has a class of 2017 students, with unhappiness levels $1, 2, \ldots, 2017$ respectively. Today in class, there is a group project and Mrs. Toad wants to split the class in exactly 15 groups. The unhappiness level of a group is the average unhappiness of its members, and the unhappiness of the class is the sum of the unhappiness of all 15 groups. What's the minimum unhappiness of the class Mrs. Toad can achieve by splitting the class into 15 groups?

15. [**9**] Start by writing the integers $1, 2, 4, 6$ on the blackboard. At each step, write the smallest positive integer n that satisfies both of the following properties on the board.

- n is larger than any integer on the board currently.
- n cannot be written as the sum of 2 distinct integers on the board.

Find the 100-th integer that you write on the board. Recall that at the beginning, there are already 4 integers on the board.

16. [**9**] Let a and b be complex numbers satisfying the two equations

$$a^3 - 3ab^2 = 36$$
$$b^3 - 3ba^2 = 28i.$$

Let M be the maximum possible magnitude of a. Find all a such that $|a| = M$.

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17. [**10**] Sean is a biologist, and is looking at a string of length 66 composed of the letters A, T, C, G. A *substring* of a string is a contiguous sequence of letters in the string. For example, the string $AGTC$ has 10 substrings: $A, G, T, C, AG, GT, TC, AGT, GTC, AGTC$. What is the maximum number of distinct substrings of the string Sean is looking at?

18. [**10**] Let $ABCD$ be a quadrilateral with side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $DA = 4$. What is the maximum possible radius of a circle inscribed in quadrilateral $ABCD$?

19. [**10**] Find (in terms of $n \geq 1$) the number of terms with odd coefficients after expanding the product:

$$\prod_{1 \leq i < j \leq n} (x_i + x_j)$$

e.g., for $n = 3$ the expanded product is given by $x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_2^2x_1 + x_3^2x_1 + x_3^2x_2 + 2x_1x_2x_3$ and so the answer would be 6.

20. [**10**] For positive integers a and N, let $r(a, N) \in \{0, 1, \ldots, N-1\}$ denote the remainder of a when divided by N. Determine the number of positive integers $n \leq 1000000$ for which

$$r(n, 1000) > r(n, 1001).$$

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21. [**12**] Let P and A denote the perimeter and area respectively of a right triangle with relatively prime integer side-lengths. Find the largest possible integral value of $\frac{P^2}{A}$

22. [**12**] Kelvin the Frog and 10 of his relatives are at a party. Every pair of frogs is either *friendly* or *unfriendly*. When 3 pairwise friendly frogs meet up, they will gossip about one another and end up in a *fight* (but stay *friendly* anyway). When 3 pairwise unfriendly frogs meet up, they will also end up in a *fight*. In all other cases, common ground is found and there is no fight. If all $\binom{11}{3}$ triples of frogs meet up exactly once, what is the minimum possible number of fights?

23. [**12**] Five points are chosen uniformly at random on a segment of length 1. What is the expected distance between the closest pair of points?

24. [**12**] At a recent math contest, Evan was asked to find $2^{2016} \pmod{p}$ for a given prime number p with $100 < p < 500$. Evan has forgotten what the prime p was, but still remembers how he solved it:

- Evan first tried taking 2016 modulo $p - 1$, but got a value e larger than 100.
- However, Evan noted that $e - \frac{1}{2}(p-1) = 21$, and then realized the answer was $-2^{21} \pmod{p}$.

What was the prime p?

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25. [**15**] Find all real numbers x satisfying the equation $x^3 - 8 = 16\sqrt[3]{x+1}$.

26. [**15**] Kelvin the Frog is hopping on a number line (extending to infinity in both directions). Kelvin starts at 0. Every minute, he has a $\frac{1}{3}$ chance of moving 1 unit left, a $\frac{1}{3}$ chance of moving 1 unit right and $\frac{1}{3}$ chance of getting eaten. Find the expected number of times Kelvin returns to 0 (not including the start) before getting eaten.

27. [**15**] Find the smallest possible value of $x + y$ where $x, y \geq 1$ and x and y are integers that satisfy $x^2 - 29y^2 = 1$

28. [**15**] Let $\ldots, a_{-1}, a_0, a_1, a_2, \ldots$ be a sequence of positive integers satisfying the following relations: $a_n = 0$ for $n < 0$, $a_0 = 1$, and for $n \geq 1$,

$$a_n = a_{n-1} + 2(n-1)a_{n-2} + 9(n-1)(n-2)a_{n-3} + 8(n-1)(n-2)(n-3)a_{n-4}.$$

Compute

$$\sum_{n\geq 0} \frac{10^n a_n}{n!}.$$

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29. [**17**] Yang has the sequence of integers $1, 2, \ldots, 2017$. He makes 2016 *swaps* in order, where a swap changes the positions of two integers in the sequence. His goal is to end with $2, 3, \ldots, 2017, 1$. How many different sequences of swaps can Yang do to achieve his goal?

30. [**17**] Consider an equilateral triangular grid G with 20 points on a side, where each row consists of points spaced 1 unit apart. More specifically, there is a single point in the first row, two points in the second row, ..., and 20 points in the last row, for a total of 210 points. Let S be a closed non-self-intersecting polygon which has 210 vertices, using each point in G exactly once. Find the sum of all possible values of the area of S.

31. [**17**] A baseball league has 6 teams. To decide the schedule for the league, for each pair of teams, a coin is flipped. If it lands head, they will play a game this season, in which one team wins and one team loses. If it lands tails, they don't play a game this season. Define the *imbalance* of this schedule to be the minimum number of teams that will end up undefeated, i.e. lose 0 games. Find the expected value of the imbalance in this league.

32. [**17**] Let a, b, c be non-negative real numbers such that $ab + bc + ca = 3$. Suppose that

$$a^3b + b^3c + c^3a + 2abc(a + b + c) = \frac{9}{2}.$$

What is the minimum possible value of $ab^3 + bc^3 + ca^3$?

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33. [**20**] Welcome to the **USAYNO**, where each question has a yes/no answer. Choose any subset of the following six problems to answer. If you answer n problems and get them **all** correct, you will receive $\max(0,(n-1)(n-2))$ points. If any of them are wrong, you will receive 0 points.

Your answer should be a six-character string containing 'Y' (for yes), 'N' (for no), or 'B' (for blank). For instance if you think 1, 2, and 6 are 'yes' and 3 and 4 are 'no', you would answer YYNNBY (and receive 12 points if all five answers are correct, 0 points if any are wrong).

(a) a, b, c, d, A, B, C, and D are positive real numbers such that $\frac{a}{b} > \frac{A}{B}$ and $\frac{c}{d} > \frac{C}{D}$. Is it necessarily true that $\frac{a+c}{b+d} > \frac{A+C}{B+D}$?

(b) Do there exist irrational numbers α and β such that the sequence $\lfloor \alpha \rfloor + \lfloor \beta \rfloor, \lfloor 2\alpha \rfloor + \lfloor 2\beta \rfloor, \lfloor 3\alpha \rfloor + \lfloor 3\beta \rfloor, \ldots$ is arithmetic?

(c) For any set of primes $\mathbb{P}$, let $S_{\mathbb{P}}$ denote the set of integers whose prime divisors all lie in $\mathbb{P}$. For instance $S_{\{2,3\}} = \{2^a 3^b \mid a, b \geq 0\} = \{1, 2, 3, 4, 6, 8, 9, 12, \ldots\}$. Does there exist a finite set of primes $\mathbb{P}$ and integer polynomials P and Q such that $\gcd(P(x), Q(y)) \in S_{\mathbb{P}}$ for all x, y?

(d) A function f is called **P-recursive** if there exists a positive integer m and real polynomials $p_0(n), p_1(n), \ldots, p_m(n)$ satisfying

$$p_m(n)f(n+m) = p_{m-1}(n)f(n+m-1) + \ldots + p_0(n)f(n)$$

for all n. Does there exist a P-recursive function f satisfying $\lim_{n\to\infty} \frac{f(n)}{n^{\sqrt{2}}} = 1$?

(e) Does there exist a **nonpolynomial** function $f : \mathbb{Z} \to \mathbb{Z}$ such that $a - b$ divides $f(a) - f(b)$ for all integers $a \neq b$?

(f) Do there exist periodic functions $f, g : \mathbb{R} \to \mathbb{R}$ such that $f(x) + g(x) = x$ for all x?

34. [**20**]

(a) Can 1000 queens be placed on a 2017×2017 chessboard such that every square is attacked by some queen? A square is attacked by a queen if it lies in the same row, column, or diagonal as the queen.

(b) A 2017×2017 grid of squares originally contains a 0 in each square. At any step, Kelvin the Frog chooses two adjacent squares (two squares are adjacent if they share a side) and increments the numbers in both of them by 1. Can Kelvin make every square contain a different power of 2?

(c) A *tournament* consists of single games between every pair of players, where each game has a winner and loser with no ties. A set of people is *dominated* if there exists a player who beats all of them. Does there exist a tournament in which every set of 2017 people is dominated?

(d) Every cell of a 19×19 grid is colored either red, yellow, green, or blue. Does there necessarily exist a rectangle whose sides are parallel to the grid, all of whose vertices are the same color?

(e) Does there exist a $c \in \mathbb{R}^+$ such that $\max(|A \cdot A|, |A + A|) \geq c|A| \log^2 |A|$ for all finite sets $A \subset \mathbb{Z}$?

(f) Can the set $\{1, 2, \ldots, 1093\}$ be partitioned into 7 subsets such that each subset is sum-free (i.e. no subset contains a, b, c with $a + b = c$)?

35. [**20**]

(a) Does there exist a finite set of points, not all collinear, such that a line between any two points in the set passes through a third point in the set?

(b) Let ABC be a triangle and P be a point. The *isogonal conjugate* of P is the intersection of the reflection of line AP over the A-angle bisector, the reflection of line BP over the B-angle bisector, and the reflection of line CP over the C-angle bisector. Clearly the incenter is its own isogonal conjugate. Does there exist another point that is its own isogonal conjugate?

(c) Let F be a convex figure in a plane, and let P be the largest pentagon that can be inscribed in F. Is it necessarily true that the area of P is at least $\frac{3}{4}$ the area of F?

(d) Is it possible to cut an equilateral triangle into 2017 pieces, and rearrange the pieces into a square?

(e) Let ABC be an acute triangle and P be a point in its interior. Let D, E, F lie on BC, CA, AB respectively so that PD bisects $\angle BPC$, PE bisects $\angle CPA$, and PF bisects $\angle APB$. Is it necessarily true that $AP + BP + CP \geq 2(PD + PE + PF)$?

(f) Let P_{2018} be the surface area of the 2018-dimensional unit sphere, and let P_{2017} be the surface area of the 2017-dimensional unit sphere. Is $P_{2018} > P_{2017}$?

36. [**20**]

(a) Does $\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \pmod{p^2}$ for all odd prime numbers p? (Note that $\frac{1}{i}$ denotes the number such that $i \cdot \frac{1}{i} \equiv 1 \pmod{p^2}$)

(b) Do there exist 2017 positive perfect cubes that sum to a perfect cube?

(c) Does there exist a right triangle with rational side lengths and area 5?

(d) A *magic square* is a 3×3 grid of numbers, all of whose rows, columns, and major diagonals sum to the same value. Does there exist a magic square whose entries are all prime numbers?

(e) Is $\prod_p \frac{p^2+1}{p^2-1} = \frac{2^2+1}{2^2-1} \cdot \frac{3^2+1}{3^2-1} \cdot \frac{5^2+1}{5^2-1} \cdot \frac{7^2+1}{7^2-1} \cdot \ldots$ a rational number?

(f) Do there exist an infinite number of pairs of *distinct* integers (a, b) such that a and b have the same set of prime divisors, and $a + 1$ and $b + 1$ also have the same set of prime divisors?

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Team

1. [**15**] Let $P(x), Q(x)$ be nonconstant polynomials with real number coefficients. Prove that if

$$\lfloor P(y) \rfloor = \lfloor Q(y) \rfloor$$

for all real numbers y, then $P(x) = Q(x)$ for all real numbers x.

2. [**25**] Does there exist a two-variable polynomial $P(x, y)$ with real number coefficients such that $P(x, y)$ is positive exactly when x and y are both positive?

3. [**30**] A polyhedron has $7n$ faces. Show that there exist $n+1$ of the polyhedron's faces that all have the same number of edges.

4. [**35**] Let $w = w_1 w_2 \dots w_n$ be a word. Define a *substring* of w to be a word of the form $w_i w_{i+1} \dots w_{j-1} w_j$, for some pair of positive integers $1 \le i \le j \le n$. Show that w has at most n distinct palindromic substrings.

   For example, *aaaaa* has 5 distinct palindromic substrings, and *abcata* has 5 (a, b, c, t, ata).

5. [**35**] Let ABC be an acute triangle. The altitudes BE and CF intersect at the orthocenter H, and point O denotes the circumcenter. Point P is chosen so that $\angle APH = \angle OPE = 90°$, and point Q is chosen so that $\angle AQH = \angle OQF = 90°$. Lines EP and FQ meet at point T. Prove that points A, T, O are collinear.

6. [**40**] Let r be a positive integer. Show that if a graph G has no cycles of length at most $2r$, then it has at most $|V|^{2016}$ cycles of length exactly $2016r$, where $|V|$ denotes the number of vertices in the graph G.

7. [**45**] Let p be a prime. A *complete residue class modulo* p is a set containing at least one element equivalent to $k \pmod{p}$ for all k.

   (a) (**20**) Show that there exists an n such that the nth row of Pascal's triangle forms a complete residue class modulo p.

   (b) (**25**) Show that there exists an $n \le p^2$ such that the nth row of Pascal's triangle forms a complete residue class modulo p.

8. [**45**] Does there exist an irrational number $\alpha > 1$ such that

$$\lfloor \alpha^n \rfloor \equiv 0 \pmod{2017}$$

for all integers $n \ge 1$?

9. [**65**] Let n be a positive odd integer greater than 2, and consider a regular n-gon $\mathcal{G}$ in the plane centered at the origin. Let a *subpolygon* $\mathcal{G}'$ be a polygon with at least 3 vertices whose vertex set is a subset of that of $\mathcal{G}$. Say $\mathcal{G}'$ is *well-centered* if its centroid is the origin. Also, say $\mathcal{G}'$ is *decomposable* if its vertex set can be written as the disjoint union of regular polygons with at least 3 vertices. Show that all well-centered subpolygons are decomposable if and only if n has at most two distinct prime divisors.

10. [**65**] Let LBC be a fixed triangle with $LB = LC$, and let A be a variable point on arc LB of its circumcircle. Let I be the incenter of $\triangle ABC$ and $\overline{AK}$ the altitude from A. The circumcircle of $\triangle IKL$ intersects lines KA and BC again at $U \ne K$ and $V \ne K$. Finally, let T be the projection of I onto line UV. Prove that the line through T and the midpoint of $\overline{IK}$ passes through a fixed point as A varies.

HMMT November 2017

November 11, 2017

General Round

1. Find the sum of all positive integers whose largest proper divisor is 55. (A proper divisor of n is a divisor that is strictly less than n.)

2. Determine the sum of all distinct real values of x such that

$$|||\cdots||x|+x|\cdots|+x|+x|=1,$$

where there are 2017 x's in the equation.

3. Find the number of integers n with $1 \le n \le 2017$ so that $(n-2)(n-0)(n-1)(n-7)$ is an integer multiple of 1001.

4. Triangle ABC has $AB = 10$, $BC = 17$, and $CA = 21$. Point P lies on the circle with diameter AB. What is the greatest possible area of APC?

5. Given that a, b, c are integers with $abc = 60$, and that complex number $\omega \neq 1$ satisfies $\omega^3 = 1$, find the minimum possible value of $\left|a + b\omega + c\omega^2\right|$.

6. A positive integer n is *magical* if

$$\left\lfloor \sqrt{\lceil \sqrt{n} \rceil} \right\rfloor = \left\lceil \sqrt{\lfloor \sqrt{n} \rfloor} \right\rceil,$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ represent the floor and ceiling function respectively. Find the number of magical integers between 1 and 10,000, inclusive.

7. Reimu has a wooden cube. In each step, she creates a new polyhedron from the previous one by cutting off a pyramid from each vertex of the polyhedron along a plane through the trisection point on each adjacent edge that is closer to the vertex. For example, the polyhedron after the first step has six octagonal faces and eight equilateral triangular faces. How many faces are on the polyhedron after the fifth step?

8. Marisa has a collection of $2^8 - 1 = 255$ distinct nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$. For each step she takes two subsets chosen uniformly at random from the collection, and replaces them with either their union or their intersection, chosen randomly with equal probability. (The collection is allowed to contain repeated sets.) She repeats this process $2^8 - 2 = 254$ times until there is only one set left in the collection. What is the expected size of this set?

9. Find the minimum possible value of

$$\sqrt{58 - 42x} + \sqrt{149 - 140\sqrt{1 - x^2}}$$

where $-1 \le x \le 1$.

10. Five equally skilled tennis players named Allen, Bob, Catheryn, David, and Evan play in a round robin tournament, such that each pair of people play exactly once, and there are no ties. In each of the ten games, the two players both have a 50% chance of winning, and the results of the games are independent. Compute the probability that there exist four distinct players P_1, P_2, P_3, P_4 such that P_i beats P_{i+1} for $i = 1, 2, 3, 4$. (We denote $P_5 = P_1$).

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1. [**2**] Suppose x is a rational number such that $x\sqrt{2}$ is also rational. Find x.

2. [**2**] Regular octagon $CHILDREN$ has area 1. Find the area of pentagon $CHILD$.

3. [**2**] The length of a rectangle is three times its width. Given that its perimeter and area are both numerically equal to $k > 0$, find k.

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4. [**3**] Alec wishes to construct a string of 6 letters using the letters A, C, G, and N, such that:

   - The first three letters are pairwise distinct, and so are the last three letters;
   - The first, second, fourth, and fifth letters are pairwise distinct.

   In how many ways can he construct the string?

5. [**3**] Define a sequence $\{a_n\}$ by $a_1 = 1$ and $a_n = (a_{n-1})! + 1$ for every $n > 1$. Find the least n for which $a_n > 10^{10}$.

6. [**3**] Lunasa, Merlin, and Lyrica each have a distinct hat. Every day, two of these three people, selected randomly, switch their hats. What is the probability that, after 2017 days, every person has their own hat back?

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7. [**3**] Compute

$$100^2 + 99^2 - 98^2 - 97^2 + 96^2 + 95^2 - 94^2 - 93^2 + \ldots + 4^2 + 3^2 - 2^2 - 1^2.$$

8. [**3**] Regular octagon $CHILDREN$ has area 1. Find the area of quadrilateral $LINE$.

9. [**3**] A malfunctioning digital clock shows the time 9 : 57 AM; however, the correct time is 10 : 10 AM. There are two buttons on the clock, one of which increases the time displayed by 9 minutes, and another which decreases the time by 20 minutes. What is the minimum number of button presses necessary to correctly set the clock to the correct time?

. . . .

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10. [**4**] Compute $\frac{x}{w}$ if $w \neq 0$ and $\frac{x+6y-3z}{-3x+4w} = \frac{-2y+z}{x-w} = \frac{2}{3}$.

11. [**4**] Find the sum of all real numbers x for which

$$\lfloor\lfloor\cdots\lfloor\lfloor\lfloor x\rfloor + x\rfloor + x\rfloor\cdots\rfloor + x\rfloor = 2017 \text{ and } \{\{\cdots\{\{\{x\} + x\} + x\}\cdots\} + x\} = \frac{1}{2017}$$

where there are 2017 x's in both equations. ($\lfloor x\rfloor$ is the integer part of x, and $\{x\}$ is the fractional part of x.) Express your sum as a mixed number.

12. [**4**] Trapezoid $ABCD$, with bases AB and CD, has side lengths $AB = 28$, $BC = 13$, $CD = 14$, and $DA = 15$. Let diagonals AC and BD intersect at P, and let E and F be the midpoints of AP and BP, respectively. Find the area of quadrilateral $CDEF$.

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13. [**5**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of V cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of 2017 cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

14. [**5**] Points A, B, C, and D lie on a line in that order such that $\frac{AB}{BC} = \frac{DA}{CD}$. If $AC = 3$ and $BD = 4$, find AD.

15. [**5**] On a 3×3 chessboard, each square contains a knight with $\frac{1}{2}$ probability. What is the probability that there are two knights that can attack each other? (In chess, a knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction.)

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16. [**7**] A *repunit* is a positive integer, all of whose digits are 1s. Let $a_1 < a_2 < a_3 < \ldots$ be a list of all the positive integers that can be expressed as the sum of distinct repunits. Compute a_{111}.

17. [**7**] A standard deck of 54 playing cards (with four cards of each of thirteen ranks, as well as two Jokers) is shuffled randomly. Cards are drawn one at a time until the first queen is reached. What is the probability that the next card is also a queen?

18. [**7**] Mr. Taf takes his 12 students on a road trip. Since it takes two hours to walk from the school to the destination, he plans to use his car to expedite the journey. His car can take at most 4 students at a time, and travels 15 times as fast as traveling on foot. If they plan their trip optimally, what is the shortest amount of time it takes for them to all reach the destination, in minutes?

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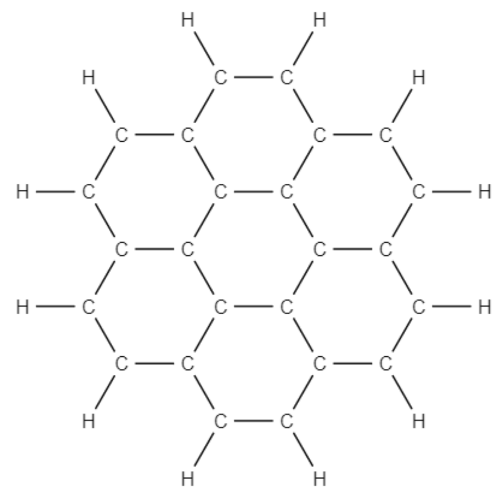
19. **[9]** The skeletal structure of coronene, a hydrocarbon with the chemical formula $C_{24}H_{12}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? (Rotations and reflections of the same arrangement are considered distinct.)

20. **[9]** Rebecca has four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a + b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R?

21. **[9]** A box contains three balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least three times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

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22. [**12**] A sequence of positive integers $a_1, a_2, \ldots, a_{2017}$ has the property that for all integers m where $1 \leq m \leq 2017$, $3(\sum_{i=1}^{m} a_i)^2 = \sum_{i=1}^{m} a_i^3$. Compute a_{1337}.

23. [**12**] A string of digits is defined to be *similar* to another string of digits if it can be obtained by reversing some contiguous substring of the original string. For example, the strings 101 and 110 are similar, but the strings 3443 and 4334 are not. (Note that a string is always similar to itself.) Consider the string of digits

$$S = 01234567890123456789012345678901234567890123456789,$$

consisting of the digits from 0 to 9 repeated five times. How many distinct strings are similar to S?

24. [**12**] Triangle ABC has side lengths $AB = 15, BC = 18, CA = 20$. Extend CA and CB to points D and E respectively such that $DA = AB = BE$. Line AB intersects the circumcircle of CDE at P and Q. Find the length of PQ.

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Déjà vu?

25. [**15**] Fisica and Ritmo discovered a piece of Notalium shaped like a rectangular box, and wanted to find its volume. To do so, Fisica measured its three dimensions using a ruler with infinite precision, multiplied the results and rounded the product to the nearest cubic centimeter, getting a result of 2017 cubic centimeters. Ritmo, on the other hand, measured each dimension to the nearest centimeter and multiplied the rounded measurements, getting a result of V cubic centimeters. Find the positive difference between the least and greatest possible positive values for V.

26. [**15**] Points A, B, C, D lie on a circle in that order such that $\dfrac{AB}{BC} = \dfrac{DA}{CD}$. If $AC = 3$ and $BD = BC = 4$, find AD.

27. [**15**] On a 3×3 chessboard, each square contains a Chinese knight with $\frac{1}{2}$ probability. What is the probability that there are two Chinese knights that can attack each other? (In Chinese chess, a Chinese knight can attack any piece which is two squares away from it in a particular direction and one square away in a perpendicular direction, under the condition that there is no other piece immediately adjacent to it in the first direction.)

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28. [**19**] Compute the number of functions $f : \{1, 2, \ldots, 9\} \to \{1, 2, \ldots, 9\}$ which satisfy $f(f(f(f(f(x))))) = x$ for each $x \in \{1, 2, \ldots, 9\}$.

29. [**19**] Consider a sequence x_n such that $x_1 = x_2 = 1$, $x_3 = \frac{2}{3}$. Suppose that $x_n = \frac{x_{n-1}^2 x_{n-2}}{2x_{n-2}^2 - x_{n-1}x_{n-3}}$ for all $n \geq 4$. Find the least n such that $x_n \leq \frac{1}{10^6}$.

30. [**19**] Given complex number z, define sequence $z_0, z_1, z_2, \ldots$ as $z_0 = z$ and $z_{n+1} = 2z_n^2 + 2z_n$ for $n \geq 0$. Given that $z_{10} = 2017$, find the minimum possible value of $|z|$.

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31. [**24**] In unit square $ABCD$, points E, F, G are chosen on side BC, CD, DA respectively such that AE is perpendicular to EF and EF is perpendicular to FG. Given that $GA = \frac{404}{1331}$, find all possible values of the length of BE.

32. [**24**] Let P be a polynomial with integer coefficients such that $P(0) + P(90) = 2018$. Find the least possible value for $|P(20) + P(70)|$.

33. [**24**] Tetrahedron $ABCD$ with volume 1 is inscribed in circumsphere ω such that $AB = AC = AD = 2$ and $BC \cdot CD \cdot DB = 16$. Find the radius of ω.

. . . .

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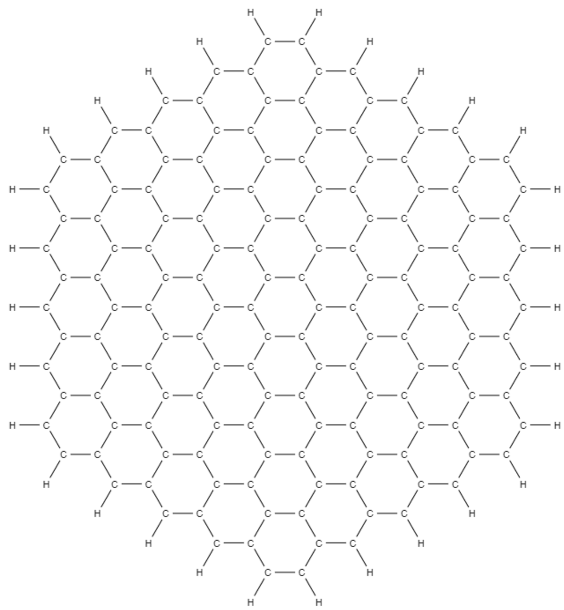
Déjà vu??

34. [**30**] The skeletal structure of circumcircumcircumcoronene, a hydrocarbon with the chemical formula $C_{150}H_{30}$, is shown below.



Each line segment between two atoms is at least a single bond. However, since each carbon (C) requires exactly four bonds connected to it and each hydrogen (H) requires exactly one bond, some of the line segments are actually double bonds. How many arrangements of single/double bonds are there such that the above requirements are satisfied? (Rotations and reflections of the same arrangement are considered distinct.)

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor, 0\right)$ points.

35. [**30**] Rebecca has twenty-four resistors, each with resistance 1 ohm. Every minute, she chooses any two resistors with resistance of a and b ohms respectively, and combine them into one by one of the following methods:

- Connect them in series, which produces a resistor with resistance of $a+b$ ohms;
- Connect them in parallel, which produces a resistor with resistance of $\frac{ab}{a+b}$ ohms;
- Short-circuit one of the two resistors, which produces a resistor with resistance of either a or b ohms.

Suppose that after twenty-three minutes, Rebecca has a single resistor with resistance R ohms. How many possible values are there for R?

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\left|\log_{\log_2 C}\frac{A}{C}\right|\right)\right\rfloor, 0\right)$ points.

36. [**30**] A box contains twelve balls, each of a different color. Every minute, Randall randomly draws a ball from the box, notes its color, and then *returns it to the box.* Consider the following two conditions:

(1) Some ball has been drawn at least twelve times (not necessarily consecutively).

(2) Every ball has been drawn at least once.

What is the probability that condition (1) is met *before* condition (2)?

If the correct answer is C and your answer is A, you get $\max\left(\left\lfloor 30\left(1-\frac{1}{2}\left|\log_2 A-\log_2 C\right|\right)\right\rfloor, 0\right)$ points.

HMMT November 2017

November 11, 2017

Team Round

1. [**15**] A positive integer k is called *powerful* if there are distinct positive integers p, q, r, s, t such that $p^2, q^3, r^5, s^7, t^{11}$ all divide k. Find the smallest powerful integer.

2. [**20**] How many sequences of integers $(a_1, \dots, a_7)$ are there for which $-1 \le a_i \le 1$ for every i, and
$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5 + a_5a_6 + a_6a_7 = 4?$$

3. [**25**] Michael writes down all the integers between 1 and N inclusive on a piece of paper and discovers that exactly 40% of them have leftmost digit 1. Given that $N > 2017$, find the smallest possible value of N.

4. [**30**] An equiangular hexagon has side lengths $1, 1, a, 1, 1, a$ in that order. Given that there exists a circle that intersects the hexagon at 12 distinct points, we have $M < a < N$ for some real numbers M and N. Determine the minimum possible value of the ratio $\frac{N}{M}$.

5. [**35**] Ashwin the frog is traveling on the xy-plane in a series of $2^{2017} - 1$ steps, starting at the origin. At the n^{th} step, if n is odd, then Ashwin jumps one unit to the right. If n is even, then Ashwin jumps m units up, where m is the greatest integer such that 2^m divides n. If Ashwin begins at the origin, what is the area of the polygon bounded by Ashwin's path, the line $x = 2^{2016}$, and the x-axis?

6. [**40**] Consider five-dimensional Cartesian space
$$\mathbb{R}^5 = \{(x_1, x_2, x_3, x_4, x_5) \mid x_i \in \mathbb{R}\},$$
and consider the hyperplanes with the following equations:
   - $x_i = x_j$ for every $1 \le i < j \le 5$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = -1$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = 0$;
   - $x_1 + x_2 + x_3 + x_4 + x_5 = 1$.

   Into how many regions do these hyperplanes divide $\mathbb{R}^5$?

7. [**50**] There are 12 students in a classroom; 6 of them are Democrats and 6 of them are Republicans. Every hour the students are randomly separated into four groups of three for political debates. If a group contains students from both parties, the minority in the group will change his/her political alignment to that of the majority at the end of the debate. What is the expected amount of time needed for all 12 students to have the same political alignment, in hours?

8. [**55**] Find the number of quadruples (a, b, c, d) of integers with absolute value at most 5 such that
$$(a^2 + b^2 + c^2 + d^2)^2 = (a + b + c + d)(a - b + c - d)((a - c)^2 + (b - d)^2).$$

9. [**60**] Let A, B, C, D be points chosen on a circle, in that order. Line BD is reflected over lines AB and DA to obtain lines ℓ_1 and ℓ_2 respectively. If lines ℓ_1, ℓ_2, and AC meet at a common point and if $AB = 4, BC = 3, CD = 2$, compute the length DA.

10. [**70**] Yannick has a bicycle lock with a 4-digit passcode whose digits are between 0 and 9 inclusive. (Leading zeroes are allowed.) The dials on the lock is currently set at 0000. To unlock the lock, every second he picks a contiguous set of dials, and increases or decreases all of them by one, until the dials are set to the passcode. For example, after the first second the dials could be set to $1100, 0010$, or 9999, but not 0909 or 0190. (The digits on each dial are cyclic, so increasing 9 gives 0, and decreasing 0 gives 9.) Let the *complexity* of a passcode be the minimum number of seconds he needs to unlock the lock. What is the maximum possible complexity of a passcode, and how many passcodes have this maximum complexity? Express the two answers as an ordered pair.

HMMT November 2017

November 11, 2017

Theme Round

Based on a true story.

1. **T**wo ordered pairs (a, b) and (c, d), where a, b, c, d are real numbers, form a basis of the coordinate plane if $ad \neq bc$. Determine the number of ordered quadruples (a, b, c, d) of integers between 1 and 3 inclusive for which (a, b) and (c, d) form a basis for the coordinate plane.

2. **H**orizontal parallel segments $AB = 10$ and $CD = 15$ are the bases of trapezoid $ABCD$. Circle γ of radius 6 has center within the trapezoid and is tangent to sides AB, BC, and DA. If side CD cuts out an arc of γ measuring $120°$, find the area of $ABCD$.

3. **E**milia wishes to create a basic solution with 7% hydroxide (OH) ions. She has three solutions of different bases available: 10% rubidium hydroxide (Rb(OH)), 8% cesium hydroxide (Cs(OH)), and 5% francium hydroxide (Fr(OH)). (The Rb(OH) solution has both 10% Rb ions and 10% OH ions, and similar for the other solutions.) Since francium is highly radioactive, its concentration in the final solution should not exceed 2%. What is the highest possible concentration of rubidium in her solution?

4. **M**ary has a sequence $m_2, m_3, m_4, \ldots$, such that for each $b \geq 2$, m_b is the least positive integer m for which none of the base-b logarithms $\log_b(m)$, $\log_b(m + 1)$, $\ldots$, $\log_b(m + 2017)$ are integers. Find the largest number in her sequence.

5. **E**ach of the integers $1, 2, \ldots, 729$ is written in its base-3 representation without leading zeroes. The numbers are then joined together in that order to form a continuous string of digits: $12101112202122\ldots$. How many times in this string does the substring 012 appear?

6. **R**thea, a distant planet, is home to creatures whose DNA consists of two (distinguishable) strands of bases with a fixed orientation. Each base is one of the letters H, M, N, T, and each strand consists of a sequence of five bases, thus forming five pairs. Due to the chemical properties of the bases, each pair must consist of distinct bases. Also, the bases H and M cannot appear next to each other on the same strand; the same is true for N and T. How many possible DNA sequences are there on Rthea?

7. **O**n a blackboard a stranger writes the values of $s_7(n)^2$ for $n = 0, 1, \ldots, 7^{20} - 1$, where $s_7(n)$ denotes the sum of digits of n in base 7. Compute the average value of all the numbers on the board.

8. **U**ndecillion years ago in a galaxy far, far away, there were four space stations in the three-dimensional space, each pair spaced 1 light year away from each other. Admiral Ackbar wanted to establish a base somewhere in space such that the sum of squares of the distances from the base to each of the stations does not exceed 15 square light years. (The sizes of the space stations and the base are negligible.) Determine the volume, in cubic light years, of the set of all possible locations for the Admiral's base.

9. **N**ew this year at HMNT: the exciting game of *RNG baseball*! In RNG baseball, a team of infinitely many people play on a square field, with a base at each vertex; in particular, one of the bases is called the *home base*. Every turn, a new player stands at home base and chooses a number n uniformly at random from $\{0, 1, 2, 3, 4\}$. Then, the following occurs:

   - If $n > 0$, then the player and everyone else currently on the field moves (counterclockwise) around the square by n bases. However, if in doing so a player returns to or moves past the home base, he/she leaves the field immediately and the team scores one point.
   - If $n = 0$ (a strikeout), then the game ends immediately; the team does not score any more points.

   What is the expected number of points that a given team will score in this game?

10. **D**enote $\phi = \frac{1+\sqrt{5}}{2}$ and consider the set of all finite binary strings without leading zeroes. Each string S has a "base-ϕ" value $p(S)$. For example, $p(1101) = \phi^3 + \phi^2 + 1$. For any positive integer n, let $f(n)$ be the number of such strings S that satisfy $p(S) = \frac{\phi^{48n}-1}{\phi^{48}-1}$. The sequence of fractions $\frac{f(n+1)}{f(n)}$ approaches a real number c as n goes to infinity. Determine the value of c.

HMIC 2017

April 10, 2017

HMIC

1. [**6**] Kevin and Yang are playing a game. Yang has $2017 + \binom{2017}{2}$ cards with their front sides face down on the table. The cards are constructed as follows:

   - For each $1 \le n \le 2017$, there is a blue card with n written on the back, and a fraction $\frac{a_n}{b_n}$ written on the front, where $\gcd(a_n, b_n) = 1$ and $a_n, b_n > 0$.
   - For each $1 \le i < j \le 2017$, there is a red card with (i, j) written on the back, and a fraction $\frac{a_i + a_j}{b_i + b_j}$ written on the front.

   It is given no two cards have equal fractions. In a turn Kevin can pick any two cards and Yang tells Kevin which card has the larger fraction on the front. Show that, in fewer than 10000 turns, Kevin can determine which red card has the largest fraction out of all of the red cards.

2. [**7**] Let $S = \{1, 2 \ldots, n\}$ for some positive integer n, and let A be an n-by-n matrix having as entries only ones and zeros. Define an infinite sequence $\{x_i\}_{i \ge 0}$ to be *strange* if:

   - $x_i \in S$ for all i,
   - $a_{x_k x_{k+1}} = 1$ for all k, where a_{ij} denotes the element in the i^{th} row and j^{th} column of A.

   Prove that the set of strange sequences is empty if and only if A is nilpotent, i.e. $A^m = 0$ for some integer m.

3. [**9**] Let $v_1, v_2, \ldots, v_m$ be vectors in $\mathbb{R}^n$, such that each has a strictly positive first coordinate. Consider the following process. Start with the zero vector $w = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Every round, choose an i such that $1 \le i \le m$ and $w \cdot v_i \le 0$, and then replace w with $w + v_i$.

   Show that there exists a constant C such that regardless of your choice of i at each step, the process is guaranteed to terminate in C rounds. The constant C may depend on the vectors $v_1, \ldots, v_m$.

4. [**9**] Let G be a weighted bipartite graph $A \cup B$, with $|A| = |B| = n$. In other words, each edge in the graph is assigned a positive integer value, called its *weight*. Also, define the weight of a perfect matching in G to be the sum of the weights of the edges in the matching.

   Let G' be the graph with vertex set $A \cup B$, and contains the edge e if and only if e is part of some minimum weight perfect matching in G.

   Show that all perfect matchings in G' have the same weight.

5. [**11**] Let S be the set $\{-1, 1\}^n$, that is, n-tuples such that each coordinate is either -1 or 1. For

   $$s = (s_1, s_2, \ldots, s_n), t = (t_1, t_2, \ldots, t_n) \in \{-1, 1\}^n,$$

   define $s \odot t = (s_1 t_1, s_2 t_2, \ldots, s_n t_n)$.

   Let c be a positive constant, and let $f : S \to \{-1, 1\}$ be a function such that there are at least $(1 - c) \cdot 2^{2n}$ pairs (s, t) with $s, t \in S$ such that $f(s \odot t) = f(s)f(t)$. Show that there exists a function f' such that $f'(s \odot t) = f'(s)f'(t)$ for all $s, t \in S$ and $f(s) = f'(s)$ for at least $(1 - 10c) \cdot 2^n$ values of $s \in S$.

HMMT February 2016

February 20, 2016

Algebra

1. Let z be a complex number such that $|z| = 1$ and $|z - 1.45| = 1.05$. Compute the real part of z.

2. For which integers $n \in \{1, 2, \ldots, 15\}$ is $n^n + 1$ a prime number?

3. Let A denote the set of all integers n such that $1 \leq n \leq 10000$, and moreover the sum of the decimal digits of n is 2. Find the sum of the squares of the elements of A.

4. Determine the remainder when
$$\sum_{i=0}^{2015} \left\lfloor \frac{2^i}{25} \right\rfloor$$
is divided by 100, where $\lfloor x \rfloor$ denotes the largest integer not greater than x.

5. An infinite sequence of real numbers $a_1, a_2, \ldots$ satisfies the recurrence
$$a_{n+3} = a_{n+2} - 2a_{n+1} + a_n$$
for every positive integer n. Given that $a_1 = a_3 = 1$ and $a_{98} = a_{99}$, compute $a_1 + a_2 + \cdots + a_{100}$.

6. Call a positive integer $N \geq 2$ "special" if for every k such that $2 \leq k \leq N$, N can be expressed as a sum of k positive integers that are relatively prime to N (although not necessarily relatively prime to each other). How many special integers are there less than 100?

7. Determine the smallest positive integer $n \geq 3$ for which
$$A \equiv 2^{10n} \pmod{2^{170}}$$
where A denotes the result when the numbers 2^{10}, 2^{20}, $\ldots$, 2^{10n} are written in decimal notation and concatenated (for example, if $n = 2$ we have $A = 10241048576$).

8. Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the number of integers $2 \leq n \leq 50$ such that n divides $\phi^!(n) + 1$.

9. For any positive integer n, S_n be the set of all permutations of $\{1, 2, 3, \ldots, n\}$. For each permutation $\pi \in S_n$, let $f(\pi)$ be the number of ordered pairs (j, k) for which $\pi(j) > \pi(k)$ and $1 \leq j < k \leq n$. Further define $g(\pi)$ to be the number of positive integers $k \leq n$ such that $\pi(k) \equiv k \pm 1 \pmod{n}$. Compute
$$\sum_{\pi \in S_{999}} (-1)^{f(\pi)+g(\pi)}.$$

10. Let a, b and c be positive real numbers such that
$$a^2 + ab + b^2 = 9$$
$$b^2 + bc + c^2 = 52$$
$$c^2 + ca + a^2 = 49.$$
Compute the value of $\dfrac{49b^2 - 33bc + 9c^2}{a^2}$.

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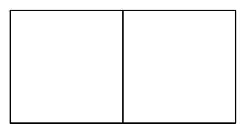
Combinatorics

1. For positive integers n, let S_n be the set of integers x such that n distinct lines, no three concurrent, can divide a plane into x regions (for example, $S_2 = \{3, 4\}$, because the plane is divided into 3 regions if the two lines are parallel, and 4 regions otherwise). What is the minimum i such that S_i contains at least 4 elements?

2. Starting with an empty string, we create a string by repeatedly appending one of the letters H, M, T with probabilities $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$, respectively, until the letter M appears twice consecutively. What is the expected value of the length of the resulting string?

3. Find the number of ordered pairs of integers (a, b) such that a, b are divisors of 720 but ab is not.

4. Let R be the rectangle in the Cartesian plane with vertices at $(0, 0), (2, 0), (2, 1)$, and $(0, 1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



How many subsets of these seven edges form a connected figure?

5. Let a, b, c, d, e, f be integers selected from the set $\{1, 2, \ldots, 100\}$, uniformly and at random with replacement. Set
$$M = a + 2b + 4c + 8d + 16e + 32f.$$
What is the expected value of the remainder when M is divided by 64?

6. Define the sequence $a_1, a_2 \ldots$ as follows: $a_1 = 1$ and for every $n \geq 2$,
$$a_n = \begin{cases} n-2 & \text{if } a_{n-1} = 0 \\ a_{n-1} - 1 & \text{if } a_{n-1} \neq 0 \end{cases}$$
A non-negative integer d is said to be *jet-lagged* if there are non-negative integers r, s and a positive integer n such that $d = r + s$ and that $a_{n+r} = a_n + s$. How many integers in $\{1, 2, \ldots, 2016\}$ are jet-lagged?

7. Kelvin the Frog has a pair of standard fair 8-sided dice (each labelled from 1 to 8). Alex the sketchy Kat also has a pair of fair 8-sided dice, but whose faces are labelled differently (the integers on each Alex's dice need not be distinct). To Alex's dismay, when both Kelvin and Alex roll their dice, the probability that they get any given sum is equal!

   Suppose that Alex's two dice have a and b total dots on them, respectively. Assuming that $a \neq b$, find all possible values of $\min\{a, b\}$.

8. Let X be the collection of all functions $f : \{0, 1, \ldots, 2016\} \to \{0, 1, \ldots, 2016\}$. Compute the number of functions $f \in X$ such that
$$\max_{g \in X} \left(\min_{0 \leq i \leq 2016} \big(\max(f(i), g(i)) \big) - \max_{0 \leq i \leq 2016} \big(\min(f(i), g(i)) \big) \right) = 2015.$$

9. Let $V = \{1, \ldots, 8\}$. How many permutations $\sigma : V \to V$ are automorphisms of some tree?

   (A *graph* consists of a some set of vertices and some edges between pairs of distinct vertices. It is *connected* if every two vertices in it are connected by some path of one or more edges. A *tree* G on V is a connected graph with vertex set V and exactly $|V| - 1$ edges, and an *automorphism* of G is a permutation $\sigma : V \to V$ such that vertices $i, j \in V$ are connected by an edge if and only if $\sigma(i)$ and $\sigma(j)$ are.)

10. Kristoff is planning to transport a number of indivisible ice blocks with positive integer weights from the north mountain to Arendelle. He knows that when he reaches Arendelle, Princess Anna and Queen Elsa will name an ordered pair (p, q) of nonnegative integers satisfying $p+q \leq 2016$. Kristoff must then give Princess Anna *exactly* p kilograms of ice. Afterward, he must give Queen Elsa *exactly* q kilograms of ice.

    What is the minimum number of blocks of ice Kristoff must carry to guarantee that he can always meet Anna and Elsa's demands, regardless of which p and q are chosen?

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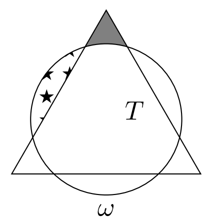
Geometry

1. Dodecagon $QWARTZSPHINX$ has all side lengths equal to 2, is not self-intersecting (in particular, the twelve vertices are all distinct), and moreover each interior angle is either $90°$ or $270°$. What are all possible values of the area of $\triangle SIX$?

2. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let H be the orthocenter of ABC. Find the distance between the circumcenters of triangles AHB and AHC.

3. In the below picture, T is an equilateral triangle with a side length of 5 and ω is a circle with a radius of 2. The triangle and the circle have the same center. Let X be the area of the shaded region, and let Y be the area of the starred region. What is $X - Y$?



4. Let ABC be a triangle with $AB = 3$, $AC = 8$, $BC = 7$ and let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Point T is selected on side BC so that $AT = TC$. The circumcircles of triangles BAT, MAN intersect at D. Compute DC.

5. Nine pairwise noncongruent circles are drawn in the plane such that any two circles intersect twice. For each pair of circles, we draw the line through these two points, for a total of $\binom{9}{2} = 36$ lines. Assume that all 36 lines drawn are distinct. What is the maximum possible number of points which lie on at least two of the drawn lines?

6. Let ABC be a triangle with incenter I, incircle γ and circumcircle Γ. Let M, N, P be the midpoints of sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ and let E, F be the tangency points of γ with $\overline{CA}$ and $\overline{AB}$, respectively. Let U, V be the intersections of line EF with line MN and line MP, respectively, and let X be the midpoint of arc $\widehat{BAC}$ of Γ. Given that $AB = 5$, $AC = 8$, and $\angle A = 60°$, compute the area of triangle XUV.

7. Let $S = \{(x, y) | x, y \in \mathbb{Z}, 0 \le x, y, \le 2016\}$. Given points $A = (x_1, y_1), B = (x_2, y_2)$ in S, define

$$d_{2017}(A, B) = (x_1 - x_2)^2 + (y_1 - y_2)^2 \pmod{2017}.$$

The points $A = (5, 5), B = (2, 6), C = (7, 11)$ all lie in S. There is also a point $O \in S$ that satisfies

$$d_{2017}(O, A) = d_{2017}(O, B) = d_{2017}(O, C).$$

Find $d_{2017}(O, A)$.

8. For $i = 0, 1, \ldots, 5$ let l_i be the ray on the Cartesian plane starting at the origin, an angle $\theta = i\frac{\pi}{3}$ counterclockwise from the positive x-axis. For each i, point P_i is chosen uniformly at random from the intersection of l_i with the unit disk. Consider the convex hull of the points P_i, which will (with probability 1) be a convex polygon with n vertices for some n. What is the expected value of n?

9. In cyclic quadrilateral $ABCD$ with $AB = AD = 49$ and $AC = 73$, let I and J denote the incenters of triangles ABD and CBD. If diagonal $\overline{BD}$ bisects $\overline{IJ}$, find the length of IJ.

10. The incircle of a triangle ABC is tangent to BC at D. Let H and Γ denote the orthocenter and circumcircle of $\triangle ABC$. The B-*mixtilinear incircle*, centered at O_B, is tangent to lines BA and BC and internally tangent to Γ. The C-*mixtilinear incircle*, centered at O_C, is defined similarly. Suppose that $\overline{DH} \perp \overline{O_B O_C}$, $AB = \sqrt{3}$ and $AC = 2$. Find BC.

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1. [**5**] Let x and y be complex numbers such that $x+y=\sqrt{20}$ and $x^2+y^2=15$. Compute $|x-y|$.

2. [**5**] Sherry is waiting for a train. Every minute, there is a 75% chance that a train will arrive. However, she is engrossed in her game of sudoku, so even if a train arrives she has a 75% chance of not noticing it (and hence missing the train). What is the probability that Sherry catches the train in the next five minutes?

3. [**5**] Let $PROBLEMZ$ be a regular octagon inscribed in a circle of unit radius. Diagonals MR, OZ meet at I. Compute LI.

4. [**5**] Consider a three-person game involving the following three types of fair six-sided dice.

   - Dice of type A have faces labelled 2, 2, 4, 4, 9, 9.
   - Dice of type B have faces labelled 1, 1, 6, 6, 8, 8.
   - Dice of type C have faces labelled 3, 3, 5, 5, 7, 7.

   All three players simultaneously choose a die (more than one person can choose the same type of die, and the players don't know one another's choices) and roll it. Then the score of a player P is the number of players whose roll is less than P's roll (and hence is either 0, 1, or 2). Assuming all three players play optimally, what is the expected score of a particular player?

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5. [**6**] Patrick and Anderson are having a snowball fight. Patrick throws a snowball at Anderson which is shaped like a sphere with a radius of 10 centimeters. Anderson catches the snowball and uses the snow from the snowball to construct snowballs with radii of 4 centimeters. Given that the total volume of the snowballs that Anderson constructs cannot exceed the volume of the snowball that Patrick threw, how many snowballs can Anderson construct?

6. [**6**] Consider a $2\times n$ grid of points and a path consisting of $2n-1$ straight line segments connecting all these $2n$ points, starting from the bottom left corner and ending at the upper right corner. Such a path is called *efficient* if each point is only passed through once and no two line segments intersect. How many efficient paths are there when $n=2016$?

7. [**6**] A contest has six problems worth seven points each. On any given problem, a contestant can score either 0, 1, or 7 points. How many possible total scores can a contestant achieve over all six problems?

8. [**6**] For each positive integer n and non-negative integer k, define $W(n,k)$ recursively by

$$W(n,k)=\begin{cases} n^n & k=0 \\ W(W(n,k-1),k-1) & k>0. \end{cases}$$

   Find the last three digits in the decimal representation of $W(555,2)$.

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9. [**7**] Victor has a drawer with two red socks, two green socks, two blue socks, two magenta socks, two lavender socks, two neon socks, two mauve socks, two wisteria socks, and 2000 copper socks, for a total of 2016 socks. He repeatedly draws two socks at a time from the drawer at random, and stops if the socks are of the same color. However, Victor is red-green colorblind, so he also stops if he sees a red and green sock.

   What is the probability that Victor stops with two socks of the same color? Assume Victor returns both socks to the drawer at each step.

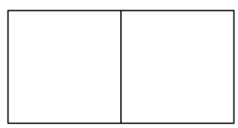
10. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let O be the circumcenter of ABC. Find the distance between the circumcenters of triangles AOB and AOC.

11. [**7**] Define $\phi^!(n)$ as the product of all positive integers less than or equal to n and relatively prime to n. Compute the remainder when

$$\sum_{\substack{2 \le n \le 50 \\ \gcd(n,50)=1}} \phi^!(n)$$

   is divided by 50.

12. [**7**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0)$, $(2,0)$, $(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown; the resulting figure has seven edges.



   Compute the number of ways to choose one or more of the seven edges such that the resulting figure is traceable without lifting a pencil. (Rotations and reflections are considered distinct.)

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13. [**9**] A right triangle has side lengths a, b, and $\sqrt{2016}$ in some order, where a and b are positive integers. Determine the smallest possible perimeter of the triangle.

14. [**9**] Let ABC be a triangle such that $AB = 13$, $BC = 14$, $CA = 15$ and let E, F be the feet of the altitudes from B and C, respectively. Let the circumcircle of triangle AEF be ω. We draw three lines, tangent to the circumcircle of triangle AEF at A, E, and F. Compute the area of the triangle these three lines determine.

15. [**9**] Compute $\tan\left(\frac{\pi}{7}\right)\tan\left(\frac{2\pi}{7}\right)\tan\left(\frac{3\pi}{7}\right)$.

16. [**9**] Determine the number of integers $2 \le n \le 2016$ such that $n^n - 1$ is divisible by 2, 3, 5, 7.

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17. [**11**] Compute the sum of all integers $1 \le a \le 10$ with the following property: there exist integers p and q such that p, q, $p^2 + a$ and $q^2 + a$ are all distinct prime numbers.

18. [**11**] Alice and Bob play a game on a circle with 8 marked points. Alice places an apple beneath one of the points, then picks five of the other seven points and reveals that none of them are hiding the apple. Bob then drops a bomb on any of the points, and destroys the apple if he drops the bomb either on the point containing the apple or on an adjacent point. Bob wins if he destroys the apple, and Alice wins if he fails. If both players play optimally, what is the probability that Bob destroys the apple?

19. [**11**] Let
$$A = \lim_{n\to\infty} \sum_{i=0}^{2016} (-1)^i \cdot \frac{\binom{n}{i}\binom{n}{i+2}}{\binom{n}{i+1}^2}$$
Find the largest integer less than or equal to $\frac{1}{A}$.

The following decimal approximation might be useful: $0.6931 < \ln(2) < 0.6932$, where ln denotes the natural logarithm function.

20. [**11**] Let ABC be a triangle with $AB = 13$, $AC = 14$, and $BC = 15$. Let G be the point on AC such that the reflection of BG over the angle bisector of $\angle B$ passes through the midpoint of AC. Let Y be the midpoint of GC and X be a point on segment AG such that $\frac{AX}{XG} = 3$. Construct F and H on AB and BC, respectively, such that $FX \parallel BG \parallel HY$. If AH and CF concur at Z and W is on AC such that $WZ \parallel BG$, find WZ.

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21. [**12**] Tim starts with a number n, then repeatedly flips a fair coin. If it lands heads he subtracts 1 from his number and if it lands tails he subtracts 2. Let E_n be the expected number of flips Tim does before his number is zero or negative. Find the pair (a, b) such that
$$\lim_{n\to\infty} (E_n - an - b) = 0.$$

22. [**12**] On the Cartesian plane $\mathbb{R}^2$, a circle is said to be *nice* if its center is at the origin $(0, 0)$ and it passes through at least one lattice point (i.e. a point with integer coordinates). Define the points $A = (20, 15)$ and $B = (20, 16)$. How many nice circles intersect the open segment AB?

For reference, the numbers 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691 are the only prime numbers between 600 and 700.

23. [**12**] Let $t = 2016$ and $p = \ln 2$. Evaluate in closed form the sum
$$\sum_{k=1}^{\infty} \left(1 - \sum_{n=0}^{k-1} \frac{e^{-t}t^n}{n!}\right) (1-p)^{k-1} p.$$

24. [**12**] Let ΔA_1B_1C be a triangle with $\angle A_1B_1C = 90^\circ$ and $\frac{CA_1}{CB_1} = \sqrt{5} + 2$. For any $i \ge 2$, define A_i to be the point on the line A_1C such that $A_iB_{i-1} \perp A_1C$ and define B_i to be the point on the line B_1C such that $A_iB_i \perp B_1C$. Let Γ_1 be the incircle of ΔA_1B_1C and for $i \ge 2$, Γ_i be the circle tangent to Γ_{i-1}, A_1C, B_1C which is smaller than Γ_{i-1}.

How many integers k are there such that the line A_1B_{2016} intersects Γ_k?

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25. [**14**] A particular coin can land on heads (H), on tails (T), or in the middle (M), each with probability $\frac{1}{3}$. Find the expected number of flips necessary to observe the contiguous sequence HMMTHMMT...HMMT, where the sequence HMMT is repeated 2016 times.

26. [**14**] For positive integers a, b, $a \uparrow\uparrow b$ is defined as follows: $a \uparrow\uparrow 1 = a$, and $a \uparrow\uparrow b = a^{a \uparrow\uparrow (b-1)}$ if $b > 1$.

    Find the smallest positive integer n for which there exists a positive integer a such that $a \uparrow\uparrow 6 \not\equiv a \uparrow\uparrow 7 \mod n$.

27. [**14**] Find the smallest possible area of an ellipse passing through $(2, 0)$, $(0, 3)$, $(0, 7)$, and $(6, 0)$.

28. [**14**] Among citizens of Cambridge there exist 8 different types of blood antigens. In a crowded lecture hall are 256 students, each of whom has a blood type corresponding to a distinct subset of the antigens; the remaining of the antigens are foreign to them.

    Quito the Mosquito flies around the lecture hall, picks a subset of the students uniformly at random, and bites the chosen students in a random order. After biting a student, Quito stores a bit of any antigens that student had. A student bitten while Quito had k blood antigen foreign to him/her will suffer for k hours. What is the expected total suffering of all 256 students, in hours?

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29. [**16**] Katherine has a piece of string that is 2016 millimeters long. She cuts the string at a location chosen uniformly at random, and takes the left half. She continues this process until the remaining string is less than one millimeter long. What is the expected number of cuts that she makes?

30. [**16**] Determine the number of triples $0 \le k, m, n \le 100$ of integers such that

$$2^m n - 2^n m = 2^k.$$

31. [**16**] For a positive integer n, denote by $\tau(n)$ the number of positive integer divisors of n, and denote by $\phi(n)$ the number of positive integers that are less than or equal to n and relatively prime to n. Call a positive integer n *good* if $\varphi(n) + 4\tau(n) = n$. For example, the number 44 is good because $\varphi(44) + 4\tau(44) = 44$.

    Find the sum of all good positive integers n.

32. [**16**] How many equilateral hexagons of side length $\sqrt{13}$ have one vertex at $(0, 0)$ and the other five vertices at lattice points?

    (A lattice point is a point whose Cartesian coordinates are both integers. A hexagon may be concave but not self-intersecting.)

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33. [**20**] **(Lucas Numbers)** The Lucas numbers are defined by $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$ for every $n \geq 0$. There are N integers $1 \leq n \leq 2016$ such that L_n contains the digit 1. Estimate N.

    An estimate of E earns $\lfloor 20 - 2|N - E| \rfloor$ or 0 points, whichever is greater.

34. [**20**] **(Caos)** A cao [sic] has 6 legs, 3 on each side. A walking pattern for the cao is defined as an ordered sequence of raising and lowering each of the legs exactly once (altogether 12 actions), starting and ending with all legs on the ground. The pattern is safe if at any point, he has at least 3 legs on the ground and not all three legs are on the same side. Estimate N, the number of safe patterns.

    An estimate of $E > 0$ earns $\left\lfloor 20 \min(N/E, E/N)^4 \right\rfloor$ points.

35. [**20**] **(Maximal Determinant)** In a 17×17 matrix M, all entries are ± 1. The maximum possible value of $|\det M|$ is N. Estimate N.

    An estimate of $E > 0$ earns $\left\lfloor 20 \min(N/E, E/N)^2 \right\rfloor$ points.

36. [**20**] **(Self-Isogonal Cubics)** Let ABC be a triangle with $AB = 2$, $AC = 3$, $BC = 4$. The *isogonal conjugate* of a point P, denoted P^*, is the point obtained by intersecting the reflection of lines PA, PB, PC across the angle bisectors of $\angle A$, $\angle B$, and $\angle C$, respectively.

    Given a point Q, let $\mathfrak{K}(Q)$ denote the unique cubic plane curve which passes through all points P such that line PP^* contains Q. Consider:

    (a) the M'Cay cubic $\mathfrak{K}(O)$, where O is the circumcenter of $\triangle ABC$,

    (b) the Thomson cubic $\mathfrak{K}(G)$, where G is the centroid of $\triangle ABC$,

    (c) the Napoleon-Feurerbach cubic $\mathfrak{K}(N)$, where N is the nine-point center of $\triangle ABC$,

    (d) the Darboux cubic $\mathfrak{K}(L)$, where L is the de Longchamps point (the reflection of the orthocenter across point O),

    (e) the Neuberg cubic $\mathfrak{K}(X_{30})$, where X_{30} is the point at infinity along line OG,

    (f) the nine-point circle of $\triangle ABC$,

    (g) the incircle of $\triangle ABC$, and

    (h) the circumcircle of $\triangle ABC$.

    Estimate N, the number of points lying on at least two of these eight curves. An estimate of E earns $\left\lfloor 20 \cdot 2^{-|N-E|/6} \right\rfloor$ points.

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Team

1. [**25**] Let a and b be integers (not necessarily positive). Prove that $a^3 + 5b^3 \neq 2016$.

2. [**25**] For positive integers n, let c_n be the smallest positive integer for which $n^{c_n} - 1$ is divisible by 210, if such a positive integer exists, and $c_n = 0$ otherwise. What is $c_1 + c_2 + \cdots + c_{210}$?

3. [**30**] Let ABC be an acute triangle with incenter I and circumcenter O. Assume that $\angle OIA = 90°$. Given that $AI = 97$ and $BC = 144$, compute the area of $\triangle ABC$.

4. [**30**] Let $n > 1$ be an odd integer. On an $n \times n$ chessboard the center square and four corners are deleted. We wish to group the remaining $n^2 - 5$ squares into $\frac{1}{2}(n^2 - 5)$ pairs, such that the two squares in each pair intersect at exactly one point (i.e. they are diagonally adjacent, sharing a single corner).

   For which odd integers $n > 1$ is this possible?

5. [**35**] Find all prime numbers p such that $y^2 = x^3 + 4x$ has exactly p solutions in integers modulo p.

   In other words, determine all prime numbers p with the following property: there exist exactly p ordered pairs of integers (x, y) such that $x, y \in \{0, 1, \ldots, p-1\}$ and

   $$p \text{ divides } y^2 - x^3 - 4x.$$

6. [**35**] A nonempty set S is called *well-filled* if for every $m \in S$, there are fewer than $\frac{1}{2}m$ elements of S which are less than m. Determine the number of well-filled subsets of $\{1, 2, \ldots, 42\}$.

7. [**40**] Let $q(x) = q^1(x) = 2x^2 + 2x - 1$, and let $q^n(x) = q(q^{n-1}(x))$ for $n > 1$. How many negative real roots does $q^{2016}(x)$ have?

8. [**40**] Compute

   $$\int_0^\pi \frac{2\sin\theta + 3\cos\theta - 3}{13\cos\theta - 5} \mathrm{d}\theta.$$

9. [**40**] Fix positive integers $r > s$, and let F be an infinite family of sets, each of size r, no two of which share fewer than s elements. Prove that there exists a set of size $r - 1$ that shares at least s elements with each set in F.

10. [**50**] Let ABC be a triangle with incenter I whose incircle is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ at D, E, F. Point P lies on $\overline{EF}$ such that $\overline{DP} \perp \overline{EF}$. Ray BP meets $\overline{AC}$ at Y and ray CP meets $\overline{AB}$ at Z. Point Q is selected on the circumcircle of $\triangle AYZ$ so that $\overline{AQ} \perp \overline{BC}$.

    Prove that P, I, Q are collinear.

HMMT November 2016

November 12, 2016

General

1. If a and b satisfy the equations $a + \frac{1}{b} = 4$ and $\frac{1}{a} + b = \frac{16}{15}$, determine the product of all possible values of ab.

2. I have five different pairs of socks. Every day for five days, I pick two socks at random without replacement to wear for the day. Find the probability that I wear matching socks on both the third day and the fifth day.

3. Let V be a rectangular prism with integer side lengths. The largest face has area 240 and the smallest face has area 48. A third face has area x, where x is not equal to 48 or 240. What is the sum of all possible values of x?

4. A rectangular pool table has vertices at $(0, 0)$ $(12, 0)$ $(0, 10)$, and $(12, 10)$. There are pockets only in the four corners. A ball is hit from $(0, 0)$ along the line $y = x$ and bounces off several walls before eventually entering a pocket. Find the number of walls that the ball bounces off of before entering a pocket.

5. Let the sequence $\{a_i\}_{i=0}^{\infty}$ be defined by $a_0 = \frac{1}{2}$ and $a_n = 1 + (a_{n-1} - 1)^2$. Find the product

$$\prod_{i=0}^{\infty} a_i = a_0 a_1 a_2 \ldots$$

6. The numbers $1, 2 \cdots 11$ are arranged in a line from left to right in a random order. It is observed that the middle number is larger than exactly one number to its left. Find the probability that it is larger than exactly one number to its right.

7. Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. The altitude from A intersects BC at D. Let ω_1 and ω_2 be the incircles of ABD and ACD, and let the common external tangent of ω_1 and ω_2 (other than BC) intersect AD at E. Compute the length of AE.

8. Let $S = \{1, 2, \ldots 2016\}$, and let f be a randomly chosen bijection from S to itself. Let n be the smallest positive integer such that $f^{(n)}(1) = 1$, where $f^{(i)}(x) = f(f^{(i-1)}(x))$. What is the expected value of n?

9. Let the sequence a_i be defined as $a_{i+1} = 2^{a_i}$. Find the number of integers $1 \leq n \leq 1000$ such that if $a_0 = n$, then 100 divides $a_{1000} - a_1$.

10. Quadrilateral $ABCD$ satisfies $AB = 8$, $BC = 5$, $CD = 17$, $DA = 10$. Let E be the intersection of AC and BD. Suppose $BE : ED = 1 : 2$. Find the area of $ABCD$.

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1. [**5**] If five fair coins are flipped simultaneously, what is the probability that at least three of them show heads?

2. [**5**] How many perfect squares divide 10^{10}?

3. [**5**] Evaluate $\frac{2016!^2}{2015!2017!}$. Here $n!$ denotes $1 \times 2 \times \cdots \times n$.

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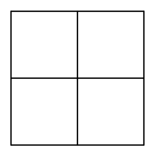
4. [**6**] A square can be divided into four congruent figures as shown:



For how many n with $1 \leq n \leq 100$ can a unit square be divided into n congruent figures?

5. [**6**] If $x + 2y - 3z = 7$ and $2x - y + 2z = 6$, determine $8x + y$.

6. [**6**] Let $ABCD$ be a rectangle, and let E and F be points on segment AB such that $AE = EF = FB$. If CE intersects the line AD at P, and PF intersects BC at Q, determine the ratio of BQ to CQ.

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7. [**7**] What is the minimum value of the product

$$\prod_{i=1}^{6} \frac{a_i - a_{i+1}}{a_{i+2} - a_{i+3}}$$

given that $(a_1, a_2, a_3, a_4, a_5, a_6)$ is a permutation of $(1, 2, 3, 4, 5, 6)$? (note $a_7 = a_1, a_8 = a_2 \cdots$)

8. [**7**] Danielle picks a positive integer $1 \le n \le 2016$ uniformly at random. What is the probability that $\gcd(n, 2015) = 1$?

9. [**7**] How many 3-element subsets of the set $\{1, 2, 3, \ldots, 19\}$ have sum of elements divisible by 4?

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10. [**8**] Michael is playing basketball. He makes 10% of his shots, and gets the ball back after 90% of his missed shots. If he does not get the ball back he stops playing. What is the probability that Michael eventually makes a shot?

11. [**8**] How many subsets S of the set $\{1, 2, \ldots, 10\}$ satisfy the property that, for all $i \in [1, 9]$, either i or $i + 1$ (or both) is in S?

12. [**8**] A positive integer $\overline{ABC}$, where A, B, C are digits, satisfies

$$\overline{ABC} = B^C - A$$

Find $\overline{ABC}$.

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13. [**9**] How many functions $f : \{0,1\}^3 \to \{0,1\}$ satisfy the property that, for all ordered triples (a_1, a_2, a_3) and (b_1, b_2, b_3) such that $a_i \geq b_i$ for all i, $f(a_1, a_2, a_3) \geq f(b_1, b_2, b_3)$?

14. [**9**] The very hungry caterpillar lives on the number line. For each non-zero integer i, a fruit sits on the point with coordinate i. The caterpillar moves back and forth; whenever he reaches a point with food, he eats the food, increasing his weight by one pound, and turns around. The caterpillar moves at a speed of 2^{-w} units per day, where w is his weight. If the caterpillar starts off at the origin, weighing zero pounds, and initially moves in the positive x direction, after how many days will he weigh 10 pounds?

15. [**9**] Let $ABCD$ be an isosceles trapezoid with parallel bases $AB = 1$ and $CD = 2$ and height 1. Find the area of the region containing all points inside $ABCD$ whose projections onto the four sides of the trapezoid lie on the segments formed by AB,BC,CD and DA.

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16. [**10**] Create a cube C_1 with edge length 1. Take the centers of the faces and connect them to form an octahedron O_1. Take the centers of the octahedron's faces and connect them to form a new cube C_2. Continue this process infinitely. Find the sum of all the surface areas of the cubes and octahedrons.

17. [**10**] Let $p(x) = x^2 - x + 1$. Let α be a root of $p(p(p(p(x))))$. Find the value of

$$(p(\alpha) - 1)p(\alpha)p(p(\alpha))p(p(p(\alpha)))$$

18. [**10**] An 8 by 8 grid of numbers obeys the following pattern:

    1) The first row and first column consist of all 1s.

    2) The entry in the ith row and jth column equals the sum of the numbers in the $(i-1)$ by $(j-1)$ sub-grid with row less than i and column less than j.

    What is the number in the 8th row and 8th column?

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19. [**11**] Let S be the set of all positive integers whose prime factorizations only contain powers of the primes 2 and 2017 (1, powers of 2, and powers of 2017 are thus contained in S). Compute $\sum_{s\in S} \frac{1}{s}$.

20. [**11**] Let $\mathcal{V}$ be the volume enclosed by the graph

$$x^{2016} + y^{2016} + z^2 = 2016$$

Find $\mathcal{V}$ rounded to the nearest multiple of ten.

21. [**11**] Zlatan has 2017 socks of various colours. He wants to proudly display one sock of each of the colours, and he counts that there are N ways to select socks from his collection for display. Given this information, what is the maximum value of N?

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22. [**12**] Let the function $f : \mathbb{Z} \to \mathbb{Z}$ take only integer inputs and have integer outputs. For any integers x and y, f satisfies

$$f(x) + f(y) = f(x+1) + f(y-1)$$

If $f(2016) = 6102$ and $f(6102) = 2016$, what is $f(1)$?

23. [**12**] Let d be a randomly chosen divisor of 2016. Find the expected value of

$$\frac{d^2}{d^2 + 2016}$$

24. [**12**] Consider an infinite grid of equilateral triangles. Each edge (that is, each side of a small triangle) is colored one of N colors. The coloring is done in such a way that any path between any two non-adjecent vertices consists of edges with at least two different colors. What is the smallest possible value of N?

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25. [**13**] Chris and Paul each rent a different room of a hotel from rooms $1 - 60$. However, the hotel manager mistakes them for one person and gives "Chris Paul" a room with Chris's and Paul's room concatenated. For example, if Chris had 15 and Paul had 9, "Chris Paul" has 159. If there are 360 rooms in the hotel, what is the probability that "Chris Paul" has a valid room?

26. [**13**] Find the number of ways to choose two nonempty subsets X and Y of $\{1, 2, \ldots, 2001\}$, such that $|Y| = 1001$ and the smallest element of Y is equal to the largest element of X.

27. [**13**] Let r_1, r_2, r_3, r_4 be the four roots of the polynomial $x^4 - 4x^3 + 8x^2 - 7x + 3$. Find the value of

$$\frac{r_1^2}{r_2^2+r_3^2+r_4^2} + \frac{r_2^2}{r_1^2+r_3^2+r_4^2} + \frac{r_3^2}{r_1^2+r_2^2+r_4^2} + \frac{r_4^2}{r_1^2+r_2^2+r_3^2}$$

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28. [**15**] The numbers $1 - 10$ are written in a circle randomly. Find the expected number of numbers which are at least 2 larger than an adjacent number.

29. [**15**] We want to design a new chess piece, the American, with the property that (i) the American can never attack itself, and (ii) if an American A_1 attacks another American A_2, then A_2 also attacks A_1. Let m be the number of squares that an American attacks when placed in the top left corner of an 8 by 8 chessboard. Let n be the maximal number of Americans that can be placed on the 8 by 8 chessboard such that no Americans attack each other, if one American must be in the top left corner. Find the largest possible value of mn.

30. [**15**] On the blackboard, Amy writes 2017 in base-a to get 133201_a. Betsy notices she can erase a digit from Amy's number and change the base to base-b such that the value of the the number remains the same. Catherine then notices she can erase a digit from Betsy's number and change the base to base-c such that the value still remains the same. Compute, in decimal, $a + b + c$.

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31. [**17**] Define a number to be an anti-palindrome if, when written in base 3 as $a_n a_{n-1} ... a_0$, then $a_i + a_{n-i} = 2$ for any $0 \leq i \leq n$. Find the number of anti-palindromes less than 3^{12} such that no two consecutive digits in base 3 are equal.

32. [**17**] Let $C_{k,n}$ denote the number of paths on the Cartesian plane along which you can travel from $(0,0)$ to (k,n), given the following rules: 1) You can only travel directly upward or directly rightward 2) You can only change direction at lattice points 3) Each horizontal segment in the path must be at most 99 units long.

    Find

$$\sum_{j=0}^{\infty} C_{100j+19,17}$$

33. [**17**] Camille the snail lives on the surface of a regular dodecahedron. Right now he is on vertex P_1 of the face with vertices P_1, P_2, P_3, P_4, P_5. This face has a perimeter of 5. Camille wants to get to the point on the dodecahedron farthest away from P_1. To do so, he must travel along the surface a distance at least L. What is L^2?

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34. [**20**] Find the sum of the ages of everyone who wrote a problem for this year's HMMT November contest. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - |X - Y|)$

35. [**20**] Find the total number of occurrences of the digits $0, 1 \ldots, 9$ in the entire guts round. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - \frac{|X-Y|}{2})$

36. [**20**] Find the number of positive integers less than 1000000 which are less than or equal to the sum of their proper divisors. If your answer is X and the actual value is Y, your score will be $\max(0, 20 - 80|1 - \frac{X}{Y}|)$ rounded to the nearest integer.

HMMT November 2016

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Team

1. [**3**] Two circles centered at O_1 and O_2 have radii 2 and 3 and are externally tangent at P. The common external tangent of the two circles intersects the line O_1O_2 at Q. What is the length of PQ?

2. [**3**] What is the smallest possible perimeter of a triangle whose side lengths are all squares of distinct positive integers?

3. [**3**] Complex number ω satisfies $\omega^5 = 2$. Find the sum of all possible values of

$$\omega^4 + \omega^3 + \omega^2 + \omega + 1.$$

4. [**5**] Meghal is playing a game with 2016 rounds $1, 2, \cdots, 2016$. In round n, two rectangular double-sided mirrors are arranged such that they share a common edge and the angle between the faces is $\frac{2\pi}{n+2}$. Meghal shoots a laser at these mirrors and her score for the round is the number of points on the two mirrors at which the laser beam touches a mirror. What is the maximum possible score Meghal could have after she finishes the game?

5. [**5**] Allen and Brian are playing a game in which they roll a 6-sided die until one of them wins. Allen wins if two consecutive rolls are equal and at most 3. Brian wins if two consecutive rolls add up to 7 and the latter is at most 3. What is the probability that Allen wins?

6. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 6$, and $AC = 7$. Let its orthocenter be H and the feet of the altitudes from A, B, C to the opposite sides be D, E, F respectively. Let the line DF intersect the circumcircle of AHF again at X. Find the length of EX.

7. [**6**] Rachel has two indistinguishable tokens, and places them on the first and second square of a 1×6 grid of squares, She can move the pieces in two ways:

   - If a token has free square in front of it, then she can move this token one square to the right
   - If the square immediately to the right of a token is occupied by the other token, then she can"leapfrog" the first token; she moves the first token two squares to the right, over the other token, so that it is on the square immediately to the right of the other token.

   If a token reaches the 6th square, then it cannot move forward any more, and Rachel must move the other one until it reaches the 5th square. How many different sequences of moves for the tokens can Rachel make so that the two tokens end up on the 5th square and the 6th square?

8. [**6**] Alex has an 20×16 grid of lightbulbs, initially all off. He has 36 switches, one for each row and column. Flipping the switch for the ith row will toggle the state of each lightbulb in the ith row (so that if it were on before, it would be off, and vice versa). Similarly, the switch for the jth column will toggle the state of each bulb in the jth column. Alex makes some (possibly empty) sequence of switch flips, resulting in some configuration of the lightbulbs and their states. How many distinct possible configurations of lightbulbs can Alex achieve with such a sequence? Two configurations are distinct if there exists a lightbulb that is on in one configuration and off in another.

9. [**7**] A cylinder with radius 15 and height 16 is inscribed in a sphere. Three congruent smaller spheres of radius x are externally tangent to the base of the cylinder, externally tangent to each other, and internally tangent to the large sphere. What is the value of x?

10. [**7**] Determine the largest integer n such that there exist monic quadratic polynomials $p_1(x)$, $p_2(x)$, $p_3(x)$ with integer coefficients so that for all integers $i \in [1, n]$ there exists some $j \in [1, 3]$ and $m \in \mathbb{Z}$ such that $p_j(m) = i$.

HMMT November 2016

November 12, 2016

Theme Round

1. DeAndre Jordan shoots free throws that are worth 1 point each. He makes 40% of his shots. If he takes two shots find the probability that he scores at least 1 point.

2. Point P_1 is located 600 miles West of point P_2. At 7:00 AM a car departs from P_1 and drives East at a speed of 50 miles per hour. At 8:00 AM another car departs from P_2 and drives West at a constant speed of x miles per hour. If the cars meet each other exactly halfway between P_1 and P_2, what is the value of x?

3. The three points A, B, C form a triangle. $AB = 4$, $BC = 5$ $AC = 6$. Let the angle bisector of $\angle A$ intersect side BC at D. Let the foot of the perpendicular from B to the angle bisector of $\angle A$ be E. Let the line through E parallel to AC meet BC at F. Compute DF.

4. A positive integer is written on each corner of a square such that numbers on opposite vertices are relatively prime while numbers on adjacent vertices are not relatively prime. What is the smallest possible value of the sum of these 4 numbers?

5. Steph Curry is playing the following game and he wins if he has exactly 5 points at some time. Flip a fair coin. If heads, shoot a 3-point shot which is worth 3 points. If tails, shoot a free throw which is worth 1 point. He makes $\frac{1}{2}$ of his 3-point shots and all of his free throws. Find the probability he will win the game. (Note he keeps flipping the coin until he has exactly 5 or goes over 5 points)

6. Let $P_1, P_2, \ldots, P_6$ be points in the complex plane, which are also roots of the equation $x^6 + 6x^3 - 216 = 0$. Given that $P_1P_2P_3P_4P_5P_6$ is a convex hexagon, determine the area of this hexagon.

7. Seven lattice points form a convex heptagon with all sides having distinct lengths. Find the minimum possible value of the sum of the squares of the sides of the heptagon.

8. Let $P_1P_2 \ldots P_8$ be a convex octagon. An integer i is chosen uniformly at random from 1 to 7, inclusive. For each vertex of the octagon, the line between that vertex and the vertex i vertices to the right is painted red. What is the expected number times two red lines intersect at a point that is not one of the vertices, given that no three diagonals are concurrent?

9. The vertices of a regular nonagon are colored such that 1) adjacent vertices are different colors and 2) if 3 vertices form an equilateral triangle, they are all different colors.

   Let m be the minimum number of colors needed for a valid coloring, and n be the total number of colorings using m colors. Determine mn. (Assume each vertex is distinguishable.)

10. We have 10 points on a line $A_1, A_2 \cdots A_{10}$ in that order. Initially there are n chips on point A_1. Now we are allowed to perform two types of moves. Take two chips on A_i, remove them and place one chip on A_{i+1}, or take two chips on A_{i+1}, remove them, and place a chip on A_{i+2} and A_i . Find the minimum possible value of n such that it is possible to get a chip on A_{10} through a sequence of moves.

HMMT Invitational Competition

April 2016

1. [**5**] Theseus starts at the point $(0, 0)$ in the plane. If Theseus is standing at the point (x, y) in the plane, he can step one unit to the north to point $(x, y + 1)$, one unit to the west to point $(x - 1, y)$, one unit to the south to point $(x, y - 1)$, or one unit to the east to point $(x + 1, y)$. After a sequence of more than two such moves, starting with a step one unit to the south (to point $(0, -1)$), Theseus finds himself back at the point $(0, 0)$. He never visited any point other than $(0, 0)$ more than once, and never visited the point $(0, 0)$ except at the start and end of this sequence of moves.

   Let X be the number of times that Theseus took a step one unit to the north, and then a step one unit to the west immediately afterward. Let Y be the number of times that Theseus took a step one unit to the west, and then a step one unit to the north immediately afterward. Prove that $|X - Y| = 1$.

2. [**7**] Let ABC be an acute triangle with circumcenter O, orthocenter H, and circumcircle Ω. Let M be the midpoint of AH and N the midpoint of BH. Assume the points M, N, O, H are distinct and lie on a circle ω. Prove that the circles ω and Ω are internally tangent to each other.

3. [**8**] Denote by $\mathbb{N}$ the positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that, for any $w, x, y, z \in \mathbb{N}$,

$$f(f(f(z)))f(wxf(yf(z))) = z^2 f(xf(y))f(w).$$

   Show that $f(n!) \geq n!$ for every positive integer n.

4. [**10**] Let P be an odd-degree integer-coefficient polynomial. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

5. [**12**] Let $S = \{a_1, \ldots, a_n\}$ be a finite set of positive integers of size $n \geq 1$, and let T be the set of all positive integers that can be expressed as sums of perfect powers (including 1) of distinct numbers in S, meaning

$$T = \left\{ \sum_{i=1}^{n} a_i^{e_i} \mid e_1, e_2, \ldots, e_n \geq 0 \right\}.$$

   Show that there is a positive integer N (only depending on n) such that T contains no arithmetic progression of length N.

HMMT February 2015

Saturday 21 February 2015

Algebra

1. Let Q be a polynomial
$$Q(x) = a_0 + a_1 x + \cdots + a_n x^n,$$
where $a_0, \ldots, a_n$ are nonnegative integers. Given that $Q(1) = 4$ and $Q(5) = 152$, find $Q(6)$.

2. The fraction $\frac{1}{2015}$ has a unique "(restricted) partial fraction decomposition" of the form
$$\frac{1}{2015} = \frac{a}{5} + \frac{b}{13} + \frac{c}{31},$$
where a, b, c are integers with $0 \le a < 5$ and $0 \le b < 13$. Find $a + b$.

3. Let p be a real number and $c \neq 0$ an integer such that
$$c - 0.1 < x^p \left(\frac{1 - (1+x)^{10}}{1 + (1+x)^{10}} \right) < c + 0.1$$
for all (positive) real numbers x with $0 < x < 10^{-100}$. (The exact value 10^{-100} is not important. You could replace it with any "sufficiently small number".)

   Find the ordered pair (p, c).

4. Compute the number of sequences of integers $(a_1, \ldots, a_{200})$ such that the following conditions hold.

   - $0 \le a_1 < a_2 < \cdots < a_{200} \le 202$.
   - There exists a positive integer N with the following property: for every index $i \in \{1, \ldots, 200\}$ there exists an index $j \in \{1, \ldots, 200\}$ such that $a_i + a_j - N$ is divisible by 203.

5. Let a, b, c be positive real numbers such that $a+b+c = 10$ and $ab+bc+ca = 25$. Let $m = \min\{ab, bc, ca\}$. Find the largest possible value of m.

6. Let a, b, c, d, e be nonnegative integers such that $625a + 250b + 100c + 40d + 16e = 15^3$. What is the maximum possible value of $a + b + c + d + e$?

7. Suppose (a_1, a_2, a_3, a_4) is a 4-term sequence of real numbers satisfying the following two conditions:

   - $a_3 = a_2 + a_1$ and $a_4 = a_3 + a_2$;
   - there exist real numbers a, b, c such that
   $$an^2 + bn + c = \cos(a_n)$$
   for all $n \in \{1, 2, 3, 4\}$.

   Compute the maximum possible value of
   $$\cos(a_1) - \cos(a_4)$$
   over all such sequences (a_1, a_2, a_3, a_4).

8. Find the number of ordered pairs of integers $(a, b) \in \{1, 2, \ldots, 35\}^2$ (not necessarily distinct) such that $ax + b$ is a "quadratic residue modulo $x^2 + 1$ and 35", i.e. there exists a polynomial $f(x)$ with integer coefficients such that either of the following **equivalent** conditions holds:

   - there exist polynomials P, Q with integer coefficients such that $f(x)^2 - (ax + b) = (x^2 + 1)P(x) + 35Q(x)$;

- or more conceptually, the remainder when (the polynomial) $f(x)^2 - (ax + b)$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with (integer) coefficients all divisible by 35.

9. Let $N = 30^{2015}$. Find the number of ordered 4-tuples of integers $(A, B, C, D) \in \{1, 2, \ldots, N\}^4$ (not necessarily distinct) such that for every integer n, $An^3 + Bn^2 + 2Cn + D$ is divisible by N.

10. Find all ordered 4-tuples of integers (a, b, c, d) (not necessarily distinct) satisfying the following system of equations:

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= c - b - 2 \\ 2ab &= a - d - 32 \\ 2ac &= 28 - a - d \\ 2ad &= b + c + 31. \end{aligned}$$

HMMT February 2015

Saturday 21 February 2015

Combinatorics

1. Evan's analog clock displays the time 12:13; the number of seconds is not shown. After 10 seconds elapse, it is still 12:13. What is the expected number of seconds until 12:14?

2. Victor has a drawer with 6 socks of 3 different types: 2 complex socks, 2 synthetic socks, and 2 trigonometric socks. He repeatedly draws 2 socks at a time from the drawer at random, and stops if the socks are of the same type. However, Victor is "synthetic-complex type-blind", so he also stops if he sees a synthetic and a complex sock.

   What is the probability that Victor stops with 2 socks of the same type? Assume Victor returns both socks to the drawer after each step.

3. Starting with the number 0, Casey performs an infinite sequence of moves as follows: he chooses a number from $\{1, 2\}$ at random (each with probability $\frac{1}{2}$) and adds it to the current number. Let p_m be the probability that Casey ever reaches the number m. Find $p_{20} - p_{15}$.

4. Alice Czarina is bored and is playing a game with a pile of rocks. The pile initially contains 2015 rocks. At each round, if the pile has N rocks, she removes k of them, where $1 \le k \le N$, with each possible k having equal probability. Alice Czarina continues until there are no more rocks in the pile. Let p be the probability that the number of rocks left in the pile after each round is a multiple of 5. If p is of the form $5^a \cdot 31^b \cdot \frac{c}{d}$, where a, b are integers and c, d are positive integers relatively prime to $5 \cdot 31$, find $a + b$.

5. For positive integers x, let $g(x)$ be the number of blocks of consecutive 1's in the binary expansion of x. For example, $g(19) = 2$ because $19 = 10011_2$ has a block of one 1 at the beginning and a block of two 1's at the end, and $g(7) = 1$ because $7 = 111_2$ only has a single block of three 1's. Compute $g(1) + g(2) + g(3) + \cdots + g(256)$.

6. Count the number of functions $f : \mathbb{Z} \to \{\text{'green'}, \text{'blue'}\}$ such that $f(x) = f(x + 22)$ for all integers x and there does **not** exist an integer y with $f(y) = f(y + 2) = \text{'green'}$.

7. 2015 people sit down at a restaurant. Each person orders a soup with probability $\frac{1}{2}$. Independently, each person orders a salad with probability $\frac{1}{2}$. What is the probability that the number of people who ordered a soup is exactly one more than the number of people who ordered a salad?

8. Let S be the set of all 3-digit numbers with all digits in the set $\{1, 2, 3, 4, 5, 6, 7\}$ (so in particular, **all three digits are nonzero**). For how many elements $\overline{abc}$ of S is it true that at least one of the (not necessarily distinct) "digit cycles"

   $$\overline{abc}, \overline{bca}, \overline{cab}$$

   is divisible by 7? (Here, $\overline{abc}$ denotes the number whose base 10 digits are a, b, and c in that order.)

9. Calvin has a bag containing 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random (without replacement), he has pulled out 5 more red balls than blue balls, what is the probability that the next ball he pulls out is red?

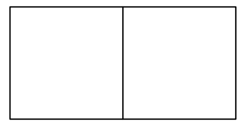
10. A group of friends, numbered $1, 2, 3, \ldots, 16$, take turns picking random numbers. Person 1 picks a number uniformly (at random) in $[0, 1]$, then person 2 picks a number uniformly (at random) in $[0, 2]$, and so on, with person k picking a number uniformly (at random) in $[0, k]$. What is the probability that the 16 numbers picked are strictly increasing?

HMMT February 2015

Saturday 21 February 2015

Geometry

1. Let R be the rectangle in the Cartesian plane with vertices at $(0,0), (2,0), (2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



   Pro selects a point P uniformly at random in the interior of R. Find the probability that the line through P with slope $\frac{1}{2}$ will pass through both unit squares.

2. Let ABC be a triangle with orthocenter H; suppose that $AB = 13$, $BC = 14$, $CA = 15$. Let G_A be the centroid of triangle HBC, and define G_B, G_C similarly. Determine the area of triangle $G_A G_B G_C$.

3. Let $ABCD$ be a quadrilateral with $\angle BAD = \angle ABC = 90°$, and suppose $AB = BC = 1$, $AD = 2$. The circumcircle of ABC meets $\overline{AD}$ and $\overline{BD}$ at points E and F, respectively. If lines AF and CD meet at K, compute EK.

4. Let $ABCD$ be a cyclic quadrilateral with $AB = 3, BC = 2, CD = 2, DA = 4$. Let lines perpendicular to $\overline{BC}$ from B and C meet $\overline{AD}$ at B' and C', respectively. Let lines perpendicular to $\overline{AD}$ from A and D meet $\overline{BC}$ at A' and D', respectively. Compute the ratio $\frac{[BCC'B']}{[DAA'D']}$, where $[\varpi]$ denotes the area of figure ϖ.

5. Let I be the set of points (x, y) in the Cartesian plane such that
$$x > \left(\frac{y^4}{9} + 2015\right)^{1/4}$$
   Let $f(r)$ denote the area of the intersection of I and the disk $x^2 + y^2 \le r^2$ of radius $r > 0$ centered at the origin $(0,0)$. Determine the minimum possible real number L such that $f(r) < Lr^2$ for all $r > 0$.

6. In triangle ABC, $AB = 2$, $AC = 1 + \sqrt{5}$, and $\angle CAB = 54°$. Suppose D lies on the extension of AC through C such that $CD = \sqrt{5} - 1$. If M is the midpoint of BD, determine the measure of $\angle ACM$, in degrees.

7. Let $ABCDE$ be a square pyramid of height $\frac{1}{2}$ with square base $ABCD$ of side length $AB = 12$ (so E is the vertex of the pyramid, and the foot of the altitude from E to $ABCD$ is the center of square $ABCD$). The faces ADE and CDE meet at an acute angle of measure α (so that $0° < \alpha < 90°$). Find $\tan \alpha$.

8. Let S be the set of **discs** D contained completely in the set $\{(x, y) : y < 0\}$ (the region below the x-axis) and centered (at some point) on the curve $y = x^2 - \frac{3}{4}$. What is the area of the union of the elements of S?

9. Let $ABCD$ be a regular tetrahedron with side length 1. Let X be the point in triangle BCD such that $[XBC] = 2[XBD] = 4[XCD]$, where $[\varpi]$ denotes the area of figure ϖ. Let Y lie on segment AX such that $2AY = YX$. Let M be the midpoint of BD. Let Z be a point on segment AM such that the lines YZ and BC intersect at some point. Find $\frac{AZ}{ZM}$.

10. Let $\mathcal{G}$ be the set of all points (x, y) in the Cartesian plane such that $0 \le y \le 8$ and
$$(x-3)^2 + 31 = (y-4)^2 + 8\sqrt{y(8-y)}.$$
   There exists a unique line ℓ of **negative slope** tangent to $\mathcal{G}$ and passing through the point $(0, 4)$. Suppose ℓ is tangent to $\mathcal{G}$ at a **unique** point P. Find the coordinates (α, β) of P.

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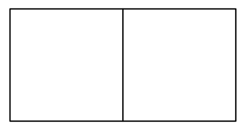
1. [**4**] Let R be the rectangle in the Cartesian plane with vertices at $(0,0),(2,0),(2,1)$, and $(0,1)$. R can be divided into two unit squares, as shown.



The resulting figure has 7 segments of unit length, connecting neighboring lattice points (those lying on or inside R). Compute the number of paths from $(0,1)$ (the upper left corner) to $(2,0)$ (the lower right corner) along these 7 segments, where each segment can be used at most once.

2. [**4**] Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle ACD = \angle ADE = 90^\circ$ and $AB = BC = CD = DE = 1$. Compute AE.

3. [**4**] Find the number of pairs of union/intersection operations $(\Box_1, \Box_2) \in \{\cup, \cap\}^2$ satisfying the following condition: for any sets S, T, function $f : S \to T$, and subsets X, Y, Z of S, we have equality of **sets**
$$f(X)\Box_1(f(Y)\Box_2 f(Z)) = f\left(X\Box_1(Y\Box_2 Z)\right),$$
where $f(X)$ denotes the *image* of X: the **set** $\{f(x) : x \in X\}$, which is a subset of T. The images $f(Y)$ (of Y) and $f(Z)$ (of Z) are similarly defined.

4. [**4**] Consider the function $z(x,y)$ describing the paraboloid
$$z = (2x-y)^2 - 2y^2 - 3y.$$
Archimedes and Brahmagupta are playing a game. Archimedes first chooses x. Afterwards, Brahmagupta chooses y. Archimedes wishes to minimize z while Brahmagupta wishes to maximize z. Assuming that Brahmagupta will play optimally, what value of x should Archimedes choose?

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5. [**5**] Let $\mathcal{H}$ be the unit hypercube of dimension 4 with a vertex at (x, y, z, w) for each choice of $x, y, z, w \in \{0, 1\}$. (Note that $\mathcal{H}$ has $2^4 = 16$ vertices.) A bug starts at the vertex $(0, 0, 0, 0)$. In how many ways can the bug move to $(1, 1, 1, 1)$ (the opposite corner of $\mathcal{H}$) by taking exactly 4 steps along the edges of $\mathcal{H}$?

6. [**5**] Let D be a regular ten-sided polygon with edges of length 1. A triangle T is defined by choosing three vertices of D and connecting them with edges. How many different (non-congruent) triangles T can be formed?

7. [**5**] Let $\mathcal{C}$ be a cube of side length 2. We color each of the faces of $\mathcal{C}$ blue, then subdivide it into $2^3 = 8$ unit cubes. We then randomly rearrange these cubes (possibly with rotation) to form a new 3-dimensional cube.

   What is the probability that its exterior is still completely blue?

8. [**5**] Evaluate
$$\sin(\arcsin(0.4) + \arcsin(0.5)) \cdot \sin(\arcsin(0.5) - \arcsin(0.4)),$$
   where for $x \in [-1, 1]$, $\arcsin(x)$ denotes the unique real number $y \in [-\pi, \pi]$ such that $\sin(y) = x$.

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9. [**6**] Let a, b, c be integers. Define $f(x) = ax^2 + bx + c$. Suppose there exist pairwise distinct integers u, v, w such that $f(u) = 0$, $f(v) = 0$, and $f(w) = 2$. Find the maximum possible value of the discriminant $b^2 - 4ac$ of f.

10. [**6**] Let $b(x) = x^2 + x + 1$. The polynomial $x^{2015} + x^{2014} + \cdots + x + 1$ has a unique "base $b(x)$" representation

$$x^{2015} + x^{2014} + \cdots + x + 1 = \sum_{k=0}^{N} a_k(x) b(x)^k,$$

where

- N is a nonnegative integer;
- each "digit" $a_k(x)$ (for $0 \le k \le N$) is either the zero polynomial (i.e. $a_k(x) = 0$) or a nonzero polynomial of degree less than $\deg b = 2$; and
- the "leading digit $a_N(x)$" is nonzero (i.e. not the zero polynomial).

Find $a_N(0)$ (the "leading digit evaluated at 0").

11. [**6**] Find

$$\sum_{k=0}^{\infty} \left\lfloor \frac{1 + \sqrt{\frac{2000000}{4^k}}}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

12. [**6**] For integers a, b, c, d, let $f(a, b, c, d)$ denote the number of ordered pairs of integers $(x, y) \in \{1, 2, 3, 4, 5\}^2$ such that $ax + by$ and $cx + dy$ are both divisible by 5. Find the sum of all possible values of $f(a, b, c, d)$.

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13. [**8**] Let $P(x) = x^3 + ax^2 + bx + 2015$ be a polynomial all of whose roots are integers. Given that $P(x) \geq 0$ for all $x \geq 0$, find the sum of all possible values of $P(-1)$.

14. [**8**] Find the smallest integer $n \geq 5$ for which there exists a set of n distinct pairs $(x_1, y_1), \ldots, (x_n, y_n)$ of positive integers with $1 \leq x_i, y_i \leq 4$ for $i = 1, 2, \ldots, n$, such that for any indices $r, s \in \{1, 2, \ldots, n\}$ (not necessarily distinct), there exists an index $t \in \{1, 2, \ldots, n\}$ such that 4 divides $x_r + x_s - x_t$ and $y_r + y_s - y_t$.

15. [**8**] Find the maximum possible value of $H \cdot M \cdot M \cdot T$ over all ordered triples (H, M, T) of integers such that $H \cdot M \cdot M \cdot T = H + M + M + T$.

16. [**8**] Determine the number of **unordered** triples of **distinct** points in the $4 \times 4 \times 4$ lattice grid $\{0, 1, 2, 3\}^3$ that are collinear in $\mathbb{R}^3$ (i.e. there exists a line passing through the three points).

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17. [**11**] Find the least positive integer $N > 1$ satisfying the following two properties:
   - There exists a positive integer a such that $N = a(2a - 1)$.
   - The sum $1 + 2 + \cdots + (N - 1)$ is divisible by k for every integer $1 \leq k \leq 10$.

18. [**11**] Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function such that for any integers x, y, we have

$$f(x^2 - 3y^2) + f(x^2 + y^2) = 2(x + y)f(x - y).$$

   Suppose that $f(n) > 0$ for all $n > 0$ and that $f(2015) \cdot f(2016)$ is a perfect square. Find the minimum possible value of $f(1) + f(2)$.

19. [**11**] Find the smallest positive integer n such that the polynomial $(x + 1)^n - 1$ is "divisible by $x^2 + 1$ modulo 3", or more precisely, either of the following **equivalent** conditions holds:
   - there exist polynomials P, Q with integer coefficients such that $(x+1)^n - 1 = (x^2+1)P(x) + 3Q(x)$;
   - or more conceptually, the remainder when (the polynomial) $(x + 1)^n - 1$ is divided by (the polynomial) $x^2 + 1$ is a polynomial with (integer) coefficients all divisible by 3.

20. [**11**] What is the largest real number θ less than π (i.e. $\theta < \pi$) such that

$$\prod_{k=0}^{10} \cos(2^k\theta) \neq 0$$

   and

$$\prod_{k=0}^{10} \left(1 + \frac{1}{\cos(2^k\theta)}\right) = 1?$$

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21. [**14**] Define a sequence $a_{i,j}$ of integers such that $a_{1,n} = n^n$ for $n \geq 1$ and $a_{i,j} = a_{i-1,j} + a_{i-1,j+1}$ for all $i, j \geq 1$. Find the last (decimal) digit of $a_{128,1}$.

22. [**14**] Let $A_1, A_2, \ldots, A_{2015}$ be distinct points on the unit circle with center O. For every two distinct integers i, j, let P_{ij} be the midpoint of A_i and A_j. Find the smallest possible value of

$$\sum_{1 \leq i < j \leq 2015} OP_{ij}^2.$$

23. [**14**] Let $S = \{1, 2, 4, 8, 16, 32, 64, 128, 256\}$. A subset P of S is called *squarely* if it is nonempty and the sum of its elements is a perfect square. A squarely set Q is called *super squarely* if it is not a proper subset of any squarely set. Find the number of super squarely sets.

    (A set A is said to be a *proper subset* of a set B if A is a subset of B and $A \neq B$.)

24. [**14**] $ABCD$ is a cyclic quadrilateral with sides $AB = 10$, $BC = 8$, $CD = 25$, and $DA = 12$. A circle ω is tangent to segments DA, AB, and BC. Find the radius of ω.

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25. [**17**] Let $r_1, \ldots, r_n$ be the distinct real zeroes of the equation

$$x^8 - 14x^4 - 8x^3 - x^2 + 1 = 0.$$

Evaluate $r_1^2 + \cdots + r_n^2$.

26. [**17**] Let $a = \sqrt{17}$ and $b = i\sqrt{19}$, where $i = \sqrt{-1}$. Find the maximum possible value of the ratio $|a - z|/|b - z|$ over all complex numbers z of magnitude 1 (i.e. over the unit circle $|z| = 1$).

27. [**17**] Let a, b be integers chosen independently and uniformly at random from the set $\{0, 1, 2, \ldots, 80\}$. Compute the expected value of the remainder when the binomial coefficient $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ is divided by 3. (Here $\binom{0}{0} = 1$ and $\binom{a}{b} = 0$ whenever $a < b$.)

28. [**17**] Let w, x, y, and z be positive real numbers such that

$$\begin{aligned} 0 &\neq \cos w \cos x \cos y \cos z \\ 2\pi &= w + x + y + z \\ 3 \tan w &= k(1 + \sec w) \\ 4 \tan x &= k(1 + \sec x) \\ 5 \tan y &= k(1 + \sec y) \\ 6 \tan z &= k(1 + \sec z). \end{aligned}$$

(Here $\sec t$ denotes $\frac{1}{\cos t}$ when $\cos t \neq 0$.) Find k.

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29. [**20**] Let ABC be a triangle whose incircle has center I and is tangent to $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, at D, E, F. Denote by X the midpoint of major arc $\widehat{BAC}$ of the circumcircle of ABC. Suppose P is a point on line XI such that $\overline{DP} \perp \overline{EF}$.

    Given that $AB = 14$, $AC = 15$, and $BC = 13$, compute DP.

30. [**20**] Find the sum of squares of all **distinct** complex numbers x satisfying the equation

$$0 = 4x^{10} - 7x^9 + 5x^8 - 8x^7 + 12x^6 - 12x^5 + 12x^4 - 8x^3 + 5x^2 - 7x + 4.$$

31. [**20**] Define a *power cycle* to be a set S consisting of the nonnegative integer powers of an integer a, i.e. $S = \{1, a, a^2, \dots\}$ for some integer a. What is the minimum number of power cycles required such that given any odd integer n, there exists some integer k in one of the power cycles such that $n \equiv k \pmod{1024}$?

32. [**20**] A wealthy king has his blacksmith fashion him a large cup, whose inside is a cone of height 9 inches and base diameter 6 inches (that is, the opening at the top of the cup is 6 inches in diameter). At one of his many feasts, he orders the mug to be filled to the brim with cranberry juice.

    For each positive integer n, the king stirs his drink vigorously and takes a sip such that the height of fluid left in his cup after the sip goes down by $\frac{1}{n^2}$ inches. Shortly afterwards, while the king is distracted, the court jester adds pure Soylent to the cup until it's once again full. The king takes sips precisely every minute, and his first sip is exactly one minute after the feast begins.

    As time progresses, the amount of juice consumed by the king (in cubic inches) approaches a number r. Find r.

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33. [**25**] Let N denote the sum of the decimal digits of $\binom{1000}{100}$. Estimate the value of N. If your answer is a positive integer A written fully in decimal notation (for example, 521495223), your score will be the greatest integer not exceeding $25 \cdot (0.99)^{|A-N|}$. Otherwise, your score will be zero.

34. [**25**] For an integer n, let $f(n)$ denote the number of pairs (x, y) of integers such that $x^2 + xy + y^2 = n$. Compute the sum
$$\sum_{n=1}^{10^6} nf(n).$$
Write your answer in the form $a \cdot 10^b$, where b is an integer and $1 \leq a < 10$ is a decimal number.

    If your answer is written in this form, your score will be $\max\{0, 25 - \lfloor 100|\log_{10}(A/N)|\rfloor\}$, where $N = a \cdot 10^b$ is your answer to this problem and A is the actual answer. Otherwise, your score will be zero.

35. [**25**] Let P denote the set of all subsets of $\{1, \ldots, 23\}$. A subset $S \subseteq P$ is called *good* if whenever A, B are sets in S, the set $(A \setminus B) \cup (B \setminus A)$ is also in S. (Here, $A \setminus B$ denotes the set of all elements in A that are not in B, and $B \setminus A$ denotes the set of all elements in B that are not in A.) What **fraction** of the good subsets of P have between 2015 and 3015 elements, inclusive?

    If your answer is a decimal number or a fraction (of the form m/n, where m and n are positive integers), then your score on this problem will be equal to $\max\{0, 25 - \lfloor 1000|A - N|\rfloor\}$, where N is your answer and A is the actual answer. Otherwise, your score will be zero.

36. [**25**] A prime number p is *twin* if at least one of $p+2$ or $p-2$ is prime and *sexy* if at least one of $p+6$ and $p-6$ is prime.

    How many sexy twin primes (i.e. primes that are both twin and sexy) are there less than 10^9? Express your answer as a positive integer N in decimal notation; for example, 521495223. If your answer is in this form, your score for this problem will be $\max\left\{0, 25 - \lfloor \frac{1}{10000}|A - N|\rfloor\right\}$, where A is the actual answer to this problem. Otherwise, your score will be zero.

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Saturday 21 February 2015

Team

For any geometry problem (e.g. Problem 2, 4, or 5), your team's solution should include diagrams as appropriate (sufficiently large, in-scale, clearly labeled, etc.). Failure to meet any of these requirements will result in a 2-point automatic deduction. Thanks in advance for your effort and understanding, and we hope you enjoy the problems!

1. [**5**] The complex numbers x, y, z satisfy

$$\begin{aligned} xyz &= -4 \\ (x+1)(y+1)(z+1) &= 7 \\ (x+2)(y+2)(z+2) &= -3. \end{aligned}$$

Find, with proof, the value of $(x+3)(y+3)(z+3)$.

2. [**10**] Let P be a (non-self-intersecting) polygon in the plane. Let $C_1, \ldots, C_n$ be circles in the plane whose interiors cover the interior of P. For $1 \le i \le n$, let r_i be the radius of C_i. Prove that there is a single circle of radius $r_1 + \cdots + r_n$ whose interior covers the interior of P.

3. [**15**] Let $z = a+bi$ be a complex number with integer real and imaginary parts $a, b \in \mathbb{Z}$, where $i = \sqrt{-1}$ (i.e. z is a Gaussian integer). If p is an odd prime number, show that the real part of $z^p - z$ is an integer divisible by p.

4. [**15**] (Convex) quadrilateral $ABCD$ with $BC = CD$ is inscribed in circle Ω; the diagonals of $ABCD$ meet at X. Suppose $AD < AB$, the circumcircle of triangle BCX intersects segment AB at a point $Y \neq B$, and **ray** $\overrightarrow{CY}$ meets Ω again at a point $Z \neq C$. Prove that ray $\overrightarrow{DY}$ bisects angle ZDB.

(We have only included the conditions $AD < AB$ and that Z lies on **ray** $\overrightarrow{CY}$ for everyone's convenience. With slight modifications, the problem holds in general. But we will only grade your team's solution in this special case.)

5. [**20**] For a convex quadrilateral P, let D denote the sum of the lengths of its diagonals and let S denote its perimeter. Determine, with proof, all possible values of $\frac{S}{D}$.

6. [**30**] \$indy has \$100 in pennies (worth \$0.01 each), nickels (worth \$0.05 each), dimes (worth \$0.10 each), and quarters (worth \$0.25 each). Prove that she can split her coins into two piles, each with total value exactly \$50.

7. [**35**] Let $f : [0, 1] \to \mathbb{C}$ be a *nonconstant* complex-valued function on the real interval $[0, 1]$. Prove that there exists $\epsilon > 0$ (possibly depending on f) such that for any polynomial P with complex coefficients, there exists a complex number z with $|z| \le 1$ such that $|f(|z|) - P(z)| \ge \epsilon$.

8. [**40**] Let π be a permutation of $\{1, 2, \ldots, 2015\}$. With proof, determine the maximum possible number of ordered pairs $(i, j) \in \{1, 2, \ldots, 2015\}^2$ with $i < j$ such that $\pi(i) \cdot \pi(j) > i \cdot j$.

9. [**40**] Let $z = e^{\frac{2\pi i}{101}}$ and let $\omega = e^{\frac{2\pi i}{10}}$. Prove that

$$\prod_{a=0}^{9} \prod_{b=0}^{100} \prod_{c=0}^{100} (\omega^a + z^b + z^c)$$

is an integer and find (with proof) its remainder upon division by 101.

10. [**40**] The sequences of real numbers $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ satisfy $a_{n+1} = (a_{n-1} - 1)(b_n + 1)$ and $b_{n+1} = a_n b_{n-1} - 1$ for $n \ge 2$, with $a_1 = a_2 = 2015$ and $b_1 = b_2 = 2013$. Evaluate, with proof, the infinite sum

$$\sum_{n=1}^{\infty} b_n \left(\frac{1}{a_{n+1}} - \frac{1}{a_{n+3}} \right).$$

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Individual

1. Find the number of triples (a, b, c) of positive integers such that $a + ab + abc = 11$.

2. Let a and b be real numbers randomly (and independently) chosen from the range $[0, 1]$. Find the probability that a, b and 1 form the side lengths of an obtuse triangle.

3. Neo has an infinite supply of red pills and blue pills. When he takes a red pill, his weight will double, and when he takes a blue pill, he will lose one pound. If Neo originally weighs one pound, what is the minimum number of pills he must take to make his weight 2015 pounds?

4. Chords AB and CD of a circle are perpendicular and intersect at a point P. If $AP = 6, BP = 12$, and $CD = 22$, find the area of the circle.

5. Let S be a subset of the set $\{1, 2, 3, \ldots, 2015\}$ such that for any two elements $a, b \in S$, the difference $a - b$ does not divide the sum $a + b$. Find the maximum possible size of S.

6. Consider all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f(f(x) + 2x + 20) = 15.$$

Call an integer n *good* if $f(n)$ can take any integer value. In other words, if we fix n, for any integer m, there exists a function f such that $f(n) = m$. Find the sum of all good integers x.

7. Let $\triangle ABC$ be a right triangle with right angle C. Let I be the incenter of ABC, and let M lie on AC and N on BC, respectively, such that M, I, N are collinear and $\overline{MN}$ is parallel to AB. If $AB = 36$ and the perimeter of CMN is 48, find the area of ABC.

8. Let $ABCD$ be a quadrilateral with an inscribed circle ω that has center I. If $IA = 5, IB = 7, IC = 4, ID = 9$, find the value of $\frac{AB}{CD}$.

9. Rosencrantz plays $n \leq 2015$ games of question, and ends up with a win rate $\left(\text{i.e. } \frac{\#\text{ of games won}}{\#\text{ of games played}}\right)$ of k. Guildenstern has also played several games, and has a win rate less than k. He realizes that if, after playing some more games, his win rate becomes higher than k, then there must have been some point in time when Rosencrantz and Guildenstern had the exact same win-rate. Find the product of all possible values of k.

10. Let N be the number of functions f from $\{1, 2, \ldots, 101\} \to \{1, 2, \ldots, 101\}$ such that $f^{101}(1) = 2$. Find the remainder when N is divided by 103.

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1. [**5**] Farmer Yang has a 2015×2015 square grid of corn plants. One day, the plant in the very center of the grid becomes diseased. Every day, every plant adjacent to a diseased plant becomes diseased. After how many days will all of Yangs corn plants be diseased?

2. [**5**] The three sides of a right triangle form a geometric sequence. Determine the ratio of the length of the hypotenuse to the length of the shorter leg.

3. [**5**] A parallelogram has 2 sides of length 20 and 15. Given that its area is a positive integer, find the minimum possible area of the parallelogram.

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4. [**6**] Eric is taking a biology class. His problem sets are worth 100 points in total, his three midterms are worth 100 points each, and his final is worth 300 points. If he gets a perfect score on his problem sets and scores $60\%, 70\%$, and 80% on his midterms respectively, what is the minimum possible percentage he can get on his final to ensure a passing grade? (Eric passes if and only if his overall percentage is at least 70%).

5. [**6**] James writes down three integers. Alex picks some two of those integers, takes the average of them, and adds the result to the third integer. If the possible final results Alex could get are 42, 13, and 37, what are the three integers James originally chose?

6. [**6**] Let AB be a segment of length 2 with midpoint M. Consider the circle with center O and radius r that is externally tangent to the circles with diameters AM and BM and internally tangent to the circle with diameter AB. Determine the value of r.

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7. [**7**] Let n be the smallest positive integer with exactly 2015 positive factors. What is the sum of the (not necessarily distinct) prime factors of n? For example, the sum of the prime factors of 72 is $2 + 2 + 2 + 3 + 3 = 14$.

8. [**7**] For how many pairs of nonzero integers (c, d) with $-2015 \leq c, d \leq 2015$ do the equations $cx = d$ and $dx = c$ both have an integer solution?

9. [**7**] Find the smallest positive integer n such that there exists a complex number z, with positive real and imaginary part, satisfying $z^n = (\overline{z})^n$.

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10. [**8**] Call a string of letters S an *almost palindrome* if S and the reverse of S differ in exactly two places. Find the number of ways to order the letters in $HMMTTHEMETEAM$ to get an almost palindrome.

11. [**8**] Find all integers n, not necessarily positive, for which there exist positive integers a, b, c satisfying $a^n + b^n = c^n$.

12. [**8**] Let a and b be positive real numbers. Determine the minimum possible value of

$$\sqrt{a^2+b^2}+\sqrt{(a-1)^2+b^2}+\sqrt{a^2+(b-1)^2}+\sqrt{(a-1)^2+(b-1)^2}$$

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13. [**9**] Consider a 4×4 grid of squares, each of which are originally colored red. Every minute, Piet can jump on one of the squares, changing the color of it and any adjacent squares (two squares are adjacent if they share a side) to blue. What is the minimum number of minutes it will take Piet to change the entire grid to blue?

14. [**9**] Let ABC be an acute triangle with orthocenter H. Let D, E be the feet of the A, B-altitudes respectively. Given that $AH = 20$ and $HD = 15$ and $BE = 56$, find the length of BH.

15. [**9**] Find the smallest positive integer b such that 1111_b (1111 in base b) is a perfect square. If no such b exists, write "No solution".

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16. [**10**] For how many triples (x, y, z) of integers between -10 and 10 inclusive do there exist reals a, b, c that satisfy

$$ab = x$$
$$ac = y$$
$$bc = z?$$

17. [**10**] Unit squares $ABCD$ and $EFGH$ have centers O_1 and O_2 respectively, and are originally situated such that B and E are at the same position and C and H are at the same position. The squares then rotate clockwise about their centers at the rate of one revolution per hour. After 5 minutes, what is the area of the intersection of the two squares?

18. [**10**] A function f satisfies, for all nonnegative integers x and y:

- $f(0, x) = f(x, 0) = x$
- If $x \geq y \geq 0$, $f(x, y) = f(x - y, y) + 1$
- If $y \geq x \geq 0$, $f(x, y) = f(x, y - x) + 1$

Find the maximum value of f over $0 \leq x, y \leq 100$.

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19. [**11**] Each cell of a 2×5 grid of unit squares is to be colored white or black. Compute the number of such colorings for which no 2×2 square is a single color.

20. [**11**] Let n be a three-digit integer with nonzero digits, not all of which are the same. Define $f(n)$ to be the greatest common divisor of the six integers formed by any permutation of ns digits. For example, $f(123) = 3$, because $\gcd(123, 132, 213, 231, 312, 321) = 3$. Let the maximum possible value of $f(n)$ be k. Find the sum of all n for which $f(n) = k$.

21. [**11**] Consider a 2×2 grid of squares. Each of the squares will be colored with one of 10 colors, and two colorings are considered equivalent if one can be rotated to form the other. How many distinct colorings are there?

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22. [**12**] Find all the roots of the polynomial $x^5 - 5x^4 + 11x^3 - 13x^2 + 9x - 3$.

23. [**12**] Compute the smallest positive integer n for which

$$0 < \sqrt[4]{n} - \lfloor \sqrt[4]{n} \rfloor < \frac{1}{2015}.$$

24. [**12**] Three ants begin on three different vertices of a tetrahedron. Every second, they choose one of the three edges connecting to the vertex they are on with equal probability and travel to the other vertex on that edge. They all stop when any two ants reach the same vertex at the same time. What is the probability that all three ants are at the same vertex when they stop?

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25. [**13**] Let ABC be a triangle that satisfies $AB = 13, BC = 14, AC = 15$. Given a point P in the plane, let P_A, P_B, P_C be the reflections of A, B, C across P. Call P *good* if the circumcircle of $P_AP_BP_C$ intersects the circumcircle of ABC at exactly 1 point. The locus of good points P encloses a region $\mathcal{S}$. Find the area of $\mathcal{S}$.

26. [**13**] Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a *continuous* function satisfying $f(xy) = f(x) + f(y) + 1$ for all positive reals x, y. If $f(2) = 0$, compute $f(2015)$.

27. [**13**] Let $ABCD$ be a quadrilateral with $A = (3, 4), B = (9, -40), C = (-5, -12), D = (-7, 24)$. Let P be a point in the plane (not necessarily inside the quadrilateral). Find the minimum possible value of $AP + BP + CP + DP$.

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28. [**15**] Find the shortest distance between the lines $\frac{x+2}{2} = \frac{y-1}{3} = \frac{z}{1}$ and $\frac{x-3}{-1} = \frac{y}{1} = \frac{z+1}{2}$

29. [**15**] Find the largest real number k such that there exists a sequence of positive reals $\{a_i\}$ for which $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n^k}$ does not.

30. [**15**] Find the largest integer n such that the following holds: there exists a set of n points in the plane such that, for any choice of three of them, some two are unit distance apart.

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31. [**17**] Two random points are chosen on a segment and the segment is divided at each of these two points. Of the three segments obtained, find the probability that the largest segment is more than three times longer than the smallest segment.

32. [**17**] Find the sum of all positive integers $n \leq 2015$ that can be expressed in the form $\lceil \frac{x}{2} \rceil + y + xy$, where x and y are positive integers.

33. [**17**] How many ways are there to place four points in the plane such that the set of pairwise distances between the points consists of exactly 2 elements? (Two configurations are the same if one can be obtained from the other via rotation and scaling.)

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34. [**20**] Let n be the **second** smallest integer that can be written as the sum of two positive cubes in two different ways. Compute n. If your guess is a, you will receive $\max\left(25 - 5 \cdot \max\left(\frac{a}{n}, \frac{n}{a}\right), 0\right)$ points, rounded up.

35. [**20**] Let n be the smallest positive integer such that any positive integer can be expressed as the sum of n integer 2015th powers. Find n. If your answer is a, your score will be $\max(20 - \frac{1}{5}|\log_{10} \frac{a}{n}|, 0)$, rounded up.

36. [**20**] Consider the following seven false conjectures with absurdly high counterexamples. Pick any subset of them, and list their labels in order of their smallest counterexample (the smallest n for which the conjecture is false) from smallest to largest. For example, if you believe that the below list is already ordered by counterexample size, you should write "PECRSGA".

   - P. (**Polya's conjecture**) For any integer n, at least half of the natural numbers below n have an odd number of prime factors.
   - E. (**Euler's conjecture**) There is no perfect cube n that can be written as the sum of three positive cubes.
   - C. (**Cyclotomic**) The polynomial with minimal degree whose roots are the primitive nth roots of unity has all coefficients equal to -1, 0, or 1.

- R. (**Prime race**) For any integer n, there are more primes below n equal to $2 \pmod 3$ than there are equal to $1 \pmod 3$.
- S. (**Seventeen conjecture**) For any integer n, $n^{17}+9$ and $(n+1)^{17}+9$ are relatively prime.
- G. (**Goldbach's (other) conjecture**) Any odd composite integer n can be written as the sum of a prime and twice a square.
- A. (**Average square**) Let $a_1 = 1$ and $a_{k+1} = \frac{1+a_1^2+a_2^2+\ldots+a_k^2}{k}$. Then a_n is an integer for any n.

If your answer is a list of $4 \le n \le 7$ labels in the correct order, your score will be $(n-2)(n-3)$. Otherwise, it will be 0.

HMMT November 2015

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Team

1. [**3**] Triangle ABC is isosceles, and $\angle ABC = x^\circ$. If the sum of the possible measures of $\angle BAC$ is 240°, find x.

2. [**3**] Bassanio has three red coins, four yellow coins, and five blue coins. At any point, he may give Shylock any two coins of different colors in exchange for one coin of the other color; for example, he may give Shylock one red coin and one blue coin, and receive one yellow coin in return. Bassanio wishes to end with coins that are all the same color, and he wishes to do this while having as many coins as possible. How many coins will he end up with, and what color will they be?

3. [**3**] Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x, and let $\{x\}$ denote the fractional part of x. For example, $\lfloor \pi \rfloor = 3$, and $\{\pi\} = 0.14159\ldots$, while $\lfloor 100 \rfloor = 100$ and $\{100\} = 0$. If n is the largest solution to the equation $\frac{\lfloor n \rfloor}{n} = \frac{2015}{2016}$, compute $\{n\}$.

4. [**5**] Call a set of positive integers *good* if there is a partition of it into two sets S and T, such that there do not exist three elements $a, b, c \in S$ such that $a^b = c$ and such that there do not exist three elements $a, b, c \in T$ such that $a^b = c$ (a and b need not be distinct). Find the smallest positive integer n such that the set $\{2, 3, 4, \ldots, n\}$ is *not* good.

5. [**5**] Kelvin the Frog is trying to hop across a river. The river has 10 lilypads on it, and he must hop on them in a specific order (the order is unknown to Kelvin). If Kelvin hops to the wrong lilypad at any point, he will be thrown back to the wrong side of the river and will have to start over. Assuming Kelvin is infinitely intelligent, what is the minimum number of hops he will need to guarantee reaching the other side?

6. [**5**] Marcus and four of his relatives are at a party. Each pair of the five people are either *friends* or *enemies*. For any two enemies, there is no person that they are both friends with. In how many ways is this possible?

7. [**6**] Let $ABCD$ be a convex quadrilateral whose diagonals AC and BD meet at P. Let the area of triangle APB be 24 and let the area of triangle CPD be 25. What is the minimum possible area of quadrilateral $ABCD$?

8. [**6**] Find **any** quadruple of positive integers (a, b, c, d) satisfying $a^3 + b^4 + c^5 = d^{11}$ and $abc < 10^5$.

9. [**7**] A graph consists of 6 vertices. For each pair of vertices, a coin is flipped, and an edge connecting the two vertices is drawn if and only if the coin shows heads. Such a graph is *good* if, starting from any vertex V connected to at least one other vertex, it is possible to draw a path starting and ending at V that traverses each edge exactly once. What is the probability that the graph is good?

10. [**7**] A number n is *bad* if there exists some integer c for which $x^x \equiv c \pmod{n}$ has no integer solutions for x. Find the number of bad integers between 2 and 42 inclusive.

HMMT November 2015

November 14, 2015

Theme round

1. Consider a 1×1 grid of squares. Let A, B, C, D be the vertices of this square, and let E be the midpoint of segment CD. Furthermore, let F be the point on segment BC satisfying $BF = 2CF$, and let P be the intersection of lines AF and BE. Find $\frac{AP}{PF}$.

2. Consider a 2×2 grid of squares. David writes a positive integer in each of the squares. Next to each row, he writes the product of the numbers in the row, and next to each column, he writes the product of the numbers in each column. If the sum of the eight numbers he writes down is 2015, what is the minimum possible sum of the four numbers he writes in the grid?

3. Consider a 3×3 grid of squares. A circle is inscribed in the lower left corner, the middle square of the top row, and the rightmost square of the middle row, and a circle O with radius r is drawn such that O is externally tangent to each of the three inscribed circles. If the side length of each square is 1, compute r.

4. Consider a 4×4 grid of squares. Aziraphale and Crowley play a game on this grid, alternating turns, with Aziraphale going first. On Aziraphales turn, he may color any uncolored square red, and on Crowleys turn, he may color any uncolored square blue. The game ends when all the squares are colored, and Aziraphales score is the area of the largest closed region that is entirely red. If Aziraphale wishes to maximize his score, Crowley wishes to minimize it, and both players play optimally, what will Aziraphales score be?

5. Consider a 5×5 grid of squares. Vladimir colors some of these squares red, such that the centers of any four red squares do **not** form an axis-parallel rectangle (i.e. a rectangle whose sides are parallel to those of the squares). What is the maximum number of squares he could have colored red?

6. Consider a 6×6 grid of squares. Edmond chooses four of these squares uniformly at random. What is the probability that the centers of these four squares form a square?

7. Consider a 7×7 grid of squares. Let $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{1, 2, 3, 4, 5, 6, 7\}$ be a function; in other words, $f(1), f(2), \ldots, f(7)$ are each (not necessarily distinct) integers from 1 to 7. In the top row of the grid, the numbers from 1 to 7 are written in order; in every other square, $f(x)$ is written where x is the number above the square. How many functions have the property that the bottom row is identical to the top row, and no other row is identical to the top row?

8. Consider an 8×8 grid of squares. A rook is placed in the lower left corner, and every minute it moves to a square in the same row or column with equal probability (the rook must move; i.e. it cannot stay in the same square). What is the expected number of minutes until the rook reaches the upper right corner?

9. Consider a 9×9 grid of squares. Haruki fills each square in this grid with an integer between 1 and 9, inclusive. The grid is called a *super-sudoku* if each of the following three conditions hold:

   - Each column in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each row in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.
   - Each 3×3 subsquare in the grid contains each of the numbers $1, 2, 3, 4, 5, 6, 7, 8, 9$ exactly once.

   How many possible super-sudoku grids are there?

10. Consider a 10×10 grid of squares. One day, Daniel drops a burrito in the top left square, where a wingless pigeon happens to be looking for food. Every minute, if the pigeon and the burrito are in the same square, the pigeon will eat 10% of the burrito's original size and accidentally throw it into a random square (possibly the one it is already in). Otherwise, the pigeon will move to an adjacent square, decreasing the distance between it and the burrito. What is the expected number of minutes before the pigeon has eaten the entire burrito?

HMMT February 2015

Saturday 21 February 2015

HMIC
Testing window: April 1–5

1. **[20]** Let S be the set of positive integers n such that the inequality

$$\phi(n) \cdot \tau(n) \geq \sqrt{\frac{n^3}{3}}$$

holds, where $\phi(n)$ is the number of positive integers $k \leq n$ that are relatively prime to n, and $\tau(n)$ is the number of positive divisors of n. Prove that S is finite.

2. **[25]** Let m, n be positive integers with $m \geq n$. Let S be the set of pairs (a, b) of relatively prime positive integers such that $a, b \leq m$ and $a + b > m$.

   For each pair $(a, b) \in S$, consider the nonnegative integer solution (u, v) to the equation $au - bv = n$ chosen with $v \geq 0$ minimal, and let $I(a, b)$ denote the (open) interval $(v/a, u/b)$.

   Prove that $I(a, b) \subseteq (0, 1)$ for every $(a, b) \in S$, and that any fixed irrational number $\alpha \in (0, 1)$ lies in $I(a, b)$ for exactly n distinct pairs $(a, b) \in S$.

3. **[30]** Let M be a 2014×2014 invertible matrix, and let $\mathcal{F}(M)$ denote the set of matrices whose rows are a permutation of the rows of M. Find the number of matrices $F \in \mathcal{F}(M)$ such that $\det(M + F) \neq 0$.

4. **[35]** Prove that there exists a positive integer N such that for any positive integer $n \geq N$, there are at least 2015 non-empty subsets S of $\{n^2 + 1, n^2 + 2, \ldots, n^2 + 3n\}$ with the property that the product of the elements of S is a perfect square.

5. **[40]** Let $\omega = e^{2\pi i/5}$ be a primitive fifth root of unity. Prove that there do not exist integers a, b, c, d, k with $k > 1$ such that

$$(a + b\omega + c\omega^2 + d\omega^3)^k = 1 + \omega.$$

HMMT 2014

THE HARVARD-MIT MATHEMATICS TOURNAMENT

ALGEBRA

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

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Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

HMMT 2014

Saturday 22 February 2014

Algebra

1. Given that x and y are nonzero real numbers such that $x + \frac{1}{y} = 10$ and $y + \frac{1}{x} = \frac{5}{12}$, find all possible values of x.

2. Find the integer closest to
$$\frac{1}{\sqrt[4]{5^4+1} - \sqrt[4]{5^4-1}}.$$

3. Let
$$A = \frac{1}{6}\left((\log_2(3))^3 - (\log_2(6))^3 - (\log_2(12))^3 + (\log_2(24))^3\right).$$
Compute 2^A.

4. Let b and c be real numbers, and define the polynomial $P(x) = x^2 + bx + c$. Suppose that $P(P(1)) = P(P(2)) = 0$, and that $P(1) \neq P(2)$. Find $P(0)$.

5. Find the sum of all real numbers x such that $5x^4 - 10x^3 + 10x^2 - 5x - 11 = 0$.

6. Given that w and z are complex numbers such that $|w + z| = 1$ and $|w^2 + z^2| = 14$, find the smallest possible value of $|w^3 + z^3|$. Here, $|\cdot|$ denotes the absolute value of a complex number, given by $|a + bi| = \sqrt{a^2 + b^2}$ whenever a and b are real numbers.

7. Find the largest real number c such that
$$\sum_{i=1}^{101} x_i^2 \geq cM^2$$
whenever $x_1, \ldots, x_{101}$ are real numbers such that $x_1 + \cdots + x_{101} = 0$ and M is the median of $x_1, \ldots, x_{101}$.

8. Find all real numbers k such that $r^4 + kr^3 + r^2 + 4kr + 16 = 0$ is true for exactly one real number r.

9. Given that a, b, and c are complex numbers satisfying
$$a^2 + ab + b^2 = 1 + i$$
$$b^2 + bc + c^2 = -2$$
$$c^2 + ca + a^2 = 1,$$
compute $(ab + bc + ca)^2$. (Here, $i = \sqrt{-1}$.)

10. For an integer n, let $f_9(n)$ denote the number of positive integers $d \leq 9$ dividing n. Suppose that m is a positive integer and $b_1, b_2, \ldots, b_m$ are real numbers such that $f_9(n) = \sum_{j=1}^{m} b_j f_9(n-j)$ for all $n > m$. Find the smallest possible value of m.

HMMT 2014
Saturday 22 February 2014

Algebra

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Name ______________________ Team ID# ________

Organization ______________________ Team ______________________

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Score: ____________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

COMBINATORICS

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Enjoy!

HMMT 2014

Saturday 22 February 2014

Combinatorics

1. There are 100 students who want to sign up for the class Introduction to Acting. There are three class sections for Introduction to Acting, each of which will fit exactly 20 students. The 100 students, including Alex and Zhu, are put in a lottery, and 60 of them are randomly selected to fill up the classes. What is the probability that Alex and Zhu end up getting into the same section for the class?

2. There are 10 people who want to choose a committee of 5 people among them. They do this by first electing a set of 1, 2, 3, or 4 committee leaders, who then choose among the remaining people to complete the 5-person committee. In how many ways can the committee be formed, assuming that people are distinguishable? (Two committees that have the same members but different sets of leaders are considered to be distinct.)

3. Bob writes a random string of 5 letters, where each letter is either A, B, C, or D. The letter in each position is independently chosen, and each of the letters A, B, C, D is chosen with equal probability. Given that there are at least two A's in the string, find the probability that there are at least three A's in the string.

4. Find the number of triples of sets (A, B, C) such that:

   (a) $A, B, C \subseteq \{1, 2, 3, \ldots, 8\}$.

   (b) $|A \cap B| = |B \cap C| = |C \cap A| = 2$.

   (c) $|A| = |B| = |C| = 4$.

   Here, $|S|$ denotes the number of elements in the set S.

5. Eli, Joy, Paul, and Sam want to form a company; the company will have 16 shares to split among the 4 people. The following constraints are imposed:

   - Every person must get a positive integer number of shares, and all 16 shares must be given out.
   - No one person can have more shares than the other three people combined.

   Assuming that shares are indistinguishable, but people are distinguishable, in how many ways can the shares be given out?

6. We have a calculator with two buttons that displays an integer x. Pressing the first button replaces x by $\lfloor \frac{x}{2} \rfloor$, and pressing the second button replaces x by $4x + 1$. Initially, the calculator displays 0. How many integers less than or equal to 2014 can be achieved through a sequence of arbitrary button presses? (It is permitted for the number displayed to exceed 2014 during the sequence. Here, $\lfloor y \rfloor$ denotes the greatest integer less than or equal to the real number y.)

7. Six distinguishable players are participating in a tennis tournament. Each player plays one match of tennis against every other player. There are no ties in this tournament—each tennis match results in a win for one player and a loss for the other. Suppose that whenever A and B are players in the tournament such that A wins strictly more matches than B over the course of the tournament, it is also true that A wins the match against B in the tournament. In how many ways could the tournament have gone?

8. The integers $1, 2, \ldots, 64$ are written in the squares of a 8×8 chess board, such that for each $1 \leq i < 64$, the numbers i and $i + 1$ are in squares that share an edge. What is the largest possible sum that can appear along one of the diagonals?

9. There is a heads up coin on every integer of the number line. Lucky is initially standing on the zero point of the number line facing in the positive direction. Lucky performs the following procedure: he looks at the coin (or lack thereof) underneath him, and then,

- If the coin is heads up, Lucky flips it to tails up, turns around, and steps forward a distance of one unit.
- If the coin is tails up, Lucky picks up the coin and steps forward a distance of one unit facing the same direction.
- If there is no coin, Lucky places a coin heads up underneath him and steps forward a distance of one unit facing the same direction.

He repeats this procedure until there are 20 coins anywhere that are tails up. How many times has Lucky performed the procedure when the process stops?

10. An *up-right path* from $(a, b) \in \mathbb{R}^2$ to $(c, d) \in \mathbb{R}^2$ is a finite sequence $(x_1, y_1), \ldots, (x_k, y_k)$ of points in $\mathbb{R}^2$ such that $(a, b) = (x_1, y_1)$, $(c, d) = (x_k, y_k)$, and for each $1 \leq i < k$ we have that either $(x_{i+1}, y_{i+1}) = (x_i + 1, y_i)$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$. Two up-right paths are said to intersect if they share any point.

    Find the number of pairs (A, B) where A is an up-right path from $(0, 0)$ to $(4, 4)$, B is an up-right path from $(2, 0)$ to $(6, 4)$, and A and B do not intersect.

HMMT 2014
Saturday 22 February 2014
Combinatorics

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Name ______________________________ Team ID# __________

Organization ______________________ Team ______________________

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Score: ____________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

GEOMETRY

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Enjoy!

HMMT 2014

Saturday 22 February 2014

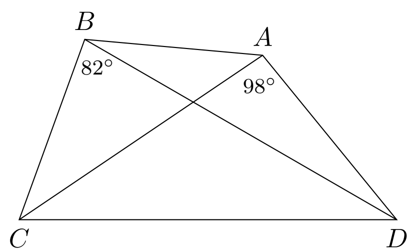
Geometry

1. Let O_1 and O_2 be concentric circles with radii 4 and 6, respectively. A chord AB is drawn in O_1 with length 2. Extend AB to intersect O_2 in points C and D. Find CD.

2. Point P and line ℓ are such that the distance from P to ℓ is 12. Given that T is a point on ℓ such that $PT = 13$, find the radius of the circle passing through P and tangent to ℓ at T.

3. ABC is a triangle such that $BC = 10$, $CA = 12$. Let M be the midpoint of side AC. Given that BM is parallel to the external bisector of $\angle A$, find area of triangle ABC. (Lines AB and AC form two angles, one of which is $\angle BAC$. The *external bisector* of $\angle A$ is the line that bisects the other angle.)

4. In quadrilateral $ABCD$, $\angle DAC = 98°, \angle DBC = 82°, \angle BCD = 70°$, and $BC = AD$. Find $\angle ACD$.



5. Let $\mathcal{C}$ be a circle in the xy plane with radius 1 and center $(0, 0, 0)$, and let P be a point in space with coordinates $(3, 4, 8)$. Find the largest possible radius of a sphere that is contained entirely in the slanted cone with base $\mathcal{C}$ and vertex P.

6. In quadrilateral $ABCD$, we have $AB = 5, BC = 6, CD = 5, DA = 4$, and $\angle ABC = 90°$. Let AC and BD meet at E. Compute $\frac{BE}{ED}$.

7. Triangle ABC has sides $AB = 14, BC = 13$, and $CA = 15$. It is inscribed in circle Γ, which has center O. Let M be the midpoint of AB, let B' be the point on Γ diametrically opposite B, and let X be the intersection of AO and MB'. Find the length of AX.

8. Let ABC be a triangle with sides $AB = 6$, $BC = 10$, and $CA = 8$. Let M and N be the midpoints of BA and BC, respectively. Choose the point Y on ray CM so that the circumcircle of triangle AMY is tangent to AN. Find the area of triangle NAY.

9. Two circles are said to be *orthogonal* if they intersect in two points, and their tangents at either point of intersection are perpendicular. Two circles ω_1 and ω_2 with radii 10 and 13, respectively, are externally tangent at point P. Another circle ω_3 with radius $2\sqrt{2}$ passes through P and is orthogonal to both ω_1 and ω_2. A fourth circle ω_4, orthogonal to ω_3, is externally tangent to ω_1 and ω_2. Compute the radius of ω_4.

10. Let ABC be a triangle with $AB = 13, BC = 14$, and $CA = 15$. Let Γ be the circumcircle of ABC, let O be its circumcenter, and let M be the midpoint of minor arc $\widehat{BC}$. Circle ω_1 is internally tangent to Γ at A, and circle ω_2, centered at M, is externally tangent to ω_1 at a point T. Ray AT meets segment BC at point S, such that $BS - CS = 4/15$. Find the radius of ω_2.

HMMT 2014

Saturday 22 February 2014

Geometry

PUT LABEL HERE

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Organization ____________________ Team ____________________

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Score: ____________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

GUTS

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HMMT 2014, 22 FEBRUARY 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# __________

1. [**4**] Compute the prime factorization of 159999.

2. [**4**] Let $x_1, \ldots, x_{100}$ be defined so that for each i, x_i is a (uniformly) random integer between 1 and 6 inclusive. Find the expected number of integers in the set $\{x_1, x_1 + x_2, \ldots, x_1 + x_2 + \ldots + x_{100}\}$ that are multiples of 6.

3. [**4**] Let $ABCDEF$ be a regular hexagon. Let P be the circle inscribed in $\triangle BDF$. Find the ratio of the area of circle P to the area of rectangle $ABDE$.

4. [**4**] Let D be the set of divisors of 100. Let Z be the set of integers between 1 and 100, inclusive. Mark chooses an element d of D and an element z of Z uniformly at random. What is the probability that d divides z?

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HMMT 2014, 22 FEBRUARY 2014 — GUTS ROUND

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5. [**5**] If four fair six-sided dice are rolled, what is the probability that the lowest number appearing on any die is exactly 3?

6. [**5**] Find all integers n for which $\dfrac{n^3+8}{n^2-4}$ is an integer.

7. [**5**] The Evil League of Evil is plotting to poison the city's water supply. They plan to set out from their headquarters at $(5,1)$ and put poison in two pipes, one along the line $y = x$ and one along the line $x = 7$. However, they need to get the job done quickly before Captain Hammer catches them. What's the shortest distance they can travel to visit both pipes and then return to their headquarters?

8. [**5**] The numbers $2^0, 2^1, \cdots, 2^{15}, 2^{16} = 65536$ are written on a blackboard. You repeatedly take two numbers on the blackboard, subtract one from the other, erase them both, and write the result of the subtraction on the blackboard. What is the largest possible number that can remain on the blackboard when there is only one number left?

HMMT 2014, 22 FEBRUARY 2014 — GUTS ROUND

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9. [**6**] Compute the side length of the largest cube contained in the region

$$\{(x, y, z) : x^2 + y^2 + z^2 \leq 25 \text{ and } x \geq 0\}$$

of three-dimensional space.

10. [**6**] Find the number of nonempty sets $\mathcal{F}$ of subsets of the set $\{1, \ldots, 2014\}$ such that:
    (a) For any subsets $S_1, S_2 \in \mathcal{F}$, $S_1 \cap S_2 \in \mathcal{F}$.
    (b) If $S \in \mathcal{F}$, $T \subseteq \{1, \ldots, 2014\}$, and $S \subseteq T$, then $T \in \mathcal{F}$.

11. [**6**] Two fair octahedral dice, each with the numbers 1 through 8 on their faces, are rolled. Let N be the remainder when the product of the numbers showing on the two dice is divided by 8. Find the expected value of N.

12. [**6**] Find a nonzero monic polynomial $P(x)$ with integer coefficients and minimal degree such that $P(1 - \sqrt[3]{2} + \sqrt[3]{4}) = 0$. (A polynomial is called *monic* if its leading coefficient is 1.)

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13. [**8**] An auditorium has two rows of seats, with 50 seats in each row. 100 indistinguishable people sit in the seats one at a time, subject to the condition that each person, except for the first person to sit in each row, must sit to the left or right of an occupied seat, and no two people can sit in the same seat. In how many ways can this process occur?

14. [**8**] Let $ABCD$ be a trapezoid with $AB \parallel CD$ and $\angle D = 90°$. Suppose that there is a point E on CD such that $AE = BE$ and that triangles AED and CEB are similar, but not congruent. Given that $\frac{CD}{AB} = 2014$, find $\frac{BC}{AD}$.

15. [**8**] Given a regular pentagon of area 1, a *pivot line* is a line not passing through any of the pentagon's vertices such that there are 3 vertices of the pentagon on one side of the line and 2 on the other. A *pivot point* is a point inside the pentagon with only finitely many non-pivot lines passing through it. Find the area of the region of pivot points.

16. [**8**] Suppose that x and y are positive real numbers such that $x^2 - xy + 2y^2 = 8$. Find the maximum possible value of $x^2 + xy + 2y^2$.

Organization ____________________________ Team ____________________________ Team ID# __________

17. [**11**] Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:

    (a) $f(1) = 1$.

    (b) $f(a) \leq f(b)$ whenever a and b are positive integers with $a \leq b$.

    (c) $f(2a) = f(a) + 1$ for all positive integers a.

    How many possible values can the 2014-tuple $(f(1), f(2), ..., f(2014))$ take?

18. [**11**] Find the number of ordered quadruples of positive integers (a, b, c, d) such that a, b, c, and d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

19. [**11**] Let $ABCD$ be a trapezoid with $AB \parallel CD$. The bisectors of $\angle CDA$ and $\angle DAB$ meet at E, the bisectors of $\angle ABC$ and $\angle BCD$ meet at F, the bisectors of $\angle BCD$ and $\angle CDA$ meet at G, and the bisectors of $\angle DAB$ and $\angle ABC$ meet at H. Quadrilaterals $EABF$ and $EDCF$ have areas 24 and 36, respectively, and triangle ABH has area 25. Find the area of triangle CDG.

20. [**11**] A deck of 8056 cards has 2014 ranks numbered 1–2014. Each rank has four suits—hearts, diamonds, clubs, and spades. Each card has a rank and a suit, and no two cards have the same rank and the same suit. How many subsets of the set of cards in this deck have cards from an odd number of distinct ranks?

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HMMT 2014, 22 FEBRUARY 2014 — GUTS ROUND

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21. [**14**] Compute the number of ordered quintuples of nonnegative integers $(a_1, a_2, a_3, a_4, a_5)$ such that $0 \leq a_1, a_2, a_3, a_4, a_5 \leq 7$ and 5 divides $2^{a_1} + 2^{a_2} + 2^{a_3} + 2^{a_4} + 2^{a_5}$.

22. [**14**] Let ω be a circle, and let $ABCD$ be a quadrilateral inscribed in ω. Suppose that BD and AC intersect at a point E. The tangent to ω at B meets line AC at a point F, so that C lies between E and F. Given that $AE = 6, EC = 4, BE = 2$, and $BF = 12$, find DA.

23. [**14**] Let $S = \{-100, -99, -98, \ldots, 99, 100\}$. Choose a 50-element subset T of S at random. Find the expected number of elements of the set $\{|x| : x \in T\}$.

24. [**14**] Let $A = \{a_1, a_2, \ldots, a_7\}$ be a set of distinct positive integers such that the mean of the elements of any nonempty subset of A is an integer. Find the smallest possible value of the sum of the elements in A.

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25. [**17**] Let ABC be an equilateral triangle of side length 6 inscribed in a circle ω. Let A_1, A_2 be the points (distinct from A) where the lines through A passing through the two trisection points of BC meet ω. Define B_1, B_2, C_1, C_2 similarly. Given that $A_1, A_2, B_1, B_2, C_1, C_2$ appear on ω in that order, find the area of hexagon $A_1A_2B_1B_2C_1C_2$.

26. [**17**] For $1 \leq j \leq 2014$, define

$$b_j = j^{2014} \prod_{i=1, i \neq j}^{2014} (i^{2014} - j^{2014})$$

where the product is over all $i \in \{1, \ldots, 2014\}$ except $i = j$. Evaluate

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_{2014}}.$$

27. [**17**] Suppose that $(a_1, \ldots, a_{20})$ and $(b_1, \ldots, b_{20})$ are two sequences of integers such that the sequence $(a_1, \ldots, a_{20}, b_1, \ldots, b_{20})$ contains each of the numbers $1, \ldots, 40$ exactly once. What is the maximum possible value of the sum

$$\sum_{i=1}^{20} \sum_{j=1}^{20} \min(a_i, b_j)?$$

28. [**17**] Let $f(n)$ and $g(n)$ be polynomials of degree 2014 such that $f(n) + (-1)^n g(n) = 2^n$ for $n = 1, 2, \ldots, 4030$. Find the coefficient of x^{2014} in $g(x)$.

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Organization ______________________ Team ______________________ Team ID# __________

29. [**20**] Natalie has a copy of the unit interval $[0, 1]$ that is colored white. She also has a black marker, and she colors the interval in the following manner: at each step, she selects a value $x \in [0, 1]$ uniformly at random, and

    (a) If $x \leq \frac{1}{2}$ she colors the interval $[x, x + \frac{1}{2}]$ with her marker.

    (b) If $x > \frac{1}{2}$ she colors the intervals $[x, 1]$ and $[0, x - \frac{1}{2}]$ with her marker.

    What is the expected value of the number of steps Natalie will need to color the entire interval black?

30. [**20**] Let ABC be a triangle with circumcenter O, incenter I, $\angle B = 45°$, and $OI \parallel BC$. Find $\cos \angle C$.

31. [**20**] Compute
$$\sum_{k=1}^{1007} \left(\cos \left(\frac{\pi k}{1007} \right) \right)^{2014}.$$

32. [**20**] Find all ordered pairs (a, b) of complex numbers with $a^2 + b^2 \neq 0$, $a + \frac{10b}{a^2+b^2} = 5$, and $b + \frac{10a}{a^2+b^2} = 4$.

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33. [**25**] An *up-right path* from $(a,b) \in \mathbb{R}^2$ to $(c,d) \in \mathbb{R}^2$ is a finite sequence $(x_1,y_1),\dots,(x_k,y_k)$ of points in $\mathbb{R}^2$ such that $(a,b)=(x_1,y_1)$, $(c,d)=(x_k,y_k)$, and for each $1 \le i < k$ we have that either $(x_{i+1},y_{i+1})=(x_i+1,y_i)$ or $(x_{i+1},y_{i+1})=(x_i,y_i+1)$.

    Let S be the set of all up-right paths from $(-400,-400)$ to $(400,400)$. What fraction of the paths in S do not contain any point (x,y) such that $|x|,|y| \le 10$? Express your answer as a decimal number between 0 and 1.

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1-10|C-A|),0\} \rceil$.

34. [**25**] Consider a number line, with a lily pad placed at each integer point. A frog is standing at the lily pad at the point 0 on the number line, and wants to reach the lily pad at the point 2014 on the number line. If the frog stands at the point n on the number line, it can jump directly to either point $n+2$ or point $n+3$ on the number line. Each of the lily pads at the points $1,\cdots,2013$ on the number line has, independently and with probability $1/2$, a snake. Let p be the probability that the frog can make some sequence of jumps to reach the lily pad at the point 2014 on the number line, without ever landing on a lily pad containing a snake. What is $p^{1/2014}$? Express your answer as a decimal number.

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1-20|C-A|),0\} \rceil$.

35. [**25**] How many times does the letter "e" occur in all problem statements in this year's HMMT February competition?

    If C is the actual answer to this question and A is your answer, then your score on this problem is $\lceil \max\{25(1-|\log_2(C/A)|),0\} \rceil$.

36. [**25**] We have two concentric circles C_1 and C_2 with radii 1 and 2, respectively. A random chord of C_2 is chosen. What is the probability that it intersects C_1?

    Your answer to this problem must be expressed in the form $\frac{m}{n}$, where m and n are positive integers. If your answer is in this form, your score for this problem will be $\lfloor \frac{25 \cdot X}{Y} \rfloor$, where X is the total number of teams who submit the answer $\frac{m}{n}$ (including your own team), and Y is the total number of teams who submit a valid answer. Otherwise, your score is 0. (Your answer is *not* graded based on correctness, whether your fraction is in lowest terms, whether it is at most 1, etc.)

HMMT 2014, 22 FEBRUARY 2014 — GUTS ROUND

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17. [**11**] ____________________

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21. [**14**] ____________________

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33. [**25**] ____________________

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36. [**25**] ____________________

HMMT 2014

THE HARVARD-MIT MATHEMATICS TOURNAMENT

TEAM

This test consists of ten problems to be solved by a team in one hour. The problems are unequally weighted with point values as shown in brackets. They are *not* necessarily in order of difficulty, though harder problems are generally worth more points. We do not expect most teams to get through all the problems.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted. If there is a chalkboard in the room, you may write on it *during* the round only, *not* before it starts.

Numerical answers should, where applicable, be simplified as much as reasonably possible and must be exact unless otherwise specified. Correct mathematical notation must be used. Your proctor *cannot* assist you in interpreting or solving problems but has our phone number and may help with any administrative difficulties.

All problems require full written proof/justification. Even if the answer is numerical, you must prove your result.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

HMMT 2014

Saturday 22 February 2014

Team

1. [**10**] Let ω be a circle, and let A and B be two points in its interior. Prove that there exists a circle passing through A and B that is contained in the interior of ω.

2. [**15**] Let $a_1, a_2, \ldots$ be an infinite sequence of integers such that a_i divides a_{i+1} for all $i \geq 1$, and let b_i be the remainder when a_i is divided by 210. What is the maximal number of distinct terms in the sequence $b_1, b_2, \ldots$?

3. [**15**] There are n girls $G_1, \ldots, G_n$ and n boys $B_1, \ldots, B_n$. A pair (G_i, B_j) is called *suitable* if and only if girl G_i is willing to marry boy B_j. Given that there is exactly one way to pair each girl with a distinct boy that she is willing to marry, what is the maximal possible number of suitable pairs?

4. [**20**] Compute
$$\sum_{k=0}^{100} \left\lfloor \frac{2^{100}}{2^{50} + 2^k} \right\rfloor.$$
(Here, if x is a real number, then $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.)

5. [**25**] Prove that there exists a nonzero complex number c and a real number d such that
$$\left| \left| \frac{1}{1+z+z^2} \right| - \left| \frac{1}{1+z+z^2} - c \right| \right| = d$$
for all z with $|z| = 1$ and $1 + z + z^2 \neq 0$. (Here, $|z|$ denotes the absolute value of the complex number z, so that $|a + bi| = \sqrt{a^2 + b^2}$ for real numbers a, b.)

6. [**25**] Let n be a positive integer. A sequence $(a_0, \ldots, a_n)$ of integers is *acceptable* if it satisfies the following conditions:

   (a) $0 = |a_0| < |a_1| < \cdots < |a_{n-1}| < |a_n|$.

   (b) The sets $\{|a_1 - a_0|, |a_2 - a_1|, \ldots, |a_{n-1} - a_{n-2}|, |a_n - a_{n-1}|\}$ and $\{1, 3, 9, \ldots, 3^{n-1}\}$ are equal.

   Prove that the number of acceptable sequences of integers is $(n+1)!$.

7. [**30**] Find the maximum possible number of diagonals of equal length in a convex hexagon.

8. [**35**] Let ABC be an acute triangle with circumcenter O such that $AB = 4$, $AC = 5$, and $BC = 6$. Let D be the foot of the altitude from A to BC, and E be the intersection of AO with BC. Suppose that X is on BC between D and E such that there is a point Y on AD satisfying $XY \parallel AO$ and $YO \perp AX$. Determine the length of BX.

9. [**35**] For integers $m, n \geq 1$, let $A(n, m)$ be the number of sequences $(a_1, \cdots, a_{nm})$ of integers satisfying the following two properties:

   (a) Each integer k with $1 \leq k \leq n$ occurs exactly m times in the sequence $(a_1, \cdots, a_{nm})$.

   (b) If i, j, and k are integers such that $1 \leq i \leq nm$ and $1 \leq j \leq k \leq n$, then j occurs in the sequence $(a_1, \cdots, a_i)$ at least as many times as k does.

   For example, if $n = 2$ and $m = 5$, a possible sequence is $(a_1, \cdots, a_{10}) = (1, 1, 2, 1, 2, 2, 1, 2, 1, 2)$. On the other hand, the sequence $(a_1, \cdots, a_{10}) = (1, 2, 1, 2, 2, 1, 1, 1, 2, 2)$ does not satisfy property (2) for $i = 5$, $j = 1$, and $k = 2$.

   Prove that $A(n, m) = A(m, n)$.

10. [**40**] Fix a positive real number $c > 1$ and positive integer n. Initially, a blackboard contains the numbers $1, c, \ldots, c^{n-1}$. Every minute, Bob chooses two numbers a, b on the board and replaces them with $ca + c^2b$. Prove that after $n - 1$ minutes, the blackboard contains a single number no less than

$$\left(\frac{c^{n/L} - 1}{c^{1/L} - 1}\right)^L,$$

where $\phi = \frac{1+\sqrt{5}}{2}$ and $L = 1 + \log_\phi(c)$.

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Saturday 22 February 2014

Team

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1. [**10**]

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6. [**25**]

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7. [**30**]

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9. [**35**]

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10. [**40**]

HMMT November 2014

Saturday 15 November 2014

General Test

1. Two circles ω and γ have radii 3 and 4 respectively, and their centers are 10 units apart. Let x be the shortest possible distance between a point on ω and a point on γ, and let y be the longest possible distance between a point on ω and a point on γ. Find the product xy.

2. Let ABC be a triangle with $\angle B = 90^\circ$. Given that there exists a point D on AC such that $AD = DC$ and $BD = BC$, compute the value of the ratio $\frac{AB}{BC}$.

3. Compute the greatest common divisor of $4^8 - 1$ and $8^{12} - 1$.

4. In rectangle $ABCD$ with area 1, point M is selected on $\overline{AB}$ and points X, Y are selected on $\overline{CD}$ such that $AX < AY$. Suppose that $AM = BM$. Given that the area of triangle MXY is $\frac{1}{2014}$, compute the area of trapezoid $AXYB$.

5. Mark and William are playing a game with a stored value. On his turn, a player may either multiply the stored value by 2 and add 1 or he may multiply the stored value by 4 and add 3. The first player to make the stored value exceed 2^{100} wins. The stored value starts at 1 and Mark goes first. Assuming both players play optimally, what is the maximum number of times that William can make a move?

   (By optimal play, we mean that on any turn the player selects the move which leads to the best possible outcome given that the opponent is also playing optimally. If both moves lead to the same outcome, the player selects one of them arbitrarily.)

6. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The angle bisector of C intersects side AB at X. Points M and N are drawn on sides BC and AC, respectively, such that $\overline{XM} \parallel \overline{AC}$ and $\overline{XN} \parallel \overline{BC}$. Compute the length MN.

7. Consider the set of 5-tuples of positive integers at most 5. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *perfect* if for any distinct indices i, j, k, the three numbers a_i, a_j, a_k do *not* form an arithmetic progression (in any order). Find the number of perfect 5-tuples.

8. Let a, b, c, x be reals with $(a+b)(b+c)(c+a) \neq 0$ that satisfy

$$\frac{a^2}{a+b} = \frac{a^2}{a+c} + 20, \quad \frac{b^2}{b+c} = \frac{b^2}{b+a} + 14, \quad \text{and} \quad \frac{c^2}{c+a} = \frac{c^2}{c+b} + x.$$

   Compute x.

9. For any positive integers a and b, define $a \oplus b$ to be the result when adding a to b in binary (base 2), neglecting any carry-overs. For example, $20 \oplus 14 = 10100_2 \oplus 1110_2 = 11010_2 = 26$. (The operation $\oplus$ is called the *exclusive or.*) Compute the sum

$$\sum_{k=0}^{2^{2014}-1} \left(k \oplus \left\lfloor \frac{k}{2} \right\rfloor \right).$$

   Here $\lfloor x \rfloor$ is the greatest integer not exceeding x.

10. Suppose that m and n are integers with $1 \leq m \leq 49$ and $n \geq 0$ such that m divides $n^{n+1} + 1$. What is the number of possible values of m?

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1. [**5**] Solve for x in the equation $20 \cdot 14 + x = 20 + 14 \cdot x$.

2. [**5**] Find the area of a triangle with side lengths 14, 48, and 50.

3. [**5**] Victoria wants to order at least 550 donuts from Dunkin' Donuts for the HMMT 2014 November contest. However, donuts only come in multiples of twelve. Assuming every twelve donuts cost \$7.49, what is the minimum amount Victoria needs to pay, in dollars? (Because HMMT is affiliated with MIT, the purchase is tax exempt. Moreover, because of the size of the order, there is no delivery fee.)

HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

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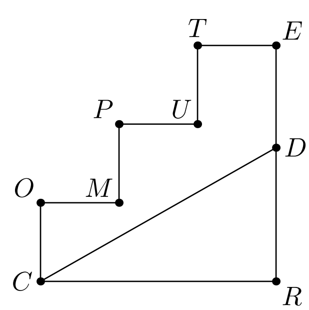
4. [**6**] How many two-digit prime numbers have the property that both digits are also primes?

5. [**6**] Suppose that x, y, z are real numbers such that

$$x = y + z + 2, \quad y = z + x + 1, \quad \text{and} \quad z = x + y + 4.$$

Compute $x + y + z$.

6. [**6**] In the octagon $COMPUTER$ exhibited below, all interior angles are either 90° or 270° and we have $CO = OM = MP = PU = UT = TE = 1$.



Point D (not to scale in the diagram) is selected on segment RE so that polygons $COMPUTED$ and CDR have the same area. Find DR.

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7. [**7**] Let $ABCD$ be a quadrilateral inscribed in a circle with diameter $\overline{AD}$. If $AB = 5$, $AC = 6$, and $BD = 7$, find CD.

8. [**7**] Find the number of digits in the decimal representation of 2^{41}.

9. [**7**] Let f be a function from the nonnegative integers to the positive reals such that $f(x+y) = f(x) \cdot f(y)$ holds for all nonnegative integers x and y. If $f(19) = 524288k$, find $f(4)$ in terms of k.

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10. [**8**] Let ABC be a triangle with $CA = CB = 5$ and $AB = 8$. A circle ω is drawn such that the interior of triangle ABC is completely contained in the interior of ω. Find the smallest possible area of ω.

11. [**8**] How many integers n in the set $\{4, 9, 14, 19, \ldots, 2014\}$ have the property that the sum of the decimal digits of n is even?

12. [**8**] Sindy writes down the positive integers less than 200 in increasing order, but skips the multiples of 10. She then alternately places $+$ and $-$ signs before each of the integers, yielding an expression $+1-2+3-4+5-6+7-8+9-11+12-\cdots-199$. What is the value of the resulting expression?

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13. [**9**] Let ABC be a triangle with $AB = AC = \frac{25}{14}BC$. Let M denote the midpoint of $\overline{BC}$ and let X and Y denote the projections of M onto $\overline{AB}$ and $\overline{AC}$, respectively. If the areas of triangle ABC and quadrilateral $AXMY$ are both positive integers, find the minimum possible sum of these areas.

14. [**9**] How many ways can the eight vertices of a three-dimensional cube be colored red and blue such that no two points connected by an edge are both red? Rotations and reflections of a given coloring are considered distinct.

15. [**9**] Carl is on a vertex of a regular pentagon. Every minute, he randomly selects an adjacent vertex (each with probability $\frac{1}{2}$) and walks along the edge to it. What is the probability that after 10 minutes, he ends up where he had started?

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16. [**10**] A particular coin has a $\frac{1}{3}$ chance of landing on heads (H), $\frac{1}{3}$ chance of landing on tails (T), and $\frac{1}{3}$ chance of landing vertically in the middle (M). When continuously flipping this coin, what is the probability of observing the continuous sequence HMMT before HMT?

17. [**10**] Let ABC be a triangle with $AB = AC = 5$ and $BC = 6$. Denote by ω the circumcircle of ABC. We draw a circle Ω which is externally tangent to ω as well as to the lines AB and AC (such a circle is called an A-*mixtilinear excircle*). Find the radius of Ω.

18. [**10**] For any positive integer x, define Accident(x) to be the set of ordered pairs (s, t) with $s \in \{0, 2, 4, 5, 7, 9, 11\}$ and $t \in \{1, 3, 6, 8, 10\}$ such that $x + s - t$ is divisible by 12. For any nonnegative integer i, let a_i denote the number of $x \in \{0, 1, \ldots, 11\}$ for which $|\text{Accident}(x)| = i$. Find

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2.$$

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19. [**11**] Let a sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_0 = \sqrt{2}, a_1 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 1$. The sequence of remainders when $a_0, a_1, a_2, \cdots$ are divided by 2014 is eventually periodic with some minimal period p (meaning that $a_m = a_{m+p}$ for all sufficiently large integers m, and p is the smallest such positive integer). Find p.

20. [**11**] Determine the number of sequences of sets $S_1, S_2, \ldots, S_{999}$ such that

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{999} \subseteq \{1, 2, \ldots, 999\}.$$

Here $A \subseteq B$ means that all elements of A are also elements of B.

21. [**11**] If you flip a fair coin 1000 times, what is the expected value of the product of the number of heads and the number of tails?

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22. [**12**] Evaluate the infinite sum

$$\sum_{n=2}^{\infty} \log_2 \left(\frac{1 - \frac{1}{n}}{1 - \frac{1}{n+1}} \right).$$

23. [**12**] Seven little children sit in a circle. The teacher distributes pieces of candy to the children in such a way that the following conditions hold.

- Every little child gets at least one piece of candy.
- No two little children have the same number of pieces of candy.
- The numbers of candy pieces given to any two adjacent little children have a common factor other than 1.
- There is no prime dividing every little child's number of candy pieces.

What is the smallest number of pieces of candy that the teacher must have ready for the little children?

24. [**12**] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. We construct isosceles right triangle ACD with $\angle ADC = 90^\circ$, where D, B are on the same side of line AC, and let lines AD and CB meet at F. Similarly, we construct isosceles right triangle BCE with $\angle BEC = 90^\circ$, where E, A are on the same side of line BC, and let lines BE and CA meet at G. Find $\cos \angle AGF$.

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Organization ______________________ Team ______________________ Team ID# __________

25. [**13**] What is the smallest positive integer n which cannot be written in any of the following forms?

- $n = 1 + 2 + \cdots + k$ for a positive integer k.
- $n = p^k$ for a prime number p and integer k.
- $n = p + 1$ for a prime number p.
- $n = pq$ for some distinct prime numbers p and q

26. [**13**] Consider a permutation $(a_1, a_2, a_3, a_4, a_5)$ of $\{1, 2, 3, 4, 5\}$. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *flawless* if for all $1 \le i < j < k \le 5$, the sequence (a_i, a_j, a_k) is *not* an arithmetic progression (in that order). Find the number of flawless 5-tuples.

27. [**13**] In triangle ABC, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2, respectively.

If triangle ABC has sides of length 5, 12, and 13, find the area of the triangle determined by lines A_1C_2, B_1A_2 and C_1B_2.

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28. [**15**] Let x be a complex number such that $x + x^{-1}$ is a root of the polynomial $p(t) = t^3 + t^2 - 2t - 1$. Find all possible values of $x^7 + x^{-7}$.

29. [**15**] Let ω be a fixed circle with radius 1, and let BC be a fixed chord of ω such that $BC = 1$. The locus of the incenter of ABC as A varies along the circumference of ω bounds a region $\mathcal{R}$ in the plane. Find the area of $\mathcal{R}$.

30. [**15**] Suppose we keep rolling a fair 2014-sided die (whose faces are labelled $1, 2, \ldots, 2014$) until we obtain a value less than or equal to the previous roll. Let E be the expected number of times we roll the die. Find the nearest integer to $100E$.

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31. [**17**] Flat Albert and his buddy Mike are watching the game on Sunday afternoon. Albert is drinking lemonade from a two-dimensional cup which is an isosceles triangle whose height and base measure 9cm and 6cm; the opening of the cup corresponds to the base, which points upwards. Every minute after the game begins, the following takes place: if n minutes have elapsed, Albert stirs his drink vigorously and takes a sip of height $\frac{1}{n^2}$ cm. Shortly afterwards, while Albert is busy watching the game, Mike adds cranberry juice to the cup until it's once again full in an attempt to create Mike's cranberry lemonade. Albert takes sips precisely every minute, and his first sip is exactly one minute after the game begins.

    After an infinite amount of time, let A denote the amount of cranberry juice that has been poured (in square centimeters). Find the integer nearest $\frac{27}{\pi^2}A$.

32. [**17**] Let $f(x) = x^2 - 2$, and let f^n denote the function f applied n times. Compute the remainder when $f^{24}(18)$ is divided by 89.

33. [**17**] How many ways can you remove one tile from a 2014×2014 grid such that the resulting figure can be tiled by 1×3 and 3×1 rectangles?

    Warning: The next set of three problems will consist of estimation problems.

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34. [**20**] Let M denote the number of positive integers which divide 2014!, and let N be the integer closest to $\ln(M)$. Estimate the value of N. If your answer is a positive integer A, your score on this problem will be the larger of 0 and $\left\lfloor 20 - \frac{1}{8}|A - N| \right\rfloor$. Otherwise, your score will be zero.

35. [**20**] Ten points are equally spaced on a circle. A *graph* is a set of segments (possibly empty) drawn between pairs of points, so that every two points are joined by either zero or one segments. Two graphs are considered the same if we can obtain one from the other by rearranging the points.

Let N denote the number of graphs with the property that for any two points, there exists a path from one to the other among the segments of the graph. Estimate the value of N. If your answer is a positive integer A, your score on this problem will be the larger of 0 and $\lfloor 20 - 5|\ln(A/N)| \rfloor$. Otherwise, your score will be zero.

36. [**20**] Pick a subset of at least four of the following geometric theorems, order them from earliest to latest by publication date, and write down their **labels** (a single capital letter) in that order. If a theorem was discovered multiple times, use the publication date corresponding to the geometer for which the theorem is named.

C. (**Ceva**) Three cevians AD, BE, CF of a triangle ABC are concurrent if and only if $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$.

E. (**Euler**) In a triangle ABC with incenter I and circumcenter O, we have $IO^2 = R(R-2r)$, where r is the inradius and R is the circumradius of ABC.

H. (**Heron**) The area of a triangle ABC is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

M. (**Menelaus**) If D, E, F lie on lines BC, CA, AB, then they are collinear if and only if $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$, where the ratios are directed.

P. (**Pascal**) Intersections of opposite sides of cyclic hexagons are collinear.

S. (**Stewart**) Let ABC be a triangle and D a point on BC. Set $m = BD$, $n = CD$, $d = AD$. Then $man + dad = bmb + cnc$.

V. (**Varignon**) The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

If your answer is a list of $4 \leq N \leq 7$ labels in a correct order, your score will be $(N-2)(N-3)$. Otherwise, your score will be zero.

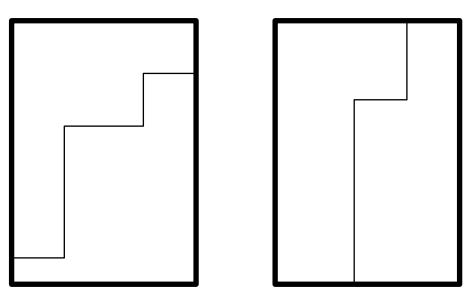
HMMT November 2014
Saturday 15 November 2014
Team Round

Mins and Maxes

1. **[3]** What is the smallest positive integer n which cannot be written in any of the following forms?
   - $n = 1 + 2 + \cdots + k$ for a positive integer k.
   - $n = p^k$ for a prime number p and integer k.
   - $n = p + 1$ for a prime number p.
2. **[5]** Let $f(x) = x^2 + 6x + 7$. Determine the smallest possible value of $f(f(f(f(x))))$ over all real numbers x.
3. **[5]** The side lengths of a triangle are distinct positive integers. One of the side lengths is a multiple of 42, and another is a multiple of 72. What is the minimum possible length of the third side?

Enumeration

4. **[3]** How many ways are there to color the vertices of a triangle red, green, blue, or yellow such that no two vertices have the same color? Rotations and reflections are considered distinct.
5. **[5]** Let A, B, C, D, E be five points on a circle; some segments are drawn between the points so that each of the $\binom{5}{2} = 10$ pairs of points is connected by either zero or one segments. Determine the number of sets of segments that can be drawn such that:
   - It is possible to travel from any of the five points to any other of the five points along drawn segments.
   - It is possible to divide the five points into two nonempty sets S and T such that each segment has one endpoint in S and the other endpoint in T.
6. **[6]** Find the number of strictly increasing sequences of nonnegative integers with the following properties:
   - The first term is 0 and the last term is 12. In particular, the sequence has at least two terms.
   - Among any two consecutive terms, exactly one of them is even.
7. **[7]** Sammy has a wooden board, shaped as a rectangle with length 2^{2014} and height 3^{2014}. The board is divided into a grid of unit squares. A termite starts at either the left or bottom edge of the rectangle, and walks along the gridlines by moving either to the right or upwards, until it reaches an edge opposite the one from which the termite started. Depicted below are two possible paths of the termite.

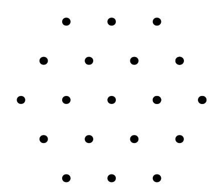


The termite's path dissects the board into two parts. Sammy is surprised to find that he can still arrange the pieces to form a new rectangle not congruent to the original rectangle. This rectangle has perimeter P. How many possible values of P are there?

Hexagons

8. [**3**] Let $\mathcal{H}$ be a regular hexagon with side length one. Peter picks a point P uniformly and at random within $\mathcal{H}$, then draws the largest circle with center P that is contained in $\mathcal{H}$. What is this probability that the radius of this circle is less than $\frac{1}{2}$?

9. [**5**] How many lines pass through exactly two points in the following hexagonal grid?



10. [**8**] Let $ABCDEF$ be a convex hexagon with the following properties.

    (a) $\overline{AC}$ and $\overline{AE}$ trisect $\angle BAF$.

    (b) $\overline{BE} \parallel \overline{CD}$ and $\overline{CF} \parallel \overline{DE}$.

    (c) $AB = 2AC = 4AE = 8AF$.

    Suppose that quadrilaterals $ACDE$ and $ADEF$ have area 2014 and 1400, respectively. Find the area of quadrilateral $ABCD$.

HMMT November 2014

Saturday 15 November 2014

Theme Round

Townspeople and Goons

In the city of Lincoln, there is an empty jail, at least two *townspeople* and at least one *goon*. A game proceeds over several days, starting with morning.

- Each morning, one randomly selected unjailed person is placed in jail. If at this point all goons are jailed, and at least one townsperson remains, then the townspeople win. If at this point all townspeople are jailed and at least one goon remains, then the goons win.
- Each evening, if there is at least one goon and at least one townsperson not in jail, then one randomly selected townsperson is jailed. If at this point there are at least as many goons remaining as townspeople remaining, then the goons win.

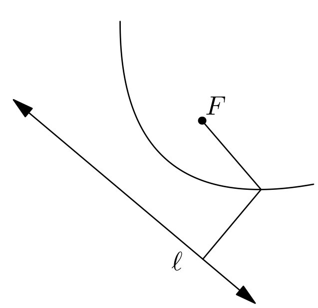
The game ends immediately after any group wins.

1. Find the probability that the townspeople win if there are initially two townspeople and one goon.
2. Find the smallest positive integer n such that, if there are initially $2n$ townspeople and 1 goon, then the probability the townspeople win is greater than 50%.
3. Find the smallest positive integer n such that, if there are initially $n+1$ townspeople and n goons, then the probability the townspeople win is less than 1%.
4. Suppose there are initially 1001 townspeople and two goons. What is the probability that, when the game ends, there are exactly 1000 people in jail?
5. Suppose that there are initially eight townspeople and one goon. One of the eight townspeople is named *Jester*. If Jester is sent to jail during some morning, then the game ends immediately in his sole victory. (However, the Jester does not win if he is sent to jail during some night.)

   Find the probability that only the Jester wins.

Parabolas

A *parabola* is the set of points in the plane equidistant from a point F (its *focus*) and a line ℓ (its directrix).



Parabolas can be represented as polynomial equations of degree two, such as $y = x^2$.

6. Let $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ be pairwise distinct parabolas in the plane. Find the maximum possible number of intersections between two or more of the $\mathcal{P}_i$. In other words, find the maximum number of points that can lie on two or more of the parabolas $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$.

7. Let $\mathcal{P}$ be a parabola with focus F and directrix ℓ. A line through F intersects $\mathcal{P}$ at two points A and B. Let D and C be the feet of the altitudes from A and B onto ℓ, respectively. Given that $AB = 20$ and $CD = 14$, compute the area of $ABCD$.

8. Consider the parabola consisting of the points (x, y) in the real plane satisfying

$$(y + x) = (y - x)^2 + 3(y - x) + 3.$$

Find the minimum possible value of y.

9. In equilateral triangle ABC with side length 2, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2, respectively.

   Find the perimeter of the triangle formed by lines A_1A_2, B_1B_2, C_1C_2.

10. Let z be a complex number and k a positive integer such that z^k is a positive real number other than 1. Let $f(n)$ denote the real part of the complex number z^n. Assume the parabola $p(n) = an^2 + bn + c$ intersects $f(n)$ four times, at $n = 0, 1, 2, 3$. Assuming the smallest possible value of k, find the largest possible value of a.

HMMT 2014

Saturday 22 February 2014

HMIC

1. [**20**] Consider a regular n-gon with $n > 3$, and call a line *acceptable* if it passes through the interior of this n-gon. Draw m different acceptable lines, so that the n-gon is divided into several smaller polygons.

   (a) Prove that there exists an m, depending only on n, such that any collection of m acceptable lines results in one of the smaller polygons having 3 or 4 sides.

   (b) Find the smallest possible m which guarantees that at least one of the smaller polygons will have 3 or 4 sides.

2. [**25**] 2014 triangles have non-overlapping interiors contained in a circle of radius 1. What is the largest possible value of the sum of their areas?

3. [**30**] Fix positive integers m and n. Suppose that $a_1, a_2, \ldots, a_m$ are reals, and that pairwise distinct vectors $v_1, \ldots, v_m \in \mathbb{R}^n$ satisfy

$$\sum_{j \neq i} a_j \frac{v_j - v_i}{\|v_j - v_i\|^3} = 0$$

   for $i = 1, 2, \ldots, m$.

   Prove that

$$\sum_{1 \leq i < j \leq m} \frac{a_i a_j}{\|v_j - v_i\|} = 0.$$

4. [**35**] Let ω be a root of unity and f be a polynomial with integer coefficients. Show that if $|f(\omega)| = 1$, then $f(\omega)$ is also a root of unity.

5. [**40**] Let n be a positive integer, and let A and B be $n \times n$ matrices with complex entries such that $A^2 = B^2$. Show that there exists an $n \times n$ invertible matrix S with complex entries that satisfies $S(AB - BA) = (BA - AB)S$.

THE HARVARD-MIT MATHEMATICS TOURNAMENT

ALGEBRA TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013

Saturday 16 February 2013

Algebra Test

1. Let x and y be real numbers with $x > y$ such that $x^2y^2 + x^2 + y^2 + 2xy = 40$ and $xy + x + y = 8$. Find the value of x.

2. Let $\{a_n\}_{n\geq 1}$ be an arithmetic sequence and $\{g_n\}_{n\geq 1}$ be a geometric sequence such that the first four terms of $\{a_n + g_n\}$ are 0, 0, 1, and 0, in that order. What is the 10th term of $\{a_n + g_n\}$?

3. Let S be the set of integers of the form $2^x + 2^y + 2^z$, where x, y, z are pairwise distinct non-negative integers. Determine the 100th smallest element of S.

4. Determine all real values of A for which there exist distinct complex numbers x_1, x_2 such that the following three equations hold:

$$x_1(x_1 + 1) = A$$
$$x_2(x_2 + 1) = A$$
$$x_1^4 + 3x_1^3 + 5x_1 = x_2^4 + 3x_2^3 + 5x_2.$$

5. Let a and b be real numbers, and let $r, s,$ and t be the roots of $f(x) = x^3 + ax^2 + bx - 1$. Also, $g(x) = x^3 + mx^2 + nx + p$ has roots r^2, s^2, and t^2. If $g(-1) = -5$, find the maximum possible value of b.

6. Find the number of integers n such that

$$1 + \left\lfloor \frac{100n}{101} \right\rfloor = \left\lceil \frac{99n}{100} \right\rceil.$$

7. Compute

$$\sum_{a_1=0}^{\infty} \sum_{a_2=0}^{\infty} \cdots \sum_{a_7=0}^{\infty} \frac{a_1 + a_2 + \cdots + a_7}{3^{a_1+a_2+\cdots+a_7}}.$$

8. Let x, y be complex numbers such that $\frac{x^2+y^2}{x+y} = 4$ and $\frac{x^4+y^4}{x^3+y^3} = 2$. Find all possible values of $\frac{x^6+y^6}{x^5+y^5}$.

9. Let z be a non-real complex number with $z^{23} = 1$. Compute

$$\sum_{k=0}^{22} \frac{1}{1 + z^k + z^{2k}}.$$

10. Let N be a positive integer whose decimal representation contains 11235 as a contiguous substring, and let k be a positive integer such that $10^k > N$. Find the minimum possible value of

$$\frac{10^k - 1}{\gcd(N, 10^k - 1)}.$$

HMMT 2013

Saturday 16 February 2013

Algebra Test

PUT LABEL HERE

Name ______________________ Team ID# ________

Organization ______________________ Team ______________________

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9. ______________
10. ______________

Score: __________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

COMBINATORICS TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

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Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

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Enjoy!

HMMT 2013

Saturday 16 February 2013

Combinatorics Test

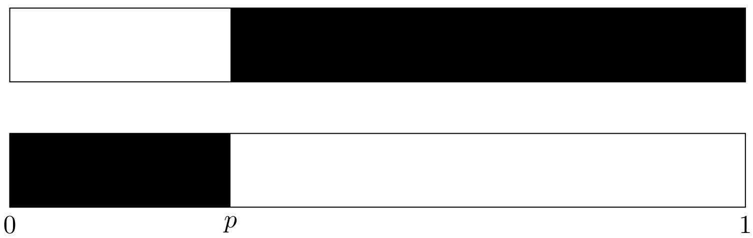
1. A standard 52-card deck contains cards of 4 suits and 13 numbers, with exactly one card for each pairing of suit and number. If Maya draws two cards with replacement from this deck, what is the probability that the two cards have the same suit or have the same number, but not both?

2. If Alex does not sing on Saturday, then she has a 70% chance of singing on Sunday; however, to rest her voice, she never sings on both days. If Alex has a 50% chance of singing on Sunday, find the probability that she sings on Saturday.

3. On a game show, Merble will be presented with a series of 2013 marbles, each of which is either red or blue on the outside. Each time he sees a marble, he can either keep it or pass, but cannot return to a previous marble; he receives 3 points for keeping a red marble, loses 2 points for keeping a blue marble, and gains 0 points for passing. All distributions of colors are equally likely and Merble can only see the color of his current marble. If his goal is to end with exactly one point and he plays optimally, what is the probability that he fails?

4. How many orderings $(a_1, \ldots, a_8)$ of $(1, 2, \ldots, 8)$ exist such that $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 = 0$?

5. At a certain chocolate company, each bar is 1 unit long. To make the bars more interesting, the company has decided to combine dark and white chocolate pieces. The process starts with two bars, one completely dark and one completely white. At each step of the process, a new number p is chosen uniformly at random between 0 and 1. Each of the two bars is cut p units from the left, and the pieces on the left are switched: each is grafted onto the opposite bar where the other piece of length p was previously attached. For example, the bars might look like this after the first step:



Each step after the first operates on the bars resulting from the previous step. After a total of 100 steps, what is the probability that on each bar, the chocolate 1/3 units from the left is the same type of chocolate as that 2/3 units from the left?

6. Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

7. A single-elimination ping-pong tournament has 2^{2013} players, seeded in order of ability. If the player with seed x plays the player with seed y, then it is possible for x to win if and only if $x \leq y + 3$. For how many players P it is possible for P to win? (In each round of a single elimination tournament, the remaining players are randomly paired up; each player plays against the other player in his pair, with the winner from each pair progressing to the next round and the loser eliminated. This is repeated until there is only one player remaining.)

8. It is known that exactly one of the three (distinguishable) musketeers stole the truffles. Each musketeer makes one statement, in which he either claims that one of the three is guilty, or claims that one of the three is innocent. It is possible for two or more of the musketeers to make the same statement. After hearing their claims, and knowing that exactly one musketeer lied, the inspector is able to deduce who stole the truffles. How many ordered triplets of statements could have been made?

9. Given a permutation σ of $\{1, 2, \ldots, 2013\}$, let $f(\sigma)$ to be the number of fixed points of σ – that is, the number of $k \in \{1, 2, \ldots, 2013\}$ such that $\sigma(k) = k$. If S is the set of all possible permutations σ, compute

$$\sum_{\sigma \in S} f(\sigma)^4.$$

(Here, a *permutation* σ is a bijective mapping from $\{1, 2, \ldots, 2013\}$ to $\{1, 2, \ldots, 2013\}$.)

10. Rosencrantz and Guildenstern each start with \$2013 and are flipping a fair coin. When the coin comes up heads Rosencrantz pays Guildenstern \$1 and when the coin comes up tails Guildenstern pays Rosencrantz \$1. Let $f(n)$ be the number of dollars Rosencrantz is ahead of his starting amount after n flips. Compute the expected value of $\max\{f(0), f(1), f(2), \ldots, f(2013)\}$.

HMMT 2013
Saturday 16 February 2013
Combinatorics Test

PUT LABEL HERE

Name ____ Team ID# ________

Organization __________________________ Team ____________________

1. ________________
2. ________________
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Score: ____________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

GEOMETRY TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013

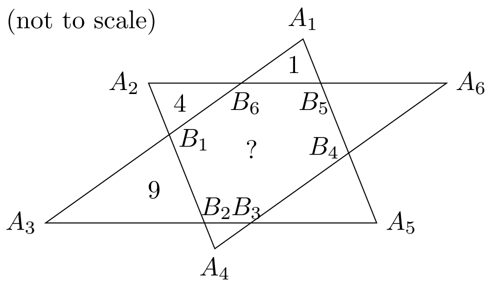
Saturday 16 February 2013

Geometry Test

1. Jarris the triangle is playing in the (x, y) plane. Let his maximum y coordinate be k. Given that he has side lengths 6, 8, and 10 and that no part of him is below the x-axis, find the minimum possible value of k.

2. Let $ABCD$ be an isosceles trapezoid such that $AD = BC$, $AB = 3$, and $CD = 8$. Let E be a point in the plane such that $BC = EC$ and $AE \perp EC$. Compute AE.

3. Let $A_1A_2A_3A_4A_5A_6$ be a convex hexagon such that $A_iA_{i+2} \parallel A_{i+3}A_{i+5}$ for $i = 1, 2, 3$ (we take $A_{i+6} = A_i$ for each i). Segment A_iA_{i+2} intersects segment $A_{i+1}A_{i+3}$ at B_i, for $1 \leq i \leq 6$, as shown. Furthermore, suppose that $\triangle A_1A_3A_5 \cong \triangle A_4A_6A_2$. Given that $[A_1B_5B_6] = 1$, $[A_2B_6B_1] = 4$, and $[A_3B_1B_2] = 9$ (by $[XYZ]$ we mean the area of $\triangle XYZ$), determine the area of hexagon $B_1B_2B_3B_4B_5B_6$.



4. Let ω_1 and ω_2 be circles with centers O_1 and O_2, respectively, and radii r_1 and r_2, respectively. Suppose that O_2 is on ω_1. Let A be one of the intersections of ω_1 and ω_2, and B be one of the two intersections of line O_1O_2 with ω_2. If $AB = O_1A$, find all possible values of $\frac{r_1}{r_2}$.

5. In triangle ABC, $\angle A = 45°$ and M is the midpoint of $\overline{BC}$. $\overline{AM}$ intersects the circumcircle of ABC for the second time at D, and $AM = 2MD$. Find $\cos \angle AOD$, where O is the circumcenter of ABC.

6. Let $ABCD$ be a quadrilateral such that $\angle ABC = \angle CDA = 90°$, and $BC = 7$. Let E and F be on BD such that AE and CF are perpendicular to BD. Suppose that $BE = 3$. Determine the product of the smallest and largest possible lengths of DF.

7. Let ABC be an obtuse triangle with circumcenter O such that $\angle ABC = 15°$ and $\angle BAC > 90°$. Suppose that AO meets BC at D, and that $OD^2 + OC \cdot DC = OC^2$. Find $\angle C$.

8. Let $ABCD$ be a convex quadrilateral. Extend line CD past D to meet line AB at P and extend line CB past B to meet line AD at Q. Suppose that line AC bisects $\angle BAD$. If $AD = \frac{7}{4}$, $AP = \frac{21}{2}$, and $AB = \frac{14}{11}$, compute AQ.

9. Pentagon $ABCDE$ is given with the following conditions:

   (a) $\angle CBD + \angle DAE = \angle BAD = 45°$, $\angle BCD + \angle DEA = 300°$

   (b) $\frac{BA}{DA} = \frac{2\sqrt{2}}{3}$, $CD = \frac{7\sqrt{5}}{3}$, and $DE = \frac{15\sqrt{2}}{4}$

   (c) $AD^2 \cdot BC = AB \cdot AE \cdot BD$

   Compute BD.

10. Triangle ABC is inscribed in a circle ω. Let the bisector of angle A meet ω at D and BC at E. Let the reflections of A across D and C be D' and C', respectively. Suppose that $\angle A = 60°$, $AB = 3$, and $AE = 4$. If the tangent to ω at A meets line BC at P, and the circumcircle of APD' meets line BC at F (other than P), compute FC'.

HMMT 2013
Saturday 16 February 2013

Geometry Test

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Name ______________________ Team ID# ________

Organization ______________________ Team ______________________

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10. ____________

Score: ____________

HMMT 2013, 16 FEBRUARY 2013 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# __________

1. [**4**] Arpon chooses a positive real number k. For each positive integer n, he places a marker at the point (n, nk) in the (x, y) plane. Suppose that two markers whose x coordinates differ by 4 have distance 31. What is the distance between the markers at $(7, 7k)$ and $(19, 19k)$?

2. [**4**] The real numbers x, y, z satisfy $0 \leq x \leq y \leq z \leq 4$. If their squares form an arithmetic progression with common difference 2, determine the minimum possible value of $|x - y| + |y - z|$.

3. [**4**] Find the rightmost non-zero digit of the expansion of $(20)(13!)$.

4. [**4**] Spencer is making burritos, each of which consists of one wrap and one filling. He has enough filling for up to four beef burritos and three chicken burritos. However, he only has five wraps for the burritos; in how many orders can he make exactly five burritos?

HMMT 2013, 16 FEBRUARY 2013 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# __________

5. [**5**] Rahul has ten cards face-down, which consist of five distinct pairs of matching cards. During each move of his game, Rahul chooses one card to turn face-up, looks at it, and then chooses another to turn face-up and looks at it. If the two face-up cards match, the game ends. If not, Rahul flips both cards face-down and keeps repeating this process. Initially, Rahul doesn't know which cards are which. Assuming that he has perfect memory, find the smallest number of moves after which he can guarantee that the game has ended.

6. [**5**] Let R be the region in the Cartesian plane of points (x, y) satisfying $x \geq 0$, $y \geq 0$, and $x + y + \lfloor x \rfloor + \lfloor y \rfloor \leq 5$. Determine the area of R.

7. [**5**] Find the number of positive divisors d of $15! = 15 \cdot 14 \cdots 2 \cdot 1$ such that $\gcd(d, 60) = 5$.

8. [**5**] In a game, there are three indistinguishable boxes; one box contains two red balls, one contains two blue balls, and the last contains one ball of each color. To play, Raj first predicts whether he will draw two balls of the same color or two of different colors. Then, he picks a box, draws a ball at random, looks at the color, and replaces the ball in the same box. Finally, he repeats this; however, the boxes are not shuffled between draws, so he can determine whether he wants to draw again from the same box. Raj wins if he predicts correctly; if he plays optimally, what is the probability that he will win?

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9. [**6**] I have 8 unit cubes of different colors, which I want to glue together into a $2 \times 2 \times 2$ cube. How many distinct $2 \times 2 \times 2$ cubes can I make? Rotations of the same cube are not considered distinct, but reflections are.

10. [**6**] Wesyu is a farmer, and she's building a cao (a relative of the cow) pasture. She starts with a triangle $A_0A_1A_2$ where angle A_0 is $90°$, angle A_1 is $60°$, and A_0A_1 is 1. She then extends the pasture. First, she extends A_2A_0 to A_3 such that $A_3A_0 = \frac{1}{2}A_2A_0$ and the new pasture is triangle $A_1A_2A_3$. Next, she extends A_3A_1 to A_4 such that $A_4A_1 = \frac{1}{6}A_3A_1$. She continues, each time extending A_nA_{n-2} to A_{n+1} such that $A_{n+1}A_{n-2} = \frac{1}{2^n-2}A_nA_{n-2}$. What is the smallest K such that her pasture never exceeds an area of K?

11. [**6**] Compute the prime factorization of 1007021035035021007001. (You should write your answer in the form $p_1^{e_1}p_2^{e_2}\dots p_k^{e_k}$, where $p_1, \dots, p_k$ are distinct prime numbers and $e_1, \dots, e_k$ are positive integers.)

12. [**6**] For how many integers $1 \le k \le 2013$ does the decimal representation of k^k end with a 1?

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HMMT 2013, 16 FEBRUARY 2013 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# __________

13. [**8**] Find the smallest positive integer n such that $\frac{5^{n+1}+2^{n+1}}{5^n+2^n} > 4.99$.

14. [**8**] Consider triangle ABC with $\angle A = 2\angle B$. The angle bisectors from A and C intersect at D, and the angle bisector from C intersects $\overline{AB}$ at E. If $\frac{DE}{DC} = \frac{1}{3}$, compute $\frac{AB}{AC}$.

15. [**8**] Tim and Allen are playing a match of *tenus*. In a match of *tenus*, the two players play a series of games, each of which is won by one of the two players. The match ends when one player has won exactly two more games than the other player, at which point the player who has won more games wins the match. In odd-numbered games, Tim wins with probability $3/4$, and in the even-numbered games, Allen wins with probability $3/4$. What is the expected number of games in a match?

16. [**8**] The walls of a room are in the shape of a triangle ABC with $\angle ABC = 90°$, $\angle BAC = 60°$, and $AB = 6$. Chong stands at the midpoint of BC and rolls a ball toward AB. Suppose that the ball bounces off AB, then AC, then returns exactly to Chong. Find the length of the path of the ball.

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HMMT 2013, 16 FEBRUARY 2013 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# __________

17. [**11**] The lines $y = x$, $y = 2x$, and $y = 3x$ are the three medians of a triangle with perimeter 1. Find the length of the longest side of the triangle.

18. [**11**] Define the sequence of positive integers $\{a_n\}$ as follows. Let $a_1 = 1, a_2 = 3$, and for each $n > 2$, let a_n be the result of expressing a_{n-1} in base $n-1$, then reading the resulting numeral in base n, then adding 2 (in base n). For example, $a_2 = 3_{10} = 11_2$, so $a_3 = 11_3 + 2_3 = 6_{10}$. Express a_{2013} in base ten.

19. [**11**] An isosceles trapezoid $ABCD$ with bases AB and CD has $AB = 13$, $CD = 17$, and height 3. Let E be the intersection of AC and BD. Circles Ω and ω are circumscribed about triangles ABE and CDE. Compute the sum of the radii of Ω and ω.

20. [**11**] The polynomial $f(x) = x^3 - 3x^2 - 4x + 4$ has three real roots r_1, r_2, and r_3. Let $g(x) = x^3 + ax^2 + bx + c$ be the polynomial which has roots s_1, s_2, and s_3, where $s_1 = r_1 + r_2z + r_3z^2$, $s_2 = r_1z + r_2z^2 + r_3$, $s_3 = r_1z^2 + r_2 + r_3z$, and $z = \frac{-1+i\sqrt{3}}{2}$. Find the real part of the sum of the coefficients of $g(x)$.

HMMT 2013, 16 FEBRUARY 2013 — GUTS ROUND

Organization __________________________ Team __________________________ Team ID# __________

21. [**14**] Find the number of positive integers $j \leq 3^{2013}$ such that

$$j = \sum_{k=0}^{m} \left((-1)^k \cdot 3^{a_k} \right)$$

for some strictly increasing sequence of nonnegative integers $\{a_k\}$. For example, we may write $3 = 3^1$ and $55 = 3^0 - 3^3 + 3^4$, but 4 cannot be written in this form.

22. [**14**] Sherry and Val are playing a game. Sherry has a deck containing 2011 red cards and 2012 black cards, shuffled randomly. Sherry flips these cards over one at a time, and before she flips each card over, Val guesses whether it is red or black. If Val guesses correctly, she wins 1 dollar; otherwise, she loses 1 dollar. In addition, Val must guess red exactly 2011 times. If Val plays optimally, what is her expected profit from this game?

23. [**14**] Let $ABCD$ be a parallelogram with $AB = 8$, $AD = 11$, and $\angle BAD = 60°$. Let X be on segment CD with $CX/XD = 1/3$ and Y be on segment AD with $AY/YD = 1/2$. Let Z be on segment AB such that AX, BY, and DZ are concurrent. Determine the area of triangle XYZ.

24. [**14**] Given a point p and a line segment l, let $d(p, l)$ be the distance between them. Let A, B, and C be points in the plane such that $AB = 6$, $BC = 8$, $AC = 10$. What is the area of the region in the (x, y)-plane formed by the ordered pairs (x, y) such that there exists a point P inside triangle ABC with $d(P, AB) + x = d(P, BC) + y = d(P, AC)$?

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Organization ____________________________ Team ____________________________ Team ID# ___________

25. [**17**] The sequence (z_n) of complex numbers satisfies the following properties:
   - z_1 and z_2 are not real.
   - $z_{n+2} = z_{n+1}^2 z_n$ for all integers $n \geq 1$.
   - $\dfrac{z_{n+3}}{z_n^2}$ is real for all integers $n \geq 1$.
   - $\left|\dfrac{z_3}{z_4}\right| = \left|\dfrac{z_4}{z_5}\right| = 2$.

   Find the product of all possible values of z_1.

26. [**17**] Triangle ABC has perimeter 1. Its three altitudes form the side lengths of a triangle. Find the set of all possible values of $\min(AB, BC, CA)$.

27. [**17**] Let W be the hypercube $\{(x_1, x_2, x_3, x_4) \mid 0 \leq x_1, x_2, x_3, x_4 \leq 1\}$. The intersection of W and a hyperplane parallel to $x_1 + x_2 + x_3 + x_4 = 0$ is a non-degenerate 3-dimensional polyhedron. What is the maximum number of faces of this polyhedron?

28. [**17**] Let $z_0 + z_1 + z_2 + \cdots$ be an infinite complex geometric series such that $z_0 = 1$ and $z_{2013} = \frac{1}{2013^{2013}}$. Find the sum of all possible sums of this series.

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Organization ____________________________ Team ____________________________ Team ID# ___________

29. [**20**] Let $A_1, A_2, ..., A_m$ be finite sets of size 2012 and let $B_1, B_2, ..., B_m$ be finite sets of size 2013 such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Find the maximum value of m.

30. [**20**] How many positive integers k are there such that

$$\frac{k}{2013}(a+b) = lcm(a,b)$$

has a solution in positive integers (a, b)?

31. [**20**] Let $ABCD$ be a quadrilateral inscribed in a unit circle with center O. Suppose that $\angle AOB = \angle COD = 135°$, $BC = 1$. Let B' and C' be the reflections of A across BO and CO respectively. Let H_1 and H_2 be the orthocenters of $AB'C'$ and BCD, respectively. If M is the midpoint of OH_1, and O' is the reflection of O about the midpoint of MH_2, compute OO'.

32. [**20**] For an even positive integer n Kevin has a tape of length $4n$ with marks at $-2n, -2n+1, \ldots, 2n-1, 2n$. He then randomly picks n points in the set $-n, -n+1, -n+2, \ldots, n-1, n$, and places a stone on each of these points. We call a stone 'stuck' if it is on $2n$ or $-2n$, or either all the points to the right, or all the points to the left, all contain stones. Then, every minute, Kevin shifts the unstuck stones in the following manner:

• He picks an unstuck stone uniformly at random and then flips a fair coin.

• If the coin came up heads, he then moves that stone and every stone in the largest contiguous set containing that stone one point to the left. If the coin came up tails, he moves every stone in that set one point right instead.

• He repeats until all the stones are stuck.

Let p_k be the probability that at the end of the process there are exactly k stones in the right half. Evaluate

$$\frac{p_{n-1} - p_{n-2} + p_{n-3} - \ldots + p_3 - p_2 + p_1}{p_{n-1} + p_{n-2} + p_{n-3} + \ldots + p_3 + p_2 + p_1}$$

in terms of n.

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Organization ____________________ Team ____________________ Team ID# __________

33. [**25**] Compute the value of $1^{25}+2^{24}+3^{23}+\ldots+24^2+25^1$. If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor 25\min\left(\left(\frac{A}{C}\right)^2,\left(\frac{C}{A}\right)^2\right)\right\rfloor$.

34. [**25**] For how many unordered sets $\{a,b,c,d\}$ of positive integers, none of which exceed 168, do there exist integers w,x,y,z such that $(-1)^w a+(-1)^x b+(-1)^y c+(-1)^z d=168$? If your answer is A and the correct answer is C, then your score on this problem will be $\lfloor 25e^{-3\frac{|C-A|}{C}}\rfloor$.

35. [**25**] Let P be the number to partition 2013 into an ordered tuple of prime numbers? What is $\log_2(P)$? If your answer is A and the correct answer is C, then your score on this problem will be $\left\lfloor \frac{125}{2}\left(\min\left(\frac{C}{A},\frac{A}{C}\right)-\frac{3}{5}\right)\right\rfloor$ or zero, whichever is larger.

36. [**24**] (Mathematicians A to Z) Below are the names of 26 mathematicians, one for each letter of the alphabet. Your answer to this question should be a subset of $\{A,B,\cdots,Z\}$, where each letter represents the corresponding mathematician. If two mathematicians in your subset have birthdates that are within 20 years of each other, then your score is 0. Otherwise, your score is $\max(3(k-3),0)$ where k is the number of elements in your subset.

Niels Abel	Isaac Netwon
Étienne Bézout	Nicole Oresme
Augustin-Louis Cauchy	Blaise Pascal
René Descartes	Daniel Quillen
Leonhard Euler	Bernhard Riemann
Pierre Fatou	Jean-Pierre Serre
Alexander Grothendieck	Alan Turing
David Hilbert	Stanislaw Ulam
Kenkichi Iwasawa	John Venn
Carl Jacobi	Andrew Wiles
Andrey Kolmogorov	Leonardo Ximenes
Joseph-Louis Lagrange	Shing-Tung Yau
John Milnor	Ernst Zermelo

2013 HMT

THE HARVARD-MIT MATHEMATICS TOURNAMENT

TEAM ROUND

This test consists of ten problems to be solved by a team in one hour. The problems are unequally weighted with point values as shown in brackets. They are *not* necessarily in order of difficulty, though harder problems are generally worth more points. We do not expect most teams to get through all the problems.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted. If there is a chalkboard in the room, you may write on it *during* the round only, *not* before it starts.

Numerical answers should, where applicable, be simplified as much as reasonably possible and must be exact unless otherwise specified. Correct mathematical notation must be used. Your proctor *cannot* assist you in interpreting or solving problems but has our phone number and may help with any administrative difficulties.

All problems require full written proof/justification. Even if the answer is numerical, you must prove your result.

If you believe the test contains an error, please submit your protest in writing to Lobby 10 during lunchtime.

Enjoy!

HMMT 2013
Saturday 16 February 2013

Team Round

Organization ____

Team ________________________________ Team ID# ________

1. [**10**]

HMMT 2013
Saturday 16 February 2013

Team Round

Organization ______________________________

Team ______________________ Team ID# ________

2. [**15**]

HMMT 2013
Saturday 16 February 2013

Team Round

Organization ____

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3. [**15**]

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Team Round

Organization ______________________________

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4. [**20**]

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Team Round

Organization ____

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5. [**25**]

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Team Round

Organization ____

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6. [**25**]

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7. [**30**]

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Team ________________________________ Team ID# ________

8. [**35**]

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Team Round

Organization ____

Team ________________________________ Team ID# ________

9. [**35**]

HMMT 2013
Saturday 16 February 2013

Team Round

Organization ____

Team ________________________________ Team ID# ________

10. [**40**]

HMMT 2013

Saturday 16 February 2013

Team Round

1. [**10**] Let a and b be real numbers such that $\frac{ab}{a^2+b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2-b^2|}{a^2+b^2}$.

2. [**15**] A cafe has 3 tables and 5 individual counter seats. People enter in groups of size between 1 and 4, inclusive, and groups never share a table. A group of more than 1 will always try to sit at a table, but will sit in counter seats if no tables are available. Conversely, a group of 1 will always try to sit at the counter first. One morning, M groups consisting of a total of N people enter and sit down. Then, a single person walks in, and realizes that all the tables and counter seats are occupied by some person or group. What is the minimum possible value of $M + N$?

3. [**15**] Let ABC be a triangle with circumcenter O such that $AC = 7$. Suppose that the circumcircle of AOC is tangent to BC at C and intersects the line AB at A and F. Let FO intersect BC at E. Compute BE.

4. [**20**] Let a_1, a_2, a_3, a_4, a_5 be real numbers whose sum is 20. Determine with proof the smallest possible value of

$$\sum_{1 \le i < j \le 5} \lfloor a_i + a_j \rfloor.$$

5. [**25**] Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

   1. Choose a row, and subtract 1 from each entry.
   2. Chooses a column, and add 1 to each entry.

   He would like to get an array where all integers are divisible by 2013. On how many arrays is this possible?

6. [**25**] Let triangle ABC satisfy $2BC = AB + AC$ and have incenter I and circumcircle ω. Let D be the intersection of AI and ω (with A, D distinct). Prove that I is the midpoint of AD.

7. [**30**] There are n children and n toys such that each child has a strict preference ordering on the toys. We want to distribute the toys: say a distribution A dominates a distribution $B \neq A$ if in A, each child receives at least as preferable of an toy as in B. Prove that if some distribution is not dominated by any other, then at least one child gets his/her favorite toy in that distribution.

8. [**35**] Let points A and B be on circle ω centered at O. Suppose that ω_A and ω_B are circles not containing O which are internally tangent to ω at A and B, respectively. Let ω_A and ω_B intersect at C and D such that D is inside triangle ABC. Suppose that line BC meets ω again at E and let line EA intersect ω_A at F. If $FC \perp CD$, prove that O, C, and D are collinear.

9. [**35**] Let m be an odd positive integer greater than 1. Let S_m be the set of all non-negative integers less than m which are of the form $x + y$, where $xy - 1$ is divisible by m. Let $f(m)$ be the number of elements of S_m.

   (a) Prove that $f(mn) = f(m)f(n)$ if m, n are relatively prime odd integers greater than 1.

   (b) Find a closed form for $f(p^k)$, where $k > 0$ is an integer and p is an odd prime.

10. [**40**] Chim Tu has a large rectangular table. On it, there are finitely many pieces of paper with non-overlapping interiors, each one in the shape of a convex polygon. At each step, Chim Tu is allowed to slide one piece of paper in a straight line such that its interior does not touch any other piece of paper during the slide. Can Chim Tu always slide all the pieces of paper off the table in finitely many steps?

HMMT 2013

Saturday 16 February 2013

Team Round

1. [**10**] Let a and b be real numbers such that $\frac{ab}{a^2+b^2} = \frac{1}{4}$. Find all possible values of $\frac{|a^2-b^2|}{a^2+b^2}$.

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HMMT 2013

Saturday 16 February 2013

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HMMT November 2013

Saturday 9 November 2013

General Test

1. [**2**] What is the smallest non-square positive integer that is the product of four prime numbers (not necessarily distinct)?

2. [**3**] Plot points A, B, C at coordinates $(0,0)$, $(0,1)$, and $(1,1)$ in the plane, respectively. Let S denote the union of the two line segments AB and BC. Let X_1 be the area swept out when Bobby rotates S *counterclockwise* 45 degrees about point A. Let X_2 be the area swept out when Calvin rotates S *clockwise* 45 degrees about point A. Find $\frac{X_1+X_2}{2}$.

3. [**4**] A 24-hour digital clock shows times $h:m:s$, where h, m, and s are integers with $0 \leq h \leq 23$, $0 \leq m \leq 59$, and $0 \leq s \leq 59$. How many times $h:m:s$ satisfy $h+m=s$?

4. [**4**] A 50-card deck consists of 4 cards labeled "i" for $i = 1, 2, \ldots, 12$ and 2 cards labeled "13". If Bob randomly chooses 2 cards from the deck without replacement, what is the probability that his 2 cards have the same label?

5. [**5**] Let ABC be an isosceles triangle with $AB = AC$. Let D and E be the midpoints of segments AB and AC, respectively. Suppose that there exists a point F on ray $\overrightarrow{DE}$ outside of ABC such that triangle BFA is similar to triangle ABC. Compute $\frac{AB}{BC}$.

6. [**5**] Find the number of positive integer divisors of 12! that leave a remainder of 1 when divided by 3.

7. [**6**] Find the largest real number λ such that $a^2 + b^2 + c^2 + d^2 \geq ab + \lambda bc + cd$ for all real numbers a, b, c, d.

8. [**6**] How many of the first 1000 positive integers can be written as the sum of finitely many distinct numbers from the sequence $3^0, 3^1, 3^2, \ldots$?

9. [**7**] Let ABC be a triangle and D a point on BC such that $AB = \sqrt{2}$, $AC = \sqrt{3}$, $\angle BAD = 30°$, and $\angle CAD = 45°$. Find AD.

10. [**8**] How many functions $f : \{1, 2, \ldots, 2013\} \to \{1, 2, \ldots, 2013\}$ satisfy $f(j) < f(i) + j - i$ for all integers i, j such that $1 \leq i < j \leq 2013$?

. . . .

1. [**5**] Evaluate $2 + 5 + 8 + \cdots + 101$.

2. [**5**] Two fair six-sided dice are rolled. What is the probability that their sum is at least 10?

3. [**5**] A square is inscribed in a circle of radius 1. Find the perimeter of the square.

. . . .

HMMT NOVEMBER 2013, 9 NOVEMBER 2013 — GUTS ROUND

4. [**6**] Find the minimum possible value of $(x^2 + 6x + 2)^2$ over all real numbers x.

5. [**6**] How many positive integers less than 100 are relatively prime to 200? (Two numbers are *relatively prime* if their greatest common factor is 1.)

6. [**6**] A right triangle has area 5 and a hypotenuse of length 5. Find its perimeter.

. . . .

HMMT NOVEMBER 2013, 9 NOVEMBER 2013 — GUTS ROUND

7. [**7**] Marty and three other people took a math test. Everyone got a non-negative integer score. The average score was 20. Marty was told the average score and concluded that everyone else scored below average. What was the minimum possible score Marty could have gotten in order to definitively reach this conclusion?

8. [**7**] Evaluate the expression

$$\cfrac{1}{2 - \cfrac{1}{2 - \cfrac{1}{2 - \cdots \cfrac{1}{2 - \cfrac{1}{2}}}}},$$

where the digit 2 appears 2013 times.

9. [**7**] Find the remainder when $1^2 + 3^2 + 5^2 + \cdots + 99^2$ is divided by 1000.

. . . .

10. [**8**] How many pairs of real numbers (x, y) satisfy the equation

$$y^4 - y^2 = xy^3 - xy = x^3y - xy = x^4 - x^2 = 0?$$

11. [**8**] David has a unit triangular array of 10 points, 4 on each side. A *looping path* is a sequence $A_1, A_2, \ldots, A_{10}$ containing each of the 10 points exactly once, such that A_i and A_{i+1} are adjacent (exactly 1 unit apart) for $i = 1, 2, \ldots, 10$. (Here $A_{11} = A_1$.) Find the number of looping paths in this array.

12. [**8**] Given that $62^2 + 122^2 = 18728$, find positive integers (n, m) such that $n^2 + m^2 = 9364$.

. . . .

13. [**9**] Let $S = \{1, 2, \ldots, 2013\}$. Find the number of ordered triples (A, B, C) of subsets of S such that $A \subseteq B$ and $A \cup B \cup C = S$.

14. [**9**] Find all triples of positive integers (x, y, z) such that $x^2 + y - z = 100$ and $x + y^2 - z = 124$.

15. [**9**] Find all real numbers x between 0 and 360 such that $\sqrt{3}\cos 10^\circ = \cos 40^\circ + \sin x^\circ$.

. . . .

16. [**10**] A bug is on one exterior vertex of solid S, a $3 \times 3 \times 3$ cube that has its center $1 \times 1 \times 1$ cube removed, and wishes to travel to the opposite exterior vertex. Let O denote the *outer* surface of S (formed by the surface of the $3 \times 3 \times 3$ cube). Let $L(S)$ denote the length of the shortest path through S. (Note that such a path cannot pass through the missing center cube, which is empty space.) Let $L(O)$ denote the length of the shortest path through O. What is the ratio $\frac{L(S)}{L(O)}$?

17. [**10**] Find the sum of $\frac{1}{n}$ over all positive integers n with the property that the decimal representation of $\frac{1}{n}$ terminates.

18. [**10**] The rightmost nonzero digit in the decimal expansion of 101! is the same as the rightmost nonzero digit of $n!$, where n is an integer greater than 101. Find the smallest possible value of n.

19. [**11**] Let p, q, r, s be distinct primes such that $pq - rs$ is divisible by 30. Find the minimum possible value of $p + q + r + s$.

20. [**11**] There exist unique nonnegative integers A, B between 0 and 9, inclusive, such that

$$(1001 \cdot A + 110 \cdot B)^2 = 57,108,249.$$

Find $10 \cdot A + B$.

21. [**11**] Suppose A, B, C, and D are four circles of radius $r > 0$ centered about the points $(0, r)$, $(r, 0)$, $(0, -r)$, and $(-r, 0)$ in the plane. Let O be a circle centered at $(0, 0)$ with radius $2r$. In terms of r, what is the area of the union of circles A, B, C, and D subtracted by the area of circle O that is not contained in the union of A, B, C, and D?

(The *union* of two or more regions in the plane is the set of points lying in at least one of the regions.)

22. [**12**] Let S be a subset of $\{1, 2, 3, \ldots, 12\}$ such that it is impossible to partition S into k disjoint subsets, each of whose elements sum to the same value, for any integer $k \geq 2$. Find the maximum possible sum of the elements of S.

23. [**12**] The number $989 \cdot 1001 \cdot 1007 + 320$ can be written as the product of three distinct primes p, q, r with $p < q < r$. Find (p, q, r).

24. [**12**] Find the number of subsets S of $\{1, 2, \ldots 6\}$ satisfying the following conditions:

- S is non-empty.
- No subset of S has the property that the sum of its elements is 10.

. . . .

25. [**13**] Let a, b be positive reals with $a > b > \frac{1}{2}a$. Place two squares of side lengths a, b next to each other, such that the larger square has lower left corner at $(0, 0)$ and the smaller square has lower left corner at $(a, 0)$. Draw the line passing through $(0, a)$ and $(a + b, 0)$. The region in the two squares lying *above* the line has area 2013. If (a, b) is the unique pair maximizing $a + b$, compute $\frac{a}{b}$.

26. [**13**] Trapezoid $ABCD$ is inscribed in the parabola $y = x^2$ such that $A = (a, a^2)$, $B = (b, b^2)$, $C = (-b, b^2)$, and $D = (-a, a^2)$ for some positive reals a, b with $a > b$. If $AD + BC = AB + CD$, and $AB = \frac{3}{4}$, what is a?

27. [**13**] Find all triples of real numbers (a, b, c) such that $a^2 + 2b^2 - 2bc = 16$ and $2ab - c^2 = 16$.

. . . .

28. [**15**] Triangle ABC has $AB = 4$, $BC = 3$, and a right angle at B. Circles ω_1 and ω_2 of equal radii are drawn such that ω_1 is tangent to AB and AC, ω_2 is tangent to BC and AC, and ω_1 is tangent to ω_2. Find the radius of ω_1.

29. [**15**] Let $\triangle XYZ$ be a right triangle with $\angle XYZ = 90°$. Suppose there exists an infinite sequence of equilateral triangles $X_0Y_0T_0, X_1Y_1T_1, \ldots$ such that $X_0 = X, Y_0 = Y$, X_i lies on the segment XZ for all $i \geq 0$, Y_i lies on the segment YZ for all $i \geq 0$, X_iY_i is perpendicular to YZ for all $i \geq 0$, T_i and Y are separated by line XZ for all $i \geq 0$, and X_i lies on segment $Y_{i-1}T_{i-1}$ for $i \geq 1$.

    Let $\mathcal{P}$ denote the union of the equilateral triangles. If the area of $\mathcal{P}$ is equal to the area of XYZ, find $\frac{XY}{YZ}$.

30. [**15**] Find the number of ordered triples of integers (a, b, c) with $1 \leq a, b, c \leq 100$ and $a^2b + b^2c + c^2a = ab^2 + bc^2 + ca^2$.

31. [**17**] Chords $\overline{AB}$ and $\overline{CD}$ of circle ω intersect at E such that $AE = 8$, $BE = 2$, $CD = 10$, and $\angle AEC = 90°$. Let R be a rectangle inside ω with sides parallel to $\overline{AB}$ and $\overline{CD}$, such that no point in the interior of R lies on $\overline{AB}$, $\overline{CD}$, or the boundary of ω. What is the maximum possible area of R?

32. [**17**] Suppose that x and y are chosen randomly and uniformly from $(0, 1)$. What is the probability that $\left\lfloor \sqrt{\frac{x}{y}} \right\rfloor$ is even? *Hint:* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

33. [**17**] On each side of a 6 by 8 rectangle, construct an equilateral triangle with that side as one edge such that the interior of the triangle intersects the interior of the rectangle. What is the total area of all regions that are contained in exactly 3 of the 4 equilateral triangles?

34. [**20**] Find the number of positive integers less than 1000000 that are divisible by some perfect cube greater than 1. Your score will be $\max\left\{0, \lfloor 20 - 200|1 - \frac{k}{S}|\rfloor\right\}$, where k is your answer and S is the actual answer.

35. [**20**] Consider the following 4 by 4 grid with one corner square removed:

You may start at any square in this grid and at each move, you may either stop or travel to an adjacent square (sharing a side, not just a corner) that you have not already visited (the square you start at is automatically marked as visited). Determine the distinct number of paths you can take. Your score will be $\max\left\{0, \lfloor 20 - 200|1 - \frac{k}{S}|\rfloor\right\}$, where k is your answer and S is the actual answer.

36. [**20**] Pick a subset of at least four of the following seven numbers, order them from least to greatest, and write down their **labels** (corresponding letters from A through G) in that order: (A) π; (B) $\sqrt{2} + \sqrt{3}$; (C) $\sqrt{10}$; (D) $\frac{355}{113}$; (E) $16\tan^{-1}\frac{1}{5} - 4\tan^{-1}\frac{1}{240}$; (F) $\ln(23)$; and (G) $2^{\sqrt{e}}$. If the ordering of the numbers you picked is correct and you picked at least 4 numbers, then your score for this problem will be $(N-2)(N-3)$, where N is the size of your subset; otherwise, your score is 0.

HMMT November 2013

Saturday 9 November 2013

Team Round

Expected Value

1. [**3**] Tim the Beaver can make three different types of geometrical figures: squares, regular hexagons, and regular octagons. Tim makes a random sequence $F_0, F_1, F_2, F_3, \ldots$ of figures as follows:

   - F_0 is a square.
   - For every positive integer i, F_i is randomly chosen to be one of the 2 figures *distinct from* F_{i-1} (each chosen with equal probability $\frac{1}{2}$).
   - Tim takes 4 seconds to make squares, 6 to make hexagons, and 8 to make octagons. He makes one figure after another, with no breaks in between.

   Suppose that exactly 17 seconds after he starts making F_0, Tim is making a figure with n sides. What is the expected value of n?

2. [**4**] Gary plays the following game with a fair n-sided die whose faces are labeled with the positive integers between 1 and n, inclusive: if $n = 1$, he stops; otherwise he rolls the die, and starts over with a k-sided die, where k is the number his n-sided die lands on. (In particular, if he gets $k = 1$, he will stop rolling the die.) If he starts out with a 6-sided die, what is the expected number of rolls he makes?

3. [**6**] The digits $1, 2, 3, 4, 5, 6$ are randomly chosen (without replacement) to form the three-digit numbers $M = \overline{ABC}$ and $N = \overline{DEF}$. For example, we could have $M = 413$ and $N = 256$. Find the expected value of $M \cdot N$.

Power of a Point

4. [**4**] Consider triangle ABC with side lengths $AB = 4$, $BC = 7$, and $AC = 8$. Let M be the midpoint of segment AB, and let N be the point on the interior of segment AC that also lies on the circumcircle of triangle MBC. Compute BN.

5. [**4**] In triangle ABC, $\angle BAC = 60°$. Let ω be a circle tangent to segment AB at point D and segment AC at point E. Suppose ω intersects segment BC at points F and G such that F lies in between B and G. Given that $AD = FG = 4$ and $BF = \frac{1}{2}$, find the length of CG.

6. [**6**] Points A, B, C lie on a circle ω such that BC is a diameter. AB is extended past B to point B' and AC is extended past C to point C' such that line $B'C'$ is parallel to BC and tangent to ω at point D. If $B'D = 4$ and $C'D = 6$, compute BC.

7. [**7**] In equilateral triangle ABC, a circle ω is drawn such that it is tangent to all three sides of the triangle. A line is drawn from A to point D on segment BC such that AD intersects ω at points E and F. If $EF = 4$ and $AB = 8$, determine $|AE - FD|$.

Periodicity

8. [**2**] Define the sequence $\{x_i\}_{i \geq 0}$ by $x_0 = x_1 = x_2 = 1$ and $x_k = \frac{x_{k-1} + x_{k-2} + 1}{x_{k-3}}$ for $k > 2$. Find x_{2013}.

9. [**7**] For an integer $n \geq 0$, let $f(n)$ be the smallest possible value of $|x + y|$, where x and y are integers such that $3x - 2y = n$. Evaluate $f(0) + f(1) + f(2) + \cdots + f(2013)$.

10. [**7**] Let $\omega = \cos \frac{2\pi}{727} + i \sin \frac{2\pi}{727}$. The imaginary part of the complex number

$$\prod_{k=8}^{13} \left(1 + \omega^{3^{k-1}} + \omega^{2 \cdot 3^{k-1}}\right)$$

is equal to $\sin \alpha$ for some angle α between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive. Find α.

HMMT November 2013

Saturday 9 November 2013

Theme Round

Traveling

1. [**2**] Two cars are driving directly towards each other such that one is twice as fast as the other. The distance between their starting points is 4 miles. When the two cars meet, how many miles is the faster car from its starting point?

2. [**4**] You are standing at a pole and a snail is moving directly away from the pole at 1 cm/s. When the snail is 1 meter away, you start "Round 1". In Round n ($n \geq 1$), you move directly toward the snail at $n+1$ cm/s. When you reach the snail, you immediately turn around and move back to the starting pole at $n+1$ cm/s. When you reach the pole, you immediately turn around and Round $n+1$ begins.

   At the start of Round 100, how many **meters** away is the snail?

3. [**5**] Let ABC be a triangle with $AB = 5$, $BC = 4$, and $CA = 3$. Initially, there is an ant at each vertex. The ants start walking at a rate of 1 unit per second, in the direction $A \to B \to C \to A$ (so the ant starting at A moves along ray $\overrightarrow{AB}$, etc.). For a positive real number t less than 3, let $A(t)$ be the area of the triangle whose vertices are the positions of the ants after t seconds have elapsed. For what positive real number t less than 3 is $A(t)$ minimized?

4. [**7**] There are 2 runners on the perimeter of a regular hexagon, initially located at adjacent vertices. Every second, each of the runners independently moves either one vertex to the left, with probability $\frac{1}{2}$, or one vertex to the right, also with probability $\frac{1}{2}$. Find the probability that after a 2013 second run (in which the runners switch vertices 2013 times each), the runners end up at adjacent vertices once again.

5. [**7**] Let ABC be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Company XYZ wants to locate their base at the point P in the plane minimizing the total distance to their workers, who are located at vertices A, B, and C. There are 1, 5, and 4 workers at A, B, and C, respectively. Find the minimum possible total distance Company XYZ's workers have to travel to get to P.

Bases

Many of you may be familiar with the decimal (or base 10) system. For example, when we say 2013_{10}, we really mean $2\cdot10^3+0\cdot10^2+1\cdot10^1+3\cdot10^0$. Similarly, there is the binary (base 2) system. For example, $11111011101_2 = 1\cdot2^{10}+1\cdot2^9+1\cdot2^8+1\cdot2^7+1\cdot2^6+0\cdot2^5+1\cdot2^4+1\cdot2^3+1\cdot2^2+0\cdot2^1+1\cdot2^0 = 2013_{10}$.

In general, if we are given a string $(a_na_{n-1}\dots a_0)_b$ in base b (the subscript b means that we are in base b), then it is equal to $\sum_{i=0}^{n} a_ib^i$.

It turns out that for every positive integer $b > 1$, every positive integer k has a unique base b representation. That is, for every positive integer k, there exists a unique n and digits $0 \leq a_0, \dots, a_n < b$ such that $(a_na_{n-1}\dots a_0)_b = k$.

We can adapt this to bases $b < -1$. It actually turns out that if $b < -1$, every *nonzero* integer has a unique base b representation. That is, for every nonzero integer k, there exists a unique n and digits $0 \leq a_0, \dots, a_n < |b|$ such that $(a_na_{n-1}\dots a_0)_b = k$. The next five problems involve base -4.

Note: Unless otherwise stated, express your answers in base 10.

6. [**2**] Evaluate 1201201_{-4}.

7. [**3**] Express -2013 in base -4.

8. [**5**] Let $b(n)$ be the number of digits in the base -4 representation of n. Evaluate $\sum_{i=1}^{2013} b(i)$.

9. [**7**] Let N be the largest positive integer that can be expressed as a 2013-digit base -4 number. What is the remainder when N is divided by 210?

10. [**8**] Find the sum of all positive integers n such that there exists an integer b with $|b| \neq 4$ such that the base -4 representation of n is the same as the base b representation of n.

HMMT 2013

Saturday 16 February 2013

HMIC

1. [**5**] Let $\mathcal{S}$ be a set of size n, and k be a positive integer. For each $1 \le i \le kn$, there is a subset $S_i \subset \mathcal{S}$ such that $|S_i| = 2$. Furthermore, for each $e \in \mathcal{S}$, there are exactly $2k$ values of i such that $e \in S_i$. Show that it is possible to choose one element from S_i for each $1 \le i \le kn$ such that every element of $\mathcal{S}$ is chosen exactly k times.

2. [**7**] Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for all real numbers x, y,

$$(x - y)(f(x) - f(y)) = f(x - f(y))f(f(x) - y).$$

3. [**8**] Triangle ABC is inscribed in a circle ω such that $\angle A = 60^\circ$ and $\angle B = 75^\circ$. Let the bisector of angle A meet BC and ω at E and D, respectively. Let the reflections of A across D and C be D' and C', respectively. If the tangent to ω at A meets line BC at P, and the circumcircle of APD' meets line AC at $F \ne A$, prove that the circumcircle of $C'FE$ is tangent to BC at E.

4. [**10**] A subset $U \subset \mathbb{R}$ is *open* if for any $x \in U$, there exist real numbers a, b such that $x \in (a, b) \subset U$. Suppose $S \subset \mathbb{R}$ has the property that any open set intersecting $(0, 1)$ also intersects S. Let T be a countable collection of open sets containing S. Prove that the intersection of all of the sets of T is not a countable subset of $\mathbb{R}$.

   (A set Γ is *countable* if there exists a bijective function $f : \Gamma \to \mathbb{Z}$.)

5. [**12**]

   (a) Given a finite set X of points in the plane, let $f_X(n)$ be the largest possible area of a polygon with at most n vertices, all of which are points of X. Prove that if m, n are integers with $m \ge n > 2$, then $f_X(m) + f_X(n) \ge f_X(m + 1) + f_X(n - 1)$.

   (b) Let P_0 be a 1-by-2 rectangle (including its interior), and inductively define the polygon P_i to be the result of folding P_{i-1} over some line that cuts P_{i-1} into two **connected** parts. The *diameter* of a polygon P_i is the maximum distance between two points of P_i. Determine the smallest possible diameter of P_{2013}.

   (In other words, given a polygon P_{i-1}, a fold of P_{i-1} consists of a line l dividing P_{i-1} into two connected parts A and B, and the folded polygon $P_i = A \cup B_l$, where B_l is the reflection of B over the line l.)

THE HARVARD-MIT MATHEMATICS TOURNAMENT

ALGEBRA TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems will be weighted with point values after the contest based on how many competitors solve each problem. There is no penalty for guessing.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted other than the official translation sheets. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

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Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to Science Center 109 during lunchtime.

Enjoy!

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Algebra Test

1. Let f be the function such that

$$f(x) = \begin{cases} 2x & \text{if } x \leq \frac{1}{2} \\ 2 - 2x & \text{if } x > \frac{1}{2} \end{cases}$$

What is the total length of the graph of $\underbrace{f(f(\ldots f}_{2012 f's}(x)\ldots))$ from $x = 0$ to $x = 1$?

2. You are given an unlimited supply of red, blue, and yellow cards to form a hand. Each card has a point value and your score is the sum of the point values of those cards. The point values are as follows: the value of each red card is 1, the value of each blue card is equal to twice the number of red cards, and the value of each yellow card is equal to three times the number of blue cards. What is the maximum score you can get with fifteen cards?

3. Given points a and b in the plane, let $a \oplus b$ be the unique point c such that abc is an equilateral triangle with a, b, c in the clockwise orientation.

   Solve $(x \oplus (0,0)) \oplus (1,1) = (1,-1)$ for x.

4. During the weekends, Eli delivers milk in the complex plane. On Saturday, he begins at z and delivers milk to houses located at z^3, z^5, $z^7, \ldots,$ z^{2013}, in that order; on Sunday, he begins at 1 and delivers milk to houses located at z^2, z^4, $z^6, \ldots,$ z^{2012}, in that order. Eli always walks directly (in a straight line) between two houses. If the distance he must travel from his starting point to the last house is $\sqrt{2012}$ on both days, find the real part of z^2.

5. Find all ordered triples (a, b, c) of positive reals that satisfy: $\lfloor a \rfloor bc = 3, a\lfloor b \rfloor c = 4$, and $ab\lfloor c \rfloor = 5$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

6. Let $a_0 = -2$, $b_0 = 1$, and for $n \geq 0$, let

$$a_{n+1} = a_n + b_n + \sqrt{a_n^2 + b_n^2},$$
$$b_{n+1} = a_n + b_n - \sqrt{a_n^2 + b_n^2}.$$

   Find a_{2012}.

7. Let $\otimes$ be a binary operation that takes two positive real numbers and returns a positive real number. Suppose further that $\otimes$ is continuous, commutative ($a \otimes b = b \otimes a$), distributive across multiplication ($a \otimes (bc) = (a \otimes b)(a \otimes c)$), and that $2 \otimes 2 = 4$. Solve the equation $x \otimes y = x$ for y in terms of x for $x > 1$.

8. Let $x_1 = y_1 = x_2 = y_2 = 1$, then for $n \geq 3$ let $x_n = x_{n-1}y_{n-2} + x_{n-2}y_{n-1}$ and $y_n = y_{n-1}y_{n-2} - x_{n-1}x_{n-2}$. What are the last two digits of $|x_{2012}|$?

9. How many real triples (a, b, c) are there such that the polynomial $p(x) = x^4 + ax^3 + bx^2 + ax + c$ has exactly three distinct roots, which are equal to $\tan y$, $\tan 2y$, and $\tan 3y$ for some real y?

10. Suppose that there are 16 variables $\{a_{i,j}\}_{0 \leq i,j \leq 3}$, each of which may be 0 or 1. For how many settings of the variables $a_{i,j}$ do there exist positive reals $c_{i,j}$ such that the polynomial

$$f(x, y) = \sum_{0 \leq i,j \leq 3} a_{i,j} c_{i,j} x^i y^j$$

   $(x, y \in \mathbb{R})$ is bounded below?

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Algebra Test

Name ______________________________ Team ID# ________

School ____________________ Team ____________________

1. ______________
2. ______________
3. ______________
4. ______________
5. ______________
6. ______________
7. ______________
8. ______________
9. ______________
10. ______________

Score: __________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

COMBINATORICS TEST

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Enjoy!

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

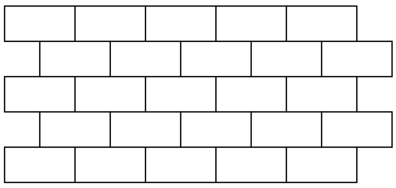
Combinatorics Test

1. In the game of Minesweeper, a number on a square denotes the number of mines that share at least one vertex with that square. A square with a number may not have a mine, and the blank squares are undetermined. How many ways can the mines be placed in this configuration?

	2		1		2

2. Brian has a 20-sided die with faces numbered from 1 to 20, and George has three 6-sided dice with faces numbered from 1 to 6. Brian and George simultaneously roll all their dice. What is the probability that the number on Brian's die is larger than the sum of the numbers on George's dice?

3. In the figure below, how many ways are there to select 5 bricks, one in each row, such that any two bricks in adjacent rows are adjacent?



4. A frog is at the point $(0,0)$. Every second, he can jump one unit either up or right. He can only move to points (x,y) where x and y are not both odd. How many ways can he get to the point $(8,14)$?

5. Dizzy Daisy is standing on the point $(0,0)$ on the xy-plane and is trying to get to the point $(6,6)$. She starts facing rightward and takes a step 1 unit forward. On each subsequent second, she either takes a step 1 unit forward or turns 90 degrees counterclockwise then takes a step 1 unit forward. She may never go on a point outside the square defined by $|x| \leq 6, |y| \leq 6$, nor may she ever go on the same point twice. How many different paths may Daisy take?

6. For a permutation σ of $1,2,\ldots,7$, a *transposition* is a swapping of two elements. (For instance, we could apply a transposition to the permutation $3,7,1,4,5,6,2$ and get $3,7,6,4,5,1,2$ by swapping the 1 and the 6.)

   Let $f(\sigma)$ be the minimum number of transpositions necessary to turn σ into the permutation $1,2,3,4,5,6,7$. Find the sum of $f(\sigma)$ over all permutations σ of $1,2,\ldots,7$.

7. You are repeatedly flipping a fair coin. What is the expected number of flips until the first time that your previous 2012 flips are 'HTHT...HT'?

8. How many ways can one color the squares of a 6x6 grid red and blue such that the number of red squares in each row and column is exactly 2?

9. A parking lot consists of 2012 parking spots equally spaced in a line, numbered 1 through 2012. One by one, 2012 cars park in these spots under the following procedure: the first car picks from the 2012 spots uniformly randomly, and each following car picks uniformly randomly among all possible choices which maximize the minimal distance from an already parked car. What is the probability that the last car to park must choose spot 1?

10. Jacob starts with some complex number x_0 other than 0 or 1. He repeatedly flips a fair coin. If the n^{th} flip lands heads, he lets $x_n = 1 - x_{n-1}$, and if it lands tails he lets $x_n = \frac{1}{x_{n-1}}$. Over all possible choices of x_0, what are all possible values of the probability that $x_{2012} = x_0$?

15$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Combinatorics Test

Name ______________________________ Team ID# ________

School ____________________ Team ____________________

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________
8. ____________
9. ____________
10. ____________

Score: ____________

THE HARVARD-MIT MATHEMATICS TOURNAMENT

GEOMETRY TEST

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Enjoy!

15^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 11 February 2012

Geometry Test

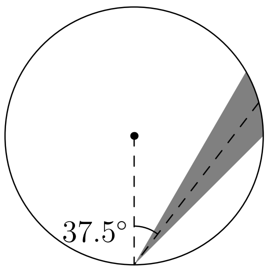
1. ABC is an isosceles triangle with $AB = 2$ and $\measuredangle ABC = 90°$. D is the midpoint of BC and E is on AC such that the area of $AEDB$ is twice the area of ECD. Find the length of DE.

2. Let ABC be a triangle with $\angle A = 90°$, $AB = 1$, and $AC = 2$. Let ℓ be a line through A perpendicular to BC, and let the perpendicular bisectors of AB and AC meet ℓ at E and F, respectively. Find the length of segment EF.

3. Let ABC be a triangle with incenter I. Let the circle centered at B and passing through I intersect side AB at D and let the circle centered at C passing through I intersect side AC at E. Suppose DE is the perpendicular bisector of AI. What are all possible measures of angle BAC in degrees?

4. There are circles ω_1 and ω_2. They intersect in two points, one of which is the point A. B lies on ω_1 such that AB is tangent to ω_2. The tangent to ω_1 at B intersects ω_2 at C and D, where D is the closer to B. AD intersects ω_1 again at E. If $BD = 3$ and $CD = 13$, find EB/ED.

5. A mouse lives in a circular cage with completely reflective walls. At the edge of this cage, a small flashlight with vertex on the circle whose beam forms an angle of 15° is centered at an angle of 37.5° away from the center. The mouse will die in the dark. What fraction of the total area of the cage can keep the mouse alive?



6. Triangle ABC is an equilateral triangle with side length 1. Let $X_0, X_1, \ldots$ be an infinite sequence of points such that the following conditions hold:

   - X_0 is the center of ABC
   - For all $i \geq 0$, X_{2i+1} lies on segment AB and X_{2i+2} lies on segment AC.
   - For all $i \geq 0$, $\measuredangle X_i X_{i+1} X_{i+2} = 90°$.
   - For all $i \geq 1$, X_{i+2} lies in triangle $AX_i X_{i+1}$.

   Find the maximum possible value of $\sum_{i=0}^{\infty} |X_i X_{i+1}|$, where $|PQ|$ is the length of line segment PQ.

7. Let S be the set of the points $(x_1, x_2, \ldots, x_{2012})$ in 2012-dimensional space such that $|x_1| + |x_2| + \cdots + |x_{2012}| \leq 1$. Let T be the set of points in 2012-dimensional space such that $\max_{i=1}^{2012} |x_i| = 2$. Let p be a randomly chosen point on T. What is the probability that the closest point in S to p is a vertex of S?

8. Hexagon $ABCDEF$ has a circumscribed circle and an inscribed circle. If $AB = 9$, $BC = 6$, $CD = 2$, and $EF = 4$. Find $\{DE, FA\}$.

9. Let O, O_1, O_2, O_3, O_4 be points such that O_1, O, O_3 and O_2, O, O_4 are collinear in that order, $OO_1 = 1, OO_2 = 2, OO_3 = \sqrt{2}, OO_4 = 2$, and $\measuredangle O_1 O O_2 = 45°$. Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the circles with respective centers O_1, O_2, O_3, O_4 that go through O. Let A be the intersection of ω_1 and ω_2, B be the intersection of ω_2 and ω_3, C be the intersection of ω_3 and ω_4, and D be the intersection of ω_4 and ω_1, with A, B, C, D all distinct from O. What is the largest possible area of a convex quadrilateral $P_1P_2P_3P_4$ such that P_i lies on O_i and that A, B, C, D all lie on its perimeter?

10. Let C denote the set of points $(x, y) \in \mathbb{R}^2$ such that $x^2 + y^2 \leq 1$. A sequence $A_i = (x_i, y_i) | i \geq 0$ of points in $\mathbb{R}^2$ is 'centric' if it satisfies the following properties:

- $A_0 = (x_0, y_0) = (0, 0)$, $A_1 = (x_1, y_1) = (1, 0)$.
- For all $n \geq 0$, the circumcenter of triangle $A_n A_{n+1} A_{n+2}$ lies in C.

Let K be the maximum value of $x_{2012}^2 + y_{2012}^2$ over all centric sequences. Find all points (x, y) such that $x^2 + y^2 = K$ and there exists a centric sequence such that $A_{2012} = (x, y)$.

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Geometry Test

Name ____________________________________ Team ID# ________

School ________________________ Team ________________________

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10. _______________

Score: __________

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Guts

1. [**2**] Square $ABCD$ has side length 2, and X is a point outside the square such that $AX = XB = \sqrt{2}$. What is the length of the longest diagonal of pentagon $AXBCD$?

2. [**2**] Let $a_0, a_1, a_2, \ldots$ denote the sequence of real numbers such that $a_0 = 2$ and $a_{n+1} = \frac{a_n}{1+a_n}$ for $n \geq 0$. Compute a_{2012}.

3. [**2**] Suppose x and y are real numbers such that $-1 < x < y < 1$. Let G be the sum of the geometric series whose first term is x and whose ratio is y, and let G' be the sum of the geometric series whose first term is y and ratio is x. If $G = G'$, find $x + y$.

4. [**2**] Luna has an infinite supply of red, blue, orange, and green socks. She wants to arrange 2012 socks in a line such that no red sock is adjacent to a blue sock and no orange sock is adjacent to a green sock. How many ways can she do this?

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5. [**3**] Mr. Canada chooses a positive real a uniformly at random from $(0, 1]$, chooses a positive real b uniformly at random from $(0, 1]$, and then sets $c = a/(a + b)$. What is the probability that c lies between $1/4$ and $3/4$?

6. [**3**] Let rectangle $ABCD$ have lengths $AB = 20$ and $BC = 12$. Extend ray BC to Z such that $CZ = 18$. Let E be the point in the interior of $ABCD$ such that the perpendicular distance from E to $\overline{AB}$ is 6 and the perpendicular distance from E to $\overline{AD}$ is 6. Let line EZ intersect AB at X and CD at Y. Find the area of quadrilateral $AXYD$.

7. [**3**] M is an 8×8 matrix. For $1 \leq i \leq 8$, all entries in row i are at least i, and all entries on column i are at least i. What is the minimum possible sum of the entries of M?

8. [**3**] Amy and Ben need to eat 1000 total carrots and 1000 total muffins. The muffins can not be eaten until all the carrots are eaten. Furthermore, Amy can not eat a muffin within 5 minutes of eating a carrot and neither can Ben. If Amy eats 40 carrots per minute and 70 muffins per minute and Ben eats 60 carrots per minute and 30 muffins per minute, what is the minimum number of minutes it will take them to finish the food?

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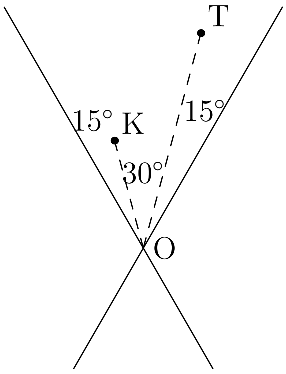
9. [**5**] Given $\triangle ABC$ with $AB < AC$, the altitude AD, angle bisector AE, and median AF are drawn from A, with D, E, F all lying on $\overline{BC}$. If $\measuredangle BAD = 2\measuredangle DAE = 2\measuredangle EAF = \measuredangle FAC$, what are all possible values of $\measuredangle ACB$?

10. [**5**] Let P be a polynomial such that $P(x) = P(0) + P(1)x + P(2)x^2$ and $P(-1) = 1$. Compute $P(3)$.

11. [**5**] Knot is on an epic quest to save the land of Hyruler from the evil Gammadorf. To do this, he must collect the two pieces of the Lineforce, then go to the Temple of Lime. As shown on the figure, Knot starts on point K, and must travel to point T, where $OK = 2$ and $OT = 4$. However, he must first reach both solid lines in the figure below to collect the pieces of the Lineforce. What is the minimal distance Knot must travel to do so?



12. [**5**] Knot is ready to face Gammadorf in a card game. In this game, there is a deck with twenty cards numbered from 1 to 20. Each player starts with a five card hand drawn from this deck. In each round, Gammadorf plays a card in his hand, then Knot plays a card in his hand. Whoever played a card with greater value gets a point. At the end of five rounds, the player with the most points wins. If Gammadorf starts with a hand of $1, 5, 10, 15, 20$, how many five-card hands of the fifteen remaining cards can Knot draw which always let Knot win (assuming he plays optimally)?

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13. [**7**] Niffy's favorite number is a positive integer, and Stebbysaurus is trying to guess what it is. Niffy tells her that when expressed in decimal without any leading zeros, her favorite number satisfies the following:

- Adding 1 to the number results in an integer divisible by 210.
- The sum of the digits of the number is twice its number of digits.
- The number has no more than 12 digits.
- The number alternates in even and odd digits.

Given this information, what are all possible values of Niffy's favorite number?

14. [**7**] Let triangle ABC have $AB = 5$, $BC = 6$, and $AC = 7$, with circumcenter O. Extend ray AB to point D such that $BD = 5$, and extend ray BC to point E such that $OD = OE$. Find CE.

15. [**7**] Let $f(x) = x^2 + ax + b$ and $g(x) = x^2 + cx + d$ be two distinct real polynomials such that the x-coordinate of the vertex of f is a root of g, the x-coordinate of the vertex of g is a root of f and both f and g have the same minimum value. If the graphs of the two polynomials intersect at the point $(2012, -2012)$, what is the value of $a + c$?

16. [**7**] Let A, B, C, and D be points randomly selected independently and uniformly within the unit square. What is the probability that the six lines $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, and $\overline{CD}$ all have positive slope?

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17. [**11**] Mark and William are playing a game. Two walls are placed 1 meter apart, with Mark and William each starting an orb at one of the walls. Simultaneously, they release their orbs directly toward the other. Both orbs are enchanted such that, upon colliding with each other, they instantly reverse direction and go at double their previous speed. Furthermore, Mark has enchanted his orb so that when it collides with a wall it instantly reverses direction and goes at double its previous speed (William's reverses direction at the same speed). Initially, Mark's orb is moving at $\frac{1}{1000}$ meters/s, and William's orb is moving at 1 meter/s. Mark wins when his orb passes the halfway point between the two walls. How fast, in meters/s, is his orb going when this first happens?

18. [**11**] Let x and y be positive real numbers such that $x^2 + y^2 = 1$ and $(3x - 4x^3)(3y - 4y^3) = -\frac{1}{2}$. Compute $x + y$.

19. [**11**] Given that P is a real polynomial of degree at most 2012 such that $P(n) = 2^n$ for $n = 1, 2, \ldots, 2012$, what choice(s) of $P(0)$ produce the minimal possible value of $P(0)^2 + P(2013)^2$?

20. [**11**] Let n be the maximum number of bishops that can be placed on the squares of a 6×6 chessboard such that no two bishops are attacking each other. Let k be the number of ways to put n bishops on an 6×6 chessboard such that no two bishops are attacking each other. Find $n + k$. (Two bishops are considered to be attacking each other if they lie on the same diagonal. Equivalently, if we label the squares with coordinates (x, y), with $1 \le x, y \le 6$, then the bishops on (a, b) and (c, d) are attacking each other if and only if $|a - c| = |b - d|$.)

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21. [**13**] Let N be a three-digit integer such that the difference between any two positive integer factors of N is divisible by 3. Let $d(N)$ denote the number of positive integers which divide N. Find the maximum possible value of $N \cdot d(N)$.

22. [**13**] For each positive integer n, there is a circle around the origin with radius n. Rainbow Dash starts off somewhere on the plane, but not on a circle. She takes off in some direction in a straight path. She moves $\frac{\sqrt{5}}{5}$ units before crossing a circle, then $\sqrt{5}$ units, then $\frac{3\sqrt{5}}{5}$ units. What distance will she travel before she crosses another circle?

23. [**13**] Points X and Y are inside a unit square. The score of a vertex of the square is the minimum distance from that vertex to X or Y. What is the minimum possible sum of the scores of the vertices of the square?

24. [**13**] Franklin has four bags, numbered 1 through 4. Initially, the first bag contains fifteen balls, numbered 1 through 15, and the other bags are empty. Franklin randomly pulls a pair of balls out of the first bag, throws away the ball with the lower number, and moves the ball with the higher number into the second bag. He does this until there is only one ball left in the first bag. He then repeats this process in the second and third bag until there is exactly one ball in each bag. What is the probability that ball 14 is in one of the bags at the end?

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25. [**17**] FemtoPravis is walking on an 8×8 chessboard that wraps around at its edges (so squares on the left edge of the chessboard are adjacent to squares on the right edge, and similarly for the top and bottom edges). Each femtosecond, FemtoPravis moves in one of the four diagonal directions uniformly at random. After 2012 femtoseconds, what is the probability that FemtoPravis is at his original location?

26. [**17**] Suppose ABC is a triangle with circumcenter O and orthocenter H such that A, B, C, O, and H are all on distinct points with integer coordinates. What is the second smallest possible value of the circumradius of ABC?

27. [**17**] Let S be the set $\{1, 2, \ldots, 2012\}$. A perfectutation is a bijective function h from S to itself such that there exists an $a \in S$ such that $h(a) \neq a$, and that for any pair of integers $a \in S$ and $b \in S$ such that $h(a) \neq a$, $h(b) \neq b$, there exists a positive integer k such that $h^k(a) = b$. Let n be the number of ordered pairs of perfectutations (f, g) such that $f(g(i)) = g(f(i))$ for all $i \in S$, but $f \neq g$. Find the remainder when n is divided by 2011.

28. [**17**] Alice is sitting in a teacup ride with infinitely many layers of spinning disks. The largest disk has radius 5. Each succeeding disk has its center attached to a point on the circumference of the previous disk and has a radius equal to 2/3 of the previous disk. Each disk spins around its center (relative to the disk it is attached to) at a rate of $\pi/6$ radians per second. Initially, at $t = 0$, the centers of the disks are aligned on a single line, going outward. Alice is sitting at the limit point of all these disks. After 12 seconds, what is the length of the trajectory that Alice has traced out?

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29. [**19**] Consider the cube whose vertices are the eight points (x, y, z) for which each of x, y, and z is either 0 or 1. How many ways are there to color its vertices black or white such that, for any vertex, if all of its neighbors are the same color then it is also that color? Two vertices are neighbors if they are the two endpoints of some edge of the cube.

30. [**19**] You have a twig of length 1. You repeatedly do the following: select two points on the twig independently and uniformly at random, make cuts on these two points, and keep only the largest piece. After 2012 repetitions, what is the expected length of the remaining piece?

31. [**19**] Let S_7 denote all the permutations of $1, 2, \ldots, 7$. For any $\pi \in S_7$, let $f(\pi)$ be the smallest positive integer i such that $\pi(1), \pi(2), \ldots, \pi(i)$ is a permutation of $1, 2, \ldots, i$. Compute $\sum_{\pi \in S_7} f(\pi)$.

32. [**19**] Let S be a set of size 3. How many collections T of subsets of S have the property that for any two subsets $U \in T$ and $V \in T$, both $U \cap V$ and $U \cup V$ are in T?

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33. [**23**] Compute the decimal expansion of $\sqrt{\pi}$. Your score will be $\min(23, k)$, where k is the number of consecutive correct digits immediately following the decimal point in your answer.

For each of the remaining three problems, it is difficult to obtain an exact answer. Instead, give an interval $[L, U]$, where L and U are positive real numbers *written in decimal.* If $[L, U]$ contains the answer to the problem, you will receive credit based off of how close L and U are. Otherwise, you will receive no credit.

34. [**23**] Let Q be the product of the sizes of all the non-empty subsets of $\{1, 2, \ldots, 2012\}$, and let $M = \log_2(\log_2(Q))$. Give lower and upper bounds L and U for M. If $0 < L \leq M \leq U$, then your score will be $\min(23, \left\lfloor \frac{23}{3(U-L)} \right\rfloor)$. Otherwise, your score will be 0.

35. [**23**] Let N be the number of distinct roots of $\prod_{k=1}^{2012}(x^k - 1)$. Give lower and upper bounds L and U on N. If $0 < L \leq N \leq U$, then your score will be $\left\lfloor \frac{23}{(U/L)^{1.7}} \right\rfloor$. Otherwise, your score will be 0.

36. [**23**] Maria is hopping up a flight of stairs with 100 steps. At every hop, she advances some integer number of steps. Each hop she makes has fewer steps. However, the positive difference between the length of consecutive hops decreases. Let P be the number of distinct ways she can hop up the stairs. Find lower and upper bounds L and U for P. If $0 < L \leq P \leq U$, your score will be $\left\lfloor \frac{23}{\sqrt{U/L}} \right\rfloor$. Otherwise, your score will be 0.

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School ____________________________ Team ____________________________ Team ID# __________

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School ____________________________ Team ____________________________ Team ID# __________

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15$^{\text{TH}}$ ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

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15$^{\text{TH}}$ ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

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15$^{\text{TH}}$ ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

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15$^{\text{TH}}$ ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 11 FEBRUARY 2012— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

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36. [**23**] ____________________

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Team A

1. [**20**] Let ABC be a triangle. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

2. [**20**]

   (a) For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for all $1 \le i \le n$, we have that exactly one of $f(g(i)) = i$ and $g(f(i)) = i$ holds?

   (b) What if f, g must be permutations?

3. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player somewhere on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any orthogonally adjacent square. Show that Bob has a winning strategy for this game.

   (b) Suppose instead that a token can legally move to any square which shares a vertex with the square it is currently on. Show that Alice has a winning strategy for this game.

4. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

5. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza: mushrooms, olives, green peppers, cheese, and pepperoni. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes 5 pizzas, each with some subset of the five toppings, then for each of those Arno tells her if that pizza has any toppings he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she can then make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

6. [**30**] It has recently been discovered that the right triangle with vertices (0,0), (0,2012), and (2012,0) is a giant pond home to many frogs. Frogs have the special ability that, if they are at a lattice point (x, y), they can hop to any of the three lattice points $(x+1, y+1)$, $(x-2, y+1)$, $(x+1, y-2)$, assuming the given lattice point lies in or on the boundary of the triangle.

   Frog Jeff starts at the corner $(0, 0)$, while Frog Kenny starts at the corner $(0, 2012)$. Show that the set of points Jeff can reach is equal in size to the set of points that Kenny can reach.

7. [**30**] Five points are chosen on a sphere of radius 1. What is the maximum possible volume of their convex hull?

8. [**30**] For integer $m, n \ge 1$, let $A(m, n)$ denote the number of functions $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ such that $f(j) - f(i) \ge j - i$ for all $1 \le i < j \le n$, and let $B(m, n)$ denote the number of functions $g : \{0, 1, \ldots, 2n + m\} \to \{0, 1, \ldots, m\}$ such that $g(0) = 0$, $g(2n + m) = m$, and $|g(i) - g(i - 1)| = 1$ for all $1 \le i \le 2n + m$. Prove that $A(m, n) = B(m, n)$.

   For the remaining problems, let $\varphi(k)$ denote the number of positive integers less than or equal to k that are relatively prime to k.

9. [**40**] For any positive integer n, let $N = \phi(1) + \phi(2) + ... + \phi(n)$. Show that there exists a sequence

$$a_1, a_2, ... a_N$$

containing exactly $\phi(k)$ instances of k for all positive integers $k \leq n$ such that

$$\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + ... + \frac{1}{a_N a_1} = 1.$$

10. [**40**] For positive odd integer n, let $f(n)$ denote the number of matrices A satisfying the following conditions:

- A is $n \times n$.
- Each row and column contains each of $1, 2, \ldots, n$ exactly once in some order.
- $A^T = A$. (That is, the element in row i and column j is equal to the one in row j and column i, for all $1 \leq i, j \leq n$.)

Prove that $f(n) \geq \frac{n!(n-1)!}{\varphi(n)}$, where φ is the totient function.

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Team B

For problems 1-5, only a short answer is required.

1. [**10**] Triangle ABC has $AB = 5$, $BC = 3\sqrt{2}$, and $AC = 1$. If the altitude from B to AC and the angle bisector of angle A intersect at at D, what is BD?

2. [**10**] You are given two line segments of length 2^n for each integer $0 \le n \le 10$. How many distinct nondegenerate triangles can you form with three of the segments? Two triangles are considered distinct if they aren't congruent.

3. [**10**] Mac is trying to fill 2012 barrels with apple cider. He starts with 0 energy. Every minute, he may rest, gaining 1 energy, or if he has n energy, he may expend k energy $(1 \le k \le n)$ to fill up to $n(k+1)$ barrels with cider. What is the minimal number of minutes he needs to fill all the barrels?

4. [**10**] A restaurant has some number of seats, arranged in a line. Its customers are in parties arranged in a queue. To seat its customers, the restaurant takes the next party in the queue and attempts to see all of the party's member(s) in a contiguous block of unoccupied seats. If one or more such blocks exist, then the restaurant places the party in an arbitrarily selected block; otherwise, the party leaves.

   Suppose the queue has parties of sizes $6, 4, 2, 5, 3, 1$ from front to back, and all seats are initially empty. What is the minimal number of seats the restaurant needs to guarantee that it will seat all of these customers?

5. [**10**] Steph and Jeff each start with the number 4, and Travis is flipping coins. Every time he flips a heads, Steph replaces her number x with $2x - 1$, and Jeff replaces his number y with $y + 8$. Every time he flips a tails, Steph replaces her number x with $\frac{x+1}{2}$, and Jeff replaces his number y with $y - 3$. After some (positive) number of coin flips, Steph and Jeff miraculously end up with the same number below 2012. How many times was a coin flipped?

For problems 6-10, justification is required.

6. [**20**] Let ABC be a triangle. Let the angle bisector of $\angle A$ and the perpendicular bisector of BC intersect at D. Then let E and F be points on AB and AC such that DE and DF are perpendicular to AB and AC, respectively. Prove that $BE = CF$.

7. [**20**]

   (a) For what positive integers n do there exist functions $f, g : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that for all $1 \le i \le n$, we have that exactly one of $f(g(i)) = i$ and $g(f(i)) = i$ holds?

   (b) What if f, g must be permutations?

8. [**20**] Alice and Bob are playing a game of Token Tag, played on an 8×8 chessboard. At the beginning of the game, Bob places a token for each player somewhere on the board. After this, in every round, Alice moves her token, then Bob moves his token. If at any point in a round the two tokens are on the same square, Alice immediately wins. If Alice has not won by the end of 2012 rounds, then Bob wins.

   (a) Suppose that a token can legally move to any orthogonally adjacent square. Show that Bob has a winning strategy for this game.

   (b) Suppose instead that a token can legally move to any square which shares a vertex with the square it is currently on. Show that Alice has a winning strategy for this game.

9. [**20**] Let ABC be a triangle with $AB < AC$. Let M be the midpoint of BC. Line l is drawn through M so that it is perpendicular to AM, and intersects line AB at point X and line AC at point Y. Prove that $\angle BAC = 90^\circ$ if and only if quadrilateral $XBYC$ is cyclic.

10. [**20**] Purineqa is making a pizza for Arno. There are five toppings that she can put on the pizza: mushrooms, olives, green peppers, cheese, and pepperoni. However, Arno is very picky and only likes some subset of the five toppings. Purineqa makes 5 pizzas, each with some subset of the five toppings, then for each of those Arno tells her if that pizza has any toppings he does not like. Purineqa chooses these pizzas such that no matter which toppings Arno likes, she can then make him a sixth pizza with all the toppings he likes and no others. What are all possible combinations of the five initial pizzas for this to be the case?

GENERAL TEST

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems are unequally weighted with point values as shown in brackets on the answer form. The maximum possible score is 50 points. There is no point penalty for guessing, though in the case of a tie it is slightly more advantageous not to answer than to answer incorrectly.

No translators, books, notes, slide rules, calculators, abaci, or other computational aids are permitted. Similarly, graph paper, rulers, protractors, compasses, and other drawing aids are not permitted.

Answers should be simplified as much as is reasonably possible and must be exact unless otherwise specified. Rational numbers should be written in lowest terms, although denominators of irrationals need not be rationalized. An nth root should be simplified so that the radicand is not divisible by the nth power of any prime.

Correct mathematical notation must be used. No partial credit will be given unless otherwise specified.

If you believe the test contains an error, please submit your protest in writing to the Science Center Lobby during lunchtime.

Enjoy!

HMMT November 2012

Saturday 10 November 2012

General Test

1. **[3]** What is the sum of all of the distinct prime factors of $25^3 - 27^2$?

2. **[3]** Let $Q(x) = x^2 + 2x + 3$, and suppose that $P(x)$ is a polynomial such that

$$P(Q(x)) = x^6 + 6x^5 + 18x^4 + 32x^3 + 35x^2 + 22x + 8.$$

Compute $P(2)$.

3. **[3]** $ABCD$ is a rectangle with $AB = 20$ and $BC = 3$. A circle with radius 5, centered at the midpoint of DC, meets the rectangle at four points: W, X, Y, and Z. Find the area of quadrilateral $WXYZ$.

4. **[4]** If you roll four fair 6-sided dice, what is the probability that at least three of them will show the same value?

5. **[4]** How many ways are there to arrange three indistinguishable rooks on a 6×6 board such that no two rooks are attacking each other? (Two rooks are attacking each other if and only if they are in the same row or the same column.)

6. **[5]** $ABCD$ is a parallelogram satisfying $AB = 7$, $BC = 2$, and $\angle DAB = 120°$. Parallelogram $ECFA$ is contained in $ABCD$ and is similar to it. Find the ratio of the area of $ECFA$ to the area of $ABCD$.

7. **[6]** Find the number of ordered 2012-tuples of integers $(x_1, x_2, \ldots, x_{2012})$, with each integer between 0 and 2011 inclusive, such that the sum $x_1 + 2x_2 + 3x_3 + \cdots + 2012x_{2012}$ is divisible by 2012.

8. **[7]** Let n be the 200th smallest positive real solution to the equation $x - \frac{\pi}{2} = \tan x$. Find the greatest integer that does not exceed $\frac{n}{2}$.

9. **[7]** Consider triangle ABC where $BC = 7$, $CA = 8$, and $AB = 9$. D and E are the midpoints of BC and CA, respectively, and AD and BE meet at G. The reflection of G across D is G', and $G'E$ meets CG at P. Find the length PG.

10. **[8]** Let α and β be reals. Find the least possible value of

$$(2\cos\alpha + 5\sin\beta - 8)^2 + (2\sin\alpha + 5\cos\beta - 15)^2.$$

HMMT November 2012
Saturday 10 November 2012

General Test

Name ______________________________ Team ID# ________

School ______________________ Team ______________________

1. [**3**] ______________
2. [**3**] ______________
3. [**3**] ______________
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9. [**7**] ______________
10. [**8**] ______________

Score: ____________

HMMT November 2012

Saturday 10 November 2012

Guts Round

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1. [**5**] Find the number of prime numbers less than 30.

2. [**5**] Albert is very hungry, and goes to his favorite burger shop to buy a meal, which consists of a burger, a side, and a drink. Given that there are 5 different types of burgers offered, 3 different types of sides, and 12 different types of drinks, find the number of meals Albert could get.

3. [**5**] Find the area of the region between two concentric circles that have radii 100 and 99.

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4. [**6**] $ABCD$ is a concave quadrilateral such that $\angle CBA = 50^\circ$, $\angle BAD = 80^\circ$, $\angle ADC = 30^\circ$, and $CB = CD$. Find $\angle CBD$.

5. [**6**] a and b are complex numbers such that $2a + 3b = 10$ and $4a^2 + 9b^2 = 20$. Find ab.

6. [**6**] Given the following formulas:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$
$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2,$$

find

$$(1^3 + 3 \cdot 1^2 + 3 \cdot 1) + (2^3 + 3 \cdot 2^2 + 3 \cdot 2) + \cdots + (99^3 + 3 \cdot 99^2 + 3 \cdot 99).$$

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7. [**7**] Consider the sequence given by $a_0 = 1, a_1 = 1 + 3, a_2 = 1 + 3 + 3^2, a_3 = 1 + 3 + 3^2 + 3^3, \ldots$. Find the number of terms among $a_0, a_1, a_2, \ldots, a_{2012}$ that are divisible by 7.

8. [**7**] Three cards are drawn from the top of a shuffled standard 52-card deck. Find the probability that they are all of different suits. (A suit is either spades, clubs, hearts, or diamonds.)

9. [**7**] How many sets consist of distinct composite numbers that add up to 23?

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

10. [**8**] Consider the sequence defined by $a_0 = 0, a_1 = 1$, and $a_n = 2013a_{n-1} + 2012a_{n-2} + n$ for $n \geq 2$. Find the remainder when a_{2012} is divided by 2012.

11. [**8**] Find the smallest positive integer n such that the number of zeroes that $n!$ ends with is a positive multiple of 5.

12. [**8**] Three circles of radius 1 are drawn, whose centers form the vertices of an equilateral triangle of side length 1. Find the area of the region common to at least two of the circles.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

13. [**9**] Triangle ABC has a perimeter of 18. $\angle A = 60°$, and $BC = 7$. Find the area of the triangle.

14. [**9**] Mark would like to place the numbers 1 to 7 on a circle such that the sequences along both arcs going from 1 to 7 are increasing. For example, one arc could be 1, 2, 4, 7 and the other could be 1, 3, 5, 6, 7. In how many distinct ways can Mark place the numbers? Two arrangements are distinct if and only if one cannot be rotated to match the other.

15. [**9**] Find the area of the region in the xy-plane consisting of all points (a, b) such that the quadratic $ax^2 + 2(a + b - 7)x + 2b = 0$ has fewer than two real solutions for x.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

16. [**10**] Yuhang is making a bracelet for his one true love using beads. How many distinct bracelets can be made from 2 red beads, 2 green beads, and 2 blue beads, if two bracelets are distinct if and only if one cannot be made into the other through rotations and reflections?

17. [**10**] Given that $x = \ln 30$, $y = \ln 360$, and $z = \ln 270$, and that there are rational numbers p, q, r such that $\ln 5400 = px + qy + rz$, find the ordered triple (p, q, r).

18. [**10**] Let $\triangle ABC$ be a triangle with $AC = 1$ and $\angle ABC$ obtuse. Let D and E be points on AC such that $\angle DBC = \angle ABE = 90°$. If $AD = DE = EC$, find $AB + AC$.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

19. [**11**] Find the number of triples of nonnegative integers (x, y, z) such that $15x + 21y + 35z = 525$.

20. [**11**] An elementary school teacher is taking a class of 20 students on a field trip. To make sure her students don't get lost, she uses the buddy system—using the complete class roster, she pairs the students into 10 pairs and leaves if no person reports that someone from his or her pair is missing. In particular, the teacher will not notice if both students from a pair go missing. Suppose each student independently gets lost with probability $\frac{1}{10}$. Given that the teacher leaves, what is the probability that no student got lost?

21. [**11**] ABC is a triangle with $AB = 7$, $BC = 10$, and $CA = 13$. Point D lies on segment BC such that $DC = 2BD$. Point E lies on segment AD such that $AE = 4ED$. Point F lies on segment AC such that $FC = 5AF$. Find the area of $\triangle EFC$.

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22. [**12**] Alice generates an infinite decimal by rolling a fair 6-sided die with the numbers 1, 2, 3, 5, 7, and 9 infinitely many times, and appending the resulting numbers after a decimal point. What is the expected value of her number?

23. [**12**] ABC is a triangle with $AB = 4$, $BC = 6$, and $CA = 7$. Let Ω be the circumcircle of ABC; the tangent to Ω passing through A meets the extension of side BC at P. Find the length PB.

24. [**12**] 12 children sit around a circle. Find the number of ways we can distribute 4 pieces of candy among them such that each child gets at most one piece of candy and no two adjacent children both get a piece of candy.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

25. [**13**] Triangle ABC satisfies $AB = 8$, $BC = 9$, and $CA = 10$. Evaluate $\dfrac{\sin^2 B + \sin^2 C - \sin^2 A}{\sin B \cdot \sin C}$.

26. [**13**] $x_1, x_2, x_3, \ldots$ is a sequence of real numbers satisfying $x_1 = 1$, $x_2 = 2$, and $x_{n+1} = 2x_n - x_{n-1} + 2^n$ for $n \geq 2$. Find x_{2012}.

27. [**13**] Find the number of ordered triples of positive integers (a, b, c) satisfying $a^2 - b^2 + ac - bc = 2012$.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

28. [**15**] Find the set of all possible values that can be attained by the expression $\dfrac{ab + b^2}{a^2 + b^2}$, where a and b are positive real numbers. Express your answer in interval notation.

29. [**15**] Find the sum of the real values of x satisfying $(x + 1)(2x + 1)(3x + 1)(4x + 1) = 16x^4$.

30. [**15**] A monkey forms a string of letters by repeatedly choosing one of the letters a, b, or c to type at random. Find the probability that he first types the string aaa before he first types the string abc.

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HMMT NOVEMBER 2012, 10 NOVEMBER 2012 — GUTS ROUND

31. [**17**] Let $ABCD$ be a regular tetrahedron with $AB = 1$, and let P be the center of face BCD. Let M be the midpoint of segment AP. Ray BM meets face ACD at Q. Determine the area of triangle QDC.

32. [**17**] Define $f(n)$ to be the remainder when n^{n^n} is divided by 23 for each positive integer n. Find the smallest positive integer k such that $f(n + k) = f(n)$ for all positive integers n.

33. [**17**] You are playing pool on a 1×1 table, and the cue ball is at the bottom left corner of the square. Find the smallest angle larger than $45°$ at which you could shoot the ball such that the ball bounces against walls exactly 2012 times before arriving at a vertex of the square.

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34. [**20**] For a positive integer n, let $\tau(n)$ be the number of divisors of n. Determine $\sum_{n=1}^{2012} \tau(n)$. Your score will be $\max\left\{0, \left\lfloor 20\left(1 - \frac{|S-k|}{S}\right)^3 \right\rfloor\right\}$, where k is your answer and S is the actual answer.

35. [**20**] A regular 2012-gon has a circumcircle with radius 1. Compute the area of the 2012-gon. Your score will be $\min(20, \lfloor \frac{k^2}{5} \rfloor)$, where k is the number of consecutive correct digits immediately following the decimal point of your answer.

36. [**20**] Let π_1 and π_2 be permutations of the numbers from 1 through 7. Call π_1 *superior to* π_2 if the sum of all i such that $\pi_1(i) > \pi_2(i)$ exceeds the sum of all i such that $\pi_2(i) > \pi_1(i)$. Write down a permutation of the integers from 1 through 7. Let N be the total number of answers submitted for this problem, and let n be the number of submitted answers your answer is superior to. Your score will be $\lceil 20\frac{n}{N} \rceil$.

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HMMT November 2012

Saturday 10 November 2012

Team Round

Divisors

In this section, the word *divisor* is used to refer to a *positive* divisor of an integer; that is, a divisor of a positive integer n is a positive integer d such that $\frac{n}{d}$ is an integer.

1. [**3**] Find the number of integers between 1 and 200 inclusive whose distinct prime divisors sum to 16. (For example, the sum of the distinct prime divisors of 12 is $2 + 3 = 5$.)

2. [**5**] Find the number of ordered triples of divisors (d_1, d_2, d_3) of 360 such that $d_1 d_2 d_3$ is also a divisor of 360.

3. [**6**] Find the largest integer less than 2012 all of whose divisors have at most two 1's in their binary representations.

Permutations

A *permutation* π is defined as a function from a set of integers to itself that rearranges the elements of the set. For example, a possible permutation of the numbers from 1 through 4 is the function π given by $\pi(1) = 2$, $\pi(2) = 4$, $\pi(3) = 3$, $\pi(4) = 1$.

4. [**3**] Let π be a permutation of the numbers from 2 through 2012. Find the largest possible value of $\log_2 \pi(2) \cdot \log_3 \pi(3) \cdots \log_{2012} \pi(2012)$.

5. [**4**] Let π be a randomly chosen permutation of the numbers from 1 through 2012. Find the probability that $\pi(\pi(2012)) = 2012$.

6. [**6**] Let π be a permutation of the numbers from 1 through 2012. What is the maximum possible number of integers n with $1 \le n \le 2011$ such that $\pi(n)$ divides $\pi(n+1)$?

7. [**8**] Let $A_1 A_2 \ldots A_{100}$ be the vertices of a regular 100-gon. Let π be a randomly chosen permutation of the numbers from 1 through 100. The segments $A_{\pi(1)}A_{\pi(2)}, A_{\pi(2)}A_{\pi(3)}, \ldots, A_{\pi(99)}A_{\pi(100)}, A_{\pi(100)}A_{\pi(1)}$ are drawn. Find the expected number of pairs of line segments that intersect at a point in the interior of the 100-gon.

Circumcircles

The *circumcircle* of a triangle is the circle passing through all three vertices of the triangle.

8. [**4**] ABC is a triangle with $AB = 15$, $BC = 14$, and $CA = 13$. The altitude from A to BC is extended to meet the circumcircle of ABC at D. Find AD.

9. [**5**] Triangle ABC satisfies $\angle B > \angle C$. Let M be the midpoint of BC, and let the perpendicular bisector of BC meet the circumcircle of $\triangle ABC$ at a point D such that points A, D, C, and B appear on the circle in that order. Given that $\angle ADM = 68^\circ$ and $\angle DAC = 64^\circ$, find $\angle B$.

10. [**6**] Triangle ABC has $AB = 4$, $BC = 5$, and $CA = 6$. Points A', B', C' are such that $B'C'$ is tangent to the circumcircle of $\triangle ABC$ at A, $C'A'$ is tangent to the circumcircle at B, and $A'B'$ is tangent to the circumcircle at C. Find the length $B'C'$.

THEME ROUND

This test consists of 10 short-answer problems to be solved individually in 50 minutes. Problems are unequally weighted with point values as shown in brackets on the answer form. The maximum possible score is 50 points. There is no point penalty for guessing, though in the case of a tie it is slightly more advantageous not to answer than to answer incorrectly.

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Enjoy!

HMMT November 2012

Saturday 10 November 2012

Theme Round

Power Towers

In a tower of exponents, the exponents are computed from the top down. For example, $2^{2^3} = 2^8 = 256$.

1. [**3**] If $4^{4^4} = \sqrt[128]{2^{2^{2^n}}}$, find n.

2. [**4**] If $x^x = 2012^{2012^{2013}}$, find x.

3. [**4**] Find the smallest positive integer n such that $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n} > 3^{3^{3^3}}$. (The notation $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ is used to denote a power tower with n 2's. For example, $\underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$ with $n = 4$ would equal $2^{2^{2^2}}$.)

4. [**7**] Find the sum of all real solutions for x to the equation $(x^2+2x+3)^{(x^2+2x+3)^{(x^2+2x+3)}} = 2012$.

5. [**7**] Given any positive integer, we can write the integer in base 12 and add together the digits of its base 12 representation. We perform this operation on the number $7^{6^{5^{4^{3^{2^{1}}}}}}$ repeatedly until a single base 12 digit remains. Find this digit.

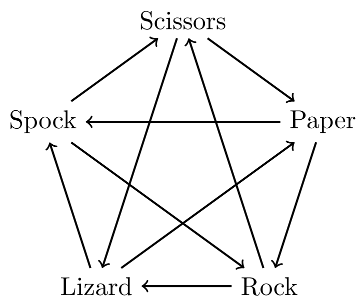
Rock-paper-scissors

In the game of rock-paper-scissors, two players each choose one of rock, paper, or scissors to play. Rock beats scissors, scissors beats paper, and paper beats rock. If the players play the same thing, the match is considered a draw.

6. [**3**] A rectangular piece of paper with vertices $ABCD$ is being cut by a pair of scissors. The pair of scissors starts at vertex A, and then cuts along the angle bisector of DAB until it reaches another edge of the paper. One of the two resulting pieces of paper has 4 times the area of the other piece. What is the ratio of the longer side of the original paper to the shorter side?

7. [**4**] The game of rock-scissors is played just like rock-paper-scissors, except that neither player is allowed to play paper. You play against a poorly-designed computer program that plays rock with 50% probability and scissors with 50% probability. If you play optimally against the computer, find the probability that after 8 games you have won at least 4.

8. [**4**] In the game of rock-paper-scissors-lizard-Spock, rock defeats scissors and lizard, paper defeats rock and Spock, scissors defeats paper and lizard, lizard defeats paper and Spock, and Spock defeats rock and scissors, as shown in the below diagram. As before, if two players choose the same move, then there is a draw. If three people each play a game of rock-paper-scissors-lizard-Spock at the same time by choosing one of the five moves at random, what is the probability that one player beats the other two?



9. [**6**] 64 people are in a single elimination rock-paper-scissors tournament, which consists of a 6-round knockout bracket. Each person has a different rock-paper-scissors skill level, and in any game, the person with the higher skill level will always win. For how many players P is it possible that P wins the first four rounds that he plays?

   (A *6-round knockout bracket* is a tournament which works as follows:

   (a) In the first round, all 64 competitors are paired into 32 groups, and the two people in each group play each other. The winners advance to the second round, and the losers are eliminated.

   (b) In the second round, the remaining 32 players are paired into 16 groups. Again, the winner of each group proceeds to the next round, while the loser is eliminated.

   (c) Each round proceeds in a similar way, eliminating half of the remaining players. After the sixth round, only one player will not have been eliminated. That player is declared the champion.)

10. [**8**] In a game of rock-paper-scissors with n people, the following rules are used to determine a champion:

   (a) In a round, each person who has not been eliminated randomly chooses one of rock, paper, or scissors to play.

   (b) If at least one person plays rock, at least one person plays paper, and at least one person plays scissors, then the round is declared a tie and no one is eliminated. If everyone makes the same move, then the round is also declared a tie.

   (c) If exactly two moves are represented, then everyone who made the losing move is eliminated from playing in all further rounds (for example, in a game with 8 people, if 5 people play rock and 3 people play scissors, then the 3 who played scissors are eliminated).

   (d) The rounds continue until only one person has not been eliminated. That person is declared the champion and the game ends.

   If a game begins with 4 people, what is the expected value of the number of rounds required for a champion to be determined?

HMMT November 2012

Saturday 10 November 2012

Theme Round

Name ______________________________ Team ID# ________

School ______________________ Team ______________________

1. [**3**] ______________
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3. [**4**] ______________
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8. [**4**] ______________
9. [**6**] ______________
10. [**8**] ______________

Score: ____________

14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

1. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

2. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(0) = 0$, $f(1) = 1$, and $|f'(x)| \le 2$ for all real numbers x. If a and b are real numbers such that the set of possible values of $\int_0^1 f(x)\,dx$ is the open interval (a, b), determine $b - a$.

4. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

5. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

6. Let $a \star b = ab + a + b$ for all integers a and b. Evaluate $1 \star (2 \star (3 \star (4 \star \ldots (99 \star 100) \ldots)))$.

7. Let $f : [0, 1) \to \mathbb{R}$ be a function that satisfies the following condition: if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{10^n} = .a_1a_2a_3\ldots$$

is the decimal expansion of x and there does not exist a positive integer k such that $a_n = 9$ for all $n \ge k$, then

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{10^{2n}}.$$

Determine $f'\left(\frac{1}{3}\right)$.

8. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

9. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

10. For all real numbers x, let

$$f(x) = \frac{1}{\sqrt[2011]{1 - x^{2011}}}.$$

Evaluate $(f(f(\ldots(f(2011))\ldots)))^{2011}$, where f is applied 2010 times.

11. Evaluate $\displaystyle\int_1^{\infty} \left(\frac{\ln x}{x}\right)^{2011} dx$.

12. Let $f(x) = x^2 + 6x + c$ for all real numbers x, where c is some real number. For what values of c does $f(f(x))$ have exactly 3 distinct real roots?

13. Sarah and Hagar play a game of darts. Let O_0 be a circle of radius 1. On the nth turn, the player whose turn it is throws a dart and hits a point p_n randomly selected from the points of O_{n-1}. The player then draws the largest circle that is centered at p_n and contained in O_{n-1}, and calls this circle O_n. The player then colors every point that is inside O_{n-1} but not inside O_n her color. Sarah goes first, and the two players alternate turns. Play continues indefinitely. If Sarah's color is red, and Hagar's color is blue, what is the expected value of the area of the set of points colored red?

14. How many polynomials P with integer coefficients and degree at most 5 satisfy $0 \leq P(x) < 120$ for all $x \in \{0, 1, 2, 3, 4, 5\}$?

15. Let $f : [0, 1] \to [0, 1]$ be a continuous function such that $f(f(x)) = 1$ for all $x \in [0, 1]$. Determine the set of possible values of $\displaystyle\int_0^1 f(x)\, dx$.

16. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

17. Let $f : (0, 1) \to (0, 1)$ be a differentiable function with a continuous derivative such that for every positive integer n and odd positive integer $a < 2^n$, there exists an odd positive integer $b < 2^n$ such that $f\left(\frac{a}{2^n}\right) = \frac{b}{2^n}$. Determine the set of possible values of $f'\left(\frac{1}{2}\right)$.

18. Let $z = \cos \dfrac{2\pi}{2011} + i \sin \dfrac{2\pi}{2011}$, and let

$$P(x) = x^{2008} + 3x^{2007} + 6x^{2006} + \ldots \frac{2008 \cdot 2009}{2} x + \frac{2009 \cdot 2010}{2}$$

for all complex numbers x. Evaluate $P(z)P(z^2)P(z^3) \ldots P(z^{2010})$.

19. Let

$$F(x) = \frac{1}{(2 - x - x^5)^{2011}},$$

and note that F may be expanded as a power series so that $F(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Find an ordered pair of positive real numbers (c, d) such that $\displaystyle\lim_{n\to\infty} \frac{a_n}{n^d} = c$.

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7. Find all integers x such that $2x^2 + x - 6$ is a positive integral power of a prime positive integer.

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1. A classroom has 30 students and 30 desks arranged in 5 rows of 6. If the class has 15 boys and 15 girls, in how many ways can the students be placed in the chairs such that no boy is sitting in front of, behind, or next to another boy, and no girl is sitting in front of, behind, or next to another girl?

2. Let ABC be a triangle such that $AB = 7$, and let the angle bisector of $\angle BAC$ intersect line BC at D. If there exist points E and F on sides AC and BC, respectively, such that lines AD and EF are parallel and divide triangle ABC into three parts of equal area, determine the number of possible integral values for BC.

3. Let a, b, and c be positive real numbers. Determine the largest total number of real roots that the following three polynomials may have among them: $ax^2 + bx + c$, $bx^2 + cx + a$, and $cx^2 + ax + b$.

4. Josh takes a walk on a rectangular grid of n rows and 3 columns, starting from the bottom left corner. At each step, he can either move one square to the right or simultaneously move one square to the left and one square up. In how many ways can he reach the center square of the topmost row?

5. Let H be a regular hexagon of side length x. Call a hexagon in the same plane a "distortion" of H if and only if it can be obtained from H by translating each vertex of H by a distance strictly less than 1. Determine the smallest value of x for which every distortion of H is necessarily convex.

6. Nathaniel and Obediah play a game in which they take turns rolling a fair six-sided die and keep a running tally of the sum of the results of all rolls made. A player wins if, after he rolls, the number on the running tally is a multiple of 7. Play continues until either player wins, or else indefinitely. If Nathaniel goes first, determine the probability that he ends up winning.

7. Let $ABCDEF$ be a regular hexagon of area 1. Let M be the midpoint of DE. Let X be the intersection of AC and BM, let Y be the intersection of BF and AM, and let Z be the intersection of AC and BF. If $[P]$ denotes the area of polygon P for any polygon P in the plane, evaluate $[BXC] + [AYF] + [ABZ] - [MXZY]$.

8. The integers from 1 to n are written in increasing order from left to right on a blackboard. David and Goliath play the following game: starting with David, the two players alternate erasing any two consecutive numbers and replacing them with their sum or product. Play continues until only one number on the board remains. If it is odd, David wins, but if it is even, Goliath wins. Find the 2011th smallest positive integer greater than 1 for which David can guarantee victory.

9. Let $ABCD$ be a square of side length 13. Let E and F be points on rays AB and AD, respectively, so that the area of square $ABCD$ equals the area of triangle AEF. If EF intersects BC at X and $BX = 6$, determine DF.

10. Mike and Harry play a game on an 8×8 board. For some positive integer k, Mike chooses k squares and writes an M in each of them. Harry then chooses $k + 1$ squares and writes an H in each of them. After Harry is done, Mike wins if there is a sequence of letters forming "HMM" or "MMH," when read either horizontally or vertically, and Harry wins otherwise. Determine the smallest value of k for which Mike has a winning strategy.

11. Let $ABCDEF$ be a convex equilateral hexagon such that lines BC, AD, and EF are parallel. Let H be the orthocenter of triangle ABD. If the smallest interior angle of the hexagon is 4 degrees, determine the smallest angle of the triangle HAD in degrees.

12. The ordered pairs $(2011, 2), (2010, 3), (2009, 4), \ldots, (1008, 1005), (1007, 1006)$ are written from left to right on a blackboard. Every minute, Elizabeth selects a pair of adjacent pairs (x_i, y_i) and (x_j, y_j), with (x_i, y_i) left of (x_j, y_j), erases them, and writes $\left(\frac{x_i y_i x_j}{y_j}, \frac{x_i y_i y_j}{x_j}\right)$ in their place. Elizabeth continues this process until only one ordered pair remains. How many possible ordered pairs (x, y) could appear on the blackboard after the process has come to a conclusion?

13. Let $ABCD$ be a cyclic quadrilateral, and suppose that $BC = CD = 2$. Let I be the incenter of triangle ABD. If $AI = 2$ as well, find the minimum value of the length of diagonal BD.

14. Let $A = \{1, 2, \ldots, 2011\}$. Find the number of functions f from A to A that satisfy $f(n) \leq n$ for all n in A and attain exactly 2010 distinct values.

15. Let $f(x) = x^2 - r_2 x + r_3$ for all real numbers x, where r_2 and r_3 are some real numbers. Define a sequence $\{g_n\}$ for all nonnegative integers n by $g_0 = 0$ and $g_{n+1} = f(g_n)$. Assume that $\{g_n\}$ satisfies the following three conditions: (i) $g_{2i} < g_{2i+1}$ and $g_{2i+1} > g_{2i+2}$ for all $0 \leq i \leq 2011$; (ii) there exists a positive integer j such that $g_{i+1} > g_i$ for all $i > j$, and (iii) $\{g_n\}$ is unbounded. If A is the greatest number such that $A \leq |r_2|$ for any function f satisfying these properties, find A.

16. Let $ABCD$ be a quadrilateral inscribed in the unit circle such that $\angle BAD$ is 30 degrees. Let m denote the minimum value of $CP + PQ + CQ$, where P and Q may be any points lying along rays AB and AD, respectively. Determine the maximum value of m.

17. Let n be an odd positive integer, and suppose that n people sit on a committee that is in the process of electing a president. The members sit in a circle, and every member votes for the person either to his/her immediate left, or to his/her immediate right. If one member wins more votes than all the other members do, he/she will be declared to be the president; otherwise, one of the the members who won at least as many votes as all the other members did will be randomly selected to be the president. If Hermia and Lysander are two members of the committee, with Hermia sitting to Lysander's left and Lysander planning to vote for Hermia, determine the probability that Hermia is elected president, assuming that the other $n - 1$ members vote randomly.

18. Collinear points A, B, and C are given in the Cartesian plane such that $A = (a, 0)$ lies along the x-axis, B lies along the line $y = x$, C lies along the line $y = 2x$, and $AB/BC = 2$. If $D = (a, a)$, the circumcircle of triangle ADC intersects $y = x$ again at E, and ray AE intersects $y = 2x$ at F, evaluate AE/EF.

19. Alice and Bob play a game in which two thousand and eleven 2011×2011 grids are distributed between the two of them, 1 to Bob, and the other 2010 to Alice. They go behind closed doors and fill their grid(s) with the numbers $1, 2, \ldots, 2011^2$ so that the numbers across rows (left-to-right) and down columns (top-to-bottom) are strictly increasing. No two of Alice's grids may be filled identically. After the grids are filled, Bob is allowed to look at Alice's grids and then swap numbers on his own grid, two at a time, as long as the numbering remains legal (i.e. increasing across rows and down columns) after each swap. When he is done swapping, a grid of Alice's is selected at random. If there exist two integers in the same column of this grid that occur in the same row of Bob's grid, Bob wins. Otherwise, Alice wins. If Bob selects his initial grid optimally, what is the maximum number of swaps that Bob may need in order to guarantee victory?

20. Let ω_1 and ω_2 be two circles that intersect at points A and B. Let line l be tangent to ω_1 at P and to ω_2 at Q so that A is closer to PQ than B. Let points R and S lie along rays PA and QA, respectively, so that $PQ = AR = AS$ and R and S are on opposite sides of A as P and Q. Let O be the circumcenter of triangle ASR, and let C and D be the midpoints of major arcs AP and AQ, respectively. If $\angle APQ$ is 45 degrees and $\angle AQP$ is 30 degrees, determine $\angle COD$ in degrees.

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1. [**4**] Let ABC be a triangle with area 1. Let points D and E lie on AB and AC, respectively, such that DE is parallel to BC and $DE/BC = 1/3$. If F is the reflection of A across DE, find the area of triangle FBC.

2. [**4**] Let $a \star b = \sin a \cos b$ for all real numbers a and b. If x and y are real numbers such that $x \star y - y \star x = 1$, what is the maximum value of $x \star y + y \star x$?

3. [**4**] Evaluate $2011 \times 20122012 \times 201320132013 - 2013 \times 20112011 \times 201220122012$.

4. [**4**] Let p be the answer to this question. If a point is chosen uniformly at random from the square bounded by $x = 0$, $x = 1$, $y = 0$, and $y = 1$, what is the probability that at least one of its coordinates is greater than p?

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5. [**5**] Rachelle picks a positive integer a and writes it next to itself to obtain a new positive integer b. For instance, if $a = 17$, then $b = 1717$. To her surprise, she finds that b is a multiple of a^2. Find the product of all the possible values of $\frac{b}{a^2}$.

6. [**5**] Square $ABCD$ is inscribed in circle ω with radius 10. Four additional squares are drawn inside ω but outside $ABCD$ such that the lengths of their diagonals are as large as possible. A sixth square is drawn by connecting the centers of the four aforementioned small squares. Find the area of the sixth square.

7. [**6**] For any positive real numbers a and b, define $a \circ b = a + b + 2\sqrt{ab}$. Find all positive real numbers x such that $x^2 \circ 9x = 121$.

8. [**6**] Find the smallest k such that for any arrangement of 3000 checkers in a 2011×2011 checkerboard, with at most one checker in each square, there exist k rows and k columns for which every checker is contained in at least one of these rows or columns.

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9. [**6**] Segments AA', BB', and CC', each of length 2, all intersect at a point O. If $\angle AOC' = \angle BOA' = \angle COB' = 60^\circ$, find the maximum possible value of the sum of the areas of triangles AOC', BOA', and COB'.

10. [**6**] In how many ways can one fill a 4×4 grid with a 0 or 1 in each square such that the sum of the entries in each row, column, and long diagonal is even?

11. [**8**] Rosencrantz and Guildenstern play a game in which they repeatedly flip a fair coin. Let $a_1 = 4$, $a_2 = 3$, and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 3$. On the nth flip, if the coin is heads, Rosencrantz pays Guildenstern a_n dollars, and, if the coin is tails, Guildenstern pays Rosencrantz a_n dollars. If play continues for 2010 turns, what is the probability that Rosencrantz ends up with more money than he started with?

12. [**8**] A sequence of integers $\{a_i\}$ is defined as follows: $a_i = i$ for all $1 \leq i \leq 5$, and $a_i = a_1 a_2 \cdots a_{i-1} - 1$ for all $i > 5$. Evaluate $a_1 a_2 \cdots a_{2011} - \sum_{i=1}^{2011} a_i^2$.

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13. [**8**] Let a,b, and c be the side lengths of a triangle, and assume that $a \le b$ and $a \le c$. Let $x = \frac{b+c-a}{2}$. If r and R denote the inradius and circumradius, respectively, find the minimum value of $\frac{ax}{rR}$.

14. [**8**] Danny has a set of 15 pool balls, numbered $1, 2, \ldots, 15$. In how many ways can he put the balls in 8 indistinguishable bins such that the sum of the numbers of the balls in each bin is 14, 15, or 16?

15. [**10**] Find all irrational numbers x such that $x^3 - 17x$ and $x^2 + 4x$ are both rational numbers.

16. [**10**] Let R be a semicircle with diameter XY. A trapezoid $ABCD$ in which AB is parallel to CD is circumscribed about R such that AB contains XY. If $AD = 4$, $CD = 5$, and $BC = 6$, determine AB.

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17. [**10**] Given positive real numbers x, y, and z that satisfy the following system of equations:

$$\begin{aligned} x^2 + y^2 + xy &= 1, \\ y^2 + z^2 + yz &= 4, \\ z^2 + x^2 + zx &= 5, \end{aligned}$$

find $x + y + z$.

18. [**10**] In how many ways can each square of a 4×2011 grid be colored red, blue, or yellow such that no two squares that are diagonally adjacent are the same color?

19. [**12**] Find the least positive integer N with the following property: If all lattice points in $[1, 3] \times [1, 7] \times [1, N]$ are colored either black or white, then there exists a rectangular prism, whose faces are parallel to the xy, xz, and yz planes, and whose eight vertices are all colored in the same color.

20. [**12**] Let $ABCD$ be a quadrilateral circumscribed about a circle with center O. Let O_1, O_2, O_3, and O_4 denote the circumcenters of $\triangle AOB$, $\triangle BOC$, $\triangle COD$, and $\triangle DOA$. If $\angle A = 120^\circ, \angle B = 80^\circ$, and $\angle C = 45^\circ$, what is the acute angle formed by the two lines passing through O_1O_3 and O_2O_4?

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21. [**12**] Let $ABCD$ be a quadrilateral inscribed in a circle with center O. Let P denote the intersection of AC and BD. Let M and N denote the midpoints of AD and BC. If $AP = 1$, $BP = 3$, $DP = \sqrt{3}$, and AC is perpendicular to BD, find the area of triangle MON.

22. [**12**] Find the number of ordered triples (a, b, c) of pairwise distinct integers such that $-31 \le a, b, c \le 31$ and $a + b + c > 0$.

23. [**14**] Let S be the set of points (x, y, z) in $\mathbb{R}^3$ such that x, y, and z are positive integers less than or equal to 100. Let f be a bijective map between S and the $\{1, 2, \ldots, 1000000\}$ that satisfies the following property: if $x_1 \le x_2$, $y_1 \le y_2$, and $z_1 \le z_2$, then $f(x_1, y_1, z_1) \le f(x_2, y_2, z_2)$. Define

$$A_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(i, j, k),$$

$$B_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, i, k),$$

$$\text{and } C_i = \sum_{j=1}^{100} \sum_{k=1}^{100} f(j, k, i).$$

Determine the minimum value of $A_{i+1} - A_i + B_{j+1} - B_j + C_{k+1} - C_k$.

24. [**14**] In how many ways may thirteen beads be placed on a circular necklace if each bead is either blue or yellow and no two yellow beads may be placed in adjacent positions? (Beads of the same color are considered to be identical, and two arrangements are considered to be the same if and only if each can be obtained from the other by rotation).

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25. [**14**] Let n be an integer greater than 3. Let R be the set of lattice points (x, y) such that $0 \leq x, y \leq n$ and $|x - y| \leq 3$. Let A_n be the number of paths from $(0, 0)$ to (n, n) that consist only of steps of the form $(x, y) \to (x, y+1)$ and $(x, y) \to (x+1, y)$ and are contained entirely within R. Find the smallest positive real number that is greater than $\frac{A_{n+1}}{A_n}$ for all n.

26. [**14**] In how many ways can 13 bishops be placed on an 8×8 chessboard such that (i) a bishop is placed on the second square in the second row, (ii) at most one bishop is placed on each square, (iii) no bishop is placed on the same diagonal as another bishop, and (iv) every diagonal contains a bishop? (For the purposes of this problem, consider all diagonals of the chessboard to be diagonals, not just the main diagonals).

27. [**16**] Find the number of polynomials $p(x)$ with integer coefficients satisfying $p(x) \geq \min\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ and $p(x) \leq \max\{2x^4 - 6x^2 + 1, 4 - 5x^2\}$ for all $x \in \mathbb{R}$.

28. [**16**] Let ABC be a triangle, and let points P and Q lie on BC such that P is closer to B than Q is. Suppose that the radii of the incircles of triangles ABP, APQ, and AQC are all equal to 1, and that the radii of the corresponding excircles opposite A are 3, 6, and 5, respectively. If the radius of the incircle of triangle ABC is $\frac{3}{2}$, find the radius of the excircle of triangle ABC opposite A.

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29. [**16**] Let ABC be a triangle such that $AB = AC = 182$ and $BC = 140$. Let X_1 lie on AC such that $CX_1 = 130$. Let the line through X_1 perpendicular to BX_1 at X_1 meet AB at X_2. Define $X_2, X_3, \ldots,$ as follows: for n odd and $n \geq 1$, let X_{n+1} be the intersection of AB with the perpendicular to $X_{n-1}X_n$ through X_n; for n even and $n \geq 2$, let X_{n+1} be the intersection of AC with the perpendicular to $X_{n-1}X_n$ through X_n. Find $BX_1 + X_1X_2 + X_2X_3 + \ldots$.

30. [**16**] How many ways are there to color the vertices of a $2n$-gon with three colors such that no vertex has the same color as its either of its two neighbors or the vertex directly across from it?

31. [**18**] Let $A = \{1, 2, 3, \ldots, 9\}$. Find the number of bijective functions $f : A \to A$ for which there exists at least one $i \in A$ such that
$$|f(i) - f^{-1}(i)| > 1.$$

32. [**18**] Let p be a prime positive integer. Define a mod-p recurrence of degree n to be a sequence $\{a_k\}_{k\geq 0}$ of numbers modulo p satisfying a relation of the form $a_{i+n} = c_{n-1}a_{i+n-1} + \ldots + c_1a_{i+1} + c_0a_i$ for all $i \geq 0$, where $c_0, c_1, \ldots, c_{n-1}$ are integers and $c_0 \not\equiv 0 \pmod{p}$. Compute the number of distinct linear recurrences of degree at most n in terms of p and n.

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33. [**25**] Find the number of sequences consisting of 100 R's and 2011 S's that satisfy the property that among the first k letters, the number of S's is strictly more than 20 times the number of R's for all $1 \leq k \leq 2111$.

34. [**25**] Let $w = w_1, w_2, \ldots, w_6$ be a permutation of the integers $\{1, 2, \ldots, 6\}$. If there do not exist indices $i < j < k$ such that $w_i < w_j < w_k$ or indices $i < j < k < l$ such that $w_i > w_j > w_k > w_l$, then w is said to be *exquisite*. Find the number of exquisite permutations.

35. [**25**] An *independent set* of a graph G is a set of vertices of G such that no two vertices among these are connected by an edge. If G has 2000 vertices, and each vertex has degree 10, find the maximum possible number of independent sets that G can have.

36. [**25**] An *ordering* of a set of n elements is a bijective map between the set and $\{1, 2, \ldots, n\}$. Call an ordering ρ of the 10 unordered pairs of distinct integers from the set $\{1, 2, 3, 4, 5\}$ admissible if, for any $1 \leq a < b < c \leq 5$, either $p(\{a, b\}) < p(\{a, c\}) < p(\{b, c\})$ or $p(\{b, c\}) < p(\{a, c\}) < p(\{a, b\})$. Find the total number of admissible orderings.

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Team Round A

Warm-Up [50]

1. [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. Alice goes first, and the players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $0 < y - x < 1$.

   (a) [**10**] If the first player to write a number greater than or equal to 2010 wins, determine, with proof, who has the winning strategy.

   (b) [**10**] If the first player to write a number greater than or equal to 2010 on her 2011th turn or later wins (if a player writes a number greater than or equal to 2010 on her 2010th turn or earlier, she loses immediately), determine, with proof, who has the winning strategy.

2. [**15**] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left, the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?

3. [**15**] Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n-1$, where, for all positive integers k, $\varphi(k)$ denotes the number of positive integers less than or equal to k that are relatively prime to k. Prove that $a_n \geq 2^{n-1}$.

Complex Numbers [35]

4. [**15**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = |a+b+c| = 1$. If $|a-b| = |a-c|$ and $b \neq c$, prove that $|a+b||a+c| = 2$.

5. [**20**] Let a and b be positive real numbers. Define two sequences of real numbers $\{a_n\}$ and $\{b_n\}$ for all positive integers n by $(a+bi)^n = a_n + b_n i$. Prove that

$$\frac{|a_{n+1}| + |b_{n+1}|}{|a_n| + |b_n|} \geq \frac{a^2+b^2}{a+b}$$

for all positive integers n.

Coin Flipping [75]

In a one-player game, the player begins with $4m$ fair coins. On each of m turns, the player takes 4 unused coins, flips 3 of them randomly to heads or tails, and then selects whether the 4th one is heads or tails (these four coins are then considered used). After m turns, when the sides of all $4m$ coins have been determined, if half the coins are heads and half are tails, the player wins; otherwise, the player loses.

6. [**10**] Prove that whenever the player must choose the side of a coin, the optimal strategy is to choose heads if more coins have been determined tails than heads and to choose tails if more coins have been determined heads than tails.

7. [**15**] Let T denote the number of coins determined tails and H denote the number of coins determined heads at the end of the game. Let $p_m(k)$ be the probability that $|T - H| = k$ after a game with $4m$ coins, assuming that the player follows the optimal strategy outlined in problem 6. Clearly $p_m(k) = 0$ if k is an odd integer, so we need only consider the case when k is an even integer. (By definition, $p_0(0) = 1$ and $p_0(k) = 0$ for $k \geq 1$). Prove that $p_m(0) \geq p_{m+1}(0)$ for all nonnegative integers m.

8. (a) [**5**] Find a_1, a_2, and a_3, so that the following equation holds for all $m \geq 1$:

$$p_m(0) = a_1 p_{m-1}(0) + a_2 p_{m-1}(2) + a_3 p_{m-1}(4)$$

   (b) [**5**] Find b_1, b_2, b_3, and b_4, so that the following equation holds for all $m \geq 1$:

$$p_m(2) = b_1 p_{m-1}(0) + b_2 p_{m-1}(2) + b_3 p_{m-1}(4) + b_4 p_{m-1}(6)$$

   (c) [**5**] Find c_1, c_2, c_3, and c_4, so that the following equation holds for all $m \geq 1$ and $j \geq 2$:

$$p_m(2j) = c_1 p_{m-1}(2j-2) + c_2 p_{m-1}(2j) + c_3 p_{m-1}(2j+2) + c_4 p_{m-1}(2j+4)$$

9. [**15**] We would now like to examine the behavior of $p_m(k)$ as m becomes arbitrarily large; specifically, we would like to discern whether $\lim_{m\to\infty} p_m(0)$ exists and, if it does, to determine its value. Let $\lim_{m\to\infty} p_m(k) = A_k$.

   (a) [**5**] Prove that $\frac{2}{3} p_m(k) \geq p_m(k+2)$ for all m and k.

   (b) [**10**] Prove that A_0 exists and that $A_0 > 0$. Feel free to assume the result of analysis that a non-increasing sequence of real numbers that is bounded below by a constant c converges to a limit that is greater than or equal to c.

10. [**20**] Once it has been demonstrated that $\lim_{n\to\infty} p_n(0)$ exists and is greater than 0, it follows that $\lim_{n\to\infty} p_n(k)$ exists and is greater than 0 for all even positive integers k and that $\sum_{k=0}^{\infty} A_{2k} = 1$. It also follows that $A_0 = a_1 A_0 + a_2 A_2 + a_3 A_4$, $A_2 = b_1 A_0 + b_2 A_2 + b_3 A_4 + b_4 A_6$, and $A_{2j} = c_1 A_{2j-2} + c_2 A_{2j} + c_3 A_{2j+2} + c_4 A_{2j+4}$ for all positive integers $j \geq 2$, where a_1, a_2, a_3, b_1, b_2, b_3, b_4, and c_1, c_2, c_3, c_4 are the constants you found in problem 8. Assuming these results, determine, with proof, the value of A_0.

Geometry [90]

11. [**20**] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC. Let the circumcircle of triangle AOM intersect ω again at D. If H is the orthocenter of triangle ABC, prove that $\angle DAH = \angle MAO$.

12. [**70**] Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C, respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF.

Let ABC be a triangle, and let E and F be the feet of the altitudes from B and C, respectively. If A is not a right angle, prove that the circumcenter of triangle AEF lies on the incircle of triangle ABC if and only if the incenter of triangle ABC lies on the circumcircle of triangle AEF.

Last Writes [110]

13. [**30**] Given positive integers a and b such that $a > b$, define a sequence of ordered pairs (a_l, b_l) for nonnegative integers l by $a_0 = a$, $b_0 = b$, and $(a_{l+1}, b_{l+1}) = (b_l, a_l \mod b_l)$, where, for all positive integers x and y, $x \mod y$ is defined to be the remainder left by x upon division by y. Define $f(a, b)$ to be the smallest positive integer j such that $b_j = 0$. Given a positive integer n, define $g(n)$ to be $\max_{1 \le k \le n-1} f(n, k)$.

    (a) [**15**] Given a positive integer m, what is the smallest positive integer n_m such that $g(n_m) = m$?

    (b) [**15**] What is the second smallest?

14. [**25**] Given a positive integer n, a sequence of integers $a_1, a_2, \ldots, a_r$, where $0 \le a_i \le k$ for all $1 \le i \le r$, is said to be a "*k-representation*" of n if there exists an integer c such that

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{r} a_i k^{c-i} = n.$$

Prove that every positive integer n has a *k-representation*, and that the *k-representation* is unique if and only if 0 does not appear in the base-k representation of $n - 1$.

15. [**55**] Denote $\{1, 2, ..., n\}$ by $[n]$, and let S be the set of all permutations of $[n]$. Call a subset T of S *good* if every permutation σ in S may be written as $t_1 t_2$ for elements t_1 and t_2 of T, where the product of two permutations is defined to be their composition. Call a subset of U of S *extremely good* if every permutation σ in S may be written as $s^{-1}us$ for elements s of S and u of U. Let τ be the smallest value of $|T|/|S|$ for all good subsets T, and let υ be the smallest value of $|U|/|S|$ for all extremely good subsets U. Prove that $\sqrt{\upsilon} \ge \tau$.

14^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 12 February 2011

Team Round B

Think Carefully! [55]

The problems in this section require only short answers.

1. [**55**] Tom, Dick, and Harry play a game in which they each pick an integer between 1 and 2011. Tom picks a number first and informs Dick and Harry of his choice. Then Dick picks a different number and informs Harry of his choice. Finally, Harry picks a number different from both Tom's and Dick's. After all the picks are complete, an integer is randomly selected between 1 and 2011. The player whose number is closest wins 2 dollars, unless there is a tie, in which case each of the tied players wins 1 dollar. If Tom knows that Dick and Harry will each play optimally and select randomly among equally optimal choices, there are two numbers Tom can pick to maximize his expected profit; what are they?

Complex Numbers [90]

The problems in this section require only short answers.

The *norm* of a complex number $z = a + bi$, denoted by $|z|$, is defined to be $\sqrt{a^2 + b^2}$. In the following problems, it may be helpful to note that the norm is multiplicative and that it obeys the triangle inequality. In other words, please observe that for all complex numbers x and y, $|xy| = |x||y|$ and $|x + y| \leq |x| + |y|$. (You may verify these facts for yourself if you like).

2. [**20**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = |a + b + c| = 1$. If $|a - b| = |a - c|$ and $b \neq c$, evaluate $|a + b||a + c|$.

3. [**30**] Let x and y be complex numbers such that $|x| = |y| = 1$.

   (a) [**15**] Determine the maximum value of $|1 + x| + |1 + y| - |1 + xy|$.

   (b) [**15**] Determine the maximum value of $|1+x|+|1+xy|+|1+xy^2|+\ldots+|1+xy^{2011}|-1006|1+y|$.

4. [**40**] Let a, b, and c be complex numbers such that $|a| = |b| = |c| = 1$. If

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 1$$

as well, determine the product of all possible values of $|a + b + c|$.

Warm Up Your Proof Skills! [40]

The problems in this section require complete proofs.

5. (a) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $0 < y - x < 1$. The first player to write a number greater than or equal to 2010 wins. If Alice goes first, determine, with proof, who has the winning strategy.

   (b) [**20**] Alice and Barbara play a game on a blackboard. At the start, zero is written on the board. The players alternate turns. On her turn, each player replaces x – the number written on the board – with any real number y, subject to the constraint that $y - x \in (0, 1)$. The first player to write a number greater than or equal to 2010 on her 2011th turn or later wins. If a player writes a number greater than or equal to 2010 on her 2010th turn or before, she loses immediately. If Alice goes first, determine, with proof, who has the winning strategy.

The Euler Totient Function [40]

The problems in this section require complete proofs.

Euler's *totient* function, denoted by φ, is a function whose domain is the set of positive integers. It is especially important in number theory, so it is often discussed on the radio or on national TV. (Just kidding). But what is it, exactly? For all positive integers k, $\varphi(k)$ is defined to be the number of positive integers less than or equal to k that are relatively prime to k. It turns out that φ is what you would call a *multiplicative function*, which means that if a and b are relatively prime positive integers, $\varphi(ab) = \varphi(a)\varphi(b)$. Unfortunately, the proof of this result is highly nontrivial. However, there is much more than that to φ, as you are about to discover!

6. [**10**] Let n be a positive integer such that $n > 2$. Prove that $\varphi(n)$ is even.
7. [**10**] Let n be an even positive integer. Prove that $\varphi(n) \leq \frac{n}{2}$.
8. [**20**] Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be a set of positive integers such that $a_1 = 2$ and $a_m = \varphi(a_{m+1})$ for all $1 \leq m \leq n-1$. Prove that $a_n \geq 2^{n-1}$.

Introduction to the Symmedian [70]

The problems in this section require complete proofs.

If A, B, and C are three points in the plane that do not all lie on the same line, the symmedian from A in triangle ABC is defined to be the reflection of the median from A in triangle ABC about the bisector of angle A. Like the φ function, it turns out that the symmedian satisfies some interesting properties, too. For instance, just like how the medians from A, B, and C all intersect at the centroid of triangle ABC, the symmedians from A, B, and C all intersect at what is called (no surprises here) the symmedian point of triangle ABC. The proof of this fact is not easy, but it is unremarkable. In this section, you will investigate some surprising alternative constructions of the symmedian.

9. [**25**] Let ABC be a non-isosceles, non-right triangle, let ω be its circumcircle, and let O be its circumcenter. Let M be the midpoint of segment BC. Let the tangents to ω at B and C intersect at X. Prove that $\angle OAM = \angle OXA$. (Hint: use SAS similarity).
10. [**15**] Let the circumcircle of triangle AOM intersect ω again at D. Prove that points A, D, and X are collinear.
11. [**10**] Let H be the intersection of the three altitudes of triangle ABC. (This point is usually called the orthocenter). Prove that $\angle DAH = \angle MAO$.
12. [**10**] Prove that line AD is the symmedian from A in triangle ABC by showing that $\angle DAB = \angle MAC$.
13. [**10**] Prove that line AD is also the symmedian from D in triangle DBC.

Last Writes [65]

The problems in this section require complete proofs.

14. [**25**] Rachel and Brian are playing a game in a grid with 1 row of 2011 squares. Initially, there is one white checker in each of the first two squares from the left, and one black checker in the third square from the left. At each stage, Rachel can choose to either run or fight. If Rachel runs, she moves the black checker moves 1 unit to the right, and Brian moves each of the white checkers one unit to the right. If Rachel chooses to fight, she pushes the checker immediately to the left of the black checker 1 unit to the left; the black checker is moved 1 unit to the right, and Brian places a new white checker in the cell immediately to the left of the black one. The game ends when the black checker reaches the last cell. How many different final configurations are possible?
15. [**40**] On Facebook, there is a group of people that satisfies the following two properties: (i) there exists a positive integers k such that any subset of $2k-1$ people in the group contains a subset of k people in the group who are all friends with each other, and (ii) every member of the group has 2011 friends or fewer.
    (a) [**15**] If $k = 2$, determine, with proof, the maximum number of people the group may contain.
    (b) [**25**] If $k = 776$, determine, with proof, the maximum number of people the group may contain.

4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

General Test

1. [**3**] Find all ordered pairs of real numbers (x, y) such that $x^2y = 3$ and $x + xy = 4$.

2. [**3**] Let ABC be a triangle, and let D, E, and F be the midpoints of sides BC, CA, and AB, respectively. Let the angle bisectors of $\angle FDE$ and $\angle FBD$ meet at P. Given that $\angle BAC = 37°$ and $\angle CBA = 85°$, determine the degree measure of $\angle BPD$.

3. [**4**] Alberto, Bernardo, and Carlos are collectively listening to three different songs. Each is simultaneously listening to exactly two songs, and each song is being listened to by exactly two people. In how many ways can this occur?

4. [**4**] Determine the remainder when

$$2^{\frac{1\cdot 2}{2}} + 2^{\frac{2\cdot 3}{2}} + \cdots + 2^{\frac{2011\cdot 2012}{2}}$$

is divided by 7.

5. [**5**] Find all real values of x for which

$$\frac{1}{\sqrt{x} + \sqrt{x-2}} + \frac{1}{\sqrt{x+2} + \sqrt{x}} = \frac{1}{4}.$$

6. [**5**] Five people of heights 65, 66, 67, 68, and 69 inches stand facing forwards in a line. How many orders are there for them to line up, if no person can stand immediately before or after someone who is exactly 1 inch taller or exactly 1 inch shorter than himself?

7. [**5**] Determine the number of angles θ between 0 and 2π, other than integer multiples of $\pi/2$, such that the quantities $\sin\theta$, $\cos\theta$, and $\tan\theta$ form a geometric sequence in some order.

8. [**6**] Find the number of integers x such that the following three conditions all hold:

   - x is a multiple of 5
   - $121 < x < 1331$
   - When x is written as an integer in base 11 with no leading 0s (i.e. no 0s at the very left), its rightmost digit is strictly greater than its leftmost digit.

9. [**7**] Let P and Q be points on line l with $PQ = 12$. Two circles, ω and Ω, are both tangent to l at P and are externally tangent to each other. A line through Q intersects ω at A and B, with A closer to Q than B, such that $AB = 10$. Similarly, another line through Q intersects Ω at C and D, with C closer to Q than D, such that $CD = 7$. Find the ratio AD/BC.

10. [**8**] Let $r_1, r_2, \ldots, r_7$ be the distinct complex roots of the polynomial $P(x) = x^7 - 7$. Let

$$K = \prod_{1\le i<j\le 7} (r_i + r_j),$$

that is, the product of all numbers of the form $r_i + r_j$, where i and j are integers for which $1 \le i < j \le 7$. Determine the value of K^2.

4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Guts Round

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Round 1

1. [**5**] Determine the remainder when $1 + 2 + \cdots + 2014$ is divided by 2012.

2. [**5**] Let $ABCD$ be a rectangle with $AB = 6$ and $BC = 4$. Let E be the point on BC with $BE = 3$, and let F be the point on segment AE such that F lies halfway between the segments AB and CD. If G is the point of intersection of DF and BC, find BG.

3. [**5**] Let x be a real number such that $2^x = 3$. Determine the value of 4^{3x+2}.

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4^{TH} ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011 — GUTS ROUND

Round 2

4. [**6**] Determine which of the following numbers is smallest in value: $54\sqrt{3}, 144, 108\sqrt{6} - 108\sqrt{2}$.

5. [**6**] Charlie folds an $\frac{17}{2}$-inch by 11-inch piece of paper in half twice, each time along a straight line parallel to one of the paper's edges. What is the smallest possible perimeter of the piece after two such folds?

6. [**6**] To survive the coming Cambridge winter, Chim Tu doesn't wear one T-shirt, but instead wears up to FOUR T-shirts, all in different colors. An *outfit* consists of three or more T-shirts, put on one on top of the other in some order, such that two outfits are distinct if the sets of T-shirts used are different or the sets of T-shirts used are the same but the order in which they are worn is different. Given that Chim Tu changes his outfit every three days, and otherwise never wears the same outfit twice, how many days of winter can Chim Tu survive? (Needless to say, he only has four t-shirts.)

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4^{TH} ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011 — GUTS ROUND

Round 3

7. [**7**] How many ordered triples of positive integers (a, b, c) are there for which $a^4 b^2 c = 54000$?

8. [**7**] Let a, b, c be not necessarily distinct integers between 1 and 2011, inclusive. Find the smallest possible value of $\dfrac{ab + c}{a + b + c}$.

9. [**7**] Unit circle Ω has points X, Y, Z on its circumference so that XYZ is an equilateral triangle. Let W be a point other than X in the plane such that triangle WYZ is also equilateral. Determine the area of the region inside triangle WYZ that lies outside circle Ω.

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Round 4

10. [**8**] Determine the number of integers D such that whenever a and b are both real numbers with $-1/4 < a, b < 1/4$, then $|a^2 - Db^2| < 1$.

11. [**8**] For positive integers m, n, let $\gcd(m, n)$ denote the largest positive integer that is a factor of both m and n. Compute

$$\sum_{n=1}^{91} \gcd(n, 91).$$

12. [**8**] Joe has written 5 questions of different difficulties for a test with problems numbered 1 though 5. He wants to make sure that problem i is harder than problem j whenever $i - j \geq 3$. In how many ways can he order the problems for his test?

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Round 5

13. [**8**] Tac is dressing his cat to go outside. He has four indistinguishable socks, four indistinguishable shoes, and 4 indistinguishable show-shoes. In a hurry, Tac randomly pulls pieces of clothing out of a door and tries to put them on a random one of his cat's legs; however, Tac never tries to put more than one of each type of clothing on each leg of his cat. What is the probability that, after Tac is done, the snow-shoe on each of his cat's legs is on top of the shoe, which is on top of the sock?

14. [**8**] Let $AMOL$ be a quadrilateral with $AM = 10$, $MO = 11$, and $OL = 12$. Given that the perpendicular bisectors of sides AM and OL intersect at the midpoint of segment AO, find the length of side LA.

15. [**8**] For positive integers n, let $L(n)$ be the largest factor of n other than n itself. Determine the number of ordered pairs of composite positive integers (m, n) for which $L(m)L(n) = 80$.

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Round 6

16. [**10**] A small fish is holding 17 cards, labeled 1 through 17, which he shuffles into a random order. Then, he notices that although the cards are not currently sorted in ascending order, he can sort them into ascending order by removing one card and putting it back in a *different* position (at the beginning, between some two cards, or at the end). In how many possible orders could his cards currently be?

17. [**10**] For a positive integer n, let $p(n)$ denote the product of the positive integer factors of n. Determine the number of factors n of 2310 for which $p(n)$ is a perfect square.

18. [**10**] Consider a cube $ABCDEFGH$, where $ABCD$ and $EFGH$ are faces, and segments AE, BF, CG, DH are edges of the cube. Let P be the center of face $EFGH$, and let O be the center of the cube. Given that $AG = 1$, determine the area of triangle AOP.

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Round 7

19. [**10**] Let $ABCD$ be a rectangle with $AB = 3$ and $BC = 7$. Let W be a point on segment AB such that $AW = 1$. Let X, Y, Z be points on segments BC, CD, DA, respectively, so that quadrilateral $WXYZ$ is a rectangle, and $BX < XC$. Determine the length of segment BX.

20. [**10**] The UEFA Champions League playoffs is a 16-team soccer tournament in which Spanish teams always win against non-Spanish teams. In each of 4 rounds, each remaining team is *randomly* paired against one other team; the winner advances to the next round, and the loser is permanently knocked out of the tournament. If 3 of the 16 teams are Spanish, what is the probability that there are 2 Spanish teams in the final round?

21. [**10**] Let $P(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$ be a polynomial with roots r_1, r_2, r_3, r_4. Let Q be the quartic polynomial with roots $r_1^2, r_2^2, r_3^2, r_4^2$, such that the coefficient of the x^4 term of Q is 1. Simplify the quotient $Q(x^2)/P(x)$, leaving your answer in terms of x. (You may assume that x is not equal to any of r_1, r_2, r_3, r_4).

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4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011 — GUTS ROUND

Round 8

22. [**12**] Let ABC be a triangle with $AB = 23$, $BC = 24$, and $CA = 27$. Let D be the point on segment AC such that the incircles of triangles BAD and BCD are tangent. Determine the ratio CD/DA.

23. [**12**] Let $N = \overline{5AB37C2}$, where A, B, C are digits between 0 and 9, inclusive, and N is a 7-digit positive integer. If N is divisible by 792, determine all possible ordered triples (A, B, C).

24. [**12**] Three not necessarily distinct positive integers between 1 and 99, inclusive, are written in a row on a blackboard. Then, the numbers, without including any leading zeros, are concatenated to form a new integer N. For example, if the integers written, in order, are 25, 6, and 12, then $N = 25612$ (and not $N = 250612$). Determine the number of possible values of N.

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Round 9

The answers to the following three problems are mutually dependent, although your answer to each will be graded independently. Let A be the answer to problem 25, B the answer to problem 26, and C be the answer to problem 27.

25. [**12**] Let XYZ be an equilateral triangle, and let K, L, M be points on sides XY, YZ, ZX, respectively, such that $XK/KY = B$, $YL/LZ = 1/C$, and $ZM/MX = 1$. Determine the ratio of the area of triangle KLM to the area of triangle XYZ.

26. [**12**] Determine the positive real value of x for which

$$\sqrt{2 + AC + 2Cx} + \sqrt{AC - 2 + 2Ax} = \sqrt{2(A + C)x + 2AC}.$$

27. [**12**] In-Young generates a string of B zeroes and ones using the following method:

- First, she flips a fair coin. If it lands heads, her first digit will be a 0, and if it lands tails, her first digit will be a 1.
- For each subsequent bit, she flips an unfair coin, which lands heads with probability A. If the coin lands heads, she writes down the number (zero or one) different from previous digit, while if the coin lands tails, she writes down the previous digit again.

What is the expected value of the number of zeroes in her string?

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4^{TH} ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011 — GUTS ROUND

Round 10

28. [**14**] Determine the value of

$$\sum_{k=1}^{2011} \frac{k-1}{k!(2011-k)!}.$$

29. [**14**] Let ABC be a triangle with $AB = 4$, $BC = 8$, and $CA = 5$. Let M be the midpoint of BC, and let D be the point on the circumcircle of ABC so that segment AD intersects the interior of ABC, and $\angle BAD = \angle CAM$. Let AD intersect side BC at X. Compute the ratio AX/AD.

30. [**14**] Let S be a set of consecutive positive integers such that for any integer n in S, the sum of the digits of n is not a multiple of 11. Determine the largest possible number of elements of S.

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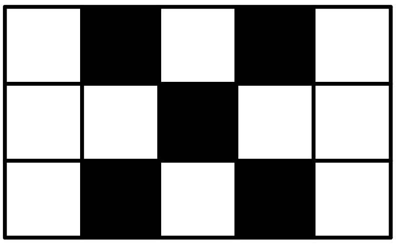
Round 11

31. [**17**] Each square in a 3×10 grid is colored black or white. Let N be the number of ways this can be done in such a way that no five squares in an 'X' configuration (as shown by the black squares below) are all white or all black. Determine $\sqrt{N}$.



32. [**17**] Find all real numbers x satisfying

$$x^9 + \frac{9}{8}x^6 + \frac{27}{64}x^3 - x + \frac{219}{512} = 0.$$

33. [**17**] Let ABC be a triangle with $AB = 5$, $BC = 8$, and $CA = 7$. Let Γ be a circle internally tangent to the circumcircle of ABC at A which is also tangent to segment BC. Γ intersects AB and AC at points D and E, respectively. Determine the length of segment DE.

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Round 12

34. [**20**] The integer 843301 is prime. The *primorial* of a prime number p, denoted $p\#$, is defined to be the product of all prime numbers less than or equal to p. Determine the number of digits in $843301\#$. Your score will be

$$\max\left\{\left\lfloor 60\left(\frac{1}{3}-\left|\ln\left(\frac{A}{d}\right)\right|\right)\right\rfloor, 0\right\},$$

where A is your answer and d is the actual answer.

35. [**20**] Let G be the number of Google hits of "guts round" at 10:31PM on October 31, 2011. Let B be the number of Bing hits of "guts round" at the same time. Determine B/G. Your score will be

$$\max\left(0, \left\lfloor 20\left(1-\frac{20|a-k|}{k}\right)\right\rfloor\right),$$

where k is the actual answer and a is your answer.

36. [**20**] Order any subset of the following twentieth century mathematical achievements chronologically, from earliest to most recent. If you correctly place at least six of the events in order, your score will be $2(n-5)$, where n is the number of events in your sequence; otherwise, your score will be zero. Note: if you order any number of events with one error, your score will be zero.

A). Axioms for Set Theory published by Zermelo
B). Category Theory introduced by Mac Lane and Eilenberg
C). Collatz Conjecture proposed
D). Erdos number defined by Goffman
E). First United States delegation sent to International Mathematical Olympiad
F). Four Color Theorem proven with computer assistance by Appel and Haken
G). Harvard-MIT Math Tournament founded
H). Hierarchy of grammars described by Chomsky
I). Hilbert Problems stated
J). Incompleteness Theorems published by Godel
K). Million dollar prize for Millennium Problems offered by Clay Mathematics Institute
L). Minimum number of shuffles needed to randomize a deck of cards established by Diaconis
M). Nash Equilibrium introduced in doctoral dissertation
N). Proof of Fermat's Last Theorem completed by Wiles
O). Quicksort algorithm invented by Hoare

Write your answer as a list of letters, without any commas or parentheses.

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Saturday 12 November 2011

Guts Round

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

1. **[5]** ____________________
2. **[5]** ____________________
3. **[5]** ____________________

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School ____________________ Team ____________________ Team ID# __________

4. **[6]** ____________________
5. **[6]** ____________________
6. **[6]** ____________________

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School ____________________ Team ____________________ Team ID# __________

7. **[7]** ____________________
8. **[7]** ____________________
9. **[7]** ____________________

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School ____________________ Team ____________________ Team ID# __________

10. **[8]** ____________________
11. **[8]** ____________________
12. **[8]** ____________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

13. **[8]** ____________________
14. **[8]** ____________________
15. **[8]** ____________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ______________________ Team ______________________ Team ID# ________

16. [**10**] ________________

17. [**10**] ________________

18. [**10**] ________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ______________________ Team ______________________ Team ID# ________

19. [**10**] ________________

20. [**10**] ________________

21. [**10**] ________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ______________________ Team ______________________ Team ID# ________

22. [**12**] ________________

23. [**12**] ________________

24. [**12**] ________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ______________________ Team ______________________ Team ID# ________

25. [**12**] ________________

26. [**12**] ________________

27. [**12**] ________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ______________________ Team ______________________ Team ID# ________

28. [**14**] ________________

29. [**14**] ________________

30. [**14**] ________________

4TH ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 12 NOVEMBER 2011— GUTS ROUND

School ____________________ Team ____________________ Team ID# __________

31. [**17**] ____________________

32. [**17**] ____________________

33. [**17**] ____________________

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School ____________________ Team ____________________ Team ID# __________

34. [**20**] ____________________

35. [**20**] ____________________

36. [**20**] ____________________

4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Team Round

p-Polynomials

1. [**2**] Find the number of positive integers x less than 100 for which

$$3^x + 5^x + 7^x + 11^x + 13^x + 17^x + 19^x$$

is prime.

2. [**4**] Determine the set of all real numbers p for which the polynomial $Q(x) = x^3 + px^2 - px - 1$ has three distinct real roots.

3. [**6**] Find the sum of the coefficients of the polynomial $P(x) = x^4 - 29x^3 + ax^2 + bx + c$, given that $P(5) = 11$, $P(11) = 17$, and $P(17) = 23$.

4. [**7**] Determine the number of quadratic polynomials $P(x) = p_1x^2 + p_2x - p_3$, where p_1, p_2, p_3 are not necessarily distinct (positive) prime numbers less than 50, whose roots are distinct rational numbers.

C Coloring

5. [**3**] Sixteen wooden Cs are placed in a 4-by-4 grid, all with the same orientation, and each is to be colored either red or blue. A *quadrant operation* on the grid consists of choosing one of the four two-by-two subgrids of Cs found at the corners of the grid and moving each C in the subgrid to the adjacent square in the subgrid that is 90 degrees away in the clockwise direction, without changing the orientation of the C. Given that two colorings are the considered same if and only if one can be obtained from the other by a series of quadrant operations, determine the number of distinct colorings of the Cs.

C	C	C	C
C	C	C	C
C	C	C	C
C	C	C	C

6. [**5**] Ten Cs are written in a row. Some Cs are upper-case and some are lower-case, and each is written in one of two colors, green and yellow. It is given that there is at least one lower-case C, at least one green C, and at least one C that is both upper-case and yellow. Furthermore, no lower-case C can be followed by an upper-case C, and no yellow C can be followed by a green C. In how many ways can the Cs be written?

7. [**7**] Julia is learning how to write the letter C. She has 6 differently-colored crayons, and wants to write Cc Cc Cc Cc Cc. In how many ways can she write the ten Cs, in such a way that each upper case C is a different color, each lower case C is a different color, and in each pair the upper case C and lower case C are different colors?

[GG]eometry

8. [**4**] Let $G, A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, B_5$ be ten points on a circle such that $GA_1A_2A_3A_4$ is a regular pentagon and $GB_1B_2B_3B_4B_5$ is a regular hexagon, and B_1 lies on minor arc GA_1. Let B_5B_3 intersect B_1A_2 at G_1, and let B_5A_3 intersect GB_3 at G_2. Determine the degree measure of $\angle GG_2G_1$.

9. [**4**] Let ABC be a triangle with $AB = 9$, $BC = 10$, and $CA = 17$. Let B' be the reflection of the point B over the line CA. Let G be the centroid of triangle ABC, and let G' be the centroid of triangle $AB'C$. Determine the length of segment GG'.

10. [**8**] Let $G_1G_2G_3$ be a triangle with $G_1G_2 = 7$, $G_2G_3 = 13$, and $G_3G_1 = 15$. Let G_4 be a point outside triangle $G_1G_2G_3$ so that ray $\overrightarrow{G_1G_4}$ cuts through the interior of the triangle, $G_3G_4 = G_4G_2$, and $\angle G_3G_1G_4 = 30°$. Let G_3G_4 and G_1G_2 meet at G_5. Determine the length of segment G_2G_5.

4^{th} Annual Harvard-MIT November Tournament

Saturday 12 November 2011

Theme Round

Fish

1. [**3**] Five of James' friends are sitting around a circular table to play a game of Fish. James chooses a place between two of his friends to pull up a chair and sit. Then, the six friends divide themselves into two disjoint teams, with each team consisting of three consecutive players at the table. If the order in which the three members of a team sit does not matter, how many possible (unordered) pairs of teams are possible?

2. [**3**] In a game of Fish, R2 and R3 are each holding a positive number of cards so that they are collectively holding a total of 24 cards. Each player gives an integer estimate for the number of cards he is holding, such that each estimate is an integer between 80% of his actual number of cards and 120% of his actual number of cards, inclusive. Find the smallest possible sum of the two estimates.

3. [**5**] In preparation for a game of Fish, Carl must deal 48 cards to 6 players. For each card that he deals, he runs through the entirety of the following process:

   1. He gives a card to a random player.
   2. A player Z is randomly chosen from the set of players who have at least as many cards as every other player (i.e. Z has the most cards or is tied for having the most cards).
   3. A player D is randomly chosen from the set of players other than Z who have at most as many cards as every other player (i.e. D has the fewest cards or is tied for having the fewest cards).
   4. Z gives one card to D.

   He repeats steps 1-4 for each card dealt, including the last card. After all the cards have been dealt, what is the probability that each player has exactly 8 cards?

4. [**6**] Toward the end of a game of Fish, the 2 through 7 of spades, inclusive, remain in the hands of three distinguishable players: DBR, RB, and DB, such that each player has at least one card. If it is known that DBR either has more than one card or has an even-numbered spade, or both, in how many ways can the players' hands be distributed?

5. [**8**] For any finite sequence of positive integers π, let $S(\pi)$ be the number of strictly increasing sub-sequences in π with length 2 or more. For example, in the sequence $\pi = \{3, 1, 2, 4\}$, there are five increasing sub-sequences: $\{3, 4\}$, $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, and $\{1, 2, 4\}$, so $S(\pi) = 5$. In an eight-player game of Fish, Joy is dealt six cards of distinct values, which she puts in a random order π from left to right in her hand. Determine

$$\sum_{\pi} S(\pi),$$

   where the sum is taken over all possible orders π of the card values.

Circles and Tangents

6. [**3**] Let ABC be an equilateral triangle with $AB = 3$. Circle ω with diameter 1 is drawn inside the triangle such that it is tangent to sides AB and AC. Let P be a point on ω and Q be a point on segment BC. Find the minimum possible length of the segment PQ.

7. [**4**] Let XYZ be a triangle with $\angle XYZ = 40^\circ$ and $\angle YZX = 60^\circ$. A circle Γ, centered at the point I, lies inside triangle XYZ and is tangent to all three sides of the triangle. Let A be the point of tangency of Γ with YZ, and let ray $\overrightarrow{XI}$ intersect side YZ at B. Determine the measure of $\angle AIB$.

8. [**5**] Points D, E, F lie on circle O such that the line tangent to O at D intersects ray $\overrightarrow{EF}$ at P. Given that $PD = 4$, $PF = 2$, and $\angle FPD = 60°$, determine the area of circle O.

9. [**6**] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let D be the foot of the altitude from A to BC. The inscribed circles of triangles ABD and ACD are tangent to AD at P and Q, respectively, and are tangent to BC at X and Y, respectively. Let PX and QY meet at Z. Determine the area of triangle XYZ.

10. [**7**] Let Ω be a circle of radius 8 centered at point O, and let M be a point on Ω. Let S be the set of points P such that P is contained within Ω, or such that there exists some rectangle $ABCD$ containing P whose center is on Ω with $AB = 4$, $BC = 5$, and $BC \parallel OM$. Find the area of S.

13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Algebra Subject Test

1. [**3**] Suppose that x and y are positive reals such that
$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$
Find x.

2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

3. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

4. [**4**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

5. [**5**] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

6. [**5**] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

7. [**5**] Let a, b, c, x, y, and z be complex numbers such that
$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$
If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.

8. [**6**] How many polynomials of degree exactly 5 with real coefficients send the set $\{1, 2, 3, 4, 5, 6\}$ to a permutation of itself?

9. [**7**] Let $f(x) = cx(x-1)$, where c is a positive real number. We use $f^n(x)$ to denote the polynomial obtained by composing f with itself n times. For every positive integer n, all the roots of $f^n(x)$ are real. What is the smallest possible value of c?

10. [**8**] Let $p(x)$ and $q(x)$ be two cubic polynomials such that $p(0) = -24$, $q(0) = 30$, and
$$p(q(x)) = q(p(x))$$
for all real numbers x. Find the ordered pair $(p(3), q(6))$.

13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Calculus Subject Test

1. [**3**] Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.

2. [**3**] Let f be a function such that $f(0) = 1$, $f'(0) = 2$, and

$$f''(t) = 4f'(t) - 3f(t) + 1$$

for all t. Compute the 4th derivative of f, evaluated at 0.

3. [**4**] Let p be a monic cubic polynomial such that $p(0) = 1$ and such that all the zeros of $p'(x)$ are also zeros of $p(x)$. Find p. Note: monic means that the leading coefficient is 1.

4. [**4**] Compute $\lim_{n\to\infty} \frac{\sum_{k=1}^{n} |\cos(k)|}{n}$.

5. [**4**] Let the functions $f(\alpha, x)$ and $g(\alpha)$ be defined as

$$f(\alpha, x) = \frac{\left(\frac{x}{2}\right)^{\alpha}}{x-1} \qquad\qquad g(\alpha) = \left.\frac{d^4 f}{dx^4}\right|_{x=2}$$

Then $g(\alpha)$ is a polynomial in α. Find the leading coefficient of $g(\alpha)$.

6. [**5**] Let $f(x) = x^3 - x^2$. For a given value of c, the graph of $f(x)$, together with the graph of the line $c + x$, split the plane up into regions. Suppose that c is such that exactly two of these regions have finite area. Find the value of c that minimizes the sum of the areas of these two regions.

7. [**6**] Let a_1, a_2, and a_3 be nonzero complex numbers with non-negative real and imaginary parts. Find the minimum possible value of

$$\frac{|a_1 + a_2 + a_3|}{\sqrt[3]{|a_1 a_2 a_3|}}.$$

8. [**6**] Let $f(n) = \sum_{k=2}^{\infty} \frac{1}{k^n \cdot k!}$. Calculate $\sum_{n=2}^{\infty} f(n)$.

9. [**7**] Let $x(t)$ be a solution to the differential equation

$$\left(x + x'\right)^2 + x \cdot x'' = \cos t$$

with $x(0) = x'(0) = \sqrt{\frac{2}{5}}$. Compute $x\left(\frac{\pi}{4}\right)$.

10. [**8**] Let $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. Then there exists constants γ, c, and d such that

$$f(n) = \ln(n) + \gamma + \frac{c}{n} + \frac{d}{n^2} + O\left(\frac{1}{n^3}\right),$$

where the $O(\frac{1}{n^3})$ means terms of order $\frac{1}{n^3}$ or lower. Compute the ordered pair (c, d).

13^{th}Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

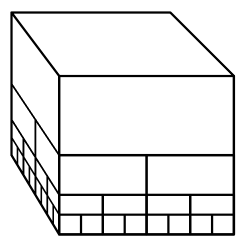
Combinatorics Subject Test

1. [**2**] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

2. [**3**] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

3. [**4**] How many ways are there to choose 2010 functions $f_1, \ldots, f_{2010}$ from $\{0, 1\}$ to $\{0, 1\}$ such that $f_{2010} \circ f_{2009} \circ \cdots \circ f_1$ is constant? Note: a function g is constant if $g(a) = g(b)$ for all a, b in the domain of g.

4. [**4**] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the kth layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



5. [**5**] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

6. [**5**] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y), it can move to $(x-1, y-1)$, $(x-1, y+1)$, $(x+1, y-1)$, or $(x+1, y+1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

7. [**6**] For each integer x with $1 \leq x \leq 10$, a point is randomly placed at either $(x, 1)$ or $(x, -1)$ with equal probability. What is the expected area of the convex hull of these points? Note: the convex hull of a finite set is the smallest convex polygon containing it.

8. [**6**] How many functions f from $\{-1005, \ldots, 1005\}$ to $\{-2010, \ldots, 2010\}$ are there such that the following two conditions are satisfied?

   - If $a < b$ then $f(a) < f(b)$.
   - There is no n in $\{-1005, \ldots, 1005\}$ such that $|f(n)| = |n|$.

9. [**7**] Rosencrantz and Guildenstern are playing a game where they repeatedly flip coins. Rosencrantz wins if 1 heads followed by 2009 tails appears. Guildenstern wins if 2010 heads come in a row. They will flip coins until someone wins. What is the probability that Rosencrantz wins?

10. [**8**] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?

13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

General Test, Part 1

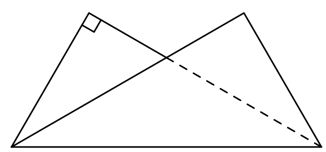
1. [**3**] Suppose that x and y are positive reals such that

$$x - y^2 = 3, \quad x^2 + y^4 = 13.$$

Find x.

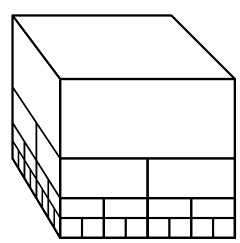
2. [**3**] Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. How many (potentially empty) subsets T of S are there such that, for all x, if x is in T and $2x$ is in S then $2x$ is also in T?

3. [**4**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



4. [**4**] Let $S_0 = 0$ and let S_k equal $a_1 + 2a_2 + \ldots + ka_k$ for $k \geq 1$. Define a_i to be 1 if $S_{i-1} < i$ and -1 if $S_{i-1} \geq i$. What is the largest $k \leq 2010$ such that $S_k = 0$?

5. [**4**] Manya has a stack of $85 = 1 + 4 + 16 + 64$ blocks comprised of 4 layers (the kth layer from the top has 4^{k-1} blocks; see the diagram below). Each block rests on 4 smaller blocks, each with dimensions half those of the larger block. Laura removes blocks one at a time from this stack, removing only blocks that currently have no blocks on top of them. Find the number of ways Laura can remove precisely 5 blocks from Manya's stack (the order in which they are removed matters).



6. [**5**] John needs to pay 2010 dollars for his dinner. He has an unlimited supply of 2, 5, and 10 dollar notes. In how many ways can he pay?

7. [**6**] Suppose that a polynomial of the form $p(x) = x^{2010} \pm x^{2009} \pm \cdots \pm x \pm 1$ has no real roots. What is the maximum possible number of coefficients of -1 in p?

8. [**6**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

9. [**7**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

10. [**8**] Let a, b, c, x, y, and z be complex numbers such that

$$a = \frac{b+c}{x-2}, \quad b = \frac{c+a}{y-2}, \quad c = \frac{a+b}{z-2}.$$

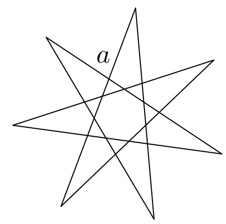
If $xy + yz + zx = 67$ and $x + y + z = 2010$, find the value of xyz.

13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

General Test, Part 2

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



2. [**3**] The *rank* of a rational number q is the unique k for which $q = \frac{1}{a_1} + \cdots + \frac{1}{a_k}$, where each a_i is the smallest positive integer such that $q \geq \frac{1}{a_1} + \cdots + \frac{1}{a_i}$. Let q be the largest rational number less than $\frac{1}{4}$ with rank 3, and suppose the expression for q is $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}$. Find the ordered triple (a_1, a_2, a_3).

3. [**4**] How many positive integers less than or equal to 240 can be expressed as a sum of distinct factorials? Consider 0! and 1! to be distinct.

4. [**4**] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

5. [**5**] Suppose that there exist nonzero complex numbers a, b, c, and d such that k is a root of both the equations $ax^3 + bx^2 + cx + d = 0$ and $bx^3 + cx^2 + dx + a = 0$. Find all possible values of k (including complex values).

6. [**5**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

7. [**5**] Suppose that x and y are complex numbers such that $x + y = 1$ and that $x^{20} + y^{20} = 20$. Find the sum of all possible values of $x^2 + y^2$.

8. [**6**] An ant starts out at $(0, 0)$. Each second, if it is currently at the square (x, y), it can move to $(x-1, y-1)$, $(x-1, y+1)$, $(x+1, y-1)$, or $(x+1, y+1)$. In how many ways can it end up at $(2010, 2010)$ after 4020 seconds?

9. [**7**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

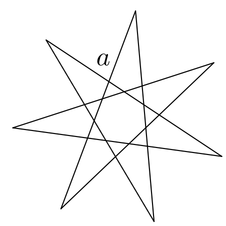
10. [**8**] In a 16×16 table of integers, each row and column contains at most 4 distinct integers. What is the maximum number of distinct integers that there can be in the whole table?

13th Annual Harvard-MIT Mathematics Tournament

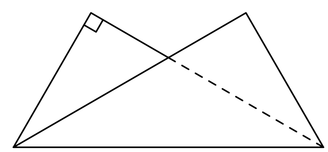
Saturday 20 February 2010

Geometry Subject Test

1. [**3**] Below is pictured a regular seven-pointed star. Find the measure of angle a in radians.



2. [**3**] A rectangular piece of paper is folded along its diagonal (as depicted below) to form a non-convex pentagon that has an area of $\frac{7}{10}$ of the area of the original rectangle. Find the ratio of the longer side of the rectangle to the shorter side of the rectangle.



3. [**4**] For $0 \leq y \leq 2$, let D_y be the half-disk of diameter 2 with one vertex at $(0, y)$, the other vertex on the positive x-axis, and the curved boundary further from the origin than the straight boundary. Find the area of the union of D_y for all $0 \leq y \leq 2$.

4. [**4**] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

5. [**4**] A sphere is the set of points at a fixed positive distance r from its center. Let $\mathcal{S}$ be a set of 2010-dimensional spheres. Suppose that the number of points lying on every element of $\mathcal{S}$ is a finite number n. Find the maximum possible value of n.

6. [**5**] Three unit circles ω_1, ω_2, and ω_3 in the plane have the property that each circle passes through the centers of the other two. A square S surrounds the three circles in such a way that each of its four sides is tangent to at least one of ω_1, ω_2 and ω_3. Find the side length of the square S.

7. [**6**] You are standing in an infinitely long hallway with sides given by the lines $x = 0$ and $x = 6$. You start at $(3, 0)$ and want to get to $(3, 6)$. Furthermore, at each instant you want your distance to $(3, 6)$ to either decrease or stay the same. What is the area of the set of points that you could pass through on your journey from $(3, 0)$ to $(3, 6)$?

8. [**6**] Let O be the point $(0, 0)$. Let A, B, C be three points in the plane such that $AO = 15$, $BO = 15$, and $CO = 7$, and such that the area of triangle ABC is maximal. What is the length of the shortest side of ABC?

9. [**7**] Let $ABCD$ be a quadrilateral with an inscribed circle centered at I. Let CI intersect AB at E. If $\angle IDE = 35°$, $\angle ABC = 70°$, and $\angle BCD = 60°$, then what are all possible measures of $\angle CDA$?

10. [**8**] Circles ω_1 and ω_2 intersect at points A and B. Segment PQ is tangent to ω_1 at P and to ω_2 at Q, and A is closer to PQ than B. Point X is on ω_1 such that $PX \parallel QB$, and point Y is on ω_2 such that $QY \parallel PB$. Given that $\angle APQ = 30°$ and $\angle PQA = 15°$, find the ratio AX/AY.

13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Guts Round

. . . .

1. [**4**] If $A = 10^9 - 987654321$ and $B = \frac{123456789+1}{10}$, what is the value of $\sqrt{AB}$?

2. [**4**] Suppose that x, y, and z are non-negative real numbers such that $x + y + z = 1$. What is the maximum possible value of $x + y^2 + z^3$?

3. [**4**] In a group of people, there are 13 who like apples, 9 who like blueberries, 15 who like cantaloupe, and 6 who like dates. (A person can like more than 1 kind of fruit.) Each person who likes blueberries also likes exactly one of apples and cantaloupe. Each person who likes cantaloupe also likes exactly one of blueberries and dates. Find the minimum possible number of people in the group.

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13^{th} ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 20 FEBRUARY 2010 — GUTS ROUND

4. [**5**] To set up for a Fourth of July party, David is making a string of red, white, and blue balloons. He places them according to the following rules:

   - No red balloon is adjacent to another red balloon.
   - White balloons appear in groups of exactly two, and groups of white balloons are separated by at least two non-white balloons.
   - Blue balloons appear in groups of exactly three, and groups of blue balloons are separated by at least three non-blue balloons.

   If David uses over 600 balloons, determine the smallest number of red balloons that he can use.

5. [**5**] You have a length of string and 7 beads in the 7 colors of the rainbow. You place the beads on the string as follows – you randomly pick a bead that you haven't used yet, then randomly add it to either the left end or the right end of the string. What is the probability that, at the end, the colors of the beads are the colors of the rainbow in order? (The string cannot be flipped, so the red bead must appear on the left side and the violet bead on the right side.)

6. [**5**] How many different numbers are obtainable from five 5s by first concatenating some of the 5s, then multiplying them together? For example, we could do $5 \cdot 55 \cdot 55$, $555 \cdot 55$, or 55555, but not $5 \cdot 5$ or 2525.

. . . .

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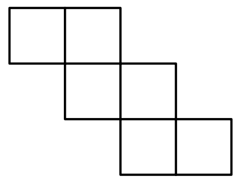
7. [**6**] What are the last 8 digits of

$$11 \times 101 \times 1001 \times 10001 \times 100001 \times 1000001 \times 111?$$

8. [**6**] Each square in the following hexomino has side length 1. Find the minimum area of any rectangle that contains the entire hexomino.



9. [**6**] Indecisive Andy starts out at the midpoint of the 1-unit-long segment $\overline{HT}$. He flips 2010 coins. On each flip, if the coin is heads, he moves halfway towards endpoint H, and if the coin is tails, he moves halfway towards endpoint T. After his 2010 moves, what is the expected distance between Andy and the midpoint of $\overline{HT}$?

. . . .

10. [**7**] Let ABC be a triangle with $AB = 8$, $BC = 15$, and $AC = 17$. Point X is chosen at random on line segment AB. Point Y is chosen at random on line segment BC. Point Z is chosen at random on line segment CA. What is the expected area of triangle XYZ?

11. [**7**] From the point (x, y), a *legal move* is a move to $(\frac{x}{3} + u, \frac{y}{3} + v)$, where u and v are real numbers such that $u^2 + v^2 \leq 1$. What is the area of the set of points that can be reached from $(0, 0)$ in a finite number of legal moves?

12. [**7**] How many different collections of 9 letters are there? A letter can appear multiple times in a collection. Two collections are equal if each letter appears the same number of times in both collections.

. . . .

13. [**8**] A triangle in the xy-plane is such that when projected onto the x-axis, y-axis, and the line $y = x$, the results are line segments whose endpoints are $(1, 0)$ and $(5, 0)$, $(0, 8)$ and $(0, 13)$, and $(5, 5)$ and $(7.5, 7.5)$, respectively. What is the triangle's area?

14. [**8**] In how many ways can you fill a 3×3 table with the numbers 1 through 9 (each used once) such that all pairs of adjacent numbers (sharing one side) are relatively prime?

15. [**8**] Pick a random integer between 0 and 4095, inclusive. Write it in base 2 (without any leading zeroes). What is the expected number of consecutive digits that are not the same (that is, the expected number of occurrences of either 01 or 10 in the base 2 representation)?

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16. [**9**] Jessica has three marbles colored red, green, and blue. She randomly selects a non-empty subset of them (such that each subset is equally likely) and puts them in a bag. You then draw three marbles from the bag with replacement. The colors you see are red, blue, red. What is the probability that the only marbles in the bag are red and blue?

17. [**9**] An ant starts at the origin, facing in the positive x-direction. Each second, it moves 1 unit forward, then turns counterclockwise by $\sin^{-1}(\frac{3}{5})$ degrees. What is the least upper bound on the distance between the ant and the origin? (The least upper bound is the smallest real number r that is at least as big as every distance that the ant ever is from the origin.)

18. [**9**] Find two lines of symmetry of the graph of the function $y = x + \frac{1}{x}$. Express your answer as two equations of the form $y = ax + b$.

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13^{th} ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 20 FEBRUARY 2010 — GUTS ROUND

19. [**10**] A 5-dimensional ant starts at one vertex of a 5-dimensional hypercube of side length 1. A *move* is when the ant travels from one vertex to another vertex at a distance of $\sqrt{2}$ away. How many ways can the ant make 5 moves and end up on the same vertex it started at?

20. [**10**] Find the volume of the set of points (x, y, z) satisfying

$$\begin{aligned} x, y, z &\geq 0 \\ x + y &\leq 1 \\ y + z &\leq 1 \\ z + x &\leq 1 \end{aligned}$$

21. [**10**] Let $\triangle ABC$ be a scalene triangle. Let h_a be the locus of points P such that $|PB - PC| = |AB - AC|$. Let h_b be the locus of points P such that $|PC - PA| = |BC - BA|$. Let h_c be the locus of points P such that $|PA - PB| = |CA - CB|$. In how many points do all of h_a, h_b, and h_c concur?

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13^{th} ANNUAL HARVARD-MIT MATHEMATICS TOURNAMENT, 20 FEBRUARY 2010 — GUTS ROUND

22. [**12**] You are the general of an army. You and the opposing general both have an equal number of troops to distribute among three battlefields. Whoever has more troops on a battlefield always wins (you win ties). An *order* is an ordered triple of non-negative real numbers (x, y, z) such that $x + y + z = 1$, and corresponds to sending a fraction x of the troops to the first field, y to the second, and z to the third. Suppose that you give the order $\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$ and that the other general issues an order chosen uniformly at random from all possible orders. What is the probability that you win two out of the three battles?

23. [**12**] In the country of Francisca, there are 2010 cities, some of which are connected by roads. Between any two cities, there is a unique path which runs along the roads and which does not pass through any city twice. What is the maximum possible number of cities in Francisca which have at least 3 roads running out of them?

24. [**12**] Define a sequence of polynomials as follows: let $a_1 = 3x^2 - x$, let $a_2 = 3x^2 - 7x + 3$, and for $n \geq 1$, let $a_{n+2} = \frac{5}{2}a_{n+1} - a_n$. As n tends to infinity, what is the limit of the sum of the roots of a_n?

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25. [**15**] How many functions $f : \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ have the property that $f(\{1,2,3\})$ and $f(f(\{1,2,3\}))$ are disjoint?

26. [**15**] Express the following in closed form, as a function of x:

$$\sin^2(x)+\sin^2(2x)\cos^2(x)+\sin^2(4x)\cos^2(2x)\cos^2(x)+\cdots+\sin^2(2^{2010}x)\cos^2(2^{2009}x)\cdots\cos^2(2x)\cos^2(x).$$

27. [**15**] Suppose that there are real numbers $a, b, c \geq 1$ and that there are positive reals x, y, z such that

$$\begin{aligned} a^x + b^y + c^z &= 4 \\ xa^x + yb^y + zc^z &= 6 \\ x^2a^x + y^2b^y + z^2c^z &= 9. \end{aligned}$$

What is the maximum possible value of c?

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28. [**18**] Danielle Bellatrix Robinson is organizing a poker tournament with 9 people. The tournament will have 4 rounds, and in each round the 9 players are split into 3 groups of 3. During the tournament, each player plays every other player exactly once. How many different ways can Danielle divide the 9 people into three groups in each round to satisfy these requirements?

29. [**18**] Compute the remainder when

$$\sum_{k=1}^{30303} k^k$$

is divided by 101.

30. [**18**] A monomial term $x_{i_1}x_{i_2}\ldots x_{i_k}$ in the variables $x_1, x_2, \ldots x_8$ is *square-free* if $i_1, i_2, \ldots i_k$ are distinct. (A constant term such as 1 is considered square-free.) What is the sum of the coefficients of the square-free terms in the following product?

$$\prod_{1\leq i<j\leq 8}(1 + x_ix_j)$$

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31. [**21**] In the Democratic Republic of Irun, 5 people are voting in an election among 5 candidates. If each person votes for a single candidate at random, what is the expected number of candidates that will be voted for?

32. [**21**] There are 101 people participating in a Secret Santa gift exchange. As usual each person is randomly assigned another person for whom (s)he has to get a gift, such that each person gives and receives exactly one gift and no one gives a gift to themself. What is the probability that the first person neither gives gifts to or receives gifts from the second or third person? Express your answer as a decimal rounded to five decimal places.

33. [**21**] Let $a_1 = 3$, and for $n > 1$, let a_n be the largest real number such that

$$4\left(a_{n-1}^2 + a_n^2\right) = 10a_{n-1}a_n - 9.$$

What is the largest positive integer less than a_8?

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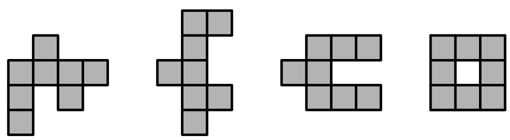
Note: Problems 34 and 36 are estimation problems. You will receive a number of points (between 0 and 25) based on how close your answer is to the correct answer. Throughout, we will let C denote the correct answer and A denote your answer. If a described scoring method would ever assign a negative number of points, you will receive zero points instead.

34. [**25**] 3000 people each go into one of three rooms randomly. What is the most likely value for the maximum number of people in any of the rooms? Your score for this problem will be 0 if you write down a number less than or equal to 1000. Otherwise, it will be $25 - 27\frac{|A-C|}{\min(A,C)-1000}$.

35. [**25**] Call an positive integer *almost-square* if it can be written as $a \cdot b$, where a and b are integers and $a \leq b \leq \frac{4}{3}a$. How many almost-square positive integers are less than or equal to 1000000? Your score will be equal to $25 - 65\frac{|A-C|}{\min(A,C)}$.

36. [**25**] Consider an infinite grid of unit squares. An *n-omino* is a subset of n squares that is connected. Below are depicted examples of 8-ominoes. Two n-ominoes are considered equivalent if one can be obtained from the other by translations and rotations. What is the number of distinct 15-ominoes? Your score will be equal to $25 - 13|\ln(A) - \ln(C)|$.



13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Team Round A

1. You are trying to sink a submarine. Every second, you launch a missile at a point of your choosing on the x-axis. If the submarine is at that point at that time, you sink it. A *firing sequence* is a sequence of real numbers that specify where you will fire at each second. For example, the firing sequence $2, 3, 5, 6, \ldots$ means that you will fire at 2 after one second, 3 after two seconds, 5 after three seconds, 6 after four seconds, and so on.

   (a) [**5**] Suppose that the submarine starts at the origin and travels along the positive x-axis with an (unknown) positive integer velocity. Show that there is a firing sequence that is guaranteed to hit the submarine eventually.

   (b) [**10**] Suppose now that the submarine starts at an unknown integer point on the non-negative x-axis and again travels with an unknown positive integer velocity. Show that there is still a firing sequence that is guaranteed to hit the submarine eventually.

2. [**15**] Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number or 1, at which point that player loses. For which values of N does player 1 have a winning strategy?

3. [**15**] Call a positive integer in base 10 *k-good* if we can split it into two integers y and z, such that y is all digits on the left and z is all digits on the right, and such that $y = k \cdot z$. For example, 2010 is 2-good because we can split it into 20 and 10 and $20 = 2 \cdot 10$. 20010 is also 2-good, because we can split it into 20 and 010. In addition, it is 20-good, because we can split it into 200 and 10.

   Show that there exists a 48-good perfect square.

4. [**20**] Let
$$e^x + e^y = A$$
$$xe^x + ye^y = B$$
$$x^2e^x + y^2e^y = C$$
$$x^3e^x + y^3e^y = D$$
$$x^4e^x + y^4e^y = E.$$
   Prove that if A, B, C, and D are all rational, then so is E.

5. [**20**] Show that, for every positive integer n, there exists a monic polynomial of degree n with integer coefficients such that the coefficients are decreasing and the roots of the polynomial are all integers.

6. [**20**] Let S be a convex set in the plane with a finite area a. Prove that either $a = 0$ or S is bounded. Note: a set is bounded if it is contained in a circle of finite radius. Note: a set is *convex* if, whenever two points A and B are in the set, the line segment between them is also in the set.

7. [**25**] Point P lies inside a convex pentagon $AFQDC$ such that $FPDQ$ is a parallelogram. Given that $\angle FAQ = \angle PAC = 10°$, and $\angle PFA = \angle PDC = 15°$. What is $\angle AQC$?

8. [**30**] A knight moves on a two-dimensional grid. From any square, it can move 2 units in one axis-parallel direction, then move 1 unit in an orthogonal direction, the way a regular knight moves in a game of chess. The knight starts at the origin. As it moves, it keeps track of a number t, which is initially 0. When the knight lands at the point (a, b), the number is changed from x to $ax + b$.

   Show that, for any integers a and b, it is possible for the knight to land at the points $(1, a)$ and $(-1, a)$ with t equal to b.

9. [**30**] Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$ be a polynomial with complex coefficients such that $a_i \neq 0$ for all i. Prove that $|r| \leq 2 \max_{i=0}^{n-1} |\frac{a_{i-1}}{a_i}|$ for all roots r of all such polynomials p. Here we let $|z|$ denote the absolute value of the complex number z.

10. Call an $2n$-digit base-10 number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$-digit number is special. (We allow leading zeros in $2n$-digit numbers).

    (a) [**20**] The sequence p_n converges to a constant c. Find c.

    (b) [**45**] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d. Find r and d.

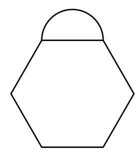
13^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 20 February 2010

Team Round B

1. [**10**] How many ways are there to place pawns on an 8×8 chessboard, so that there is at most 1 pawn in each horizontal row? Express your answer in the form $p_1^{e_1} \cdot p_2^{e_2} \cdots$, where the p_i are distinct primes and the e_i are positive integers.

2. [**10**] In the following figure, a regular hexagon of side length 1 is attached to a semicircle of diameter 1. What is the longest distance between any two points in the figure?



3. [**15**] Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, where each a_i is either 1 or -1. Let r be a root of p. If $|r| > \frac{15}{8}$, what is the minimum possible value of n?

4. [**20**] Find all 4-digit integers of the form *aabb* (when written in base 10) that are perfect squares.

5. [**25**] Compute
$$\sum_{n=1}^{98} \frac{2}{\sqrt{n} + \sqrt{n+2}} + \frac{1}{\sqrt{n+1} + \sqrt{n+2}}.$$

6. [**25**] Into how many regions can a circle be cut by 10 parabolas?

7. [**30**] Evaluate
$$\sum_{k=1}^{2010} \cos^2(k)$$

8. [**30**] Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2 (a proper divisor of N is a positive divisor of N that is not equal to 1 or N). Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number, at which point that player loses. For how many values of N between 2 and 100 inclusive does player 1 have a winning strategy?

9. [**35**] Let S be the set of ordered pairs of integers (x, y) with $1 \leq x \leq 5$ and $1 \leq y \leq 3$. How many subsets R of S have the property that all the points of R lie on the graph of a single cubic? A cubic is a polynomial of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c, and d are real numbers (meaning that a is allowed to be 0).

10. Call an $2n$-digit number *special* if we can split its digits into two sets of size n such that the sum of the numbers in the two sets is the same. Let p_n be the probability that a randomly-chosen $2n$-digit number is special (we will allow leading zeros in the number).

    (a) [**25**] The sequence p_n converges to a constant c. Find c.

    (b) [**30**] Let $q_n = p_n - c$. There exists a unique positive constant r such that $\frac{q_n}{r^n}$ converges to a constant d. Find r and d.

3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

General Test

1. [**2**] Jacob flips five coins, exactly three of which land heads. What is the probability that the first two are both heads?

2. [**3**] How many sequences $a_1, a_2, \ldots, a_8$ of zeroes and ones have $a_1a_2 + a_2a_3 + \cdots + a_7a_8 = 5$?

3. [**3**] Triangle ABC has $AB = 5$, $BC = 7$, and $CA = 8$. New lines not containing but parallel to AB, BC, and CA are drawn tangent to the incircle of ABC. What is the area of the hexagon formed by the sides of the original triangle and the newly drawn lines?

4. [**4**] An ant starts at the point $(1, 0)$. Each minute, it walks from its current position to one of the four adjacent lattice points until it reaches a point (x, y) with $|x| + |y| \geq 2$. What is the probability that the ant ends at the point $(1, 1)$?

5. [**5**] A polynomial P is of the form $\pm x^6 \pm x^5 \pm x^4 \pm x^3 \pm x^2 \pm x \pm 1$. Given that $P(2) = 27$, what is $P(3)$?

6. [**5**] What is the sum of the positive solutions to $2x^2 - x\lfloor x \rfloor = 5$, where $\lfloor x \rfloor$ is the largest integer less than or equal to x?

7. [**6**] What is the remainder when $(1 + x)^{2010}$ is divided by $1 + x + x^2$?

8. [**7**] Two circles with radius one are drawn in the coordinate plane, one with center $(0, 1)$ and the other with center $(2, y)$, for some real number y between 0 and 1. A third circle is drawn so as to be tangent to both of the other two circles as well as the x axis. What is the smallest possible radius for this third circle?

9. [**7**] What is the sum of all numbers between 0 and 511 inclusive that have an even number of 1s when written in binary?

10. [**8**] You are given two diameters AB and CD of circle Ω with radius 1. A circle is drawn in one of the smaller sectors formed such that it is tangent to AB at E, tangent to CD at F, and tangent to Ω at P. Lines PE and PF intersect Ω again at X and Y. What is the length of XY, given that $AC = \frac{2}{3}$?

3^{rd}Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Guts Round

1. [**5**] David, Delong, and Justin each showed up to a problem writing session at a random time during the session. If David arrived before Delong, what is the probability that he also arrived before Justin?

2. [**5**] A circle of radius 6 is drawn centered at the origin. How many squares of side length 1 and integer coordinate vertices intersect the interior of this circle?

3. [**5**] Jacob flipped a fair coin five times. In the first three flips, the coin came up heads exactly twice. In the last three flips, the coin also came up heads exactly twice. What is the probability that the third flip was heads?

4. [**6**] Let x be a real number. Find the maximum value of $2^{x(1-x)}$.

5. [**6**] An icosahedron is a regular polyhedron with twenty faces, all of which are equilateral triangles. If an icosahedron is rotated by θ degrees around an axis that passes through two opposite vertices so that it occupies exactly the same region of space as before, what is the smallest possible positive value of θ?

6. [**6**] How many ordered pairs (S, T) of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ are there whose union contains exactly three elements?

7. [**7**] Let $f(x, y) = x^2 + 2x + y^2 + 4y$. Let (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4) be the vertices of a square with side length one and sides parallel to the coordinate axes. What is the minimum value of $f(x_1, y_1) + f(x_2, y_2) + f(x_3, y_3) + f(x_4, y_4)$?

8. [**7**] What is the sum of all four-digit numbers that are equal to the cube of the sum of their digits (leading zeros are not allowed)?

9. [**7**] How many functions $f : \{1, 2, \ldots, 10\} \to \{1, 2, \ldots, 10\}$ satisfy the property that $f(i) + f(j) = 11$ for all values of i and j such that $i + j = 11$.

10. [**8**] What is the smallest integer greater than 10 such that the sum of the digits in its base 17 representation is equal to the sum of the digits in its base 10 representation?

11. [**8**] How many nondecreasing sequences $a_1, a_2, \ldots, a_{10}$ are composed entirely of at most three distinct numbers from the set $\{1, 2, \ldots, 9\}$ (so $1, 1, 1, 2, 2, 2, 3, 3, 3, 3$ and $2, 2, 2, 2, 5, 5, 5, 5, 5, 5$ are both allowed)?

12. [**8**] An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point $(5, 5)$?

13. [**8**] How many sequences of ten binary digits are there in which neither two zeroes nor three ones ever appear in a row?

14. [**8**] The positive integer i is chosen at random such that the probability of a positive integer k being chosen is $\frac{3}{2}$ times the probability of $k+1$ being chosen. What is the probability that the i^{th} digit after the decimal point of the decimal expansion of $\frac{1}{7}$ is a 2?

15. [**8**] Distinct points A, B, C, D are given such that triangles ABC and ABD are equilateral and both are of side length 10. Point E lies inside triangle ABC such that $EA = 8$ and $EB = 3$, and point F lies inside triangle ABD such that $FD = 8$ and $FB = 3$. What is the area of quadrilateral $AEFD$?

16. [**9**] Triangle ABC is given in the plane. Let AD be the angle bisector of $\angle BAC$; let BE be the altitude from B to AD, and let F be the midpoint of AB. Given that $AB = 28, BC = 33, CA = 37$, what is the length of EF?

17. [**9**] A triangle with side lengths $5, 7, 8$ is inscribed in a circle C. The diameters of C parallel to the sides of lengths 5 and 8 divide C into four sectors. What is the area of either of the two smaller ones?

18. [**9**] Jeff has a 50 point quiz at 11 am. He wakes up at a random time between 10 am and noon, then arrives at class 15 minutes later. If he arrives on time, he will get a perfect score, but if he arrives more than 30 minutes after the quiz starts, he will get a 0, but otherwise, he loses a point for each minute he's late (he can lose parts of one point if he arrives a nonintegral number of minutes late). What is Jeff's expected score on the quiz?

19. [**11**] How many 8-digit numbers begin with 1, end with 3, and have the property that each successive digit is either one more or two more than the previous digit, considering 0 to be one more than 9?

20. [**11**] Given a permutation π of the set $\{1, 2, \ldots, 10\}$, define a rotated cycle as a set of three integers i, j, k such that $i < j < k$ and $\pi(j) < \pi(k) < \pi(i)$. What is the total number of rotated cycles over all permutations π of the set $\{1, 2, \ldots, 10\}$?

21. [**11**] George, Jeff, Brian, and Travis decide to play a game of hot potato. They begin by arranging themselves clockwise in a circle in that order. George and Jeff both start with a hot potato. On his turn, a player gives a hot potato (if he has one) to a randomly chosen player among the other three (if a player has two hot potatoes on his turn, he only passes one). If George goes first, and play proceedes clockwise, what is the probability that Travis has a hot potato after each player takes one turn?

22. [**12**] Let $g_1(x) = \frac{1}{3}(1 + x + x^2 + \cdots)$ for all values of x for which the right hand side converges. Let $g_n(x) = g_1(g_{n-1}(x))$ for all integers $n \geq 2$. What is the largest integer r such that $g_r(x)$ is defined for some real number x?

23. [**12**] Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers such that for integers $n > 2$, $a_n = 3a_{n-1} - 2a_{n-2}$. How many such sequences $\{a_n\}$ are there such that $a_{2010} \leq 2^{2012}$?

24. [**12**] Let $P(x)$ be a polynomial of degree at most 3 such that $P(x) = \frac{1}{1+x+x^2}$ for $x = 1, 2, 3, 4$. What is $P(5)$?

25. [**14**] Triangle ABC is given with $AB = 13$, $BC = 14$, $CA = 15$. Let E and F be the feet of the altitudes from B and C, respectively. Let G be the foot of the altitude from A in triangle AFE. Find AG.

26. [**14**] w, x, y, z are real numbers such that

$$\begin{aligned} w + x + y + z &= 5 \\ 2w + 4x + 8y + 16z &= 7 \\ 3w + 9x + 27y + 81z &= 11 \\ 4w + 16x + 64y + 256z &= 1 \end{aligned}$$

What is the value of $5w + 25x + 125y + 625z$?

27. [**14**] Let $f(x) = -x^2 + 10x - 20$. Find the sum of all 2^{2010} solutions to $\underbrace{f(f(\ldots(x)\ldots))}_{2010 fs} = 2$.

28. [**17**] In the game of set, each card has four attributes, each of which takes on one of three values. A set deck consists of one card for each of the 81 possible four-tuples of attributes. Given a collection of 3 cards, call an attribute *good* for that collection if the three cards either all take on the same value of that attribute or take on all three different values of that attribute. Call a collection of 3 cards *two-good* if exactly two attributes are good for that collection. How many two-good collections of 3 cards are there? The order in which the cards appear does not matter.

29. [**17**] In the game of Galactic Dominion, players compete to amass cards, each of which is worth a certain number of points. Say you are playing a version of this game with only two kinds of cards, planet cards and hegemon cards. Each planet card is worth 2010 points, and each hegemon card is

worth four points per planet card held. You start with no planet cards and no hegemon cards, and, on each turn, starting at turn one, you take either a planet card or a hegemon card, whichever is worth more points given the hand you currently hold. Define a sequence $\{a_n\}$ for all positive integers n by setting a_n to be 0 if on turn n you take a planet card and 1 if you take a hegemon card. What is the smallest value of N such that the sequence $a_N, a_{N+1}, \ldots$ is necessarily periodic (meaning that there is a positive integer k such that $a_{n+k} = a_n$ for all $n \geq N$)?

30. [**17**] In the game of projective set, each card contains some nonempty subset of six distinguishable dots. A projective set deck consists of one card for each of the 63 possible nonempty subsets of dots. How many collections of five cards have an even number of each dot? The order in which the cards appear does not matter.

31. [**20**] What is the perimeter of the triangle formed by the points of tangency of the incircle of a 5-7-8 triangle with its sides?

32. [**20**] Let T be the set of numbers of the form $2^a 3^b$ where a and b are integers satisfying $0 \leq a, b \leq 5$. How many subsets S of T have the property that if n is in S then all positive integer divisors of n are in S?

33. [**20**] Convex quadrilateral $BCDE$ lies in the plane. Lines EB and DC intersect at A, with $AB = 2$, $AC = 5$, $AD = 200$, $AE = 500$, and $\cos \angle BAC = \frac{7}{9}$. What is the largest number of nonoverlapping circles that can lie in quadrilateral $BCDE$ such that all of them are tangent to both lines BE and CD?

34. [**25**] Estimate the sum of all the prime numbers less than $1,000,000$. If the correct answer is X and you write down A, your team will receive $\min\left(\lfloor \frac{25X}{A} \rfloor, \lfloor \frac{25A}{X} \rfloor\right)$ points, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

35. [**25**] A mathematician M' is called a descendent of mathematician M if there is a sequence of mathematicians $M = M_1, M_2, \ldots, M_k = M'$ such that M_i was M_{i+1}'s doctoral advisor for all i. Estimate the number of descendents that the mathematician who has had the largest number of descendents has had, according to the Mathematical Genealogy Project. Note that the Mathematical Genealogy Project has records dating back to the 1300s. If the correct answer is X and you write down A, your team will receive $\max\left(25 - \lfloor \frac{|X-A|}{100} \rfloor, 0\right)$ points, where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

36. [**25**] Paul Erdős was one of the most prolific mathematicians of all time and was renowned for his many collaborations. The Erdős number of a mathematician is defined as follows. Erdős has an Erdős number of 0, a mathematician who has coauthored a paper with Erdős has an Erdős number of 1, a mathematician who has not coauthored a paper with Erdős, but has coauthored a paper with a mathematician with Erdős number 1 has an Erdős number of 2, etc. If no such chain exists between Erdős and another mathematician, that mathematician has an Erdős number of infinity. Of the mathematicians with a finite Erdős number (including those who are no longer alive), what is their average Erdős number according to the Erdős Number Project? If the correct answer is X and you write down A, your team will receive $\max(25 - \lfloor 100|X - A| \rfloor, 0)$ points where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

3rd Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Team Round

Polyhedron Hopping

1. [**3**] Travis is hopping around on the vertices of a cube. Each minute he hops from the vertex he's currently on to the other vertex of an edge that he is next to. After four minutes, what is the probability that he is back where he started?

2. [**6**] In terms of k, for $k > 0$ how likely is he to be back where he started after $2k$ minutes?

3. [**3**] While Travis is having fun on cubes, Sherry is hopping in the same manner on an octahedron. An octahedron has six vertices and eight regular triangular faces. After five minutes, how likely is Sherry to be one edge away from where she started?

4. [**6**] In terms of k, for $k > 0$, how likely is it that after k minutes Sherry is at the vertex opposite the vertex where she started?

Circles in Circles

5. [**4**] Circle O has chord AB. A circle is tangent to O at T and tangent to AB at X such that $AX = 2XB$. What is $\frac{AT}{BT}$?

6. [**6**] AB is a diameter of circle O. X is a point on AB such that $AX = 3BX$. Distinct circles ω_1 and ω_2 are tangent to O at T_1 and T_2 and to AB at X. The lines T_1X and T_2X intersect O again at S_1 and S_2. What is the ratio $\frac{T_1T_2}{S_1S_2}$?

7. [**7**] ABC is a right triangle with $\angle A = 30^\circ$ and circumcircle O. Circles ω_1, ω_2, and ω_3 lie outside ABC and are tangent to O at T_1, T_2, and T_3 respectively and to AB, BC, and CA at S_1, S_2, and S_3, respectively. Lines T_1S_1, T_2S_2, and T_3S_3 intersect O again at A', B', and C', respectively. What is the ratio of the area of $A'B'C'$ to the area of ABC?

Linear? What's The Problem?

A function $f(x_1, x_2, \ldots, x_n)$ is said to be linear in each of its variables if it is a polynomial such that no variable appears with power higher than one in any term. For example, $1 + x + xy$ is linear in x and y, but $1 + x^2$ is not. Similarly, $2x + 3yz$ is linear in x, y, and z, but xyz^2 is not.

8. [**4**] A function $f(x, y)$ is linear in x and in y. $f(x, y) = \frac{1}{xy}$ for $x, y \in \{3, 4\}$. What is $f(5, 5)$?

9. [**5**] A function $f(x, y, z)$ is linear in x, y, and z such that $f(x, y, z) = \frac{1}{xyz}$ for $x, y, z \in \{3, 4\}$. What is $f(5, 5, 5)$?

10. [**6**] A function $f(x_1, x_2, \ldots, x_n)$ is linear in each of the x_i and $f(x_1, x_2, \ldots, x_n) = \frac{1}{x_1 x_2 \cdots x_n}$ when $x_i \in \{3, 4\}$ for all i. In terms of n, what is $f(5, 5, \ldots, 5)$?

3^{rd} Annual Harvard-MIT November Tournament

Sunday 7 November 2010

Theme Round

StarCraft

1. [**3**] 16 progamers are playing in a single elimination tournament. Each player has a different skill level and when two play against each other the one with the higher skill level will always win. Each round, each progamer plays a match against another and the loser is eliminated. This continues until only one remains. How many different progamers can reach the round that has 2 players remaining?

2. [**4**] 16 progamers are playing in another single elimination tournament. Each round, each of the remaining progamers plays against another and the loser is eliminated. Additionally, each time a progamer wins, he will have a ceremony to celebrate. A player's first ceremony is ten seconds long, and afterward each ceremony is ten seconds longer than the last. What is the total length in seconds of all the ceremonies over the entire tournament?

3. [**5**] Dragoons take up 1×1 squares in the plane with sides parallel to the coordinate axes such that the interiors of the squares do not intersect. A dragoon can fire at another dragoon if the difference in the x-coordinates of their centers and the difference in the y-coordinates of their centers are both at most 6, regardless of any dragoons in between. For example, a dragoon centered at $(4, 5)$ can fire at a dragoon centered at the origin, but a dragoon centered at $(7, 0)$ can not. A dragoon cannot fire at itself. What is the maximum number of dragoons that can fire at a single dragoon simultaneously?

4. [**5**] A zerg player can produce one zergling every minute and a protoss player can produce one zealot every 2.1 minutes. Both players begin building their respective units immediately from the beginning of the game. In a fight, a zergling army overpowers a zealot army if the ratio of zerglings to zealots is more than 3. What is the total amount of time (in minutes) during the game such that at that time the zergling army would overpower the zealot army?

5. [**7**] There are 111 StarCraft progamers. The StarCraft team SKT starts with a given set of eleven progamers on it, and at the end of each season, it drops a progamer and adds a progamer (possibly the same one). At the start of the second season, SKT has to field a team of five progamers to play the opening match. How many different lineups of five players could be fielded if the order of players on the lineup matters?

Unfair Coins

6. [**4**] When flipped, a coin has a probability p of landing heads. When flipped twice, it is twice as likely to land on the same side both times as it is to land on each side once. What is the larger possible value of p?

7. [**4**] George has two coins, one of which is fair and the other of which always comes up heads. Jacob takes one of them at random and flips it twice. Given that it came up heads both times, what is the probability that it is the coin that always comes up heads?

8. [**5**] Allison has a coin which comes up heads $\frac{2}{3}$ of the time. She flips it 5 times. What is the probability that she sees more heads than tails?

9. [**6**] Newton and Leibniz are playing a game with a coin that comes up heads with probability p. They take turns flipping the coin until one of them wins with Newton going first. Newton wins if he flips a heads and Leibniz wins if he flips a tails. Given that Newton and Leibniz each win the game half of the time, what is the probability p?

10. [**7**] Justine has a coin which will come up the same as the last flip $\frac{2}{3}$ of the time and the other side $\frac{1}{3}$ of the time. She flips it and it comes up heads. She then flips it 2010 more times. What is the probability that the last flip is heads?

12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Algebra Test

1. [**3**] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

2. [**3**] Let S be the sum of all the real coefficients of the expansion of $(1 + ix)^{2009}$. What is $\log_2(S)$?

3. [**4**] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

4. [**4**] Suppose a, b and c are integers such that the greatest common divisor of $x^2 + ax + b$ and $x^2 + bx + c$ is $x + 1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of $x^2 + ax + b$ and $x^2 + bx + c$ is $x^3 - 4x^2 + x + 6$. Find $a + b + c$.

5. [**4**] Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

6. [**5**] Let x and y be positive real numbers and θ an angle such that $\theta \neq \frac{\pi}{2}n$ for any integer n. Suppose
$$\frac{\sin\theta}{x} = \frac{\cos\theta}{y}$$
and
$$\frac{\cos^4\theta}{x^4} + \frac{\sin^4\theta}{y^4} = \frac{97\sin 2\theta}{x^3y + y^3x}.$$
Compute $\frac{x}{y} + \frac{y}{x}$.

7. [**5**] Simplify the product
$$\prod_{m=1}^{100}\prod_{n=1}^{100}\frac{x^{n+m} + x^{n+m+2} + x^{2n+1} + x^{2m+1}}{x^{2n} + 2x^{n+m} + x^{2m}}.$$
Express your answer in terms of x.

8. [**7**] If a, b, x and y are real numbers such that $ax + by = 3$, $ax^2 + by^2 = 7$, $ax^3 + by^3 = 16$, and $ax^4 + by^4 = 42$, find $ax^5 + by^5$.

9. [**7**] Let $f(x) = x^4 + 14x^3 + 52x^2 + 56x + 16$. Let z_1, z_2, z_3, z_4 be the four roots of f. Find the smallest possible value of $|z_a z_b + z_c z_d|$ where $\{a, b, c, d\} = \{1, 2, 3, 4\}$.

10. [**8**] Let $f(x) = 2x^3 - 2x$. For what positive values of a do there exist distinct b, c, d such that $(a, f(a))$, $(b, f(b))$, $(c, f(c))$, $(d, f(d))$ is a rectangle?

12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Calculus Test

1. [**3**] Let f be a differentiable real-valued function defined on the positive real numbers. The tangent lines to the graph of f always meet the y-axis 1 unit lower than where they meet the function. If $f(1) = 0$, what is $f(2)$?

2. [**3**] The differentiable function $F : \mathbb{R} \to \mathbb{R}$ satisfies $F(0) = -1$ and
$$\frac{d}{dx}F(x) = \sin(\sin(\sin(\sin(x)))) \cdot \cos(\sin(\sin(x))) \cdot \cos(\sin(x)) \cdot \cos(x).$$
Find $F(x)$ as a function of x.

3. [**4**] Compute e^A where A is defined as
$$\int_{3/4}^{4/3} \frac{2x^2 + x + 1}{x^3 + x^2 + x + 1}\,dx.$$

4. [**4**] Let P be a fourth degree polynomial, with derivative P', such that $P(1) = P(3) = P(5) = P'(7) = 0$. Find the real number $x \neq 1, 3, 5$ such that $P(x) = 0$.

5. [**4**] Compute
$$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{3} + 4h\right) - 4\sin\left(\frac{\pi}{3} + 3h\right) + 6\sin\left(\frac{\pi}{3} + 2h\right) - 4\sin\left(\frac{\pi}{3} + h\right) + \sin\left(\frac{\pi}{3}\right)}{h^4}.$$

6. [**5**] Let $p_0(x), p_1(x), p_2(x), \ldots$ be polynomials such that $p_0(x) = x$ and for all positive integers n, $\frac{d}{dx}p_n(x) = p_{n-1}(x)$. Define the function $p(x) : [0, \infty) \to \mathbb{R}$ by $p(x) = p_n(x)$ for all $x \in [n, n+1)$. Given that $p(x)$ is continuous on $[0, \infty)$, compute
$$\sum_{n=0}^{\infty} p_n(2009).$$

7. [**5**] A line in the plane is called *strange* if it passes through $(a, 0)$ and $(0, 10 - a)$ for some a in the interval $[0, 10]$. A point in the plane is called *charming* if it lies in the first quadrant and also lies below some strange line. What is the area of the set of all charming points?

8. [**7**] Compute
$$\int_1^{\sqrt{3}} x^{2x^2+1} + \ln\left(x^{2x^{2x^2+1}}\right) dx.$$

9. [**7**] Let $\mathcal{R}$ be the region in the plane bounded by the graphs of $y = x$ and $y = x^2$. Compute the volume of the region formed by revolving $\mathcal{R}$ around the line $y = x$.

10. [**8**] Let a and b be real numbers satisfying $a > b > 0$. Evaluate
$$\int_0^{2\pi} \frac{1}{a + b\cos(\theta)}\,d\theta.$$
Express your answer in terms of a and b.

12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Combinatorics Test

1. [**3**] How many ways can the integers from -7 to 7 inclusive be arranged in a sequence such that the absolute value of the numbers in the sequence does not decrease?

2. [**3**] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

3. [**4**] How many rearrangements of the letters of "HMMTHMMT" do not contain the substring "HMMT"? (For instance, one such arrangement is HMMHMTMT.)

4. [**4**] How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

5. [**4**] Let $s(n)$ denote the number of 1's in the binary representation of n. Compute
$$\frac{1}{255} \sum_{0 \le n < 16} 2^n (-1)^{s(n)}.$$

6. [**5**] How many sequences of 5 positive integers (a, b, c, d, e) satisfy $abcde \le a + b + c + d + e \le 10$?

7. [**5**] Paul fills in a 7×7 grid with the numbers 1 through 49 in a random arrangement. He then erases his work and does the same thing again, to obtain two different random arrangements of the numbers in the grid. What is the expected number of pairs of numbers that occur in either the same row as each other or the same column as each other in both of the two arrangements?

8. [**7**] There are 5 students on a team for a math competition. The math competition has 5 subject tests. Each student on the team must choose 2 distinct tests, and each test must be taken by exactly two people. In how many ways can this be done?

9. [**7**] The squares of a 3×3 grid are filled with positive integers such that 1 is the label of the upper-leftmost square, 2009 is the label of the lower-rightmost square, and the label of each square divides the one directly to the right of it and the one directly below it. How many such labelings are possible?

10. [**8**] Given a rearrangement of the numbers from 1 to n, each pair of consecutive elements a and b of the sequence can be either increasing (if $a < b$) or decreasing (if $b < a$). How many rearrangements of the numbers from 1 to n have exactly two increasing pairs of consecutive elements? Express your answer in terms of n.

12^{th} Annual Harvard-MIT Mathematics Tournament

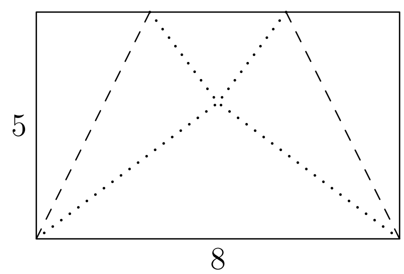
Saturday 21 February 2009

Individual Round: General Test, Part 1

1. [**2**] If a and b are positive integers such that $a^2 - b^4 = 2009$, find $a + b$.

2. [**2**] Suppose N is a 6-digit number having base-10 representation $\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}\,\underline{f}$. If N is 6/7 of the number having base-10 representation $\underline{d}\,\underline{e}\,\underline{f}\,\underline{a}\,\underline{b}\,\underline{c}$, find N.

3. [**3**] A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



4. [**3**] If $\tan x + \tan y = 4$ and $\cot x + \cot y = 5$, compute $\tan(x + y)$.

5. [**4**] Two jokers are added to a 52 card deck and the entire stack of 54 cards is shuffled randomly. What is the expected number of cards that will be strictly between the two jokers?

6. [**4**] The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

7. [**5**] Let $s(n)$ denote the number of 1's in the binary representation of n. Compute
$$\frac{1}{255} \sum_{0 \le n < 16} 2^n (-1)^{s(n)}.$$

8. [**5**] Let a, b, and c be the 3 roots of $x^3 - x + 1 = 0$. Find $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$.

9. [**6**] How many functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ satisfy $f(f(x)) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$?

10. [**6**] A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD. Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω. Let N' be the point of tangency of $B'C'$ to ω. If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

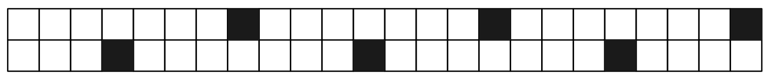
12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: General Test, Part 2

1. [**2**] How many ways can the integers from -7 to 7 be arranged in a sequence such that the absolute values of the numbers in the sequence are nonincreasing?

2. [**2**] How many ways can you tile the white squares of following 2×24 grid with dominoes? (A domino covers two adjacent squares, and a tiling is a non-overlapping arrangement of dominoes that covers every white square and does not intersect any black square.)



3. [**3**] Let S be the sum of all the real coefficients of the expansion of $(1+ix)^{2009}$. What is $\log_2(S)$?

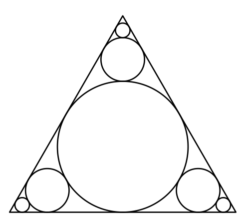
4. [**3**] A torus (donut) having inner radius 2 and outer radius 4 sits on a flat table. What is the radius of the largest spherical ball that can be placed on top of the center torus so that the ball still touches the horizontal plane? (If the $x-y$ plane is the table, the torus is formed by revolving the circle in the $x-z$ plane centered at $(3,0,1)$ with radius 1 about the z axis. The spherical ball has its center on the z-axis and rests on either the table or the donut.)

5. [**4**] Suppose a, b and c are integers such that the greatest common divisor of x^2+ax+b and x^2+bx+c is $x+1$ (in the set of polynomials in x with integer coefficients), and the least common multiple of x^2+ax+b and x^2+bx+c is x^3-4x^2+x+6. Find $a+b+c$.

6. [**4**] In how many ways can you rearrange the letters of "HMMTHMMT" such that the substring "HMMT" does not appear? (For instance, one such rearrangement is HMMHMTMT.)

7. [**5**] Let F_n be the Fibonacci sequence, that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. Compute $\sum_{n=0}^{\infty} F_n/10^n$.

8. [**5**] The incircle ω of equilateral triangle ABC has radius 1. Three smaller circles are inscribed tangent to ω and the sides of ABC, as shown. Three smaller circles are then inscribed tangent to the previous circles and to each of two sides of ABC. This process is repeated an infinite number of times. What is the total length of the circumferences of all the circles?



9. [**6**] How many sequences of 5 positive integers (a,b,c,d,e) satisfy $abcde \le a+b+c+d+e \le 10$?

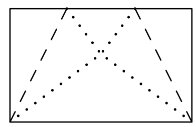
10. [**6**] Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called "awesome" if P is the center of a parallelogram whose vertices all lie on the boundary of T. What is the area of the set of awesome points?

12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Individual Round: Geometry Test

1. [**3**] A rectangular piece of paper with side lengths 5 by 8 is folded along the dashed lines shown below, so that the folded flaps just touch at the corners as shown by the dotted lines. Find the area of the resulting trapezoid.



2. [**3**] The corner of a unit cube is chopped off such that the cut runs through the three vertices adjacent to the vertex of the chosen corner. What is the height of the cube when the freshly-cut face is placed on a table?

3. [**4**] Let T be a right triangle with sides having lengths 3, 4, and 5. A point P is called *awesome* if P is the center of a parallelogram whose vertices all lie on the boundary of T. What is the area of the set of awesome points?

4. [**4**] A *kite* is a quadrilateral whose diagonals are perpendicular. Let kite $ABCD$ be such that $\angle B = \angle D = 90^\circ$. Let M and N be the points of tangency of the incircle of $ABCD$ to AB and BC respectively. Let ω be the circle centered at C and tangent to AB and AD. Construct another kite $AB'C'D'$ that is similar to $ABCD$ and whose incircle is ω. Let N' be the point of tangency of $B'C'$ to ω. If $MN' \parallel AC$, then what is the ratio of $AB : BC$?

5. [**4**] Circle B has radius $6\sqrt{7}$. Circle A, centered at point C, has radius $\sqrt{7}$ and is contained in B. Let L be the locus of centers C such that there exists a point D on the boundary of B with the following property: if the tangents from D to circle A intersect circle B again at X and Y, then XY is also tangent to A. Find the area contained by the boundary of L.

6. [**5**] Let ABC be a triangle in the coordinate plane with vertices on lattice points and with $AB = 1$. Suppose the perimeter of ABC is less than 17. Find the largest possible value of $1/r$, where r is the inradius of ABC.

7. [**5**] In triangle ABC, D is the midpoint of BC, E is the foot of the perpendicular from A to BC, and F is the foot of the perpendicular from D to AC. Given that $BE = 5$, $EC = 9$, and the area of triangle ABC is 84, compute $|EF|$.

8. [**7**] Triangle ABC has side lengths $AB = 231, BC = 160$, and $AC = 281$. Point D is constructed on the opposite side of line AC as point B such that $AD = 178$ and $CD = 153$. Compute the distance from B to the midpoint of segment AD.

9. [**7**] Let ABC be a triangle with $AB = 16$ and $AC = 5$. Suppose the bisectors of angles $\angle ABC$ and $\angle BCA$ meet at point P in the triangle's interior. Given that $AP = 4$, compute BC.

10. [**8**] Points A and B lie on circle ω. Point P lies on the extension of segment AB past B. Line ℓ passes through P and is tangent to ω. The tangents to ω at points A and B intersect ℓ at points D and C respectively. Given that $AB = 7, BC = 2$, and $AD = 3$, compute BP.

12^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 21 February 2009

Guts Round

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12th HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

1. [**5**] Compute
$$1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \cdots + 19 \cdot 20^2.$$

2. [**5**] Given that $\sin A + \sin B = 1$ and $\cos A + \cos B = 3/2$, what is the value of $\cos(A - B)$?

3. [**5**] Find all pairs of integer solutions (n, m) to
$$2^{3^n} = 3^{2^m} - 1.$$

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12th HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

4. [**6**] Simplify: $i^0 + i^1 + \cdots + i^{2009}$.

5. [**6**] In how many distinct ways can you color each of the vertices of a tetrahedron either red, blue, or green such that no face has all three vertices the same color? (Two colorings are considered the same if one coloring can be rotated in three dimensions to obtain the other.)

6. [**6**] Let ABC be a right triangle with hypotenuse AC. Let B' be the reflection of point B across AC, and let C' be the reflection of C across AB'. Find the ratio of $[BCB']$ to $[BC'B']$.

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12th HARVARD-MIT MATHEMATICS TOURNAMENT, 21 FEBRUARY 2009 — GUTS ROUND

7. [**6**] How many perfect squares divide $2^3 \cdot 3^5 \cdot 5^7 \cdot 7^9$?

8. [**6**] Which is greater, $\log_{2008}(2009)$ or $\log_{2009}(2010)$?

9. [**6**] An icosidodecahedron is a convex polyhedron with 20 triangular faces and 12 pentagonal faces. How many vertices does it have?

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10. [**7**] Let a, b, and c be real numbers. Consider the system of simultaneous equations in x and y:

$$\begin{aligned} ax + by &= c - 1 \\ (a+5)x + (b+3)y &= c + 1 \end{aligned}$$

Determine the value(s) of c, in terms of a, such that the system has a solution for any a and b.

11. [**7**] There are 2008 distinct points on a circle. If you connect two of these points to form a line and then connect another two points (distinct from the first two) to form another line, what is the probability that the two lines intersect inside the circle?

12. [**7**] Bob is writing a sequence of letters of the alphabet, each of which can be either uppercase or lowercase, according to the following two rules:

- If he had just written an **uppercase** letter, he can either write the same letter in **lowercase** after it, or the **next** letter of the alphabet in **uppercase**.
- If he had just written a **lowercase** letter, he can either write the same letter in **uppercase** after it, or the **preceding** letter of the alphabet in **lowercase**.

For instance, one such sequence is $aAaABCDdcbBC$. How many sequences of 32 letters can he write that start at (lowercase) a and end at (lowercase) z? (The alphabet contains 26 letters from a to z, without wrapping around, so that z does not precede a.)

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13. [**8**] How many ordered quadruples (a, b, c, d) of four distinct numbers chosen from the set $\{1, 2, 3, \ldots, 9\}$ satisfy $b < a$, $b < c$, and $d < c$?

14. [**8**] Compute

$$\sum_{k=1}^{2009} k\left(\left\lfloor \frac{2009}{k} \right\rfloor - \left\lfloor \frac{2008}{k} \right\rfloor\right).$$

Here $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

15. [**8**] Stan has a stack of 100 blocks and starts with a score of 0, and plays a game in which he iterates the following two-step procedure:

(a) Stan picks a stack of blocks and splits it into 2 smaller stacks each with a positive number of blocks, say a and b. (The order in which the new piles are placed does not matter.)

(b) Stan adds the product of the two piles' sizes, ab, to his score.

The game ends when there are only 1-block stacks left. What is the expected value of Stan's score at the end of the game?

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16. [**9**] A spider is making a web between $n > 1$ distinct leaves which are equally spaced around a circle. He chooses a leaf to start at, and to make the base layer he travels to each leaf one at a time, making straight lines of silk from one leaf to another, such that no two of the lines of silk cross each other and he visits every leaf exactly once. In how many ways can the spider make the base layer of the web? Express your answer in terms of n.

17. [**9**] How many positive integers $n \leq 2009$ have the property that $\lfloor \log_2(n) \rfloor$ is odd?

18. [**9**] Find the positive integer n such that $n^3 + 2n^2 + 9n + 8$ is the cube of an integer.

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19. [**10**] Shelly writes down a vector $v = (a, b, c, d)$, where $0 < a < b < c < d$ are integers. Let $\sigma(v)$ denote the set of 24 vectors whose coordinates are a, b, c, and d in some order. For instance, $\sigma(v)$ contains (b, c, d, a). Shelly notes that there are 3 vectors in $\sigma(v)$ whose sum is of the form (s, s, s, s) for some s. What is the smallest possible value of d?

20. [**10**] A positive integer is called *jubilant* if the number of 1's in its binary representation is even. For example, $6 = 110_2$ is a jubilant number. What is the 2009th smallest jubilant number?

21. [**10**] Simplify
$$2\cos^2(\ln(2009)i) + i\sin(\ln(4036081)i).$$

. . . .

22. [**10**] A circle having radius r_1 centered at point N is externally tangent to a circle of radius r_2 centered at M. Let l and j be the two common external tangent lines to the two circles. A circle centered at P with radius r_2 is externally tangent to circle N at the point at which l coincides with circle N, and line k is externally tangent to P and N such that points M, N, and P all lie on the same side of k. For what ratio r_1/r_2 are j and k parallel?

23. [**10**] The roots of $z^6 + z^4 + z^2 + 1 = 0$ are the vertices of a convex polygon in the complex plane. Find the sum of the squares of the side lengths of the polygon.

24. [**10**] Compute, in terms of n,
$$\sum_{k=0}^{n} \binom{n-k}{k} 2^k.$$
Note that whenever $s < t$, $\binom{s}{t} = 0$.

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25. [**12**] Four points, A, B, C, and D, are chosen randomly on the circumference of a circle with independent uniform probability. What is the expected number of sides of triangle ABC for which the projection of D onto the line containing the side lies between the two vertices?

26. [**12**] Define the sequence $\{x_i\}_{i\geq 0}$ by $x_0 = 2009$ and $x_n = -\frac{2009}{n}\sum_{k=0}^{n-1} x_k$ for all $n \geq 1$. Compute the value of $\sum_{k=0}^{2009} 2^n x_n$.

27. [**12**] Circle Ω has radius 5. Points A and B lie on Ω such that chord AB has length 6. A unit circle ω is tangent to chord AB at point T. Given that ω is also internally tangent to Ω, find $AT \cdot BT$.

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28. [**15**] The vertices of a regular hexagon are labeled $\cos(\theta), \cos(2\theta), \ldots, \cos(6\theta)$. For every pair of vertices, Benjamin draws a blue line through the vertices if one of these functions can be expressed as a polynomial function of the other (that holds for all real θ), and otherwise Roberta draws a red line through the vertices. In the resulting graph, how many triangles whose vertices lie on the hexagon have at least one red and at least one blue edge?

29. [**15**] The average of a set of distinct primes is 27. What is the largest prime that can be in this set?

30. [**15**] Let f be a polynomial with integer coefficients such that the greatest common divisor of all its coefficients is 1. For any $n \in \mathbb{N}$, $f(n)$ is a multiple of 85. Find the smallest possible degree of f.

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31. [**18**] How many ways are there to win tic-tac-toe in $\mathbb{R}^n$? (That is, how many lines pass through three of the lattice points $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$ with each coordinate a_i in $\{1, 2, 3\}$?) Express your answer in terms of n.

32. [**18**] Circle Ω has radius 13. Circle ω has radius 14 and its center P lies on the boundary of circle Ω. Points A and B lie on Ω such that chord AB has length 24 and is tangent to ω at point T. Find $AT \cdot BT$.

33. [**18**] Let m be a positive integer. Let $d(n)$ denote the number of divisors of n, and define the function

$$F(x) = \sum_{n=1}^{105^m} \frac{d(n)}{n^x}.$$

Define the numbers $a(n)$ to be the positive integers for which

$$F(x)^2 = \sum_{n=1}^{105^{2m}} \frac{a(n)}{n^x}$$

for all real x. Express $a(105^m)$ in terms of m.

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34. [$\leq$ **25**] How many integer lattice points (points of the form (m, n) for integers m and n) lie inside or on the boundary of the disk of radius 2009 centered at the origin?

If your answer is higher than the correct answer, you will receive 0 points. If your answer is d less than the correct answer, your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$.

35. [$\leq$ **25**] **Von Neumann's Poker:** The first step in Von Neumann's game is selecting a random number on $[0, 1]$. To generate this number, Chebby uses the factorial base: the number $0.A_1A_2A_3A_4\ldots$ stands for $\sum_{n=0}^{\infty} \frac{A_n}{(n+1)!}$, where each A_n is an integer between 0 and n, inclusive.

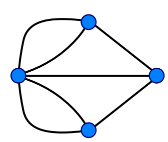
Chebby has an infinite number of cards labeled $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$. He begins by putting cards $\mathbf{0}$ and $\mathbf{1}$ into a hat and drawing randomly to determine A_1. The card assigned A_1 does not get reused. Chebby then adds in card $\mathbf{2}$ and draws for A_2, and continues in this manner to determine the random number. At each step, he only draws one card from two in the hat.

Unfortunately, this method does not result in a uniform distribution. What is the expected value of Chebby's final number?

Your score on this problem will be the larger of 0 and $\lfloor 25(1 - d) \rfloor$, where d is the positive difference between your answer and the correct answer.

36. [$\leq$ **25**] **Euler's Bridge:** The following figure is the graph of the city of Konigsburg in 1736 - vertices represent sections of the cities, edges are bridges. An *Eulerian path* through the graph is a path which moves from vertex to vertex, crossing each edge exactly once. How many ways could World War II bombers have knocked out some of the bridges of Konigsburg such that the Allied victory parade could trace an Eulerian path through the graph? (The order in which the bridges are destroyed matters.)

Your score on this problem will be the larger of 0 and $25 - \lfloor d/10 \rfloor$, where d is the positive difference between your answer and the correct answer.



12^{th} Annual Harvard-MIT Math Tournament

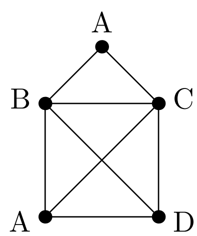
Saturday 21 February 2009

Team Round - Division A

A *graph* consists of a set of vertices and a set of edges connecting pairs of vertices. If vertices u and v are connected by an edge, we say they are *adjacent*. The *degree* of a vertex is the number of vertices adjacent to it.

A *coloring* of a graph is an assignment of a color to each vertex of the graph. A coloring is *good* if no two adjacent vertices are the same color. The *chromatic number* of a graph is the minimum number of colors needed in any good coloring of that graph.

For example, the chromatic number of the following graph is 4. The good coloring shown below uses four colors, and there is no good coloring using only three colors because four of the vertices are mutually adjacent.



1. [**8**] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n-gon is a set of $n-3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n-2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n-3$ diagonals are the edges.

   For a fixed n-gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

2. (a) [**4**] Let P be a graph with one vertex v_n for each positive integer n. If $a < b$, then an edge connects vertices v_a and v_b if and only if $\frac{b}{a}$ is a prime number. What is the chromatic number of P? Prove your answer.

   (b) [**6**] Let T be a graph with one vertex v_n for every integer n. An edge connects v_a and v_b if $|a-b|$ is a power of two. What is the chromatic number of T? Prove your answer.

3. A graph is *finite* if it has a finite number of vertices.

   (a) [**6**] Let G be a finite graph in which every vertex has degree k. Prove that the chromatic number of G is at most $k+1$.

   (b) [**10**] In terms of n, what is the minimum number of edges a finite graph with chromatic number n could have? Prove your answer.

4. A *k-clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent.

   (a) [**6**] Find a graph with chromatic number 3 that does not contain any 3-cliques.

(b) [**10**] Prove that, for all $n > 3$, there exists a graph with chromatic number n that does not contain any n-cliques.

5. The *size* of a finite graph is the number of vertices in the graph.

   (a) [**15**] Show that, for any $n > 2$, and any positive integer N, there are finite graphs with size at least N and with chromatic number n such that removing any vertex (and all its incident edges) from the graph decreases its chromatic number.

   (b) [**15**] Show that, for any positive integers n and r, there exists a positive integer N such that for any finite graph having size at least N and chromatic number equal to n, it is possible to remove r vertices (and all their incident edges) in such a way that the remaining vertices form a graph with chromatic number at least $n - 1$.

6. For any set of graphs $G_1, G_2, \ldots, G_n$ all having the same set of vertices V, define their *overlap*, denoted $G_1 \cup G_2 \cup \cdots \cup G_n$, to be the graph having vertex set V for which two vertices are adjacent in the overlap if and only if they are adjacent in at least one of the graphs G_i. In other words, the edge set of the overlap of the graphs G_i is the union of their edge sets.

   (a) [**10**] Let G and H be graphs having the same vertex set and let a be the chromatic number of G and b the chromatic number of H. Find, in terms of a and b, the largest possible chromatic number of $G \cup H$. Prove your answer.

   (b) [**10**] Suppose G is a graph with chromatic number n. Suppose there exist k graphs $G_1, G_2, \ldots, G_k$ having the same vertex set as G such that $G_1 \cup G_2 \cup \cdots \cup G_k = G$ and each G_i has chromatic number at most 2. Show that $k \geq \lceil \log_2(n) \rceil$.

7. [**20**] Let n be a positive integer. Let V_n be the set of all sequences of 0's and 1's of length n. Define G_n to be the graph having vertex set V_n, such that two sequences are adjacent in G_n if and only if they differ in either 1 or 2 places. For instance, if $n = 3$, the sequences $(1, 0, 0)$, $(1, 1, 0)$, and $(1, 1, 1)$ are mutually adjacent, but $(1, 0, 0)$ is not adjacent to $(0, 1, 1)$.

   Show that, if $n + 1$ is not a power of 2, then the chromatic number of G_n is at least $n + 2$.

8. [**30**] Two colorings are *distinct* if there is no way to relabel the colors to transform one into the other. Equivalently, they are distinct if and only if there is some pair of vertices which are the same color in one coloring but different colors in the other. For what pairs (n, k) of positive integers does there exist a finite graph with chromatic number n which has exactly k distinct good colorings using n colors?

12^{th} Annual Harvard-MIT Math Tournament

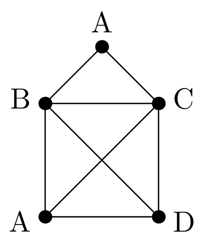
Saturday 21 February 2009

Team Round - Division B

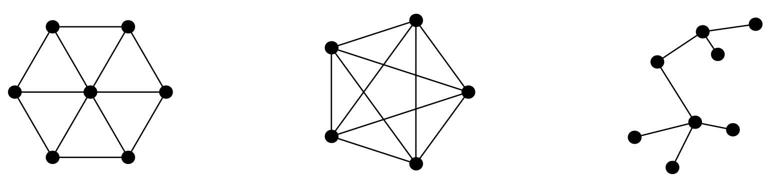
A *graph* consists of a set of vertices and a set of edges connecting pairs of vertices. If vertices u and v are connected by an edge, we say they are *adjacent*. The *degree* of a vertex u is the number of vertices v which are adjacent to u. A *finite* graph has a finite number of vertices.

A *coloring* of a graph is an assignment of a color to each vertex of the graph. A coloring is *good* if no two adjacent vertices are the same color. The *chromatic number* of a graph is the minimum number of colors needed in any good coloring of that graph.

For example, the chromatic number of the following graph is 4. The good coloring shown below uses four colors, and there is no good coloring using only three colors because four of the vertices are mutually adjacent.



1. [**6**] What are the chromatic numbers of each of the three graphs shown below? Draw a coloring having the minimum number of colors for each. Label the vertices with symbols to indicate the color of each vertex. (For example, you may mark a vertex "R" if you wish to indicate that it is colored red.)



2. [**6**] In a *connected* graph, it is possible to reach any vertex from any other vertex by following the edges. A *tree* is a connected graph with n vertices and $n-1$ edges for some positive integer n. Suppose $n \geq 2$. What is the chromatic number of a tree having n vertices? Prove your answer.

3. [**8**] Let $n \geq 3$ be a positive integer. A *triangulation* of a convex n-gon is a set of $n-3$ of its diagonals which do not intersect in the interior of the polygon. Along with the n sides, these diagonals separate the polygon into $n-2$ disjoint triangles. Any triangulation can be viewed as a graph: the vertices of the graph are the corners of the polygon, and the n sides and $n-3$ diagonals are the edges.

   For a fixed n-gon, different triangulations correspond to different graphs. Prove that all of these graphs have the same chromatic number.

4. [**10**] Let G be a finite graph in which every vertex has degree less than or equal to k. Prove that the chromatic number of G is less than or equal to $k+1$.

5. [**10**] A *k-clique* of a graph is a set of k vertices such that all pairs of vertices in the clique are adjacent. The *clique number* of a graph is the size of the largest clique in the graph. Does there exist a graph which has a clique number smaller than its chromatic number?

6. (a) [**5**] If a single vertex is removed from a finite graph, show that the graph's chromatic number cannot decrease by more than 1.

   (b) [**15**] Show that, for any $n > 2$, there are infinitely many graphs with chromatic number n such that removing any vertex from the graph decreases its chromatic number.

2^{nd} Annual Harvard-MIT November Tournament

Saturday 7 November 2009

General Test

1. [**2**] Evaluate the sum:

$$11^2 - 1^2 + 12^2 - 2^2 + 13^2 - 3^2 + \ldots + 20^2 - 10^2.$$

2. [**3**] Given that $a+b+c=5$ and that $1 \le a,b,c \le 2$, what is the minimum possible value of $\frac{1}{a+b} + \frac{1}{b+c}$?

3. [**3**] What is the period of the function $f(x) = \cos(\cos(x))$?

4. [**4**] How many subsets A of $\{1,2,3,4,5,6,7,8,9,10\}$ have the property that no two elements of A sum to 11?

5. [**5**] A polyhedron has faces that are all either triangles or squares. No two square faces share an edge, and no two triangular faces share an edge. What is the ratio of the number of triangular faces to the number of square faces?

6. [**5**] Find the maximum value of $x+y$, given that $x^2 + y^2 - 3y - 1 = 0$.

7. [**6**] There are 15 stones placed in a line. In how many ways can you mark 5 of these stones so that there are an odd number of stones between any two of the stones you marked?

8. [**7**] Let $\triangle ABC$ be an equilateral triangle with height 13, and let O be its center. Point X is chosen at random from all points inside $\triangle ABC$. Given that the circle of radius 1 centered at X lies entirely inside $\triangle ABC$, what is the probability that this circle contains O?

9. [**7**] A set of points is *convex* if the points are the vertices of a convex polygon (that is, a non-self-intersecting polygon with all angles less than or equal to $180°$). Let S be the set of points (x,y) such that x and y are integers and $1 \le x,y \le 26$. Find the number of ways to choose a convex subset of S that contains exactly 98 points.

10. [**8**] Compute

$$\prod_{n=0}^{\infty}\left(1 - \left(\frac{1}{2}\right)^{3^n} + \left(\frac{1}{4}\right)^{3^n}\right).$$

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1. [**5**] If $f(x) = x/(x+1)$, what is $f(f(f(f(2009))))$?

2. [**5**] A knight begins on the lower-left square of a standard chessboard. How many squares could the knight end up at after exactly 2009 legal knight's moves? (A knight's move is 2 squares either horizontally or vertically, followed by 1 square in a direction perpendicular to the first.)

3. [**5**] Consider a square, inside which is inscribed a circle, inside which is inscribed a square, inside which is inscribed a circle, and so on, with the outermost square having side length 1. Find the difference between the sum of the areas of the squares and the sum of the areas of the circles.

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

4. [**6**] A cube has side length 1. Find the product of the lengths of the diagonals of this cube (a diagonal is a line between two vertices that is not an edge).

5. [**6**] Tanks has a pile of 5 blue cards and 5 red cards. Every morning, he takes a card and throws it down a well. What is the probability that the first card he throws down and the last card he throws down are the same color?

6. [**6**] Find the last two digits of 1032^{1032}. Express your answer as a two-digit number.

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7. [**7**] A *computer program* is a function that takes in 4 bits, where each bit is either a 0 or a 1, and outputs TRUE or FALSE. How many computer programs are there?

8. [**7**] The angles of a convex n-sided polygon form an arithmetic progression whose common difference (in degrees) is a non-zero integer. Find the largest possible value of n for which this is possible. (A polygon is convex if its interior angles are all less than $180°$.)

9. [**7**] Daniel wrote all the positive integers from 1 to n inclusive on a piece of paper. After careful observation, he realized that the sum of all the digits that he wrote was exactly 10,000. Find n.

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10. [**8**] Admiral Ackbar needs to send a 5-character message through hyperspace to the Rebels. Each character is a lowercase letter, and the same letter may appear more than once in a message. When the message is beamed through hyperspace, the characters come out in a random order. Ackbar chooses his message so that the Rebels have at least a $\frac{1}{2}$ chance of getting the same message he sent. How many distinct messages could he send?

11. [**8**] Lily and Sarah are playing a game. They each choose a real number at random between -1 and 1. They then add the squares of their numbers together. If the result is greater than or equal to 1, Lily wins, and if the result is less than 1, Sarah wins. What is the probability that Sarah wins?

12. [**8**] Let ω be a circle of radius 1 centered at O. Let B be a point on ω, and let l be the line tangent to ω at B. Let A be on l such that $\angle AOB = 60°$. Let C be the foot of the perpendicular from B to OA. Find the length of line segment OC.

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13. [**8**] 8 students are practicing for a math contest, and they divide into pairs to take a practice test. In how many ways can they be split up?

14. [**8**] Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$ be a polynomial whose roots are all negative integers. If $a + b + c + d = 2009$, find d.

15. [**8**] The curves $x^2 + y^2 = 36$ and $y = x^2 - 7$ intersect at four points. Find the sum of the squares of the x-coordinates of these points.

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16. [**9**] Pick a random digit in the decimal expansion of $\frac{1}{99999}$. What is the probability that it is 0?

17. [**9**] A circle passes through the points $(2, 0)$ and $(4, 0)$ and is tangent to the line $y = x$. Find the sum of all possible values for the y-coordinate of the center of the circle.

18. [**9**] Let f be a function that takes in a triple of integers and outputs a real number. Suppose that f satisfies the equations

$$\begin{aligned} f(a,b,c) &= \frac{f(a+1,b,c)+f(a-1,b,c)}{2} \\ f(a,b,c) &= \frac{f(a,b+1,c)+f(a,b-1,c)}{2} \\ f(a,b,c) &= \frac{f(a,b,c+1)+f(a,b,c-1)}{2} \end{aligned}$$

for all integers a, b, c. What is the minimum number of triples at which we need to evaluate f in order to know its value everywhere?

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19. [**11**] You are trapped in ancient Japan, and a giant enemy crab is approaching! You must defeat it by cutting off its two claws and six legs and attacking its weak point for massive damage. You cannot cut off any of its claws until you cut off at least three of its legs, and you cannot attack its weak point until you have cut off all of its claws and legs. In how many ways can you defeat the giant enemy crab? (Note that the legs are distinguishable, as are the claws.)

20. [**11**] Consider an equilateral triangle and a square both inscribed in a unit circle such that one side of the square is parallel to one side of the triangle. Compute the area of the convex heptagon formed by the vertices of both the triangle and the square.

21. [**11**] Let $f(x) = x^2 + 2x + 1$. Let $g(x) = f(f(\cdots f(x)))$, where there are 2009 fs in the expression for $g(x)$. Then $g(x)$ can be written as

$$g(x) = x^{2^{2009}} + a_{2^{2009}-1}x^{2^{2009}-1} + \cdots + a_1x + a_0,$$

where the a_i are constants. Compute $a_{2^{2009}-1}$.

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22. [**12**] Five cards labeled A, B, C, D, and E are placed consecutively in a row. How many ways can they be re-arranged so that no card is moved more than one position away from where it started? (Not moving the cards at all counts as a valid re-arrangement.)

23. [**12**] Let $a_0, a_1, \ldots$ be a sequence such that $a_0 = 3$, $a_1 = 2$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$. Find

$$\sum_{n=0}^{8} \frac{a_n}{a_{n+1}a_{n+2}}.$$

24. [**12**] Penta chooses 5 of the vertices of a unit cube. What is the maximum possible volume of the figure whose vertices are the 5 chosen points?

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25. [**14**] Find all solutions to $x^4 + 2x^3 + 2x^2 + 2x + 1 = 0$ (including non-real solutions).

26. [**14**] In how many ways can the positive integers from 1 to 100 be arranged in a circle such that the sum of every two integers placed opposite each other is the same? (Arrangements that are rotations of each other count as the same.) Express your answer in the form $a! \cdot b^c$.

27. [**14**] $ABCD$ is a regular tetrahedron of volume 1. Maria glues regular tetrahedra $A'BCD$, $AB'CD$, $ABC'D$, and $ABCD'$ to the faces of $ABCD$. What is the volume of the tetrahedron $A'B'C'D'$?

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2nd ANNUAL HARVARD-MIT NOVEMBER TOURNAMENT, 7 NOVEMBER 2009 — GUTS ROUND

28. [**17**] Six men and their wives are sitting at a round table with 12 seats. These men and women are very jealous — no man will allow his wife to sit next to any man except for himself, and no woman will allow her husband to sit next to any woman except for herself. In how many distinct ways can these 12 people be seated such that these conditions are satisfied? (Rotations of a valid seating are considered distinct.)

29. [**17**] For how many integer values of b does there exist a polynomial function with integer coefficients such that $f(2) = 2010$ and $f(b) = 8$?

30. [**17**] Regular hexagon $ABCDEF$ has side length 2. A laser beam is fired inside the hexagon from point A and hits $\overline{BC}$ at point G. The laser then reflects off $\overline{BC}$ and hits the midpoint of $\overline{DE}$. Find BG.

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31. [**20**] There are two buildings facing each other, each 5 stories high. How many ways can Kevin string ziplines between the buildings so that:

    (a) each zipline starts and ends in the middle of a floor.

    (b) ziplines can go up, stay flat, or go down, but can't touch each other (this includes touching at their endpoints).

    Note that you can't string a zipline between two floors of the same building.

32. [**20**] A circle ω_1 of radius 15 intersects a circle ω_2 of radius 13 at points P and Q. Point A is on line PQ such that P is between A and Q. R and S are the points of tangency from A to ω_1 and ω_2, respectively, such that the line AS does not intersect ω_1 and the line AR does not intersect ω_2. If $PQ = 24$ and $\angle RAS$ has a measure of 90°, compute the length of AR.

33. [**20**] Compute

$$\sum_{n=2009}^{\infty} \frac{1}{\binom{n}{2009}}.$$

    Note that $\binom{n}{k}$ is defined as $\frac{n!}{k!(n-k)!}$.

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34. [**25**] How many hits does "3.1415" get on Google? Quotes are for clarity only, and not part of the search phrase. Also note that Google does not search substrings, so a webpage with 3.14159 on it will not match 3.1415. If A is your answer, and S is the correct answer, then you will get $\max(25 - |\ln(A) - \ln(S)|, 0)$ points, rounded to the nearest integer.

35. [**25**] Call an integer $n > 1$ *radical* if $2^n - 1$ is prime. What is the 20th smallest radical number? If A is your answer, and S is the correct answer, you will get $\max\left(25\left(1 - \frac{|A-S|}{S}\right), 0\right)$ points, rounded to the nearest integer.

36. [**25**] Write down a pair of integers (a, b), where $-100000 < a < b < 100000$. You will get $\max(25, k)$ points, where k is the number of other teams' pairs that you interleave. (Two pairs (a, b) and (c, d) of integers *interleave* each other if $a < c < b < d$ or $c < a < d < b$.)

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2^{nd} Annual Harvard-MIT November Tournament

Saturday 7 November 2009

Team Round

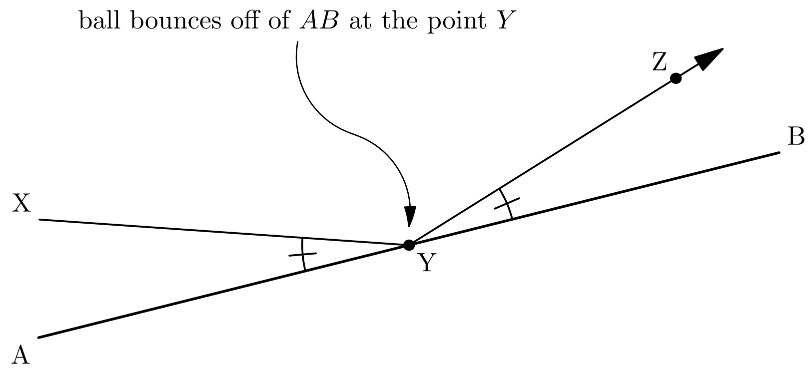
Down the Infinite Corridor

Consider an isosceles triangle T with base 10 and height 12. Define a sequence $\omega_1, \omega_2, \ldots$ of circles such that ω_1 is the incircle of T and ω_{i+1} is tangent to ω_i and both legs of the isosceles triangle for $i > 1$.

1. [**3**] Find the radius of ω_1.
2. [**3**] Find the ratio of the radius of ω_{i+1} to the radius of ω_i.
3. [**3**] Find the total area contained in all the circles.

Bouncy Balls

In the following problems, you will consider the trajectories of balls moving and bouncing off of the boundaries of various containers. The balls are small enough that you can treat them as points. Let us suppose that a ball starts at a point X, strikes a boundary (indicated by the line segment AB) at Y, and then continues, moving along the ray YZ. Balls always bounce in such a way that $\angle XYA = \angle BYZ$. This is indicated in the above diagram.



Balls bounce off of boundaries in the same way light reflects off of mirrors - if the ball hits the boundary at point P, the trajectory after P is the reflection of the trajectory before P through the perpendicular to the boundary at P.

A ball inside a rectangular container of width 7 and height 12 is launched from the lower-left vertex of the container. It first strikes the right side of the container after traveling a distance of $\sqrt{53}$ (and strikes no other sides between its launch and its impact with the right side).

4. [**2**] Find the height at which the ball first contacts the right side.
5. [**3**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

Now a ball is launched from a vertex of an equilateral triangle with side length 5. It strikes the opposite side after traveling a distance of $\sqrt{19}$.

6. [**4**] Find the distance from the ball's point of first contact with a wall to the nearest vertex.
7. [**4**] How many times does the ball bounce before it returns to a vertex? (The final contact with a vertex does not count as a bounce.)

In this final problem, a ball is again launched from the vertex of an equilateral triangle with side length 5.

8. [**6**] In how many ways can the ball be launched so that it will return again to a vertex for the first time after 2009 bounces?

Super Mario 64!

Mario is once again on a quest to save Princess Peach. Mario enters Peach's castle and finds himself in a room with 4 doors. This room is the first in a sequence of 2 indistinugishable rooms. In each room, 1 door leads to the next room in the sequence (or, for the second room, into Bowser's level), while the other 3 doors lead to the first room.

9. [**3**] Suppose that in every room, Mario randomly picks a door to walk through. What is the expected number of doors (not including Mario's initial entrance to the first room) through which Mario will pass before he reaches Bowser's level?

10. [**4**] Suppose that instead there are 6 rooms with 4 doors. In each room, 1 door leads to the next room in the sequence (or, for the last room, Bowser's level), while the other 3 doors lead to the first room. Now what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

11. [**5**] In general, if there are d doors in every room (but still only 1 correct door) and r rooms, the last of which leads into Bowser's level, what is the expected number of doors through which Mario will pass before he reaches Bowser's level?

2^{nd} Annual Harvard-MIT November Tournament

Saturday 7 November 2009

Theme Round

Shortest Paths

1. [**3**] Paul starts with the number 19. In one step, he can add 1 to his number, divide his number by 2, or divide his number by 3. What is the minimum number of steps Paul needs to get to 1?

2. [**4**] You start with a number. Every second, you can add or subtract any number of the form $n!$ to your current number to get a new number. In how many ways can you get from 0 to 100 in 4 seconds? ($n!$ is defined as $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$, so $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, etc.)

3. [**5**] Let C be the circle of radius 12 centered at $(0, 0)$. What is the length of the shortest path in the plane between $(8\sqrt{3}, 0)$ and $(0, 12\sqrt{2})$ that does not pass through the interior of C?

4. [**6**] You are given a 5×6 checkerboard with squares alternately shaded black and white. The bottom-left square is white. Each square has side length 1 unit. You can normally travel on this board at a speed of 2 units per second, but while you travel through the interior (not the boundary) of a black square, you are slowed down to 1 unit per second. What is the shortest time it takes to travel from the bottom-left corner to the top-right corner of the board?

5. [**7**] The following grid represents a mountain range; the number in each cell represents the height of the mountain located there. Moving from a mountain of height a to a mountain of height b takes $(b-a)^2$ time. Suppose that you start on the mountain of height 1 and that you can move up, down, left, or right to get from one mountain to the next. What is the minimum amount of time you need to get to the mountain of height 49?

1	3	6	10	15	21	28
2	5	9	14	20	27	34
4	8	13	19	26	33	39
7	12	18	25	32	38	43
11	17	24	31	37	42	46
16	23	30	36	41	45	48
22	29	35	40	44	47	49

Five Guys

6. [**3**] There are five guys named Alan, Bob, Casey, Dan, and Eric. Each one either always tells the truth or always lies. You overhear the following discussion between them:

```
Alan: "All of us are truth-tellers."
Bob: "No, only Alan and I are truth-tellers."
Casey: "You are both liars."
Dan: "If Casey is a truth-teller, then Eric is too."
Eric: "An odd number of us are liars."
```

Who are the liars?

7. [**4**] Five guys are eating hamburgers. Each one puts a top half and a bottom half of a hamburger bun on the grill. When the buns are toasted, each guy randomly takes two pieces of bread off of the grill. What is the probability that each guy gets a top half and a bottom half?

8. [**5**] A single burger is not enough to satisfy a guy's hunger. The five guys go to Five Guys' Restaurant, which has 20 different meals on the menu. Each meal costs a different integer dollar amount between \$1 and \$20. The five guys have \$20 to split between them, and they want to use all the money to order five different meals. How many sets of five meals can the guys choose?

9. [**6**] Five guys each have a positive integer (the integers are not necessarily distinct). The greatest common divisor of any two guys' numbers is always more than 1, but the greatest common divisor of all the numbers is 1. What is the minimum possible value of the product of the numbers?

10. [**7**] Five guys join five girls for a night of bridge. Bridge games are always played by a team of two guys against a team of two girls. The guys and girls want to make sure that every guy and girl play against each other an equal number of times. Given that at least one game is played, what is the least number of games necessary to accomplish this?

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Algebra Test

1. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

2. [**3**] Let $f(n)$ be the number of times you have to hit the $\sqrt{\ }$ key on a calculator to get a number less than 2 starting from n. For instance, $f(2) = 1, f(5) = 2$. For how many $1 < m < 2008$ is $f(m)$ odd?

3. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \le 2$ has exactly one solution in x.

4. [**4**] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

for all real numbers x, y. Determine the value of $f(10)$.

5. [**5**] Let $f(x) = x^3 + x + 1$. Suppose g is a cubic polynomial such that $g(0) = -1$, and the roots of g are the squares of the roots of f. Find $g(9)$.

6. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

7. [**5**] Compute $\displaystyle\sum_{n=1}^{\infty} \sum_{k=1}^{n-1} \frac{k}{2^{n+k}}$.

8. [**6**] Compute $\arctan(\tan 65^\circ - 2\tan 40^\circ)$. (Express your answer in degrees.)

9. [**7**] Let S be the set of points (a, b) with $0 \le a, b \le 1$ such that the equation

$$x^4 + ax^3 - bx^2 + ax + 1 = 0$$

has at least one real root. Determine the area of the graph of S.

10. [**8**] Evaluate the infinite sum

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{5^n}.$$

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Calculus Test

1. [**3**] Let $f(x) = 1 + x + x^2 + \cdots + x^{100}$. Find $f'(1)$.

2. [**3**] Let ℓ be the line through $(0,0)$ and tangent to the curve $y = x^3 + x + 16$. Find the slope of ℓ.

3. [**4**] Find all $y > 1$ satisfying $\displaystyle\int_1^y x \ln x \, dx = \frac{1}{4}$.

4. [**4**] Let a, b be constants such that $\displaystyle\lim_{x \to 1} \frac{(\ln(2-x))^2}{x^2 + ax + b} = 1$. Determine the pair (a, b).

5. [**4**] Let $f(x) = \sin^6\left(\frac{x}{4}\right) + \cos^6\left(\frac{x}{4}\right)$ for all real numbers x. Determine $f^{(2008)}(0)$ (i.e., f differentiated 2008 times and then evaluated at $x = 0$).

6. [**5**] Determine the value of $\displaystyle\lim_{n \to \infty} \sum_{k=0}^{n} \binom{n}{k}^{-1}$.

7. [**5**] Find p so that $\lim_{x \to \infty} x^p \left(\sqrt[3]{x+1} + \sqrt[3]{x-1} - 2\sqrt[3]{x}\right)$ is some non-zero real number.

8. [**7**] Let $T = \displaystyle\int_0^{\ln 2} \frac{2e^{3x} + e^{2x} - 1}{e^{3x} + e^{2x} - e^x + 1} \, dx$. Evaluate e^T.

9. [**7**] Evaluate the limit $\displaystyle\lim_{n \to \infty} n^{-\frac{1}{2}\left(1 + \frac{1}{n}\right)} \left(1^1 \cdot 2^2 \cdot \cdots \cdot n^n\right)^{\frac{1}{n^2}}$.

10. [**8**] Evaluate the integral $\displaystyle\int_0^1 \ln x \ln(1-x) \, dx$.

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: Combinatorics Test

1. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes from a $3 \times 3 \times 1$ block (the order is irrelevant) with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

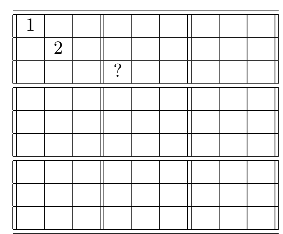
2. [**3**] Let $S = \{1, 2, \ldots, 2008\}$. For any nonempty subset $A \subset S$, define $m(A)$ to be the median of A (when A has an even number of elements, $m(A)$ is the average of the middle two elements). Determine the average of $m(A)$, when A is taken over all nonempty subsets of S.

3. [**4**] Farmer John has 5 cows, 4 pigs, and 7 horses. How many ways can he pair up the animals so that every pair consists of animals of different species? (Assume that all animals are distinguishable from each other.)

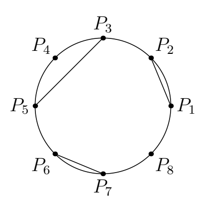
4. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

5. [**5**] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of non-negative integers in S less than 2008.

6. [**5**] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?



7. [**6**] Let $P_1, P_2, \ldots, P_8$ be 8 distinct points on a circle. Determine the number of possible configurations made by drawing a set of line segments connecting pairs of these 8 points, such that: (1) each P_i is the endpoint of at most one segment and (2) two no segments intersect. (The configuration with no edges drawn is allowed. An example of a valid configuration is shown below.)



8. [**6**] Determine the number of ways to select a sequence of 8 sets $A_1, A_2, \ldots, A_8$, such that each is a subset (possibly empty) of $\{1, 2\}$, and A_m contains A_n if m divides n.

9. [**7**] On an infinite chessboard (whose squares are labeled by (x, y), where x and y range over all integers), a king is placed at $(0, 0)$. On each turn, it has probability of 0.1 of moving to each of the four edge-neighboring squares, and a probability of 0.05 of moving to each of the four diagonally-neighboring squares, and a probability of 0.4 of not moving. After 2008 turns, determine the probability that the king is on a square with both coordinates even. An exact answer is required.

10. [**7**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: General Test, Part 1

1. [**2**] Let $ABCD$ be a unit square (that is, the labels A, B, C, D appear in that order around the square). Let X be a point outside of the square such that the distance from X to AC is equal to the distance from X to BD, and also that $AX = \frac{\sqrt{2}}{2}$. Determine the value of CX^2.

2. [**3**] Find the smallest positive integer n such that $107n$ has the same last two digits as n.

3. [**3**] There are 5 dogs, 4 cats, and 7 bowls of milk at an animal gathering. Dogs and cats are distinguishable, but all bowls of milk are the same. In how many ways can every dog and cat be paired with either a member of the other species or a bowl of milk such that all the bowls of milk are taken?

4. [**3**] Positive real numbers x, y satisfy the equations $x^2 + y^2 = 1$ and $x^4 + y^4 = \frac{17}{18}$. Find xy.

5. [**4**] The function f satisfies

$$f(x) + f(2x + y) + 5xy = f(3x - y) + 2x^2 + 1$$

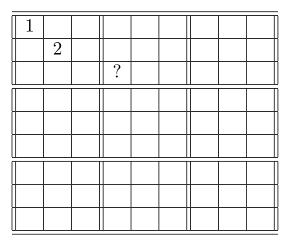
for all real numbers x, y. Determine the value of $f(10)$.

6. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

7. [**5**] The equation $x^3 - 9x^2 + 8x + 2 = 0$ has three real roots p, q, r. Find $\frac{1}{p^2} + \frac{1}{q^2} + \frac{1}{r^2}$.

8. [**5**] Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x + 1 \in S$. Determine the number of positive integers in S less than 2008.

9. [**5**] A *Sudoku matrix* is defined as a 9×9 array with entries from $\{1, 2, \ldots, 9\}$ and with the constraint that each row, each column, and each of the nine 3×3 boxes that tile the array contains each digit from 1 to 9 exactly once. A Sudoku matrix is chosen at random (so that every Sudoku matrix has equal probability of being chosen). We know two of squares in this matrix, as shown. What is the probability that the square marked by ? contains the digit 3?



10. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Individual Round: General Test, Part 2

1. [**2**] Four students from Harvard, one of them named Jack, and five students from MIT, one of them named Jill, are going to see a Boston Celtics game. However, they found out that only 5 tickets remain, so 4 of them must go back. Suppose that at least one student from each school must go see the game, and at least one of Jack and Jill must go see the game, how many ways are there of choosing which 5 people can see the game?

2. [**2**] Let ABC be an equilateral triangle. Let Ω be a circle inscribed in ABC and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3. [**3**] A $3 \times 3 \times 3$ cube composed of 27 unit cubes rests on a horizontal plane. Determine the number of ways of selecting two distinct unit cubes (order is irrelevant) from a $3 \times 3 \times 1$ block with the property that the line joining the centers of the two cubes makes a 45° angle with the horizontal plane.

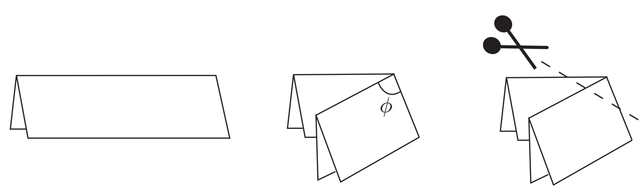
4. [**3**] Suppose that a, b, c, d are real numbers satisfying $a \geq b \geq c \geq d \geq 0$, $a^2 + d^2 = 1$, $b^2 + c^2 = 1$, and $ac + bd = 1/3$. Find the value of $ab - cd$.

5. [**4**] Kermit the frog enjoys hopping around the infinite square grid in his backyard. It takes him 1 Joule of energy to hop one step north or one step south, and 1 Joule of energy to hop one step east or one step west. He wakes up one morning on the grid with 100 Joules of energy, and hops till he falls asleep with 0 energy. How many different places could he have gone to sleep?

6. [**4**] Determine all real numbers a such that the inequality $|x^2 + 2ax + 3a| \leq 2$ has exactly one solution in x.

7. [**5**] A *root of unity* is a complex number that is a solution to $z^n = 1$ for some positive integer n. Determine the number of roots of unity that are also roots of $z^2 + az + b = 0$ for some integers a and b.

8. [**5**] A piece of paper is folded in half. A second fold is made such that the angle marked below has measure ϕ ($0^\circ < \phi < 90^\circ$), and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144^\circ$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

9. [**6**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10. [**6**] Determine the number of 8-tuples of nonnegative integers $(a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying $0 \leq a_k \leq k$, for each $k = 1, 2, 3, 4$, and $a_1 + a_2 + a_3 + a_4 + 2b_1 + 3b_2 + 4b_3 + 5b_4 = 19$.

11^{th} Annual Harvard-MIT Mathematics Tournament

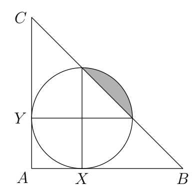
Saturday 23 February 2008

Individual Round: Geometry Test

1. [**3**] How many different values can $\angle ABC$ take, where A,B,C are distinct vertices of a cube?

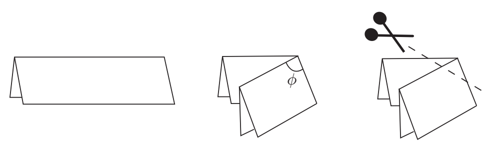
2. [**3**] Let ABC be an equilateral triangle. Let Ω be its incircle (circle inscribed in the triangle) and let ω be a circle tangent externally to Ω as well as to sides AB and AC. Determine the ratio of the radius of Ω to the radius of ω.

3. [**4**] Let ABC be a triangle with $\angle BAC = 90°$. A circle is tangent to the sides AB and AC at X and Y respectively, such that the points on the circle diametrically opposite X and Y both lie on the side BC. Given that $AB = 6$, find the area of the portion of the circle that lies outside the triangle.



4. [**4**] In a triangle ABC, take point D on BC such that $DB = 14, DA = 13, DC = 4$, and the circumcircle of ADB is congruent to the circumcircle of ADC. What is the area of triangle ABC?

5. [**5**] A piece of paper is folded in half. A second fold is made at an angle ϕ ($0° < \phi < 90°$) to the first, and a cut is made as shown below.



When the piece of paper is unfolded, the resulting hole is a polygon. Let O be one of its vertices. Suppose that all the other vertices of the hole lie on a circle centered at O, and also that $\angle XOY = 144°$, where X and Y are the the vertices of the hole adjacent to O. Find the value(s) of ϕ (in degrees).

6. [**5**] Let ABC be a triangle with $\angle A = 45°$. Let P be a point on side BC with $PB = 3$ and $PC = 5$. Let O be the circumcenter of ABC. Determine the length OP.

7. [**6**] Let C_1 and C_2 be externally tangent circles with radius 2 and 3, respectively. Let C_3 be a circle internally tangent to both C_1 and C_2 at points A and B, respectively. The tangents to C_3 at A and B meet at T, and $TA = 4$. Determine the radius of C_3.

8. [**6**] Let ABC be an equilateral triangle with side length 2, and let Γ be a circle with radius $\frac{1}{2}$ centered at the center of the equilateral triangle. Determine the length of the shortest path that starts somewhere on Γ, visits all three sides of ABC, and ends somewhere on Γ (not necessarily at the starting point). Express your answer in the form of $\sqrt{p} - q$, where p and q are rational numbers written as reduced fractions.

9. [**7**] Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, determine the value of $(DP/DQ)^2$.

10. [**7**] Let ABC be a triangle with $BC = 2007$, $CA = 2008$, $AB = 2009$. Let ω be an excircle of ABC that touches the line segment BC at D, and touches extensions of lines AC and AB at E and F, respectively (so that C lies on segment AE and B lies on segment AF). Let O be the center of ω. Let ℓ be the line through O perpendicular to AD. Let ℓ meet line EF at G. Compute the length DG.

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Guts Round

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1. [**5**] Determine all pairs (a, b) of real numbers such that $10, a, b, ab$ is an arithmetic progression.

2. [**5**] Given right triangle ABC, with $AB = 4, BC = 3$, and $CA = 5$. Circle ω passes through A and is tangent to BC at C. What is the radius of ω?

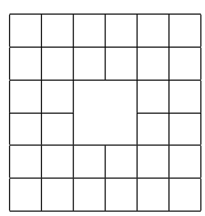
3. [**5**] How many ways can you color the squares of a 2×2008 grid in 3 colors such that no two squares of the same color share an edge?

. . . .

4. [**6**] Find the real solution(s) to the equation $(x + y)^2 = (x + 1)(y - 1)$.

5. [**6**] A Vandal and a Moderator are editing a Wikipedia article. The article originally is error-free. Each day, the Vandal introduces one new error into the Wikipedia article. At the end of the day, the moderator checks the article and has a $2/3$ chance of catching each individual error still in the article. After 3 days, what is the probability that the article is error-free?

6. [**6**] Determine the number of non-degenerate rectangles whose edges lie completely on the grid lines of the following figure.



. . . .

7. [**6**] Given that $x + \sin y = 2008$ and $x + 2008 \cos y = 2007$, where $0 \leq y \leq \pi/2$, find the value of $x + y$.

8. [**6**] Trodgor the dragon is burning down a village consisting of 90 cottages. At time $t = 0$ an angry peasant arises from each cottage, and every 8 minutes (480 seconds) thereafter another angry peasant spontaneously generates from each non-burned cottage. It takes Trodgor 5 seconds to either burn a peasant or to burn a cottage, but Trodgor cannot begin burning cottages until all the peasants around him have been burned. How many **seconds** does it take Trodgor to burn down the entire village?

9. [**6**] Consider a circular cone with vertex V, and let ABC be a triangle inscribed in the base of the cone, such that AB is a diameter and $AC = BC$. Let L be a point on BV such that the volume of the cone is 4 times the volume of the tetrahedron $ABCL$. Find the value of BL/LV.

. . . .

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10. [**7**] Find the number of subsets S of $\{1, 2, \dots 63\}$ the sum of whose elements is 2008.

11. [**7**] Let $f(r) = \sum_{j=2}^{2008} \frac{1}{j^r} = \frac{1}{2^r} + \frac{1}{3^r} + \dots + \frac{1}{2008^r}$. Find $\sum_{k=2}^{\infty} f(k)$.

12. [**7**] Suppose we have an (infinite) cone $\mathcal{C}$ with apex A and a plane π. The intersection of π and $\mathcal{C}$ is an ellipse $\mathcal{E}$ with major axis BC, such that B is closer to A than C, and $BC = 4$, $AC = 5$, $AB = 3$. Suppose we inscribe a sphere in each part of $\mathcal{C}$ cut up by $\mathcal{E}$ with both spheres tangent to $\mathcal{E}$. What is the ratio of the radii of the spheres (smaller to larger)?

. . . .

13. [**8**] Let $P(x)$ be a polynomial with degree 2008 and leading coefficient 1 such that

$$P(0) = 2007, P(1) = 2006, P(2) = 2005, \dots, P(2007) = 0.$$

Determine the value of $P(2008)$. You may use factorials in your answer.

14. [**8**] Evaluate the infinite sum $\sum_{n=1}^{\infty} \frac{n}{n^4+4}$.

15. [**8**] In a game show, Bob is faced with 7 doors, 2 of which hide prizes. After he chooses a door, the host opens three other doors, of which one is hiding a prize. Bob chooses to switch to another door. What is the probability that his new door is hiding a prize?

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16. [**9**] Point A lies at $(0, 4)$ and point B lies at $(3, 8)$. Find the x-coordinate of the point X on the x-axis maximizing $\angle AXB$.

17. [**9**] Solve the equation

$$\sqrt{x + \sqrt{4x + \sqrt{16x + \sqrt{\dots + \sqrt{4^{2008}x + 3}}}}} - \sqrt{x} = 1.$$

Express your answer as a reduced fraction with the numerator and denominator written in their prime factorization.

18. [**9**] Let ABC be a right triangle with $\angle A = 90^\circ$. Let D be the midpoint of AB and let E be a point on segment AC such that $AD = AE$. Let BE meet CD at F. If $\angle BFC = 135^\circ$, determine BC/AB.

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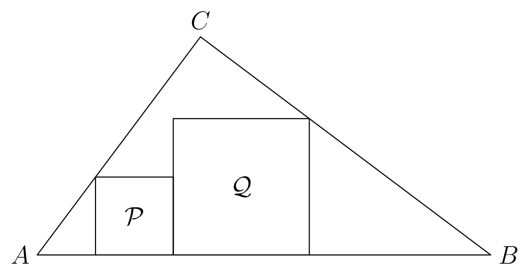
11$^{\text{th}}$ HARVARD-MIT MATHEMATICS TOURNAMENT, 23 FEBRUARY 2008 — GUTS ROUND

19. [**10**] Let $ABCD$ be a regular tetrahedron, and let O be the centroid of triangle BCD. Consider the point P on AO such that P minimizes $PA + 2(PB + PC + PD)$. Find $\sin \angle PBO$.

20. [**10**] For how many ordered triples (a, b, c) of positive integers are the equations $abc + 9 = ab + bc + ca$ and $a + b + c = 10$ satisfied?

21. [**10**] Let ABC be a triangle with $AB = 5$, $BC = 4$ and $AC = 3$. Let $\mathcal{P}$ and $\mathcal{Q}$ be squares inside ABC with disjoint interiors such that they both have one side lying on AB. Also, the two squares each have an edge lying on a common line perpendicular to AB, and $\mathcal{P}$ has one vertex on AC and $\mathcal{Q}$ has one vertex on BC. Determine the minimum value of the sum of the areas of the two squares.



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11$^{\text{th}}$ HARVARD-MIT MATHEMATICS TOURNAMENT, 23 FEBRUARY 2008 — GUTS ROUND

22. [**10**] For a positive integer n, let $\theta(n)$ denote the number of integers $0 \le x < 2010$ such that $x^2 - n$ is divisible by 2010. Determine the remainder when $\sum_{n=0}^{2009} n \cdot \theta(n)$ is divided by 2010.

23. [**10**] Two mathematicians, Kelly and Jason, play a cooperative game. The computer selects some secret positive integer $n < 60$ (both Kelly and Jason know that $n < 60$, but that they don't know what the value of n is). The computer tells Kelly the unit digit of n, and it tells Jason the number of divisors of n. Then, Kelly and Jason have the following dialogue:

    Kelly: I don't know what n is, and I'm sure that you don't know either. However, I know that n is divisible by at least two different primes.

    Jason: Oh, then I know what the value of n is.

    Kelly: Now I also know what n is.

    Assuming that both Kelly and Jason speak truthfully and to the best of their knowledge, what are all the possible values of n?

24. [**10**] Suppose that ABC is an isosceles triangle with $AB = AC$. Let P be the point on side AC so that $AP = 2CP$. Given that $BP = 1$, determine the maximum possible area of ABC.

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25. [**12**] Alice and the Cheshire Cat play a game. At each step, Alice either (1) gives the cat a penny, which causes the cat to change the number of (magic) beans that Alice has from n to $5n$ or (2) gives the cat a nickel, which causes the cat to give Alice another bean. Alice wins (and the cat disappears) as soon as the number of beans Alice has is greater than 2008 and has last two digits 42. What is the minimum number of cents Alice can spend to win the game, assuming she starts with 0 beans?

26. [**12**] Let $\mathcal{P}$ be a parabola, and let V_1 and F_1 be its vertex and focus, respectively. Let A and B be points on $\mathcal{P}$ so that $\angle AV_1B = 90°$. Let $\mathcal{Q}$ be the locus of the midpoint of AB. It turns out that $\mathcal{Q}$ is also a parabola, and let V_2 and F_2 denote its vertex and focus, respectively. Determine the ratio F_1F_2/V_1V_2.

27. [**12**] Cyclic pentagon $ABCDE$ has a right angle $\angle ABC = 90°$ and side lengths $AB = 15$ and $BC = 20$. Supposing that $AB = DE = EA$, find CD.

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28. [**15**] Let P be a polyhedron where every face is a regular polygon, and every edge has length 1. Each vertex of P is incident to two regular hexagons and one square. Choose a vertex V of the polyhedron. Find the volume of the set of all points contained in P that are closer to V than to any other vertex.

29. [**15**] Let (x, y) be a pair of real numbers satisfying

$$56x + 33y = \frac{-y}{x^2+y^2}, \qquad \text{and} \qquad 33x - 56y = \frac{x}{x^2+y^2}.$$

Determine the value of $|x| + |y|$.

30. [**15**] Triangle ABC obeys $AB = 2AC$ and $\angle BAC = 120°$. Points P and Q lie on segment BC such that

$$AB^2 + BC \cdot CP = BC^2$$
$$3AC^2 + 2BC \cdot CQ = BC^2$$

Find $\angle PAQ$ in degrees.

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31. [**18**] Let $\mathcal{C}$ be the hyperbola $y^2 - x^2 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points (P_n) on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing passing through P_n, then P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)

    Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

32. [**18**] Cyclic pentagon $ABCDE$ has side lengths $AB = BC = 5$, $CD = DE = 12$, and $AE = 14$. Determine the radius of its circumcircle.

33. [**18**] Let a, b, c be nonzero real numbers such that $a + b + c = 0$ and $a^3 + b^3 + c^3 = a^5 + b^5 + c^5$. Find the value of $a^2 + b^2 + c^2$.

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34. **Who Wants to Be a Millionaire.** In 2000, the Clay Mathematics Institute named seven *Millennium Prize Problems*, with each carrying a prize of \$1 Million for its solution. Write down the name of ONE of the seven Clay Millennium Problems. If your submission is incorrect or misspelled, then your submission is disqualified. If another team wrote down the same Millennium Problem as you, then you get 0 points, otherwise you get 20 points.

35. **NUMB3RS.** The RSA Factoring Challenge, which ended in 2007, challenged computational mathematicians to factor extremely large numbers that were the product of two prime numbers. The largest number successfully factored in this challenge was RSA-640, which has 193 decimal digits and carried a prize of $\$20,000$. The next challenge number carried prize of $\$30,000$, and contains N decimal digits. Your task is to submit a guess for N. Only the team(s) that have the closest guess(es) receives points. If k teams all have the closest guesses, then each of them receives $\lceil \frac{20}{k} \rceil$ points.

36. **The History Channel.** Below is a list of famous mathematicians. Your task is to list a subset of them in the chronological order of their birth dates. Your submission should be a sequence of letters. If your sequence is not in the correct order, then you get 0 points. Otherwise your score will be $\min\{\max\{5(N-4), 0\}, 25\}$, where N is the number of letters in your sequence.

    (A) Niels Abel (B) Arthur Cayley (C) Augustus De Morgan (D) Gustav Dirichlet (E) Leonhard Euler (F) Joseph Fourier (G) Évariste Galois (H) Carl Friedrich Gauss (I) Marie-Sophie Germain (J) Joseph Louis Lagrange (K) Pierre-Simon Laplace (L) Henri Poincaré (N) Bernhard Riemann

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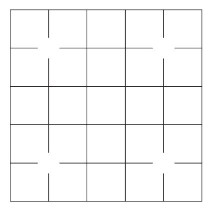
11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: A Division

Lattice Walks [90]

1. [**20**] Determine the number of ways of walking from $(0, 0)$ to $(5, 5)$ using only up and right unit steps such that the path does not pass through any of the following points: $(1, 1), (1, 4), (4, 1), (4, 4)$.



2. [**20**] Let $n > 2$ be a positive integer. Prove that there are $\frac{1}{2}(n-2)(n+1)$ ways to walk from $(0, 0)$ to $(n, 2)$ using only up and right unit steps such that the walk never visits the line $y = x$ after it leaves the origin.

3. [**20**] Let $n > 4$ be a positive integer. Determine the number of ways to walk from $(0, 0)$ to $(n, 2)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 2$ except at the start and at the end.

4. [**30**] Let $n > 6$ be a positive integer. Determine the number of ways to walk from $(0, 0)$ to $(n, 3)$ using only up and right unit steps such that the path does not meet the lines $y = x$ or $y = x - n + 3$ except at the start and at the end.

Lattice and Centroids [130]

A d-dimensional *lattice point* is a point of the form $(x_1, x_2, \ldots, x_d)$ where $x_1, x_2, \ldots, x_d$ are all integers. For a set of d-dimensional points, their *centroid* is the point found by taking the coordinate-wise average of the given set of points.

Let $f(n, d)$ denote the minimal number f such that any set of f lattice points in the d-dimensional Euclidean space contains a subset of size n whose centroid is also a lattice point.

5. [**10**] Let S be a set of 5 points in the 2-dimensional lattice. Show that we can always choose a pair of points in S whose midpoint is also a lattice point.

6. [**10**] Construct a set of 2^d d-dimensional lattice points so that for any two chosen points A, B, the line segment AB does not pass through any other lattice point.

7. [**35**] Show that for positive integers n and d,

$$(n-1)2^d + 1 \leq f(n, d) \leq (n-1)n^d + 1.$$

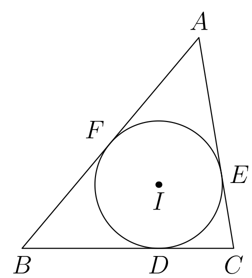
8. [**40**] Show that for positive integers n_1, n_2 and d,

$$f(n_1 n_2, d) \leq f(n_1, d) + n_1 \left(f(n_2, d) - 1\right).$$

9. [**35**] Determine, with proof, a simple closed-form expression for $f(2^a, d)$.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. On the circumcircle of ABC, let A' be the midpoint of arc BC (not containing A).
    (a) [**10**] Show that A, I, A' are collinear.
    (b) [**20**] Show that A' is the circumcenter of BIC.

11. [**30**] Let lines BI and EF meet at K. Show that I, K, E, C, D are concyclic.

12. [**40**] Let K be as in the previous problem. Let M be the midpoint of BC and N the midpoint of AC. Show that K lies on line MN.

13. [**40**] Let M be the midpoint of BC, and T diametrically opposite to D on the incircle of ABC. Show that DT, AM, EF are concurrent.

14. [**40**] Let P be a point inside the incircle of ABC. Let lines DP, EP, FP meet the incircle again at D', E', F'. Show that AD', BE', CF' are concurrent.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point. A set of points are *concyclic* if they lie on a common circle.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.

- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

- *"Trig" Ceva* states that given ABC a triangle, and points X, Y, Z inside the triangle, the lines AX, BY, CZ are concurrent if and only if

$$\frac{\sin \angle BAX}{\sin \angle XAC} \cdot \frac{\sin \angle CBY}{\sin \angle YBA} \cdot \frac{\sin \angle ACZ}{\sin \angle ZCB} = 1.$$

11^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 23 February 2008

Team Round: B Division

Tropical Mathematics [95]

For real numbers x and y, let us consider the two operations $\oplus$ and $\odot$ defined by

$$x \oplus y = \min(x, y) \qquad \text{and} \qquad x \odot y = x + y.$$

We also include ∞ in our set, and it satisfies $x \oplus \infty = x$ and $x \odot \infty = \infty$ for all x. When unspecified, $\odot$ precedes $\oplus$ in the order of operations.

1. [**10**] (Distributive law) Prove that $(x \oplus y) \odot z = x \odot z \oplus y \odot z$ for all $x, y, z \in \mathbb{R} \cup \{\infty\}$.

2. [**10**] (Freshman's Dream) Let z^n denote $z \odot z \odot z \odot \cdots \odot z$ with z appearing n times. Prove that $(x \oplus y)^n = x^n \oplus y^n$ for all $x, y \in \mathbb{R} \cup \{\infty\}$ and positive integer n.

3. [**35**] By a *tropical polynomial* we mean a function of the form

   $$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

   where exponentiation is as defined in the previous problem.

   Let p be a tropical polynomial. Prove that

   $$p\left(\frac{x+y}{2}\right) \geq \frac{p(x) + p(y)}{2}$$

   for all $x, y \in \mathbb{R} \cup \{\infty\}$. (This means that all tropical polynomials are concave.)

4. [**40**] (Fundamental Theorem of Algebra) Let p be a tropical polynomial:

   $$p(x) = a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0, \quad a_n \neq \infty$$

   Prove that we can find $r_1, r_2, \ldots, r_n \in \mathbb{R} \cup \{\infty\}$ so that

   $$p(x) = a_n \odot (x \oplus r_1) \odot (x \oplus r_2) \odot \cdots \odot (x \oplus r_n)$$

   for all x.

Juggling [125]

A *juggling sequence* of *length* n is a sequence $j(\cdot)$ of n nonnegative integers, usually written as a string

$$j(0)j(1)\ldots j(n-1)$$

such that the mapping $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(t) = t + j(\overline{t})$$

is a permutation of the integers. Here $\bar{t}$ denotes the remainder of t when divided by n. In this case, we say that f is the corresponding *juggling pattern.*

For a juggling pattern f (or its corresponding juggling sequence), we say that it has b *balls* if the permutation induces b infinite orbits on the set of integers. Equivalently, b is the maximum number such that we can find a set of b integers $\{t_1, t_2, \ldots, t_b\}$ so that the sets $\{t_i, f(t_i), f(f(t_i)), f(f(f(t_i))), \ldots\}$ are all infinite and mutually disjoint (i.e. non-overlapping) for $i = 1, 2, \ldots, b$. (This definition will become clear in a second.)

Now is probably a good time to pause and think about what all this has to do with juggling. Imagine that we are juggling a number of balls, and at time t, we toss a ball from our hand up to a height $j(\bar{t})$. This ball stays up in the air for $j(\bar{t})$ units of time, so that it comes back to our hand at time $f(t) = t + j(\bar{t})$. Then, the juggling pattern presents a simplified model of how balls are juggled (for instance, we ignore information such as which hand we use to toss the ball). A throw height of 0 (i.e., $j(\bar{t}) = 0$ and $f(t) = t$) represents that no thrown takes place at time t, which could correspond to an empty hand. Then, b is simply the minimum number of balls needed to carry out the juggling.

The following graphical representation may be helpful to you. On a horizontal line, an curve is drawn from t to $f(t)$. For instance, the following diagram depicts the juggling sequence 441 (or the juggling sequences 414 and 144). Then b is simply the number of contiguous "paths" drawn, which is 3 in this case.

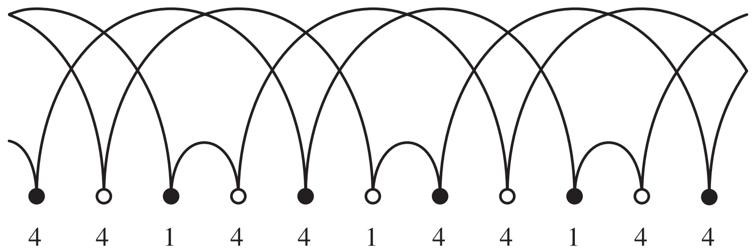


Figure 1: Juggling diagram of 441.

5. [**10**] Prove that 572 is not a juggling sequence.

6. [**40**] Suppose that $j(0)j(1)\cdots j(n-1)$ is a valid juggling sequence. For $i = 0, 1, \ldots, n-1$, Let a_i denote the remainder of $j(i) + i$ when divided by n. Prove that $(a_0, a_1, \ldots, a_{n-1})$ is a permutation of $(0, 1, \ldots, n-1)$.

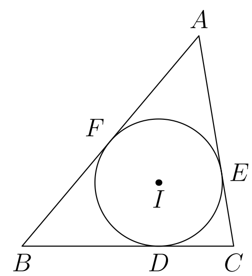
7. [**30**] Determine the number of juggling sequences of length n with exactly 1 ball.

8. [**40**] Prove that the number of balls b in a juggling sequence $j(0)j(1)\cdots j(n-1)$ is simply the average
$$b = \frac{j(0) + j(1) + \cdots + j(n-1)}{n}.$$

9. [**5**] Show that the converse of the previous statement is false by providing a non-juggling sequence $j(0)j(1)j(2)$ of length 3 where the average $\frac{1}{3}(j(0) + j(1) + j(2))$ is an integer. Show that your example works.

Incircles [180]

In the following problems, ABC is a triangle with incenter I. Let D, E, F denote the points where the incircle of ABC touches sides BC, CA, AB, respectively.



At the end of this section you can find some terminology and theorems that may be helpful to you.

10. [**15**] Let a, b, c denote the side lengths of BC, CA, AB. Find the lengths of AE, BF, CD in terms of a, b, c.

11. [**15**] Show that lines AD, BE, CF pass through a common point.

12. [**35**] Show that the incenter of triangle AEF lies on the incircle of ABC.

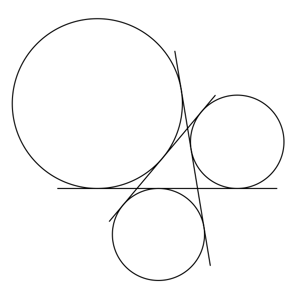
13. [**35**] Let A_1, B_1, C_1 be the incenters of triangle AEF, BDF, CDE, respectively. Show that A_1D, B_1E, C_1F all pass through the orthocenter of $A_1B_1C_1$.

14. [**40**] Let X be the point on side BC such that $BX = CD$. Show that the excircle ABC opposite of vertex A touches segment BC at X.

15. [**40**] Let X be as in the previous problem. Let T be the point diametrically opposite to D on on the incircle of ABC . Show that A, T, X are collinear.

Glossary and some possibly useful facts

- A set of points is *collinear* if they lie on a common line. A set of lines is *concurrent* if they pass through a common point.
- Given ABC a triangle, the three angle bisectors are concurrent at the *incenter* of the triangle. The incenter is the center of the *incircle*, which is the unique circle inscribed in ABC, tangent to all three sides.
- The *excircles* of a triangle ABC are the three circles on the exterior the triangle but tangent to all three lines AB, BC, CA.



- The *orthocenter* of a triangle is the point of concurrency of the three altitudes.
- *Ceva's theorem* states that given ABC a triangle, and points X, Y, Z on sides BC, CA, AB, respectively, the lines AX, BY, CZ are concurrent if and only if

$$\frac{BX}{XB} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

1^{st} Annual Harvard-MIT November Tournament

Saturday 8 November 2008

Individual Round

1. [**2**] Find the minimum of $x^2 - 2x$ over all real numbers x.

2. [**3**] What is the units digit of 7^{2009}?

3. [**3**] How many diagonals does a regular undecagon (11-sided polygon) have?

4. [**4**] How many numbers between 1 and $1,000,000$ are perfect squares but not perfect cubes?

5. [**5**] Joe has a triangle with area $\sqrt{3}$. What's the smallest perimeter it could have?

6. [**5**] We say "s grows to r" if there exists some integer $n > 0$ such that $s^n = r$. Call a real number r "sparse" if there are only finitely many real numbers s that grow to r. Find all real numbers that are sparse.

7. [**6**] Find all ordered pairs (x, y) such that
$$(x - 2y)^2 + (y - 1)^2 = 0.$$

8. [**7**] How many integers between 2 and 100 inclusive *cannot* be written as $m \cdot n$, where m and n have no common factors and neither m nor n is equal to 1? Note that there are 25 primes less than 100.

9. [**7**] Find the product of all real x for which
$$2^{3x+1} - 17 \cdot 2^{2x} + 2^{x+3} = 0.$$

10. [**8**] Find the largest positive integer n such that $n^3 + 4n^2 - 15n - 18$ is the cube of an integer.

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

1. [**5**] Find the sum of all solutions for x:

$$\begin{aligned} xy &= 1 \\ x + y &= 3 \end{aligned}$$

2. [**5**] Evaluate the sum
$$1 - 2 + 3 - 4 + \cdots + 2007 - 2008.$$

3. [**5**] What is the largest x such that x^2 divides $24 \cdot 35 \cdot 46 \cdot 57$?

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

4. [**6**] What is the smallest prime divisor of $5^{7^{10^{7^{10}}}} + 1$?

5. [**6**] What is the sum of all integers x such that $|x + 2| \leq 10$?

6. [**6**] Sarah is deciding whether to visit Russia or Washington, DC for the holidays. She makes her decision by rolling a regular 6-sided die. If she gets a 1 or 2, she goes to DC. If she rolls a $3, 4$, or 5, she goes to Russia. If she rolls a 6, she rolls again. What is the probability that she goes to DC?

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

7. [**7**] Compute $2009^2 - 2008^2$.

8. [**7**] Alice rolls two octahedral dice with the numbers $2, 3, 4, 5, 6, 7, 8, 9$. What's the probability the two dice sum to 11?

9. [**7**] Let $a_0 = \frac{6}{7}$, and
$$a_{n+1} = \begin{cases} 2a_n & \text{if } a_n < \frac{1}{2} \\ 2a_n - 1 & \text{if } a_n \geq \frac{1}{2}. \end{cases}$$
Find a_{2008}.

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

10. [**8**] Find the sum of all positive integers n such that n divides $n^2 + n + 2$.

11. [**8**] Al has a rectangle of integer side lengths a and b, and area 1000. What is the smallest perimeter it could have?

12. [**8**] Solve the following system of equations for w.

$$\begin{aligned} 2w + x + y + z &= 1 \\ w + 2x + y + z &= 2 \\ w + x + 2y + z &= 2 \\ w + x + y + 2z &= 1. \end{aligned}$$

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13. [**9**] Find the number of distinct primes dividing $1 \cdot 2 \cdot 3 \cdots 9 \cdot 10$.

14. [**9**] You have a 2×3 grid filled with integers between 1 and 9. The numbers in each row and column are distinct, the first row sums to 23, and the columns sum to 14, 16, and 17 respectively.

	14	16	17
23	a	b	c
	x	y	z

What is $x + 2y + 3z$?

15. [**9**] A cat is going up a stairwell with ten stairs. However, instead of walking up the stairs one at a time, the cat jumps, going either two or three stairs up at each step (though if necessary, it will just walk the last step). How many different ways can the cat go from the bottom to the top?

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

16. [**10**] If p and q are positive integers and $\frac{2008}{2009} < \frac{p}{q} < \frac{2009}{2010}$, what is the minimum value of p?

17. [**10**] Determine the last two digits of 17^{17}, written in base 10.

18. [**10**] Find the coefficient of x^6 in the expansion of

$$(x+1)^6 \cdot \sum_{i=0}^{6} x^i$$

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1st HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

19. [**11**] Let P be a polynomial with $P(1) = P(2) = \cdots = P(2007) = 0$ and $P(0) = 2009!$. $P(x)$ has leading coefficient 1 and degree 2008. Find the largest root of $P(x)$.

20. [**11**] You have a die with faces labelled 1 through 6. On each face, you draw an arrow to an adjacent face, such that if you start on a face and follow the arrows, after 6 steps you will have passed through every face once and will be back on your starting face. How many ways are there to draw the arrows so that this is true?

21. [**11**] Call a number *overweight* if it has at least three positive integer divisors (including 1 and the number), and call a number *obese* if it has at least four positive integer divisors (including 1 and the number). How many positive integers between 1 and 200 are overweight, but not obese?

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22. [**12**] Sandra the Maverick has 5 pairs of shoes in a drawer, each pair a different color. Every day for 5 days, Sandra takes two shoes out and throws them out the window. If they are the same color, she treats herself to a practice problem from a past HMMT. What is the expected value (average number) of practice problems she gets to do?

23. [**12**] If x and y are real numbers such that $\frac{(x-4)^2}{4}+\frac{y^2}{9}=1$, find the largest possible value of $\frac{x^2}{4}+\frac{y^2}{9}$.

24. [**12**] Let $f(x)=\frac{1}{1-x}$. Let $f^{k+1}(x)=f(f^k(x))$, with $f^1(x)=f(x)$. What is $f^{2008}(2008)$?

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1[st] HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

25. [**13**] Evaluate the sum

$$\cos\left(\frac{2\pi}{18}\right)+\cos\left(\frac{4\pi}{18}\right)+\cdots+\cos\left(\frac{34\pi}{18}\right).$$

26. [**13**] John M. is sitting at $(0,0)$, looking across the aisle at his friends sitting at (i,j) for each $1\leq i\leq 10$ and $0\leq j\leq 5$. Unfortunately, John can only see a friend if the line connecting them doesn't pass through any other friend. How many friends can John see?

27. [**13**] $ABCDE$ is a regular pentagon inscribed in a circle of radius 1. What is the area of the set of points inside the circle that are farther from A than they are from any other vertex?

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1[st] HARVARD-MIT NOVEMBER TOURNAMENT, 8 NOVEMBER 2008 — GUTS ROUND

28. [**14**] Johnny the grad student is typing all the integers from 1 to ∞, in order. The 2 on his computer is broken however, so he just skips any number with a 2. What's the 2008th number he types?

29. [**14**] Let $p(x)$ be the polynomial of degree 4 with roots $1,2,3,4$ and leading coefficient 1. Let $q(x)$ be the polynomial of degree 4 with roots $1,\frac{1}{2},\frac{1}{3}$, and $\frac{1}{4}$ and leading coefficient 1. Find $\lim_{x\to 1}\frac{p(x)}{q(x)}$.

30. [**14**] Alice has an equilateral triangle ABC of area 1. Put D on BC, E on CA, and F on AB, with $BD=DC$, $CE=2EA$, and $2AF=FB$. Note that AD,BE, and CF pass through a single point M. What is the area of triangle EMC?

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31. [**15**] Find the sum of all primes p for which there exists a prime q such that $p^2 + pq + q^2$ is a square.

32. [**15**] Pirate ships Somy and Lia are having a tough time. At the end of the year, they are both one pillage short of the minimum required for maintaining membership in the Pirate Guild, so they decide to pillage each other to bring their counts up. Somy by tradition only pillages $28 \cdot 3^k$ coins for integers k, and Lia by tradition only pillages $82 \cdot 3^j$ coins for integers j. Note that each pillage can have a different k or j. Soma and Lia work out a system where Somy pillages Lia n times, Lia pillages Somy n times, and after both sets of pillages Somy and Lia are financially even.

    What is the smallest n can be?

33. [**15**] The polynomial $ax^2 - bx + c$ has two distinct roots p and q, with a, b, and c positive integers and with $0 < p, q < 1$. Find the minimum possible value of a.

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34. [**20**] How many hits did *math tournament* get on Google the morning of November 8, 2008? If you submit integer N, and the correct answer is A, you will receive $\lfloor 20 \cdot \min\{\frac{N}{A}, \frac{A}{N}\}\rfloor$ points for this problem.

35. [**25**] Find $\max\{\text{Perimeter}(T)\}$ for T a triangle contained in a regular heptagon (7-sided figure) of unit edge length. Write your answer N to 2 places after the decimal. If the correct answer rounded to 2 decimal places is A, you will receive 0 points if $N > A$ and $\lfloor \max\{0, 25 - 50 \cdot (A - N)\}\rfloor$ points otherwise.

36. [**25**] How many numbers less than 1,000,000 are the product of exactly 2 distinct primes? You will receive $\lfloor 25 - 50 \cdot |\frac{N}{A} - 1|\rfloor$ points, if you submit N and the correct answer is A.

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Team Round

Unit Fractions [100]

A *unit fraction* is a fraction of the form $\frac{1}{n}$, where n is a positive integer. In this problem, you will find out how rational numbers can be expressed as sums of these unit fractions. Even if you do not solve a problem, you may apply its result to later problems.

We say we *decompose* a rational number q into unit fractions if we write q as a sum of 2 or more **distinct** unit fractions. In particular, if we write q as a sum of k distinct unit fractions, we say we have decomposed q into k fractions. As an example, we can decompose $\frac{2}{3}$ into 3 fractions: $\frac{2}{3} = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}$.

1. (a) Decompose 1 into unit fractions.
   (b) Decompose $\frac{1}{4}$ into unit fractions.
   (c) Decompose $\frac{2}{5}$ into unit fractions.

2. Explain how any unit fraction $\frac{1}{n}$ can be decomposed into other unit fractions.

3. (a) Write 1 as a sum of 4 distinct unit fractions.
   (b) Write 1 as a sum of 5 distinct unit fractions.
   (c) Show that, for any integer $k > 3$, 1 can be decomposed into k unit fractions.

4. Say that $\frac{a}{b}$ is a positive rational number in simplest form, with $a \neq 1$. Further, say that n is an integer such that:
$$\frac{1}{n} > \frac{a}{b} > \frac{1}{n+1}$$
Show that when $\frac{a}{b} - \frac{1}{n+1}$ is written in simplest form, its numerator is smaller than a.

5. **An aside: the sum of all the unit fractions**
It is possible to show that, given any real M, there exists a positive integer k large enough that:
$$\sum_{n=1}^{k} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} \ldots > M$$
Note that this statement means that the infinite harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n}$, grows without bound, or diverges. For the specific example M = 5, find a value of k, *not necessarily the smallest*, such that the inequality holds. Justify your answer.

6. Now, using information from problems 4 and 5, prove that the following method to decompose any positive rational number will always terminate:

Step 1. Start with the fraction $\frac{a}{b}$. Let t_1 be the largest unit fraction $\frac{1}{n}$ which is less than or equal to $\frac{a}{b}$.

Step 2. If we have already chosen t_1 through t_k, and if $t_1 + t_2 + \ldots + t_k$ is still less than $\frac{a}{b}$, then let t_{k+1} be the largest unit fraction less than both t_k and $\frac{a}{b}$.

Step 3. If $t_1 + \ldots + t_{k+1}$ equals $\frac{a}{b}$, the decomposition is found. Otherwise, repeat step 2.

Why does this method never result in an infinite sequence of t_i?

Juicy Numbers [100]

A *juicy number* is an integer $j > 1$ for which there is a sequence $a_1 < a_2 < \ldots < a_k$ of positive integers such that $a_k = j$ and such that the sum of the reciprocals of all the a_i is 1. For example, 6 is a juicy number because $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, but 2 is not juicy.

In this part, you will investigate some of the properties of juicy numbers. Remember that if you do not solve a question, you can still use its result on later questions.

1. Explain why 4 is not a juicy number.
2. It turns out that 6 is the smallest juicy integer. Find the next two smallest juicy numbers, and show a decomposition of 1 into unit fractions for each of these numbers. You do not need to prove that no smaller numbers are juicy.
3. Let p be a prime. Given a sequence of positive integers b_1 through b_n, exactly one of which is divisible by p, show that when
$$\frac{1}{b_1} + \frac{1}{b_2} + \ldots + \frac{1}{b_n}$$
is written as a fraction in lowest terms, then its denominator is divisible by p. Use this fact to explain why no prime p is ever juicy.
4. Show that if j is a juicy integer, then $2j$ is juicy as well.
5. Prove that the product of two juicy numbers (not necessarily distinct) is always a juicy number. Hint: if j_1 and j_2 are the two numbers, how can you change the decompositions of 1 ending in $\frac{1}{j_1}$ or $\frac{1}{j_2}$ to make them end in $\frac{1}{j_1 j_2}$?

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Saturday 8 November 2008

Theme Round

Triangular Areas [**25**]

Triangles have many properties. Calculate some things about their area.

1. [**3**] A triangle has sides of length 9, 40, and 41. What is its area?
2. [**4**] Let ABC be a triangle, and let M be the midpoint of side AB. If AB is 17 units long and CM is 8 units long, find the maximum possible value of the area of ABC.
3. [**5**] Let DEF be a triangle and H the foot of the altitude from D to EF. If $DE = 60$, $DF = 35$, and $DH = 21$, what is the difference between the minimum and the maximum possible values for the area of DEF?
4. [**6**] Right triangle XYZ, with hypotenuse YZ, has an incircle of radius $\frac{3}{8}$ and one leg of length 3. Find the area of the triangle.
5. [**7**] A triangle has altitudes of length 15, 21, and 35. Find its area.

Chessboards [**25**]

Joe B. is playing with some chess pieces on a 6×6 chessboard. Help him find out some things.

1. [**3**] Joe B. first places the black king in one corner of the board. In how many of the 35 remaining squares can he place a white bishop so that it does not check the black king?
2. [**4**] Joe B. then places a white king in the opposite corner of the board. How many total ways can he place one black bishop and one white bishop so that neither checks the king of the opposite color?
3. [**5**] Joe B. now clears the board. How many ways can he place 3 white rooks and 3 black rooks on the board so that no two rooks of **opposite** color can attack each other?
4. [**6**] Joe B. is frustrated with chess. He breaks the board, leaving a 4×4 board, and throws 3 black knights and 3 white kings at the board. Miraculously, they all land in distinct squares! What is the expected number of checks in the resulting position? (Note that a knight can administer multiple checks and a king can be checked by multiple knights.)
5. [**7**] Suppose that at some point Joe B. has placed 2 black knights on the original board, but gets bored of chess. He now decides to cover the 34 remaining squares with 17 dominos so that no two overlap and the dominos cover the entire rest of the board. For how many initial arrangements of the two pieces is this possible?

Note: Chess is a game played with pieces of two colors, black and white, that players can move between squares on a rectangular grid. Some of the pieces move in the following ways:

- **Bishop:** This piece can move any number of squares diagonally if there are no other pieces along its path.
- **Rook:** This piece can move any number of squares either vertically or horizontally if there are no other pieces along its path.
- **Knight:** This piece can move either two squares along a row and one square along a column or two squares along a column and one square along a row.
- **King:** This piece can move to any open adjacent square (including diagonally).

If a piece can move to a square occupied by a king of the opposite color, we say that it is *checking* the king.

If a piece moves to a square occupied by another piece, this is called *attacking*.

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Saturday 24 February 2007

Individual Round: Algebra Test

1. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor.$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

2. [**3**] Two reals x and y are such that $x - y = 4$ and $x^3 - y^3 = 28$. Compute xy.

3. [**4**] Three real numbers x, y, and z are such that $(x+4)/2 = (y+9)/(z-3) = (x+5)/(z-5)$. Determine the value of x/y.

4. [**4**] Compute
$$\frac{2^3-1}{2^3+1} \cdot \frac{3^3-1}{3^3+1} \cdot \frac{4^3-1}{4^3+1} \cdot \frac{5^3-1}{5^3+1} \cdot \frac{6^3-1}{6^3+1}.$$

5. [**5**] A convex quadrilateral is determined by the points of intersection of the curves $x^4 + y^4 = 100$ and $xy = 4$; determine its area.

6. [**5**] Consider the polynomial $P(x) = x^3 + x^2 - x + 2$. Determine all real numbers r for which there exists a complex number z not in the reals such that $P(z) = r$.

7. [**5**] An infinite sequence of positive real numbers is defined by $a_0 = 1$ and $a_{n+2} = 6a_n - a_{n+1}$ for $n = 0, 1, 2, \cdots$ Find the possible value(s) of a_{2007}.

8. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,
$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$
Compute the value of $f(2007)$.

9. [**7**] The complex numbers $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the four distinct roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set
$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

10. [**8**] The polynomial $f(x) = x^{2007} + 17x^{2006} + 1$ has distinct zeroes $r_1, \ldots, r_{2007}$. A polynomial P of degree 2007 has the property that $P\left(r_j + \frac{1}{r_j}\right) = 0$ for $j = 1, \ldots, 2007$. Determine the value of $P(1)/P(-1)$.

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Individual Round: Calculus Test

1. [**3**] Compute:

$$\lim_{x \to 0} \quad \frac{x^2}{1 - \cos(x)}$$

2. [**3**] Determine the real number a having the property that $f(a) = a$ is a relative minimum of $f(x) = x^4 - x^3 - x^2 + ax + 1$.

3. [**4**] Let a be a positive real number. Find the value of a such that the definite integral

$$\int_a^{a^2} \frac{\mathrm{d}x}{x + \sqrt{x}}$$

achieves its smallest possible value.

4. [**4**] Find the real number α such that the curve $f(x) = e^x$ is tangent to the curve $g(x) = \alpha x^2$.

5. [**5**] The function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x^2)f''(x) = f'(x)f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, determine $f'(1) + f''(1)$.

6. [**5**] The elliptic curve $y^2 = x^3 + 1$ is tangent to a circle centered at $(4, 0)$ at the point (x_0, y_0). Determine the sum of all possible values of x_0.

7. [**5**] Compute

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1) \cdot (n+1)!}.$$

8. [**6**] Suppose that ω is a primitive 2007^{th} root of unity. Find $\left(2^{2007} - 1\right) \sum_{j=1}^{2006} \frac{1}{2 - \omega^j}$.

For this problem only, you may express your answer in the form $m \cdot n^k + p$, where m, n, k, and p are positive integers. Note that a number z is a *primitive* n^{th} *root of unity* if $z^n = 1$ and n is the smallest number amongst $k = 1, 2, \ldots, n$ such that $z^k = 1$.

9. [**7**] g is a twice differentiable function over the positive reals such that

$$\begin{aligned} g(x) + 2x^3 g'(x) + x^4 g''(x) &= 0 \quad \text{for all positive reals } x. && (1) \\ \lim_{x \to \infty} x g(x) &= 1 && (2) \end{aligned}$$

Find the real number $\alpha > 1$ such that $g(\alpha) = 1/2$.

10. [**8**] Compute

$$\int_0^{\infty} \frac{e^{-x} \sin(x)}{x} dx$$

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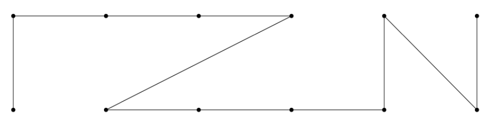
Individual Round: Combinatorics Test

1. [**3**] A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Biff and Jacob must serve together or not at all, and Alice and Jane refuse to serve with each other?

2. [**3**] How many 5-digit numbers $\overline{abcde}$ exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$.)

3. [**4**] Jack, Jill, and John play a game in which each randomly picks and then *replaces* a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

4. [**4**] On the Cartesian grid, Johnny wants to travel from $(0, 0)$ to $(5, 1)$, and he wants to pass through all twelve points in the set $S = \{(i, j) \mid 0 \leq i \leq 1, 0 \leq j \leq 5, i, j \in \mathbb{Z}\}$. Each step, Johnny may go from one point in S to another point in S by a line segment connecting the two points. How many ways are there for Johnny to start at $(0, 0)$ and end at $(5, 1)$ so that he never crosses his own path?

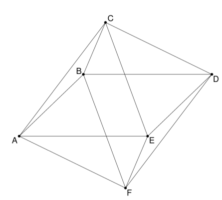


5. [**5**] Determine the number of ways to select a positive number of squares on an 8×8 chessboard such that no two lie in the same row or the same column and no chosen square lies to the left of and below another chosen square.

6. [**5**] Kevin has four red marbles and eight blue marbles. He arranges these twelve marbles randomly, in a ring. Determine the probability that no two red marbles are adjacent.

7. [**5**] Forty two cards are labeled with the natural numbers 1 through 42 and randomly shuffled into a stack. One by one, cards are taken off of the top of the stack until a card labeled with a prime number is removed. How many cards are removed on average?

8. [**6**] A set of six edges of a regular octahedron is called *Hamiltonian cycle* if the edges in some order constitute a single continuous loop that visits each vertex exactly once. How many ways are there to partition the twelve edges into two Hamiltonian cycles?



9. [**7**] Let S denote the set of all triples (i, j, k) of positive integers where $i + j + k = 17$. Compute

$$\sum_{(i,j,k) \in S} ijk.$$

10. [**8**] A subset S of the nonnegative integers is called *supported* if it contains 0, and $k + 8, k + 9 \in S$ for all $k \in S$. How many supported sets are there?

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Individual Round: General Test, Part 1

1. [**2**] Michael has 16 white socks, 3 blue socks, and 6 red socks in a drawer. Ever the lazy college student, he has overslept and is late for his favorite team's season-opener. Because he is now in such a rush to get from Harvard to Foxborough, he randomly takes socks from the drawer (one at a time) until he has a pair of the same color. What is the largest number of socks he could possibly withdraw in this fashion?

2. [**2**] Rectangle $ABCD$ has side lengths $AB = 12$ and $BC = 5$. Let P and Q denote the midpoints of segments AB and DP, respectively. Determine the area of triangle CDQ.

3. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

4. [**3**] Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?

5. [**4**] Find the smallest positive integer that is twice a perfect square and three times a perfect cube.

6. [**4**] The positive integer n is such that the numbers 2^n and 5^n start with the same digit when written in decimal notation; determine this common leading digit.

7. [**4**] Jack, Jill, and John play a game in which each randomly picks and then replaces a card from a standard 52 card deck, until a spades card is drawn. What is the probability that Jill draws the spade? (Jack, Jill, and John draw in that order, and the game repeats if no spade is drawn.)

8. [**5**] Determine the largest positive integer n such that there exist positive integers x, y, z so that

$$n^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + 3x + 3y + 3z - 6$$

9. [**6**] I have four distinct rings that I want to wear on my right hand hand (five distinct fingers.) One of these rings is a Canadian ring that must be worn on a finger by itself, the rest I can arrange however I want. If I have two or more rings on the same finger, then I consider different orders of rings along the same finger to be different arrangements. How many different ways can I wear the rings on my fingers?

10. [**7**] $\alpha_1, \alpha_2, \alpha_3$, and α_4 are the complex roots of the equation $x^4 + 2x^3 + 2 = 0$. Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

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Individual Round: General Test, Part 2

1. [**2**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**2**] A parallelogram has 3 of its vertices at (1,2), (3,8), and (4,1). Compute the sum of all possible x coordinates of the 4th vertex.

3. [**3**] Compute
$$\left\lfloor \frac{2007! + 2004!}{2006! + 2005!} \right\rfloor .$$
(Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

4. [**3**] Three brothers Abel, Banach, and Gauss each have portable music players that can share music with each other. Initially, Abel has 9 songs, Banach has 6 songs, and Gauss has 3 songs, and none of these songs are the same. One day, Abel flips a coin to randomly choose one of his brothers and he adds all of that brother's songs to his collection. The next day, Banach flips a coin to randomly choose one of his brothers and he adds all of that brother's collection of songs to his collection. Finally, each brother randomly plays a song from his collection with each song in his collection being equally likely to be chosen. What is the probability that they all play the same song?

5. [**4**] A best of 9 series is to be played between two teams. That is, the first team to win 5 games is the winner. One of the teams, the Mathletes, has a 2/3 chance of winning any given game. What is the probability that the winner is determined in the 7th game?

6. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

7. [**4**] My friend and I are playing a game with the following rules: If one of us says an integer n, the opponent then says an integer of their choice between $2n$ and $3n$, inclusive. Whoever first says 2007 or greater loses the game, and their opponent wins. I must begin the game by saying a positive integer less than 10. With how many of them can I guarantee a win?

8. [**5**] Compute the number of sequences of numbers $a_1, a_2, \ldots, a_{10}$ such that
$$\begin{aligned} &\text{I. } a_i = 0 \text{ or } 1 \text{ for all } i \\ &\text{II. } a_i \cdot a_{i+1} = 0 \text{ for } i = 1, 2, \ldots, 9 \\ &\text{III. } a_i \cdot a_{i+2} = 0 \text{ for } i = 1, 2, \ldots, 8. \end{aligned}$$

9. [**6**] Let $A := \mathbb{Q} \setminus \{0, 1\}$ denote the set of all rationals other than 0 and 1. A function $f : A \to \mathbb{R}$ has the property that for all $x \in A$,
$$f(x) + f\left(1 - \frac{1}{x}\right) = \log |x|.$$
Compute the value of $f(2007)$.

10. [**7**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.

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Individual Round: Geometry Test

1. [**3**] A cube of edge length $s > 0$ has the property that its surface area is equal to the sum of its volume and five times its edge length. Compute all possible values of s.

2. [**3**] A, B, C, and D are points on a circle, and segments $\overline{AC}$ and $\overline{BD}$ intersect at P, such that $AP = 8$, $PC = 1$, and $BD = 6$. Find BP, given that $BP < DP$.

3. [**4**] Circles ω_1, ω_2, and ω_3 are centered at M, N, and O, respectively. The points of tangency between ω_2 and ω_3, ω_3 and ω_1, and ω_1 and ω_2 are tangent at A, B, and C, respectively. Line MO intersects ω_3 and ω_1 again at P and Q respectively, and line AP intersects ω_2 again at R. Given that ABC is an equilateral triangle of side length 1, compute the area of PQR.

4. [**4**] Circle ω has radius 5 and is centered at O. Point A lies outside ω such that $OA = 13$. The two tangents to ω passing through A are drawn, and points B and C are chosen on them (one on each tangent), such that line BC is tangent to ω and ω lies outside triangle ABC. Compute $AB + AC$ given that $BC = 7$.

5. [**5**] Five marbles of various sizes are placed in a conical funnel. Each marble is in contact with the adjacent marble(s). Also, each marble is in contact all around the funnel wall. The smallest marble has a radius of 8, and the largest marble has a radius of 18. What is the radius of the middle marble?

6. [**5**] Triangle ABC has $\angle A = 90°$, side $BC = 25, AB > AC$, and area 150. Circle ω is inscribed in ABC, with M its point of tangency on AC. Line BM meets ω a second time at point L. Find the length of segment BL.

7. [**5**] Convex quadrilateral $ABCD$ has sides $AB = BC = 7$, $CD = 5$, and $AD = 3$. Given additionally that $m\angle ABC = 60°$, find BD.

8. [**6**] $ABCD$ is a convex quadrilateral such that $AB < AD$. The diagonal $\overline{AC}$ bisects $\angle BAD$, and $m\angle ABD = 130°$. Let E be a point on the interior of $\overline{AD}$, and $m\angle BAD = 40°$. Given that $BC = CD = DE$, determine $m\angle ACE$ in degrees.

9. [**7**] $\triangle ABC$ is right angled at A. D is a point on AB such that $CD = 1$. AE is the altitude from A to BC. If $BD = BE = 1$, what is the length of AD?

10. [**8**] $ABCD$ is a convex quadrilateral such that $AB = 2, BC = 3, CD = 7$, and $AD = 6$. It also has an incircle. Given that $\angle ABC$ is right, determine the radius of this incircle.

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Guts Round

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Note that there are just 36 problems in the Guts round this year.

1. [**5**] Define the sequence of positive integers a_n recursively by $a_1 = 7$ and $a_n = 7^{a_{n-1}}$ for all $n \geq 2$. Determine the last two digits of a_{2007}.

2. [**5**] A candy company makes 5 colors of jellybeans, which come in equal proportions. If I grab a random sample of 5 jellybeans, what is the probability that I get exactly 2 distinct colors?

3. [**5**] The equation $x^2 + 2x = i$ has two complex solutions. Determine the product of their real parts.

. . . .

4. [**6**] A sequence consists of the digits $122333444455555\ldots$ such that the each positive integer n is repeated n times, in increasing order. Find the sum of the 4501st and 4052nd digits of this sequence.

5. [**6**] Compute the largest positive integer such that $\frac{2007!}{2007^n}$ is an integer.

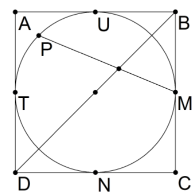
6. [**6**] There are three video game systems: the Paystation, the WHAT, and the ZBoz2π, and none of these systems will play games for the other systems. Uncle Riemann has three nephews: Bernoulli, Galois, and Dirac. Bernoulli owns a Paystation and a WHAT, Galois owns a WHAT and a ZBoz2π, and Dirac owns a ZBoz2π and a Paystation. A store sells 4 different games for the Paystation, 6 different games for the WHAT, and 10 different games for the ZBoz2π. Uncle Riemann does not understand the difference between the systems, so he walks into the store and buys 3 random games (not necessarily distinct) and randomly hands them to his nephews. What is the probability that each nephew receives a game he can play?

. . . .

7. [**7**] A student at Harvard named Kevin
Was counting his stones by 11
He messed up n times
And instead counted 9s
And wound up at 2007.

How many values of n could make this limerick true?

8. [**7**] A circle inscribed in a square,
Has two chords as shown in a pair.
It has radius 2,
And P bisects TU.
The chords' intersection is where?



Answer the question by giving the distance of the point of intersection from the center of the circle.

9. [**7**] I ponder some numbers in bed,
All products of three primes I've said,
Apply ϕ they're still fun:
now Elev'n cubed plus one.
What numbers could be in my head?

$n = 37^2 \cdot 3 \ldots$
$\phi(n) =$
$11^3 + 1$?

. . . .

10. [**8**] Let A_{12} denote the answer to problem 12. There exists a unique triple of digits (B, C, D) such that $10 > A_{12} > B > C > D > 0$ and

$$\overline{A_{12}BCD} - \overline{DCBA_{12}} = \overline{BDA_{12}C},$$

where $\overline{A_{12}BCD}$ denotes the four digit base 10 integer. Compute $B + C + D$.

11. [**8**] Let A_{10} denote the answer to problem 10. Two circles lie in the plane; denote the lengths of the internal and external tangents between these two circles by x and y, respectively. Given that the product of the radii of these two circles is $15/2$, and that the distance between their centers is A_{10}, determine $y^2 - x^2$.

12. [**8**] Let A_{11} denote the answer to problem 11. Determine the smallest prime p such that the arithmetic sequence $p, p + A_{11}, p + 2A_{11}, \ldots$ begins with the largest possible number of primes.

There is just one triple of possible (A_{10}, A_{11}, A_{12}) of answers to these three problems. Your team will receive credit only for answers matching these. (So, for example, submitting a wrong answer for problem 11 will not alter the correctness of your answer to problem 12.)

. . . .

13. [**9**] Determine the largest integer n such that $7^{2048} - 1$ is divisible by 2^n.

14. [**9**] We are given some similar triangles. Their areas are $1^2, 3^2, 5^2 \ldots$, and 49^2. If the smallest triangle has a perimeter of 4, what is the sum of all the triangles' perimeters?

15. [**9**] Points A, B, and C lie in that order on line ℓ, such that $AB = 3$ and $BC = 2$. Point H is such that CH is perpendicular to ℓ. Determine the length CH such that $\angle AHB$ is as large as possible.

. . . .

16. [**10**] Let ABC be a triangle with $AB = 7, BC = 9$, and $CA = 4$. Let D be the point such that $AB \parallel CD$ and $CA \parallel BD$. Let R be a point within triangle BCD. Lines ℓ and m going through R are parallel to CA and AB respectively. Line ℓ meets AB and BC at P and P' respectively, and m meets CA and BC at Q and Q' respectively. If S denotes the largest possible sum of the areas of triangles $BPP', RP'Q'$, and CQQ', determine the value of S^2.

17. [**10**] During the regular season, Washington Redskins achieve a record of 10 wins and 6 losses. Compute the probability that their wins came in three streaks of consecutive wins, assuming that all possible arrangements of wins and losses are equally likely. (For example, the record LLWWWWWWLWWL-WWWLL contains three winning streaks, while WWWWWWWLLLLLLWWW has just two.)

18. [**10**] Convex quadrilateral $ABCD$ has right angles $\angle A$ and $\angle C$ and is such that $AB = BC$ and $AD = CD$. The diagonals AC and BD intersect at point M. Points P and Q lie on the circumcircle of triangle AMB and segment CD, respectively, such that points P, M, and Q are collinear. Suppose that $m\angle ABC = 160°$ and $m\angle QMC = 40°$. Find $MP \cdot MQ$, given that $MC = 6$.

. . . .

19. [**10**] Define $x \star y = \frac{\sqrt{x^2+3xy+y^2-2x-2y+4}}{xy+4}$. Compute

$$((\cdots((2007 \star 2006) \star 2005) \star \cdots) \star 1).$$

20. [**10**] For a a positive real number, let x_1, x_2, x_3 be the roots of the equation $x^3 - ax^2 + ax - a = 0$. Determine the smallest possible value of $x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3$.

21. [**10**] Bob the bomb-defuser has stumbled upon an active bomb. He opens it up, and finds the red and green wires conveniently located for him to cut. Being a seasoned member of the bomb-squad, Bob quickly determines that it is the green wire that he should cut, and puts his wirecutters on the green wire. But just before he starts to cut, the bomb starts to count down, ticking every second. Each time the bomb ticks, starting at time $t = 15$ seconds, Bob panics and has a certain chance to move his wirecutters to the other wire. However, he is a rational man even when panicking, and has a $\frac{1}{2t^2}$ chance of switching wires at time t, regardless of which wire he is about to cut. When the bomb ticks at $t = 1$, Bob cuts whatever wire his wirecutters are on, without switching wires. What is the probability that Bob cuts the green wire?

. . . .

22. [**12**] The sequence $\{a_n\}_{n\geq 1}$ is defined by $a_{n+2} = 7a_{n+1} - a_n$ for positive integers n with initial values $a_1 = 1$ and $a_2 = 8$. Another sequence, $\{b_n\}$, is defined by the rule $b_{n+2} = 3b_{n+1} - b_n$ for positive integers n together with the values $b_1 = 1$ and $b_2 = 2$. Find $\gcd(a_{5000}, b_{501})$.

23. [**12**] In triangle ABC, $\angle ABC$ is obtuse. Point D lies on side AC such that $\angle ABD$ is right, and point E lies on side AC between A and D such that BD bisects $\angle EBC$. Find CE, given that $AC = 35, BC = 7$, and $BE = 5$.

24. [**12**] Let x, y, n be positive integers with $n > 1$. How many ordered triples (x, y, n) of solutions are there to the equation $x^n - y^n = 2^{100}$?

. . . .

25. [**12**] Two real numbers x and y are such that $8y^4 + 4x^2y^2 + 4xy^2 + 2x^3 + 2y^2 + 2x = x^2 + 1$. Find all possible values of $x + 2y^2$.

26. [**12**] $ABCD$ is a cyclic quadrilateral in which $AB = 4, BC = 3, CD = 2$, and $AD = 5$. Diagonals AC and BD intersect at X. A circle ω passes through A and is tangent to BD at X. ω intersects AB and AD at Y and Z respectively. Compute YZ/BD.

27. [**12**] Find the number of 7-tuples $(n_1, \ldots, n_7)$ of integers such that

$$\sum_{i=1}^{7} n_i^6 = 96957.$$

. . . .

28. [**15**] Compute the circumradius of cyclic hexagon $ABCDEF$, which has side lengths $AB = BC = 2, CD = DE = 9$, and $EF = FA = 12$.

29. [**15**] A sequence $\{a_n\}_{n\geq 1}$ of positive reals is defined by the rule $a_{n+1}a_{n-1}^5 = a_n^4 a_{n-2}^2$ for integers $n > 2$ together with the initial values $a_1 = 8$ and $a_2 = 64$ and $a_3 = 1024$. Compute

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \cdots}}}$$

30. [**15**] $ABCD$ is a cyclic quadrilateral in which $AB = 3, BC = 5, CD = 6$, and $AD = 10$. M, I, and T are the feet of the perpendiculars from D to lines AB, AC, and BC respectively. Determine the value of MI/IT.

. . . .

31. [**18**] A sequence $\{a_n\}_{n\geq 0}$ of real numbers satisfies the recursion $a_{n+1} = a_n^3 - 3a_n^2 + 3$ for all positive integers n. For how many values of a_0 does $a_{2007} = a_0$?

32. [**18**] Triangle ABC has $AB = 4, BC = 6$, and $AC = 5$. Let O denote the circumcenter of ABC. The circle Γ is tangent to and surrounds the circumcircles of triangles AOB, BOC, and AOC. Determine the diameter of Γ.

33. [**18**] Compute

$$\int_1^2 \frac{9x+4}{x^5+3x^2+x}dx.$$

(No, your TI-89 doesn't know how to do this one. Yes, the end is near.)

. . . .

34. [?] *The Game.* Eric and Greg are watching their new favorite TV show, *The Price is Right.* Bob Barker recently raised the intellectual level of his program, and he begins the latest installment with bidding on following question: How many Carmichael numbers are there less than 100,000?

    Each team is to list one nonnegative integer not greater than 100,000. Let X denote the answer to Bob's question. The teams listing N, a maximal bid (of those submitted) not greater than X, will receive N points, and all other teams will neither receive nor lose points. (A Carmichael number is an odd composite integer n such that n divides $a^{n-1} - 1$ for all integers a relatively prime to n with $1 < a < n$.)

35. [$\leq$ **25**] *The Algorithm.* There are thirteen broken computers situated at the following set S of thirteen points in the plane:

$$\begin{array}{ccc} A = (1, 10) & B = (976, 9) & C = (666, 87) \\ D = (377, 422) & E = (535, 488) & F = (775, 488) \\ G = (941, 500) & H = (225, 583) & I = (388, 696) \\ J = (3, 713) & K = (504, 872) & L = (560, 934) \\ & M = (22, 997) & \end{array}$$

    At time $t = 0$, a repairman begins moving from one computer to the next, traveling continuously in straight lines at unit speed. Assuming the repairman begins and A and fixes computers instantly, what path does he take to minimize the *total downtime* of the computers? List the points he visits in order. Your score will be $\lfloor \frac{N}{40} \rfloor$, where

$$N = 1000 + \lfloor \text{the optimal downtime} \rfloor - \lfloor \text{your downtime} \rfloor,$$

    or 0, whichever is greater. By total downtime we mean the sum

$$\sum_{P \in S} t_P,$$

    where t_P is the time at which the repairman reaches P.

36. [**25**] *The Marathon.* Let ω denote the incircle of triangle ABC. The segments BC, CA, and AB are tangent to ω at D, E, and F, respectively. Point P lies on EF such that segment PD is perpendicular to BC. The line AP intersects BC at Q. The circles ω_1 and ω_2 pass through B and C, respectively, and are tangent to AQ at Q; the former meets AB again at X, and the latter meets AC again at Y. The line XY intersects BC at Z. Given that $AB = 15, BC = 14$, and $CA = 13$, find $\lfloor XZ \cdot YZ \rfloor$.

10^{th} Annual Harvard-MIT Mathematics Tournament

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Team Round: A Division

Σ, τ, and You: Fun at Fraternities? [270]

A *number theoretic function* is a function whose domain is the set of positive integers. A *multiplicative* number theoretic function is a number theoretic function f such that $f(mn) = f(m)f(n)$ for all pairs of relatively prime positive integers m and n. Examples of multiplicative number theoretic functions include σ, τ, ϕ, and μ, defined as follows. For each positive integer n,

- The *sum-of-divisors function*, $\sigma(n)$, is the sum of all positive integer divisors of n. If $p_1, \ldots, p_i$ are distinct primes and $e_1, \ldots, e_i$ are positive integers,

$$\sigma\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} \left(1 + p_k + \cdots + p_k^{e_k}\right) = \prod_{k=1}^{i} \frac{p_k^{e_k+1} - 1}{p_k - 1}.$$

- The *divisor function*, $\tau(n)$, is the number of positive integer divisors of n. It can be computed by the formula
$$\tau\left(p_1^{e_1} \cdots p_i^{e_i}\right) = (e_1 + 1) \cdots (e_i + 1),$$
where $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ are as above.

- Euler's *totient* function, $\phi(n)$, is the number of positive integers $k \leq n$ such that k and n are relatively prime. For $p_1, \ldots, p_i$ and $e_1, \ldots, e_i$ as above, the phi function satisfies

$$\phi\left(p_1^{e_1} \cdots p_i^{e_i}\right) = \prod_{k=1}^{i} p_k^{e_k - 1}(p_k - 1).$$

- The *Möbius function*, $\mu(n)$, is equal to either 1, -1, or 0. An integer is called *square-free* if it is not divisible by the square of any prime. If n is a square-free positive integer having an even number of distinct prime divisors, $\mu(n) = 1$. If n is a square-free positive integer having an odd number of distinct prime divisors, $\mu(n) = -1$. Otherwise, $\mu(n) = 0$.

The Möbius function has a number of peculiar properties. For example, if f and g are number theoretic functions such that
$$g(n) = \sum_{d|n} f(d),$$
for all positive integers n, then
$$f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right).$$
This is known as *Möbius inversion.* In proving the following problems, *you may use any of the preceding assertions without proving them. You may also cite the results of previous problems, even if you were unable to prove them.*

1. [**15**] Evaluate the functions $\phi(n), \sigma(n)$, and $\tau(n)$ for $n = 12, n = 2007$, and $n = 2^{2007}$.

2. [**20**] Solve for the positive integer(s) n such that $\phi(n^2) = 1000\phi(n)$.

3. [**25**] Prove that for every integer n greater than 1,
$$\sigma(n)\phi(n) \leq n^2 - 1.$$
When does equality hold?

4. [**25**] Let F and G be two multiplicative functions, and define for positive integers n,

$$H(n) = \sum_{d|n} F(d)G\left(\frac{n}{d}\right).$$

The number theoretic function H is called the *convolution* of F and G. Prove that H is multiplicative.

5. [**30**] Prove the identity

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d)\right)^2.$$

6. [**25**] Show that for positive integers n,

$$\sum_{d|n} \phi(d) = n.$$

7. [**25**] Show that for positive integers n,

$$\sum_{d|n} \frac{\mu(d)}{d} = \frac{\phi(n)}{n}.$$

8. [**30**] Determine with proof, a simple closed form expression for

$$\sum_{d|n} \phi(d)\tau\left(\frac{n}{d}\right).$$

9. [**35**] Find all positive integers n such that

$$\sum_{k=1}^{n} \phi(k) = \frac{3n^2+5}{8}.$$

10. [**40**] Find all pairs (n, k) of positive integers such that

$$\sigma(n)\phi(n) = \frac{n^2}{k}.$$

Grab Bag - Miscellaneous Problems [130]

11. [**30**] Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$\begin{aligned} f(x)f(y) &= f(x) + f(y) - f(xy) \\ 1 + f(x+y) &= f(xy) + f(x)f(y) \end{aligned}$$

for all rational numbers x, y.

12. [**30**] Let $ABCD$ be a cyclic quadrilateral, and let P be the intersection of its two diagonals. Points R, S, T, and U are feet of the perpendiculars from P to sides AB, BC, CD, and AD, respectively. Show that quadrilateral $RSTU$ is bicentric if and only if $AC \perp BD$. (Note that a quadrilateral is called *inscriptible* if it has an incircle; a quadrilateral is called *bicentric* if it is both cyclic and inscriptible.)

13. [**30**] Find all nonconstant polynomials $P(x)$, with real coefficients and having only real zeros, such that $P(x+1)P(x^2-x+1) = P(x^3+1)$ for all real numbers x.

14. [**40**] Find an explicit, closed form formula for

$$\sum_{k=1}^{n} \frac{k \cdot (-1)^k \cdot \binom{n}{k}}{n+k+1}.$$

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Team Round: B Division

Compute $(x-a)(x-b)\cdots(x-z)$ - Short Answer [200]

For this section, your team should give only the anwers to the problems.

1. [**20**] Find the sum of the positive integer divisors of 2^{2007}.
2. [**20**] The four sides of quadrilateral $ABCD$ are equal in length. Determine the perimeter of $ABCD$ given that it has area 120 and $AC = 10$.
3. [**20**] Five people are crowding into a booth against a wall at a noisy restaurant. If at most three can fit on one side, how many seating arrangements accomodate them all?
4. [**20**] Thomas and Michael are just two people in a large pool of well qualified candidates for appointment to a problem writing committee for a prestigious college math contest. It is 40 times more likely that both will serve if the size of the committee is increased from its traditional 3 members to a whopping n members. Determine n. (Each person in the pool is equally likely to be chosen.)
5. [**20**] The curves $y = x^2(x-3)^2$ and $y = (x^2-1)(x-2)$ intersect at a number of points in the real plane. Determine the sum of the x-coordinates of these points of intersection.
6. [**20**] Andrew has a fair six sided die labeled with 1 through 6 as usual. He tosses it repeatedly, and on every third roll writes down the number facing up as long as it is not the 6. He stops as soon as the last two numbers he has written down are squares or one is a prime and the other is a square. What is the probability that he stops after writing squares consecutively?
7. [**20**] Three positive reals x, y, and z are such that

$$\begin{aligned} x^2 + 2(y-1)(z-1) &= 85 \\ y^2 + 2(z-1)(x-1) &= 84 \\ z^2 + 2(x-1)(y-1) &= 89. \end{aligned}$$

   Compute $x + y + z$.

8. [**20**] Find the *positive* real number(s) x such that $\frac{1}{2}\left(3x^2-1\right) = \left(x^2-50x-10\right)\left(x^2+25x+5\right)$.
9. [**20**] Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$, and $AD = 4$. Determine AC/BD.
10. [**20**] A positive real number x is such that

$$\sqrt[3]{1-x^3} + \sqrt[3]{1+x^3} = 1.$$

   Find x^2.

Adult Acorns - Gee, I'm a Tree! [200]

In this section of the team round, your team will derive some basic results concerning *tangential* quadrilaterals. Tangential quadrilaterals have an *incircle*, or a circle lying within them that is tangent to all four sides. If a quadrilateral has an incircle, then the center of this circle is the *incenter* of the quadrilateral. As you shall see, tangential quadrilaterals are related to cyclic quadrilaterals. For reference, a review of cyclic quadrilaterals is given at the end of this section.

Your answers for this section of the team test should be proofs. *Note that you may use any standard facts about cyclic quadrilaterals, such as those listed at the end of this test, without proving them. Additionally, you may cite the results of previous problems, even if you were unable to prove them.*

For these problems, $ABCD$ is a tangential quadrilateral having incenter I. For the first three problems, the point P is constructed such that triangle PAB is similar to triangle IDC and lies outside $ABCD$.

1. [**30**] Show that $PAIB$ is cyclic by proving that $\angle IAP$ is supplementary to $\angle PBI$.

2. [**40**] Show that triangle PAI is similar to triangle BIC. Then conclude that

$$PA = \frac{PI}{BC} \cdot BI.$$

3. [**25**] Deduce from the above that

$$\frac{BC}{AD} \cdot \frac{AI}{BI} \cdot \frac{DI}{CI} = 1.$$

4. [**25**] Show that $AB + CD = AD + BC$. Use the above to conclude that for some positive number α,

$$AB = \alpha \cdot \left(\frac{AI}{CI} + \frac{BI}{DI}\right) \qquad BC = \alpha \cdot \left(\frac{BI}{DI} + \frac{CI}{AI}\right)$$
$$CD = \alpha \cdot \left(\frac{CI}{AI} + \frac{DI}{BI}\right) \qquad DA = \alpha \cdot \left(\frac{DI}{BI} + \frac{AI}{CI}\right).$$

5. [**40**] Show that

$$AB \cdot BC = BI^2 + \frac{AI \cdot BI \cdot CI}{DI}.$$

6. [**40**] Let the incircle of $ABCD$ be tangent to sides AB, BC, CD, and AD at points P, Q, R, and S, respectively. Show that $ABCD$ is cyclic if and only if $PR \perp QS$.

A brief review of cyclic Quadrilaterals.

The following discussion of cyclic quadrilaterals is included for reference. Any of the results given here may be cited without proof in your writeups.

A *cyclic quadrilateral* is a quadrilateral whose four vertices lie on a circle called the *circumcircle* (the circle is unique if it exists.) If a quadrilateral has a circumcircle, then the center of this circumcircle is called the *circumcenter* of the quadrilateral. For a convex quadrilateral $ABCD$, the following are equivalent:

- Quadrilateral $ABCD$ is cyclic;
- $\angle ABD = \angle ACD$ (or $\angle BCA = \angle BDA$, etc.);
- Angles $\angle ABC$ and $\angle CDA$ are *supplementary*, that is, $m\angle ABC + m\angle CDA = 180°$ (or angles $\angle BCD$ and $\angle BAD$ are supplementary);

Cyclic quadrilaterals have a number of interesting properties. A cyclic quadrilateral $ABCD$ satisfies

$$AC \cdot BD = AB \cdot CD + AD \cdot BC,$$

a result known as *Ptolemy's theorem.* Another result, typically called *Power of a Point*, asserts that given a circle ω, a point P anywhere in the plane of ω, and a line ℓ through P intersecting ω at points A and B, the value of $AP \cdot BP$ is independent of ℓ; i.e., if a second line ℓ' through P intersects ω at A' and B', then $AP \cdot BP = A'P \cdot B'P$. This second theorem is proved via similar triangles. Say P lies outside of ω, that ℓ and ℓ' are as before and that A and A' lie on segments BP and $B'P$ respectively. Then triangle $AA'P$ is similar to triangle $B'BP$ because the triangles share an angle at P and we have

$$m\angle AA'P = 180° - m\angle B'A'A = m\angle ABB' = m\angle PBB'.$$

The case where $A = B$ is valid and describes the tangents to ω. A similar proof works for P inside ω.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Algebra Test

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

2. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

3. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

4. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = a_2 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$. Find
$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}.$$

5. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between 0° and 180° inclusive. At how many times during that day are the angles on the two clocks equal?

6. Let a, b, c be the roots of $x^3 - 9x^2 + 11x - 1 = 0$, and let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$. Find $s^4 - 18s^2 - 8s$.

7. Let
$$f(x) = x^4 - 6x^3 + 26x^2 - 46x + 65.$$
Let the roots of $f(x)$ be $a_k + ib_k$ for $k = 1, 2, 3, 4$. Given that the a_k, b_k are all integers, find $|b_1| + |b_2| + |b_3| + |b_4|$.

8. Solve for all complex numbers z such that $z^4 + 4z^2 + 6 = z$.

9. Compute the value of the infinite series
$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n^4 + 4)}$$

10. Determine the maximum value attained by
$$\frac{x^4 - x^2}{x^6 + 2x^3 - 1}$$
over real numbers $x > 1$.

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Individual Round: Calculus Test

1. A nonzero polynomial $f(x)$ with real coefficients has the property that $f(x) = f'(x)f''(x)$. What is the leading coefficient of $f(x)$?

2. Compute $\displaystyle\lim_{x\to 0} \frac{e^{x\cos x} - 1 - x}{\sin(x^2)}$.

3. At time 0, an ant is at $(1,0)$ and a spider is at $(-1,0)$. The ant starts walking counterclockwise along the unit circle, and the spider starts creeping to the right along the x-axis. It so happens that the ant's horizontal speed is always half the spider's. What will the shortest distance ever between the ant and the spider be?

4. Compute $\displaystyle\sum_{k=1}^{\infty} \frac{k^4}{k!}$.

5. Compute $\displaystyle\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$.

6. A triangle with vertices at $(1003,0)$, $(1004,3)$, and $(1005,1)$ in the xy-plane is revolved all the way around the y-axis. Find the volume of the solid thus obtained.

7. Find all positive real numbers c such that the graph of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 - cx$ has the property that the circle of curvature at any local extremum is centered at a point on the x-axis.

8. Compute $\displaystyle\int_0^{\pi/3} x \tan^2(x)dx$.

9. Compute the sum of all real numbers x such that
$$2x^6 - 3x^5 + 3x^4 + x^3 - 3x^2 + 3x - 1 = 0.$$

10. Suppose f and g are differentiable functions such that
$$xg(f(x))f'(g(x))g'(x) = f(g(x))g'(f(x))f'(x)$$
for all real x. Moreover, f is nonnegative and g is positive. Furthermore,
$$\int_0^a f(g(x))dx = 1 - \frac{e^{-2a}}{2}$$
for all reals a. Given that $g(f(0)) = 1$, compute the value of $g(f(4))$.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Individual Round: Combinatorics Test

1. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

2. Compute
$$\sum_{n_{60}=0}^{2} \sum_{n_{59}=0}^{n_{60}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2} \sum_{n_0=0}^{n_1} 1.$$

3. A moth starts at vertex A of a certain cube and is trying to get to vertex B, which is opposite A, in five or fewer "steps," where a step consists in traveling along an edge from one vertex to another. The moth will stop as soon as it reaches B. How many ways can the moth achieve its objective?

4. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

5. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

6. For how many ordered triplets (a, b, c) of positive integers less than 10 is the product $a \times b \times c$ divisible by 20?

7. Let n be a positive integer, and let Pushover be a game played by two players, standing squarely facing each other, pushing each other, where the first person to lose balance loses. At the HMPT, 2^{n+1} competitors, numbered 1 through 2^{n+1} clockwise, stand in a circle. They are equals in Pushover: whenever two of them face off, each has a 50% probability of victory. The tournament unfolds in $n+1$ rounds. In each round, the referee randomly chooses one of the surviving players, and the players pair off going clockwise, starting from the chosen one. Each pair faces off in Pushover, and the losers leave the circle. What is the probability that players 1 and 2^n face each other in the last round? Express your answer in terms of n.

8. In how many ways can we enter numbers from the set $\{1, 2, 3, 4\}$ into a 4×4 array so that all of the following conditions hold?

   (a) Each row contains all four numbers.

   (b) Each column contains all four numbers.

   (c) Each "quadrant" contains all four numbers. (The quadrants are the four corner 2×2 squares.)

9. Eight celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

10. Somewhere in the universe, n students are taking a 10-question math competition. Their collective performance is called *laughable* if, for some pair of questions, there exist 57 students such that either all of them answered both questions correctly or none of them answered both questions correctly. Compute the smallest n such that the performance is necessarily laughable.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

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Individual Round: General Test, Part 1

1. How many positive integers x are there such that $3x$ has 3 digits and $4x$ has four digits?

2. What is the probability that two cards randomly selected (without replacement) from a standard 52-card deck are neither of the same value nor the same suit?

3. A square and an equaliteral triangle together have the property that the area of each is the perimeter of the other. Find the square's area.

4. Find
$$\frac{\sqrt{31+\sqrt{31+\sqrt{31+\ldots}}}}{\sqrt{1+\sqrt{1+\sqrt{1+\ldots}}}}.$$

5. In the plane, what is the length of the shortest path from $(-2,0)$ to $(2,0)$ that avoids the interior of the unit circle (i.e., circle of radius 1) centered at the origin?

6. Six celebrities meet at a party. It so happens that each celebrity shakes hands with exactly two others. A fan makes a list of all unordered pairs of celebrities who shook hands with each other. If order does not matter, how many different lists are possible?

7. The train schedule in Hummut is hopelessly unreliable. Train A will enter Intersection X from the west at a random time between 9:00 am and 2:30 pm; each moment in that interval is equally likely. Train B will enter the same intersection from the north at a random time between 9:30 am and 12:30 pm, independent of Train A; again, each moment in the interval is equally likely. If each train takes 45 minutes to clear the intersection, what is the probability of a collision today?

8. A dot is marked at each vertex of a triangle ABC. Then, 2, 3, and 7 more dots are marked on the sides AB, BC, and CA, respectively. How many triangles have their vertices at these dots?

9. Take a unit sphere S, i.e., a sphere with radius 1. Circumscribe a cube C about S, and inscribe a cube D in S, so that every edge of cube C is parallel to some edge of cube D. What is the shortest possible distance from a point on a face of C to a point on a face of D?

10. A positive integer n is called "flippant" if n does not end in 0 (when written in decimal notation) and, moreover, n and the number obtained by reversing the digits of n are both divisible by 7. How many flippant integers are there between 10 and 1000?

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Individual Round: General Test, Part 2

1. Larry can swim from Harvard to MIT (with the current of the Charles River) in 40 minutes, or back (against the current) in 45 minutes. How long does it take him to *row* from Harvard to MIT, if he rows the return trip in 15 minutes? (Assume that the speed of the current and Larry's swimming and rowing speeds relative to the current are all constant.) Express your answer in the format mm:ss.

2. Find
$$\frac{2^2}{2^2-1}\cdot\frac{3^2}{3^2-1}\cdot\frac{4^2}{4^2-1}\cdot\cdots\cdot\frac{2006^2}{2006^2-1}.$$

3. Let C be the unit circle. Four distinct, smaller congruent circles C_1, C_2, C_3, C_4 are internally tangent to C such that C_i is externally tangent to C_{i-1} and C_{i+1} for $i = 1, \ldots, 4$ where C_5 denotes C_1 and C_0 represents C_4. Compute the radius of C_1.

4. Vernonia High School has 85 seniors, each of whom plays on at least one of the school's three varsity sports teams: football, baseball, and lacrosse. It so happens that 74 are on the football team; 26 are on the baseball team; 17 are on both the football and lacrosse teams; 18 are on both the baseball and football teams; and 13 are on both the baseball and lacrosse teams. Compute the number of seniors playing all three sports, given that twice this number are members of the lacrosse team.

5. If a, b are nonzero real numbers such that $a^2 + b^2 = 8ab$, find the value of $\left|\frac{a+b}{a-b}\right|$.

6. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

7. What is the smallest positive integer n such that n^2 and $(n+1)^2$ both contain the digit 7 but $(n+2)^2$ does not?

8. Six people, all of different weights, are trying to build a human pyramid: that is, they get into the formation

   A
   B C
   D E F

   We say that someone not in the bottom row is "supported by" each of the two closest people beneath her or him. How many different pyramids are possible, if nobody can be supported by anybody of lower weight?

9. Tim has a working analog 12-hour clock with two hands that run continuously (instead of, say, jumping on the minute). He also has a clock that runs really slow—at half the correct rate, to be exact. At noon one day, both clocks happen to show the exact time. At any given instant, the hands on each clock form an angle between $0°$ and $180°$ inclusive. At how many times during that day are the angles on the two clocks equal?

10. Fifteen freshmen are sitting in a circle around a table, but the course assistant (who remains standing) has made only six copies of today's handout. No freshman should get more than one handout, and any freshman who does not get one should be able to read a neighbor's. If the freshmen are distinguishable but the handouts are not, how many ways are there to distribute the six handouts subject to the above conditions?

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Individual Round: Geometry Test

1. Octagon $ABCDEFGH$ is equiangular. Given that $AB = 1$, $BC = 2$, $CD = 3$, $DE = 4$, and $EF = FG = 2$, compute the perimeter of the octagon.

2. Suppose ABC is a scalene right triangle, and P is the point on hypotenuse $\overline{AC}$ such that $\angle ABP = 45°$. Given that $AP = 1$ and $CP = 2$, compute the area of ABC.

3. Let A, B, C, and D be points on a circle such that $AB = 11$ and $CD = 19$. Point P is on segment AB with $AP = 6$, and Q is on segment CD with $CQ = 7$. The line through P and Q intersects the circle at X and Y. If $PQ = 27$, find XY.

4. Let ABC be a triangle such that $AB = 2, CA = 3$, and $BC = 4$. A semicircle with its diameter on $\overline{BC}$ is tangent to $\overline{AB}$ and $\overline{AC}$. Compute the area of the semicircle.

5. Triangle ABC has side lengths $AB = 2\sqrt{5}$, $BC = 1$, and $CA = 5$. Point D is on side AC such that $CD = 1$, and F is a point such that $BF = 2$ and $CF = 3$. Let E be the intersection of lines AB and DF. Find the area of $CDEB$.

6. A circle of radius t is tangent to the hypotenuse, the incircle, and one leg of an isosceles right triangle with inradius $r = 1 + \sin\frac{\pi}{8}$. Find rt.

7. Suppose $ABCD$ is an isosceles trapezoid in which $\overline{AB} \parallel \overline{CD}$. Two mutually externally tangent circles ω_1 and ω_2 are inscribed in $ABCD$ such that ω_1 is tangent to $\overline{AB}, \overline{BC}$, and $\overline{CD}$ while ω_2 is tangent to $\overline{AB}, \overline{DA}$, and $\overline{CD}$. Given that $AB = 1, CD = 6$, compute the radius of either circle.

8. Triangle ABC has a right angle at B. Point D lies on side BC such that $3\angle BAD = \angle BAC$. Given $AC = 2$ and $CD = 1$, compute BD.

9. Four spheres, each of radius r, lie inside a regular tetrahedron with side length 1 such that each sphere is tangent to three faces of the tetrahedron and to the other three spheres. Find r.

10. Triangle ABC has side lengths $AB = 65$, $BC = 33$, and $AC = 56$. Find the radius of the circle tangent to sides AC and BC and to the circumcircle of triangle ABC.

IX^{th} Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Guts Round

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IX^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 25 FEBRUARY 2006 — GUTS ROUND

1. **[5]** A bear walks one mile south, one mile east, and one mile north, only to find itself where it started. Another bear, more energetic than the first, walks two miles south, two miles east, and two miles north, only to find itself where it started. However, the bears are *not* white and did *not* start at the north pole. At most how many miles apart, to the nearest .001 mile, are the two bears' starting points?

2. **[5]** Compute the positive integer less than 1000 which has exactly 29 positive proper divisors. (Here we refer to positive integer divisors other than the number itself.)

3. **[5]** At a nursey, 2006 babies sit in a circle. Suddenly each baby pokes the baby immediately to either its left or its right, with equal probability. What is the expected number of unpoked babies?

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IX^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 25 FEBRUARY 2006 — GUTS ROUND

4. **[6]** Ann and Anne are in bumper cars starting 50 meters apart. Each one approaches the other at a constant ground speed of 10 km/hr. A fly starts at Ann, flies to Anne, then back to Ann, and so on, back and forth until it gets crushed when the two bumper cars collide. When going from Ann to Anne, the fly flies at 20 km/hr; when going in the opposite direction the fly flies at 30 km/hr (thanks to a breeze). How many meters does the fly fly?

5. **[6]** Find the number of solutions in positive integers $(k; a_1, a_2, \ldots, a_k; b_1, b_2, \ldots, b_k)$ to the equation $a_1(b_1) + a_2(b_1 + b_2) + \cdots + a_k(b_1 + b_2 + \cdots + b_k) = 7$.

6. **[6]** Suppose ABC is a triangle such that $AB = 13, BC = 15$, and $CA = 14$. Say D is the midpoint of $\overline{BC}$, E is the midpoint of $\overline{AD}$, F is the midpoint of $\overline{BE}$, and G is the midpoint of $\overline{DF}$. Compute the area of triangle EFG.

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IX^{th} HARVARD-MIT MATHEMATICS TOURNAMENT, 25 FEBRUARY 2006 — GUTS ROUND

7. **[6]** Find all real numbers x such that $x^2 + \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{x}{3} \right\rfloor = 10$.

8. **[6]** How many ways are there to label the faces of a regular octahedron with the integers 1–8, using each exactly once, so that any two faces that share an edge have numbers that are relatively prime? Physically realizable rotations are considered indistinguishable, but physically unrealizable reflections are considered different.

9. **[6]** Four unit circles are centered at the vertices of a unit square, one circle at each vertex. What is the area of the region common to all four circles?

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10. **[7]** Let $f(x) = x^2 - 2x$. How many distinct real numbers c satisfy $f(f(f(f(c)))) = 3$?

11. **[7]** Find all positive integers $n > 1$ for which $\frac{n^2+7n+136}{n-1}$ is the square of a positive integer.

12. **[7]** For each positive integer n let S_n denote the set $\{1, 2, 3, \ldots, n\}$. Compute the number of triples of subsets A, B, C of S_{2006} (not necessarily nonempty or proper) such that A is a subset of B and $S_{2006} - A$ is a subset of C.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

13. **[7]** Let Z be as in problem 15. Let X be the greatest integer such that $|XZ| \leq 5$. Find X.

14. **[7]** Let X be as in problem 13. Let Y be the number of ways to order X crimson flowers, X scarlet flowers, and X vermillion flowers in a row so that no two flowers of the same hue are adjacent. (Flowers of the same hue are mutually indistinguishable.) Find Y.

15. **[7]** Let Y be as in problem 14. Find the maximum Z such that three circles of radius $\sqrt{Z}$ can simultaneously fit inside an equilateral triangle of area Y without overlapping each other.

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16. **[8]** A sequence $a_1, a_2, a_3, \ldots$ of positive reals satisfies $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine all a_1 such that $a_i = \frac{\sqrt{6}+\sqrt{2}}{4}$ for some positive integer i.

17. **[8]** Begining at a vertex, an ant is crawls between the vertices of a regular octahedron. After reaching a vertex, it randomly picks a neighboring vertex (sharing an edge) and walks to that vertex along the adjoining edge (with all possibilities equally likely.) What is the probability that after walking along 2006 edges, the ant returns to the vertex where it began?

18. **[8]** Cyclic quadrilateral $ABCD$ has side lengths $AB = 1, BC = 2, CD = 3$ and $DA = 4$. Points P and Q are the midpoints of $\overline{BC}$ and $\overline{DA}$. Compute PQ^2.

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19. **[8]** Let ABC be a triangle with $AB = 2, CA = 3, BC = 4$. Let D be the point diametrically opposite A on the circumcircle of ABC, and let E lie on line AD such that D is the midpoint of $\overline{AE}$. Line l passes through E perpendicular to $\overline{AE}$, and F and G are the intersections of the extensions of $\overline{AB}$ and $\overline{AC}$ with l. Compute FG.

20. **[8]** Compute the number of real solutions (x, y, z, w) to the system of equations:

$$x = z + w + zwx \qquad z = x + y + xyz$$
$$y = w + x + wxy \qquad w = y + z + yzw$$

21. **[8]** Find the smallest positive integer k such that $z^{10} + z^9 + z^6 + z^5 + z^4 + z + 1$ divides $z^k - 1$.

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22. **[9]** Let $f(x)$ be a degree 2006 polynomial with complex roots $c_1, c_2, \ldots, c_{2006}$, such that the set $\{|c_1|, |c_2|, \ldots, |c_{2006}|\}$ consists of exactly 1006 distinct values. What is the minimum number of real roots of $f(x)$?

23. **[9]** Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers defined by $a_0 = 21, a_1 = 35$, and $a_{n+2} = 4a_{n+1} - 4a_n + n^2$ for $n \geq 2$. Compute the remainder obtained when a_{2006} is divided by 100.

24. **[9]** Two 18-24-30 triangles in the plane share the same circumcircle as well as the same incircle. What's the area of the region common to both the triangles?

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25. **[9]** Points A, C, and B lie on a line in that order such that $AC = 4$ and $BC = 2$. Circles ω_1, ω_2, and ω_3 have $\overline{BC}, \overline{AC}$, and $\overline{AB}$ as diameters. Circle Γ is externally tangent to ω_1 and ω_2 at D and E respectively, and is internally tangent to ω_3. Compute the circumradius of triangle CDE.

26. **[9]** Let $a \geq b \geq c$ be real numbers such that

$$\begin{aligned} a^2bc + ab^2c + abc^2 + 8 &= a + b + c \\ a^2b + a^2c + b^2c + b^2a + c^2a + c^2b + 3abc &= -4 \\ a^2b^2c + ab^2c^2 + a^2bc^2 &= 2 + ab + bc + ca \end{aligned}$$

If $a + b + c > 0$, then compute the integer nearest to a^5.

27. **[9]** Let N denote the number of subsets of $\{1, 2, 3, \ldots, 100\}$ that contain more prime numbers than multiples of 4. Compute the largest integer k such that 2^k divides N.

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28. **[10]** A pebble is shaped as the intersection of a cube of side length 1 with the solid sphere tangent to all of the cube's edges. What is the surface area of this pebble?

29. **[10]** Find the area in the first quadrant bounded by the hyperbola $x^2 - y^2 = 1$, the x-axis, and the line $3x = 4y$.

30. **[10]** ABC is an acute triangle with incircle ω. ω is tangent to sides $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. P is a point on the altitude from A such that Γ, the circle with diameter $\overline{AP}$, is tangent to ω. Γ intersects $\overline{AC}$ and $\overline{AB}$ at X and Y respectively. Given $XY = 8$, $AE = 15$, and that the radius of Γ is 5, compute $BD \cdot DC$.

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The problems in this batch all depend on each other. If you solve them correctly, you will produce a triple of mutually consistent answers. There is only one such triple. Your score will be determined by how many of your answers match that triple.

31. **[10]** Let A be as in problem 33. Let W be the sum of all positive integers that divide A. Find W.

32. **[10]** In the alphametic $WE \times EYE = SCENE$, each different letter stands for a different digit, and no word begins with a 0. The W in this problem has the same value as the W in problem 31. Find S.

33. **[10]** Let W, S be as in problem 32. Let A be the least positive integer such that an acute triangle with side lengths S, A, and W exists. Find A.

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34. **[12]** In bridge, a standard 52-card deck is dealt in the usual way to 4 players. By convention, each hand is assigned a number of "points" based on the formula

$$4 \times (\#\text{ A's}) + 3 \times (\#\text{ K's}) + 2 \times (\#\text{ Q's}) + 1 \times (\#\text{ J's}).$$

Given that a particular hand has exactly 4 cards that are A, K, Q, or J, find the probability that its point value is 13 or higher.

35. **[12]** A sequence is defined by $A_0 = 0, A_1 = 1, A_2 = 2$, and, for integers $n \geq 3$,

$$A_n = \frac{A_{n-1} + A_{n-2} + A_{n-3}}{3} + \frac{1}{n^4 - n^2}$$

Compute $\lim_{N\to\infty} A_N$.

36. **[12]** Four points are independently chosen uniformly at random from the interior of a regular dodecahedron. What is the probability that they form a tetrahedron whose interior contains the dodecahedron's center?

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37. **[15]** Compute $\displaystyle\sum_{n=1}^{\infty} \frac{2n+5}{2^n \cdot (n^3+7n^2+14n+8)}$

38. **[15]** Suppose ABC is a triangle with incircle ω, and ω is tangent to $\overline{BC}$ and $\overline{CA}$ at D and E respectively. The bisectors of $\angle A$ and $\angle B$ intersect line DE at F and G respectively, such that $BF = 1$ and $FG = GA = 6$. Compute the radius of ω.

39. **[15]** A *fat coin* is one which, when tossed, has a 2/5 probability of being heads, 2/5 of being tails, and 1/5 of landing on its edge. Mr. Fat starts at 0 on the real line. Every minute, he tosses a fat coin. If it's heads, he moves left, decreasing his coordinate by 1; if it's tails, he moves right, increasing his coordinate by 1. If the coin lands on its edge, he moves back to 0. If Mr. Fat does this *ad infinitum*, what fraction of his time will he spend at 0?

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40. **[18]** Compute $\displaystyle\sum_{k=1}^{\infty} \frac{3k+1}{2k^3+k^2} \cdot (-1)^{k+1}$.

41. **[18]** Let Γ denote the circumcircle of triangle ABC. Point D is on $\overline{AB}$ such that $\overline{CD}$ bisects $\angle ACB$. Points P and Q are on Γ such that $\overline{PQ}$ passes through D and is perpendicular to $\overline{CD}$. Compute PQ, given that $BC = 20, CA = 80, AB = 65$.

42. **[18]** Suppose hypothetically that a certain, very corrupt political entity in a universe holds an election with two candidates, say A and B. A total of 5,825,043 votes are cast, but, in a sudden rainstorm, all the ballots get soaked. Undaunted, the election officials decide to guess what the ballots say. Each ballot has a 51% chance of being deemed a vote for A, and a 49% chance of being deemed a vote for B. The probability that B will win is 10^{-X}. What is X *rounded to the nearest* 10?

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43. Write down at least one, and up to ten, different 3-digit prime numbers. If you somehow fail to do this, we will ignore your submission for this problem. Otherwise, you're entered into a game with other teams. In this game, you start with 10 points, and each number you write down is like a bet: if no one else writes that number, you gain 1 point, but if anyone else writes that number, you lose 1 point. Thus, your score on this problem can be anything from 0 to 20.

44. On the Euclidean plane are given 14 points:

$A = (0, 428)$	$B = (9, 85)$	$C = (42, 865)$	$D = (192, 875)$
$E = (193, 219)$	$F = (204, 108)$	$G = (292, 219)$	$H = (316, 378)$
$I = (375, 688)$	$J = (597, 498)$	$K = (679, 766)$	$L = (739, 641)$
	$M = (772, 307)$	$N = (793, 0)$	

A fly starts at A, visits all the other points, and comes back to A in such a way as to minimize the total distance covered. What path did the fly take? Give the names of the points it visits in order. Your score will be

$$20 + \lfloor \text{the optimal distance} \rfloor - \lfloor \text{your distance} \rfloor$$

or 0, whichever is greater.

45. On your answer sheet, *clearly* mark at least seven points, as long as

(i) No three are collinear.

(ii) No seven form a convex heptagon.

Please do not cross out any points; erase if you can do so neatly. If the graders deem that your paper is too messy, or if they determine that you violated one of those conditions, your submission for this problem will be disqualified. Otherwise, your score will be the number of points you marked minus 6, even if you actually violated one of the conditions but were able to fool the graders.

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Team Round A

Mobotics [120]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged at the points of a certain grid (to be specified later). Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction. At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

1. **[15]** For this problem, our lawn consists of a row of n clumps of grass. This row runs in an east-west direction. In our formation, each mobot may be oriented toward the north, south, east, or west. One example of an allowable formation if $n = 6$ is symbolized below:

   $$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \downarrow$$

   (The mobot on the third clump will move westward, mowing the first three clumps. Each of the last three clumps is mowed by a different mobot.) Here's another allowable formation for $n = 6$, considered different from the first:

   $$\cdot \quad \cdot \leftarrow \quad \uparrow \quad \uparrow \quad \rightarrow$$

   Compute the number of different allowable formations for any given n.

2. **[25]** For this problem, our lawn is an $m \times n$ rectangular grid of clumps, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

   $$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m\}.$$

   However, mobots are now allowed to be oriented to go either north or east only. So one allowable formation for $m = 2$, $n = 3$ might be as follows:

   $$\begin{array}{ccc} \cdot & \rightarrow & \cdot \\ \uparrow & \rightarrow & \cdot \end{array}$$

   Prove that the number of allowable formations for given m and n is $\dfrac{(m+n)!}{m!\,n!}$.

3. **[40]** In this problem, we stipulate that $m \ge n$, and the lawn is shaped differently. The clumps are now at the lattice points in a trapezoid:

   $$\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x < n \text{ and } 0 \le y < m + 1 - n + x\},$$

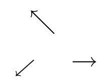
   As in problem 2, mobots can be set to move either north or east. For given m and n, determine with proof the number of allowable formations.

4. **[15]** In this problem and the next, the lawn consists of points in a triangular grid of size n, so that for $n = 3$ the lawn looks like

   $$\begin{array}{ccccc} & & \cdot & & \\ & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot \end{array}$$

Mobots are allowed to be oriented to the east, 30° west of north, or 30° west of south. Under these conditions, for any given n, what is the minimum number of mobots needed to now the lawn?

5. **[25]** With the same lawn and the same allowable mobot orientations as in the previous problem, let us call a formation "happy" if it is invariant under 120° rotations. (A rotation applies both to the positions of the mobots and to their orientations.) An example of a happy formation for $n = 2$ might be



Find the number of happy formations for a given n.

Polygons [110]

6. **[15]** Let n be an integer at least 5. At most how many diagonals of a regular n-gon can be simultaneously drawn so that no two are parallel? Prove your answer.

7. **[25]** Given a convex n-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal either in the interior of the n-gon or at a vertex? Prove your answer.

8. **[15]** Given a regular n-gon with sides of length 1, what is the smallest radius r such that there is a non-empty intersection of n circles of radius r centered at the vertices of the n-gon? Give r as a formula in terms of n. Be sure to prove your answer.

9. **[40]** Let $n \geq 3$ be a positive integer. Prove that given any n angles $0 < \theta_1, \theta_2, \ldots, \theta_n < 180°$, such that their sum is $180(n-2)$ degrees, there exists a convex n-gon having exactly those angles, in that order.

10. **[15]** Suppose we have an n-gon such that each interior angle, measured in degrees, is a positive integer. Suppose further that all angles are less than 180°, and that all angles are different sizes. What is the maximum possible value of n? Prove your answer.

What do the following problems have in common? [170]

11. **[15]** The lottery cards of a certain lottery contain all nine-digit numbers that can be formed with the digits 1, 2 and 3. There is exactly one number on each lottery card. There are only red, yellow and blue lottery cards. Two lottery numbers that differ from each other in all nine digits always appear on cards of different color. Someone draws a red card and a yellow card. The red card has the number 122 222 222 and the yellow card has the number 222 222 222. The first prize goes to the lottery card with the number 123 123 123. What color(s) can it possibly have? Prove your answer.

12. **[25]** A $3 \times 3 \times 3$ cube is built from 27 unit cubes. Suddenly five of those cubes mysteriously teleport away. What is the minimum possible surface area of the remaining solid? Prove your answer.

13. **[40]** Having lost a game of checkers and my temper, I dash all the pieces to the ground but one. This last checker, which is perfectly circular in shape, remains completely on the board, and happens to cover equal areas of red and black squares. Prove that the center of this piece must lie on a boundary between two squares (or at a junction of four).

14. **[40]** A number n is called *bummed out* if there is exactly one ordered pair of positive integers (x, y) such that

$$\lfloor x^2/y \rfloor + \lfloor y^2/x \rfloor = n.$$

Find all bummed out numbers.

15. **[50]** Find, with proof, all positive integer palindromes whose square is also a palindrome.

IX$^{\text{th}}$ Annual Harvard-MIT Mathematics Tournament

Saturday 25 February 2006

Team Round B

Mobotics [135]

Spring is finally here in Cambridge, and it's time to mow our lawn. For the purpose of these problems, our lawn consists of little *clumps* of grass arranged in an $m \times n$ rectangular grid, that is, with m rows running east-west and n columns running north-south. To be even more explicit, we might say our clumps are at the lattice points

$$\{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x < n \text{ and } 0 \leq y < m\}.$$

Our machinery consists of a fleet of identical *mowbots* (or "mobots" for short). A mobot is a lawn-mowing machine. To mow our lawn, we begin by choosing a *formation*: we place as many mobots as we want at various clumps of grass and orient each mobot's head in a certain direction, either north or east (*not* south or west). At the blow of a whistle, each mobot starts moving in the direction we've chosen, mowing every clump of grass in its path (including the clump it starts on) until it goes off the lawn.

Because the spring is so young, our lawn is rather delicate. Consequently, we want to make sure that every clump of grass is mowed once and only once. We will not consider formations that do not meet this criterion.

One more thing: two formations are considered "different" if there exists a clump of grass for which either (1) for exactly one of the formations does a mobot starts on that clump, or (2) there are mobots starting on this clump for both the formations, but they're oriented in different directions.

As an example, one allowable formation for $m = 2$, $n = 3$ might be as follows:

$\cdot$	$\rightarrow$	$\cdot$
$\uparrow$	$\rightarrow$	$\cdot$

1. **[25]** Prove that the maximum number of mobots you need to mow your lawn is $m + n - 1$.

2. **[40]** Prove that the minimum number of mobots you need to mow your lawn is $\min\{m, n\}$.

3. **[15]** Prove that, given any formation, each mobot may be colored in one of three colors — say, white, black, and blue — such that no two adjacent clumps of grass are mowed by different mobots of the same color. Two clumps of grass are adjacent if the distance between them is 1. In your proof, you may use the Four-Color Theorem if you're familiar with it.

4. **[15]** For $n = m = 4$, find a formation with 6 mobots for which there are exactly 12 ways to color the mobots in three colors as in problem 3. (No proof is necessary.)

5. **[40]** For $n, m \geq 3$, prove that a formation has exactly six possible colorings satisfying the conditions in problem 3 if and only if there is a mobot that starts at $(1, 1)$.

Polygons [130]

6. **[15]** Suppose we have a regular hexagon and draw all its sides and diagonals. Into how many regions do the segments divide the hexagon? (No proof is necessary.)

7. **[25]** Suppose we have an octagon with all angles of $135°$, and consecutive sides of alternating length 1 and $\sqrt{2}$. We draw all its sides and diagonals. Into how many regions do the segments divide the octagon? (No proof is necessary.)

8. **[25]** A regular 12-sided polygon is inscribed in a circle of radius 1. How many chords of the circle that join two of the vertices of the 12-gon have lengths whose squares are rational? (No proof is necessary.)

9. **[25]** Show a way to construct an equiangular hexagon with side lengths 1, 2, 3, 4, 5, and 6 (not necessarily in that order).

10. **[40]** Given a convex n-gon, $n \geq 4$, at most how many diagonals can be drawn such that each drawn diagonal intersects every other drawn diagonal strictly in the interior of the n-gon? Prove that your answer is correct.

What do the following problems have in common? [135]

11. **[15]** Find the largest positive integer n such that $1! + 2! + 3! + \cdots + n!$ is a perfect square. Prove that your answer is correct.

12. **[15]** Find all ordered triples (x, y, z) of positive reals such that $x + y + z = 27$ and $x^2 + y^2 + z^2 - xy - yz - zx = 0$. Prove that your answer is correct.

13. **[25]** Four circles with radii $1, 2, 3$, and r are externally tangent to one another. Compute r. (No proof is necessary.)

14. **[40]** Find the prime factorization of

$$2006^2 \cdot 2262 - 669^2 \cdot 3599 + 1593^2 \cdot 1337.$$

(No proof is necessary.)

15. **[40]** Let a, b, c, d be real numbers so that c, d are not both 0. Define the function

$$m(x) = \frac{ax + b}{cx + d}$$

on all real numbers x except possibly $-d/c$, in the event that $c \neq 0$. Suppose that the equation $x = m(m(x))$ has at least one solution that is not a solution of $x = m(x)$. Find all possible values of $a + d$. Prove that your answer is correct.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Algebra Subject Test

1. How many real numbers x are solutions to the following equation?
$$|x-1| = |x-2| + |x-3|$$

2. How many real numbers x are solutions to the following equation?
$$2003^x + 2004^x = 2005^x$$

3. Let x, y, and z be distinct real numbers that sum to 0. Find the maximum possible value of
$$\frac{xy+yz+zx}{x^2+y^2+z^2}.$$

4. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

5. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

6. Find the sum of the x-coordinates of the distinct points of intersection of the plane curves given by $x^2 = x + y + 4$ and $y^2 = y - 15x + 36$.

7. Let x be a positive real number. Find the maximum possible value of
$$\frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}.$$

8. Compute
$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}.$$

9. The number 27,000,001 has exactly four prime factors. Find their sum.

10. Find the sum of the absolute values of the roots of $x^4 - 4x^3 - 4x^2 + 16x - 8 = 0$.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Calculus Subject Test

1. Let $f(x) = x^3 + ax + b$, with $a \neq b$, and suppose the tangent lines to the graph of f at $x = a$ and $x = b$ are parallel. Find $f(1)$.

2. A plane curve is parameterized by $x(t) = \int_t^\infty \frac{\cos u}{u}\, du$ and $y(t) = \int_t^\infty \frac{\sin u}{u}\, du$ for $1 \leq t \leq 2$. What is the length of the curve?

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\int_0^1 f(x)f'(x)dx = 0$ and $\int_0^1 f(x)^2 f'(x)dx = 18$. What is $\int_0^1 f(x)^4 f'(x)dx$?

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x)^2 = f(x)f''(x)$ for all x. Suppose $f(0) = 1$ and $f^{(4)}(0) = 9$. Find all possible values of $f'(0)$.

5. Calculate
$$\lim_{x \to 0^+} \left(x^{x^x} - x^x \right).$$

6. The graph of $r = 2 + \cos 2\theta$ and its reflection over the line $y = x$ bound five regions in the plane. Find the area of the region containing the origin.

7. Two ants, one starting at $(-1, 1)$, the other at $(1, 1)$, walk to the right along the parabola $y = x^2$ such that their midpoint moves along the line $y = 1$ with constant speed 1. When the left ant first hits the line $y = \frac{1}{2}$, what is its speed?

8. If f is a continuous real function such that $f(x - 1) + f(x + 1) \geq x + f(x)$ for all x, what is the minimum possible value of $\int_1^{2005} f(x)dx$?

9. Compute
$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}.$$

10. Let $f : \mathbf{R} \to \mathbf{R}$ be a smooth function such that $f'(x) = f(1 - x)$ for all x and $f(0) = 1$. Find $f(1)$.

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: Combinatorics Subject Test

1. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

2. How many nonempty subsets of $\{1, 2, 3, \ldots, 12\}$ have the property that the sum of the largest element and the smallest element is 13?

3. The Red Sox play the Yankees in a best-of-seven series that ends as soon as one team wins four games. Suppose that the probability that the Red Sox win Game n is $\frac{n-1}{6}$. What is the probability that the Red Sox will win the series?

4. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

5. Doug and Ryan are competing in the 2005 Wiffle Ball Home Run Derby. In each round, each player takes a series of swings. Each swing results in either a home run or an out, and an out ends the series. When Doug swings, the probability that he will hit a home run is $1/3$. When Ryan swings, the probability that he will hit a home run is $1/2$. In one round, what is the probability that Doug will hit more home runs than Ryan hits?

6. Three fair six-sided dice, each numbered 1 through 6, are rolled. What is the probability that the three numbers that come up can form the sides of a triangle?

7. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

8. Every second, Andrea writes down a random digit uniformly chosen from the set $\{1, 2, 3, 4\}$. She stops when the last two numbers she has written sum to a prime number. What is the probability that the last number she writes down is 1?

9. Eight coins are arranged in a circle heads up. A move consists of flipping over two adjacent coins. How many different sequences of six moves leave the coins alternating heads up and tails up?

10. You start out with a big pile of 3^{2004} cards, with the numbers $1, 2, 3, \ldots, 3^{2004}$ written on them. You arrange the cards into groups of three any way you like; from each group, you keep the card with the largest number and discard the other two. You now again arrange these 3^{2003} remaining cards into groups of three any way you like, and in each group, keep the card with the smallest number and discard the other two. You now have 3^{2002} cards, and you again arrange these into groups of three and keep the largest number in each group. You proceed in this manner, alternating between keeping the largest number and keeping the smallest number in each group, until you have just one card left.

    How many different values are possible for the number on this final card?

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 1

1. How many real numbers x are solutions to the following equation?

$$|x-1| = |x-2| + |x-3|$$

2. A true-false test has ten questions. If you answer five questions "true" and five "false," your score is guaranteed to be at least four. How many answer keys are there for which this is true?

3. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

4. Find all real solutions to $x^3 + (x+1)^3 + (x+2)^3 = (x+3)^3$.

5. In how many ways can 4 purple balls and 4 green balls be placed into a 4×4 grid such that every row and column contains one purple ball and one green ball? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

6. In an election, there are two candidates, A and B, who each have 5 supporters. Each supporter, independent of other supporters, has a $\frac{1}{2}$ probability of voting for his or her candidate and a $\frac{1}{2}$ probability of being lazy and not voting. What is the probability of a tie (which includes the case in which no one votes)?

7. If $a, b, c > 0$, what is the smallest possible value of $\left\lfloor \frac{a+b}{c} \right\rfloor + \left\lfloor \frac{b+c}{a} \right\rfloor + \left\lfloor \frac{c+a}{b} \right\rfloor$? (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

8. Ten positive integers are arranged around a circle. Each number is one more than the greatest common divisor of its two neighbors. What is the sum of the ten numbers?

9. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

10. What is the smallest integer x larger than 1 such that x^2 ends in the same three digits as x does?

Harvard-MIT Mathematics Tournament

February 19, 2005

Individual Round: General Test, Part 2

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Find three real numbers $a < b < c$ satisfying:

$$\begin{aligned} a+b+c &= 21/4 \\ 1/a+1/b+1/c &= 21/4 \\ abc &= 1. \end{aligned}$$

3. Working together, Jack and Jill can paint a house in 3 days; Jill and Joe can paint the same house in 4 days; or Joe and Jack can paint the house in 6 days. If Jill, Joe, and Jack all work together, how many days will it take them?

4. In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?

5. You and I play the following game on an 8×8 square grid of boxes: Initially, every box is empty. On your turn, you choose an empty box and draw an X in it; if any of the four adjacent boxes are empty, you mark them with an X as well. (Two boxes are adjacent if they share an edge.) We alternate turns, with you moving first, and whoever draws the last X wins. How many choices do you have for a first move that will enable you to guarantee a win no matter how I play?

6. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

7. Three distinct lines are drawn in the plane. Suppose there exist exactly n circles in the plane tangent to all three lines. Find all possible values of n.

8. What is the maximum number of bishops that can be placed on an 8×8 chessboard such that at most three bishops lie on any diagonal?

9. In how many ways can the cells of a 4×4 table be filled in with the digits $1, 2, \ldots, 9$ so that each of the 4-digit numbers formed by the columns is divisible by each of the 4-digit numbers formed by the rows?

10. Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many positive integers less than 2005 can be expressed in the form $\lfloor x \lfloor x \rfloor \rfloor$ for some positive real x?

Harvard-MIT Mathematics Tournament
February 19, 2005

Individual Round: Geometry Subject Test

1. The volume of a cube (in cubic inches) plus three times the total length of its edges (in inches) is equal to twice its surface area (in square inches). How many inches long is its long diagonal?

2. Let $ABCD$ be a regular tetrahedron with side length 2. The plane parallel to edges AB and CD and lying halfway between them cuts $ABCD$ into two pieces. Find the surface area of one of these pieces.

3. Let $ABCD$ be a rectangle with area 1, and let E lie on side CD. What is the area of the triangle formed by the centroids of triangles ABE, BCE, and ADE?

4. Let XYZ be a triangle with $\angle X = 60^\circ$ and $\angle Y = 45^\circ$. A circle with center P passes through points A and B on side XY, C and D on side YZ, and E and F on side ZX. Suppose $AB = CD = EF$. Find $\angle XPY$ in degrees.

5. A cube with side length 2 is inscribed in a sphere. A second cube, with faces parallel to the first, is inscribed between the sphere and one face of the first cube. What is the length of a side of the smaller cube?

6. A triangular piece of paper of area 1 is folded along a line parallel to one of the sides and pressed flat. What is the minimum possible area of the resulting figure?

7. Let $ABCD$ be a tetrahedron such that edges AB, AC, and AD are mutually perpendicular. Let the areas of triangles ABC, ACD, and ADB be denoted by x, y, and z, respectively. In terms of x, y, and z, find the area of triangle BCD.

8. Let T be a triangle with side lengths 26, 51, and 73. Let S be the set of points inside T which do not lie within a distance of 5 of any side of T. Find the area of S.

9. Let AC be a diameter of a circle ω of radius 1, and let D be the point on AC such that $CD = 1/5$. Let B be the point on ω such that DB is perpendicular to AC, and let E be the midpoint of DB. The line tangent to ω at B intersects line CE at the point X. Compute AX.

10. Let AB be the diameter of a semicircle Γ. Two circles, ω_1 and ω_2, externally tangent to each other and internally tangent to Γ, are tangent to the line AB at P and Q, respectively, and to semicircular arc AB at C and D, respectively, with $AP < AQ$. Suppose F lies on Γ such that $\angle FQB = \angle CQA$ and that $\angle ABF = 80^\circ$. Find $\angle PDQ$ in degrees.

Harvard-MIT Mathematics Tournament

February 19, 2005

Guts Round

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1. [**5**] Find the largest positive integer n such that $1+2+3+\cdots+n^2$ is divisible by $1+2+3+\cdots+n$.

2. [**5**] Let x, y, and z be positive real numbers such that $(x \cdot y) + z = (x+z) \cdot (y+z)$. What is the maximum possible value of xyz?

3. [**5**] Find the sum

$$\frac{2^1}{4^1-1} + \frac{2^2}{4^2-1} + \frac{2^4}{4^4-1} + \frac{2^8}{4^8-1} + \cdots .$$

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4. [**6**] What is the probability that in a randomly chosen arrangement of the numbers and letters in "HMMT2005," one can read either "HMMT" or "2005" from left to right? (For example, in "5HM0M20T," one can read "HMMT.")

5. [**6**] For how many integers n between 1 and 2005, inclusive, is $2 \cdot 6 \cdot 10 \cdots (4n-2)$ divisible by $n!$?

6. [**6**] Let $m \circ n = (m+n)/(mn+4)$. Compute $((\cdots((2005 \circ 2004) \circ 2003) \circ \cdots \circ 1) \circ 0)$.

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7. [**6**] Five people of different heights are standing in line from shortest to tallest. As it happens, the tops of their heads are all collinear; also, for any two successive people, the horizontal distance between them equals the height of the shorter person. If the shortest person is 3 feet tall and the tallest person is 7 feet tall, how tall is the middle person, in feet?

8. [**6**] Let $ABCD$ be a convex quadrilateral inscribed in a circle with shortest side AB. The ratio $[BCD]/[ABD]$ is an integer (where $[XYZ]$ denotes the area of triangle XYZ.) If the lengths of AB, BC, CD, and DA are distinct integers no greater than 10, find the largest possible value of AB.

9. [**6**] Farmer Bill's 1000 animals — ducks, cows, and rabbits — are standing in a circle. In order to feel safe, every duck must either be standing next to at least one cow or between two rabbits. If there are 600 ducks, what is the least number of cows there can be for this to be possible?

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10. [**7**] You are given a set of cards labeled from 1 to 100. You wish to make piles of three cards such that in any pile, the number on one of the cards is the product of the numbers on the other two cards. However, no card can be in more than one pile. What is the maximum number of piles you can form at once?

11. [**7**] The Dingoberry Farm is a 10 mile by 10 mile square, broken up into 1 mile by 1 mile patches. Each patch is farmed either by Farmer Keith or by Farmer Ann. Whenever Ann farms a patch, she also farms all the patches due west of it and all the patches due south of it. Ann puts up a scarecrow on each of her patches that is adjacent to exactly two of Keith's patches (and nowhere else). If Ann farms a total of 30 patches, what is the largest number of scarecrows she could put up?

12. [**7**] Two vertices of a cube are given in space. The locus of points that could be a third vertex of the cube is the union of n circles. Find n.

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13. [**7**] Triangle ABC has $AB = 1$, $BC = \sqrt{7}$, and $CA = \sqrt{3}$. Let ℓ_1 be the line through A perpendicular to AB, ℓ_2 the line through B perpendicular to AC, and P the point of intersection of ℓ_1 and ℓ_2. Find PC.

14. [**7**] Three noncollinear points and a line ℓ are given in the plane. Suppose no two of the points lie on a line parallel to ℓ (or ℓ itself). There are exactly n lines perpendicular to ℓ with the following property: the three circles with centers at the given points and tangent to the line all concur at some point. Find all possible values of n.

15. [**7**] Let S be the set of lattice points inside the circle $x^2 + y^2 = 11$. Let M be the greatest area of any triangle with vertices in S. How many triangles with vertices in S have area M?

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16. [**8**] A regular octahedron has a side length of 1. What is the distance between two opposite faces?

17. [**8**] Compute

$$2\sqrt{2\sqrt[3]{2\sqrt[4]{2\sqrt[5]{2\cdots}}}}.$$

18. [**8**] If a, b, and c are random real numbers from 0 to 1, independently and uniformly chosen, what is the average (expected) value of the smallest of a, b, and c?

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19. [**8**] Regular tetrahedron $ABCD$ is projected onto a plane sending A, B, C, and D to A', B', C', and D' respectively. Suppose $A'B'C'D'$ is a convex quadrilateral with $A'B' = B'C$' and $C'D' = D'A'$, and suppose that the area of $A'B'C'D' = 4$. Given these conditions, the set of possible lengths of AB consists of all real numbers in the interval $[a, b)$. Compute b.

20. [**8**] If n is a positive integer, let $s(n)$ denote the sum of the digits of n. We say that n is *zesty* if there exist positive integers x and y greater than 1 such that $xy = n$ and $s(x)s(y) = s(n)$. How many zesty two-digit numbers are there?

21. [**8**] In triangle ABC with altitude AD, $\angle BAC = 45°$, $DB = 3$, and $CD = 2$. Find the area of triangle ABC.

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22. [**9**] Find

$$\{\ln(1 + e)\} + \{\ln(1 + e^2)\} + \{\ln(1 + e^4)\} + \{\ln(1 + e^8)\} + \cdots ,$$

where $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.

23. [**9**] The sides of a regular hexagon are trisected, resulting in 18 points, including vertices. These points, starting with a vertex, are numbered clockwise as $A_1, A_2, \ldots, A_{18}$. The line segment $A_k A_{k+4}$ is drawn for $k = 1, 4, 7, 10, 13, 16$, where indices are taken modulo 18. These segments define a region containing the center of the hexagon. Find the ratio of the area of this region to the area of the large hexagon.

24. [**9**] In the base 10 arithmetic problem $HMMT + GUTS = ROUND$, each distinct letter represents a different digit, and leading zeroes are not allowed. What is the maximum possible value of $ROUND$?

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25. **[9]** An ant starts at one vertex of a tetrahedron. Each minute it walks along a random edge to an adjacent vertex. What is the probability that after one hour the ant winds up at the same vertex it started at?

26. **[9]** In triangle ABC, $AC = 3AB$. Let AD bisect angle A with D lying on BC, and let E be the foot of the perpendicular from C to AD. Find $[ABD]/[CDE]$. (Here, $[XYZ]$ denotes the area of triangle XYZ).

27. **[9]** In a chess-playing club, some of the players take lessons from other players. It is possible (but not necessary) for two players both to take lessons from each other. It so happens that for any three distinct members of the club, A, B, and C, exactly one of the following three statements is true: A takes lessons from B; B takes lessons from C; C takes lessons from A. What is the largest number of players there can be?

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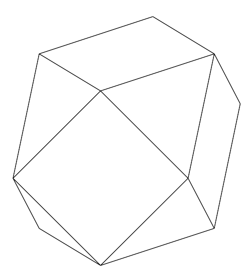
28. **[10]** There are three pairs of real numbers (x_1, y_1), (x_2, y_2), and (x_3, y_3) that satisfy both $x^3 - 3xy^2 = 2005$ and $y^3 - 3x^2y = 2004$. Compute $\left(1 - \frac{x_1}{y_1}\right)\left(1 - \frac{x_2}{y_2}\right)\left(1 - \frac{x_3}{y_3}\right)$.

29. **[10]** Let $n > 0$ be an integer. Each face of a regular tetrahedron is painted in one of n colors (the faces are not necessarily painted different colors.) Suppose there are n^3 possible colorings, where rotations, but not reflections, of the same coloring are considered the same. Find all possible values of n.

30. **[10]** A cuboctahedron is a polyhedron whose faces are squares and equilateral triangles such that two squares and two triangles alternate around each vertex, as shown.



What is the volume of a cuboctahedron of side length 1?

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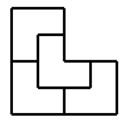
31. **[10]** The L shape made by adjoining three congruent squares can be subdivided into four smaller L shapes.



Each of these can in turn be subdivided, and so forth. If we perform 2005 successive subdivisions, how many of the 4^{2005} L's left at the end will be in the same orientation as the original one?

32. **[10]** Let $a_1 = 3$, and for $n \geq 1$, let $a_{n+1} = (n+1)a_n - n$. Find the smallest $m \geq 2005$ such that $a_{m+1} - 1 \mid a_m^2 - 1$.

33. **[10]** Triangle ABC has incircle ω which touches AB at C_1, BC at A_1, and CA at B_1. Let A_2 be the reflection of A_1 over the midpoint of BC, and define B_2 and C_2 similarly. Let A_3 be the intersection of AA_2 with ω that is closer to A, and define B_3 and C_3 similarly. If $AB = 9$, $BC = 10$, and $CA = 13$, find $[A_3B_3C_3]/[ABC]$. (Here $[XYZ]$ denotes the area of triangle XYZ.)

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 19, 2005 — GUTS ROUND

34. **[12]** A regular octahedron $ABCDEF$ is given such that AD, BE, and CF are perpendicular. Let G, H, and I lie on edges AB, BC, and CA respectively such that $\frac{AG}{GB} = \frac{BH}{HC} = \frac{CI}{IA} = \rho$. For some choice of $\rho > 1$, GH, HI, and IG are three edges of a regular icosahedron, eight of whose faces are inscribed in the faces of $ABCDEF$. Find ρ.

35. **[12]** Let $p = 2^{24036583} - 1$, the largest prime currently known. For how many positive integers c do each of the quadratics $\pm x^2 \pm px \pm c$ have rational roots?

36. **[12]** One hundred people are in line to see a movie. Each person wants to sit in the front row, which contains one hundred seats, and each has a favorite seat, chosen randomly. They enter the row one at a time from the far right. As they walk, if they reach their favorite seat, they sit, but to avoid stepping over people, if they encounter a person already seated, they sit to that person's right. If the seat furthest to the right is already taken, they sit in a different row. What is the most likely number of people that will get to sit in the first row?

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37. **[15]** Let $a_1, a_2, \ldots, a_{2005}$ be real numbers such that

$$\begin{aligned} a_1 \cdot 1 + a_2 \cdot 2 + a_3 \cdot 3 + \cdots + a_{2005} \cdot 2005 &= 0 \\ a_1 \cdot 1^2 + a_2 \cdot 2^2 + a_3 \cdot 3^2 + \cdots + a_{2005} \cdot 2005^2 &= 0 \\ a_1 \cdot 1^3 + a_2 \cdot 2^3 + a_3 \cdot 3^3 + \cdots + a_{2005} \cdot 2005^3 &= 0 \\ &\vdots \\ a_1 \cdot 1^{2004} + a_2 \cdot 2^{2004} + a_3 \cdot 3^{2004} + \cdots + a_{2005} \cdot 2005^{2004} &= 0 \end{aligned}$$

and

$$a_1 \cdot 1^{2005} + a_2 \cdot 2^{2005} + a_3 \cdot 3^{2005} + \cdots + a_{2005} \cdot 2005^{2005} = 1.$$

What is the value of a_1?

38. **[15]** In how many ways can the set of ordered pairs of integers be colored red and blue such that for all a and b, the points (a, b), $(-1-b, a+1)$, and $(1-b, a-1)$ are all the same color?

39. **[15]** How many regions of the plane are bounded by the graph of

$$x^6 - x^5 + 3x^4y^2 + 10x^3y^2 + 3x^2y^4 - 5xy^4 + y^6 = 0?$$

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 19, 2005 — GUTS ROUND

40. **[18]** In a town of n people, a governing council is elected as follows: each person casts one vote for some person in the town, and anyone that receives at least five votes is elected to council. Let $c(n)$ denote the expected number of people elected to council if everyone votes randomly. Find $\lim_{n\to\infty} c(n)/n$.

41. **[18]** There are 42 stepping stones in a pond, arranged along a circle. You are standing on one of the stones. You would like to jump among the stones so that you move counterclockwise by either 1 stone or 7 stones at each jump. Moreover, you would like to do this in such a way that you visit each stone (except for the starting spot) exactly once before returning to your initial stone for the first time. In how many ways can you do this?

42. **[18]** In how many ways can 6 purple balls and 6 green balls be placed into a 4×4 grid such that every row and column contains two balls of one color and one ball of the other color? Only one ball may be placed in each box, and rotations and reflections of a single configuration are considered different.

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43. Write down an integer N between 0 and 20 inclusive. If at least N teams write down N, your score is N; otherwise it is 0.

44. Write down a set S of positive integers, all greater than 1, whose product is P, such that for each $x \in S$, x is a proper divisor of $(P/x)+1$. Your score is $2n$, where $n = |S|$.

45. A *binary word* is a finite sequence of 0's and 1's. A *square subword* is a subsequence consisting of two identical chunks next to each other. For example, the word 100101011 contains the square subwords 00, 0101 (twice), 1010, and 11.

    Find a long binary word containing a small number of square subwords. Specifically, write down a binary word of any length $n \leq 50$. Your score will be $\max\{0, n - s\}$, where s is the number of occurrences of square subwords. (That is, each different square subword will be counted according to the number of times it appears.)

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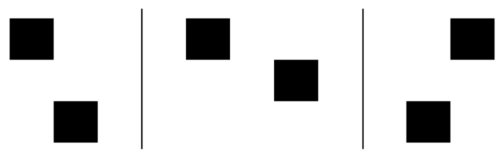
Team Round A

Disconnected Domino Rally [**175**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Prove that for any two types of dominoes, there exists a rectangle that can be tiled by dominoes of either type.

2. [**25**] Suppose $0 < a \leq b$ and $4 \nmid mn$. Prove that the number of ways in which an $m \times n$ rectangle can be partitioned into dominoes of type (a, b) is even.

3. [**10**] Show that no rectangle of the form $1 \times k$ or $2 \times n$, where $4 \nmid n$, is $(1, 2)$-tileable.

4. [**35**] Show that all other rectangles of even area are $(1, 2)$-tileable.

5. [**25**] Show that for b even, there exists some M such that for every $n > M$, a $2b \times n$ rectangle is $(1, b)$-tileable.

6. [**40**] Show that for b even, there exists some M such that for every $m, n > M$ with mn even, an $m \times n$ rectangle is $(1, b)$-tileable.

7. [**25**] Prove that neither of the previous two problems holds if b is odd.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$. You may use the following facts: if $\omega \neq 1$, then

$$1 + \omega + \omega^2 + \cdots + \omega^{k-1} = 0,$$

and if $1, \omega, \ldots, \omega^{k-1}$ are distinct, then

$$(x^k - 1) = (x - 1)(x - \omega)(x - \omega^2) \cdots (x - \omega^{k-1}).$$

8. **[25]** Suppose x is a fifth root of unity. Find, in radical form, all possible values of

$$2x + \frac{1}{1+x} + \frac{x}{1+x^2} + \frac{x^2}{1+x^3} + \frac{x^3}{1+x^4}.$$

9. **[25]** Let $A_1A_2\ldots A_k$ be a regular k-gon inscribed in a circle of radius 1, and let P be a point lying on or inside the circumcircle. Find the maximum possible value of $(PA_1)(PA_2)\cdots(PA_k)$.

10. **[25]** Let P be a regular k-gon inscribed in a circle of radius 1. Find the sum of the squares of the lengths of all the sides and diagonals of P.

11. **[25]** Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial with real coefficients, $a_n \neq 0$. Suppose every root of P is a root of unity, but $P(1) \neq 0$. Show that the coefficients of P are symmetric; that is, show that $a_n = a_0, a_{n-1} = a_1, \ldots$

Early Re-tile-ment **[125]**

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that two sets S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

12. **[20]** Suppose the elements of A are either bounded below or bounded above. Show that if S tiles A, then it does so uniquely, i.e., there is a unique tiling of A by S.

13. **[35]** Let B be a set of integers either bounded below or bounded above. Then show that if S tiles all other integers $\mathbf{Z}\backslash B$, then S tiles all integers $\mathbf{Z}$.

14. **[35]** Suppose S tiles the natural numbers $\mathbf{N}$. Show that S tiles the set $\{1, 2, \ldots, k\}$ for some positive integer k.

15. **[35]** Suppose S tiles $\mathbf{N}$. Show that S is symmetric; that is, if $-S = \{-s_n, \ldots, -s_0\}$, show that $S \sim -S$.

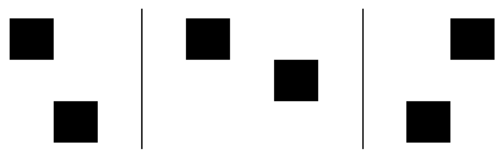
Harvard-MIT Mathematics Tournament

February 19, 2005

Team Round B

Disconnected Domino Rally [**150**]

On an infinite checkerboard, the union of any two distinct unit squares is called a *(disconnected) domino*. A domino is said to be of *type* (a, b), with $a \leq b$ integers not both zero, if the centers of the two squares are separated by a distance of a in one orthogonal direction and b in the other. (For instance, an ordinary connected domino is of type $(0, 1)$, and a domino of type $(1, 2)$ contains two squares separated by a knight's move.)



Each of the three pairs of squares above forms a domino of type $(1, 2)$.

Two dominoes are said to be *congruent* if they are of the same type. A rectangle is said to be (a, b)-*tileable* if it can be partitioned into dominoes of type (a, b).

1. [**15**] Let $0 < m \leq n$ be integers. How many different (i.e., noncongruent) dominoes can be formed by choosing two squares of an $m \times n$ array?

2. [**10**] What are the dimensions of the rectangle of smallest area that is (a, b)-tileable?

3. [**20**] Prove that every (a, b)-tileable rectangle contains a rectangle of these dimensions.

4. [**30**] Prove that an $m \times n$ rectangle is (b, b)-tileable if and only if $2b \mid m$ and $2b \mid n$.

5. [**35**] Prove that an $m \times n$ rectangle is $(0, b)$-tileable if and only if $2b \mid m$ or $2b \mid n$.

6. [**40**] Let k be an integer such that $k \mid a$ and $k \mid b$. Prove that if an $m \times n$ rectangle is (a, b)-tileable, then $2k \mid m$ or $2k \mid n$.

An Interlude — Discovering One's Roots [**100**]

A *kth root of unity* is any complex number ω such that $\omega^k = 1$.

7. [**15**] Find a real, irreducible quartic polynomial with leading coefficient 1 whose roots are all twelfth roots of unity.

8. [**25**] Let x and y be two kth roots of unity. Prove that $(x + y)^k$ is real.

9. [**30**] Let x and y be two distinct roots of unity. Prove that $x + y$ is also a root of unity if and only if $\frac{y}{x}$ is a cube root of unity.

10. [**30**] Let x, y, and z be three roots of unity. Prove that $x + y + z$ is also a root of unity if and only if $x + y = 0$, $y + z = 0$, or $z + x = 0$.

Early Re-tile-ment [**150**]

Let $S = \{s_0, \ldots, s_n\}$ be a finite set of integers, and define $S + k = \{s_0 + k, \ldots, s_n + k\}$. We say that S and T are *equivalent*, written $S \sim T$, if $T = S + k$ for some k. Given a (possibly infinite) set of integers A, we say that S *tiles* A if A can be partitioned into subsets equivalent to S. Such a partition is called a *tiling* of A by S.

11. [**20**] Find all sets S with minimum element 1 that tile $A = \{1, \ldots, 12\}$.

12. [**35**] Let A be a finite set with more than one element. Prove that the number of nonequivalent sets S which tile A is always even.

13. [**25**] Exhibit a set S which tiles the integers $\mathbf{Z}$ but not the natural numbers $\mathbf{N}$.

14. [**30**] Suppose that S tiles the set of all integer cubes. Prove that S has only one element.

15. [**40**] Suppose that S tiles the set of odd prime numbers. Prove that S has only one element.

Harvard-MIT Mathematics Tournament
February 28, 2004

Individual Round: Algebra Subject Test

1. How many ordered pairs of integers (a,b) satisfy all of the following inequalities?

$$\begin{aligned} a^2+b^2 &< 16 \\ a^2+b^2 &< 8a \\ a^2+b^2 &< 8b \end{aligned}$$

2. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

3. Compute:

$$\left\lfloor \frac{2005^3}{2003 \cdot 2004} - \frac{2003^3}{2004 \cdot 2005} \right\rfloor.$$

4. Evaluate the sum

$$\frac{1}{2\lfloor\sqrt{1}\rfloor+1} + \frac{1}{2\lfloor\sqrt{2}\rfloor+1} + \frac{1}{2\lfloor\sqrt{3}\rfloor+1} + \cdots + \frac{1}{2\lfloor\sqrt{100}\rfloor+1}.$$

5. There exists a positive real number x such that $\cos(\tan^{-1}(x)) = x$. Find the value of x^2.

6. Find all real solutions to $x^4 + (2-x)^4 = 34$.

7. If x, y, k are positive reals such that

$$3 = k^2\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) + k\left(\frac{x}{y} + \frac{y}{x}\right),$$

find the maximum possible value of k.

8. Let x be a real number such that $x^3 + 4x = 8$. Determine the value of $x^7 + 64x^2$.

9. A sequence of positive integers is defined by $a_0 = 1$ and $a_{n+1} = a_n^2 + 1$ for each $n \geq 0$. Find $\gcd(a_{999}, a_{2004})$.

10. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Calculate the quotient $P(1)/P(-1)$.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Calculus Subject Test

1. Let $f(x) = \sin(\sin x)$. Evaluate $\lim_{h\to 0} \frac{f(x+h)-f(h)}{x}$ at $x = \pi$.

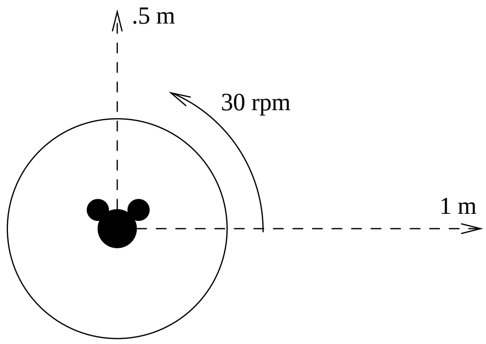
2. Suppose the function $f(x) - f(2x)$ has derivative 5 at $x = 1$ and derivative 7 at $x = 2$. Find the derivative of $f(x) - f(4x)$ at $x = 1$.

3. Find $\lim_{x\to\infty}(\sqrt[3]{x^3+x^2} - \sqrt[3]{x^3-x^2})$.

4. Let $f(x) = \cos(\cos(\cos(\cos(\cos(\cos(\cos(\cos x)))))))$, and suppose that the number a satisfies the equation $a = \cos a$. Express $f'(a)$ as a polynomial in a.

5. A mouse is sitting in a toy car on a negligibly small turntable. The car cannot turn on its own, but the mouse can control when the car is launched and when the car stops (the car has brakes). When the mouse chooses to launch, the car will immediately leave the turntable on a straight trajectory at 1 meter per second.

   Suddenly someone turns on the turntable; it spins at 30 rpm. Consider the set S of points the mouse can reach in his car within 1 second after the turntable is set in motion. (For example, the arrows in the figure below represent two possible paths the mouse can take.) What is the area of S, in square meters?



6. For $x > 0$, let $f(x) = x^x$. Find all values of x for which $f(x) = f'(x)$.

7. Find the area of the region in the xy-plane satisfying $x^6 - x^2 + y^2 \le 0$.

8. If x and y are real numbers with $(x+y)^4 = x - y$, what is the maximum possible value of y?

9. Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

   converges for $c > c_0$ and diverges for $0 < c < c_0$.

10. Let $P(x) = x^3 - \frac{3}{2}x^2 + x + \frac{1}{4}$. Let $P^{[1]}(x) = P(x)$, and for $n \ge 1$, let $P^{[n+1]}(x) = P^{[n]}(P(x))$. Evaluate $\int_0^1 P^{[2004]}(x)\,dx$.

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Combinatorics Subject Test

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

2. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

3. A class of 10 students took a math test. Each problem was solved by exactly 7 of the students. If the first nine students each solved 4 problems, how many problems did the tenth student solve?

4. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

5. A best-of-9 series is to be played between two teams; that is, the first team to win 5 games is the winner. The Mathletes have a chance of $2/3$ of winning any given game. What is the probability that exactly 7 games will need to be played to determine a winner?

6. A committee of 5 is to be chosen from a group of 9 people. How many ways can it be chosen, if Bill and Karl must serve together or not at all, and Alice and Jane refuse to serve with each other?

7. We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traversing every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?

8. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

9. A classroom consists of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and not which student sits at each desk), how many possible arrangements are there?

10. In a game similar to three card monte, the dealer places three cards on the table: the queen of spades and two red cards. The cards are placed in a row, and the queen starts in the center; the card configuration is thus RQR. The dealer proceeds to move. With each move, the dealer randomly switches the center card with one of the two edge cards (so the configuration after the first move is either RRQ or QRR). What is the probability that, after 2004 moves, the center card is the queen?

Harvard-MIT Mathematics Tournament
February 28, 2004

Individual Round: General Test, Part 1

1. There are 1000 rooms in a row along a long corridor. Initially the first room contains 1000 people and the remaining rooms are empty. Each minute, the following happens: for each room containing more than one person, someone in that room decides it is too crowded and moves to the next room. All these movements are simultaneous (so nobody moves more than once within a minute). After one hour, how many different rooms will have people in them?

2. What is the largest whole number that is equal to the product of its digits?

3. Suppose f is a function that assigns to each real number x a value $f(x)$, and suppose the equation

$$f(x_1 + x_2 + x_3 + x_4 + x_5) = f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5) - 8$$

holds for all real numbers x_1, x_2, x_3, x_4, x_5. What is $f(0)$?

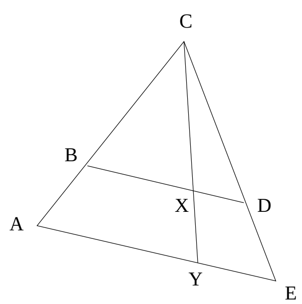
4. How many ways can you mark 8 squares of an 8×8 chessboard so that no two marked squares are in the same row or column, and none of the four corner squares is marked? (Rotations and reflections are considered different.)

5. A rectangle has perimeter 10 and diagonal $\sqrt{15}$. What is its area?

6. Find the ordered quadruple of digits (A, B, C, D), with $A > B > C > D$, such that

$$\begin{array}{rl} & ABCD \\ - & \underline{DCBA} \\ = & BDAC. \end{array}$$

7. Let ACE be a triangle with a point B on segment AC and a point D on segment CE such that BD is parallel to AE. A point Y is chosen on segment AE, and segment CY is drawn. Let X be the intersection of CY and BD. If $CX = 5$, $XY = 3$, what is the ratio of the area of trapezoid $ABDE$ to the area of triangle BCD?

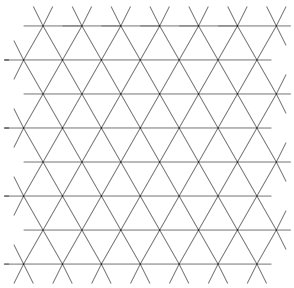


8. You have a 10×10 grid of squares. You write a number in each square as follows: you write $1, 2, 3, \ldots, 10$ from left to right across the top row, then $11, 12, \ldots, 20$ across the second row, and so on, ending with a 100 in the bottom right square. You then write a second number in each square, writing $1, 2, \ldots, 10$ in the first column (from top to bottom), then $11, 12, \ldots, 20$ in the second column, and so forth.

   When this process is finished, how many squares will have the property that their two numbers sum to 101?

9. Urn A contains 4 white balls and 2 red balls. Urn B contains 3 red balls and 3 black balls. An urn is randomly selected, and then a ball inside of that urn is removed. We then repeat the process of selecting an urn and drawing out a ball, without returning the first ball. What is the probability that the first ball drawn was red, given that the second ball drawn was black?

10. A floor is tiled with equilateral triangles of side length 1, as shown. If you drop a needle of length 2 somewhere on the floor, what is the largest number of triangles it could end up intersecting? (Only count the triangles whose interiors are met by the needle — touching along edges or at corners doesn't qualify.)



Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: General Test, Part 2

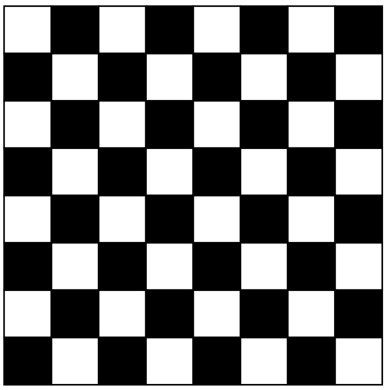
1. Find the largest number n such that $(2004!)!$ is divisible by $((n!)!)!$.

2. Andrea flips a fair coin repeatedly, continuing until she either flips two heads in a row (the sequence HH) or flips tails followed by heads (the sequence TH). What is the probability that she will stop after flipping HH?

3. How many ordered pairs of integers (a, b) satisfy all of the following inequalities?

$$\begin{aligned} a^2 + b^2 &< 16 \\ a^2 + b^2 &< 8a \\ a^2 + b^2 &< 8b \end{aligned}$$

4. A horse stands at the corner of a chessboard, a white square. With each jump, the horse can move either two squares horizontally and one vertically or two vertically and one horizontally (like a knight moves). The horse earns two carrots every time it lands on a black square, but it must pay a carrot in rent to rabbit who owns the chessboard for every move it makes. When the horse reaches the square on which it began, it can leave. What is the maximum number of carrots the horse can earn without touching any square more than twice?



5. Eight strangers are preparing to play bridge. How many ways can they be grouped into two bridge games — that is, into unordered pairs of unordered pairs of people?

6. a and b are positive integers. When written in binary, a has 2004 1's, and b has 2005 1's (not necessarily consecutive). What is the smallest number of 1's $a + b$ could possibly have?

7. Farmer John is grazing his cows at the origin. There is a river that runs east to west 50 feet north of the origin. The barn is 100 feet to the south and 80 feet to the east of the origin. Farmer John leads his cows to the river to take a swim, then the cows leave the river from the same place they entered and Farmer John leads them to the barn. He does this using the shortest path possible, and the total distance he travels is d feet. Find the value of d.

8. A freight train leaves the town of Jenkinsville at 1:00 PM traveling due east at constant speed. Jim, a hobo, sneaks onto the train and falls asleep. At the same time, Julie leaves Jenkinsville on her bicycle, traveling along a straight road in a northeasterly direction (but not due northeast) at 10 miles per hour. At 1:12 PM, Jim rolls over in his sleep and falls from the train onto the side of the tracks. He wakes up and immediately begins walking at 3.5 miles per hour directly towards the road on which Julie is riding. Jim reaches the road at 2:12 PM, just as Julie is riding by. What is the speed of the train in miles per hour?

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

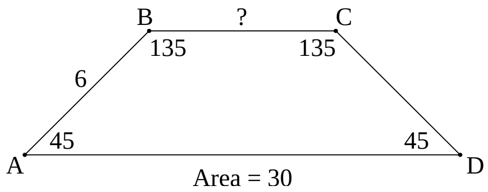
10. A *lattice point* is a point whose coordinates are both integers. Suppose Johann walks in a line from the point $(0, 2004)$ to a random lattice point in the interior (not on the boundary) of the square with vertices $(0, 0), (0, 99), (99, 99), (99, 0)$. What is the probability that his path, including the endpoints, contains an even number of lattice points?

Harvard-MIT Mathematics Tournament

February 28, 2004

Individual Round: Geometry Subject Test

1. In trapezoid $ABCD$, AD is parallel to BC. $\angle A = \angle D = 45°$, while $\angle B = \angle C = 135°$. If $AB = 6$ and the area of $ABCD$ is 30, find BC.



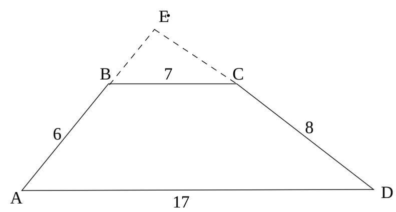
2. A parallelogram has 3 of its vertices at (1, 2), (3,8), and (4, 1). Compute the sum of the possible x-coordinates for the 4th vertex.

3. A swimming pool is in the shape of a circle with diameter 60 ft. The depth varies linearly along the east-west direction from 3 ft at the shallow end in the east to 15 ft at the diving end in the west (this is so that divers look impressive against the sunset) but does not vary at all along the north-south direction. What is the volume of the pool, in ft^3?

4. P is inside rectangle $ABCD$. $PA = 2, PB = 3$, and $PC = 10$. Find PD.

5. Find the area of the region of the xy-plane defined by the inequality $|x|+|y|+|x+y| \leq 1$.

6. In trapezoid $ABCD$ shown, AD is parallel to BC, and $AB = 6, BC = 7, CD = 8, AD = 17$. If sides AB and CD are extended to meet at E, find the resulting angle at E (in degrees).



7. Yet another trapezoid $ABCD$ has AD parallel to BC. AC and BD intersect at P. If $[ADP]/[BCP] = 1/2$, find $[ADP]/[ABCD]$. (Here the notation $[P_1 \cdots P_n]$ denotes the area of the polygon $P_1 \cdots P_n$.)

8. A triangle has side lengths 18, 24, and 30. Find the area of the triangle whose vertices are the incenter, circumcenter, and centroid of the original triangle.

9. Given is a regular tetrahedron of volume 1. We obtain a second regular tetrahedron by reflecting the given one through its center. What is the volume of their intersection?

10. Right triangle XYZ has right angle at Y and $XY = 228$, $YZ = 2004$. Angle Y is trisected, and the angle trisectors intersect XZ at P and Q so that X, P, Q, Z lie on XZ in that order. Find the value of $(PY + YZ)(QY + XY)$.

Harvard-MIT Mathematics Tournament

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Guts Round

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1. **[5]** Find the value of

$$\binom{6}{1}2^1+\binom{6}{2}2^2+\binom{6}{3}2^3+\binom{6}{4}2^4+\binom{6}{5}2^5+\binom{6}{6}2^6.$$

2. **[5]** If the three points

$$(1,a,b)$$
$$(a,2,b)$$
$$(a,b,3)$$

are collinear (in 3-space), what is the value of $a+b$?

3. **[5]** If the system of equations

$$\begin{aligned}|x+y| &= 99\\ |x-y| &= c\end{aligned}$$

has exactly two real solutions (x,y), find the value of c.

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4. **[6]** A tree grows in a rather peculiar manner. Lateral cross-sections of the trunk, leaves, branches, twigs, and so forth are circles. The trunk is 1 meter in diameter to a height of 1 meter, at which point it splits into two sections, each with diameter .5 meter. These sections are each one meter long, at which point they each split into two sections, each with diameter .25 meter. This continues indefinitely: every section of tree is 1 meter long and splits into two smaller sections, each with half the diameter of the previous.

   What is the total volume of the tree?

5. **[6]** Augustin has six $1\times 2\times \pi$ bricks. He stacks them, one on top of another, to form a tower six bricks high. Each brick can be in any orientation so long as it rests flat on top of the next brick below it (or on the floor). How many distinct heights of towers can he make?

6. **[6]** Find the smallest integer n such that $\sqrt{n+99}-\sqrt{n}<1$.

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7. **[6]** Find the shortest distance from the line $3x+4y=25$ to the circle $x^2+y^2=6x-8y$.

8. **[6]** I have chosen five of the numbers $\{1,2,3,4,5,6,7\}$. If I told you what their product was, that would not be enough information for you to figure out whether their sum was even or odd. What is their product?

9. **[6]** A positive integer n is *picante* if $n!$ ends in the same number of zeroes whether written in base 7 or in base 8. How many of the numbers $1,2,\ldots,2004$ are picante?

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10. **[7]** Let $f(x)=x^2+x^4+x^6+x^8+\cdots$, for all real x such that the sum converges. For how many real numbers x does $f(x)=x$?

11. **[7]** Find all numbers n with the following property: there is exactly one set of 8 different positive integers whose sum is n.

12. **[7]** A convex quadrilateral is drawn in the coordinate plane such that each of its vertices (x,y) satisfies the equations $x^2+y^2=73$ and $xy=24$. What is the area of this quadrilateral?

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13. **[7]** Find all positive integer solutions (m,n) to the following equation:

$$m^2=1!+2!+\cdots+n!.$$

14. **[7]** If $a_1=1, a_2=0$, and $a_{n+1}=a_n+\frac{a_{n+2}}{2}$ for all $n\geq 1$, compute a_{2004}.

15. **[7]** A regular decagon $A_0A_1A_2\cdots A_9$ is given in the plane. Compute $\angle A_0A_3A_7$ in degrees.

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16. [**8**] An n-string is a string of digits formed by writing the numbers $1, 2, \ldots, n$ in some order (in base ten). For example, one possible 10-string is

$$35728910461$$

What is the smallest $n > 1$ such that there exists a palindromic n-string?

17. [**8**] Kate has four red socks and four blue socks. If she randomly divides these eight socks into four pairs, what is the probability that none of the pairs will be mismatched? That is, what is the probability that each pair will consist either of two red socks or of two blue socks?

18. [**8**] On a spherical planet with diameter $10,000$ km, powerful explosives are placed at the north and south poles. The explosives are designed to vaporize all matter within $5,000$ km of ground zero and leave anything beyond $5,000$ km untouched. After the explosives are set off, what is the new surface area of the planet, in square kilometers?

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 28, 2004 — GUTS ROUND

19. [**8**] The Fibonacci numbers are defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If the number

$$\frac{F_{2003}}{F_{2002}} - \frac{F_{2004}}{F_{2003}}$$

is written as a fraction in lowest terms, what is the numerator?

20. [**8**] Two positive rational numbers x and y, when written in lowest terms, have the property that the sum of their numerators is 9 and the sum of their denominators is 10. What is the largest possible value of $x + y$?

21. [**8**] Find all ordered pairs of integers (x, y) such that $3^x 4^y = 2^{x+y} + 2^{2(x+y)-1}$.

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 28, 2004 — GUTS ROUND

22. [**9**] I have written a strictly increasing sequence of six positive integers, such that each number (besides the first) is a multiple of the one before it, and the sum of all six numbers is 79. What is the largest number in my sequence?

23. [**9**] Find the largest integer n such that $3^{512} - 1$ is divisible by 2^n.

24. [**9**] We say a point is *contained* in a square if it is in its interior or on its boundary. Three unit squares are given in the plane such that there is a point contained in all three. Furthermore, three points A, B, C, are given, each contained in at least one of the squares. Find the maximum area of triangle ABC.

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 28, 2004 — GUTS ROUND

25. [**9**] Suppose $x^3 - ax^2 + bx - 48$ is a polynomial with three positive roots p, q, and r such that $p < q < r$. What is the minimum possible value of $1/p + 2/q + 3/r$?

26. [**9**] How many of the integers $1, 2, \ldots, 2004$ can be represented as $(mn+1)/(m+n)$ for positive integers m and n?

27. [**9**] A regular hexagon has one side along the diameter of a semicircle, and the two opposite vertices on the semicircle. Find the area of the hexagon if the diameter of the semicircle is 1.

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 28, 2004 — GUTS ROUND

28. [**10**] Find the value of

$$\binom{2003}{1} + \binom{2003}{4} + \binom{2003}{7} + \cdots + \binom{2003}{2002}.$$

29. [**10**] A regular dodecahedron is projected orthogonally onto a plane, and its image is an n-sided polygon. What is the smallest possible value of n?

30. [**10**] We have an n-gon, and each of its vertices is labeled with a number from the set $\{1, \ldots, 10\}$. We know that for any pair of distinct numbers from this set there is at least one side of the polygon whose endpoints have these two numbers. Find the smallest possible value of n.

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31. **[10]** P is a point inside triangle ABC, and lines AP, BP, CP intersect the opposite sides BC, CA, AB in points D, E, F, respectively. It is given that $\angle APB = 90°$, and that $AC = BC$ and $AB = BD$. We also know that $BF = 1$, and that $BC = 999$. Find AF.

32. **[10]** Define the sequence $b_0, b_1, \ldots, b_{59}$ by

$$b_i = \begin{cases} 1 & \text{if i is a multiple of 3} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\{a_i\}$ be a sequence of elements of $\{0, 1\}$ such that

$$b_n \equiv a_{n-1} + a_n + a_{n+1} \pmod 2$$

for $0 \leq n \leq 59$ ($a_0 = a_{60}$ and $a_{-1} = a_{59}$). Find all possible values of $4a_0 + 2a_1 + a_2$.

33. **[10]** A plane P slices through a cube of volume 1 with a cross-section in the shape of a regular hexagon. This cube also has an inscribed sphere, whose intersection with P is a circle. What is the area of the region inside the regular hexagon but outside the circle?

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34. **[12]** Find the number of 20-tuples of integers $x_1, \ldots, x_{10}, y_1, \ldots, y_{10}$ with the following properties:

- $1 \leq x_i \leq 10$ and $1 \leq y_i \leq 10$ for each i;
- $x_i \leq x_{i+1}$ for $i = 1, \ldots, 9$;
- if $x_i = x_{i+1}$, then $y_i \leq y_{i+1}$.

35. **[12]** There are eleven positive integers n such that there exists a convex polygon with n sides whose angles, in degrees, are unequal integers that are in arithmetic progression. Find the sum of these values of n.

36. **[12]** For a string of P's and Q's, the *value* is defined to be the product of the positions of the P's. For example, the string $PPQPQQ$ has value $1 \cdot 2 \cdot 4 = 8$.

Also, a string is called *antipalindromic* if writing it backwards, then turning all the P's into Q's and vice versa, produces the original string. For example, $PPQPQQ$ is antipalindromic.

There are 2^{1002} antipalindromic strings of length 2004. Find the sum of the reciprocals of their values.

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37. **[15]** Simplify $\prod_{k=1}^{2004} \sin(2\pi k/4009)$.

38. **[15]** Let $S = \{p_1 p_2 \cdots p_n \mid p_1, p_2, \ldots, p_n \text{ are distinct primes and } p_1, \ldots, p_n < 30\}$. Assume 1 is in S. Let a_1 be an element of S. We define, for all positive integers n:

$$a_{n+1} = a_n/(n+1) \quad \text{if } a_n \text{ is divisible by } n+1;$$

$$a_{n+1} = (n+2)a_n \quad \text{if } a_n \text{ is not divisible by } n+1.$$

How many distinct possible values of a_1 are there such that $a_j = a_1$ for infinitely many j's?

39. **[15]** You want to arrange the numbers $1, 2, 3, \ldots, 25$ in a sequence with the following property: if n is divisible by m, then the nth number is divisible by the mth number. How many such sequences are there?

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HARVARD-MIT MATHEMATICS TOURNAMENT, FEBRUARY 28, 2004 — GUTS ROUND

40. **[18]** You would like to provide airline service to the 10 cities in the nation of Schizophrenia, by instituting a certain number of two-way routes between cities. Unfortunately, the government is about to divide Schizophrenia into two warring countries of five cities each, and you don't know which cities will be in each new country. All airplane service between the two new countries will be discontinued. However, you want to make sure that you set up your routes so that, for any two cities in the same new country, it will be possible to get from one city to the other (without leaving the country).

    What is the minimum number of routes you must set up to be assured of doing this, no matter how the government divides up the country?

41. **[18]** A tetrahedron has all its faces triangles with sides $13, 14, 15$. What is its volume?

42. **[18]** S is a set of complex numbers such that if $u, v \in S$, then $uv \in S$ and $u^2 + v^2 \in S$. Suppose that the number N of elements of S with absolute value at most 1 is finite. What is the largest possible value of N?

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43. Write down an integer from 0 to 20 inclusive. This problem will be scored as follows: if N is the second-largest number from among the responses submitted, then each team that submits N gets N points, and everyone else gets zero. (If every team picks the same number then nobody gets any points.)

44. Shown on your answer sheet is a 20×20 grid. Place as many queens as you can so that each of them attacks at most one other queen. (A queen is a chess piece that can move any number of squares horizontally, vertically, or diagonally.) It's not very hard to get 20 queens, so you get no points for that, but you get 5 points for each further queen beyond 20. You can mark the grid by placing a dot in each square that contains a queen.

45. A *binary string of length n* is a sequence of n digits, each of which is 0 or 1. The *distance* between two binary strings of the same length is the number of positions in which they disagree; for example, the distance between the strings 01101011 and 00101110 is 3 since they differ in the second, sixth, and eighth positions.

    Find as many binary strings of length 8 as you can, such that the distance between any two of them is at least 3. You get one point per string.

Harvard-MIT Mathematics Tournament
February 28, 2004

Team Round

A Build-It-Yourself Table [**150 points**]

An infinite table of nonnegative integers is constructed as follows: in the top row, some number is 1 and all other numbers are 0's; in each subsequent row, every number is the sum of some two of the three closest numbers in the preceding row. An example of such a table is shown below.

$$\begin{array}{ccccccccccc} \cdots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 2 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 1 & 3 & 3 & 2 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 2 & 4 & 4 & 6 & 3 & 2 & 0 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

The top row (with the one 1) is called row 0; the next row is row 1; the next row is row 2, and so forth.

Note that the following problems require you to prove the statements for *every* table that can be constructed by the process described above, not just for the example shown.

1. [**10**] Show that any number in row n (for $n > 0$) is at most 2^{n-1}.

2. [**20**] What is the earliest row in which the number 2004 may appear?

3. [**35**] Let
$$S(n,r) = \binom{n-1}{r-1} + \binom{n-1}{r} + \binom{n-1}{r+1} + \cdots + \binom{n-1}{n-1}$$
for all $n, r > 0$, and in particular $S(n,r) = 0$ if $r > n > 0$. Prove that the number in row n of the table, r columns to the left of the 1 in the top row, is at most $S(n,r)$. (**Hint**: First prove that $S(n-1,r-1) + S(n-1,r) = S(n,r)$.)

4. [**25**] Show that the sum of all the numbers in row n is at most $(n+2)2^{n-1}$.

A pair of successive numbers in the same row is called a *switch pair* if one number in the pair is even and the other is odd.

5. [**15**] Prove that the number of switch pairs in row n is at most twice the number of odd numbers in row n.

6. [**20**] Prove that the number of odd numbers in row n is at most twice the number of switch pairs in row $n-1$.

7. [**25**] Prove that the number of switch pairs in row n is at most twice the number of switch pairs in row $n-1$.

Written In The Stars [**125 points**]

Suppose S is a finite set with a binary operation $\star$ — that is, for any elements a, b of S, there is defined an element $a \star b$ of S. It is given that $(a \star b) \star (a \star b) = b \star a$ for all $a, b \in S$.

8. [**20**] Prove that $a \star b = b \star a$ for all $a, b \in S$.

Let T be the set of elements of the form $a \star a$ for $a \in S$.

9. [**15**] If b is any element of T, prove that $b \star b = b$.

Now suppose further that $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in S$. (Thus we can write an expression like $a \star b \star c \star d$ without ambiguity.)

10. [**25**] Let a be an element of T. Let the *image* of a be the set of all elements of T that can be represented as $a \star b$ for some $b \in T$. Prove that if c is in the image of a, then $a \star c = c$.

11. [**40**] Prove that there exists an element $a \in T$ such that the equation $a \star b = a$ holds for all $b \in T$.

12. [**25**] Prove that there exists an element $a \in S$ such that the equation $a \star b = a$ holds for all $b \in S$.

Sigma City [**125 points**]

13. [**25**] Let n be a positive odd integer. Prove that

$$\lfloor \log_2 n \rfloor + \lfloor \log_2 (n/3) \rfloor + \lfloor \log_2 (n/5) \rfloor + \lfloor \log_2 (n/7) \rfloor + \cdots + \lfloor \log_2 (n/n) \rfloor = (n-1)/2.$$

Let $\sigma(n)$ denote the sum of the (positive) divisors of n, including 1 and n itself.

14. [**30**] Prove that
$$\sigma(1) + \sigma(2) + \sigma(3) + \cdots + \sigma(n) \leq n^2$$
for every positive integer n.

15. [**30**] Prove that
$$\frac{\sigma(1)}{1} + \frac{\sigma(2)}{2} + \frac{\sigma(3)}{3} + \cdots + \frac{\sigma(n)}{n} \leq 2n$$
for every positive integer n.

16. [**40**] Now suppose again that n is odd. Prove that

$$\sigma(1)\lfloor \log_2 n \rfloor + \sigma(3)\lfloor \log_2 (n/3) \rfloor + \sigma(5)\lfloor \log_2 (n/5) \rfloor + \cdots + \sigma(n)\lfloor \log_2 (n/n) \rfloor < n^2/8.$$

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Algebra Subject Test

1. Find the smallest value of x such that $a \geq 14\sqrt{a} - x$ for all nonnegative a.

2. Compute $\frac{\tan^2(20°)-\sin^2(20°)}{\tan^2(20°)\sin^2(20°)}$.

3. Find the smallest n such that $n!$ ends in 290 zeroes.

4. Simplify: $2\sqrt{1.5+\sqrt{2}} - (1.5+\sqrt{2})$.

5. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, ..., n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

6. Let $a_1 = 1$, and let $a_n = \lfloor n^3/a_{n-1} \rfloor$ for $n > 1$. Determine the value of a_{999}.

7. Let a, b, c be the three roots of $p(x) = x^3 + x^2 - 333x - 1001$. Find $a^3 + b^3 + c^3$.

8. Find the value of $\frac{1}{3^2+1} + \frac{1}{4^2+2} + \frac{1}{5^2+3} + \cdots$.

9. For how many integers n, for $1 \leq n \leq 1000$, is the number $\frac{1}{2}\binom{2n}{n}$ even?

10. Suppose $P(x)$ is a polynomial such that $P(1) = 1$ and

$$\frac{P(2x)}{P(x+1)} = 8 - \frac{56}{x+7}$$

for all real x for which both sides are defined. Find $P(-1)$.

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Calculus Subject Test

1. A point is chosen randomly with uniform distribution in the interior of a circle of radius 1. What is its expected distance from the center of the circle?

2. A particle moves along the x-axis in such a way that its velocity at position x is given by the formula $v(x) = 2 + \sin x$. What is its acceleration at $x = \frac{\pi}{6}$?

3. What is the area of the region bounded by the curves $y = x^{2003}$ and $y = x^{1/2003}$ and lying above the x-axis?

4. The sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies $\lim_{n\to\infty}(x_{2n} + x_{2n+1}) = 315$ and $\lim_{n\to\infty}(x_{2n} + x_{2n-1}) = 2003$. Evaluate $\lim_{n\to\infty}(x_{2n}/x_{2n+1})$.

5. Find the minimum distance from the point $(0, 5/2)$ to the graph of $y = x^4/8$.

6. For n an integer, evaluate

$$\lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2 - 0^2}} + \frac{1}{\sqrt{n^2 - 1^2}} + \cdots + \frac{1}{\sqrt{n^2 - (n-1)^2}}\right).$$

7. For what value of $a > 1$ is

$$\int_a^{a^2} \frac{1}{x} \log \frac{x-1}{32}\, dx$$

minimum?

8. A right circular cone with a height of 12 inches and a base radius of 3 inches is filled with water and held with its vertex pointing downward. Water flows out through a hole at the vertex at a rate in cubic inches per second numerically equal to the height of the water in the cone. (For example, when the height of the water in the cone is 4 inches, water flows out at a rate of 4 cubic inches per second.) Determine how many seconds it will take for all of the water to flow out of the cone.

9. Two differentiable real functions $f(x)$ and $g(x)$ satisfy

$$\frac{f'(x)}{g'(x)} = e^{f(x)-g(x)}$$

for all x, and $f(0) = g(2003) = 1$. Find the largest constant c such that $f(2003) > c$ for all such functions f, g.

10. Evaluate

$$\int_{-\infty}^{\infty} \frac{1-x^2}{1+x^4}\, dx.$$

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: Combinatorics Subject Test

1. You have 2003 switches, numbered from 1 to 2003, arranged in a circle. Initially, each switch is either ON or OFF, and all configurations of switches are equally likely. You perform the following operation: for each switch S, if the two switches next to S were initially in the same position, then you set S to ON; otherwise, you set S to OFF. What is the probability that all switches will now be ON?

2. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

3. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

4. In a certain country, there are 100 senators, each of whom has 4 aides. These senators and aides serve on various committees. A committee may consist either of 5 senators, of 4 senators and 4 aides, or of 2 senators and 12 aides. Every senator serves on 5 committees, and every aide serves on 3 committees. How many committees are there altogether?

5. We wish to color the integers $1, 2, 3, \ldots, 10$ in red, green, and blue, so that no two numbers a and b, with $a - b$ odd, have the same color. (We do not require that all three colors be used.) In how many ways can this be done?

6. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

7. You have infinitely many boxes, and you randomly put 3 balls into them. The boxes are labeled $1, 2, \ldots$. Each ball has probability $1/2^n$ of being put into box n. The balls are placed independently of each other. What is the probability that some box will contain at least 2 balls?

8. For any subset $S \subseteq \{1, 2, \ldots, 15\}$, a number n is called an "anchor" for S if n and $n + |S|$ are both members of S, where $|S|$ denotes the number of members of S. Find the average number of anchors over all possible subsets $S \subseteq \{1, 2, \ldots, 15\}$.

9. At a certain college, there are 10 clubs and some number of students. For any two different students, there is some club such that exactly one of the two belongs to that club. For any three different students, there is some club such that either exactly one or all three belong to that club. What is the largest possible number of students?

10. A calculator has a display, which shows a nonnegative integer N, and a button, which replaces N by a random integer chosen uniformly from the set $\{0, 1, \ldots, N-1\}$, provided that $N > 0$. Initially, the display holds the number $N = 2003$. If the button is pressed repeatedly until $N = 0$, what is the probability that the numbers $1, 10, 100$, and 1000 will each show up on the display at some point?

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 1

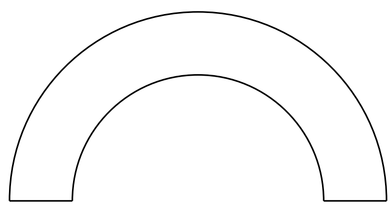
1. 10 people are playing musical chairs with n chairs in a circle. They can be seated in 7! ways (assuming only one person fits on each chair, of course), where different arrangements of the same people on chairs, even rotations, are considered different. Find n.

2. $OPEN$ is a square, and T is a point on side NO, such that triangle TOP has area 62 and triangle TEN has area 10. What is the length of a side of the square?

3. There are 16 members on the Height-Measurement Matching Team. Each member was asked, "How many other people on the team — not counting yourself — are exactly the same height as you?" The answers included six 1's, six 2's, and three 3's. What was the sixteenth answer? (Assume that everyone answered truthfully.)

4. How many 2-digit positive integers have an even number of positive divisors?

5. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?



6. In how many ways can 3 bottles of ketchup and 7 bottles of mustard be arranged in a row so that no bottle of ketchup is immediately between two bottles of mustard? (The bottles of ketchup are mutually indistinguishable, as are the bottles of mustard.)

7. Find the real value of x such that $x^3 + 3x^2 + 3x + 7 = 0$.

8. A broken calculator has the $+$ and $\times$ keys switched. For how many ordered pairs (a, b) of integers will it correctly calculate $a + b$ using the labelled $+$ key?

9. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

10. Bessie the cow is trying to navigate her way through a field. She can travel only from lattice point to adjacent lattice point, can turn only at lattice points, and can travel only to the east or north. (A lattice point is a point whose coordinates are both integers.) $(0, 0)$ is the southwest corner of the field. $(5, 5)$ is the northeast corner of the field. Due to large rocks, Bessie is unable to walk on the points $(1, 1)$, $(2, 3)$, or $(3, 2)$. How many ways are there for Bessie to travel from $(0, 0)$ to $(5, 5)$ under these constraints?

Harvard-MIT Mathematics Tournament

March 15, 2003

Individual Round: General Test, Part 2

1. A compact disc has the shape of a circle of diameter 5 inches with a 1-inch-diameter circular hole in the center. Assuming the capacity of the CD is proportional to its area, how many inches would need to be added to the outer diameter to double the capacity?

2. You have a list of real numbers, whose sum is 40. If you replace every number x on the list by $1-x$, the sum of the new numbers will be 20. If instead you had replaced every number x by $1+x$, what would the sum then be?

3. How many positive rational numbers less than π have denominator at most 7 when written in lowest terms? (Integers have denominator 1.)

4. In triangle ABC with area 51, points D and E trisect AB and points F and G trisect BC. Find the largest possible area of quadrilateral $DEFG$.

5. You are given a 10×2 grid of unit squares. Two different squares are adjacent if they share a side. How many ways can one mark exactly nine of the squares so that no two marked squares are adjacent?

6. The numbers 112, 121, 123, 153, 243, 313, and 322 are among the rows, columns, and diagonals of a 3×3 square grid of digits (rows and diagonals read left-to-right, and columns read top-to-bottom). What 3-digit number completes the list?

7. Daniel and Scott are playing a game where a player wins as soon as he has two points more than his opponent. Both players start at par, and points are earned one at a time. If Daniel has a 60% chance of winning each point, what is the probability that he will win the game?

8. If $x \geq 0$, $y \geq 0$ are integers, randomly chosen with the constraint $x + y \leq 10$, what is the probability that $x + y$ is even?

9. In a classroom, 34 students are seated in 5 rows of 7 chairs. The place at the center of the room is unoccupied. A teacher decides to reassign the seats such that each student will occupy a chair adjacent to his/her present one (i.e. move one desk forward, back, left or right). In how many ways can this reassignment be made?

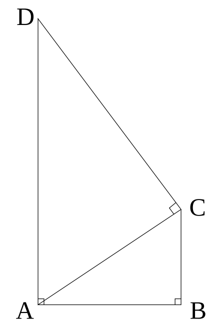
10. Several positive integers are given, not necessarily all different. Their sum is 2003. Suppose that n_1 of the given numbers are equal to 1, n_2 of them are equal to 2, $\ldots$, n_{2003} of them are equal to 2003. Find the largest possible value of

$$n_2 + 2n_3 + 3n_4 + \cdots + 2002n_{2003}.$$

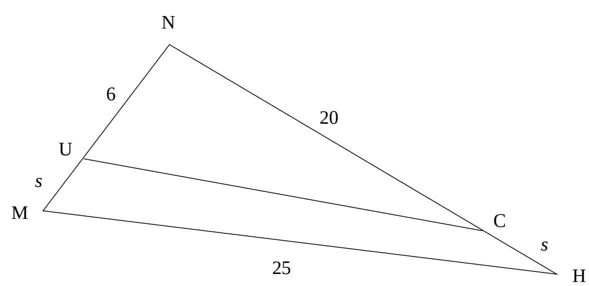
Harvard-MIT Mathematics Tournament
March 15, 2003

Individual Round: Geometry Subject Test

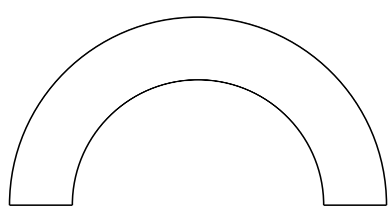
1. AD and BC are both perpendicular to AB, and CD is perpendicular to AC. If $AB = 4$ and $BC = 3$, find CD.



2. As shown, U and C are points on the sides of triangle MNH such that $MU = s$, $UN = 6$, $NC = 20$, $CH = s$, $HM = 25$. If triangle UNC and quadrilateral $MUCH$ have equal areas, what is s?



3. A room is built in the shape of the region between two semicircles with the same center and parallel diameters. The farthest distance between two points with a clear line of sight is 12m. What is the area (in m^2) of the room?



4. Farmer John is inside of an ellipse with reflective sides, given by the equation $x^2/a^2 + y^2/b^2 = 1$, with $a > b > 0$. He is standing at the point $(3, 0)$, and he shines a laser pointer in the y-direciton. The light reflects off the ellipse and proceeds directly toward Farmer Brown, traveling a distance of 10 before reaching him. Farmer John then spins around in a circle; wherever he points the laser, the light reflects off the wall and hits Farmer Brown. What is the ordered pair (a, b)?

5. Consider a 2003-gon inscribed in a circle and a triangulation of it with diagonals intersecting only at vertices. What is the smallest possible number of obtuse triangles in the triangulation?

6. Take a clay sphere of radius 13, and drill a circular hole of radius 5 through its center. Take the remaining "bead" and mold it into a new sphere. What is this sphere's radius?

7. Let $RSTUV$ be a regular pentagon. Construct an equilateral triangle PRS with point P inside the pentagon. Find the measure (in degrees) of angle PTV.

8. Let ABC be an equilateral triangle of side length 2. Let ω be its circumcircle, and let $\omega_A, \omega_B, \omega_C$ be circles congruent to ω centered at each of its vertices. Let R be the set of all points in the plane contained in exactly two of these four circles. What is the area of R?

9. In triangle ABC, $\angle ABC = 50^\circ$ and $\angle ACB = 70^\circ$. Let D be the midpoint of side BC. A circle is tangent to BC at B and is also tangent to segment AD; this circle instersects AB again at P. Another circle is tangent to BC at C and is also tangent to segment AD; this circle intersects AC again at Q. Find $\angle APQ$ (in degrees).

10. Convex quadrilateral $MATH$ is given with $HM/MT = 3/4$, and $\angle ATM = \angle MAT = \angle AHM = 60^\circ$. N is the midpoint of MA, and O is a point on TH such that lines MT, AH, NO are concurrent. Find the ratio HO/OT.

Harvard-MIT Mathematics Tournament

March 15, 2003

Guts Round

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

1. [**5**] Simplify $\sqrt[2003]{2\sqrt{11}-3\sqrt{5}} \cdot \sqrt[4006]{89+12\sqrt{55}}$.

2. [**5**] The graph of $x^4 = x^2y^2$ is a union of n different lines. What is the value of n?

3. [**5**] If a and b are positive integers that can each be written as a sum of two squares, then ab is also a sum of two squares. Find the smallest positive integer c such that $c = ab$, where $a = x^3 + y^3$ and $b = x^3 + y^3$ each have solutions in integers (x, y), but $c = x^3 + y^3$ does not.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

4. [**6**] Let $z = 1 - 2i$. Find $\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \cdots$.

5. [**6**] Compute the surface area of a cube inscribed in a sphere of surface area π.

6. [**6**] Define the Fibonacci numbers by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For how many n, $0 \leq n \leq 100$, is F_n a multiple of 13?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

7. [**6**] a and b are integers such that $a + \sqrt{b} = \sqrt{15 + \sqrt{216}}$. Compute a/b.

8. [**6**] How many solutions in nonnegative integers (a, b, c) are there to the equation

$$2^a + 2^b = c! \quad ?$$

9. [**6**] For x a real number, let $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. How many solutions are there to the equation

$$f(f(f(f(x)))) = x?$$

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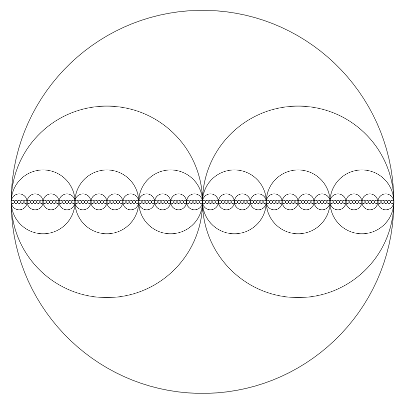
HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

10. **[7]** Suppose that A, B, C, D are four points in the plane, and let Q, R, S, T, U, V be the respective midpoints of AB, AC, AD, BC, BD, CD. If $QR = 2001, SU = 2002, TV = 2003$, find the distance between the midpoints of QU and RV.

11. **[7]** Find the smallest positive integer n such that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$ is divisible by 100.

12. **[7]** As shown in the figure, a circle of radius 1 has two equal circles whose diameters cover a chosen diameter of the larger circle. In each of these smaller circles we similarly draw three equal circles, then four in each of those, and so on. Compute the area of the region enclosed by a positive even number of circles.



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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

13. **[7]** If $xy = 5$ and $x^2 + y^2 = 21$, compute $x^4 + y^4$.

14. **[7]** A positive integer will be called "sparkly" if its smallest (positive) divisor, other than 1, equals the total number of divisors (including 1). How many of the numbers $2, 3, \ldots, 2003$ are sparkly?

15. **[7]** The product of the digits of a 5-digit number is 180. How many such numbers exist?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

16. [**8**] What fraction of the area of a regular hexagon of side length 1 is within distance $\frac{1}{2}$ of at least one of the vertices?

17. [**8**] There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is called a "hub" if it is directly connected to every other city. What is the largest possible number of hubs?

18. [**8**] Find the sum of the reciprocals of all the (positive) divisors of 144.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

19. [**8**] Let r, s, t be the solutions to the equation $x^3 + ax^2 + bx + c = 0$. What is the value of $(rs)^2 + (st)^2 + (rt)^2$ in terms of a, b, and c?

20. [**8**] What is the smallest number of regular hexagons of side length 1 needed to completely cover a disc of radius 1?

21. [**8**] r and s are integers such that

$$3r \geq 2s - 3 \text{ and } 4s \geq r + 12.$$

What is the smallest possible value of r/s?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

22. [**9**] There are 100 houses in a row on a street. A painter comes and paints every house red. Then, another painter comes and paints every third house (starting with house number 3) blue. Another painter comes and paints every fifth house red (even if it is already red), then another painter paints every seventh house blue, and so forth, alternating between red and blue, until 50 painters have been by. After this is finished, how many houses will be red?

23. [**9**] How many lattice points are enclosed by the triangle with vertices $(0, 99)$, $(5, 100)$, and $(2003, 500)$? Don't count boundary points.

24. [**9**] Compute the radius of the inscribed circle of a triangle with sides 15, 16, and 17.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

25. [**9**] Let ABC be an isosceles triangle with apex A. Let I be the incenter. If $AI = 3$ and the distance from I to BC is 2, then what is the length of BC?

26. [**9**] Find all integers x such that $x^2 + 6x + 28$ is a perfect square.

27. [**9**] The rational numbers x and y, when written in lowest terms, have denominators 60 and 70, respectively. What is the smallest possible denominator of $x + y$?

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

28. [**10**] A point in three-space has distances $2, 6, 7, 8, 9$ from five of the vertices of a regular octahedron. What is its distance from the sixth vertex?

29. [**10**] A *palindrome* is a positive integer that reads the same backwards as forwards, such as 82328. What is the smallest 5-digit palindrome that is a multiple of 99?

30. [**10**] The sequence $a_1, a_2, a_3, \ldots$ of real numbers satisfies the recurrence

$$a_{n+1} = \frac{a_n^2 - a_{n-1} + 2a_n}{a_{n-1} + 1}.$$

Given that $a_1 = 1$ and $a_9 = 7$, find a_5.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

31. [**10**] A cylinder of base radius 1 is cut into two equal parts along a plane passing through the center of the cylinder and tangent to the two base circles. Suppose that each piece's surface area is m times its volume. Find the greatest lower bound for all possible values of m as the height of the cylinder varies.

32. [**10**] If x, y, and z are real numbers such that $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, find the maximum possible value of $x - y + z$.

33. [**10**] We are given triangle ABC, with $AB = 9$, $AC = 10$, and $BC = 12$, and a point D on BC. B and C are reflected in AD to B' and C', respectively. Suppose that lines BC' and $B'C$ never meet (i.e., are parallel and distinct). Find BD.

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HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

34. **[12]** $OKRA$ is a trapezoid with OK parallel to RA. If $OK = 12$ and RA is a positive integer, how many integer values can be taken on by the length of the segment in the trapezoid, parallel to OK, through the intersection of the diagonals?

35. **[12]** A certain lottery has tickets labeled with the numbers $1, 2, 3, \ldots, 1000$. The lottery is run as follows: First, a ticket is drawn at random. If the number on the ticket is odd, the drawing ends; if it is even, another ticket is randomly drawn (without replacement). If this new ticket has an odd number, the drawing ends; if it is even, another ticket is randomly drawn (again without replacement), and so forth, until an odd number is drawn. Then, every person whose ticket number was drawn (at any point in the process) wins a prize.

    You have ticket number 1000. What is the probability that you get a prize?

36. **[12]** A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples? (A whole uncut apple counts as one piece.)

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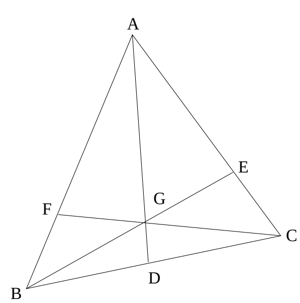
HARVARD-MIT MATHEMATICS TOURNAMENT, MARCH 15, 2003 — GUTS ROUND

37. **[15]** A *quagga* is an extinct chess piece whose move is like a knight's, but much longer: it can move 6 squares in any direction (up, down, left, or right) and then 5 squares in a perpendicular direction. Find the number of ways to place 51 quaggas on an 8×8 chessboard in such a way that no quagga attacks another. (Since quaggas are naturally belligerent creatures, a quagga is considered to attack quaggas on any squares it can move to, as well as any other quaggas on the same square.)

38. **[15]** Given are real numbers x, y. For any pair of real numbers a_0, a_1, define a sequence by $a_{n+2} = xa_{n+1} + ya_n$ for $n \geq 0$. Suppose that there exists a fixed nonnegative integer m such that, for every choice of a_0 and a_1, the numbers a_m, a_{m+1}, a_{m+3}, in this order, form an arithmetic progression. Find all possible values of y.

39. **[15]** In the figure, if $AE = 3$, $CE = 1$, $BD = CD = 2$, and $AB = 5$, find AG.



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40. [**18**] All the sequences consisting of five letters from the set $\{T, U, R, N, I, P\}$ (with repetitions allowed) are arranged in alphabetical order in a dictionary. Two sequences are called "anagrams" of each other if one can be obtained by rearranging the letters of the other. How many pairs of anagrams are there that have exactly 100 other sequences between them in the dictionary?

41. [**18**] A hotel consists of a 2×8 square grid of rooms, each occupied by one guest. All the guests are uncomfortable, so each guest would like to move to one of the adjoining rooms (horizontally or vertically). Of course, they should do this simultaneously, in such a way that each room will again have one guest. In how many different ways can they collectively move?

42. [**18**] A tightrope walker stands in the center of a rope of length 32 meters. Every minute she walks forward one meter with probability $3/4$ and backward one meter with probability $1/4$. What is the probability that she reaches the end in front of her before the end behind her?

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43. Write down an integer N between 0 and 10, inclusive. You will receive N points — unless some other team writes down the same N, in which case you receive nothing.

44. A *partition* of a number n is a sequence of positive integers, arranged in nonincreasing order, whose sum is n. For example, $n = 4$ has 5 partitions: $1 + 1 + 1 + 1 = 2 + 1 + 1 = 2 + 2 = 3 + 1 = 4$. Given two different partitions of the same number, $n = a_1 + a_2 + \cdots + a_k = b_1 + b_2 + \cdots + b_l$, where $k \le l$, the first partition is said to *dominate* the second if all of the following inequalities hold:

$$\begin{aligned} a_1 &\ge b_1; \\ a_1 + a_2 &\ge b_1 + b_2; \\ a_1 + a_2 + a_3 &\ge b_1 + b_2 + b_3; \\ &\vdots \\ a_1 + a_2 + \cdots + a_k &\ge b_1 + b_2 + \cdots + b_k. \end{aligned}$$

Find as many partitions of the number $n = 20$ as possible such that none of the partitions dominates any other. Your score will be the number of partitions you find. If you make a mistake and one of your partitions does dominate another, your score is the largest m such that the first m partitions you list constitute a valid answer.

45. Find a set S of positive integers such that no two distinct subsets of S have the same sum. Your score will be $\lfloor 20(2^n/r - 2) \rfloor$, where n is the number of elements in the set S, and r is the largest element of S (assuming, of course, that this number is nonnegative).

Hej då!

Harvard-MIT Mathematics Tournament
March 15, 2003

Team Round

Completions and Configurations

Given a set A and a nonnegative integer k, the *k-completion* of A is the collection of all k-element subsets of A, and a *k-configuration* of A is any subset of the k-completion of A (including the empty set and the entire k-completion). For instance, the 2-completion of $A = \{1,2,3\}$ is $\{\{1,2\},\{1,3\},\{2,3\}\}$, and the 2-configurations of A are

$\{\}$	$\{\{1,2\}\}$
$\{\{1,3\}\}$	$\{\{2,3\}\}$
$\{\{1,2\},\{1,3\}\}$	$\{\{1,2\},\{2,3\}\}$
$\{\{1,3\},\{2,3\}\}$	$\{\{1,2\},\{1,3\},\{2,3\}\}$

The *order* of an element a of A with respect to a given k-configuration of A is the number of subsets in the k-configuration that contain a. A k-configuration of a set A is *consistent* if the order of every element of A is the same, and the *order* of a consistent k-configuration is this common value.

1. (a) **[10]** How many k-configurations are there of a set that has n elements?

   (b) **[10]** How many k-configurations that have m elements are there of a set that has n elements?

2. **[15]** Suppose A is a set with n elements, and k is a divisor of n. Find the number of consistent k-configurations of A of order 1.

3. (a) **[15]** Let $A_n = \{a_1, a_2, a_3, \ldots, a_n, b\}$, for $n \geq 3$, and let C_n be the 2-configuration consisting of $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq n-1$, $\{a_1, a_n\}$, and $\{a_i, b\}$ for $1 \leq i \leq n$. Let $S_e(n)$ be the number of subsets of C_n that are consistent of order e. Find $S_e(101)$ for $e = 1, 2$, and 3.

   (b) **[20]** Let $A = \{V, W, X, Y, Z, v, w, x, y, z\}$. Find the number of subsets of the 2-configuration

   $$\{\ \{V,W\},\{W,X\},\{X,Y\},\{Y,Z\},\{Z,V\},\ \{v,x\},\{v,y\},\{w,y\},\{w,z\},\{x,z\},$$
   $$\{V,v\},\{W,w\},\{X,x\},\{Y,y\},\{Z,z\}\ \}$$

   that are consistent of order 1.

   (c) **[30]** Let $A = \{a_1, b_1, a_2, b_2, \ldots, a_{10}, b_{10}\}$, and consider the 2-configuration C consisting of $\{a_i, b_i\}$ for all $1 \leq i \leq 10$, $\{a_i, a_{i+1}\}$ for all $1 \leq i \leq 9$, and $\{b_i, b_{i+1}\}$ for all $1 \leq i \leq 9$. Find the number of subsets of C that are consistent of order 1.

Define a k-configuration of A to be *m-separable* if we can label each element of A with an integer from 1 to m (inclusive) so that there is no element E of the k-configuration all of whose elements are assigned the same integer. If C is any subset of A, then C is *m-separable* if we can assign an integer from 1 to m to each element of C so that there is no element E of the k-configuration such that $E \subseteq C$ and all elements of E are assigned the same integer.

4. (a) **[15]** Suppose A has n elements, where $n \geq 2$, and C is a 2-configuration of A that is not m-separable for any $m < n$. What is (in terms of n) the smallest number of elements that C can have?

   (b) **[15]** Show that every 3-configuration of an n-element set A is m-separable for every integer $m \geq n/2$.

   (c) **[25]** Fix $k \geq 2$, and suppose A has k^2 elements. Show that any k-configuration of A with fewer than $\binom{k^2-1}{k-1}$ elements is k-separable.

5. **[30]** Let $B_k(n)$ be the largest number of elements in a 2-separable k-configuration of a set with $2n$ elements ($2 \leq k \leq n$). Find a closed-form expression (i.e. an expression not involving any sums or products with an variable number of terms) for $B_k(n)$.

6. **[40]** Prove that any 2-configuration containing e elements is m-separable for some $m \leq \frac{1}{2} + \sqrt{2e + \frac{1}{4}}$.

A *cell* of a 2-configuration of a set A is a nonempty subset C of A such that

i. for any two distinct elements a, b of C, there exists a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration, and

ii. if a is an element of C and b is an element of A but not of C, there does NOT exist a sequence $c_0, c_1, \ldots, c_n$ of elements of A with $c_0 = a, c_n = b$, and such that $\{c_0, c_1\}, \{c_1, c_2\}, \ldots, \{c_{n-1}, c_n\}$ are all elements of the 2-configuration.

Also, we define a 2-configuration of A to be *barren* if there is no subset $\{a_0, a_1, \ldots, a_n\}$ of A, with $n \geq 2$, such that $\{a_0, a_1\}, \{a_1, a_2\}, \ldots, \{a_{n-1}, a_n\}$ and $\{a_n, a_0\}$ are all elements of the 2-configuration.

7. **[20]** Show that, given any 2-configuration of a set A, every element of A belongs to exactly one cell.

8. (a) **[15]** Given a set A with $n \geq 1$ elements, find the number of consistent 2-configurations of A of order 1 with exactly 1 cell.

   (b) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of A of order 2 with exactly 1 cell.

   (c) **[25]** Given a set A with 10 elements, find the number of consistent 2-configurations of order 2 with exactly 2 cells.

9. (a) **[15]** Show that if every cell of a 2-configuration of a finite set A is m-separable, then the whole 2-configuration is m-separable.

   (b) **[30]** Show that any barren 2-configuration of a finite set A is 2-separable.

10. **[45]** Show that every consistent 2-configuration of order 4 on a finite set A has a subset that is a consistent 2-configuration of order 2.

Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Advanced Topics**

1. Eight knights are randomly placed on a chessboard (not necessarily on distinct squares). A knight on a given square attacks all the squares that can be reached by moving either (1) two squares up or down followed by one squares left or right, or (2) two squares left or right followed by one square up or down. Find the probability that every square, occupied or not, is attacked by some knight.

2. A certain cafeteria serves ham and cheese sandwiches, ham and tomato sandwiches, and tomato and cheese sandwiches. It is common for one meal to include multiple types of sandwiches. On a certain day, it was found that 80 customers had meals which contained both ham and cheese; 90 had meals containing both ham and tomatoes; 100 had meals containing both tomatoes and cheese. 20 customers' meals included all three ingredients. How many customers were there?

3. How many four-digit numbers are there in which at least one digit occurs more than once?

4. Two fair coins are simultaneously flipped. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that the other coin also came up heads on this last flip?

5. Determine the number of subsets S of $\{1, 2, 3, \ldots, 10\}$ with the following property: there exist integers $a < b < c$ with $a \in S, b \notin S, c \in S$.

6. In how many ways can the numbers $1, 2, \ldots, 2002$ be placed at the vertices of a regular 2002-gon so that no two adjacent numbers differ by more than 2? (Rotations and reflections are considered distinct.)

7. A manufacturer of airplane parts makes a certain engine that has a probability p of failing on any given flight. Their are two planes that can be made with this sort of engine, one that has 3 engines and one that has 5. A plane crashes if more than half its engines fail. For what values of p do the two plane models have the same probability of crashing?

8. Given a 9×9 chess board, we consider all the rectangles whose edges lie along grid lines (the board consists of 81 unit squares, and the grid lines lie on the borders of the unit squares). For each such rectangle, we put a mark in every one of the unit squares inside it. When this process is completed, how many unit squares will contain an even number of marks?

9. Given that a, b, c are positive real numbers and $\log_a b + \log_b c + \log_c a = 0$, find the value of $(\log_a b)^3 + (\log_b c)^3 + (\log_c a)^3$.

10. One fair die is rolled; let a denote the number that comes up. We then roll a dice; let the sum of the resulting a numbers be b. Finally, we roll b dice, and let c be the sum of the resulting b numbers. Find the expected (average) value of c.

Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Algebra**

1. Nine nonnegative numbers have average 10. What is the greatest possible value for their median?

2. p and q are primes such that the numbers $p+q$ and $p+7q$ are both squares. Find the value of p.

3. Real numbers a, b, c satisfy the equations $a+b+c=26, 1/a+1/b+1/c=28$. Find the value of
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{a}{c}+\frac{c}{b}+\frac{b}{a}.$$

4. If a positive integer multiple of 864 is picked randomly, with each multiple having the same probability of being picked, what is the probability that it is divisible by 1944?

5. Find the greatest common divisor of the numbers $2002+2, 2002^2+2, 2002^3+2, \ldots$.

6. Find the sum of the even positive divisors of 1000.

7. The real numbers x, y, z, w satisfy
$$\begin{aligned} 2x+y+z+w &= 1 \\ x+3y+z+w &= 2 \\ x+y+4z+w &= 3 \\ x+y+z+5w &= 25. \end{aligned}$$
Find the value of w.

8. Determine the value of the sum
$$\frac{3}{1^2\cdot 2^2}+\frac{5}{2^2\cdot 3^2}+\frac{7}{3^2\cdot 4^2}+\cdots+\frac{29}{14^2\cdot 15^2}.$$

9. For any positive integer n, let $f(n)$ denote the number of 1's in the base-2 representation of n. For how many values of n with $1 \le n \le 2002$ do we have $f(n)=f(n+1)$?

10. Determine the value of
$$2002+\frac{1}{2}(2001+\frac{1}{2}(2000+\cdots+\frac{1}{2}(3+\frac{1}{2}\cdot 2))\cdots).$$

Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Calculus**

1. Two circles have centers that are d units apart, and each has diameter $\sqrt{d}$. For any d, let $A(d)$ be the area of the smallest circle that contains both of these circles. Find $\lim\limits_{d\to\infty} \frac{A(d)}{d^2}$.

2. Find $\lim\limits_{h\to 0} \frac{x^2-(x+h)^2}{h}$.

3. We are given the values of the differentiable real functions f, g, h, as well as the derivatives of their pairwise products, at $x = 0$:

$$f(0) = 1;\ g(0) = 2;\ h(0) = 3;\ (gh)'(0) = 4;\ (hf)'(0) = 5;\ (fg)'(0) = 6.$$

Find the value of $(fgh)'(0)$.

4. Find the area of the region in the first quadrant $x > 0, y > 0$ bounded above the graph of $y = \arcsin(x)$ and below the graph of the $y = \arccos(x)$.

5. What is the minimum vertical distance between the graphs of $2 + \sin(x)$ and $\cos(x)$?

6. Determine the positive value of a such that the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0), (a, 0), (0, a^2 + 1)$, and $(a, a^2 + 1)$.

7. Denote by $\langle x\rangle$ the fractional part of the real number x (for instance, $\langle 3.2\rangle = 0.2$). A positive integer N is selected randomly from the set $\{1, 2, 3, \ldots, M\}$, with each integer having the same probability of being picked, and $\left\langle \frac{87}{303}N\right\rangle$ is calculated. This procedure is repeated M times and the average value $A(M)$ is obtained. What is $\lim\limits_{M\to\infty} A(M)$?

8. Evaluate $\displaystyle\int_0^{(\sqrt{2}-1)/2} \frac{dx}{(2x+1)\sqrt{x^2+x}}$.

9. Suppose f is a differentiable real function such that $f(x) + f'(x) \le 1$ for all x, and $f(0) = 0$. What is the largest possible value of $f(1)$? (Hint: consider the function $e^x f(x)$.)

10. A continuous real function f satisfies the identity $f(2x) = 3f(x)$ for all x. If $\int_0^1 f(x)\,dx = 1$, what is $\int_1^2 f(x)\,dx$?

Harvard-MIT Math Tournament
March 17, 2002
Individual General Test: **Part 1**

1. What is the maximum number of lattice points (i.e. points with integer coordinates) in the plane that can be contained strictly inside a circle of radius 1?

2. Eight knights are randomly placed on a chessboard (not necessarily on distinct squares). A knight on a given square attacks all the squares that can be reached by moving either (1) two squares up or down followed by one squares left or right, or (2) two squares left or right followed by one square up or down. Find the probability that every square, occupied or not, is attacked by some knight.

3. How many triples (A, B, C) of positive integers (positive integers are the numbers $1, 2, 3, 4, \ldots$) are there such that $A + B + C = 10$, where order does not matter (for instance the triples $(2, 3, 5)$ and $(3, 2, 5)$ are considered to be the same triple) and where two of the integers in a triple could be the same (for instance $(3, 3, 4)$ is a valid triple).

4. We call a set of professors and committees on which they serve a *university* if

(1) given two distinct professors there is one and only one committee on which they both serve,

(2) given any committee, C, and any professor, P, not on that committee, there is exactly one committee on which P serves and no professors on committee C serve, and

(3) there are at least two professors on each committee; there are at least two committees.

What is the smallest number of committees a university can have?

5. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

6. A man, standing on a lawn, is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result?

7. A circle is inscribed in a square dartboard. If a dart is thrown at the dartboard and hits the dartboard in a random location, with all locations having the same probability of being hit, what is the probability that it lands within the circle?

8. Count the number of triangles with positive area whose vertices are points whose (x, y)-coordinates lie in the set $\{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)\}$.

9. Real numbers a, b, c satisfy the equations $a + b + c = 26, 1/a + 1/b + 1/c = 28$. Find the value of

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + \frac{a}{c} + \frac{c}{b} + \frac{b}{a}.$$

10. A certain cafeteria serves ham and cheese sandwiches, ham and tomato sandwiches, and tomato and cheese sandwiches. It is common for one meal to include multiple types of sandwiches. On a certain day, it was found that 80 customers had meals which contained both ham and cheese; 90 had meals containing both ham and tomatoes; 100 had meals containing both tomatoes and cheese. 20 customers' meals included all three ingredients. How many customers were there?

Harvard-MIT Math Tournament
March 17, 2002
Individual General Test: **Part 2**

1. The squares of a chessboard are numbered from left to right and top to bottom (so that the first row reads $1, 2, \ldots, 8$, the second reads $9, 10, \ldots, 16$, and so forth). The number 1 is on a black square. How many black squares contain odd numbers?

2. You are in a completely dark room with a drawer containing 10 red, 20 blue, 30 green, and 40 khaki socks. What is the smallest number of socks you must randomly pull out in order to be sure of having at least one of each color?

3. Solve for x in $3 = \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$.

4. Dan is holding one end of a 26 inch long piece of light string that has a heavy bead on it with each hand (so that the string lies along two straight lines). If he starts with his hands together at the start and leaves his hands at the same height, how far does he need to pull his hands apart so that the bead moves upward by 8 inches?

5. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

6. Nine nonnegative numbers have average 10. What is the greatest possible value for their median?

7. p and q are primes such that the numbers $p + q$ and $p + 7q$ are both squares. Find the value of p.

8. Two fair coins are simultaneously flipped. This is done repeatedly until at least one of the coins comes up heads, at which point the process stops. What is the probability that the other coin also came up heads on this last flip?

9. A and B are two points on a circle with center O, and C lies outside the circle, on ray AB. Given that $AB = 24, BC = 28, OA = 15$, find OC.

10. How many four-digit numbers are there in which at least one digit occurs more than once?

Harvard-MIT Math Tournament
March 17, 2002
Individual Subject Test: **Geometry**

1. A man, standing on a lawn, is wearing a circular sombrero of radius 3 feet. Unfortunately, the hat blocks the sunlight so effectively that the grass directly under it dies instantly. If the man walks in a circle of radius 5 feet, what area of dead grass will result?

2. Dan is holding one end of a 26 inch long piece of light string that has a heavy bead on it with each hand (so that the string lies along straight lines). If he starts with his hands together at the start and leaves his hands at the same height, how far does he need to pull his hands apart so that the bead moves upward by 8 inches?

3. A square and a regular hexagon are drawn with the same side length. If the area of the square is $\sqrt{3}$, what is the area of the hexagon?

4. We call a set of professors and committees on which they serve a *university* if

(1) given two distinct professors there is one and only one committee on which they both serve,

(2) given any committee, C, and any professor, P, not on that committee, there is exactly one committee on which P serves and no professors on committee C serve, and

(3) there are at least two professors on each committee; there are at least two committees.

What is the smallest number of committees a university can have?

5. Consider a square of side length 1. Draw four lines that each connect a midpoint of a side with a corner not on that side, such that each midpoint and each corner is touched by only one line. Find the area of the region completely bounded by these lines.

6. If we pick (uniformly) a random square of area 1 with sides parallel to the $x-$ and $y-$axes that lies entirely within the 5-by-5 square bounded by the lines $x = 0, x = 5, y = 0, y = 5$ (the corners of the square need not have integer coordinates), what is the probability that the point $(x, y) = (4.5, 0.5)$ lies within the square of area 1?

7. Equilateral triangle ABC of side length 2 is drawn. Three squares external to the triangle, $ABDE$, $BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that contains these squares?

8. Equilateral triangle ABC of side length 2 is drawn. Three squares containing the triangle, $ABDE$, $BCFG$, and $CAHI$, are drawn. What is the area of the smallest triangle that contains these squares?

9. A and B are two points on a circle with center O, and C lies outside the circle, on ray AB. Given that $AB = 24, BC = 28, OA = 15$, find OC.

10. Let $\triangle ABC$ be equilateral, and let D, E, F be points on sides BC, CA, AB respectively, with $FA = 9, AE = EC = 6, CD = 4$. Determine the measure (in degrees) of $\angle DEF$.

1. [**4**] An (l, a)-*design* of a set is a collection of subsets of that set such that each subset contains exactly l elements and that no two of the subsets share more than a elements. How many (2,1)-designs are there of a set containing 8 elements?

2. [**5**] A *lattice point* in the plane is a point of the form (n, m), where n and m are integers. Consider a set S of lattice points. We construct the *transform* of S, denoted by S', by the following rule: the pair (n, m) is in S' if and only if any of $(n, m-1)$, $(n, m+1)$, $(n-1, m)$, $(n+1, m)$, and (n, m) is in S. How many elements are in the set obtained by successively transforming $\{(0, 0)\}$ 14 times?

3. [**5**] How many elements are in the set obtained by transforming $\{(0, 0), (2, 0)\}$ 14 times?

4. [**4**] How many ways are there of using diagonals to divide a regular 6-sided polygon into triangles such that at least one side of each triangle is a side of the original polygon and that each vertex of each triangle is a vertex of the original polygon?

5. [**6**] Two 4×4 squares are randomly placed on an 8×8 chessboard so that their sides lie along the grid lines of the board. What is the probability that the two squares overlap?

6. [$\pm$6] Find all values of x that satisfy $x = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$ (be careful; this is tricky).

7. [**5**] A rubber band is 4 inches long. An ant begins at the left end. Every minute, the ant walks one inch along rightwards along the rubber band, but then the band is stretched (uniformly) by one inch. For what value of n will the ant reach the right end during the nth minute?

8. [**7**] Draw a square of side length 1. Connect its sides' midpoints to form a second square. Connect the midpoints of the sides of the second square to form a third square. Connect the midpoints of the sides of the third square to form a fourth square. And so forth. What is the sum of the areas of all the squares in this infinite series?

9. [**4**] Find all values of x with $0 \leq x < 2\pi$ that satisfy $\sin x + \cos x = \sqrt{2}$.

10. **[6]** The mathematician John is having trouble remembering his girlfriend Alicia's 7-digit phone number. He remembers that the first four digits consist of one 1, one 2, and two 3s. He also remembers that the fifth digit is either a 4 or 5. While he has no memory of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. If this is all the information he has, how many phone numbers does he have to try if he is to make sure he dials the correct number?

11. **[7]** How many real solutions are there to the equation

$$||||x|-2|-2|-2|=||||x|-3|-3|-3| \ ?$$

12. [± 7] This question forms a three question multiple choice test. After each question, there are 4 choices, each preceded by a letter. Please write down your answer as the ordered triple (letter of the answer of Question #1, letter of the answer of Question #2, letter of the answer of Question #3). If you find that all such ordered triples are logically impossible, then write "no answer" as your answer. If you find more than one possible sets of answers, then provide all ordered triples as your answer.

When we refer to "the correct answer to Question X" it is the actual answer, not the letter, to which we refer. When we refer to "the letter of the correct answer to question X" it is the letter contained in parentheses that precedes the answer to which we refer.

You are given the following condition: No two correct answers to questions on the test may have the same letter.

Question 1. If a fourth question were added to this test, and if the letter of its correct answer were (C), then:

(A) This test would have no logically possible set of answers.
(B) This test would have one logically possible set of answers.
(C) This test would have more than one logically possible set of answers.
(D) This test would have more than one logically possible set of answers.

Question 2. If the answer to Question 2 were "Letter (D)" and if Question 1 were not on this multiple-choice test (still keeping Questions 2 and 3 on the test), then the letter of the answer to Question 3 would be:

(A) Letter (B)
(B) Letter (C)
(C) Letter (D)
(D) Letter (A)

Question 3. Let $P_1 = 1$. Let $P_2 = 3$. For all $i > 2$, define $P_i = P_{i-1}P_{i-2} - P_{i-2}$. Which is a factor of P_{2002}?

(A) 3
(B) 4
(C) 7
(D) 9

13. [**7**] A *domino* is a 1-by-2 or 2-by-1 rectangle. A *domino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping dominoes. For instance, there is one domino tiling of a 2-by-1 rectangle and there are 2 tilings of a 2-by-2 rectangle (one consisting of two horizontal dominoes and one consisting of two vertical dominoes). How many domino tilings are there of a 2-by-10 rectangle?

14. [**7**] An *omino* is a 1-by-1 square or a 1-by-2 horizontal rectangle. An *omino tiling* of a region of the plane is a way of covering it (and only it) by ominoes. How many omino tilings are there of a 2-by-10 horizontal rectangle?

15. [**7**] How many sequences of 0s and 1s are there of length 10 such that there are no three 0s or 1s consecutively anywhere in the sequence?

16. [**5**] Divide an m-by-n rectangle into mn nonoverlapping 1-by-1 squares. A *polyomino* of this rectangle is a subset of these unit squares such that for any two unit squares S, T in the polyomino, either

(1) S and T share an edge or

(2) there exists a positive integer n such that the polyomino contains unit squares $S_1, S_2, S_3, \ldots, S_n$ such that S and S_1 share an edge, S_n and T share an edge, and for all positive integers $k < n$, S_k and S_{k+1} share an edge.

We say a polyomino of a given rectangle *spans* the rectangle if for each of the four edges of the rectangle the polyomino contains a square whose edge lies on it.

What is the minimum number of unit squares a polyomino can have if it spans a 128-by-343 rectangle?

17. [± 8] Find the number of *pentominoes* (5-square polyominoes) that span a 3-by-3 rectangle, where polyominoes that are flips or rotations of each other are considered the same polyomino.

18. [± 5] Call the pentominoes found in the last problem *square pentominoes*. Just like dominos and ominos can be used to tile regions of the plane, so can square pentominoes. In particular, a *square pentomino tiling* of a region of the plane is a way of covering it (and only it) completely by nonoverlapping square pentominoes. How many square pentomino tilings are there of a 12-by-12 rectangle?

19. [**8**] For how many integers a ($1 \le a \le 200$) is the number a^a a square?

20. [**7**] The Antarctican language has an alphabet of just 16 letters. Interestingly, every word in the language has exactly 3 letters, and it is known that no word's first letter equals any word's last letter (for instance, if the alphabet were $\{a, b\}$ then aab and aaa could not both be words in the language because a is the first letter of a word and the last letter of a word; in fact, just aaa alone couldn't be in the language). Given this, determine the maximum possible number of words in the language.

21. [**7**] The Dyslexian alphabet consists of consonants and vowels. It so happens that a finite sequence of letters is a word in Dyslexian precisely if it alternates between consonants and vowels (it may begin with either). There are 4800 five-letter words in Dyslexian. How many letters are in the alphabet?

22. [**5**] A *path* of length n is a sequence of points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with integer coordinates such that for all i between 1 and $n-1$ inclusive, either

(1) $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$ (in which case we say the ith step is *rightward*) or

(2) $x_{i+1} = x_i$ and $y_{i+1} = y_i + 1$ (in which case we say that the ith step is *upward*).

This path is said to *start* at (x_1, y_1) and *end* at (x_n, y_n). Let $P(a, b)$, for a and b nonnegative integers, be the number of paths that start at $(0, 0)$ and end at (a, b).

Find $\sum_{i=0}^{10} P(i, 10 - i)$.

23. [**5**] Find $P(7, 3)$.

24. [**7**] A *restricted path* of length n is a path of length n such that for all i between 1 and $n-2$ inclusive, if the ith step is upward, the $i+1$st step must be rightward.

Find the number of restricted paths that start at $(0, 0)$ and end at $(7, 3)$.

25. [±4] A math professor stands up in front of a room containing 100 very smart math students and says, "Each of you has to write down an integer between 0 and 100, inclusive, to guess 'two-thirds of the average of all the responses.' Each student who guesses the highest integer that is not higher than two-thirds of the average of all responses will receive a prize." If among all the students it is common knowledge that everyone will write down the best response, and there is no communication between students, what single integer should each of the 100 students write down?

26. [±4] Another professor enters the same room and says, "Each of you has to write down an integer between 0 and 200. I will then compute X, the number that is 3 greater than half the average of all the numbers that you will have written down. Each student who writes down the number closest to X (either above or below X) will receive a prize." One student, who misunderstood the question, announces to the class that he will write the number 107. If among the other 99 students it is common knowledge that all 99 of them will write down the best response, and there is no further communication between students, what single integer should each of the 99 students write down?

27. [7] Consider the two hands of an analog clock, each of which moves with constant angular velocity. Certain positions of these hands are possible (e.g. the hour hand halfway between the 5 and 6 and the minute hand exactly at the 6), while others are impossible (e.g. the hour hand exactly at the 5 and the minute hand exactly at the 6). How many different positions are there that would remain possible if the hour and minute hands were switched?

28. [6] Count how many 8-digit numbers there are that contain exactly four nines as digits.

29. [8] A sequence $s_0, s_1, s_2, s_3, \ldots$ is defined by $s_0 = s_1 = 1$ and, for every positive integer n, $s_{2n} = s_n$, $s_{4n+1} = s_{2n+1}$, $s_{4n-1} = s_{2n-1} + s_{2n-1}^2/s_{n-1}$. What is the value of s_{1000}?

30. [9] A conical flask contains some water. When the flask is oriented so that its base is horizontal and lies at the bottom (so that the vertex is at the top), the water is 1 inch deep. When the flask is turned upside-down, so that the vertex is at the bottom, the water is 2 inches deep. What is the height of the cone?

31. [**6**] Express, as concisely as possible, the value of the product

$$(0^3 - 350)(1^3 - 349)(2^3 - 348)(3^3 - 347)\cdots(349^3 - 1)(350^3 - 0).$$

32. [**9**] Two circles have radii 13 and 30, and their centers are 41 units apart. The line through the centers of the two circles intersects the smaller circle at two points; let A be the one outside the larger circle. Suppose B is a point on the smaller circle and C a point on the larger circle such that B is the midpoint of AC. Compute the distance AC.

33. [**8**] The expression $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Find the value of

$$\left\lfloor \frac{2002!}{2001! + 2000! + 1999! + \cdots + 1!} \right\rfloor.$$

34. [**7**] Points P and Q are 3 units apart. A circle centered at P with a radius of $\sqrt{3}$ units intersects a circle centered at Q with a radius of 3 units at points A and B. Find the area of quadrilateral APBQ.

35. [$\pm$7] Suppose a, b, c, d are real numbers such that

$$|a - b| + |c - d| = 99; \qquad |a - c| + |b - d| = 1.$$

Determine all possible values of $|a - d| + |b - c|$.

36. [$\pm$6] Find the set consisting of all real values of x such that the three numbers $2^x, 2^{x^2}, 2^{x^3}$ form a non-constant arithmetic progression (in that order).

37. [8] Call a positive integer "mild" if its base-3 representation never contains the digit 2. How many values of n $(1 \leq n \leq 1000)$ have the property that n and n^2 are both mild?

38. [6] Massachusetts Avenue is ten blocks long. One boy and one girl live on each block. They want to form friendships such that each boy is friends with exactly one girl and vice-versa. Nobody wants a friend living more than one block away (but they may be on the same block). How many pairings are possible?

39. [7] In the x-y plane, draw a circle of radius 2 centered at $(0, 0)$. Color the circle red above the line $y = 1$, color the circle blue below the line $y = -1$, and color the rest of the circle white. Now consider an arbitrary straight line at distance 1 from the circle. We color each point P of the line with the color of the closest point to P on the circle. If we pick such an arbitrary line, randomly oriented, what is the probability that it contains red, white, and blue points?

40. [9] Find the volume of the three-dimensional solid given by the inequality $\sqrt{x^2 + y^2} + |z| \leq 1$.

41. [9] For any integer n, define $\lfloor n \rfloor$ as the greatest integer less than or equal to n. For any positive integer n, let

$$f(n) = \lfloor n \rfloor + \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \cdots + \left\lfloor \frac{n}{n} \right\rfloor .$$

For how many values of $n, 1 \leq n \leq 100$, is $f(n)$ odd?

42. [± 10] Find all the integers $n > 1$ with the following property: the numbers $1, 2, \ldots, n$ can be arranged in a line so that, of any two adjacent numbers, one is divisible by the other.

43. **[9]** Given that a, b, c are positive integers satisfying

$$a + b + c = \gcd(a, b) + \gcd(b, c) + \gcd(c, a) + 120,$$

determine the maximum possible value of a.

44. **[5]** The unknown real numbers x, y, z satisfy the equations

$$\frac{x+y}{1+z} = \frac{1-z+z^2}{x^2-xy+y^2}; \qquad \frac{x-y}{3-z} = \frac{9+3z+z^2}{x^2+xy+y^2}.$$

Find x.

45. **[9]** Find the number of sequences $a_1, a_2, \ldots, a_{10}$ of positive integers with the property that $a_{n+2} = a_{n+1} + a_n$ for $n = 1, 2, \ldots, 8$, and $a_{10} = 2002$.

46. [± 6] Points A, B, C in the plane satisfy $\overline{AB} = 2002, \overline{AC} = 9999$. The circles with diameters AB and AC intersect at A and D. If $\overline{AD} = 37$, what is the shortest distance from point A to line BC?

47. **[9]** The real function f has the property that, whenever a, b, n are positive integers such that $a + b = 2^n$, the equation $f(a) + f(b) = n^2$ holds. What is $f(2002)$?

48. **[9]** A *permutation* of a finite set is a one-to-one function from the set to itself; for instance, one permutation of $\{1, 2, 3, 4\}$ is the function π defined such that $\pi(1) = 1$, $\pi(2) = 3$, $\pi(3) = 4$, and $\pi(4) = 2$. How many permutations π of the set $\{1, 2, \ldots, 10\}$ have the property that $\pi(i) \neq i$ for each $i = 1, 2, \ldots, 10$, but $\pi(\pi(i)) = i$ for each i?

49. **[7]** Two integers are *relatively prime* if they don't share any common factors, i.e. if their greatest common divisor is 1. Define $\varphi(n)$ as the number of positive integers that are less than n and relatively prime to n. Define $\varphi_d(n)$ as the number of positive integers that are less than dn and relatively prime to n.

What is the least n such that $\varphi_x(n) = 64000$, where $x = \varphi_y(n)$, where $y = \varphi(n)$?

50. **[6]** Give the set of all positive integers n such that $\varphi(n) = 2002^2 - 1$.

51. **[10]** Define $\varphi^k(n)$ as the number of positive integers that are less than or equal to n/k and relatively prime to n. Find $\phi^{2001}(2002^2 - 1)$. (Hint: $\phi(2003) = 2002$.)

52. [±8] Let $ABCD$ be a quadrilateral, and let E, F, G, H be the respective midpoints of AB, BC, CD, DA. If $EG = 12$ and $FH = 15$, what is the maximum possible area of $ABCD$?

53. [**10**] ABC is a triangle with points E, F on sides AC, AB, respectively. Suppose that BE, CF intersect at X. It is given that $AF/FB = (AE/EC)^2$ and that X is the midpoint of BE. Find the ratio CX/XF.

54. [**10**] How many pairs of integers (a, b), with $1 \leq a \leq b \leq 60$, have the property that b is divisible by a and $b + 1$ is divisible by $a + 1$?

55. [**10**] A sequence of positive integers is given by $a_1 = 1$ and $a_n = \gcd(a_{n-1}, n) + 1$ for $n > 1$. Calculate a_{2002}.

56. [±6] x, y are positive real numbers such that $x + y^2 = xy$. What is the smallest possible value of x?

57. [**9**] How many ways, without taking order into consideration, can 2002 be expressed as the sum of 3 positive integers (for instance, $1000 + 1000 + 2$ and $1000 + 2 + 1000$ are considered to be the same way)?

58. [**8**] A sequence is defined by $a_0 = 1$ and $a_n = 2^{a_{n-1}}$ for $n \geq 1$. What is the last digit (in base 10) of a_{15}?

59. [**7**] Determine the value of

$$1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - 4 \cdot 5 + \cdots + 2001 \cdot 2002.$$

60. [**10**] A 5×5 square grid has the number -3 written in the upper-left square and the number 3 written in the lower-right square. In how many ways can the remaining squares be filled in with integers so that any two adjacent numbers differ by 1, where two squares are adjacent if they share a common edge (but not if they share only a corner)?

61. Bob Barker went back to school for a PhD in math, and decided to raise the intellectual level of *The Price is Right* by having contestants guess how many objects exist of a certain type, without going over. The number of points you will get is the percentage of the correct answer, divided by 10, with no points for going over (i.e. a maximum of 10 points).

Let's see the first object for our contestants...a *table* of shape $(5, 4, 3, 2, 1)$ is an arrangement of the integers 1 through 15 with five numbers in the top row, four in the next, three in the next, two in the next, and one in the last, such that each row and each column is increasing (from left to right, and top to bottom, respectively). For instance:

1	2	3	4	5
6	7	8	9	
10	11	12		
13	14			
15				

is one table. How many tables are there?

62. Our next object up for bid is an arithmetic progression of primes. For example, the primes 3, 5, and 7 form an arithmetic progression of length 3. What is the largest possible length of an arithmetic progression formed of positive primes less than 1,000,000? Be prepared to justify your answer.

63. Our third and final item comes to us from Germany, I mean Geometry. It is known that a regular n-gon can be constructed with straightedge and compass if n is a prime that is 1 plus a power of 2. It is also possible to construct a $2n$-gon whenever an n-gon is constructible, or a $p_1 p_2 \cdots p_m$-gon where the p_i's are distinct primes of the above form. What is really interesting is that these conditions, together with the fact that we can construct a square, is that they give us all constructible regular n-gons. What is the largest n less than 4,300,000,000 such that a regular n-gon is constructible?

Help control the pet population. Have your pets spayed or neutered. Bye-bye.

Team Event
HMMT 2002

Palindromes. A *palindrome* is a positive integer n not divisible by 10 such that if you write the decimal digits of n in reverse order, the number you get is n itself. For instance, the numbers 4 and 25752 are palindromes.

1. [15] Determine the number of palindromes that are less than 1000.

2. [30] Determine the number of four-digit integers n such that n and $2n$ are both palindromes.

3. [40] Suppose that a positive integer n has the property that n, $2n$, $3n$, …, $9n$ are all palindromes. Prove that the decimal digits of n are all zeros or ones.

Floor functions. The notation $\lfloor x \rfloor$ stands for the largest integer less than or equal to x.

4. [15] Let n be an integer. Prove that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n+1}{2} \right\rfloor = n.$$

5. [20] Prove for integers n that

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n+1}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

In problems 6–7 you may use without proof the known summations

$$\sum_{n=1}^{L} n = n(n+1)/2 \quad \text{and} \quad \sum_{n=1}^{L} n^3 = n^2(n+1)^2/4 \quad \text{for positive integers } L.$$

6. [20] For positive integers L, let $S_L = \sum_{n=1}^{L} \lfloor n/2 \rfloor$. Determine all L for which S_L is a square number.

7. [45] Let $T_L = \sum_{n=1}^{L} \lfloor n^3/9 \rfloor$ for positive integers L. Determine all L for which T_L is a square number.

Luck of the dice. Problems 8–12 concern a two-player game played on a board consisting of fourteen spaces in a row. The leftmost space is labeled *START*, and the rightmost space is labeled *END*. Each of the twelve other squares, which we number 1 through 12 from left to right, may be blank or may be labeled with an arrow pointing to the right. The term *blank square* will refer to one of these twelve squares that is not labeled with an arrow. The set of blank squares on the board will be called a *board configuration*; the board below uses the configuration $\{1, 2, 3, 4, 7, 8, 10, 11, 12\}$.

START					$\Rightarrow$	$\Rightarrow$			$\Rightarrow$				*END*
	1	2	3	4	5	6	7	8	9	10	11	12	

For $i \in \{1, 2\}$, player i has a die that produces each integer from 1 to s_i with probability $1/s_i$. Here s_1 and s_2 are positive integers fixed before the game begins. The game rules are as follows:

1. The players take turns alternately, and player 1 takes the first turn.
2. On each of his turns, player i rolls his die and moves his piece to the right by the number of squares that he rolled. If his move ends on a square marked with an arrow, he moves his piece forward another s_i squares. If that move ends on an arrow, he moves another s_i squares, repeating until his piece comes to rest on a square without an arrow.
3. If a player's move would take him past the *END* square, instead he lands on the *END* square.
4. Whichever player reaches the *END* square first wins.

As an example, suppose that $s_1 = 3$ and the first player is on square 4 in the sample board shown above. If the first player rolls a 2, he moves to square 6, then to square 9, finally coming to rest on square 12. If the second player does not reach the *END* square on her next turn, the first player will necessarily win on his next turn, as he must roll at least a 1.

8. [35] In this problem only, assume that $s_1 = 4$ and that exactly one board square, say square number n, is marked with an arrow. Determine all choices of n that maximize the average distance in squares the first player will travel in his first two turns.

9. [30] In this problem suppose that $s_1 = s_2$. Prove that for each board configuration, the first player wins with probability strictly greater than $\frac{1}{2}$.

10. [30] Exhibit a configuration of the board and a choice of s_1 and s_2 so that $s_1 > s_2$, yet the *second* player wins with probability strictly greater than $\frac{1}{2}$.

11. [55] In this problem assume $s_1 = 3$ and $s_2 = 2$. Determine, with proof, the nonnegative integer k with the following property:

1. For every board configuration with strictly fewer than k blank squares, the first player wins with probability strictly greater than $\frac{1}{2}$; but
2. there exists a board configuration with exactly k blank squares for which the second player wins with probability strictly greater than $\frac{1}{2}$.

12. [65] Now suppose that before the game begins, the players choose the initial game state as follows:

1. The first player chooses s_1 subject to the constraint that $2 \leq s_1 \leq 5$; then
2. the second player chooses s_2 subject to the constraint that $2 \leq s_2 \leq 5$ and then specifies the board configuration.

Prove that the second player can always make her decisions so that she will win the game with probability strictly greater than $\frac{1}{2}$.

Advanced Topics Test

Harvard-MIT Math Tournament

March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

2. Find $\log_n\left(\frac{1}{2}\right)\log_{n-1}\left(\frac{1}{3}\right)\cdots\log_2\left(\frac{1}{n}\right)$ in terms of n.

3. Calculate the sum of the coefficients of $P(x)$ if $(20x^{27} + 2x^2 + 1)P(x) = 2001x^{2001}$.

4. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n. (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

6. There are two red, two black, two white, and a positive but unknown number of blue socks in a drawer. It is empirically determined that if two socks are taken from the drawer without replacement, the probability they are of the same color is $\frac{1}{5}$. How many blue socks are there in the drawer?

7. Order these four numbers from least to greatest: $5^{56}, 10^{51}, 17^{35}, 31^{28}$.

8. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

9. Find the real solutions of $(2x + 1)(3x + 1)(5x + 1)(30x + 1) = 10$.

10. Alex picks his favorite point (x, y) in the first quadrant on the unit circle $x^2 + y^2 = 1$, such that a ray from the origin through (x, y) is θ radians counterclockwise from the positive x-axis. He then computes $\cos^{-1}\left(\frac{4x+3y}{5}\right)$ and is surprised to get θ. What is $\tan(\theta)$?

Algebra Test

Harvard-MIT Math Tournament

March 3, 2001

1. Find $x - y$, given that $x^4 = y^4 + 24$, $x^2 + y^2 = 6$, and $x + y = 3$.

2. Find $(x+1)(x^2+1)(x^4+1)(x^8+1)\cdots$, where $|x| < 1$.

3. How many times does 24 divide into 100! (factorial)?

4. Given that $7,999,999,999$ has at most two prime factors, find its largest prime factor.

5. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

6. What is the last digit of $1^1 + 2^2 + 3^3 + \cdots + 100^{100}$?

7. A polynomial P has four roots, $\frac{1}{4}, \frac{1}{2}, 2, 4$. The product of the roots is 1, and $P(1) = 1$. Find $P(0)$.

8. How many integers between 1 and 2000 inclusive share no common factors with 2001?

9. Find the number of positive integer solutions to $n^x + n^y = n^z$ with $n^z < 2001$.

10. Find the real solutions of $(2x+1)(3x+1)(5x+1)(30x+1) = 10$.

Calculus Test

Harvard-MIT Math Tournament
March 3, 2001

1. A sequence of ants walk from $(0,0)$ to $(1,0)$ in the plane. The nth ant walks along n semicircles of radius $\frac{1}{n}$ with diameters lying along the line from $(0,0)$ to $(1,0)$. Let L_n be the length of the path walked by the nth ant. Compute $\lim_{n\to\infty} L_n$.

2. The polynomial $3x^5 - 250x^3 + 735x$ is interesting because it has the maximum possible number of relative extrema and points of inflection at integer lattice points for a quintic polynomial. What is the sum of the x-coordinates of these points?

3. A balloon that blows up in the shape of a perfect cube is being blown up at a rate such that at time t fortnights, it has surface area $6t$ square furlongs. At how many cubic furlongs per fortnight is the air being pumped in when the surface area is 144 square furlongs?

4. What is the size of the largest rectangle that can be drawn inside of a 3-4-5 right triangle with one of the rectangle's sides along one of the legs of the triangle?

5. Same as question **4**, but now we want one of the rectangle's sides to be along the hypotenuse.

6. The graph of $x^2 - (y-1)^2 = 1$ has one tangent line with positive slope that passes through $(x,y) = (0,0)$. If the point of tangency is (a,b), find $\sin^{-1}(\frac{a}{b})$ in radians.

7. Find the coefficient of x^{12} in the Maclaurin series (i.e. Taylor series around $x=0$) for $\frac{1}{1-3x+2x^2}$.

8. Evaluate $\sum_{n=0}^{\infty} \cot^{-1}(n^2+n+1)$.

9. On the planet Lemniscate, the people use the elliptic table of elements, a far more advanced version of our periodic table. They're not very good at calculus, though, so they've asked for your help. They know that Kr is somewhat radioactive and deteriorates into Pl, a very unstable element that deteriorates to form the stable element As. They started with a block of Kr of size 10 and nothing else. (Their units don't translate into English, sorry.) and nothing else. At time t, they let $x(t)$ be the amount of Kr, $y(t)$ the amount of Pl, and $z(t)$ the amount of As. They know that $x'(t) = -x$, and that, in the absence of Kr, $y'(t) = -2y$. Your job is to find at what time t the quantity of Pl will be largest. You should assume that the entire amount of Kr that deteriorates has turned into Pl.

10. Evaluate the definite integral $\int\limits_{-1}^{+1} \frac{2u^{332}+u^{998}+4u^{1664}\sin u^{691}}{1+u^{666}} du$.

General Test, First Half

Harvard-MIT Math Tournament

March 3, 2001

1. What is the last digit of $17^{103} + 5$?

2. Find $x + y$, given that $x^2 - y^2 = 10$ and $x - y = 2$.

3. There are some red and blue marbles in a box. We are told that there are twelve more red marbles than blue marbles, and we experimentally determine that when we pick a marble randomly we get a blue marble one quarter of the time. How many marbles are there in the box?

4. Find $a + b + c + d + e$ if

$$\begin{gathered} 3a + 2b + 4d = 10, \\ 6a + 5b + 4c + 3d + 2e = 8, \\ a + b + 2c + 5e = 3, \\ 2c + 3d + 3e = 4, \text{ and} \\ a + 2b + 3c + d = 7. \end{gathered}$$

5. What is the sum of the coefficients of the expansion $(x + 2y - 1)^6$?

6. A right triangle has a hypotenuse of length 2, and one of its legs has length 1. The altitude to its hypotenuse is drawn. What is the area of the rectangle whose diagonal is this altitude?

7. Find $(x + 1)(x^2 + 1)(x^4 + 1)(x^8 + 1) \cdots$, where $|x| < 1$.

8. How many times does 24 divide into 100!?

9. Boris was given a Connect Four game set for his birthday, but his color-blindness makes it hard to play the game. Still, he enjoys the shapes he can make by dropping checkers into the set. If the number of shapes possible modulo (horizontal) flips about the vertical axis of symmetry is expressed as $9(1 + 2 + \cdots + n)$, find n. (Note: the board is a vertical grid with seven columns and eight rows. A checker is placed into the grid by dropping it from the top of a column, and it falls until it hits either the bottom of the grid or another checker already in that column. Also, $9(1 + 2 + \cdots + n)$ is the number of shapes possible, with two shapes that are horizontal flips of each other counted as one. In other words, the shape that consists solely of 3 checkers in the rightmost row and the shape that consists solely of 3 checkers in the leftmost row are to be considered the same shape.)

10. Find the 6-digit number beginning and ending in the digit 2 that is the product of three consecutive even integers.

General Test Solutions (Second Half)

Harvard-MIT Math Tournament

March 3, 2001

1. A circle of radius 3 crosses the center of a square of side length 2. Find the difference between the areas of the nonoverlapping portions of the figures.

2. Call three sides of an opaque cube adjacent if someone can see them all at once. Draw a plane through the centers of each triple of adjacent sides of a cube with edge length 1. Find the volume of the closed figure bounded by the resulting planes.

3. Find x if $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$.

4. Some students are taking a math contest, in which each student takes one of four tests. One third of the students take one test, one fourth take another test, one fifth take the next test, and 26 students take the last test. How many students are taking the contest in total?

5. What is the area of a square inscribed in a semicircle of radius 1, with one of its sides flush with the diameter of the semicircle?

6. You take a wrong turn on the way to MIT and end up in Transylvania, where 99% of the inhabitants are vampires and the rest are regular humans. For obvious reasons, you want to be able to figure out who's who. On average, nine-tenths of the vampires are correctly identified as vampires and nine-tenths of humans are correct identified as humans. What is the probability that someone identified as a human is actually a human?

7. A real numbers x is randomly chosen in the interval $\left[-15\frac{1}{2}, 15\frac{1}{2}\right]$. Find the probability that the closest integer to x is odd.

8. A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square.

9. Two circles are concentric. The area of the ring between them is A. In terms of A, find the length of the longest chord contained entirely within the ring.

10. Find the volume of the tetrahedron with vertices $(5, 8, 10), (10, 10, 17), (4, 45, 46), (2, 5, 4)$.

Geometry Test

Harvard-MIT Math Tournament

March 3, 2001

1. A circle of radius 3 crosses the center of a square of side length 2. Find the positive difference between the areas of the nonoverlapping portions of the figures.

2. Call three sides of an opaque cube adjacent if someone can see them all at once. Draw a plane through the centers of each triple of adjacent sides of a cube with edge length 1. Find the volume of the closed figure bounded by the resulting planes.

3. Square $ABCD$ is drawn. Isosceles Triangle CDE is drawn with E a right angle. Square $DEFG$ is drawn. Isosceles triangle FGH is drawn with H a right angle. This process is repeated infinitely so that no two figures overlap each other. If square $ABCD$ has area 1, compute the area of the entire figure.

4. A circle has two parallel chords of length x that are x units apart. If the part of the circle included between the chords has area $2+\pi$, find x.

5. Find the volume of the tetrahedron with vertices $(5,8,10)$, $(10,10,17)$, $(4,45,46)$, $(2,5,4)$.

6. A point on a circle inscribed in a square is 1 and 2 units from the two closest sides of the square. Find the area of the square.

7. Equilateral triangle ABC with side length 1 is drawn. A square is drawn such that its vertex at A is opposite to its vertex at the midpoint of BC. Find the area enclosed within the intersection of the insides of the triangle and square. Hint: $\sin 75 = \frac{\sqrt{2}(\sqrt{3}+1)}{4}$.

8. Point D is drawn on side BC of equilateral triangle ABC, and AD is extended past D to E such that angles EAC and EBC are equal. If $BE = 5$ and $CE = 12$, determine the length of AE.

9. Parallelogram $AECF$ is inscribed in square $ABCD$. It is reflected across diagonal AC to form another parallelogram $AE'CF'$. The region common to both parallelograms has area m and perimeter n. Compute the value of $\frac{m}{n^2}$ if $AF : AD = 1 : 4$.

10. A is the center of a semicircle, with radius AD lying on the base. B lies on the base between A and D, and E is on the circular portion of the semicircle such that EBA is a right angle. Extend EA through A to C, and put F on line CD such that EBF is a line. Now $EA = 1$, $AC = \sqrt{2}$, $BF = \frac{2-\sqrt{2}}{4}$, $CF = \frac{2\sqrt{5}+\sqrt{10}}{4}$, and $DF = \frac{2\sqrt{5}-\sqrt{10}}{4}$. Find DE.

Guts Round Questions

Harvard-MIT Math Tournament

March 3, 2001

1. [**5**] January 3, 1911 was an *odd date* as its abbreviated representation, 1/3/1911, can be written using only odd digits (note all four digits are written for the year). To the nearest month, how many months will have elapsed between the most recent odd date and the next odd date (today is 3/3/2001, an even date).

2. [**4**] Ken is the best sugar cube retailer in the nation. Trevor, who loves sugar, is coming over to make an order. Ken knows Trevor cannot afford more than 127 sugar cubes, but might ask for any number of cubes less than or equal to that. Ken prepares seven cups of cubes, with which he can satisfy any order Trevor might make. How many cubes are in the cup with the most sugar?

3. [**7**] Find the number of triangulations of a general convex 7-gon into 5 triangles by 4 diagonals that do not intersect in their interiors.

4. [**7**] Find $\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)$.

5. [$\pm$ **6**] Let ABC be a triangle with incenter I and circumcenter O. Let the circumradius be R. What is the least upper bound of all possible values of IO?

6. [**8**] Six students taking a test sit in a row of seats with aisles only on the two sides of the row. If they finish the test at random times, what is the probability that some student will have to pass by another student to get to an aisle?

7. [**5**] Suppose a, b, c, d, and e are objects that we can multiply together, but the multiplication doesn't necessarily satisfy the associative law, i.e. $(xy)z$ does not necessarily equal $x(yz)$. How many different ways are there to interpret the product $abcde$?

8. [**10**] Compute $1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1)n$.

9. [**5**] Suppose x satisfies $x^3 + x^2 + x + 1 = 0$. What are all possible values of $x^4 + 2x^3 + 2x^2 + 2x + 1$?

10. [**8**] Two concentric circles have radii r and $R > r$. Three new circles are drawn so that they are each tangent to the big two circles and tangent to the other two new circles. Find $\frac{R}{r}$.

11. [**8**] 12 points are placed around the circumference of a circle. How many ways are there to draw 6 non-intersecting chords joining these points in pairs?

12. [$\pm$ **6**] How many distinct sets of 8 positive odd integers sum to 20?

13. [**5**] Find the number of real zeros of $x^3 - x^2 - x + 2$.

14. [**8**] Find the exact value of $1 + \frac{1}{1+\frac{2}{1+\frac{1}{1+\frac{2}{1+\cdots}}}}$.

15. [**6**] A beaver walks from $(0, 0)$ to $(4, 4)$ in the plane, walking one unit in the positive x direction or one unit in the positive y direction at each step. Moreover, he never goes to a point (x, y) with $y > x$. How many different paths can he walk?

16. [**6**] After walking so much that his feet get really tired, the beaver staggers so that, at each step, his coordinates change by either $(+1, +1)$ or $(+1, -1)$. Now he walks from $(0, 0)$ to $(8, 0)$ without ever going below the x-axis. How many such paths are there?

17. [**4**] Frank and Joe are playing ping pong. For each game, there is a 30% chance that Frank wins and a 70% chance Joe wins. During a match, they play games until someone wins a total of 21 games. What is the expected value of number of games played per match?

18. [**5**] Find the largest prime factor of $-x^{10} - x^8 - x^6 - x^4 - x^2 - 1$, where $x = 2i$, $i = \sqrt{-1}$.

19. [**9**] Calculate $\sum_{n=1}^{2001} n^3$.

20. [$\pm$ **4**] Karen has seven envelopes and seven letters of congratulations to various HMMT coaches. If she places the letters in the envelopes at random with each possible configuration having an equal probability, what is the probability that exactly six of the letters are in the correct envelopes?

21. [**10**] Evaluate $\sum_{i=1}^{\infty} \frac{(i+1)(i+2)(i+3)}{(-2)^i}$.

22. [**6**] A man is standing on a platform and sees his train move such that after t seconds it is $2t^2 + d_0$ feet from his original position, where d_0 is some number. Call the smallest (constant) speed at which the man have to run so that he catches the train v. In terms of n, find the nth smallest value of d_0 that makes v a perfect square.

23. [**5**] Alice, Bob, and Charlie each pick a 2-digit number at random. What is the probability that all of their numbers' tens' digits are different from each others' tens' digits and all of their numbers' ones digits are different from each others' ones' digits?

24. [**6**] Square $ABCD$ has side length 1. A dilation is performed about point A, creating square $AB'C'D'$. If $BC' = 29$, determine the area of triangle BDC'.

25. [$\pm$ **10**] What is the remainder when 100! is divided by 101?

26. [**6**] A circle with center at O has radius 1. Points P and Q outside the circle are placed such that PQ passes through O. Tangent lines to the circle through P hit the circle at P_1 and P_2, and tangent lines to the circle through Q hit the circle at Q_1 and Q_2. If $\angle P_1PP_2 = 45°$ and $angleQ_1QQ_2 = 30°$, find the minimum possible length of arc P_2Q_2.

27. [**5**] Mona has 12 match sticks of length 1, and she has to use them to make regular polygons, with each match being a side or a fraction of a side of a polygon, and no two matches overlapping or crossing each other. What is the smallest total area of the polygons Mona can make?

28. [**4**] How many different combinations of 4 marbles can be made from 5 indistinguishable red marbles, 4 indistinguishable blue marbles, and 2 indistinguishable black marbles?

29. [**10**] Count the number of sequences a_1, a_2, a_3, a_4, a_5 of integers such that $a_i \leq 1$ for all i and all partial sums (a_1, $a_1 + a_2$, etc.) are non-negative.

30. [$\pm$ **4**] How many roots does $\arctan x = x^2 - 1.6$ have, where the arctan function is defined in the range $-\frac{pi}{2} < \arctan x < \frac{pi}{2}$?

31. [**5**] If two fair dice are tossed, what is the probability that their sum is divisible by 5?

32. [**10**] Count the number of permutations $a_1a_2\ldots a_7$ of 1234567 with longest decreasing subsequence of length at most two (i.e. there does not exist $i<j<k$ such that $a_i>a_j>a_k$).

33. [$\pm$ **5**] A line of soldiers 1 mile long is jogging. The drill sergeant, in a car, moving at twice their speed, repeatedly drives from the back of the line to the front of the line and back again. When each soldier has marched 15 miles, how much mileage has been added to the car, to the nearest mile?

34. [**8**] Find all the values of m for which the zeros of $2x^2-mx-8$ differ by $m-1$.

35. [**7**] Find the largest integer that divides m^5-5m^3+4m for all $m\geq 5$.
. $5! = \boxed{120}$.

36. [**4**] Count the number of sequences $1\leq a_1\leq a_2\leq\cdots\leq a_5$ of integers with $a_i\leq i$ for all i.

37. [**5**] Alex and Bob have 30 matches. Alex picks up somewhere between one and six matches (inclusive), then Bob picks up somewhere between one and six matches, and so on. The player who picks up the last match wins. How many matches should Alex pick up at the beginning to guarantee that he will be able to win?

38. [**9**] The cafeteria in a certain laboratory is open from noon until 2 in the afternoon every Monday for lunch. Two professors eat 15 minute lunches sometime between noon and 2. What is the probability that they are in the cafeteria simultaneously on any given Monday?

39. [**9**] What is the remainder when 2^{2001} is divided by 2^7-1?

40. [**5**] A product of five primes is of the form ABC,ABC, where A, B, and C represent digits. If one of the primes is 491, find the product ABC,ABC.

41. [**4**] If $\left(a+\frac{1}{a}\right)^2=3$, find $\left(a+\frac{1}{a}\right)^3$ in terms of a.

42. [**10**] Solve $x=\sqrt{x-\frac{1}{x}}+\sqrt{1-\frac{1}{x}}$ for x.

43. [**4**] When a single number is added to each member of the seqence 20, 50, 100, the sequence becomes expressable as x, ax, a^2x. Find a.

44. [**7**] Through a point in the interior of a triangle ABC, three lines are drawn, one parallel to each side. These lines divide the sides of the triangle into three regions each. Let a, b, and c be the lengths of the sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively, and let a', b', and c' be the lengths of the middle regions of the sides opposite $\angle A$, $\angle B$, and $\angle C$, respectively. Find the numerical value of $a'/a + b'/b + c'/c$.

45. [**4**] A stacking of circles in the plane consists of a base, or some number of unit circles centered on the x-axis in a row without overlap or gaps, and circles above the x-axis that must be tangent to two circles below them (so that if the ends of the base were secured and gravity were applied from below, then nothing would move). How many stackings of circles in the plane have 4 circles in the base?

46. [$\pm$ **5**] Draw a rectangle. Connect the midpoints of the opposite sides to get 4 congruent rectangles. Connect the midpoints of the lower right rectangle for a total of 7 rectangles. Repeat this process infinitely. Let n be the minimum number of colors we can assign to the rectangles so that no two rectangles sharing an edge have the same color and m be the minimum number of colors we can assign to the rectangles so that no two rectangles sharing a corner have the same color. Find the ordered pair (n, m).

47. [**7**] For the sequence of numbers $n_1, n_2, n_3, \ldots$, the relation $n_i = 2n_{i-1} + a$ holds for all $i > 1$. If $n_2 = 5$ and $n_8 = 257$, what is n_5?

48. [**8**] What is the smallest positive integer x for which $x^2 + x + 41$ is not a prime?

49. [**5**] If $\frac{1}{9}$ of 60 is 5, what is $\frac{1}{20}$ of 80?

50. [**9**] The Fibonacci sequence $F_1, F_2, F_3, \ldots$ is defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$. Find the least positive integer t such that for all $n > 0$, $F_n = F_{n+t}$.

51. [**5**] Some people like to write with larger pencils than others. Ed, for instance, likes to write with the longest pencils he can find. However, the halls of MIT are of limited height L and width L. What is the longest pencil Ed can bring through the halls so that he can negotiate a square turn?

52. [**6**] Find all ordered pairs (m, n) of integers such that $231m^2 = 130n^2$.

53. [7] Find the sum of the infinite series $\frac{1}{3^2-1^2}\left(\frac{1}{1^2}-\frac{1}{3^2}\right)+\frac{1}{5^2-3^2}\left(\frac{1}{3^2}-\frac{1}{5^2}\right)+\frac{1}{7^2-5^2}\left(\frac{1}{5^2}-\frac{1}{7^2}\right)+\cdots$.

54. [10] The set of points (x_1, x_2, x_3, x_4) in $\mathbf{R}^4$ such that $x_1 \geq x_2 \geq x_3 \geq x_4$ is a cone (or hypercone, if you insist). Into how many regions is this cone sliced by the hyperplanes $x_i - x_j = 1$ for $1 \leq i < j \leq n$?

55. [7] How many multiples of 7 between 10^6 and 10^9 are perfect squares?

56. [6] A triangle has sides of length 888, 925, and $x > 0$. Find the value of x that minimizes the area of the circle circumscribed about the triangle.

57. [5] Let $x = 2001^{1002} - 2001^{-1002}$ and $y = 2001^{1002} + 2001^{-1002}$. Find $x^2 - y^2$.

58. [9] Let (x, y) be a point in the cartesian plane, $x, y > 0$. Find a formula in terms of x and y for the minimal area of a right triangle with hypotenuse passing through (x, y) and legs contained in the x and y axes.

59. [10] Trevor and Edward play a game in which they take turns adding or removing beans from a pile. On each turn, a player must either add or remove the largest perfect square number of beans that is in the heap. The player who empties the pile wins. For example, if Trevor goes first with a pile of 5 beans, he can either add 4 to make the total 9, or remove 4 to make the total 1, and either way Edward wins by removing all the beans. There is no limit to how large the pile can grow; it just starts with some finite number of beans in it, say fewer than 1000.

Before the game begins, Edward dispatches a spy to find out how many beans will be in the opening pile, call this n, then "graciously" offers to let Trevor go first. Knowing that the first player is more likely to win, but not knowing n, Trevor logically but unwisely accepts, and Edward goes on to win the game. Find a number n less than 1000 that would prompt this scenario, assuming both players are perfect logicians. A correct answer is worth the nearest integer to $\log_2(n-4)$ points.

60. [∞] Find an n such that $n! - (n-1)! + (n-2)! - (n-3)! + \cdots \pm 1!$ is prime. Be prepared to justify your answer for $\begin{cases} n, & n \leq 25 \\ \left[\frac{n+225}{10}\right], & n > 25 \end{cases}$ points, where $[N]$ is the greatest integer less than N.

Team Round Solutions
Harvard-MIT Math Tournament
March 3, 2001

1. How many digits are in the base two representation of 10! (factorial)?

2. On a certain unidirectional highway, trucks move steadily at 60 miles per hour spaced 1/4 of a mile apart. Cars move steadily at 75 miles per hour spaced 3 seconds apart. A lone sports car weaving through traffic at a steady forward speed passes two cars between each truck it passes. How quickly is it moving in miles per hour?

3. What is the 18th digit after the decimal point of $\frac{10000}{9899}$?

4. P is a polynomial. When P is divided by $x-1$, the remainder is -4. When P is divided by $x-2$, the remainder is -1. When P is divided by $x-3$, the remainder is 4. Determine the remainder when P is divided by $x^3-6x^2+11x-6$.

5. Find all x between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ such that $1-\sin^4 x-\cos^2 x=\frac{1}{16}$.

6. What is the radius of the smallest sphere in which 4 spheres of radius 1 will fit?

7. The Fibonacci numbers are defined by $F_1=F_2=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n\geq 1$. The Lucas numbers are defined by $L_1=1$, $L_2=2$, and $L_{n+2}=L_{n+1}+L_n$ for $n\geq 1$. Calculate $\frac{\prod_{n=1}^{15}\frac{F_{2n}}{F_n}}{\prod_{n=1}^{13}L_n}$.

8. Express $\frac{\sin 10+\sin 20+\sin 30+\sin 40+\sin 50+\sin 60+\sin 70+\sin 80}{\cos 5\cos 10\cos 20}$ without using trigonometric functions.

9. Compute $\sum_{i=1}^{\infty}\frac{ai}{a^i}$ for $a>1$.

10. Define a monic irreducible polynomial with integral coefficients to be a polynomial with leading coefficient 1 that cannot be factored, and the prime factorization of a polynomial with leading coefficient 1 as the factorization into monic irreducible polynomials. How many not necessarily distinct monic irreducible polynomials are there in the prime factorization of $(x^8 + x^4 + 1)(x^8 + x + 1)$ (for instance, $(x + 1)^2$ has two prime factors)?

11. Define $a? = (a - 1)/(a + 1)$ for $a \neq -1$. Determine all real values N for which $(N?)? = \tan 15$.

12. All subscripts in this problem are to be considered modulo 6, that means for example that ω_7 is the same as ω_1. Let $\omega_1, \ldots \omega_6$ be circles of radius r, whose centers lie on a regular hexagon of side length 1. Let P_i be the intersection of ω_i and ω_{i+1} that lies further from the center of the hexagon, for $i = 1, \ldots 6$. Let Q_i, $i = 1 \ldots 6$, lie on ω_i such that Q_i, P_i, Q_{i+1} are colinear. Find the number of possible values of r.

Advanced Topics
Rice Mathematics Tournament 2000

1. How many different ways are there to paint the sides of a tetrahedron with exactly 4 colors? Each side gets its own color, and two colorings are the same if one can be rotated to get the other.

2. Simplify $\left(\frac{-1+i\sqrt{3}}{2}\right)^6 + \left(\frac{-1-i\sqrt{3}}{2}\right)^6$ to the form $a + bi$.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2+2n}$.

4. Five positive integers from 1 to 15 are chosen without replacement. What is the probability that their sum is divisible by 3?

5. Find all 3-digit numbers which are the sums of the cubes of their digits.

6. 6 people each have a hat. If they shuffle their hats and redistribute them, what is the probability that exactly one person gets their own hat back?

7. Assume that a, b, c, d are positive integers, and $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2+c^2} - \sqrt{b^2+d^2} = 15$. Find $ac + bd - ad - bc$.

8. How many non-isomorphic graphs with 9 vertices, with each vertex connected to exactly 6 other vertices, are there? (Two graphs are isomorphic if one can relabel the vertices of one graph to make all edges be exactly the same.)

9. The Cincinnati Reals are playing the Houston Alphas in the last game of the Swirled Series. The Alphas are leading by 1 run in the bottom of the 9th (last) inning, and the Reals are at bat. Each batter has a $\frac{1}{3}$ chance of hitting a single and a $\frac{2}{3}$ chance of making an out. If the Reals hit 5 or more singles before they make 3 outs, they will win. If the Reals hit exactly 4 singles before making 3 outs, they will tie the game and send it into extra innings, and they will have a $\frac{3}{5}$ chance of eventually winning the game (since they have the added momentum of coming from behind). If the Reals hit fewer than 4 singles, they will LOSE! What is the probability that the Alphas hold off the Reals and win, sending the packed Alphadome into a frenzy? Express the answer as a fraction.

10. I call two people A and B and think of a natural number n. Then I give the number n to A and the number $n+1$ to B. I tell them that they have both been given natural numbers, and further that they are consecutive natural numbers. However, I don't tell A what B's number is and vice versa. I start by asking A if he knows B's number. He says "no". Then I ask B if he knows A's number, and he says "no" too. I go back to A and ask, and so on. A and B can both hear each other's responses. Do I ever get a "yes" in response? If so, who responds first with "yes" and how many times does he say "no" before this? Assume that both A and B are very intelligent and logical. You may need to consider multiple cases.

Algebra
Rice Mathematics Tournament 2000

1. How many integers x satisfy $|x| + 5 < 7$ and $|x - 3| > 2$?

2. Evaluate $2000^3 - 1999 \cdot 2000^2 - 1999^2 \cdot 2000 + 1999^3$.

3. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores?

4. What is the fewest number of multiplications required to reach x^{2000} from x, using only previously generated powers of x? For example, $x \to x^2 \to x^4 \to x^8 \to x^{16} \to x^{32} \to x^{64} \to x^{128} \to x^{256} \to x^{512} \to x^{1024} \to x^{1536} \to x^{1792} \to x^{1920} \to x^{1984} \to x^{2000}$ uses 15 multiplications.

5. A jacket was originally priced \$100. The price was reduced by 10% three times and increased by 10% four times in some order. To the nearest cent, what was the final price?

6. Barbara, Edward, Abhinav, and Alex took turns writing this test. Working alone, they could finish it in 10, 9, 11, and 12 days, respectively. If only one person works on the test per day, and nobody works on it unless everyone else has spent at least as many days working on it, how many days (an integer) did it take to write this test?

7. A number n is called multiplicatively perfect if the product of all the positive divisors of n is n^2. Determine the number of positive multiplicatively perfect numbers less than 100.

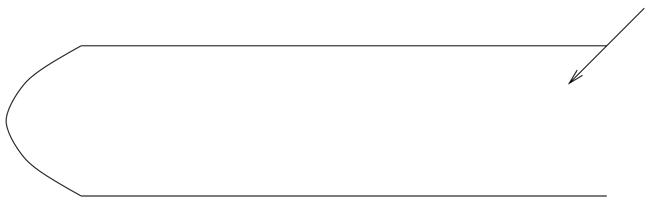
8. A man has three daughters. The product of their ages is 168, and he remembers that the sum of their ages is the number of trees in his yard. He counts the trees but cannot determine any of their ages. What are all possible ages of his oldest daughter?

9. $\frac{a}{c} = \frac{b}{d} = \frac{3}{4}$, $\sqrt{a^2 + c^2} - \sqrt{b^2 + d^2} = 15$. Find $ac + bd - ad - bc$.

10. Find the smallest positive integer a such that $x^4 + a^2$ is not prime for any integer x.

Calculus
Rice Mathematics Tournament 2000

1. Find the slope of the tangent at the point of inflection of $y = x^3 - 3x^2 + 6x + 2000$.
2. Karen is attempting to climb a rope that is not securely fastened. If she pulls herself up x feet at once, then the rope slips x^3 feet down. How many feet at a time must she pull herself up to climb as efficiently as possible?
3. A rectangle of length $\frac{1}{4}\pi$ and height 4 is bisected by the x-axis and is in the first and fourth quadrants. The graph of $y = \sin(x) + C$ divides the area of the square in half. What is C?
4. For what value of x $(0 < x < \frac{\pi}{2})$ does $\tan x + \cot x$ achieve its minimum?
5. For $-1 < x < 1$, let $f(x) = \sum_{i=1}^{\infty} \frac{x^i}{i}$. Find a closed form expression (a closed form expression is one not involving summation) for f.
6. A hallway of width 6 feet meets a hallway of width $6\sqrt{5}$ feet at right angles. Find the length of the longest pipe that can be carried horizontally around this corner.
7. An **envelope** of a set of lines is a curve tangent to all of them. What is the envelope of the family of lines $y = \frac{2}{x_0} + x(1 - \frac{1}{x_0^2})$, with x_0 ranging over the positive real numbers? *Hint*: slope at a point P on a curve is $\frac{dy}{dx}|_P$.
8. Find $\int_0^{\frac{\pi}{2}} \ln \sin \theta d\theta$.
9. Let $f(x) = \sqrt{x + \sqrt{0 + \sqrt{x + \sqrt{0 + \sqrt{x + \ldots}}}}}$. If $f(a) = 4$, then find $f'(a)$.
10. A mirror is constructed in the shape of y equals $\pm\sqrt{x}$ for $0 \leq x \leq 1$, and ± 1 for $1 < x < 9$. A ray of light enters at (10,1) with slope 1. How many times does it bounce before leaving?



General
Rice Mathematics Tournament 2000

1. If $a = 2b + c$, $b = 2c + d$, $2c = d + a - 1$, $d = a - c$, what is b?

2. The temperatures f°F and c°C are equal when $f = \frac{9}{5}c + 32$. What temperature is the same in both °F and °C?

3. A twelve foot tree casts a five foot shadow. How long is Henry's shadow (at the same time of day) if he is five and a half feet tall?

4. Tickets for the football game are \$10 for students and \$15 for non-students. If 3000 fans attend and pay \$36250, how many students went?

5. Find the interior angle between two sides of a regular octagon (degrees).

6. Three cards, only one of which is an ace, are placed face down on a table. You select one, but do not look at it. The dealer turns over one of the other cards, which is not the ace (if neither are, he picks one of them randomly to turn over). You get a chance to change your choice and pick either of the remaining two face-down cards. If you selected the cards so as to maximize the chance of finding the ace on the second try, what is the probability that you selected it on the

   (a) first try?

   (b) second try?

7. Find $[\sqrt{19992000}]$ where $[x]$ is the greatest integer less than or equal to x.

8. Bobo the clown was juggling his spherical cows again when he realized that when he drops a cow is related to how many cows he started off juggling. If he juggles 1, he drops it after 64 seconds. When juggling 2, he drops one after 55 seconds, and the other 55 seconds later. In fact, he was able to create the following table:

cows started juggling	1	2	3	4	5	6	7	8	9	10	11
seconds he drops after	64	55	47	40	33	27	22	18	14	13	12
cows started juggling	12	13	14	15	16	17	18	19	20	21	22
seconds he drops after	11	10	9	8	7	6	5	4	3	2	1

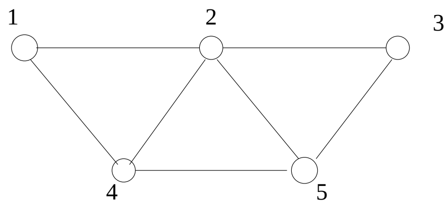
   He can only juggle up to 22 cows. To juggle the cows the longest, what number of cows should he start off juggling? How long (in minutes) can he juggle for?

9. Edward's formula for the stock market predicts correctly that the price of HMMT is directly proportional to a secret quantity x and inversely proportional to y, the number of hours he slept the night before. If the price of HMMT is \$12 when $x = 8$ and $y = 4$, how many dollars does it cost when $x = 4$ and $y = 8$?

10. Bob has a 12 foot by 20 foot garden. He wants to put fencing around it to keep out the neighbor's dog. Normal fenceposts cost \$2 each while strong ones cost \$3 each. If he needs one fencepost for every 2 feet and has \$70 to spend on the fenceposts, what is the largest number of strong fenceposts he can buy?

11. If $a@b = \frac{a+b}{a-b}$, find n such that $3@n = 3$.

12. In 2020, the United States admits North Mathematica as the 51st state. It consists of 5 islands joined by bridges as shown. Is it possible to cross all the bridges without doubling over? If so, what is the difference (positive) between the number of the start island and the number of the end island?



13. How many permutations of 123456 have exactly one number in the correct place?

14. The author of this question was born on April 24, 1977. What day of the week was that?

15. Which is greater: $(3^5)^{(5^3)}$ or $(5^3)^{(3^5)}$?

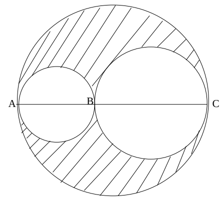
16. Joe bikes x miles East at 20 mph to his friend's house. He then turns South and bikes x miles at 20 mph to the store. Then, Joe turns East again and goes to his grandma's house at 14 mph. On this last leg, he has to carry flour he bought for her at the store. Her house is 2 more miles from the store than Joe's friend's house is from the store. Joe spends a total of 1 hour on the bike to get to his grandma's house. If Joe then rides straight home in his grandma's helicopter at 78 mph, how many minutes does it take Joe to get home from his grandma's house?

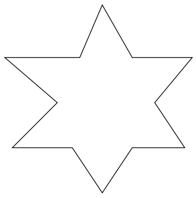
17. In how many distinct ways can the letters of STANTON be arranged?

18. You use a lock with four dials, each of which is set to a number between 0 and 9 (inclusive). You can never remember your code, so normally you just leave the lock with each dial one higher than the correct value. Unfortunately, last night someone changed all the values to 5. All you remember about your code is that none of the digits are prime, 0, or 1, and that the average value of the digits is 5. How many combinations will you have to try?

19. Eleven pirates find a treasure chest. When they split up the coins in it, they find that there are 5 coins left. They throw one pirate overboard and split the coins again, only to find that there are 3 coins left over. So, they throw another pirate over and try again. This time, the coins split evenly. What is the least number of coins there could have been?

20. Given: AC has length 5, semicircle AB has radius 1, semicircle BC has diameter 3. What percent of the the big circle is shaded?





21. Find the area of the six-pointed star if all edges are of length s, all acute angles are 60° and all obtuse angles are 240°.

22. An equilateral triangle with sides of length 4 has an isosceles triangle with the same base and half the height cut out of it. Find the remaining area.

23. What are the last two digits of $7^{7^{7^7}}$?

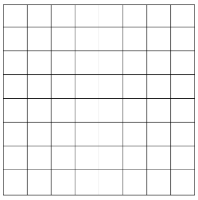
24. Peter is randomly filling boxes with candy. If he has 10 pieces of candy and 5 boxes in a row labeled A, B, C, D, and E, how many ways can he distribute the candy so that no two adjacent boxes are empty?

25. How many points does one have to place on a unit square to guarantee that two of them are strictly less than 1/2 unit apart?

26. Janet is trying to find Tim in a Cartesian forest. Janet is $5\sqrt{2}$ miles from (0,0), $\sqrt{41}$ miles from (1,0), and $\sqrt{61}$ miles from (0,1). Tim is $\sqrt{65}$ miles from (0,0), $2\sqrt{13}$ miles from (1,0), and $\sqrt{58}$ miles from (0,1). How many miles apart are Janet and Tim?

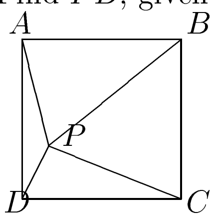
Geometry
Rice Mathematics Tournament 2000

1. How many rectangles are there on an $8x8$ checkerboard?

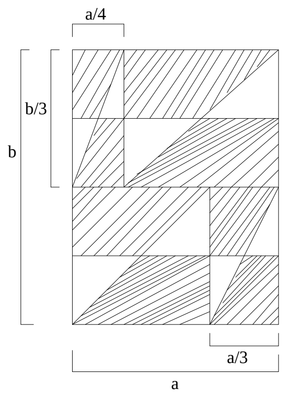


2. In a triangle the sum of squares of the sides is 96. What is the maximum possible value of the sum of the medians?

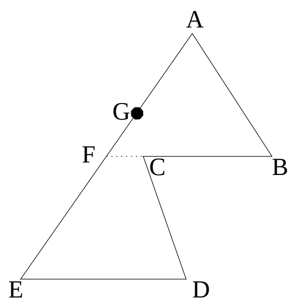
3. Find PB, given that $PA = 15, PC = 20, PD = 7$, and $ABCD$ is a square.



4. Find the total area of the non-triangle regions in the figure below (the shaded area).



5. Side $\overline{AB} = 3$. $\triangle ABF$ is an equilateral triangle. Side $\overline{DE} = \overline{AB} = \overline{AF} = \overline{GE}$. $\angle FED = 60\circ$. $\overline{FG} = 1$. Calculate the area of $ABCDE$.

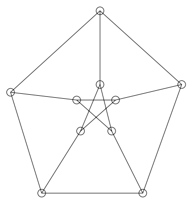


6. What is the area of the largest circle contained in an equilateral triangle of area $8\sqrt{3}$?

7. Let ABC be a triangle inscribed in the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. If its centroid is the origin (0,0), find its area.

8. A sphere is inscribed inside a pyramid with a square as a base whose height is $\frac{\sqrt{15}}{2}$ times the length of one edge of the base. A cube is inscribed inside the sphere. What is the ratio of the volume of the pyramid to the volume of the cube?

9. How many hexagons are in the figure below with vertices on the given vertices? (Note that a hexagon need not be convex, and edges may cross!)



10. Let C_1 and C_2 be two concentric reflective hollow metal spheres of radius R and $R\sqrt{3}$ respectively. From a point P on the surface of C_2, a ray of light is emitted inward at 30° from the radial direction. The ray eventually returns to P. How many **total** reflections off of C_1 and C_2 does it take?

Guts
HMMT 2000

School: __________________________

Problem Gu1 [4]

The sum of 3 real numbers is known to be zero. If the sum of their cubes is π^e, what is their product equal to?

____

Problem Gu2 [5]

If $X = 1 + x + x^2 + x^3 + \ldots$ and $Y = 1 + y + y^2 + y^3 + \ldots$, what is $1 + xy + x^2y^2 + x^3y^3 + \ldots$ in terms of X and Y only?

____

School: __________________________

Problem Gu3 [±7]

Using 3 colors, red, blue and yellow, how many different ways can you color a cube (modulo rigid rotations)?

____

Problem Gu4 [5]

Let ABC be a triangle and H be its orthocentre. If it is given that B is $(0, 0)$, C is $(1, 2)$ and H is $(5, 0)$, find A.

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School: __________________________

Problem Gu5 [3]

Find all natural numbers n such that n equals the cube of the sum of its digits.

____

Problem Gu6 [±10]

If integers m, n, k satisfy $m^2 + n^2 + 1 = kmn$, what values can k have?

____

School: ____________________

Problem Gu7 [7]

Suppose you are give a fair coin and a sheet of paper with the polynomial x^m written on it. Now for each toss of the coin, if heads show up, you much erase the polynomial x^r (where r is going to change with time – initially it is m) written on the paper and replace it by x^{r-1}. If tails show up, replace it by x^{r+1}. What is the expected value of the polynomial I get after m such tosses? (Note: this is a different concept from the most probable value)

____

Problem Gu8 [±4]

Johny's father tells him : "I am twice as old as you will be seven years from the time I was thrice as old as you were". What is Johny's age?

____

School: ____________________

Problem Gu9 [6]

A cubic polynomial f satisfies $f(0) = 0, f(1) = 1, f(2) = 2, f(3) = 4$. What is $f(5)$?

____

Problem Gu10 [7]

What is the total surface area of an ice cream cone, radius R, height H, with a spherical scoop of ice cream of radius r on top? (Given $R < r$)

____

School: ____________________

Problem Gu11 [6]

Let M be the maximum possible value of $x_1x_2 + x_2x_3 + ... + x_5x_1$ where $x1, x2, ..., x5$ is a permutation of $(1, 2, 3, 4, 5)$ and let N be the number of permutations for which this maximum is attained. Evaluate $M + N$.

____

Problem Gu12 [9]

Calculate the number of ways of choosing 4 numbers from the set $\{1, 2, ..., 11\}$ such that at least 2 of the numbers are consecutive.

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School: _________________________

Problem Gu13 [±4]

Determine the remainder when $(x^4 - 1)(x^2 - 1)$ is divided by $1 + x + x^2$.

____

Problem Gu14 [7]

$ABCD$ is a cyclic quadrilateral inscribed in a circle of radius 5, with $AB = 6, BC = 7, CD = 8$. Find AD.

____

School: _________________________

Problem Gu15 [8]

Find the number of ways of filling a 8×8 grid with 0's and X's so that the number of 0's in each row and each column is odd.

____

Problem Gu16 [5]

Solve for real x, y:

$$\begin{aligned} x + y &= 2 \\ x^5 + y^5 &= 82 \end{aligned}$$

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School: _________________________

Problem Gu17 [5]

Find the highest power of 3 dividing $\binom{666}{333}$

____

Problem Gu18 [±5]

What is the value of $\sum_{n=1}^{\infty} (\tan^{-1}\sqrt{n} - \tan^{-1}\sqrt{n+1})$?

____

School: ____________________________

Problem Gu19 [3]

Define $a * b = \frac{a-b}{1-ab}$. What is $(1 * (2 * (3 * ...(n * (n+1))...)))$?

____

Problem Gu20 [6]

What is the minimum possible perimeter of a triangle two of whose sides are along the x-and y-axes and such that the third contains the point (1,2)?

____

School: ____________________________

Problem Gu21 [8]

How many ways can you color a necklace of 7 beads with 4 colors so that no two adjacent beads have the same color?

____

Problem Gu22 [6]

Find the smallest n such that 2^{2000} divides n!

____

School: ____________________________

Problem Gu23 [5]

How many 7-digit numbers with distinct digits can be made that are divisible by 3?

____

Problem Gu24 [±3]

At least how many moves must a knight make to get from one corner of a chessboard to the opposite corner?

____

School: ______________________

Problem Gu25 [4]

Find the next number in the sequence 131, 111311, 311321, 1321131211,

____

Problem Gu26 [5]

What are the last 3 digits of $1! + 2! + ... + 100!$

____

School: ______________________

Problem Gu27 [±6]

What is the smallest number that can be written as a sum of 2 squares in 3 ways?

____

Problem Gu28 [8]

What is the smallest possible volume to surface ratio of a solid cone with height = 1 unit?

____

School: ______________________

Problem Gu29 [±9]

What is the value of $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+...}}}}}$

____

Problem Gu30 [7]

$ABCD$ is a unit square. If $\angle PAC = \angle PCD$, find the length BP.

____

School: ______________________

Problem Gu31 [10]

Given collinear points A, B, C such that $AB = BC$. How can you construct a point D on AB such that $AD = 2DB$, using only a straightedge? (You are not allowed to measure distances)

____

Problem Gu32 [7]

How many (nondegenerate) tetrahedrons can be formed from the vertices of an n-dimensional hypercube?

____

School: ______________________

Problem Gu33 [±5]

Characterise all numbers that cannot be written as a sum of 1 or more consecutive odd numbers.

____

Problem Gu34 [±6]

What is the largest n such that $n! + 1$ is a square?

____

School: ______________________

Problem Gu35 [4]

If $1 + 2x + 3x^2 + ... = 9$, find x.

____

Problem Gu36 [6]

If, in a triangle of sides a, b, c, the incircle has radius $\frac{b+c-a}{2}$, what is the magnitude of angle A?

____

School: ________________________

Problem Gu37 [9]

A cone with semivertical angle 30° is half filled with water. What is the angle it must be tilted by so that water starts spilling?

____

Problem Gu38 [4]

What is the largest number you can write with three 3's and three 8's, using only symbols +,-,/,× and exponentiation?

____

School: ________________________

Problem Gu39 [±8]

If $r = 1/3$, what is the value, rounded to 100 decimal digits, of

$$\sum_{n=0}^{7} \frac{2^n}{1 + x^{2^n}}$$

____

Problem Gu40 [±10]

Let $\phi(n)$ denote the number of positive integers less than equal to n and relatively prime to n. find all natural numbers n and primes p such that $\phi(n) = \phi(np)$.

____

School: ________________________

Problem Gu41 [7]

A observes a building of height h at an angle of inclination α from a point on the ground. After walking a distance a toward it, the angle is now 2α, and walking a further distance b causes to increase to 3α. Find h in terms of a and b.

____

Problem Gu42 [4]

A $n \times n$ magic square contains numbers from 1 to n^2 such that the sum of every row and every column is the same. What is this sum?

____

School: ______________________

Problem Gu43 [6]

Box A contains 3 black and 4 blue marbles. Box B has 7 black and 1 blue, whereas Box C has 2 black, 3 blue and 1 green marble. I close my eyes and pick two marbles from 2 different boxes. If it turns out that I get 1 black and 1 blue marble, what is the probability that the black marble is from box A and the blue one is from C?

____

Problem Gu44 [6]

A function $f : Z \to Z$ satisfies

$$\begin{aligned} f(x+4) - f(x) &= 8x + 20 \\ f(x^2-1) &= (f(x)-x)^2 + x^2 - 2 \end{aligned} \tag{1}$$

Find $f(0)$ and $f(1)$.

____

School: ______________________

Problem Gu45 [7]

Find all positive integers x for which there exists a positive integer y such that $\binom{x}{y} = 1999000$

____

Problem Gu46 [6]

For what integer values of n is $1 + n + n^2/2 + ... + n^n/n!$ an integer?

____

School: ______________________

Problem Gu47

Find an $n < 100$ such that $n \cdot 2^n - 1$ is prime. Score will be $n - 5$ for correct n, $5 - n$ for incorrect n (0 points for answer < 5).

____

Oral Event
HMMT 2000

1. [25] Find all integer solutions to $m^2 = n^6 + 1$.

2. [30] How many positive solutions are there to $x^{10} + 7x^9 + 14x^8 + 1729x^7 - 1379x^6 = 0$? How many positive integer solutions ?

3. [35] Suppose the positive integers a, b, c satisfy $a^n + b^n = c^n$, where n is a positive integer greater than 1. Prove that $a, b, c > n$. (Note: Fermat's Last Theorem may *not* be used)

4. [40] On an $n \times n$ chessboard, numbers are written on each square so that the number in a square is the average of the numbers on the adjacent squares. Show that all the numbers are the same.

5. [45] Show that it is impossible to find a triangle in the plane with all integer coordinates such that the lengths of the sides are all odd.

6. [45] Prove that every multiple of 3 can be written as a sum of four cubes (positive or negative).

7. [45] A regular tetrahedron of volume 1 is filled with water of total volume 7/16. Is it possible that the center of the tetrahedron lies on the surface of the water? How about in a cube of volume 1?

8. [55] f is a polynomial of degree n with integer coefficients and $f(x) = x^2 + 1$ for $x = 1, 2, ..., n$. What are the possible values for $f(0)$?

9. [60] Let $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ and $\vec{v_5}$ be vectors in three dimensions. Show that for some i, j in $1, 2, 3, 4, 5$, $\vec{v_i} \cdot \vec{v_j} \geq 0$.

10. [75] 23 frat brothers are sitting in a circle. One, call him Alex, starts with a gallon of water. On the first turn, Alex gives each person in the circle some rational fraction of his water. On each subsequent turn, every person with water uses the same scheme as Alex did to distribute his water, but in relation to themselves. For instance, suppose Alex gave 1/2 and 1/6 of his water to his left and right neighbors respectively on the first turn and kept 1/3 for himself. On each subsequent turn everyone gives 1/2 and 1/6 of the water they started the turn with to their left and right neighbours (resp.) and keep the final third for themselves. After 23 turns, Alex again has a gallon of water. What possibilities are there for the scheme he used in the first turn? (Note: you may find it useful to know that $1 + x + x^2 + ... + x^{22}$ has no polynomial factors with rational coefficients.)

Power Test
Rice Mathematics Tournament 2000

1. A derangement of a string of distinct elements is a rearrangement of the string such that no element appears in its original position. For example, BCA is a derangement of ABC. D_n represents the number of derangements of any string composed of n distinct elements. $D_2 = 1$ and $D_3 = 2$.

   (a) What are D_4 and D_5?

   (b) How many derangments are there of the string ABCDEFG?

   (c) Find a recursive relationship for D_n in terms of the previous two terms (D_{n-1} and D_{n-2}).

   (d) Find a recursive relationship for D_n in terms of only the previous term, D_{n-1}.

2. Find the number of 3-letter "words" that use letters from the 10-letter set $\{A, B, C, ..., J\}$ in which all letters are different and the letters appear in alphabetical order.

3. Assume that a hand of thirteen cards is dealt from a randomized deck of 52 cards. Let A be the probability that the hand contains two aces. Let B be the probability that it contains two aces if you already know it contains at least one ace. Let C be the probability that it contains at least two aces if you already know it contains an ace of hearts. Write down the inequality relationship between A,B and C, i.e. one possibility is $A = B > C$.

4. Find the number of rearrangments of 12345 (including 12345) such that none of the following is true: 1 is in position 5, 2 is in position 1, 3 is in position 2, 4 is in position 4, and 5 is in position 3.

5. Assume that nobody has a birthday on February 29th. How large must a group be so that there is a greater than 50% chance that at least 2 members have the same birthday?

6. Find the number of combinations of length k that use elements from a set of n distinct elements, allowing repetition.

7. Find the number of combinations of length k that use elements from a given set of n distinct elements, allowing repetition and with no missing elements. (Obviously, k must be greater than n)

8. An election takes place between two candidates. Candidate A wins by a vote of 1032 to 971. If the votes are counted one at a time and in a random order, determine the probability that the winner was never behind at any point in the counting.

9. Evaluate the sum $\binom{100}{0} + \frac{1}{2}\binom{100}{1} + \frac{1}{3}\binom{100}{2} + \ldots + \frac{1}{101}\binom{100}{100}$.

10. Find the number of distributions of a given set of m identical balls into a given set of n distinct boxes.

11. Find the number of distributions of a given set of m distinct balls into a given set of n distinct boxes if each box must contain a specific number of balls (m_i is the number of balls to be put into box i). Please state the answer with only factorials (not in combinatorial notation).

12. Find the coefficient of X^2Y^3 in each of the following.
    (a) $(X + Y + 1)^7$
    (b) $(X^2 + Y - 1)^7$

13. Find the number of "words" of length m from a set of n letters, if each letter must occur at least once in each word.

14. Find the number of ways to distribute seven distinct balls into three distinct boxes if each box must contain a different number of balls, allowing an empty box.

15. How many ways can a class of 10 students be divided into two groups of 3 and 1 group of 4?

16. Find the number of subsets A of the set of digits $\{0, 1, 2, 3, \ldots, 9\}$ such that A contains no two consecutive digits. Hint: Find a better statement of the problem; find a recursive formula, and then attempt to solve the problem for the number of digits given.

17. If we are trying to find the number of words of length m from a given set of n letters, with each letter occuring at least once in each word, let us call the answer $T(m, n)$. This is equivalent to finding the number of distribution of a set of m distinct balls into a set of n distinct boxes, if no boxes can be empty. $T(m, n)$ is the sum of all possible partitions of the balls (i.e. we sum all possible ways of putting the balls into boxes (4 in box 1, 2 in box 2, 1 in box 3 for example)). More precisely, if we call m_i to be the number of balls in box i, then $T(m, n) = \sum_{m_1,m_2,m_3>=1}^{m_1+m_2+m_3=m} \frac{m!}{m_1!m_2!m_3!...m_n!}$. For example, $T(3, 2) = \frac{3!}{1!2!} + \frac{3!}{2!1!} = 3 + 3 = 6$. Find a recursive pattern for $T(m, n)$ in terms of previous terms (previous meaning a smaller m, a smaller n, or both). Hint: set up a sort of "Pascal's Triangle" for $T(m, n)$. Prove your answer using words.

18. You have an infinite number of 1 cent, 2 cent, and 5 cent stamps. You are trying to post a letter that requires n cents of postage stamps, where $n > 8$. Let $a(n)$ be the number of sequences of stamps that give exactly the required postage of n cents (i.e. order matters). Find $a(n)$ in terms of previous terms of the sequence of a's, using as few previous terms as possible.

19. Suppose we have n lines in a plane in general position, which means that none are parallel to each other and that no three of these lines intersect at a single point. Find the number of regions that these lines divide the plane into...
    (a) in a recursive form.
    (b) in a nonrecursive formula.

20. Find the 2000th positive integer that is not the difference between any two integer squares.

Team Test
Rice Mathematics Tournament 2000

1. You are given a number, and round it to the nearest thousandth, round this result to nearest hundredth, and round this result to the nearest tenth. If the final result is .7, what is the smallest number you could have been given? As is customary, 5's are always rounded up. Give the answer as a decimal.
2. The price of a gold ring in a certain universe is proportional to the square of its purity and the cube of its diameter. The purity is inversely proportional to the square of the depth of the gold mine and directly proportional to the square of the price, while the diameter is determined so that it is proportional to the cube root of the price and also directly proportional to the depth of the mine. How does the price vary solely in terms of the depth of the gold mine?
3. Find the sum of all integers from 1 to 1000 inclusive which contain at least one 7 in their digits, i.e. find $7 + 17 + ... + 979 + 987 + 997$.
4. All arrangements of letters VNNWHTAAIE are listed in lexicographic (dictionary) order. If AAEHINNTVW is the first entry, what entry number is VANNAWHITE?
5. Given $\cos(\alpha+\beta)+\sin(\alpha-\beta)=0$, $\tan\beta=\frac{1}{2000}$, find $\tan\alpha$.
6. If α is a root of $x^3 - x - 1 = 0$, compute the value of $\alpha^{10} + 2\alpha^8 - \alpha^7 - 3\alpha^6 - 3\alpha^5 + 4\alpha^4 + 2\alpha^3 - 4\alpha^4 - 6\alpha - 17$.
7. 8712 is an integral multiple of its reversal, 2178, as 8712=4*2178. Find another 4-digit number which is a non-trivial integral multiple of its reversal.
8. A woman has \$1.58 in pennies, nickels, dimes, quarters, half-dollars and silver dollars. If she has a different number of coins of each denomination, how many coins does she have?
9. Find all positive primes of the form $4x^4 + 1$, for x an integer.
10. How many times per day do at least two of the three hands on a clock coincide?
11. Find all polynomials $f(x)$ with integer coefficients such that the coefficients of both $f(x)$ and $[f(x)]^3$ lie in the set $\{0, 1, -1\}$
12. At a dance, Abhinav starts from point $(a, 0)$ and moves along the negative x direction with speed v_a, while Pei-Hsin starts from $(0, b)$ and glides in the negative y-direction with speed v_b. What is the distance of closest approach between the two?
13. Let $P_1, P_2, \ldots, P_n$ be a convex n-gon. If all lines P_iP_j are joined, what is the maximum possible number of intersections in terms of n obtained from strictly inside the polygon?
14. Define a sequence $<x_n>$ of real numbers by specifying an initial x_0 and by the recurrence $x_{n+1} = \frac{1+x_n}{1-x_n}$. Find x_n as a function of x_0 and n, in closed form. There may be multiple cases.
15. $\lim_{n\to\infty} nr\sqrt[2]{1-\cos\frac{2\pi}{n}} = ?$

Advanced Topics

Harvard-MIT Math Tournament
February 27, 1999

1. One of the receipts for a math tournament showed that 72 identical trophies were purchased for \$-99.9-, where the first and last digits were illegible. How much did each trophy cost?

2. Stacy has d dollars. She enters a mall with 10 shops and a lottery stall. First she goes to the lottery and her money is doubled, then she goes into the first shop and spends 1024 dollars. After that she alternates playing the lottery and getting her money doubled (Stacy always wins) then going into a new shop and spending \$1024. When she comes out of the last shop she has no money left. What is the minimum possible value of d?

3. An unfair coin has the property that when flipped four times, it has the same nonzero probability of turning up 2 heads and 2 tails (in any order) as 3 heads and 1 tail (in any order). What is the probability of getting a head in any one flip?

4. You are given 16 pieces of paper numbered 16, 15, ..., 2, 1 in that order. You want to put them in the order 1, 2, ..., 15, 16 switching only two adjacent pieces of paper at a time. What is the minimum number of switches necessary?

5. For any finite set S, let $f(S)$ be the sum of the elements of S (if S is empty then $f(S) = 0$). Find the sum over all subsets E of S of $\frac{f(E)}{f(S)}$ for $S = \{1, 2, ..., 1999\}$.

6. Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?

7. Find an ordered pair (a, b) of real numbers for which $x^2 + ax + b$ has a non-real root whose cube is 343.

8. Let C be a circle with two diameters intersecting at an angle of 30 degrees. A circle S is tangent to both diameters and to C, and has radius 1. Find the largest possible radius of C.

9. As part of his effort to take over the world, Edward starts producing his own currency. As part of an effort to stop Edward, Alex works in the mint and produces 1 counterfeit coin for every 99 real ones. Alex isn't very good at this, so none of the counterfeit coins are the right weight. Since the mint is not perfect, each coin is weighed before leaving. If the coin is not the right weight, then it is sent to a lab for testing. The scale is accurate 95% of the time, 5% of all the coins minted are sent to the lab, and the lab's test is accurate 90% of the time. If the lab says a coin is counterfeit, what is the probability that it really is?

10. Find the minimum possible value of the largest of xy, $1 - x - y + xy$, and $x + y - 2xy$ if $0 \leq x \leq y \leq 1$.

Algebra

Harvard-MIT Math Tournament
February 27, 1999

1. If $a@b = \frac{a^3-b^3}{a-b}$, for how many real values of a does $a@1 = 0$?

2. For what single digit n does 91 divide the 9-digit number $12345n789$?

3. Alex is stuck on a platform floating over an abyss at 1 ft/s. An evil physicist has arranged for the platform to fall in (taking Alex with it) after traveling 100ft. One minute after the platform was launched, Edward arrives with a second platform capable of floating all the way across the abyss. He calculates for 5 seconds, then launches the second platform in such a way as to maximize the time that one end of Alex's platform is between the two ends of the new platform, thus giving Alex as much time as possible to switch. If both platforms are 5 ft long and move with constant velocity once launched, what is the speed of the second platform (in ft/s)?

4. Find all possible values of $\frac{d}{a}$ where $a^2 - 6ad + 8d^2 = 0$, $a \neq 0$.

5. You are trapped in a room with only one exit, a long hallway with a series of doors and land mines. To get out you must open all the doors and disarm all the mines. In the room is a panel with 3 buttons, which conveniently contains an instruction manual. The red button arms a mine, the yellow button disarms two mines and closes a door, and the green button opens two doors. Initially 3 doors are closed and 3 mines are armed. The manual warns that attempting to disarm two mines or open two doors when only one is armed/closed will reset the system to its initial state. What is the minimum number of buttons you must push to get out?

6. Carl and Bob can demolish a building in 6 days, Anne and Bob can do it in 3, Anne and Carl in 5. How many days does it take all of them working together if Carl gets injured at the end of the first day and can't come back? Express your answer as a fraction in lowest terms.

7. Matt has somewhere between 1000 and 2000 pieces of paper he's trying to divide into piles of the same size (but not all in one pile or piles of one sheet each). He tries 2, 3, 4, 5, 6, 7, and 8 piles but ends up with one sheet left over each time. How many piles does he need?

8. If $f(x)$ is a monic quartic polynomial such that $f(-1) = -1$, $f(2) = -4$, $f(-3) = -9$, and $f(4) = -16$, find $f(1)$.

9. How many ways are there to cover a 3×8 rectangle with 12 identical dominoes?

10. Pyramid $EARLY$ is placed in (x, y, z) coordinates so that $E = (10, 10, 0)$, $A = (10, -10, 0)$, $R = (-10, -10, 0)$, $L = (-10, 10, 0)$, and $Y = (0, 0, 10)$. Tunnels are drilled through the pyramid in such a way that one can move from (x, y, z) to any of the 9 points $(x, y, z-1)$, $(x \pm 1, y, z-1)$, $(x, y \pm 1, z-1)$,$(x \pm 1, y \pm 1, z-1)$. Sean starts at Y and moves randomly down to the base of the pyramid, choosing each of the possible paths with probability $\frac{1}{9}$ each time. What is the probability that he ends up at the point $(8, 9, 0)$?

Calculus

Harvard-MIT Math Tournament
February 27, 1999

1. Find all twice differentiable functions $f(x)$ such that $f''(x) = 0$, $f(0) = 19$, and $f(1) = 99$.

2. A rectangle has sides of length $\sin x$ and $\cos x$ for some x. What is the largest possible area of such a rectangle?

3. Find

$$\int_{-4\pi\sqrt{2}}^{4\pi\sqrt{2}} \left(\frac{\sin x}{1+x^4} + 1\right) dx.$$

4. f is a continuous real-valued function such that $f(x+y) = f(x)f(y)$ for all real x, y. If $f(2) = 5$, find $f(5)$.

5. Let $f(x) = x + \cfrac{1}{2x + \cfrac{1}{2x + \cfrac{1}{2x + \ddots}}}$ for $x > 0$. Find $f(99)f'(99)$.

6. Evaluate $\frac{d}{dx}(\sin x - \frac{4}{3}\sin^3 x)$ when $x = 15$.

7. If a right triangle is drawn in a semicircle of radius $1/2$ with one leg (not the hypotenuse) along the diameter, what is the triangle's maximum possible area?

8. A circle is randomly chosen in a circle of radius 1 in the sense that a point is randomly chosen for its center, then a radius is chosen at random so that the new circle is contained in the original circle. What is the probability that the new circle contains the center of the original circle?

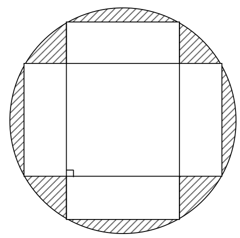
9. What fraction of the Earth's volume lies above the 45 degrees north parallel? You may assume the Earth is a perfect sphere. The volume in question is the smaller piece that we would get if the sphere were sliced into two pieces by a plane.

10. Let A_n be the area outside a regular n-gon of side length 1 but inside its circumscribed circle, let B_n be the area inside the n-gon but outside its inscribed circle. Find the limit as n tends to infinity of $\frac{A_n}{B_n}$.

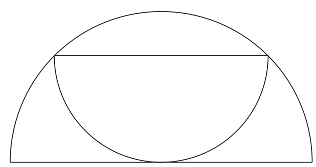
Geometry

Harvard-MIT Math Tournament
February 27, 1999

1. Two 10×24 rectangles are inscribed in a circle as shown. Find the shaded area.



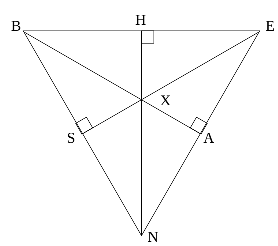
2. A semicircle is inscribed in a semicircle of radius 2 as shown. Find the radius of the smaller semicircle.



3. In a cube with side length 6, what is the volume of the tetrahedron formed by any vertex and the three vertices connected to that vertex by edges of the cube?

4. A cross-section of a river is a trapezoid with bases 10 and 16 and slanted sides of length 5. At this section the water is flowing at π mph. A little ways downstream is a dam where the water flows through 4 identical circular holes at 16 mph. What is the radius of the holes?

5. In triangle BEN shown below with its altitudes intersecting at X, $NA = 7, EA = 3, AX = 4$, and $NS = 8$. Find the area of BEN.



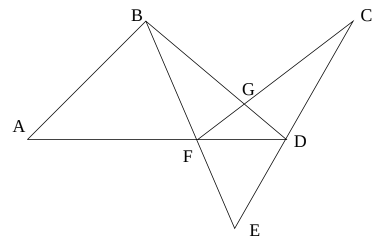
6. A sphere of radius 1 is covered in ink and rolling around between concentric spheres of radii 3 and 5. If this process traces a region of area 1 on the larger sphere, what is the area of the region traced on the smaller sphere?

7. A dart is thrown at a square dartboard of side length 2 so that it hits completely randomly. What is the probability that it hits closer to the center than any corner, but within a distance 1 of a corner?

8. Squares $ABKL, BCMN, CAOP$ are drawn externally on the sides of a triangle ABC. The line segments KL, MN, OP, when extended, form a triangle $A'B'C'$. Find the area of $A'B'C'$ if ABC is an equilateral triangle of side length 2.

9. A regular tetrahedron has two vertices on the body diagonal of a cube with side length 12. The other two vertices lie on one of the face diagonals not intersecting that body diagonal. Find the side length of the tetrahedron.

10. In the figure below, $AB = 15, BD = 18, AF = 15, DF = 12, BE = 24$, and $CF = 17$. Find $BG : FG$.



Oral Event

Harvard-MIT Math Tournament
February 27, 1999

1. [25] Start with an angle of 60° and bisect it, then bisect the lower 30° angle, then the upper 15° angle, and so on, always alternating between the upper and lower of the previous two angles constructed. This process approaches a limiting line that divides the original 60° angle into two angles. Find the measure (degrees) of the smaller angle.

2. [25] Alex, Pei-Hsin, and Edward got together before the contest to send a mailing to all the invited schools. Pei-Hsin usually just stuffs the envelopes, but if Alex leaves the room she has to lick them as well and has a 25% chance of dying from an allergic reaction before he gets back. Licking the glue makes Edward a bit psychotic, so if Alex leaves the room there is a 20% chance that Edward will kill Pei-Hsin before she can start licking envelopes. Alex leaves the room and comes back to find Pei-Hsin dead. What is the probability that Edward was responsible?

3. [30] If x, y, and z are distinct positive integers such that $x^2 + y^2 = z^3$, what is the smallest possible value of $x + y + z$.

4. [35] Evaluate $\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$, where $\cos\theta = \frac{1}{5}$.

5. [45] Let, r be the inradius of triangle ABC. Take a point D on side BC, and let r_1 and r_2 be the inradii of triangles ABD and ACD. Prove that r, r_1, and r_2 can always be the side lengths of a triangle.

6. [45] You want to sort the numbers 5 4 3 2 1 using block moves. In other words, you can take any set of numbers that appear consecutively and put them back in at any spot as a block. For example, *6 5 3* 4 2 1 $\rightarrow$ 4 2 *6 5 3* 1 is a valid block move for 6 numbers. What is the minimum number of block moves necessary to get 1 2 3 4 5?

7. [55] Evaluate $\sum_{n=1}^{\infty} \frac{n^5}{n!}$.

8. [55] What is the smallest square-free composite number that can divide a number of the form $4242\ldots42 \pm 1$?

9. [60] You are somewhere on a ladder with 5 rungs. You have a fair coin and an envelope that contains either a double-headed coin or a double-tailed coin, each with probability 1/2. Every minute you flip a coin. If it lands heads you go up a rung, if it lands tails you go down a rung. If you move up from the top rung you win, if you move down from the bottom rung you lose. You can open the envelope at any time, but if you do then you must immediately flip that coin once, after which you can use it or the fair coin whenever you want. What is the best strategy (i.e. on what rung(s) should you open the envelope)?

10. [75] A, B, C, D, and E are relatively prime integers (i.e., have no single common factor) such that the polynomials $5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E$ and $10Ax^3 + 6Bx^2 + 3Cx + D$ together have 7 distinct integer roots. What are all possible values of A? Your team has been given a sealed envelope that contains a hint for this problem. If you open the envelope, the value of this problem decreases by 20 points. To get full credit, give the sealed envelope to the judge before presenting your solution.

Team

Harvard-MIT Math Tournament
February 27, 1999

1. A combination lock has a 3 number combination, with each number an integer between 0 and 39 inclusive. Call the numbers n_1, n_2, and n_3. If you know that n_1 and n_3 leave the same remainder when divided by 4, and n_2 and $n_1 + 2$ leave the same remainder when divided by 4, how many possible combinations are there?

2. A ladder is leaning against a house with its lower end 15 feet from the house. When the lower end is pulled 9 feet farther from the house, the upper end slides 13 feet down. How long is the ladder (in feet)?

3. How many non-empty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ have exactly k elements and do not contain the element k for some $k = 1, 2, ..., 8$.

4. Consider the equation $FORTY + TEN + TEN = SIXTY$, where each of the ten letters represents a distinct digit from 0 to 9. Find all possible values of $SIXTY$.

5. If a and b are randomly selected real numbers between 0 and 1, find the probability that the nearest integer to $\frac{a-b}{a+b}$ is odd.

6. Reduce the number $\sqrt[3]{2+\sqrt{5}} + \sqrt[3]{2-\sqrt{5}}$.

7. Let $\frac{1}{1-x-x^2-x^3} = \sum_{i=0}^{\infty} a_n x^n$, for what positive integers n does $a_{n-1} = n^2$?

8. Find all the roots of $(x^2 + 3x + 2)(x^2 - 7x + 12)(x^2 - 2x - 1) + 24 = 0$.

9. Evaluate $\sum_{n=2}^{17} \frac{n^2+n+1}{n^4+2n^3-n^2-2n}$.

10. If 5 points are placed in the plane at lattice points (i.e. points (x, y) where x and y are both integers) such that no three are collinear, then there are 10 triangles whose vertices are among these points. What is the minimum possible number of these triangles that have area greater than 1/2?

11. Circles C_1, C_2, C_3 have radius 1 and centers O, P, Q respectively. C_1 and C_2 intersect at A, C_2 and C_3 intersect at B, C_3 and C_1 intersect at C, in such a way that $\angle APB = 60°$, $\angle BQC = 36°$, and $\angle COA = 72°$. Find angle ABC (degrees).

12. A fair coin is flipped every second and the results are recorded with 1 meaning heads and 0 meaning tails. What is the probability that the sequence 10101 occurs before the first occurance of the sequence 010101?

1998 HMMT Advanced Topics Event

1. Evaluate $\sin(1998°+237°)\sin(1998°-1653°)$.

2. How many values of x, $-19 < x < 98$ satisfy $\cos^2 x + 2\sin^2 x = 1$?

3. Find the sum of the infinite series $1 + 2(\frac{1}{1998}) + 3(\frac{1}{1998})^2 + 4(\frac{1}{1998})^3 + K$

4. Find the range of $f(A) = \dfrac{\sin A(3\cos^2 A + \cos^4 A + 3\sin^2 A + \sin^2 A\cos^2 A)}{\tan A(\sec A - \sin A\tan A)}$ if $A \neq \dfrac{n\pi}{2}$.

5. How many positive integers less than 1998 are relatively prime to 1547? (Two integers are relatively prime if they have no common factors besides 1.)

6. In the diagram below, how many distinct paths are there from January 1 to December 31, moving from one adjacent dot to the next either to the right, down, or diagonally down to the right?

Jan. 1 ->	*	*	*	*	*	*	*	*	*	*	
	*	*		*	*		*	*		*	
	*	*	*	*	*	*	*	*	*	*	
	*			*			*	*		*	
	*	*	*	*	*	*	*	*	*	*	<-Dec. 31

7. The Houson Association of Mathematics Educators decides to hold a grand forum on mathematics education and invites a number of politicians from around the United States to participate. Around lunch time the politicians decide to play a game. In this game, players can score 19 points for pegging the coordinator of the gathering with a spit ball, 9 points for downing an entire cup of the forum's interpretation of coffee, or 8 points for quoting more than three consecutive words from the speech Senator Bobbo delivered before lunch. What is the product of the two greatest scores that a player *cannot* score in this game?

8. Given any two positive real numbers x and y, then $x \Diamond y$ is a positive real number defined in terms of x and y by some fixed rule. Suppose the operation $x \Diamond y$ satisfies the equations

$(x \cdot y)\Diamond y = x(y \Diamond y)$ and $(x \Diamond 1)\Diamond x = x \Diamond 1$ for all $x, y > 0$. Given that $1 \Diamond 1 = 1$, find $19 \Diamond 98$.

9. Bob's Rice ID number has six digits, each a number from 1 to 9, and any digit can be used any number of times. The ID number satifies the following property: the first two digits is a number divisible by 2, the first three digits is a number divisible by 3, etc. so that the ID number itself is divisible by 6. One ID number that satifies this condition is 123252. How many different possibilities are there for Bob's ID number?

10. In the fourth annual Swirled Series, the Oakland Alphas are playing the San Francisco Gammas. The first game is played in San Francisco and succeeding games alternate in location. San Francisco has a 50% chance of winning their home games, while Oakland has a probability of 60% of winning at home. Normally, the series will stretch on forever until one team gets a three game lead, in which case they are declared the winners. However, after each game in San Francisco there is a 50% chance of an earthquake, which will cause the series to end with the team that has won more games declared the winner. What is the probability that the Gammas will win?

1998 HMMT Algebra Event

1. The cost of 3 hamburgers, 5 milk shakes, and 1 order of fries at a certain fast food restaurant is \$23.50. At the same restaurant, the cost of 5 hamburgers, 9 milk shakes, and 1 order of fries is \$39.50. What is the cost of 2 hamburgers, 2 milk shakes and 2 orders of fries at this restaurant?

2. Bobbo starts swimming at 2 feet/s across a 100 foot wide river with a current of 5 feet/s. Bobbo doesn't know that there is a waterfall 175 feet from where he entered the river. He realizes his predicament midway across the river. What is the minimum speed that Bobbo must increase to make it to the other side of the river safely?

3. Find the sum of every even positive integer less than 233 not divisible by 10.

4. Given that r and s are relatively prime positive integers such that $\frac{r}{s} = \frac{2(\sqrt{2}+\sqrt{10})}{5(\sqrt{(3+\sqrt{5})}}$, find r and s.

5. A man named Juan has three rectangular solids, each having volume 128. Two of the faces of one solid have areas 4 and 32. Two faces of another solid have areas 64 and 16. Finally, two faces of the last solid have areas 8 and 32. What is the minimum possible exposed surface area of the tallest tower Juan can construct by stacking his solids one on top of the other, face to face? (Assume that the base of the tower is not exposed.)

6. How many pairs of positive integers (a,b) with $a \le b$ satisfy $\frac{1}{a} + \frac{1}{b} = \frac{1}{6}$?

7. Given that three roots of $f(x) = x^4 + ax^2 + bx + c$ are 2, -3, and 5, what is the value of $a+b+c$?

8. Find the set of solutions for x in the inequality $\frac{x+1}{x+2} > \frac{3x+4}{2x+9}$ when $x \ne -2, x \ne -\frac{9}{2}$.

9. Suppose $f(x)$ is a rational function such that $3f(\frac{1}{x}) + \frac{2f(x)}{x} = x^2$ for $x \ne 0$. Find $f(-2)$.

10. G. H. Hardy once went to visit Srinivasa Ramanujan in the hospital, and he started the conversation with: "I came here in taxi-cab number 1729. That number seems dull to me, which I hope isn't a bad omen." "Nonsense," said Ramanujan. "The number isn't dull at all. It's quite interesting. It's the smallest number that can be expressed as the sum of two cubes in two different ways." Ramujan had immediately seen that

$1729 = 12^3 + 1^3 = 10^3 + 9^3$. What is the smallest positive integer representable as the sum of the cubes of *three* positive integers in two different ways?

1998 Harvard/MIT Math Tournament

CALCULUS Answer Sheet

Name: ______________________________

School: ______________________ **Grade:** _____

1	____________	**6**	____________
2	____________	**7**	____________
3	____________	**8**	____________
4	____________	**9**	____________
5	____________	**10**	____________

TOTAL: ________

CALCULUS

Question One. [3 points]

Farmer Tim is lost in the densely-forested Cartesian plane. Starting from the origin he walks a sinusoidal path in search of home; that is, after t minutes he is at position $(t, \sin t)$.

Five minutes after he sets out, Alex enters the forest at the origin and sets out in search of Tim. He walks in such a way that after he has been in the forest for m minutes, his position is $(m, \cos t)$.

What is the greatest distance between Alex and Farmer Tim while they are walking in these paths?

Question Two. [3 points]

A cube with sides 1m in length is filled with water, and has a tiny hole through which the water drains into a cylinder of radius 1m.

If the water level in the cube is falling at a rate of $1\ {}^{cm}\!/_{s}$, at what rate is the water level in the cylinder rising?

Question Three. [4 points]

Find the area of the region bounded by the graphs of $y = x^2$, $y = x$, and $x = 2$.

Question Four. [4 points]

Let $f(x) = 1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \ldots$, for $-1 \leq x \leq 1$. Find $\sqrt{e^{\int_0^1 f(x)\,dx}}$.

Question Five. [5 points]

Evaluate $\lim_{x \to 1} x^{\frac{x}{\sin(1-x)}}$.

Question Six. [5 points]

Edward, the author of this test, had to escape from prison to work in the grading room today. He stopped to rest at a place 1,875 feet from the prison and was spotted by a guard with a crossbow.

The guard fired an arrow with an initial velocity of $100\,{}^{ft}\!/_{s}$. At the same time, Edward started running away with an acceleration of $1\,{}^{ft}\!/_{s^2}$. Assuming that air resistance causes the arrow to decelerate at $1\,{}^{ft}\!/_{s^2}$ and that it does hit Edward, how fast was the arrow moving at the moment of impact (in ${}^{ft}\!/_{s}$)?

Question Seven. [5 points]

A parabola is inscribed in equilateral triangle *ABC* of side length 1 in the sense that *AC* and *BC* are tangent to the parabola at *A* and *B*, respectively:

Find the area between *AB* and the parabola.

Question Eight. [6 points]

Find the slopes of all lines passing through the origin and tangent to the curve $y^2 = x^3 + 39x - 35$.

Question Nine. [7 points]

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n \cdot 2^{n-1}}$

Question Ten. [8 points]

Let *S* be the locus of all points (x, y) in the first quadrant such that $\frac{x}{t} + \frac{y}{1-t} = 1$ for some *t* with $0<t<1$. Find the area of *S*.

1998 Harvard/MIT Math Tournament

GEOMETRY Answer Sheet

Name: ______________________________

School: ____________________ **Grade:** ______

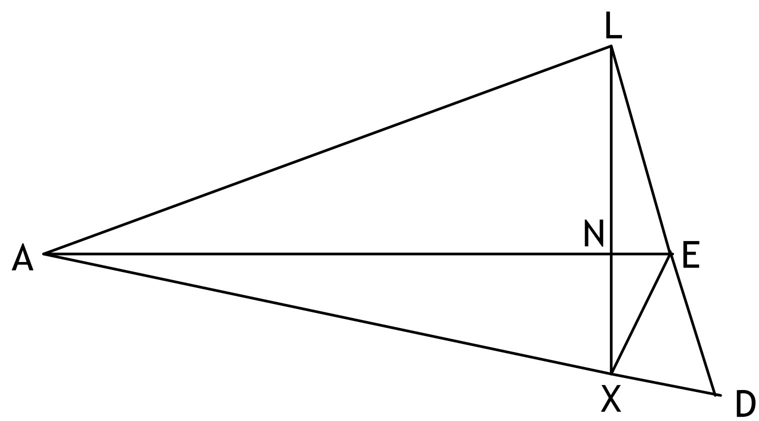
1	____________	**7**	____________
2	____________	**8**	____________
3	____________	**9**	____________
4	____________	**10a**	____________
5	____________	**10b**	____________
6	____________	**10c**	____________

TOTAL: ________

GEOMETRY

Question One. [3 points]

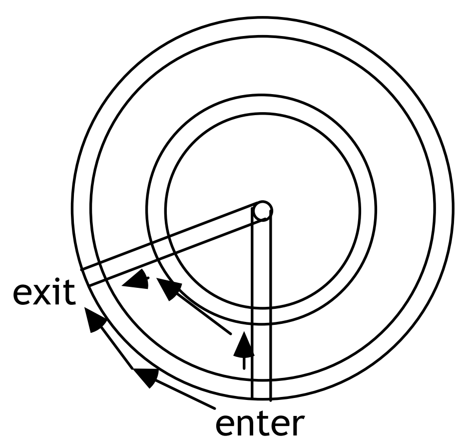
Quadrilateral *ALEX*, pictured below (but not necessarily to scale!) can be inscribed in a circle; with $m\angle LAX = 20°$ and $m\angle AXE = 100°$:



Calculate $m\angle EDX$.

Question Two. [3 points]

Anne and Lisa enter a park that has two concentric circular paths joined by two radial paths, one of which is at the point where they enter. Anne goes in to the inner circle along the first radial path, around by the shorter way to the second radial path and out along it to the exit. Walking at a the same rate, Lisa goes around the outer circle to the exit, taking the shorter of the two directions around the park.



They arrive at the exit at the same time. The radial paths meet at the center of the park; what is the angle between them?

Question Three. [4 points]

MD is a chord of length $\sqrt{2}$ in a circle of radius 1, and *L* is chosen on the circle so that the area of triangle *MLD* is the maximized.

Find $m\angle MLD$.

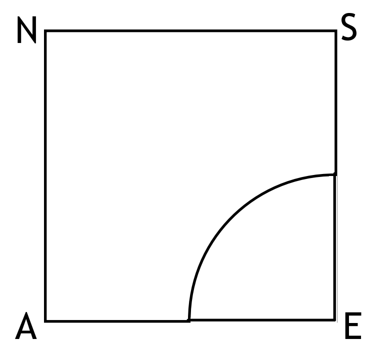
Question Four. [4 points]

A cube with side length 100cm is filled with water and has a hole through which the water drains into a cylinder of radius 100cm. If the water level in the cube is falling at a rate of $1\ {}^{cm}\!/_{s}$, how fast is the water level in the cylinder rising?

Question Five. [5 points]

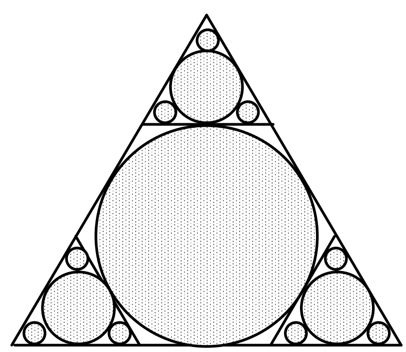
Square *SEAN* has side length 2 and a quarter-circle of radius 1 around *E* is cut out.

Find the radius of the largest circle that can be inscribed in the remaining figure.



Question Six. [5 points]

A circle is inscribed in an equilateral triangle of side length 1. Tangents to the circle are drawn that cut off equilateral triangles at each corner. Circles are inscribed in each of these equilateral triangles. If this process is repeated infinitely many times, what is the sum of the areas of all the circles?



Question Seven. [5 points]

Pyramid *EARLY* has rectangular base *EARL* and apex *Y*, and all of its edges are of integer length. The four edges from the apex have lengths 1, 4, 7, 8 (in no particular order), and *EY* is perpendicular to *YR*.

Find the area of rectangle *EARL*.

Question Eight. [6 points]

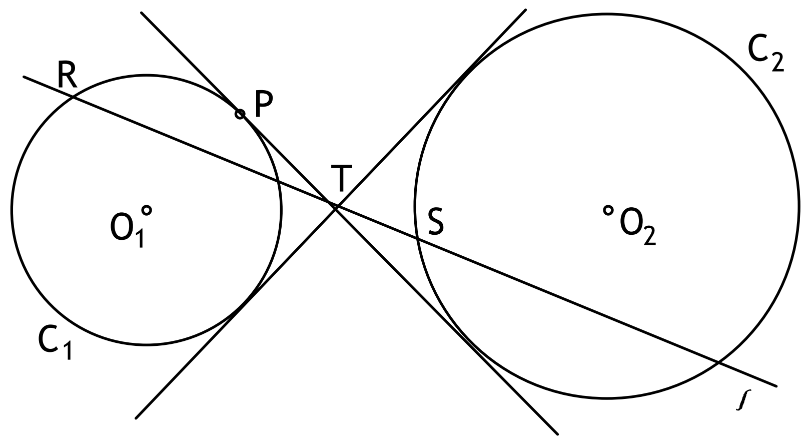
It is not possible to construct a segment of length π using a straightedge, compass, and a given segment of length 1. The following construction, given in 1685 by Adam Kochansky, yields a segment whose length agrees with π to five decimal places:

Construct a circle of radius 1 and call its center *O*. Construct a diameter *AB* of this circle and a line ℓ tangent to the circle at *A*. Next, draw a circle with radius 1 centered at *A*, and call one of the intersections with the original circle *C*. Now from *C* draw an arc of radius 1 intersecting the circle around *A* at *D*, where *D* lies outside of the circle centered at *O*. Draw *OD* and let *E* be its point of intersection with ℓ. Construct *H* on *AE* such that *A* is between *H* and *E*, and *HE*=3.

The distance between *B* and *H* is then close to π; calculate its exact value.

Question Nine. [7 points]

Let T be the intersection of the common internal tangents of circles C_1, C_2 with centers O_1, O_2 respectively. Let P be one of the points of tangency on C_1 and let line ℓ bisect angle O_1TP. Label the intersection of ℓ with C_1 that is farthest from T, R, and label the intersection of ℓ with C_2 that is closest to T, S. If C_1 has radius 4, C_2 has radius 6, and $O_1O_2 = 20$, calculate $(TR)(TS)$.



Question Ten [8 points].

Lukas is playing pool on a table shaped like an equilateral triangle. The pockets are at the corners of the triangle and are labeled C, H, and T. Each side of the table is 16 feet long. Lukas shoots a ball from corner C of the table in such a way that on the second bounce, the ball hits 2 feet away from him along side CH.

a. [5 points]

How many times will the ball bounce before hitting a pocket?

b. [2 points]

Which pocket will the ball hit?

c. [1 points]

How far will the ball travel before hitting the pocket?

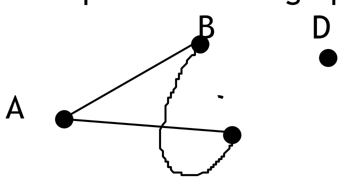
<u>Power Question - Coloring Graphs</u>

Perhaps you have heard of the Four Color Theorem (if not, don't panic!), which essentially says that any map (e.g. a map of the United States) can be colored with four or fewer colors without giving neighboring regions (e.g. Massachusetts and New Hampshire) the same color. It was, until 22 years ago, one of the most famous unsolved problems in mathematics. The only known proof, however, is so long that it requires a computer to carry it out. This power question will lead you through the basic definitions and theorems of graph theory necessary to prove the Five Color Theorem, a weaker and much easier (though still quite challenging) statement.

<u>Part I - Graphs</u>

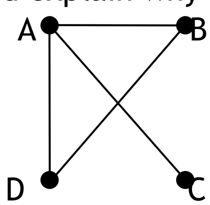
Definitions: A *graph* is a collection of points and lines (or curves), called *vertices* and *edges*, respectively, where each edge connects exactly two distinct vertices and any two vertices are connected by at most one edge. We say that two vertices are *adjacent* if they are connected by an edge.
Notation: For this problem we will denote vertices by capital letters and the edge connecting vertices X and Y by XY. Vertices will be drawn as dots so that they will be distinguishable from edge crossings. We will denote graphs by capital script letters, usually G or H.

Example: Here is a graph with vertices A, B, C, D, and edges AB, AC, BC.

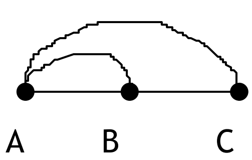


a. Which of the following are graphs? List the vertices and edges of each graph and explain why the others are not graphs. [1 point each]

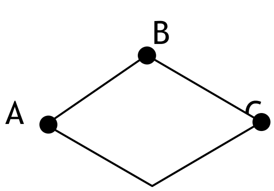
i.



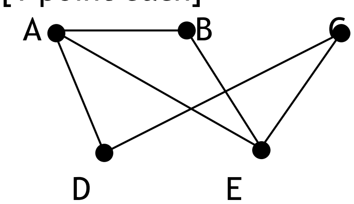
ii.



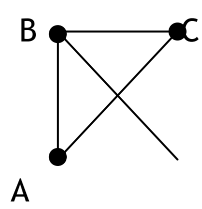
iii.



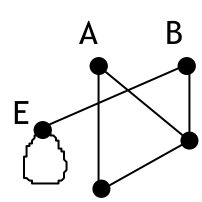
iv.



v.



vi.

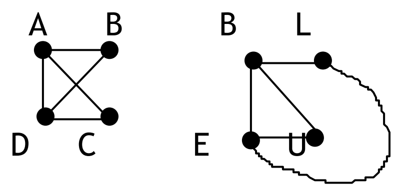


C

D

Definition: Graph G is said to be *isomorphic* (from Greek roots meaning “same structure”) to H, written G ≅ H, if the vertices of G can be given the names of the vertices of H in such a way that G and H have the same edges. This relabling is called an *isomorphism* and is reversible, thus implying H ≅ G as well.

Example:



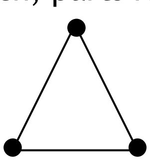
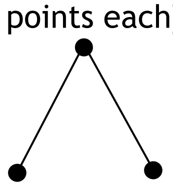
G H

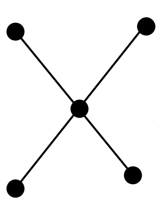
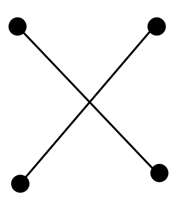
G ≅ H since we can relable the vertices of G as follows: A↔B, B↔L, C↔U, D↔E (we write the double arrows to emphasize the reversibility of the isomorphism).

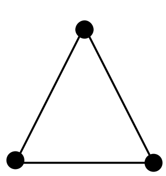
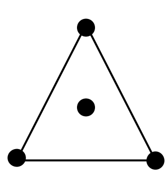
b. [1 point each]
i. List the edges of H.
ii. Show that H ≅ G in the example by writing an isomorphism from H to G.

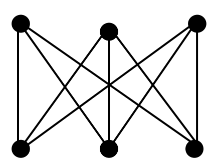
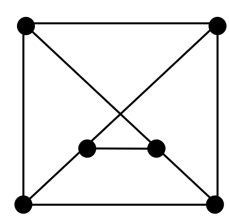
From now on the diagrams in the problem will not have the vertices explicitly labled unless necessary. Thus to show two unlabled graphs are isomorphic we can arbitrarily lable the vertices of one and then show that the other can be labled with the same names to yield the same edges.

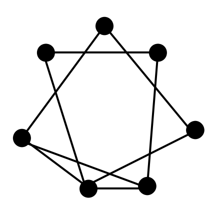
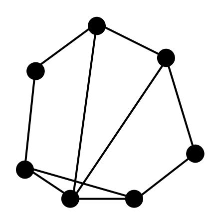
c. For each of the following pairs of graphs, state whether or not they are isomorphic (you do not need to justify your answer). [parts *i-iii* are 1 point each, parts *iv-vi* are 3 points each]

i.  and 

ii.  and 

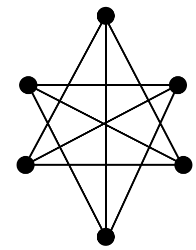
iii.  

iv.  and 

v.  and 

vi.

and



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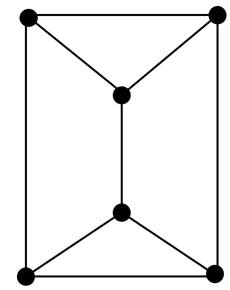
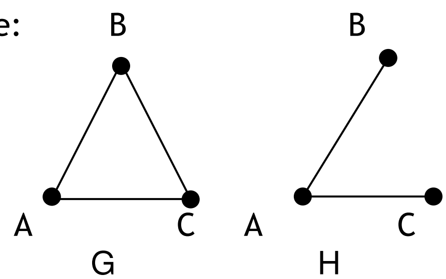


A couple more definitions are best given in this section, though there won't be any questions specifically about them until later.

Definitions: A graph H is a *subgraph* of a graph G if every vertex of H is a vertex of G and every edge of H is an edge of G.

The *degree* of a vertex is the number of edges connected to it.

Example:



H is a subgraph of G. Vertex B has degree 2 in G and degree 1 in H.

<u>Part II - Planar Graphs</u>

There are not many nontrivial properties possessed by all graphs, so we will restrict our attention to graphs with certain nice properties that we can exploit to prove cool theorems. So now we need one more round of definitions, then the real fun begins!

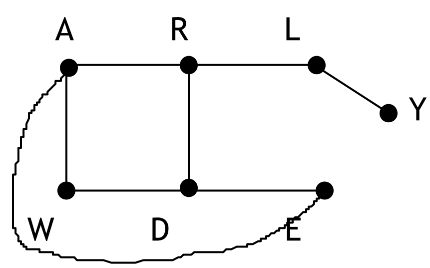
Definitions: A graph is *planar* if it is isomorphic to a graph that has been drawn in a plane without edge crossings. Otherwise a graph is nonplanar. It shouldn't be too hard to see that every subgraph of a planar graph is planar.

A *walk* in a graph is a sequence A_1 A_2 ... A_n of not necessarily distinct vertices in which A_k is adjacent to A_{k+1} for k = 1, 2, ..., n-1.

A graph is *connected* if every pair of vertices is joined by a walk, or equivalently if there is a walk that passes through every vertex at least once. Otherwise a graph is said to be disconnected.

When a planar graph is actually drawn in a plane without edge crossings, it cuts the plane into regions called *faces* of the graph.

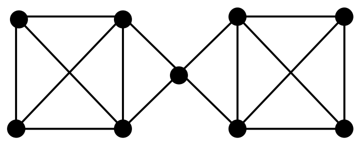
Example:



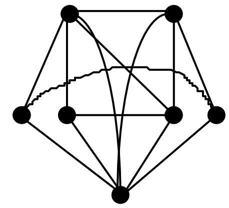
Is connected since the walk EDWARDEARLY passes through every vertex at least once. It has 3 faces (don't forget the exterior region when counting).

a. How many vertices, edges, and faces do each of the following graphs have? [3 points each]

i.

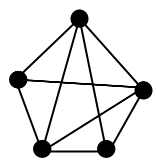


ii.

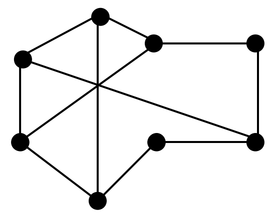


b. For each of the following graphs, state whether or not it is planar. You do not need to justify your answer. [3 points each]

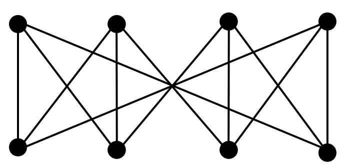
i.



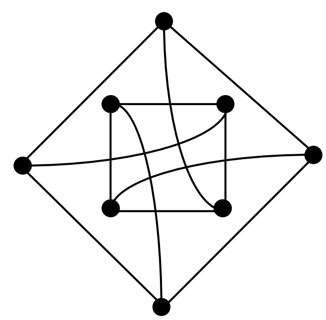
iii.



ii.



iv.

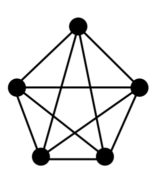


Euler's formula states if a connected planar graph has v vertices, e edges, and f faces, then v-e+f=2. Proving this would go beyond the scope of this problem, so you may take this formula as given.

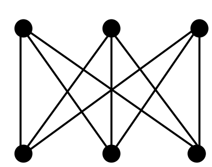
c. [15 points] Prove that if G is planar and connected with v≥3, then $\frac{3}{2}f \le e \le 3v-6$.

d. Prove that the following graphs are nonplanar.

i. [5 points]



ii. [10 points]

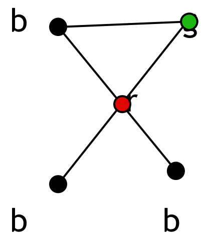


e. [12 points] Prove that every planar graph has at least one vertex of degree 5 or less.

Part III - Coloring

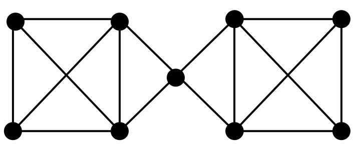
Now we can begin talking about coloring planar graphs.
Definition: We say a graph has been *colored* if a color has been assigned to each vertex in such a way that adjacent vertices have different colors.
The *chromatic number* χ (the Greek letter chi, pronounced like sky without the s) of a graph is the smallest number of colors with which it can be colored.

Example: Here is a graph with chromatic number 3, colored in black, green, and red.

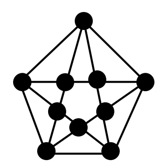


a. Calculate χ for each of the following graphs. [3 points each]

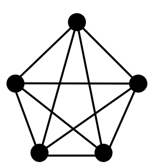
i.



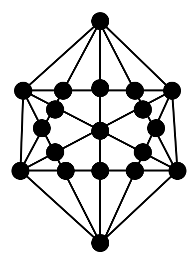
iii.



ii.

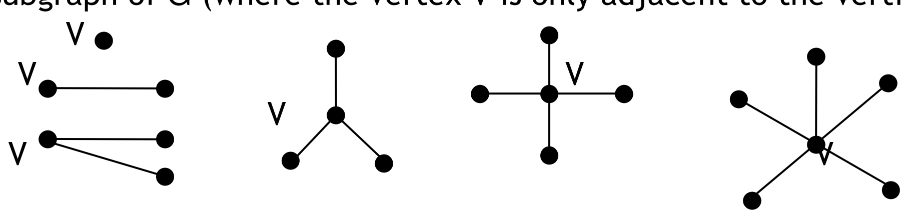


iv.



b. [14 points] Given a graph G, let $\bar{\chi}$ be the chromatic number of the graph obtained by taking the vertices of G and drawing edges only between those that are not adjacent in G. Prove that $\chi + \bar{\chi} \geq 2\sqrt{v}$.

c. Assume G is a planar graph with χ>5.
i. [5 points] Prove that at least one of the following must be isomorphic to a subgraph of G (where the vertex V is only adjacent to the vertices shown).



ii. [20 points] Prove that if one of the above is a subgraph of G (where the vertex V is only adjacent to the vertices shown) then we can remove one vertex (and all edges touching it) from G to obtain a graph with fewer vertices which also has $\chi>5$ (you may use the Jordan Curve Theorem, which states that a closed loop that does not intersect itself divides the plane into an inside and outside, and if a continuous curve joins a point on the inside to one on the outside then it must cross the loop).
iii. [5 points] Prove the Five Color Theorem: Every planar graph has $\chi\leq5$.