# Chapter 1

# The Problems

## 1.1 Individual Speed Test

**Morning, January 30, 2010**

*There are 20 problems, worth 3 points each, and 20 minutes to solve as many problems as possible.*

1. Evaluate $\dfrac{\sqrt{2}\cdot\sqrt{6}}{\sqrt{3}}$.

2. If 6% of a number is 1218, what is 18% of that number?

3. What is the median of $\{42, 9, 8, 4, 5, 1, 13666, 3\}$?

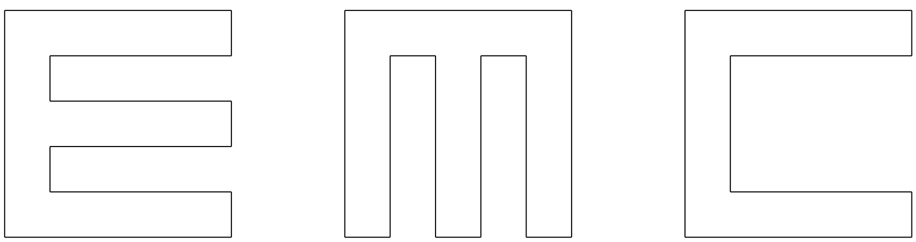
4. Define the operation $\heartsuit$ so that $i \heartsuit u = 5i - 2u$. What is $3 \heartsuit 4$?

5. How many 0.2-inch by 1-inch by 1-inch gold bars can fit in a 15-inch by 12-inch by 9-inch box?

6. A tetrahedron is a triangular pyramid. What is the sum of the number of edges, faces, and vertices of a tetrahedron?

7. Ron has three blue socks, four white socks, five green socks, and two black socks in a drawer. Ron takes socks out of his drawer blindly and at random. What is the least number of socks that Ron needs to take out to guarantee he will be able to make a pair of matching socks?

8. One segment with length 6 and some segments with lengths 10, 8, and 2 form the three letters in the diagram shown below. Compute the sum of the perimeters of the three figures.



9. How many integer solutions are there to the inequality $|x - 6| \leq 4$?

10. In a land for bad children, the flavors of ice cream are grass, dirt, earwax, hair, and dust-bunny. The cones are made out of granite, marble, or pumice, and can be topped by hot lava, chalk, or ink. How many ice cream cones can the evil confectioners in this ice-cream land make? (Every ice cream cone consists of one scoop of ice cream, one cone, and one topping.)

11. Compute the sum of the prime divisors of $245 + 452 + 524$.

12. In quadrilateral $SEAT$, $SE = 2$, $EA = 3$, $AT = 4$, $\angle EAT = \angle SET = 90^\circ$. What is the area of the quadrilateral?

13. What is the angle, in degrees, formed by the hour and minute hands on a clock at 10:30 AM?

14. Three numbers are randomly chosen without replacement from the set $\{101, 102, 103, \ldots, 200\}$. What is the probability that these three numbers are the side lengths of a triangle?

15. John takes a 30-mile bike ride over hilly terrain, where the road always either goes uphill or downhill, and is never flat. If he bikes a total of 20 miles uphill, and he bikes at 6 mph when he goes uphill, and 24 mph when he goes downhill, what is his average speed, in mph, for the ride?

16. How many distinct six-letter words (not necessarily in any language known to man) can be formed by rearranging the letters in EXETER? (You should include the word EXETER in your count.)

17. A pie has been cut into eight slices of different sizes. Snow White steals a slice. Then, the seven dwarfs (Sneezy, Sleepy, Dopey, Doc, Happy, Bashful, Grumpy) take slices one by one according to the alphabetical order of their names, but each dwarf can only take a slice next to one that has already been taken. In how many ways can this pie be eaten by these eight persons?

18. Assume that $n$ is a positive integer such that the remainder of $n$ is 1 when divided by 3, is 2 when divided by 4, is 3 when divided by 5, ..., and is 8 when divided by 10. What is the smallest possible value of $n$?

19. Find the sum of all positive four-digit numbers that are perfect squares and that have remainder 1 when divided by 100.

20. A coin of radius 1 cm is tossed onto a plane surface that has been tiled by equilateral triangles with side length $20\sqrt{3}$ cm. What is the probability that the coin lands within one of the triangles?

## 1.2 Individual Accuracy Test

**Morning, January 30, 2010**

*There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.*

1. Calculate $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)^2$.

2. Find the $2010^{\text{th}}$ digit after the decimal point in the expansion of $\frac{1}{7}$.

3. If you add 1 liter of water to a solution consisting of acid and water, the new solutions will contain of 30% water. If you add another 5 liters of water to the new solution, it will contain $36\frac{4}{11}\%$ water. Find the number of liters of acid in the original solution.

4. John places 5 indistinguishable blue marbles and 5 indistinguishable red marbles into two distinguishable buckets such that each bucket has at least one blue marble and one red marble. How many distinguishable marble distributions are possible after the process is completed?

5. In quadrilateral $PEAR$, $PE = 21$, $EA = 20$, $AR = 15$, $RE = 25$, and $AP = 29$. Find the area of the quadrilateral.

6. Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region $\mathcal{R}$ consists of points lying inside the square but outside of the semicircles. The area of $\mathcal{R}$ can be written in the form $a - b\pi$, where $a$ and $b$ are positive rational numbers. Compute $a + b$.

7. Let $x$ and $y$ be two numbers satisfying the relations $x \geq 0$, $y \geq 0$, and $3x + 5y = 7$. What is the maximum possible value of $9x^2 + 25y^2$?

8. In the Senate office in Exie-land, there are 6 distinguishable senators and 6 distinguishable interns. Some senators and an equal number of interns will attend a convention. If at least one senator must attend, how many combinations of senators and interns can attend the convention?

9. Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$.

10. Segment $EA$ has length 1. Region $\mathcal{R}$ consists of points $P$ in the plane such that $\angle PEA \geq 120^\circ$ and $PE < \sqrt{3}$. If point $X$ is picked randomly from the region $\mathcal{R}$, the probability that $AX < \sqrt{3}$ can be written in the form $a - \frac{\sqrt{b}}{c\pi}$, where $a$ is a rational number, $b$ and $c$ are positive integers, and $b$ is not divisible by the square of a prime. Find the ordered triple $(a, b, c)$.

## 1.3 Team Round

**Morning, January 30, 2010**

*There are 15 problems, worth 20 points each, and 30 minutes to solve as many problems as possible.*

1. A very large lucky number $N$ consists of eighty-eight 8s in a row. Find the remainder when this number $N$ is divided by 6.

2. If 3 chickens can lay 9 eggs in 4 days, how many chickens does it take to lay 180 eggs in 8 days?

3. Find the ordered pair $(x, y)$ of real numbers satisfying the conditions $x > y$, $x+y = 10$, and $xy = -119$.

4. There is pair of similar triangles. One triangle has side lengths 4, 6, and 9. The other triangle has side lengths 8, 12 and $x$. Find the sum of two possible values of $x$.

5. If $x^2 + \frac{1}{x^2} = 3$, there are two possible values of $x + \frac{1}{x}$. What is the smaller of the two values?

6. Three flavors (chocolate strawberry, vanilla) of ice cream are sold at Brian's ice cream shop. Brian's friend Zerg gets a coupon for 10 free scoops of ice cream. If the coupon requires Zerg to choose an even number of scoops of each flavor of ice cream, how many ways can he choose his ice cream scoops? (For example, he could have 6 scoops of vanilla and 4 scoops of chocolate. The order in which Zerg eats the scoops does not matter.)

7. David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs $m$ feet of rope. Given that $m = a\pi + b$, where $a$ and $b$ are rational numbers, find sum $a + b$.

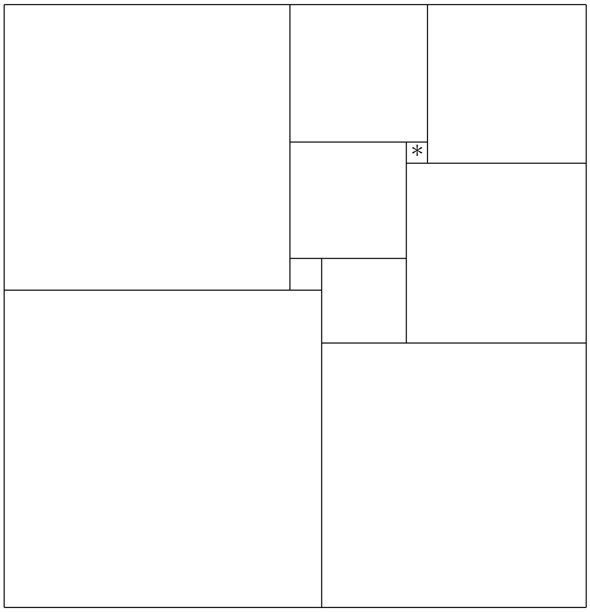
8. Segment $AB$ is the diameter of a semicircle of radius 24. A beam of light is shot from a point $12\sqrt{3}$ from the center of the semicircle, and perpendicular to $AB$. How many times does it reflect off the semicircle before hitting $AB$ again?

9. A cube is inscribed in a sphere of radius 8. A smaller sphere is inscribed in the same sphere such that it is externally tangent to one face of the cube and internally tangent to the larger sphere. The maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere can be written in the form $\frac{a-\sqrt{b}}{36}$, where $a$ and $b$ are positive integers. Find the product $ab$.

10. How many ordered pairs $(x, y)$ of integers are there such that $2xy + x + y = 52$?

11. Three musketeers looted a caravan and walked off with a chest full of coins. During the night, the first musketeer divided the coins into three equal piles, with one coin left over. He threw it into the ocean and took one of the piles for himself, then went back to sleep. The second musketeer woke up an hour later. He divided the remaining coins into three equal piles, and threw out the one coin that was left over. He took one of the piles and went back to sleep. The third musketeer woke up and divided the remaining coins into three equal piles, threw out the extra coin, and took one pile for himself. The next morning, the three musketeers gathered around to divide the coins into three equal piles. Strangely enough, they had one coin left over this time as well. What is the minimum number of coins that were originally in the chest?

12. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is $2 \times 2$ (marked with $*$). What is the area of the rectangle (which looks rather like a square itself)?




13. Let $A = (3, 2)$, $B = (0, 1)$, and $P$ be on the line $x + y = 0$. What is the minimum possible value of $AP + BP$?

14. Mr. Mustafa the number man got a $6 \times x$ rectangular chess board for his birthday. Because he was bored, he wrote the numbers 1 to $6x$ starting in the upper left corner and moving across row by row (so the number $x+1$ is in the $2^{\text{nd}}$ row, $1^{\text{st}}$ column). Then, he wrote the same numbers starting in the upper left corner and moving down each column (so the number 7 appears in the $1^{\text{st}}$ row, $2^{\text{nd}}$ column). He then added up the two numbers in each of the cells and found that some of the sums were repeated. Given that $x$ is less than or equal to 100, how many possibilities are there for $x$?

15. Six congruent equilateral triangles are arranged in the plane so that every triangle shares at least one whole edge with some other triangle. Find the number of distinct arrangements. (Two arrangements are considered the same if one can be rotated and/or reflected onto another.)

## 1.4 Guts Round

**Afternoon, January 30, 2010**

### 1.4.1 Round 1

1. [5pts] Define the operation $\clubsuit$ so that $a \clubsuit b = a^b + b^a$. Then, if $2 \clubsuit b = 32$, what is $b$?

2. [5pts] A square is changed into a rectangle by increasing two of its sides by $p\%$ and decreasing the two other sides by $p\%$. The area is then reduced by $1\%$. What is the value of $p$?

3. [5pts] What is the sum, in degrees, of the internal angles of a heptagon?

4. [5pts] How many integers in between $\sqrt{47}$ and $\sqrt{8283}$ are divisible by 7?

### 1.4.2 Round 2

5. [8pts] Some mutant green turkeys and pink elephants are grazing in a field. Mutant green turkeys have six legs and three heads. Pink elephants have 4 legs and 1 head. There are 100 legs and 37 heads in the field. How many animals are grazing?

6. [8pts] Let $A = (0, 0)$, $B = (6, 8)$, $C = (20, 8)$, $D = (14, 0)$, $E = (21, -10)$, and $F = (7, -10)$. Find the area of the hexagon $ABCDEF$.

7. [8pts] In Moscow, three men, Oleg, Igor, and Dima, are questioned on suspicion of stealing Vladimir Putin's blankie. It is known that each man either always tells the truth or always lies. They make the following statements:

   (a) Oleg: I am innocent!
   (b) Igor: Dima stole the blankie!
   (c) Dima: I am innocent!
   (d) Igor: I am guilty!
   (e) Oleg: Yes, Igor is indeed guilty!

   If exactly one of Oleg, Igor, and Dima is guilty of the theft, who is the thief??

8. [8pts] How many 11-letter sequences of E's and M's have at least as many E's as M's?

### 1.4.3 Round 3

9. [11pts] John is entering the following summation $31 + 32 + 33 + 34 + 35 + 36 + 37 + 38 + 39$ in his calculator. However, he accidently leaves out a plus sign and the answer becomes 3582. What is the number that comes before the missing plus sign?

10. [11pts] Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where $a$ and $b$ are integers. What is $a + b$?

11. [11pts] Alice has a rectangular room with 6 outlets lined up on one wall and 6 lamps lined up on the opposite wall. She has 6 distinct power cords (red, blue, green, purple, black, yellow). If the red and green power cords cannot cross, how many ways can she plug in all six lamps?

12. [11pts] Tracy wants to jump through a line of 12 tiles on the floor by either jumping onto the next block, or jumping onto the block two steps ahead. An example of a path through the 12 tiles may be: 1 step, 2 steps, 2 steps, 2 steps, 1 step, 2 steps, 2 steps. In how many ways can Tracy jump through these 12 tiles?

### 1.4.4 Round 4

13. [14pts] What is the units digit of the number $(2^1 + 1)(2^2 - 1)(2^3 + 1)(2^4 - 1) \dots (2^{2010} - 1)$?

14. [14pts] Mr. Fat noted that on January 2, 2010, the display of the day is 01/02/2010, and the sequence 01022010 is a palindrome (a number that reads the same forwards and backwards). How many days does Mr. Fat need to wait between this palindrome day and the last palindrome day of this decade?

15. [14pts] Farmer Tim has a 30-meter by 30-meter by $30\sqrt{2}$-meter triangular barn. He ties his goat to the corner where the two shorter sides meet with a 60-meter rope. What is the area, in square meters, of the land where the goat can graze, given that it cannot get inside the barn?

16. [14pts] In triangle $ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Point $P$ lies inside the triangle and the distances from $P$ to two of the sides of the triangle are 1 and 2. What is the maximum distance from $P$ to the third side of the triangle?

### 1.4.5 Round 5

17. [17pts] Let $Z$ be the answer to the third question on this guts quadruplet. If $x^2 - 2x = Z - 1$, find the positive value of $x$.

18. [17pts] Let $X$ be the answer to the first question on this guts quadruplet. To make a FATRON2012, a cubical steel body as large as possible is cut out from a solid sphere of diameter $X$. A TAFTRON2013 is created by cutting a FATRON2012 into 27 identical cubes, with no material wasted. What is the length of one edge of a TAFTRON2013?

19. [17pts] Let $Y$ be the smallest integer greater than the answer to the second question on this guts quadruplet. Fred posts two distinguishable sheets on the wall. Then, $Y$ people walk into the room. Each of the $Y$ people signs up on 0, 1, or 2 of the sheets. Given that there are at least two people in the room other than Fred, how many possible pairs of lists can Fred have?

20. [17pts] Let $A$, $B$, $C$, be the respective answers to the first, second, and third questions on this guts quadruplet. At the Robot Design Convention and Showcase, a series of robots are programmed such that each robot shakes hands exactly once with every other robot of the same height. If the heights of the 16 robots are 4, 4, 4, 5, 5, 7, 17, 17, 17, 34, 34, 42, 100, $A$, $B$, and $C$ feet, how many handshakes will take place?

### 1.4.6 Round 6

21. [20pts] Determine the number of ordered triples $(p, q, r)$ of primes with $1 < p < q < r < 100$ such that $q - p = r - q$.

22. [20pts] For numbers $a$, $b$, $c$, $d$ such that $0 \leq a, b, c, d \leq 10$, find the minimum value of $ab + bc + cd + da - 5a - 5b - 5c - 5d$.

23. [20pts] Daniel has a task to measure 1 gram, 2 grams, 3 grams, 4 grams , ..., all the way up to $n$ grams. He goes into a store and buys a scale and six weights of his choosing (so that he knows the value for each weight that he buys). If he can place the weights on either side of the scale, what is the maximum value of $n$?

24. [20pts] Given a Rubik's cube, what is the probability that at least one face will remain unchanged after a random sequence of three moves? (A Rubik's cube is a 3 by 3 by 3 cube with each face starting as a different color. The faces (3 by 3) can be freely turned. A move is defined in this problem as a 90 degree rotation of one *face either clockwise or counter-clockwise. The center square on each face–six in total–is fixed.)*

# Chapter 3

# The Solutions

## 3.1 Individual Speed Test Solutions

1. Evaluate $\dfrac{\sqrt{2}\cdot\sqrt{6}}{\sqrt{3}}$.

   **Solution.** The answer is $(\sqrt{2})^2 = \boxed{2}$.

2. If 6% of a number is 1218, what is 18% of that number?

   **Solution.** The answer is $1218 \cdot 3 = \boxed{3654}$.

3. What is the median of $\{42, 9, 8, 4, 5, 1, 13666, 3\}$?

   **Solution.** The answer is $\dfrac{5+8}{2} = \boxed{\dfrac{13}{2}}$. We can arrange these elements in increasing order: 1, 3, 4, 5, 8, 9, 42, 13666. Since there are an even number of values, the median is the average of the two middle numbers 5 and 8.

4. Define the operation $\heartsuit$ so that $i \heartsuit u = 5i - 2u$. What is $3 \heartsuit 4$?

   **Solution.** The answer is $5 \cdot 3 - 2 \cdot 4 = \boxed{7}$.

5. How many 0.2-inch by 1-inch by 1-inch gold bars can fit in a 15-inch by 12-inch by 9-inch box?

   **Solution.** The answer is $15 \cdot 5 \cdot 12 \cdot 9 = 3^4 \cdot 5^2 \cdot 2^2 = \boxed{8100}$.

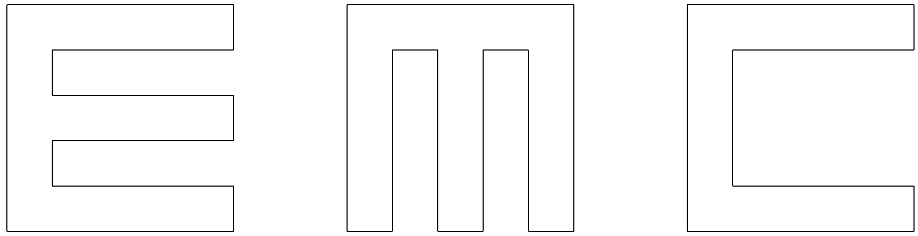
6. A tetrahedron is a triangular pyramid. What is the sum of the number of edges, faces, and vertices of a tetrahedron?

   **Solution.** The answer is $6 + 4 + 4 = \boxed{14}$.

7. Ron has three blue socks, four white socks, five green socks, and two black socks in a drawer. Ron takes socks out of his drawer blindly and at random. What is the least number of socks that Ron needs to take out to guarantee he will be able to make a pair of matching socks?

   **Solution.** The answer is $\boxed{5}$. There are only four different colors. Among any five socks, at least two of them are of the same color.

8. One segment with length 6 and some segments with lengths 10, 8, and 2 form the three letters in the diagram shown below. Compute the sum of the perimeters of the three figures.



   **Solution.** The answer is $30 \cdot 4 + 10 \cdot 8 = \boxed{200}$. For each letter, we can rearrange segments with lengths less than 8 to form a square of side 10. The sum of the perimeters of the three squares is 120. There are 10 additional segments of length 8.

9. How many integer solutions are there to the inequality $|x-6| \leq 4$?

   **Solution.** The answer is $\boxed{9}$. The possible values of $x-6$ are $0, \pm 1, \pm 2, \pm 3, \pm 4$.

10. In a land for bad children, the flavors of ice cream are grass, dirt, earwax, hair, and dust-bunny. The cones are made out of granite, marble, or pumice, and can be topped by hot lava, chalk, or ink. How many ice cream cones can the evil confectioners in this ice-cream land make? (Every ice cream cone consists of one scoop of ice cream, one cone, and one topping.)

    **Solution.** The answer is $5 \cdot 3 \cdot 3 = \boxed{45}$.

11. Compute the sum of the prime divisors of $245 + 452 + 524$.

    **Solution.** The answer is $11 + 3 + 37 = \boxed{51}$. Note that $245 + 452 + 524 = 222 + 444 + 555 = (2 + 4 + 5) \cdot 111 = 11 \cdot 3 \cdot 37$.

12. In quadrilateral $SEAT$, $SE = 2$, $EA = 3$, $AT = 4$, $\angle EAT = \angle SET = 90°$. What is the area of the quadrilateral?

    **Solution.** The answer is $5+6 = \boxed{11}$. The quadrilateral is the union of two right triangles $EAT$ (with area 6 and $ET = 5$) and $SET$ (with area 5).

13. What is the angle, in degrees, formed by the hour and minute hands on a clock at 10:30 AM?

    **Solution.** The answer is $\boxed{135}$.

14. Three numbers are randomly chosen without replacement from the set $\{101, 102, 103, \ldots, 200\}$. What is the probability that these three numbers are the side lengths of a triangle?

    **Solution.** The answer is $\boxed{1}$. Note that the sum of any two numbers in the set is greater than any number in the set.

15. John takes a 30-mile bike ride over hilly terrain, where the road always either goes uphill or downhill, and is never flat. If he bikes a total of 20 miles uphill, and he bikes at 6 mph when he goes uphill, and 24 mph when he goes downhill, what is his average speed, in mph, for the ride?

    **Solution.** The answer is $\dfrac{30}{\frac{20}{6} + \frac{10}{24}} = \boxed{8}$.

16. How many distinct six-letter words (not necessarily in any language known to man) can be formed by rearranging the letters in EXETER? (You should include the word EXETER in your count.)

    **Solution.** The answer is $\dfrac{6!}{3!} = \boxed{120}$.

17. A pie has been cut into eight slices of different sizes. Snow White steals a slice. Then, the seven dwarfs (Sneezy, Sleepy, Dopey, Doc, Happy, Bashful, Grumpy) take slices one by one according to the alphabetical order of their names, but each dwarf can only take a slice next to one that has already been taken. In how many ways can this pie be eaten by these eight persons?

    **Solution.** The answer is $8 \cdot 2^6 = 2^9 = \boxed{512}$. Snow White has 8 choices. Each of Bashful, Doc, Dopey, Grumpy, Happy, and Sleepy has two choices. Then, there is only one slice left for Sneezy to eat.

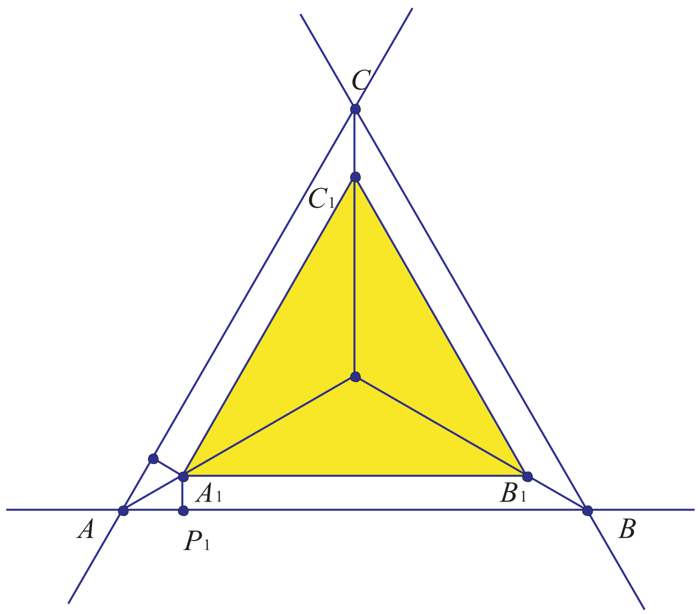
18. Assume that $n$ is a positive integer such that the remainder of $n$ is 1 when divided by 3, is 2 when divided by 4, is 3 when divided by 5, ..., and is 8 when divided by 10. What is the smallest possible value of $n$?

    **Solution.** The answer is $\text{lcm}(3, 4\ldots, 10) - 2 = 5\cdot7\cdot8\cdot9-2 = \boxed{2518}$. Adding 2 to the desired number yields a number that is divisible by each of $3, 4, \ldots, 10$.

19. Find the sum of all positive four-digit numbers that are perfect squares and that have remainder 1 when divided by 100.

    **Solution.** The answer is $49^2 + 51^2 + 99^2 = 2401 + 2601 + 9801 = \boxed{14803}$. Let $\overline{abcd}$ be such a number. Then $\overline{abcd} = (10x \pm 1)^2 = 100x^2 \pm 20x + 1$, implying that $x = 5$ or $x = 10$.

20. A coin of radius 1 cm is tossed onto a plane surface that has been tiled by equilateral triangles with side length $20\sqrt{3}$ cm. What is the probability that the coin lands within one of the triangles?




    **Solution.** The answer is $\dfrac{(18\sqrt{3})^2}{(20\sqrt{3})^2} = \boxed{\dfrac{81}{100}}$. Assume that the center of the coin lies within the boundary of triangle $ABC$. In order for the coin to lie completely within the triangle, the center of the circle must lie in triangle $A_1B_1C_1$, where the distances between pairs of parallel lines $A_1B_1$ and $AB$, $B_1C_1$ and $BC$, and $C_1A_1$ and $CA$ are all equal to 1. Let $P_1$ be the foot of the perpendicular from $A_1$ to line $AB$. It is not difficult to see that $A_1P_1 = 1$ and $AP_1 = \sqrt{3}$, from which it follows that $A_1B_1 = 18\sqrt{3}$ and the ratio between the areas of triangles $A_1B_1C_1$ and $ABC$ is equal to $\frac{(18\sqrt{3})^2}{(20\sqrt{3})^2}$.

## 3.2 Individual Accuracy Test Solutions

1. Calculate $\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)^2$.

   **Solution.** The answer is $\left(\frac{13}{12}\right)^2 = \boxed{\frac{169}{144}}$.

2. Find the $2010^{\text{th}}$ digit after the decimal point in the expansion of $\frac{1}{7}$.

   **Solution.** The answer is $\boxed{7}$. Note that $\frac{1}{7} = 0.\overline{142857}$ and 2010 is divisible by 6.

3. If you add 1 liter of water to a solution consisting of acid and water, the new solutions will contain of 30% water. If you add another 5 liters of water to the new solution, it will contain $36\frac{4}{11}\%$ water. Find the number of liters of acid in the original solution.

   **Solution.** The answer is $\boxed{35}$. Assume that there are $x$ liters of water and $y$ liters of acid in the original solution. We have $\frac{x+1}{x+y+1} = 30\%$ and $\frac{x+6}{x+y+6} = 36\frac{4}{11}\%$, from which it follows that $x = 14$, $y = 35$.

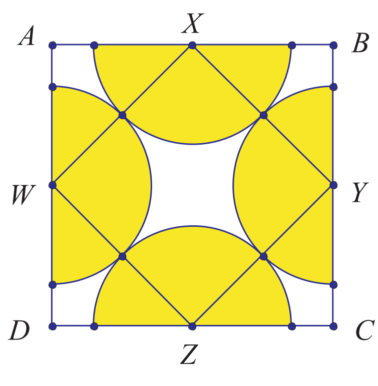
4. John places 5 indistinguishable blue marbles and 5 indistinguishable red marbles into two distinguishable buckets such that each bucket has at least one blue marble and one red marble. How many distinguishable marble distributions are possible after the process is completed?

   **Solution.** The answer is $4 \cdot 4 = \boxed{16}$. The first bucket can have $x$ (with $x = 0, 1, 2$, or 3) blue marbles and $y$ (with $y = 0, 1, 2$, or 3) red marbles, and the second bucket will have $4 - x$ blue marbles and $4 - y$ red marbles.

5. In quadrilateral $PEAR$, $PE = 21$, $EA = 20$, $AR = 15$, $RE = 25$, and $AP = 29$. Find the area of the quadrilateral.

   **Solution.** The answer is $\frac{21+15}{2} \cdot 20 = \boxed{360}$. Because $15^2 + 20^2 = 25^2$ and $21^2 + 20^2 = 29^2$, triangles $APE$ and $ERA$ are right triangles (with $\angle EAR = \angle PEA = 90°$). Consequently, $PEAR$ is a right trapezoid with bases $PE = 21$ and $AR = 15$ and height $EA = 20$.

6. Four congruent semicircles are drawn within the boundary of a square with side length 1. The center of each semicircle is the midpoint of a side of the square. Each semicircle is tangent to two other semicircles. Region $\mathcal{R}$ consists of points lying inside the square but outside of the semicircles. The area of $\mathcal{R}$ can be written in the form $a - b\pi$, where $a$ and $b$ are positive rational numbers. Compute $a + b$.



**Solution.** The answer is $1 + \dfrac{1}{4} = \boxed{\dfrac{5}{4}}$. Let $ABCD$ denote the unit square, and let $X, Y, Z, W$ be the midpoints of sides $AB, BC, CD, DA$, respectively. Then $AX = AW = \frac{1}{2}$ and $WX = \frac{\sqrt{2}}{2}$. The radii of the semicircles are equal to $r = \frac{WX}{2} = \frac{\sqrt{2}}{4}$. The area of the region covered by the semicircles is equal to $4(\frac{1}{2}\pi r^2) = \frac{\pi}{4}$ and the area of region $\mathcal{R}$ is equal to $1 - \frac{\pi}{4}$, implying that $(a, b) = \left(1, \frac{1}{4}\right)$.

7. Let $x$ and $y$ be two numbers satisfying the relations $x \geq 0$, $y \geq 0$, and $3x + 5y = 7$. What is the maximum possible value of $9x^2 + 25y^2$?

   **Solution.** The answer is $\boxed{49}$. We have $9x^2 + 25y^2 = (3x + 5y)^2 - 30xy = 49 - 30xy \leq 49$ for nonnegative numbers $x$ and $y$.

8. In the Senate office in Exie-land, there are 6 distinguishable senators and 6 distinguishable interns. Some senators and an equal number of interns will attend a convention. If at least one senator must attend, how many combinations of senators and interns can attend the convention?

   **Solution.** The answer is $\boxed{923}$. If there are $i$, $1 \leq i \leq 6$, senators attending the convention, then there are $i$ interns attending the convention. Hence is the answer is

$$\binom{6}{1}^2 + \binom{6}{2}^2 + \binom{6}{3}^2 + \binom{6}{4}^2 + \binom{6}{5}^2 + \binom{6}{6}^2 = 36 + 225 + 400 + 225 + 36 + 1 = 923.$$

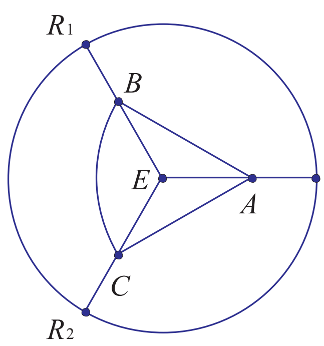
   This is a special case of Vandermonde's identity – the answer is equal to $\binom{12}{6} - 1$. Indeed, we can choose any 6 people ($x$ senators interns and $6 - x$ interns) and we will then send the $x$ chosen senators and $x$ non-chosen interns to the convention. The only case we have to exclude is the case when 0 senators and 6 interns are chosen.

9. Evaluate $(1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2)$.

   **Solution.** The answer is $2008 + 2009^2 - 2010^2 = \boxed{-2011}$. The key fact is $n^2 - (n+1)^2 - (n+2)^2 + (n+3)^2 = 4$ for all numbers $n$. Thus,

$$\begin{aligned} & (1^2 - 3^2 + 5^2 - 7^2 + 9^2 - \cdots + 2009^2) - (2^2 - 4^2 + 6^2 - 8^2 + 10^2 - \cdots + 2010^2) \\ = \ & (1^2 - 2^2 - 3^2 + 4^2) + \cdots + (2005^2 - 2006^2 - 2007^2 + 2008^2) + 2009^2 - 2010^2. \end{aligned}$$

10. Segment $EA$ has length 1. Region $\mathcal{R}$ consists of points $P$ in the plane such that $\angle PEA \geq 120^\circ$ and $PE < \sqrt{3}$. If point $X$ is picked randomly from the region $\mathcal{R}$, the probability that $AX < \sqrt{3}$ can be written in the form $a - \frac{\sqrt{b}}{c\pi}$, where $a$ is a rational number, $b$ and $c$ are positive integers, and $b$ is not divisible by the square of a prime. Find the ordered triple $(a, b, c)$.



**Solution.** The answer is $\boxed{\left(\frac{1}{2}, 3, 2\right)}$.

Let $\omega_1$ denote the circle centered at $E$ with radius $\sqrt{3}$. Points $R_1$ and $R_2$ lie on $\omega_1$ with $\angle R_1EA = \angle R_2EA = 120°$. Region $\mathcal{R}$ is the circular sector centered $R_1ER_2$. The area of region $\mathcal{R}$ is equal to $\pi$. Let $\omega_2$ denote the circle centered at $A$ with radius $\sqrt{3}$, and let $\omega_2$ intersects segments $ER_1$ and $ER_2$ are $B$ and $C$, respectively. Let $\mathcal{R}_1$ denote the region enclosed by arc $\widehat{BC}$ and segments $EB$ and $EC$. Then a point $X$ is in region $\mathcal{R}$ if and only if $X$ lies in $\mathcal{R}_1$; that is, The probability $p$ that $X$ lies $\mathcal{R}_1$ is equal to ratio of the area of $\mathcal{R}_1$ to that of $\mathcal{R}$. It is not difficult to see that $EA = EB = EC = 1$, the area of circular sector $BCA$ is $\frac{\pi}{2}$, the triangles $EAB$ and $EAC$ have the same area $\frac{\sqrt{3}}{4}$, the area of region $\mathcal{R}_1 = \frac{\pi}{2} - \frac{\sqrt{3}}{2}$, and

$$p = \frac{\frac{\pi}{2} - \frac{\sqrt{3}}{2}}{\pi} = \frac{1}{2} - \frac{\sqrt{3}}{2\pi}.$$

## 3.3 Team Round Solutions

1. A very large lucky number $N$ consists of eighty-eight 8s in a row. Find the remainder when this number $N$ is divided by 6.

   **Solution.** The answer is $\boxed{2}$. Because 888 is divisible by 6, $N-8$ is divisible by 6.

2. If 3 chickens can lay 9 eggs in 4 days, how many chickens does it take to lay 180 eggs in 8 days?

   **Solution.** The answer is $\boxed{30}$. If 3 chickens can lay 9 eggs in 4 days, then 3 chickens can lay 18 eggs in 8 days, and then 30 chickens can lay 180 eggs in 8 days.

3. Find the ordered pair $(x, y)$ of real numbers satisfying the conditions $x > y$, $x+y = 10$, and $xy = -119$.

   **Solution.** The answer is $\boxed{(17, -7)}$ We know that $(x-y)^2 = (x+y)^2 - 4xy = 100 - 4(-119) = 576$, so $(x-y) = 24$ and this gives us $(x, y) = (17, -7)$.

4. There is pair of similar triangles. One triangle has side lengths 4, 6, and 9. The other triangle has side lengths 8, 12 and $x$. Find the sum of two possible values of $x$.

   **Solution.** The answer is $\dfrac{16}{3} + 18 = \boxed{\dfrac{70}{3}}$. We have either $4 : 6 : 9 = x : 8 : 12$ or $4 : 6 : 9 = 8 : 12 : x$. In the former case, $x = \frac{16}{3}$. In the latter case, $x = 18$.

5. If $x^2 + \frac{1}{x^2} = 3$, there are two possible values of $x + \frac{1}{x}$. What is the smaller of the two values?

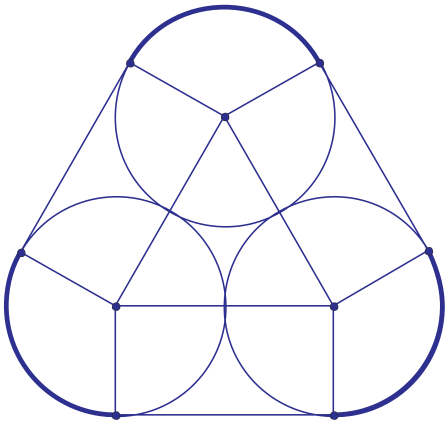
   **Solution.** The answer is $\boxed{-\sqrt{5}}$. We have $5 = x^2 + \frac{1}{x^2} + 2 = \left(x + \frac{1}{x}\right)^2$.

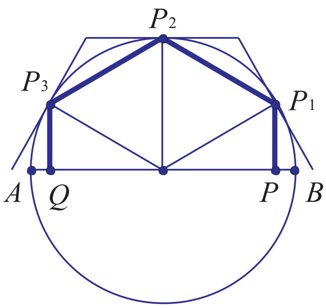
6. Three flavors (chocolate strawberry, vanilla) of ice cream are sold at Brian's ice cream shop. Brian's friend Zerg gets a coupon for 10 free scoops of ice cream. If the coupon requires Zerg to choose an even number of scoops of each flavor of ice cream, how many ways can he choose his ice cream scoops? (For example, he could have 6 scoops of vanilla and 4 scoops of chocolate. The order in which Zerg eats the scoops does not matter.)

   **Solution.** The answer is $\boxed{21}$. Assume that Zerg has $2x$ scoops of chocolate ice cream, $2y$ scoops of strawberry ice cream, and $2z$ scoops of vanilla ice cream, where $x, y, z$ are nonnegative integers. Then $2x + 2y + 2z = 10$ or $x + y + z = 5$, which has $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$ ordered triples of solutions in nonnegative integers.

7. David decides he wants to join the West African Drumming Ensemble, and thus he goes to the store and buys three large cylindrical drums. In order to ensure none of the drums drop on the way home, he ties a rope around all of the drums at their mid sections so that each drum is next to the other two. Suppose that each drum has a diameter of 3.5 feet. David needs $m$ feet of rope. Given that $m = a\pi + b$, where $a$ and $b$ are rational numbers, find sum $a + b$.

   **Solution.** The answer is $3.5 + 10.5 = \boxed{14}$. The rope band is formed by six parts – three circular parts and three linear parts. Because three drums are congruent to each other, by symmetry, the three circular parts are congruent to each other and three linear parts are congruent to each other. (See the left-hand side figure shown below.) Moving along this band exactly once, one turns around 360 degrees. Because there is no changing of direction along the linear parts, each circular part of the band is one third of a full circle with diameter 3.5 feet. The remaining three parts are line segments. The length of each of the linear parts of the band is equal to the distance between the center of the drums. Therefore, the total length of the rope is $10.5 + 3.5\pi$.

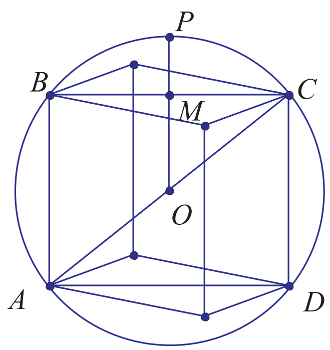


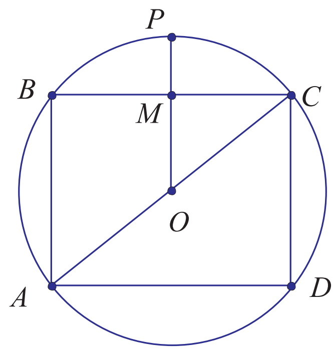



8. Segment $AB$ is the diameter of a semicircle of radius 24. A beam of light is shot from a point $12\sqrt{3}$ from the center of the semicircle, and perpendicular to $AB$. How many times does it reflect off the semicircle before hitting $AB$ again?

   **Solution.** The answer is $\boxed{3}$. Note that the beam of light will trace a half of a regular hexagon. (See the right-hand side figure shown above. The path of the light beam is $P \to P_1 \to P_2 \to P_3 \to Q$.

9. A cube is inscribed in a sphere of radius 8. A smaller sphere is inscribed in the same sphere such that it is externally tangent to one face of the cube and internally tangent to the larger sphere. The maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere can be written in the form $\frac{a-\sqrt{b}}{36}$, where $a$ and $b$ are positive integers. Find the product $ab$.




   **Solution.** The answer is $9 \cdot 75 = \boxed{675}$. Let $s$ denote the side length of the cube. The length of a face diagonal $BC$ is $s\sqrt{2}$ and the length of an interior diagonal $AC$ is $s\sqrt{3}$. Note that $AC$ is also a diameter of the sphere. Hence, $s\sqrt{3} = 16$ or $s = \frac{16}{\sqrt{3}}$. Let $M$ be the midpoint of segment $BC$. The maximum value of the diameter of the smaller sphere is equal to $PM = OP - OM = r - \frac{s}{2} = 8 - \frac{8}{\sqrt{3}}$, and the maximum value of the ratio of the volume of the smaller sphere to the volume of the larger sphere is equal to

$$\left(\frac{4-\frac{4}{\sqrt{3}}}{8}\right)^3 = \left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^3 = \frac{6\sqrt{3}-10}{24\sqrt{3}} = \frac{9-5\sqrt{3}}{36} = \frac{9-\sqrt{75}}{36}.$$

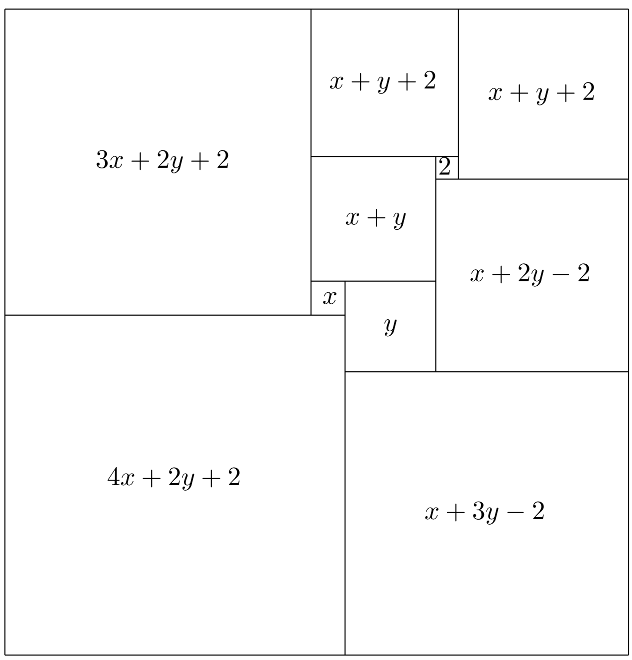
10. How many ordered pairs $(x, y)$ of integers are there such that $2xy + x + y = 52$?

    **Solution.** The answer is $\boxed{16}$. We can write the given function as $4xy + 2x + 2y + 1 = 105$ or $(2x + 1)(2y + 1) = 105$. Note that $105 = 3 \cdot 5 \cdot 7$ has a total of $2 \cdot 2 \cdot 2 \cdot 2 = 16$ (positive and negative) integer divisors.

11. Three musketeers looted a caravan and walked off with a chest full of coins. During the night, the first musketeer divided the coins into three equal piles, with one coin left over. He threw it into the ocean and took one of the piles for himself, then went back to sleep. The second musketeer woke up an hour later. He divided the remaining coins into three equal piles, and threw out the one coin that was left over. He took one of the piles and went back to sleep. The third musketeer woke up and divided the remaining coins into three equal piles, threw out the extra coin, and took one pile for himself. The next morning, the three musketeers gathered around to divide the coins into three equal piles. Strangely enough, they had one coin left over this time as well. What is the minimum number of coins that were originally in the chest?

    **Solution.** The answer is $\boxed{79}$. Assume that there are $x$ coins in the beginning. The first musketeer threw away a coin, took $\frac{x-1}{3}$ coins and left $\frac{2x-2}{3}$ coins. The second musketeer threw away one coin, took $\frac{2x-5}{9}$ coins and left $\frac{4x-10}{9}$ coins. The third musketeer threw away one coin, took $\frac{4x-19}{27}$ coins and left $\frac{8x-38}{27}$ coins. They then threw away one coin and each got $\frac{8x-65}{81}$ coins. Thus, $8x - 65$ must be a multiple of 81. Because $8x - 65 = 8x + 16 - 81 = 8(x + 2) - 81$, the minimum value of $x$ for $8x - 65$ being a multiple of 81 is $x = 79$.

12. The diagram shows a rectangle that has been divided into ten squares of different sizes. The smallest square is $2 \times 2$ (marked with $*$). What is the area of the rectangle (which looks rather like a square itself)?



**Solution.** The answer is $55 \cdot 57 = \boxed{3135}$. We assume that the two center squares have side lengths $x$ and $y$, respectively. We can then express all the side lengths in terms of $x$ and $y$. (See the figure shown above.) Because the opposite sides of the rectangle have the same lengths, we obtain the system of equations

$$\begin{cases} (3x+2y+2)+(x+y+2)+(x+y+4) = (4x+2y+2)+(x+3y-2), \\ (3x+2y+2)+(4x+2y+2) = (x+y+4)+(x+2y-2)+(x+3y-2) \end{cases}$$

or

$$\begin{cases} 5x+4y+8 = 5x+5y, \\ 7x+4y+4 = 3x+6y \end{cases}$$

from which it follows that $(x, y) = (3, 8)$ and that it is a $55 \times 57$ rectangle.

13. Let $A = (3, 2)$, $B = (0, 1)$, and $P$ be on the line $x + y = 0$. What is the minimum possible value of $AP + BP$?

    **Solution.** The answer is $\boxed{2\sqrt{5}}$. Set $C = (-1, 0)$. Note that $B$ and $C$ are reflections of each other across the line $x + y = 0$ and that points $A$ and $C$ lie on opposite sides of the line. Hence $AP + BP = AP + PC \le AC = 2\sqrt{5}$.

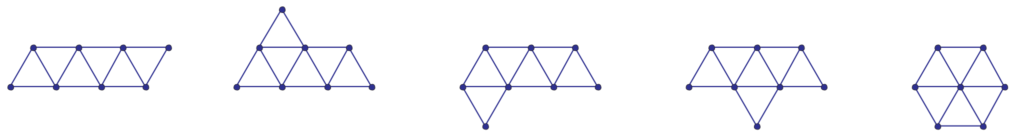
14. Mr. Mustafa the number man got a $6 \times x$ rectangular chess board for his birthday. Because he was bored, he wrote the numbers 1 to $6x$ starting in the upper left corner and moving across row by row (so the number $x+1$ is in the $2^{\text{nd}}$ row, $1^{\text{st}}$ column). Then, he wrote the same numbers starting in the upper left corner and moving down each column (so the number 7 appears in the $1^{\text{st}}$ row, $2^{\text{nd}}$ column). He then added up the two numbers in each of the cells and found that some of the sums were repeated. Given that $x$ is less than or equal to 100, how many possibilities are there for $x$?

    **Solution.** The answer is $\boxed{14}$. Let $(i, j)$ denote the cell in the $i^{\text{th}}$ row and $j^{\text{th}}$ column. Let $a_{i,j}$ denote the number written in the cell $(i, j)$ in the first scheme, and let $b_{i,j}$ denote the number written in the cell $(i, j)$ in the second scheme. Then we have $a_{i,j} = (i-1)x + j$ and $b_{i,j} = 6(j-1) + i$. The sum of the numbers written in the cell $(i, j)$ is equal to $a_{i,j} + b_{i,j} = (i-1)x + j + 6(j-1) + i = i(x+1) + 7j - x - 6$. Assume that the sums of the numbers written in cells $(i_1, j_1)$ and $(i_2, j_2)$ are equal. We have $i_1(x+1) + 7j_1 - x - 6 = i_2(x+1) + 7j_2 - x - 6$ or $(i_1 - i_2)(x+1) = 7(j_2 - j_1)$. Because $i_1$ and $i_2$ are row numbers, their positive difference is less than 6. In particular, $i_1 - i_2$ is not divisible by 7. Hence $x + 1$ must be a multiple of 7. The possible values of $x$ are $6, 13, 20, \ldots, 97$. It is easy to check that each of these values indeed works.

15. Six congruent equilateral triangles are arranged in the plane so that every triangle shares at least one whole edge with some other triangle. Find the number of distinct arrangements. (Two arrangements are considered the same if one can be rotated and/or reflected onto another.)

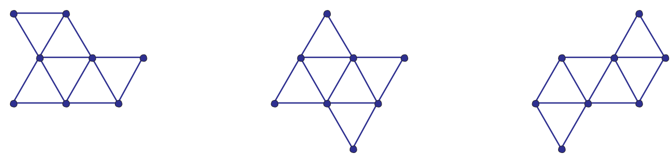
    **Solution.** The answer is $\boxed{12}$. For an arrangement of this six equilateral triangles, we define its *diameter* as the maximum number of triangles appeared in row in t his arrangement. In any arrangement, some two triangles must be in a row, and by symmetry, a third triangle must be attached to these two triangle to form a row of three triangle.

Hence the diameter is equal to 3, 4, 5, or 6. If the diameter is equal to 3, 5, or 6, it is not difficult to see the following 5 arrangements:



If the diameter is equal to 4, we have 7 arrangements. There are 4 arrangements with diameter 4 and two additional triangles on the same side of the diameter.

There are 3 arrangements with diameter 4 and two additional triangles on the opposite sides of the diameter.



## 3.4 Guts Round Solutions

1. [5pts] Define the operation $\clubsuit$ so that $a \clubsuit b = a^b + b^a$. Then, if $2 \clubsuit b = 32$, what is $b$?

   **Solution.** The answer is $\boxed{4}$. Note that $2 \clubsuit b = 2^b + b^2 = 2^4 + 4^2$. (Why is $b = 4$ the unique solution?)

2. [5pts] A square is changed into a rectangle by increasing two of its sides by $p\%$ and decreasing the two other sides by $p\%$. The area is then reduced by 1%. What is the value of $p$?

   **Solution.** The answer is $\boxed{10}$. We have $(1-p\%)(1+p\%) = 1 - 1\%$; that is, $(p\%)^2 = 1\%$ or $p\% = 10\%$.

3. [5pts] What is the sum, in degrees, of the internal angles of a heptagon?

   **Solution.** The answer is $180 \cdot 5 = \boxed{900}$. The heptagon can be dissected into 5 triangles.

4. [5pts] How many integers in between $\sqrt{47}$ and $\sqrt{8283}$ are divisible by 7?

   **Solution.** The answer is $\boxed{13}$. Note that $\sqrt{47} < \sqrt{49} = 7 \cdot 1$ and $\sqrt{8283} > \sqrt{8281} = 91 = 7 \cdot 13$.

5. [8pts] Some mutant green turkeys and pink elephants are grazing in a field. Mutant green turkeys have six legs and three heads. Pink elephants have 4 legs and 1 head. There are 100 legs and 37 heads in the field. How many animals are grazing?

   **Solution.** The answer is $8 + 13 = \boxed{21}$. Assume that there are $t$ turkeys and $e$ elephants. Then $6t + 4e = 100$ and $3t + e = 37$, from which it follows that $(t, e) = (8, 13)$.

6. [8pts] Let $A = (0, 0)$, $B = (6, 8)$, $C = (20, 8)$, $D = (14, 0)$, $E = (21, -10)$, and $F = (7, -10)$. Find the area of the hexagon $ABCDEF$.

   **Solution.** The answer is $14 \cdot 18 = \boxed{252}$. The hexagon $ABCDEF$ can be dissected into two parallelograms $ABCD$ and $ADEF$. The area of parallelogram $ABCD$ is equal $14 \cdot 8$ and the area of parallelogram $ADEF$ is equal to $14 \cdot 10$.

7. [8pts] In Moscow, three men, Oleg, Igor, and Dima, are questioned on suspicion of stealing Vladimir Putin's blankie. It is known that each man either always tells the truth or always lies. They make the following statements:

   (a) Oleg: I am innocent!
   (b) Igor: Dima stole the blankie!
   (c) Dima: I am innocent!
   (d) Igor: I am guilty!
   (e) Oleg: Yes, Igor is indeed guilty!

   If exactly one of Oleg, Igor, and Dima is guilty of the theft, who is the thief?

   **Solution.** The answer is $\boxed{\text{Oleg}}$. Igor made two contradicting statements. Hence Igor always lies and Igor is not guilty, from which it follows that Oleg lies and he is guilty.

8. [8pts] How many 11-letter sequences of E's and M's have at least as many E's as M's?

**Solution.** The answer is $\dfrac{2^{11}}{2} = \boxed{1024}$. There are $2^{11}$ sequence of E's and M's. By symmetry, half them have more E's than M's. (Because 11 is odd, we cannot have equal number of E's and Ms in a 11-letter sequence of E's and M's.)

9. [11pts] John is entering the following summation $31+32+33+34+35+36+37+38+39$ in his calculator. However, he accidently leaves out a plus sign and the answer becomes 3582. What is the number that comes before the missing plus sign?

**Solution.** The answer is $\boxed{33}$. If the $+$ in between the numbers $\overline{3a}$ and $\overline{3b}$ is left out, we entered $\overline{3a3b}$ and increased the sum by $\overline{3a3b} - (\overline{3a} + \overline{3b}) = 100 \cdot \overline{3a} + ol3b - (\overline{3a} + \overline{3b}) = 99 \cdot \overline{3a}$. Note that

$$3582 - (31+32+33+34+35+36+37+38+39) = 3582 - 315 = 3267 = 99 \cdot 33.$$

10. [11pts] Two circles of radius 6 intersect such that they share a common chord of length 6. The total area covered may be expressed as $a\pi + \sqrt{b}$, where $a$ and $b$ are integers. What is $a+b$?

**Solution.** The answer is $60 + 972 = \boxed{1032}$. The region covered by the two circles can be dissected into two congruent equilateral triangles of side length 6 and two congruent circular sectors of radius 6 and central angle $300^\circ$. Hence the total area covered is equal to

$$\frac{2 \cdot \frac{5}{6} \cdot 36\pi + 2 \cdot 36 \cdot \sqrt{3}}{4} = 60\pi + 18\sqrt{3} = 60\pi + \sqrt{972}.$$

11. [11pts] Alice has a rectangular room with 6 outlets lined up on one wall and 6 lamps lined up on the opposite wall. She has 6 distinct power cords (red, blue, green, purple, black, yellow). If the red and green power cords cannot cross, how many ways can she plug in all six lamps?

**Solution.** The answer is $\dfrac{6! \cdot 6!}{2} = \boxed{259200}$.

12. [11pts] Tracy wants to jump through a line of 12 tiles on the floor by either jumping onto the next block, or jumping onto the block two steps ahead. An example of a path through the 12 tiles may be: 1 step, 2 steps, 2 steps, 2 steps, 1 step, 2 steps, 2 steps. In how many ways can Tracy jump through these 12 tiles?

**Solution.** The answer is $\boxed{233}$. For $n \geq 1$, let $f_n$ denote the number of ways Tracy can jump through through a line of $n$ tiles. There are two choices for Tracy's first jump – jumping to the next block or jumping onto the block two steps ahead. In the former case, she has $f_{n-1}$ ways to jump through the rest of $n-1$ tiles. In the later case, she has $f_{n-2}$ ways to jump through the rest of $n-2$ tiles. Therefore, $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$. It is easy to compute that $f_1 = 1$, $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, $f_7 = 21$, $f_8 = 34$, $f_9 = 55$, $f_{10} = 89$, $f_{11} = 144$, and $f_{12} = 233$. (This is the famous Fibonacci sequence.)

13. [14pts] What is the units digit of the number $(2^1+1)(2^2-1)(2^3+1)(2^4-1)\ldots(2^{2010}-1)$?

**Solution.** The answer is $\boxed{5}$. All the multiplicands are odd and the units digit of $2^4 - 1$ is 5.

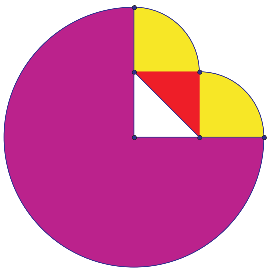
14. [14pts] Mr. Fat noted that on January 2, 2010, the display of the day is 01/02/2010, and the sequence 01022010 is a palindrome (a number that reads the same forwards and backwards). How many days does Mr. Fat need to wait between this palindrome day and the last palindrome day of this decade?

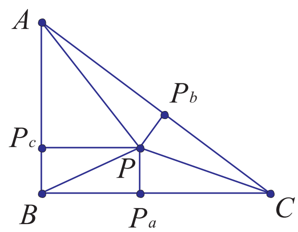
**Solution.** The answer is $2 \cdot 365 - 30 - 31 = \boxed{669}$. The last palindrome day of this decade is 11/02/2011, which is exactly 22 months (a November and December short of two full years) away from 01/02/2010.

15. [14pts] Farmer Tim has a 30-meter by 30-meter by $30\sqrt{2}$-meter triangular barn. He ties his goat to the corner where the two shorter sides meet with a 60-meter rope. What is the area, in square meters, of the land where the goat can graze, given that it cannot get inside the barn?

**Solution.** The answer is $\boxed{3150\pi + 450}$. The grazing area consists of one 270° circular sector region of radius 60 meters, two 90° circular sector regions of radius 30 meters, and one 30-meter by 30-meter by $30\sqrt{2}$-meter triangular region. (See the left-hand side figure shown below.) The total area is equal to

$$\frac{3}{4} \cdot 3600\pi + 2 \cdot \frac{1}{4} 900\pi + 450 = 3150\pi + 450.$$






16. [14pts] In triangle $ABC$, $AB = 3$, $BC = 4$, and $CA = 5$. Point $P$ lies inside the triangle and the distances from $P$ to two of the sides of the triangle are 1 and 2. What is the maximum distance from $P$ to the third side of the triangle?

**Solution.** The answer is $\boxed{\dfrac{2}{5}}$. Let $P_a, P_b, P_c$ be the feet of the perpendiculars from $P$ to sides $BC, CA, AB$, respectively. Set $x = PP_a$, $y = PP_b$, and $z = PP_c$. Because the sum of the areas of triangles $PAB, PBC, PCA$ is equal to the area of triangle $ABC$, we have $4x + 5y + 3z = 12$. Two of $x, y, z$ take the values 1 and 2, and we want to maximize the third. If $y = 2$, then $4x + 5y + 3z \geq 13$, which is not possible. If $y = 1$, then $4x + 3z = 7$, implying that $z = 2$ and $x = \frac{1}{4}$. If $y$ is unknown, then $5y = 12 - (4x + 3z)$ is equal to either 2 (with $(x, z) = (1, 2)$) or 1 (with $(x, z) = (2, 1)$), implying that $y = \frac{2}{5}$ or $y = \frac{1}{5}$.

17. [17pts] Let $Z$ be the answer to the third question on this guts quadruplet. If $x^2 - 2x = Z - 1$, find the positive value of $x$.

**Solution.** The answer is $\boxed{17}$. We can rearrange the given equation to get $x^2 - 2x + 1 = Z$, or $(x-1)^2 = Z$, so $x = \sqrt{Z} + 1$.

18. [17pts] Let $X$ be the answer to the first question on this guts quadruplet. To make a FATRON2012, a cubical steel body as large as possible is cut out from a solid sphere of diameter $X$. A TAFTRON2013 is created by cutting a FATRON2012 into 27 identical cubes, with no material wasted. What is the length of one edge of a TAFTRON2013?

**Solution.** The answer is $\boxed{\dfrac{17\sqrt{3}}{9}}$. We see that each side of the FATRON2012 has side length $\frac{X}{\sqrt{3}}$. Thus, each side of the TAFTRON2013 is $\frac{1}{3} \cdot \frac{X}{\sqrt{3}} = \frac{X\sqrt{3}}{9}$.

19. [17pts] Let $Y$ be the smallest integer greater than the answer to the second question on this guts quadruplet. Fred posts two distinguishable sheets on the wall. Then, $Y$ people walk into the room. Each of the $Y$ people signs up on 0, 1, or 2 of the sheets. Given that there are at least two people in the room other than Fred, how many possible pairs of lists can Fred have?

**Solution.** The answer is $\boxed{256}$.

We know that each person can sign up on neither of the sheets, only the first sheet, only the second sheet, or both sheets. Thus, there are $Z = 4^Y$ possible sets of lists.

By the definitions of $X$ and $Y$, we obtain that

$$Y - 1 \leq \frac{X\sqrt{3}}{9} = \frac{(2^Y + 1)\sqrt{3}}{9} = \frac{2^Y + 1}{3\sqrt{3}} \leq Y$$

or

$$3\sqrt{3}Y - 3\sqrt{3} - 1 \leq 2^Y \leq 3\sqrt{3}Y - 1, \tag{3.1}$$

implying that $Y = 4$ (and consequently, $Z = 4^4 = 256$, $X = 17$).

Intuitively, because the exponential function $2^Y$ grows much faster than the liner function $3\sqrt{3}Y - 1$, there are very few integer values of $Y$ satisfying (3.1). It is not difficult to check that $Y = 4$ is the only integer answer to (3.1) such that $Y$ is at least 2.

Rigorously, we can show that $2^n > 6n > 6n - 1 > 3\sqrt{3}n - 1$ for $n \geq 6$. (Hence the only possible values of $Y$ satisfying (3.1) are $Y = 1, 2, 3, 4, 5$. By a quick check, we can conclude that $Y = 4$ is the only solution for $Y$ at least 2.) To show that $2^n > 6n$ for $n \geq 6$, we induct on $n$. For $n = 6$, this is clearly true. Assume that $2^n > 6n$ is true for some integer $n = k \geq 6$; that is, $2^k > 6k$ and $k \geq 6$. Then for $n = k + 1$, we have $2^{k+1} = 2 \cdot 2^k > 2 \cdot 6k > 6(k + 1)$.

20. [17pts] Let $A$, $B$, $C$, be the respective answers to the first, second, and third questions on this guts quadruplet. At the Robot Design Convention and Showcase, a series of robots are programmed such that each robot shakes hands exactly once with every other robot of the same height. If the heights of the 16 robots are 4, 4, 4, 5, 5, 7, 17, 17, 17, 34, 34, 42, 100, $A$, $B$, and $C$ feet, how many handshakes will take place?

**Solution.** The answer is $\binom{3}{2} + \binom{2}{2} + \binom{4}{2} + \binom{2}{2} = \boxed{11}$. We know that $A = 17$, $B = \frac{17\sqrt{3}}{9}$, and $C = 256$. Thus, we have 3 robots of height 4, 2 robots of height 5, 4 robots of height 17, and 2 robots of height 34.

21. [20pts] Determine the number of ordered triples $(p, q, r)$ of primes with $1 < p < q < r < 100$ such that $q - p = r - q$.

**Solution.** The answer is $10 + 22 + 14 = \boxed{46}$. Let $d = q - p = r - q$ be the common difference. Because both $r$ and $q$ must be odd, $d$ must be even.

If $d$ is not a multiple of 3, then one of $p, q, r$ is a multiple of 3. Because $p < q < r$ are primes, we must have $p = 3$. It is not difficult to list all 10 such triples, namely, $(3, 5, 7)$, $(3, 7, 11)$, $(3, 11, 19)$, $(3, 13, 23)$, $(3, 17, 31)$, $(3, 23, 46)$, $(3, 31, 59)$, $(3, 37, 71)$, $(3, 41, 79)$, $(3, 43, 83)$,

If $d$ is a multiple of 3, then $d$ is a multiple of 6. We classify the primes greater than 3 and less than 100 modulo 6:

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $6k-1$ | 5 | 11 | 17 | 23 | 29 | | 41 | 47 | 53 | 59 | | 71 | | 83 | 89 | |
| $6k+1$ | 7 | 13 | 19 | | 31 | 37 | 43 | | | 61 | 67 | 73 | 79 | | | 97 |

Note that the primes $6k_1 \pm 1, 6k_2 \pm 1, 6k_3 \pm 1$ form an arithmetic progression if and only if the numbers $k_1, k_2, k_3$ form an arithmetic progression.

To count the 3-term arithmetic progressions of primes in the from $6k - 1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 14, 15\}$. There are 22 such triples, namely,

$$(1,2,3), (1,3,5), (1,4,7), (1,5,9), (1,8,15); (2,3,4), (2,5,8), (2,7,12), (2,8,14);$$
$$(3,4,5), (3,5,7), (3,9,15); (4,7,10), (4,8,12), (4,9,14); (5,7,9), (5,10,15);$$
$$(7,8,9); (8,9,10), (8,10,12); (9,12,15); (10,12,14).$$

To count the 3-term arithmetic progressions of primes in the from $6k + 1$, we count all the 3-term arithmetic progressions in the set $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 16\}$. There are 14 such triples, namely,

$$(1,2,3), (1,3,5), (1,6,11), (1,7,13); (2,6,10), (2,7,12);$$
$$(3,5,7), (3,7,11); (5,6,7); (6,11,16); (7,10,13); (10,11,12), (10,13,16); (11,12,13).$$

22. [20pts] For numbers $a$, $b$, $c$, $d$ such that $0 \leq a, b, c, d \leq 10$, find the minimum value of $ab + bc + cd + da - 5a - 5b - 5c - 5d$.

**Solution.** The answer is $\boxed{-100}$. We have

$$\begin{aligned} & ab + bc + cd + da - 5a - 5b - 5c - 5d = (a + c)(b + d) - 5(a + c) - 5(b + d) \\ = \; & (a + c - 5)(b + d - 5) - 25 \geq (-5) \cdot 15 - 25 = -100. \end{aligned}$$

23. [20pts] Daniel has a task to measure 1 gram, 2 grams, 3 grams, 4 grams , ..., all the way up to $n$ grams. He goes into a store and buys a scale and six weights of his choosing (so that he knows the value for each weight that he buys). If he can place the weights on either side of the scale, what is the maximum value of $n$?

**Solution.** The answer is $\frac{3^6-1}{2} = \boxed{364}$. We see that we can put each weight on either the right side of the scale, the left side of the scale, or neither, for $3^6$ possible combinations. However, we need to exclude the case where there are no weights on the scale, leaving $3^6-1$ ways. Without loss of generality, we always want to weight our object on the right side of the scale, so we always want the placement of the weights to leave the left side heavier; by symmetry, this only happens in half the cases we are currently counting, so we have at most $\frac{3^6-1}{2}$ different masses we can weight. We can check that using the set $\{3^0, 3^1, 3^2, 3^3, 3^4, 3^5\}$ will let us weight any mass between 1 and 364: The weight $3^0$ will let us weight objects of mass 1. Then, adding the weight $3^1$ will then let us weight objects from $3^1 - 3^0$ to $3^1 + 3^0$ (from 2 to 4). Adding the weight of $3^2$ will then let us weight objects from $3^2 - (3^1 + 3^0)$ to $3^2 + 3^1 + 3^0$ (from 5 to 13), and so on.

24. [20pts] Given a Rubik's cube, what is the probability that at least one face will remain unchanged after a random sequence of three moves? (A Rubik's cube is a 3 by 3 by 3 cube with each face starting as a different color. The faces (3 by 3) can be freely turned. A move is defined in this problem as a 90 degree rotation of one *face either clockwise or counter-clockwise. The center square on each face–six in total–is fixed.)*

**Solution.** The answer is $\boxed{\frac{2}{9}}$.

For each move, we can select one of the six faces and make either a 90 degree clockwise rotation or a 90 degree counter-clockwise rotation, implying that there are 12 distinct options. Thus, there are $12^3$ distinct sequence of three moves.

We consider those sequences of three moves that leave at least one face unchanged. We consider three pairs of parallel faces $A_1$ and $A_2$, $B_1$ and $B_2$, and $C_1$ and $C_2$. The key observation is that, after three moves, if one face in a pair is unchanged, then so does the other face in the pair. Indeed, by symmetry, we may assume without loss of generality that $A_1$ is unchanged. We categorize the moves into two types:

- moves on faces $A_1$ and $A_2$. There are 4 such moves, and they are all denoted by $T$;
- moves on the other four faces. There are 8 such moves, and they are all denoted by $S$.

For $A_1$ to be left unchanged, we have either three $T$ moves or a $T$ move followed by a $S$ move and then its inverse move $S-1$ or a $S$ move, its inverse move $S^{-1}$, and then a $T$ move. In each case, face $A_2$ is also unchanged.

In summary, we can choose one of the three pairs of parallel faces to be unchanged. For each chosen pair of parallel faces, there are $4^3 + 4 \cdot 8 \cdot 1 + 8 \cdot 1 \cdot 4 = 2 \cdot 4^3$ distinct sequence of three moves to have the chosen pair of faces unchanged. Hence the desired probability is equal to

$$\frac{3 \cdot 2 \cdot 4^3}{12^3} = \frac{3 \cdot 2}{3^3} = \frac{2}{9}.$$

# Exeter Math Club Contest
# January 29, 2011

# Contents

# Organizing Information

- *Tournament Directors* In Young Cho, Shijie (Joy) Zheng
- *Tournament Supervisor* Zuming Feng
- *System Administrator and Webmaster* Albert Chu
- *Problem Committee* Ravi Bajaj, Chong Gu, Ravi Jagadeesan, Gwangseung (Eric) Kim, Yong Wook (Spencer) Kwon, Ray Li, Abraham Shin, Dai Yang, David Yang
- *Solutions Editors* Ravi Bajaj, Ravi Jagadeesan, Yong Wook (Spencer) Kwon, Ray Li, Dai Yang
- *Problem reviewers* Zuming Feng, Chris Jeuell, Richard Parris, Allen Yuan
- *Problem Contributors* Ravi Bajaj, Mickey Chao, Ravi Charan, Jiapei Chen, In Young Cho, Albert Chu, Mihail Eric, Zuming Feng, Jiexiong (Chelsea) Ge, Ravi Jagadeesan, Harini Kannan, Gwangseung (Eric) Kim, Adisa Kruayatidee Yong Wook (Spencer) Kwon, Harlin Lee, Harold Li, Ray Li, Dan Lu, Abraham Shin, Anubhav Sinha, Potcharapol (Neung) Suteparuk, David Xiao, Lealia Xiong, Dai Yang, David Yang, Will Zhang, Shijie (Joy) Zheng
- *Treasurer* Chong Gu
- *Publicity* Claudia Feng, Jiexiong (Chelsea) Ge, Chieh-Ming (Jamin) Liu
- *Participating Organizations* Acton, Bigelow Middle School, Chesapeake Science Point Charter School, Century Chinese Language School, Debrown, Eaglebrook School, Fudan School, Homeschool, Honey Creek School, IDEAMath, Jonas Clarke Middle School, Lincoln Middle School, Math Girls, R.J. Grey Jr. High School, Roxbury Latin Middle School, The Sage School, Sycamore School, William Diamond Middle School
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.
- *Tournament Sponors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

# Contest day information

- *Proctors*

| Proctor | Team 1 (code) | Team 2 (code) | Location |
|---|---|---|---|
| Ted Lee *(Head Proctor)* | Clarke A (7) | Sycamore-Honey Creek (175) | ACD 008 (Swift) |
| Will Zhang *(Head Proctor)* | Clarke B (11) | Dolphin (19) | ACD 007 (Seidenberg) |
| Gwangseung (Eric) Kim | Panda (23) | Bigelow (31) | ACD 109 (Marshall) |
| Payut (Paul) Pantawongdecha | Penguin (29) | Lincoln (41) | ACD 004 (Feng) |
| Theo Motzkin | Eaglebrook 1 (43) | debrown (101) | ACD 102 (Girard) |
| Lucy Yu | Eaglebrook 2 (47) | Fudan 1 (166) | ACD 105 (Kaminski) |
| Harlin Lee | Infinity 1 (132) | Sycamore (134) | ACD 108 (Keeble) |
| Harini Kannan | Infinity 2 (133) | Clarke C (136) | ACD 009 (Coogan) |
| Jiapei Chen | Clarke D (137) | Fudan 2 (167) | ACD 104 (Spanier) |
| Adisa Kruayatidee | Infinity 3 (141) | Fudan 3 (168) | ACD 106 (Chen) |
| Shi-Fan Chen | Infinity 4 (142) | Fudan 4 (169) | ACD 021 (Stahr) |
| Theo Gao | Honey Creek (143) | Liu-hui (152) | ACD 103 (Wolfson) |
| Ki Hong Ahn | Zu-Chong-zhi (153) | WMD (156) | ACD 006 (Parris) |
| Eugenia Kang | R J Grey (160) | Math Girls (162) | ACD 206 (Bergofsky) |
| Kaitlin Kimberling | Epic Ninja Cows (164) | AHA (172) | ACD 207 (Holden) |
| Claudia Feng | Sage 2 (165) | Diamond (174) | ACD 107 (Mallinson) |
| Elizabeth Gong | Infinity Indiv. (176) | | ACD 126 (Golay) |

- *Head Graders* Albert Chu, Yong Wook (Spencer) Kwon,
- *Graders* Ravi Charan, Andrew Holzman, Ravi Jagadeesan, Daniel Li, Ray Li, Abraham Shin, David Xiao, Dai Yang, David Yang
- *Judges* Zuming Feng, Greg Spanier
- *Runners* Michelle (Hannah) Jung *(Head Runner)*, Jiexiong (Chelsea) Ge, Lealia Xiong
- *Breakfast and Lunch Setup* Jiexiong (Chelsea) Ge, Yvonne Guu, Chanthana Kruayatidee
- *Lunch Vouchers and Coaches/Parents Room* Ravi Bajaj, Chong Gu

# Chapter 1

# EMC$^2$ 2011 Problems

## 1.1 Individual Speed Test

**Morning, January 29, 2011**

*There are 20 problems, worth 3 points each, and 20 minutes to solve as many problems as possible.*

1. Euclid eats $\frac{1}{7}$ of a pie in 7 seconds. Euler eats $\frac{1}{5}$ of an identical pie in 10 seconds. Who eats faster?

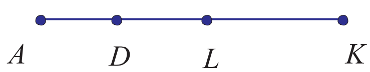
2. Given that $\pi = 3.1415926\ldots$, compute the circumference of a circle of radius 1. Express your answer as a decimal rounded to the nearest hundred thousandth (i.e. 1.234562 and 1.234567 would be rounded to 1.23456 and 1.23457, respectively).

3. Alice bikes to Wonderland, which is 6 miles from her house. Her bicycle has two wheels, and she also keeps a spare tire with her. If each of the three tires must be used for the same number of miles, for how many miles will each tire be used?

4. Simplify $\frac{2010 \cdot 2010}{2011}$ to a mixed number. (For example, $2\frac{1}{2}$ is a mixed number while $\frac{5}{2}$ and 2.5 are not.)

5. There are currently 175 problems submitted for EMC$^2$. Chris has submitted 51 of them. If nobody else submits any more problems, how many more problems must Chris submit so that he has submitted $\frac{1}{3}$ of the problems?

6. As shown in the diagram below, points $D$ and $L$ are located on segment $AK$, with $D$ between $A$ and $L$, such that $\frac{AD}{DK} = \frac{1}{3}$ and $\frac{DL}{LK} = \frac{5}{9}$. What is $\frac{DL}{AK}$?




7. Find the number of possible ways to order the letters $G, G, e, e, e$ such that two neighboring letters are never $G$ and $e$ in that order.

8. Find the number of odd composite integers between 0 and 50.

9. Bob tries to remember his 2-digit extension number. He knows that the number is divisible by 5 and that the first digit is odd. How many possibilities are there for this number?

10. Al walks 1 mile due north, then 2 miles due east, then 3 miles due south, and then 4 miles due west. How far, in miles, is he from his starting position? (Assume that the Earth is flat.)

11. When $n$ is a positive integer, $n!$ denotes the product of the first $n$ positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$.

12. Sam's phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged \$4.88 for an 11-minute call and \$6.00 for a 19-minute call, how much would he be charged for a 15-minute call?

13. For a positive integer $n$, let $s_n$ be the sum of the $n$ smallest primes. Find the least $n$ such that $s_n$ is a perfect square (the square of an integer).

14. Find the remainder when $2011^{2011}$ is divided by 7.

15. Let $a, b, c$, and $d$ be 4 positive integers, each of which is less than 10, and let $e$ be their least common multiple. Find the maximum possible value of $e$.

16. Evaluate $100 - 1 + 99 - 2 + 98 - 3 + \cdots + 52 - 49 + 51 - 50$.

17. There are 30 basketball teams in the Phillips Exeter Dorm Basketball League. In how ways can 4 teams be chosen for a tournament if the two teams Soule Internationals and Abbot United cannot be chosen at the same time?

18. The numbers $1, 2, 3, 4, 5, 6$ are randomly written around a circle. What is the probability that there are four neighboring numbers such that the sum of the middle two numbers is less than the sum of the other two?

19. What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$?

20. Rhombus $ABCD$ has vertices $A = (-12, -4)$, $B = (6, b)$, $C = (c, -4)$ and $D = (d, -28)$, where $b$, $c$, and $d$ are integers. Find a constant $m$ such that the line $y = mx$ divides the rhombus into two regions of equal area.

## 1.2 Individual Accuracy Test

**Morning, January 29, 2011**

*There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.*

1. What is the maximum number of points of intersection between a square and a triangle, assuming that no side of the triangle is parallel to any side of the square?

2. Two angles of an isosceles triangle measure $80^\circ$ and $x^\circ$. What is the sum of all the possible values of $x$?

3. Let $p$ and $q$ be prime numbers such that $p+q$ and $p+7q$ are both perfect squares. Find the value of $pq$.

4. Anna, Betty, Carly, and Danielle are four pit bulls, each of which is either wearing or not wearing lipstick. The following three facts are true:

   (1) Anna is wearing lipstick if Betty is wearing lipstick.
   (2) Betty is wearing lipstick only if Carly is also wearing lipstick.
   (3) Carly is wearing lipstick if and only if Danielle is wearing lipstick

   The following five statements are each assigned a certain number of points:

   (a) Danielle is wearing lipstick if and only if Carly is wearing lipstick. (This statement is assigned 1 point.)
   (b) If Anna is wearing lipstick, then Betty is wearing lipstick. (This statement is assigned 6 points.)
   (c) If Betty is wearing lipstick, then both Anna and Danielle must be wearing lipstick. (This statement is assigned 10 points.)
   (d) If Danielle is wearing lipstick, then Anna is wearing lipstick. (This statement is assigned 12 points.)
   (e) If Betty is wearing lipstick, then Danielle is wearing lipstick. (This statement is assigned 14 points.)

   What is the sum of the points assigned to the statements that *must* be true? (For example, if only statements (a) and (d) are true, then the answer would be $1+12=13$.)

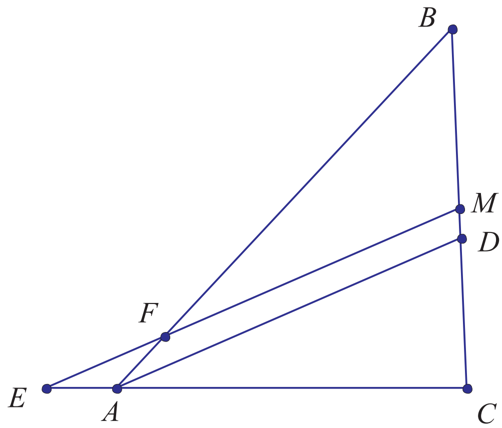
5. Let $f(x)$ and $g(x)$ be functions such that $f(x)=4x+3$ and $g(x)=\dfrac{x+1}{4}$. Evaluate $g(f(g(f(42))))$.

6. Let $A, B, C$, and $D$ be consecutive vertices of a regular polygon. If $\angle ACD = 120^\circ$, how many sides does the polygon have?

7. Fred and George have a fair 8-sided die with the numbers $0, 1, 2, 9, 2, 0, 1, 1$ written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

8. Find the smallest *positive* integer $t$ such that $(23t)^3-(20t)^3-(3t)^3$ is a perfect square.

9. In triangle $ABC$, $AC = 8$ and $AC < AB$. Point $D$ lies on side $BC$ with $\angle BAD = \angle CAD$. Let $M$ be the midpoint of $BC$. The line passing through $M$ parallel to $AD$ intersects lines $AB$ and $AC$ at $F$ and $E$, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment $BC$? (See the following diagram.)




10. There are 2011 evenly spaced points marked on a circular table. Three segments are randomly drawn between pairs of these points such that no two segments share an endpoint on the circle. What is the probability that each of these segments intersects the other two? (See the following diagram.)

## 1.3 Team Test

**Morning, January 29, 2011**

*There are 15 problems, worth 20 points each, and 30 minutes to solve as many problems as possible.*

1. Velociraptor A is located at $x = 10$ on the number line and runs at 4 units per second. Velociraptor B is located at $x = -10$ on the number line and runs at 3 units per second. If the velociraptors run towards each other, at what point do they meet?

2. Let $n$ be a positive integer. There are $n$ non-overlapping circles in a plane with radii 1, 2, $\dots$, $n$. The total area that they enclose is at least 100. Find the minimum possible value of $n$.

3. How many integers between 1 and 50, inclusive, are divisible by 4 but not 6?

4. Let $a \star b = 1 + \dfrac{b}{a}$. Evaluate $((((((1 \star 1) \star 1) \star 1) \star 1) \star 1) \star 1) \star 1$.

5. In acute triangle $ABC$, $D$ and $E$ are points inside triangle $ABC$ such that $DE \parallel BC$, $B$ is closer to $D$ than it is to $E$, $\angle AED = 80^\circ$, $\angle ABD = 10^\circ$, and $\angle CBD = 40^\circ$. Find the measure of $\angle BAE$, in degrees.

6. Al is at $(0, 0)$. He wants to get to $(4, 4)$, but there is a building in the shape of a square with vertices at $(1, 1)$, $(1, 2)$, $(2, 2)$, and $(2, 1)$. Al cannot walk inside the building. If Al is not restricted to staying on grid lines, what is the shortest distance he can walk to get to his destination?

7. Point $A = (1, 211)$ and point $B = (b, 2011)$ for some integer $b$. For how many values of $b$ is the slope of $AB$ an integer?

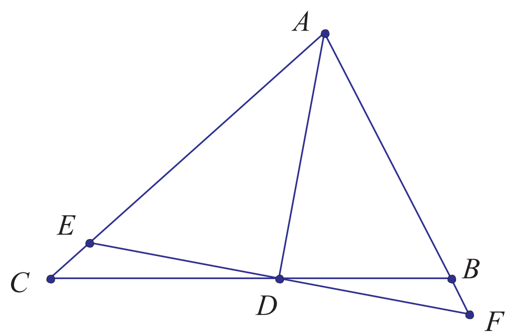
8. A palindrome is a number that reads the same forwards and backwards. For example, $1, 11$ and $141$ are all palindromes. How many palindromes between 1 and 1000 are divisible by 11?

9. Suppose $x, y, z$ are real numbers that satisfy:

$$\begin{aligned} x + y - z &= 5 \\ y + z - x &= 7 \\ z + x - y &= 9 \end{aligned}$$

   Find $x^2 + y^2 + z^2$.

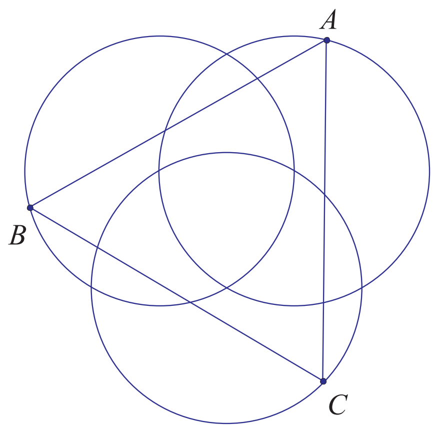
10. In triangle $ABC$, $AB = 3$ and $AC = 4$. The bisector of angle $A$ meets $BC$ at $D$. The line through $D$ perpendicular to $AD$ intersects lines $AB$ and $AC$ at $F$ and $E$, respectively. Compute $EC - FB$. (See the following diagram.)



11. Bob has a six-sided die with a number written on each face such that the sums of the numbers written on each pair of opposite faces are equal to each other. Suppose that the numbers 109, 131, and 135 are written on three faces which share a corner. Determine the maximum possible sum of the numbers on the three remaining faces, given that all three are positive primes less than 200.

12. Let $d$ be a number chosen at random from the set $\{142, 143, \ldots, 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length $d$ is an integer?

13. There are 3 congruent circles such that each circle passes through the centers of the other two. Suppose that $A$, $B$, and $C$ are points on the circles such that each circle has exactly one of $A$, $B$, or $C$ on it and triangle $ABC$ is equilateral. Find the ratio of the maximum possible area of $ABC$ to the minimum possible area of $ABC$. (See the following diagram.)




14. Let $k$ and $m$ be constants such that for all triples $(a, b, c)$ of positive real numbers,

$$\sqrt{\frac{4}{a^2} + \frac{36}{b^2} + \frac{9}{c^2} + \frac{k}{ab}} = \left|\frac{2}{a} + \frac{6}{b} + \frac{3}{c}\right| \quad \text{if and only if} \quad am^2 + bm + c = 0.$$

Find $k$.

15. A bored student named Abraham is writing $n$ numbers $a_1, a_2, \ldots, a_n$. The value of each number is either 1, 2, or 3; that is, $a_i$ is 1, 2 or 3 for $1 \le i \le n$. Abraham notices that the ordered triples

$$(a_1, a_2, a_3), \quad (a_2, a_3, a_4), \quad \ldots, \quad (a_{n-2}, a_{n-1}, a_n), \quad (a_{n-1}, a_n, a_1), \quad (a_n, a_1, a_2)$$

are distinct from each other. What is the maximum possible value of $n$? Give the answer $n$, along with an example of such a sequence. Write your answer as an ordered pair. (For example, if the answer were 5, you might write $(5, 12311)$.)

# 1.4 Guts Test

**Afternoon, January 29, 2011**

### 1.4.1 Round 1

1. [3pts] In order to make good salad dressing, Bob needs a 0.9% salt solution. If soy sauce is 15% salt, how much water, in mL, does Bob need to add to 3 mL of pure soy sauce in order to have a good salad dressing?

2. [3pts] Alex the Geologist is buying a canteen before he ventures into the desert. The original cost of a canteen is \$20, but Alex has two coupons. One coupon is \$3 off and the other is 10% off the entire remaining cost. Alex can use the coupons in any order. What is the least amount of money he could pay for the canteen?

3. [3pts] Steve and Yooni have six distinct teddy bears to split between them, including exactly 1 blue teddy bear and 1 green teddy bear. How many ways are there for the two to divide the teddy bears, if Steve gets the blue teddy bear and Yooni gets the green teddy bear? (The two do not necessarily have to get the same number of teddy bears, but each teddy bear must go to a person.)

### 1.4.2 Round 2

4. [5pts] In the currency of Mathamania, 5 wampas are equal to 3 kabobs and 10 kabobs are equal to 2 jambas. How many jambas are equal to twenty-five wampas?

5. [5pts] A sphere has a volume of $81\pi$. A new sphere with the same center is constructed with a radius that is $\frac{1}{3}$ the radius of the original sphere. Find the volume, in terms of $\pi$, of the region between the two spheres.

6. [5pts] A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?

### 1.4.3 Round 3

7. [6pts] Nick multiplies two consecutive positive integers to get $4^5 - 2^5$. What is the smaller of the two numbers?

8. [6 pts] In rectangle $ABCD$, $E$ is a point on segment $CD$ such that $\angle EBC = 30°$ and $\angle AEB = 80°$. Find $\angle EAB$, in degrees.

9. [6pts] Mary's secret garden contains clones of Homer Simpson and WALL-E. A WALL-E clone has 4 legs. Meanwhile, Homer Simpson clones are human and therefore have 2 legs each. A Homer Simpson clone always has 5 donuts, while a WALL-E clone has 2. In Mary's secret garden, there are 184 donuts and 128 legs. How many WALL-E clones are there?

### 1.4.4 Round 4

10. [7pts] Including Richie, there are 6 students in a math club. Each day, Richie hangs out with a different group of club mates, each of whom gives him a dollar when he hangs out with them. How many dollars will Richie have by the time he has hung out with every possible group of club mates?

11. [7pts] There are seven boxes in a line: three empty, three holding \$10 each, and one holding the jackpot of \$1,000,000. From the left to the right, the boxes are numbered 1, 2, 3, 4, 5, 6 and 7, in that order. You are told the following:

    - No two adjacent boxes hold the same contents.
    - Box 4 is empty.
    - There is one more \$10 prize to the right of the jackpot than there is to the left.

    Which box holds the jackpot?

12. [7pts] Let $a$ and $b$ be real numbers such that $a + b = 8$. Let $c$ be the minimum possible value of $x^2 + ax + b$ over all real numbers $x$. Find the maximum possible value of $c$ over all such $a$ and $b$.

### 1.4.5 Round 5

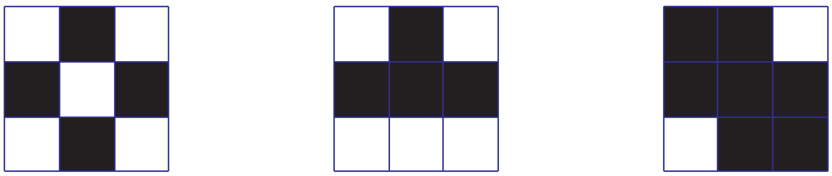
13. [9pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let $M$ be the midpoint of $CD$, and $P$ be a point on $BM$ such that $BP = BC$. Find the area of $ABPD$.

14. [9pts] The number 19 has the following properties:

    - It is a 2-digit positive integer.
    - It is the two leading digits of a 4-digit perfect square, because $1936 = 44^2$.

    How many numbers, including 19, satisfy these two conditions?

15. [9pts] In a $3 \times 3$ grid, each unit square is colored either black or white. A coloring is considered "nice" if there is at most one white square in each row or column. What is the total number of nice colorings? Rotations and reflections of a coloring are considered distinct. (For example, in the three squares shown below, only the rightmost one has a nice coloring.)



### 1.4.6 Round 6

16. [11pts] Let $a_1, a_2, \ldots, a_{2011}$ be a sequence of numbers such that $a_1 = 2011$ and $a_1 + a_2 + \cdots + a_n = n^2 \cdot a_n$ for $n = 1, 2, \ldots 2011$. (That is, $a_1 = 1^2 \cdot a_1$, $a_1 + a_2 = 2^2 \cdot a_2, \ldots$) Compute $a_{2011}$.

17. [11pts] Three rectangles, with dimensions $3 \times 5$, $4 \times 2$, and $6 \times 4$, are each divided into unit squares which are alternately colored black and white like a checkerboard. Each rectangle is cut along one of its diagonals into two triangles. For each triangle, let $m$ be the total black area and $n$ the total white area. Find the maximum value of $|m - n|$ for the 6 triangles.

18. [11pts] In triangle $ABC$, $\angle BAC = 90°$, and the length of segment $AB$ is 2011. Let $M$ be the midpoint of $BC$ and $D$ the midpoint of $AM$. Let $E$ be the point on segment $AB$ such that $EM \parallel CD$. What is the length of segment $BE$?

### 1.4.7 Round 7

19. [12pts] How many integers from 1 to 100, inclusive, can be expressed as the difference of two perfect squares? (For example, $3 = 2^2 - 1^2$).

20. [12pts] In triangle $ABC$, $\angle ABC = 45°$ and $\angle ACB = 60°$. Let $P$ and $Q$ be points on segment $BC$, $F$ a point on segment $AB$, and $E$ a point on segment $AC$ such that $FQ \parallel AC$ and $EP \parallel AB$. Let $D$ be the foot of the altitude from $A$ to $BC$. The lines $AD$, $FQ$, and $PE$ form a triangle. Find the positive difference, in degrees, between the largest and smallest angles of this triangle.

21. [12pts] For real number $x$, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to $x$. For example, $\lceil 3 \rceil = 3$ and $\lceil 2.5 \rceil = 3$. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer $n$ greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where $k$ is a positive integer less than or equal to 2011.

### 1.4.8 Round 8

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

22. [14pts] Let $W$ be the answer to problem 24 in this guts round. Let $f(a) = \frac{1}{1 - \frac{1}{1 - \frac{1}{a}}}$. Determine $|f(2) + \cdots + f(W)|$.

23. [14pts] Let $X$ be the answer to problem 22 in this guts round. How many odd perfect squares are less than $8X$?

24. [14pts] Let $Y$ be the answer to problem 23 in this guts round. What is the maximum number of points of intersections of two regular $(Y - 5)$-sided polygons, if no side of the first polygon is parallel to any side of the second polygon?

### 1.4.9 Round 9

25. [16pts] Cross country skiers $s_1, s_2, s_3, \ldots, s_7$ start a race one by one in that order. While each skier skis at a constant pace, the skiers do not all ski at the same rate. In the course of the race, each skier either overtakes another skier or is overtaken by another skier exactly two times. Find all the possible orders in which they can finish. Write each possible finish as an ordered septuplet $(a, b, c, d, e, f, g)$ where $a, b, c, d, e, f, g$ are the numbers 1-7 in some order. (So $a$ finishes first, $b$ finishes second, etc.)

26. [16pts] Archie the Alchemist is making a list of all the elements in the world, and the proportion of earth, air, fire, and water needed to produce each. He writes the proportions in the form E:A:F:W. If each of the letters represents a whole number from 0 to 4, inclusive, how many different elements can Archie list? Note that if Archie lists wood as 2:0:1:2, then 4:0:2:4 would also produce wood. In addition, 0:0:0:0 does not produce an element.

27. [16pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let $M$ be the midpoint of $CD$, and $P$ be the point on $BM$ such that $DP = DA$. Find the area of quadrilateral $ABPD$.

### 1.4.10 Round 10

28. [17pts] David the farmer has an infinitely large grass-covered field which contains a straight wall. He ties his cow to the wall with a rope of integer length. The point where David ties his rope to the wall divides the wall into two parts of length $a$ and $b$, where $a > b$ and both are integers. The rope is shorter than the wall but is longer than $a$. Suppose that the cow can reach grass covering an area of $\frac{165\pi}{2}$. Find the ratio $\frac{a}{b}$. You may assume that the wall has 0 width.

29. [17pts] Let $S$ be the number of ordered quintuples $(a, b, x, y, n)$ of positive integers such that

$$\begin{aligned}\frac{a}{x} + \frac{b}{y} &= \frac{1}{n}, \\ abn &= 2011^{2011}.\end{aligned}$$

Compute the remainder when $S$ is divided by 2012.

30. [17pts] Let $n$ be a positive integer. An $n \times n$ square grid is formed by $n^2$ unit squares. Each unit square is then colored either red or blue such that each row or column has exactly 10 blue squares. A *move* consists of choosing a row or a column, and recolor each unit square in the chosen row or column – if it is red, we recolor it blue, and if it is blue, we recolor it red. Suppose that it is possible to obtain fewer than $10n$ blue squares after a sequence of finite number of moves. Find the maximum possible value of $n$.

# Chapter 2

# EMC$^2$ 2011 Solutions

## 2.1 Individual Speed Test Solutions

1. Euclid eats $\frac{1}{7}$ of a pie in 7 seconds. Euler eats $\frac{1}{5}$ of an identical pie in 10 seconds. Who eats faster?

   **Solution.** The answer is $\boxed{\text{Euclid}}$. Euclid eats at $\frac{1}{49}$ pies per second; Euler eats at $\frac{1}{50}$ pies per second.

2. Given that $\pi = 3.1415926\ldots$, compute the circumference of a circle of radius 1. Express your answer as a decimal rounded to the nearest hundred thousandth (i.e. 1.234562 and 1.234567 would be rounded to 1.23456 and 1.23457, respectively).

   **Solution.** The answer is $\boxed{6.28319}$. The circumference is $2\pi$.

3. Alice bikes to Wonderland, which is 6 miles from her house. Her bicycle has two wheels, and she also keeps a spare tire with her. If each of the three tires must be used for the same number of miles, for how many miles will each tire be used?

   **Solution.** The answer is $\boxed{4}$. The tires combined travel a total of $6 \cdot 2 = 12$ tire-miles. Alice has 3 tires, so the answer is $\frac{12}{3} = 4$ miles.

4. Simplify $\frac{2010 \cdot 2010}{2011}$ to a mixed number. (For example, $2\frac{1}{2}$ is a mixed number while $\frac{5}{2}$ and 2.5 are not.)

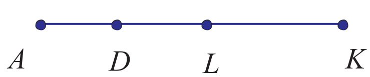
   **Solution.** The answer is $\boxed{2009\frac{1}{2011}}$.

   We have
   $$\frac{2010 \cdot 2010}{2011} = \frac{2010(2011-1)}{2011} = 2010 - \frac{2010}{2011} = \boxed{2009\frac{1}{2011}}.$$

5. There are currently 175 problems submitted for EMC$^2$. Chris has submitted 51 of them. If nobody else submits any more problems, how many more problems must Chris submit so that he has submitted $\frac{1}{3}$ of the problems?

   **Solution.** The answer is $\boxed{11}$. Suppose that Chris submits $x$ more problems. Then,
   $$51 + x = \frac{(175 + x)}{3},$$
   implying that $x = 11$.

6. As shown in the diagram below, points $D$ and $L$ are located on segment $AK$, with $D$ between $A$ and $L$, such that $\frac{AD}{DK} = \frac{1}{3}$ and $\frac{DL}{LK} = \frac{5}{9}$. What is $\frac{DL}{AK}$?



**Solution.** The answer is $\boxed{\frac{15}{56}}$. Without loss of generality, let $AK = 1$. Then, we know that $AD = 1/4$ and $DK = 3/4$. It follows that

$$\frac{DL}{AK} = DL = \frac{5}{14} \cdot DK = \frac{5}{14} \cdot \frac{3}{4} = \frac{15}{56}.$$

7. Find the number of possible ways to order the letters $G, G, e, e, e$ such that two neighboring letters are never $G$ and $e$ in that order.

   **Solution.** The answer is $\boxed{1}$. The only arrangement is $eeGGG$.

8. Find the number of odd composite integers between 0 and 50.

   **Solution.** The answer is $\boxed{10}$. The numbers are $9, 15, 21, 25, 27, 33, 35, 39, 45, 49$.

9. Bob tries to remember his 2-digit extension number. He knows that the number is divisible by 5 and that the first digit is odd. How many possibilities are there for this number?

   **Solution.** The answer is $\boxed{10}$. The first digit is one of $1, 3, 5, 7, 9$ and the last digit is one of $0, 5$ so there are $5 \cdot 2 = 10$ possible extensions.

10. Al walks 1 mile due north, then 2 miles due east, then 3 miles due south, and then 4 miles due west. How far, in miles, is he from his starting position? (Assume that the Earth is flat.)

    **Solution.** The answer is $\boxed{2\sqrt{2}}$. Al's final position is 2 miles south and 2 miles east of his initial position, so the distance is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$.

11. When $n$ is a positive integer, $n!$ denotes the product of the first $n$ positive integers; that is, $n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n$. Given that $7! = 5040$, compute $8! + 9! + 10!$.

    **Solution.** The answer is $\boxed{4032000}$. The answer is

    $$8! + 9! + 10! = 8!(1 + 9 + 9 \cdot 10) = 8! \cdot 100 = 7! \cdot 800 = 4032000.$$

12. Sam's phone company charges him a per-minute charge as well as a connection fee (which is the same for every call) every time he makes a phone call. If Sam was charged \$4.88 for an 11-minute call and \$6.00 for a 19-minute call, how much would he be charged for a 15-minute call?

    **Solution.** The answer is $\boxed{\$5.44}$. Suppose that he is charged $m$ dollars per minute and $n$ dollars for the initial connection fee. Then, we want

    $$15m + n = \frac{30m + 2n}{2} = \frac{(11m + n) + (19m + n)}{2} = \frac{4.88 + 6.00}{2} = 5.44.$$

13. For a positive integer $n$, let $s_n$ be the sum of the $n$ smallest primes. Find the least $n$ such that $s_n$ is a perfect square (the square of an integer).

    **Solution.** The answer is $\boxed{9}$. It is routine to check that

    $$(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9) = (2, 5, 10, 17, 28, 41, 58, 77, 100).$$

    **Note:** The second least $n$ such that $s_n$ is a perfect square is $n = 2474$ with $s_{2474} = 25633969 = 5063^2$. Can you prove that no consecutive terms in the sequence $s_1, s_2, \ldots$ are both perfect squares?

14. Find the remainder when $2011^{2011}$ is divided by 7.

    **Solution.** The answer is $\boxed{2}$. Because 2011 has remainder 2 when it is divided by 7, we need to find the remainder when $2^{2011}$ is divided by 7. It is easy to check that, when divided by 7, the respective remainders of $2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \ldots$ are $2, 4, 1, 2, 4, 1, \ldots$. The remainder of $2^{2011}$ is the $2011^{\text{th}}$ term in this sequence and it is equal to 2.

15. Let $a, b, c$, and $d$ be 4 positive integers, each of which is less than 10, and let $e$ be their least common multiple. Find the maximum possible value of $e$.

    **Solution.** The answer is $\boxed{2520}$. The largest power of 2 possible is $2^3$, and similarly the largest powers of 3, 5, and 7 are $3^2$, $5^1$, and $7^1$, respectively. Thus, the answer is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$, achieved with $\{a, b, c, d\} = \{5, 7, 8, 9\}$.

16. Evaluate $100 - 1 + 99 - 2 + 98 - 3 + \cdots + 52 - 49 + 51 - 50$.

    **Solution.** The answer is $\boxed{2500}$. We know that

    $$100-1+99-2+98-3+\cdots+52-49+51-50 = (100-50)+(99-49)+\cdots+(51-1) = 50\cdot 50 = 2500.$$

17. There are 30 basketball teams in the Phillips Exeter Dorm Basketball League. In how ways can 4 teams be chosen for a tournament if the two teams Soule Internationals and Abbot United cannot be chosen at the same time?

    **Solution.** The answer is $\boxed{27027}$. If neither team is chosen, then there are $\binom{28}{4} = 20475$ choices for the tournament. If one of them is chosen, then there are 2 choices for which one is chosen and $\binom{28}{3}$ choices for the other teams, resulting in 6552 choices. The answer is $20475 + 6552 = 27027$.

18. The numbers $1, 2, 3, 4, 5, 6$ are randomly written around a circle. What is the probability that there are four neighboring numbers such that the sum of the middle two numbers is less than the sum of the other two?

    **Solution.** The answer is $\boxed{1}$. We consider the neighboring numbers $a, b, 1, c, d$ (in that order). If $b > c$, then $(b, 1, c, d)$ has the property that $1 + c < d + b$; if $b < c$, then $(a, b, 1, c)$ has the property that $1 + b < a + c$.

19. What is the largest positive 2-digit factor of $3^{2^{2011}} - 2^{2^{2011}}$?

    **Solution.** The answer is $\boxed{97}$. Note that

    $$\begin{aligned} 3^{2^{2011}} - 2^{2^{2011}} &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2010}} - 2^{2^{2010}}) \\ &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2009}} + 2^{2^{2009}})(3^{2^{2009}} - 2^{2^{2009}}) \\ &\vdots \\ &= (3^{2^{2010}} + 2^{2^{2010}})(3^{2^{2009}} + 2^{2^{2009}})\cdots(3^{2^3} + 2^{2^3})(3^{2^2} + 2^{2^2})(3^{2^2} - 2^{2^2}). \end{aligned}$$

    Hence $3^{2^2} + 2^{2^2} = 97$ divides $3^{2^{2011}} - 2^{2^{2011}}$. On the other hand, because $3^{2^{2011}} - 2^{2^{2011}}$ is divisible by neither 2 nor 3, it is divisible by neither 98 nor 99.

20. Rhombus $ABCD$ has vertices $A = (-12, -4)$, $B = (6, b)$, $C = (c, -4)$ and $D = (d, -28)$, where $b$, $c$, and $d$ are integers. Find a constant $m$ such that the line $y = mx$ divides the rhombus into two regions of equal area.

**Solution.** The answer is $\boxed{-\frac{2}{3}}$. Let $G$ be the intersection of diagonals $AC$ and $BD$. Because $G$ lies on $AC$, we know that $G = (x, -4)$ for some real number $x$. Note that the diagonals of a rhombus are perpendicular to each other. Because $AC$ is parallel to the $x$-axis, it follows that $BD$ is parallel to the $y$-axis. Hence $G = (6, -4)$. Note also that a rhombus has $180^\circ$ rotational symmetry about $G$. Therefore, the line passing through the origin and $G$ will divide the rhombus into two regions of equal area and $m = -\frac{4}{6} = -\frac{2}{3}$.

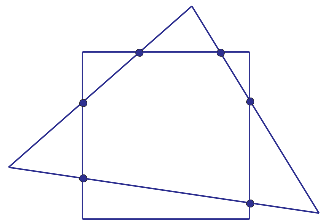
**Note:** Is such a line unique?

## 2.2 Individual Accuracy Test Solutions

**Morning, January 29, 2011**

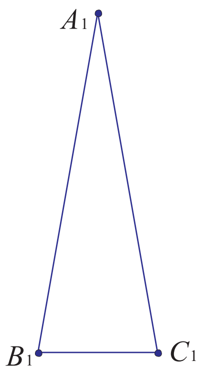
*There are 10 problems, worth 9 points each, and 30 minutes to solve as many problems as possible.*

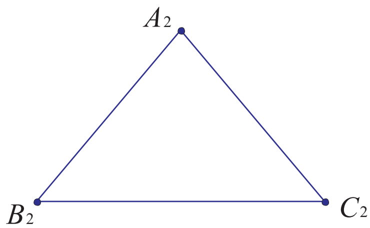
1. What is the maximum number of points of intersection between a square and a triangle, assuming that no side of the triangle is parallel to any side of the square?



**Solution.** The answer is $\boxed{6}$. (See the proceeding diagram.) Each side of the triangle can intersect the square at no more than 2 points. This means there can be at most $3 \cdot 2 = 6$ intersections, which can be achieved.

2. Two angles of an isosceles triangle measure $80^\circ$ and $x^\circ$. What is the sum of all the possible values of $x$?




**Solution.** The answer is $\boxed{150}$. There are two possible triangles. (See the diagrams shown above.) In triangle $A_1B_1C_1$, $A_1B_1 = A_1C_1$, $\angle A_1 = 20^\circ$, $\angle B_1 = \angle C_1 = 80^\circ$. In triangle $A_2B_2C_2$, $A_2B_2 = A_2C_2$, $\angle A_2 = 80^\circ$, $\angle B_1 = \angle C_1 = 50^\circ$. Hence the possible values of $x$ are 20, 50, and 80, with their sum equal to 150.

3. Let $p$ and $q$ be prime numbers such that $p+q$ and $p+7q$ are both perfect squares. Find the value of $pq$.

**Solution.** The answer is $\boxed{4}$. Write $p+q = a^2$, $p+7q = b^2$ so that $b^2 - a^2 = (b-a)(b+a) = 6q$. Since $(b+a) = (b-a)+2a$, $b+a$ and $b-a$ have the same parity - in other words, $b-a$ and $b+a$ are either both even or both odd. Because $(b-a)(b+a) = 6q$ is even, $b-a$ and $b+a$ are both even. This means $q$ is even; that is, $q = 2$. Then $(b-a)(b+a) = 12$ with $b-a = 2$ and $b+a = 6$, implying that $b = 4$, $a = 2$. This means $p = 2$ as well, so $pq = 4$.

4. Anna, Betty, Carly, and Danielle are four pit bulls, each of which is either wearing or not wearing lipstick. The following three facts are true:

   (1) Anna is wearing lipstick if Betty is wearing lipstick.

   (2) Betty is wearing lipstick only if Carly is also wearing lipstick.

   (3) Carly is wearing lipstick if and only if Danielle is wearing lipstick

   The following five statements are each assigned a certain number of points:

   (a) Danielle is wearing lipstick if and only if Carly is wearing lipstick. (This statement is assigned 1 point.)

   (b) If Anna is wearing lipstick, then Betty is wearing lipstick. (This statement is assigned 6 points.)

   (c) If Betty is wearing lipstick, then both Anna and Danielle must be wearing lipstick. (This statement is assigned 10 points.)

   (d) If Danielle is wearing lipstick, then Anna is wearing lipstick. (This statement is assigned 12 points.)

   (e) If Betty is wearing lipstick, then Danielle is wearing lipstick. (This statement is assigned 14 points.)

   What is the sum of the points assigned to the statements that *must* be true? (For example, if only statements (a) and (d) are true, then the answer would be $1 + 12 = 13$.)

   **Solution.** The answer is $\boxed{25}$. Statement (a) is true because it is equivalent to fact (3). By fact (1), if Betty did not wear lipstick, Anna still could be wearing lipstick. Hence statement (b) is not always true. If Betty wears lipstick, then Anna wears lipstick by fact (1) and Carly wears lipstick by fact (2), implying that Danielle wears lipstick by fact (3). Hence statement (c) is true, from which it follows that statment (e) is true. Finally, statement (d) is not always true. We can have Danielle wearing lipstick, Carly wearing lipstick, Betty not wearing lipstick, and Anna not wearing lipstick. Therefore statements (a), (c), (e) are true and the total points is $1 + 10 + 14 = 25$.

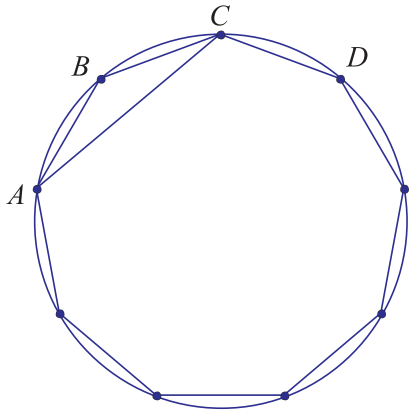
5. Let $f(x)$ and $g(x)$ be functions such that $f(x) = 4x + 3$ and $g(x) = \dfrac{x+1}{4}$. Evaluate $g(f(g(f(42))))$.

   **Solution.** The answer is $\boxed{44}$. We have

$$g(f(x)) = g(4x+3) = \frac{4x+3}{4} + \frac{1}{4} = \frac{4x+4}{4} = x + 1.$$

   It then follows that $g(f(g(f(42)))) = g(f(43)) = 44$.

6. Let $A, B, C$, and $D$ be consecutive vertices of a regular polygon. If $\angle ACD = 120^\circ$, how many sides does the polygon have?




**Solution.** The answer is $\boxed{9}$. In isosceles triangle $ABC$, we set

$$\angle BAC = \angle BCA = x°.$$

Then it follows that

$$\angle ABC = 180° - 2x°$$

and

$$\angle BCD = 120° + x°$$

. Because this is a regular polygon, we have

$$\angle ABC = \angle BCD$$

or

$$180° - 2x° = 120° + x°$$

; that is, $x = 20°$. It follows that each interior angle of the polygon is $140°$; that is, each exterior angle is $40°$. Because the sum of the exterior angles of a polygon is $360°$, we conclude that this polygon has 9 sides.

7. Fred and George have a fair 8-sided die with the numbers $0, 1, 2, 9, 2, 0, 1, 1$ written on the sides. If Fred and George each roll the die once, what is the probability that Fred rolls a larger number than George?

**Solution.** The answer is $\boxed{\dfrac{23}{64}}$. The probability of both getting a 0 is equal to $(\left(\dfrac{1}{4}\right)^2 = \dfrac{1}{16}$. The probability of both getting a 1 is equal to $\left(\dfrac{1}{4}\right)^2 = \dfrac{1}{16}$. The probability of both getting a 2 is equal to $\left(\dfrac{3}{8}\right)^2 = \dfrac{9}{64}$. The probability of both getting a 9 is equal to $\left(\dfrac{1}{8}\right)^2 = \dfrac{1}{64}$. Hence the probability of both getting the same number is

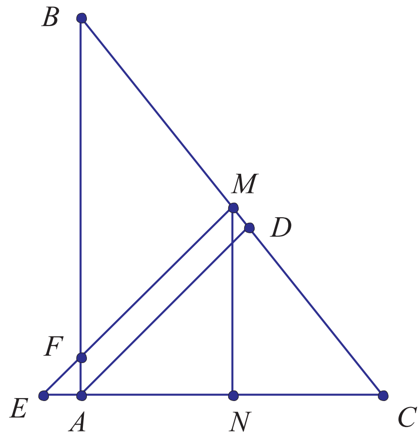
$$\frac{1}{16} + \frac{1}{16} + \frac{9}{64} + \frac{1}{64} = \frac{9}{32}.$$

It follows that the probability of them getting distinct numbers is $1 - \dfrac{9}{32} = \dfrac{23}{32}$. If they do not get the same number then Freds number will be larger one-half of the time. Therefore the answer is $\left(\frac{23}{32}\right) \cdot \left(\frac{1}{2}\right) = \frac{23}{64}$.

8. Find the smallest *positive* integer $t$ such that $(23t)^3 - (20t)^3 - (3t)^3$ is the square of an integer.

   **Solution.** The answer is $\boxed{115}$. Note that $(23t)^3 - (20t)^3 - (3t)^3 = t^3(23^3 - 20^3 - 3^3)$. Because $(x+y)^3 = x^3+y^3+3xy(x+y)$, we have $23^3-20^3-3^3 = (20+3)^3-20^3-3^3 = 3\cdot 20\cdot 3\cdot(20+3) = 2^2\cdot 3^2\cdot 5\cdot 23$. Hence
   $$(23t)^3 - (20t)^3 - (3t)^3 = t^3(23^3 - 20^3 - 3^3) = t^2 \cdot 2^2 \cdot 3^2 \cdot 5 \cdot 23 \cdot t.$$
   Therefore, $(23t)^3 - (20t)^3 - (3t)^3$ is a perfect square if and only if $5 \cdot 23 \cdot t$ is a perfect square. The minimum positive value of such $t$ is clearly $t = 5 \cdot 23 = 115$.

9. In triangle $ABC$, $AC = 8$ and $AC < AB$. Point $D$ lies on side $BC$ with $\angle BAD = \angle CAD$. Let $M$ be the midpoint of $BC$. The line passing through $M$ parallel to $AD$ intersects lines $AB$ and $AC$ at $F$ and $E$, respectively. If $EF = \sqrt{2}$ and $AF = 1$, what is the length of segment $BC$?

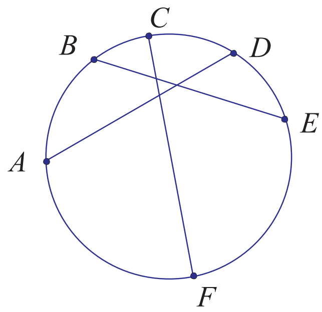



   **Solution.** The answer is $\boxed{2\sqrt{41}}$. (See the proceeding diagram.)Because $EM \parallel AD$, $\angle AEF = \angle CAD$ and $\angle FAD = \angle AFE$. By the given condition, we have $\angle CAD = \angle BAD = \angle FAD$. Combining the last two equations yields $\angle AEF = \angle AFE$; that is, $AEF$ is an isosceles triangle with $AE = AF = 1$. In addition, because $AE = AF = 1$ and $EF = \sqrt{2}$, we conclude that $\angle AEF$ is an isosceles right triangle with $\angle EAF = 90°$ and $\angle AEF = \angle AFE = 45°$. Let $N$ be the midpoint of segment $AC$. Then $MN$ is the midline of triangle $ABC$. In particular, $MN \parallel AB$, from which it follows that $MNE$ is an isosceles triangle with $MN = NE = NA + AE = 4 + 1 = 5$. It follows that $MC = \sqrt{NC^2 + MN^2} = \sqrt{4^2 + 5^2} = \sqrt{41}$ and $BC = 2MC = 2\sqrt{41}$.

10. There are 2011 evenly spaced points marked on a circular table. Three segments are randomly drawn between pairs of these points such that no two segments share an endpoint on the circle. What is the probability that each of these segments intersects the other two?

    **Solution.** The answer is $\boxed{\dfrac{1}{15}}$. We note that it does not matter *which* six points on the circle are used as endpoints; what matters is the manner in which segments are drawn between them. Let our 6 endpoints be $A, B, C, D, E, F$ around the circle in that order. (See the following diagram.)

    We will show that $A$ must be connected to $D$, $B$ to $E$, and $C$ to $F$. If $A$ is connected to $B, C, E$, or $F$, then there are three points on one side of the segment with an endpoint at $A$. It follows that there must be a segment among this points; however, this segments would not intersect the one passing through $A$, which is a contradiction. Hence, $A$ must be connected to $D$. Similarly, $B$ must be connected to $E$ and $C$ to $F$.




The probability that $A$ is connected to $D$ is $\frac{1}{5}$. If $A$ is connected to $D$, then the probability that $B$ is connected to $E$ is $\frac{1}{3}$. If both those connections occur, then $C$ is automatically connected to $F$. Thus, the probability of getting a set of segments which pairwise intersect is $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$.

## 2.3 Team Test Solutions

1. Velociraptor A is located at $x = 10$ on the number line and runs at 4 units per second. Velociraptor B is located at $x = -10$ on the number line and runs at 3 units per second. If the velociraptors run towards each other, at what point do they meet?

   **Solution.** The answer is $\boxed{-\frac{10}{7}}$. Suppose the velociraptors collide at $a$. The distances from $a$ to the starting points of velociraptors A and B are $10 - a$ and $a + 10$, respectively. Because the time it takes for the velociraptors to reach this point is the same, we know that

   $$\frac{a+10}{3} = \frac{10-a}{4},$$

   so $a = \frac{-10}{7}$.

2. Let $n$ be a positive integer. There are $n$ non-overlapping circles in a plane with radii 1, 2, ..., $n$. The total area that they enclose is at least 100. Find the minimum possible value of $n$.

   **Solution.** The answer is $\boxed{5}$. If $n = 4$, then the total area is $\pi\cdot(1+4+9+16) = 30\pi < 30\cdot 3.2 = 96 < 100$. If $n = 5$, the total area is $\pi \cdot (1 + 4 + 9 + 16 + 25) = 55\pi > 100$.

3. How many integers between 1 and 50, inclusive, are divisible by 4 but not 6?

   **Solution.** The answer is $\boxed{8}$. These integers are $4, 8, 16, 20, 28, 32, 40, 44$.

4. Let $a \star b = 1 + \frac{b}{a}$. Evaluate $((((((1 \star 1) \star 1) \star 1) \star 1) \star 1) \star 1) \star 1$.

   **Solution.** The answer is $\boxed{\frac{34}{21}}$.

   $$\begin{aligned}
   1 \star 1 &= 2 \\
   2 \star 1 &= \frac{3}{2} \\
   \frac{3}{2} \star 1 &= \frac{5}{3} \\
   \frac{5}{3} \star 1 &= \frac{8}{5} \\
   \frac{8}{5} \star 1 &= \frac{13}{8} \\
   \frac{13}{8} \star 1 &= \frac{21}{13} \\
   \frac{21}{13} \star 1 &= \frac{34}{21}
   \end{aligned}$$

   Alternatively, note that if $x = \frac{F_n}{F_{n-1}}$, where $F_n$ is the $n^{\text{th}}$ Fibonacci number (that is, $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for $n \geq 0$), then

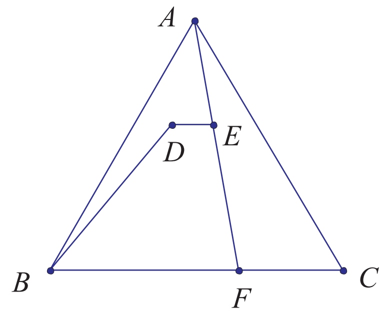
   $$x \star 1 = 1 + \frac{F_{n-1}}{F_n} = \frac{F_{n-1} + F_n}{F_n} = \frac{F_{n+1}}{F_n}.$$

Note also that

$$1 \star 1 = \frac{2}{1} = \frac{F_3}{F_2},$$

from which it follows that the number we want is $\dfrac{F_9}{F_8} = \dfrac{34}{21}$.

5. In acute triangle $ABC$, $D$ and $E$ are points inside triangle $ABC$ such that $DE \parallel BC$, $B$ is closer to $D$ than it is to $E$, $\angle AED = 80^\circ$, $\angle ABD = 10^\circ$, and $\angle CBD = 40^\circ$. Find the measure of $\angle BAE$, in degrees.




**Solution.** The answer is $\boxed{50^\circ}$. Extend $AE$ to meet $BC$ at $F$. The angles of $\triangle ABF$ sum to $180^\circ$. Therefore,

$$\begin{aligned}\angle BAE &= 180^\circ - \angle ABF - \angle AFB = 180^\circ - (\angle ABD + \angle DBC) - \angle AED \\ &= 180^\circ - (10^\circ + 40^\circ) - 80^\circ = 50^\circ.\end{aligned}$$

6. Al is at $(0, 0)$. He wants to get to $(4, 4)$, but there is a building in the shape of a square with vertices at $(1, 1)$, $(1, 2)$, $(2, 2)$, and $(2, 1)$. Al cannot walk inside the building. If Al is not restricted to staying on grid lines, what is the shortest distance he can walk to get to his destination?

**Solution.** The answer is $\boxed{\sqrt{5} + \sqrt{13}}$. One possible shortest path is to go from $(0, 0)$ to $(1, 2)$ and then to $(4, 4)$. From $(0, 0)$ to $(1, 2)$, the distance is $\sqrt{1^2 + 2^2} = \sqrt{5}$. From $(1, 2)$ to $(4, 4)$, the distance is $\sqrt{3^2 + 2^2} = \sqrt{13}$. The total length is $\sqrt{5} + \sqrt{13}$.

(To show that this is actually the shortest path: If we put a rubber band around nails at the points $(0, 0)$ and $(4, 4)$ and we stretch the rubber band so that it does not pass through the building, then if we let go of the rubber band, it will hit the building at $(1, 2)$. Hence the distance is composed of two straight line segments as computed above.)

7. Point $A = (1, 211)$ and point $B = (b, 2011)$ for some integer $b$. For how many values of $b$ is the slope of $AB$ an integer?

**Solution.** The answer is $\boxed{72}$. The slope $m = \dfrac{2011 - 211}{b - 1} = \dfrac{1800}{b - 1}$ is an integer, so $(b - 1)$ divides 1800. If $x$ is a positive factor of 1800, then $1 + x$ and $1 - x$ are solutions for $b$, so each positive factor of 1800 should be counted twice. Because, $2^3 \cdot 3^2 \cdot 5^2 = 1800$, we have $\cdot(4 \cdot 3 \cdot 3) = 36$ positive factors of 1800 and therefore $2 \cdot 36 = 72$ possible values of $b$.

8. A palindrome is a number that reads the same forwards and backwards. For example, 1, 11 and 141 are all palindromes. How many palindromes between 1 and 1000 are divisible by 11?

**Solution.** The answer is $\boxed{17}$. We have three cases; the palindrome has either 1, 2 or 3 digits.

- *Case 1* The palindrome has 1 digit. There are no one-digit palindromes that are divisible by 11.
- *Case 2* The palindrome has 2 digits. There are 9 two-digit palindromes, and all are divisible by 11.
- *Case 3* The palindrome has 3 digits. Note that for any given hundreds digit, there is at most one palindrome divisible by 11 (because if $a$ is a palindrome divisible by 11, then the next palindrome divisible by 11 is at least $a + 121$). Thus, we can try to find the palindrome for each hundreds digit. The three-digit palindromes are $979, 858, 737, 616, 484, 363, 242, 121$, for a total of 8.

The total number of palindromes is therefore $9 + 8 = 17$.

9. Suppose $x, y, z$ are real numbers that satisfy:

$$\begin{aligned} x + y - z &= 5 \\ y + z - x &= 7 \\ z + x - y &= 9. \end{aligned}$$

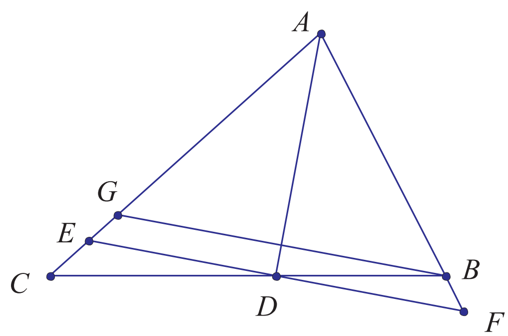
Find $x^2 + y^2 + z^2$.

**Solution.** The answer is $\boxed{149}$. Adding the equations gives $x + y + z = 21$, and subtracting each original equation from this yields

$$2z = 16 \quad 2x = 14 \quad 2y = 12.$$

Hence $(x, y, z) = (7, 6, 8)$ and $x^2 + y^2 + z^2 = 7^2 + 6^2 + 8^2 = 149$.

10. In triangle $ABC$, $AB = 3$ and $AC = 4$. The bisector of angle $A$ meets $BC$ at $D$. The line through $D$ perpendicular to $AD$ intersects lines $AB$ and $AC$ at $F$ and $E$, respectively. Compute $EC - FB$. (See the diagram shown below.)




**Solution.** The answer is $\boxed{\frac{1}{7}}$. Let $G$ be the reflection of $B$ over line $AD$. Then, $AG = AB = 3$ and $BF = GE$, so we find that $EC + EG = AC - AG = 4 - 3 = 1$. Because $GB \parallel EF$ and $AD$ is the angle bisector of $\angle A$, we have

$$\frac{CE}{EG} = \frac{CD}{DB} = \frac{CA}{AB} = \frac{4}{3}.$$

Then, $EC = \frac{4}{7}$ and $BF = GE = \frac{3}{7}$, so $EC - FB = \frac{1}{7}$.

11. Bob has a six-sided die with a number written on each face such that the sums of the numbers written on each pair of opposite faces are equal to each other. Suppose that the numbers 109, 131, and 135 are written on three faces which share a corner. Determine the maximum possible sum of the numbers on the three remaining faces, given that all three are positive primes less than 200.

**Solution.** The answer is $\boxed{39}$. Let $p, q, r$ be the primes opposite 109, 131, 135, respectively. Because 109, 131, and 135 have distinct remainders when divided by 3, it follows that $p, q$, and $r$ have distinct remainders when divided by 3. We conclude that exactly one of $p$, $q$, $r$ is 3. Because $r$ is opposite the largest number, we have $r = 3$. This yields $p = 29$, and $q = 7$, so $p + q + r = 29 + 7 + 3 = 39$.

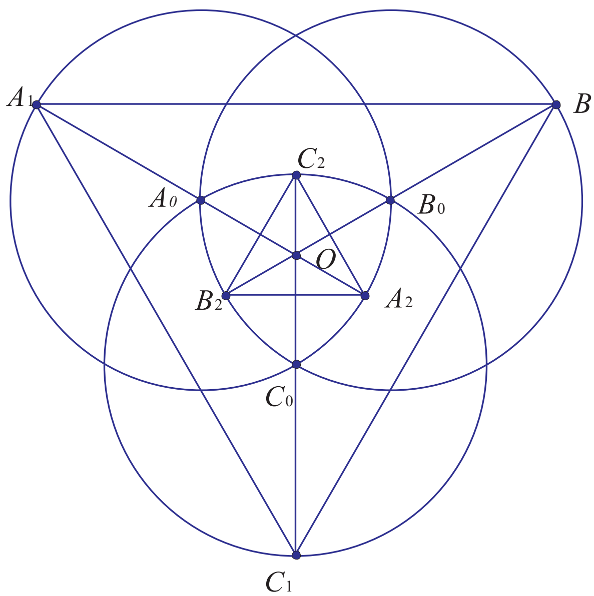
12. Let $d$ be a number chosen at random from the set $\{142, 143, \ldots, 198\}$. What is the probability that the area of a rectangle with perimeter 400 and diagonal length $d$ is an integer?

**Solution.** The answer is $\boxed{\frac{29}{57}}$. Let $x, y$ be the side lengths of the rectangle, and $K$ be the area. Hence

$$\begin{aligned} xy &= K \\ x + y &= 200 \\ x^2 + y^2 &= d^2 \end{aligned}$$

Squaring the second equation gives $200 = x^2 + y^2 + 2xy = d^2 + 2K$. It follows that $K$ is an integer if and only if $d$ is even. The set $\{142, 143, \ldots, 198\}$ has 29 even elements out of 57 total elements, so the probability that the area of the rectangle is an integer is $\frac{29}{57}$.

13. There are 3 congruent circles such that each circle passes through the centers of the other two. Suppose $A$, $B$, and $C$ are points on the circles such that each circle has exactly one of $A$, $B$, or $C$ on it and triangle $ABC$ is equilateral. Find the ratio of the maximum possible area of $ABC$ to the minimum possible area of $ABC$.



**Solution.** The answer is $\boxed{7+4\sqrt{3}}$. Without loss of generality, let the radius of each circle be 1. Let $A_0, B_0, C_0$ be the centers of the circles, $A_1B_1C_1$ be the largest triangle, and $A_2B_2C_2$ be the smallest triangle, with $A_0$ the center of the circle containing $A_1$ and $A_2$, and similarly for $B_0$ and $C_0$. Note that both triangles share a circumcenter. Let this be $O$. Note that $A_0B_0C_0$ is an equilateral triangle with sidelength 1 and circumradius $\frac{\sqrt{3}}{3}$. To find the ratio of the areas of the triangles, we can square the ratio of their circumradii. This is equal to:

$$\frac{(1-C_0O)^2}{(1+C_0O)^2} = \frac{(1+(\frac{\sqrt{3}}{3}))^2}{(1-(\frac{\sqrt{3}}{3}))^2} = 7+4\sqrt{3}.$$

14. Let $k$ and $m$ be constants such that for all triples $(a, b, c)$ of positive real numbers,

$$\sqrt{\frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{k}{ab}} = \left|\frac{2}{a}+\frac{6}{b}+\frac{3}{c}\right| \quad \text{if and only if} \quad am^2+bm+c=0.$$

Find $k$.

**Solution.** The answer is $\boxed{20}$. We square both sides of the first equation and simplify to get

$$\frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{k}{ab} = \frac{4}{a^2}+\frac{36}{b^2}+\frac{9}{c^2}+\frac{24}{ab}+\frac{36}{bc}+\frac{12}{ca}$$

or

$$0 = \frac{24-k}{ab}+\frac{12}{ac}+\frac{36}{bc}.$$

Multiplying both sides of the last equation by $abc$ yields $36a+12b+(24-k)c=0$. Solving this equation in $c$ gives $c = \frac{12(3a+b)}{24-k}$.

Hence, for all triples $(a, b, c)$ of positive real numbers,

$$c = \frac{12(3a+b)}{24-k} \quad \text{if and only if} \quad c = -am^2 - bm;$$

that is,

$$\frac{12(3a+b)}{24-k} = -am^2 - bm$$

or

$$a\left(\frac{36}{24-k} - m^2\right) + b\left(\frac{12}{24-k} - m\right) = 0.$$

Because this equation holds for all real numbers $a$ and $b$, the coefficients of $a$ and $b$ must be 0; that is

$$\frac{36}{24-k} = m^2 \quad \text{and} \quad \frac{12}{24-k} = m.$$

Dividing the first equation by the second equation yields $m=3$, from which it follows that $k=20$.

15. A bored student named Abraham is writing $n$ numbers $a_1, a_2, \ldots, a_n$. The value of each number is either 1, 2, or 3; that is, $a_i$ is 1, 2 or 3 for $1 \le i \le n$. Abraham notices that the ordered triples

$$(a_1, a_2, a_3), \quad (a_2, a_3, a_4), \quad \ldots, \quad (a_{n-2}, a_{n-1}, a_n), \quad (a_{n-1}, a_n, a_1), \quad (a_n, a_1, a_2)$$

are distinct from each other. What is the maximum possible value of $n$? Give the answer $n$, along with an example of such a sequence. Write your answer as an ordered pair. (For example, if the answer were 5, you might write $(5, 12311)$.)

**Solution.** A possible answer is $\boxed{(27, 122322213211131123331332312)}$. Note that there are $3^3 = 27$ distinct possible sequences of 1, 2, 3. If we use $n$ numbers, the resulting sequence has $n$ ordered triples, which must be distinct, so $n \leq 27$. For $n = 27$, we consider the following sequence:

$$122322213211131123331332312$$

which gives no repeated ordered triples.

**Note:** Can you find other examples (that are not circular permutations of the current example) to support the conclusion $n = 27$?

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. [3pts] In order to make good salad dressing, Bob needs a 0.9% salt solution. If soy sauce is 15% salt, how much water, in mL, does Bob need to add to 3 mL of pure soy sauce in order to have a good salad dressing?

   **Solution.** The answer is $\boxed{47}$. Let $x$ be the amount of water Bob adds to the solution. Because the amount of salt remains the same, we have $3 \cdot 0.15 = (3 + x) \cdot .009$, implying that $x = 47$.

2. [3pts] Alex the Geologist is buying a canteen before he ventures into the desert. The original cost of a canteen is \$20, but Alex has two coupons. One coupon is \$3 off and the other is 10% off the entire remaining cost. Alex can use the coupons in any order. What is the least amount of money he could pay for the canteen?

   **Solution.** The answer is $\boxed{\$15}$. If he uses the first coupon and then the second he pays $(20 - 3) \cdot 0.9 = \$15.30$. However, if he uses the second and then the first, he only has to pay $(20 \cdot 0.9) - 3 = \$15$.

3. [3pts] Steve and Yooni have six distinct teddy bears to split between them, including exactly 1 blue teddy bear and 1 green teddy bear. How many ways are there for the two to divide the teddy bears, if Steve gets the blue teddy bear and Yooni gets the green teddy bear? (The two do not necessarily have to get the same number of teddy bears, but each teddy bear must go to a person.)

   **Solution.** The answer is $\boxed{16}$. We know that Steve gets the blue teddy bear and Yooni gets the green one. Then Yooni and Steve have to divide the 4 remaining teddy bears. Now, each bear can go to two possible people (Yooni or Steve), and we have four bears, so we have $2 \cdot 2 \cdot 2 \cdot 2 = 16$ ways to distribute them.

### 2.4.2 Round 2

4. [5pts] In the currency of Mathamania, 5 wampas are equal to 3 kabobs and 10 kabobs are equal to 2 jambas. How many jambas are equal to twenty-five wampas?

   **Solution.** The answer is $\boxed{3}$. The exchange rate for wampas to kabobs is $\frac{3}{5}$, and the rate for kabobs to jambas is $\frac{2}{10} = \frac{1}{5}$, so 25 wampas is equivalent to $25 \cdot \frac{3}{5} \cdot \frac{1}{5} = 3$ jambas.

5. [5pts] A sphere has a volume of $81\pi$. A new sphere with the same center is constructed with a radius that is $\frac{1}{3}$ the radius of the original sphere. Find the volume, in terms of $\pi$, of the region between the two spheres.

   **Solution.** The answer is $\boxed{78\pi}$. The ratio of the radius of the new sphere to that of the original sphere is 1:3, so the ratio of their volumes is $1 : 3^3$, or $1 : 27$. Therefore the volume of the new sphere is $\frac{1}{27} \cdot 81\pi = 3\pi$. The volume between the two spheres is thus $81\pi - 3\pi = 78\pi$.

6. [5pts] A frog is located at the origin. It makes four hops, each of which moves it either 1 unit to the right or 1 unit to the left. If it also ends at the origin, how many 4-hop paths can it take?

   **Solution.** The answer is $\boxed{6}$. Since the frog ends at the origin, it must move an equal number of right and left steps. Thus, it must choose 2 steps out of 4 to go to the right, so there are $\binom{4}{2} = 6$ paths total.

### 2.4.3 Round 3

7. [6pts] Nick multiplies two consecutive positive integers to get $4^5 - 2^5$. What is the smaller of the two numbers?

**Solution.** The answer is $\boxed{31}$. We see that $4^5 - 2^5 = (2^5) \cdot (2^5 - 1) = 32 \cdot 31$.

8. [6 pts] In rectangle $ABCD$, $E$ is a point on segment $CD$ such that $\angle EBC = 30^\circ$ and $\angle AEB = 80^\circ$. Find $\angle EAB$, in degrees.

**Solution.** The answer is $\boxed{40^\circ}$. We note that $\angle ABE = 90^\circ - 30^\circ = 60^\circ$. It follows that $\angle EAB = 180^\circ - 60^\circ - 40^\circ = 40^\circ$.

9. [6pts] Mary's secret garden contains clones of Homer Simpson and WALL-E. A WALL-E clone has 4 legs. Meanwhile, Homer Simpson clones are human and therefore have 2 legs each. A Homer Simpson clone always has 5 donuts, while a WALL-E clone has 2. In Mary's secret garden, there are 184 donuts and 128 legs. How many WALL-E clones are there?

**Solution.** The answer is $\boxed{17}$. Let $H$ be the number of Homer clones, and let $W$ be the number of WALL-E clones. By the given conditions, we have $4W + 2H = 128$ legs and $2W + 5H = 184$ donuts. Solving these equations yields $H = 30$ and $W = 17$.

### 2.4.4 Round 4

10. [7pts] Including Richie, there are 6 students in a math club. Each day, Richie hangs out with a different group of club mates, each of whom gives him a dollar when he hangs out with them. How many dollars will Richie have by the time he has hung out with every possible group of club mates?

**Solution.** The answer is $\boxed{80}$. Since each person other than Richie can either be in a group or not be in a group, there are $2^5 = 32$ groups. An individual student (other than Richie) is equally likely to be or not to be in one of these groups; that is, an individual student (other than Richie) is in exactly $\frac{32}{2} = 16$ of these groups, and thus will contribute \$16. There are five students, so the answer is 80.

11. [7pts] There are seven boxes in a line: three empty, three holding \$10 each, and one holding the jackpot of $\$1,000,000$. From the left to the right, the boxes are numbered 1, 2, 3, 4, 5, 6 and 7, in that order. You are told the following:

- No two adjacent boxes hold the same contents.
- Box 4 is empty.
- There is one more \$10 prize to the right of the jackpot than there is to the left.

Which box holds the jackpot?

**Solution.** The answer is $\boxed{\text{Box 3}}$. There are two \$10 boxes are to the right of the jackpot, and one to the left, so the jackpot cannot be in boxes 1, 6, or 7. We know that box 4 is empty so the jackpot is in box 2, 3, or 5. If the jackpot is in box 5, then boxes 6 and 7 would both hold \$10 prizes, which is impossible. If the jackpot is in box 2, then box 1 must hold \$10, but then there is no way to arrange the other two empty boxes without violating the first condition. Hence the jackpot must be in box 3 with arrangement (\$10, empty, Jackpot, empty, \$10, empty, \$10).

12. [7pts] Let $a$ and $b$ be real numbers such that $a+b=8$. Let $c$ be the minimum possible value of $x^2+ax+b$ over all real numbers $x$. Find the maximum possible value of $c$ over all such $a$ and $b$.

**Solution.** The answer is $\boxed{9}$. By substituting $b=8-a$ and completing the square (for $x$), we have

$$x^2+ax+b=x^2+ax+8-a=\left(x+\frac{a}{2}\right)^2+8-a-\frac{a^2}{4}\geq 8-a-\frac{a^2}{4}.$$

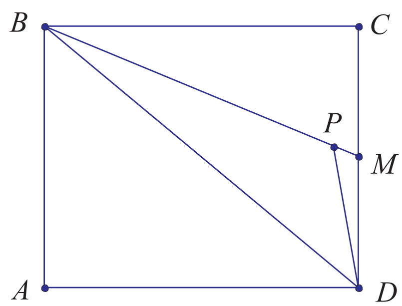
Hence for a fixed pair $(a,b)$, $c=8-a-\dfrac{a^2}{4}$. Completing the square (for $a$) yields

$$c=8-a-\frac{a^2}{4}=-\frac{(a+2)^2}{4}+9,$$

from which it follows that the maximum possible value of $c$ is 9 with $a=-2$ and $b=10$.

### 2.4.5 Round 5

13. [9pts] Let $ABCD$ be a rectangle with $AB=10$ and $BC=12$. Let $M$ be the midpoint of $CD$, and $P$ be the point on $BM$ such that $BP=BC$. Find the area of $ABPD$.




**Solution.** The answer is $\boxed{\dfrac{1140}{13}}$. (See the diagram shown above.) We will use $[\mathcal{R}]$ to denote the area of a region $\mathcal{R}$. (*This notation also applies to later solutions.*) We have $[APDB]=[ABCD]-[MBC]-[DPM]$. The area of $MBC$ is $\frac{1}{2}\cdot\frac{10}{2}\cdot 12=30$. Note that $MBC$ is a 5-12-13 triangle, so $BM=13$, $BP=12$, and $PM=1$. Because triangles $DPM$ and $DBM$ share a common altitude from $D$, the area of $DPM$ is $\frac{PM}{BM}\cdot[DMB]=\frac{1}{13}[DMB]$. In addition, $[DMB]=\frac{1}{2}[DBC]$, because both triangles share a common altitude. Finally, because $[DBC]=\frac{1}{2}\cdot 10\cdot 12=60$, it follows that $[DMB]=30$ and $[DPM]=\frac{30}{13}$. Thus, the area of quadrilateral $APDB$ is

$$[ABCD]-[DPM]-[MBC]=120-\frac{30}{13}-30=\frac{1140}{13}.$$

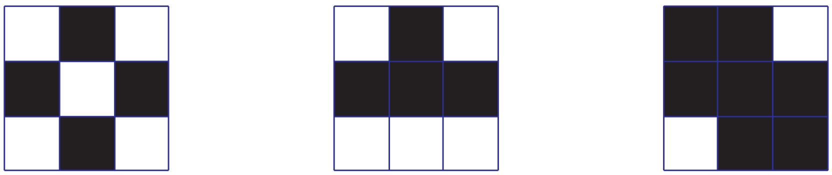
14. [9pts] The number 19 has the following properties:

- It is a 2-digit positive integer.
- It is the two leading digits of a 4-digit perfect square, because $1936=44^2$.

How many numbers, including 19, satisfy these two conditions?

**Solution.** The answer is $\boxed{65}$. The numbers $51^2, 52^2, \ldots, 99^2$ each have distinct leading digit pairs, because $(k+1)^2 - k^2 = 2k+1 > 100$ for $k > 50$, so the difference between consecutive perfect squares in this set is greater than 100. The numbers $32^2, 33^2, ..., 50^2$ cover each of the leading digit pairs 10 through 25, inclusive (because the difference between consecutive perfect squares here is less than 100), so we have $16 + 49 = 65$ numbers with the requisite properties.

15. [9pts] In a $3 \times 3$ grid, each unit square is colored either black or white. A coloring is considered "nice" if there is at most one white square in each row or column. What is the total number of nice colorings? Rotations and reflections of a coloring are considered distinct. (For example, in the three squares shown below, only the rightmost one has a nice coloring.)



**Solution.** The answer is $\boxed{34}$. Consider a few cases based on the number of white squares. There is one configuration for zero white squares. If we add one white square, we always satisfy the condition, so there are 9 ways to do this. For two white squares, there are four possible squares where the second white square could be placed. Because the order of the white squares doesn't matter, the number of configurations with two white squares is then $\frac{9 \cdot 4}{2!} = 18$. Similarly, the number of configurations for three white squares is $\frac{9 \cdot 4}{3!} = 6$, since the order doesn't matter and there is only possible space for the last white square after the first two have been fixed. The total is thus $1 + 9 + 18 + 6 = 34$ nice colorings.

### 2.4.6 Round 6

16. [11pts] Let $a_1, a_2, \ldots, a_{2011}$ be a sequence of numbers such that $a_1 = 2011$ and $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for $n = 1, 2, \ldots, 2011$. (That is, $a_1 = 1^2 \cdot a_1$, $a_1 + a_2 = 2^2 \cdot a_2$, etc.) Compute $a_{2011}$.

**Solution.** The answer is $\boxed{\frac{1}{1006}}$. Because $a_1 + a_2 + \cdots + a_{n-1} + a_n = n^2 a_n$, it follows that $(n-1)^2 a_{n-1} + a_n = n^2 a_n$, from which it follows that $a_n = \frac{n-1}{n+1} \cdot a_{n-1}$. Thus,

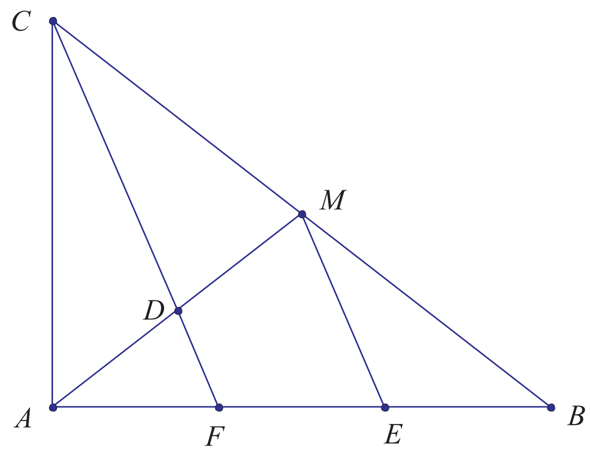
$$a_n = a_1 \cdot \frac{1}{3} \cdot \frac{2}{4} \cdot \frac{3}{5} \cdots \frac{n-1}{n+1}.$$

Most of the terms in this product cancel, leaving $a_{2011} = 2011 \cdot \frac{1 \cdot 2}{2011 \cdot 2012} = \frac{1}{1006}$.

17. [11pts] Three rectangles, with dimensions $3 \times 5$, $4 \times 2$, and $6 \times 4$, are each divided into unit squares which are alternately colored black and white like a checkerboard. Each rectangle is cut along one of its diagonals into two triangles. For each triangle, let $m$ be the total black area and $n$ the total white area. Find the maximum value of $|m - n|$ for the 6 triangles.

**Solution.** The answer is $\boxed{\frac{1}{2}}$. Each of the three rectangles has either odd by odd dimensions or even by even dimensions. Whenever a diagonal is drawn, the two triangles produced are congruent in shape and color. Thus, because an even by even rectangle has the same number of black and white squares total, $|m - n|$ for triangles produced from even by even rectangles will always equal 0. Meanwhile, because an odd by odd rectangle has 1 more square of one color than of the other, $|m - n| = \frac{1}{2}$ for triangles produced from odd by odd rectangles.

18. [11pts] In triangle $ABC$, $\angle BAC = 90°$, and the length of segment $AB$ is 2011. Let $M$ be the midpoint of $BC$ and $D$ the midpoint of $AM$. Let $E$ be the point on segment $AB$ such that $EM||CD$. What is the length of segment $BE$?




**Solution.** The answer is $\boxed{\frac{2011}{3}}$. (See the diagram shown above.) Let $CD$ intersect $AB$ at $F$. We can see because $AD = DM$ and triangles $AFD$ and $AEM$ are similar to each other, it follows that $AF = FE$. Similarly, because $BM = MC$ and triangles $BEM$ and $BFC$ are similar to each other, it follows that $FE = EB$. Because $AF = FE = EB$, we have $EB = \frac{2011}{3}$.

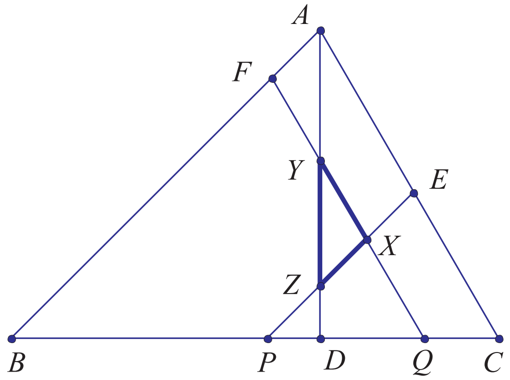
### 2.4.7 Round 7

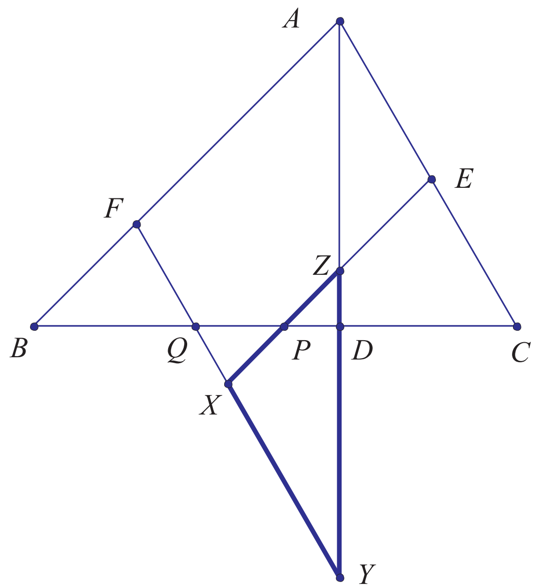
19. [12pts] How many integers from 1 to 100, inclusive, can be expressed as the difference of two perfect squares? (For example, $3 = 2^2 - 1^2$).

**Solution.** The answer is $\boxed{75}$. A difference of integer squares $a^2-b^2$ can be factored as $(a+b)(a-b)$, the product of two integers of the same parity. Thus, for an even integer to be a difference of integer squares, it must be a product of two even integers, and therefore must be divisible by 4. Conversely, any integer multiple of 4 may be expressed as the difference of two integer squares, because $4n = (n+1)^2-(n-1)^2$. Any odd number may be expressed as the difference of two integer squares, because $2n+1 = (n+1)^2-n^2$. There are 25 multiples of 4 from 1 to 100 and 50 odd numbers. The answer is therefore $25 + 50 = 75$.

20. [12pts] In triangle $ABC$, $\angle ABC = 45°$ and $\angle ACB = 60°$. Let $P$ and $Q$ be points on segment $BC$, $F$ a point on segment $AB$, and $E$ a point on segment $AC$ such that $FQ \parallel AC$ and $EP \parallel AB$. Let $D$ be

the foot of the altitude from $A$ to $BC$. The lines $AD$, $FQ$, and $PE$ form a triangle. Find the positive difference, in degrees, between the largest and smallest angles of this triangle.




**Solution.** The answer is $\boxed{75^\circ}$. Let $FQ$ and $EP$ intersect at $X$. Then $\angle PXQ = A = 180-60-45 = 75$. Then the largest angle of the triangle formed is the angle adjacent to $\angle PXQ$, or $\angle 180^\circ - \angle PXQ = 105^\circ$. The other angles are $90^\circ - \angle B = 45^\circ$ and $90^\circ - 60^\circ = 30^\circ$, so the difference is $105^\circ - 30^\circ = 75^\circ$. (See the diagrams above for two of the few possible configurations of the triangle. It is important to note that the any resulting triangles $XYZ$ are always similar to each other, because the angles between respective parallel lines in different configurations remain the same.)

21. [12pts] For real number $x$, $\lceil x \rceil$ is equal to the smallest integer larger than or equal to $x$. For example, $\lceil 3 \rceil = 3$ and $\lceil 2.5 \rceil = 3$. Let $f(n)$ be a function such that $f(n) = \left\lceil \frac{n}{2} \right\rceil + f\left(\left\lceil \frac{n}{2} \right\rceil\right)$ for every integer $n$ greater than 1. If $f(1) = 1$, find the maximum value of $f(k) - k$, where $k$ is a positive integer less than or equal to 2011.

**Solution.** The answer is $\boxed{10}$. Let $g(n) = f(n) - n$. We see that

$$f(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n}{4} \right\rceil + \cdots + \left\lceil \frac{n}{2^j} \right\rceil,$$

where $2^j$ is the smallest power of 2 such that $2^j > n$, and that

$$n = \left(\frac{n}{2} + \frac{n}{4} + \cdots + \frac{n}{2^{j-1}} + \frac{n}{2^j}\right) + \frac{n}{2^j}.$$

Then,

$$g(n) = f(n) - n = \left[\left(\left\lceil \frac{n}{2} \right\rceil - \frac{n}{2}\right) + \left(\left\lceil \frac{n}{4} \right\rceil - \frac{n}{4}\right) + \cdots + \left(\left\lceil \frac{n}{2^j} \right\rceil - \frac{n}{2^j}\right)\right] - \frac{n}{2^j}$$

Note that the quantity

$$\left\lceil \frac{n}{k} \right\rceil - \frac{n}{k}$$

is maximized if and only if $n$ is congruent to 1 modulo $k$. We will show that the maximum value of $g(n)$ over $[2^i, 2^{i+1}-1]$ is achieved at $n = 2^i + 1$, for all $i$. Therefore, to maximize $g(n) + \frac{n}{2^j}$, we want

$$n \equiv 1 \pmod 2, \quad n \equiv 1 \pmod 4, \quad n \equiv 1 \pmod 8, \quad \ldots, \quad n \equiv 1 \pmod{2^j}.$$

As long as $\frac{n}{2^i}$ is minimized, $g(n)$ is maximized over $[2^i, 2^{i+1}-1]$ at $n = 2^i + 1$. Thus, we only have to check $n = 2^i$ for a potentially larger value of $g(n)$. Let $a_k = f(2^k)$. Then, we have $a_0 = 1$, $a_k = 2^{k-1} + a_{k-1}$. We can prove by induction that $a_k = 2^k$ for all $k$. So, $g(2^k) = 0$, for all $k$. Let $b_k = g(2^k + 1)$. Then, we have $b_0 = 0$,

$$b_k = 2^{k-1} + 1 + (b_{k-1} + 2^{k-1} + 1) - 2^k - 1 = 2^k + 1 - 2^k + b_{k-1} = 1 + b_{k-1}.$$

Thus, $b_k = k$ for all $k > 0$, by induction on $k$. Because $g(2^k + 1) = k$, we have proved that the maximum value of $g(n)$ over $[2^i, 2^{i+1}-1]$ is $g(2^i + 1) = i$. In addition, because $2^{10} < 2011 < 2^{11}$, the maximum value of $g(n)$ is achieved when $n = 1025 = 2^{10} + 1$, in which case $g(1025) = 10$.

### 2.4.8 Round 8

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

22. [14pts] Let $W$ be the answer to problem 24 in this guts round. Let $f(a) = \frac{1}{1-\frac{1}{1-\frac{1}{a}}}$. Determine $|f(2) + \cdots + f(W)|$.

**Solution.** The answer is $\boxed{66}$. We have

$$f(a) = \frac{1}{1-\frac{1}{1-\frac{1}{a}}} = \frac{1}{1-\frac{a}{a-1}} = 1 - a.$$

Hence

$$|f(1) + f(2) + \cdots + f(W)| = |W - (1 + 2 + \cdot + W)| = |W - \frac{W(W+1)}{2}| = \frac{W(W-1)}{2}.$$

For $W = 12$, the answer is 66.

23. [14pts] Let $X$ be the answer to problem 22 in this guts round. How many odd perfect squares are less than $8X$?

**Solution.** The answer is $\boxed{11}$. It is given from problem 22 that $X = \frac{W(W-1)}{2}$, so $8X = 4W(W-1)$. Note that $(2W-2)^2 < 4W(W-1) < (2W-1)^2$, so the number of odd perfect squares less than 8X is $W - 1$. For $W = 12$, the answer is 11.

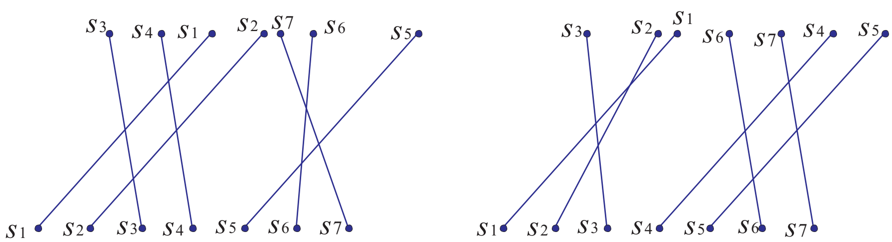
24. [14pts] Let $Y$ be the answer to problem 23 in this guts round. What is the maximum number of points of intersections of two regular $(Y-5)$-sided polygons, if no side of the first polygon is parallel to any side of the second polygon?

**Solution.** The answer is $\boxed{12}$. You are given that $Y = W - 1$ from problem 23. The maximum number of intersections of two regular $k$-sided polygons is $2(Y-5) = 2(W-6) = W$ or $W = 12$.

### 2.4.9 Round 9

25. [16pts] Cross country skiers $s_1, s_2, s_3, \ldots, s_7$ start a race one by one in that order. While each skier skis at a constant pace, the skiers do not all ski at the same rate. In the course of the race, each skier either overtakes another skier or is overtaken by another skier exactly two times. Find all the possible orders in which they can finish. Write each possible finish as an ordered septuplet $(a, b, c, d, e, f, g)$ where $a, b, c, d, e, f, g$ are the numbers 1-7 in some order. (So $a$ finishes first, $b$ finishes second, etc.)

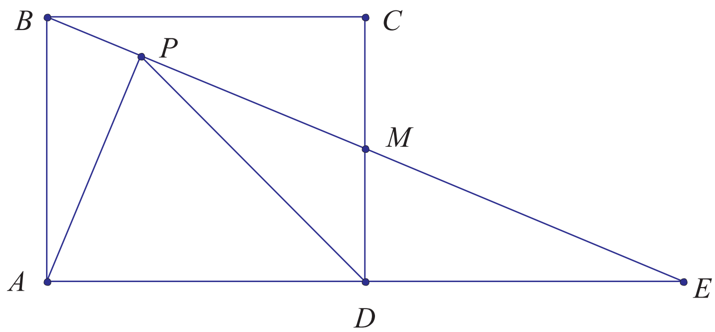
**Solution.** The answers are $\boxed{(s_3, s_2, s_1, s_6, s_7, s_4, s_5)}$ and $\boxed{(s_3, s_4, s_1, s_2, s_7, s_6, s_5)}$. Clearly the $s_7$ must finish 5$^{\text{th}}$, and $s_1$ must finish 3$^{\text{rd}}$. If the $s_6$ finishes 7$^{\text{th}}$, then he has only been passed by $s_7$, so he must finish 6$^{\text{th}}$ or 4$^{\text{th}}$. In both cases $s_5$ must finish 7$^{\text{th}}$, and similarly $s_3$ must finish 1$^{\text{st}}$. If he finishes 6$^{\text{th}}$, we must only determine the place of $s_4$ and $s_2$. Because $s_4$ cannot be overtaken by $s_7, s_6, s_5$ (because these have already overtaken or been overtaken twice), $s_4$ cannot place 4$^{\text{th}}$, because it has not overtaken or been overtaken. Then $s_4$ must place 2nd and $s_2 2$ must place 4$^{\text{th}}$ - so $(s_3, s_2, s_1, s_6, s_7, s_4, s_5)$ is a solution. When $s_6$ places 4$^{\text{th}}$, $s_4$ must place 6$^{\text{th}}$, or another skier will have to be overtaken more than two times. This means, $s_2$ places 2nd, so $(s_3, s_4, s_1, s_2, s_7, s_6, s_5)$ is the other solution. (Please see the diagrams shown below for a geometric interpretation of the solution.)




26. [16pts] Archie the Alchemist is making a list of all the elements in the world, and the proportion of earth, air, fire, and water needed for producing each. He writes the proportions in the form E:A:F:W (E for Earth, A for Air, F for Fire, and W for water). If each of the letters represents a whole number from 0 to 4, inclusive, how many different elements can Archie list? Note that if Archie lists Wood as 2:0:1:2, then 4:0:2:4 would also produce wood, and that 0:0:0:0 does not produce an element.

**Solution.** The answer is $\boxed{529}$. Note that, except for $0:0:0:0$, there are $5^4 - 1 = 624$ proportions if we ignore the condition that multiples of a proportion produce the same element. However, there are two cases of overcounting. Firstly, doubling: $2:0:1:2$ and $4:0:2:4$ produce the same element. In this case, there are $3^4 - 1 = 80$ overcounts. Secondly, tripling and quadrupling: $(1:0:1:1 = 4:0:4:4 = 3:0:3:3)$. Here, there are $2^4 - 1 = 15$ overcounts from the tripling. The quadruples have already been counted by the doubling case. Hence, there are $624 - 80 - 15 = 529$ proportions.

27. [16pts] Let $ABCD$ be a rectangle with $AB = 10$ and $BC = 12$. Let $M$ be the midpoint of $CD$, and $P$ be the point on $BM$ such that $DP = DA$. Find the area of quadrilateral $ABPD$.




**Solution.** The answer is $\boxed{\frac{11640}{169}}$. Extend segment $BM$ through $M$ to meet ray $AD$ at point $E$. It is easy to see that triangle $BCM$ is congruent to triangle $EDM$. In particular, we conclude that $DE = BC = AD = DP$. This means that, in triangle $APE$, median $PD$ is equal to half of the side $AE$; that is, $APE$ is a right triangle with $\angle APE = 90°$. Note that both triangles $APB$ and $APE$ are similar to triangle $BCM$, which is a 5-12-13 triangle. Note also that $[APD] = \frac{[APE]}{2}$ because $AD = DE$. Thus,

$$[ABP] = \left(\frac{AB}{BM}\right)^2 \cdot [BCM] = \frac{100[BCM]}{169}$$

and

$$[APD] = \frac{[APE]}{2} = \frac{1}{2} \cdot \left(\frac{AE}{BM}\right)^2 \cdot [BCM] = \frac{288[BCM]}{169}.$$

It follows that $[ABPD] = \frac{388[BCM]}{169} = \frac{388 \cdot 30}{169} = \frac{11640}{169}$.

### 2.4.10 Round 10

28. [17pts] David the farmer has an infinitely large grass-covered field which contains a straight wall. He ties his cow to the wall with a rope of integer length. The point where David ties his rope to the wall divides the wall into two parts of length $a$ and $b$, where $a > b$ and both are integers. The rope is shorter than the wall but is longer than $a$. Suppose that the cow can reach grass covering an area of $\frac{165\pi}{2}$. Find the ratio $\frac{a}{b}$. You may assume that the wall has 0 width.

**Solution.** The answer is $\boxed{\frac{9}{2}}$. The cow eats the grass in front of the wall with the area of $\frac{1}{2} \cdot \pi \cdot \ell^2$, and eats the grass behind the wall with the area of

$$\frac{1}{2} \cdot \pi \cdot (\ell - a)^2 + \frac{1}{2} \cdot \pi \cdot (\ell - b)^2.$$

Therefore $\ell^2 + (\ell - a)^2 + (\ell - b)^2 = 165$. Note that $165 = 10^2 + 7^2 + 4^2$ or $165 = 10^2 + 8^2 + 1^2$; that is, $(\ell, a, b) = (10, 6, 3)$ or $(\ell, a, b) = (10, 9, 2)$. Because $a + b > \ell$, we must have $(\ell, a, b) = (10, 9, 2)$ and $\frac{a}{b} = \frac{9}{2}$.

29. [17pts] Let $S$ be the number of ordered quintuples $(a,b,x,y,n)$ of positive integers such that

$$\begin{aligned}\frac{a}{x}+\frac{b}{y} &= \frac{1}{n}, \\ abn &= 2011^{2011}.\end{aligned}$$

Compute the remainder when $S$ is divided by 2012.

**Solution.** The answer is $\boxed{1006}$. Because 2011 is a prime, by the second given equation, we may set $n = 2011^m, a = 2011^k, b = 2011^{2011-m-k}$ for some nonnegative integers $m, k$. Clearly, the possible values of $m$ are $0,1,2,\ldots,2011$. For every $0 \le m \le 2011$, there are $2012-m$ possible values (namely, $0,1,\ldots,2011-m$) for $k$; that is, for each fixed $n = 2011^m$, there are $2012-m$ triples $(a,b,n) = \left(2011^k, 2011^{2011-m-k}, 2011^m\right)$ of positive integers satisfy the second given equation. For a fixed such triple $(a,b,n)$, we claim that there are $2012+m$ pairs $(x,y)$ of positive integers satisfy the first given equation. With this claim, we have established the fact that for each $0 \le m \le 2011$, there are $(2012-m)(2012+m) = 2012^2 - m^2$ quintuples $(a,b,x,y,n)$ of positive integers satisfy the given system of the equation; that is,

$$\begin{aligned} S &= \left(2012^2-0^2\right)+\left(2012^2-1^2\right)+\cdots+\left(2012^2-2011^2\right) = 2012^3-\left(1^2+2^2\cdots+2011^2\right) \\ &= 2012^3 - \frac{2011\cdot 2012\cdot(2\cdot 2011+1)}{6}.\end{aligned}$$

It is easy to check that $S$ has remainder 2006 when divided by 2012.

To complete our solution, it suffices to establish our claim. Multiplying both sides of the first given equation by $abn$ yields $ayn + bxn = xy$ or $xy - bnx - any = 0$. Adding $abn^2$ to both sides the last equation gives $xy - bnx - any + abn^2 = abn^2$ or

$$(x-an)(y-bn) = 2011^{2011+m}.$$

Because 2011 is a prime, we may set $(x-an, y-bn) = \left(2011^\ell, 2011^{2011+m-\ell}\right)$ or

$$(x,y) = \left(2011^\ell + an, 2011^{2011+m-\ell}+bn\right).$$

Clearly, there are exactly $2012+m$ possible values for $\ell$, from which our claim follows.

30. [17pts] Let $n$ be a positive integer. An $n \times n$ square grid is formed by $n^2$ unit squares. Each unit square is then colored either red or blue such that each row or column has exactly 10 blue squares. A *move* consists of choosing a row or a column, and recolor each unit square in the chosen row or column – if it is red, we recolor it blue, and if it is blue, we recolor it red. Suppose that it is possible to obtain fewer than $10n$ blue squares after a sequence of finite number of moves. Find the maximum possible value of $n$.

**Solution.** The answer is $\boxed{39}$. First, we prove $n < 40$. Assume that we make moves on $a$ rows and $b$ columns. For each unit square, its final color depends only on the parity of the moves on the row and the column it lies on. Hence the order of the moves does not matter, and we may assume that each row or column is chosen at most once. It follows that there are $f(a,b) = (n-a)b + (n-b)a$ squares have changed their color, and we call these squares *chameleons*. Because $f(a,b) = f(n-a, n-b)$, we may assume that $a + b \le n$. For the final resulting board to have fewer than $10n$ blue squares, more than half of the chameleons were blue at the beginning stage. On the other hand, at the beginning stage, at most $10a + 10b$ blue squares lie on the chosen rows and columns where the moves were made, and only they can be the candidates for the chameleons that were blue at the beginning stage. Therefore, we have

$$\frac{1}{2}[(n-a)b+(n-b)a] < 10(a+b)$$

Multiplying both sides of the last inequality by 2 and rearranging the terms gives

$$n(a+b) < 20(a+b)+2ab \quad \text{or} \quad 2n(a+b) < 40(a+b)+4ab$$

Because $(a-b)^2 \geq 0$ or $(a+b)^2 \geq 4ab$, we have

$$2n(a+b) < 40(a+b)+4ab \leq 40(a+b)+(a+b)^2 \quad \text{or} \quad 2n < 40+(a+b).$$

Because $a+b \leq n$, we conclude that $2n < 40+(a+b) \leq 40+n$, implying that $n < 40$.

For $n = 39$, we consider the $39 \times 39$ board with a $19 \times 29$ all red sub-board on the top left, a $19 \times 10$ all blue sub-board on the top right, a $20 \times 10$ all blue sub-board on the bottom left, and a $20 \times 19$ all red sub-board on the bottom right. We can make moves on each of the bottom 20 rows and right 19 columns to obtain a board with $19 + 20 \times 10 = 371$ blue unit squares.

# Exeter Math Club Contest
# January 28, 2012

# Contents

## Organizing Information

- *Tournament Directors* Yong Wook (Spencer) Kwon, Abraham Shin
- *Tournament Supervisor* Zuming Feng
- *System Administrator and Webmaster* Albert Chu
- *Head Problem Writers* Ravi Bajaj, Dai Yang
- *Problem Committee* Ravi Bajaj, Ravi Jagadeesan, Yong Wook (Spencer) Kwon, Ray Li, Abraham Shin, Alex Song, Dai Yang, David Yang
- *Solutions Editors* Ravi Bajaj, Ravi Jagadeesan, Yong Wook (Spencer) Kwon, Ray Li, Alex Song, Dai Yang
- *Problem Reviewers* Zuming Feng, Chris Jeuell, Richard Parris
- *Problem Contributors* Ravi Bajaj, Priyanka Boddu, Mickey Chao, Bofan Chen, Jiapei Chen, Vivian Chen, Albert Chu, Trang Duong, Mihail Eric, Claudia Feng, Chelsea Ge, Mark Huang, Ravi Jagadeesan, Spencer Kwon, Jay Lee, Ray Li, Calvin Luo, Abraham Shin, Alex Song, Elizabeth Wei, Dai Yang, David Yang, Grace Yin, Will Zhang, Joy Zheng
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- *Publicity* Jiexiong (Chelsea) Ge, Claudia Feng
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.
- *Tournament Sponsors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

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- *Graders* Ravi Bajaj, Ravi Charan, Bofan Chen, In Young Cho, Albert Chu, Ravi Jagadeesan, Spencer Kwon, Ray Li, David Rush, Alex Song, David Xiao, Dai Yang, Allen Yuan, Will Zhang, Joy Zheng
- *Judges* Zuming Feng, Greg Spanier
- *Runners* Michelle (Hannah) Jung *(Head Runner)*, Mickey Chao, Max Vachon, Elizabeth Wei

# Chapter 1

# EMC$^2$ 2012 Problems

## 1.1 Individual Speed Test

**Morning, January 28, 2012**

*There are 20 problems, worth 3 points each, to be solved in 20 minutes.*

1. Evaluate
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}.$$

2. A regular hexagon and a regular $n$-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the $n$-sided polygon is $2:1$, what is $n$?

3. How many nonzero digits are there in the decimal representation of $2\cdot 10\cdot 500\cdot 2500$?

4. When the numerator of a certain fraction is increased by 2012, the value of the fraction increases by 2. What is the denominator of the fraction?

5. Sam did the computation $1-10\cdot a+22$, where $a$ is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1-10)\cdot(a+22)$ instead. Luckily, he still ended up with the right answer. What is $a$?

6. Let $n! = n\cdot(n-1)\cdots 2\cdot 1$. For how many integers $n$ between 1 and 100 inclusive is $n!$ divisible by 36?

7. Simplify the expression $\sqrt{\dfrac{3\cdot 27^3}{27\cdot 3^3}}$.

8. Four points $A, B, C, D$ lie on a line in that order such that $\dfrac{AB}{CB}=\dfrac{AD}{CD}$. Let $M$ be the midpoint of segment $AC$. If $AB=6, BC=2$, compute $MB\cdot MD$.

9. Allan has a deck with 8 cards, numbered $1,1,2,2,3,3,4,4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly 2 cards?

10. Starting from the sequence $(3,4,5,6,7,8,\dots)$, one applies the following operation repeatedly. In each operation, we change the sequence
$$(a_1,a_2,a_3,\dots,a_{a_1-1},a_{a_1},a_{a_1+1},\dots)$$
to the sequence
$$(a_2,a_3,\dots,a_{a_1},a_1,a_{a_1+1},\dots).$$
(In other words, for a sequence starting with $x$, we shift each of the next $x-1$ term to the left by one, and put $x$ immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4,5,3,6,7,8,\dots)$, and after two operations, the sequence becomes $(5,3,6,4,7,8,\dots)$. How many operations will it take to obtain a sequence of the form $(7,\dots)$ (that is, a sequence starting with 7)?

11. How many ways are there to place 4 balls into a $4\times 6$ grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

12. Point $P$ lies inside triangle $ABC$ such that $\angle PBC = 30^\circ$ and $\angle PAC = 20^\circ$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

13. What is the largest prime factor of $9^3 - 4^3$?

14. Joey writes down the numbers 1 through 10 and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to 47?

15. In the coordinate plane, a *lattice point* is a point whose coordinates are *integers*. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most 3 units away from the origin. How many piles of grass can she eat?

16. A book has 1000 pages numbered $1, 2, \ldots, 1000$. The pages are numbered so that pages 1 and 2 are back to back on a single sheet, pages 3 and 4 are back to back on the next sheet, and so on, with pages 999 and 1000 being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages 9 and 10, and pages 89 and 90.)

17. Find a pair of integers $(a, b)$ for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$.

18. Find all ordered pairs $(x, y)$ of real numbers satisfying

$$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 = 0 \\ 2x^2 - 2y^2 - x + y + 21 = 0 \end{cases}$$

19. There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

20. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

## 1.2 Individual Accuracy Test

**Morning, January 28, 2012**

*There are 10 problems, worth 9 points each, to be solved in 30 minutes.*

1. An 18oz glass of apple juice is 6% sugar and a 6oz glass of orange juice is 12% sugar. The two glasses are poured together to create a cocktail. What percent of the cocktail is sugar?

2. Find the number of positive numbers that can be expressed as the difference of two integers between $-2$ and 2012 inclusive.

3. An *annulus* is defined as the region between two concentric circles. Suppose that the inner circle of an annulus has radius 2 and the outer circle has radius 5. Find the probability that a randomly chosen point in the annulus is at most 3 units from the center.

4. Ben and Jerry are walking together inside a train tunnel when they hear a train approaching. They decide to run in opposite directions, with Ben heading towards the train and Jerry heading away from the train. As soon as Ben finishes his 1200 meter dash to the outside, the front of the train enters the tunnel. Coincidentally, Jerry also barely survives, with the front of the train exiting the tunnel as soon as he does. Given that Ben and Jerry both run at 1/9 of the train's speed, how long is the tunnel in meters?

5. Let $ABC$ be an isosceles triangle with $AB = AC = 9$ and $\angle B = \angle C = 75^\circ$. Let $DEF$ be another triangle congruent to $ABC$. The two triangles are placed together (without overlapping) to form a quadrilateral, which is cut along one of its diagonals into two triangles. Given that the two resulting triangles are incongruent, find the area of the larger one.

6. There is an infinitely long row of boxes, with a Ditto in one of them. Every minute, each existing Ditto clones itself, and the clone moves to the box to the right of the original box, while the original Ditto does not move. Eventually, one of the boxes contains over 100 Dittos. How many Dittos are in that box when this first happens?

7. Evaluate
$$26 + 36 + 998 + 26 \cdot 36 + 26 \cdot 998 + 36 \cdot 998 + 26 \cdot 36 \cdot 998.$$

8. There are 15 students in a school. Every two students are either friends or not friends. Among every group of three students, either all three are friends with each other, or exactly one pair of them are friends. Determine the minimum possible number of friendships at the school.

9. Let $f(x) = \sqrt{2x + 1 + 2\sqrt{x^2 + x}}$. Determine the value of
$$\frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(24)}.$$

10. In square $ABCD$, points $E$ and $F$ lie on segments $AD$ and $CD$, respectively. Given that $\angle EBF = 45^\circ$, $DE = 12$, and $DF = 35$, compute $AB$.

## 1.3 Team Test

**Morning, January 28, 2012**

*There are 15 problems, worth 20 points each, to be solved in 45 minutes.*

1. The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

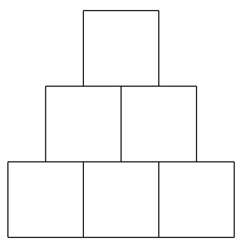
2. When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\frac{2}{3}$, and the probability that Bob does not win is $\frac{3}{4}$. What is the probability that they tie?

3. Find the sum of the $a+b$ values over all pairs of integers $(a, b)$ such that $1 \le a < b \le 10$. That is, compute the sum

$$(1+2)+(1+3)+(1+4)+\cdots+(2+3)+(2+4)+\cdots+(9+10).$$

4. A $3 \times 11$ cm rectangular box has one vertex at the origin, and the other vertices are above the $x$-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the $y$-coordinate of the highest point on the box, in centimeters?

5. Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?



6. Suppose that a right triangle has sides of lengths $\sqrt{a+b\sqrt{3}}$, $\sqrt{3+2\sqrt{3}}$, and $\sqrt{4+5\sqrt{3}}$, where $a, b$ are positive integers. Find all possible ordered pairs $(a, b)$.

7. Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

8. Compute the number of ordered pairs $(a, b)$ of positive integers less than or equal to 100, such that $a^b - 1$ is a multiple of 4.

9. In triangle $ABC$, $\angle C = 90^\circ$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$. If $\angle BIP = \angle PBI = \angle CAB = m^\circ$ for some positive integer $m$, find the sum of all possible values of $m$.

10. Bob has 2 identical red coins and 2 identical blue coins, as well as 4 distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

11. Albert takes a $4 \times 4$ checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a $4 \times 4$ grid.)

12. Let $S$ be the set of points $(x, y)$ with $0 \leq x \leq 5, 0 \leq y \leq 5$ where $x$ and $y$ are integers. Let $T$ be the set of all points in the plane that are the midpoints of two distinct points in $S$. Let $U$ be the set of all points in the plane that are the midpoints of two distinct points in $T$. How many distinct points are in $U$? (Note: The points in $T$ and $U$ do not necessarily have integer coordinates.)

13. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

14. Let $a, b, c, d, e, f$ be integers (not necessarily distinct) between $-100$ and 100, inclusive, such that $a+b+c+d+e+f = 100$. Let $M$ and $m$ be the maximum and minimum possible values, respectively, of
$$abc + bcd + cde + def + efa + fab + ace + bdf.$$
Find $\dfrac{M}{m}$.

15. In quadrilateral $ABCD$, diagonal $AC$ bisects diagonal $BD$. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find $DA$.

## 1.4 Guts Test

**Afternoon, January 28, 2012**

*There are 24 problems, with varying point values, to be solved in 75 minutes.*

### 1.4.1 Round 1

1. [6pts] Ravi has a bag with 100 slips of paper in it. Each slip has one of the numbers 3, 5, or 7 written on it. Given that half of the slips have the number 3 written on them, and the average of the values on all the slips is 4.4, how many slips have 7 written on them?

2. [6pts] In triangle $ABC$, point $D$ lies on side $AB$ such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle $BCD$.

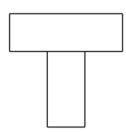
3. [6pts] Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

### 1.4.2 Round 2

4. [8pts] David is evaluating the terms in the sequence $a_n = (n + 1)^3 - n^3$ for $n = 1, 2, 3, \ldots$ (that is, $a_1 = 2^3 - 1^3, a_2 = 3^3 - 2^3, a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is *composite* if it has a divisor other than 1 and itself.)

5. [8pts] Find the sum of all positive integers strictly less than 100 that are not divisible by 3.

6. [8pts] In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).



### 1.4.3 Round 3

7. [10pts] Fresh Mann is a 9th grader at Euclid High School. Fresh Mann thinks that the word *vertices* is the plural of the word *vertice*. Indeed, vertices is the plural of the word *vertex*. Using all the letters in the word *vertice*, he can make $m$ 7-letter sequences. Using all the letters in the word *vertex*, he can make $n$ 6-letter sequences. Find $m - n$.

8. [10pts] Fresh Mann is given the following expression in his Algebra 1 class:

$$101 - 102 = 1.$$

Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

9. [10pts] Fresh Mann said, "The function $f(x) = ax^2+bx+c$ passes through 6 points. Their $x$-coordinates are consecutive positive integers, and their $y$-coordinates are $34, 55, 84, 119, 160$, and $207$, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct $y$-coordinate. Find the corrected value.

### 1.4.4 Round 4

10. [12pts] An assassin is trying to find his target's hotel room number, which is a three-digit positive integer. He knows the following clues about the number:

    (a) The sum of any two digits of the number is divisible by the remaining digit.

    (b) The number is divisible by 3, but if the first digit is removed, the remaining two-digit number is not.

    (c) The middle digit is the only digit that is a perfect square.

    Given these clues, what is a possible value for the room number?

11. [12pts] Find a positive real number $r$ that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

12. [12pts] Find the largest integer $n$ such that there exist integers $x$ and $y$ between 1 and 20 inclusive with

$$\left|\frac{21}{19} - \frac{x}{y}\right| < \frac{1}{n}.$$

### 1.4.5 Round 5

13. [14pts] A unit square is rotated 30° counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

14. [14pts] Suppose points $A$ and $B$ lie on a circle of radius 4 with center $O$, such that $\angle AOB = 90°$. The perpendicular bisectors of segments $OA$ and $OB$ divide the interior of the circle into four regions. Find the area of the smallest region.

15. [14pts] Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90°$. There is a point $I$ inside the quadrilateral that is equidistant from all the sides. Find $AI$.

### 1.4.6 Round 6

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

16. [16pts] Let $C$ be the answer to problem 18. Compute

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{C^2}\right).$$

17. [16pts] Let $A$ be the answer to problem 16. Let $PQRS$ be a square, and let point $M$ lie on segment $PQ$ such that $MQ = 7PM$ and point $N$ lie on segment $PS$ such that $NS = 7PN$. Segments $MS$ and $NQ$ meet at point $X$. Given that the area of quadrilateral $PMXN$ is $A - \frac{1}{2}$, find the side length of the square.

18. [16pts] Let $B$ be the answer to problem 17 and let $N = 6B$. Find the number of ordered triples $(a, b, c)$ of integers between 0 and $N - 1$, inclusive, such that $a + b + c$ is divisible by $N$.

### 1.4.7 Round 7

19. [16pts] Let $k$ be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{\text{Seven 7s}}$. What is the largest prime factor of the number consisting of $k$ 7's written in a row?

20. [16pts] Suppose that $E = 7^7$, $M = 7$, and $C = 7 \cdot 7 \cdot 7$. The characters $E, M, C, C$ are arranged randomly in the following blanks.

$$\_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

21. [16pts] During a recent math contest, Sophy Moore made the mistake of thinking that 133 is a prime number. Fresh Mann replied, "To test whether a number is divisible by 3, we just need to check whether the sum of the digits is divisible by 3. By the same reasoning, to test whether a number is divisible by 7, we just need to check that the sum of the digits is a multiple of 7, so 133 is clearly divisible by 7." Although his general principle is false, 133 is indeed divisible by 7. How many three-digit numbers are divisible by 7 and have the sum of their digits divisible by 7?

### 1.4.8 Round 8

22. [18pts] A *look-and-say sequence* is defined as follows: starting from an initial term $a_1$, each subsequent term $a_k$ is found by reading the digits of $a_{k-1}$ from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

    If $a_1$ is a random two-digit positive integer, find the probability that $a_4$ is at least six digits long.

23. [18pts] In triangle $ABC$, $\angle C = 90^\circ$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$, and $\angle BIP = \angle PBI = \angle CAB$. If $\dfrac{AP}{BC} = k$, express $\dfrac{IP}{CP}$ in terms of $k$.

24. [18pts] A subset of $\{1, 2, 3, \ldots, 30\}$ is called *delicious* if it does not contain an element that is 3 times another element. A subset is called *super delicious* if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.

## 1.5 Puzzle Test

**Afternoon, January 28, 2012**

*There are a total of 100 boxes to be filled in, worth 1 point each, to be completed in 25 minutes.*

### 1.5.1 The Grid

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 |
| 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 |
| 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 |
| 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 |
| 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 |
| 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 |
| 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 |

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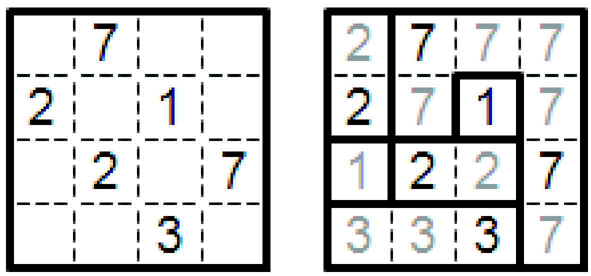
### 1.5.2 Fillomino

This puzzle round is a **fillomino puzzle** with clues presented in the form of math problems.

A fillomino is a puzzle on an $n \times n$ grid. The goal is to fill a positive integer into each square of the grid such that, if one were to draw a boundary between every pair of neighboring squares with distinct numbers, we would partition the grid into polyominoes (i.e. connected groups of adjacent cells). The number in each square should tell how many cells are in the polyomino that contains it. Below, we present a fillomino puzzle (from http://mellowmelon.wordpress.com/) along with its only solution.




As you can see, in a conventional fillomino, some clues are given, and the puzzle solver must use logical reasoning to determine what numbers go in the other squares. However, for this round, you are initially given a blank grid, and the clues are given in the form of math problems.

### 1.5.3 The Format

The clues are split into two categories. To determine the unique solution to the fillomino puzzle, your team will only need to solve the problems from section 1.5.4. However, if you get stuck or cannot solve all of those problems, additional clues are available in section 1.5.5. Note that even if you solve all of the problems in the two sections, there will still be empty spaces in the grid that will need to be filled in with logical deduction. Your score for this section will be the number of correct squares that you have filled in.

**Teamwork and speed are critical during this round.** You are not expected to solve all the problems, and it is important to find a balance between solving problems and logically deducing the puzzle. As always, it is important to have fun.

### 1.5.4 Clues

Each problem will be assigned a *clue-value*, and the answer to each problem will be an integer between 0 and 99 inclusive. Each answer corresponds to one of the 100 boxes in the grid shown on the previous page. The clue-value for a given problem is the number that goes inside the box corresponding to the problem's answer. For example if the answer to a problem with clue-value 2 is 27, then the number in box 27 is a 2. No two problems will have the same answer, ensuring that no clues contradict each other.

#### Clue-value 1 Problems

- What is the smallest even two-digit positive integer whose tens and units digits are equal?
- A class has 5 boys and 5 girls. How many ways are there to chose two presidents from the class, one boy and one girl?
- What is the smallest composite integer that is 1 more than the 4th power of a positive integer?

#### Clue-value 2 Problems

- What is the smallest 2-digit prime number?
- What is the smallest prime number whose digits sum to 10?
- How many ways are there to arrange Al, Bob, Carl, and David in a line such that at least one person is standing in front a person whose name comes later in the alphabet?
- What is the average of the five largest two-digit multiples of 3?
- Let $N$ be the largest 3 digit perfect square. Find $\frac{N-21}{10}$.
- Evaluate $3^4 + 2^4 + 1^4$.

#### Clue-value 3 Problems

- A square is inscribed in a circle of area $\dfrac{\pi}{2}$. What is the area of the square?
- Two lattice points with different $x$ and $y$-coordinates are chosen, one on the line $y = x + 1$, the other on the line $y = x$. If the slope of the line through these two points is an integer, what is the slope?
- Pete has 3 slips of paper. For each slip, he randomly chooses one of $1, 2$, or 3, each with equal probability, and writes it on the slip. (The numbers are chosen independently for each slip.) Let $p$ be the probability that the sum of the numbers on the slips equals 9. What is $\dfrac{1}{p}$?
- A bug is moving along a number line at a constant rate. At time 0, the bug is at 3. At time 2, the bug is at 13. Where is the bug at time 10?
- Let $G$ denote the greatest common divisor and $L$ the least common multiple of 6 and 159. What is the remainder when $L \cdot G$ is divided by 100?

## Clue-value 4 Problems

- In rectangle $ABCD$, $AB = 1$ and $BC = 2$. Let $M$ be the midpoint of segment $AD$. Compute the remainder when the degree measure of $\angle BMD$. is divided by 100.
- Let $x$ be the answer to this problem. Compute $\dfrac{x+86}{3}$.
- How many ways are there to chose a two person team from a group of 10 people?
- How many ways are there to select a president and vice-president from a group of 9 people containing Newt and Rom, if Newt cannot be vice president when Rom is president?

## Clue-value 5 Problems

- Find the smallest positive odd number that leaves a remainder of 2 upon division by 3 and a remainder of 4 upon division by 5.
- A rectangular box has sides $9, 12$, and 36. Compute the length of the longest diagonal of the box.
- What is the $7^{\text{th}}$ smallest positive integer with an odd number of positive integer factors?
- How many integers are between $\dfrac{444}{239}$ and $\dfrac{239}{4}$?

## Clue-value 6 Problems

- When $\dfrac{1}{93} + \dfrac{1}{186}$ is simplified and reduced to lowest terms, what is the denominator of the resulting fraction?
- Find the remainder when $3 \cdot 6 \cdot 37$ is divided by 100.
- Let $A$ be the surface area of a cube with side length 5 units. Compute $\frac{A}{2}$.
- The probability that it will be sunny on a given day is twice the probability that it will be hot. If the probability that it will not be hot that day is 57%, then the probability that it will be sunny is $x\%$. What is $x$?

## Clue-value 9 Problems

- When 4 standard, six-sided dice are tossed, how many possible values are there for their sum?
- If $4^{x-1} = 2$, compute $4^{x+1} - 1$.
- Two sides of a right triangle are 9 and 40, and all the sides are integers. What is the length of the third side?
- When a number $n$ is increased by 50%, the result is 35 more than when $n$ is decreased by $33\dfrac{1}{3}\%$. Find $n$.

### Clue-value 16 Problems

- When evaluated as a decimal number, the quantity $(2^2)(3^3)(4^4)(5^5)(6^6)(7^7)(8^8)(9^9)(10^{10})$ ends with a string of $n$ consecutive zeroes. Determine $n$.
- Let $S$ be the set of all nonnegative integers whose base-3 representation contains no twos. How many elements of $S$ are less than $2 + 3 + 3^2 + 3^3 + 3^4$?

### Clue-value 27 Problems

- In my (rather large) drawer, I have 2012 red socks, and $2012^{2012}$ blue socks. How many socks do I need to draw in order to guarantee that I have two socks of the same color?
- If $a@b = ab - a$ and $a\&b = 2a + b$, evaluate $3@(3\&14)$
- A frog is located on a number line. Each second it jumps either 1 unit backwards or 3 units forward. At how many positions could it be after 96 jumps?

## 1.5.5 Optional Clues

### Clue-value 1 Problems

- Suppose Albert bikes $x$ mph when going uphill, and $y$ mph when going downhill, where $x$ and $y$ are positive integers with $y > x$. During one trip, Albert biked uphill for some distance and then back downhill that same distance. His average speed on this trip was more than 15 miles per hour. For another trip, he biked uphill for some amount of time and then back downhill for the same amount of his time. His average speed was an integer between 1 and 14 miles per hour inclusive. For how many pairs of positive integers $(x, y)$ could this have happened?

### Clue-value 2 Problems

- Find the smallest positive integer $d$ greater than 1 such that $2d-1$ and $5d-1$ are both perfect squares.
- What is the largest two-digit factor of $\underbrace{11...11}_{18\ 1\text{'s}}$?

### Clue-value 3 Problems

- Let $f(x) = \frac{x+1}{6}, g(x) = 3x - 2$ and $h(x) = 2x + 3$. Evaluate $h(g(f(h(g(f(17))))))$.
- Al has an 8 by 8 chessboard. Initially all the squares are white. First, he colors all the squares on the two main diagonals black. Next, he picks a row and colors all white squares in that row black. Finally, he picks a column and colors all white squares in that column black. What is the maximum possible number of remaining white squares?
- Let $N$ be a two-digit prime number such that when the digits of $N$ are reversed, the resulting number is a multiple of 5. There are two possible values of $N$, and let the smaller possibility be $x$. Find $x - 1$.

## Clue-value 4 Problems

- The Fibonacci Sequence is defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. Let $F_k$ be the smallest term greater than 1 in the Fibonacci Sequence that is a perfect square. Determine the remainder when $F_k$ is divided by 100.

- A palindrome is a number that reads the same forwards and backwards. The product of the digits of a 3-digit palindrome, $N$, is 36. It is known that $N$ is not a perfect cube, and the remainder when $N$ is divided by 100 is not a perfect square. What is the remainder when $N$ is divided by 100?

## Clue-value 6 Problems

- Al had 4 test scores whose average was 81 points. Recently he took another test, and now the average of his 5 tests is 80 points. What was his score on the last test?

- A number is called *almost-prime* if it is not prime but not divisible by 2,3, or 5. There are three 2-digit almost-prime numbers. What is the second-smallest one?

- How many paths of length 6 are there from the point $(0, 0, 0)$ to the point $(2, 2, 2)$ in space if one can only move between lattice points separated by a distance of 1?

## Clue-value 9 Problems

- Five circles with distinct radii are drawn in the plane. What is the maximum possible number of distinct points of intersection between the circles?

## Clue-value 16 Problems

- Let $ABCD$ be a rectangle with $AB = 1, BC = 42$. Let $E, F, G, H$ be the midpoints of segments $AB, BC, CD, DA$, respectively. Let $P$ be the intersection of $DE$ and $BH$ and $Q$ be the intersection of $DF$ and $BG$. Compute the area of quadrilateral $BPDQ$.

- How many ways are there to choose an ordered quadruple of positive integers $(a, b, c, d)$ such that $a + b + c + d = 9$?

- An ordered quintuple of nonnegative integers $(a, b, c, d, e)$ is called *good* if it satisfies $2 \geq a \geq b \geq c \geq d \geq e \geq 0$. Define the *goodness* of a good quintuple as the product of all its elements. What is the sum of the goodness of all good quintuples?

- A *lucky number* is a number whose digits are all either 4 or 7 (ignoring leading zeros). The smallest lucky number is 4. What is the remainder when the $2013^{\text{th}}$ smallest lucky number is divided by 100?

## Clue-value 27 Problems

- How many ways are there to make 39 cents using pennies, nickels, and dimes, given that at least one of each coin must be used, and the total value of the dimes used is strictly less than the total value of the other coins used?

- Evaluate the sum

$$(1) + (1+2) + (1+2+3) + \cdots + (1+2+3+4+5+6+7)$$

- Let $ABCD$ be a quadrilateral with $AB = AC = AD = 12$, $BC = CD$, and $\angle BAC = \angle CAD = 30°$. What is the area of $ABCD$?

- Let N be a two-digit multiple of 5 such that when the digits of N are reversed, the resulting number is a two digit prime. There are two possible values of N. What is the larger one?

# Chapter 2

# EMC$^2$ 2012 Solutions

## 2.1 Individual Speed Test Solutions

1. Evaluate
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}.$$

   **Solution.** The answer is $\boxed{\frac{7}{120}}$.

   Combining the fractions gives
$$\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}=\frac{5}{2\cdot 3\cdot 4\cdot 5}+\frac{2}{2\cdot 3\cdot 4\cdot 5}=\frac{5+2}{2\cdot 3\cdot 4\cdot 5}=\frac{7}{2\cdot 3\cdot 4\cdot 5}=\frac{7}{120}.$$

2. A regular hexagon and a regular $n$-sided polygon have the same perimeter. If the ratio of the side length of the hexagon to the side length of the $n$-sided polygon is $2:1$, what is $n$?

   **Solution.** The answer is $\boxed{12}$.

   Let the side length of the regular hexagon be $2x$. Then the side length of the $n$-sided polygon is $x$. Because the regular hexagon and the regular $n$-sided polygon have the same perimeter, we have $nx = 6\cdot 2x$ or $n = 12$.

3. How many nonzero digits are there in the decimal representation of $2\cdot 10\cdot 500\cdot 2500$?

   **Solution.** The answer is $\boxed{2}$.

   We have $2\cdot 10\cdot 500\cdot 2500 = 2\cdot 5\cdot 25\cdot 100000 = 25000000$.

4. When the numerator of a certain fraction is increased by 2012, the value of the fraction increases by 2. What is the denominator of the fraction?

   **Solution.** The answer is $\boxed{1006}$.

   Let $x$ be the numerator and $y$ be the denominator of the original fraction. We have
$$\frac{x}{y}+2=\frac{x+2012}{y} \quad \text{or} \quad \frac{x+2y}{y}=\frac{x+2012}{y}$$
   from which it follows that $y = 1006$.

5. Sam did the computation $1 - 10\cdot a + 22$, where $a$ is some real number, except he messed up his order of operations and computed the multiplication last; that is, he found the value of $(1-10)\cdot(a+22)$ instead. Luckily, he still ended up with the right answer. What is $a$?

   **Solution.** The answer is $\boxed{221}$.

   Simplifying both sides of the equation yields $23 - 10a = -9a - 198$. Adding $10a + 198$ to both sides, we have $a = 221$.

6. Let $n! = n\cdot(n-1)\cdots 2\cdot 1$. For how many integers $n$ between 1 and 100 inclusive is $n!$ divisible by 36?

   **Solution.** The answer is $\boxed{95}$.

   Note that $6!$ divides $n!$ for all $n \geq 6$. Because 36 does not divide $5! = 120$ and 36 does divide $6! = 720$, the possible values of $n$ are $6,\ldots,100$.

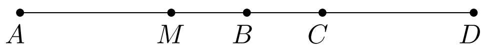
7. Simplify the expression $\sqrt{\dfrac{3 \cdot 27^3}{27 \cdot 3^3}}$.

   **Solution.** The answer is $\boxed{9}$.

   We have
$$\sqrt{\frac{3 \cdot 27^3}{27 \cdot 3^3}} = \sqrt{\frac{27^2}{3^2}} = \sqrt{9^2} = 9.$$

8. Four points $A, B, C, D$ lie on a line in that order such that $\dfrac{AB}{CB} = \dfrac{AD}{CD}$. Let $M$ be the midpoint of segment $AC$. If $AB = 6, BC = 2$, compute $MB \cdot MD$.

   **Solution.** The answer is $\boxed{16}$.




   We have $\frac{AB}{CB} = 3$. Let the length of segment $CD$ be $x$. Hence
$$3 = \frac{AB}{CB} = \frac{AD}{CD} = \frac{8+x}{x} \quad \text{or} \quad x = 4.$$
   We conclude that $AM = MC = \frac{AC}{2} = 4$, $MB = AB - AM = 2$, $MD = MC + CD = 8$, and $MB \cdot MD = 2 \cdot 8 = 16$.

9. Allan has a deck with 8 cards, numbered $1, 1, 2, 2, 3, 3, 4, 4$. He pulls out cards without replacement, until he pulls out an even numbered card, and then he stops. What is the probability that he pulls out exactly 2 cards?

   **Solution.** The answer is $\boxed{\dfrac{2}{7}}$.

   If Allan pulls out exactly 2 cards, his first card must be odd and his second card must be even. The probability that the first card is odd is $\frac{1}{2}$, and given that the first card is odd, the probability that the second card is even is $\frac{4}{7}$. Thus the probability that Allan stops after the second draw is $\frac{1}{2} \cdot \frac{4}{7} = \frac{2}{7}$.

10. Starting from the sequence $(3, 4, 5, 6, 7, 8, \dots)$, one applies the following operation repeatedly. In each operation, we change the sequence
$$(a_1, a_2, a_3, \dots, a_{a_1 - 1}, a_{a_1}, a_{a_1 + 1}, \dots)$$
   to the sequence
$$(a_2, a_3, \dots, a_{a_1}, a_1, a_{a_1+1}, \dots).$$
   (In other words, for a sequence starting with $x$, we shift each of the next $x - 1$ term to the left by one, and put $x$ immediately to the right of these numbers, and keep the rest of the terms unchanged. For example, after one operation, the sequence is $(4, 5, 3, 6, 7, 8, \dots)$, and after two operations, the sequence becomes $(5, 3, 6, 4, 7, 8, \dots)$. How many operations will it take to obtain a sequence of the form $(7, \dots)$ (that is, a sequence starting with 7)?

   **Solution.** The answer is $\boxed{7}$.

   We obtain the following sequences in order: $(4, 5, 3, 6, 7, 8, \dots)$, $(5, 3, 6, 4, 7, 8, \dots)$, $(3, 6, 4, 7, 5, 8, \dots)$, $(6, 4, 3, 7, 5, 8, \dots)$, $(4, 3, 7, 5, 8, 6, \dots)$, $(3, 7, 5, 4, 8, 6, \dots)$, $(7, 5, 3, 4, 8, 6, \dots)$.

11. How many ways are there to place 4 balls into a $4 \times 6$ grid such that no column or row has more than one ball in it? (Rotations and reflections are considered distinct.)

    **Solution.** The answer is $\boxed{360}$.

    Note there must be exactly 1 ball in each row. There are six ways to place a ball in the first row, and after that is placed, five in the second row. Likewise there are four ways for placement in the third row and three in the fourth row. Thus there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways in total.

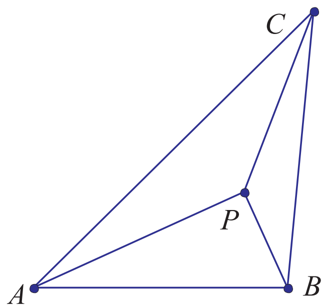
12. Point $P$ lies inside triangle $ABC$ such that $\angle PBC = 30^\circ$ and $\angle PAC = 20^\circ$. If $\angle APB$ is a right angle, find the measure of $\angle BCA$ in degrees.

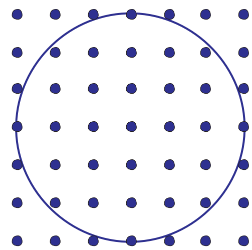
    **Solution.** The answer is $\boxed{40^\circ}$.

    Construct segment $CP$. In triangles $ACP$ and $BCP$, we have $\angle ACP + \angle CPA = 180^\circ - \angle PAC = 160^\circ$ and $\angle BCP + \angle CPB = 180^\circ - \angle PBC = 150^\circ$. Adding the last two equations gives

$$\begin{aligned} 310^\circ &= \angle ACP + \angle CPA + \angle BCP + \angle CPB = (\angle ACP + \angle BCP) + (\angle CPA + \angle CPB) \\ &= \angle ACB + (360^\circ - \angle APB), \end{aligned}$$

    from which it follows that $\angle ACB = 310^\circ - (360^\circ - 90^\circ) = 40^\circ$. (Please see the left-hand side diagram shown below.)



13. What is the largest prime factor of $9^3 - 4^3$?

    **Solution.** The answer is $\boxed{19}$.

    We have $9^3 - 4^3 = 729 - 64 = 665 = 5 \cdot 133 = 5 \cdot 7 \cdot 19$.

14. Joey writes down the numbers 1 through 10 and crosses one number out. He then adds the remaining numbers. What is the probability that the sum is less than or equal to 47?

    **Solution.** The answer is $\boxed{\dfrac{3}{10}}$.

    Let the number that Joey crossed out be $x$. Therefore sum of the numbers which remain is equal to $(1 + 2 + \cdots + 10) - x = 55 - x$. We want $55 - x \le 47$, or $8 \le x$, which happens with probability $\frac{3}{10}$.

15. In the coordinate plane, a *lattice point* is a point whose coordinates are *integers*. There is a pile of grass at every lattice point in the coordinate plane. A certain cow can only eat piles of grass that are at most 3 units away from the origin. How many piles of grass can she eat?

**Solution.** The answer is $\boxed{29}$.

The points in the coordinate plane the cow can reach are $(\pm 3, 0), (0, \pm 3)$, and $(x, y)$ with the possible values of $x$ and $y$ being $-2, -1, 0, 1, 2$. (Please see the right-hand side diagram shown above.)

16. A book has 1000 pages numbered $1, 2, \ldots, 1000$. The pages are numbered so that pages 1 and 2 are back to back on a single sheet, pages 3 and 4 are back to back on the next sheet, and so on, with pages 999 and 1000 being back to back on the last sheet. How many pairs of pages that are back to back (on a single sheet) share no digits in the same position? (For example, pages 9 and 10, and pages 89 and 90.)

    **Solution.** The answer is $\boxed{23}$.

    From 1 to 10, there are 5 pairs of pages that do not share any digits: $(1, 2), (3, 4), (5, 6), (7, 8), (9, 10)$. Any other pairs of back-to-back pages that share no digits in the same position must differ in their tens digit. There are 99 such pairs, i.e. $19-20, 29-30, \ldots, 999-1000$. However, among these 99 pairs, 81 have a repeat in the hundreds digit: $(\overline{k09}, \overline{k10}), (\overline{k29}, \overline{k30}), \ldots, (\overline{k89}, \overline{k90})$, where $1 \leq k \leq 9$ ($\overline{kab}$ denotes the integer with hundreds' digit $k$, tens' digit $a$ and units' digit $b$). Thus, the total number of pairs of back-to-back pages with no shared digits in the same position is $5 + (99 - 81) = 23$.

17. Find a pair of integers $(a, b)$ for which $\dfrac{10^a}{a!} = \dfrac{10^b}{b!}$ and $a < b$.

    **Solution.** The answer is $\boxed{(9, 10)}$.

    We have
    $$\frac{10!}{9!} = 10 = \frac{10^{10}}{10^9}.$$

18. Find all ordered pairs $(x, y)$ of real numbers satisfying
    $$\begin{cases} -x^2 + 3y^2 - 5x + 7y + 4 & = & 0 \\ 2x^2 - 2y^2 - x + y + 21 & = & 0 \end{cases}$$

    **Solution.** The answer is $\boxed{(3, -4)}$.

    Adding the two given equations, we get $x^2 + y^2 - 6x + 8y + 25 = 0$ or $(x - 3)^2 + (y + 4)^2 = 0$, by completing the squares. Hence $(x, y) = (3, -4)$ is the only possible solution of the system, and it is easy to check that it indeed is a solution.

19. There are six blank fish drawn in a line on a piece of paper. Lucy wants to color them either red or blue, but will not color two adjacent fish red. In how many ways can Lucy color the fish?

    **Solution.** The answer is $\boxed{21}$.

    Let $R_n$ denote the number of ways to color $n$ fish such that the last fish is colored red, and let $B_n$ denote the number of ways to do so with the last fish colored blue. Then, we have $R_1 = 1$, $B_1 = 1$, and for $n \geq 1$,
    $$\begin{aligned} R_{n+1} &= B_n \\ B_{n+1} &= R_n + B_n. \end{aligned}$$

    (From the first equation, we have $R_n = B_{n-1}$. Substituting the last equation into the second equation in the above system gives $B_{n+1} = R_n + B_n = B_{n-1} + B_n$, implying that the sequence $B_n$ is the *Fibonacci* sequence.) For our current problem, we can simply evaluate $B_2 = R_1 + B_1 = 2$, $R_2 = B_1 = 1$, $B_3 = R_2 + B_2 = 3$, $R_3 = B_2 = 2$, $B_4 = R_3 + B_3 = 5$, and so on, to obtain $B_7 = 21$.

20. There are sixteen 100-gram balls and sixteen 99-gram balls on a table (the balls are visibly indistinguishable). You are given a balance scale with two sides that reports which side is heavier or that the two sides have equal weights. A weighing is defined as reading the result of the balance scale: For example, if you place three balls on each side, look at the result, then add two more balls to each side, and look at the result again, then two weighings have been performed. You wish to pick out two different sets of balls (from the 32 balls) with equal numbers of balls in them but different total weights. What is the minimal number of weighings needed to ensure this?

    **Solution.** The answer is $\boxed{1}$.

    It is clear that one has to use at least one weighing. We claim that it is also enough.

    Split the thirty-two balls into three sets $S_1$, $S_2$, and $S_3$ of 11, 11, and 10 balls, respectively. Weigh sets $S_1$ and $S_2$ against each other. If the total weights are not equal, we are done. Otherwise, discard one ball from $S_1$ to form a new set $T_1$ of 10 balls. We claim that $T_1$ and $S_3$ have different weights. If not, then they have the same number of 10-gram balls, say, $n$. Then $S_1$ and $S_2$ either each had $n$ 10-gram balls or each had $n+1$ 10-gram balls. This would imply that 16 equals $3n$ or $3n+2$, both of which are impossible.

## 2.2 Individual Accuracy Test Solutions

1. An 18oz glass of apple juice is 6% sugar and a 6oz glass of orange juice is 12% sugar. The two glasses are poured together to create a cocktail. What percent of the cocktail is sugar?

   **Solution.** The answer is $\boxed{7.5\%}$.

   There is $0.06 \cdot 18 = 1.08$oz of sugar in the glass of apple juice and $0.12 \cdot 6 = 0.72$oz of sugar in the glass of orange juice. Hence, there is 1.80oz of sugar in the cocktail, which yields an answer of $\frac{1.80}{24} = 7.5\%$.

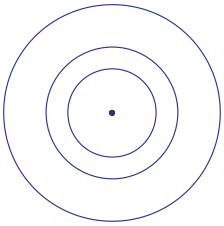
2. Find the number of positive numbers that can be expressed as the difference of two integers between $-2$ and 2012 inclusive.

   **Solution.** The answer is $\boxed{2014}$.

   Every positive integer between $2012 - 2011 = 1$ and $2012 - (-2) = 2014$ inclusive can be expressed as a difference between two integers in our set, and no positive integer greater than 2014 can be expressed.

3. An *annulus* is defined as the region between two concentric circles. Suppose that the inner circle of an annulus has radius 2 and the outer circle has radius 5. Find the probability that a randomly chosen point in the annulus is at most 3 units from the center.

   **Solution.** The answer is $\boxed{\dfrac{5}{21}}$.



   The total area of the annulus (the region between the inner circle and the outer circle in the diagram shown above) is $\pi \cdot 5^2 - \pi \cdot 2^2 = 21\pi$. The region of the annulus that is within 3 units from the center is another annulus; that is, the region between the two concentric circles of radius 2 (the inner circle) and radius 3 (the middle circle). It has area $\pi \cdot 3^2 - \pi \cdot 2^2 = 5\pi$. It follows that the probability that a randomly chosen point is within 3 units of the center is $\frac{5\pi}{21\pi} = \frac{5}{21}$.

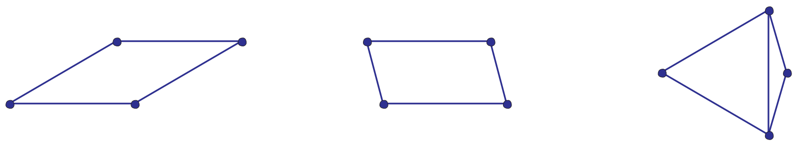
4. Ben and Jerry are walking together inside a train tunnel when they hear a train approaching. They decide to run in opposite directions, with Ben heading towards the train and Jerry heading away from the train. As soon as Ben finishes his 1200 meter dash to the outside, the front of the train enters the tunnel. Coincidentally, Jerry also barely survives, with the front of the train exiting the tunnel as soon as he does. Given that Ben and Jerry both run at 1/9 of the train's speed, how long is the tunnel in meters?

   **Solution.** The answer is $\boxed{2700}$.

   Suppose that the tunnel has length $\ell$ meters. In the time that the train takes to reach the north entrance, Ben runs 1200 meters, so the train was $1200 \cdot 9 = 10800$ km away from the north entrance. Therefore, in the time that Jerry takes to run $\ell - 1200$ meters to the south entrance, the train moves $10800 + \ell$ meters. Thus, we have $9(\ell - 1200) = 10800 + \ell$, implying that $\ell = 2700$.

5. Let $ABC$ be an isosceles triangle with $AB = AC = 9$ and $\angle B = \angle C = 75^\circ$. Let $DEF$ be another triangle congruent to $ABC$. The two triangles are placed together (without overlapping) to form a quadrilateral, which is cut along one of its diagonals into two triangles. Given that the two resulting triangles are incongruent, find the area of the larger one.

   **Solution.** The answer is $\boxed{\dfrac{81\sqrt{3}}{4}}$.



   The two isosceles triangles are congruent to each other. To stack the two triangles together to form a quadrilateral, we must match one side of a triangle with its corresponding side in the other triangle. There are three ways to do so. (Please see the diagrams shown above.) Two of the resulting quadrilaterals are parallelograms and the other is a kite (shown in the above right-hand side diagram). In this kite, one of its diagonals cuts the quadrilateral into two incongruent triangles, with the larger triangle being an equilateral triangle with side length 9 and area $\frac{81\sqrt{3}}{4}$.

6. There is an infinitely long row of boxes, with a Ditto in one of them. Every minute, each existing Ditto clones itself, and the clone moves to the box to the right of the original box, while the original Ditto does not move. Eventually, one of the boxes contains over 100 Dittos. How many Dittos are in that box when this first happens?

   **Solution.** The answer is $\boxed{126}$.

   Call the box in which the single Ditto starts out box 0 and call the initial time 0. Now, let $a_{b,t}$ denote the number of Dittos in box $b$ and time $t$. It is not difficult to see that

$$a_{b,t} = a_{b-1,t-1} + a_{b,t-1},$$

   because the first term is the number of new Dittos that come from the box to the left of box $b$ and the second term is the number of Dittos that stay in box $b$. We also know that $a_{b,0} = 1$ if $b = 0$ and $a_{b,0} = 0$ otherwise. Thus, we notice that the terms of this sequence are the entries on Pascal's triangle: $a_{b,t}$ satisfies the same recursion as the binomial coefficients (Pascal's Identity), and it clearly satisfies the initial conditions as well. It follows that $a_{b,t} = \binom{t}{b}$. Now, the first value of $t$ for which there is a $b$ such that $\binom{t}{b} > 100$ is $t = 9$, so we have $\binom{9}{4} = \binom{9}{5} = 126$, as claimed.

7. Evaluate

$$26 + 36 + 998 + 26 \cdot 36 + 26 \cdot 998 + 36 \cdot 998 + 26 \cdot 36 \cdot 998.$$

   **Solution.** The answer is $\boxed{998000}$.

   Let $p$ denote the desired quantity. We note that $p$ resembles the following identity:

$$(1 + a)(1 + b)(1 + c) - 1 = a + b + c + ab + bc + ca + abc.$$

   Setting $(a, b, c) = (26, 36, 998)$ in the above identity gives

$$p + 1 = (1 + 26)(1 + 36)(1 + 998) = 27 \cdot 37 \cdot 999 = 999^2.$$

   Hence $p = 999^2 - 1 = (999 + 1)(999 - 1) = 1000 \cdot 998 = 998000.$

8. There are 15 students in a school. Every two students are either friends or not friends. Among every group of three students, either all three are friends with each other, or exactly one pair of them are friends. Determine the minimum possible number of friendships at the school.

**Solution.** The answer is $\boxed{49}$.

This can be achieved by forming a set of 7 students who are all friends with each other, and separate set of 8 students who are all friends with each other, with no pairs of friends between these two sets. It is not difficult to see that this construction satisfies the conditions of the problem, and it consists of $\binom{7}{2}+\binom{8}{2}=49$ pairs of friends. We will now show that there must be at least 49 friendships.

Let $s_1,\ldots,s_{15}$ denote these 15 students. There must a nonzero number of friendships, so assume that $s_1$ has friends and his friends are $s_2,\ldots,s_a$ for some $2\le a\le 15$. Consider three students $s_1$ and his friend $s_m,s_n$ (that is, $2\le m,n\le a$). We already have two pairs of friends $(s_1,s_m)$ and $(s_1,s_n)$. Hence $s_m$ and $s_n$ must also be friends. Therefore, all of $s_1,\ldots,s_a$ are friends of each other.

If $a\ge 14$, then we have at least 14 students who are all friends with each other, and thus at least $\binom{14}{2}>49$ pairs of friends.

Now we assume that $a<14$; that is, at least two people (namely, $s_{a+1},\ldots,s_{15}$) are not friends with $s_1$. We consider three students $s_1$, $s_i$ and $s_j$ that are not friends with $s_1$ (that is, $a<i,j\le 15$). Because there is at least one pair of friends among these three students, $s_i$ and $s_j$ must be friends with each other. Hence all the students that are not friends with $s_1$ are friends with one another. Therefore, we have at least

$$\binom{j}{2}+\binom{15-j}{2}=\frac{a(a-1)+(15-a)(14-a)}{2}=a^2-15a+105$$

pairs of friends. For integers $a$, the quadratic expression $a^2-15a+105$ obtains its minimum value of 49 when $a=7$ or $a=8$.

**Note:** This problem and its solution are motivated by graph theory. Every student can be treated as a vertex, and every friendship can be treated as an edge. Then we are given the condition that every set of three vertices has an odd number of edges among them, and we want to minimize the number of edges in the graph. The example given is the disjoint union of two complete graphs with sizes as equal as possible.

9. Let $f(x)=\sqrt{2x+1+2\sqrt{x^2+x}}$. Determine the value of

$$\frac{1}{f(1)}+\frac{1}{f(2)}+\frac{1}{f(3)}+\cdots+\frac{1}{f(24)}.$$

**Solution.** The answer is $\boxed{4}$.

Note that, for $x\ge 0$,

$$\begin{aligned} f(x) &= \sqrt{2x+1+2\sqrt{x^2+x}}=\sqrt{x+(x+1)+2\sqrt{x(x+1)}} \\ &= \sqrt{(\sqrt{x}+\sqrt{x+1})^2}=\sqrt{x}+\sqrt{x+1}. \end{aligned}$$

Then, we have

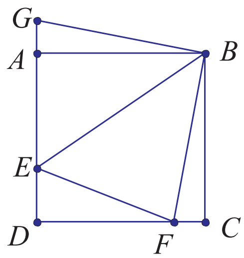
$$\frac{1}{f(x)}=\frac{1}{\sqrt{x}+\sqrt{x+1}}=\sqrt{x+1}-\sqrt{x}$$

by rationalizing the denominator. Therefore, we can evaluate the desired expression as a telescoping sum, which yields

$$\begin{aligned}\frac{1}{f(1)}+\frac{1}{f(2)}+\cdots+\frac{1}{f(24)} &= (\sqrt{2}-\sqrt{1})+(\sqrt{3}-\sqrt{2})+\cdots+(\sqrt{25}-\sqrt{24})=\sqrt{25}-\sqrt{1}\\ &= 5-1=4.\end{aligned}$$

10. In square $ABCD$, points $E$ and $F$ lie on segments $AD$ and $CD$, respectively. Given that $\angle EBF = 45^\circ$, $DE = 12$, and $DF = 35$, compute $AB$.

**Solution.** The answer is $\boxed{42}$.

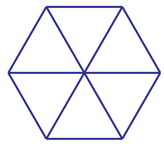



Extend segment $DA$ through $A$ to $G$ such that $AG = CF$. Then right triangles $BAG$ and $BCF$ are congruent to each other. It follows that $BG = BF$ and $\angle GBA = \angle FBC$, which implies that $\angle GBE = \angle GBA + \angle ABE = \angle FBC + \angle ABE = \angle ABC - \angle EBF = 45^\circ$. We conclude that triangles $BEG$ and $BEF$ are congruent to each other by SAS conditions ($BG = BF$, $\angle GBE = \angle FBE = 45^\circ$, $BE = BE$). In right triangle $DEF$, we have $EF = 37$. Hence $DG = DE + EG = DE + EF$ and $DG = AD + AG = AD + CF = AD + CD - DF = 2AD - DF$, from which it follows that $2AD - DF = DE + EF$ or $AD = \frac{DE+EF+FD}{2} = 42$.

## 2.3 Team Test Solutions

1. The longest diagonal of a regular hexagon is 12 inches long. What is the area of the hexagon, in square inches?

   **Solution.** The answer is $\boxed{54\sqrt{3}}$.

   As shown in the diagram above, we can divide the given regular hexagon into six equilateral triangles, each of which has side length 6. The area of the hexagon is $6 \cdot 9\sqrt{3} = 54\sqrt{3}$.

2. When Al and Bob play a game, either Al wins, Bob wins, or they tie. The probability that Al does not win is $\frac{2}{3}$, and the probability that Bob does not win is $\frac{3}{4}$. What is the probability that they tie?

   **Solution.** The answer is $\boxed{\dfrac{5}{12}}$.

   The probability that Al wins is $\frac{1}{3}$ and the probability that Bob wins is $\frac{1}{4}$. Hence, the probability that one of them wins is $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$. Thus, the probability that neither of them wins is $\frac{5}{12}$.

3. Find the sum of the $a + b$ values over all pairs of integers $(a, b)$ such that $1 \le a < b \le 10$. That is, compute the sum

   $$(1 + 2) + (1 + 3) + (1 + 4) + \cdots + (2 + 3) + (2 + 4) + \cdots + (9 + 10).$$

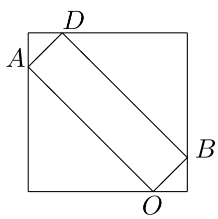
   **Solution.** The answer is $\boxed{495}$.

   Each number between 1 and 10 appears a total of 9 times over all the pairs $(a, b)$. Therefore, $(1+2)+(1+3)+(1+4)+\cdots+(2+3)+(2+4)+\cdots+(9+10) = 9(1+2+\cdots+10) = 9 \cdot 55 = 495$.

4. A $3 \times 11$ cm rectangular box has one vertex at the origin, and the other vertices are above the $x$-axis. Its sides lie on the lines $y = x$ and $y = -x$. What is the $y$-coordinate of the highest point on the box, in centimeters?

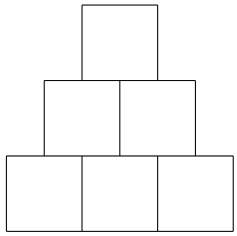
   **Solution.** The answer is $\boxed{7\sqrt{2}}$.

   Suppose a bug starts at the origin and walks up the edges of the box until it reaches the highest point. For each centimeter that the bug travels up an edge, its height increases by $\frac{1}{\sqrt{2}}$ centimeters. Since the bug travels a total of 14 centimeters before it goes down, its final height is $\frac{14}{\sqrt{2}} = 7\sqrt{2}$.

In the coordinate plane, one possible configuration, with $O = (0, 0)$, $A = \left(-\frac{11}{\sqrt{2}}, \frac{11}{\sqrt{2}}\right)$, $B = \left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)$, and $C = \left(-\frac{8}{\sqrt{2}}, \frac{14}{\sqrt{2}}\right)$, is shown below. It is not difficult to find the second possible configuration.

5. Six blocks are stacked on top of each other to create a pyramid, as shown below. Carl removes blocks one at a time from the pyramid, until all the blocks have been removed. He never removes a block until all the blocks that rest on top of it have been removed. In how many different orders can Carl remove the blocks?



**Solution.** The answer is $\boxed{16}$.

In the first move, Carl must take the top block. Consider the next two moves that Carl makes. He can either remove both blocks in the second row or remove two blocks from the same side to obtain a 3-block pyramid. We now check these two cases separately.

- *Case 1:* If he removes the second row, he can do this in 2 ways. He can remove the remaining 3 blocks in the bottom row in $3! = 6$ ways, for a total of $2 \cdot 6 = 12$ orders in which he can remove the blocks.
- *Case 2:* If he removes two blocks from the same side, he can do this in 2 ways, either by removing the right side or the left side. For the remaining 3-block pyramid, he must remove the top block first, and then remove the remaining two blocks in $2! = 2$ ways for a total of $2 \cdot 2 = 4$ orders.

Adding the results of these two cases, we see that there are $12 + 4 = 16$ orders possible.

6. Suppose that a right triangle has sides of lengths $\sqrt{a + b\sqrt{3}}$, $\sqrt{3 + 2\sqrt{3}}$, and $\sqrt{4 + 5\sqrt{3}}$, where $a, b$ are positive integers. Find all possible ordered pairs $(a, b)$.

**Solution.** The answers are $\boxed{(1,3) \text{ and } (7,7)}$.

Because $\sqrt{4 + 5\sqrt{3}} > \sqrt{3 + 2\sqrt{3}}$, either $\sqrt{4 + 5\sqrt{3}}$ or $\sqrt{a + b\sqrt{3}}$ is the hypotenuse.

If $\sqrt{4 + 5\sqrt{3}}$ is the hypotenuse, we have, by Pythagorean Theorem,

$$a + b\sqrt{3} + 3 + 2\sqrt{3} = (a + 3) + (b + 2)\sqrt{3} = 4 + 5\sqrt{3}$$

which gives $(a, b) = (1, 3)$.

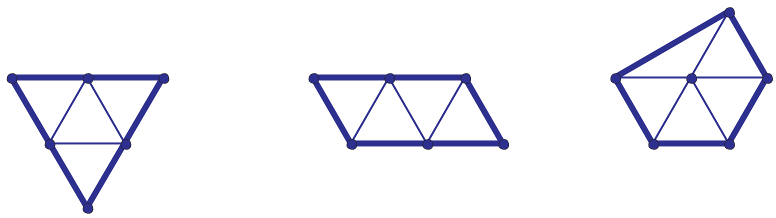
If $\sqrt{a + b\sqrt{3}}$ is the hypotenuse, we have

$$3 + 2\sqrt{3} + 4 + 5\sqrt{3} = 7 + 7\sqrt{3} = a + b\sqrt{3},$$

which gives $(a, b) = (7, 7)$, so the only ordered pairs of $(a, b)$ are $(1, 3)$ and $(7, 7)$.

7. Farmer Chong Gu glues together 4 equilateral triangles of side length 1 such that their edges coincide. He then drives in a stake at each vertex of the original triangles and puts a rubber band around all the stakes. Find the minimum possible length of the rubber band.

**Solution.** The answer is $\boxed{4+\sqrt{3}}$.



Let $T$ be the polygon consisting of the union of the 4 equilateral triangles. As shown in the diagrams above, there are three possible shapes of $T$. The left-hand side and the middle polygons can be surrounded by a rubber band of length 6. However, the surrounding rubber band for the right-hand polygon is a pentagon with four sides of length 1 and a fifth side of length $\sqrt{3}$, which gives the minimum perimeter of $4+\sqrt{3}$. (As a side note, the polygon the rubber band forms is called the *convex hull* of polygon $T$.)

8. Compute the number of ordered pairs $(a,b)$ of positive integers less than or equal to 100, such that $a^b - 1$ is a multiple of 4.

**Solution.** The answer is $\boxed{3750}$.

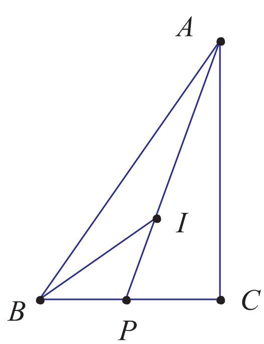
We consider the cases $a$ is even, $a$ is of the form $4n+1$, and $a$ is of the form $4n+3$.

- *Case 1:* If $a$ is even, $a^b - 1$ is always odd. Hence there are no solutions when $a$ is even.
- *Case 2:* If $a$ is of the form $4n+1$, where $n$ is an integer, 4 is a factor of $a^b - 1$ for all integers $b$, which gives $25 \cdot 100 = 2500$ pairs.
- *Case 3:* If $a$ is of the form $4n+3$, where $n$ is an integer, 4 is a factor of $a^b - 1$ only for even $b$, which gives $25 \cdot 50 = 1250$ pairs.

This gives us a total of $2500 + 1250 = 3750$ pairs.

9. In triangle $ABC$, $\angle C = 90^\circ$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$. If $\angle BIP = \angle PBI = \angle CAB = m^\circ$ for some positive integer $m$, find the sum of all possible values of $m$.

**Solution.** The answer is $\boxed{525}$.



Let $\angle CAP = x^\circ$ and $\angle PAB = y^\circ$. Then $x + y = m$. We have $\angle IBP = \angle BIP = (x + y)^\circ$. Note that $\angle BIP$ is an exterior angle of triangle $ABI$ and hence it is equal to the sum of its two remote angles $ABI$ and $BAI$. It follows that $\angle ABI = \angle BIP - \angle BAI = m^\circ - y^\circ = x^\circ$, and so $\angle ABC = \angle ABI + \angle IBP = (2x + y)^\circ$. Because $\angle C = 90^\circ$, we have $90^\circ = \angle ABC + \angle CBA = (2x + y)^\circ + (x + y)^\circ = (3x + 2y)^\circ$ or

$$90 = 3x + 2y = 2m + x.$$

Because $m$ is an integer, $x = 90 - 2m$ is an even integer and $2y = 90 - 3x$ is a multiple of 3. Hence we can write $x = 2x_1$ and $y = 3y_1$ for some positive integers $x_1$ and $y_1$. Substituting into $90 = 3x + 2y$ yields $90 = 6x_1 + 6y_1$ or $15 = x_1 + y_1$, with possible values for $y_1$ being $1, 2, \ldots, 14$. Note that $m = x + y = 2x_1 + 3y_1 = 30 + y_1$. The possible values of $m$ are $31, 32, \ldots, 44$ and their sum is equal to $31 + 32 + \cdots + 44 = \frac{31+44}{2} \cdot 14 = 525$.

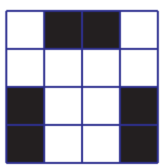
10. Bob has 2 identical red coins and 2 identical blue coins, as well as 4 distinguishable buckets. He places some, but not necessarily all, of the coins into the buckets such that no bucket contains two coins of the same color, and at least one bucket is not empty. In how many ways can he do this?

**Solution.** The answer is $\boxed{120}$.

Notice that the placement of the red coins does not affect the placement of the blue coins and vise versa. Thus, we can consider the placement of the red coins independently of the blue coins. There is $\binom{4}{0} = 1$ way to place 0 red coins so that no two are in the same bucket. There are $\binom{4}{1} = 4$ ways to place 1 red coin so that no two are in the same bucket. There are $\binom{4}{2} = 6$ ways to place 2 red coins so that no two are in the same bucket. Thus, there are a total of $1 + 4 + 6 = 11$ valid ways to place red coins, and symmetrically, 11 valid ways to place blue coins. Consequently, there are $11 \cdot 11 = 121$ total valid configurations, but we must exclude the case in which all the buckets are empty, which yields a total of 120 ways.

11. Albert takes a $4 \times 4$ checkerboard and paints all the squares white. Afterward, he wants to paint some of the square black, such that each square shares an edge with an odd number of black squares. Help him out by drawing one possible configuration in which this holds. (Note: the answer sheet contains a $4 \times 4$ grid.)

**Solution.** One possible answer is the following:



There are two ways to motivate the above configuration. One is to note that every black square must be adjacent to another black square. Thus, if we color two adjacent squares black, and color none of their other neighbors, each of the black squares is adjacent to exactly one other black square, which satisfies the problem conditions. Therefore, it makes sense to tile the board with $1 \times 2$ black dominoes instead of unit squares, in such a way that every unit square is either contained in a domino or adjacent to exactly one domino. From there, we can easily check that there are four ways to place dominoes on a $4 \times 4$ board in this manner, which are the picture above and its rotations.

The other approach is to divide the large square into two sets of 8 unit squares each in the manner of a chessboard, such that no two adjacent squares belong to the same set. Marking the unit squares in one set only affects the unit squares in the other set, so we only have to find a valid way to color half of the unit squares, then reflect our coloring horizontally across the midline to complete the picture. After this simplification, it is easier to find the above configuration through experimentation.

**Query:** Prove that any $2n \times 2n$ board can be colored in this manner.

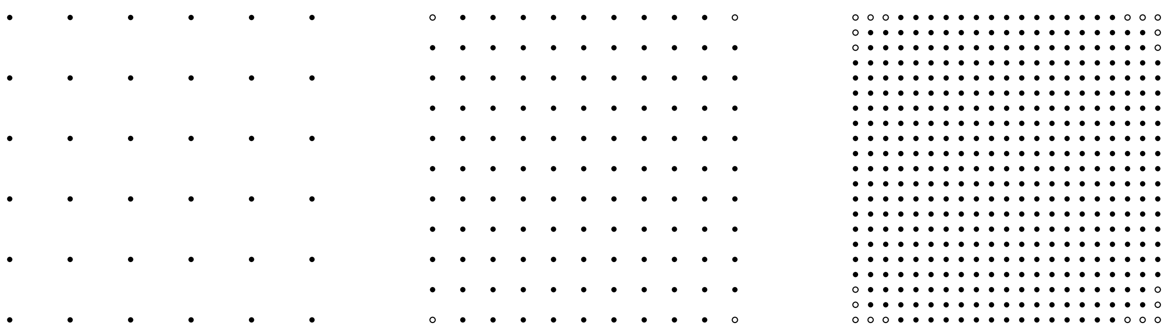
12. Let $S$ be the set of points $(x, y)$ with $0 \le x \le 5, 0 \le y \le 5$ where $x$ and $y$ are integers. Let $T$ be the set of all points in the plane that are the midpoints of two distinct points in $S$. Let $U$ be the set of all points in the plane that are the midpoints of two distinct points in $T$. How many distinct points are in $U$? (Note: The points in $T$ and $U$ do not necessarily have integer coordinates.)

**Solution.** The answer is $\boxed{421}$.

Initially, the points in $S$ form a $6 \times 6$ grid. (refer to diagrams below for $S, T, U$) After we replace these points by all their possible midpoints to get $T$, we obtain an $11 \times 11$ grid of points, but with one point missing from each corner, because these points are not midpoints of distinct points in $S$.

Now we replace these points by all their possible midpoints again to get $U$. Note that $U$ is a $21 \times 21$ grid with 5 points missing from each corner of $U$. Thus, there are $4 \cdot 5 = 20$ total points missing from the $21 \times 21$ grid. Hence, the total number of points in $U$ is $441 - 20 = 421$.

We have included a diagram of $S$, $T$, and $U$ in that order to help visualize the problem. Note that the white circles indicate the points in the corner excluded from the entire grid after each iteration.



13. In how many ways can one express 6036 as the sum of at least two (not necessarily positive) consecutive integers?

**Solution.** The answer is $\boxed{7}$.

The sum of $b$ consecutive integers starting from $a$ is

$$a + (a+1) + (a+2) + \cdots + (a+b-1) = \frac{(2a+b-1)(b)}{2} = 6036$$

Note that $b \ge 2$, and $2a+b-1$ and $b$ differ by an odd number $2a-1$, so one of $b$ and $2a+b-1$ is odd. It follows that one of $2a+b-1$ and $b$ must be an odd factor of $6036 \cdot 2 = 8 \cdot 3 \cdot 503$. Thus, one of them must be a factor of $3 \cdot 503$. There are 4 factors of $3 \cdot 503$, so there are $4 \cdot 2$ ways to pick $(2a+b-1, b)$. However, because $b$ must be at least 2, we must exclude the case where $(2a+b-1, b) = (12072, 1)$, yielding a total of 7 solutions.

14. Let $a, b, c, d, e, f$ be integers (not necessarily distinct) between $-100$ and $100$, inclusive, such that $a+b+c+d+e+f = 100$. Let $M$ and $m$ be the maximum and minimum possible values, respectively, of
$$abc + bcd + cde + def + efa + fab + aec + bdf.$$
Find $\dfrac{M}{m}$.

**Solution.** The answer is $\boxed{-\dfrac{1}{9}}$.

Note that
$$S = abc + bcd + cde + def + efa + fab + ace + bdf = (a+d)(b+e)(c+f).$$
Let $a+d = x$, $b+e = y$, $c+f = z$ and we have $S = xyz$, $x+y+z = 100$, and $-200 \le x, y, z \le 200$.

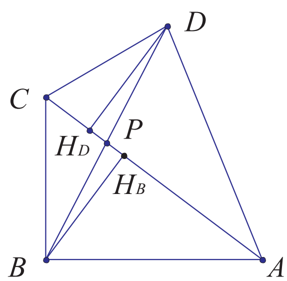
First, we will minimize $xyz$. The sum of $x, y, z$ is positive, but their product is minimized at a negative value. Consequently, $xyz$ is minimized when one of $x, y, z$ is negative and the other two are positive. Assume without loss of generality $x$ is negative. For any given $x$, we can maximize $yz$ by setting $y = z$, hence we can minimize $xyz$ by setting $y = z$. Then $xyz$ becomes $x \cdot \left(\frac{100-x}{2}\right)^2$. When $x \le 0$, the expression $\left(\frac{100-x}{2}\right)^2$ increases as $x$ decreases, so $xyz$ decreases as $x$ decreases. Thus, $xyz$ is minimized when $x$ is minimized at $-200$. This gives $y = z = 150$, for a minimum of $m = (-200) \cdot 150 \cdot 150$.

Now, we will maximize $xyz$. If $x, y, z$ are all positive, $xyz \le 33 \cdot 33 \cdot 34$ by the AM-GM inequality. If $x$ and $y$ are negative and $z$ is positive, $xyz$ is maximized when $x = -50$, $y = -50$, and $z = 200$ by a reasoning similar to the minimization argument, giving $xyz = (-50) \cdot (-50) \cdot 200$, which is clearly larger than $33 \cdot 33 \cdot 34$. Thus, $M = (-50) \cdot (-50) \cdot 200$.

Therefore, $\frac{M}{m} = \frac{(-50)\cdot(-50)\cdot 200}{-200\cdot 150\cdot 150} = -\frac{1}{9}$.

15. In quadrilateral $ABCD$, diagonal $AC$ bisects diagonal $BD$. Given that $AB = 20$, $BC = 15$, $CD = 13$, $AC = 25$, find $DA$.

**Solution.** The answer is $\boxed{4\sqrt{34}}$.




As shown in the diagram above, $P$ is the intersection of diagonals $AC$ and $BD$, and $H_B$ and $H_D$ are the feet of the perpendiculars from $B$ and $D$ to line $AC$, respectively. Because $BP = PD$, it follows that right triangles $BPH_B$ and $DPH_D$ are congruent. In particular, we have $BH_B = DH_D$. Because $AB^2 + BC^2 = 15^2 + 20^2 = 25^2 = CA^2$, $ABC$ is a right triangle with area 150. It follows that $\frac{BH_B \cdot AC}{2} = 150$, which yields that $DH_D = BH_B = 12$. In right triangle $CDH_D$, we have $CD = 13$ and $DH_D = 12$, and thus $CH_D = 5$. Hence $AH_D = AC - CH_D = 20$. Therefore, in right triangle $ADH_D$, we have $AD^2 = DH_D^2 + AH_D^2 = 12^2 + 20^2$, from which it follows that $AD = 4\sqrt{34}$.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. [6pts] Ravi has a bag with 100 slips of paper in it. Each slip has one of the numbers 3, 5, or 7 written on it. Given that half of the slips have the number 3 written on them, and the average of the values on all the slips is 4.4, how many slips have 7 written on them?

   **Solution.** The answer is $\boxed{20}$.

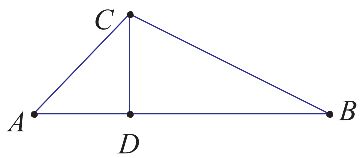
   Let $x$ be the number of slips with 7 written on them. Then, there are $50 - x$ slips with 5 written on them. The average of the numbers on the slips is

   $$4.4 = \frac{50 \cdot 3 + (50 - x) \cdot 5 + x \cdot 7}{100} = \frac{400 + 2x}{100}.$$

   This yields $x = 20$, and it follows that there are 20 slips with 7 written on them.

2. [6pts] In triangle $ABC$, point $D$ lies on side $AB$ such that $AB \perp CD$. It is given that $\frac{CD}{BD} = \frac{1}{2}$, $AC = 29$, and $AD = 20$. Find the area of triangle $BCD$.

   **Solution.** The answer is $\boxed{441}$.

   In right triangle $ADC$, we have $CD = 21$. The area of triangle $BCD$ is equal to

   $$\frac{BD \cdot CD}{2} = CD^2 = 441.$$

3. [6pts] Compute $(123 + 4)(123 + 5) - 123 \cdot 132$.

   **Solution.** The answer is $\boxed{20}$.

   Expanding the expression, we have

   $$\begin{aligned}(123+4)(123+5) - 123 \cdot 132 &= 123 \cdot 123 + 4 \cdot 123 + 5 \cdot 123 + 4 \cdot 5 - 123(123 + 9) \\ &= 123 \cdot 123 + 9 \cdot 123 + 20 - 123 \cdot 123 - 9 \cdot 123 = 20.\end{aligned}$$

### 2.4.2 Round 2

4. [8pts] David is evaluating the terms in the sequence $a_n = (n+1)^3 - n^3$ for $n = 1, 2, 3, \ldots$ (that is, $a_1 = 2^3 - 1^3, a_2 = 3^3 - 2^3, a_3 = 4^3 - 3^3$, and so on). Find the first composite number in the sequence. (An positive integer is *composite* if it has a divisor other than 1 and itself.)

   **Solution.** The answer is $\boxed{91}$.

   The first four terms of the sequence, $a_1 = 7, a_2 = 19, a_3 = 37, a_4 = 61$ are all prime. The fifth term $a_5 = 91 = 13 \cdot 7$ is the first composite number in the sequence.

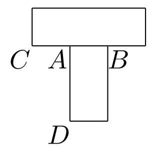
5. [8pts] Find the sum of all positive integers strictly less than 100 that are not divisible by 3.

**Solution.** The answer is $\boxed{3267}$.

Each term $n$ can be paired with $99 - n$, because if $n$ is a multiple of 3, so is $99 - n$, and vice versa. There are 66 positive integers strictly less than 100 that are not divisible by 3. Hence, their sum is

$$1 + 2 + 4 + 5 + 7 + 8 + \cdots + 97 + 98 = \frac{99 \cdot 66}{2} = 3267.$$

6. [8pts] In how many ways can Alex draw the diagram below without lifting his pencil or retracing a line? (Two drawings are different if the order in which he draws the edges is different, or the direction in which he draws an edge is different).




**Solution.** The answer is $\boxed{12}$.

Such a drawing must either start at corner $A$ and end at corner $B$ or start at $B$ and end at $A$. By symmetry, half of the drawings start at $A$ and half start at $B$. There are three possible ways to go from $A$ to $B$ without retracing: (1) via segment $AB$; (2) go around the upper rectangle via corner $C$; (3) go around the lower rectangle via corner $D$. A drawing starting from $A$ is a permutation of (1), (2), and (3). Hence the answer is $2 \cdot 3! = 12$.

**Note:** These drawings are called *Eulerian paths/cycles* in the subject of *Graph Theory*, first studied by one of the most celebrated mathematician Euler when he was walking on the bridges in the city of Konigsburg in 1736.

### 2.4.3 Round 3

7. [10pts] Fresh Mann is a 9th grader at Euclid High School. Fresh Mann thinks that the word *vertices* is the plural of the word *vertice*. Indeed, vertices is the plural of the word *vertex*. Using all the letters in the word *vertice*, he can make $m$ 7-letter sequences. Using all the letters in the word *vertex*, he can make $n$ 6-letter sequences. Find $m - n$.

**Solution.** The answer is $\boxed{2160}$

For 7-letter sequences, we can place the five letters excluding e in $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ ways. After having placed these letters, there is only one way to place the e's, so $m = 2520$. Similarly, we can place the four letters excluding e in the six-letter sequence in $6 \cdot 5 \cdot 4 \cdot 3$ ways, so $n = 360$. This yields $m - n = 2160$.

8. [10pts] Fresh Mann is given the following expression in his Algebra 1 class:

$$101 - 102 = 1.$$

Fresh Mann is allowed to move some of the digits in this (incorrect) equation to make it into a correct equation. What is the minimal number of digits Fresh Mann needs to move?

**Solution.** The answer is $\boxed{1}$.

Moving the 2, we have $101 - 10^2 = 1$.

9. [10pts] Fresh Mann said, "The function $f(x) = ax^2+bx+c$ passes through 6 points. Their $x$-coordinates are consecutive positive integers, and their $y$-coordinates are $34, 55, 84, 119, 160$, and $207$, respectively." Sophy Moore replied, "You've made an error in your list," and replaced one of Fresh Mann's numbers with the correct $y$-coordinate. Find the corrected value.

**Solution.** The answer is $\boxed{32}$.

We define

$$g(x+1) = f(x+1) - f(x) = a(x+1)^2 + b(x+1) + c - ax^2 - bx - c = 2ax + a + b.$$

We then define

$$h(x+1) = g(x+1) - g(x) = 2a(x+1) + a + b - 2ax - a - b = 2a.$$

Notice that $h(x)$ is constant.

Let the $x$-values that correspond to $35, 55, 84, 119, 160$, and $207$ be $n, n+1, n+2, n+3, n+4$, and $n+5$, respectively. Then we have $h(n+3) = h(n+4) = h(n+5) = 6$, and $h(n+2) = 9$. Consequently, there are no errors in $f(n+1), f(n+2), f(n+3), f(n+4)$, and $f(n+5)$. For $h(n+2)$ to equal 6, we must have

$$g(n+2) - g(n+1) = 29 - (55 - f(n)) = 6$$

Solving for $f(n)$ yields $f(n) = 32$.

### 2.4.4 Round 4

10. [12pts] An assassin is trying to find his target's hotel room number, which is a three-digit positive integer. He knows the following clues about the number:

    (a) The sum of any two digits of the number is divisible by the remaining digit.
    (b) The number is divisible by 3, but if the first digit is removed, the remaining two-digit number is not.
    (c) The middle digit is the only digit that is a perfect square.

    Given these clues, what is a possible value for the room number?

**Solution.** The answer is either $\boxed{213}$ or $\boxed{246}$.

It is easy to check that all three conditions are satisfied for both values.

11. [12pts] Find a positive real number $r$ that satisfies

$$\frac{4+r^3}{9+r^6} = \frac{1}{5-r^3} - \frac{1}{9+r^6}.$$

**Solution.** The answer is $\boxed{\sqrt{2}}$.

Simplifying, we have

$$\frac{5+r^3}{9+r^6} = \frac{1}{5-r^3},$$

and cross-multiplying yields, $25 - r^6 = 9 + r^6$ or $16 = 2r^6$. It follows that $r^6 = 8$, or $r = \sqrt[6]{8} = \sqrt{2}$.

12. [12pts] Find the largest integer $n$ such that there exist integers $x$ and $y$ between 1 and 20 inclusive with
$$\left|\frac{21}{19} - \frac{x}{y}\right| < \frac{1}{n}.$$

**Solution.** The answer is $\boxed{189}$.
When $x = 11$ and $y = 10$, we have
$$\left|\frac{21}{19} - \frac{11}{10}\right| = \frac{1}{190} < \frac{1}{189},$$
and hence $n = 189$ satisfies the problem conditions. We will show that
$$\frac{1}{190} \le \left|\frac{21}{19} - \frac{x}{y}\right| = \left|\frac{21y - 19x}{19y}\right|.$$
for all integers $x, y$ between 1 and 20 inclusive by casework. Because $0 < x < 21$, $19x$ is not a multiple of $x$. In particular, we have $21y - 19x \neq 0$. We consider the cases $|21y - 19x| \ge 2$, $21y - 19x = 1$, and $21y - 19x = -1$.

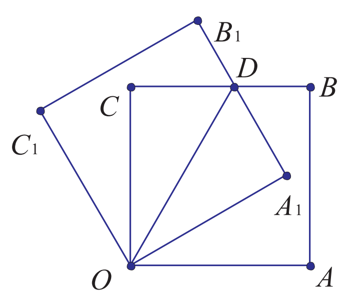
- *Case 1:* If $|21y - 19x| \ge 2$, then we have
$$\left|\frac{21}{19} - \frac{x}{y}\right| = \left|\frac{21y - 19x}{19y}\right| \ge \frac{2}{19y} \ge \frac{2}{19 \cdot 20} = \frac{1}{190}.$$
- *Case 2:* If $21y - 19x = 1$, then 21 divides $19x + 1$ from which it follows that 21 divides $-2x + 1$. The only possible value for $x$ between 1 and 20 inclusive is $x = 11$, which yields $y = 10$ and $\left|\frac{21}{19} - \frac{x}{y}\right| = \frac{1}{190}$.
- *Case 3:* If $21y - 19x = -1$, then 21 divides $19x - 1$, from which it follows that 21 divides $-2x - 1$. The only possible value for $x$ in the required range is $x = 10$, which yields $y = 9$ and $\left|\frac{21}{19} - \frac{x}{y}\right| = \frac{1}{171}$.

Combining the above, we have $\left|\frac{21}{19} - \frac{x}{y}\right| \ge \frac{1}{190}$ for all integers $x$,$y$ between 1 and 20 inclusive.

### 2.4.5 Round 5

13. [14pts] A unit square is rotated 30° counterclockwise about one of its vertices. Determine the area of the intersection of the original square with the rotated one.

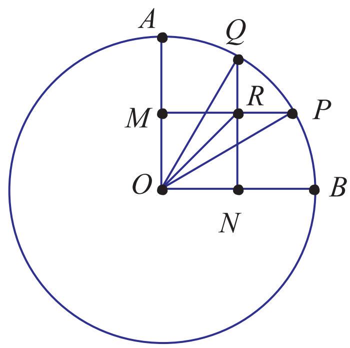
**Solution.** The answer is $\boxed{\dfrac{\sqrt{3}}{3}}$.



The diagram has reflective symmetry over line $OD$. We have $\angle A_1OA = 30°$, and hence we have $\angle COA_1 = 60°$. By symmetry, we have $\angle COD = 30°$, and thus triangle $COD$ is a $30° - 60° - 90°$ right triangle. We have $CD = \frac{\sqrt{3}}{3}$, which yields that the area of triangle $COD$ is $\frac{\sqrt{3}}{6}$. By symmetry the area of triangle $DOA_1$ is also $\frac{\sqrt{3}}{6}$, and it follows that the area of intersection is $\frac{\sqrt{3}}{3}$.

14. [14pts] Suppose points $A$ and $B$ lie on a circle of radius 4 with center $O$, such that $\angle AOB = 90°$. The perpendicular bisectors of segments $OA$ and $OB$ divide the interior of the circle into four regions. Find the area of the smallest region.

**Solution.** The answer is $\boxed{\frac{4\pi}{3} + 4 - 4\sqrt{3}}$.



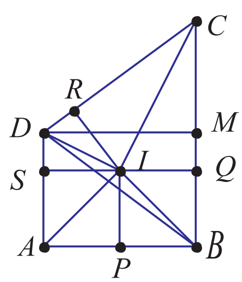

Because $Q$ lies on the perpendicular bisector of segment $OB$, we have $BQ = QO$ and because $Q$ lies on the circle, we have $QO = OB$. Thus triangle $QOB$ is equilateral, and hence we have $\angle QOB = 60°$. Similarly, we have $\angle AOP = 60°$. Furthermore, we have $QN = QO\frac{\sqrt{3}}{2} = 2\sqrt{3}$. Because $\angle AOB = 90°$, we have $\angle QOP = 30°$, from which it follows that the area of sector $QOP$ is $\frac{16\pi}{12} = \frac{4\pi}{3}$. We have $OM = ON = 2$, and because $\angle RMO = \angle MON = \angle ONR = 90°$, quadrilateral $RMON$ is a square. Hence, we have $RN = 2$ and it follows that $QR = 2\sqrt{3} - 2$. Thus, the area of triangle $QOR$ is

$$\frac{ON \cdot QR}{2} = \frac{2(2\sqrt{3} - 2)}{2} = 2\sqrt{3} - 2,$$

which by symmetry equals the area of triangle $ROP$. The area of the desired region is the area of sector $QOP$ minus the sum of the areas of triangles $QOR$ and $ROP$, so the answer is $\frac{4\pi}{3} + 4 - 4\sqrt{3}$.

15. [14pts] Let $ABCD$ be a quadrilateral such that $AB = 4$, $BC = 6$, $CD = 5$, $DA = 3$, and $\angle DAB = 90°$. There is a point $I$ inside the quadrilateral that is equidistant from all the sides. Find $AI$.

**Solution.** The answer is $\boxed{2\sqrt{2}}$.



Let the distance from $I$ to each side be $r$. Because triangle $ABD$ is right, we have $BD = 5$, which yields that triangle $BCD$ is isosceles. Let $M$ be the midpoint of segment $BC$, and we have $BM = MC = 3$, from which it follows that $DM = 4$. Hence, $ADMB$ is a rectangle, so lines $AD$ and $BC$ are parallel. Because $I$ is equidistant from $AD$ and $BC$ and $AB = 4$, we have that $r = 2$.

Let $P, Q, R, S$ be the feet of the perpendiculars from $I$ to $AB, BC, CD, DA$, respectively. We have that $ASIP$ is a square with side length $r$. Hence, we have $AI = r\sqrt{2} = 2\sqrt{2}$.

### 2.4.6 Round 6

The answer to each of the three questions in this round depends on the answer to one of the other questions. There is only one set of correct answers to these problems; however, each question will be scored independently, regardless of whether the answers to the other questions are correct.

16. [16pts] Let $C$ be the answer to problem 18. Compute

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right).$$

**Solution.** The answer is $\boxed{\frac{37}{72}}$. For every $t \neq 0$, we have that

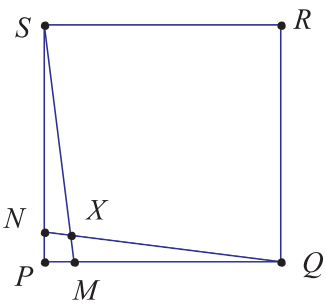
$$1-\frac{1}{t^2} = \frac{t^2-1}{t} = \frac{t-1}{t}\cdot\frac{t+1}{t}.$$

This yields

$$\begin{aligned}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{C^2}\right) &= \left(\frac{1}{2}\cdot\frac{3}{2}\right)\left(\frac{2}{3}\cdot\frac{4}{3}\right)\cdots\left(\frac{C-1}{C}\cdot\frac{C+1}{C}\right)\\ &= \frac{1}{2}\left(\frac{3}{2}\cdot\frac{2}{3}\right)\left(\frac{4}{3}\cdot\frac{3}{4}\right)\cdots\left(\frac{C}{C-1}\cdot\frac{C-1}{C}\right)\frac{C+1}{C}\\ &= \frac{1}{2}\cdot\frac{C+1}{C} = \frac{C+1}{2C}.\end{aligned}$$

17. [16pts] Let $A$ be the answer to problem 16. Let $PQRS$ be a square, and let point $M$ lie on segment $PQ$ such that $MQ = 7PM$ and point $N$ lie on segment $PS$ such that $NS = 7PN$. Segments $MS$ and $NQ$ meet at point $X$. Given that the area of quadrilateral $PMXN$ is $A - \frac{1}{2}$, find the side length of the square.

**Solution.** The answer is $\boxed{1}$.



By symmetry, triangles $XNP$ and $XPM$ have the same area, and hence they must each have area $\frac{2A-1}{4}$. The ratio of the area of triangle $XNP$ to the area of triangle $XSP$ is $\frac{NP}{SP} = \frac{1}{8}$, and it follows that triangle $XSP$ has area $4A-2$. Hence, triangle $MSP$ has area $4A-2+\frac{2A-1}{4} = \frac{18A-9}{4}$. Because $\frac{PN}{PS} = \frac{1}{8}$, the area of triangle $QSP$ is $36A-18$, from which it follows that the area of square $PQRS$ is $72A-36$. Thus, the side length of the square is $\sqrt{72A-36}$.

18. [16pts] Let $B$ be the answer to problem 17 and let $N = 6B$. Find the number of ordered triples $(a, b, c)$ of integers between 0 and $N-1$, inclusive, such that $a+b+c$ is divisible by $N$.

**Solution.** The answer is $\boxed{36}$. For every choice of integers $a, b$ between 0 and $N-1$ inclusive, there is a unique integer $c$ between 0 and $N-1$ inclusive such that $a+b+c$ is divisible by $N$. Because there are $N$ choices for each of $a$ and $b$, the number of ordered triples $(a, b, c)$ with $a+b+c$ divisible by $N$ is $N^2 = 36B^2$.

**Solution (to the system).** We have

$$\begin{aligned} A &= \frac{C+1}{2C} \\ B &= \sqrt{72A-36} \\ C &= 36B^2. \end{aligned}$$

Substituting the expression for $A$ into the expression for $B$, we have

$$B = \sqrt{72\left(\frac{C+1}{2C} - 36\right)} = \sqrt{72 \cdot \frac{1}{2C}} = \sqrt{\frac{36}{C}}.$$

and hence we have

$$C = 36B^2 = \frac{36^2}{C}.$$

This yields that $C = 36$, from which it follows that $A = \frac{37}{72}$ and $B = 1$.

### 2.4.7 Round 7

19. [16pts] Let $k$ be the units digit of $\underbrace{7^{7^{7^{7^{7^{7^{7}}}}}}}_{\text{Seven 7s}}$. What is the largest prime factor of the number consisting of $k$ 7's written in a row?

**Solution.** The answer is $\boxed{37}$.

Because $7^{7^{7^{7^{7}}}}$ is odd, the remainder of $7^{7^{7^{7^{7^{7}}}}}$ on division by 4 is 3. We have that $7^4$ has a units digit of 1, and hence we have that $7^{7^{7^{7^{7^{7^{7}}}}}}$ has the same units digit as $7^3 = 343$, and it follows that $k = 3$. We have that $777 = 3 \cdot 7 \cdot 37$, and thus the largest prime divisor is 37.

20. [16pts] Suppose that $E = 7^7$, $M = 7$, and $C = 7 \cdot 7 \cdot 7$. The characters $E, M, C, C$ are arranged randomly in the following blanks.

$$\_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$$

Then one of the multiplication signs is chosen at random and changed to an equals sign. What is the probability that the resulting equation is true?

**Solution.** The answer is $\boxed{\dfrac{1}{6}}$.

Suppose that the side of the equation containing $E$ also contains one of the other letters. Then, that side will be at least $7E = 7^8$ while the other side will be at most $C^2 = 7^6$, so the two sides cannot be equal. Thus, the only way in which the equation can be true is if $E$ is at one of the ends and the $=$ sign is right next to it, and it is clear that any such configuration is a true equation. Because the probability that $E$ is placed at one of the ends is $\frac{1}{2}$ and the probability that the $=$ sign is next to it is $\frac{1}{3}$, the answer is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

21. [16pts] During a recent math contest, Sophy Moore made the mistake of thinking that 133 is a prime number. Fresh Mann replied, "To test whether a number is divisible by 3, we just need to check whether the sum of the digits is divisible by 3. By the same reasoning, to test whether a number is divisible by 7, we just need to check that the sum of the digits is a multiple of 7, so 133 is clearly divisible by 7." Although his general principle is false, 133 is indeed divisible by 7. How many three-digit numbers are divisible by 7 and have the sum of their digits divisible by 7?

**Solution.** The answer is $\boxed{18}$.

Suppose that $n$ is a three-digit number with hundreds digit $a$, tens digit $b$, and units digit $c$. The first condition is that $a + b + c$ is divisible by 7; and the condition that $n$ is divisible by 7 is equivalent to $2a + 3b + c$ being divisible by 7. The two conditions are equivalent to $a + 2b$ being divisible by 7 and $b - c$ being divisible by 7.

We consider the cases $a = 1$ or 8, $a = 2$ or 9, $a = 3$, $a = 4$, $a = 5$, $a = 6$, and $a = 7$.

- *Case 1:* We have $a = 1$ or $a = 8$. Then, we must have $b = c = 3$, which yields 2 solutions.
- *Case 2:* We have $a = 2$ or $a = 9$. Then, we must have $b = c = 6$, which yields 2 solutions.
- *Case 3:* We have $a = 3$. Then, we can have $b = 2$ or $b = 9$ and $c = 2$ or $c = 9$, which yields 4 solutions.
- *Case 4:* We have $a = 4$. Then, we must have $b = c = 5$, which yields 1 solution.
- *Case 5:* We have $a = 5$. Then, we can have $b = 1$ or $b = 8$ and $c = 1$ or $c = 8$, which yields 4 solutions.
- *Case 6:* We have $a = 6$. Then, we must have $b = c = 4$, which yields 1 solution.
- *Case 7:* We have $a = 7$. Then, we can have $b = 0$ or $b = 7$ and $c = 0$ or $c = 7$, which yields 4 solutions.

There are a total of $2 + 2 + 4 + 1 + 4 + 1 + 4 = 18$ solutions.

### 2.4.8 Round 8

22. [18pts] A *look-and-say sequence* is defined as follows: starting from an initial term $a_1$, each subsequent term $a_k$ is found by reading the digits of $a_{k-1}$ from left to right and specifying the number of times each digit appears consecutively. For example, 4 would be succeeded by 14 ("One four."), and 31337 would be followed by 13112317 ("One three, one one, two three, one seven.")

If $a_1$ is a random two-digit positive integer, find the probability that $a_4$ is at least six digits long.

**Solution.** The answer is $\boxed{\dfrac{89}{90}}$.

We claim that $a_1 = 22$ is the only starting value such that $a_4$ has less than six digits. If the starting value is 22, then it is easy to see that $a_2 = a_3 = a_4 = 22$. Suppose that $a_1$ has a tens digit of $x$ and a units digit of $y$. We write $a_1 = \overline{xy}$.

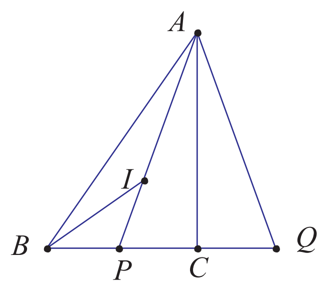
We consider the cases $x = y = 1$, $x = y \neq 1$ or 2, $x \neq y$ and $x, y \neq 1$, $x = 1$ and $y \neq 1$, and $x \neq 1$ and $y = 1$.

- *Case 1:* We have $x = y = 1$. Then, the sequence is $a_2 = 21$, $a_3 = 1211$ and $a_4 = 111221$, which has 6 digits.
- *Case 2:* We have $x = y \neq 1, 2$. Then, the sequence is $a_2 = \overline{2x}$, $a_3 = \overline{121x}$ and $a_4 = \overline{1112111x}$, which has 8 digits.
- *Case 3:* We have $x \neq y$ and $x, y \neq 1$. Then, the sequence is $a_2 = \overline{1x1y}$, $a_3 = \overline{111x111y}$ and $a_4 = \overline{311x311y}$, which has 8 digits.
- *Case 4:* We have $x = 1$ and $y \neq 1$. Then, the sequence is $a_2 = \overline{111y}$, $a_3 = \overline{311y}$ and $a_4 = \overline{13211y}$, which has 6 digits.
- *Case 5:* We have $x \neq 1$ and $y = 1$. Then, the sequence is $a_2 = \overline{1x11}$, $a_3 = \overline{111x21}$. If $x = 2$, then we have $a_4 = 312221$, which has 6 digits; if $x \neq 2$, then we have $a_4 = \overline{311x1211}$, which has 8 digits.

Combining the above, we have that 22 is the only choice of $a_1$ such that $a_4$ has less than 6 digits, and it follows that the probability that $a_4$ has at least 6 digits is $\frac{89}{90}$.

23. [18pts] In triangle $ABC$, $\angle C = 90^\circ$. Point $P$ lies on segment $BC$ and is not $B$ or $C$. Point $I$ lies on segment $AP$, and $\angle BIP = \angle PBI = \angle CAB$. If $\dfrac{AP}{BC} = k$, express $\dfrac{IP}{CP}$ in terms of $k$.

**Solution.** The answer is $\boxed{\dfrac{2-k}{k-1}}$.




Let $\angle PBI = \theta$ and let $Q$ be the reflection of point $P$ over line $AC$. We have $QA = AP$, which yields that $\angle AQP = \angle APQ$. By exterior angles on triangle $PIB$, we have $\angle APQ = \angle PBI + \angle BIP = 2\theta$. Additionally, we have

$$\angle QBA = \angle CBA = 90^\circ - \angle BAC = 90^\circ - \theta.$$

Because the angles of a triangle sum to $180^\circ$, we have

$$\angle BAQ = 180^\circ - 2\theta - (90^\circ - \theta) = 90 - \theta = \angle QBA,$$

from which it follows that $BQ = QA = AP$. Because $\angle BIP = \angle PBI$, we have $BP = PI$.

Let $BP = x$ and $PC = y$; we have $CQ = PC = y$. This yields that $BC = x + y$ and $BQ = x + 2y$. We have

$$\frac{x+2y}{x+y} = \frac{BQ}{BC} = \frac{AP}{BC} = k.$$

Hence, we have $x + 2y = kx + ky$ or $y(2-k) = x(k-1)$ or $\frac{2-k}{k-1} = \frac{x}{y}$. It follows that

$$\frac{IP}{CP} = \frac{BP}{PC} = \frac{x}{y} = \frac{2-k}{k-1}.$$

24. [18pts] A subset of $\{1, 2, 3, \ldots, 30\}$ is called *delicious* if it does not contain an element that is 3 times another element. A subset is called *super delicious* if it is delicious and no delicious set has more elements than it has. Determine the number of super delicious subsets.

**Solution.** The answer is $\boxed{96}$.

We partition the set $\{1, 2, 3, \cdots, 30\}$ into subsets:

$$\{1,3,9,27\},\{2,6,18\},\{4,12\},\{5,15\},\{7,21\},\{8,24\},\{10,30\},\{11\},\{13\},\{14\},\{16\},\{17\},\{19\},$$
$$\{20\},\{22\},\{23\},\{25\},\{26\},\{28\},\{29\}.$$

If $A$ is delicious, then it can contain at most 1 member of each listed 1 or 2-element subset and at most 2 members of each listed 3 or 4-element subset. Thus $A$ contains at most $2\cdot2+18\cdot1 = 22$ elements. If $A$ is super delicious, then $A$ must contain the given maximum possible number of elements for each listed subset. There are 3 ways to choose 2 elements of the 4-element listed subset: $\{1,9\},\{1,27\},\{3,27\}$, and there are 2 ways to choose the an element of each 2-element subset. There is only 1 way to choose 2 elements of the 3-element subset; we must choose 2 and 18. Thus there are $3\cdot2^5 = 96$ super delicious sets $A$.

## 2.5 Puzzle Round Answer

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 3 | 3 | 27 | 27 | 27 | 27 | 27 | 27 | 2 |
| 2 | 2 | 3 | 2 | 16 | 16 | 16 | 3 | 27 | 2 |
| 9 | 9 | 1 | 2 | 16 | 1 | 16 | 3 | 27 | 5 |
| 9 | 9 | 16 | 16 | 16 | 4 | 16 | 3 | 27 | 5 |
| 9 | 9 | 9 | 4 | 4 | 4 | 16 | 27 | 27 | 5 |
| 9 | 9 | 3 | 3 | 3 | 16 | 16 | 27 | 5 | 5 |
| 6 | 6 | 6 | 16 | 16 | 16 | 6 | 27 | 27 | 27 |
| 6 | 4 | 27 | 27 | 16 | 6 | 6 | 6 | 6 | 27 |
| 6 | 4 | 1 | 27 | 27 | 27 | 6 | 27 | 27 | 27 |
| 6 | 4 | 4 | 2 | 2 | 27 | 27 | 27 | 2 | 2 |

# Exeter Math Club Contest
# January 26, 2013

# Table of Contents

## Organizing Information

- *Tournament Directors* Albert Chu, Ray Li
- *Tournament Supervisor* Zuming Feng
- *System Administrator and Webmaster* Albert Chu
- *Head Problem Writers* Ravi Jagadeesan, David Yang
- *Problem Committee* Leigh Marie Braswell, Vahid Fazel-Rezai, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei
- *Solutions Editors* Leigh Marie Braswell, Vahid Fazel-Rezai, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei
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- *Treasurer* Jiexiong (Chelsea) Ge
- *Publicity* Claudia Feng
- *Primary Tournament Sponsor* We would like to thank Jane Street Capital for their generous support of this competition.

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- *Tournament Sponsors* We would also like to thank the Phillips Exeter Academy Math Department and Art of Problem Solving for their support.

# Contest day information

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- *Graders* Leigh Marie Braswell, Jiapei Chen, Albert Chu, Vahid Fazel-Rezai, Claudia Feng, Jiexiong (Chelsea) Ge, Chad Qian, Zhuo Qun (Alex) Song, Kuo-An (Andy) Wei, Dai Yang, David Yang
- *Judges* Zuming Feng, Greg Spanier
- *Head Runner* Mickey Chao
- *Runners* Janet Chen, Calvin Luo, Max Vachon Lizzie Wei

# Chapter 1

# EMC$^2$ 2013 Problems

## 1.1 Individual Speed Test

**Morning, January 26, 2013**

*There are 20 problems, worth 3 points each, to be solved in 20 minutes.*

1. Determine how many digits the number $10^{10}$ has.

2. Let $ABC$ be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$. Compute $\angle CAB$ in degrees.

3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

4. Determine the smallest perfect square greater than 2400.

5. At 12:34 and 12:43, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute $a$.

7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

9. Albert wants to make $2 \times 3$ wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one $4 \times 7$ sheet of paper.

10. Let $ABC$ be an equilateral triangle, and let $D$ be a point inside $ABC$. Let $E$ be a point such that $ADE$ is an equilateral triangle and suppose that segments $DE$ and $AB$ intersect at point $F$. Given that $\angle CAD = 15^\circ$, compute $\angle DFB$ in degrees.

11. A *palindrome* is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An *almost-palindrome* is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

12. Determine the smallest positive integer $n$ such that the sum of the digits of $11^n$ is not $2^n$.

13. Determine the minimum number of breaks needed to divide an $8 \times 4$ bar of chocolate into $1 \times 1$ pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is $x = (t - 2012)^2 - 2012(t - 2012) - 2013$. Find the number of positive integer values of $t$ at which time the particle lies in the negative half of the number line (strictly to the left of 0).

15. Let $A$ be a vertex of a unit cube and let $B, C$, and $D$ be the vertices adjacent to $A$. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

16. In equilateral triangle $ABC$, points $P$ and $R$ lie on segment $AB$, points $I$ and $M$ lie on segment $BC$, and points $E$ and $S$ lie on segment $CA$ such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PS = 2$, $RI = 3$, and $ME = 5$, compute the area of hexagon $PRIMES$.

17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

18. Fresh Mann thinks that the expressions $2\sqrt{x^2-4}$ and $2(\sqrt{x^2}-\sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers $x$. Find all real numbers $x$ such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

19. Let $m$ be the positive integer such that a $3 \times 3$ chessboard can be tiled by at most $m$ pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are $n$ ways to tile the chessboard with $m$ pairwise incongruent rectangles with integer side lengths. Find the product $mn$.

20. Let $ABC$ be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle $XYZ$ is said to be *friendly* if it intersects triangle $ABC$ and it is a translation of triangle $ABC$. Let $S$ be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of $S$ to the area of triangle $ABC$.

## 1.2 Individual Accuracy Test

**Morning, January 26, 2013**

*There are 10 problems, worth 9 points each, to be solved in 30 minutes.*

1. Find the largest possible number of consecutive 9's in which an integer between 10,000,000 and 13,371,337 can end. For example, 199 ends in two 9's, while 92,999 ends in three 9's.

2. Let $ABCD$ be a square of side length 2. Equilateral triangles $ABP$, $BCQ$, $CDR$, and $DAS$ are constructed inside the square. Compute the area of quadrilateral $PQRS$.

3. Evaluate the expression $7 \cdot 11 \cdot 13 \cdot 1003 - 3 \cdot 17 \cdot 59 \cdot 331$.

4. Compute the number of positive integers $c$ such that there is a non-degenerate obtuse triangle with side lengths $21, 29$, and $c$.

5. Consider a $5 \times 5$ board, colored like a chessboard, such that the four corners are black. Determine the number of ways to place 5 rooks on black squares such that no two of the rooks attack one another, given that the rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

6. Let $ABCD$ be a trapezoid of height 6 with bases $AB$ and $CD$. Suppose that $AB = 2$ and $CD = 3$, and let $F$ and $G$ be the midpoints of segments $AD$ and $BC$, respectively. If diagonals $AC$ and $BD$ intersect at point $E$, compute the area of triangle $FGE$.

7. A *regular octahedron* is a solid with eight faces that are congruent equilateral triangles. Suppose that an ant is at the center of one face of a regular octahedron of edge length 10. The ant wants to walk along the surface of the octahedron to reach the center of the opposite face. (Two faces of an octahedron are said to be opposite if they do not share a vertex.) Determine the minimum possible distance that the ant must walk.

8. Let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, and $D_1D_2D_3$ be triangles in the plane. All the sides of the four triangles are extended into lines. Determine the maximum number of pairs of these lines that can meet at $60^\circ$ angles.

9. For an integer $n$, let $f_n(x)$ denote the function $f_n(x) = \sqrt{x^2 - 2012x + n} + 1006$. Determine all positive integers $a$ such that $f_a(f_{2012}(x)) = x$ for all $x \geq 2012$.

10. Determine the number of ordered triples of integers $(a, b, c)$ such that $(a + b)(b + c)(c + a) = 1800$.

## 1.3 Team Test

**Morning, January 26, 2013**

*There are 10 problems, worth 30 points each, to be solved in 45 minutes.*

1. Determine the number of ways to place 4 rooks on a $4 \times 4$ chessboard such that:

   (a) no two rooks attack one another, and

   (b) the main diagonal (the set of squares marked $X$ below) does not contain any rooks.

| | | | |
|---|---|---|---|
| $X$ | | | |
| | $X$ | | |
| | | $X$ | |
| | | | $X$ |

   The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives $i$ erasers to student $i$; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

3. Let $ABC$ be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \geq 1$, $x_{n+1} = 2 - \dfrac{1}{x_n}$. Compute the product $x_1 x_2 x_3 \cdots x_{2013}$.

5. In equilateral triangle $ABC$, points $P$ and $R$ lie on segment $AB$, points $I$ and $M$ lie on segment $BC$, and points $E$ and $S$ lie on segment $CA$ such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

6. Let $f(a, b) = \frac{a^2}{a+b}$. Let $A$ denote the sum of $f(i, j)$ over all pairs of integers $(i, j)$ with $1 \leq i < j \leq 10$; that is,
$$A = (f(1,2) + f(1,3) + \cdots + f(1,10)) + (f(2,3) + f(2,4) + \cdots + f(2,10)) + \cdots + f(9,10).$$
   Similarly, let $B$ denote the sum of $f(i, j)$ over all pairs of integers $(i, j)$ with $1 \leq j < i \leq 10$; that is,
$$B = (f(2,1) + f(3,1) + \cdots + f(10,1)) + (f(3,2) + f(4,2) + \cdots + f(10,2)) + \cdots + f(10,9).$$
   Compute $B - A$.

7. Fresh Mann has a pile of seven rocks with weights 1, 1, 2, 4, 8, 16, and 32 pounds and some integer $X$ between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly $X$ pounds. Given that he can do so in more than one way, determine the sum of all possible values of $X$. (The two 1-pound rocks are indistinguishable.)

8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point $P$ lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30^\circ$. Given that $CP = 2$ and $CD = 3$, compute $CA$.

9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \geq 1$,

$$x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_n x_{n+1}}.$$

Compute $x_{100}$.

10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.

# 1.4 Guts Test

**Afternoon, January 26, 2013**

*There are 24 problems, with varying point values, to be solved in 75 minutes.*

## 1.4.1 Round 1

1. [6pts] Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

2. [6pts] A 4-meter long stick is rotated $60^\circ$ about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

3. [6pts] Let
$$f(x) = 2x(x-1)^2 + x^3(x-2)^2 + 10(x-1)^3(x-2).$$
Compute $f(0) + f(1) + f(2)$.

## 1.4.2 Round 2

4. [8pts] Twenty boxes with weights $10, 20, 30, \ldots, 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

5. [8pts] Let $ABC$ be a right triangle with a right angle at $A$, and let $D$ be the foot of the perpendicular from vertex $A$ to side $BC$. If $AB = 5$ and $BC = 7$, compute the length of segment $AD$.

6. [8pts] There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

### 1.4.3 Round 3

7. [10pts] A line of slope $-4$ forms a right triangle with the positive $x$ and $y$ axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

8. [10pts] Let $ABC$ be a right triangle with a right angle at $B$, $AB = 9$, and $BC = 7$. Suppose that point $P$ lies on segment $AB$ with $AP = 3$ and that point $Q$ lies on ray $BC$ with $BQ = 11$. Let segments $AC$ and $PQ$ intersect at point $X$. Compute the positive difference between the areas of triangles $APX$ and $CQX$.

9. [10pts] Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

### 1.4.4 Round 4

10. [12pts] The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13$. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers; for example, $7 = 5 + 2$. Express 121 as the sum of nonconsecutive, distinct, positive Fibonacci numbers. (It is not permitted to use both a 2 and a 1 in the expression.)

11. [12pts] There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

12. [12pts] Let $ABC$ be an acute triangle, and let $D$ and $E$ be the feet of the altitudes to $BC$ and $CA$, respectively. Suppose that segments $AD$ and $BE$ intersect at point $H$ with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

### 1.4.5 Round 5

13. [14pts] In coordinate space, a *lattice point* is a point all of whose coordinates are integers. The lattice points $(x, y, z)$ in three-dimensional space satisfying $0 \le x, y, z \le 5$ are colored in $n$ colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of $n$.

14. [14pts] Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

15. [14pts] Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles $ABP$, $BCQ$, $CDR$, and $DAS$ are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

### 1.4.6 Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

16. [16pts] Let $C$ be the answer to problem 18. Suppose that $x$ and $y$ are real numbers with $y > 0$ and

$$\begin{cases} x + y = C \\ x + \frac{1}{y} = -2. \end{cases}$$

Compute $y + \frac{1}{y}$.

17. [16pts] Let $A$ be the answer to problem 16. Let $PQR$ be a triangle with $\angle PQR = 90^\circ$, and let $X$ be the foot of the perpendicular from point $Q$ to segment $PR$. Given that $QX = A$, determine the minimum possible area of triangle $PQR$.

18. [16pts] Let $B$ be the answer to problem 17 and let $K = 36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in $K$ days.

### 1.4.7 Round 7

19. [16pts] Find the number of positive integers $a$ such that all roots of $x^2 + ax + 100$ are real and the sum of their squares is at most 2013.

20. [16pts] Determine all values of $k$ such that the system of equations

$$\begin{cases} y = x^2 - kx + 1 \\ x = y^2 - ky + 1 \end{cases}$$

has a real solution.

21. [16pts] Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

### 1.4.8 Round 8

22. [18pts] A sequence that contains the numbers $1, 2, 3, \ldots, n$ exactly once each is said to be a *permutation* of length $n$. A permutation $w_1 w_2 w_3 \cdots w_n$ is said to be *sad* if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation **3142756** is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

23. [18pts] Let $ABC$ be a triangle with $AB = 39, BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

24. [18pts] On a strange planet, there are $n$ cities. Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of $n$. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of $n$.

# Chapter 2

# EMC$^2$ 2013 Solutions

## 2.1 Individual Speed Test Solutions

1. Determine how many digits the number $10^{10}$ has.

   **Solution.** The answer is $\boxed{11}$.

   The number is written as a 1 followed by ten 0's, for a total of 11 digits.

2. Let $ABC$ be a triangle with $\angle ABC = 60^\circ$ and $\angle BCA = 70^\circ$. Compute $\angle CAB$ in degrees.

   **Solution.** The answer is $\boxed{50^\circ}$.

   The sum of the three angles in a triangle is $180^\circ$, and thus $\angle CAB = 180^\circ - 60^\circ - 70^\circ = 50^\circ$.

3. Given that $x : y = 2012 : 2$ and $y : z = 1 : 2013$, compute $x : z$. Express your answer as a common fraction.

   **Solution.** The answer is $\boxed{\dfrac{1006}{2013}}$.

   We have
   $$\frac{x}{z} = \frac{x}{y} \cdot \frac{y}{z} = \frac{2012}{2} \cdot \frac{1}{2013} = \frac{1006}{2013}.$$

4. Determine the smallest perfect square greater than 2400.

   **Solution.** The answer is $\boxed{2401}$.

   Note that $2401 = 49^2$.

5. At 12:34 and 12:43, the time contains four consecutive digits. Find the next time after 12:43 that the time contains four consecutive digits on a 24-hour digital clock.

   **Solution.** The answer is $\boxed{13:02 \text{ or } 1:02}$.

   12:03, 12:30, 12:34, and 12:43 are the only times between 12:00 and 12:59, inclusive, that satisfy the condition, and 13:02 is the first time after 12:59 that satisfies the condition.

6. Given that $\sqrt{3^a \cdot 9^a \cdot 3^a} = 81^2$, compute $a$.

   **Solution.** The answer is $\boxed{4}$.

   We have $9^4 = 81^2 = \sqrt{3^a \cdot 9^a \cdot 3^a} = \sqrt{81^a} = 9^a$, which implies that $a = 4$.

7. Find the number of positive integers less than 8888 that have a tens digit of 4 and a units digit of 2.

   **Solution.** The answer is $\boxed{89}$.

   We want to find the number of positive integers less than 8888 that end in 42. The minimum number that satisfies this conditions is 42, which we can write as 0042, and the maximum is 8842. The two digits in front of the 42 can range from 00 to 88, so there are a total of $88 + 1 = 89$ numbers.

8. Find the sum of the distinct prime divisors of $1 + 2012 + 2013 + 2011 \cdot 2013$.

**Solution.** The answer is $\boxed{75}$.

Grouping terms together appropriately, we have

$$1 + 2012 + 2013 + 2011 \cdot 2013 = 2013 + 2013 + 2011 \cdot 2013 = 2013 \cdot 2013 = 2013^2.$$

Because $2013 = 3 \cdot 11 \cdot 61$, the prime divisors of $2013^2$ are 3, 11, and 61, which sum to 75.

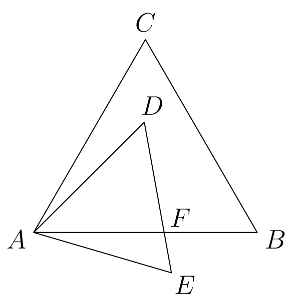
9. Albert wants to make $2 \times 3$ wallet sized prints for his grandmother. Find the maximum possible number of prints Albert can make using one $4 \times 7$ sheet of paper.

**Solution.** The answer is $\boxed{4}$.

The area of the sheet of paper is $4 \cdot 7 = 28$, while the area of each wallet sized print is $2 \cdot 3 = 6$. We can cut out at most $\frac{28}{6} = 4\frac{2}{3}$ prints, which implies that we can make at most 4 prints. We can place four $2 \times 3$ prints in a $4 \times 6$ sheet of paper, which in turn fits inside the $4 \times 7$ sheet of paper.

10. Let $ABC$ be an equilateral triangle, and let $D$ be a point inside $ABC$. Let $E$ be a point such that $ADE$ is an equilateral triangle and suppose that segments $DE$ and $AB$ intersect at point $F$. Given that $\angle CAD = 15°$, compute $\angle DFB$ in degrees.

**Solution.** The answer is $\boxed{105°}$.




Because $\angle CAD = 15°$, we have $\angle DAB = 45°$. Considering triangle $ADF$, the fact that the angles of a triangle sum to $180°$ implies that $\angle DFA = 180° - 45° - 60° = 75°$, which implies that $\angle DFB = 105°$.

11. A *palindrome* is a number that reads the same forwards and backwards; for example, 1221 is a palindrome. An *almost-palindrome* is a number that is not a palindrome but whose first and last digits are equal; for example, 1231 and 1311 are an almost-palindromes, but 1221 is not. Compute the number of 4-digit almost-palindromes.

**Solution.** The answer is $\boxed{810}$.

Because the first digit cannot be 0, there are 9 choices for the first digit, which is the same as the last digit. The second digit can be any number from 0 to 9, so there are 10 choices for the second digit. Because the middle digits must be different, there are only 9 choices for the third digit. Therefore, there are a total of $9 \times 10 \times 9 = 810$ almost-palindromes.

12. Determine the smallest positive integer $n$ such that the sum of the digits of $11^n$ is not $2^n$.

**Solution.** The answer is $\boxed{5}$.

For $n = 1, 2, 3, 4$, we have that $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, $11^4 = 14641$, and in all these cases, the sum of the digits of $11^n$ is $2^n$. For $n = 5$, we have $11^5 = 161051$, whose digits sum to $14 \neq 2^4$.

**Remark:** In fact, for $n \geq 5$, the sum of the digits of $11^n$ is strictly less than $2^n$. Indeed, by the Binomial Theorem, we have

$$2^n = (1+1)^n = \binom{n}{n} + \binom{n}{n-1} + \binom{n}{n-2} + \cdots + \binom{n}{0},$$

and by the Binomial Theorem again we have

$$11^n = (10+1)^n = \binom{n}{n}10^n + \binom{n}{n-1}10^{n-1} + \binom{n}{n-2}10^{n-2} + \cdots + \binom{n}{0}10^0.$$

The above expression for $11^n$ is the sum of $2^n$ powers of 10. When we add the powers of 10 and write the result in base 10, some powers of 10 may be grouped together and therefore contribute only 1 to the sum of the digits of $2^n$. If this occurs, the sum of the digits of $11^n$ will be strictly less than $2^n$. Because $\binom{n}{2} \geq 10$ for $n \geq 5$ and $\binom{n}{k} < 10$ for all $0 \leq k \leq n \leq 4$, this grouping occurs for $n \geq 5$.

13. Determine the minimum number of breaks needed to divide an $8 \times 4$ bar of chocolate into $1 \times 1$ pieces. (When a bar is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

    **Solution.** The answer is $\boxed{5}$.

    Each break multiplies the number of pieces by a factor of at most 2. The final configuration is composed of $32 = 2^5$ pieces, which implies that at least 5 breaks are required. To complete the process in five breaks, we can break the $8 \times 4$ block into two $4 \times 4$ blocks of chocolate, and then stack the two pieces. Repeating this process gives thirty-two $1 \times 1$ pieces after 5 breaks.

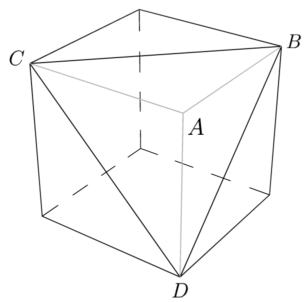
14. A particle starts moving on the number line at a time $t = 0$. Its position on the number line, as a function of time, is $x = (t-2012)^2 - 2012(t-2012) - 2013$. Find the number of positive integer values of $t$ at which time the particle lies in the negative half of the number line (strictly to the left of 0).

    **Solution.** The answer is $\boxed{2013}$.

    Let $s = t - 2012$. Substituting $s$ into the original equation, we have $x = s^2 - 2012s - 2013 = (s+1)(s-2013)$. The value of $x$ is negative for $-1 < s < 2013$, and thus $s$ can range from 0 to 2012, inclusive. Therefore, thetr are $4024 - 2012 + 1 = 2013$ values of $s$ which correspond to 2013 values of $t$.

15. Let $A$ be a vertex of a unit cube and let $B, C$, and $D$ be the vertices adjacent to $A$. The tetrahedron $ABCD$ is cut off the cube. Determine the surface area of the remaining solid.

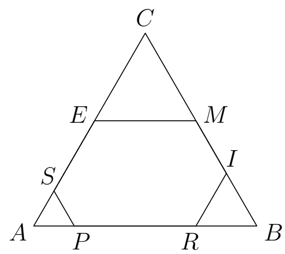
    **Solution.** The answer is $\boxed{\dfrac{9+\sqrt{3}}{2}}$.




The surface area of a unit cube is $6\cdot(1\cdot 1)=6$. When the tetrahedron $ABCD$ is cut away, the triangles $ABC$, $ABD$, and $ACD$ are removed from the surface area, but the equilateral triangle $BCD$ is added to the surface area. The total area of the triangles $ABC$, $ABD$, and $ACD$ is $3\cdot(\frac{1\cdot 1}{2})=\frac{3}{2}$. By the Pythagorean Theorem, we have $BC=CD=DB=\sqrt{2}$, which implies that triangle $BCD$ has area $\frac{(\sqrt{2})^2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}$. Thus, the surface area of the solid is $6-\frac{3}{2}+\frac{\sqrt{3}}{2}=\frac{9+\sqrt{3}}{2}$.

16. In equilateral triangle $ABC$, points $P$ and $R$ lie on segment $AB$, points $I$ and $M$ lie on segment $BC$, and points $E$ and $S$ lie on segment $CA$ such that $PRIMES$ is a equiangular hexagon. Given that $AB=11$, $PS=2$, $RI=3$, and $ME=5$, compute the area of hexagon $PRIMES$.

**Solution.** The answer is $\boxed{\frac{83\sqrt{3}}{4}}$.




Because $PRIMES$ is equiangular, the opposite sides of $PRIMES$ must be pairwise parallel; namely, $SP$ and $IM$ are parallel, $PR$ and $ME$ are parallel, and $RI$ and $ES$ are parallel. The first statement implies that $SP$ is parallel to $BC$, which implies that triangles $ABC$ and $APS$ are similar. Analogously, the triangle $ABC$ is also similar to the triangles $RBI$ and $EMC$, which implies that all three triangles are equilateral. Given that $PS=2$, we can compute the area of triangle $APS$: It is $\frac{PS^2\sqrt{3}}{4}=\sqrt{3}$. Similarly, the area of triangle $RBI$ is $\frac{RI^2\sqrt{3}}{4}=\frac{9\sqrt{3}}{4}$, and the area of triangle $EMC$ is $\frac{ME^2\sqrt{3}}{4}=\frac{25\sqrt{3}}{4}$. We can find the area of the hexagon $PRIMES$ by subtracting the area of the three triangles $APS$, $BIR$, $CME$ from the area of $ABC$. Therefore, $PRIMES$ has area $\frac{121\sqrt{3}}{4}-\sqrt{3}-\frac{9\sqrt{3}}{4}-\frac{25\sqrt{3}}{4}=\frac{83\sqrt{3}}{4}$.

17. Find the smallest odd positive integer with an odd number of positive integer factors, an odd number of distinct prime factors, and an odd number of perfect square factors.

**Solution.** The answer is $\boxed{81}$.

If a positive integer $n$ has an odd number of positive integer factors, it is a perfect square. Indeed, if we pair a divisor $a$ of $n$ with $\frac{n}{a}$, the only factor that could be paired with itself is $\sqrt{n}$ (the converse also holds). Therefore, a positive integer $n$ satisfying the problem's conditions must be a square. Furthermore, a square $b^2$ divides $n$ if and only if $\sqrt{b}$ divides $\sqrt{n}$, and thus $\sqrt{n}$ must have an odd number of factors, which implies that $\sqrt{n}$ is itself a perfect square. It follows that $n$ must be a perfect fourth power. The conditions that $n$ is odd and $n$ has an odd number of prime factors yield that $3^4 = 81$, which has one prime factor, is the smallest possible value of $n$.

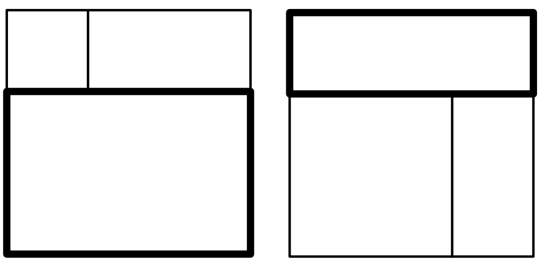
18. Fresh Mann thinks that the expressions $2\sqrt{x^2-4}$ and $2(\sqrt{x^2}-\sqrt{4})$ are equivalent to each other, but the two expressions are not equal to each other for most real numbers $x$. Find all real numbers $x$ such that $2\sqrt{x^2-4} = 2(\sqrt{x^2}-\sqrt{4})$.

    **Solution.** The answer is $\boxed{-2 \text{ and } 2}$.

    Notice that if the condition holds for a value $x$, it also holds for $-x$ because $x^2 = (-x)^2$. Thus, we may restrict our consideration to nonnegative values of $x$, where the condition reduces to $2\sqrt{x^2-4} = 2x-4$. Dividing both sides by 2 and then squaring, we get: $x^2-4 = (x-2)^2 = x^2-4x+4$, which is equivalent to $4x = 8$, or $x = 2$. Substituting into the original equation, $x = 2$ indeed satisfies the condition. Thus, we have shown that the only nonnegative value of $x$ satisfying the condition is 2, and thus, the only values of $x$ over the real numbers are $-2$ and 2.

19. Let $m$ be the positive integer such that a $3 \times 3$ chessboard can be tiled by at most $m$ pairwise incongruent rectangles with integer side lengths. If rotations and reflections of tilings are considered distinct, suppose that there are $n$ ways to tile the chessboard with $m$ pairwise incongruent rectangles with integer side lengths. Find the product $mn$.

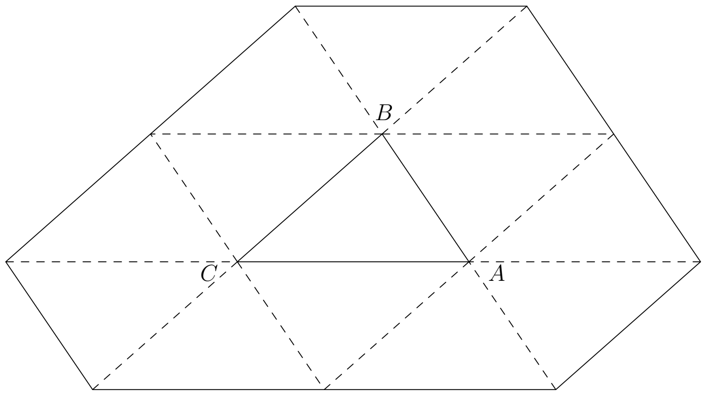
    **Solution.** The answer is $\boxed{48}$.



First of all, we establish that the rectangles must have sides parallel to those of the chessboard because they must combine to tile the entire chessboard. Next, the vertices of the rectangle must be intersections of lines of the chessboard, because the rectangles have integer side lengths. Finally, we note that we cannot have four rectangles because for each value of $n$ between 1 and 3, inclusive, there is only one rectangle up to congruence with an area $n$, which implies that four potential pairwise incongruent rectangles have an area of at least $1+2+3+4 = 10$, larger than the area of the chessboard. With this information in mind, we attempt to construct all possible tilings with 3 rectangles. First note that if three vertices of the chessboard are vertices of the same rectangle, then that rectangle must be the entire chessboard. Thus, when tilling the chessboard with 3 rectangles, there must be one rectangle for which two vertices are also vertices of the chessboard. This rectangle can be either a $1 \times 3$ or a $2 \times 3$ rectangle, and can share any of the four sides of the chessboard. If the rectangle is a $1 \times 3$, then we are left to split the remaining $2 \times 3$ into 2 rectangles, for which there are only two ways. Similarly, if the rectangle is a $2 \times 3$, then we are left to split the remaining $1 \times 3$ into 2 rectangles, for which there are also only two ways. Thus, $m = 3$, and $n = 4 \cdot (2+2) = 16$, implying that $mn = 48$.

20. Let $ABC$ be a triangle with $AB = 4$, $BC = 5$, and $CA = 6$. A triangle $XYZ$ is said to be *friendly* if it intersects triangle $ABC$ and it is a translation of triangle $ABC$. Let $S$ be the set of points in the plane that are inside some friendly triangle. Compute the ratio of the area of $S$ to the area of triangle $ABC$.

**Solution.** The answer is $\boxed{13}$.




See the figure above for a graph of the region $S$. The region can be tiled by 13 copies of triangle $ABC$.

## 2.2 Individual Accuracy Test Solutions

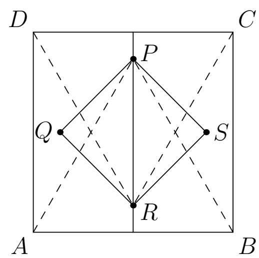
1. Find the largest possible number of consecutive 9's in which an integer between 10,000,000 and 13,371,337 can end. For example, 199 ends in two 9's, while 92,999 ends in three 9's.

   **Solution.** The answer is $\boxed{6}$.

   The two smallest positive integers that end in at least seven consecutive nines are 9,999,999 and 19,999,999. Because $9,999,999 < 10,000,000 < 13,371,337 < 19,999,999$, an integer between 10,000,000 and 13,371,337 cannot end in seven or more consecutive 9's. There are three integers in these bounds that end in six consecutive 9's: 10,999,999; 11,999,999; and 12,999,999.

2. Let $ABCD$ be a square of side length 2. Equilateral triangles $ABP$, $BCQ$, $CDR$, and $DAS$ are constructed inside the square. Compute the area of quadrilateral $PQRS$.

   **Solution.** The answer is $\boxed{8-4\sqrt{3}}$.




   Because triangle $ABP$ is equilateral, the distance from point $P$ to line $AB$ is $\frac{\sqrt{3}}{2}\cdot AB = \sqrt{3}$. Similarly, the distance from point $R$ to line $CD$ is $\sqrt{3}$, and the distance between lines $AB$ and $CD$ is 2, which implies that $PR = \sqrt{3}+\sqrt{3}-2 = 2\sqrt{3}-2$. Similarly, we have $QS = 2\sqrt{3}-2$. By symmetry, quadrilateral $PQRS$ is a rhombus. Therefore, the area of rhombus $PQRS$ is $\frac{PR\cdot QS}{2} = \frac{(2\sqrt{3}-2)\cdot(2\sqrt{3}-2)}{2} = \frac{16-8\sqrt{3}}{2} = 8-4\sqrt{3}$.

3. Evaluate the expression $7\cdot 11\cdot 13\cdot 1003 - 3\cdot 17\cdot 59\cdot 331$.

   **Solution.** The answer is $\boxed{8024}$.

   Notice that $1003 = 17\cdot 59$. The expression can be factored as

$$1003(7\cdot 11\cdot 13 - 3\cdot 331) = 1003\cdot 8 = 8024.$$

4. Compute the number of positive integers $c$ such that there is a non-degenerate obtuse triangle with side lengths $21, 29$, and $c$.

   **Solution.** The answer is $\boxed{25}$.

   Consider a triangle with sides $x \le y \le z$. By the triangle inequality, $x+y > z$. The triangle is obtuse if and only if $x^2+y^2 < z^2$. We do casework to determine all possible values of $c$.

   - *Case 1:* $c \le 29$. The following two inequalities must be satisfied:

$$21 + c > 29$$
$$21^2 + c^2 < 29^2.$$

     From the first we find that $c > 8$. The second yields $c < 20$. There are 11 possible values of $c$ between 9 and 19 inclusive.

- *Case 2:* $c > 29$. The following two inequalities must be satisfied:

$$21 + 29 > c$$
$$21^2 + 29^2 < c^2.$$

From the first we find that $c < 50$. The second yields $c > \sqrt{1282} > 35$. There are 14 possible values of $c$ between 36 and 49 inclusive.

Combining the two cases, there are $11 + 14 = 25$ possible integers $c$.

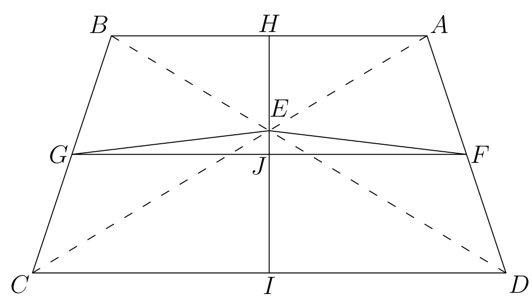
5. Consider a $5 \times 5$ board, colored like a chessboard, such that the four corners are black. Determine the number of ways to place 5 rooks on black squares such that no two of the rooks attack one another, given that the rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

**Solution.** The answer is $\boxed{12}$.

We start by placing the first rook in the leftmost column of the board. There are three choices of a black square. There are two choices of a placement for the in the second column. Neither of these black squares is in a row that could contain the first rook, so they are both valid possibilities. The third column has three black squares. However, there are only two options for the placement of the rook in the third column, since the first rook already occupies the row of one of the three black squares. Similarly, there is $2 - 1 = 1$ option for the rook in the fourth column, as the second rook is already in the row of one of the two black squares in the fourth column. With four rows and four columns already occupied, the fifth rook's position is necessarily on a black square and is determined. Thus, there are $3 \cdot 2 \cdot 2 \cdot 1 \cdot 1 = 12$ ways to place the rooks.

6. Let $ABCD$ be a trapezoid of height 6 with bases $AB$ and $CD$. Suppose that $AB = 2$ and $CD = 3$, and let $F$ and $G$ be the midpoints of segments $AD$ and $BC$, respectively. If diagonals $AC$ and $BD$ intersect at point $E$, compute the area of triangle $FGE$.

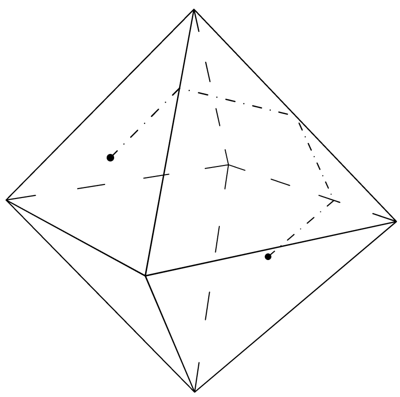
**Solution.** The answer is $\boxed{\dfrac{3}{4}}$.

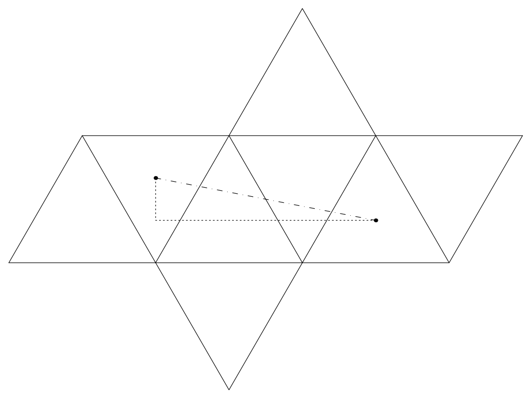



The diagram is not drawn to scale. Because $FG$ is a midline of the trapezoid, it is parallel to the bases $AB$ and $CD$. Let the line perpendicular to the bases passing through $E$ intersect line $AB$ at $H$, line $FG$ at $J$, and line $DC$ at $I$. Because $HI$ is a height of the trapezoid, we have $HI = 6$. Because $FG$ is a midline of the trapezoid, $HJ = JI = 3$. From the similarity $ABE \sim CDE$, it follows that $\frac{EH}{EI} = \frac{AB}{CD} = \frac{2}{3}$. Thus we have $3 \cdot EH = 2 \cdot EI$, which implies that $6 = EH + EI = \frac{5}{3}EI$, or $EI = \frac{18}{5}$, and therefore we have $EJ = EI - JI = \frac{18}{5} - 3 = \frac{3}{5}$. It follows that the area of triangle $FGE$ is $\frac{EJ \cdot FG}{2} = \frac{\frac{3}{5} \cdot \frac{5}{2}}{2} = \frac{3}{4}$.

7. A *regular octahedron* is a solid with eight faces that are congruent equilateral triangles. Suppose that an ant is at the center of one face of a regular octahedron of edge length 10. The ant wants to walk along the surface of the octahedron to reach the center of the opposite face. (Two faces of an octahedron are said to be opposite if they do not share a vertex.) Determine the minimum possible distance that the ant must walk.

**Solution.** The answer is $\boxed{\dfrac{10\sqrt{21}}{3}}$.



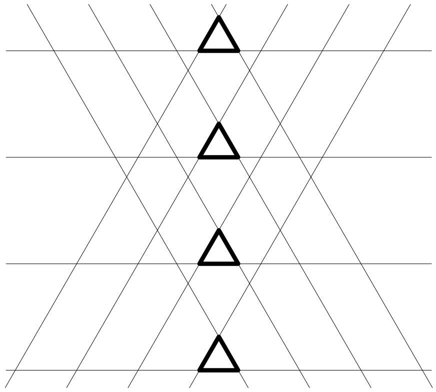


First, we cut the octahedron out into a net, as shown in the right figure above. Considering two opposite faces, we see that they are the outer triangles of a parallelogram made out of four equilateral triangles. The starting and ending points of the ant are the centers of the triangles, and thus we can form a right triangle whose hypotenuse is the line connecting the two points. The base has length $10 + \frac{10}{2} = 10 + 5 = 15$, and the height has length $5\sqrt{3} - 2 \cdot \frac{5}{\sqrt{3}} = \frac{5}{\sqrt{3}}$. Therefore, the hypotenuse has length $\sqrt{15^2 + \frac{5}{\sqrt{3}}^2} = \sqrt{225 + \frac{25}{3}} = \sqrt{\frac{700}{3}} = \sqrt{\frac{2100}{9}} = \frac{10\sqrt{21}}{3}$.

8. Let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, and $D_1D_2D_3$ be triangles in the plane. All the sides of the four triangles are extended into lines. Determine the maximum number of pairs of these lines that can meet at $60^\circ$ angles.

**Solution.** The answer is $\boxed{48}$.

There are two possible slopes that will intersect any given line at a $60^\circ$ angle, either at $60^\circ$ or $120^\circ$ counterclockwise from the line. Therefore, given any line $\ell$ in the place and any triangle $PQR$, at most two of the extensions of the sides of triangle $PQR$ can intersect $\ell$ at a $60^\circ$ angle. Letting $\ell$ range over all extensions of the sides of the four triangles and letting $PQR$ range over the four triangles yields at most $2 \cdot 12 \cdot 4$ intersections at $60^\circ$ angles. However, every intersection will be counted twice; for example, if $A_1A_2$ and $B_1B_2$ intersect at a $60^\circ$ angle, then this will be counted for both $\ell = A_1A_2$ and $\ell = B_1B_2$. Therefore, there are at most $12 \cdot 4 = 48$ such intersections. When the four triangles are congruent equilateral triangles that are oriented in the same way, the diagram below verifies that 48 intersections at $60^\circ$ angles are possible.



9. For an integer $n$, let $f_n(x)$ denote the function $f_n(x) = \sqrt{x^2 - 2012x + n} + 1006$. Determine all positive integers $a$ such that $f_a(f_{2012}(x)) = x$ for all $x \geq 2012$.

   **Solution.** The answer is $\boxed{2,022,060}$.

   By completing the square, we have

$$f_a(x) = \sqrt{(x - 1006)^2 + n - 1006^2} + 1006.$$

   Simplifying $f_a(f_{2012}(x))$, we have

$$\begin{aligned} f_a(f_{2012}(x)) &= \sqrt{(f_{2012}(x) - 1006)^2 + a - 1006^2} + 1006 \\ &= \sqrt{\left(\sqrt{(x - 1006)^2 + 2012 - 1006^2}\right)^2 + a - 1006^2} + 1006 \\ &= \sqrt{(x - 1006)^2 + a + 2012 - 2 \cdot 1006^2} + 1006. \end{aligned}$$

   In order for $f_a(f_{2012}(x)) = x$, the expression inside the square root must equal $(x - 1006)^2$, which implies that $a + 2012 - 2 \cdot 1006^2 = 0$ or $a = 2 \cdot 1006^2 - 2012 = 2,022,060$.

10. Determine the number of ordered triples of integers $(a, b, c)$ such that $(a + b)(b + c)(c + a) = 1800$.

    **Solution.** The answer is $\boxed{576}$.

    Let $a + b = x, b + c = y$, and $a + c = z$. Consider the system of equations

$$\begin{cases} a + b &= x \\ b + c &= y \\ a + c &= z. \end{cases}$$

    By adding the three equations and dividing by 2, we find that

$$a + b + c = \frac{x + y + z}{2}.$$

    Combined with the first equation, we have $c = \frac{y+z-x}{2}$, and similarly $a = \frac{x+z-y}{2}$ and $b = \frac{x+y-z}{2}$. Therefore, for each possible ordered triple $(x, y, z)$, there exists a unique ordered triple $(a, b, c)$, and $a, b,$and $c$ are integers when $x + y + z$ is even. Thus, $x, y$, and $z$ must all be even, or exactly two of them must be odd.

- *Case 1:* $x, y$, and $z$ are all even. Because $1800 = 2^3 \cdot 3^2 \cdot 5^2$, and each of $x, y$, and $z$ must have a factor of 2, the only factors that remain to be divided among $x, y$, and $z$ are those of $3^2 \cdot 5^2$. The possible unordered triples, dividing $x, y$,and $z$ by 2 and ignoring possible negative signs, are (225, 1, 1), (75, 3, 1), (45, 5, 1), (25, 9, 1), (25, 3, 3), (15, 3, 5), (15, 15, 1), and (9, 5, 5). In each unordered triple of three distinct numbers, there are $(6)(4) = 24$ ordered triples, since there are 6 ways to assign the numbers to $x, y$, and $z$ and $\binom{3}{2} + 1 = 4$ ways to distribute either two or zero negative signs among $x, y$, and$z$. In each unordered triple of two distinct numbers, there are $(3)(4) = 12$ ordered triples, since there are 3 ways to assign the numbers to $x, y, z$ and $\binom{3}{2} + 1 = 4$ ways to distribute either two or zero negative signs among $x, y$, and$z$. There are four unordered triples of three distinct numbers and four unordered triples of two distinct numbers for a total of $(4)(24) + (4)(12) = 144$ ordered triples in this case.
- *Case 2:* Exactly two of $x, y$, and $z$ are odd. Then one of $x, y$, and $z$ must contain all three factors of 2. Once again, the only factors that remain to be divided among $x, y$, and$z$ are those of $3^2 \cdot 5^2$. The possible unordered triples, disregarding the factor of 8 in one of $x, y$, and$z$, are the same as in *Case 1*, as are the number of ways to assign the numbers to $x, y$, and$z$. There are 3 choices for which of $x, y$, and$z$ contains the factor of 8, so there are $3 \cdot 144 = 432$ ordered triples in this case.

Thus, there are a total of $144 + 432 = 576$ ordered triples of integers $(a, b, c)$.

## 2.3 Team Test Solutions

1. Determine the number of ways to place 4 rooks on a $4 \times 4$ chessboard such that:

   (a) no two rooks attack one another, and

   (b) the main diagonal (the set of squares marked $X$ below) does not contain any rooks.

| | | | |
|---|---|---|---|
| $X$ | | | |
| | $X$ | | |
| | | $X$ | |
| | | | $X$ |

   The rooks are indistinguishable and the board cannot be rotated. (Two rooks attack each other if they are in the same row or column.)

   **Solution.** The answer is $\boxed{9}$.

   To form a valid board, the rook in the leftmost column can occupy one of 3 places. The rook in the column that has an X in the row of the leftmost rook also has 3 possible places. From there, the placements of the remaining two rooks are determined. The number of boards is thus $3 \cdot 3 = 9$.

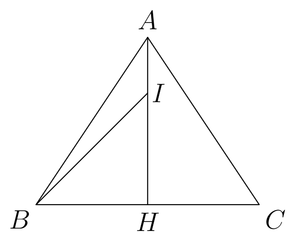
2. Seven students, numbered 1 to 7 in counter-clockwise order, are seated in a circle. Fresh Mann has 100 erasers, and he wants to distribute them to the students, albeit unfairly. Starting with person 1 and proceeding counter-clockwise, Fresh Mann gives $i$ erasers to student $i$; for example, he gives 1 eraser to student 1, then 2 erasers to student 2, et cetera. He continues around the circle until he does not have enough erasers to give to the next person. At this point, determine the number of erasers that Fresh Mann has.

   **Solution.** The answer is $\boxed{1}$.

   Fresh Mann distributes $1 + 2 + 3 + \cdots + 7 = \frac{7 \cdot 8}{2} = 28$ erasers each time he walks around the circle. Therefore, after walking around the circle three times, he will have $100 - 3 \cdot 28 = 16$ erasers left. Then, he distributes $1 + 2 + 3 + 4 + 5 = \frac{5 \cdot 6}{2} = 15$ erasers to the first five students. At this point, he has 1 eraser left, but he needs 6 erasers to continue. Thus, he stops with 1 eraser left.

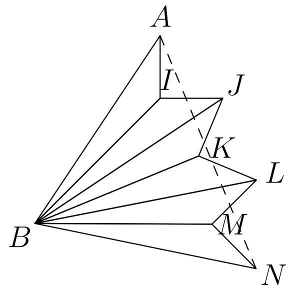
3. Let $ABC$ be a triangle with $AB = AC = 17$ and $BC = 24$. Approximate $\angle ABC$ to the nearest multiple of 10 degrees.

   **Solution.** The answer is $\boxed{50^\circ}$.




   The diagram is not drawn to scale. Let $H$ be the foot of the altitude from point $A$ to segment $BC$. Because $\triangle ABC$ is isosceles, $BH = CH = 12$ holds. The Pythagorean theorem then implies that $AH = \sqrt{AB^2 - BH^2} = \sqrt{145}$. Note that

Now construct a point $I$ such that $I$ is on segment $AH$, and $HI = 12$. It is clear that $\triangle HIB$ is a $45° - 45° - 90°$ right triangle, and thus $\angle IBH = 45°$. We have shown that $\angle ABC > 45°$, and it is intuitively clear that $\angle ABC < 55°$, which implies that the answer is $50°$. The remainder of the solution is devoted to a careful proof of the fact that $\angle ABI < 10°$, which implies that $\angle ABC < 55°$.




Let $J$ be the reflection of $A$ across line $BI$, let $K$ be the reflection of $I$ across line $BJ$, let $L$ be the reflection of $J$ across line $BK$, let $M$ be the reflection of $K$ across line $BL$, and finally, let $N$ be the reflection of $L$ across line $BM$. Due to the reflections, we have the following congruences:

$$\triangle ABI \cong \triangle JBI \cong \triangle JBK \cong \triangle LBK \cong \triangle LBM \cong \triangle NBM.$$

Hence, we have $\angle ABN = 6 \cdot \angle ABI$.

Furthermore, because

$$AN < AI + IJ + JK + KL + LM + MN = 6\sqrt{145} - 72,$$

we have that $AB > AN$. One may easily verify that $17 > 6\sqrt{145} - 72$. (Because $89^2 = 7921 > 5220 = 36 \cdot 145$, we have that $89 > 6\sqrt{145}$.) Since triangle $ABN$ is isosceles, we have $\angle ABN < 60°$, and thus $\angle ABI < 10°$, as claimed.

**Remark:** The ratio $\frac{17}{12}$ is extremely close to $\sqrt{2}$. In fact, one can verify that among all rational numbers of denominator at most 12, $\frac{17}{12}$ is closest to $\sqrt{2}$.

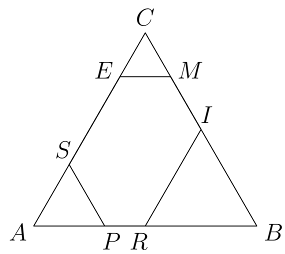
4. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = \frac{3}{5}$ and for $n \geq 1$, $x_{n+1} = 2 - \dfrac{1}{x_n}$. Compute the product $x_1x_2x_3 \cdots x_{2013}$.

   **Solution.** The answer is $\boxed{-\dfrac{4021}{5}}$.

   Consider the new sequence $a_n = x_1x_2 \cdots x_n$. Then, let $a_0 = 1$. We have that $a_1 = \frac{3}{5}$. Multiplying the recursive relation for $\{x_n\}$ by $x_1x_2 \cdots x_n$, we have $a_{n+1} = 2a_n - a_{n-1}$, or $a_{n+1} - a_n = a_n - a_{n-1}$. Thus, $\{a_n\}$ is an arithmetic sequence with initial term $a_0 = 1$ and common difference $\frac{3}{5} - 1 = -\frac{2}{5}$, which implies that $x_1x_2x_3 \cdots x_{2013} = a_{2013} = 1 - 2013 \cdot \frac{2}{5} = -\frac{4021}{5}$.

5. In equilateral triangle $ABC$, points $P$ and $R$ lie on segment $AB$, points $I$ and $M$ lie on segment $BC$, and points $E$ and $S$ lie on segment $CA$ such that $PRIMES$ is a equiangular hexagon. Given that $AB = 11$, $PR = 2$, $IM = 3$, and $ES = 5$, compute the area of hexagon $PRIMES$.

**Solution.** The answer is $\boxed{\dfrac{289\sqrt{3}}{16}}$.




**Note:** *For this solution and all succeeding solutions, the area of a polygon $\mathcal{P}$ will be denoted by $[\mathcal{P}]$.*

Because hexagon $PRIMES$ is equiangular, all of its internal angles are $120°$, which implies that all the internal angles of triangles $APS$, $BIR$, $CEM$ are $60°$. It follows that triangles $APS$, $BIR$, and $CEM$ are equilateral. Let $AP = AS = x, BI = BR = y$, and $CE = CM = z$. Then, we have $11 = AB = AP + PR + RB = x + y + 2$, which implies that $x + y = 9$. Similarly, we have $y + z = 8$ and $z + x = 6$. Adding the three equations together and dividing by 2 yields that $x + y + z = \frac{23}{2}$. It follows that $x = (x + y + z) - (y + z) = \frac{23}{2} - 8 = \frac{7}{2}$, and similarly we have $y = \frac{23}{2} - 6 = \frac{11}{2}$ and $z = \frac{23}{2} - 9 = \frac{5}{2}$. Therefore, the area of hexagon $PRIMES$ is

$$\begin{aligned}[ABC] - ([APS] + [BIR] + [CEM]) &= \frac{\sqrt{3}}{4}\left(AB^2 - (x^2 + y^2 + z^2)\right) = \frac{\sqrt{3}}{4}\left(121 - \frac{195}{4}\right) \\ &= \frac{\sqrt{3}}{4} \cdot \frac{289}{4} = \frac{289\sqrt{3}}{16}.\end{aligned}$$

6. Let $f(a, b) = \frac{a^2}{a+b}$. Let $A$ denote the sum of $f(i, j)$ over all pairs of integers $(i, j)$ with $1 \le i < j \le 10$; that is,

$$A = (f(1, 2) + f(1, 3) + \cdots + f(1, 10)) + (f(2, 3) + f(2, 4) + \cdots + f(2, 10)) + \cdots + f(9, 10).$$

Similarly, let $B$ denote the sum of $f(i, j)$ over all pairs of integers $(i, j)$ with $1 \le j < i \le 10$; that is,

$$B = (f(2, 1) + f(3, 1) + \cdots + f(10, 1)) + (f(3, 2) + f(4, 2) + \cdots + f(10, 2)) + \cdots + f(10, 9).$$

Compute $B - A$.

**Solution.** The answer is $\boxed{165}$.

Define $g(a, b) = f(b, a) - f(a, b) = \frac{b^2}{a+b} - \frac{a^2}{a+b} = b - a$. The value $B - A$ is then the sum of $g(i, j)$ over all pairs of integers $(i, j)$ with $1 \le i < j \le 10$. For $k$ an integer between 1 and 9, inclusive, $k$ appears in the sum as $g(i, j)$ for $10 - k$ pairs $(i, j)$, namely $(1, k + 1), (2, k + 2), \ldots, (10 - k, 10)$. Therefore, we have $B - A = 1 \cdot 9 + 2 \cdot 8 + \cdots + 9 \cdot 1 = 165$.

7. Fresh Mann has a pile of seven rocks with weights 1, 1, 2, 4, 8, 16, and 32 pounds and some integer $X$ between 1 and 64, inclusive. He would like to choose a set of the rocks whose total weight is exactly $X$ pounds. Given that he can do so in more than one way, determine the sum of all possible values of $X$. (The two 1-pound rocks are indistinguishable.)

**Solution.** The answer is $\boxed{992}$.

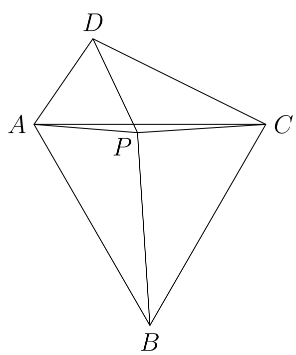
Clearly, 64 can only be achieved in one way: pick all the rocks.

With the seven rocks form a "big pile" consisting of rocks with weights 1, 2, 4, 8, 16, and 32 pounds and a "little pile" consisting of just the other 1-pound rock. The numbers 1 to 63 have a unique 7-digit representation in base two, and thus there is exactly one way to pick from only the big pile to achieve every total weight between 1 and 63. It follows that any other choice of rocks must include the 1-pound rock from the little pile. All odd total weights from 1 to 63 must be achieved using exactly one 1-pound rock, and using the 1-pound rock from the little pile is equivalent to using the 1-pound rock from the big pile, which yields the same set of rocks. Thus, each odd weight can be achieved in only one way.

However, for any even total weight of $Y$ pounds, there is a unique way to achieve $Y - 1$ pounds using a set of rocks solely from the big pile, including the 1-pound rock (since $Y - 1$ is odd). Using these rocks in addition to the 1-pound rock from the little pile, we can form a new set of rocks with total weight $Y$ (which uses both 1-pound rocks). Therefore, the numbers that can be represented in more than one way are the even numbers that are between 2 and 62, inclusive, which sum to $2 + 4 + \cdots + 62 = 2(1 + 2 + \cdots + 31) = 31 \cdot 32 = 992$.

8. Let $ABCD$ be a convex quadrilateral with $AB = BC = CA$. Suppose that point $P$ lies inside the quadrilateral with $AP = PD = DA$ and $\angle PCD = 30°$. Given that $CP = 2$ and $CD = 3$, compute $CA$.

**Solution.** The answer is $\boxed{\sqrt{13}}$.




Draw segment $BP$. Because triangles $ABC$ and $APD$ are equilateral, we have that $\angle PAB = 60° - \angle CAP = \angle DAC$. Therefore, $AB = AC$ and $AP = AD$ imply that triangles $ABP$ and $ACD$ are congruent by SAS. It follows that $BP = 3$, and $\angle APB = \angle ADC$. Because the angles of a convex quadrilateral sum to $360°$, we have

$$\angle BPC = 360° - \angle APB - \angle CPA = 360° - \angle ADC - \angle CPA = \angle DAP + \angle PCD = 90°.$$

By the Pythagorean Theorem, we have $CA = CB = \sqrt{2^2 + 3^2} = \sqrt{13}$.

9. Define a sequence of rational numbers $\{x_n\}$ by $x_1 = 2$, $x_2 = \frac{13}{2}$, and for $n \geq 1$,

$$x_{n+2} = 3 - \frac{3}{x_{n+1}} + \frac{1}{x_n x_{n+1}}.$$

Compute $x_{100}$.

**Solution.** The answer is $\boxed{\dfrac{49601}{48610}}$.

This problem is similar to problem 3 of the Team Round. Once again, let $a_n = x_1x_2x_3\cdots x_n$, and let $a_0 = 1$. We have that $a_1 = 2$ and $a_2 = 13$. Multiplying the recursion by $x_1x_2x_3\cdots x_{n+1}$ and rearranging the terms, we have $a_{n+2} - 3a_{n+1} + 3a_n - a_{n-1} = 0$. We claim that there are real numbers $a, b, c$ such that $a_n = an^2 + bn + c$ for all $n$. Assuming that this holds, the initial values of $a_n$ yield that $(a, b, c) = (5, -4, 1)$, which would imply that

$$x_{2013} = \frac{a_{2013}}{a_{2012}} = \frac{5\cdot 100^2 - 4\cdot 100 + 1}{5\cdot 99^2 - 4\cdot 99 + 1} = \frac{49601}{48610}.$$

Consider the sequence $a'_n = 5n^2 - 4n + 1$, and we will prove that $a_n = a'_n$ for all $n$. This clearly holds for $n = 0, 1,$ and 2. Then, it suffices to prove that $a'_{m+2} - 3a'_{m+1} + 3a'_m - a'_{m-1} = 0$ for all $m \geq 1$, which would imply that for all $n \geq 3$, $a_n = a'_n$. We compute the above expression progressively; we have $a'_m - a'_{m-1} = 5(m^2 - (m-1)^2) - 4 = 10m - 9$. This implies that $a'_{m+1} - 2a'_m + a'_{m-1} = (a'_{m+1} - a'_m) + (a'_m - a'_{m-1}) = 10$, which yields that $a'_{m+2} - 3a'_{m+1} + 3a'_m - a'_{m-1} = (a'_{m+2} - 2a'_{m+1} + a'_m) - (a'_{m+1} - 2a'_m + a'_{m-1}) = 0$, as claimed.

**Remark:** For a sequence $\{a_n\}$ define the sequence $\Delta\{a_n\} = \{a_1 - a_0, a_2 - a_1, \ldots\}$. The operator $\Delta$ is called the *finite difference operator*. We showed that the sequences satisfying $\Delta^3\{a_n\} = \{0, 0, \ldots\}$ are exactly those that satisfy $a_n = f(n)$ for some quadratic function $n$. In fact, $\Delta^{k+1}\{a_n\} = \{0, 0, \ldots\}$ holds if and only if $a_n = f(n)$ for some polynomial $f$ of degree at most $k$.

10. Ten prisoners are standing in a line. A prison guard wants to place a hat on each prisoner. He has two colors of hats, red and blue, and he has 10 hats of each color. Determine the number of ways in which the prison guard can place hats such that among any set of consecutive prisoners, the number of prisoners with red hats and the number of prisoners with blue hats differ by at most 2.

**Solution.** The answer is $\boxed{94}$.

For $0 \leq n \leq 10$, let $f(n)$ be the number of prisoners among the first $n$ prisoners who are wearing red hats minus the number wearing blue hats. For the rest of this problem, assume that $a < b$. Notice that $f(b) - f(a)$ is the number of prisoners wearing red hats minus the number of prisoners wearing blue hats, among the prisoners $a+1, a+2, a+3, \ldots, b$. It follows from the definition of $f$ that $|f(n) - f(n-1)| = 1$ for all $n$. We want to find the number of hat configurations such that $f$ takes on at most three consecutive values: indeed, if $|f(b) - f(a)| > 2$, then among the prisoners $a+1, a+2, a+3, \ldots, b$, the number of prisoners with red hats and the number of prisoners with blue hats differ by more than 2.

We claim that there are 32 choices of $f$ such that $-2 \leq f(n) \leq 0$. Indeed, if $f(n) = 0$ or $f(n) = -2$, then we must have $f(n+1) = -1$, and if $f(n) = -1$, we have two choices for $f(n+1)$. It follows that we make five choices in determining $f(1), f(2), f(3), \ldots, f(10)$, which yields $2^5 = 32$ functions $f$. Similarly, there are 32 functions $f$ with $-1 \leq f(n) \leq 1$ and 32 functions $f$ with $0 \leq f(n) \leq 2$. However, we double-count two functions: the function

$$g(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ -1 & \text{otherwise} \end{cases}$$

is counted both in the case $-2 \leq f(n) \leq 0$ and the case $-1 \leq f(n) \leq 1$, while $-g(n)$ is counted both in the case $-1 \leq f(n) \leq 1$ and the case $0 \leq f(n) \leq 2$, and these are the only functions that are counted more than once. Thus, there are $3 \cdot 32 - 2 = 94$ functions $f$, which yields 94 hat configurations.

# 2.4 Guts Test Solutions

## 2.4.1 Round 1

1. [6pts] Five girls and three boys are sitting in a room. Suppose that four of the children live in California. Determine the maximum possible number of girls that could live somewhere outside California.

**Solution.** The answer is $\boxed{4}$.

Because there are eight students in total, four of whom live in California, exactly four must live outside California. Hence at most four girls live outside California. It is indeed possible for four girls to live outside California when one girl and three boys live in California, and thus the answer is 4.

2. [6pts] A 4-meter long stick is rotated $60^\circ$ about a point on the stick 1 meter away from one of its ends. Compute the positive difference between the distances traveled by the two endpoints of the stick, in meters.

**Solution.** The answer is $\boxed{\dfrac{2\pi}{3}}$.

The endpoint of the stick that is closer to the center of rotation traces out a $60^\circ$ arc of a circle of radius 1 meter, and thus the point travels $\frac{1}{6} \cdot 2 \cdot \pi$ meters. The farther end traces out a $60^\circ$ arc of radius of 3 meters, and therefore the point travels $\frac{1}{6} \cdot 2 \cdot 3 \cdot \pi$ meters. The difference is $\frac{6\pi}{6} - \frac{2\pi}{6} = \frac{2\pi}{3}$.

3. [6pts] Let

$$f(x) = 2x(x-1)^2 + x^3(x-2)^2 + 10(x-1)^3(x-2).$$

Compute $f(0) + f(1) + f(2)$.

**Solution.** The answer is $\boxed{25}$.

Conveniently, when we input the values into the function, exactly two of the three terms always become 0. Therefore, we have $f(0) = (10)(-1)^3(-2) = 20$, $f(1) = (1)^3(-1)^2 = 1$, and $f(2) = (2)(2)(1) = 4$, which implies that $f(0) + f(1) + f(2) = 20 + 1 + 4 = 25$.

## 2.4.2 Round 2

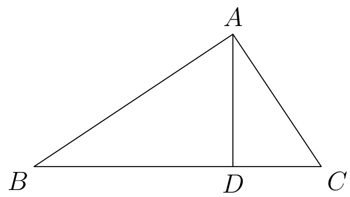
4. [8pts] Twenty boxes with weights $10, 20, 30, \ldots, 200$ pounds are given. One hand is needed to lift a box for every 10 pounds it weighs. For example, a 40 pound box needs four hands to be lifted. Determine the number of people needed to lift all the boxes simultaneously, given that no person can help lift more than one box at a time.

**Solution.** The answer is $\boxed{110}$.

The 10 pound box needs one hand to lift, which requires one person, while the 20 pound box needs two hands, which also requires one person. The 30 and 40 pound boxes need 3 and 4 hands respectively, so each needs 2 people to lift. In general, the $10(2k-1)$ pound and $10(2k)$ pound boxes each require $k$ people to lift. Therefore, the answer is is $1 + 1 + 2 + 2 + 3 + 3 + \cdots + 10 + 10 = 2\left(\frac{10\cdot 11}{2}\right) = 110$.

5. [8pts] Let $ABC$ be a right triangle with a right angle at $A$, and let $D$ be the foot of the perpendicular from vertex $A$ to side $BC$. If $AB = 5$ and $BC = 7$, compute the length of segment $AD$.

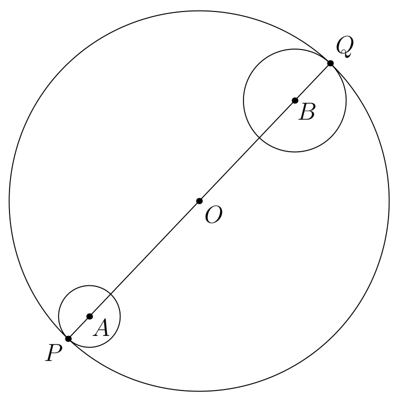
**Solution.** The answer is $\boxed{\dfrac{10\sqrt{6}}{7}}$.




By the Pythagorean Theorem, we have $AC = \sqrt{7^2 - 5^2} = \sqrt{24} = 2\sqrt{6}$. We compute the area of triangle $ABC$ in two ways; it equals $\frac{1}{2}AB \cdot AC = 5\sqrt{6}$, and it equals $\frac{1}{2}AD \cdot BC = \frac{7}{2}AD$. Setting the two expressions equal yields that $AD = \frac{10\sqrt{6}}{7}$.

6. [8pts] There are two circular ant holes in the coordinate plane. One has center $(0, 0)$ and radius 3, and the other has center $(20, 21)$ and radius 5. Albert wants to cover both of them completely with a circular bowl. Determine the minimum possible radius of the circular bowl.

**Solution.** The answer is $\boxed{\dfrac{37}{2}}$.




Let the centers $A = (0, 0)$ and $B = (20, 21)$. Let line $AB$ intersect circle $A$ at $P$ and circle $B$ at $Q$ so that both centers lie between $P$ and $Q$. Note that $PQ = PA + AB + BQ = 3 + \sqrt{20^2 + 21^2} + 5 = 37$, so the radius of the bowl must be at least $\frac{37}{2}$. Conversely, a circle centered at the midpoint of segment $PQ$ of radius $\frac{37}{2}$ covers both holes.

### 2.4.3 Round 3

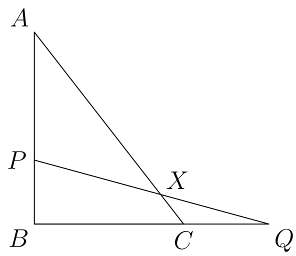
7. [10pts] A line of slope $-4$ forms a right triangle with the positive $x$ and $y$ axes. If the area of the triangle is 2013, find the square of the length of the hypotenuse of the triangle.

**Solution.** The answer is $\boxed{\dfrac{34221}{2}}$.

Let the line intersect the $x$ axis at $(k, 0)$. Because it has slope $-4$, the line must intersect the $y$ axis at $(0, 4k)$. The right triangle then has legs of lengths $k$ and $4k$. Because the area of the triangle is 2013, we have $\frac{k \cdot 4k}{2} = 2013$, or $k^2 = \frac{2013}{2}$. By the Pythagorean Theorem, the square of the length of the hypotenuse is the sum of the squares of its legs, which is $k^2 + (4k)^2 = 17k^2 = \frac{34221}{2}$.

8. [10pts] Let $ABC$ be a right triangle with a right angle at $B$, $AB = 9$, and $BC = 7$. Suppose that point $P$ lies on segment $AB$ with $AP = 3$ and that point $Q$ lies on ray $BC$ with $BQ = 11$. Let segments $AC$ and $PQ$ intersect at point $X$. Compute the positive difference between the areas of triangles $APX$ and $CQX$.

**Solution.** The answer is $\boxed{\frac{3}{2}}$.




Note that the area of triangle $PQB$ is $\frac{1}{2}BP \cdot BQ = 33$ and the area of triangle $ABC$ is $\frac{1}{2}AB \cdot BC = \frac{63}{2}$. It follows that

$$\begin{aligned}[CQX] - [APX] &= ([PQB] - [BPXC]) + ([ABC] - [BPXC]) \\ &= [PQB] - [ABC] \\ &= 33 - \frac{63}{2} = \frac{3}{2}.\end{aligned}$$

9. [10pts] Fresh Mann and Sophy Moore are racing each other in a river. Fresh Mann swims downstream, while Sophy Moore swims $\frac{1}{2}$ mile upstream and then travels downstream in a boat. They start at the same time, and they reach the finish line 1 mile downstream of the starting point simultaneously. If Fresh Mann and Sophy Moore both swim at 1 mile per hour in still water and the boat travels at 10 miles per hour in still water, find the speed of the current.

**Solution.** The answer is $\boxed{\frac{7}{29}}$.

Let the speed of the current be $c$ miles per hour. Then, it takes $\frac{1}{1+c}$ hours for Fresh Mann to swim 1 mile downstream at a rate of $1 + c$ miles per hour, so it must also take Sophy Moore the same amount of time to complete her journey. Her first leg swimming upstream 0.5 miles at $1 - c$ miles per hour takes $\frac{0.5}{1-c}$ hours and her second leg boating downstream 1.5 miles takes $\frac{1.5}{10+c}$, so we must have

$$\frac{1}{1+c} = \frac{0.5}{1-c} + \frac{1.5}{10+c}.$$

We solve this equation to find $c$:

$$\begin{aligned}
\frac{2}{1+c} &= \frac{1}{1-c} + \frac{3}{10+c} \\
2(1-c)(10+c) &= (1+c)(10+c) + 3(1+c)(1-c) \\
20 - 18c - 2c^2 &= (10 + 11c + c^2) + (3 - 3c^2) \\
29c &= 7 \\
c &= \frac{7}{29}.
\end{aligned}$$

### 2.4.4 Round 4

10. [12pts] The *Fibonacci numbers* are defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. The first few terms of the Fibonacci sequence are $0, 1, 1, 2, 3, 5, 8, 13$. Every positive integer can be expressed as the sum of nonconsecutive, distinct, positive Fibonacci numbers; for example, $7 = 5 + 2$. Express 121 as the sum of nonconsecutive, distinct, positive Fibonacci numbers. (It is not permitted to use both a 2 and a 1 in the expression.)

**Solution.** The answer is $\boxed{89 + 21 + 8 + 3}$.

The Fibonacci numbers less than 121 are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, and 89 in order. We consider whether we can include or exclude certain numbers, starting from 89 and working downward.

If we don't use 89, the largest sum we can get is $55 + 21 + 8 + 3 + 1 = 88 < 121$, and thus we must use 89. The remaining Fibonacci numbers must sum to 32. We cannot include 34 or 55 because they are both greater than 32. If we don't use 21, then the total sum is at most $89 + 13 + 5 + 2 + 1 = 110 < 121$, and thus we have to use 21. The remaining of the summands must sum $32 - 21 = 11$.

We cannot include 13 because it is too large. If we don't use 8, then the total sum is at most $89 + 21 + 5 + 3 + 1 = 119 < 121$, and thus use 8. This leaves $11 - 8 = 3$ for the numbers 1, 1, 2, and 3. Since 2 and 1 are consecutive, we must use the 3. This yields the unique expression $121 = 89 + 21 + 8 + 3$.

**Remark:** Zeckendorff's Theorem states that every positive integer can be expressed as a sum of nonconsecutive, distinct, positive Fibonacci numbers, and that such an expression is unique. The solution presented above follows the theorem's classical proof.

11. [12pts] There is a rectangular box of surface area 44 whose space diagonals have length 10. Find the sum of the lengths of all the edges of the box.

**Solution.** The answer is $\boxed{48}$.

Let the rectangular box have edge lengths $a$, $b$, and $c$. Then, the surface area is

$$2ab + 2bc + 2ca = 44$$

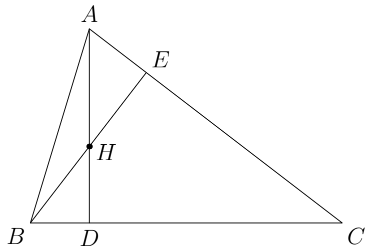
and the space diagonal is $\sqrt{a^2 + b^2 + c^2} = 10$, so

$$a^2 + b^2 + c^2 = 100.$$

Adding the two equations yields $a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = (a + b + c)^2 = 144$, which implies that $a + b + c = \sqrt{144} = 12$. The sum of the lengths of the edges is $4(a + b + c) = 48$.

12. [12pts] Let $ABC$ be an acute triangle, and let $D$ and $E$ be the feet of the altitudes to $BC$ and $CA$, respectively. Suppose that segments $AD$ and $BE$ intersect at point $H$ with $AH = 20$ and $HD = 13$. Compute $BD \cdot CD$.

    **Solution.** The answer is $\boxed{429}$.




We have $AD = AH + HD = 33$. Triangles $BDH$ and $ADC$ are similar by AA similarity, from which it follows that $\frac{CD}{AD} = \frac{HD}{BD}$. Cross multiplying, we have

$$BD \cdot CD = AD \cdot HD = 33 \cdot 13 = 429.$$

### 2.4.5 Round 5

13. [14pts] In coordinate space, a *lattice point* is a point all of whose coordinates are integers. The lattice points $(x, y, z)$ in three-dimensional space satisfying $0 \le x, y, z \le 5$ are colored in $n$ colors such that any two points that are $\sqrt{3}$ units apart have different colors. Determine the minimum possible value of $n$.

    **Solution.** The answer is $\boxed{2}$.

    The points $(0, 0, 0)$ and $(1, 1, 1)$ are $\sqrt{3}$ units apart, and thus they must have different colors, which implies that $n \ge 2$. To construct a coloring for $n = 2$, color the points with $z = 0, 2, 4$ black and color the points with $z = 1, 3, 5$ white. The distance between any two lattice points is of the form $\sqrt{a^2 + b^2 + c^2}$ for some integers $a, b, c$, and $\sqrt{3}$ can only be achieved when $a = b = c = 1$. However, if the $z$-coordinates of two points differ by 1, then the points are different colors in the given coloring.

    Another possible construction is to color a point $(x, y, z)$ black if $x+y+z$ is odd and white if $x+y+z$ is even.

14. [14pts] Determine the number of ways to express 121 as a sum of strictly increasing positive Fibonacci numbers.

    **Solution.** The answer is $\boxed{8}$.

    Let $(f_1, f_2, \dots, f_{10}) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89)$ be the first ten Fibonacci numbers (excluding the first 1 because it is repeated). We have some observations:

    (a) If 89 is one of the summands, neither 34 nor 55 can be chosen (otherwise the sum will be over 121). If 89 is not one of the summands, 34 and 55 are both required to be chosen (otherwise the sum will be less than 121). Because $34 + 55 = 89$, we know that the numbers chosen from $f_1, f_2, \dots, f_7$ must add up to $121 - 89 = 32$.

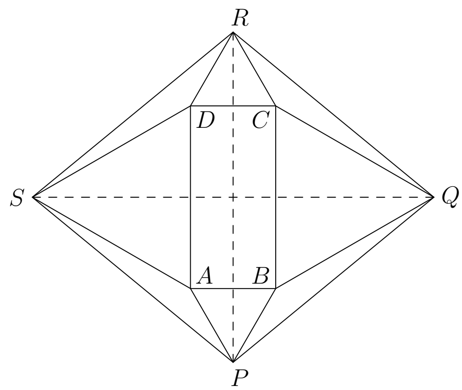
(b) If $f_7 = 21$ is not chosen, then because $f_1 + f_2 + f_3 + f_4 + f_5 + f_6 = 32$, and thus we must use all of $f_1, \ldots, f_6$.

(c) If $f_7 = 21$ is chosen, then $f_6 = 13$ can not be one of the summands because $21 + 13 = 34 > 32$. We can list out the remaining ways to express 32: $(1, 2, 3, 5, 21)$, $(1, 2, 8, 21)$, and $(3, 8, 21)$.

From part $(a)$, we have two ways to choose summands form the first 89 of the sum. For either case, we can form the remaining 32 without $f_7 = 21$ in one way and with $f_7 = 21$ in three ways. This gives a total of $2 \cdot (3 + 1) = 8$ solutions.

15. [14pts] Let $ABCD$ be a rectangle with $AB = 7$ and $BC = 15$. Equilateral triangles $ABP$, $BCQ$, $CDR$, and $DAS$ are constructed outside the rectangle. Compute the area of quadrilateral $PQRS$.

**Solution.** The answer is $\boxed{210 + 137\sqrt{3}}$.




By symmetry, quadrilateral $PQRS$ is a rhombus. The heights of equilateral triangles $ABP$ and $CDR$ are both $h_1 = \frac{7\sqrt{3}}{2}$, and the heights of equilateral triangles $BCQ$ and $DAS$ are both $h_2 = \frac{15\sqrt{3}}{2}$. Therefore, we have $PR = 15 + 2h_1 = 15 + 7\sqrt{3}$ and $QS = 7 + 2h_2 = 7 + 15\sqrt{3}$. The area of $PQRS$ is $\frac{1}{2}PR \cdot QS = \frac{(7+15\sqrt{3})(15+7\sqrt{3})}{2} = 210 + 137\sqrt{3}$.

### 2.4.6 Round 6

Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.

16. [16pts] Let $C$ be the answer to problem 18. Suppose that $x$ and $y$ are real numbers with $y > 0$ and

$$\begin{cases} x + y = C \\ x + \frac{1}{y} = -2. \end{cases}$$

Compute $y + \frac{1}{y}$.

**Solution.** The answer is $\boxed{\dfrac{\sqrt{85}}{3}}$.

Subtracting the two equations, we have $y - \frac{1}{y} = C + 2$. Squaring then yields that $y^2 - 2 + \frac{1}{y^2} = (C+2)^2$, from which it follows that $y^2 + 2 + \frac{1}{y^2} = (C+2)^2 + 4$. Taking square roots of both sides yields that $y + \frac{1}{y} = \sqrt{(C+2)^2 + 4}$ because $y > 0$. Substituting $C = \frac{1}{3}$ from problem 18 yields an answer of $\frac{\sqrt{85}}{3}$.

17. [16pts] Let $A$ be the answer to problem 16. Let $PQR$ be a triangle with $\angle PQR = 90^\circ$, and let $X$ be the foot of the perpendicular from point $Q$ to segment $PR$. Given that $QX = A$, determine the minimum possible area of triangle $PQR$.

    **Solution.** The answer is $\boxed{\frac{85}{9}}$.

    Because $\angle PQX = \angle QRX$ and $\angle QPX = \angle RQX$, triangles $PQX$ and $QRX$ are similar, which implies that $\frac{PX}{XQ} = \frac{XQ}{XR}$. It follows that $PX \cdot XR = XQ^2 = A^2$. Because the square of any real number is nonnegative, we have

$$\begin{aligned} \left(\sqrt{PX} - \sqrt{XR}\right)^2 &\geq 0 \\ PX - 2\sqrt{PX \cdot XR} + XR &\geq 0 \\ PR = PX + XR &\geq 2\sqrt{A^2} \end{aligned}$$

    Therefore, the area of $\triangle PQR$ is $\frac{1}{2}PR \cdot QX \leq \frac{1}{2}2A \cdot A = A^2$. Equality is achieved at $PX = XR$, and therefore the answer is $A^2$. Substituting $A = \frac{1}{3}$ from problem 16 yields an answer of $\frac{85}{9}$.

18. [16pts] Let $B$ be the answer to problem 17 and let $K = 36B$. Alice, Betty, and Charlize are identical triplets, only distinguishable by their hats. Every day, two of them decide to exchange hats. Given that they each have their own hat today, compute the probability that Alice will have her own hat in $K$ days.

    **Solution.** The answer is $\boxed{\frac{1}{3}}$.

    Each day, one of the following three exchanges happens: Alice and Betty switch hats, Betty and Charlize switch hats, or Charlize and Alice switch hats. If Alice has her own hat on day $K - 1$, there is a $\frac{1}{3}$ probability of Alice keeping her own hat on day $K$, namely when Betty and Charlize switch hats that day. If Alice doesn't have her own hat on day $K - 1$, there is also a $\frac{1}{3}$ probability that she gets her hat on day $K$, namely when Alice switches with the girl who had her hat on day $K - 1$. Thus, Alice has a $\frac{1}{3}$ chance of having her own hat on day $K$. This value is independent of $B$, provided that $K > 0$. Indeed, $B$ is the area of a non-degenerate triangle, which must be positive.

### 2.4.7 Round 7

19. [16pts] Find the number of positive integers $a$ such that all roots of $x^2 + ax + 100$ are real and the sum of their squares is at most 2013.

    **Solution.** The answer is $\boxed{28}$.

    Let $x^2 + ax + 100 = (x - m)(x - n)$ where $m$ and $n$ are the real roots of the original quadratic equation. After expanding the right-hand side, we equate the linear and constant terms, which yields $m + n = -a$

and $mn = 100$. Thus, we have $a^2 - 200 = (m+n)^2 - 2mn = m^2 + n^2 \geq 2013$, according to the problem's condition. Therefore, we have $a^2 \geq 2213$. Because $m$ and $n$ are real, the discriminant is nonnegative, which implies that $a^2 - 400 \geq 0$, or $a^2 \geq 400$. As a result, we have $400 \leq a^2 \geq 2213$, or $20 \leq a \leq 47$, which yields 28 possible values for a. Conversely, if $20 \leq a \leq 47$, then the discriminant is positive and $m^2 + n^2 \leq 2013$.

20. [16pts] Determine all values of $k$ such that the system of equations

$$\begin{cases} y = x^2 - kx + 1 \\ x = y^2 - ky + 1 \end{cases}$$

has a real solution.

**Solution.** The answer is $\boxed{k \leq -3 \text{ or } k \geq 1}$.

The graphs of the two equations are parabolas that are reflections of one another over the line $y = x$. If the parabola $y = x^2 - kx + 1$ does not intersect the line $y = x$, then the parabola lies completely above the line, which implies that the graph of $x = y^2 - ky + 1$ lies below the line. Therefore, the two parabolas cannot intersect. Conversely, if the parabola $y = x^2 - kx + 1$ intersects the line $y = x$ at a point, then that point lies on the other parabola as well. It follows that the two parabolas intersect if and only if the quadratic equation $x = x^2 - kx + 1$ has a real solution. This is equivalent to $x^2 - (k+1)x + 1 = 0$ having a real solution, which occurs if and only if the discriminant $\Delta = (k+1)^2 - 4$ is nonnegative. Therefore, the two parabolas intersect if and only if $|k+1| \geq 2$, which occurs when $k \geq 1$ and when $k \leq -3$.

21. [16pts] Determine the minimum number of cuts needed to divide an $11 \times 5 \times 3$ block of chocolate into $1 \times 1 \times 1$ pieces. (When a block is broken into pieces, it is permitted to rotate some of the pieces, stack some of the pieces, and break any set of pieces along a vertical plane simultaneously.)

**Solution.** The answer is $\boxed{9}$.

We make the following more general claim: Given an $a \times b \times c$ block, at least $p + q + r$ cuts are needed to break it into unit cubes, where $p$, $q$, and $r$ are integers such that $2^{p-1} < a \leq 2^p$, $2^{q-1} < b \leq 2^q$, and $2^{r-1} < c \leq 2^r$.

We prove the claim by induction on $a + b + c$.. For the base case of a $1 \times 1 \times 1$ block, we have $a = b = c = 1$, which implies that $p + q + r = 0$; indeed, zero cuts are required to turn this block into a unit cube.

Now, we assume that our claim is true for all blocks with sum of dimensions less than or equal to $n - 1$, and consider an $a \times b \times c$ block with $a + b + c = n$. Note that the first cut will produce two blocks, both having two dimensions the same as the original and the other dimension split. Without loss of generality, suppose that the edge of length $a$ is split. Then, one of the two blocks has dimensions $d \times b \times c$ with $d \geq \frac{a}{2}$. Let $s$ be the integer such that $2^{s-1} < d \leq 2^s$. Because $d \geq \frac{a}{2}$, we have $d > 2^{p-2}$, which implies that $s > p - 2$. By the inductive hypothesis, at least $s + q + r \geq p + q + r - 1$ cuts are required to divide the $d \times b \times c$ block into unit cubes. In addition to the initial cut, this implies at least $p + q + r$ cuts are required to divide the $a \times b \times c$ block into unit cubes, which completes the proof of the inductive step.

Applying the above claim to our problem, we see that it takes at least $4 + 3 + 2 = 9$ cuts to fully divide an $11 \times 5 \times 3$ block. We can also show that this minimum is attainable, by the following process:

(a) Cut the $11 \times 5 \times 3$ into three blocks of $11 \times 5 \times 1$. This uses two cuts.

(b) Stack the $11 \times 5 \times 1$ blocks together, and now cut each of them into two $11 \times 2 \times 1$ blocks and one $11 \times 1 \times 1$ block. This uses two cuts.

(c) Stack the six resulting $11 \times 2 \times 1$ blocks together, and cut them to form twelve $11 \times 1 \times 1$ blocks , for a grand total of fifteen $11 \times 1 \times 1$ blocks. This uses one cut.

(d) Stack all fifteen blocks together, and cut each of them into two $4 \times 1 \times 1$ blocks and one $3 \times 1 \times 1$ block. This uses two cuts.

(e) Using only one cut, we splice the thirty $4 \times 1 \times 1$ blocks into sixty $2 \times 1 \times 1$ blocks, and the fifteen $3 \times 1 \times 1$ blocks into fifteen $2 \times 1 \times 1$ blocks and fifteen $1 \times 1 \times 1$ blocks.

(f) Finally, cut the seventy-five $2 \times 1 \times 1$ blocks into one hundred and fifty $1 \times 1 \times 1$ blocks.

This process uses a total of $2+2+1+2+1+1 = 9$ cuts, which implies that 9 is indeed the minimum possible number of cuts.

### 2.4.8 Round 8

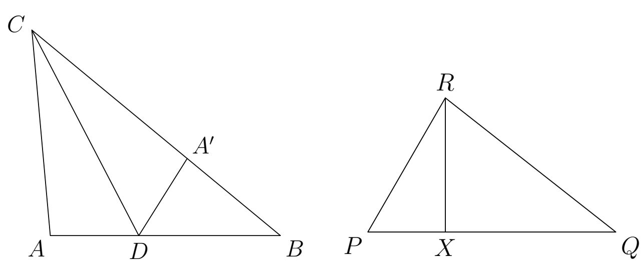
22. [18pts] A sequence that contains the numbers $1, 2, 3, \ldots, n$ exactly once each is said to be a *permutation* of length $n$. A permutation $w_1 w_2 w_3 \cdots w_n$ is said to be *sad* if there are indices $i < j < k$ such that $w_j > w_k$ and $w_j > w_i$. For example, the permutation 3142**7**5**6** is sad because $7 > 6$ and $7 > 1$. Compute the number of permutations of length 11 that are not sad.

**Solution.** The answer is $\boxed{1024}$.

Call a permutation *happy* if it is not sad. We can construct a happy sequence of $n$ numbers by adding the number $n$ to a happy sequence of $n-1$ numbers. Since $n$ is greater than any number in the permutation of length $n-1$, $n$ must be inserted at either the start or end; otherwise the two numbers adjacent to $n$ will both be smaller than $n$, which would yield a sad permutation. Thus, for each permutation of length $n-1$, there are two permutations of length $n$. Since there is 1 happy permutation of length 1, there are 2 happy permutations of length 2, 4 of length 3, and so on until $2^{10} = 1024$ happy permutations of length 11.

23. [18pts] Let $ABC$ be a triangle with $AB = 39, BC = 56$, and $CA = 35$. Compute $\angle CAB - \angle ABC$ in degrees.

**Solution.** The answer is $\boxed{60^\circ}$.




Let $D$ be the foot of the angle bisector from $C$ to $AB$, and let $A'$ be the reflection of $A$ over line $CD$. Triangles $CAD$ and $CA'D$ are congruent, which implies that $\angle CAB = \angle DA'C$. Furthermore, by exterior angles, we have $\angle DA'C = \angle A'DB + \angle ABC$, which implies that $\angle CAB - \angle ABC = \angle AD'B$. By the Angle-Bisector Theorem, we have $\frac{AD}{DB} = \frac{AC}{CB} = \frac{5}{8}$, which implies that $A'D = AD = 15$ and

$DB = 24$. In addition, we have $BA' = BC - A'C = 56 - 35 = 21$. Therefore, triangle $A'BC$ is similar to a triangle with side lengths 5, 7, and 8. Let $PQR$ be a triangle with $PQ = 8$, $QR = 7$, and $RP = 5$. Then, we have $\angle AD'B = \angle RPQ$. Let $X$ be the foot of the perpendicular from point $R$ to side $PQ$; the area of triangle $ABC$ is $\frac{8XR}{2} = 4XR$. However, by Heron's Formula, the area of triangle $PQR$ is $\sqrt{10 \cdot 5 \cdot 3 \cdot 2} = 10\sqrt{3}$, which implies that $XR = \frac{5\sqrt{3}}{2}$. Therefore, $RPX$ is a $30 - 60 - 90$ right triangle, which implies that $\angle RPQ = 60°$. It follows that $\angle CAB = \angle ABC = 60°$.

24. [18pts] On a strange planet, there are $n$ cities. Between any pair of cities, there can either be a one-way road, two one-way roads in different directions, or no road at all. Every city has a name, and at the source of every one-way road, there is a signpost with the name of the destination city. In addition, the one-way roads only intersect at cities, but there can be bridges to prevent intersections at non-cities. Fresh Mann has been abducted by one of the aliens, but Sophy Moore knows that he is in Rome, a city that has no roads leading out of it. Also, there is a direct one-way road leading from each other city to Rome. However, Rome is the secret police's name for the so-described city; its official name, the name appearing on the labels of the one-way roads, is unknown to Sophy Moore. Sophy Moore is currently in Athens and she wants to head to Rome in order to rescue Fresh Mann, but she does not know the value of $n$. Assuming that she tries to minimize the number of roads on which she needs to travel, determine the maximum possible number of roads that she could be forced to travel in order to find Rome. Express your answer as a function of $n$.

**Solution.** The answer is $\boxed{n-1}$.

We will first show that Sophy Moore can guarantee to have arrived at Rome by traveling in $n - 1$ roads, and then giving an example of a road map where Sophy Moore is required to use at least $n - 1$ moves to arrive at Rome.

In order to arrive at Rome in at most $n - 1$ moves, Sophy Moore can employ the following strategy: At every city, she looks at the labels of all the one-way roads on which she could travel. If there are no one-way roads, then she has arrived at Rome, because otherwise there would be a one-way road to Rome. If there are some one-way roads, then she then chooses one of the destination cities to which she has not yet been; she can always do this so long as she has not yet arrived at Rome, because until her arrival at Rome, there will always be a one-way road to Rome in the set of destination cities to which she has not yet been. Because this guarantees that she visits each city at most once, she will eventually visit Rome. In addition, because there are only. This guarantees that she visits each city at most once, and thus she will arrive at Rome after visiting at most n cities, or traveling on at most $n - 1$ roads.

A possible map configuration that forces Sophy Moore to use $n - 1$ roads to visit Rome is as follows: Name the cities $1, 2, 3, ..., n$; where 1 is Athens and $n$ is Rome. Between every two cities, there is a one-way road leading from the city with smaller number to the city with larger number. Notice that if Sophy Moore is, at any point in time, at the city named $m$, $(1 \le m \le n - 1)$, then the cities $m + 1, m + 2, \ldots, n$ will all look identical to her, because they appear identical from the perspective of all cities named $1, 2, \ldots, m$, and if Sophy Moore is currently at the city named $m$, then she has only been to cities in the set $\{1, 2, \ldots, m\}$. Thus we can assume, under worst-case scenario, that she travels to the city named $m + 1$. This means that she will need to travel $m - 1$ roads under this road configuration and worst-case scenario in order to arrive at Rome.

Because Sophy Moore has a strategy which allows her to arrive at Rome within $n-1$ road travels, and there can be a road configuration for which she could potentially be required to use $n-1$ road travels, the answer is $n - 1$.

# Exeter Math Club Competition
# January 25, 2014

## Table of Contents

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- **Judges** Zuming Feng, Greg Spanier

# Chapter 1

# EMC$^2$ 2014 Problems

## 1.1 Speed Test

*There are 25 problems, worth 2 points each, to be solved in 30 minutes.*

1. Chad, Ravi, Kevin, and Meena are four of the 551 residents of Chadwick, Illinois. Expressing your answer to the nearest percent, how much of the population do they represent?

2. Points $A$, $B$, and $C$ are on a line for which $AB = 625$ and $BC = 256$. What is the sum of all possible values of the length $AC$?

3. An increasing arithmetic sequence has first term 2014 and common difference 1337. What is the least odd term of this sequence?

4. How many non-congruent scalene triangles with integer side lengths have two sides with lengths 3 and 4?

5. Let $a$ and $b$ be real numbers for which the function $f(x) = ax^2 + bx + 3$ satisfies $f(0) + 2^0 = f(1) + 2^1 = f(2) + 2^2$. What is $f(0)$?

6. A *pentomino* is a set of five planar unit squares that are joined edge to edge. Two pentominoes are considered the same if and only if one can be rotated and translated to be identical to the other. We say that a pentomino is *compact* if it can fit within a 2 by 3 rectangle. How many distinct compact pentominoes exist?

7. Consider a hexagon with interior angle measurements of 91, 101, 107, 116, 152, and 153 degrees. What is the average of the interior angles of this hexagon, in degrees?

8. What is the smallest positive number that is either one larger than a perfect cube and one less than a perfect square, or vice versa?

9. What is the first time after 4:56 (a.m.) when the 24-hour expression for the time has three consecutive digits that form an increasing arithmetic sequence with difference 1? (For example, 23:41 is one of those moments, while 23:12 is not.)

10. Chad has trouble counting. He wants to count from 1 to 100, but cannot pronounce the word "three," so he skips every number containing the digit three. If he tries to count up to 100 anyway, how many numbers will he count?

11. In square $ABCD$, point $E$ lies on side $BC$ and point $F$ lies on side $CD$ so that triangle $AEF$ is equilateral and inside the square. Point $M$ is the midpoint of segment $EF$, and $P$ is the point other than $E$ on $AE$ for which $PM = FM$. The extension of segment $PM$ meets segment $CD$ at $Q$. What is the measure of $\angle CQP$, in degrees?

12. One apple is five cents cheaper than two bananas, and one banana is seven cents cheaper than three peaches. How much cheaper is one apple than six peaches, in cents?

13. How many ordered pairs of integers $(a, b)$ exist for which $|a|$ and $|b|$ are at most 3, and $a^3 - a = b^3 - b$?

14. Five distinct boys and four distinct girls are going to have lunch together around a table. They decide to sit down one by one under the following conditions: no boy will sit down when more boys than girls are already seated, and no girl will sit down when more girls than boys are already seated. How many possible sequences of taking seats exist?

15. Jordan is swimming laps in a pool. For each lap after the first, the time it takes her to complete is five seconds more than that of the previous lap. Given that she spends 10 minutes on the first six laps, how long does she spend on the next six laps, in minutes?

16. Chad decides to go to trade school to ascertain his potential in carpentry. Chad is assigned to cut away all the vertices of a wooden regular tetrahedron with sides measuring four inches. Each vertex is cut away by a plane which passes through the three midpoints of the edges adjacent to that vertex. What is the surface area of the resultant solid, in square inches?

    Note: A *tetrahedron* is a solid with four triangular faces. In a *regular* tetrahedron, these faces are all equilateral triangles.

17. Chad and Jordan independently choose two-digit positive integers. The two numbers are then multiplied together. What is the probability that the result has a units digit of zero?

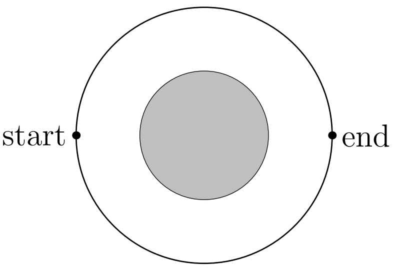
18. For art class, Jordan needs to cut a circle out of the coordinate grid. She would like to find a circle passing through at least 16 lattice points so that her cut is accurate. What is the smallest possible radius of her circle?

    Note: A lattice point is defined as one whose coordinates are both integers. For example, (5, 8) is a lattice point whereas (3.5, 5) is not.

19. Chad's ant Arctica is on one of the eight corners of Chad's toolbox, which measures two decimeters in width, three decimeters in length, and four decimeters in height. One day, Arctica wanted to go to the opposite corner of this box. Assuming she can only crawl on the surface of the toolbox, what is the shortest distance she has to crawl to accomplish this task, in decimeters? (You may assume that the toolbox is floating in the Exeter Space Station, so that Arctica can crawl on all six faces.)

20. Jordan is counting numbers for fun. She starts with the number 1, and then counts onward, skipping any number that is a divisor of the product of all previous numbers she has said. For example, she starts by counting 1, 2, 3, 4, 5, but skips 6, a divisor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. What is the $20^{\text{th}}$ number she counts?

21. Chad and Jordan are having a race in the lake shown below. The lake has a diameter of four kilometers and there is a circular island in the middle of the lake with a diameter of two kilometers. They start at one point on the edge of the lake and finish at the diametrically opposite point. Jordan makes the trip only by swimming in the water, while Chad swims to the island, runs across it, and then continues swimming. They both take the fastest possible route and, amazingly, they tie! Chad swims at two kilometers an hour and runs at five kilometers an hour. At what speed does Jordan swim?




22. Cameron has stolen Chad's barrel of oil and is driving it around on a truck on the coordinate grid on his truck. Cameron is a bad truck driver, so he can only move the truck forward one kilometer at a

time along one of the gridlines. In fact, Cameron is so bad at driving the truck that between every two one-kilometer movements, he has to turn exactly 90 degrees. After 50 one-kilometer movements, given that Cameron's first one-kilometer movement was westward, how many points he could be on?

23. Let $a, b$, and $c$ be distinct nonzero base ten digits. Assume there exist integers $x$ and $y$ for which $\overline{abc} \cdot \overline{cb} = 100x^2 + 1$ and $\overline{acb} \cdot \overline{bc} = 100y^2 + 1$. What is the minimum value of the number $\overline{abbc}$?

    Note: The notation $\overline{pqr}$ designates the number whose hundreds digit is $p$, tens digit is $q$, and units digit is $r$, not the product $p \cdot q \cdot r$.

24. Let $r_1, r_2, r_3, r_4$ and $r_5$ be the five roots of the equation $x^5 - 4x^4 + 3x^2 - 2x + 1 = 0$. What is the product of $(r_1 + r_2 + r_3 + r_4), (r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_5), (r_1 + r_3 + r_4 + r_5)$, and $(r_2 + r_3 + r_4 + r_5)$?

25. Chad needs seven apples to make an apple strudel for Jordan. He is currently at 0 on the metric number line. Every minute, he randomly moves one meter in either the positive or the negative direction with equal probability. Arctica's parents are located at $+4$ and $-4$ on the number line. They will bite Chad for kidnapping Arctica if he walks onto those numbers. Also, there is one apple located at each integer between $-3$ and 3, inclusive. Whenever Chad lands on an integer with an unpicked apple, he picks it. What is the probability that Chad picks all the apples without getting bitten by Arctica's parents?

## 1.2 Accuracy Test

*There are 8 problems, worth 5 points each, to be solved in 40 minutes.*

1. Chad lives on the third floor of an apartment building with ten floors. He leaves his room and goes up two floors, goes down four floors, goes back up five floors, and finally goes down one floor, where he finds Jordan's room. On which floor does Jordan live?

2. A real number $x$ satisfies the equation $2014x + 1337 = 1337x + 2014$. What is $x$?

3. Given two points on the plane, how many distinct regular hexagons include both of these points as vertices?

4. Jordan has six different files on her computer and needs to email them to Chad. The sizes of these files are 768, 1024, 2304, 2560, 4096, and 7680 kilobytes. Unfortunately, the email server holds a limit of $S$ kilobytes on the total size of the attachments per email, where $S$ is a positive integer. It is additionally given that all of the files are indivisible. What is the maximum value of $S$ for which it will take Jordan at least three emails to transmit all six files to Chad?

5. If real numbers $x$ and $y$ satisfy $(x + 2y)^2 + 4(x + 2y + 2 - xy) = 0$, what is $x + 2y$?

6. While playing table tennis against Jordan, Chad came up with a new way of scoring. After the first point, the score is regarded as a ratio. Whenever possible, the ratio is reduced to its simplest form. For example, if Chad scores the first two points of the game, the score is reduced from 2:0 to 1:0. If later in the game Chad has 5 points and Jordan has 9, and Chad scores a point, the score is automatically reduced from 6:9 to 2:3. Chad's next point would tie the game at 1:1. Like normal table tennis, a player wins if he or she is the first to obtain 21 points. However, he or she does not win if after his or her receipt of the $21^{\text{st}}$ point, the score is immediately reduced. Chad and Jordan start at 0:0 and finish the game using this rule, after which Jordan notes a curiosity: the score was never reduced. How many possible games could they have played? Two games are considered the same if and only if they include the exact same sequence of scoring.

7. For a positive integer $m$, we define $m$ as a *factorial number* if and only if there exists a positive integer $k$ for which $m = k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$. We define a positive integer $n$ as a *Thai number* if and only if $n$ can be written as both the sum of two factorial numbers and the product of two factorial numbers. What is the sum of the five smallest Thai numbers?

8. Chad and Jordan are in the Exeter Space Station, which is a triangular prism with equilateral bases. Its height has length one decameter and its base has side lengths of three decameters. To protect their station against micrometeorites, they install a force field that contains all points that are within one decameter of any point of the surface of the station. What is the volume of the set of points within the force field and outside the station, in cubic decameters?

## 1.3 Team Test

*There are 10 problems, worth 18 points each, to be solved in 45 minutes.*

1. What is the units digit of the product of the first seven primes?

2. In triangle $ABC$, $\angle BAC$ is a right angle and $\angle ACB$ measures 34 degrees. Let $D$ be a point on segment $BC$ for which $AC = CD$, and let the angle bisector of $\angle CBA$ intersect line $AD$ at $E$. What is the measure of $\angle BED$?

3. Chad numbers five paper cards on one side with each of the numbers from 1 through 5. The cards are then turned over and placed in a box. Jordan takes the five cards out in random order and again numbers them from 1 through 5 on the other side. When Chad returns to look at the cards, he deduces with great difficulty that the probability that exactly two of the cards have the same number on both sides is $p$. What is $p$?

4. Only one real value of $x$ satisfies the equation $kx^2 + (k + 5)x + 5 = 0$. What is the product of all possible values of $k$?

5. On the Exeter Space Station, where there is no effective gravity, Chad has a geometric model consisting of 125 wood cubes measuring 1 centimeter on each edge arranged in a 5 by 5 by 5 cube. An aspiring carpenter, he practices his trade by drawing the projection of the model from three views: front, top, and side. Then, he removes some of the original 125 cubes and redraws the three projections of the model. He observes that his three drawings after removing some cubes are identical to the initial three. What is the maximum number of cubes that he could have removed? (Keep in mind that the cubes could be suspended without support.)

6. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for $E$ and $m$ in the equation. Then, Cameron solves for the absolute value of $c$. What is the probability that Cameron's result is a rational number?

7. Let $CDE$ be a triangle with side lengths $EC = 3$, $CD = 4$, and $DE = 5$. Suppose that points $A$ and $B$ are on the perimeter of the triangle such that line $AB$ divides the triangle into two polygons of equal area and perimeter. What are all the possible values of the length of segment $AB$?

8. Chad and Jordan are raising bacteria as pets. They start out with one bacterium in a Petri dish. Every minute, each existing bacterium turns into 0, 1, 2 or 3 bacteria, with equal probability for each of the four outcomes. What is the probability that the colony of bacteria will eventually die out?

9. Let $a = w + x, b = w + y, c = x + y, d = w + z, e = x + z$, and $f = y + z$. Given that $af = be = cd$ and

$$(x - y)(x - z)(x - w) + (y - x)(y - z)(y - w) + (z - x)(z - y)(z - w) + (w - x)(w - y)(w - z) = 1,$$

what is

$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) - ab - ac - ad - ae - bc - bd - bf - ce - cf - de - df - ef?$$

10. If $a$ and $b$ are integers at least 2 for which $a^b - 1$ strictly divides $b^a - 1$, what is the minimum possible value of $ab$?

Note: If $x$ and $y$ are integers, we say that $x$ *strictly* divides $y$ if $x$ divides $y$ and $|x| \neq |y|$.

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. [4] What is $2 + 22 + 1 + 3 - 31 - 3$?

2. [4] Let $ABCD$ be a rhombus. Given $AB = 5$, $AC = 8$, and $BD = 6$, what is the perimeter of the rhombus?

3. [4] There are 2 hats on a table. The first hat has 3 red marbles and 1 blue marble. The second hat has 2 red marbles and 4 blue marbles. Jordan picks one of the hats randomly, and then randomly chooses a marble from that hat. What is the probability that she chooses a blue marble?

### 1.4.2 Round 2

4. [5] There are twelve students seated around a circular table. Each of them has a slip of paper that they may choose to pass to either their clockwise or counterclockwise neighbor. After each person has transferred their slip of paper once, the teacher observes that no two students exchanged papers. In how many ways could the students have transferred their slips of paper?

5. [5] Chad wants to test David's mathematical ability by having him perform a series of arithmetic operations at lightning-speed. He starts with the number of cubic centimeters of silicon in his 3D printer, which is 109. He has David perform all of the following operations in series each second:

   - Double the number
   - Subtract 4 from the number
   - Divide the number by 4
   - Subtract 5 from the number
   - Double the number
   - Subtract 4 from the number

   Chad instructs David to shout out after three seconds the result of three rounds of calculations. However, David computes too slowly and fails to give an answer in three seconds. What number should David have said to Chad?

6. [5] Points $D$, $E$, and $F$ lie on sides $BC$, $CA$, and $AB$ of triangle $ABC$, respectively, such that the following length conditions are true: $CD = AE = BF = 2$ and $BD = CE = AF = 4$. What is the area of triangle $ABC$?

### 1.4.3 Round 3

7. [6] In the *2, 3, 5, 7 game*, players count the positive integers, starting with 1 and increasing, which do not contain the digits 2, 3, 5, and 7, and also are not divisible by the numbers 2, 3, 5, and 7. What is the fifth number counted?

8. [6] If $A$ is a real number for which
$$19 \cdot A = \frac{2014!}{1! \cdot 2! \cdot 2013!},$$
what is $A$?

   Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

9. [6] What is the smallest number that can be written as both $x^3 + y^2$ and $z^3 + w^2$ for positive integers $x$, $y$, $z$, and $w$ with $x \neq z$?

### 1.4.4 Round 4

*Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.*

*In addition, it is given that the answer to each of the following problems is a positive integer less than or equal to the problem number.*

10. [7] Let $B$ be the answer to problem 11 and let $C$ be the answer to problem 12. What is the sum of a side length of a square with perimeter $B$ and a side length of a square with area $C$?

11. [7] Let $A$ be the answer to problem 10 and let $C$ be the answer to problem 12. What is $(C-1)(A+1) - (C+1)(A-1)$?

12. [7] Let $A$ be the answer to problem 10 and let $B$ be the answer to problem 11. Let $x$ denote the positive difference between $A$ and $B$. What is the sum of the digits of the positive integer $9x$?

### 1.4.5 Round 5

13. [8] Five different schools are competing in a tournament where each pair of teams plays at most once. Four pairs of teams are randomly selected and play against each other. After these four matches, what is the probability that Chad's and Jordan's respective schools have played against each other, assuming that Chad and Jordan come from different schools?

14. [8] A square of side length 1 and a regular hexagon are both circumscribed by the same circle. What is the side length of the hexagon?

15. [8] From the list of integers $1, 2, 3, \ldots, 30$, Jordan can pick at least one pair of distinct numbers such that none of the 28 other numbers are equal to the sum or the difference of this pair. Of all possible such pairs, Jordan chooses the pair with the least sum. Which two numbers does Jordan pick?

### 1.4.6 Round 6

16. [9] What is the sum of all two-digit integers with no digit greater than four whose squares also have no digit greater than four?

17. [9] Chad marks off ten points on a circle. Then, Jordan draws five chords under the following constraints:

    - Each of the ten points is on exactly one chord.
    - No two chords intersect.
    - There do not exist (potentially non-consecutive) points $A, B, C, D, E$, and $F$, in that order around the circle, for which $AB$, $CD$, and $EF$ are all drawn chords.

    In how many ways can Jordan draw these chords?

18. [9] Chad is thirsty. He has 109 cubic centimeters of silicon and a 3D printer with which he can print a cup to drink water in. He wants a silicon cup whose exterior is cubical, with five square faces and an

open top, that can hold exactly 234 cubic centimeters of water when filled to the rim in a rectangular-box-shaped cavity. Using all of his silicon, he prints a such cup whose thickness is the same on the five faces. What is this thickness, in centimeters?

### 1.4.7 Round 7

19. [10] Jordan wants to create an equiangular octagon whose side lengths are exactly the first 8 positive integers, so that each side has a different length. How many such octagons can Jordan create?

20. [10] There are two positive integers on the blackboard. Chad computes the sum of these two numbers and tells it to Jordan. Jordan then calculates the sum of the greatest common divisor and the least common multiple of the two numbers, and discovers that her result is exactly 3 times as large as the number Chad told her. What is the smallest possible sum that Chad could have said?

21. [10] Chad uses *yater* to measure distances, and knows the conversion factor from yaters to meters precisely. When Jordan asks Chad to convert yaters into meters, Chad only gives Jordan the result rounded to the nearest integer meters. At Jordan's request, Chad converts 5 yaters into 8 meters and 7 yaters into 12 meters. Given this information, how many possible numbers of meters could Jordan receive from Chad when requesting to convert 2014 yaters into meters?

### 1.4.8 Round 8

22. [11] Jordan places a rectangle inside a triangle with side lengths 13, 14, and 15 so that the vertices of the rectangle all lie on sides of the triangle. What is the maximum possible area of Jordan's rectangle?

23. [11] Hoping to join Chad and Jordan in the Exeter Space Station, there are 2014 prospective astronauts of various nationalities. It is given that 1006 of the astronaut applicants are American and that there are a total of 64 countries represented among the applicants. The applicants are to group into 1007 pairs with no pair consisting of two applicants of the same nationality. Over all possible distributions of nationalities, what is the maximum number of possible ways to make the 1007 pairs of applicants? Express your answer in the form $a \cdot b!$, where $a$ and $b$ are positive integers and $a$ is not divisible by

$b+1$.

Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

24. [11] We say a polynomial $P$ in $x$ and $y$ is $n$-good if $P(x, y) = 0$ for all integers $x$ and $y$, with $x \neq y$, between 1 and $n$, inclusive. We also define the complexity of a polynomial to be the maximum sum of exponents of $x$ and $y$ across its terms with nonzero coefficients. What is the minimal complexity of a nonzero 4-good polynomial? In addition, give an example of a 4-good polynomial attaining this minimal complexity.

**Chapter 2**

# EMC$^2$ 2014 Solutions

## 2.1 Speed Test Solutions

1. Chad, Ravi, Kevin, and Meena are four of the 551 residents of Chadwick, Illinois. Expressing your answer to the nearest percent, how much of the population do they represent?

   **Solution.** The answer is $\boxed{1\%}$.

   We can see that
   $$551 \cdot 1\% > 500 \cdot 0.01 = 5 > 4,$$
   so 1% of the population is more than 4. This means that the four people represent less than 1% of the population. Also,
   $$4 > 3 = 600 \cdot 0.005 > 551 \cdot 0.5\%,$$
   so the percentage is between 0.5% and 1%. Rounding to the nearest percent yields 1%.

2. Points $A$, $B$, and $C$ are on a line for which $AB = 625$ and $BC = 256$. What is the sum of all possible values of the length $AC$?

   **Solution.** The answer is $\boxed{1250}$.

   If $C$ is the middle point, then $AC = AB - BC$. If $B$ is the middle point, then $AC = AB + BC$. If $A$ is the middle point, then we have that $BC = BA + AC$, which results in a negative value of $AC$. Therefore there are two possible values of $AC$, which sum to $(AB - BC) + (AB + BC) = 2 \cdot AB = 2 \cdot 625 = 1250$.

3. An increasing arithmetic sequence has first term 2014 and common difference 1337. What is the least odd term of this sequence?

   **Solution.** The answer is $\boxed{3351}$.

   The first term 2014 is not odd, but the second term $2014 + 1337 = 3351$ is odd. Because the sequence is increasing, all other terms are greater than 3351, and therefore 3351 is the answer.

4. How many non-congruent scalene triangles with integer side lengths have two sides with lengths 3 and 4?

   **Solution.** The answer is $\boxed{3}$.

   The length of the third side of the triangle must be between 2 and 6, inclusive; otherwise, the longest side of the triangle would be at least the sum of the other two, and the triangle could not be constructed. However, we must also exclude 3 and 4 as possibilities because the triangle is scalene, which only leaves 2, 5, and 6 as possibilities for the length of the third side. All three of these triangles are valid.

5. Let $a$ and $b$ be real numbers for which the function $f(x) = ax^2 + bx + 3$ satisfies $f(0) + 2^0 = f(1) + 2^1 = f(2) + 2^2$. What is $f(0)$?

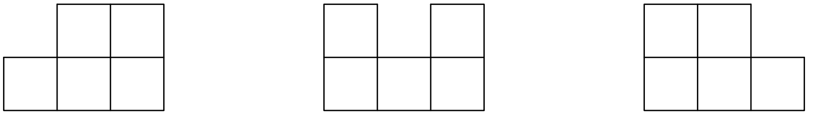
   **Solution.** The answer is $\boxed{3}$.

   We have $f(0) = a(0)^2 + b(0) + 3 = 3$.

6. A *pentomino* is a set of five planar unit squares that are joined edge to edge. Two pentominoes are considered the same if and only if one can be rotated and translated to be identical to the other. We say that a pentomino is *compact* if it can fit within a 2 by 3 rectangle. How many distinct compact pentominoes exist?

   **Solution.** The answer is $\boxed{3}$.

   We can get 3 distinct compact pentominoes by removing any square in the top row (with 3 squares) in a 2 by 3 rectangle, as shown below. The 3 configurations with bottom squares removed are the same as the previous 3 under a 180° rotation.



7. Consider a hexagon with interior angle measurements of 91, 101, 107, 116, 152, and 153 degrees. What is the average of the interior angles of this hexagon, in degrees?

   **Solution.** The answer is $\boxed{120^\circ}$.

   We can sum the six numbers to get a total sum of 720°, so the average is $\frac{720^\circ}{6} = 120^\circ$.

   **Alternate solution:** The sum of all of the interior angles of any hexagon is $(6-2)\cdot 180^\circ = 720^\circ$. Therefore the average is $\frac{720^\circ}{6} = 120^\circ$.

8. What is the smallest positive number that is either one larger than a perfect cube and one less than a perfect square, or vice versa?

   **Solution.** The answer is $\boxed{26}$.

   The first few squares are $1, 4, 9, 16, 25, 36, \ldots$, and the first few cubes are $1, 8, 27, 64, \ldots$. We can see that 26 is the first number between a square and a cube.

9. What is the first time after 4:56 (a.m.) when the 24-hour expression for the time has three consecutive digits that form an increasing arithmetic sequence with difference 1? (For example, 23:41 is one of those moments, while 23:12 is not.)

   **Solution.** The answer is $\boxed{10{:}12}$.

   All times after 4:56 and before 10:00 have three digits. Additionally, their second digit is at most 5, so if they satisfy the desired conditions their first digit must be at most 4. It is clear that none of 4:57, 4:58, and 4:59 satisfies the desired conditions. Therefore, the next time with three consecutive digits that form an increasing arithmetic sequence must be after 10:00. Note that 10:12 is a time with the desired conditions, and it is easy to check that the times after 10:00 and before 10:12 do not contain arithmetic sequences, so 10:12 is the answer.

10. Chad has trouble counting. He wants to count from 1 to 100, but cannot pronounce the word "three," so he skips every number containing the digit three. If he tries to count up to 100 anyway, how many numbers will he count?

**Solution.** The answer is $\boxed{81}$.

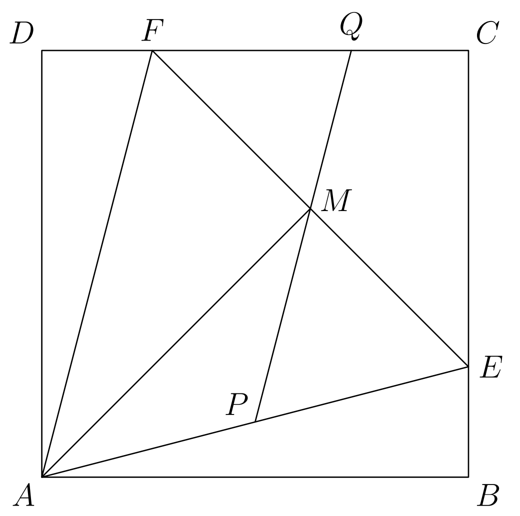
Chad can count 100, so we can replace 100 with the two-digit number 00. We can also add a 0 to the front of all one-digit numbers to make all of the 100 numbers have two digits. There are 9 possibilities for each digit of the numbers Chad can count because he cannot count 3, so there are $9 \cdot 9 = 81$ numbers that Chad can count.

**Alternate solution:** There are 10 numbers that have 3 as their tens digit and 10 numbers that have 3 as their ones digit. However, these 20 numbers include 33 twice, so there are $10 + 10 - 1 = 19$ numbers that have at least one 3 as a digit. Chad cannot say these numbers, so he says $100 - 19 = 81$ numbers.

11. In square $ABCD$, point $E$ lies on side $BC$ and point $F$ lies on side $CD$ so that triangle $AEF$ is equilateral and inside the square. Point $M$ is the midpoint of segment $EF$, and $P$ is the point other than $E$ on $AE$ for which $PM = FM$. The extension of segment $PM$ meets segment $CD$ at $Q$. What is the measure of $\angle CQP$, in degrees?

    **Solution.** The answer is $\boxed{105}$.

    The diagram is as follows:




Because $M$ is the midpoint of $EF$, we have $EM = FM = PM$. Therefore, $\triangle EPM$ is isosceles with $\angle EPM = \angle PEM = 60°$.

Also, $ABCD$ is a square, so $AB = AD$, and $\triangle AEF$ is equilateral, so $AE = AF$. Then, by the Pythagorean Theorem,

$$BE = \sqrt{AE^2 - AB^2} = \sqrt{AF^2 - AD^2} = DF.$$

Combining these, we have $\triangle ABE \cong \triangle ADF$, so $\angle BAE = \angle DAF$. Considering $\angle BAD$, we have $\angle BAE + \angle EAF + \angle FAD = 90°$, or $\angle BAE = \frac{90° - 60°}{2} = 15°$ using the fact that $\angle EAF = 60°$. Then, considering $\triangle ABE$, we have $\angle AEB = 90° - \angle BAE = 75°$. Finally, $\angle CEP = 180° - \angle BEA = 105°$.

Now, consider quadrilateral $CQPE$, whose interior angles should sum to $360°$. That is,

$$\begin{aligned}\angle CQP + \angle QPE + \angle PEC + \angle ECQ &= 360° \\ \angle CQP + 60° + 105° + 90° &= 360° \\ \angle CQP &= 105°,\end{aligned}$$

because $\angle ECQ = 90°$.

**Alternate solution:** As in the previous solution, we can find that $\angle AEB = \angle AFD = 75°$ and $\angle EPM = 60°$. Because $\angle EAF = 60° = \angle EPM$, lines $AF$ and $PQ$ are parallel. Therefore, $\angle CQP = \angle CFA = 180° - \angle DFA = 180° - 75° = 105°$.

12. One apple is five cents cheaper than two bananas, and one banana is seven cents cheaper than three peaches. How much cheaper is one apple than six peaches, in cents?

    **Solution.** The answer is $\boxed{19}$.

    Suppose the prices of an apple, a banana, and a peach are $A$, $B$, and $P$ cents, respectively. We have

    $$\begin{aligned}2B &= A + 5 \\ 3P &= B + 7,\end{aligned}$$

    so the answer is

    $$\begin{aligned}6P - A &= 2(3P) - A \\ &= 2(B + 7) - A \\ &= 2B + 14 - A \\ &= (A + 5) + 14 - A \\ &= 19.\end{aligned}$$

13. How many ordered pairs of integers $(a, b)$ exist for which $|a|$ and $|b|$ are at most 3, and $a^3 - a = b^3 - b$?

    **Solution.** The answer is $\boxed{13}$.

    We can create a table of values of $n^3 - n$ for when $n$ is between $-3$ and 3:

    | $n$ | $n^3 - n$ |
    |---|---|
    | $-3$ | $-24$ |
    | $-2$ | $-6$ |
    | $-1$ | 0 |
    | 0 | 0 |
    | 1 | 0 |
    | 2 | 6 |
    | 3 | 24 |

    So, the solutions for $(a, b)$ are $(-3, -3)$, $(-2, -2)$, $(2, 2)$, $(3, 3)$ and all $(a, b)$ in which $a$ and $b$ are both any of $-1$, 0, or 1. Thus, in total there are $4 + 3 \cdot 3 = 13$ solutions.

14. Five distinct boys and four distinct girls are going to have lunch together around a table. They decide to sit down one by one under the following conditions: no boy will sit down when more boys than girls are already seated, and no girl will sit down when more girls than boys are already seated. How many possible sequences of taking seats exist?

**Solution.** The answer is $\boxed{46080}$.

Pair the first person to sit with the second, the third with the fourth, and so on, excluding the last person. In order to satisfy the conditions, there must be one boy and one girl in each pair. So, in the four pairs, there are exactly 4 boys and 4 girls, hence the last person to be seated must be a boy. There are 2 possibilities for the order of genders in each pair, which yields a total of $2^4 = 16$ gender arrangements. Then, considering each gender separately, there are five choices for the first boy to sit down, four choices for the second, and so on. Similarly, there are four choices for the first girl to sit down, three choices for the second, and so on. So the total number of sequences is $16 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) = 16 \cdot 120 \cdot 24 = 46080$.

15. Jordan is swimming laps in a pool. For each lap after the first, the time it takes her to complete is five seconds more than that of the previous lap. Given that she spends 10 minutes on the first six laps, how long does she spend on the next six laps, in minutes?

**Solution.** The answer is $\boxed{13}$.

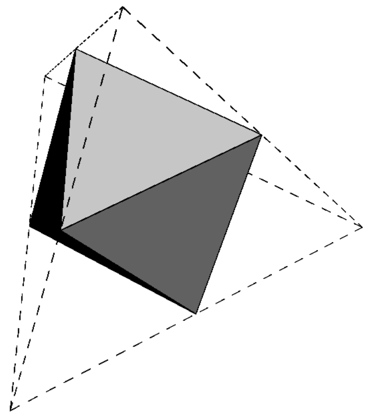
Match the $n$-th lap with the $(n+6)$-th lap for each positive integer $n \leq 6$. In each pair, the latter lap takes $6 \cdot 5 = 30$ seconds longer to finish than the former lap, so it would take $6 \cdot 30 = 180$ seconds more to finish the next 6 laps. Therefore, the answer is $10 + \frac{180}{60} = 13$ minutes.

16. Chad decides to go to trade school to ascertain his potential in carpentry. Chad is assigned to cut away all the vertices of a wooden regular tetrahedron with sides measuring four inches. Each vertex is cut away by a plane which passes through the three midpoints of the edges adjacent to that vertex. What is the surface area of the resultant solid, in square inches?

Note: A *tetrahedron* is a solid with four triangular faces. In a *regular* tetrahedron, these faces are all equilateral triangles.

**Solution.** The answer is $\boxed{8\sqrt{3}}$.

After each cut, a new equilateral triangular face with side length 2 inches is created. After all four cuts, each original face is also an equilateral triangle with side length 2 inches.



Each of these triangles has area $\frac{2^2 \cdot \sqrt{3}}{4} = \sqrt{3}$ square inches. The resultant solid consists of eight of these triangles, so its total surface area is $8\sqrt{3}$ square inches.

**Alternate solution:** Each of the four cuts adds an equilateral triangle with side length 2 inches to the surface area, but also takes away three such equilateral triangles. The new surface area can be computed by adding to and subtracting from the surface area of the original solid. The uncut tetrahedron has four equilateral triangles each with area $4\sqrt{3}$, so the final surface area is $4 \cdot 4\sqrt{3} + 4 \cdot \sqrt{3} - 4 \cdot 3 \cdot \sqrt{3} = 8\sqrt{3}$.

17. Chad and Jordan independently choose two-digit positive integers. The two numbers are then multiplied together. What is the probability that the result has a units digit of zero?

**Solution.** The answer is $\boxed{\frac{27}{100}}$.

For each given digit between 0 and 9, inclusive, there are exactly 9 two-digit numbers ending in that digit. Because this is the same for all digits, the problem can be regarded as randomly choosing two one-digit integers from 0 to 9. We can see that there are a total of $10 \cdot 10 = 100$ possible choices for the two units digits.

Now suppose the product of the two numbers ends in zero. If Chad chooses 2, 4, 6, or 8, Jordan must choose 0 or 5. If Chad chooses 5, Jordan's number is one of 0, 2, 4, 6 or 8. If Chad's number is 0, Jordan can choose any of the ten digits. Finally, if Chad chooses 1, 3, 7, or 9, Jordan can only choose 0. Therefore, there are $4 \cdot 2 + 1 \cdot 5 + 1 \cdot 10 + 4 \cdot 1 = 27$ ways for Chad and Jordan to have numbers with product ending in 0. Thus, the desired probability is $\frac{27}{100}$.

**Alternate solution:** We assign to each two-digit integer an ordered pair $(a, b)$ as follows: the remainder upon division of the number by 2 is $a$, and upon division by 5 is $b$. The product of the two numbers has units digit 0 if and only if at least one of the $a$ values and at least one of the $b$ values are 0, as the number must be divisible by both 2 and 5 to have units digit 0. We treat each value independently. For the $a$-values, the probability that one value is 0 is $\frac{1}{2}$, and that it is not 0 is the same. Thus, the probability that at least one of the two numbers has $a$-value 0 is $1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$. Similarly, the probability that one $b$-value is 0 is $\frac{1}{5}$, so the probability that at least one of the two numbers has $b$-value 0 is $1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25}$. Multiplying the two together gives an answer of $\frac{27}{100}$.

Note that this solution requires that the $a$-value and $b$-value are indeed probabilistically independent of each other. However, this is also easy to verify because there exist an equal number of integers assigned to each of the ten possible pairs $(a, b)$.

18. For art class, Jordan needs to cut a circle out of the coordinate grid. She would like to find a circle passing through at least 16 lattice points so that her cut is accurate. What is the smallest possible radius of her circle?

    Note: A lattice point is defined as one whose coordinates are both integers. For example, (5, 8) is a lattice point whereas (3.5, 5) is not.

    **Solution.** The answer is $\boxed{\frac{\sqrt{130}}{2}}$.

    Let the smallest possible radius be $r$ and the center of one circle containing 16 lattice points with radius $r$ have center $(p, q)$.

    First, suppose $r < \frac{15}{2}$. Then the diameter is less than 15, so any two points on the circle are less than 15 units apart. Thus, it is impossible for all 16 lattice points on the circle to have different $x$-coordinates, as the leftmost and rightmost points would then be at least 15 units apart. So, the circle contains two points $(a, b)$ and $(a, c)$, where $a$, $b$, and $c$ are integers, with the same $x$-coordinate. Then, $(p, q)$ must lie on the perpendicular bisector of the line segment between these points, so $q = \frac{b+c}{2}$. By a similar argument, we find that $p$ is an integer divided by two. Without loss of generality, we may assume $(p, q) = \left(\frac{1}{2}, \frac{1}{2}\right)$ because a circle with different center could be translated to satisfy this.

    Now, suppose a lattice point $(m, n)$ is on the circle. Then, $\left(m - \frac{1}{2}\right)^2 + \left(n - \frac{1}{2}\right)^2 = r^2$. Also, we have $\left(m - \frac{1}{2}\right)^2 = \left((1 - m) - \frac{1}{2}\right)^2$ and $\left(n - \frac{1}{2}\right)^2 = \left((1 - n) - \frac{1}{2}\right)^2$. So, $(m, n)$, $(m, (1 - n))$, $((1 - m), n)$, and $((1 - m), (1 - n))$ are all on the circle. In addition, the four points produced by switching the $x$- and $y$-coordinates of those points are also on the circle. Note that at least one of the points $(s, t)$ in such a family has the property that $s \geq t \geq 1$. Thus, we can count the number of lattice points on the circle by only considering those with this property. If $s = t$, there are only 4 distinct points to be counted; otherwise, there are 8. However, note that the circle may only contain one lattice point $(u, u)$ with $u \geq 1$, in which case there still must be at least two other points $(s_1, t_1)$ and $(s_2, t_2)$ on the circle with $s_i > t_i \geq 1$ for $i = 1, 2$ if the circle is to have more than 12 lattice points.

    For the circle to contain 16 lattice points, we must have $\left(s_1 - \frac{1}{2}\right)^2 + \left(t_1 - \frac{1}{2}\right) = r^2$ and $\left(s_2 - \frac{1}{2}\right)^2 + \left(t_2 - \frac{1}{2}\right) = r^2$. If we substitute $x_i = 2s_i - 1$ and $y_i = 2t_i - 1$ for $i = 1, 2$, so that $x_i$ and $y_i$ are positive integers, we obtain
    $$x_1^2 + y_1^2 = x_2^2 + y_2^2 = (2r)^2.$$
    Thus, we must find the smallest number expressible as the sum of two perfect squares in two ways. We can check that $65 = 1^2 + 8^2 = 4^2 + 7^2$ satisfies this and no smaller integer does. This means $(2r)^2 = 65$, or $r = \frac{\sqrt{130}}{2}$. Indeed, the circle given by
    $$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{65}{2}$$
    contains 16 lattice points.

    Note that we do not need to check the cases in which $r \geq \frac{15}{2}$ because such an $r$ cannot be the smallest possible radius.

19. Chad's ant Arctica is on one of the eight corners of Chad's toolbox, which measures two decimeters in width, three decimeters in length, and four decimeters in height. One day, Arctica wanted to go to the opposite corner of this box. Assuming she can only crawl on the surface of the toolbox, what is the shortest distance she has to crawl to accomplish this task, in decimeters? (You may assume that the toolbox is floating in the Exeter Space Station, so that Arctica can crawl on all six faces.)

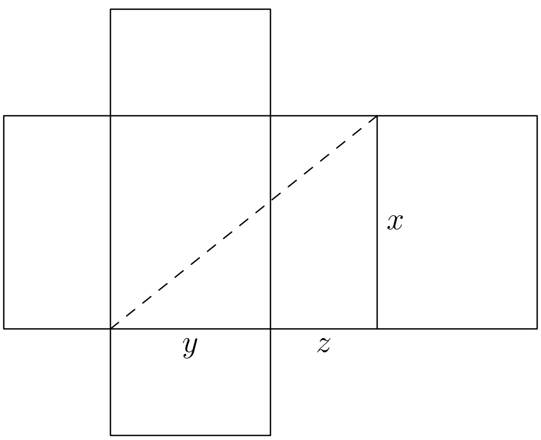
**Solution.** The answer is $\boxed{\sqrt{41}}$.

Given a path from one corner to the opposite corner, observe that the length of the path does not change if we unfold the box.

After unfolding, the shortest path is a straight line between $A$ and $B$, where $A$ and $B$ are opposite corners. Depending on how we unfold, $AB$ can have different lengths, so we want the shortest possible value of $AB$. From the Pythagorean Theorem, we have

$$AB = \sqrt{x^2 + (y+z)^2},$$

where $x$, $y$ and $z$ are the side lengths of the box, in some order.




This is minimized when $x$ is the longest side. Therefore, our answer is

$$\sqrt{4^2 + (3+2)^2} = \boxed{\sqrt{41}}.$$

20. Jordan is counting numbers for fun. She starts with the number 1, and then counts onward, skipping any number that is a divisor of the product of all previous numbers she has said. For example, she starts by counting 1, 2, 3, 4, 5, but skips 6, a divisor of $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. What is the $20^{\text{th}}$ number she counts?

    **Solution.** The answer is $\boxed{47}$.

    We can show that the only integers that are counted are either 1, a prime, or a prime raised to the $2^n$-th power, where $n$ is a positive integer.

    It's easy to show that 1 and any prime will be counted, as a prime $p$ wouldn't be a divisor of $(p-1) \cdot (p-2) \cdots 1$. By playing with smaller numbers we can arrive at the result regarding $2^n$-th powers of primes, but a rigorous proof is included below. Using this claim, we can list the numbers that Jordan counts: 1, 2, 3, 4, 5, 7, 9, 11, 13, 16, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47, and so on. Therefore, the $20^{\text{th}}$ number would be 47.

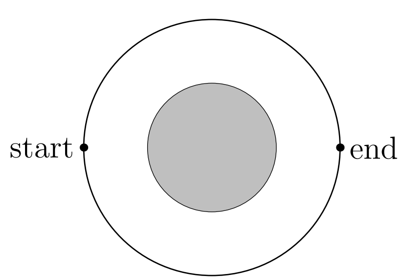
    We now prove that the only numbers other than 1 and primes that are counted are primes raised to the $2^n$-th power. No numbers with at least 2 distinct prime factors are counted, because if there exists such a number $m = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ (where $p_1, p_2, \cdots, p_k$ are all distinct primes and $e_1, e_2, \cdots, e_k$ are all positive integers) then for each integer $i$ from 1 to $k$, either $p_i^{e_i}$ is counted or the product of all the numbers counted from 1 to $p_i^{e_i} - 1$ is divisible by $p_i^{e_i}$. Therefore, $m$ divides the product of all counted numbers between 1 and $(m-1)$, inclusive, and would not be counted.

We establish the last part by induction, considering the $k$-th instance of a power of a prime $p$ being counted and inducting on $k$. The base case for $k = 1$ of $p^{2^0} = p$ was already observed above. Now assume that the only powers of $p$ that were counted are $p^1, p^2, p^4, \ldots, p^{2^{k-1}}$. Then, the next power of $p$ that will be the smallest one that does not divide

$$\begin{aligned} p^1 \cdot p^2 \cdot p^4 \cdots p^{2^{k-1}} &= p^{1+2+4+\cdots+2^{k-1}} \\ &= p^{2^k - 1}. \end{aligned}$$

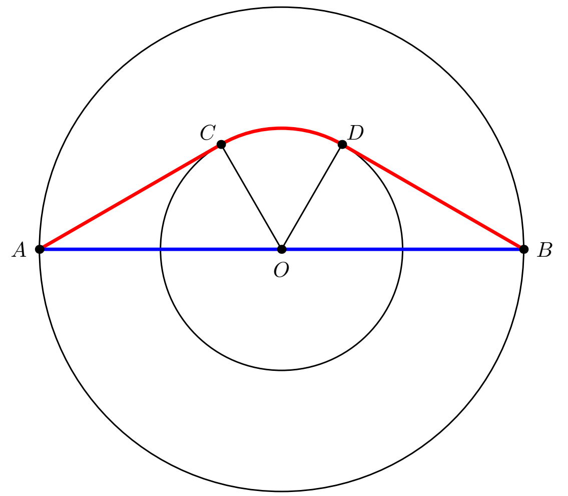
Therefore, the next power of $p$ to be counted is $p^{2^k}$, proving the claim.

21. Chad and Jordan are having a race in the lake shown below. The lake has a diameter of four kilometers and there is a circular island in the middle of the lake with a diameter of two kilometers. They start at one point on the edge of the lake and finish at the diametrically opposite point. Jordan makes the trip only by swimming in the water, while Chad swims to the island, runs across it, and then continues swimming. They both take the fastest possible route and, amazingly, they tie! Chad swims at two kilometers an hour and runs at five kilometers an hour. At what speed does Jordan swim?




**Solution.** The answer is $\boxed{\frac{30\sqrt{3}+5\pi}{21}}$.

In the diagram below, Chad's path is shown in blue and Jordan's in red:




Chad swims a total of 2 kilometers and runs 2 kilometers, so he takes $\frac{2}{2} + \frac{2}{5} = \frac{7}{5}$ hours to finish the race. Because Chad and Jordan finish at the same time, this is also the amount of time Jordan requires to swim across the lake.

The distance Jordan swims is $AC + \widehat{CD} + BD$. Because $\triangle ACO$ and $\triangle BDO$ are 30-60-90 triangles, $AC = BD = \sqrt{3}$. Also, $\angle AOC = \angle BOD = 60^\circ$, so $\angle COD = 180^\circ - 60^\circ - 60^\circ = 60^\circ$ as well. Therefore, $\widehat{CD} = \frac{60^\circ}{360^\circ} \cdot 2\pi = \frac{\pi}{3}$. Summing these lengths and dividing by the time, we obtain the speed at which Jordan swims:

$$\frac{\sqrt{3} + \frac{\pi}{3} + \sqrt{3}}{\frac{7}{5}} = \boxed{\frac{30\sqrt{3} + 5\pi}{21}}.$$

22. Cameron has stolen Chad's barrel of oil and is driving it around on a truck on the coordinate grid on his truck. Cameron is a bad truck driver, so he can only move the truck forward one kilometer at a time along one of the gridlines. In fact, Cameron is so bad at driving the truck that between every two one-kilometer movements, he has to turn exactly 90 degrees. After 50 one-kilometer movements, given that Cameron's first one-kilometer movement was westward, how many points he could be on?

**Solution.** The answer is $\boxed{650}$.

Number the movements from 1 to 50, and let Cameron start at $(0, 0)$ on the coordinate plane. Note that odd-numbered movements only affect the $x$-coordinate, while even-numbered movements only affect the $y$-coordinate. This means the $x$- and $y$-coordinates are independent of each other, so our answer will be the product of the number of possible $x$ values and the number of possible $y$ values. Each movement either increases or decreases the coordinate it affects by one. We have no choice about Cameron's first movement, since he starts facing west and moves to $(-1, 0)$. Thus we have 24 $x$-coordinate movements and 25 $y$-coordinate movements to consider. Observe that after 24 $x$-coordinate movements, Cameron's $x$-coordinate must be odd, and must be between $-25$ and 23. This yields $\frac{23-(-25)}{2} + 1 = 25$ possible $x$-coordinates. After 25 $y$-coordinate movements, Cameron's $y$-coordinate must be odd as well, and must be between $-25$ and 25. Thus Cameron can end up at $\frac{25-(-25)}{2} + 1 = 26$ different $y$-coordinates. Multiplying these two values gives us $25 \cdot 26 = 650$ as our answer.

23. Let $a, b$, and $c$ be distinct nonzero base ten digits. Assume there exist integers $x$ and $y$ for which $\overline{abc} \cdot \overline{cb} = 100x^2 + 1$ and $\overline{acb} \cdot \overline{bc} = 100y^2 + 1$. What is the minimum value of the number $\overline{abbc}$?

    Note: The notation $\overline{pqr}$ designates the number whose hundreds digit is $p$, tens digit is $q$, and units digit is $r$, not the product $p \cdot q \cdot r$.

    **Solution.** The answer is $\boxed{1337}$.

    Note that both $\overline{abc} \cdot \overline{cb}$ and $\overline{acb} \cdot \overline{bc}$ have 01 as their last two digits, which means $bc$ ends in 1. This can only happen in four ways, namely $(b, c) = (1, 1), (3, 7), (7, 3)$, and $(9, 9)$.

    If $(b, c) = (1, 1)$, then the last two digits of both products would be the same as those of $\overline{bc} \times \overline{cb} = 11 \times 11 = 121$, which means both products end with 21, not 01.

    For $(b, c) = (3, 7)$ and $(b, c) = (7, 3)$, a valid value of $a$ for one case is also valid for the other case. Clearly $\overline{a337} < \overline{a773}$, so we only consider the case where $b = 3$ and $c = 7$. We have $37 \cdot 73 = 2701$, so $x^2 = 73a + 27$ and $y^2 = 37a + 27$. Trying $a = 1$ yields $x = 10$ and $y = 8$, and no smaller value of $a$ is allowed, so the smallest value of $\overline{abbc}$ in this case is 1337.

    If $(b, c) = (9, 9)$, then $\overline{abbc} \geq 1999 > 1337$ because $a \geq 1$. So we do not need to consider this case, as it cannot achieve the minimum.

    Therefore, the answer is 1337.

24. Let $r_1, r_2, r_3, r_4$ and $r_5$ be the five roots of the equation $x^5 - 4x^4 + 3x^2 - 2x + 1 = 0$. What is the product of $(r_1 + r_2 + r_3 + r_4), (r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_5), (r_1 + r_3 + r_4 + r_5)$, and $(r_2 + r_3 + r_4 + r_5)$?

    **Solution.** The answer is $\boxed{41}$.

    Let $s = r_1 + r_2 + r_3 + r_4 + r_5$ and $f(x) = x^5 - 4x^4 + 3x^2 - 2x + 1$. Note that $f(x)$ can also be written as $(x - r_1)(x - r_2)(x - r_3)(x - r_4)(x - r_5)$, whose expansion has an $x^4$ term with coefficient $-(r_1 + r_2 + r_3 + r_4 + r_5) = -s$. Therefore, we must have $-s = -4$, or $s = 4$.

    Now, we can rewrite the desired product of the quantities as

$$\begin{aligned}(s - r_1)(s - r_2)(s - r_3)(s - r_4)(s - r_5) &= f(s) \\ &= f(4) \\ &= 4^5 - 4 \cdot 4^4 + 3 \cdot 4^2 - 2 \cdot 4 + 1 \\ &= 41.\end{aligned}$$

25. Chad needs seven apples to make an apple strudel for Jordan. He is currently at 0 on the metric number line. Every minute, he randomly moves one meter in either the positive or the negative direction with equal probability. Arctica's parents are located at $+4$ and $-4$ on the number line. They will bite Chad for kidnapping Arctica if he walks onto those numbers. Also, there is one apple located at each integer between $-3$ and 3, inclusive. Whenever Chad lands on an integer with an unpicked apple, he picks it. What is the probability that Chad picks all the apples without getting bitten by Arctica's parents?

    **Solution.** The answer is $\boxed{\frac{1}{7}}$.

    At some point in time, Chad will reach the apple at $-3$ or 3 (as he can only get bitten after passing one of these points). By symmetry, we may assume that he reaches the apple at $-3$ first. Then, we may rephrase the probability as that of Chad reaching either 3 before reaching $-4$. Shifting the numbers by 4 to the right, this could be rephrased as Chad starting at the number 1 and needing to reach 7 before 0.

We use the process of mathematical induction to show that the probability of Chad reaching $n$ before 0 when starting at 1 is precisely $\frac{1}{n}$. This statement is clearly true for $n = 2$. Assume the probability that Chad reaches $n - 1$ before 0 when starting at 1 is $\frac{1}{n-1}$. Let the probability that he reaches $n$ before 0 be $p$. Since he must first reach $n - 1$ to ever reach $n$, we have that $p = \frac{1}{n-1} \cdot (1 - p)$, where $1 - p$ is the probability that Chad reaches $n$ before 0 when starting at $n - 1$, as the situation is exactly reversed. Solving, we arrive at the conclusion that $p = \frac{1}{n}$, completing the induction.

Thus, the answer is $\frac{1}{7}$.

**Alternate solution:** Again, we can reduce the problem to finding the probability that, starting at 1, Chad reaches 7 before 0. Note that because moving in either direction is equally likely, the expected value of Chad's position is 1 at all times. In particular, when Chad first steps on one of 0 and 7, his expected position is again 1. If his actual position is 7 with probability $p$, then his expected position can also be written as $7 \cdot +0 \cdot (1 - p) = 1$, so $p = \frac{1}{7}$.

## 2.2 Accuracy Test Solutions

1. Chad lives on the third floor of an apartment building with ten floors. He leaves his room and goes up two floors, goes down four floors, goes back up five floors, and finally goes down one floor, where he finds Jordan's room. On which floor does Jordan live?

   **Solution.** The answer is $\boxed{5}$.

   Chad finds Jordan's room at floor number $3 + 2 - 4 + 5 - 1 = 5$.

2. A real number $x$ satisfies the equation $2014x + 1337 = 1337x + 2014$. What is $x$?

   **Solution.** The answer is $\boxed{1}$.

   Rearranging the terms yields

$$\begin{aligned} 2014x - 1337x &= 2014 - 1337 \\ 677x &= 677 \\ x &= 1. \end{aligned}$$

   **Alternate solution:** Note that the given equation is linear, so it must have at most one solution. Rewriting the equation as

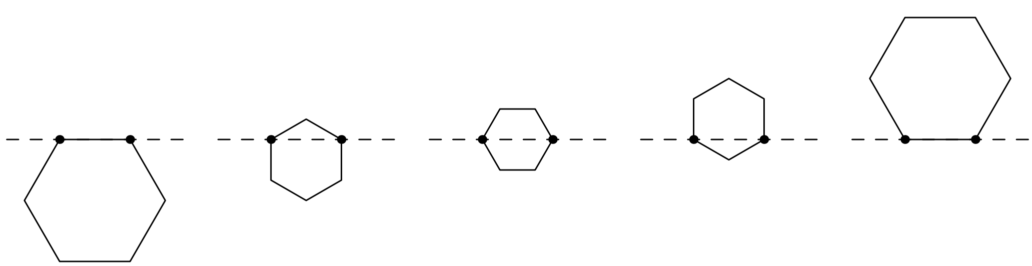
$$2014(x - 1) = 1337(x - 1),$$

   it is clear that $x = 1$ is a solution, and it must be the only solution.

3. Given two points on the plane, how many distinct regular hexagons include both of these points as vertices?

   **Solution.** The answer is $\boxed{5}$.

   Consider the line going through the two points. This line could split the other four points of the hexagon so that there are 0, 1, 2, 3, or 4 points above the line. Each way yields one distinct hexagon, as shown below.



   There are five ways in which the line can split the other four points so there are five possible hexagons.

4. Jordan has six different files on her computer and needs to email them to Chad. The sizes of these files are 768, 1024, 2304, 2560, 4096, and 7680 kilobytes. Unfortunately, the email server holds a limit of $S$ kilobytes on the total size of the attachments per email, where $S$ is a positive integer. It is additionally given that all of the files are indivisible. What is the maximum value of $S$ for which it will take Jordan at least three emails to transmit all six files to Chad?

**Solution.** The answer is $\boxed{9471}$.

Noticing that all the file sizes are multiples of 256 kilobytes, we define one chadbyte to be 256 kilobytes so that the files have sizes 3, 4, 9, 10, 16, and 30 chadbytes. Thus, the total size of all files is $3+4+9+10+16+30 = 72$ chadbytes.

Note that $S$ must be smaller than the minimum email size limit for which two emails can send all files. This minimum limit must therefore be at least 36 chadbytes, because otherwise, two completely full emails would not be able send all 72 chadbytes. If two emails of size 36 chadbytes could send all of the files, the sizes of the other files on the same email as the 30 chadbyte file would sum to 6 chadbytes, which is clearly impossible. All file sizes have an integral number of chadbytes, so we next check if two emails with sizes at most 37 chadbytes can send all files. Indeed, sending the files of sizes 3, 4, and 30 chadbytes in one email and the rest in the other, achieves this.

Therefore, the email size limit must be smaller than 37 chadbytes to require three emails. Converting back to kilobytes, this means $S < 37 \cdot 256 = 9472$ kilobytes. Because $S$ is an integer, the maximum possible value of $S$ is 9471 kilobytes.

5. If real numbers $x$ and $y$ satisfy $(x+2y)^2 + 4(x+2y+2-xy) = 0$, what is $x+2y$?

   **Solution.** The answer is $\boxed{-4}$.

   Expanding and rearranging the equation yields

$$\begin{aligned}\left(x^2+4xy+4y^2\right)+(4x+8y+8-4xy)&=0\\ \left(x^2+4x+4\right)+4\left(y^2+2y+1\right)&=0\\ (x+2)^2+4\,(y+1)^2&=0\end{aligned}$$

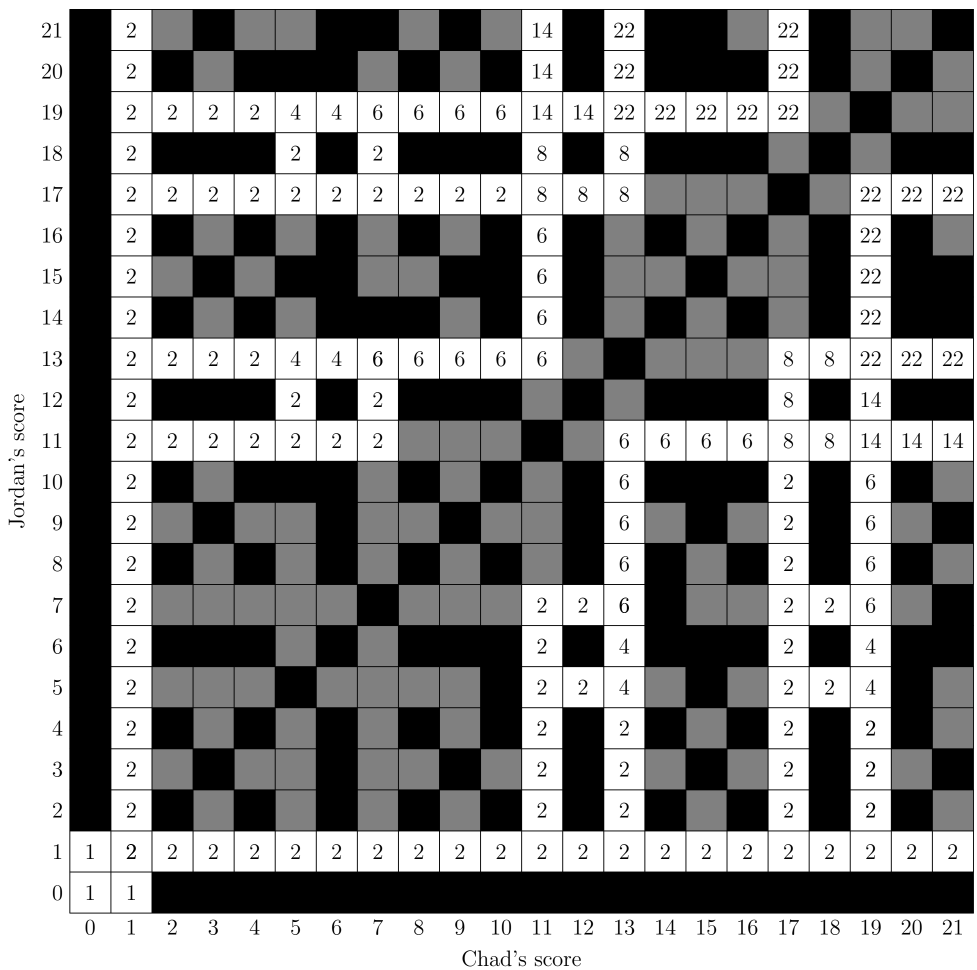
   Squares of real numbers are nonnegative, so the only way the equation can be satisfied is if $(x+2)^2 = 0$ and $(y+1)^2 = 0$. Therefore, $x = -2$ and $y = -1$, so $x + 2y = -4$.

6. While playing table tennis against Jordan, Chad came up with a new way of scoring. After the first point, the score is regarded as a ratio. Whenever possible, the ratio is reduced to its simplest form. For example, if Chad scores the first two points of the game, the score is reduced from 2:0 to 1:0. If later in the game Chad has 5 points and Jordan has 9, and Chad scores a point, the score is automatically reduced from 6:9 to 2:3. Chad's next point would tie the game at 1:1. Like normal table tennis, a player wins if he or she is the first to obtain 21 points. However, he or she does not win if after his or her receipt of the 21$^{\text{st}}$ point, the score is immediately reduced. Chad and Jordan start at 0:0 and finish the game using this rule, after which Jordan notes a curiosity: the score was never reduced. How many possible games could they have played? Two games are considered the same if and only if they include the exact same sequence of scoring.

   **Solution.** The answer is $\boxed{120}$.

   We create a 22 by 22 grid in which the rows and columns are each numbered from 0 to 21, and each cell represents the score of the game. So, for example, the cell in column 11 and row 5 represents Chad having 11 points and Jordan having 5. Thus, a game without any score reductions is represented by a path from the cell in row 0 and column 0 to any cell in row 21 or column 21 that does not pass through a cell in which the score is reduced. Note that one point is added to one person's score each rally, so moves along the grid consist of either one step in the positive row direction or one step in the positive column direction.

We color black the cells at which the score would be reduced, so that a path may not pass through a black square. We also color grey those cells which are not reachable from the starting score of 0:0, or do not have a valid path to either row 21 or column 21. The remaining squares are colored white. The diagram below shows the resulting grid:




In addition to the colors, we write in each white cell the number of ways in which that score could be reached. We compute these numbers by starting with 1 in the cell in row 0 and column 0. Then, for each empty white cell we write in it the sum of the numbers below it and to its left, if those cells contain numbers. This gives the desired number because a score can only be reached by either increasing Chad's score by one or increasing Jordan's score by one. So, a 1 would be written for the scores 0:1 and 1:0, and then a 2 for the score 1:1, and so on.

Finally, we count the total number of valid games by adding the number of ways one of the players' scores could reach 21. That is, we sum the numbers in row 21 and column 21, which yields $2 \cdot (2 + 14 + 22 + 22) = 120$.

7. For a positive integer $m$, we define $m$ as a *factorial number* if and only if there exists a positive integer $k$ for which $m = k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$. We define a positive integer $n$ as a *Thai number* if and only if $n$ can be written as both the sum of two factorial numbers and the product of two factorial numbers. What is the sum of the five smallest Thai numbers?

   **Solution.** The answer is $\boxed{210}$.

   Notice that 2 is a factorial number, so all numbers of the form $2m$, where $m$ is a factorial number, are Thai numbers: $2m$ is the sum of $m$ and $m$ as well as the product of 2 and $m$. The first few factorial numbers are 1, 2, 6, 24, and 120, from which we can find five Thai numbers: 2, 4, 12, 48, and 240.

   However, we must check if these are the smallest five. We can do this by examining the possible products of two factorial numbers less than 240, and determine if two factorial numbers sum to each product. The products in which the smaller factorial number is 2 were already examined above. Note that no two factorial numbers sum to another factorial number (other than $1 + 1 = 2$, which was accounted for above), so smaller factorial number in the product cannot be 1. The smaller number also cannot be larger than 6, because then the product would be at least $24 \cdot 24 > 240$. If the smaller number is 6, we must check $6 \cdot 6 = 36$, which is not a sum of two factorial numbers, and $6 \cdot 24 = 144$, which is $120 + 24$. Hence, 144 is also a Thai number.

   The smallest five Thai numbers are 2, 4, 12, 48, and 144, which sum to 210.

8. Chad and Jordan are in the Exeter Space Station, which is a triangular prism with equilateral bases. Its height has length one decameter and its base has side lengths of three decameters. To protect their station against micrometeorites, they install a force field that contains all points that are within one decameter of any point of the surface of the station. What is the volume of the set of points within the force field and outside the station, in cubic decameters?

   **Solution.** The answer is $\boxed{\frac{9\sqrt{3}}{2} + 9 + \frac{41\pi}{6}}$.

   We can split the shape between the Exeter Space Station and the force field into five types of regions:

   - Three rectangular prisms along the three rectangular faces of the Exeter Space Station, with a height of 1, length of 3, and width of 1, each have volume $1 \cdot 3 \cdot 1 = 3$, for a total volume of 9 for this type.
   - Along each of the two triangular faces of the station is a region of the force field in the shape of a triangular prism with height 1 and base side length 3. The total volume for the two regions of this type is $2 \cdot \left(\frac{3^2\sqrt{3}}{4} \cdot 1\right) = \frac{9\sqrt{3}}{2}$.
   - The third type of region consists of six shapes that are quarter sectors of a cylinder with height 3 and radius 1, formed along the edges of the Exeter Space Station that have length 3. This type of region has a total volume of $6 \cdot \left(3 \cdot \frac{\pi}{4}\right) = \frac{9\pi}{2}$.
   - The fourth set of regions consists of three shapes that are each $\frac{360^\circ - 90^\circ - 90^\circ - 60^\circ}{360^\circ} = \frac{120^\circ}{360^\circ} = \frac{1}{3}$ of a cylinder with height 1 and radius 1, found along the edges of the Exeter Space Station with length 1. This type of region has a total volume of $3 \cdot \frac{\pi}{3} = \pi$.
   - The fifth category consists of six regions that have the shape of a $120^\circ$ slice of a sphere cut in half perpendicular to the the exposed diameter, and are located near the six corners of the station. Together, these regions have volume $6 \cdot \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{4\pi}{3}\right) = \frac{4}{3}\pi$.

Summing these values, the total desired volume is $9+\frac{9\sqrt{3}}{2}+\frac{9\pi}{2}+\pi+\frac{4\pi}{3}=\frac{9\sqrt{3}}{2}+9+\frac{41\pi}{6}$.

## 2.3 Team Test Solutions

1. What is the units digit of the product of the first seven primes?

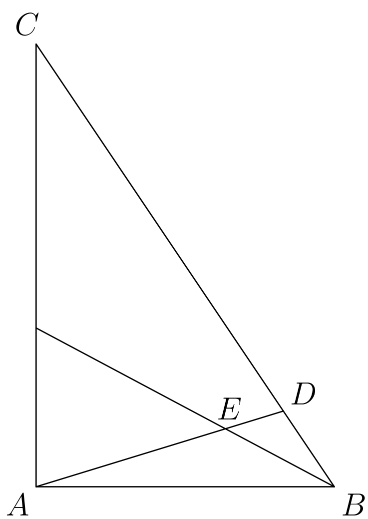
   **Solution.** The answer is $\boxed{0}$.

   We know that the first three primes are 2, 3, 5, so the product of the first seven primes is divisible by 2 and 5, and therefore is a multiple of 10. Hence, the units digit must be 0.

2. In triangle $ABC$, $\angle BAC$ is a right angle and $\angle ACB$ measures 34 degrees. Let $D$ be a point on segment $BC$ for which $AC = CD$, and let the angle bisector of $\angle CBA$ intersect line $AD$ at $E$. What is the measure of $\angle BED$?

   **Solution.** The answer is $\boxed{45^\circ}$.

   The diagram is as follows:




   Note that $AC = CD$ means $\triangle ACD$ is isosceles, so $\angle ADC = \frac{180^\circ - \angle ACD}{2} = \frac{180^\circ - 34^\circ}{2} = 73^\circ$. Therefore, $\angle BDE = 180^\circ - \angle ADC = 107^\circ$. Also, because $BE$ bisects $\angle ABC$, we have $\angle DBE = \frac{90^\circ - \angle ACB}{2} = 28^\circ$. Finally, considering $\triangle BED$, we have $\angle BED = 180^\circ - \angle BDE - \angle DBE = 180^\circ - 107^\circ - 28^\circ = 45^\circ$.

3. Chad numbers five paper cards on one side with each of the numbers from 1 through 5. The cards are then turned over and placed in a box. Jordan takes the five cards out in random order and again numbers them from 1 through 5 on the other side. When Chad returns to look at the cards, he deduces with great difficulty that the probability that exactly two of the cards have the same number on both sides is $p$. What is $p$?

   **Solution.** The answer is $\boxed{\frac{1}{6}}$.

   There are $\binom{5}{2} = 10$ possibilities for the set two numbers that are written on the pair of cards that have the same number on both sides. For each of those possibilities, we have 3 mismatched numbers $a$, $b$, and $c$ written on the front of three cards with 2 ways to write numbers on the backs, in order: $b$,

$c$, and $a$; or $c$, $a$, and $b$. In total, there are $5! = 120$ ways to write the five numbers on the back of the cards, so the probability is $p = \frac{10\cdot 2}{120} = \frac{1}{6}$.

4. Only one real value of $x$ satisfies the equation $kx^2 + (k+5)x + 5 = 0$. What is the product of all possible values of $k$?

   **Solution.** The answer is $\boxed{0}$.

   If $k = 0$, the equation becomes $5x + 5 = 0$, which clearly has exactly one solution, $x = -1$. Because 0 is among the possible values of $k$, the desired product is 0.

   While it doesn't affect the answer, we can find all other possible values for $k \neq 0$ by noting that for the given equation to have one solution, the discriminant of the quadratic, $(k+5)^2 - 4(k)(5) = (k-5)^2$, must be 0. Therefore, $k = 5$ is the only other possible value of $k$.

5. On the Exeter Space Station, where there is no effective gravity, Chad has a geometric model consisting of 125 wood cubes measuring 1 centimeter on each edge arranged in a 5 by 5 by 5 cube. An aspiring carpenter, he practices his trade by drawing the projection of the model from three views: front, top, and side. Then, he removes some of the original 125 cubes and redraws the three projections of the model. He observes that his three drawings after removing some cubes are identical to the initial three. What is the maximum number of cubes that he could have removed? (Keep in mind that the cubes could be suspended without support.)

   **Solution.** The answer is $\boxed{100}$.

   Note that in each view, 25 cubes are observed, so there must be at least 25 cubes in the model with some cubes removed. We can show that 25 is indeed the minimum number of cubes required, which would make the number of cubes removed and the answer 100, by constructing a valid model with 25 cubes, as follows.

   Group the 125 original cubes into five layers of 5 by 5 squares, and number the layers from 1 to 5, so that only one layers is visible when the original cube is viewed from the top. Then, remove all cubes except those specified in the diagram below:

| | | | | |
|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 1 |
| 3 | 4 | 5 | 1 | 2 |
| 4 | 5 | 1 | 2 | 3 |
| 5 | 1 | 2 | 3 | 4 |
| 1 | 2 | 3 | 4 | 5 |

   In the grid, each cell represents one cube. The number is the layers number of the cell and the position in the grid is the same as the position of the cube as viewed from the top.

   This construction is clearly valid for the top view, as 25 cubes are visible. For the other views, we must check that there is one of each number (which represents row number in the 5 by 5 by 5 arrangement) in each row and column of the grid. This is indeed true, so our construction is valid.

**Alternate solution:** By the same reasoning as the previous solution, we must have at least 25 cubes, but we provide a different description of a valid construction for removing 100 cubes.

Align the original 5 by 5 by 5 cube on a three-dimensional coordinate grid so that there is a cube at each point $(x, y, z)$ for all $x$, $y$, and $z$ between 0 and 4, inclusive. Then, remove all cubes except those for which the sum of their coordinates is divisible by 5. For each of the three views to have 25 cubes, there must be a cube located at one of the five points with two fixed coordinates. For example, there must be a cube at a point $(1, 3, z)$, for some $z$, if a cube is observed in the corresponding spot when viewed parallel to the $z$-axis. Our construction satisfies this because for any given $x$ and $y$, exactly one of $(x, y, -x-y)$, $(x, y, 5-x-y)$, and $(x, y, 10-x-y)$ is kept. The same is true for the other two views, so this construction satisfies the conditions. There will be 25 cubes remaining because for each choice of $x$ and $y$, there is a unique $z$ for which $5|x+y+z$.

6. Eric, Meena, and Cameron are studying the famous equation $E = mc^2$. To memorize this formula, they decide to play a game. Eric and Meena each randomly think of an integer between 1 and 50, inclusively, and substitute their numbers for $E$ and $m$ in the equation. Then, Cameron solves for the absolute value of $c$. What is the probability that Cameron's result is a rational number?

**Solution.** The answer is $\boxed{\frac{69}{1250}}$.

There exist integers $p$, $q$, $r$, and $s$ such that $E = p \cdot q^2$ and $m = r \cdot s^2$ and $p$ and $r$ have no perfect square divisors other than 1. Then, we have

$$c = \sqrt{\frac{E}{m}} = \frac{\sqrt{p \cdot q^2}}{\sqrt{r \cdot s^2}} = \frac{q}{s} \cdot \sqrt{\frac{p}{r}},$$

so $c$ is rational if and only if $\sqrt{\frac{p}{r}}$ is rational. Suppose $\sqrt{\frac{p}{r}} = \frac{a}{b}$, where $a$ and $b$ are relatively prime integers. Therefore,

$$pb^2 = ra^2,$$

so $a^2 \mid p$ and $b^2 \mid r$. Thus, if $c$ is rational we must have $a = b = 1$, which means $p = r$.

Now, for each possible value $p$, any choice for $E$ and $m$ from the numbers between 1 and 50 that are exactly $p$ times a perfect square produces a rational $c$. We proceed with casework on the value of $p$:

- If $p = 1$, we choose $E$ and $m$ from 1, 4, 9, 16, 25, 36, and 49, which we can do in $7^2 = 49$ ways.
- If $p = 2$, we can choose $E$ and $m$ from 2, 8, 18, 32, and 50, in $5^2 = 25$ ways.
- If $p = 3$, we can choose $E$ and $m$ from 3, 12, 27, and 48 in $4^2 = 16$ ways.
- If $p = 5$, we can choose $E$ and $m$ from 5, 20, and 45 in $3^2 = 9$ ways.
- For $p = 6$, 7, 10, and 11, we can choose $E$ and $m$ from $p$ and $4p$ in $2^2 = 4$ ways for each $p$.
- For each of the 23 values of $p$ between 12 and 50 that do not have perfect square divisors, there is one choice for each of $E$ and $m$, namely $p$.

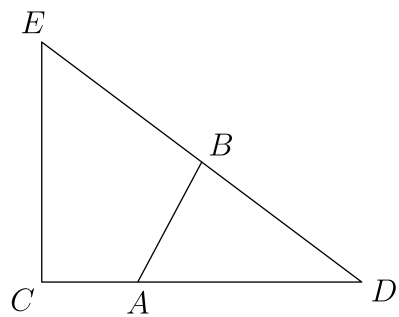
In total, there are $49+25+16+9+4\cdot4+23\cdot1 = 138$ possibilities for $E$ and $m$ so that $c$ is rational. Also, $50^2 = 2500$ ways exist to assign values to $E$ and $m$ without considering $c$, so the desired probability is $\frac{138}{2500} = \frac{69}{1250}$.

7. Let $CDE$ be a triangle with side lengths $EC = 3$, $CD = 4$, and $DE = 5$. Suppose that points $A$ and $B$ are on the perimeter of the triangle such that line $AB$ divides the triangle into two polygons of equal area and perimeter. What are all the possible values of the length of segment $AB$?

**Solution.** The answer is $\boxed{2\sqrt{3}}$.

Note that there are three possible configurations for $AB$, depending on which side contains neither $A$ nor $B$. We consider each separately.

Suppose $CE$ does not contain either endpoint of $AB$. Then the diagram looks like the following:




Let $DA = p$ and $DB = q$. We have

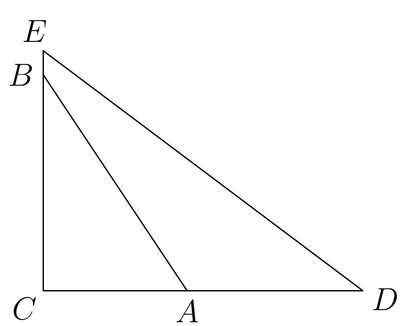
$$[DAB] = \frac{p}{4} \cdot [CBD] = \frac{q}{5} \cdot \frac{p}{4} \cdot [CDE] = \frac{pq}{20} \cdot 2[DAB],$$

or $pq = 10$. Also, due to the perimeter condition, we have

$$p + q + AB = AB + BE + EC + CA = AB + (12 - p - q),$$

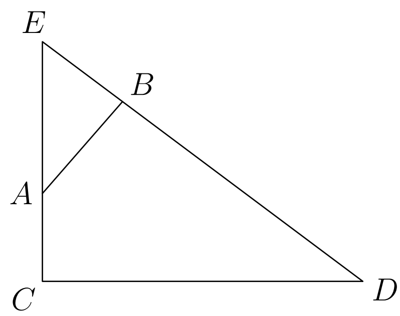
so $p+q = 6$. Substituting for $q$, this yields $p(6-p) = 10$, or $(p-3)^2+1 = 0$, which has no real solutions.

Now consider the configuration as depicted in the diagram below:




Let $CA = r$ and $CB = s$. Using a similar argument as above, we can derive the equations $r + s = 6$ and $rs = \frac{3 \cdot 4}{2} = 6$. Solving for $r$ and $s$ yields $(r, s) = (3 + \sqrt{3}, 3 - \sqrt{3}), (3 - \sqrt{3}, 3 + \sqrt{3})$. However, $3 + \sqrt{3} > 4 > 3$, so in either case either $A$ or $B$ lies outside the perimeter of $\triangle CDE$, so this configuration does not produce a valid $AB$.

Finally, consider the third possible arrangement:




Let $BE = x$ and $CE = y$. Again, we can find that $x + y = 6$ and $xy = \frac{15}{2}$. Solving yields $\{x, y\} = \left\{3 + \frac{\sqrt{6}}{2}, 3 - \frac{\sqrt{6}}{2}\right\}$ (either order for $x$ and $y$ is valid). By the Law of Cosines, we find that

$$\begin{aligned} AB &= \sqrt{\left(3+\frac{\sqrt{6}}{2}\right)^2 + \left(3-\frac{\sqrt{6}}{2}\right)^2 - 2\left(3+\frac{\sqrt{6}}{2}\right)\left(3-\frac{\sqrt{6}}{2}\right)\cos\angle CED} \\ &= \sqrt{2\cdot 3^2 + 2\cdot\left(\frac{\sqrt{6}}{2}\right)^2 - 2\left(9-\frac{6}{4}\right)\frac{3}{5}} \\ &= \sqrt{21-9} \\ &= 2\sqrt{3}, \end{aligned}$$

which is the only possible value of $AB$.

8. Chad and Jordan are raising bacteria as pets. They start out with one bacterium in a Petri dish. Every minute, each existing bacterium turns into 0, 1, 2 or 3 bacteria, with equal probability for each of the four outcomes. What is the probability that the colony of bacteria will eventually die out?

**Solution.** The answer is $\boxed{\sqrt{2}-1}$.

Suppose that the probability of a colony eventually dying out is $x$. Then for a colony with $n$ bacteria, the probability of eventually dying out is $x^n$ because we can consider each single bacterium as a separate colony. Comparing bacteria before and after the first minute, we have

$$\begin{aligned} x &= \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot x + \frac{1}{4}\cdot x^2 + \frac{1}{4}\cdot x^3 \\ x^3 + x^2 - 3x + 1 &= 0 \\ (x-1)(x^2+2x-1) &= 0 \\ x &= 1, -1-\sqrt{2}, \text{ or } -1+\sqrt{2} \end{aligned}$$

We claim that $x$ cannot be 1, which would mean that all colonies eventually die out. The number of bacteria in the colony is, on average, multiplied by $\frac{0+1+2+3}{4} = 1.5$ every minute, which means in general the bacteria do not die out. (A more rigorous line of reasoning is included below.) Because $x$ is not negative, the only valid solution is $x = \sqrt{2} - 1$.

To show that $x$ cannot be 1, we show that it is at most $\sqrt{2}-1$. Let $x_n$ be the probability that a colony of one bacteria will die out after at most $n$ minutes. Then, we have the relation

$$x_{n+1} = \frac{1}{4}\left(1 + x_n + x_n^2 + x_n^3\right).$$

We claim that $x_n \le \sqrt{2}-1$ for all $n$, which we will prove using induction.

It is clear that $x_1 = \frac{1}{4} \le \sqrt{2}-1$. Now, assume $x_k \le \sqrt{2}-1$ for some $k$. We have

$$\begin{aligned} x_{k+1} &\le \frac{1}{4}\left(1 + x_k + x_k^2 + x_k^3\right) \\ &\le \frac{1}{4}\left(1 + \left(\sqrt{2}-1\right) + \left(\sqrt{2}-1\right)^2 + \left(\sqrt{2}-1\right)^3\right) \\ &= \sqrt{2}-1, \end{aligned}$$

which completes the proof that $x_n \le \sqrt{2}-1$ for all $n$.

Now, we note that as $n$ becomes large, $x_n$ approaches $x$. Using formal notation, this is $x = \lim_{n\to\infty} x_n \le \sqrt{2}-1$, so $x$ cannot be 1.

9. Let $a = w+x, b = w+y, c = x+y, d = w+z, e = x+z$, and $f = y+z$. Given that $af = be = cd$ and

$$(x-y)(x-z)(x-w) + (y-x)(y-z)(y-w) + (z-x)(z-y)(z-w) + (w-x)(w-y)(w-z) = 1,$$

what is

$$2(a^2+b^2+c^2+d^2+e^2+f^2) - ab - ac - ad - ae - bc - bd - bf - ce - cf - de - df - ef?$$

**Solution.** The answer is $\boxed{3}$.

Let $af = be = cd = p$ and $a+f = b+e = c+d = q$. Then $a$ and $f$ are distinct roots of the polynomial $t^2 - qt + p = 0$. Similarly, so are $b$ and $e$, as well as $c$ and $d$. Therefore, $\{a,f\} = \{b,e\} = \{c,d\}$.

As $x$, $y$, $z$, and $w$ are symmetric, we may assume without loss of generality that $x \le y \le z \le w$. By the second equation, $x$, $y$, $z$, and $w$ are not all equal, so $x \ne w$. Then $c = x+y$ cannot equal $b = w+y$, so $c = e$, which means $x+y = x+z$, or $y = z$. Now, if $c = a$, we have $y = z = w$; otherwise $c = f$ and $x = y = z$. In the first case, the given equation yields $(x-y)^3 = 1$, implying that $x - y = 1$, contradicting $x \le y$. In the second case, $(w-x)^3 = 1$, implying that $w - x = 1$. Thus, we must have $x = y = z = w - 1$.

Now, the value we need to compute simplifies to

$$\begin{aligned} &\frac{1}{2}\cdot\big[(a-b)^2 + (a-c)^2 + (a-d)^2 + (a-e)^2 + (b-c)^2 + (b-d)^2 \\ &\quad + (b-f)^2 + (c-e)^2 + (c-f)^2 + (d-e)^2 + (d-f)^2 + (e-f)^2\big] \\ &\quad = (x-y)^2 + (x-z)^2 + (x-w)^2 + (y-z)^2 + (y-w)^2 + (z-w)^2 \\ &\quad = 3. \end{aligned}$$

10. If $a$ and $b$ are integers at least 2 for which $a^b - 1$ strictly divides $b^a - 1$, what is the minimum possible value of $ab$?

Note: If $x$ and $y$ are integers, we say that $x$ *strictly* divides $y$ if $x$ divides $y$ and $|x| \ne |y|$.

**Solution.** The answer is $\boxed{32}$.

We claim that the pair $(a, b) = (16, 2)$, with product 32, satisfies the above conditions, and that no pair with smaller product can satisfy those conditions.

If $(a, b) = (16, 2)$, we have $a^b - 1 = 16^2 - 1 = 2^8 - 1$, while $b^a - 1 = 2^{16} - 1 = (2^8 - 1)(2^8 + 1)$, so it is clear that $16^2 - 1$ strictly divides $2^{16} - 1$.

We prove that no pair with smaller product can satisfy the conditions.

If $3 \le a \le b$, we have $a^b - 1 \ge b^a - 1$, contrary to the strict divisibility condition. Therefore, either $a = 2$ or $a > b$. If $a = 2$, we again have $a^b - 1 \ge b^a - 1$ for $b \ge 4$, and we can check that $b = 2$ and $b = 3$ do not satisfy the conditions. Thus, we may limit our considerations to when $a > b$. In addition, $b^2 < ab < 32$, so $b < 6$.

When $b = 2$, we have that $a^2 - 1$ strictly divides $2^a - 1$. Note that $a$ must be even, because otherwise an even number would divide an odd number. It is easy to check that none of $a = 6$, 8, 10, 12, and 14 satisfies the conditions.

When $b = 3$, we can similarly check the values for $a$: The values $a = 4$, 5, 6, 7, 8, 9, and 10 do not satisfy the conditions.

When $b = 4$, we have that $a$ is even, but $a = 4$ does not satisfy the conditions.

When $b = 5$, we have that $a = 6$ does not satisfy the conditions.

We have accounted for all cases where $ab \le 31$, so we are done.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. What is $2 + 22 + 1 + 3 - 31 - 3$?

   **Solution.** The answer is $\boxed{-6}$.

   Canceling out two terms, we have $(2 + 22 + 1 - 31) + (3 - 3) = -6$.

2. Let $ABCD$ be a rhombus. Given $AB = 5$, $AC = 8$, and $BD = 6$, what is the perimeter of the rhombus?

   **Solution.** The answer is $\boxed{20}$.

   Because $ABCD$ is a rhombus, all four of its sides have the same length. We are given that one side, $AB$, has length 5, so the perimeter is $4 \cdot 5 = 20$.

3. There are 2 hats on a table. The first hat has 3 red marbles and 1 blue marble. The second hat has 2 red marbles and 4 blue marbles. Jordan picks one of the hats randomly, and then randomly chooses a marble from that hat. What is the probability that she chooses a blue marble?

   **Solution.** The answer is $\boxed{\frac{11}{24}}$.

   There is a $\frac{1}{2}$ probability that Jordan chooses the first hat. If she chooses the first hat, there is a $\frac{1}{4}$ probability that she chooses a blue marble. Otherwise, she can choose the second hat with probability $\frac{1}{2}$, in which case there is a $\frac{4}{6}$ probability that she chooses a blue marble. Therefore, the total probability that she chooses a blue marble is $\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{4}{6} = \frac{3}{24} + \frac{8}{24} = \frac{11}{24}$.

### 2.4.2 Round 2

4. There are twelve students seated around a circular table. Each of them has a slip of paper that they may choose to pass to either their clockwise or counterclockwise neighbor. After each person has transferred their slip of paper once, the teacher observes that no two students exchanged papers. In how many ways could the students have transferred their slips of paper?

   **Solution.** The answer is $\boxed{2}$.

   Suppose one of the students passes to the left. Then the student to that student's left must have also passed to the left, and then the student to this student's left must have also passed the left, and so on. Therefore, all students passed their slip to their left. The same applies if the original student passed the slip to the right, so in total there are 2 ways to pass the slips.

5. Chad wants to test David's mathematical ability by having him perform a series of arithmetic operations at lightning-speed. He starts with the number of cubic centimeters of silicon in his 3D printer, which is 109. He has David perform all of the following operations in series each second:

   - Double the number
   - Subtract 4 from the number

- Divide the number by 4
- Subtract 5 from the number
- Double the number
- Subtract 4 from the number

Chad instructs David to shout out after three seconds the result of three rounds of calculations. However, David computes too slowly and fails to give an answer in three seconds. What number should David have said to Chad?

**Solution.** The answer is $\boxed{61}$.

Suppose David starts the series of operations with a number $n$. After the six operations, the number is

$$2\left(\frac{2n-4}{4}-5\right)-4=n-16,$$

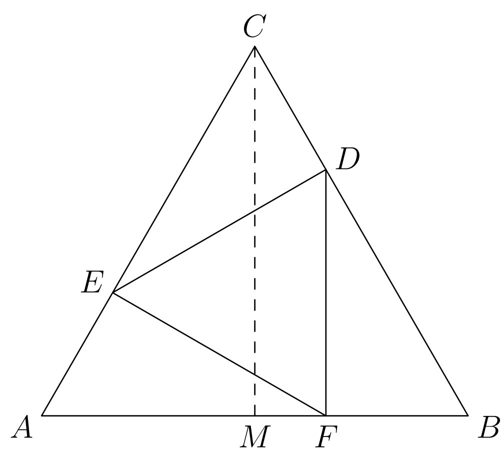
so the series of operations is equivalent to subtracting 16 from the number.

In each of the three rounds of calculations, 16 is subtracted from David's number. Thus, Davids number, originally 109, becomes $109-16\cdot 3=61$.

6. Points $D$, $E$, and $F$ lie on sides $BC$, $CA$, and $AB$ of triangle $ABC$, respectively, such that the following length conditions are true: $CD=AE=BF=2$ and $BD=CE=AF=4$. What is the area of triangle $ABC$?

**Solution.** The answer is $\boxed{9\sqrt{3}}$.

The diagram is as follows:




Adding $CD=AE=BF=2$ and $BD=CE=AF=4$, we have $BC=AC=AB=2+4=6$. This means that $\triangle ABC$ is an equilateral triangle with side length 6.

To find the area of $\triangle ABC$, we find the lengths of an altitude and its corresponding base. Let $M$ be the point on $AB$ such that $CM$ and $AB$ are perpendicular. Then, we have $\angle CMA=\angle CMB=90°$ and $\angle CAM=\angle CBM=60°$. Together with $CM=CM$, this shows that $\triangle CAM\cong\triangle CBM$, so $AM=BM=\frac{AB}{2}=3$. By the Pythagorean Theorem, we then have $CM=\sqrt{6^2-3^2}=3\sqrt{3}$. Thus, the area of $\triangle ABC$ is $\frac{AB\cdot CM}{2}=\frac{6\cdot 3\sqrt{3}}{2}=9\sqrt{3}$.

### 2.4.3 Round 3

7. In the *2, 3, 5, 7 game*, players count the positive integers, starting with 1 and increasing, which do not contain the digits 2, 3, 5, and 7, and also are not divisible by the numbers 2, 3, 5, and 7. What is the fifth number counted?

   **Solution.** The answer is $\boxed{61}$.

   Following the problem's conditions, we find that the first five numbers counted are 1, 11, 19, 41, and 61.

8. If $A$ is a real number for which
$$19 \cdot A = \frac{2014!}{1! \cdot 2! \cdot 2013!},$$
   what is $A$?

   Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

   **Solution.** The answer is $\boxed{53}$.

   We have
$$A = \frac{2014!}{2 \cdot 19 \cdot 2013!} = \frac{2014 \cdot 2013!}{2 \cdot 19 \cdot 2013!} = \frac{2014}{2 \cdot 19} = 53.$$

9. What is the smallest number that can be written as both $x^3 + y^2$ and $z^3 + w^2$ for positive integers $x$, $y$, $z$, and $w$ with $x \neq z$?

   **Solution.** The answer is $\boxed{17}$.

   We can see that $17 = 3^2 + 2^3 = 4^2 + 1^3$ satisfies the conditions. After checking that all numbers less than 17 cannot be expressed in this form in two ways, we conclude that 17 is the answer.

### 2.4.4 Round 4

*Each of the three problems in this round depends on the answer to one of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.*

*In addition, it is given that the answer to each of the following problems is a positive integer less than or equal to the problem number.*

10. Let $B$ be the answer to problem 11 and let $C$ be the answer to problem 12. What is the sum of a side length of a square with perimeter $B$ and a side length of a square with area $C$?

    **Solution.** The answer is $\boxed{5}$.

    The perimeter of a square is four times its side length, so the side length of the first square is $\frac{B}{4}$. In the second square, $C$ must be the square of the side length, so the side length is $\sqrt{C}$. Adding, the answer to this problem is $A = \frac{B}{4} + \sqrt{C}$.

    After determining $C$, the equations from problems 10 and 11 form a system of equations, which when solved yields $A = 5$.

11. Let $A$ be the answer to problem 10 and let $C$ be the answer to problem 12. What is $(C-1)(A+1)-(C+1)(A-1)$?

    **Solution.** The answer is $\boxed{8}$.

    If the answer to this problem is $B$, we have

$$\begin{aligned} B &= (C-1)(A+1)-(C+1)(A-1) \\ &= (CA-A+C-1)-(CA+A-C-1) \\ &= 2(C-A). \end{aligned}$$

    After determining $C$, the equations from problems 10 and 11 form a system of equations, which when solved yields $B = 8$.

12. Let $A$ be the answer to problem 10 and let $B$ be the answer to problem 11. Let $x$ denote the positive difference between $A$ and $B$. What is the sum of the digits of the positive integer $9x$?

    **Solution.** The answer is $\boxed{9}$.

    Using the divisibility rule for 9, we can conclude that because $9x$ is divisible by 9, the sum of the digits of $9x$ must also be divisible by 9. The only positive integer less than or equal to 12 that is divisible by 9 is 9, so the answer to this question is 9.

    The divisibility rule for 9 states that a number is divisible by 9 if and only if the sum of its digits is also divisible by 9. This is true because

$$\overline{a_n a_{n-1} \cdots a_1 a_0} = 10^n \cdot a_n + \cdots + 10 \cdot a_1 + a_0$$

    and $10^n, \cdots, 1$ all have remainder 1 when divided by 9, so we can rewrite the equation as

$$\overline{a_n a_{n-1} \cdots a_1 a_0} = 9\left(\frac{10^n - 1}{9} a_n + \frac{10^{n-1} - 1}{9} a_{n-1} + \cdots\right) + a_n + \cdots + a_0.$$

    From this it is clear that $\overline{a_n a_{n-1} \cdots a_1 a_0}$ and $a_n + \cdots + a_0$ have the same remainder when divided by 9.

### 2.4.5 Round 5

13. Five different schools are competing in a tournament where each pair of teams plays at most once. Four pairs of teams are randomly selected and play against each other. After these four matches, what is the probability that Chad's and Jordan's respective schools have played against each other, assuming that Chad and Jordan come from different schools?

    **Solution.** The answer is $\boxed{\frac{2}{5}}$.

    In this round robin, there are $\binom{5}{2} = 10$ possible matches. The probability that the first match is not between Jordan's and Chad's teams is $\frac{9}{10}$. Similarly, the probability that Chad doesn't play Jordan in the second round is $\frac{8}{9}$. Considering all four matches, the probability that the two schools that

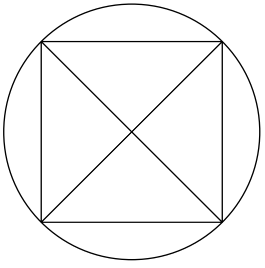
Chad and Jordan come from have not played against each other is $\frac{9}{10} \cdot \frac{8}{9} \cdot \frac{7}{8} \cdot \frac{6}{7} = \frac{6}{10} = \frac{3}{5}$. Therefore, the probability that they have played against each other is $1 - \frac{3}{5} = \frac{2}{5}$.

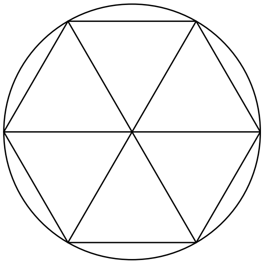
**Alternate solution:** There are a total of $\binom{5}{2} = 10$ pairs of teams, so there are a total of $\binom{10}{4} = 210$ possible sets of four pairs of teams. If Chad's and Jordan's respective schools play against each other, there will be a total of $\binom{9}{3} = 84$ choices for the other three pairs of teams, since Chad's and Jordan's schools can no longer play against each other. Therefore, the probability that they play against each other is $\frac{84}{210} = \frac{2}{5}$.

14. A square of side length 1 and a regular hexagon are both circumscribed by the same circle. What is the side length of the hexagon?

**Solution.** The answer is $\boxed{\frac{\sqrt{2}}{2}}$.

If we connect the center of the circle with all vertices on the square and the hexagon, we get 4 right isosceles triangles in the square and 6 equilateral triangles in the hexagon.





Because the side length of the square is 1, the radius of the circle would be $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Because the side length of the hexagon is equal to the radius, it would also be $\frac{\sqrt{2}}{2}$.

15. From the list of integers $1, 2, 3, \ldots, 30$, Jordan can pick at least one pair of distinct numbers such that none of the 28 other numbers are equal to the sum or the difference of this pair. Of all possible such pairs, Jordan chooses the pair with the least sum. Which two numbers does Jordan pick?

**Solution.** The answer is $\boxed{\text{11 and 22}}$.

Call the numbers Jordan picks $x$ and $y$ with $x > y$. Because $x - y$ is an integer between 1 and $x$, and no integers other than $y$ could be equal to this value, we must have $x - y = y$, or $x = 2y$. Also, $x + y > x > y$, so $x + y$ must be greater than 30; otherwise an integer in this set would be equal to this sum. So we also have $2y + y > 30$, or $y \geq 11$. Thus, the sum is $x + y = 2y + y \geq 33$, so the smallest possible sum is 33, which is produced by the pair $(x, y) = (22, 11)$.

### 2.4.6 Round 6

16. What is the sum of all two-digit integers with no digit greater than four whose squares also have no digit greater than four?

**Solution.** The answer is $\boxed{106}$.

The first digit of such a number cannot be greater than four, so it must be 1, 2, 3, or 4. Similarly, the last digit is limited to 0, 1, 2, 3, and 4. However, if the last digit is 3 or 4, then the last digit of the square is greater than four. Thus, we only have to check the 12 numbers with first digits 1, 2, 3, and 4, and last digits 0, 1, and 2. We find that six of these numbers have squares with no digits greater than four: $10^2 = 100$, $11^2 = 121$, $12^2 = 144$, $20^2 = 400$, $21^2 = 441$, and $32^2 = 1024$. The desired sum is $10 + 11 + 12 + 20 + 21 + 32 = 106$.

17. Chad marks off ten points on a circle. Then, Jordan draws five chords under the following constraints:

- Each of the ten points is on exactly one chord.
- No two chords intersect.
- There do not exist (potentially non-consecutive) points $A, B, C, D, E$, and $F$, in that order around the circle, for which $AB$, $CD$, and $EF$ are all drawn chords.

In how many ways can Jordan draw these chords?

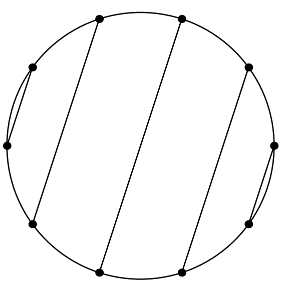
**Solution.** The answer is $\boxed{5}$.

Note that each chord must divide the other eight points of the circle into two groups each with an even number of points. Otherwise, there would be an unpaired point on one side of that chord that must be the endpoint of a different drawn chord that intersects that chord. Thus, each chord cuts off 0, 2, or 4 other points on the side with fewer points.

Consider the number of chords between pairs of adjacent points. If three or more such chords exist, they clearly contradict the third constraint. Thus, at most two of the chords connect adjacent points.

Now consider the chords that cut off 2 points. For each of these chords, the two points cut off are adjacent and must be connected by a chord. Therefore, we cannot have more than two chords cutting off 2 points, as then we will also have more than two chords cutting off 0 points, which we previously stated was contradictory to the constraints. As a result, there can only be at most 2 chords cutting off 2 points.

Because there are 5 chords in total, there must be at least one chord splitting the other eight points into two groups of four. This chord can be chosen in 5 ways. Once it is chosen, the four points on either side can either be connected by two chords each cutting off 0 points, or one chord cutting off 2 and one chord cutting off 0 points. However, connecting either group of four with two chords that join adjacent points produces three chords that connect adjacent points, as there must be at least one such chord on the other side of the longest chord. Thus, the two chords on either side have to cut off 2 and 0 points, respectively, and there is only one way to draw the five chords in each of the 5 ways to choose the first and longest chord.



18. Chad is thirsty. He has 109 cubic centimeters of silicon and a 3D printer with which he can print a cup to drink water in. He wants a silicon cup whose exterior is cubical, with five square faces and an open top, that can hold exactly 234 cubic centimeters of water when filled to the rim in a rectangular-box-shaped cavity. Using all of his silicon, he prints a such cup whose thickness is the same on the five faces. What is this thickness, in centimeters?

**Solution.** The answer is $\boxed{\frac{1}{2}}$.

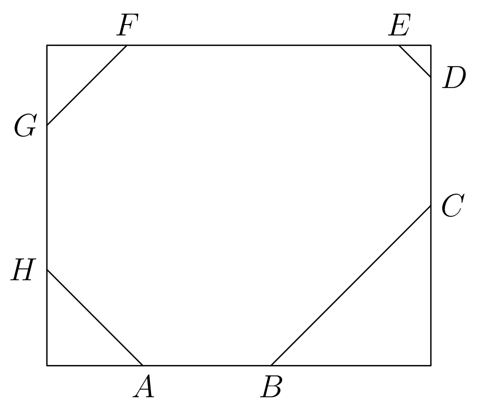
The total volume of the cup (including the cavity) is $109 + 234 = 343 = 7^3$, so the cup has an outer side length of 7 centimeters. Suppose the thickness of the cup is $x$ centimeters. Then, we have $(7-x)(7-2x)(7-2x) = 234 = 6.5 \cdot 6 \cdot 6$. Hence, $x = \frac{1}{2}$ is a solution, and our answer. Furthermore, we can see that the expression $(7-x)(7-2x)(7-2x)$ is strictly decreasing when $x$ is within the limits of the problem; that is, $0 \le x \le \frac{7}{2}$, so no other sensible solutions exist.

### 2.4.7 Round 7

19. Jordan wants to create an equiangular octagon whose side lengths are exactly the first 8 positive integers, so that each side has a different length. How many such octagons can Jordan create?

**Solution.** The answer is $\boxed{0}$.

We know that all interior angles of such an octagon would measure $135°$. Call the octagon $ABCDEFGH$ and extend sides $AB$, $CD$, $EF$, and $GH$ into lines, making a rectangle which consists of the octagon and 4 isosceles right triangles.




Suppose the lengths of $AB$, $BC$, $CD$, $DE$, $EF$, $FG$, $GH$, $HA$ are $a$, $b$, $c$, $d$, $e$, $f$, $g$, and $h$, respectively. These 8 sides all have integer lengths and none of them are equal. Consider the two sides of the rectangle that contain $AB$ and $EF$. These two sides have equal length, so

$$a + \frac{\sqrt{2}}{2}(h+b) = e + \frac{\sqrt{2}}{2}(d+f).$$

Since $a$, $h$, $b$, $e$, $d$, $f$ are all integers and $\frac{\sqrt{2}}{2}$ is irrational, we see that the two sides must have equal rational parts and equal irrational parts. Therefore we get $a = e$, which contradicts the fact that no two of the 8 sides are equal. Thus Jordan cannot create any such octagons.

20. There are two positive integers on the blackboard. Chad computes the sum of these two numbers and tells it to Jordan. Jordan then calculates the sum of the greatest common divisor and the least common multiple of the two numbers, and discovers that her result is exactly 3 times as large as the number Chad told her. What is the smallest possible sum that Chad could have said?

    **Solution.** The answer is $\boxed{12}$.

    Suppose the two integers on the board are $a$ and $b$, with $a \geq b$. Let their greatest common divisor be $d$. The two integers can then be expressed as $a = pd, b = qd$, where $p, q$ are relatively prime integers and $p \geq q$. So the least common multiple of $a$ and $b$ is $pqd$. Therefore, we have

$$\begin{aligned} pqd + d &= 3(pd + qd) \\ pq - 3p - 3q + 1 &= 0 \\ (p-3)(q-3) &= 8. \end{aligned}$$

    Because $p \geq q$ we have $p - 3 \geq q - 3$, so the possible solutions are $(p-3, q-3) = (8,1), (4,2), (-2,-4)$, and $(-1,-8)$. We see that the last two make $q$ negative, which is invalid. The other two yield $(p,q) = (11,4)$, and $(7,5)$. Since the sum of the two integers is $(p+q)d$, the minimum is achieved when $d = 1$ and $p + q$ is the smallest. We have $11 + 4 > 7 + 5 = 12$, so the smallest sum is 12.

21. Chad uses *yater* to measure distances, and knows the conversion factor from yaters to meters precisely. When Jordan asks Chad to convert yaters into meters, Chad only gives Jordan the result rounded to the nearest integer meters. At Jordan's request, Chad converts 5 yaters into 8 meters and 7 yaters into 12 meters. Given this information, how many possible numbers of meters could Jordan receive from Chad when requesting to convert 2014 yaters into meters?

    **Solution.** The answer is $\boxed{116}$.

    Suppose 1 yater is exactly $y$ meters long. From the information we know, we have:

$$7.5 \leq 5y < 8.5$$

$$11.5 \leq 7y < 12.5$$

    Simplifying, we have $\frac{3}{2} \leq y < \frac{17}{10}$ and $\frac{23}{14} \leq y < \frac{25}{14}$. Because $\frac{17}{10} < \frac{25}{14}$ and $\frac{3}{2} < \frac{23}{14}$, we can combine the inequalities:

$$\frac{23}{14} \leq y < \frac{17}{10}$$

    Therefore,

$$\frac{23}{14} \cdot 2014 \leq 2014y < \frac{17}{10} \cdot 2014.$$

    Now, $\frac{23}{14} \cdot 2014 = \frac{23 \cdot 1007}{7} = \frac{23161}{7} = 3308\frac{5}{7}$, and $\frac{17}{10} \cdot 2014 = \frac{34238}{10} = 3423.8 = 3423\frac{4}{5}$, so

$$3308.5 < \frac{23}{14} \cdot 2014 \leq 2014y < \frac{17}{10} \cdot 2014 < 3424.5.$$

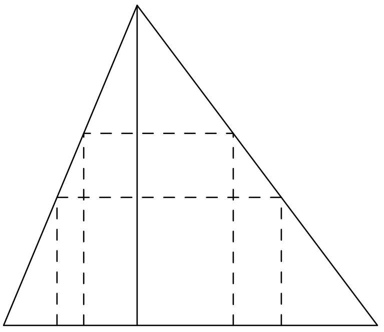
    We see that, when rounded to the nearest integer, $2014y$ can be any integer between 3309 and 3424, inclusive. Thus, the number of possible responses Chad could give is $3424 - 3309 + 1 = 116$.

### 2.4.8 Round 8

22. Jordan places a rectangle inside a triangle with side lengths 13, 14, and 15 so that the vertices of the rectangle all lie on sides of the triangle. What is the maximum possible area of Jordan's rectangle?

**Solution.** The answer is $\boxed{42}$.

The answer is 42. We claim that no matter which side of the triangle is shared with the rectangle, the maximum area of it is half the area of the triangle as long as the triangle is acute.



Suppose the base of the triangle is $b$ and the height is $h$. Let the height of the rectangle be $kh$, where $0 < k < 1$, so the small triangle above the rectangle has height $(1-k)h$, and, by similar triangles, the base of the small triangle, which coincides with the top side of the rectangle, is $(1-k)b$. it follows that the area of the rectangle is $k(1-k)bh$. We see that

$$k(1-k) = -k^2 + k = -(k - \frac{1}{2})^2 + \frac{1}{4} \leq \frac{1}{4}.$$

So the maximum area of the rectangle is $\frac{1}{4}bh$, which is half of $\frac{1}{2}bh$, the area of the triangle. In addition, this optimal area can be achieved by using $k = \frac{1}{2}$.

Now it suffices to find the area of the triangle. Draw an altitude to the side with length 14, which divides it into two parts with lengths $x$ and $(14-x)$, with $x$ adjacent to the side with length 13. Letting the altitude be $y$, we have

$$y^2 = 13^2 - x^2 = 15^2 - (14-x)^2.$$

Solving the equation between the second and third expressions, we have $x = 5$, and therefore $h = 12$. So, the area of the triangle is $\frac{12 \cdot 14}{2} = 84$, and the maximum area of the rectangle is $\frac{84}{2} = 42$.

23. Hoping to join Chad and Jordan in the Exeter Space Station, there are 2014 prospective astronauts of various nationalities. It is given that 1006 of the astronaut applicants are American and that there are a total of 64 countries represented among the applicants. The applicants are to group into 1007 pairs with no pair consisting of two applicants of the same nationality. Over all possible distributions of nationalities, what is the maximum number of possible ways to make the 1007 pairs of applicants? Express your answer in the form $a \cdot b!$, where $a$ and $b$ are positive integers and $a$ is not divisible by $b + 1$.

    Note: The expression $k!$ denotes the product $k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1$.

**Solution.** The answer is $\boxed{499968 \cdot 1006!}$.

Suppose the number of astronauts from countries other than America are $n_1, n_2, \ldots, n_{63}$. Clearly, the 1006 Americans must all be in different pairs, which leaves us 1 pair with two non-Americans, and 1006 pairs with one non-American. Note that once astronauts are assigned to the pair with two non-Americans, there are 1006! ways to assign astronauts to the remaining 1006 pairs, as there are 1006 ways to pair the first non-American with an American, 1005 ways to pair the second, and so on. Thus, we aim to maximize the number of ways to assign astronauts to the pair with two non-Americans.

We know that if the one of them is from the $i$-th nation, there would be $n_i \cdot (1008 - n_i) = 1008n_i + {n_i}^2$ ways to assign the pair. Adding this up for all 63 values of $i$ and dividing it by 2 (because each way is counted twice), we have

$$\frac{1008(n_1 + n_2 + \cdots + n_{63}) - ({n_1}^2 + {n_2}^2 + \cdots + {n_{63}}^2)}{2} = \frac{1008 \cdot 1008 - ({n_1}^2 + {n_2}^2 + \cdots + {n_{63}}^2)}{2}$$

ways to do so. It remains to minimize the sum of the squares of $n_i$. In order to minimize this, we evenly distribute 1008 to all nationalities (a difference of 1 is allowed in case even distribution is impossible). This is optimal because if we have two integers $a$ and $b$ among $n_i$ with $a - b > 1$, we can replace them with $a - 1$ and $b + 1$, keeping the same sum but smaller sum of squares, because

$$(a-1)^2 + (b+1)^2 = (a^2 + b^2) + 2(b - a) + 2 > a^2 + b^2 - 2 + 2 = a^2 + b^2.$$

Because $\frac{1008}{63} = 16$, the maximum number of ways to assign astronauts to the pair with two non-Americans is

$$\frac{1008 \cdot 1008 - 63 \cdot 16^2}{2} = 1008 \cdot \frac{1008 - 16}{2} = 499968.$$

Multiplying by 1006!, as noted above, and noticing that 499968 is not a multiple of 1007, we get an answer of $499968 \cdot 1006!$.

24. We say a polynomial $P$ in $x$ and $y$ is $n$-good if $P(x, y) = 0$ for all integers $x$ and $y$, with $x \neq y$, between 1 and $n$, inclusive. We also define the complexity of a polynomial to be the maximum sum of exponents of $x$ and $y$ across its terms with nonzero coefficients. What is the minimal complexity of a nonzero 4-good polynomial? In addition, give an example of a 4-good polynomial attaining this minimal complexity.

**Solution.** The answer is $\boxed{\text{3 (and example must be provided)}}$.

We first show that there exists a polynomial of complexity 3 satisfying the above conditions, and then that no polynomial of lower complexity can satisfy those conditions.

Consider the polynomial $f(x, y) = (x + y - 5)((x - 2.5)^2 + (y - 2.5)^2 - 2.5)$. At each of the points (1, 4), (2, 3), (3, 2), and (4, 1), the first term $x + y - 5$ is equal to 0, and at the remaining eight points the second term $(x - 2.5)^2 + (y - 2.5)^2 - 2.5$ is equal to 0. Thus, at each of the twelve points, $f(x, y)$ is equal to 0.

Next, assume that a polynomial $g(x, y)$ with complexity at most 2 exists satisfying the conditions. Consider the one-variable polynomial $g(x, 1)$. Since $g(x, y)$ has complexity at most 2, $g(x, 1)$ has degree at most 2. In addition to this, $g(x, 1)$ has roots at $x = 2, 3, 4$. Since the number of roots is greater than the degree, $g(x, 1)$ must be the zero polynomial. Now, let $g(x, y) = (y - 1)(h(x, y)) + i(x)$ for polynomials $h$ and $i$; these can be found by long division by $y - 1$. Since $g(x, 1) = i(x)$ is the zero polynomial, we have that $y-1$ divides $g(x,y)$. By a similar argument, $y-2, y-3$, and $y-4$ each divide $g(x,y)$. Hence, $g(x,y) = (y-1)(y-2)(y-3)(y-4)k(x,y)$. This has a complexity of at least 4, which contradicts our original assumption. Thus, we have ascertained that no such polynomial $g(x,y)$ exists.

# Exeter Math Club Competition
# January 24, 2015

## Table of Contents

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Jane Street

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- **Judges** Zuming Feng, Greg Spanier

# Chapter 1

# EMC$^2$ 2015 Problems

## 1.1 Speed Test

*There are 20 problems, worth 3 points each, to be solved in 25 minutes.*

1. Matt has a twenty dollar bill and buys two items worth \$7.99 each. How much change does he receive, in dollars?

2. The sum of two distinct numbers is equal to the positive difference of the two numbers. What is the product of the two numbers?

3. Evaluate
$$\frac{1+2+3+4+5+6+7}{8+9+10+11+12+13+14}.$$

4. A sphere with radius $r$ has volume $2\pi$. Find the volume of a sphere with diameter $r$.

5. Yannick ran 100 meters in 14.22 seconds. Compute his average speed in meters per second, rounded to the nearest integer.

6. The mean of the numbers $2, 0, 1, 5$, and $x$ is an integer. Find the smallest possible positive integer value for $x$.

7. Let $f(x) = \sqrt{2^2 - x^2}$. Find the value of $f(f(f(f(f(-1)))))$.

8. Find the smallest positive integer $n$ such that 20 divides $15n$ and 15 divides $20n$.

9. A circle is inscribed in equilateral triangle $ABC$. Let $M$ be the point where the circle touches side $AB$ and let $N$ be the second intersection of segment $CM$ and the circle. Compute the ratio $\frac{MN}{CN}$.

10. Four boys and four girls line up in a random order. What is the probability that both the first and last person in line is a girl?

11. Let $k$ be a positive integer. After making $k$ consecutive shots successfully, Andy's overall shooting accuracy increased from 65% to 70%. Determine the minimum possible value of $k$.

12. In square $ABCD$, $M$ is the midpoint of side $CD$. Points $N$ and $P$ are on segments $BC$ and $AB$ respectively such that $\angle AMN = \angle MNP = 90^\circ$. Compute the ratio $\frac{AP}{PB}$.

13. Meena writes the numbers 1, 2, 3, and 4 in some order on a blackboard, such that she cannot swap two numbers and obtain the sequence $1, 2, 3, 4$. How many sequences could she have written?

14. Find the smallest positive integer $N$ such that $2N$ is a perfect square and $3N$ is a perfect cube.

15. A polyhedron has 60 vertices, 150 edges, and 92 faces. If all of the faces are either regular pentagons or equilateral triangles, how many of the 92 faces are pentagons?

16. All positive integers relatively prime to 2015 are written in increasing order. Let the twentieth number be $p$. The value of
$$\frac{2015}{p} - 1$$
can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Compute $a + b$.

17. Five red lines and three blue lines are drawn on a plane. Given that $x$ pairs of lines of the same color intersect and $y$ pairs of lines of different colors intersect, find the maximum possible value of $y - x$.

18. In triangle $ABC$, where $AC > AB$, $M$ is the midpoint of $BC$ and $D$ is on segment $AC$ such that $DM$ is perpendicular to $BC$. Given that the areas of $MAD$ and $MBD$ are 5 and 6, respectively, compute the area of triangle $ABC$.

19. For how many ordered pairs $(x, y)$ of integers satisfying $0 \leq x, y \leq 10$ is $(x + y)^2 + (xy - 1)^2$ a prime number?

20. A solitaire game is played with 8 red, 9 green, and 10 blue cards. Totoro plays each of the cards exactly once in some order, one at a time. When he plays a card of color $c$, he gains a number of points equal to the number of cards that are *not* of color $c$ in his hand. Find the maximum number of points that he can obtain by the end of the game.

## 1.2 Accuracy Test

*There are 10 problems, worth 9 points each, to be solved in 45 minutes.*

1. A number of Exonians took a math test. If all of their scores were positive integers and the mean of their scores was 8.6, find the minimum possible number of students.

2. Find the least composite positive integer that is not divisible by any of 3, 4, and 5.

3. Five checkers are on the squares of an $8 \times 8$ checkerboard such that no two checkers are in the same row or the same column. How many squares on the checkerboard share neither a row nor a column with any of the five checkers?

4. Let the operation $x@y$ be $y - x$. Compute $((\cdots((1@2)@3)@\cdots@2013)@2014)@2015$.

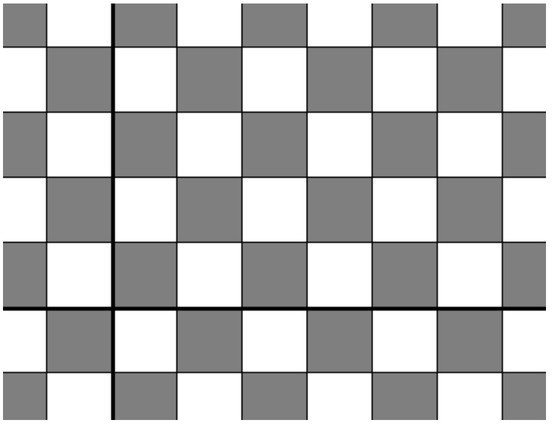
5. In a town, each family has either one or two children. According to a recent survey, 40% of the children in the town have a sibling. What fraction of the families in the town have two children?

6. Equilateral triangles $ABE$, $BCF$, $CDG$ and $DAH$ are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by an edge. What is the area of the region in which she can stand?

7. Let $S$ be the string 0101010101010. Determine the number of substrings containing an odd number of 1's. (A substring is defined by a pair of (not necessarily distinct) characters of the string and represents the characters between, inclusively, the two elements of the string.)

8. Let the positive divisors of $n$ be $d_1$, $d_2$, $\ldots$ in increasing order. If $d_6 = 35$, determine the minimum possible value of $n$.

9. The unit squares on the coordinate plane that have four lattice point vertices are colored black or white, as on a chessboard, shown on the diagram below.



   For an ordered pair $(m, n)$, let $OXZY$ be the rectangle with vertices $O = (0, 0)$, $X = (m, 0)$, $Z = (m, n)$ and $Y = (0, n)$. How many ordered pairs $(m, n)$ of nonzero integers exist such that rectangle $OXZY$ contains exactly 32 black squares?

10. In triangle $ABC$, $AB = 2BC$. Given that $M$ is the midpoint of $AB$ and $\angle MCA = 60^\circ$, compute $\frac{CM}{AC}$.

## 1.3 Team Test

*There are 15 problems, worth 20 points each, to be solved in 45 minutes.*

1. Nicky is studying biology and has a tank of 17 lizards. In one day, he can either remove 5 lizards or add 2 lizards to his tank. What is the minimum number of days necessary for Nicky to get rid of all of the lizards from his tank?

2. What is the maximum number of spheres with radius 1 that can fit into a sphere with radius 2?

3. A positive integer $x$ is *sunny* if $3x$ has more digits than $x$. If all sunny numbers are written in increasing order, what is the 50th number written?

4. Quadrilateral $ABCD$ satisfies $AB = 4, BC = 5, DA = 4, \angle DAB = 60^\circ$, and $\angle ABC = 150^\circ$. Find the area of $ABCD$.

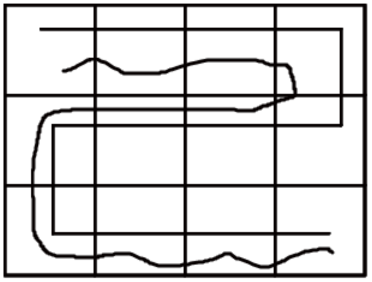
5. Totoro wants to cut a 3 meter long bar of mixed metals into two parts with equal monetary value. The left meter is bronze, worth 10 zoty per meter, the middle meter is silver, worth 25 zoty per meter, and the right meter is gold, worth 40 zoty per meter. How far, in meters, from the left should Totoro make the cut?

6. If the numbers $x_1, x_2, x_3, x_4$, and $x_5$ are a permutation of the numbers $1, 2, 3, 4$, and $5$, compute the maximum possible value of $|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5|$.

7. In a $3 \times 4$ grid of 12 squares, find the number of paths from the top left corner to the bottom right corner that satisfy the following two properties:

   - The path passes through each square exactly once.
   - Consecutive squares share a side.

   Two paths are considered distinct if and only if the order in which the twelve squares are visited is different. For instance, in the diagram below, the two paths drawn are considered the same.



8. Scott, Demi, and Alex are writing a computer program that is 25 lines long. Since they are working together on one computer, only one person may type at a time. To encourage collaboration, no person can type two lines in a row, and everyone must type something. If Scott takes 10 seconds to type one line, Demi takes 15 seconds, and Alex takes 20 seconds, at least how long, in seconds, will it take them to finish the program?

9. A hand of four cards of the form $(c, c, c + 1, c + 1)$ is called a *tractor*. Vinjai has a deck consisting of four of each of the numbers 7, 8, 9 and 10. If Vinjai shuffles and draws four cards from his deck, compute the probability that they form a tractor.

10. The parabola $y = 2x^2$ is the wall of a fortress. Totoro is located at $(0, 4)$ and fires a cannonball in a straight line at the closest point on the wall. Compute the $y$-coordinate of the point on the wall that the cannonball hits.

11. How many ways are there to color the squares of a 10 by 10 grid with black and white such that in each row and each column there are exactly two black squares and between the two black squares in a given row or column there are exactly 4 white squares? Two configurations that are the same under rotations or reflections are considered different.

12. In rectangle $ABCD$, points $E$ and $F$ are on sides $AB$ and $CD$, respectively, such that $AE = CF > AD$ and $\angle CED = 90°$. Lines $AF, BF, CE$ and $DE$ enclose a rectangle whose area is 24% of the area of $ABCD$. Compute $\frac{BF}{CE}$.

13. Link cuts trees in order to complete a quest. He must cut 3 Fenwick trees, 3 Splay trees and 3 KD trees. If he must also cut 3 trees of the same type in a row at some point during his quest, in how many ways can he cut the trees and complete the quest? (Trees of the same type are indistinguishable.)

14. Find all ordered pairs $(a, b)$ of positive integers such that

$$\sqrt{64a + b^2} + 8 = 8\sqrt{a} + b.$$

15. Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle BCD = 108°$, $\angle CDE = 168°$ and $AB = BC = CD = DE$. Find the measure of $\angle AEB$.

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. [5] Alec rated the movie *Frozen* 1 out of 5 stars. At least how many ratings of 5 out of 5 stars does Eric need to collect to make the average rating for *Frozen* greater than or equal to 4 out of 5 stars?

2. [5] Bessie shuffles a standard 52-card deck and draws five cards without replacement. She notices that all five of the cards she drew are red. If she draws one more card from the remaining cards in the deck, what is the probability that she draws another red card?

3. [5] Find the value of $121 \cdot 1020304030201$.

### 1.4.2 Round 2

4. [7] Find the smallest positive integer $c$ for which there exist positive integers $a$ and $b$ such that $a \neq b$ and $a^2 + b^2 = c$.

5. [7] A semicircle with diameter $AB$ is constructed on the outside of rectangle $ABCD$ and has an arc length equal to the length of $BC$. Compute the ratio of the area of the rectangle to the area of the semicircle.

6. [7] There are 10 monsters, each with 6 units of health. On turn $n$, you can attack one monster, reducing its health by $n$ units. If a monster's health drops to 0 or below, the monster dies. What is the minimum number of turns necessary to kill all of the monsters?

### 1.4.3 Round 3

7. [9] It is known that 2 students make up 5% of a class, when rounded to the nearest percent. Determine the number of possible class sizes.

8. [9] At 17:10, Totoro hopped onto a train traveling from Tianjin to Urumuqi. At 14:10 that same day, a train departed Urumuqi for Tianjin, traveling at the same speed as the 17:10 train. If the duration of a one-way trip is 13 hours, then how many hours after the two trains pass each other would Totoro reach Urumuqi?

9. [9] Chad has 100 cookies that he wants to distribute among four friends. Two of them, Jeff and Qiao, are rivals; neither wants the other to receive more cookies than they do. The other two, Jim and Townley, don't care about how many cookies they receive. In how many ways can Chad distribute all 100 cookies to his four friends so that everyone is satisfied? (Some of his four friends may receive zero cookies.)

### 1.4.4 Round 4

10. [11] Compute the smallest positive integer with at least four two-digit positive divisors.

11. [11] Let $ABCD$ be a trapezoid such that $AB$ is parallel to $CD$, $BC = 10$ and $AD = 18$. Given that the two circles with diameters $BC$ and $AD$ are tangent, find the perimeter of $ABCD$.

12. [11] How many length ten strings consisting of only $A$s and $B$s contain neither "$BAB$" nor "$BBB$" as a substring?

### 1.4.5 Round 5

*Each of the three problems in this round depends on the answer to two of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.*

13. [13] Let $B$ be the answer to problem 14, and let $C$ be the answer to problem 15. A quadratic function $f(x)$ has two real roots that sum to $2^{10} + 4$. After translating the graph of $f(x)$ left by $B$ units and down by $C$ units, the new quadratic function also has two real roots. Find the sum of the two real roots of the new quadratic function.

14. [13] Let $A$ be the answer to problem 13, and let $C$ be the answer to problem 15. In the interior of angle $NOM = 45°$, there is a point $P$ such that $\angle MOP = A°$ and $OP = C$. Let $X$ and $Y$ be the reflections of $P$ over $MO$ and $NO$, respectively. Find $(XY)^2$.

15. [13] Let $A$ be the answer to problem 13, and let $B$ be the answer to problem 14. Totoro hides a guava at point $X$ in a flat field and a mango at point $Y$ different from $X$ such that the length $XY$ is $B$. He wants to hide a papaya at point $Z$ such that $YZ$ has length $A$ and the distance $ZX$ is a nonnegative integer. In how many different locations can he hide the papaya?

### 1.4.6 Round 6

16. [15] Let $ABCD$ be a trapezoid such that $AB$ is parallel to $CD$, $AB = 4$, $CD = 8$, $BC = 5$, and $AD = 6$. Given that point $E$ is on segment $CD$ and that $AE$ is parallel to $BC$, find the ratio between the area of trapezoid $ABCD$ and the area of triangle $ABE$.

17. [15] Find the maximum possible value of the greatest common divisor of $\overline{MOO}$ and $\overline{MOOSE}$, given that $S$, $O$, $M$, and $E$ are some nonzero digits. (The digits $S$, $O$, $M$, and $E$ are not necessarily pairwise distinct.)

18. [15] Suppose that 125 politicians sit around a conference table. Each politician either always tells the truth or always lies. (Statements of a liar are never completely true, but can be partially true.) Each politician now claims that the two people beside them are both liars. Suppose that the greatest possible number of liars is $M$ and that the least possible number of liars is $N$. Determine the ordered pair $(M, N)$.

### 1.4.7 Round 7

19. [18] Define a *lucky* number as a number that only contains 4s and 7s in its decimal representation. Find the sum of all three-digit lucky numbers.

20. [18] Let line segment $AB$ have length 25 and let points $C$ and $D$ lie on the same side of line $AB$ such that $AC = 15$, $AD = 24$, $BC = 20$, and $BD = 7$. Given that rays $AC$ and $BD$ intersect at point $E$, compute $EA + EB$.

21. [18] A $3 \times 3$ grid is filled with positive integers and has the property that each integer divides both the integer directly above it and directly to the right of it. Given that the number in the top-right corner is 30, how many distinct grids are possible?

### 1.4.8 Round 8

22. [22] Define a sequence of positive integers $s_1$, $s_2$, $\ldots$, $s_{10}$ to be *terrible* if the following conditions are satisfied for any pair of positive integers $i$ and $j$ satisfying $1 \le i < j \le 10$:

    - $s_i > s_j$
    - $j - i + 1$ divides the quantity $s_i + s_{i+1} + \cdots + s_j$

    Determine the minimum possible value of $s_1 + s_2 + \cdots + s_{10}$ over all terrible sequences.

23. [22] The four points $(x, y)$ that satisfy $x = y^2 - 37$ and $y = x^2 - 37$ form a convex quadrilateral in the coordinate plane. Given that the diagonals of this quadrilateral intersect at point $P$, find the coordinates of $P$ as an ordered pair.

24. [22] Consider a non-empty set of segments of length 1 in the plane which do not intersect except at their endpoints. (In other words, if point $P$ lies on distinct segments $a$ and $b$, then $P$ is an endpoint of both $a$ and $b$.) This set is called *3-amazing* if each endpoint of a segment is the endpoint of exactly three segments in the set. Find the smallest possible size of a 3-amazing set of segments.

# Chapter 2

# EMC$^2$ 2015 Solutions

## 2.1 Speed Test Solutions

1. Matt has a twenty dollar bill and buys two items worth \$7.99 each. How much change does he receive, in dollars?

   **Solution.** The answer is $\boxed{\$4.02}$.

   The two items in total cost $2 \times \$7.99 = \$15.98$. The change received is then $\$20.00 - \$15.98 = \$4.02$.

2. The sum of two distinct numbers is equal to the positive difference of the two numbers. What is the product of the two numbers?

   **Solution.** The answer is $\boxed{0}$.

   Let the two numbers be $x$ and $y$ with $x$ being the greater one; then, we have that $x + y = x - y$, which implies $y = 0$. As a result, we have that $xy = 0$.

3. Evaluate

   $$\frac{1+2+3+4+5+6+7}{8+9+10+11+12+13+14}.$$

   **Solution.** The answer is $\boxed{\frac{4}{11}}$.

   Both the numerator and denominator are arithmetic series with length seven, so each has a value of 7 times its middle term. Thus our answer is the ratio of their middle terms, or $\frac{4}{11}$.

4. A sphere with radius $r$ has volume $2\pi$. Find the volume of a sphere with diameter $r$.

   **Solution.** The answer is $\boxed{\frac{\pi}{4}}$.

   A sphere with diameter $r$ has radius $\frac{r}{2}$. Then the volume of this sphere is $(\frac{1}{2})^3 = \frac{1}{8}$ of the volume of the original sphere, or $\frac{2\pi}{8} = \frac{\pi}{4}$.

5. Yannick ran 100 meters in 14.22 seconds. Compute his average speed in meters per second, rounded to the nearest integer.

   **Solution.** The answer is $\boxed{7}$.

   We have

   $$14.22 \times 6.5 < 15 \times 6.5 < 100 < 14 \times 7.5 < 14.22 \times 7.5.$$

   So Yannick's average speed is strictly between 6.5 and 7.5 meters per second, and therefore the answer is 7 meters per second.

6. The mean of the numbers $2, 0, 1, 5$, and $x$ is an integer. Find the smallest possible positive integer value for $x$.

   **Solution.** The answer is $\boxed{2}$.

   We need $2 + 0 + 1 + 5 + x = x + 8$ to a multiple of 5, so we can conclude that $x = 2$.

7. Let $f(x) = \sqrt{2^2 - x^2}$. Find the value of $f(f(f(f(f(-1)))))$.

**Solution.** The answer is $\boxed{\sqrt{3}}$.

We have that

$$f(f(x)) = \sqrt{2^2 - (\sqrt{2^2 - x^2})^2} = \sqrt{x^2} = |x|.$$

Therefore $f(f(f(f(f(-1))))) = f(1) = \sqrt{4-1} = \sqrt{3}$.

8. Find the smallest positive integer $n$ such that 20 divides $15n$ and 15 divides $20n$.

   **Solution.** The answer is $\boxed{12}$.

   20 dividing $15n$ is equivalent to 4 dividing $3n$, and 15 dividing $20n$ is equivalent to 3 dividng $4n$. Thus $n$ must be a multiple of 12, so the smallest possible value of $n$ is 12.

9. A circle is inscribed in equilateral triangle $ABC$. Let $M$ be the point where the circle touches side $AB$ and let $N$ be the second intersection of segment $CM$ and the circle. Compute the ratio $\frac{MN}{CN}$.

   **Solution.** The answer is $\boxed{2}$.

   Let $O$ be the center of the circle. Since it is an equilateral triangle, $M$ is the midpoint of $AB$ and $CM$ goes through $O$. Notice that $\angle OAM = \angle MCA = 30^\circ$, so $\frac{AM}{MO} = \frac{CM}{AM} = \sqrt{3}$ and $\frac{CM}{MO} = 3$. Therefore we have $CN = NO = OM$ and $\frac{NM}{CN} = 2$.

10. Four boys and four girls line up in a random order. What is the probability that both the first and last person in line is a girl?

    **Solution.** The answer is $\boxed{\frac{3}{14}}$.

    When lining up the 8 people, we can always determine the first and last person before ordering the other 6 people. There are $8 \times 7 = 56$ ways to choose two people, and $4 \times 3 = 12$ ways chooses 2 girls. So the probablility is $\frac{12}{56} = \frac{3}{14}$.

11. Let $k$ be a positive integer. After making $k$ consecutive shots successfully, Andy's overall shooting accuracy increased from 65% to 70%. Determine the minimum possible value of $k$.

    **Solution.** The answer is $\boxed{10}$.

    Assume that Andy had made $a$ shots before the $k$ shots and $0.65a$ of them were successful. So we have $\frac{0.65a+k}{a+k} = 0.7$. Solving the equation for $a$, we get $a = 6k$, and therefore $0.65a = 3.9k$. It suffices to make $3.9k$ an integer, so the minimum value of $k = 10$.

12. In square $ABCD$, $M$ is the midpoint of side $CD$. Points $N$ and $P$ are on segments $BC$ and $AB$ respectively such that $\angle AMN = \angle MNP = 90^\circ$. Compute the ratio $\frac{AP}{PB}$.

    **Solution.** The answer is $\boxed{\frac{5}{3}}$.

    Assume the side length of the square is 1. Since $\angle AMN = \angle MNP = \angle D = \angle C = \angle B = 90^\circ$, we have $\angle DAM = \angle CMN = \angle BNP$ by simple angle-chasing, and therefore triangle $DAM, CMN, BNP$ are all similar to each other. It is clear that $AD = 1, DM = \frac{1}{2}$. From this we have $\frac{AD}{DM} = \frac{MC}{CN} = \frac{NB}{BP} = 2$. Then $MC = \frac{1}{2}, CN = \frac{1}{4}, NB = \frac{3}{4}, BP = \frac{3}{8}, PA = \frac{5}{8}$, and finally $\frac{AP}{PB} = \frac{5}{3}$.

13. Meena writes the numbers 1, 2, 3, and 4 in some order on a blackboard, such that she cannot swap two numbers and obtain the sequence $1, 2, 3, 4$. How many sequences could she have written?

    **Solution.** The answer is $\boxed{18}$.

    Observe that the condition makes the problem equivalent to counting the number of permutations that can be obtained by swapping two distinct elements of $(1, 2, 3, 4)$. This is because we can always undo/redo the swap to move between the permutation we're counting and $(1, 2, 3, 4)$. Since we all we need to do is choose two of the four elements to swap, the number of ways is $\binom{4}{2} = 6$. Since the total number of permutations is $4! = 24$, our answer is $24 - 6 = 18$.

14. Find the smallest positive integer $N$ such that $2N$ is a perfect square and $3N$ is a perfect cube.

    **Solution.** The answer is $\boxed{72}$.

    Assume $N = 2^a 3^b P$, where $a, b$ are nonnegative integers and $P$ is an integer that is a multiple of neither 2 or 3. From the problem statement we see that $P$ must be an integer to the sixth power, so we can safely assume that it is at least 1. Also since $2^{a+1}3^b$ is a perfect square, and $2^a 3^{b+1}$ is a perfect cube, we see that $a + 1, b$ are even and $a, b + 1$ are multiples of 3.It is easy to see that $a$ is at least 3 and $b$ is at least 2, so $N$ is at least $2^3 \cdot 3^2 = 72$. Indeed, $2 \times 72 = 144 = 12^2$ and $3 \times 72 = 216 = 6^3$, so $N = 72$ satisfies the requirements.

15. A polyhedron has 60 vertices, 150 edges, and 92 faces. If all of the faces are either regular pentagons or equilateral triangles, how many of the 92 faces are pentagons?

    **Solution.** The answer is $\boxed{12}$.

    Since each edge of the polyhedron is an edge of exactly two faces, the total number of edges of all the faces must be $2 \times 150 = 300$. Letting $t$ be the number of triangular faces and $p$ the number of pentagonal faces, we have a system $\{t+p = 92, 3t+5p = 300\}$ from which we find that $t = 80$ and $p = 12$.

    *Note:* The polyhedron described in the problem is known as a snub dodecahedron.

16. All positive integers relatively prime to 2015 are written in increasing order. Let the twentieth number be $p$. The value of
$$\frac{2015}{p} - 1$$
can be expressed as $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Compute $a + b$.

    **Solution.** The answer is $\boxed{2015}$.

    The fraction we want to compute is equivalent to $\frac{2015-p}{p}$. Since $p$ is coprime with 2015, by Euler's algorithm it is clear that $\gcd(2015 - p, p) = \gcd(2015, p) = 1$. Therefore $a = 2015 - p$ and $b = p$, and $a + b = 2015$, as desired.

17. Five red lines and three blue lines are drawn on a plane. Given that $x$ pairs of lines of the same color intersect and $y$ pairs of lines of different colors intersect, find the maximum possible value of $y - x$.

    **Solution.** The answer is $\boxed{15}$.

The maximum possible value of $y$ is $5 \cdot 3 = 15$, since each blue line can intersect at most five red lines. The minimum possible value of $x$ is 0, which occurs when all lines of the same color are parallel. This implies that $y - x \le 15$. Note we can achieve this bound by making all red lines parallel to the $x$-axis and all blue lines parallel to the $y$-axis on the coordinate plane. Thus our answer is 15.

18. In triangle $ABC$, where $AC > AB$, $M$ is the midpoint of $BC$ and $D$ is on segment $AC$ such that $DM$ is perpendicular to $BC$. Given that the areas of $MAD$ and $MBD$ are 5 and 6, respectively, compute the area of triangle $ABC$.

    **Solution.** The answer is $\boxed{22}$.

    Since $M$ is the midpoint of $BC$, we have $[MCD] = [MBD] = 6$, and $[ABM] = [ACM] = [MAD] + [MCD] = 5 + 6 = 11$, and finally $[ABC] = [ABM] + [ACM] = 11 + 11 = 22$.

19. For how many ordered pairs $(x, y)$ of integers satisfying $0 \le x, y \le 10$ is $(x+y)^2 + (xy-1)^2$ a prime number?

    **Solution.** The answer is $\boxed{10}$.

    We can expand and factor the given expression to obtain

    $$(x+y)^2 + (xy-1)^2 = x^2y^2 + x^2 + y^2 + 1 = (x^2+1)(y^2+1).$$

    For this value to be prime, exactly one of $x^2+1$ and $y^2+1$ must be 1, so one of $x$ and $y$ is 0. Without loss of generality, let $x = 0$, Then we know that $y \le 10$ and $y^2+1$ is a prime. If $y^2+1$ is even, then it must be 2, giving us the solution $(0, 1)$. Otherwise, $y^2+1$ is odd, and $y$ is even.

    We can now check all the even numbers less than or equal to 10: $2^2+1 = 5$ is prime. $4^2+1 = 17$ is prime. $6^2+1 = 37$ is prime. $8^2+1 = 65 = 5 \cdot 13$ is not prime. $10^2+1 = 101$ is prime. This gives us a total of 5 solutions of the form $(0, y)$. If we swap $x$ and $y$, we get another 5 solutions, yielding a final answer of 10.

20. A solitaire game is played with 8 red, 9 green, and 10 blue cards. Totoro plays each of the cards exactly once in some order, one at a time. When he plays a card of color $c$, he gains a number of points equal to the number of cards that are *not* of color $c$ in his hand. Find the maximum number of points that he can obtain by the end of the game.

    **Solution.** The answer is $\boxed{242}$.

    I claim that regardless of how the cards are played, the total number of points rewarded starting with $r$ red cards, $g$ green cards, and $b$ blue cards is always $rg + gb + br$. I prove this by induction with base case $(r, b, g) = (0, 0, 0)$. Assume that for all ordered triples $(r, g, b)$ with $r + g + b \le s$ that the number of points achieved is always $rg + gb + br$.

    Now consider an ordered triple $(r', g', b')$ with $r' + g' + b' = s + 1$. If a blue card is played first, generating $r' + g'$ points, then by the induction hypothesis the total number of points is

    $$r'g' + g'(b'-1) + (b'-1) + r' + g' = r'g' + g'b' + b'r'.$$

    One can verify that playing a red or green card generates the same total, thus completing the induction argument. Therefore the total number of points he can get is $8 \cdot 9 + 9 \cdot 10 + 10 \cdot 8 = 242$.

## 2.2 Accuracy Test Solutions

1. A number of Exonians took a math test. If all of their scores were positive integers and the mean of their scores was 8.6, find the minimum possible number of students.

   **Solution.** The answer is $\boxed{5}$.

   Observe that the sum of all scores is $8.6n = \frac{43n}{5}$. Since this is an integer, $n$ must be divisible by 5 and therefore at least 5. One example where $n = 5$ works is when the students' scores are $8, 8, 9, 9$ and 9, respectively.

2. Find the least composite positive integer that is not divisible by any of 3, 4, and 5.

   **Solution.** The answer is $\boxed{14}$.

   For an integer $n$ to be composite, it must be expressible as the product of 2 integers greater than 1. One of these integers can be 2, but not both, because otherwise $n$ would be a multiple of 4. Thus the second integer is at least 7, giving us $n = 14$. It is easy to check that none of the positive integers below 14 satisfy the requirements.

3. Five checkers are on the squares of an $8 \times 8$ checkerboard such that no two checkers are in the same row or the same column. How many squares on the checkerboard share neither a row nor a column with any of the five checkers?

   **Solution.** The answer is $\boxed{9}$.

   It is clear that the checkers occupy 5 distinct rows and 5 distinct columns. Thus 3 rows and 3 columns are unoccupied. Since each pair of an unoccupied row and an unoccupied column correspond to exactly one desired square, we have $3 \cdot 3 = 9$ such squares in total.

4. Let the operation $x@y$ be $y - x$. Compute $((\cdots((1@2)@3)@\cdots@2013)@2014)@2015$.

   **Solution.** The answer is $\boxed{1008}$.

   Since $x@y = y - x$, we can rewrite the original expression as

   $$2015 - (2014 - (\cdots - (3 - (2 - 1))\cdots)) = 2015 - 2014 + 2013 - \cdots + 3 - 2 + 1.$$

   Therefore, the original expression is equal to

   $$(2015 - 2014) + (2013 - 2012) + \cdots + (3 - 2) + 1 = \frac{2014}{2} + 1 = 1008.$$

5. In a town, each family has either one or two children. According to a recent survey, 40% of the children in the town have a sibling. What fraction of the families in the town have two children?

   **Solution.** The answer is $\boxed{\frac{1}{4}}$.

   Suppose that there are $n$ children in the town. Then $0.4n$ children are from families with two children and $0.6n$ children are from families with one child. Therefore, there are $\frac{0.4n}{2} = 0.2n$ families with two children and $0.6n$ families with one child, for a total $0.2n + 0.6n = 0.8n$ families. Thus the desired fraction is $\frac{0.2n}{0.8n} = \frac{1}{4}$.

6. Equilateral triangles $ABE$, $BCF$, $CDG$ and $DAH$ are constructed outside the unit square $ABCD$. Eliza wants to stand inside octagon $AEBFCGDH$ so that she can see every point in the octagon without being blocked by an edge. What is the area of the region in which she can stand?

   **Solution.** The answer is $\boxed{\frac{3+\sqrt{3}}{3}}$.

   For the whole octagon to be visible to her, observe that Eliza only has to be able to see each point on the perimeter of the octagon. In order to do so, she must be inside the extensions of each acute angle of the octagon. (The acute angles we are referring to are $\angle AEB$, $\angle BFC$, $\angle CGD$ and $\angle DHA$.) Note that this is sufficient, since each point on the perimeter belongs to at least one such acute angle. Thus our answer is the area of the intersection of all four of these acute angles. We can cut this region into square $ABCD$ and four smaller triangles. It is not difficult to see that the rays intersect at the center of each equilateral triangle, so each smaller triangle has area $\frac{1}{3}$ that of an equilateral triangle. Each equilateral triangle has area $\frac{\sqrt{3}}{4}$, so we can sum these parts and obtain

$$1 + 4 \cdot \frac{1}{3} \cdot \frac{\sqrt{3}}{4} = \frac{3+\sqrt{3}}{3}$$

   as our answer.

7. Let $S$ be the string 0101010101010. Determine the number of substrings containing an odd number of 1's. (A substring is defined by a pair of (not necessarily distinct) characters of the string and represents the characters between, inclusively, the two elements of the string.)

   **Solution.** The answer is $\boxed{48}$.

   Call a substring *minimal* if it starts and ends with 1. We see that adding a 0 to either the front, back, or both of the minimal substring produces another substring with the same number of ones. Since there are 4 substrings corresponding to a single minimal substring, we only need to find the number of minimal substrings with an odd number of ones.

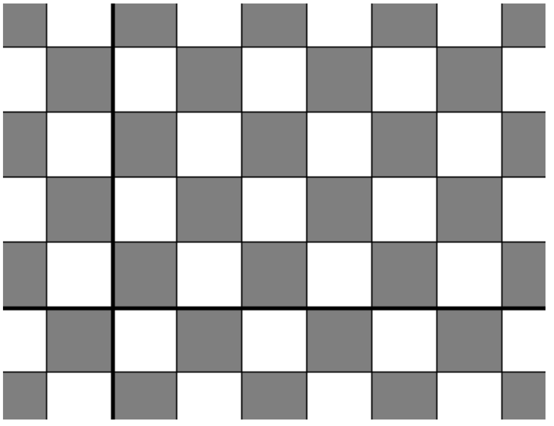
   It is not difficult to see that there are 6 ways to choose a minimal substring with 1 one, 4 ways to choose one with 3 ones and 2 ways to choose one with 5 ones. Therefore there are $6 + 4 + 2 = 12$ minimal substrings with desired property and thus $4 \cdot 12 = 48$ such substrings in total.

8. Let the positive divisors of $n$ be $d_1$, $d_2$, $\dots$ in increasing order. If $d_6 = 35$, determine the minimum possible value of $n$.

   **Solution.** The answer is $\boxed{1925}$.

   Since 35 is a divisor of $n$, $1, 5, 7$ are also divisors of $n$. We need to find exactly 2 more divisors that are less than 35. If $n$ is divisible by another prime factor $p$ that is 2 or 3, then $p, 5p, 7p$ are all divisors of $n$ and 35 can't be the sixth smallest divisor of $n$. If $n$ is divisible by $5^2$, then 25 is a divisor of $n$. If $n$ is divisible by $7^2$, then no new divisors less than 35 will be produced. If $n$ is divisible by another prime factor $p$ that is greater than 7 and less than 35, then $p$ will be the only new factor less than 35. We see then that making $n$ divisible by $5^2, 7, 11$ is the optimal strategy and thus the smallest $n$ is $5^2 \cdot 7 \cdot 11 = 1925$.

9. The unit squares on the coordinate plane that have four lattice point vertices are colored black or white, as on a chessboard, shown on the diagram below.



For an ordered pair $(m, n)$, let $OXZY$ be the rectangle with vertices $O = (0, 0)$, $X = (m, 0)$, $Z = (m, n)$ and $Y = (0, n)$. How many ordered pairs $(m, n)$ of nonzero integers exist such that rectangle $OXZY$ contains exactly 32 black squares?

**Solution.** The answer is $\boxed{48}$.

We have the following cases:

*Case 1: The area is even.* In this case, the number of black squares is always equal to the number of white squares. Thus if there are 32 black squares, there must also be 32 white squares, which implies the area of $OXZY$ is 64. Therefore $|mn| = 64$, which means the possible ordered pairs $(m, n)$ are $(\pm1, \pm64)$, $(\pm2, \pm32)$, $(\pm4, \pm16)$, $(\pm8, \pm8)$, $(\pm16, \pm4)$, $(\pm32, \pm2)$ and $(\pm64, \pm1)$, giving us $7 \cdot 4 = 28$ possibilities.

*Case 2: Both sides have odd length and the four corner squares are all black.* Observe that the number of black squares is one more than the number of white squares. Thus when there are 32 black squares, there must be 31 white squares, giving us an area of 63. In this case, point $Z$ must be in the first or third quadrant, so $m$ and $n$ must have the same sign. Now, we can compute the first-quadrant pairs $(m, n)$ to be $(1, 63)$, $(3, 21)$, $(7, 9)$, $(9, 7)$, $(21, 3)$ and $(63, 1)$. The reflections of these points over the origin into the third quadrant also yield solutions, thus there are a total $6 \cdot 2 = 12$ pairs for this case.

*Case 3: Both sides have odd length and the four corner squares are all white.* This case is essentially the opposite of Case 2. The number of white squares is one more than the number of black squares. Thus when there are 32 black squares, there must be 33 white squares, giving us an area of 65 for $OXZY$. In this case, point $Z$ must be in the second or fourth quadrant, so $m$ and $n$ must have opposite signs. Observe that the second quadrant pairs $(m, n)$ are $(-1, 65)$, $(-5, 13)$, $(-13, 5)$ and $(-65, 1)$. Since these points and their reflections over the origin into the fourth quadrant are all solutions, we have $4 \cdot 2 = 8$ pairs here.

Summing the results for all the cases, we obtain a total of $28 + 12 + 8 = 48$ ordered pairs.

10. In triangle $ABC$, $AB = 2BC$. Given that $M$ is the midpoint of $AB$ and $\angle MCA = 60^\circ$, compute $\frac{CM}{AC}$.

**Solution.** The answer is $\boxed{\frac{1}{3}}$.

Since $M$ is the midpoint of $AB$ and $AB = 2BC$, we know that $AM = MB = BC$. Draw $BX$ perpendicular to $MC$ at $X$ and extend $CM$ to a point $Y$ such that $MX = MY$. Now, observe that $CX = MX$ and $\triangle AYM \cong \triangle BXM$. Therefore, $\angle AYM = 90^\circ$ and $CA = 2CY$. Since $X$ and $M$ trisect $CY$, $CY = \frac{3}{2}CM$, which implies $CA = 3CM$ and $\frac{CM}{AC} = \frac{1}{3}$ as desired.

## 2.3 Team Test Solutions

1. Nicky is studying biology and has a tank of 17 lizards. In one day, he can either remove 5 lizards or add 2 lizards to his tank. What is the minimum number of days necessary for Nicky to get rid of all of the lizards from his tank?

   **Solution.** The answer is $\boxed{9}$.

   Assume for $x$ days he removed 5 lizards and for $y$ days he added 2 lizards. So we have $5x - 2y = 17$ or $x = \frac{17+2y}{5}$. The minimum positive integer value of $x$ and $y$ is achieved when $y = 4$, and thus $x = 5, x + y = 9$, so Nicky needs 9 days to get rid of all 17 lizards.

2. What is the maximum number of spheres with radius 1 that can fit into a sphere with radius 2?

   **Solution.** The answer is $\boxed{2}$.

   It's easy to see that 2 unit spheres can fit inside the big sphere by putting the two centers along a diameter. Now we show that we cannot fit 3 or more inside. Clearly if we want to fit three inside the optimal situation is when they are mutually tangent and therefore the centers are vertices of an equilateral triangle with side length 2. By cutting through the plane with the three centers, we get three mutually tangent unit circles and we see that they cannot fit inside a circle with radius 2 (in fact, the smallest circle that encloses the three unit circles has a radius of $\frac{2}{\sqrt{3}} + 1$ and we leave the proof to the readers), which is the largest cross-section of the large sphere. Therefore we can't fit three spheres in a sphere with radius 2.

3. A positive integer $x$ is *sunny* if $3x$ has more digits than $x$. If all sunny numbers are written in increasing order, what is the 50th number written?

   **Solution.** The answer is $\boxed{77}$.

   The one-digit sunny numbers can be computed to be $\{4, 5, 6, 7, 8, 9\}$, and the two-digit sunny numbers are $\{34, 35, 36, ..., 99\}$. The problem is equivalent to finding the $50 - 6 = 44$-th two digit number, which is $34 + 44 - 1 = 77$.

4. Quadrilateral $ABCD$ satisfies $AB = 4, BC = 5, DA = 4, \angle DAB = 60^\circ$, and $\angle ABC = 150^\circ$. Find the area of $ABCD$.

   **Solution.** The answer is $\boxed{10 + 4\sqrt{3}}$.

   Connect $DB$. Notice that triangle $ABD$ is an equilateral triangle with area $\frac{\sqrt{3}}{4} \cdot 4^2 = 4\sqrt{3}$. Then $\angle DBC = 150^\circ - 60^\circ = 90^\circ$. Then triangle $BCD$ is an equilateral triangle with area $\frac{4 \cdot 5}{2} = 10$. Combining the two areas, we get $10 + 4\sqrt{3}$.

5. Totoro wants to cut a 3 meter long bar of mixed metals into two parts with equal monetary value. The left meter is bronze, worth 10 zoty per meter, the middle meter is silver, worth 25 zoty per meter, and the right meter is gold, worth 40 zoty per meter. How far, in meters, from the left should Totoro make the cut?

   **Solution.** The answer is $\boxed{\frac{33}{16}}$.

Note that the entire bar is worth $10 + 25 + 40 = 75$ zoty, so each part should be worth $\frac{75}{2} = 37.5$ zoty. Also since the gold part of the bar is already worth 40 zoty, we should make the cut in the gold part, and therefore the right part is entirely gold. From this we can find that the right part should be $\frac{37.5}{40} = \frac{15}{16}$ meters long, leaving the left part to be $3 - \frac{15}{16} = \frac{33}{16}$ meters long.

6. If the numbers $x_1, x_2, x_3, x_4$, and $x_5$ are a permutation of the numbers $1, 2, 3, 4$, and $5$, compute the maximum possible value of $|x_1 - x_2| + |x_2 - x_3| + |x_3 - x_4| + |x_4 - x_5|$.

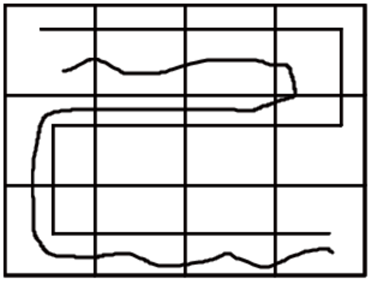
   **Solution.** The answer is $\boxed{11}$.

   If $x_i$ is between $x_{i+1}$ and $x_{i-1}$, we can remove $x_i$ from the sequence and add it to the end. Since $|x_{i-1} - x_{i+1}| = |x_{i-1} - x_i| + |x_i - x_{i+1}|$, we did not decrease the sum of difference in the middle and added a positive difference at the end, and therefore increased the desired sum. Thus we can assume the sequence alternates between increasing and decreasing. Now, we can also assume $x_1 > x_2 < x_3 > x_4 < x_5$, so the sum is equal to $x_1 - 2x_2 + 2x_3 - 2x_4 + x_5$, which is maximized when $x_3 = 5$, $x_1, x_5 = 3, 4$, and $x_2, x_4 = 1, 2$, giving a sum of 11.

7. In a $3 \times 4$ grid of 12 squares, find the number of paths from the top left corner to the bottom right corner that satisfy the following two properties:

   - The path passes through each square exactly once.
   - Consecutive squares share a side.

   Two paths are considered distinct if and only if the order in which the twelve squares are visited is different. For instance, in the diagram below, the two paths drawn are considered the same.



   **Solution.** The answer is $\boxed{4}$.

   Label the squares on each row ABCD, EFGH, IJKL from left to right. We want to find the number of paths that start from A and ends with L. We start by examining the possible moves starting from the first step.
   *Case 1*: First move is to B. Then there are still 2 possibilities: If next move is to C, then the path must go to D, H, and then GFEIJKL; if the next move is to F, then the path must go to EIJK, then GCDHL. There are 2 ways in all.
   *Case 2*: First move is to E, then we are forced to go to I, and then J. If next move is to F, then we have to go to B, C, then DHGKL; if next move is to K, then we have to go to G, F, B, C, then DHL. There are 2 ways in all in this case.
   Summing up the two cases, we have $2 + 2 = 4$ paths in total.

8. Scott, Demi, and Alex are writing a computer program that is 25 lines long. Since they are working together on one computer, only one person may type at a time. To encourage collaboration, no person can type two lines in a row, and everyone must type something. If Scott takes 10 seconds to type one line, Demi takes 15 seconds, and Alex takes 20 seconds, at least how long, in seconds, will it take them to finish the program?

   **Solution.** The answer is $\boxed{315}$.

   To minimize the total time we need to maximize the number of lines typed by Scott and minimize the number of lines typed by Alex. Since Scott cannot type twice in a row, he can type at most 13 lines of code. Thus Demi and Alex together must type at least 12 lines. Additionally, at least one of these lines must be Alex's. Thus it must take at least $13 \cdot 10 + 11 \cdot 15 + 1 \cdot 20 = 315$ seconds to finish the program. One way this can happen is SASDS...SDSDS, with Scott typing every even line, and Demi typing the rest of the lines except for one line which Alex types.

9. A hand of four cards of the form $(c, c, c+1, c+1)$ is called a *tractor*. Vinjai has a deck consisting of four of each of the numbers 7, 8, 9 and 10. If Vinjai shuffles and draws four cards from his deck, compute the probability that they form a tractor.

   **Solution.** The answer is $\boxed{\frac{27}{455}}$.

   Assume that the hand of four cards is ordered and all cards are distinct (even if they have the same number), so all tractors should be $(7,7,8,8)$, $(8,8,9,9)$, $(9,9,10,10)$, or their permutations. If Matt draws $7,7,8,8$, there are $\binom{4}{2} = 6$ ways to choose two 7s and 6 ways to choose two 8s, and there are $4! = 24$ ways to order them, so in total there are $3 \cdot 6 \cdot 6 \cdot 24$ ways to draw a tractor. On the other hand, there are $16 \cdot 15 \cdot 14 \cdot 13$ ways to draw four cards. So the probability is $\frac{3 \cdot 12 \cdot 12 \cdot 24}{16 \cdot 15 \cdot 14 \cdot 13} = \frac{27}{455}$.

10. The parabola $y = 2x^2$ is the wall of a fortress. Totoro is located at $(0,4)$ and fires a cannonball in a straight line at the closest point on the wall. Compute the $y$-coordinate of the point on the wall that the cannonball hits.

    **Solution.** The answer is $\boxed{\frac{15}{4}}$.

    The cannonball hits the fortress at a point $(x, 2x^2)$ for some real number $x$, and we wish to minimize the distance from $(0,4)$ to $(x, 2x^2)$. By the distance formula, this is

$$\sqrt{(x-0)^2 + (2x^2-4)^2} = \sqrt{4x^4 - 15x^2 + 16} = \sqrt{\left(2x^2 - \frac{15}{4}\right)^2 + \frac{31}{16}}.$$

    The expression is minimized when $y = 2x^2 = \frac{15}{4}$.

11. How many ways are there to color the squares of a 10 by 10 grid with black and white such that in each row and each column there are exactly two black squares and between the two black squares in a given row or column there are exactly 4 white squares? Two configurations that are the same under rotations or reflections are considered different.

    **Solution.** The answer is $\boxed{120}$.

    Notice that in each row there should be exactly 1 black square in the first 5 squares and 1 black square in the last 5 squares, and when one cut this row into two equal halves, the two black squares are on the same relative position of each half.

Now cut the grid into four 5 by 5 sub-grids and consider the sub-grid on the top-left corner. There should be 1 black square on each row and column in this sub-grid, so there are $5! = 120$ ways to color this sub-grid. It's then obvious that the other three sub-grids are simply the translations of this sub-grid. So there are 120 ways to color the entire grid.

12. In rectangle $ABCD$, points $E$ and $F$ are on sides $AB$ and $CD$, respectively, such that $AE = CF > AD$ and $\angle CED = 90°$. Lines $AF, BF, CE$ and $DE$ enclose a rectangle whose area is 24% of the area of $ABCD$. Compute $\frac{BF}{CE}$.

**Solution.** The answer is $\boxed{\frac{\sqrt{6}}{2}}$.

Since $AE = CF, AB = CD$, it is clear that $BE = DF$. By SAS congruence we have $ADF$ being congruent to $CBE$, and $ADE$ is congruent to $CBF$. Then we have $AF||CE, DE||BF$. Because $\angle CED = 90°$, $AF, CE$ are perpendicular to $DE, BF$ respectively. By angle-chasing we have triangle $ADE$ is similar to $DFA$. Since $AE > AD$, we get that $AD > DF$, so $CF > DF$ as well.

Assume the intersection of $AF$ and $DE$, $BF$ and $CE$ are $X$ and $Y$ respectively. It is clear that

$$\frac{XE}{DE} = \frac{CF}{CD}, \frac{YE}{CE} = \frac{DF}{CD},$$

$$\frac{XE}{DE} + \frac{YE}{CE} = \frac{CF}{CD} + \frac{DF}{CD} = 1.$$

Also because $CF > DF$, $\frac{XE}{DE} > \frac{YE}{CE}$. Let $\frac{XE}{DE} = k$, so $\frac{YE}{CE} = 1 - k$, and $k > 0.5$. From problem statement we have $\frac{[XEYF]}{[ABCD]} = 0.24$, so

$$\frac{[XEYF]}{[CDE]} = \frac{[XEYF]}{1/2[ABCD]} = 0.48,$$

$$\frac{[XEYF]}{[CDE]} = \frac{k(1-k)}{1/2}.$$

Setting the two equations equal to each other and solving for $k$, we have $k = 0.6$ (the other solution $k = 0.4$ is disregarded because $k > 0.5$.)

Noticing that $\frac{k}{1-k} = \frac{CF}{DF} = \frac{FC}{BE}$, and that

$$\frac{BF}{CE} = \frac{FC}{BC} = \frac{BC}{BE} = \sqrt{\frac{FC}{BE}},$$

which follows from the fact that triangle $FCB$ and $CBE$ are similar, we can get that $\frac{BF}{CE} = \sqrt{\frac{k}{1-k}} = \frac{\sqrt{6}}{2}$, as desired.

13. Link cuts trees in order to complete a quest. He must cut 3 Fenwick trees, 3 Splay trees and 3 KD trees. If he must also cut 3 trees of the same type in a row at some point during his quest, in how many ways can he cut the trees and complete the quest? (Trees of the same type are indistinguishable.)

**Solution.** The answer is $\boxed{366}$.

If a solution cuts a type of tree three times in a row, we say that this solution *clears* that type of tree. We can now use the principle of inclusion-exclusion to count number of ways for Link to cut the trees. We start off by estimating our answer to be $3 \cdot 7 \cdot \binom{6}{3} = 420$: 3 ways to choose which type of tree is cleared, 7 places to put this group of three cuts, and 6 choose 3 ways to cut the remaining trees. However, we count twice each solution that clears two types of tree, and thrice each solution that clears three types of tree. Now, we subtract out the solutions that clear two types of trees, or $\binom{3}{2} \cdot 4 \cdot 5 = 60$: 3 choose 2 ways to choose the trees that are cleared, 4 ways to place the first group of cleared trees and 5 ways to place the second. However, we are not done yet, since we subtracted the solutions that clear all three types of trees three times. We have to add the $3! = 6$ ways to do so back, getting an answer of $420 - 60 + 6 = 366$.

14. Find all ordered pairs $(a, b)$ of positive integers such that

$$\sqrt{64a + b^2} + 8 = 8\sqrt{a} + b.$$

**Solution.** The answer is $\boxed{(4, 12), (9, 10), (25, 9)}$.

Rewrite the equation as

$$\sqrt{64a + b^2} = 8\sqrt{a} + b - 8.$$

Squaring and cancelling terms yields

$$0 = 64 - 128\sqrt{a} - 16b + 16b\sqrt{a},$$

or $b = 8 + \frac{4}{\sqrt{a}-1}$. Notice that if $a > 25$ then $8 < b < 9$, giving no solutions. Also, if $a$ is not a perfect square, then $b$ is irrational. It remains to check $a = 1, 4, 9, 16, 25$, which give the solutions $(4, 12), (9, 10), (25, 9)$. It is not difficult to check that all three solutions work.

15. Let $ABCDE$ be a convex pentagon such that $\angle ABC = \angle BCD = 108^\circ$, $\angle CDE = 168^\circ$ and $AB = BC = CD = DE$. Find the measure of $\angle AEB$.

**Solution.** The answer is $\boxed{24^\circ}$.

Construct a point $F$ such that $ABCDF$ is a regular pentagon. Note that $\triangle FDE$ is equilateral. In addition, observe that $\angle AEB = \angle FEB - \angle FEA$. By symmetry, we know that $\angle FEB$ is half of $\angle FED$ or $30^\circ$. Because $\triangle AFE$ is isosceles, $\angle FEA = \frac{180^\circ - \angle AFE}{2} = \frac{180^\circ - 168^\circ}{2} = 6^\circ$. This allows us to compute $\angle AEB$ as $30^\circ - 6^\circ = 24^\circ$.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. [5] Alec rated the movie *Frozen* 1 out of 5 stars. At least how many ratings of 5 out of 5 stars does Eric need to collect to make the average rating for *Frozen* greater than or equal to 4 out of 5 stars?

   **Solution.** The answer is $\boxed{3}$.

   If Eric collects one 5 star vote, then the average is 3, and if he collects two, then the average is $\frac{11}{3}$. Thus he needs to collect at least 3 votes, which will bring the average rating to exactly 4.

2. [5] Bessie shuffles a standard 52-card deck and draws five cards without replacement. She notices that all five of the cards she drew are red. If she draws one more card from the remaining cards in the deck, what is the probability that she draws another red card?

   **Solution.** The answer is $\boxed{\frac{21}{47}}$.

   After Bessie draws five red cards, there are $26 - 5 = 21$ red cards among the remaining $52 - 5 = 47$ cards. Thus the probability that her next card is red is $\dfrac{21}{47}$.

3. [5] Find the value of $121 \cdot 1020304030201$.

   **Solution.** The answer is $\boxed{123456787654321}$.

   Observe that $121 = 11^2$ and $1020304030201 = 1010101^2$. Thus the product is equal to $(11 \cdot 1010101)^2 = (11111111)^2 = 123456787654321$.

### 2.4.2 Round 2

4. [7] Find the smallest positive integer $c$ for which there exist positive integers $a$ and $b$ such that $a \neq b$ and $a^2 + b^2 = c$.

   **Solution.** The answer is $\boxed{5}$.

   Assume $a > b$. Then $a \geq 2$ and $b \geq 1$, so $a^2 + b^2 \geq 5$, so the smallest such integer $c$ is $c = 5$.

5. [7] A semicircle with diameter $AB$ is constructed on the outside of rectangle $ABCD$ and has an arc length equal to the length of $BC$. Compute the ratio of the area of the rectangle to the area of the semicircle.

   **Solution.** The answer is $\boxed{4}$.

   Suppose $AB = 2r$, and the length of arc $AB$ and line segment $BC$ are both $l$. Then $[ABCD] = 2rl$, and the area of semicircle $AB$ is $\frac{rl}{2}$, so the ratio is 4.

6. [7] There are 10 monsters, each with 6 units of health. On turn $n$, you can attack one monster, reducing its health by $n$ units. If a monster's health drops to 0 or below, the monster dies. What is the minimum number of turns necessary to kill all of the monsters?

**Solution.** The answer is $\boxed{13}$.

For every turn after turn 6, the attack will kill one monster no matter what. Therefore, to achieve the minimum number of turns, we should spend the first 5 turns completely killing as many monsters as we can. Since we do 15 damage on the first 5 turns, we can kill 2 monsters, but not 3. It takes 8 more turns to kill the remaining 8 monsters, for a total of 13 turns.

### 2.4.3 Round 3

7. [9] It is known that 2 students make up 5% of a class, when rounded to the nearest percent. Determine the number of possible class sizes.

**Solution.** The answer is $\boxed{8}$.

Suppose that there are $N$ students in the class. From the problem statement we have $4.5\% \leq \frac{2}{N} < 5.5\%$. So by taking reciprocals, we get

$$\frac{100}{4.5} \geq \frac{N}{2} > \frac{100}{5.5};$$

$$\frac{400}{9} > 44 \geq N \geq 37 > \frac{400}{11}.$$

Therefore $N$ can be any integer between 37 and 44 inclusive, so there are $44 - 37 + 1 = 8$ possibilities in all.

8. [9] At 17:10, Totoro hopped onto a train traveling from Tianjin to Urumuqi. At 14:10 that same day, a train departed Urumuqi for Tianjin, traveling at the same speed as the 17:10 train. If the duration of a one-way trip is 13 hours, then how many hours after the two trains pass each other would Totoro reach Urumuqi?

**Solution.** The answer is $\boxed{8}$.

Let $t$ be the number of hours that Totoro has traveled when the two trains pass each other. Since the train from Urumuqi departs 3 hours earlier, it has been traveling for $t + 3$ hours. We are also given that the one way trip time is 13 hours, so $t + (t + 3) = 13 \to t = 5$. Thus Totoro has $13 - 5 = 8$ hours left on his train.

9. [9] Chad has 100 cookies that he wants to distribute among four friends. Two of them, Jeff and Qiao, are rivals; neither wants the other to receive more cookies than they do. The other two, Jim and Townley, don't care about how many cookies they receive. In how many ways can Chad distribute all 100 cookies to his four friends so that everyone is satisfied? (Some of his four friends may receive zero cookies.)

**Solution.** The answer is $\boxed{2601}$.

We see that Jeff and Qiao must receive the same number of cookies. Thus if each receives $k$ cookies, then there are $100 - 2k$ cookies to split between the two other friends, which we can do in $101 - 2k$ ways. By summing $101 - 2k$ over all $k$, $0 \leq k \leq 50$, we see that our answer is the sum of the first 51 odd numbers, equal to $51^2 = 2601$.

### 2.4.4 Round 4

10. [11] Compute the smallest positive integer with at least four two-digit positive divisors.

**Solution.** The answer is $\boxed{48}$.

Let the number be $N$. If $N < 50$, then its 4 two-digit divisors must be $N, \frac{N}{2}, \frac{N}{3}, \frac{N}{4}$. This requires that $N \geq 40$ and that $N$ is a multiple of 2, 3 and 4. It is easy to see that $N = 48$ is the smallest such integeer, and indeed 48, 24, 16, 12 are all its two-digit divisors.

11. [11] Let $ABCD$ be a trapezoid such that $AB$ is parallel to $CD$, $BC = 10$ and $AD = 18$. Given that the two circles with diameters $BC$ and $AD$ are tangent, find the perimeter of $ABCD$.

**Solution.** The answer is $\boxed{56}$.

Let $M$ be the midpoint of $BC$ and $N$ be the midpoint of $AD$. Because the circles are tangent, we have $MN = \frac{10}{2} + \frac{18}{2} = 14$. In addition, observe that $MN = \frac{AB+CD}{2}$. Thus the perimeter of $ABCD$ is $AB + BC + CD + AD = 10 + 18 + 2 \cdot MN = 56$.

12. [11] How many length ten strings consisting of only $A$s and $B$s contain neither "$BAB$" nor "$BBB$" as a substring?

**Solution.** The answer is $\boxed{169}$.

The condition is equivalent to the statement that no two $B$s have exactly one letter between them. Consider the two length five strings derived from the even-indexed and odd-indexed letters of the original string. Observe that neither of these two strings can have two consecutive $B$s.

Now, suppose the total number of length $n$ strings with no two consecutive $B$s is $F_n$. If the first letter is $A$, then there are $F_{n-1}$ possibilities; if the first letter is $B$, then the second letter must be $A$, leaving us with $F_{n-2}$ more possibilities. Therefore, $F_n = F_{n-1} + F_{n-2}$, which yields the Fibonacci recursion with $F_1 = 2$ and $F_2 = 3$. Thus, it is clear that $F_5 = 13$ and that the answer to this problem is $(F_5)^2 = 169$.

### 2.4.5 Round 5

*Each of the three problems in this round depends on the answer to two of the other problems. There is only one set of correct answers to these problems; however, each problem will be scored independently, regardless of whether the answers to the other problems are correct.*

13. [13] Let $B$ be the answer to problem 14, and let $C$ be the answer to problem 15. A quadratic function $f(x)$ has two real roots that sum to $2^{10} + 4$. After translating the graph of $f(x)$ left by $B$ units and down by $C$ units, the new quadratic function also has two real roots. Find the sum of the two real roots of the new quadratic function.

**Solution.** The answer is $\boxed{4}$.

Note that translating the quadratic down by $C$ does not affect the sum of the roots. However, translating the quadratic to the left by $B$ decreases the value of both roots by $B$, making the new sum $1028 - 2B$.

14. [13] Let $A$ be the answer to problem 13, and let $C$ be the answer to problem 15. In the interior of angle $NOM = 45°$, there is a point $P$ such that $\angle MOP = A°$ and $OP = C$. Let $X$ and $Y$ be the reflections of $P$ over $MO$ and $NO$, respectively. Find $(XY)^2$.

    **Solution.** The answer is $\boxed{512}$.

    Because of reflections, $\angle XOM = A°$ and $\angle NOY = (45 - A)°$, which means that $\angle XOY = \angle XOM + \angle MON + \angle NOY = 90°$. Also, $XO = YO = PO = C$, so we conclude that $(XY)^2 = (\sqrt{2}C)^2 = 2C^2$.

15. [13] Let $A$ be the answer to problem 13, and let $B$ be the answer to problem 14. Totoro hides a guava at point $X$ in a flat field and a mango at point $Y$ different from $X$ such that the length $XY$ is $B$. He wants to hide a papaya at point $Z$ such that $YZ$ has length $A$ and the distance $ZX$ is a nonnegative integer. In how many different locations can he hide the papaya?

    **Solution.** The answer is $\boxed{16}$.

    First we note that the answer to this problem, $C$ must be a nonnegative integer. We also have $B = 2C^2$ and $A = 1028 - 2B$, so $A$ and $B$ are also nonnegative integers. We also notice that $X$ different from $Y$ means that $B$ is positive. By the triangle inequality,

$$|A - B| \leq ZX \leq A + B.$$

If $ZX = |A - B|$, there is only one location $Z$ he can hide the papaya, namely, the point on $XY$ such that $YZ = A$ and $ZX = |A - B|$. Similarly, if $ZX = A + B$, there is only one location. For all

$$|A - B| < ZX < A + B,$$

there are 2 locations, reflections of each other over $XY$. In total, there are

$$2 + 2(A + B - |A - B| - 1) = 4\min(A, B)$$

locations to hide the papaya.

*Note:* It remains to solve a system of three equations generated by the answers to the problems:

$$\{A = 1028 - 2B, B = 2C^2, C = 4\min(A, B)\}.$$

If $B \geq A$, then

$$C = 4\min(A, B) = 4B \rightarrow B = 2C^2 = 2(4B)^2 = 32B^2,$$

implying that $B = 0$ or $B = \frac{1}{32}$. None of these values work, so thus $A < B$, yielding

$$C = 4A \rightarrow B = 2C^2 = 32A^2 \rightarrow A = 1028 - 2B = 1028 - 64A^2.$$

Solving the quadratic yields $A = 4$ or $A = -\frac{257}{64}$. We conclude that $A = 4, B = 512, C = 16$.

### 2.4.6 Round 6

16. [15] Let $ABCD$ be a trapezoid such that $AB$ is parallel to $CD$, $AB = 4$, $CD = 8$, $BC = 5$, and $AD = 6$. Given that point $E$ is on segment $CD$ and that $AE$ is parallel to $BC$, find the ratio between the area of trapezoid $ABCD$ and the area of triangle $ABE$.

**Solution.** The answer is $\boxed{3}$.

Suppose the height of the trapezoid is $h$. Then the altitude of triangle $ABE$ from $E$ to $AB$ also has length $h$. Thus the ratio between the area of trapezoid $ABCD$ and the area of triangle $ABE$ equals

$$\frac{(AB+CD)\cdot h/2}{AB\cdot h/2} = \frac{4+8}{4} = 3.$$

17. [15] Find the maximum possible value of the greatest common divisor of $\overline{MOO}$ and $\overline{MOOSE}$, given that $S$, $O$, $M$, and $E$ are some nonzero digits. (The digits $S$, $O$, $M$, and $E$ are not necessarily pairwise distinct.)

**Solution.** The answer is $\boxed{98}$.

Note that $\overline{MOOSE} = 100 \cdot \overline{MOO} + \overline{SE}$. Thus $\gcd(\overline{MOO}, \overline{MOOSE}) = \gcd(\overline{MOO}, \overline{SE})$. Since $M$ is a non-zero digit, $\overline{SE}$ is less than $\overline{MOO}$. This means that the gcd is at most $\overline{SE}$, which occurs when $\overline{MOO}$ is a multiple of $\overline{SE}$. Therefore, the largest possible gcd is 99. However, it is not difficult to check that 99 has no three-digit multiple that ends with two of the same digit. Looking at the next largest number, we see that we do have a solution, with $\gcd(588, 98) = 98$.

*Note:* We can check 98 and 99 by writing them as $100-k$, in which case we want $n\cdot(100-k) = \overline{n00}-nk$ to end with two of the same digit. When checking 99, $k = 1$, so $n$ has to be 1 for the last two digits to be the same, which clearly doesn't work. When checking 98, we have $k = 2$. This works if we want the last two digits to be 88: $nk = 2n = 100 - 88 \Rightarrow n = 6$.)

18. [15] Suppose that 125 politicians sit around a conference table. Each politician either always tells the truth or always lies. (Statements of a liar are never completely true, but can be partially true.) Each politician now claims that the two people beside them are both liars. Suppose that the greatest possible number of liars is $M$ and that the least possible number of liars is $N$. Determine the ordered pair $(M, N)$.

**Solution.** The answer is $\boxed{(83, 63)}$.

If a politician always tells truth, then his/her two neighbors are liars. On the other hand, if a politician is a liar, then at least one of his/her neighbors always tells truth. From this we deduce that there cannot be two truth-tellers in a row, nor can there be three liars in a row. So between two truth-tellers there is either one or two liars. So if there are $L$ liars, we require that $1 \le \frac{L}{125-L} \le 2$. Solving this inequality, we get that $\frac{250}{3} \ge L \ge \frac{125}{2}$. So $M = 83, N = 63$.

Now we need to construct the case where we achieve maximum and minimum respectively. For ease of reference label each person 1, 2, $\cdots$, 125 in clockwise order. The minimum can be achieved when 1, 3, 5, $\cdots$, 123 are all truth-tellers, for 63 liars in all. The maximum can be achieved when 1, 4, 7, $\cdots$, 124 are all truth-tellers, for 83 liars in all.

### 2.4.7 Round 7

19. [18] Define a *lucky* number as a number that only contains 4s and 7s in its decimal representation. Find the sum of all three-digit lucky numbers.

**Solution.** The answer is $\boxed{4884}$.

Observe that the average of all of the numbers is $\frac{4+7}{2} \cdot 111$. Since there are $2^3 = 8$ lucky numbers, their sum must be $8 \cdot \frac{4+7}{2} \cdot 111 = 4884$.

20. [18] Let line segment $AB$ have length 25 and let points $C$ and $D$ lie on the same side of line $AB$ such that $AC = 15$, $AD = 24$, $BC = 20$, and $BD = 7$. Given that rays $AC$ and $BD$ intersect at point $E$, compute $EA + EB$.

**Solution.** The answer is $\boxed{55}$.

It is trivial to verify that $\angle BDA = \angle ACB = 90°$. So we see that triangle $EDA$ is similar to triangle $ECB$ because they share angle $E$ and both have a right angle. Now let $EC = x, ED = y$, and we have the following set of equations (by using the pair of similar triangles):

$$\frac{x}{20} = \frac{y}{24},$$

$$\frac{x+15}{24} = \frac{y+7}{20}.$$

Solving the set of equations gives $x = 15, y = 18$, so $EA + EB = (x + 15) + (y + 7) = 55$.

21. [18] A $3 \times 3$ grid is filled with positive integers and has the property that each integer divides both the integer directly above it and directly to the right of it. Given that the number in the top-right corner is 30, how many distinct grids are possible?

**Solution.** The answer is $\boxed{6859}$.

It is clear that all integers in the grid should have no prime factors other than 2, 3, or 5, and these prime factors should only divide each of them at most once. Define a *0-1 grid* as a 3 by 3 grid filled with integers 0 or 1, with the top-right corner filled with 1, such that each integer is no larger than the one above or to the right of it. It is clear that for each prime factor, we can use a 0-1 grid to show the divisibility of each number in the grid by this factor (for example, if a number is even, it can be replaced by 1 in the 0-1 grid for the prime 2, and otherwise it can be replaced by 0 in the 0-1 grid). Also we can see that we can set up a one-to-one correspondence from the original 3 by 3 grid to an ordered triple of 0-1 grids (each 0-1 grid is for one prime factor). Therefore, it suffices to find the number of 0-1 grids and cube that number to get our desired answer.

If we observe each row of 0-1 grid, it should be one of 000, 001, 011, or 111. And a row should contain at least as many 1s as the row below it. Therefore, the number of 0-1 grids is equal to the number of ordered triples $(a, b, c)$ of integers between 0 and 3 inclusive, such that $a \geq 1$ and $a \geq b \geq c$.

It is not difficult to see that when $a = 3$, there are 10 ways to set a value for $b$ and $c$, and there are 6 and 3 ways when $a = 2$, $a = 1$ respective, so the number of such triples is $10+6+3 = 19$, and therefore there are 19 0-1 grids in all. By cubing the number, we get the answer $19^3 = 6859$.

### 2.4.8 Round 8

22. [22] Define a sequence of positive integers $s_1, s_2, \ldots, s_{10}$ to be *terrible* if the following conditions are satisfied for any pair of positive integers $i$ and $j$ satisfying $1 \leq i < j \leq 10$:

- $s_i > s_j$
- $j - i + 1$ divides the quantity $s_i + s_{i+1} + \cdots + s_j$

Determine the minimum possible value of $s_1 + s_2 + \cdots + s_{10}$ over all terrible sequences.

**Solution.** The answer is $\boxed{100}$.

The second condition is equivalent to the mean of any set of consecutive numbers in the sequence is an integer. Observe that for all integer $9 \geq i \geq 1$, we have $s_i - s_{i+1} \geq 2$. Otherwise if $s_i - s_{i+1} = 1$, 2 won't divide $s_i + s_{i+1}$, which is odd. Since $s_{10} \geq 1$, from the preceeding observation we see that $s_9, s_8, s_7, \cdots, s_1$ are at least $3, 5, 7, \cdots, 19$ respectively.

Now we show that this minimization is optimal, that is, the sequence $19, 17, \cdots, 1$ is truly terrible. Observe that $s_i = (10-i)^2 - (9-i)^2$ for all $i$ between 1 and 10 inclusive. Thus $s_i + s_{i+1} + \cdots + s_j = (10-i)^2 - (9-j)^2 = (19-i-j)(1+j-i)$, and the sum is clearly divisible by $j - i + 1$. Noting that the sum of the entire sequence is $10^2 = 100$, the desired answer is therefore 100.

23. [22] The four points $(x, y)$ that satisfy $x = y^2 - 37$ and $y = x^2 - 37$ form a convex quadrilateral in the coordinate plane. Given that the diagonals of this quadrilateral intersect at point $P$, find the coordinates of $P$ as an ordered pair.

**Solution.** The answer is $\boxed{(-\frac{1}{2}, -\frac{1}{2})}$.

Subtracting the two equations gives us $x - y = y^2 - x^2 \Leftrightarrow (x-y)(x+y-1) = 0$. Thus each of the solutions to this system satisfy at least one of $x - y = 0$ and $x + y - 1 = 0$. These, then, must be two of the lines that the vertices of the quadrilateral lie on. By symmetry, if $(x, y)$ is a solution, then $(y, x)$ is also a solution, therefore the solutions on the line $x + y = 1$ must be symmetric about $y = x$, and thus lie on different sides of the line. Because we are given that the quadrilateral formed by the solutions is convex, these lines must also be the diagonals of the quadrilateral. Hence we can solve this system of linear equations to find that the diagonals intersect at $(-\frac{1}{2}, -\frac{1}{2})$.

24. [22] Consider a non-empty set of segments of length 1 in the plane which do not intersect except at their endpoints. (In other words, if point $P$ lies on distinct segments $a$ and $b$, then $P$ is an endpoint of both $a$ and $b$.) This set is called *3-amazing* if each endpoint of a segment is the endpoint of exactly three segments in the set. Find the smallest possible size of a 3-amazing set of segments.

**Solution.** The answer is $\boxed{12}$.

Since each endpoint is an endpoint of exactly 3 segments, and each segment has 2 endpoints, if we assume that there are $2k$ endpoints there would be $\frac{2k \cdot 3}{2} = 3k$ segments. Now we claim that there doesn't exist a 3-amazing construction for $k < 4$.

First when $k = 1$, we have 2 points and 3 segments, but there can be at most 1 segment between 2 endpoints, so there's no such 3-amazing set. When $k = 2$, we have 4 points and 6 segments, implying that each two points should be 1 apart from each other, yielding the only possible configuration of a regular tetrahedron, which is not planar.

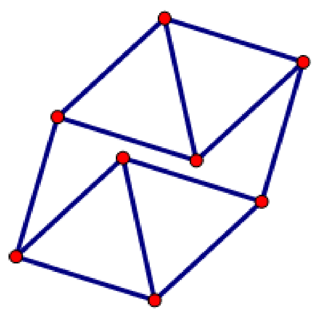
When $k = 3$, we have 6 points and 9 segments. Call these points $A, B, C, D, E, F$, and WLOG assume $A$ is connected to $B, C, D$. From the previous case we see that at least one of $BC, CD, DB$ is not 1, so WLOG assume $CD$ is not connected. We now split into cases.

*Case 1*: $BC$ and $BD$ are both connected. So $C$ and $D$ both needs to connect to one more point while $E$ and $F$ needs to connect to three more points. This implies that both $E$ and $F$ are connected to both $C$ and $D$, yielding contradiction.

*Case 2*: One of $BC, BD$ is connected. WLOG assume $BC$ is connected. So $D$ needs to be connected to both $E$ and $F$. Moreover, $B$ and $C$ needs to be connected to 1 more point, and $E$ and $F$ both needs 2. So WLOG assume $BE, CF, EF$ are connected. Therefore $ABC$ and $DEF$ are both equilateral triangles, with $AD, BE, CF$ connected. It is easy to show that this configuration must involve intersections, so it is not possible to achieve this case.

*Case 3*: Neither $BC$ not $BD$ is connected. So all of $B, C, D$ needs to be connected to $E$ and $F$. This means that $BE = BF = CE = CF = DE = DF = 1$. But given two points $E, F$ in the plane, there can be at most two distinct points $P$ such that $PE = PF = 1$, which means that at least two of $B, C, D$ are the same point, yielding contradiction.

However, there exists a construction of 3-amazing sets with 12 segments:



We showed that there are no 3-amazing sets with less than 12 segments and there is such a set with exactly 12 segments, so the answer is 12.

# Exeter Math Club Competition
## January 23, 2016

## Table of Contents

## Organizing Acknowledgments

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Jane Street

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- **Head Grader** Alec Sun
- **Graders** Alec Sun, Yuan Yao, Kevin Sun, Vinjai Vale, Ivan Borensco, Ravi Jagadeesan, Leigh Marie Braswell, Dai Yang, Ray Li, Brian Liu
- **Judges** Zuming Feng, Greg Spanier

# Chapter 1

# EMC$^2$ 2016 Problems

## 1.1 Speed Test

*There are 20 problems, worth 3 points each, to be solved in 25 minutes.*

1. Compute the value of $2 + 20 + 201 + 2016$.

2. Gleb is making a doll, whose prototype is a cube with side length 5 centimeters. If the density of the toy is 4 grams per cubic centimeter, compute its mass in grams.

3. Find the sum of 20% of 16 and 16% of 20.

4. How many times does Akmal need to roll a standard six-sided die in order to guarantee that two of the rolled values sum to an even number?

5. During a period of one month, there are ten days without rain and twenty days without snow. What is the positive difference between the number of rainy days and the number of snowy days?

6. Joanna has a fully charged phone. After using it for 30 minutes, she notices that 20 percent of the battery has been consumed. Assuming a constant battery consumption rate, for how many additional minutes can she use the phone until 20 percent of the battery remains?

7. In a square $ABCD$, points $P$, $Q$, $R$, and $S$ are chosen on sides $AB$, $BC$, $CD$, and $DA$ respectively, such that $AP = 2PB$, $BQ = 2QC$, $CR = 2RD$, and $DS = 2SA$. What fraction of square $ABCD$ is contained within square $PQRS$?

8. The sum of the reciprocals of two not necessarily distinct positive integers is 1. Compute the sum of these two positive integers.

9. In a room of government officials, two-thirds of the men are standing and 8 women are standing. There are twice as many standing men as standing women and twice as many women in total as men in total. Find the total number of government officials in the room.

10. A string of lowercase English letters is called *pseudo-Japanese* if it begins with a consonant and alternates between consonants and vowels. (Here the letter "y" is considered neither a consonant nor vowel.) How many 4-letter pseudo-Japanese strings are there?

11. In a wooden box, there are 2 identical black balls, 2 identical grey balls, and 1 white ball. Yuka randomly draws two balls in succession without replacement. What is the probability that the first ball is strictly darker than the second one?

12. Compute the real number $x$ for which
$$(x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2.$$

13. Let $ABC$ be an isosceles right triangle with $\angle C = 90^\circ$ and $AB = 2$. Let $D$, $E$, and $F$ be points outside $ABC$ in the same plane such that the triangles $DBC$, $AEC$, and $ABF$ are isosceles right triangles with hypotenuses $BC, AC$, and $AB$, respectively. Find the area of triangle $DEF$.

14. Salma is thinking of a six-digit positive integer $n$ divisible by 90. If the sum of the digits of $n$ is divisible by 5, find $n$.

15. Kiady ate a total of 100 bananas over five days. On the $(i + 1)$-th day $(1 \le i \le 4)$, he ate $i$ more bananas than he did on the $i$-th day. How many bananas did he eat on the fifth day?

16. In a unit equilateral triangle $ABC$, points $D, E$, and $F$ are chosen on sides $BC, CA$, and $AB$, respectively. If lines $DE, EF$, and $FD$ are perpendicular to $CA$, $AB$, and $BC$, respectively, compute the area of triangle $DEF$.

17. Carlos rolls three standard six-sided dice. What is the probability that the product of the three numbers on the top faces has units digit 5?

18. Find the positive integer $n$ for which $n^{n^n} = 3^{3^{82}}$.

19. John folds a rope in half five times then cuts the folded rope with four knife cuts, leaving five stacks of rope segments. How many pieces of rope does he now have?

20. An integer $n > 1$ is *conglomerate* if all positive integers less than $n$ and relatively prime to $n$ are not composite. For example, 3 is conglomerate since 1 and 2 are not composite. Find the sum of all conglomerate integers less than or equal to 200.

## 1.2 Accuracy Test

*There are 10 problems, worth 9 points each, to be solved in 45 minutes.*

1. A right triangle has a hypotenuse of length 25 and a leg of length 16. Compute the length of the other leg of this triangle.

2. Tanya has a circular necklace with 5 evenly-spaced beads, each colored red or blue. Find the number of distinct necklaces in which no two red beads are adjacent. If a necklace can be transformed into another necklace through a series of rotations and reflections, then the two necklaces are considered to be the same.

3. Find the sum of the digits in the decimal representation of $10^{2016} - 2016$.

4. Let $x$ be a real number satisfying
$$x^1 \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 = 8^7.$$
Compute the value of $x^7$.

5. What is the smallest possible perimeter of an acute, scalene triangle with integer side lengths?

6. Call a sequence $a_1, a_2, a_3, \ldots, a_n$ *mountainous* if there exists an index $t$ between 1 and $n$ inclusive such that
$$a_1 \le a_2 \le \cdots \le a_t \quad \text{and} \quad a_t \ge a_{t+1} \ge \cdots \ge a_n.$$
In how many ways can Bishal arrange the ten numbers $1, 1, 2, 2, 3, 3, 4, 4, 5$, and 5 into a mountainous sequence? (Two possible mountainous sequences are $1, 1, 2, 3, 4, 4, 5, 5, 3, 2$ and $5, 5, 4, 4, 3, 3, 2, 2, 1, 1$.)

7. Find the sum of the areas of all (non self-intersecting) quadrilaterals whose vertices are the four points $(-3, -6), (7, -1), (-2, 9)$, and $(0, 0)$.

8. Mohammed Zhang's favorite function is $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8}$. Find the minumum possible value of $f(x)$ over all real numbers $x$.

9. A segment $AB$ with length 1 lies on a plane. Find the area of the set of points $P$ in the plane for which $\angle APB$ is the second smallest angle in triangle $ABP$.

10. A binary string is a *dipalindrome* if it can be produced by writing two non-empty palindromic strings one after the other. For example, 10100100 is a dipalindrome because both 101 and 00100 are palindromes. How many binary strings of length 18 are both palindromes and dipalindromes?

## 1.3 Team Test

*There are 15 problems, worth 20 points each, to be solved in 45 minutes.*

1. Lisa is playing the piano at a tempo of 80 beats per minute. If four beats make one measure of her rhythm, how many seconds are in one measure?

2. Compute the smallest integer $n > 1$ whose base-2 and base-3 representations both do not contain the digit 0.

3. In a room of 24 people, $5/6$ of the people are old, and $5/8$ of the people are male. At least how many people are both old and male?

4. Juan chooses a random even integer from 1 to 15 inclusive, and Gina chooses a random odd integer from 1 to 15 inclusive. What is the probability that Juan's number is larger than Gina's number? (They choose all possible integers with equal probability.)

5. Set $S$ consists of all positive integers less than or equal to 2016. Let $A$ be the subset of $S$ consisting of all multiples of 6. Let $B$ be the subset of $S$ consisting of all multiples of 7. Compute the ratio of the number of positive integers in $A$ but not $B$ to the number of integers in $B$ but not $A$.

6. Three peas form a unit equilateral triangle on a flat table. Sebastian moves one of the peas a distance $d$ along the table to form a right triangle. Determine the minimum possible value of $d$.

7. Oumar is four times as old as Marta. In $m$ years, Oumar will be three times as old as Marta will be. In another $n$ years after that, Oumar will be twice as old as Marta will be. Compute the ratio $m/n$.

8. Compute the area of the smallest square in which one can inscribe two non-overlapping equilateral triangles with side length 1.

9. Teemu, Marcus, and Sander are signing documents. If they all work together, they would finish in 6 hours. If only Teemu and Sander work together, the work would be finished in 8 hours. If only Marcus and Sander work together, the work would be finished in 10 hours. How many hours would Sander take to finish signing if he worked alone?

10. Triangle $ABC$ has a right angle at $B$. A circle centered at $B$ with radius $BA$ intersects side $AC$ at a point $D$ different from $A$. Given that $AD = 20$ and $DC = 16$, find the length of $BA$.

11. A regular hexagon $H$ with side length 20 is divided completely into equilateral triangles with side length 1. How many regular hexagons with sides parallel to the sides of $H$ are formed by lines in the grid?

12. In convex pentagon $PEARL$, quadrilateral $PERL$ is a trapezoid with side $PL$ parallel to side $ER$. The areas of triangle $ERA$, triangle $LAP$, and trapezoid $PERL$ are all equal. Compute the ratio $\dfrac{PL}{ER}$.

13. Let $m$ and $n$ be positive integers with $m < n$. The first two digits after the decimal point in the decimal representation of the fraction $\frac{m}{n}$ are 74. What is the smallest possible value of $n$?

14. Define functions
$$f(x, y) = \frac{x+y}{2} - \sqrt{xy} \quad \text{and} \quad g(x, y) = \frac{x+y}{2} + \sqrt{xy}.$$
Compute
$$g\left(g\left(f\left(1,3\right),\ f\left(5,7\right)\right),\ g\left(f\left(3,5\right),\ f\left(7,9\right)\right)\right).$$

15. Natalia plants two gardens in a $5 \times 5$ grid of points. Each garden is the interior of a rectangle with vertices on grid points and sides parallel to the sides of the grid. How many unordered pairs of two non-overlapping rectangles can Nataliia choose as gardens? (The two rectangles may share an edge or part of an edge but should not share an interior point.)

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. [5] Suppose that gold satisfies the relation $p = v + v^2$, where $p$ is the price and $v$ is the volume. How many pieces of gold with volume 1 can be bought for the price of a piece with volume 2?

2. [5] Find the smallest prime number with each digit greater or equal to 8.

3. [5] What fraction of regular hexagon $ZUMING$ is covered by both quadrilateral $ZUMI$ and quadrilateral $MING$?

### 1.4.2 Round 2

4. [7] The two smallest positive integers expressible as the sum of two (not necessarily positive) perfect cubes are $1 = 1^3 + 0^3$ and $2 = 1^3 + 1^3$. Find the next smallest positive integer expressible in this form.

5. [7] In how many ways can the numbers $1, 2, 3$, and 4 be written in a row such that no two adjacent numbers differ by exactly 1?

6. [7] A real number is placed in each cell of a grid with 3 rows and 4 columns. The average of the numbers in each column is 2016, and the average of the numbers in each row is a constant $x$. Compute $x$.

### 1.4.3 Round 3

7. [9] Fardin is walking from his home to his office at a speed of 1 meter per second, expecting to arrive exactly on time. When he is halfway there, he realizes that he forgot to bring his pocketwatch, so he runs back to his house at 2 meters per second. If he now decides to travel from his home to his office at $x$ meters per second, find the minimum $x$ that will allow him to be on time.

8. [9] In triangle $ABC$, the angle bisector of $\angle B$ intersects the perpendicular bisector of $AB$ at point $P$ on segment $AC$. Given that $\angle C = 60^\circ$, determine the measure of $\angle CPB$ in degrees.

9. [9] Katie colors each of the cells of a $6 \times 6$ grid either black or white. From top to bottom, the number of black squares in each row are 1, 2, 3, 4, 5, and 6, respectively. From left to right, the number of black squares in each column are 6, 5, 4, 3, 2, and 1, respectively. In how many ways could Katie have colored the grid?

### 1.4.4 Round 4

10. [11] Lily stands at the origin of a number line. Each second, she either moves 2 units to the right or 1 unit to the left. At how many different places could she be after 2016 seconds?

11. [11] There are 125 politicians standing in a row. Each either always tells the truth or always lies. Furthermore, each politician (except the leftmost politician) claims that at least half of the people to his left always lie. Find the number of politicians that always lie.

12. [11] Two concentric circles with radii 2 and 5 are drawn on the plane. What is the side length of the largest square whose area is contained entirely by the region between the two circles?

### 1.4.5 Round 5

13. [13] Initially, the three numbers 20, 201, and 2016 are written on a blackboard. Each minute, Zhuo selects two of the numbers on the board and adds 1 to each. Find the minimum $n$ for which Zhuo can make all three numbers equal to $n$.

14. [13] Call a three-letter string *rearrangeable* if, when the first letter is moved to the end, the resulting string comes later alphabetically than the original string. For example, $AAA$ and $BAA$ are not rearrangeable, while $ABB$ is rearrangeable. How many three-letters strings with (not necessarily distinct) uppercase letters are rearrangeable?

15. [13] Triangle $ABC$ is an isosceles right triangle with $\angle C = 90^\circ$ and $AC = 1$. Points $D, E$, and $F$ are chosen on sides $BC, CA$, and $AB$, respectively, such that $AEF, BFD, CDE$, and $DEF$ are isosceles right triangles. Find the sum of all distinct possible lengths of segment $DE$.

### 1.4.6 Round 6

16. [15] Let $p$, $q$, and $r$ be prime numbers such that $pqr = 17(p + q + r)$. Find the value of the product $pqr$.

17. [15] A cylindrical cup containing some water is tilted 45 degrees from the vertical. The point on the surface of the water closest to the bottom of the cup is 6 units away. The point on the surface of the water farthest from the bottom of the cup is 10 units away. Compute the volume of the water in the cup.

18. [15] Each dot in an equilateral triangular grid with 63 rows and $2016 = \frac{1}{2} \cdot 63 \cdot 64$ dots is colored black or white. Every unit equilateral triangle with three dots has the property that exactly one of its vertices is colored black. Find all possible values of the number of black dots in the grid.

### 1.4.7 Round 7

19. [18] Tomasz starts with the number 2. Each minute, he either adds 2 to his number, subtracts 2 from his number, multiplies his number by 2, or divides his number by 2. Find the minimum number of minutes he will need in order to make his number equal 2016.

20. [18] The edges of a regular octahedron $ABCDEF$ are painted with 3 distinct colors such that no two edges with the same color lie on the same face. In how many ways can the octahedron be painted? Colorings are considered different under rotation or reflection.

21. [18] Jacob is trapped inside an equilateral triangle $ABC$ and must visit each edge of triangle $ABC$ at least once. (Visiting an edge means reaching a point on the edge.) His distances to sides $AB$, $BC$, and $CA$ are currently 3, 4, and 5, respectively. If he does not need to return to his starting point, compute the least possible distance that Jacob must travel.

### 1.4.8 Round 8

22. [22] Four integers $a, b, c$, and $d$ with $a \leq b \leq c \leq d$ satisfy the property that the product of any two of them is equal to the sum of the other two. Given that the four numbers are not all equal, determine the 4-tuple $(a, b, c, d)$.

23. [22] In equilateral triangle $ABC$, points $D, E$, and $F$ lie on sides $BC, CA$, and $AB$, respectively, such that $BD = 4$ and $CD = 5$. If $DEF$ is an isosceles right triangle with right angle at $D$, compute $EA + FA$.

24. [22] On each edge of a regular tetrahedron, four points that separate the edge into five equal segments are marked. There are sixteen planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these sixteen planes, how many new tetrahedrons are produced?

# Chapter 2

# EMC$^2$ 2016 Solutions

## 2.1 Speed Test Solutions

1. Compute the value of $2 + 20 + 201 + 2016$.

   **Solution.** The answer is $\boxed{2239}$.

   We compute $2 + 20 + 201 + 2016 = 2239$.

2. Gleb is making a doll, whose prototype is a cube with side length 5 centimeters. If the density of the toy is 4 grams per cubic centimeter, compute its mass in grams.

   **Solution.** The answer is $\boxed{500}$.

   The volume of the doll is $5^3 = 125$ cubic centimeters, so the mass is $4 \cdot 125 = 500$ grams.

3. Find the sum of 20% of 16 and 16% of 20.

   **Solution.** The answer is $\boxed{\frac{32}{5}}$.

   We compute
   $$20\% \cdot 16 + 16\% \cdot 20 = 320\% + 320\% = 640\% = \frac{32}{5}.$$

4. How many times does Akmal need to roll a standard six-sided die in order to guarantee that two of the rolled values sum to an even number?

   **Solution.** The answer is $\boxed{3}$.

   If Akmal rolls twice, then the numbers can be 1 and 2 and their sum will be odd. However when he rolls three times, by Pigeonhole Principle two of them will have the same parity and their sum will be even. Therefore the least number of rolls needed is 4.

5. During a period of one month, there are ten days without rain and twenty days without snow. What is the positive difference between the number of rainy days and the number of snowy days?

   **Solution.** The answer is $\boxed{10}$.

   Suppose a month has $n$ days. Then ten days without rain means $n - 10$ days with rain, and twenty days without snow means $n - 20$ days with snow. Therefore the difference is $(n - 10) - (n - 20) = (-10) + 20 = 10$ days.

6. Joanna has a fully charged phone. After using it for 30 minutes, she notices that 20 percent of the battery has been consumed. Assuming a constant battery consumption rate, for how many additional minutes can she use the phone until 20 percent of the battery remains?

   **Solution.** The answer is $\boxed{90}$.

   To go from 20 percent gone (or 80 percent remaining), to 20 percent remaining, Joanna needs to use 60 percent of the battery. Since using 20 percent takes 30 minutes, using 60 percent will take $\frac{30}{20} \cdot 60 = 90$ minutes.

7. In a square $ABCD$, points $P$, $Q$, $R$, and $S$ are chosen on sides $AB$, $BC$, $CD$, and $DA$ respectively, such that $AP = 2PB$, $BQ = 2QC$, $CR = 2RD$, and $DS = 2SA$. What fraction of square $ABCD$ is contained within square $PQRS$?

   **Solution.** The answer is $\boxed{\frac{5}{9}}$.

   Suppose the side length of the square $ABCD$ is 1. Then it is easy to see that the area of triangle $SAP$,$PBQ$,$QCR$,$RDS$ are all $\frac{1}{3} \cdot \frac{2}{3} \div 2 = \frac{1}{9}$, so the area of the quadrilateral $PQRS$ is $1 - 4 \cdot \frac{1}{9} = \frac{5}{9}$, and that is also the desired fraction since the whole area of $ABCD$ is 1.

8. The sum of the reciprocals of two not necessarily distinct positive integers is 1. Compute the sum of these two positive integers.

   **Solution.** The answer is $\boxed{4}$.

   Suppose the two integers are $a, b$. Then $\frac{1}{a} + \frac{1}{b} = 1$, so neither $a$ or $b$ can be 1. Hence $\frac{1}{a} + \frac{1}{b} \leq \frac{1}{2} + \frac{1}{2} = 1$, with the equality case of $a = b = 2$. Thus the answer is $a + b = 4$.

9. In a room of government officials, two-thirds of the men are standing and 8 women are standing. There are twice as many standing men as standing women and twice as many women in total as men in total. Find the total number of government officials in the room.

   **Solution.** The answer is $\boxed{72}$.

   There are $8 \cdot 2 = 16$ men standing, and thus $\frac{16}{2/3} = 24$ men in the room. Then we deduce there are $24 \cdot 2 = 48$ women, so the room contains $24 + 48 = 72$ government officials in total.

10. A string of lowercase English letters is called *pseudo-Japanese* if it begins with a consonant and alternates between consonants and vowels. (Here the letter "y" is considered neither a consonant nor vowel.) How many 4-letter pseudo-Japanese strings are there?

    **Solution.** The answer is $\boxed{10000}$.

    There are 5 vowels and $26 - 1 - 5 = 20$ consonants, so there are $20 \cdot 5 \cdot 20 \cdot 5 = 10000$ possible pseudo-Japanese strings.

11. In a wooden box, there are 2 identical black balls, 2 identical grey balls, and 1 white ball. Yuka randomly draws two balls in succession without replacement. What is the probability that the first ball is strictly darker than the second one?

    **Solution.** The answer is $\boxed{\frac{2}{5}}$.

    There are 2 pairs out of $\binom{5}{2} = 10$ pairs of balls that have the same color, so the probability of getting two balls of the same color is $\frac{1}{5}$. Since the probability of the first ball being darker than the second one is equal that of the first ball being lighter, both probabilities are $(1 - \frac{1}{5}) \div 2 = \frac{2}{5}$.

12. Compute the real number $x$ for which

$$(x+1)^2 + (x+2)^2 + (x+3)^2 = (x+4)^2 + (x+5)^2 + (x+6)^2.$$

**Solution.** The answer is $\boxed{-\frac{7}{2}}$.

Expanding both sides, we see that

$$x^2 + 2x + 1 + x^2 + 4x + 4 + x^2 + 6x + 9 = x^2 + 8x + 16 + x^2 + 10x + 25 + x^2 + 12x + 36.$$

This simplifies to $18x = -63$, or $x = -\frac{7}{2}$.

13. Let $ABC$ be an isosceles right triangle with $\angle C = 90^\circ$ and $AB = 2$. Let $D$, $E$, and $F$ be points outside $ABC$ in the same plane such that the triangles $DBC$, $AEC$, and $ABF$ are isosceles right triangles with hypotenuses $BC, AC$, and $AB$, respectively. Find the area of triangle $DEF$.

**Solution.** The answer is $\boxed{2}$.

Notice that $ACBF$ is a square with diagonal length 2 and $AEDB$ is a rectangle with length 2 and width 1. Therefore triangle $DEF$ has base $DE = 2$ and height $CF = 2$, and its area is $\frac{2 \cdot 2}{2} = 2$.

14. Salma is thinking of a six-digit positive integer $n$ divisible by 90. If the sum of the digits of $n$ is divisible by 5, find $n$.

**Solution.** The answer is $\boxed{999990}$.

Since $n$ is a multiple of 90, it ends with zero and the sum of digits is a multiple of 9. Since the sum of digits is also a multiple of 5, the sum of digits is a multiple of 45. Since there are at most five non-zero digits, each of them have to be a 9, so $n = 999990$.

15. Kiady ate a total of 100 bananas over five days. On the $(i + 1)$-th day $(1 \leq i \leq 4)$, he ate $i$ more bananas than he did on the $i$-th day. How many bananas did he eat on the fifth day?

**Solution.** The answer is $\boxed{26}$.

Suppose Kiady ate $x$ bananas on the first day, so he ate $(x+1)$ bananas on the second days, $(x+3)$ on the third day, $(x+6)$ on fourth and $(x+10)$ on fifth. Solving $x+(x+1)+(x+3)+(x+6)+(x+10) = 100$ gives $x = 16$, so on fifth day he ate $x + 10 = 26$ bananas.

16. In a unit equilateral triangle $ABC$, points $D, E$, and $F$ are chosen on sides $BC, CA$, and $AB$, respectively. If lines $DE, EF$, and $FD$ are perpendicular to $CA, AB$, and $BC$, respectively, compute the area of triangle $DEF$.

**Solution.** The answer is $\boxed{\frac{\sqrt{3}}{12}}$.

Notice that triangles $DEC$, $EFA$, $FDB$ are all 30-60-90 triangles, so $EA = 2AF, FB = 2BD, DC = 2CE$. Then $AE = BF = CD = \frac{2}{3}$ and $AF = BD = CE = \frac{1}{3}$, and thus $DE = EF = FD = \frac{\sqrt{3}}{3}$. Since the area of an equilateral triangle with side length $a$ has area $\frac{\sqrt{3}}{4}a^2$, the area of $\triangle DEF$ is equal to $\frac{\sqrt{3}}{4}(\frac{\sqrt{3}}{3})^2 = \frac{\sqrt{3}}{12}$.

17. Carlos rolls three standard six-sided dice. What is the probability that the product of the three numbers on the top faces has units digit 5?

**Solution.** The answer is $\boxed{\frac{19}{216}}$.

The unit digit of a product is a 5 if and only if all of the rolls are odd and at least one of them is a 5. There are $3^3 = 27$ ways to roll three odd numbers, and $2^3 = 8$ ways to roll three odd numbers *without* a 5, so there are $27 - 8 = 19$ ways in total, and the probability is $\frac{19}{6^3} = \frac{19}{216}$.

18. Find the positive integer $n$ for which $n^{n^n} = 3^{3^{82}}$.

    **Solution.** The answer is $\boxed{27}$.

    First note that $n$ must be a power of 3, so let $n = 3^k$. Now we have $n^{n^n} = (3^k)^{(3^k)^{3^k}} = 3^{k\cdot 3^{k\cdot 3^k}} = 3^{3^{82}}$, so $k \cdot 3^{k\cdot 3^k} = 3^{82}$, and $k$ must be a power of 3 as well. Notice that when $k = 3$, the LHS of the equation becomes $3 \cdot 3^{3\cdot 27} = 3^{82}$, which is equal to the RHS. Therefore $n = 3^3 = 27$.

19. John folds a rope in half five times then cuts the folded rope with four knife cuts, leaving five stacks of rope segments. How many pieces of rope does he now have?

    **Solution.** The answer is $\boxed{129}$.

    A rope folded in half five times has $2^5 = 32$ layers, and cutting into five equal parts requires $5 - 1 = 4$ cuts for each layer, or $4 \cdot 32 = 128$ cuts in total on the original rope. Since each cut creates one more piece, there will be $128 + 1 = 129$ disjoint pieces at the end.

20. An integer $n > 1$ is *conglomerate* if all positive integers less than $n$ and relatively prime to $n$ are not composite. For example, 3 is conglomerate since 1 and 2 are not composite. Find the sum of all conglomerate integers less than or equal to 200.

    **Solution.** The answer is $\boxed{107}$.

    If $p$ is the smallest prime not dividing $n$, then $p^2$ is the smallest composite number that is relatively prime to $n$, which means that $n$ is very composite if and only if $n < p^2$. Now we base our cases on $p$.

    If $p = 2$, then $n < 4$ and $n$ is odd. This gives $n = 3$.

    If $p = 3$, then $n < 9$, $n$ is even but not divisible by 3. This gives $n = 2, 4, 8$.

    If $p = 5$, then $n < 25$ and $n$ is a multiple of $2 \cdot 3$. This gives $n = 6, 12, 18, 24$.

    If $p = 7$, then $n < 49$ and $n$ is a multiple of $2 \cdot 3 \cdot 5$. This gives $n = 30$.

    If $p > 7$, then $n$ has to be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 = 210 > 200$, which yields no results inside the bounds.

    Therefore the sum of all possibilities is $3 + 2 + 4 + 8 + 6 + 12 + 18 + 24 + 30 = 107$.

## 2.2 Accuracy Test Solutions

1. A right triangle has a hypotenuse of length 25 and a leg of length 16. Compute the length of the other leg of this triangle.

   **Solution.** The answer is $\boxed{3\sqrt{41}}$.

   By the Pythagorean Theorem we have $16^2+x^2 = 25^2$, so the length of the other leg is $x = \sqrt{25^2 - 16^2} = \sqrt{(25-16)(25+16)} = 3\sqrt{41}$.

2. Tanya has a circular necklace with 5 evenly-spaced beads, each colored red or blue. Find the number of distinct necklaces in which no two red beads are adjacent. If a necklace can be transformed into another necklace through a series of rotations and reflections, then the two necklaces are considered to be the same.

   **Solution.** The answer is $\boxed{3}$.

   If there are no red beads there is one way (all blue). If there is one red bead then there is only one way as well, since all other ways can be rotated to make the red bead placed at the top. If there are two red beads then there is also only one way which separates the blue beads into groups of 1 and 2. Adding up all cases, we have $1+1+1 = 3$ distinct configurations.

3. Find the sum of the digits in the decimal representation of $10^{2016} - 2016$.

   **Solution.** The answer is $\boxed{18136}$.

   $10^{2016} - 2016$ is less than $10^{2016}$ so it has 2016 digits. It's easy to see that $N = \overline{999\cdots997984}$, where there are 2012 9's preceding the last four digits. Therefore the sum of digits is $2012 \cdot 9 + 7 + 9 + 8 + 4 = 18136$.

4. Let $x$ be a real number satisfying

   $$x^1 \cdot x^2 \cdot x^3 \cdot x^4 \cdot x^5 \cdot x^6 = 8^7.$$

   Compute the value of $x^7$.

   **Solution.** The answer is $\boxed{128}$.

   The LHS of the equation is equal to $x^{21}$, so $x^7 = \sqrt[3]{x^{21}} = \sqrt[3]{8^7} = (\sqrt[3]{8})^7 = 2^7 = 128$.

5. What is the smallest possible perimeter of an acute, scalene triangle with integer side lengths?

   **Solution.** The answer is $\boxed{15}$.

   When the longest side is 3, the shorter side must be at most 1 and 2, and the triangle is degenerate. When the longest side is 4, the shorter side must be at most 2 and 3, but since $2^2 + 3^2 < 4^2$ the triangle is obtuse. When the longest side is 5, the shorter side must be at most 3 and 4, but this triangle is a right triangle. When the longest side is 6, the shorter sides can be 4 and 5, and since $4^2 + 5^2 > 6^2$ the triangle is acute and its perimeter is $4 + 5 + 6 = 15$. If the sides are $3, 5, 6$ then it is obtuse since $3^2 + 5^2 < 6^2$. If the longest side is 7 or larger, the perimeter will be at least $7 + (7+1) = 15$ so we cannot improve the minimum. Therefore the shortest perimeter is 15.

6. Call a sequence $a_1, a_2, a_3, \dots, a_n$ *mountainous* if there exists an index $t$ between 1 and $n$ inclusive such that
$$a_1 \le a_2 \le \cdots \le a_t \quad \text{and} \quad a_t \ge a_{t+1} \ge \cdots \ge a_n.$$
In how many ways can Bishal arrange the ten numbers $1, 1, 2, 2, 3, 3, 4, 4, 5$, and $5$ into a mountainous sequence? (Two possible mountainous sequences are $1, 1, 2, 3, 4, 4, 5, 5, 3, 2$ and $5, 5, 4, 4, 3, 3, 2, 2, 1, 1$.)

**Solution.** The answer is $\boxed{81}$.

The two 5's have to be together in the sequence. Now we put 4's around them: there are three ways, corresponding to two 4's on the left, one 4 and no 4's. The same logic applies to 3's, 2's and 1's. So in total there are $3^4 = 81$ possible sequences.

7. Find the sum of the areas of all (non self-intersecting) quadrilaterals whose vertices are the four points $(-3, -6), (7, -1), (-2, 9)$, and $(0, 0)$.

**Solution.** The answer is $\boxed{145}$.

Let $A = (-3, -6), B = (7, -1), C = (-2, 9), O = (0, 0)$. There are three possible quadrilaterals: $ABCO, BCAO, CABO$, and each of the triangles $ABO, BCO, CAO$ are contained in exactly two of the quadrilaterals, so the sum of the areas is equal to twice the area of triangle $ABC$. By bounding the triangle with a rectangle and subtracting the three extra triangles from it, we can find the area of triangle $ABC$ to be $10 \cdot 15 - \frac{1}{2}(10 \cdot 5 + 1 \cdot 15 + 9 \cdot 10) = 150 - \frac{1}{2}(155) = \frac{145}{2}$, so the sum of the areas is equal to 145.

8. Mohammed Zhang's favorite function is $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8}$. Find the minumum possible value of $f(x)$ over all real numbers $x$.

**Solution.** The answer is $\boxed{5}$.

Observe that $f(x) = \sqrt{x^2 - 4x + 5} + \sqrt{x^2 + 4x + 8} = \sqrt{(x-2)^2 + (-1)^2} + \sqrt{(x+2)^2 + 2^2}$. If we let $A = (2, 1), B = (-2, -2), P = (x, 0)$, then $f(x) = AP + BP \le AB = \sqrt{(2+2)^2 + (1+2)^2} = 5$ by triangle inequality. The equality can be achieved when $P$ lies on segment $AB$, so the minimum is indeed 5.

9. A segment $AB$ with length 1 lies on a plane. Find the area of the set of points $P$ in the plane for which $\angle APB$ is the second smallest angle in triangle $ABP$.

**Solution.** The answer is $\boxed{\sqrt{3} + \frac{2}{3}\pi}$.

Since the largest angle in a triangle faces the longest side, and the smallest angle faces the shortest side, the original condition is equivalent to $AB$ being neither the longest nor the shortest side in the triangle $ABP$, which means either $AP > 1, BP < 1$, or $AP < 1, BP > 1$. The locus (the set of all such points) is therefore the region inside exactly one of the unit circles drawn around $A$ and $B$. Since the area in common is equal to $2(2 \cdot \frac{\pi}{6} - \frac{\sqrt{3}}{4}) = \frac{2}{3}\pi - \frac{\sqrt{3}}{2}$, the area of the desired regions is $2\pi - 2(\frac{2}{3}\pi - \frac{\sqrt{3}}{2}) = \sqrt{3} + \frac{2}{3}\pi$.

10. A binary string is a *dipalindrome* if it can be produced by writing two non-empty palindromic strings one after the other. For example, 10100100 is a dipalindrome because both 101 and 00100 are palindromes. How many binary strings of length 18 are both palindromes and dipalindromes?

**Solution.** The answer is $\boxed{36}$.

We make the claim that each dipalindrome must be comprised of at least two identical palindromes written back-to-back, and that one can only split it into two palindromes by separating between two of these copies.

*Proof:* For a string $S$, define $S'$ as the reversed version of $S$. A palindrome is therefore defined as a string $P$ that satisfies $P = P'$.

Now we prove our claim using induction. When the dipalindrome has length 2 the claim is true. Now suppose all dipalindromes with length less than or equal to $k$ satisfy this property. Consider a dipalindrome $Q$ with length $k+1$, which can be expressed as $S+T$, where $S, T$ are both palindromes. If $S$ and $T$ has the same length, then $S = T' = T = S'$ and we are done (if we can split it in another way we can simply consider $S$ and $T$ as having different length). Otherwise WLOG assume $T$ is longer than $S$, then since $Q$ itself is a palindrome, $T$ must be able to be expressed as $U + S' = U + S$ for some string $U$. But also since $Q = Q'$, $S + U + S = S' + U' + S' = S + U' + S$, so $U' = U$ and it is a palindrome. This means that $T$ itself must be a dipalindrome. By induction hypothesis $T$ is comprised of some number of identical palindromes $P$, and that $S$ must contain a smaller number of the same $P$. Therefore, $Q = S + T$ must contain only identical copies of $P$ as well and we proved our inductive step and thus proved our claim.

Now we need to find the number of dipalindromes using this property. When the string has length 18, it can be split into 2 identical palindromes of length 9, 3 palindromes of length 6, 6 palindromes of length 3, 9 palindromes of length 2, 18 palindromes of length 1. Here we need to use Principle of Inclusion-Exclusion for the case for 6 palindromes since it's accounted for twice in the 2-palindrome and 3-palindrome case. Other cases will be properly taken care of after this consideration. Making use of the fact that there are $2^{\lceil \frac{n}{2} \rceil}$ binary palindromes with length $n$, the answer will be

$$2^{\lceil \frac{9}{2} \rceil} + 2^{\lceil \frac{6}{2} \rceil} - 2^{\lceil \frac{3}{2} \rceil} = 32 + 8 - 4 = 36.$$

## 2.3 Team Test Solutions

1. Lisa is playing the piano at a tempo of 80 beats per minute. If four beats make one measure of her rhythm, how many seconds are in one measure?

   **Solution.** The answer is $\boxed{3}$.

   Lisa plays the piano at $\frac{80}{4} = 20$ measures per minute, or 60 seconds, so each measure takes $\frac{60}{20} = 3$ seconds.

2. Compute the smallest integer $n > 1$ whose base-2 and base-3 representations both do not contain the digit 0.

   **Solution.** The answer is $\boxed{7}$.

   Since base 2 uses digits 0 and 1 only, the base-2 representation of $n$ must contain only 1s. So we list $11_2 = 3, 111_2 = 7, 1111_2 = 15, \cdots$ and express each of them in base-3. We see that $3 = 10_3$ doesn't work but $7 = 21_3$ does, so the smallest integer is $n = 7$.

3. In a room of 24 people, $5/6$ of the people are old, and $5/8$ of the people are male. At least how many people are both old and male?

   **Solution.** The answer is $\boxed{11}$.

   There are $24 \cdot \frac{5}{6} = 20$ people older than 20 years old, and $24 \cdot \frac{5}{8} = 15$ people who are male. By the Principle of Inclusion-Exclusion, there are at least $20 + 15 - 24 = 11$ people who satisfy both properties.

4. Juan chooses a random even integer from 1 to 15 inclusive, and Gina chooses a random odd integer from 1 to 15 inclusive. What is the probability that Juan's number is larger than Gina's number? (They choose all possible integers with equal probability.)

   **Solution.** The answer is $\boxed{\frac{1}{2}}$.

   Suppose Juan and Gina chooses $J$ and $G$, respectively. Since each ordered pair $(J, G)$ for which $J > G$ can be paired with a pair $(16 - J, 16 - G)$ for which $16 - J < 16 - G$, the probability of Juan's number exceeding Gina's is equal to the probability of it being smaller than Gina's. Because it's impossible for $J = G$, the desired probability is $\frac{1}{2}$.

5. Set $S$ consists of all positive integers less than or equal to 2016. Let $A$ be the subset of $S$ consisting of all multiples of 6. Let $B$ be the subset of $S$ consisting of all multiples of 7. Compute the ratio of the number of positive integers in $A$ but not $B$ to the number of integers in $B$ but not $A$.

   **Solution.** The answer is $\boxed{\frac{6}{5}}$.

   Since 2016 is a multiple of both 6 and 7, we see that $|A| = \frac{1}{6}(2016), |B| = \frac{1}{7}(2016)$, and $|A \cap B| = \frac{1}{42}(2016)$. So the number of positive integers that are in $A$ but not in $B$ is equal to $(\frac{1}{6} - \frac{1}{42})(2016) = \frac{6}{42}(2016)$ and the number of integers that are in $B$ but not in $A$ is equal to $(\frac{1}{7} - \frac{1}{42})(2016) = \frac{5}{42}(2016)$. Therefore the ratio is $\frac{6/42}{5/42} = \frac{6}{5}$.

6. Three peas form a unit equilateral triangle on a flat table. Sebastian moves one of the peas a distance $d$ along the table to form a right triangle. Determine the minimum possible value of $d$.

   **Solution.** The answer is $\boxed{\frac{\sqrt{3}-1}{2}}$.

   Suppose that the three peas are $P, E, A$ respectively, and the one that Sebastian moved was $P$ (from $P$ to $P'$). Suppose the midpoint of $EA$ is $M$. If $\angle P' = 90°$, then the minimal distance is equal to $PM - P'M = \frac{\sqrt{3}-1}{2}$. If $\angle E = 90°$, then the minimal distance is equal to $EM = \frac{1}{2}$. It's easy to see that the first one is smaller, so $d_{min} = \frac{\sqrt{3}-1}{2}$.

7. Oumar is four times as old as Marta. In $m$ years, Oumar will be three times as old as Marta will be. In another $n$ years after that, Oumar will be twice as old as Marta will be. Compute the ratio $m/n$.

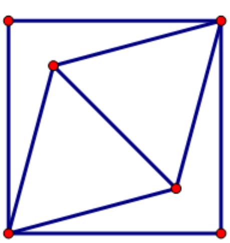
   **Solution.** The answer is $\boxed{\frac{1}{3}}$.

   Suppose that Marta is $x$ years old and Oumar is $4x$ years old. So $4x+m = 3(x+m)$ and $4x+m+n = 2(x+m+n)$. Solving the first equation gives $m = \frac{1}{2}x$, and plugging that into the second gives $n = \frac{3}{2}x$. Therefore $\frac{m}{n} = \frac{1/2}{3/2} = \frac{1}{3}$.

8. Compute the area of the smallest square in which one can inscribe two non-overlapping equilateral triangles with side length 1.

   **Solution.** The answer is $\boxed{\frac{3}{2}}$.

   No matter how one arranges the two equilateral triangles, the distance between the two farthest vertices must be at least $2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$, achieved when the two triangles share an edge. Since the square has to contain both of these vertices, the diagonal must be at least $\sqrt{3}$ long, so the area is at least $\frac{1}{2} \cdot (\sqrt{3})^2 = \frac{3}{2}$. We can easily see that this squares works, by putting the two triangles with an overlapping edge along with the diagonal.



9. Teemu, Marcus, and Sander are signing documents. If they all work together, they would finish in 6 hours. If only Teemu and Sander work together, the work would be finished in 8 hours. If only Marcus and Sander work together, the work would be finished in 10 hours. How many hours would Sander take to finish signing if he worked alone?

   **Solution.** The answer is $\boxed{\frac{120}{7}}$.

   Suppose the whole job of signing is 1 unit, and the rates of Teemu, Marcus, and Sander are $T, M, S$ respectively. We can see that $T+M+S = \frac{1}{6}, T+S = \frac{1}{8}, M+S = \frac{1}{10}$, so $S = (T+S)+(M+S)-(T+M+S) = \frac{1}{10} + \frac{1}{8} - \frac{1}{6} = \frac{7}{120}$. Therefore, it takes $\frac{1}{S} = \frac{120}{7}$ hours for Sander alone to sign all documents.

10. Triangle $ABC$ has a right angle at $B$. A circle centered at $B$ with radius $BA$ intersects side $AC$ at a point $D$ different from $A$. Given that $AD = 20$ and $DC = 16$, find the length of $BA$.

    **Solution.** The answer is $\boxed{6\sqrt{10}}$.

    Since $BA = BD$, if $H$ is the midpoint of $AD$, then $BH$ is perpendicular to $AD$. From this we can compute $AC = 20 + 16 = 36, AH = 20 \div 2 = 10$. Since $ABC$ is a right triangle, $ABC$ is similar to $AHB$, so $\frac{AB}{AH} = \frac{AC}{AB}$, or $BA = \sqrt{AC \cdot AH} = \sqrt{360} = 6\sqrt{10}$.

11. A regular hexagon $H$ with side length 20 is divided completely into equilateral triangles with side length 1. How many regular hexagons with sides parallel to the sides of $H$ are formed by lines in the grid?

    **Solution.** The answer is $\boxed{8000}$.

    Define $H_n$ as the number of vertices in a regular hexagon that has side length $n$. So $H_0 = 1, H_1 = 7, H_2 = 19, \cdots H_n = (n+1)^3 - n^3$ for all $n$. When the side length of a sub-hexagon of $H$ is 1, the center of this sub-hexagon can be any vertex except for the ones on the boundary, so there are $H_{19}$ possibilities, similarly when the side length is 2 there are $H_{18}$ possibilities, $\cdots$, and when the side length is 20 there are $H_0$ possibilities. Therefore in total the number of sub-hexagons is

    $$H_{19} + H_{18} + \cdots + H_0 = (20^3 - 19^3) + (19^3 - 18^3) + \cdots + (1^3 - 0^3) = 20^3 = 8000.$$

12. In convex pentagon $PEARL$, quadrilateral $PERL$ is a trapezoid with side $PL$ parallel to side $ER$. The areas of triangle $ERA$, triangle $LAP$, and trapezoid $PERL$ are all equal. Compute the ratio $\dfrac{PL}{ER}$.

    **Solution.** The answer is $\boxed{\frac{\sqrt{5}-1}{2}}$.

    Suppose $PL = x, RE = y$, and the heights of trapezoid $REPL$ and triangle $EAR$ (on $RE$) are $u, v$ respectively. So we have $yv = (x+y)u = x(u+v)$, which gives $yu = xv$, and $\frac{u}{v} = \frac{x}{y} = \frac{y}{x+y}$. If $\frac{x}{y} = k$ then $\frac{y}{x+y} = \frac{1}{k+1} = k$, and since $k > 0$ we can solve the quadratic equation $k^2 + k = 1$ and get $k = \frac{\sqrt{5}-1}{2}$. Since $k$ is equal to $\frac{ER}{OA}$, we get our desired ratio.

13. Let $m$ and $n$ be positive integers with $m < n$. The first two digits after the decimal point in the decimal representation of the fraction $\frac{m}{n}$ are 74. What is the smallest possible value of $n$?

    **Solution.** The answer is $\boxed{27}$.

    The original condition is equivalent to $0.74 \leq \frac{m}{n} < 0.75$, so we have that $\frac{1}{100} = 0.01 \geq \frac{3}{4} - \frac{m}{n} > 0$. Since $\frac{3}{4} - \frac{m}{n} = \frac{3n-4m}{4n}$, we need $4n \geq 100$ or $n \geq 25$. If $n = 25$ then we need $3 \cdot 25 - 4m = 1$ for some integer $m$, which is impossible. Similar situation happen with $n = 26$, but when $n = 27$, $3 \cdot 27 - 4m = 1$ has a valid solution $m = 20$, and indeed $\frac{20}{27} = 0.\overline{740}$ works. So the smallest possible value for $n$ is 27.

14. Define functions

    $$f(x,y) = \frac{x+y}{2} - \sqrt{xy} \quad \text{and} \quad g(x,y) = \frac{x+y}{2} + \sqrt{xy}.$$

    Compute

    $$g\left(g\left(f\left(1,3\right),\ f\left(5,7\right)\right),\ g\left(f\left(3,5\right),\ f\left(7,9\right)\right)\right).$$

**Solution.** The answer is $\boxed{\frac{1}{2}}$.

Notice that $f(x,y) = (\sqrt{x}-\sqrt{y})^2/2$, $g(x,y) = (\sqrt{x}+\sqrt{y})^2/2$. Therefore

$$g(f(a,b), f(c,d)) = \frac{1}{2}\left(\sqrt{\frac{(\sqrt{a}-\sqrt{b})^2}{2}} + \sqrt{\frac{(\sqrt{c}-\sqrt{d})^2}{2}}\right)^2 = \frac{1}{4}\left(\left|\sqrt{b}-\sqrt{a}\right| + \left|\sqrt{d}-\sqrt{c}\right|\right)^2$$

for all positive real numbers $a, b, c, d$. Plugging this result into the expression, we obtain

$$g\left(\frac{1}{4}\left(\sqrt{3}-\sqrt{1}+\sqrt{7}-\sqrt{5}\right)^2, \frac{1}{4}\left(\sqrt{5}-\sqrt{3}+\sqrt{9}-\sqrt{7}\right)^2\right)$$
$$= \frac{1}{2}\left(\frac{(\sqrt{3}-\sqrt{1}+\sqrt{7}-\sqrt{5}) + (\sqrt{5}-\sqrt{3}+\sqrt{9}-\sqrt{7})}{2}\right)^2 = \frac{(\sqrt{9}-\sqrt{1})^2}{8} = \frac{1}{2}.$$

15. Natalia plants two gardens in a $5 \times 5$ grid of points. Each garden is the interior of a rectangle with vertices on grid points and sides parallel to the sides of the grid. How many unordered pairs of two non-overlapping rectangles can Nataliia choose as gardens? (The two rectangles may share an edge or part of an edge but should not share an interior point.)

**Solution.** The answer is $\boxed{2550}$.

First assume the two rectangles are distinct and call them $P, Q$. Label the 5 horizontal lines 0,1,2,3,4 from top to bottom and 5 vertical lines 0,1,2,3,4 from left to right. Let $P_{h1}$, $P_{h2}$, $Q_{h1}$, $Q_{h2}$, $P_{v1}$, $P_{v2}$, $Q_{v1}$, $Q_{v2}$ be the number corresponding to the boundaries of each rectangle such that $P_{h1} < P_{h2}, Q_{h1} < Q_{h2}, P_{v1} < P_{v2}, Q_{v1} < Q_{v2}$ ($h$ stand for horizontal, and $v$ stand for vertical). Notice that a rectangle is uniquely determined by the four lines that bounds the rectangle. There are $\binom{5}{2} = 10$ ways to choose the boundaries for one rectangle from one dimension, so there are $\binom{5}{2}^2 = 100$ ways to choose a rectangle and therefore $100^2 = 10000$ ways to choose an *ordered* pair of (possibly overlapping) rectangles.

By complementary counting we only need to count the number of pairs of rectangles that do intersect. Now notice that the two rectangles intersect if and only if the intervals $[P_{h1}, P_{h2}]$ and $[Q_{h1}, Q_{h2}]$ intersect and so do $[P_{v1}, P_{v2}]$ and $[Q_{v1}, Q_{v2}]$. Here we focus on one dimension, say, horizontal. There are $10^2 = 100$ ways to choose two intervals on this dimension, and again by complementary counting we only need count the number of intervals that do *not* intersect. The four ends of the interval have to use either 4 or 3 distinct numbers, and for each subset of 4 or 3 numbers there are two ways to designate them to the two intervals such that they don't intersect, except for possibly at a common point. For example, if the numbers chosen are 0, 2, 3, 4, then the two possibilities are $(P_{h1}, P_{h2}, Q_{h1}, Q_{h2}) = (0, 2, 3, 4)$ or $(3, 4, 0, 2)$; if the numbers are 0, 1, 3 only, then the two possibilities are $(P_{h1}, P_{h2}, Q_{h1}, Q_{h2}) = (0, 1, 1, 3)$ or $(1, 3, 0, 1)$. So there will be $2(\binom{5}{4} + \binom{5}{3}) = 30$ ordered pairs of non-intersecting intervals and $100 - 30 = 70$ pairs of intersecting intervals.

This implies that there are $70^2 = 4900$ ways to choose two overlapping (possibly identical) rectangles in the grid, and $10000 - 4900 = 5100$ ways that they don't overlap.

At this point, remember that the problem asks for *unordered* pairs, and each pair so far is counted exactly twice. Therefore, we divide 5100 by 2 and get 2550 as the answer.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. [5] Suppose that gold satisfies the relation $p = v + v^2$, where $p$ is the price and $v$ is the volume. How many pieces of gold with volume 1 can be bought for the price of a piece with volume 2?

   **Solution.** The answer is $\boxed{3}$.

   The price of the gold with volume 2 has price $2 + 2^2 = 6$, and the price of the gold with volume 1 has price $1 + 1^2 = 2$, so one can buy $\frac{6}{2} = 3$ pieces of gold.

2. [5] Find the smallest prime number with each digit greater or equal to 8.

   **Solution.** The answer is $\boxed{89}$.

   The problem statement is equivalent to finding the smallest prime comprising of digit 8 and 9 only. Since the number has to be odd the unit digit has to be 9. 9 itself is obviously not a prime, but the second smallest candidate, 89, is. Therefore the answer is 89.

3. [5] What fraction of regular hexagon $ZUMING$ is covered by both quadrilateral $ZUMI$ and quadrilateral $MING$?

   **Solution.** The answer is $\boxed{\frac{1}{6}}$.

   Suppose that the center of the hexagon is $O$. Since $ZI$ and $MG$ both goes through $O$, the intersection of the two quadrilaterals is the triangle $IMO$, which is $\frac{1}{6}$ of the entire area.

### 2.4.2 Round 2

4. [7] The two smallest positive integers expressible as the sum of two (not necessarily positive) perfect cubes are $1 = 1^3 + 0^3$ and $2 = 1^3 + 1^3$. Find the next smallest positive integer expressible in this form.

   **Solution.** The answer is $\boxed{7}$.

   Notice that $7 = 2^3 + (-1)^3$, and no smaller number greater than 2 can be expressed in this form since the smallest difference of two nonnegative perfect cubes that is greater than 2 is $2^3 - 1 = 7$. Hence 7 is the smallest such number after 1 and 2.

5. [7] In how many ways can the numbers $1, 2, 3$, and 4 be written in a row such that no two adjacent numbers differ by exactly 1?

**Solution.** The answer is $\boxed{2}$.

Note that 2 and 3 can each have only one neighbor, so they cannot be in the middle. Hence the middle two can be 14 or 41, and the only valid ways are 3142 and 2413, for 2 ways in total.

6. [7] A real number is placed in each cell of a grid with 3 rows and 4 columns. The average of the numbers in each column is 2016, and the average of the numbers in each row is a constant $x$. Compute $x$.

   **Solution.** The answer is $\boxed{2016}$.

   The average of the numbers in each column is 2016, so the average of all numbers is 2016. Similarly the average of all numbers is also $x$ since the average of each row is $x$. Thus $x = 2016$.

### 2.4.3 Round 3

7. [9] Fardin is walking from his home to his office at a speed of 1 meter per second, expecting to arrive exactly on time. When he is halfway there, he realizes that he forgot to bring his pocketwatch, so he runs back to his house at 2 meters per second. If he now decides to travel from his home to his office at $x$ meters per second, find the minimum $x$ that will allow him to be on time.

   **Solution.** The answer is $\boxed{4}$.

   When he remembers his pocketwatch, Fardin has already used up half of his total time, and when he ran back home he travelled half the entire distance at double the speed, so he used another quarter of his time. When he departs again from his home, he has only $1 - \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$ of the time left, so he needs to go 4 times as fast as he started, or 4 meters per second.

8. [9] In triangle $ABC$, the angle bisector of $\angle B$ intersects the perpendicular bisector of $AB$ at point $P$ on segment $AC$. Given that $\angle C = 60^\circ$, determine the measure of $\angle CPB$ in degrees.

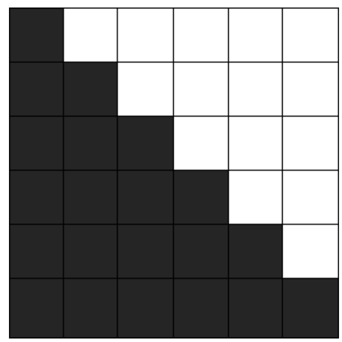
   **Solution.** The answer is $\boxed{80^\circ}$.

   From the problem statement we see that $AP = BP$ and $\angle A = \angle ABP = \angle CBP = x$, and $\angle CPB = \angle A + \angle ABP = 2x$. Since $x + 2x + 60^\circ = 180^\circ$, $x = 40^\circ$ and $\angle CPB = 80^\circ$.

9. [9] Katie colors each of the cells of a $6 \times 6$ grid either black or white. From top to bottom, the number of black squares in each row are 1, 2, 3, 4, 5, and 6, respectively. From left to right, the number of black squares in each column are 6, 5, 4, 3, 2, and 1, respectively. In how many ways could Katie have colored the grid?

**Solution.** The answer is $\boxed{1}$.

Since both the last row and first column have 6 black squares, we see that all squares on that row and column are black. Then since both the first row and last column have 1 black square, and the first square of the first row and last square of the last column are already black, all other squares on that row must be white. Similarly we can apply our argument to the row/column with 5, 2, 4, 3 black squares in that order, and find out that there is only one way Katie could have colored the grid:



### 2.4.4 Round 4

10. [11] Lily stands at the origin of a number line. Each second, she either moves 2 units to the right or 1 unit to the left. At how many different places could she be after 2016 seconds?

**Solution.** The answer is $\boxed{2017}$.

Suppose Lily moved right for $x$ seconds and left for $(2016 - x)$ seconds, so at the end she is on $x - 2(2016 - x) = 3x - 4032$. Since $x$ is an integer between 0 and 2016, there are 2017 possibilities for $x$ and thus 2017 possibilities for the final position.

11. [11] There are 125 politicians standing in a row. Each either always tells the truth or always lies. Furthermore, each politician (except the leftmost politician) claims that at least half of the people to his left always lie. Find the number of politicians that always lie.

**Solution.** The answer is $\boxed{62}$.

If the leftmost person is a liar, then the second person is telling the truth and so is the third person, then the fourth person is a liar, fifth person is a truth-teller, sixth person is a liar, etc. So there will be $\frac{125-3}{2} + 1 = 62$ liars in this case.

If the leftmost person is telling the truth, then the second person is a liar, third person is telling the truth, and the rest is identical to the first case. So there will also be 62 liars in this case.

Since there can only be 62 liars in either case, it is the only possibility and therefore our answer.

12. [11] Two concentric circles with radii 2 and 5 are drawn on the plane. What is the side length of the largest square whose area is contained entirely by the region between the two circles?

**Solution.** The answer is $\boxed{\frac{14}{5}}$.

The largest square $ABCD$ must satisfy that a side $AB$ must be the chord of the larger circle and the opposite side $CD$ must be tangent to the smaller circle. Let the midpoint of $AB, CD$ be $M, N$ respectively, then $O, M, N$ should be collinear, and $N$ is the point of tangency to the smaller circle. If the side length of $ABCD$ is $x$, then by expressing the length of $OA$ using Pythagorean Theorem, we have

$$\left(\frac{x}{2}\right)^2 + (2+x)^2 = 5^2.$$

This equation has two solutions: $x = -6$ and $x = \frac{14}{5}$. Since $x$ is positive, we disregard the negative solution and obtain $\frac{14}{5}$ as the desired side length.

### 2.4.5 Round 5

13. [13] Initially, the three numbers 20, 201, and 2016 are written on a blackboard. Each minute, Zhuo selects two of the numbers on the board and adds 1 to each. Find the minimum $n$ for which Zhuo can make all three numbers equal to $n$.

**Solution.** The answer is $\boxed{2197}$.

Suppose that Zhuo adds the 20 and 201 sides $x$ times, 20 and 2016 sides $y$ times, and 201 and 2016 sides $z$ times, then we have

$$x + y = n - 20, x + z = n - 201, y + z = n - 2016.$$

This implies that $n - 20, n - 201, n - 2016$ are three sides of a (possibly degenrate) triangle, which means $(n - 20) \geq (n - 201) + (n - 2016)$, or $n \geq 201 + 2016 - 20 = 2197$. This minimum is established when $x = 1996, y = 181, z = 0$.

14. [13] Call a three-letter string *rearrangeable* if, when the first letter is moved to the end, the resulting string comes later alphabetically than the original string. For example, $AAA$ and $BAA$ are not rearrangeable, while $ABB$ is rearrangeable. How many three-letters strings with (not necessarily distinct) uppercase letters are rearrangeable?

**Solution.** The answer is $\boxed{8775}$.

Suppose the word is $\overline{\alpha\beta\gamma}$, where $\alpha, \beta, \gamma$ represent an uppercase letter. We then treat the alphabet as base-26 integers, and therefore the problem condition is then equivalent to $\overline{\alpha\beta\gamma} < \overline{\beta\gamma\alpha}$. This requires (a) $\alpha < \beta$ or (b) $\alpha = \beta$ and $\beta < \gamma$. In case (a), there are $\binom{26}{2}$ ways to choose two different letters for $\alpha$ and $\beta$ and 26 ways to choose a letter for $\gamma$. In case (b), there are $\binom{26}{2}$ ways to choose two different letters for $\beta$ and $\omega$ and $\alpha$ will be determined by $\beta$. Therefore there are $\binom{26}{2} \cdot (26+1) = 13 \cdot 25 \cdot 27 = 8775$ possible strings.

Alternatively, since the probability that the new string comes strictly later is equal to that the new string comes strictly earlier, we only need to discount the case where the two strings are equal, which requires all three letters to be the same. Therefore there are $\frac{26^3-26}{2} = 8775$ possible strings.

15. [13] Triangle $ABC$ is an isosceles right triangle with $\angle C = 90°$ and $AC = 1$. Points $D, E$, and $F$ are chosen on sides $BC, CA$, and $AB$, respectively, such that $AEF, BFD, CDE$, and $DEF$ are isosceles right triangles. Find the sum of all distinct possible lengths of segment $DE$.

**Solution.** The answer is $\boxed{\frac{5}{6}\sqrt{2}}$.

Since $C$ is a right angle, it's clear that $CD = CE$. Now we separate into cases.

*Case 1:* $DF = EF$. This means that $\angle DFE = 90°$, and $CDFE$ is a square. It's not difficult to see that in this case $AEF$ and $BDF$ are both right isosceles triangles, and $DF = \frac{1}{2}$. Therefore $DE = \frac{\sqrt{2}}{2}$.

*Case 2:* $DF \neq EF$. WLOG assume then that $DF = DE$. Then we need $AE = EF, DF = FB$. Since $DE = \sqrt{2}CD = \sqrt{2}CE$, and $AE = EF = \sqrt{2}DE = \sqrt{2}DF = DB$, we have $AE = 2CE, BD = 2CD$, so $CE = CD = \frac{1}{3}$. Therefore $DE = \frac{\sqrt{2}}{3}$.

Adding up the two cases, we get $(\frac{1}{2} + \frac{1}{3})\sqrt{2} = \frac{5}{6}\sqrt{2}$ as the answer.

### 2.4.6 Round 6

16. [15] Let $p$, $q$, and $r$ be prime numbers such that $pqr = 17(p + q + r)$. Find the value of the product $pqr$.

**Solution.** The answer is $\boxed{646}$.

Since $pqr$ is a multiple of 17, at least one of $p, q, r$ is 17. Without loss of generality, let $r = 17$, so we have $pq = p + q + 17$, or $(p-1)(q-1) = 18$. Assuming $p \leq q$, we have that $(p-1, q-1) = (1, 18), (2, 9)$ or $(3, 6)$, corresponding to $(p, q) = (2, 19), (3, 10)$ or $(4, 7)$. Since 10 and 4 are not prime, $p, q, r$ has to be a permutation of $2, 19, 17$, and therefore their product is $2 \cdot 19 \cdot 17 = 646$.

17. [15] A cylindrical cup containing some water is tilted 45 degrees from the vertical. The point on the surface of the water closest to the bottom of the cup is 6 units away. The point on the surface of the water farthest from the bottom of the cup is 10 units away. Compute the volume of the water in the cup.

    **Solution.** The answer is $\boxed{32\pi}$.

    Since the cup is tilted exactly 45 degrees, we see that the diameter of the cup is equal to the difference between the highest point and lowest point from the bottom. which is $10-6=4$ inches, so the diameter is $\frac{4}{2}=2$ inches. Now notice that if we replace the water with ice (with equal volume), then we can place another identical copy of the ice block on top of the current one and create a cylinder of height $10+6=16$ inches. Since the entire cylinder has volume $2^2 \cdot 16 \cdot \pi = 64\pi$ cubic inches, the volume of water originally would be $64\pi \div 2 = 32\pi$ cubic inches.

18. [15] Each dot in an equilateral triangular grid with 63 rows and $2016 = \frac{1}{2} \cdot 63 \cdot 64$ dots is colored black or white. Every unit equilateral triangle with three dots has the property that exactly one of its vertices is colored black. Find all possible values of the number of black dots in the grid.

    **Solution.** The answer is $\boxed{672}$.

    Notice that knowing the configuration of a row (except for the first one with only one dot) uniquely determines the next row. If the single dot on the first row is black, then both dots on the second row must be white. On the third row the dots must be white, black, white in that order, and on fourth black, white, white, black, fifth white, white, black, white, white, etc. In general the number of black dots on each row will be $1, 0, 1, 2, 1, 2, 3, 2, 3, 4, 3, 4, \cdots, 19, 18, 19, 20, 19, 20, 21, 20, 21$, where each number repeats itself three times (except for 0 and 21) on the row with the same parity as itself, so the sum will be $3 \cdot \frac{(1+20)\cdot 20}{2} + 2 \cdot 21 = 672$.

    If the dot on the first row is white, then we can assume (without loss of generality) that the left dot on the second row is black. By a similar logic we get that the number of black dots on each row is $0, 1, 1, 1, 2, 2, 2, 3, 3, 3, \cdots, 20, 20, 20, 21, 21$, where each number between 1 and 20 inclusive repeats itself three times. It's not difficult to see that this yields the same sum. Therefore there will be 672 black dots in the grid.

    *Note:* In general, the number of black dots will be either $\lfloor \frac{N}{3} \rfloor$ or $\lceil \frac{N}{3} \rceil$, where $N$ is the number of dots in the triangular grid. A proof follows a similar line as the solution.

### 2.4.7 Round 7

19. [18] Tomasz starts with the number 2. Each minute, he either adds 2 to his number, subtracts 2 from his number, multiplies his number by 2, or divides his number by 2. Find the minimum number of minutes he will need in order to make his number equal 2016.

**Solution.** The answer is $\boxed{11}$.

Let $a_n$ be the number Tomasz has after $n$ minutes. When $n = 11$, we can make $a_0 = 2, a_1 = 4, a_2 = 8, a_3 = 16, a_4 = 32, a_5 = 64, a_6 = 128, a_7 = 126, a_8 = 252, a_9 = 504, a_{10} = 1008, a_{11} = 2016$. Now we show that $n = 10$ is impossible. In fact, since $a_1 \leq 4$, if $a_n \neq a_{n-1} \cdot 2$ for any $n > 2$, then $a_{10} \leq (4+2) \cdot 2^8 = 1536 < 2016$, but otherwise $a_{10} = 2^{10} = 2048 > 2016$, which means that it is impossible for us to achieve 2016. This means that the minimum number of minutes is indeed 11.

20. [18] The edges of a regular octahedron $ABCDEF$ are painted with 3 distinct colors such that no two edges with the same color lie on the same face. In how many ways can the octahedron be painted? Colorings are considered different under rotation or reflection.

**Solution.** The answer is $\boxed{24}$.

WLOG let the faces be $ABC, ACD, ADE, AEB, FBC, FCD, FDE, FEB$. Also let the colors be red, green and blue. Consider the coloring of edges $AB, AC, AD, AE$, which, by Pigeonhole Principle, includes two edges of the same color. However, no two edges of the same color can be the same color. This leaves us with two cases.

*Case 1:* Only two colors are used in these four edges. Then $AB, AD$ takes on a color and $AC, AE$ takes on another color. There are $3 \cdot 2 = 6$ ways to choose the two colors and WLOG suppose $AB$ is red and $AC$ is green. This forces $BC, CD, DE, EB$ to be all blue, and therefore $FB, FC, FD, FE$ alternates red and green. There are two ways to color the those four edges, so there are $6 \cdot 2 = 12$ ways in this case.

*Case 2:* All three colors are used in these four edges. Then either $AB, AD$ or $AC, AE$ takes the same color. There are two ways to choose which pair, three ways to choose the color that the pair takes, and two ways to choose the color for the remaining two edges. WLOG both $AB$ and $AD$ takes blue, $AC$ takes red and $AE$ takes green. Then $BC, CD, DE, EB$ have to be green, green, red, red respectively. Then we have $FB, FD$ having to be blue, and $FC, FE$ are red and green respectively. Since all the remaining colors are uniquely determined by the first four edges, there are $2 \cdot 3 \cdot 2 = 12$ ways in this case.

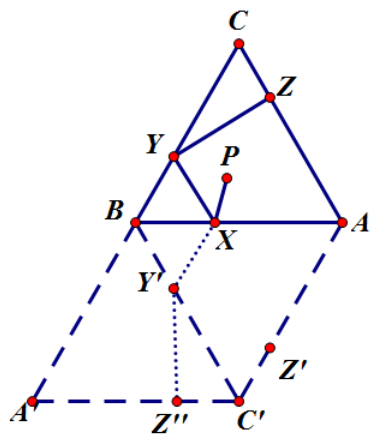
Adding up the two cases, we have $12 + 12 = 24$ ways in total.

21. [18] Jacob is trapped inside an equilateral triangle $ABC$ and must visit each edge of triangle $ABC$ at least once. (Visiting an edge means reaching a point on the edge.) His distances to sides $AB$, $BC$, and $CA$ are currently 3, 4, and 5, respectively. If he does not need to return to his starting point, compute the least possible distance that Jacob must travel.

**Solution.** The answer is $\boxed{15}$.

Suppose that Jacob starts at point $P$. Then since the area of triangle $ABC$ is equal to $\frac{1}{2}(3a+4a+5a) = \frac{1}{2}(ha)$, where $a$ is the side length of the triangle and $h$ is its height, so $h = 3+4+5 = 12$.

First assume that Jacob visits $AB$ first at point $X$, then $BC, CA$ at $Y, Z$ respectively. Reflect $C, Y, Z$ across line $AB$ to get $C', Y', Z'$, and then reflect $Z'$ across $BC'$ to get $Z''$. It is not difficult to see that $PX + XY + YZ = AX + XY' + Y'Z'' \leq PZ''$ (with equality achieved when $P, X, Y', Z''$ are all collinear and $Z''$ lies on the reflection line $AC'$ over $BC'$, which is a line that is parallel to $AB$ that is $h$ apart from it. Since $P$ is 3 away from $AB$, it is $3+h = 15$ away from the line $Z''$ is on, so $PZ'' \geq 15$, with equality case achieved when $PZ''$ is perpendicular to $AB$. Similarly we can do the case for all possible sequences of visits and get $3+h, 4+h, 5+h$ as the respective lower bounds. Therefore the minimum is $3 + h = 15$ and it is not difficult to achieve this bound when we reflect the points back onto the sides of $ABC$.




### 2.4.8 Round 8

22. [22] Four integers $a, b, c$, and $d$ with $a \le b \le c \le d$ satisfy the property that the product of any two of them is equal to the sum of the other two. Given that the four numbers are not all equal, determine the 4-tuple $(a, b, c, d)$.

**Solution.** The answer is $\boxed{(-1, -1, -1, 2)}$.

We can in fact solve the problem in real numbers. Let $S = a + b + c + d$. From the problem condition we see that $ab + cd = (c + d) + (a + b) = S$. Similarly $ac + bd = ad + bc = S$. Notice that $(a + b)(c + d) = (ac + bd) + (ad + bc) = 2S$, and similarly $(a + c)(b + d) = (a + d)(b + c) = 2S$. By Vieta's Theorem we get that all three pairs $(a + b, c + d), (a + c, b + d), (a + d, b + c)$ are solutions of the equation $x^2 - Sx + 2S = 0$, so all three pairs must be equal to $(u, v)$ or $(v, u)$ for some real number $u, v$. Since there are at most two distinct values in $\{a+b, a+c, a+d\}$, by Pigeonhole Principle at least two of $b, c, d$ are the same. WLOG let $b = c$. Then by making the same observation on $a, b, d$ gives that either there are two pairs of equal numbers among the four three of the four numbers are equal. If it's the first possibility then $a, b, c, d$ is a permutation of $p, p, q, q$ for some distinct reals $p, q$, but it's not difficult to see that $\{p + p, q + q\} \neq \{p + q, p + q\}$. This leaves the second possibility (where the numbers is a permutation of $p, p, p, q$ for $p \neq q$), which satisfy the desired property.

Now it remains to solve the following system of equations:

$$p + p = pq, p + q = p^2.$$

The first equation gives $p = 0$ or $q = 2$. If $p = 0$ then $q = 0$, which contradicts the requirement that $p \neq q$. If $q = 2$, then $p + 2 = p^2$ has two solutions $-1$ or $2$, but since $p \neq q = 2$ we are forced to have $p = -1$, and by putting the numbers in nondecreasing order we get $(-1, -1, -1, 2)$ as the one and only possible quadruple.

23. [22] In equilateral triangle $ABC$, points $D, E$, and $F$ lie on sides $BC, CA$, and $AB$, respectively, such that $BD = 4$ and $CD = 5$. If $DEF$ is an isosceles right triangle with right angle at $D$, compute $EA + FA$.

**Solution.** The answer is $\boxed{27 - 9\sqrt{3}}$.

The side length of the triangle is $4 + 5 = 9$. Rotate triangle $CDE$ by 90 degrees to $C'DF$, where $CD = C'D = 5$. $C'D$ and $BD$ are perpendicular. Draw rectangle $XFYD$ such that $X, Y$ lie on $C'D$ and $BD$ respectively. Suppose $BF = p, CE = C'F = q$, then

$$BD = BY + FX = \frac{1}{2}p + \frac{\sqrt{3}}{2}q,$$

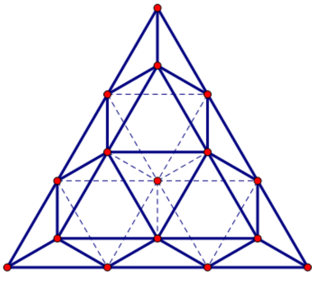
$$C'D = FY + C'X = \frac{\sqrt{3}}{2}p + \frac{1}{2}q.$$

Adding up the two equations, we have $BD + C'D = 9 = \frac{\sqrt{3}+1}{2}(p + q)$, so $p + q = 9 \div \frac{\sqrt{3}+1}{2} = 9\sqrt{3} - 9$, and thus $EA + FA = (9 - p) + (9 - q) = 18 - (p + q) = 27 - 9\sqrt{3}$.

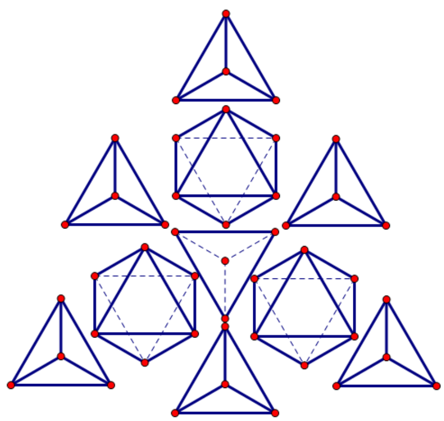
24. [22] On each edge of a regular tetrahedron, four points that separate the edge into five equal segments are marked. There are sixteen planes that are parallel to a face of the tetrahedron and pass through exactly three of the marked points. When the tetrahedron is cut along each of these sixteen planes, how many new tetrahedrons are produced?

**Solution.** The answer is $\boxed{45}$.

Consider the four cuts in the same direction, which cuts the tetrahedron into 4 frustums of different size and a unit regular tetrahedron. Now put the remaining four frustums separately such that one angle is pointing up and order them in increasing size. The $i$-th smallest frustum has a smaller base being an equilateral triangle with side length $i$ and a larger base being an equilateral triangle with side length $i + 1$. Now perform the cuts in the remaining directions on each frustum. Below is an example for the cuts on second smallest frustum.



After that, on the smaller base there are $\frac{i(i-1)}{2}$ downward-pointing triangles and each of them faces an intersection on the larger base, meaning that each correspond to a tetrahedron; on the larger base there are $\frac{(i+1)(i+2)}{2}$ upward-pointing triangles, each facing an intersection and therefore corresponding to a tetrahedron as well. The $\frac{i(i+1)}{2}$ upward-pointing triangles on the smaller base and downward-pointing triangles on the larger base correspond to each other and each pair correspond to a regular *octahedron*. For example, when we separate all the pieces in the second frustum, we should get the following pieces: (in their relative position in order to aid visualization)



Finally, by summing $\frac{i(i-1)}{2}$ and $\frac{(i+1)(i+2)}{2}$ over $i = 0, 1, 2, 3, 4$, we get that there are $(0+0+1+3+6)+(1+3+6+10+15) = 45$ tetrahedrons (and $0+1+3+6+10 = 20$ octahedrons).

# Exeter Math Club Competition
## January 21, 2017

## Table of Contents

## Organizing Acknowledgments

- **Tournament Directors** Eliza Khokhar, Yuan (Yannick) Yao
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- **Treasurer** John Wang
- **Publicity** Matt Hambacher, Fiona Ku, Geyang Qin, Jena Yun
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Jane Street

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- **Runners** Kristy Chang, Michael Chen, James Lin, Stephen Price
- **Food Czar** Geyang Qin
- **Head Grader** Qi Qi, Vinjai Vale
- **Graders** Ivan Borensco, Catherine Huang, Chad Qian, Jeffrey Qiao, Geyang Qin, Meena Jagadeesan, Alec Sun, Alex Wei
- **Judges** Zuming Feng, Greg Spanier

# Chapter 1

# EMC$^2$ 2017 Problems

## 1.1 Speed Test

*There are 20 problems, worth 3 points each, to be solved in 25 minutes.*

1. Ben was trying to solve for $x$ in the equation $6 + x = 1$. Unfortunately, he was reading upside-down and misread the equation as $1 = x + 9$. What is the positive difference between Ben's answer and the correct answer?

2. Anjali and Meili each have a chocolate bar shaped like a rectangular box. Meili's bar is four times as long as Anjali's, while Anjali's is three times as wide and twice as thick as Meili's. What is the ratio of the volume of Anjali's chocolate to the volume of Meili's chocolate?

3. For any two nonnegative integers $m, n$, not both zero, define $m?n = m^n + n^m$. Compute the value of $((2?0)?1)?7$.

4. Eliza is making an in-scale model of the Phillips Exeter Academy library, and her prototype is a cube with side length 6 inches. The real library is shaped like a cube with side length 120 feet, and it contains an entrance chamber in the front. If the chamber in Eliza's model is 0.8 inches wide, how wide is the real chamber, in feet?

5. One day, Isaac begins sailing from Marseille to New York City. On the exact same day, Evan begins sailing from New York City to Marseille along the exact same route as Isaac. If Marseille and New York are exactly 3000 miles apart, and Evan sails exactly 40 miles per day, how many miles must Isaac sail each day to meet Evan's ship in 30 days?

6. The conversion from Celsius temperature $C$ to Fahrenheit temperature $F$ is:

$$F = 1.8C + 32.$$

   If the lowest temperature at Exeter one day was 20°F, and the next day the lowest temperature was 5°C higher, what would be the lowest temperature that day, in degrees Fahrenheit?

7. In a school, 60% of the students are boys and 40% are girls. Given that 40% of the boys like math and 50% of the people who like math are girls, what percentage of girls like math?

8. Adam and Victor go to an ice cream shop. There are four sizes available (kiddie, small, medium, large) and seventeen different flavors, including three that contain chocolate. If Victor insists on getting a size at least as large as Adam's, and Adam refuses to eat anything with chocolate, how many different ways are there for the two of them to order ice cream?

9. There are 10 (not necessarily distinct) positive integers with arithmetic mean 10. Determine the maximum possible range of the integers. (The range is defined to be the nonnegative difference between the largest and smallest number within a list of numbers.)

10. Find the sum of all distinct prime factors of $11! - 10! + 9!$.

11. Inside regular hexagon $ZUMING$, construct square $FENG$. What fraction of the area of the hexagon is occupied by rectangle $FUME$?

12. How many ordered pairs $(x, y)$ of nonnegative integers satisfy the equation $4^x \cdot 8^y = 16^{10}$?

13. In triangle $ABC$ with $BC = 5$, $CA = 13$, and $AB = 12$, Points $E$ and $F$ are chosen on sides $AC$ and $AB$, respectively, such that $EF \parallel BC$. Given that triangle $AEF$ and trapezoid $EFBC$ have the same perimeter, find the length of $EF$.

14. Find the number of two-digit positive integers with exactly 6 positive divisors. (Note that 1 and $n$ are both counted among the divisors of a number $n$.)

15. How many ways are there to put two identical red marbles, two identical green marbles, and two identical blue marbles in a row such that no red marble is next to a green marble?

16. Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

17. The tiny island nation of Konistan is a cone with height twelve meters and base radius nine meters, with the base of the cone at sea level. If the sea level rises four meters, what is the surface area of Konistan that is still above water, in square meters?

18. Nicky likes to doodle. On a convex octagon, he starts from a random vertex and doodles a path, which consists of seven line segments between vertices. At each step, he chooses a vertex randomly among all unvisited vertices to visit, such that the path goes through all eight vertices and does not visit the same vertex twice. What is the probability that this path does not cross itself?

19. In a right-angled trapezoid $ABCD$, $\angle B = \angle C = 90°$, $AB = 20, CD = 17$, and $BC = 37$. A line perpendicular to $DA$ intersects segment $BC$ and $DA$ at $P$ and $Q$ respectively and separates the trapezoid into two quadrilaterals with equal area. Determine the length of $BP$.

20. A sequence of integers $a_i$ is defined by $a_1 = 1$ and $a_{i+1} = 3i - 2a_i$ for all integers $i \geq 1$. Given that $a_{15} = 5476$, compute the sum $a_1 + a_2 + a_3 + \cdots + a_{15}$.

## 1.2 Accuracy Test

*There are 10 problems, worth 9 points each, to be solved in 45 minutes.*

1. Chris goes to Matt's Hamburger Shop to buy a hamburger. Each hamburger must contain exactly one bread, one lettuce, one cheese, one protein, and at least one condiment. There are two kinds of bread, two kinds of lettuce, three kinds of cheese, three kinds of protein, and six different condiments: ketchup, mayo, mustard, dill pickles, jalapeños, and Matt's Magical Sunshine Sauce. How many different hamburgers can Chris make?

2. The degree measures of the interior angles in convex pentagon $NICKY$ are all integers and form an increasing arithmetic sequence in some order. What is the smallest possible degree measure of the pentagon's smallest angle?

3. Daniel thinks of a two-digit positive integer $x$. He swaps its two digits and gets a number $y$ that is less than $x$. If 5 divides $x - y$ and 7 divides $x + y$, find all possible two-digit numbers Daniel could have in mind.

4. At the Lio Orympics, a target in archery consists of ten concentric circles. The radii of the circles are $1, 2, 3, \ldots, 9$, and 10 respectively. Hitting the innermost circle scores the archer 10 points, the next ring is worth 9 points, the next ring is worth 8 points, all the way to the outermost ring, which is worth 1 point. If a beginner archer has an equal probability of hitting any point on the target and never misses the target, what is the probability that his total score after making two shots is even?

5. Let $F(x) = x^2 + 2x - 35$ and $G(x) = x^2 + 10x + 14$. Find all distinct real roots of $F(G(x)) = 0$.

6. One day while driving, Ivan noticed a curious property on his car's digital clock. The sum of the digits of the current hour equaled the sum of the digits of the current minute. (Ivan's car clock shows 24-hour time; that is, the hour ranges from 0 to 23, and the minute ranges from 0 to 59.) For how many possible times of the day could Ivan have observed this property?

7. Qi Qi has a set $Q$ of all lattice points in the coordinate plane whose $x$- and $y$-coordinates are between 1 and 7 inclusive. She wishes to color 7 points of the set blue and the rest white so that each row or column contains exactly 1 blue point and no blue point lies on or below the line $x + y = 5$. In how many ways can she color the points?

8. A piece of paper is in the shape of an equilateral triangle $ABC$ with side length 12. Points $A_B$ and $B_A$ lie on segment $AB$, such that $AA_B = 3$, and $BB_A = 3$. Define points $B_C$ and $C_B$ on segment $BC$ and points $C_A$ and $A_C$ on segment $CA$ similarly. Point $A_1$ is the intersection of $A_CB_C$ and $A_BC_B$. Define $B_1$ and $C_1$ similarly. The three rhombi — $AA_BA_1A_C, BB_CB_1B_A, CC_AC_1C_B$ — are cut from triangle $ABC$, and the paper is folded along segments $A_1B_1, B_1C_1, C_1A_1$, to form a tray without a top. What is the volume of this tray?

9. Define $\{x\}$ as the fractional part of $x$. Let $S$ be the set of points $(x, y)$ in the Cartesian coordinate plane such that $x + \{x\} \leq y$, $x \geq 0$, and $y \leq 100$. Find the area of $S$.

10. Nicky likes dolls. He has 10 toy chairs in a row, and he wants to put some indistinguishable dolls on some of these chairs. (A chair can hold only one doll.) He doesn't want his dolls to get lonely, so he wants each doll sitting on a chair to be adjacent to at least one other doll. How many ways are there for him to put any number (possibly none) of dolls on the chairs? Two ways are considered distinct if and only if there is a chair that has a doll in one way but does not have one in the other.

## 1.3 Team Test

*There are 15 problems, worth 20 points each, to be solved in 45 minutes.*

1. Compute $2017 + 7201 + 1720 + 172$.

2. A number is called *downhill* if its digits are distinct and in descending order. (For example, 653 and 8762 are downhill numbers, but 97721 is not.) What is the smallest downhill number greater than 86432?

3. Each vertex of a unit cube is sliced off by a planar cut passing through the midpoints of the three edges containing that vertex. What is the ratio of the number of edges to the number of faces of the resulting solid?

4. In a square with side length 5, the four points that divide each side into five equal segments are marked. Including the vertices, there are 20 marked points in total on the boundary of the square. A pair of distinct points $A$ and $B$ are chosen randomly among the 20 points. Compute the probability that $AB = 5$.

5. A positive two-digit integer is one less than five times the sum of its digits. Find the sum of all possible such integers.

6. Let
$$f(x) = 5^{4^{3^{2^x}}}.$$
Determine the greatest possible value of $L$ such that $f(x) > L$ for all real numbers $x$.

7. If $\overline{AAAA}+\overline{BB} = \overline{ABCD}$ for some distinct base-10 digits $A, B, C, D$ that are consecutive in some order, determine the value of $\overline{ABCD}$. (The notation $\overline{ABCD}$ refers to the four-digit integer with thousands digit $A$, hundreds digit $B$, tens digit $C$, and units digit $D$.)

8. A regular tetrahedron and a cube share an inscribed sphere. What is the ratio of the volume of the tetrahedron to the volume of the cube?

9. Define $\lfloor x \rfloor$ as the greatest integer less than or equal to $x$, and $\{x\} = x - \lfloor x \rfloor$ as the fractional part of $x$. If $\lfloor x^2 \rfloor = 2\lfloor x \rfloor$ and $\{x^2\} = \frac{1}{2}\{x\}$, determine all possible values of $x$.

10. Find the largest integer $N > 1$ such that it is impossible to divide an equilateral triangle of side length 1 into $N$ smaller equilateral triangles (of possibly different sizes).

11. Let $f$ and $g$ be two quadratic polynomials. Suppose that $f$ has zeroes 2 and 7, $g$ has zeroes 1 and 8, and $f - g$ has zeroes 4 and 5. What is the product of the zeroes of the polynomial $f + g$?

12. In square $PQRS$, points $A, B, C, D, E$, and $F$ are chosen on segments $PQ$, $QR$, $PR$, $RS$, $SP$, and $PR$, respectively, such that $ABCDEF$ is a regular hexagon. Find the ratio of the area of $ABCDEF$ to the area of $PQRS$.

13. For positive integers $m$ and $n$, define $f(m, n)$ to be the number of ways to distribute $m$ identical candies to $n$ distinct children so that the number of candies that any two children receive differ by at most 1. Find the number of positive integers $n$ satisfying the equation $f(2017, n) = f(7102, n)$.

14. Suppose that real numbers $x$ and $y$ satisfy the equation

$$x^4 + 2x^2y^2 + y^4 - 2x^2 + 32xy - 2y^2 + 49 = 0.$$

Find the maximum possible value of $\frac{y}{x}$.

15. A point $P$ lies inside equilateral triangle $ABC$. Let $A', B', C'$ be the feet of the perpendiculars from $P$ to $BC, AC, AB$, respectively. Suppose that $PA = 13, PB = 14$, and $PC = 15$. Find the area of $A'B'C'$.

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. **[5]** If $2m = 200cm$ and $m \neq 0$, find $c$.

2. **[5]** A right triangle has two sides of lengths 3 and 4. Find the smallest possible length of the third side.

3. **[5]** Given that $20(x + 17) = 17(x + 20)$, determine the value of $x$.

### 1.4.2 Round 2

4. **[7]** According to the Egyptian Metropolitan Culinary Community, food service is delayed on $\frac{2}{3}$ of flights departing from Cairo airport. On average, if flights with delayed food service have twice as many passengers per flight as those without, what is the probability that a passenger departing from Cairo airport experiences delayed food service?

5. **[7]** In a positive geometric sequence $\{a_n\}$, $a_1 = 9, a_9 = 25$. Find the integer $k$ such that $a_k = 15$.

6. **[7]** In the Delicate, Elegant, and Exotic Music Organization, pianist Hans is selling two types of flowers with different prices (per flower): magnolias and myosotis. His friend Alice originally plans to buy a bunch containing 50% more magnolias than myosotis for \$50, but then she realizes that if she buys 50% less magnolias and 50% more myosotis than her original plan, she would still need to pay the same amount of money. If instead she buys 50% more magnolias and 50% less myosotis than her original plan, then how much, in dollars, would she need to pay?

### 1.4.3 Round 3

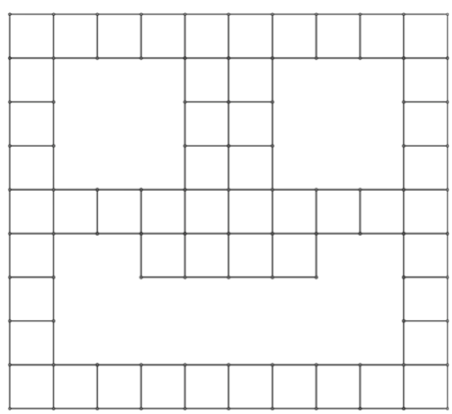
7. **[9]** In square $ABCD$, point $P$ lies on side $AB$ such that $AP = 3, BP = 7$. Points $Q, R, S$ lie on sides $BC, CD, DA$ respectively such that $PQ = PR = PS = AB$. Find the area of quadrilateral $PQRS$.

8. **[9]** Kristy is thinking of a number $n < 10^4$ and she says that 143 is one of its divisors. What is the smallest number greater than 143 that could divide $n$?

9. **[9]** A positive integer $n$ is called *special* if the product of the $n$ smallest prime numbers is divisible by the sum of the $n$ smallest prime numbers. Find the sum of the three smallest special numbers.

### 1.4.4 Round 4

10. **[11]** In the diagram below, all adjacent points connected with a segment are unit distance apart. Find the number of squares whose vertices are among the points in the diagram and whose sides coincide with the drawn segments.



11. **[11]** Geyang tells Junze that he is thinking of a positive integer. Geyang gives Junze the following clues:

    - My number has three distinct odd digits.
    - It is divisible by each of its three digits, as well as their sum.

    What is the sum of all possible values of Geyang's number?

12. **[11]** Regular octagon $ABCDEFGH$ has center $O$ and side length 2. A circle passes through $A, B$, and $O$. What is the area of the part of the circle that lies outside of the octagon?

### 1.4.5 Round 5

13. **[13]** Kelvin Amphibian, a not-frog who lives on the coordinate plane, likes jumping around. Each step, he jumps either to the spot that is 1 unit to the right and 2 units up, or the spot that is 2 units to the right and 1 unit up, from his current location. He chooses randomly among these two choices with equal probability. He starts at the origin and jumps for a long time. What is the probability that he lands on $(10, 8)$ at some time in his journey?

14. **[13]** Points $A$, $B$, $C$, and $D$ are randomly chosen on the circumference of a unit circle. What is the probability that line segments $AB$ and $CD$ intersect inside the circle?

15. **[13]** Let $P(x)$ be a quadratic polynomial with two consecutive integer roots. If it is also known that $\frac{P(2017)}{P(2016)} = \frac{2016}{2017}$, find the larger root of $P(x)$.

### 1.4.6 Round 6

16. **[15]** Let $S_n$ be the sum of reciprocals of the integers between 1 and $n$ inclusive. Find a triple $(a, b, c)$ of positive integers such that $S_{2017} \cdot S_{2017} - S_{2016} \cdot S_{2018} = \frac{S_a + S_b}{c}$.

17. **[15]** Suppose that $m$ and $n$ are both positive integers. Alec has $m$ standard 6-sided dice, each labelled 1 to 6 inclusive on the sides, while James has $n$ standard 12-sided dice, each labelled 1 to 12 inclusive on the sides. They decide to play a game with their dice. They each toss all their dice simultaneously and then compute the sum of the numbers that come up on their dice. Whoever has a higher sum wins (if the sums are equal, they tie). Given that both players have an equal chance of winning, determine the minimum possible value of $mn$.

18. **[15]** Overlapping rectangles $ABCD$ and $BEDF$ are congruent to each other and both have area 1. Given that $A, C, E, F$ are the vertices of a square, find the area of the square.

### 1.4.7 Round 7

19. **[18]** Find the number of solutions to the equation

$$|||\dots|||||x|+1|-2|+3|-4|+\cdots-98|+99|-100|=0.$$

20. **[18]** A *split* of a positive integer in base 10 is the separation of the integer into two nonnegative integers, allowing leading zeroes. For example, 2017 can be split into 2 and 017 (or 17), 20 and 17, or 201 and 7. A split is called *squarish* if both integers are nonzero perfect squares. 49 and 169 are the two smallest perfect squares that have a squarish split (4 and 9, 16 and 9 respectively). Determine all other perfect squares less than 2017 with at least one squarish split.

21. **[18]** Polynomial $f(x) = 2x^3 + 7x^2 - 3x + 5$ has zeroes $a, b$, and $c$. Cubic polynomial $g(x)$ with $x^3$-coefficient 1 has zeroes $a^2, b^2$, and $c^2$. Find the sum of coefficients of $g(x)$.

### 1.4.8 Round 8

22. **[22]** Two congruent circles, $\omega_1$ and $\omega_2$, intersect at points $A$ and $B$. The centers of $\omega_1$ and $\omega_2$ are $O_1$ and $O_2$ respectively. The arc $AB$ of $\omega_1$ that lies inside $\omega_2$ is trisected by points $P$ and $Q$, with the points lying in the order $A, P, Q, B$. Similarly, the arc $AB$ of $\omega_2$ that lies inside $\omega_1$ is trisected by points $R$ and $S$, with the points lying in the order $A, R, S, B$. Given that $PQ = 1$ and $PR = \sqrt{2}$, find the measure of $\angle AO_1B$ in degrees.

23. **[22]** How many ordered triples of $(a, b, c)$ of integers between $-10$ and 10 inclusive satisfy the equation $-abc = (a + b)(b + c)(c + a)$?

24. **[22]** For positive integers $n$ and $b$ where $b > 1$, define $s_b(n)$ as the sum of digits in the base-$b$ representation of $n$. A positive integer $p$ is said to *dominate* another positive integer $q$ if for all positive integers $n$, $s_p(n)$ is greater than or equal to $s_q(n)$. Find the number of ordered pairs $(p, q)$ of *distinct* positive integers between 2 and 100 inclusive such that $p$ dominates $q$.

# Chapter 2

# EMC$^2$ 2017 Solutions

## 2.1 Speed Test Solutions

1. Ben was trying to solve for $x$ in the equation $6 + x = 1$. Unfortunately, he was reading upside-down and misread the equation as $1 = x + 9$. What is the positive difference between Ben's answer and the correct answer?

   **Solution.** The answer is $\boxed{3}$.

   $x = -5$ and $x = -8$ are the solutions to these two equations. The positive difference is 3.

2. Anjali and Meili each have a chocolate bar shaped like a rectangular box. Meili's bar is four times as long as Anjali's, while Anjali's is three times as wide and twice as thick as Meili's. What is the ratio of the volume of Anjali's chocolate to the volume of Meili's chocolate?

   **Solution.** The answer is $\boxed{\frac{3}{2}}$.

   The volume of the chocolate bar is given by length $\cdot$ width $\cdot$ thickness. If Meili's bar has dimensions $l, w, t$, Anjali's bar must have dimensions $\frac{1}{4}l, 3w, 2t$, which yields a volume of $\frac{3}{2}lwt$. The answer is $\frac{3}{2}$.

3. For any two nonnegative integers $m, n$, not both zero, define $m?n = m^n + n^m$. Compute the value of $((2?0)?1)?7$.

   **Solution.** The answer is $\boxed{177}$.

   We compute

   $$((2?0)?1)?7 = ((2^0 + 0^2)?1)?7 = (1?1)?7 = (1^1 + 1^1)?7 = 2?7 = 2^7 + 7^2 = 128 + 49 = 177.$$

4. Eliza is making an in-scale model of the Phillips Exeter Academy library, and her prototype is a cube with side length 6 inches. The real library is shaped like a cube with side length 120 feet, and it contains an entrance chamber in the front. If the chamber in Eliza's model is 0.8 inches wide, how wide is the real chamber, in feet?

   **Solution.** The answer is $\boxed{16}$.

   120 feet is equivalent to $120 \cdot 12 = 1440$ inches. Therefore, the actual library is $\frac{1440}{6} = 240$ times bigger than the model. The size of the actual chamber is thus $240 \cdot 0.8 = 192$ inches, or $\frac{192}{12} = 16$ feet. Alternatively, since the width of the chamber of the model is $\frac{0.8}{6} = \frac{2}{15}$ of the side length of the model, the width of the real chamber is also $\frac{2}{15}$ of the real side length, which is $\frac{2}{15} \cdot 120 = 16$ feet.

5. One day, Isaac begins sailing from Marseille to New York City. On the exact same day, Evan begins sailing from New York City to Marseille along the exact same route as Isaac. If Marseille and New York are exactly 3000 miles apart, and Evan sails exactly 40 miles per day, how many miles must Isaac sail each day to meet Evan's ship in 30 days?

   **Solution.** The answer is $\boxed{60}$.

   Isaac and Evan, together, must sail the distance of exactly 3000 miles in 30 days. This means that, together, they must sail $\frac{3000}{30} = 100$ miles every day. Evan sails 40 miles a day, so Isaac must sail $100 - 40 = 60$ miles a day.

6. The conversion from Celsius temperature $C$ to Fahrenheit temperature $F$ is:

$$F = 1.8C + 32.$$

If the lowest temperature at Exeter one day was 20°F, and the next day the lowest temperature was 5°C higher, what would be the lowest temperature that day, in degrees Fahrenheit?

**Solution.** The answer is $\boxed{29}$.

From the given equation, we notice that a $x$ degree Celsius increase in temperature is equivalent to a $1.8 \cdot x$ degree Fahrenheit increase in temperature. Therefore, the increase in temperature between these two days is $1.8 \cdot 5 = 9$ degrees Fahrenheit. The lowest temperature on the second day is $20 + 9 = 29$ Fahrenheit.

7. In a school, 60% of the students are boys and 40% are girls. Given that 40% of the boys like math and 50% of the people who like math are girls, what percentage of girls like math?

**Solution.** The answer is $\boxed{60\%}$.

Because the number of boys who like math is equal to the number of girls who like math, $0.6 \cdot 0.4 = 0.4 \cdot x$, where $x$ is the fraction of girls who like math. Solving gives $x = 0.6$, so the answer is 60%.

8. Adam and Victor go to an ice cream shop. There are four sizes available (kiddie, small, medium, large) and seventeen different flavors, including three that contain chocolate. If Victor insists on getting a size at least as large as Adam's, and Adam refuses to eat anything with chocolate, how many different ways are there for the two of them to order ice cream?

**Solution.** The answer is $\boxed{2380}$.

Consider size and flavor separately. For size, if Adam chooses kiddie/small/medium/large, Victor has 4/3/2/1 choices in order to get a size at least as large as Adam's. In total, there are 10 ways for Adam and Victor to pick sizes. For flavor, Victor can pick any of the 17 flavors, and Adam can pick any of the 14 non-chocolate flavors. Therefore, there is a total of $10 \cdot 17 \cdot 14 = 2380$ different ways to order ice cream.

9. There are 10 (not necessarily distinct) positive integers with arithmetic mean 10. Determine the maximum possible range of the integers. (The range is defined to be the nonnegative difference between the largest and smallest number within a list of numbers.)

**Solution.** The answer is $\boxed{90}$.

Because the average of 10 numbers is 10, the sum of these 10 numbers must be 100. In order to maximize the range, we need to maximize the largest number and minimize the smallest number. Since all 10 numbers are positive integers, the largest integer is at most $100 - 9 \cdot 1 = 91$, and the smallest integer is at least 1, so the maximal range is $91 - 1 = 90$. This maximum is achieved when nine of the numbers are 1 and the tenth is 91.

10. Find the sum of all distinct prime factors of $11! - 10! + 9!$.

**Solution.** The answer is $\boxed{118}$.

We compute

$$11! - 10! + 9! = (11 \cdot 10 - 10 + 1) \cdot 9! = (110 - 10 + 1) \cdot 9! = 101 \cdot 9!$$

Note that 101 is prime. The distinct prime factors of this number are: 2, 3, 5, 7, 101. The sum is 118.

11. Inside regular hexagon $ZUMING$, construct square $FENG$. What fraction of the area of the hexagon is occupied by rectangle $FUME$?

**Solution.** The answer is $\boxed{\frac{6-2\sqrt{3}}{9}}$.

Suppose that the side length of the hexagon is 1, then the area of the hexagon is $6 \cdot \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{2}$. The length of $UN$ is $\sqrt{3}$, so the length of $FU$ is $\sqrt{3} - 1$, and the area of the rectangle of $FUME$ is also $\sqrt{3} - 1$. Finally, we compute the fraction $\frac{\sqrt{3}-1}{3\sqrt{3}/2} = \frac{6-2\sqrt{3}}{9}$.

12. How many ordered pairs $(x, y)$ of nonnegative integers satisfy the equation $4^x \cdot 8^y = 16^{10}$?

**Solution.** The answer is $\boxed{7}$.

We can first rewrite the equation into $2^{2x} \cdot 2^{3y} = 2^{40}$, or $2^{2x+3y} = 2^{40}$. We need the exponents to equal each other, so $2x + 3y = 40$. The solution with the largest possible value of x should be obvious: $x = 20, y = 0$. Notice that decreasing x by 3 and increasing y by 2 will always give another solution, and by doing this we can generate all solutions: $(x, y) = (20, 0), (17, 2), (14, 4), (11, 6), (8, 8), (5, 10), (2, 12)$, for a total of 7.

13. In triangle $ABC$ with $BC = 5$, $CA = 13$, and $AB = 12$, Points $E$ and $F$ are chosen on sides $AC$ and $AB$, respectively, such that $EF \parallel BC$. Given that triangle $AEF$ and trapezoid $EFBC$ have the same perimeter, find the length of $EF$.

**Solution.** The answer is $\boxed{3}$.

Notice that $\triangle ABC$ and $\triangle AFE$ are similar. Therefore, $\frac{AF}{AB} = \frac{AE}{AC} = \frac{EF}{BC}$.

Because the perimeters of triangle $AEF$ and trapezoid $EFBC$ share the segment $EF$, the perimeters being equal is equivalent to $AE + AF = FB + BC + CE$. However, the two sides of this equation together make up the perimeter of triangle $ABC$. Therefore, our equation is equivalent to $AE + AF = \frac{1}{2}(AB + AC + BC) = 15$.

Now we use our earlier similarity. Because $\frac{AF}{12} = \frac{AE}{13} = \frac{EF}{5}$, $AF = \frac{12}{5}EF$ and $AE = \frac{13}{5}EF$. Plugging back into the previous equation, we get $5EF = 15$, or $EF = 3$.

14. Find the number of two-digit positive integers with exactly 6 positive divisors. (Note that 1 and $n$ are both counted among the divisors of a number $n$.)

**Solution.** The answer is $\boxed{16}$.

If the prime factorization of $n$ is ${p_1}^{e_1}{p_2}^{e_2}...{p_k}^{e_k}$, it has $(e_1 + 1)(e_2 + 1)...(e_k + 1)$ divisors. In order for a number to have exactly 6 divisors, it must be in the form of either ${p_1}^5$ or ${p_1}^2{p_2}^1$. The first case only gives 32. The second case can be done by casework on $p_1$. If $p_1 = 2$, $p_2 = 3, 5, 7, 11, 13, 17, 19, 23$. If $p_1 = 3$, $p_2 = 2, 5, 7, 11$. If $p_1 = 5$, $p_2 = 2, 3$. If $p_1 = 7$, $p_2 = 2$. This gives a total of 16 two-digit integers.

15. How many ways are there to put two identical red marbles, two identical green marbles, and two identical blue marbles in a row such that no red marble is next to a green marble?

    **Solution.** The answer is $\boxed{12}$.

    Consider the positions of the two blue marbles (denoted by $B$). Note that we cannot have a block of three or more consecutive non-blue marbles in a row. Let $P, Q, R, S$ denote the other four marbles.

    - $BPQBRS$. $P$ and $Q$ must be the same color, and so must be $R$ and $S$. Thus there are 2 ways in this case.
    - $PBQBRS$. $R$ and $S$ must be the same color, and so must be $P$ and $Q$. Thus there are 2 ways in this case.
    - $PBQRBS$. $Q$ and $R$ must be the same color, and so must be $P$ and $S$. Thus there are 2 ways in this case.
    - $PQBBRS$. $P$ and $Q$ must be the same color, and so must be $R$ and $S$. Thus there are 2 ways in this case.
    - $PQBRBS$. This is the identical case as the second case, and thus there are 2 ways in this case.
    - $PQBRSB$. This is the identical case as the first case, and thus there are 2 ways in this case.

    Adding up all cases, we have $2 + 2 + 2 + 2 + 2 + 2 = 12$ ways in total.

16. Every day, Yannick submits 8 more problems to the EMCC problem database than he did the previous day. Every day, Vinjai submits twice as many problems to the EMCC problem database as he did the previous day. If Yannick and Vinjai initially both submit one problem to the database on a Monday, on what day of the week will the total number of Vinjai's problems first exceed the total number of Yannick's problems?

    **Solution.** The answer is $\boxed{\text{Monday}}$.

    Let the Monday when both Yannick and Vinjai submit 1 problem be day 1, then:

    On day 2, Yannick submits 9 problems, Vinjai submits 2 problems.

    On day 3, Yannick submits 17 problems, Vinjai submits 4 problems.

    On day 4, Yannick submits 25 problems, Vinjai submits 8 problems.

    On day 5, Yannick submits 33 problems, Vinjai submits 16 problems.

    On day 6, Yannick submits 41 problems, Vinjai submits 32 problems.

    On day 7, Yannick submits 49 problems, Vinjai submits 64 problems. At this point Yannick have submitted $((1 + 49) \cdot 7)/2 = 175$ problems, and Vinjai have submitted $2^7 - 1 = 127$ problems.

    On day 8, Yannick submits 57 problems, Vinjai submits 128 problems. At this point Yannick have submitted $175 + 57 = 232$ problems, and Vinjai have submited $127 + 128 = 255$ problems.

    Therefore, day 8 is the first day that Vinjai have submitted more problems than Yannick in total. Since day 1 is Monday, day 8 is Monday as well.

17. The tiny island nation of Konistan is a cone with height twelve meters and base radius nine meters, with the base of the cone at sea level. If the sea level rises four meters, what is the surface area of Konistan that is still above water, in square meters?

**Solution.** The answer is $\boxed{60\pi}$.

We first find the radius of the cone at the risen sea level, which is four meters above the base. Let this radius be $x$. Because the part of Konistan above the risen sea level is similar in shape to the entire Konistan, we can write the equation $\frac{x}{12-4} = \frac{9}{12}$, or $x = 6$. Then, we need to find the lateral area of the cone with height 8 and base radius 6. By the Pythagorean theorem, the lateral height of the cone is 10. We can use lateralSA $= \pi \cdot \text{radius} \cdot \text{lateral height}$, which comes out to $\pi \cdot 6 \cdot 10 = 60\pi$.

18. Nicky likes to doodle. On a convex octagon, he starts from a random vertex and doodles a path, which consists of seven line segments between vertices. At each step, he chooses a vertex randomly among all unvisited vertices to visit, such that the path goes through all eight vertices and does not visit the same vertex twice. What is the probability that this path does not cross itself?

**Solution.** The answer is $\boxed{\frac{4}{315}}$.

We first count the number of paths that don't cross itself. The key observation is as follows: if at any time, Nicky draws a line with unvisited vertices on both sides of the line, then the path has to cross itself.

Nicky can start at any vertex, and for every move after that except for the very last move, Nicky only has two choices for his next vertex, in order to not have unvisited vertices on both sides of his next line. In the very last move, he has no choices. Therefore, there is a total of $8 \cdot 2^6$ paths that don't self intersect.

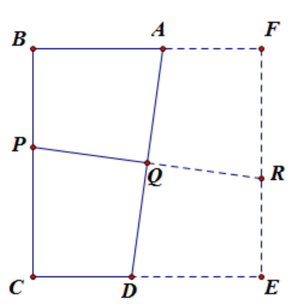
Then we count the total number of paths that Nicky can draw. He can start anywhere, and for each move afterwards, choose any of the yet unvisited vertices. There is a total of 8! total paths.

The answer is $\frac{8 \cdot 2^6}{8!} = \frac{2^6}{7!} = \frac{64}{5040} = \frac{4}{315}$.

19. In a right-angled trapezoid $ABCD$, $\angle B = \angle C = 90^\circ$, $AB = 20, CD = 17$, and $BC = 37$. A line perpendicular to $DA$ intersects segment $BC$ and $DA$ at $P$ and $Q$ respectively and separates the trapezoid into two quadrilaterals with equal area. Determine the length of $BP$.

**Solution.** The answer is $\boxed{17}$.

Construct a second copy $ADEF$ of trapezoid $ABCD$ such that $BCEF$ is a square with side length of 37, and let $R$ be the intersection of line $PQ$ with segment $EF$. When $Q$ is the midpoint of $AD$, by symmetry we can see that all four quadrilaterals $ABPQ, CDQP, DERQ, FAQR$ are congruent and therefore have the same area. If $Q$ is closer to $A$, then $ABPQ$ would have a smaller area and $CDQP$ would have a larger area, and vice versa. Therefore, $Q$ is the midpoint of $AD$ and $BP = CD = 17$.



20. A sequence of integers $a_i$ is defined by $a_1 = 1$ and $a_{i+1} = 3i - 2a_i$ for all integers $i \geq 1$. Given that $a_{15} = 5476$, compute the sum $a_1 + a_2 + a_3 + \cdots + a_{15}$.

    **Solution.** The answer is $\boxed{3756}$.

    Notice that $2a_i + a_{i+1} = 3i$ for all $i \geq 1$, and so we have

$$\begin{aligned} 3(a_1 + a_2 + \cdots + a_{15}) &= a_1 + (2a_1 + a_2) + (2a_2 + a_3) + \cdots + (2a_{14} + a_{15}) + 2a_{15} \\ &= 1 + 3(1 + 2 + 3 + \cdots + 14) + 2 \cdot 5476 = 10953 + 3 \cdot 105 = 11268 \end{aligned}$$

    Therefore $a_1 + a_2 + \cdots + a_{15} = \frac{11268}{3} = 3756$.

## 2.2 Accuracy Test Solutions

1. Chris goes to Matt's Hamburger Shop to buy a hamburger. Each hamburger must contain exactly one bread, one lettuce, one cheese, one protein, and at least one condiment. There are two kinds of bread, two kinds of lettuce, three kinds of cheese, three kinds of protein, and six different condiments: ketchup, mayo, mustard, dill pickles, jalapeños, and Matt's Magical Sunshine Sauce. How many different hamburgers can Chris make?

   **Solution.** The answer is $\boxed{2268}$.

   We first consider the number of ways to choose the set of condiments. If there are no restrictions on condiments, then there are 2 ways to determine whether each condiment is chosen or not, for $2^6 = 64$ ways. But since we cannot choose no condiments, we exclude one to get 63 ways to choose the set of condiments. Now we consider the other elements of the hamburger and multiply each of them to get the number of ways in total: $2 \cdot 2 \cdot 3 \cdot 3 \cdot 63 = 2268$ ways.

2. The degree measures of the interior angles in convex pentagon $NICKY$ are all integers and form an increasing arithmetic sequence in some order. What is the smallest possible degree measure of the pentagon's smallest angle?

   **Solution.** The answer is $\boxed{38^\circ}$.

   Suppose that the angles of the pentagon are $a-2d, a-d, a, a+d, a+2d$ degrees respectively for some positive integer $a$ and $d$. Notice that $(a-2d)+(a-d)+a+(a+d)+(a+2d) = 5a = (5-2)\cdot 180 = 540 \Rightarrow a = 108$, so we need to minimize $108-2d$ while $108+2d < 180$. The inequality implies $d < 36$, or $d \le 35 \Rightarrow 108 - 2d \ge 38$. Therefore, the smallest possible measure of the smallest angle is 38 degrees.

3. Daniel thinks of a two-digit positive integer $x$. He swaps its two digits and gets a number $y$ that is less than $x$. If 5 divides $x-y$ and 7 divides $x+y$, find all possible two-digit numbers Daniel could have in mind.

   **Solution.** The answer is $\boxed{61}$.

   Let the two digit integer $x$ be $10a+b$. From the given information, $10a+b-(10b+a) = 9(a-b)$ is divisible by 5, and $10a+b+(10b+a) = 11(a+b)$ is divisible by 7. Because $\gcd(5,9) = \gcd(11,7) = 1$, we must have that $a-b$ is divisible by 5 and $a+b$ is divisible by 7. In order for $a, b$ to have integer solutions, $a-b$ and $a+b$ must have the same parity, and $a+b \ge a-b$, so either $a-b = 5, a+b = 7$, or $a-b = 10, a+b = 14$. Only the first pair gives a valid solution of 61.

4. At the Lio Orympics, a target in archery consists of ten concentric circles. The radii of the circles are $1, 2, 3, \dots, 9$, and 10 respectively. Hitting the innermost circle scores the archer 10 points, the next ring is worth 9 points, the next ring is worth 8 points, all the way to the outermost ring, which is worth 1 point. If a beginner archer has an equal probability of hitting any point on the target and never misses the target, what is the probability that his total score after making two shots is even?

   **Solution.** The answer is $\boxed{\frac{101}{200}}$.

   We first find the probability that a single shot has an odd score. This is equivalent to the probability of hitting an area with score $1, 3, 5, 7, 9$, or the ratio of the area of the odd-score rings to the total area. The area of a ring with outer radius $a$ and inner radius $b$ is $\pi(a^2-b^2)$, and we can use this formula to calculate the sum of areas of the odd-score rings as $(10^2-9^2+8^2-7^2+\cdot+2^2-1^2)\pi = (10+9+8+\cdot+1)\pi = 55\pi$. Therefore, the probability of a single shot being odd is $\frac{55\pi}{100\pi} = \frac{11}{20}$.

For the sum of two shots to be even, they must be either both even or both odd. The probability of this happening is $(\frac{11}{20})^2 + (\frac{9}{20})^2 = \frac{121+81}{400} = \frac{101}{200}$

5. Let $F(x) = x^2 + 2x - 35$ and $G(x) = x^2 + 10x + 14$. Find all distinct real roots of $F(G(x)) = 0$.

**Solution.** The answer is $\boxed{-1, -3, -7, -9}$.

$F(x) = (x+7)(x-5)$, which means that in order for $F(G(x)) = 0$, we need $G(x) = -7$ or 5. $G(x) = -7$ factors into $(x+3)(x+7)$, and $G(x) = 5$ factors into $(x+1)(x+9)$. The roots are $-1, -3, -7, -9$.

6. One day while driving, Ivan noticed a curious property on his car's digital clock. The sum of the digits of the current hour equaled the sum of the digits of the current minute. (Ivan's car clock shows 24-hour time; that is, the hour ranges from 0 to 23, and the minute ranges from 0 to 59.) For how many possible times of the day could Ivan have observed this property?

**Solution.** The answer is $\boxed{112}$.

We consider cases based on the sum of digits in both hour and minute.

| Sum | Hour | Min | # of ways |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| 1 | 2 | 2 | 4 |
| 2 | 3 | 3 | 9 |
| 3 | 3 | 4 | 12 |
| 4 | 3 | 5 | 15 |
| 5 | 3 | 6 | 18 |
| 6 | 2 | 6 | 12 |
| 7 | 2 | 6 | 12 |
| 8 | 2 | 6 | 12 |
| 9 | 2 | 6 | 12 |
| 10 | 1 | 5 | 5 |

Summing all the totals in the right-hand column yields 112.

7. Qi Qi has a set $Q$ of all lattice points in the coordinate plane whose $x$- and $y$-coordinates are between 1 and 7 inclusive. She wishes to color 7 points of the set blue and the rest white so that each row or column contains exactly 1 blue point and no blue point lies on or below the line $x + y = 5$. In how many ways can she color the points?

**Solution.** The answer is $\boxed{486}$.

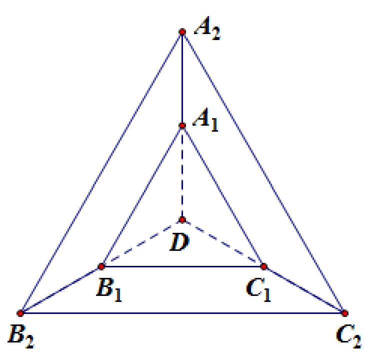
Qi Qi has to choose one of the three points $(1,5), (1,6), (1,7)$ to be colored blue, and once this point is chosen, one of the $(2,4), (2,5), (2,6), (2,7)$ cannot be blue anymore, so she still has 3 choices. We can continue this process for the first five columns, each with 3 possibilities, and then for the sixth and seventh column there are $2 \cdot 1 = 2$ ways to choose a blue point for each. Therefore, there are $3^5 \cdot 2 = 486$ ways to color the grid.

8. A piece of paper is in the shape of an equilateral triangle $ABC$ with side length 12. Points $A_B$ and $B_A$ lie on segment $AB$, such that $AA_B = 3$, and $BB_A = 3$. Define points $B_C$ and $C_B$ on segment $BC$ and points $C_A$ and $A_C$ on segment $CA$ similarly. Point $A_1$ is the intersection of $A_CB_C$ and $A_BC_B$. Define $B_1$ and $C_1$ similarly. The three rhombi — $AA_BA_1A_C, BB_CB_1B_A, CC_AC_1C_B$ — are cut from triangle $ABC$, and the paper is folded along segments $A_1B_1, B_1C_1, C_1A_1$, to form a tray without a top. What is the volume of this tray?

**Solution.** The answer is $\boxed{\frac{63}{4}\sqrt{2}}$.

By labeling equal segments, one can see that the top of the tray is in the shape of an equilateral triangle $A_2B_2C_2$ with side length 6, and the bottom of the tray $A_1B_1C_1$ is in the shape of an equilateral triangle with side length 3. The sides of the tray are trapezoids with angles 60 and 120 degrees. Imagine extending rays $A_2A_1$, $B_2B_1$, $C_2C_1$. The three rays will concur at some point $D$. This "completes" the tray into a regular tetrahedron, because all faces are equilateral triangles. In other words, the tray may be viewed as a regular tetrahedron with side length 6, with one of its corners cut off half way from the tip to the base.

Now we calculate the volume of the whole tetrahedron. Drop the altitude from $D$ to triangle $ABC$, and let the foot of perpendicular be $E$. $ABE$ is a 30-30-120 triangle, so $AE = 2\sqrt{3}$. By the pythagorean theorem, $DE = \sqrt{36-12} = \sqrt{24} = 2\sqrt{6}$. The volume of the tetrahedron is $\frac{1}{3} \cdot \text{base} \cdot \text{height} = \frac{1}{3} \cdot 9\sqrt{3} \cdot 2\sqrt{6} = 18\sqrt{2}$. The cut-off tip is simply a smaller version of this tetrahedron, and since its side length is half of the large tetrahedron, its volume is one eighth of the large tetrahedron. The remaining tray has volume that is seven-eighth of the large tetrahedron, or $\frac{7}{8} \cdot 18\sqrt{2} = \frac{63}{4}\sqrt{2}$.




9. Define $\{x\}$ as the fractional part of $x$. Let $S$ be the set of points $(x, y)$ in the Cartesian coordinate plane such that $x + \{x\} \le y$, $x \ge 0$, and $y \le 100$. Find the area of $S$.

**Solution.** The answer is $\boxed{\frac{19801}{4}}$.

Define $f(x) = x + \{x\}$. Divide the region $S$ up into vertical strips, each containing the range $a \le x < a+1$ for some nonnegative integer $a$. Within each strip, $f(x) = x + (x-a) = 2x - a$. When $0 \le a \le 98$, the region of $S$ within each strip is a trapezoid with two bases $100 - a$ and $98 - a$ and the height 1, which has area $99 - a$. When $a = 99$, the region of $S$ within that strip is a right triangle with base 1 and height $\frac{1}{2}$, which has area $\frac{1}{4}$. When $a > 99$, thre are no points in $S$ that lie in the strip. Therefore, the area of $S$ is equal to $99 + 98 + 97 + \cdots + 2 + 1 + \frac{1}{4} = \frac{99(1+99)}{2} + \frac{1}{4} = 4950.25$ or $\frac{19801}{4}$.

10. Nicky likes dolls. He has 10 toy chairs in a row, and he wants to put some indistinguishable dolls on some of these chairs. (A chair can hold only one doll.) He doesn't want his dolls to get lonely, so he wants each doll sitting on a chair to be adjacent to at least one other doll. How many ways are there for him to put any number (possibly none) of dolls on the chairs? Two ways are considered distinct if and only if there is a chair that has a doll in one way but does not have one in the other.

**Solution.** The answer is $\boxed{200}$.

We do casework on the number of "groups" of dolls there are. We'll count the number of seatings for each case by selecting the beginning and ending chair for each group.

There is one way for there to be no dolls.

If there is only one group of dolls, we only need to choose two distinct chairs. There are $\binom{10}{2} = 45$ seatings.

If there are two groups, we need to choose four distinct chairs, with at least one empty space between the second and third chair. We can solve this problem by choosing four chairs out of nine chairs, then "adding" an unchosen chairs between the second and third. Following this logic, there are $\binom{9}{4} = 126$ seatings.

If there are three groups, we continue using the logic of the previous cases and pick six chairs out of eight chairs. There are $\binom{8}{6} = 28$ seatings.

It is not possible for there to be four or more groups of dolls.

Overall, there are $1 + 45 + 126 + 28 = 200$ seatings.

Alternatively, define $A_n$ to be the number of ways to place dolls on $n$ chairs in a row such that no doll is alone. Then, By considering whether there is a doll in the first chair or not, and if so, how many dolls are there next to each other, we see that $A_n = A_{n-1} + (A_{n-3} + A_{n-4} + \cdots + A_1 + A_0 + A_0)$ for $n \geq 3$. ($A_{n-1}$ represents the case where there is no dolls on the first chair, $A_{n-3}$ represents two dolls in the first two chairs, $A_{n-4}$ represents three dolls in the first three chairs, etc. Note that we need to include the empty chair after the leading group, and there are two $A_0$ at the end of the expression because one is the case of the first $n-1$ chairs being occupied and the other is the case of all $n$ chairs being occupied.)

Starting with $A_0 = 1, A_1 = 1$, and $A_2 = 2$, we get $A_3 = 4, A_4 = 7, A_5 = 12, A_6 = 21, A_7 = 37, A_8 = 65, A_9 = 114$, and finally $A_{10} = 200$.

## 2.3 Team Test Solutions

1. Compute $2017 + 7201 + 1720 + 172$.

   **Solution.** The answer is $\boxed{11110}$.

   Notice that digits $0, 1, 2$, and $7$ appear on each place once each, so the sum is just $7777+2222+1111 = 11110$.

2. A number is called *downhill* if its digits are distinct and in descending order. (For example, 653 and 8762 are downhill numbers, but 97721 is not.) What is the smallest downhill number greater than 86432?

   **Solution.** The answer is $\boxed{86510}$.

   We cannot only alter the last two digits of the number since 432 is the largest downhill number with hundreds digit 4, so the next downhill number must have hundreds digit 5. When the hundreds digit is 5, the last two digits must be at least 10. Indeed 86510 is a downhill number, so that is the answer.

3. Each vertex of a unit cube is sliced off by a planar cut passing through the midpoints of the three edges containing that vertex. What is the ratio of the number of edges to the number of faces of the resulting solid?

   **Solution.** The answer is $\boxed{\frac{12}{7}}$.

   After the cuts are made, each edge belongs to exactly one of the six original faces, and each of these six faces has four of these edges, so there are $6 \cdot 4 = 24$ edges. As for faces, since each cut creates exactly one new face, and there are eight cuts, so we have $6 + 8 = 14$ faces. Finally, we compute the ratio $\frac{24}{14} = \frac{12}{7}$.

4. In a square with side length 5, the four points that divide each side into five equal segments are marked. Including the vertices, there are 20 marked points in total on the boundary of the square. A pair of distinct points $A$ and $B$ are chosen randomly among the 20 points. Compute the probability that $AB = 5$.

   **Solution.** The answer is $\boxed{\frac{2}{19}}$.

   There are two main ways to get $AB = 5$: one is to have segment $AB$ parallel to one of the edges, and one is to have segment $AB$ cut the square into a pentagon and a $3-4-5$ triangle. There are $6\cdot 2 = 12$ (unordered) pairs of points that satisfy the first condition, and $2 \cdot 4 = 8$ pairs that satisfy the second, for $12 + 8 = 20$ pairs in total. Since there are $\binom{20}{2} = 190$ possible pairs in total, the probability is $\frac{20}{190} = \frac{2}{19}$.

5. A positive two-digit integer is one less than five times the sum of its digits. Find the sum of all possible such integers.

   **Solution.** The answer is $\boxed{113}$.

   It is easy to see that this integer must leave a remainder of 4 when divided by 5, which means that the last digit must be 4 or 9. If the two-digit number is $\overline{a4}$, then we have $10a + 4 = 5(a + 4) - 1$, which gives $a = 3$. If the number is $\overline{a9}$, then we have $10a + 9 = 5(a + 9) - 1$, which gives $a = 7$. Therefore the two possible numbers are 34 and 79, and their sum is 113.

6. Let

$$f(x) = 5^{4^{3^{2^x}}}.$$

Determine the greatest possible value of $L$ such that $f(x) > L$ for all real numbers $x$.

**Solution.** The answer is $\boxed{625}$.

Since $2^x > 0$ for all real number $x$, we have $3^{2^x} > 3^0 = 1$, $4^{3^{2^x}} > 4^1 = 4$, and finally $5^{4^{3^{2^x}}} > 5^4 = 625$. When $x$ is very small (very negative), $2^x$ can get arbitrarily close to 0, and similarly all the expression on the left-hand side of each inequality can be arbitrarily close to the right-hand side. Therefore $L_{\max} = 625$.

7. If $\overline{AAAA} + \overline{BB} = \overline{ABCD}$ for some distinct base-10 digits $A, B, C, D$ that are consecutive in some order, determine the value of $\overline{ABCD}$. (The notation $\overline{ABCD}$ refers to the four-digit integer with thousands digit $A$, hundreds digit $B$, tens digit $C$, and units digit $D$.)

**Solution.** The answer is $\boxed{7865}$.

Clearly $B = A + 1$ because exactly one carry-over happens on the tens digit, and consequently $\overline{CD} < \overline{AA}$. In particular, since $A + B > 10$, we see that a carry-over also occurs on the ones digit, we have $D = A + B - 10 < A$ and $C = D + 1$. Therefore, the four digits satisfy $D < C < A < B$. Using the fact that $D = A + B - 10 = A + (A + 1) - 10 = 2A - 9$ and $D = A - 2$, we find that $A = 7$, and therefore $\overline{ABCD} = 7865$. It is not difficult to see that $7777 + 88 = 7865$ is a valid solution.

8. A regular tetrahedron and a cube share an inscribed sphere. What is the ratio of the volume of the tetrahedron to the volume of the cube?

**Solution.** The answer is $\boxed{\sqrt{3}}$.

Suppose that the side length of the tetrahedron $ABCD$ is $a$, and let the center of triangle $BCD$ be $H$. Then the distance from $B$ to $H$ is $\frac{1}{\sqrt{3}}a$, and thus the distance from $A$ to $H$ is $\sqrt{\frac{2}{3}}a = \frac{\sqrt{6}}{3}a$. Since the center of the inscribed sphere lies three-fourth the way from $A$ to $H$, the radius of the inscribed sphere is $\frac{\sqrt{6}}{12}a$. Consequently, the side length of the cube is $\frac{\sqrt{6}}{6}a$. The volume of the tetrahedron is $\frac{1}{3}(\frac{\sqrt{6}}{3}a \cdot (\frac{\sqrt{3}}{4}a^2)) = \frac{\sqrt{2}}{12}a^3$, and the volume of the cube is $(\frac{\sqrt{6}}{6}a)^3 = \frac{\sqrt{6}}{36}a^3$. Therefore, the ratio of the two volumes is $\frac{\sqrt{2}/12}{\sqrt{6}/36} = \sqrt{3}$.

9. Define $\lfloor x \rfloor$ as the greatest integer less than or equal to $x$, and $\{x\} = x - \lfloor x \rfloor$ as the fractional part of $x$. If $\lfloor x^2 \rfloor = 2\lfloor x \rfloor$ and $\{x^2\} = \frac{1}{2}\{x\}$, determine all possible values of $x$.

**Solution.** The answer is $\boxed{0, \frac{1}{2}, \frac{3}{2}, 2}$.

Since $\lfloor x^2 \rfloor \geq 0$, we have $\lfloor x \rfloor \geq 0$. On the other hand, if $\lfloor x \rfloor > 2$, we have $\lfloor x^2 \rfloor \geq \lfloor x \rfloor^2 > 2\lfloor x \rfloor$, so $\lfloor x \rfloor = 0, 1$ or 2.

When $\lfloor x \rfloor = 0$, we have $\lfloor x^2 \rfloor = 0$, and $x^2 = \frac{1}{2}x$, which gives $x = 0$ or $\frac{1}{2}$.

When $\lfloor x \rfloor = 1$, we have $\lfloor x^2 \rfloor = 2$, and $\{x^2\} = x^2 - 2 = \frac{1}{2}(x-1) = \frac{1}{2}\{x\}$, which gives $x = \frac{3}{2}$ or $-1$ (the second solution is discarded).

When $\lfloor x \rfloor = 2$, we have $\lfloor x^2 \rfloor = 2$, and $\{x^2\} = x^2 - 4 = \frac{1}{2}(x-2) = \frac{1}{2}\{x\}$, which gives $x = 2$ or $-\frac{3}{2}$ (the second solution is discarded). Therefore, all possible values of $x$ are $0, \frac{1}{2}, \frac{3}{2}$ or 2.

10. Find the largest integer $N > 1$ such that it is impossible to divide an equilateral triangle of side length 1 into $N$ smaller equilateral triangles (of possibly different sizes).

**Solution.** The answer is $\boxed{5}$.

We first show that it is impossible to divide an equilateral triangle into five smaller equilateral triangles. First of all, each vertex of the original triangle must belong to different smaller triangles, otherwise the triangle that contains two of these vertices will cover the entire triangle. If we cut away these three triangles, then we would either have a hexagon, a pentagon, an isosceles trapezoid, or an equilateral triangle. However, if two equilateral triangles combined form a convex figure, they must form a rhombus. This means that the remaining region cannot be divided into two equilateral triangles.

Now we show that for all $N > 5$, it is possible to do such a division. First, note that all even numbers $2n$ with $n \geq 2$ are possible: Cut out a triangle with side length $\frac{n-1}{n}$, and then divide the remaining trapezoid into $2n-1$ triangles with side length $\frac{1}{n}$. Second, given a valid division, we can always create exactly 3 more triangles by dividing a current one into four smaller congruent triangles. This covers all the odd numbers greater than or equal to 7. Therefore, whenever $N > 5$, we can always find a division with $N$ triangles.

11. Let $f$ and $g$ be two quadratic polynomials. Suppose that $f$ has zeroes 2 and 7, $g$ has zeroes 1 and 8, and $f - g$ has zeroes 4 and 5. What is the product of the zeroes of the polynomial $f + g$?

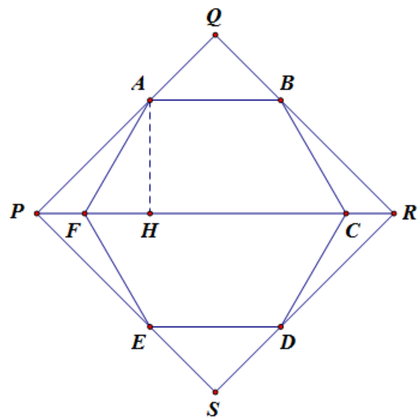
**Solution.** The answer is $\boxed{12}$.

Note that we can express $f = p(x-2)(x-7)$ and $g = q(x-1)(x-8)$ where $p$ and $q$ are two nonzero constants. The problem condition tells us that $f = g$ when $x = 4$ or $x = 5$, and plugging in one of them gives $-6p = -12q$, or $p = 2q$. Without loss of generality, assume that $q = 1$ and $p = 2$, then $f + g = 3x^2 - 27x + 36$, and by Vieta's Theorem the product of the two roots is $\frac{36}{3} = 12$.

12. In square $PQRS$, points $A, B, C, D, E$, and $F$ are chosen on segments $PQ$, $QR$, $PR$, $RS$, $SP$, and $PR$, respectively, such that $ABCDEF$ is a regular hexagon. Find the ratio of the area of $ABCDEF$ to the area of $PQRS$.

**Solution.** The answer is $\boxed{3\sqrt{3} - \frac{9}{2}}$.

Notice that the entire diagram is symmetric about both diagonals $PR$ and $QS$. Suppose that the side length of the hexagon is $a$, then the area of the hexagon is $6(\frac{\sqrt{3}}{4})a^2 = \frac{3\sqrt{3}}{2}a^2$. Let $AH$ be perpendicular to $CF$ at $H$, then $AH = \frac{\sqrt{3}}{2}a$ and $AP = \sqrt{2}AH = \frac{\sqrt{6}}{2}a$. Moreover, we have $AQ = \frac{\sqrt{2}}{2}a$, so $PQ = \frac{\sqrt{6}+\sqrt{2}}{2}a$, and the area of the square is $PQ^2 = (\sqrt{3}+2)a^2$. Therefore the desired ratio is $\frac{3\sqrt{3}/2}{\sqrt{3}+2} = 3\sqrt{3} - \frac{9}{2}$.




13. For positive integers $m$ and $n$, define $f(m, n)$ to be the number of ways to distribute $m$ identical candies to $n$ distinct children so that the number of candies that any two children receive differ by at most 1. Find the number of positive integers $n$ satisfying the equation $f(2017, n) = f(7102, n)$.

**Solution.** The answer is $\boxed{15}$.

If $m = nq + r$ where $0 \le r < n$, then we need to first give each child $q$ candies, and then give the rest to $r$ distinct children. Therefore $f(m, n) = \binom{n}{r}$. Since $\binom{n}{x} = \binom{n}{y}$ if and only if $x = y$ or $x + y = n$, we see that either $2017 \equiv 7102 \pmod{n}$ or $2017 + 7102 \equiv 0 \pmod{n}$. This means that $n$ either divides $7102 - 2017 = 5085 = 3^2 \cdot 5 \cdot 113$, or $7102 + 2017 = 9119 = 11 \cdot 829$. There are $3 \cdot 2 \cdot 2 = 12$ factors of 5085, and $2 \cdot 2 = 4$ factors of 9119, but we counted $n = 1$ twice, so there are $12 + 4 - 1 = 15$ possible values of $n$.

14. Suppose that real numbers $x$ and $y$ satisfy the equation

$$x^4 + 2x^2y^2 + y^4 - 2x^2 + 32xy - 2y^2 + 49 = 0.$$

Find the maximum possible value of $\frac{y}{x}$.

**Solution.** The answer is $\boxed{\frac{-4+\sqrt{7}}{3}}$.

We may rewrite the left-hand side into

$$\begin{aligned}(x^2 + y^2 + 7)^2 - 16x^2 + 32xy - 16y^2 &= (x^2 + y^2 + 7)^2 - (4x - 4y)^2 \\ &= (x^2 - 4x + y^2 + 4y + 7)(x^2 + 4x + y^2 - 4y + 7) \\ &= ((x - 2)^2 + (y + 2)^2 - 1)((x + 2)^2 + (y - 2)^2 - 1)\end{aligned}$$

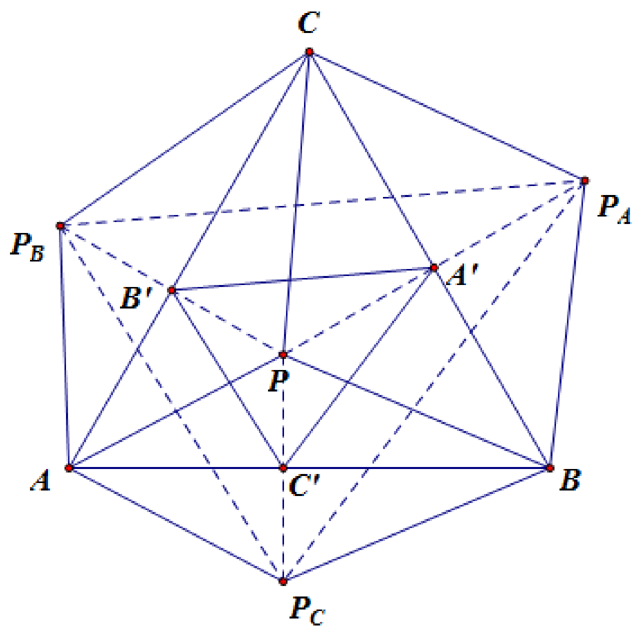
Graphically speaking, the equation is equivalent to two unit circles centered at $(2, -2)$ and $(-2, 2)$ respectively, and we aim to maximize the slope through the origin and a point on either circle. Since the two circles are symmetric about the origin, we only need to consider one of them.

WLOG we assume that $x^2 - 4x + y^2 + 4y + 7 = 0$. Set $y = kx$, and we have $(k^2+1)x^2 + 4(k-1)x + 7 = 0$. This equation has solutions if and only if $(4(k - 1))^2 - 4(7)(k^2 + 1) = 4(-3k^2 - 8k - 3) \ge 0$, or $3k^2 + 8k + 3 \le 0$. Solving this inequality, we get $\frac{-4-\sqrt{7}}{3} \le k \le \frac{-4+\sqrt{7}}{3}$, so the maximum value of $k$ is $\frac{-4+\sqrt{7}}{3}$.

15. A point $P$ lies inside equilateral triangle $ABC$. Let $A', B', C'$ be the feet of the perpendiculars from $P$ to $BC, AC, AB$, respectively. Suppose that $PA = 13, PB = 14$, and $PC = 15$. Find the area of $A'B'C'$.

    **Solution.** The answer is $\boxed{63}$.

    Reflect $P$ across sides $BC, CA, AB$ to get $P_A, P_B, P_C$ respectively.




Notice that $CP_A = CP_B = CP = 15$ and $\angle P_ACP_B = 2\angle ACB = 120°$, which implies $P_AP_B = 15\sqrt{3}$. Since $A'B'$ is the midline of the triangle $P_APP_B$, we have $A'B' = \frac{1}{2}P_AP_B = 7.5\sqrt{3}$. Similarly, we get $B'C' = 6.5\sqrt{3}$ and $C'A' = 7\sqrt{3}$. We can then use Heron's formula:

$$[A'B'C'] = \sqrt{s'(s'-a')(s'-b')(s'-c')} = \sqrt{(10.5\sqrt{3}) \cdot (4\sqrt{3}) \cdot (3.5\sqrt{3}) \cdot (3\sqrt{3})} = 63.$$

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. **[5]** If $2m = 200cm$ and $m \neq 0$, find $c$.

   **Solution.** The answer is $\boxed{\frac{1}{100}}$.

   We may divide both sides by $2m$ to get $1 = 100c$, which means $c = \frac{1}{100}$.

2. **[5]** A right triangle has two sides of lengths 3 and 4. Find the smallest possible length of the third side.

   **Solution.** The answer is $\boxed{\sqrt{7}}$.

   If the third side is the hypotenuse, then the length is $\sqrt{3^2 + 4^2} = 5$. If the third side is not the hypotenuse, then the length is $\sqrt{4^2 - 3^2} = \sqrt{7}$. Clearly the second one is smaller, so that is our answer.

3. **[5]** Given that $20(x + 17) = 17(x + 20)$, determine the value of $x$.

   **Solution.** The answer is $\boxed{0}$.

   By expanding both sides, we get $20x + 20 \cdot 17 = 17x + 17 \cdot 20$, which means $20x = 17x$. Therefore the only solution is $x = 0$.

### 2.4.2 Round 2

4. **[7]** According to the Egyptian Metropolitan Culinary Community, food service is delayed on $\frac{2}{3}$ of flights departing from Cairo airport. On average, if flights with delayed food service have twice as many passengers per flight as those without, what is the probability that a passenger departing from Cairo airport experiences delayed food service?

   **Solution.** The answer is $\boxed{\frac{4}{5}}$.

   Suppose that there are $3f$ flights, then $2f$ of them have delayed food service. If the flightss with normal food service have $p$ passengers per flight, then in total there are $(2f)(2p) + fp = 5fp$ departing passengers, among which $4fp$ of them experiences delayed food service. Therefore the probability is $\frac{4}{5}$.

5. **[7]** In a positive geometric sequence $\{a_n\}$, $a_1 = 9, a_9 = 25$. Find the integer $k$ such that $a_k = 15$.

   **Solution.** The answer is $\boxed{5}$.

   Notice that 15 is the geometric mean of 9 and 25, which means that in a geometric sequence 15 should be exactly half-way between 9 and 25. Therefore $k = \frac{1+9}{2} = 5$.

6. **[7]** In the Delicate, Elegant, and Exotic Music Organization, pianist Hans is selling two types of flowers with different prices (per flower): magnolias and myosotis. His friend Alice originally plans to buy a bunch containing 50% more magnolias than myosotis for \$50, but then she realizes that if she buys

50% less magnolias and 50% more myosotis than her original plan, she would still need to pay the same amount of money. If instead she buys 50% more magnolias and 50% less myosotis than her original plan, then how much, in dollars, would she need to pay?

**Solution.** The answer is $\boxed{50}$.

Suppose that the total price of magnolias in Alice's original plan is $M$ dollars, and the total price of myosotis is $N$ dollars. Then we have $M + N = \frac{1}{2}M + \frac{3}{2}N = 50$, which implies $M = N$, and thus $\frac{3}{2}M + \frac{1}{2}N = M + N = 50$ dollars as well.

### 2.4.3 Round 3

7. **[9]** In square $ABCD$, point $P$ lies on side $AB$ such that $AP = 3, BP = 7$. Points $Q, R, S$ lie on sides $BC, CD, DA$ respectively such that $PQ = PR = PS = AB$. Find the area of quadrilateral $PQRS$.

   **Solution.** The answer is $\boxed{50}$.

   Since $PR = AB$, we see that triangle $PQR$ has half the area of rectangle $PBCR$, and triangle $PRS$ has half the area of rectangle $PRCD$. Therefore, the quadrilateral $PQRS$ has half the area of square $ABCD$, which has side length $3 + 7 = 10$. So the area of $PQRS$ is $\frac{1}{2}(10^2) = 50$.

8. **[9]** Kristy is thinking of a number $n < 10^4$ and she says that 143 is one of its divisors. What is the smallest number greater than 143 that could divide $n$?

   **Solution.** The answer is $\boxed{154}$.

   Notice that if the next divisor of $n$ is $143 + k$, then $\text{lcm}(143, 143 + k)$ divides $n$. Since $143 = 11 \cdot 13$ we see that $\text{lcm}(143, 143 + k) = 143(143 + k) > 143^2 > 10^4$ if $k$ is divisible by neither 11 nor 13, so $k$ has to be at least 11. When $k = 11$, we see that $\text{lcm}(143, 154) = 11 \cdot 13 \cdot 14 = 2002 < 10^4$, which means that $n = 2002$ could satisfy this condition. Therefore the smallest possible number is 154.

9. **[9]** A positive integer $n$ is called *special* if the product of the $n$ smallest prime numbers is divisible by the sum of the $n$ smallest prime numbers. Find the sum of the three smallest special numbers.

   **Solution.** The answer is $\boxed{12}$.

   We proceed by trying small cases:

   When $n = 1$, 2 divides 2, so 1 is special.

   When $n = 2$, $2 + 3 = 5$ does not divide $2 \cdot 3$, so 2 is not special.

   When $n = 3$, $2 + 3 + 5 = 10$ divides $2 \cdot 3 \cdot 5$, so 3 is special.

   When $n = 4$, $2 + 3 + 5 + 7 = 17$ does not divide $2 \cdot 3 \cdot 5 \cdot 7$, so 4 is not special.

   When $n = 5$, $2 + 3 + 5 + 7 + 11 = 28$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$, so 5 is not special.

   When $n = 6$, $2 + 3 + 5 + 7 + 11 + 13 = 41$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$, so 6 is not special.

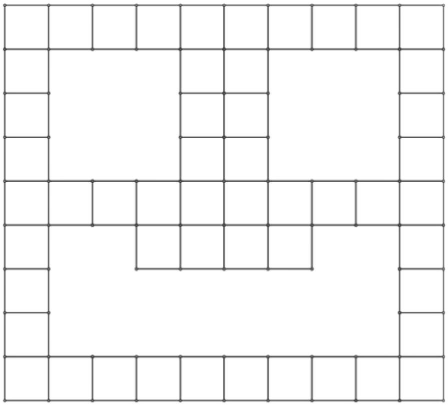
   When $n = 7$, $2 + 3 + 5 + 7 + 11 + 13 + 17 = 58$ does not divide $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$, so 7 is not special.

   When $n = 8$, $2 + 3 + 5 + 7 + 11 + 13 + 17 + 19 = 77$ divides $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so 8 is special.

   So the three smallest special numbers are 1, 3, and 8, and their sum is $1 + 3 + 8 = 12$.

### 2.4.4 Round 4

10. **[11]** In the diagram below, all adjacent points connected with a segment are unit distance apart. Find the number of squares whose vertices are among the points in the diagram and whose sides coincide with the drawn segments.



**Solution.** The answer is $\boxed{77}$.

There are 52 squares with side length 1, 7 squares with side length 2, 2 squares with side length 3, 8 squares with side length 4, 4 squares with side length 5, 2 squares with side length 8, and 2 squares with side length 9. for $52 + 7 + 2 + 8 + 4 + 2 + 2 = 77$ squares in total.

11. **[11]** Geyang tells Junze that he is thinking of a positive integer. Geyang gives Junze the following clues:

- My number has three distinct odd digits.
- It is divisible by each of its three digits, as well as their sum.

What is the sum of all possible values of Geyang's number?

**Solution.** The answer is $\boxed{1185}$.

We consider two cases:

*Case 1:* One of the digits is 5. Then since the number must be a multiple of 5, 5 must be on the units digit. Suppose that neither 3 nor 9 is in this number, then the number must be 175 or 715, but the first one is not divisible by $1 + 7 + 5 = 13$ and the second one is not divisible by 7. Therefore, one of the digits must be 3 or 9 and therefore the sum of digits must be a multiple of 3, which means that the other digit must be 1 or 7. When a digit is 3, the other can be 1 or 7. We see that 135, 315, and 735 all satisfy the requirements, but 375 is not divisible by 7. When a digit is 9, the other one can be neither 1 nor 7 because neither $1 + 9 + 5 = 15$ nor $7 + 9 + 5 = 24$ is divisible by 9.

*Case 2:* None of the digits is 5. Then we see that one of the digits must be 3 or 9, which means that the sum of digits is a multiple of 9. Since 1 and 7 are both congruent to 1 modulo 3, and we must use at least one of them, it is impossible to choose three numbers out of $\{1, 3, 7, 9\}$ whose sum is divisible by 3.

Therefore, all possible values of Geyang's number are 135, 315, and 735, and their sum is 1185.

12. **[11]** Regular octagon $ABCDEFGH$ has center $O$ and side length 2. A circle passes through $A, B$, and $O$. What is the area of the part of the circle that lies outside of the octagon?

**Solution.** The answer is $\boxed{\frac{\pi}{2} - 1}$.

Since $\angle AOB = 45°$, it follows that arc $AB$ in the circumcircle of $ABO$ has measure $90°$, which also means that the radius of this circle is $\sqrt{2}$. The area outside the octagon is a quarter-circle minus a right isosceles triangle, or $\frac{1}{4}\pi(\sqrt{2})^2 - \frac{1}{2}(\sqrt{2})^2 = \frac{\pi}{2} - 1$.

### 2.4.5 Round 5

13. **[13]** Kelvin Amphibian, a not-frog who lives on the coordinate plane, likes jumping around. Each step, he jumps either to the spot that is 1 unit to the right and 2 units up, or the spot that is 2 units to the right and 1 unit up, from his current location. He chooses randomly among these two choices with equal probability. He starts at the origin and jumps for a long time. What is the probability that he lands on $(10, 8)$ at some time in his journey?

**Solution.** The answer is $\boxed{\frac{15}{64}}$.

The sum of coordinates of Kelvin must increase by 3 each time, so if he reaches $(10, 8)$ he must reach it after exactly $\frac{10+8}{3} = 6$ jumps. Suppose that among these 6 jumps, $x$ of them are 1 unit right and 2 units up, and $y$ of them are 2 units right and 1 unit up. Then we have $x+2y = 10$ and $2x+y = 8$, which gives $x = 4, y = 2$. The number of ways to get to $(10, 8)$ is therefore $\binom{6}{2} = 15$, out of $2^6$ ways to make the 6 jumps, so the probability is $\frac{15}{64}$.

14. **[13]** Points $A$, $B$, $C$, and $D$ are randomly chosen on the circumference of a unit circle. What is the probability that line segments $AB$ and $CD$ intersect inside the circle?

**Solution.** The answer is $\boxed{\frac{1}{3}}$.

Suppose that we first choose four random points on the circle without assigning the labels. If $A$ is assigned to one of the points, then $AB$ and $CD$ intersects if and only if $B$ is assigned to the one point that is not adjacent to $A$, which happens with $\frac{1}{3}$ possibility.

15. **[13]** Let $P(x)$ be a quadratic polynomial with two consecutive integer roots. If it is also known that $\frac{P(2017)}{P(2016)} = \frac{2016}{2017}$, find the larger root of $P(x)$.

**Solution.** The answer is $\boxed{6050}$.

Suppose that the two roots of $P(x)$ are $a$ and $a + 1$, then we have $P(x) = k(x - a)(x - a - 1)$ for some nonzero constant $k$. The left-hand side of the given equation can then be expressed as $\frac{k(2017-a)(2016-a)}{k(2016-a)(2015-a)} = \frac{2017-a}{2015-a}$, which implies $2016(2015 - a) = 2017(2017 - a)$, and so $a = 2017 \cdot 2017 - 2016 \cdot 2015 = 6049$. The larger root is $a + 1 = 6050$.

### 2.4.6 Round 6

16. **[15]** Let $S_n$ be the sum of reciprocals of the integers between 1 and $n$ inclusive. Find a triple $(a, b, c)$ of positive integers such that $S_{2017} \cdot S_{2017} - S_{2016} \cdot S_{2018} = \frac{S_a + S_b}{c}$.

**Solution.** The answer is $\boxed{(2017, 1, 4070306) \text{ or } (1, 2017, 4070306)}$.

We can rewrite the left hand side as

$$S_{2017} \cdot S_{2017} - (S_{2017} - \frac{1}{2017})(S_{2017} + \frac{1}{2018}) = \frac{S_{2017}}{2017} - \frac{S_{2017}}{2018} + \frac{1}{2017 \cdot 2018} = \frac{S_{2017} + 1}{4070306}.$$

Therefore one possible triple is $(2017, 1, 4070306)$ since $S_1 = 1$. We can also swap $a$ and $b$ to get $(1, 2017, 4070306)$.

17. **[15]** Suppose that $m$ and $n$ are both positive integers. Alec has $m$ standard 6-sided dice, each labelled 1 to 6 inclusive on the sides, while James has $n$ standard 12-sided dice, each labelled 1 to 12 inclusive on the sides. They decide to play a game with their dice. They each toss all their dice simultaneously and then compute the sum of the numbers that come up on their dice. Whoever has a higher sum wins (if the sums are equal, they tie). Given that both players have an equal chance of winning, determine the minimum possible value of $mn$.

**Solution.** The answer is $\boxed{91}$.

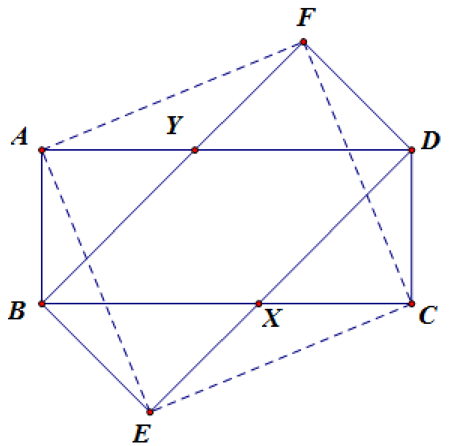
Notice that on average Alec's sum is $(\frac{1+2+3+4+5+6}{6})m = \frac{7}{2}m$ and James' sum is $(\frac{1+2+3+\cdots+12}{12})n = \frac{13}{2}n$, and the probability distribution for each person is symmetric about his average value. This means that it is equally likely for Alec to get $x$ and $7m - x$ for his sum, and equally likely for James to get $y$ and $13n - y$ for his sum.

If Alec and James have an equal chance of winning, we should be able to pair up the cases where Alec wins with the cases where James wins. This can happen if and only if they have the same average value, because only then can we have a one-to-one correspondence between $(x, y)$ and $(7m - x, 13n - y)$, with either both of them representing a tie or one representing an Alec victory and the other representing a James victory. Therefore $\frac{7}{2}m = \frac{13}{2}n$, or $m : n = 13 : 7$, and the minimum value of $mn$ is 91, achieved when $m = 13, n = 7$.

18. **[15]** Overlapping rectangles $ABCD$ and $BEDF$ are congruent to each other and both have area 1. Given that $A, C, E, F$ are the vertices of a square, find the area of the square.

**Solution.** The answer is $\boxed{\sqrt{2}}$.

Without loss of generality assume that $AB < BC$, $BC$ and $DE$ intersects at $X$, and $AD$ and $BF$ intersects at $Y$. Notice that $AB = BE$, $EX = XC$ (by symmetry), and $\angle ABE = \angle EXC$, so if $AE = EC$, then triangles $ABE$ and $EXC$ are congruent. This means that $BE = EX$, $\angle ABE = 135°$, and $BX = \sqrt{2}EX = \sqrt{2}XC$. Therefore the overlapping area $BXYD$ is equal to $\frac{\sqrt{2}}{\sqrt{2}+1} = 2 - \sqrt{2}$. Since the area of the square is equal to that of the concave octagon $ABEXCDFY$, which is $1 + 1 - (2 - \sqrt{2}) = \sqrt{2}$. Our answer is $\sqrt{2}$.




### 2.4.7 Round 7

19. **[18]** Find the number of solutions to the equation

$$|||\dots|||||x|+1|-2|+3|-4|+\cdots-98|+99|-100|=0.$$

**Solution.** The answer is $\boxed{51}$.

To simplify notation, we define $A_0=|x|, A_1=||x|+1|, A_2=|||x|+1|-2|, \dots, A_{100}=||||\dots|||||x|+1|-2|+3|-4|+\dots|+99|-100|$.

Notice that when $n$ is odd, $A_n=|A_{n-1}+n|=A_{n-1}+n$, and then $A_{n+1}=|A_n-(n+1)|=|A_{n-1}-1|$. So in general, we have $A_{2k}=|A_{2(k-1)}-1|$.

We now work backwards from $A_{100}=0$: the equation implies $A_{98}=1, A_{96}=0$ or $2, A_{94}=1$ or $3, A_{92}=0$ or $2$ or $4, A_{90}=1$ or $3$ or $5, \dots$.

Therefore, we can follow this pattern and get $A_0=0$ or $2$ or $4$ or $\dots$ or $50$. When $A_0=0$, there is 1 solution, otherwise there are 2 possible values for each $A_0$, so there are $1+2\cdot 25=51$ solutions in total.

20. **[18]** A *split* of a positive integer in base 10 is the separation of the integer into two nonnegative integers, allowing leading zeroes. For example, 2017 can be split into 2 and 017 (or 17), 20 and 17, or 201 and 7. A split is called *squarish* if both integers are nonzero perfect squares. 49 and 169 are the two smallest perfect squares that have a squarish split (4 and 9, 16 and 9 respectively). Determine all other perfect squares less than 2017 with at least one squarish split.

**Solution.** The answer is $\boxed{361,\ 1225,\ 1444,\ 1681}$.

Let's start by considering three-digit numbers greater than 169. Since the first digit or the first two digits must be a perfect square, the possibilities are: $\overline{1AB}, \overline{25A}, \overline{36A}, \overline{4AB}, \overline{49A}, \overline{64A}, \overline{81A}, \overline{9AB}$. It is not difficult to check that among all these options, $361=19^2$ is the only one that works.

Now we consider the four-digit numbers whose thousand digit is 1. If the last three digits form a perfect square $m^2$, and the four-digit number is $n^2$, then we have $n^2-m^2=1000$, or $(n-m)(n+m)=$

$1000 = 2^3 \cdot 5^3$. Keeping in mind that $n-m$ and $n+m$ have the same parity, the only solution with $m^2 < 1000$ is $(n-m, n+m) = (20, 50) \Rightarrow (n, m) = (35, 15) \Rightarrow n^2 = 1225$.

If the first two digits are 16, then similarly we need $n^2 - m^2 = 1600$, since $m^2 < 100$, the only solution is $(n-m, n+m) = (32, 50) \Rightarrow (n, m) = (41, 9) \Rightarrow n^2 = 1681$.

If the first three digits are a perfect square, then the only options are $\overline{121A}, \overline{144A}, \overline{169A}, \overline{196A}$. Among these possibilities, only $1444 = 38^2$ is a perfect square with the last digit being a perfect square as well.

There are also no perfect squares between 2000 and 2016 inclusive, so all perfect squares greater than 169 with a squarish split are 361, 1225, 1444, and 1681.

21. **[18]** Polynomial $f(x) = 2x^3 + 7x^2 - 3x + 5$ has zeroes $a, b$, and $c$. Cubic polynomial $g(x)$ with $x^3$-coefficient 1 has zeroes $a^2, b^2$, and $c^2$. Find the sum of coefficients of $g(x)$.

**Solution.** The answer is $\boxed{-\frac{143}{4}}$.

Note that we can write $f(x) = 2(x-a)(x-b)(x-c)$, and $g(x) = (x-a^2)(x-b^2)(x-c^2)$. The sum of coefficients of $g(x)$ is equal to $g(1)$, which can be rewritten as

$$\begin{aligned} g(1) &= (1-a^2)(1-b^2)(1-c^2) \\ &= (1-a)(1-b)(1-c)(1+a)(1+b)(1+c) \\ &= \frac{2(1-a)(1-b)(1-c)}{2} \cdot \frac{-2(-1-a)(-1-b)(-1-c)}{2} \\ &= -\frac{f(1)f(-1)}{4}. \end{aligned}$$

The last expression can be computed simply by plugging in $x = 1$ and $x = -1$ into $f(x)$, and we get $f(1) = 2 + 7 - 3 + 5 = 11$ and $f(-1) = -2 + 7 + 3 + 5 = 13$, so the sum of coefficients of $g(x)$ is $-\frac{11 \cdot 13}{4} = -\frac{143}{4}$.

### 2.4.8 Round 8

22. **[22]** Two congruent circles, $\omega_1$ and $\omega_2$, intersect at points $A$ and $B$. The centers of $\omega_1$ and $\omega_2$ are $O_1$ and $O_2$ respectively. The arc $AB$ of $\omega_1$ that lies inside $\omega_2$ is trisected by points $P$ and $Q$, with the points lying in the order $A, P, Q, B$. Similarly, the arc $AB$ of $\omega_2$ that lies inside $\omega_1$ is trisected by points $R$ and $S$, with the points lying in the order $A, R, S, B$. Given that $PQ = 1$ and $PR = \sqrt{2}$, find the measure of $\angle AO_1B$ in degrees.

**Solution.** The answer is $\boxed{135}$.

Since $P, Q$ trisects arc $AB$, $AP = PQ = QB = 1$, and by symmetry $AR = RS = SB = 1$. Because $PR = \sqrt{2}$, we can conclude that $APQ$ is a right isosceles triangle with right angle at $A$. Note that by symmetry $PQSR$ is a rectangle, which means $\angle RPQ = 90°$ and $\angle APQ = \angle RPQ + \angle APR = 90° + 45° = 135°$. Then, we get $\angle APO_1 = \frac{1}{2}\angle APQ = 67.5°$, $\angle AO_1P = 180° - 2 \cdot 67.5° = 45°$, and finally $\angle AO_1B = 3\angle AO_1P = 135°$.

23. **[22]** How many ordered triples of $(a, b, c)$ of integers between $-10$ and 10 inclusive satisfy the equation $-abc = (a+b)(b+c)(c+a)$?

**Solution.** The answer is $\boxed{433}$.

We can expand and rewrite the equation as

$$a^2b + b^2c + c^2a + a^2c + b^2a + c^2b + 3abc = 0,$$

which factors to

$$(a + b + c)(ab + bc + ca) = 0.$$

Now we need either $a + b + c = 0$ or $ab + bc + ca = 0$. If one of $a, b, c$ is zero (WLOG assume $c = 0$), then either $a = -b$ or $ab = 0$. The former case gives all the cases $(x, -x, 0)$ and its permutations. If $x = 0$, there is 1 possible solution, and otherwise we have $10 \cdot 6 = 60$ solutions. The latter gives all the cases $(x, 0, 0)$ and its permutations. Since $x = 0$ has been accounted before, we only need to consider the case with $x \neq 0$, which gives $20 \cdot 3 = 60$ solutions.

If none of $a, b, c$ are zero, then if $a+b+c = 0$, exactly one of them will have different signs as the other two. WLOG assume that $c$ is the only negative one, so $-c = a + b$. By caseworking on the value of $c$ between $-2$ and $-10$ inclusive, we see that there are $1 + 2 + \cdots + 9 = 45$ solutions for $a$ and $b$. Since there are 3 ways to choose the variable with the different sign and 2 ways to choose the sign, there are $3 \cdot 2 \cdot 45 = 270$ solutions in this case.

If instead $ab+bc+ca = 0$, then by dividing both sides by $abc$, we get $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. Again, exactly one of the three have different signs as the other two, and WLOG assume that $c$ is the only negative one, so $-\frac{1}{c} = \frac{1}{a} + \frac{1}{b}$. (Note that this implies $|c| < a$ and $|c| < b$, so there will be no overlap with the case $-c = a + b$.) The only possible solutions for $(a, b, c)$ are $(2, 2, -1), (4, 4, -2), (3, 6, -2), (6, 3, -2), (6, 6, -3), (8, 8, -4)$, and $(10, 10, -5)$, for 7 solutions in total. Similarly, because there are $3 \cdot 2 = 6$ ways to choose the "special" variable and its sign, there are $6 \cdot 7 = 42$ solutions in this case.

Adding up all cases, we have $1 + 60 + 60 + 270 + 42 = 433$ triples in total.

24. **[22]** For positive integers $n$ and $b$ where $b > 1$, define $s_b(n)$ as the sum of digits in the base-$b$ representation of $n$. A positive integer $p$ is said to *dominate* another positive integer $q$ if for all positive integers $n$, $s_p(n)$ is greater than or equal to $s_q(n)$. Find the number of ordered pairs $(p, q)$ of *distinct* positive integers between 2 and 100 inclusive such that $p$ dominates $q$.

**Solution.** The answer is $\boxed{16}$.

We claim that $p$ dominates $q$ if and only if $p$ is a power of $q$. In other words, there exists a positive integer $k$ such that $p = q^k$.

If $p$ dominates $q$, then by plugging in $n = p = \overline{10}_p$, we have $1 = s_p(p) \geq s_q(p) > 0$, so $s_q(p) = 1$, which happens if and only if the base-$q$ representation of $p$ is $\overline{100\ldots00}$, or $p$ is a power of $q$.

Now suppose that $p = q^k$ for some positive integer $k$. Notice that for any base-$q$ representation of a number $n$, we can break the string into several $k$-digit blocks from right to left (possibly with leading zeroes), convert each $k$-digit block to a single digit in base $p$, and put the digits back together to get the base-$p$ representation of $n$. Consider a $k$-digit block $D = \overline{d_{k-1}d_{k-2}\cdots d_0}$ in base $q$, and its base-$p$ equivalent $D' = \overline{d'}$. Since

$$s_p(D') = D' = D = d_{k-1} \cdot q^{k-1} + d_{k-2} \cdot q^{k-2} + \cdots + d_0 \cdot q^0 \geq d_{k-1} + d_{k-2} + \cdots + d_0 = s_q(D),$$

the sum of digits in each block in base $q$ is no greater than the corresponding digit in base $p$, which means that $s_p(n) \geq s_q(n)$ for all $n$.

It remains to find the number of pairs $(p, q)$ between 2 and 100 such that $p$ is a power of $q$. Since $100 \geq p \geq q^2$, we only need to consider the cases where $q \leq 10$.

If $q = 2$, $p = 4, 8, 16, 32, 64$, for 5 pairs. If $q = 3$, $p = 9, 27, 81$, for 3 pairs. If $q = 4$, $p = 16, 64$, for 2 pairs. If $q = 5, 6, 7, 8, 9, 10$, $p = q^2$ for each case, for 6 pairs.

Therefore we have $5 + 3 + 2 + 6 = 16$ pairs in total.

# Exeter Math Club Competition
# January 27, 2018

## Table of Contents

## Organizing Acknowledgments


- **Tournament Directors** James Lin, Vinjai Vale
- **Tournament Supervisor** Zuming Feng
- **System Administrator and Webmaster** Sanath Govindarajan
- **Problem Committee** Adam Bertelli, Jinpyo Hong, James Lin, Victor Luo
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- **Treasurers** Michael Chen, Sanath Govindarajan
- **Publicity** Dawson Byrd, Angelina Zhang
- **Primary Tournament Sponsor** We would like to thank Jane Street Capital for their generous support of this competition.

Jane Street

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# Contest Day Acknowledgments


- **Tournament Directors** James Lin, Vinjai Vale
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- **Runners** Kristy Chang, Michael Chen, David Anthony Bau, Li Dai
- **Head Graders** Adam Bertelli, Victor Luo, Zhuoqun "Alex" Song
- **Grading System Managers** Michael Chen, Zhuoqun "Alex" Song, Maxwell Wang
- **Graders** Adam Bertelli, Ivan Borsenco, Jinpyo Hong, Eliza Khokhar, Daniel Li, James Lin, Victor Luo, Qi Qi, Chad Qian, Zhuoqun "Alex" Song, Vinjai Vale, Daniel Whatley, Benjamin Wright, Yuan "Yannick" Yao
- **Judges** Zuming Feng, Greg Spanier

# Chapter 1

# EMC$^2$ 2018 Problems

## 1.1 Speed Test

*There are 20 problems, worth 3 points each, to be solved in 25 minutes.*

1. What is $2018 - 3018 + 4018$?

2. What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

3. What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits $a$ and $b$? Express your answer as a decimal.

4. A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where $n$ is a positive integer. How many possible values of $n$ are there?

5. When three integers are added in pairs, and the results are 20, 18, and $x$. If all three integers sum to 31, what is $x$?

6. A cube's volume in cubic inches is numerically equal to the sum of the lengths of all its edges, in inches. Find the surface area of the cube, in square inches.

7. A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

8. SeaBay, an online grocery store, offers two different types of egg cartons. Small egg cartons contain 12 eggs and cost 3 dollars, and large egg cartons contain 18 eggs and cost 4 dollars. What is the maximum number of eggs that Farmer James can buy with 10 dollars?

9. What is the sum of the 3 leftmost digits of $\underbrace{999\ldots9}_{\text{2018 9's}} \times 12$?

10. Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

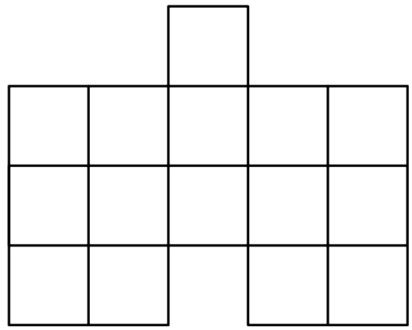
11. Farmer James is ordering takeout from Kristy's Krispy Chicken. The base cost for the dinner is \$14.40, the sales tax is 6.25%, and delivery costs \$3.00 (applied after tax). How much did Farmer James pay, in dollars?

12. Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90°$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

13. Farmer James has 6 cards with the numbers $1 - 6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

14. Farmer James has a $5 \times 5$ grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

15. How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



16. On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

17. At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

18. Let $x, y, z$ be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of $xyz$?

19. Let $p$ be a prime number and $x, y$ be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

20. Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

## 1.2 Accuracy Test

*There are 10 problems, worth 9 points each, to be solved in 45 minutes.*

1. On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

2. A triangle has side lengths $3, 4$, and $6$. A second triangle, similar to the first one, has one side of length 12. Find the sum of all possible lengths of the second triangle's longest side.

3. Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

4. Square $ABCD$ has side length 2. Circle $\omega$ is centered at $A$ with radius 2, and intersects line $AD$ at distinct points $D$ and $E$. Let $X$ be the intersection of segments $EC$ and $AB$, and let $Y$ be the intersection of the minor arc $\overset{\frown}{DB}$ with segment $EC$. Compute the length of $XY$.

5. Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and $4$, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

6. Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to $N$, inclusive. Later, during the live drawing, another group of six numbers from 1 to $N$ will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of $N$?

7. An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height $h$. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for $h$?

8. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer $n$ such that the product $a_1 a_2 a_3 \cdots a_n$ is greater than 1?

9. In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment $BC$ at $E$, and $AE$ and $BD$ intersect at $P$. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find $BC$.

10. Farmer James places the numbers $1, 2, \ldots, 9$ in a $3 \times 3$ grid such that each number appears exactly once in the grid. Let $x_i$ be the product of the numbers in row $i$, and $y_i$ be the product of the numbers in column $i$. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

## 1.3 Team Test

*There are 15 problems, worth 20 points each, to be solved in 45 minutes.*

1. Farmer James goes to Kristy's Krispy Chicken to order a crispy chicken sandwich. He can choose from 3 types of buns, 2 types of sauces, 4 types of vegetables, and 4 types of cheese. He can only choose one type of bun and cheese, but can choose any nonzero number of sauces, and the same with vegetables. How many different chicken sandwiches can Farmer James order?

2. A line with slope 2 and a line with slope 3 intersect at the point $(m, n)$, where $m, n > 0$. These lines intersect the $x$ axis at points $A$ and $B$, and they intersect the $y$ axis at points $C$ and $D$. If $AB = CD$, find $\frac{m}{n}$.

3. A multi-set of 11 positive integers has a median of 10, a unique mode of 11, and a mean of 12. What is the largest possible number that can be in this multi-set? (A multi-set is a set that allows repeated elements.)

4. Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

5. $ABCD$ is a square with side length 60. Point $E$ is on $AD$ and $F$ is on $CD$ such that $\angle BEF = 90°$. Find the minimum possible length of $CF$.

6. Farmer James makes a *trianglomino* by gluing together 5 equilateral triangles of side length 1, with adjacent triangles sharing an entire edge. Two trianglominoes are considered the same if they can be matched using only translations and rotations (but not reflections). How many distinct trianglominoes can Farmer James make?

7. Two real numbers $x$ and $y$ satisfy

$$\begin{cases} x^2 - y^2 = 2y - 2x \text{ , and} \\ x + 6 = y^2 + 2y. \end{cases}$$

   What is the sum of all possible values of $y$?

8. Let $N$ be a positive multiple of 840. When $N$ is written in base 6, it is of the form $\overline{abcdef}_6$ where $a, b, c, d, e, f$ are distinct base 6 digits. What is the smallest possible value of $N$, when written in base 6?

9. For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

   (a) If $n$ is divisible by 3, $f(n)$ is not divisible by 3,

   (b) If $n$ is not divisible by 3, $f(n)$ is divisible by 3, and

   (c) $f(f(n)) = n$ holds for exactly 8 distinct values of $n$ in $S$.

10. Regular pentagon $JAMES$ has area 1. Let $O$ lie on line $EM$ and $N$ lie on line $MA$ so that $E, M, O$ and $M, A, N$ lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of $NOM$.

11. Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up*.

12. Find the product of all integer values of $a$ such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

13. Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point $X$ is the intersection of the diagonals $AC$ and $BD$. There exist a positive real number $k$ and a point $P$ inside $ABCD$ which satisfy

$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$

where $[XYZ]$ denotes the area of triangle $XYZ$. If $PX \parallel AB$, find the value of $k$.

14. How many positive integers $n < 1000$ are there such that in base 10, every digit in $3n$ (that isn't a leading zero) is greater than the corresponding place value digit (possibly a leading zero) in $n$?

    For example, $n = 56$, $3n = 168$ satisfies this property as $1 > 0, 6 > 5$, and $8 > 6$. On the other hand, $n = 506$, $3n = 1518$ does not work because of the hundreds place.

15. Find the greatest integer that is smaller than

$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. How many distinct ways are there to *scramble* the letters in $EXETER$?

2. Given that $\dfrac{x-y}{x-z}=3$, find $\dfrac{x-z}{y-z}$.

3. When written in base 10,
$$9^9 = \overline{ABC420DEF}.$$
Find the remainder when $A+B+C+D+E+F$ is divided by 9.

### 1.4.2 Round 2

4. How many positive integers, when expressed in base 7, have exactly 3 digits, but don't contain the digit 3?

5. Pentagon $JAMES$ is such that its internal angles satisfy $\angle J = \angle A = \angle M = 90^\circ$ and $\angle E = \angle S$. If $JA = AM = 4$ and $ME = 2$, what is the area of $JAMES$?

6. Let $x$ be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of $x$?

### 1.4.3 Round 3

7. Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms $D$ and the average number of students per dorm $A$. Hen Hao rounds both $D$ and $A$ down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both $D$ and $A$ up to the nearest multiple of 10 and multiplies them, getting an estimate of $N$ students. What is the maximum possible value of $N$?

8. Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping $k$ eggs into the soup in the $k$th hour. Find the smallest positive integer $n$ so that after exactly $n$ hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

9. Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How

many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

### 1.4.4 Round 4

10. Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

11. Find the value of $\dfrac{4 \cdot 7}{\sqrt{4+\sqrt{7}}+\sqrt{4-\sqrt{7}}}$.

12. Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

### 1.4.5 Round 5

13. Find all ordered pairs of real numbers $(x, y)$ satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

14. An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

15. How many ways are there for Farmer James to color each square of a $2 \times 6$ grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

### 1.4.6 Round 6

16. In a triangle $ABC$, $\angle A = 45$, and let $D$ be the foot of the perpendicular from $A$ to segment $BC$. $BD = 2$ and $DC = 4$. Let $E$ be the intersection of the line $AD$ and the perpendicular line from $B$ to line $AC$. Find the length of $AE$.

17. Find the largest positive integer $n$ such that there exists a unique positive integer $m$ satisfying

$$\frac{1}{10} \le \frac{m}{n} \le \frac{1}{9}.$$

18. How many ordered pairs $(A, B)$ of positive integers are there such that $A + B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

### 1.4.7 Round 7

19. Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

20. $P(x)$ is a monic polynomial (a polynomial with leading coefficient 1) of degree 4, such that $P(2^n + 1) = 8^n + 1$ when $n = 1, 2, 3, 4$. Find the value of $P(1)$.

21. PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

### 1.4.8 Round 8

22. How many ways are there to *scramble* the letters in $EGGSEATER$ such that no two consecutive letters are the same?

23. Let $JAMES$ be a regular pentagon. Let $X$ be on segment $JA$ such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point $P$ on segment $AE$ such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

24. Find the minimum value of the function

$$f(x) = \left|x - \frac{1}{x}\right| + \left|x - \frac{2}{x}\right| + \left|x - \frac{3}{x}\right| + \cdots + \left|x - \frac{9}{x}\right| + \left|x - \frac{10}{x}\right|$$

over all nonzero real numbers $x$.

## 1.5 Practice Sets

*We have arranged 65 of this year's 69 EMCC problems into 11 sets, in approximately increasing order of difficulty. This sorting is based on solve rate during the contest, among other measures. The first two sets consist of 10 problems each, and the other nine consist of 5 problems each. The goal of these Practice Sets is to provide a repository that coaches can use for training at various skill levels.*

### 1.5.1 Set 1

1. (Speed #1) What is $2018 - 3018 + 4018$?

2. (Speed #2) What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

3. (Accuracy #1) On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

4. (Speed 7) A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

5. (Speed #4) A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where $n$ is a positive integer. How many possible values of $n$ are there?

6. (Guts #1) How many distinct ways are there to *scramble* the letters in $EXETER$?

7. (Speed #3) What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits $a$ and $b$? Express your answer as a decimal.

8. (Speed #11) Farmer James is ordering takeout from Kristy's Krispy Chicken. The base cost for the dinner is \$14.40, the sales tax is 6.25%, and delivery costs \$3.00 (applied after tax). How much did Farmer James pay, in dollars?

9. (Speed #8) SeaBay, an online grocery store, offers two different types of egg cartons. Small egg cartons contain 12 eggs and cost 3 dollars, and large egg cartons contain 18 eggs and cost 4 dollars. What is the maximum number of eggs that Farmer James can buy with 10 dollars?

10. (Accuracy #2) A triangle has side lengths $3, 4$, and 6. A second triangle, similar to the first one, has one side of length 12. Find the sum of all possible lengths of the second triangle's longest side.

### 1.5.2 Set 2

11. (Team #1) Farmer James goes to Kristy's Krispy Chicken to order a crispy chicken sandwich. He can choose from 3 types of buns, 2 types of sauces, 4 types of vegetables, and 4 types of cheese. He can only choose one type of bun and cheese, but can choose any nonzero number of sauces, and the same with vegetables. How many different chicken sandwiches can Farmer James order?

12. (Guts #2) Given that $\frac{x-y}{x-z} = 3$, find $\frac{x-z}{y-z}$.

13. (Guts #3) When written in base 10,
$$9^9 = \overline{ABC420DEF}.$$
Find the remainder when $A + B + C + D + E + F$ is divided by 9.

14. (Speed #5) When three integers are added in pairs, and the results are 20, 18, and $x$. If all three integers sum to 31, what is $x$?

15. (Team #3) A multi-set of 11 positive integers has a median of 10, a unique mode of 11, and a mean of 12. What is the largest possible number that can be in this multi-set? (A multi-set is a set that allows repeated elements.)

16. (Guts #5) Pentagon $JAMES$ is such that its internal angles satisfy $\angle J = \angle A = \angle M = 90°$ and $\angle E = \angle S$. If $JA = AM = 4$ and $ME = 2$, what is the area of $JAMES$?

17. (Team #6) Farmer James makes a *trianglomino* by gluing together 5 equilateral triangles of side length 1, with adjacent triangles sharing an entire edge. Two trianglominoes are considered the same if they can be matched using only translations and rotations (but not reflections). How many distinct trianglominoes can Farmer James make?

18. (Speed #6) A cube's volume in cubic inches is numerically equal to the sum of the lengths of all its edges, in inches. Find the surface area of the cube, in square inches.

19. (Guts #4) How many positive integers, when expressed in base 7, have exactly 3 digits, but don't contain the digit 3?

20. (Speed #14) Farmer James has a $5 \times 5$ grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

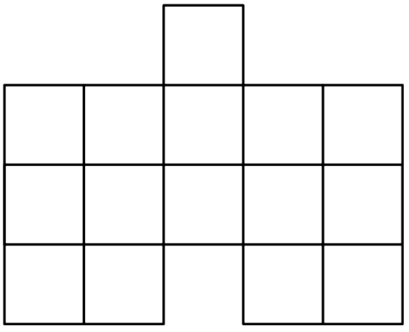
### 1.5.3 Set 3

21. (Team #4) Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

22. (Speed #9) What is the sum of the 3 leftmost digits of $\underbrace{999\ldots9}_{\text{2018 9's}} \times 12$?

23. (Speed #12) Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90°$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

24. (Speed #15) How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



25. (Guts #6) Let $x$ be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of $x$?

### 1.5.4 Set 4

26. (Speed #13) Farmer James has 6 cards with the numbers $1-6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

27. (Accuracy #3) Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

28. (Team #5) $ABCD$ is a square with side length 60. Point $E$ is on $AD$ and $F$ is on $CD$ such that $\angle BEF = 90°$. Find the minimum possible length of $CF$.

29. (Speed #17) At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

30. (Speed #10) Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

### 1.5.5 Set 5

31. (Guts #7) Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms $D$ and the average number of students per dorm $A$. Hen Hao rounds both $D$ and $A$ down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both $D$ and $A$ up to the nearest multiple of 10 and multiplies them, getting an estimate of $N$ students. What is the maximum possible value of $N$?

32. (Team #7) Two real numbers $x$ and $y$ satisfy

$$\begin{cases} x^2 - y^2 = 2y - 2x \text{ , and} \\ x + 6 = y^2 + 2y. \end{cases}$$

What is the sum of all possible values of $y$?

33. (Guts #12) Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

34. (Team #2) A line with slope 2 and a line with slope 3 intersect at the point $(m, n)$, where $m, n > 0$. These lines intersect the $x$ axis at points $A$ and $B$, and they intersect the $y$ axis at points $C$ and $D$. If $AB = CD$, find $\frac{m}{n}$.

35. (Guts #15) How many ways are there for Farmer James to color each square of a $2 \times 6$ grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

### 1.5.6 Set 6

36. (Guts #8) Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping $k$ eggs into the soup in the $k$th hour. Find the smallest positive integer $n$ so that after exactly $n$ hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

37. (Guts #11) Find the value of $\dfrac{4 \cdot 7}{\sqrt{4+\sqrt{7}}+\sqrt{4-\sqrt{7}}}$.

38. (Accuracy #4) Square $ABCD$ has side length 2. Circle $\omega$ is centered at $A$ with radius 2, and intersects line $AD$ at distinct points $D$ and $E$. Let $X$ be the intersection of segments $EC$ and $AB$, and let $Y$ be the intersection of the minor arc $\overset{\frown}{DB}$ with segment $EC$. Compute the length of $XY$.

39. (Accuracy #5) Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and 4, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

40. (Speed #16) On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

### 1.5.7 Set 7

41. (Guts #9) Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

42. (Accuracy #6) Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to $N$, inclusive. Later, during the live drawing, another group of six numbers from 1 to $N$ will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of $N$?

43. (Guts #13) Find all ordered pairs of real numbers $(x, y)$ satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

44. (Speed #20) Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

45. (Guts #16) In a triangle $ABC$, $\angle A = 45$, and let $D$ be the foot of the perpendicular from $A$ to segment $BC$. $BD = 2$ and $DC = 4$. Let $E$ be the intersection of the line $AD$ and the perpendicular line from $B$ to line $AC$. Find the length of $AE$.

### 1.5.8 Set 8

46. (Speed #18) Let $x, y, z$ be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of $xyz$?

47. (Speed #19) Let $p$ be a prime number and $x, y$ be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

48. (Guts #14) An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

49. (Guts #10) Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

50. (Accuracy #9) In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment $BC$ at $E$, and $AE$ and $BD$ intersect at $P$. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find $BC$.

### 1.5.9 Set 9

51. (Accuracy #8) Let $a_1, a_2, a_3, \ldots$ be an infinite geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer $n$ such that the product $a_1 a_2 a_3 \cdots a_n$ is greater than 1?

52. (Team #11) Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up*.

53. (Team #8) Let $N$ be a positive multiple of 840. When $N$ is written in base 6, it is of the form $\overline{abcdef}_6$ where $a, b, c, d, e, f$ are distinct base 6 digits. What is the smallest possible value of $N$, when written in base 6?

54. (Team #9) For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

    (a) If $n$ is divisible by 3, $f(n)$ is not divisible by 3,

    (b) If $n$ is not divisible by 3, $f(n)$ is divisible by 3, and

    (c) $f(f(n)) = n$ holds for exactly 8 distinct values of $n$ in $S$.

55. (Team #13) Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point $X$ is the intersection of the diagonals $AC$ and $BD$. There exist a positive real number $k$ and a point $P$ inside $ABCD$ which satisfy

$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$

where $[XYZ]$ denotes the area of triangle $XYZ$. If $PX \parallel AB$, find the value of $k$.

### 1.5.10 Set 10

56. (Team #10) Regular pentagon $JAMES$ has area 1. Let $O$ lie on line $EM$ and $N$ lie on line $MA$ so that $E, M, O$ and $M, A, N$ lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of $NOM$.

57. (Team #12) Find the product of all integer values of $a$ such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

58. (Guts #17) Find the largest positive integer $n$ such that there exists a unique positive integer $m$ satisfying
$$\frac{1}{10} \leq \frac{m}{n} \leq \frac{1}{9}.$$

59. (Accuracy #7) An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height $h$. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for $h$?

60. (Guts #18) How many ordered pairs $(A, B)$ of positive integers are there such that $A + B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

### 1.5.11 Set 11

61. (Guts #19) Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

62. (Guts #21) PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

63. (Team #15) Find the greatest integer that is smaller than
$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

64. (Accuracy #10) Farmer James places the numbers $1, 2, \ldots, 9$ in a $3 \times 3$ grid such that each number appears exactly once in the grid. Let $x_i$ be the product of the numbers in row $i$, and $y_i$ be the product of the numbers in column $i$. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

65. (Guts #23) Let $JAMES$ be a regular pentagon. Let $X$ be on segment $JA$ such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point $P$ on segment $AE$ such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

# Chapter 2

# EMC$^2$ 2018 Solutions

## 2.1 Speed Test Solutions

1. What is $2018 - 3018 + 4018$?

   **Solution.** The answer is $\boxed{3018}$.

   The expression is $2018 - 3018 + 4018 = 2018 + (4018 - 3018) = 2018 + 1000 = 3018$.

2. What is the smallest integer greater than 100 that is a multiple of both 6 and 8?

   **Solution.** The answer is $\boxed{120}$.

   The least common multiple of 6 and 8 is 24, so this number must be a multiple of 24 above 100, the smallest of which is 120.

3. What positive real number can be expressed as both $\frac{b}{a}$ and $a.b$ in base 10 for nonzero digits $a$ and $b$? Express your answer as a decimal.

   **Solution.** The answer is $\boxed{2.5}$.

   A fraction that has exactly one digit after the decimal point must have a denominator dividing 10, but can't be 1, so $a$ is either 2 or 5. If $a$ were 5, then $\frac{b}{5} = 5.b > 5$, so $b > 5 \cdot 5$, which cannot be true as $b$ is between 1 and 9. Thus $a = 2$, and any non-integer fraction with denominator 2 must end in .5, so our number is 2.5 or $\frac{5}{2}$.

4. A non-degenerate triangle has sides of lengths 1, 2, and $\sqrt{n}$, where $n$ is a positive integer. How many possible values of $n$ are there?

   **Solution.** The answer is $\boxed{7}$.

   By the triangle inequality, we have $2 - 1 < \sqrt{n} < 2 + 1$, so $1^2 < n < 3^2$, giving 7 possible integer values for $n$.

5. When three integers are added in pairs, and the results are 20, 18, and $x$. If all three integers sum to 31, what is $x$?

   **Solution.** The answer is $\boxed{24}$.

   Note that each of the 3 numbers is contained in exactly two of the pairwise sums, meaning that the sum of all 3 pairs will count each element twice. Since the 3 numbers sum to 31, we get $2 \cdot 31 = 20 + 18 + x$, therefore $x = 24$.

6. A cube's volume in cubic inches is numerically equal to the sum of the lengths of all its edges, in inches. Find the surface area of the cube, in square inches.

   **Solution.** The answer is $\boxed{72}$.

   If the side length of the cube is $s$, this tells us that $s^3 = 12s$, or $s^2 = 12$, since a cube has 12 edges. Then the surface area is just $6s^2 = 6 \cdot 12 = 72$.

7. A 12 hour digital clock currently displays $9:30$. Ignoring the colon, how many times in the next hour will the clock display a palindrome (a number that reads the same forwards and backwards)?

**Solution.** The answer is $\boxed{4}$.

The only possible palindromes between 9:30 and 10:30 are 9:39, 9:49, 9:59, and 10:01, giving 4 palindromes total.

8. SeaBay, an online grocery store, offers two different types of egg cartons. Small egg cartons contain 12 eggs and cost 3 dollars, and large egg cartons contain 18 eggs and cost 4 dollars. What is the maximum number of eggs that Farmer James can buy with 10 dollars?

   **Solution.** The answer is $\boxed{42}$.

   Since you can buy up to 2 large cartons, there are 3 cases. If you buy 2 large cartons, then you will have two dollars left, so you can buy 0 small cartons, and will have $2 \cdot 18 = 36$ eggs. If you buy 1 large carton, then you will have six dollars left, so you can buy two small cartons, and will have $18 + 2 \cdot 12 = 42$ eggs. If you don't buy any large cartons, then you can buy 3 small cartons, and will have $3 \cdot 12 = 36$ eggs. Therefore, buying 1 large and 2 small cartons gives you 42 eggs, the greatest number possible.

9. What is the sum of the 3 leftmost digits of $\underbrace{999\ldots9}_{2018\text{ 9's}} \times 12$?

   **Solution.** The answer is $\boxed{11}$.

   The given number is slightly less than $\overline{1000\ldots}$ (with many trailing 0's), so 12 times the number will be slightly less than $\overline{12000\ldots}$, so it will start with $\overline{11999\ldots}$, making the sum of the first three digits $1 + 1 + 9 = 11$.

10. Farmer James trisects the edges of a regular tetrahedron. Then, for each of the four vertices, he slices through the plane containing the three trisection points nearest to the vertex. Thus, Farmer James cuts off four smaller tetrahedra, which he throws away. How many edges does the remaining shape have?

    **Solution.** The answer is $\boxed{18}$.

    Note that each time Farmer James cuts off a small tetrahedron, none of the original edges are lost, because the middle third of the original edge is never cut off. In addition, he creates 3 new edges, the sides of the newly-created triangular face. Therefore, Farmer James gains 3 edges for each of the 4 times he cuts off a small tetrahedron, gaining 12 edges in total. Since he started with 6 edges, the resulting shape has $6 + 12 = 18$ edges in total.

11. Farmer James is ordering takeout from Kristy's Krispy Chicken. The base cost for the dinner is \$14.40, the sales tax is 6.25%, and delivery costs \$3.00 (applied after tax). How much did Farmer James pay, in dollars?

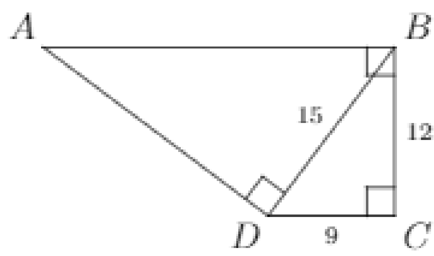
    **Solution.** The answer is $\boxed{18.30}$.

    A 6.25% sales tax is equal to an extra $\frac{1}{16}$ of the base cost, therefore the total cost is $\$14.40 \cdot \left(1 + \frac{1}{16}\right) + \$3.00 = \$15.30 + \$3.00 = \$18.30$.

12. Quadrilateral $ABCD$ has $\angle ABC = \angle BCD = \angle BDA = 90^\circ$. Given that $BC = 12$ and $CD = 9$, what is the area of $ABCD$?

**Solution.** The answer is $\boxed{204}$.

By the Pythagorean Theorem, $BD = \sqrt{9^2 + 12^2} = 15$. Also, $\angle DBA = 90^\circ - \angle CBD = \angle CDB$, so $\triangle DBA \sim \triangle CDB$ in a ratio of 5 to 3. Thus the area of $\triangle DBA$ is $\frac{25}{9}$ times the area of $\triangle CDB$, which itself has area $\frac{1}{2} \cdot 9 \cdot 12 = 54$, so the total area of $ABCD$ is $54\left(1 + \frac{25}{9}\right) = 204$.




13. Farmer James has 6 cards with the numbers $1 - 6$ written on them. He discards a card and makes a 5 digit number from the rest. In how many ways can he do this so that the resulting number is divisible by 6?

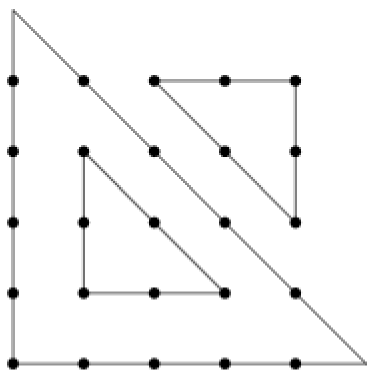
**Solution.** The answer is $\boxed{120}$.

For a number to be divisible by 6, its digits must sum to a multiple of 3, and since the numbers 1 through 6 already sum to a multiple of 3, Farmer James must discard either 3 or 6. Since the number is even, if he discards 3, there are 3 choices for the last digit: $2, 4$, and 6. The other 4 digits can then be ordered in the front in $4! = 24$ ways, giving $3 \cdot 24 = 72$ possible numbers. Similarly, if he discards 6, there are 2 choices for the last digit, 2 and 4, so there are $2 \cdot 4! = 48$ total possibilities. Therefore, Farmer James can create $72 + 48 = 120$ different numbers.

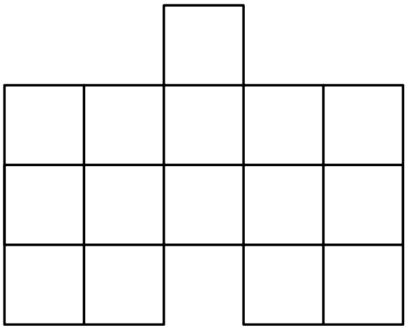
14. Farmer James has a $5 \times 5$ grid of points. What is the smallest number of triangles that he may draw such that each of these 25 points lies on the boundary of at least one triangle?

**Solution.** The answer is $\boxed{3}$.

If it were possible to cover all 25 points with 2 triangles, one of them must have at least 13 points on its boundary. It is not hard to see that the maximum number of points that can lie on any triangle is in fact 13, but this can only be achieved one way (up to rotation), and the remaining 12 points clearly cannot be covered by one triangle. They can, however, be covered by two triangles, giving a minimal covering of 3 triangles, shown below.

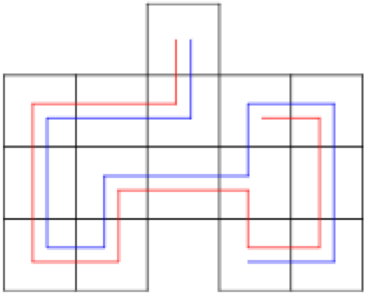


15. How many ways are there to label these 15 squares from 1 to 15 such that squares 1 and 2 are adjacent, squares 2 and 3 are adjacent, and so on?



**Solution.** The answer is $\boxed{8}$.

The top square is only adjacent to one other square, so the number that goes there should only be adjacent to one other number, meaning it is either 1 or 15. From this point, we can then draw one of two paths, times two for horizontal reflection, giving a total of $2 \cdot 2 \cdot 2 = 8$ different labellings.

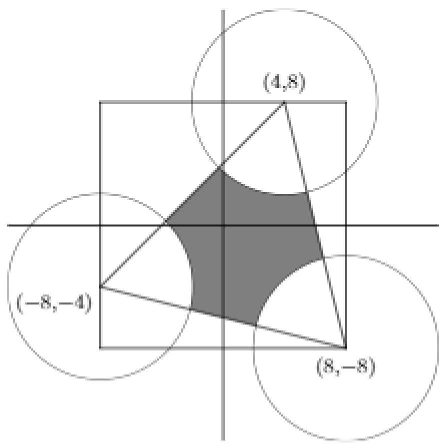


16. On Farmer James's farm, there are three henhouses located at $(4, 8), (-8, -4), (8, -8)$. Farmer James wants to place a feeding station within the triangle formed by these three henhouses. However, if the

feeding station is too close to any one henhouse, the hens in the other henhouses will complain, so Farmer James decides the feeding station cannot be within 6 units of any of the henhouses. What is the area of the region where he could possibly place the feeding station?

**Solution.** The answer is $\boxed{120 - 18\pi}$.

By not allowing a feeding station within 6 units of any vertex of this triangle, we are removing three sectors of radius 6, one from each corner. Since the sum of the angles in a triangle is 180 degrees, these sectors added together will equal the area of a semicircle with radius 6, or $18\pi$. We can then find the area of the triangle by bounding it in a larger square with side length 16, as shown, giving the area as $16 \cdot 16 - \frac{12 \cdot 12}{2} - \frac{16 \cdot 4}{2} - \frac{16 \cdot 4}{2} = 120$, thus the remaining area where the feeding station can be placed is $120 - 18\pi$.




17. At Eggs-Eater Academy, every student attends at least one of 3 clubs. 8 students attend frying club, 12 students attend scrambling club, and 20 students attend poaching club. Additionally, 10 students attend at least two clubs, and 3 students attend all three clubs. How many students are there in total at Eggs-Eater Academy?

    **Solution.** The answer is $\boxed{27}$.

    If we simply sum up the memberships of each club, we get a total of $8 + 12 + 20 = 40$ people. However, doing so counts people in exactly 2 clubs twice each, and people in all 3 clubs three times each. To fix this, we can first subtract the 10 people who go to at least 2 clubs, leaving $40 - 10 = 30$ people, and ensuring that the people who attend exactly 2 clubs are counted properly. However, those who attend all 3 clubs will still have been counted $3 - 1 = 2$ times after this, so we must additionally subtract the number of people who go to all three clubs, giving the correct value as $30 - 3 = 27$ people.

18. Let $x, y, z$ be real numbers such that $8^x = 9$, $27^y = 25$, and $125^z = 128$. What is the value of $xyz$?

    **Solution.** The answer is $\boxed{28/27}$.

    From power rules, we know that $9^{\frac{3}{2}} = 27, 25^{\frac{3}{2}} = 125$, and $128^{\frac{3}{7}} = 8$, therefore $8^{\frac{3}{2}x} = 27$, $27^{\frac{3}{2}y} = 125$, and $125^{\frac{3}{7}z} = 8$, or $\left(\left(8^{\frac{3}{2}x}\right)^{\frac{3}{2}y}\right)^{\frac{3}{7}z} = 8$, thus $\frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{7} \cdot xyz = 1$, so $xyz = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{7}{3} = \frac{28}{27}$.

19. Let $p$ be a prime number and $x, y$ be positive integers. Given that $9xy = p(p + 3x + 6y)$, find the maximum possible value of $p^2 + x^2 + y^2$.

    **Solution.** The answer is $\boxed{38}$.

    Since $9xy$ is divisible by 9, $p(p + 3x + 6y)$ must be as well. However, $p$ and $p + 3x + 6y$ differ by a multiple of 3, so in order for their product to be divisible by 3 (and furthermore by 9), both numbers must be multiples of 3, meaning $p = 3$ (as no other prime is divisible by 3). From here, we have $9xy = 9 + 9x + 18y$, or $xy - x - 2y - 1 = 0$, which we can rewrite as $(x - 2)(y - 1) = 3$. Over all 4 possible integer solutions for $x$ and $y$, the one having the largest value for $x^2 + y^2$ is $(x, y) = (5, 2)$, giving $p^2 + x^2 + y^2 = 3^2 + 5^2 + 2^2 = 38$.

20. Farmer James's hens like to drop eggs. Hen Hao drops 6 eggs uniformly at random in a unit square. Farmer James then draws the smallest possible rectangle (by area), with sides parallel to the sides of the square, that contain all 6 eggs. What is the probability that at least one of the 6 eggs is a vertex of this rectangle?

    **Solution.** The answer is $\boxed{3/5}$.

    If we look at the eggs from left to right after they have been dropped, we can assign each egg a ranking from highest to lowest in terms of their $y$ value (the probability of any two eggs having the exact same $x$ or $y$ value is 0, so we consider when this does not happen). Our bounding rectangle, in order to have the least area, will touch the highest, lowest, furthest left, and furthest right of the 6 eggs. Therefore, in order for an egg to end up in a corner, it must have two of these properties at the same time; in other words, one of the leftmost/rightmost eggs must also be the lowest/highest egg.
    We can calculate this probability by instead finding the odds that this does not happen: there is a $\frac{4}{6}$ chance that the leftmost egg is neither the highest nor lowest egg, and then a $\frac{3}{5}$ chance that the rightmost is neither of these as well: after picking the height of the leftmost egg, there are only 5 remaining ranks for height, and only 3 of these are neither the highest nor the lowest. Therefore the probability that no egg ends up in the corner is $\frac{4}{6} \cdot \frac{3}{5} = \frac{2}{5}$, meaning the complementary probability that at least one egg is in a corner is $1 - \frac{2}{5} = \frac{3}{5}$.

## 2.2 Accuracy Test Solutions

1. On SeaBay, green herring costs \$2.50 per pound, blue herring costs \$4.00 per pound, and red herring costs \$5.85 per pound. What must Farmer James pay for 12 pounds of green herring and 7 pounds of blue herring, in dollars?

   **Solution.** The answer is $\boxed{58}$.

   The total cost is \$2.50 times 12 of pounds of green herring, added to \$4.00 times 7 pounds of blue herring, or $12 \cdot \$2.50 + 7 \cdot \$4.00 = \$30 + \$28 = \$58$

2. A triangle has side lengths $3, 4$, and $6$. A second triangle, similar to the first one, has one side of length 12. Find the sum of all possible lengths of the second triangle's longest side.

   **Solution.** The answer is $\boxed{54}$.

   The second triangle is either $2, 3$, or $4$ times as large as the original, depending on whether the side of length 12 corresponds to the side of length $6, 4$, or $3$ in the original, respectively. Therefore the sum of all possible longest side lengths will just be the original longest side, 6, scaled by all three of these possibilities, giving $6 \cdot (2 + 3 + 4) = 54$.

3. Hen Hao runs two laps around a track. Her overall average speed for the two laps was 20% slower than her average speed for just the first lap. What is the ratio of Hen Hao's average speed in the first lap to her average speed in the second lap?

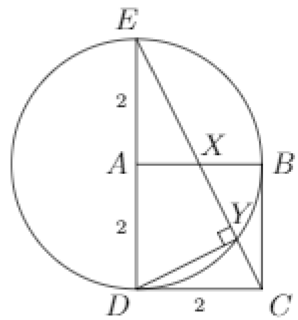
   **Solution.** The answer is $\boxed{3/2}$.

   Since we are not given any info on the speed or distance PEAcock travels, we can pick our own values, so suppose PEAcock's average speed for the first lap was 15mph, and the lap around the farm is 60 miles long. We know the overall average speed is 20% slower than 15mph, or 12mph, thus the total time taken by PEAcock would be 2 laps times $\frac{60}{12}$ hours per lap, or 10 hours. On the first lap, he travels 60 miles at 15mph, taking $\frac{60}{15} = 4$ hours, so he must have completed the second lap in $10 - 4 = 6$ hours, making his speed $\frac{60}{6} = 10$mph. The ratio of the two speeds is therefore $\frac{15}{10} = \frac{3}{2}$.

4. Square $ABCD$ has side length 2. Circle $\omega$ is centered at $A$ with radius 2, and intersects line $AD$ at distinct points $D$ and $E$. Let $X$ be the intersection of segments $EC$ and $AB$, and let $Y$ be the intersection of the minor arc $\overset{\frown}{DB}$ with segment $EC$. Compute the length of $XY$.

   **Solution.** The answer is $\boxed{3\sqrt{5}/5}$.

   First, note that $EC$, the segment containing $XY$, has length $\sqrt{2^2 + 4^2} = 2\sqrt{5}$. We can find that $\angle EYD = 90°$, since it inscribes diameter $ED$ of the circle. Thus $\angle DYC = 90°$ as well, so $\triangle DYC \sim \triangle EDC$, giving $\frac{CY}{DC} = \frac{DC}{EC} = \frac{2}{2\sqrt{5}}$, or $CY = \frac{\sqrt{5}}{5} \cdot DC = \frac{2\sqrt{5}}{5}$. We can also easily find $X$ to be the midpoint of $EC$, as $\triangle EAX \cong \triangle CBX$, so $CX = \sqrt{5}$. Thus $XY = CX - CY = \sqrt{5} - \frac{2\sqrt{5}}{5} = \frac{3\sqrt{5}}{5}$.




5. Hen Hao rolls 4 tetrahedral dice with faces labeled $1, 2, 3$, and $4$, and adds up the numbers on the faces facing down. Find the probability that she ends up with a sum that is a perfect square.

   **Solution.** The answer is $\boxed{21/128}$.

   The possible perfect squares that could be the sum are $4, 9$, and $16$. $4$ and $16$ can both only occur in one way: $1, 1, 1, 1$, and $4, 4, 4, 4$. Next, there are 4 possible (unordered) combinations of numbers that can sum to 9: $\{1, 1, 3, 4\}, \{1, 2, 2, 4\}, \{1, 2, 3, 3\}$, and $\{2, 2, 2, 3\}$. The first 3 can be reordered in 12 ways, while the last can only be reordered in 4 ways, giving 40 total ordered combinations. Finally, after adding the one combination each from 4 and 16, and dividing by the total number of combinations (which is $4^4$), we end up with a probability of $\frac{42}{4^4} = \frac{21}{128}$.

6. Let $N \geq 11$ be a positive integer. In the Eggs-Eater Lottery, Farmer James needs to choose an (unordered) group of six different integers from 1 to $N$, inclusive. Later, during the live drawing, another group of six numbers from 1 to $N$ will be randomly chosen as *winning* numbers. Farmer James notices that the probability he will choose exactly zero *winning* numbers is the same as the probability that he will choose exactly one *winning* number. What must be the value of $N$?

   **Solution.** The answer is $\boxed{47}$.

   First, there are $\binom{N}{6}$ combinations of 6 numbers that Farmer James chooses from, each with equal probability. Of these, there are $\binom{N-6}{6}$ ways to have no *winning* numbers, and $\binom{6}{1}\binom{N-6}{5}$ ways to have exactly 1 *winning* number. For the two probabilities to be equal, these two values must be equal. This can be expressed as $\frac{(N-6)!}{6!(N-12)!} = \frac{6(N-6)!}{5!(N-11)!}$, or $\frac{5!(N-11)!}{6!(N-12)!} = \frac{N-11}{6} = 6$, so $N = 47$.

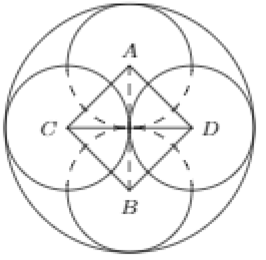
   *(Note that during the contest, the problem mistakenly stated $N \geq 6$, allowing the 5 degenerate solutions $6 \leq N \leq 10$, where both probabilities were equal to 0)*

7. An *egg plant* is a hollow cylinder of negligible thickness with radius 2 and height $h$. Inside the egg plant, there is enough space for four solid spherical eggs of radius 1. What is the minimum possible value for $h$?

   **Solution.** The answer is $\boxed{2 + \sqrt{2}}$.

   We can first place two eggs against the bottom of the cylindrical egg plant, tangent to each other, such that their point of contact lies along the central vertical axis of the cylinder. Next, we place another pair of eggs on top, such that all four eggs are pairwise tangent to each other. Let $A$ and

$B$ be the centers of the lower spheres, and $C$ and $D$ be the centers of the upper spheres. Since each pair of spheres is tangent, $ABCD$ is a regular tetrahedron of side length 2. In addition, all four points are 1 unit away from the central vertical axis of the cylinder, and the top pair and bottom pair are oriented perpendicularly, so the planar distance (if viewed from above) between any two centers from opposite pairs (ex: $A$ and $C$) would be $\sqrt{2}$. Therefore, the vertical distance between the two pairs is $\sqrt{2^2 - \sqrt{2}^2} = \sqrt{2}$. Since the bottom of the cylinder is 1 unit below the lower pair $AB$, and the top is 1 unit above the upper pair $CD$, the total height of the egg plant is $2 + \sqrt{2}$.




(birds eye view of an egg plant)

8. Let $a_1, a_2, a_3, \ldots$ be a geometric sequence of positive reals such that $a_1 < 1$ and $(a_{20})^{20} = (a_{18})^{18}$. What is the smallest positive integer $n$ such that the product $a_1 a_2 a_3 \cdots a_n$ is greater than 1?

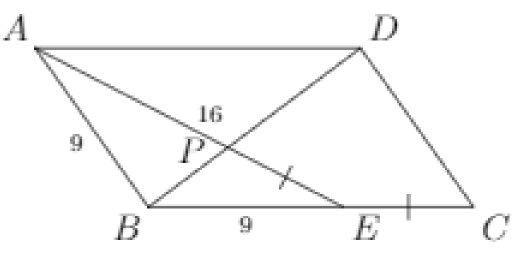
   **Solution.** The answer is $\boxed{76}$.

   If the constant ratio between consecutive terms is $r$, then we can write $a_n = a_1 \cdot r^{n-1}$, so the condition given is $(a_1)^{20} \cdot r^{19 \cdot 20} = (a_1)^{18} \cdot r^{17 \cdot 18}$, or $(a_1)^2 = r^{17 \cdot 18 - 19 \cdot 20} = r^{-74}$, thus $a_1 = r^{-37}$. Therefore, the product $a_1 a_2 \cdots a_{75}$ is $r^{(-37)+(-36)+\cdots+36+37} = 1$, and since $a_1 < 1 \implies r > 1$, we get that $a_1 a_2 \cdots a_{76} = (a_1 a_2 \cdots a_{75}) \cdot a_{76} = 1 \cdot r^{38}$ is the first product of consecutive terms that is greater than 1.

9. In parallelogram $ABCD$, the angle bisector of $\angle DAB$ meets segment $BC$ at $E$, and $AE$ and $BD$ intersect at $P$. Given that $AB = 9$, $AE = 16$, and $EP = EC$, find $BC$.

   **Solution.** The answer is $\boxed{15}$.

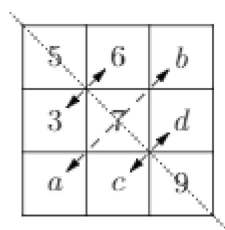
   Since $AE$ bisects $\angle DAB$ and $AD$ is parallel to $BC$, we have $\angle BAE = \angle DAE = \angle BEA$, so $BE = AB = 9$. Let $EC = EP = x$, then $AD = BC = 9 + x$, so from $\triangle APD \sim \triangle EPB$ we have $\frac{PA}{AD} = \frac{PE}{EB} \Rightarrow \frac{16-x}{9+x} = \frac{x}{9} \Rightarrow x = 6$ or $x = -24$. Since $x$ is positive, we get $BC = 9 + x = 15$.



10. Farmer James places the numbers $1, 2, \ldots, 9$ in a $3 \times 3$ grid such that each number appears exactly once in the grid. Let $x_i$ be the product of the numbers in row $i$, and $y_i$ be the product of the numbers in column $i$. Given that the unordered sets $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$ are the same, how many possible arrangements could Farmer James have made?

    **Solution.** The answer is $\boxed{432}$.

    First, note that if 5 and 7 are in the same row, one of the $x_i$ would be divisible by 35, but none of the $y_i$ would be, since 5 and 7 can't be in both the same row and the same column simultaneously. Therefore 5 and 7 are in separate rows and, by the same argument, in separate columns as well. If we then try to place these numbers, there would be 9 spots to first place the 5, and only 4 remaining spots to place the 7. Since we can reorder the rows and columns freely, assume WLOG that the 5 is in the top left, and the 7 is in the center of the grid.
    Then if 9 is placed in the same column as 7 then there will be a column product that is divisible by 63, which forces 3 and 6 to be placed on the same row as 7, and this makes all three column products divisible by 3 but one row having no multiples of 3. Similarly, we can rule out the possibility of 9 sharing a row or column with either 5 or 7. Thus the only remaining possibility is that 9 shares no column or row with 5 or 7, and WLOG it is on the bottom right of the grid.
    It is then clear that 3 and 6 must be symmetric about the diagonal containing $5, 7, 9$, so there are 6 unique ways to place the two of them. Each of these ways result in 3 and 6 sharing rows/columns with exactly two of $5, 7, 9$, so WLOG we can place 3 at the middle left, and 6 at the top center, where they will share rows/columns with 5 and 7. If we consider the row and column containing 5 here, we get $5 \cdot 3 \cdot a = 5 \cdot 6 \cdot b$, or $a = 2b$, for some $a, b \in \{1, 2, 4, 8\}$, and likewise by looking at 7 we get $c = 2d$ for the remaining two elements. There are only two ways to make such a pairing: $(a, b), (c, d) = (2, 1), (8, 4)$ or $(8, 4), (2, 1)$.
    Therefore the total number of possible grids comes out to $9 \cdot 4 \cdot 6 \cdot 2 = 432$.



## 2.3 Team Test Solutions

1. Farmer James goes to Kristy's Krispy Chicken to order a crispy chicken sandwich. He can choose from 3 types of buns, 2 types of sauces, 4 types of vegetables, and 4 types of cheese. He can only choose one type of bun and cheese, but can choose any nonzero number of sauces, and the same with vegetables. How many different chicken sandwiches can Farmer James order?

   **Solution.** The answer is $\boxed{540}$.

   There are 3 ways to choose the bun and 4 ways to choose the cheese. There are 4 types of vegetables, each of which can be on the sandwich or not, for $2^4$ choices. However, there must be a nonzero number of vegetables, so only $2^4-1$ of these choices can work. Similarly, there are $2^2-1$ sauce choices. So, there are a total of $3\cdot 4\cdot(2^2-1)\cdot(2^4-1)=540$ different sandwiches that Farmer James can order.

2. A line with slope 2 and a line with slope 3 intersect at the point $(m,n)$, where $m,n>0$. These lines intersect the $x$ axis at points $A$ and $B$, and they intersect the $y$ axis at points $C$ and $D$. If $AB=CD$, find $\frac{m}{n}$.

   **Solution.** The answer is $\boxed{1/6}$.

   We have two lines: $y-n=2(x-m)$ and $y-n=3(x-m)$. Plugging $y=0$ into both of these, we get the two x-intercepts are $(m-\frac{n}{2},0)$ and $(m-\frac{n}{3},0)$, so $AB=\frac{n}{6}$. Plugging $x=0$ into both of these, we get the two y-intercepts are $(0,n-2m)$ and $(0,n-3m)$, so $CD=m$. $AB=CD$ tells us that $\frac{m}{n}=\frac{1}{6}$.

3. A multi-set of 11 positive integers has a median of 10, a unique mode of 11, and a mean of 12. What is the largest possible number that can be in this multi-set? (A multi-set is a set that allows repeated elements.)

   **Solution.** The answer is $\boxed{71}$.

   First, since the mean of the 11 integers is 12, their sum must be $11\times 12=132$. To find the largest possible value of the largest integer, we must find the smallest possible value of the sum of the 10 smallest integers. The 5 smallest integers must be between 1 and 9, inclusive, and since the $6^{th}$ smallest must be the median, the $6^{th}$ to $10^{th}$ smallest integers are at least 10. In addition, if there are $n$ 11s, then there must be at most $n-1$ 1s, since 11 is the unique mode, and therefore $6-n$ integers that are at least 2. Therefore, the sum of the 5 smallest integers is at least $2(6-n)+1(n-1)=11-n$, and the sum of the $6^{th}$ to $10^{th}$ smallest integers is at least $50+n$. This means the total sum is at least 61, and the largest integer is at most $132-61=71$. The value 71 can be achieved with the set $\{1,1,1,2,2,10,11,11,11,11,71\}$.

4. Farmer James is swimming in the Eggs-Eater River, which flows at a constant rate of 5 miles per hour, and is recording his time. He swims 1 mile upstream, against the current, and then swims 1 mile back to his starting point, along with the current. The time he recorded was double the time that he would have recorded if he had swum in still water the entire trip. To the nearest integer, how fast can Farmer James swim in still water, in miles per hour?

   **Solution.** The answer is $\boxed{7}$.

   Suppose Farmer James's swimming speed is $x$ miles per hour. Then, when swimming upstream, he will only move at $x-5$ miles per hour, relative to the riverbank, and similarly when swimming downstream, he will move at $x+5$ miles per hour, therefore the total amount of time his round trip takes is $\frac{1}{x-5}+\frac{1}{x+5}=\frac{2x}{x^2-25}$ hours. This is twice as long as it would have taken without the river current, which would be $\frac{1}{x}+\frac{1}{x}=\frac{2}{x}$ hours, thus $2\cdot\frac{2}{x}=\frac{2x}{x^2-25}$, or $2(x^2-25)=x^2$, giving $x^2=50$. Therefore Farmer James's swimming speed is $x=5\sqrt{2}$, which is closest to 7 miles per hour.

5. $ABCD$ is a square with side length 60. Point $E$ is on $AD$ and $F$ is on $CD$ such that $\angle BEF=90°$. Find the minimum possible length of $CF$.

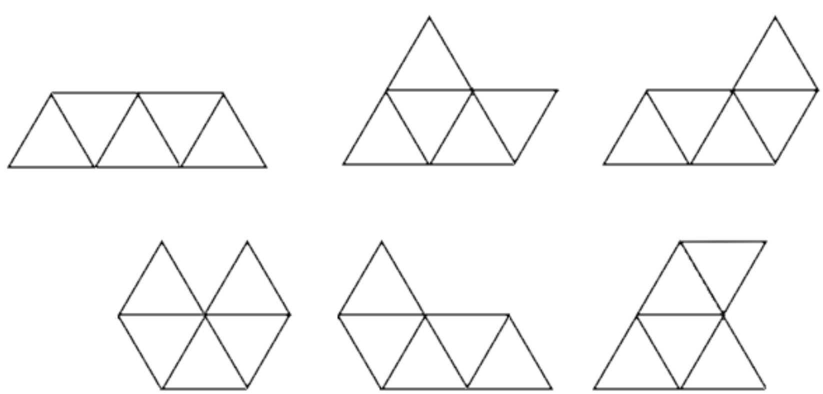
**Solution.** The answer is $\boxed{45}$.

Let $AE=x$. Then, we have $DE=60-x$, and $\triangle BAE\sim\triangle EDF$. Hence, $DF=\frac{x(60-x)}{60}$. Since $DC=60$, we wish to maximize $DF=\frac{900-(x-30)^2}{60}\le\frac{900}{60}=15$, so the minimum value of $CF$ is when $AE=30$, giving $CF=60-DF=60-15=45$.

6. Farmer James makes a *trianglomino* by gluing together 5 equilateral triangles of side length 1, with adjacent triangles sharing an entire edge. Two trianglominoes are considered the same if they can be matched using only translations and rotations (but not reflections). How many distinct trianglominoes can Farmer James make?

**Solution.** The answer is $\boxed{6}$.

There are only 6 possible distinct trianglominoes, as shown below.



7. Two real numbers $x$ and $y$ satisfy

$$\begin{cases} x^2-y^2=2y-2x \text{ , and} \\ x+6=y^2+2y. \end{cases}$$

What is the sum of all possible values of $y$?

**Solution.** The answer is $\boxed{-4}$.

Subtracting the two equations gives $0 = x^2 + x - 6 = (x+3)(x-2)$. So, $x = -3$ or $x = 2$. Plugging these both into the second equation, we have $y^2 + 2y = 3$ or $y^2 + 2y = 8$, which can be factored as $(y-1)(y+3) = 0$ and $(y+4)(y-2) = 0$ respectively, giving all possible $y$ values as $-4, -3, 1, 2$, which sum to $-4$.

8. Let $N$ be a positive multiple of 840. When $N$ is written in base 6, it is of the form $\overline{abcdef}_6$ where $a, b, c, d, e, f$ are distinct base 6 digits. What is the smallest possible value of $N$, when written in base 6?

**Solution.** The answer is $\boxed{\overline{415320}_6}$.

Since 840 is a multiple of 6, we must have $f = 0$. Hence, $(a, b, c, d, e)$ is a permutation of $(1, 2, 3, 4, 5)$. Now, dividing by 6, we get that $abcde_6$ is a multiple of 140. This means that $e$ is even, and, by the definition of base 6, that $e + 6 \cdot d + 6^2 \cdot c + 6^3 \cdot b + 6^4 \cdot a \equiv 0 \pmod 7$. We can write 6 as $-1 \pmod 7$, so we get $a + c + e \equiv b + d \pmod 7$. Note that $a + b + c + d + e = 15$, so we must have $a + c + e = 11$ and $b + d = 4$ or $a + c + e = 4$ and $b + d = 11$. The latter is impossible since $a + c + e \geq 1 + 2 + 3 = 6$, so we must have $b + d = 4$. Hence, either $b = 1$ and $d = 3$ or $d = 1$ and $b = 3$. Also, note that $abcde_6$ is divisible by 4, so $de_6$ is divisible by 4. Therefore, $e = 2$ as $14_6$ and $34_6$ are not multiples of 4. Hence, $a$ and $c$ must be 4 and 5, while $b$ and $d$ must be 1 and 3 so the minimum possible value of $abcde_6$ is $41532_6$. This is indeed a multiple of 840, so $415320_6$ is the answer.

9. For $S = \{1, 2, ..., 12\}$, find the number of functions $f : S \to S$ that satisfy the following 3 conditions:

   (a) If $n$ is divisible by 3, $f(n)$ is not divisible by 3,

   (b) If $n$ is not divisible by 3, $f(n)$ is divisible by 3, and

   (c) $f(f(n)) = n$ holds for exactly 8 distinct values of $n$ in $S$.

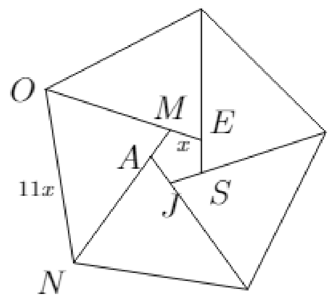
**Solution.** The answer is $\boxed{430080}$.

Note that $f(n) \in \{3, 6, 9, 12\}$ for $n \in \{1, 2, 4, 5, 7, 8, 10, 11\}$, while $f(3), f(6), f(9), f(12)$ are the other possible values. Hence, there are at most 8 possible values for $f(n)$, so these must be the values for $f(f(n))$ as well. Since $f(f(n)) = n$ for 8 values of $n$, we must have exactly 8 elements in the range of $f(n)$. From this, we know that $f(3), f(6), f(9), f(12)$ are all distinct, and that the set of $f(n) \in \{3, 6, 9, 12\}$ for $n \in \{1, 2, 4, 5, 7, 8, 10, 11\}$ contains 4 distinct elements. There are $8 \cdot 7 \cdot 6 \cdot 5 = 1680$ ways to choose the values of $f(3), f(6), f(9)$, and $f(12)$. Now, we know $f(f(n)) = n$ for $n \in \{3, 6, 9, 12, f(3), f(6), f(9), f(12)\}$, and there are 4 ways to assign values for each $f(n)$ not already assigned. Hence, the total number of ways is $1680 \cdot 4^4 = 430080$.

10. Regular pentagon $JAMES$ has area 1. Let $O$ lie on line $EM$ and $N$ lie on line $MA$ so that $E, M, O$ and $M, A, N$ lie on their respective lines in that order. Given that $MO = AN$ and $NO = 11 \cdot ME$, find the area of $NOM$.

**Solution.** The answer is $\boxed{24}$.

If we perform the given triangle construction on every side of pentagon $JAMES$, instead of just on side $MA$, we end up with an external regular pentagon with side length $NO$. Since $NO = 11 \cdot ME$, this pentagon has an area $11^2$ times the original, or 121. This pentagon's area is also composed of 5 instances of the desired triangle, and the original pentagon in the center, thus the area of the desired triangle $\triangle NOM$ is simply $\frac{121-1}{5} = 24$




11. Hen Hao is flipping a special coin, which lands on its *sunny* side and its *rainy* side each with probability $\frac{1}{2}$. Hen Hao flips her coin ten times. Given that the coin never landed with its rainy side up twice in a row, find the probability that Hen Hao's last flip had its *sunny side up*.

**Solution.** The answer is $\boxed{89/144}$.

Let $T(n)$ be the total number of arrangements for $n$ coin flips, and $S(n)$ be the number that end in a sunny side up flip. Clearly $S(n) = T(n-1)$, since you can simply add a sunny flip to the end of any sequence of flips, and removing the sunny flip from the end of a valid sequence still gives a valid sequence. To find $T(n)$, we can consider a few cases: if our sequence ends with a rainy flip, the previous $n-1$ flips only have to end in a sunny flip, so there are $S(n-1) = T(n-2)$ of these. If our sequence ends with a sunny flip, then there are $S(n) = T(n-1)$ such sequences. So we have $T(n) = T(n-1) + T(n-2)$. Starting with $T(0) = 1, T(1) = 2, T(2) = 3$, we see that this is exactly the Fibonacci recursion, giving our final answer as $\frac{S(10)}{T(10)} = \frac{T(9)}{T(10)} = \frac{89}{144}$.

12. Find the product of all integer values of $a$ such that the polynomial $x^4 + 8x^3 + ax^2 + 2x - 1$ can be factored into two non-constant polynomials with integer coefficients.

**Solution.** The answer is $\boxed{-1500}$.

We will divide this problem into two cases.

Case 1: The polynomial is the product of a linear term and a cubic (over the integers). Then, either

$$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^3 + px^2 + qx + 1)(x - 1)$$

or

$$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^3 + px^2 + qx - 1)(x + 1)$$

For the first subcase, substituting $x = 1$ gives $1 + 8 + a + 2 - 1 = 0$, so $a = -10$ is one possibility. For the second subcase, substituting $x = -1$ gives $1 - 8 + a - 2 - 1 = 0$. This gives $a = 10$ as a possibility. (It is not difficult to verify that in both subcases the factorization indeed exists).

Case 2: The polynomial is the product of two quadratics that are indecomposable over the integers. Then,

$$x^4 + 8x^3 + ax^2 + 2x - 1 = (x^2 + px + 1)(x^2 + qx - 1)$$

so $q - p = 2$ and $p + q = 8$ (from looking at the coefficients of $x$ and $x^3$). Hence, $q = 5$ and $p = 3$, so $x^4 + 8x^3 + ax^2 + 2x - 1 = (x^2 + 3x + 1)(x^2 + 5x - 1)$, which gives $a = 15$ as our final possibility.

Hence, the product of all possible values of $a$ is $-10 \cdot 10 \cdot 15 = -1500$.

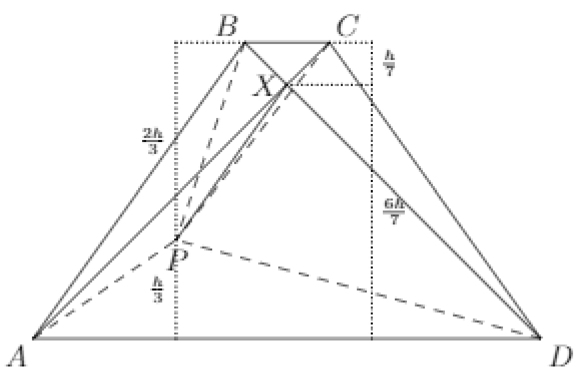
13. Isosceles trapezoid $ABCD$ has $AB = CD$ and $AD = 6BC$. Point $X$ is the intersection of the diagonals $AC$ and $BD$. There exist a positive real number $k$ and a point $P$ inside $ABCD$ which satisfy

$$[PBC] : [PCD] : [PDA] = 1 : k : 3,$$

where $[XYZ]$ denotes the area of triangle $XYZ$. If $PX \parallel AB$, find the value of $k$.

**Solution.** The answer is $\boxed{73/14}$.

Let $h$ be the height of the trapezoid, which is the distance between $AD$ and $BC$. Now, since $\dfrac{[PDA]}{[PBC]} = 3$, and $AD = 6BC$, we know that the distance from $P$ to $AD$ is $\frac{h}{3}$, while the distance from $P$ to $BC$ is $\frac{2}{3}h$. Since $PX||AB$ we know that $[PAB] = [XAB]$. We also know that $\triangle XAD \sim \triangle XCB$ with ratio 6, so the distance from $X$ to $AD$ is $\frac{6}{7}h$, while the distance from $X$ to $BC$ is $\frac{h}{7}$. Also, $[XAB] = [XCD]$ by symmetry, so $2[XAB] = [ABCD] - [XAD] - [XBC] = \frac{(AD+BC)\cdot h}{2} - \frac{AD\cdot 6h}{14} - \frac{BC\cdot h}{14} = \frac{6BC\cdot h}{7}$. Hence, $[PAB] = [XAB] = \frac{3BC\cdot h}{7}$. Also, $[PBC] = \frac{BC\cdot h}{3}$ and $[PAD] = BC \cdot h$, so $[PCD] = \frac{7BC\cdot h}{2} - \frac{3BC\cdot h}{7} - \frac{BC\cdot h}{3} - BC \cdot h = \frac{73BC\cdot h}{42}$. Hence, $k = \dfrac{[PCD]}{[PBC]} = \frac{73}{14}$.




14. How many positive integers $n < 1000$ are there such that in base 10, every digit in $3n$ (that isn't a leading zero) is greater than the corresponding place value digit (possibly a leading zero) in $n$?

For example, $n = 56$, $3n = 168$ satisfies this property as $1 > 0, 6 > 5$, and $8 > 6$. On the other hand, $n = 506$, $3n = 1518$ does not work because of the hundreds place.

**Solution.** The answer is $\boxed{88}$.

When multiplying by 3, the digits which will end up larger in the final number (without any carrying from the previous place value) are $1, 2, 3$, and 6. With carrying, however, this list changes to $0, 1, 2, 5, 6$. We can then note that, out of the 4 possibilities for a non-carried place, 3 of them will leave the next place non-carried, while 1 (the 6) will cause the next place to become carried; out of the 5 possibilities for a carried place, 3 of them will leave the next place non-carried, while the other 2 will keep the next place carried.

If we say non-carried is $N$ (note that the last digit of a number is always non-carried, since there is no place value smaller than it), and carried is $C$, we can then do casework on 3 digit numbers:

$NNN = 4 \cdot 3 \cdot 3 = 36$, since there are 3 numbers that transition from $N$ to $N$, a necessary condition

for only the last two digits.
$NCN = 4 \cdot 3 \cdot 1 = 12$, as there is 1 number that transitions from $N$ to $C$, and 3 that transition from $C$ to $N$. The remaining cases will work similarly.
$CNN = 4 \cdot 1 \cdot 3 = 12$, note that we must remove 1 from the 5 possible carried digits for this case (and the next), because it would create a leading 0.
$CCN = 4 \cdot 2 \cdot 1 = 8$
This gives a total of $36 + 12 + 12 + 8 = 68$ three digit numbers. However, we can use the same method to quickly check one and two digit numbers:
$CN = 4 \cdot 1 = 4$, $NN = 4 \cdot 3 = 12$, total is 16 for two digits.
$N = 4$, total is 4 for one digit.
Thus the final answer is $68 + 16 + 4 = 88$.

15. Find the greatest integer that is smaller than

$$\frac{2018}{37^2} + \frac{2018}{39^2} + \cdots + \frac{2018}{107^2}.$$

**Solution.** The answer is $\boxed{18}$.

We can approximate the sum as $\frac{2018}{36 \cdot 38} + \frac{2018}{38 \cdot 40} + \cdots + \frac{2018}{106 \cdot 108}$, where the error in each term will effectively be negligible (on the order of $\epsilon \leq \frac{2018}{37^2(37^2-1)} < \frac{1}{928}$ for 36 terms), as we are looking for only the greatest integer less than the sum. We can then write this modified sum as $1009\left(\frac{1}{36} - \frac{1}{38}\right) + 1009\left(\frac{1}{38} - \frac{1}{40}\right) + \cdots + 1009\left(\frac{1}{106} - \frac{1}{108}\right)$, which telescopes to $1009\left(\frac{1}{36} - \frac{1}{108}\right) = \frac{1009}{54} \approx 18.7$, making the greatest integer less than the sum 18.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. How many distinct ways are there to *scramble* the letters in $EXETER$?

   **Solution.** The answer is $\boxed{120}$.

   We can choose one of the 6 places to place the $X$, one of the remaining 5 places to place the $T$, and one of the remaining 4 places to place the $R$. The rest of the letters will be filled with $E$s, which are indistinguishable, thus the total number of orderings is $6 \cdot 5 \cdot 4 = 120$

2. Given that $\dfrac{x-y}{x-z} = 3$, find $\dfrac{x-z}{y-z}$.

   **Solution.** The answer is $\boxed{-\frac{1}{2}}$.

   By setting $(x, y, z) = (4, 1, 3)$, we get $\frac{x-y}{x-z} = \frac{4-1}{4-3} = 3$ as given, and therefore $\frac{x-z}{y-z} = \frac{4-3}{1-3} = -\frac{1}{2}$. Alternatively, since we are given $\frac{x-y}{x-z} = 3$, $\frac{z-y}{x-z} = \frac{x-y}{x-z} - \frac{x-z}{x-z} = 3 - 1 = 2$. Therefore, $\frac{y-z}{x-z} = -2$, and $\frac{x-z}{y-z} = -\frac{1}{2}$.

3. When written in base 10,
$$9^9 = \overline{ABC420DEF}.$$
   Find the remainder when $A + B + C + D + E + F$ is divided by 9.

   **Solution.** The answer is $\boxed{3}$.

   Since $9^9$ is divisible by 9, the digits must sum to a multiple of 9. $4 + 2 + 0$ leaves a remainder of 6 when divided by 9, so the sum of the remaining 6 digits, when divided by 9, must leave a remainder of 3.

### 2.4.2 Round 2

4. How many positive integers, when expressed in base 7, have exactly 3 digits, but don't contain the digit 3?

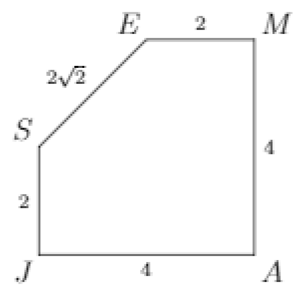
   **Solution.** The answer is $\boxed{180}$.

   The last two digits in our number can be anything except 3, giving 6 possibilities for each, while the first can be neither 0 nor 3, since numbers can't start with 0, leaving only 5 possibilities, so the total amount of possible numbers is $5 \cdot 6 \cdot 6 = 180$.

5. Pentagon $JAMES$ is such that its internal angles satisfy $\angle J = \angle A = \angle M = 90^\circ$ and $\angle E = \angle S$. If $JA = AM = 4$ and $ME = 2$, what is the area of $JAMES$?

**Solution.** The answer is $\boxed{14}$.

Since we know $\angle E = \angle S$ and the sum of the angles of a pentagon is $540°$, we get that $\angle E = \angle S = 135°$. Pentagon $JAMES$ is thus a $4 \times 4$ square with a $2 \times 2$ isosceles right triangle cut from one of the corners, as shown in the diagram below. We can calculate its area to be $4 \cdot 4 - \frac{2 \cdot 2}{2} = 14$.




6. Let $x$ be a real number such that $x = \frac{1+\sqrt{x}}{2}$. What is the sum of all possible values of $x$?

   **Solution.** The answer is $\boxed{1}$.

   We can simplify to get $2x - 1 = \sqrt{x}$, and then square both sides, noting that there will be an extraneous solution (where $2x - 1 = -\sqrt{x}$). This gives $4x^2 - 4x + 1 = x$, or $(4x - 1)(x - 1) = 0$. Plugging in these solutions yields $x = 1$ to be the only non-extraneous solution, so the answer is 1.

### 2.4.3 Round 3

7. Farmer James sends his favorite chickens, Hen Hao and PEAcock, to compete at the Fermi Estimation All Star Tournament (FEAST). The first problem at the FEAST requires the chickens to estimate the number of boarding students at Eggs-Eater Academy given the number of dorms $D$ and the average number of students per dorm $A$. Hen Hao rounds both $D$ and $A$ down to the nearest multiple of 10 and multiplies them, getting an estimate of 1200 students. PEAcock rounds both $D$ and $A$ up to the nearest multiple of 10 and multiplies them, getting an estimate of $N$ students. What is the maximum possible value of $N$?

   **Solution.** The answer is $\boxed{2600}$.

   Suppose Hen Hao rounded his numbers down to $10A$ and $10B$, where $AB = 12$. Then, we are trying to maximize $N = (10A + 10)(10B + 10)$. This is simply $100AB + 100 + 100(A + B) = 1300 + 100(A + B)$, so we want to maximize the sum of two factors of 12, which occurs at $(A, B) = (1, 12)$, giving $N = 1300 + 100 \cdot 13 = 2600$.

8. Farmer James has decided to prepare a large bowl of *egg drop soup* for the Festival of Eggs-Eater Annual Soup Tasting (FEAST). To flavor the soup, Hen Hao drops eggs into it. Hen Hao drops 1 egg into the soup in the first hour, 2 eggs into the soup in the second hour, and so on, dropping $k$ eggs into the soup in the $k$th hour. Find the smallest positive integer $n$ so that after exactly $n$ hours, Farmer James finds that the number of eggs dropped in his egg drop soup is a multiple of 200.

**Solution.** The answer is $\boxed{175}$.

After $n$ hours, Hen Hao will have dropped $\frac{n(n+1)}{2}$ eggs, so $n(n+1)$ must be a multiple of $400 = 16 \cdot 25$. Since $n$ and $n+1$ are relatively prime, either one of them is divisible by 400, giving $n = 399$ as the minimum, or one is divisible by 16, and the other is divisible by 25. By listing out the first few multiples of 25, we can see that no multiple of 16 is adjacent to any of these until 175 and 176, so $n = 175$ is minimal.

9. Farmer James decides to FEAST on Hen Hao. First, he cuts Hen Hao into 2018 pieces. Then, he eats 1346 pieces every day, and then splits each of the remaining pieces into three smaller pieces. How many days will it take Farmer James to eat Hen Hao? (If there are fewer than 1346 pieces remaining, then Farmer James will just eat all of the pieces.)

   **Solution.** The answer is $\boxed{7}$.

   Note that if Farmer James has $2019 - 3^k$ pieces at any point, he will eat 1346 of them, and then triple the remaining amount, leaving us with $3(2019 - 3^k - 1346) = 2019 - 3^{k+1}$ pieces for the next day. On the first day, we have $2018 = 2019 - 3^0$ pieces, and the first power of 3 greater than 2019 is $3^7 = 2187$, so Farmer James can FEAST for values of $k$ between 0 and 6, inclusive, giving a total of 7 days.

### 2.4.4 Round 4

10. Farmer James has three baskets, and each basket has one magical egg. Every minute, each magical egg disappears from its basket, and reappears with probability $\frac{1}{2}$ in each of the other two baskets. Find the probability that after three minutes, Farmer James has *all his eggs in one basket.*

    **Solution.** The answer is $\boxed{\frac{27}{256}}$.

    The odds of all 3 eggs ending up in any basket is equal among the 3 baskets, so WLOG we can calculate the probability they all end up in basket 1, and multiply that by 3.
    In this case, the first egg has to end up in either basket 2 or 3 after two minutes, and since it is guaranteed to already be in basket 2 or 3 after one minute, it must switch from one to the other, a move that has probability $\frac{1}{2}$ of occurring. The final jump into basket 1 also has $\frac{1}{2}$ chance of occurring, giving the first egg a total $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ probability of success.
    The second egg also cannot be in basket 1 after two minutes. If it moved to basket 1 originally in the first minute, there is no chance of this happening, whereas if it moved to basket 3, there would be a $\frac{1}{2}$ chance, so the overall probability is $\frac{3}{4}$ that the second egg will be able to make the last jump to basket 1, which again occurs with $\frac{1}{2}$ chance, giving a probability of $\frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$.
    Since the second and third eggs are symmetric, the total probability of success here is $\frac{1}{4} \cdot \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{256}$.
    Since this is only for the first basket, the final answer will be 3 times this, or $\frac{27}{256}$.

11. Find the value of $$\frac{4 \cdot 7}{\sqrt{4+\sqrt{7}} + \sqrt{4-\sqrt{7}}}.$$

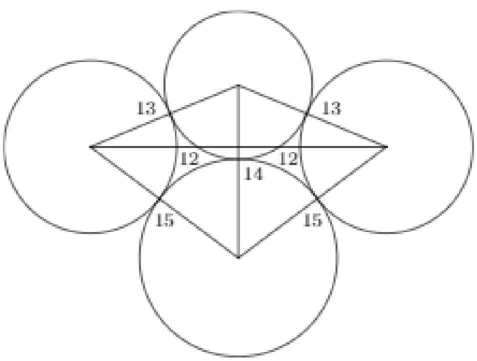
**Solution.** The answer is $\boxed{2\sqrt{14}}$.

If we square the denominator, we get $(4+\sqrt{7})+(4-\sqrt{7})+2\sqrt{(4+\sqrt{7})(4-\sqrt{7})} = 8+2\sqrt{16-7} = 14$, so the denominator is equal to $\sqrt{14}$, giving the total fraction a value of $\frac{28}{\sqrt{14}} = 2\sqrt{14}$

12. Two circles, with radius 6 and radius 8, are externally tangent to each other. Two more circles, of radius 7, are placed on either side of this configuration, so that they are both externally tangent to both of the original two circles. Out of these 4 circles, what is the maximum distance between any two centers?

    **Solution.** The answer is $\boxed{24}$.

    If we connect the centers of the mutually tangent circles with radii $6, 7$, and 8, we end up with a triangle with side lengths $13, 14, 15$. This triangle has an altitude of length 12 to the side composed of the length 6 and 8 radii, and the same configuration will exist on the other side of these two radii, forming a total diagonal distance, between the two centers of the radius 7 circles, of 24. Besides this pair, every other pair of circles is tangent, meaning the distance between the centers is the sum of the radii, which will clearly be less than 24, therefore 24 is the maximum distance.




### 2.4.5 Round 5

13. Find all ordered pairs of real numbers $(x, y)$ satisfying the following equations:

$$\begin{cases} \frac{1}{xy} + \frac{y}{x} = 2 \\ \frac{1}{xy^2} + \frac{y^2}{x} = 7. \end{cases}$$

**Solution.** The answer is $\boxed{(2, 2\pm\sqrt{3})}$.

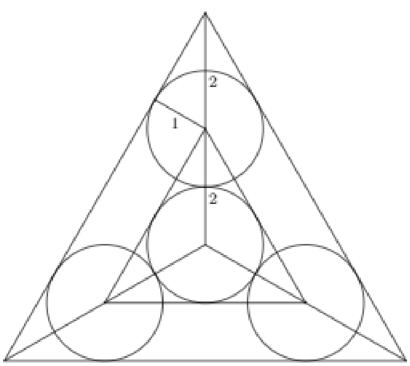
If we divide the two equations, we get $\dfrac{y^2+\frac{1}{y^2}}{y+\frac{1}{y}} = \frac{7}{2}$, so if we substitute $z = y + \frac{1}{y}$, we find $\frac{z^2-2}{z} = \frac{7}{2}$, or

$2z^2 - 7z - 4 = 0$, which factors as $(2z+1)(z-4) = 0$. Note that from the second (given) equation we have $x > 0$, and thus from the first we have $y > 0$, so $z > 0$ as well, meaning $z = 4$ is the only solution. Since the first equation can be written as $z = 2x$, we find $x = 2$, and the solutions for $y + \frac{1}{y} = 4$ are given by $y^2 - 4y + 1 = 0$, or $y = 2 \pm \sqrt{3}$, giving the solution set $(2, 2-\sqrt{3}), (2, 2+\sqrt{3})$.

14. An *egg plant* is a hollow prism of negligible thickness, with height 2 and an equilateral triangle base. Inside the egg plant, there is enough space for four spherical eggs of radius 1. What is the minimum possible volume of the egg plant?

**Solution.** The answer is $\boxed{24\sqrt{3}}$.

Since the height of the prism is 2, and the radius of each sphere is 1, the centers of all the spheres must be at height 1, otherwise they would intersect with the bottom or top of the prism. Thus this reduces to a 2-dimensional problem: We have four circles of radius one to fit into an equilateral triangle. The optimal arrangement is to have one circle in the center, and every other circle tangent to that central circle and to two other sides. The center to vertex length of this equilateral triangle is two radii plus the hypotenuse of a $30-60-90$ triangle with the short leg as a radius. So, the center to vertex length of the triangle is 4, giving the side length of the triangle as $4\sqrt{3}$, meaning the total area of the triangle is $\frac{(4\sqrt{3})^2\sqrt{3}}{4} = 12\sqrt{3}$, so the volume of the prism is $24\sqrt{3}$.



(birds eye view of an egg plant)

15. How many ways are there for Farmer James to color each square of a $2 \times 6$ grid with one of the three colors eggshell, cream, and cornsilk, so that no two adjacent squares are the same color?

**Solution.** The answer is $\boxed{1458}$.

For the first column, there are 3 ways to color the top square, and 2 remaining ways to color the bottom square, giving 6 combinations total. Then, given an existing column to the left (say with colors $A$ and $B$ as top and bottom, respectively), among all combinations of colors for the next column, only 3 of them work ($BA$, $BC$, $CA$ as top and bottom). Therefore, number of ways to color the grid is simply $6 \cdot (3^5) = 1458$ ways.

### 2.4.6 Round 6

16. In a triangle $ABC$, $\angle A = 45$, and let $D$ be the foot of the perpendicular from $A$ to segment $BC$. $BD = 2$ and $DC = 4$. Let $E$ be the intersection of the line $AD$ and the perpendicular line from $B$ to line $AC$. Find the length of $AE$.

    **Solution.** The answer is $\boxed{6}$.

    Define the foot of the perpendicular from B be F. Notice that since $\angle A = 45$, then $AF = BF$. Since $\angle AEF = \angle BED = \angle BCF$, and $\angle AFE = \angle BFC$, we have that $\triangle AFE \cong \triangle BFC$. So, $AE = BC = 6$.

17. Find the largest positive integer $n$ such that there exists a unique positive integer $m$ satisfying

$$\frac{1}{10} \leq \frac{m}{n} \leq \frac{1}{9}.$$

    **Solution.** The answer is $\boxed{161}$.

    Take the reciprocal of each half of the inequality, and multiply by $m$, to get $9m \leq n \leq 10m$. The condition requires a unique solution for $m$, meaning that the left side of this inequality cannot be satisfied by a greater value of $m$, so $n < 9(m+1)$. Similarly, we get $n > 10(m-1)$, or $9m+9 > n > 10m-10$ in total. Since the inequalities are strict, this means the interval from $10m-10$ to $9m+9$ must have width at least 2, to leave room for the intermediate value $n$, so $(9m+9)-(10m-10) \geq 2$, or $m \leq 17$. Thus the maximal $n$ occurs when $m = 17$, giving $9m+9 = 162$ and $10m-10 = 160$, so $n = 161$.

18. How many ordered pairs $(A, B)$ of positive integers are there such that $A+B = 10000$ and the number $A^2 + AB + B$ has all distinct digits in base 10?

    **Solution.** The answer is $\boxed{384}$.

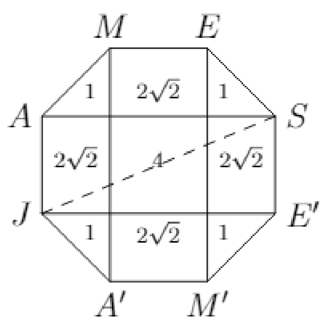
    Note that $A^2 + AB + B = 10000A + B$, so $B$ must be at least 3 digits, otherwise the sum will have two consecutive 0s. Also, since $A + B = 10000$, the last digits of each must sum to 10, while the remaining pairs of corresponding digits (which there are 3 of) must sum to 9 each. Out of the digits from 0 to 9, there are 5 such pairs that sum to 9, and 4 options for the initial pair summing to 10 (since the digits must be distinct). This initial pair also uses up one of the elements for 2 out of 5 pairs that sum to 9, leaving exactly 3 pairs, which are therefore strictly determined by this choice of initial pair. There are then $3! \cdot 2^3 = 48$ ways to permute these pairs among the place values, and then assign one of each pair to either $A$ or $B$ (note that leading 0s are no longer a concern, as both numbers will be at least 3 digits, so 0 will only occur at most once). We must also assign each digit in the initial pair to either $A$ or $B$, which can be done in 2 ways, giving the total number of ordered pairs as $4 \cdot 2 \cdot 48 = 384$.

### 2.4.7 Round 7

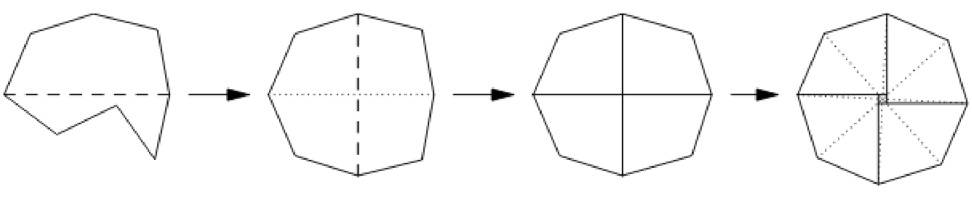
19. Pentagon $JAMES$ satisfies $JA = AM = ME = ES = 2$. Find the maximum possible area of $JAMES$.

**Solution.** The answer is $\boxed{4+4\sqrt{2}}$.

Notice that if we reflect the pentagon over side $JS$, we end up with an equilateral octagon (with no other constraints). It is then intuitive (and is proven below, for completeness) that the equilateral octagon with largest possible area is a regular one, so it remains to find the area of a regular octagon with side length 2. To do this, we can split the octagon into 9 regions, as shown. The four corner regions have an area of 1 each, the four edge regions have an area of $2\sqrt{2}$ each, and the central square has area 4, making the total area $8+8\sqrt{2}$. Since our pentagon is half of this octagon, it ends up with area $4+4\sqrt{2}$.




To show that the equilateral octagon with maximal area is in fact regular, we can observe that any for any such octagon, we can choose a diameter, pick the half with larger area, and reflect that over the diameter, replacing the other half and leaving an octagon at least as large, and with horizontal symmetry. We can then again reflect the larger half over the (now) perpendicular diameter, leaving an octagon (at least as large) with 4-way symmetry. If the two diagonals in this octagon are not the same length, we can perform a dissection, creating a new octagon that is *strictly* larger than before; it thus follows that an octagon with maximal area *must* have equal angles from the center to the vertices.



From here, each triangle formed between two adjacent vertices with the center has an angle of 45 degrees across from a side of length 1, and this has maximal area when it is isosceles, thus the maximal octagon occurs when all angles in the octagon are equal, i.e. it is regular

20. $P(x)$ is a monic polynomial (a polynomial with leading coefficient 1) of degree 4, such that $P(2^n+1) = 8^n+1$ when $n = 1, 2, 3, 4$. Find the value of $P(1)$.

**Solution.** The answer is $\boxed{1025}$.

Note that the cubic polynomial $(x-1)^3+1$ takes $2^n+1$ to $8^n+1$, therefore to get the correct polynomial, we can simply add this to the monic 4th degree polynomial which takes all given inputs to 0, or $(x-3)(x-5)(x-9)(x-17)$. Thus $P(1) = (1-1)^3+1+(1-3)(1-5)(1-9)(1-17) = 1+1024 = 1025$.

21. PEAcock and Zombie Hen Hao are at the starting point of a circular track, and start running in the same direction at the same time. PEAcock runs at a constant speed that is 2018 times faster than Zombie Hen Hao's constant speed. At some point in time, Farmer James takes a photograph of his two favorite chickens, and he notes that they are at different points along the track. Later on, Farmer James takes a second photograph, and to his amazement, PEAcock and Zombie Hen Hao have now swapped locations from the first photograph! How many distinct possibilities are there for PEAcock and Zombie Hen Hao's positions in Farmer James's first photograph? (Assume PEAcock and Zombie Hen Hao have negligible size.)

    **Solution.** The answer is $\boxed{4070306}$.

    Since PEAcock runs 2018 times faster than Hen Hao, if Hen Hao's position is the real number $0 \le x < 1$ (representing his progress along the track), then PEAcock will be at $\{2018x\}$, where $\{x\}$ denotes the fractional part of $x$. We know that, in the second photo, Hen Hao will now be at $\{2018x\}$, meaning PEAcock will end up at $\{2018\{2018x\}\}$, which must be Hen Hao's original position, giving us $x = \{2018\{2018x\}\} = \{2018^2x\}$. If $2018^2x = x + k$ for some integer $k$, then $k = (2018^2 - 1)x$, so $x = \frac{k}{2018^2-1}$, where $k$ can range from 0 to $2018^2 - 2$ inclusive, giving $2018^2 - 1$ values. However, we have not yet considered the condition that PEAcock and Hen Hao have their picture taken at distinct locations; if this is not the case, then $x = \{2018x\}$, and then again we get $x = \frac{k}{2017}$, leaving 2017 solutions that we must exclude from our original. Therefore the total number of distinct possible positions for the first photograph is $2018^2 - 1 - 2017 = 2017 \cdot 2018 = 4070306$.

### 2.4.8 Round 8

22. How many ways are there to *scramble* the letters in *EGGSEATER* such that no two consecutive letters are the same?

    **Solution.** The answer is $\boxed{10200}$.

    The only letters with more than one appearance are the three E's and two G's. (There are 9 letters in total.) We first enforce the requirement that the three E's are not next to each other, and then exclude the case where the two G's are next to each other.

    By stars and bars there are $\binom{7}{3} = 35$ ways to place the three E's in the 9-letter sequence so that they are not next to each other, so this gives $35 \cdot \frac{6!}{2} = 12600$ possible scrambles initially. If the two G's are next to each other, then we can consider them as one single letter, so by the same method we see that this accounts for $\binom{6}{3} \cdot 5! = 2400$ scrambles.

    Therefore, the total number of scrambles is $12600 - 2400 = 10200$.

23. Let $JAMES$ be a regular pentagon. Let $X$ be on segment $JA$ such that $\frac{JX}{XA} = \frac{XA}{JA}$. There exists a unique point $P$ on segment $AE$ such that $XM = XP$. Find the ratio $\frac{AE}{PE}$.

    **Solution.** The answer is $\boxed{\frac{3\sqrt{5}+7}{2}}$.

    First, we prove quadrilateral $XPMA$ is cyclic. To do this, let $P'$ be the point on line $EA$ such

that quadrilateral $XAMP'$ is cyclic. Then, $\angle P'XM = \angle P'AM = 36$, and $\angle P'MX = \angle P'AX = 72$. So, since $P'XM$ is a $36-72-72$ triangle, we know $P'$ fits the problem conditions. Since there can only be one point on line segment $AE$ that fits these conditions, we know $P' = P$. Therefore, quadrilateral $XPMA$ is cyclic. Moreover, $\triangle MXP \sim \triangle MJE$. This tells us that $\triangle MXP$ is just $\triangle MJE$ rotated and dilated about point $M$ (a spiral similarity).
Now, let the intersection of $MS$ and $AE$ be $Y$. So, $P$ in $\triangle MYE$ corresponds to $X$ in $\triangle MAJ$, and therefore $\frac{EY}{EP} = \frac{JA}{JX}$.
WLOG assume $JA = 1$. Then, the condition $\frac{JX}{XA} = \frac{XA}{JA}$ becomes $XA^2 + XA - 1 = 0$. So, $XA = \frac{\sqrt{5}-1}{2}$, so $\frac{JA}{JX} = \frac{3+\sqrt{5}}{2}$.
It is well-known that the ratio of a diagonal of a regular pentagon to its side is $\frac{\sqrt{5}+1}{2}$ (and in any other $36-36-72$ or $36-36-108$ triangle). Therefore, if $Z$ is the intersection of $AE$ and $JM$, then $YZ = \frac{EY}{\frac{\sqrt{5}+1}{2}}$, so $AE = EY \cdot (2 + \frac{2}{\sqrt{5}+1})$.
Then, $\frac{AE}{PE} = \frac{EY}{PE} \cdot (2 + \frac{2}{\sqrt{5}+1}) = \frac{3+\sqrt{5}}{2} \cdot (2 + \frac{2}{\sqrt{5}+1}) = \frac{7+3\sqrt{5}}{2}$

24. Find the minimum value of the function

$$f(x) = \left|x - \frac{1}{x}\right| + \left|x - \frac{2}{x}\right| + \left|x - \frac{3}{x}\right| + \cdots + \left|x - \frac{9}{x}\right| + \left|x - \frac{10}{x}\right|$$

over all nonzero real numbers $x$.

**Solution.** The answer is $\boxed{2\sqrt{26}}$.

Consider the 11 intervals split by the $x$ intercepts $(1, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{10})$ of the 10 absolute values inside $f(x)$. Between all 11 intervals, we can take off the absolute values, leaving a function of the form $f(x) = ax + \frac{b}{x}$. For $x \leq \sqrt{6}$, $a \leq 0$ and $b > 0$, so all these intervals are monotonically decreasing. Similarly, $x \geq \sqrt{7}$ makes $a > 0$ and $b < 0$, and each of these intervals is monotonically increasing. This function is continuous; therefore, the minimum must be between $\sqrt{6} \leq x \leq \sqrt{7}$. In this range, we $f(x) = 2x + \frac{13}{x}$, so we can apply AM-GM to give $f(x) \geq 2\sqrt{2x \cdot \frac{13}{x}} = 2\sqrt{26}$, with equality achieved when $x = \sqrt{\frac{13}{2}}$, which is indeed between $\sqrt{6}$ and $\sqrt{7}$.

# Exeter Math Club Competition
## January 26, 2019

## Table of Contents

## Organizing Acknowledgments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Jane Street

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- **Cleanup Crew** Victor Luo, Thomas Guo, Sanath Govindarajan

# Chapter 1

# EMC$^2$ 2019 Problems

## 1.1 Speed Test

*There are 20 problems, worth 3 points each, to be solved in 25 minutes.*

1. Given that $a + 19b = 3$ and $a + 1019b = 5$, what is $a + 2019b$?

2. How many multiples of 3 are there between 2019 and 2119, inclusive?

3. What is the maximum number of quadrilaterals a 12-sided regular polygon can be *quadrangulated* into? Here *quadrangulate* means to cut the polygon along lines from vertex to vertex, none of which intersect inside the polygon, to form pieces which all have exactly 4 sides.

4. What is the value of $|2\pi - 7| + |2\pi - 6|$, rounded to the nearest hundredth?

5. In the town of EMCCxeter, there is a 30% chance that it will snow on Saturday, and independently, a 40% chance that it will snow on Sunday. What is the probability that it snows exactly once that weekend, as a percentage?

6. Define $n!$ to be the product of all integers between 1 and $n$ inclusive. Compute $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$.

7. There are 2019 people standing in a row, and they are given positions $1, 2, 3, \ldots 2019$ from left to right. Next, everyone in an odd position simultaneously leaves the row, and the remaining people are assigned new positions from 1 to 1009, again from left to right. This process is then repeated until one person remains. What was this person's original position?

8. The product $1234 \times 4321$ contains exactly one digit not in the set $\{1, 2, 3, 4\}$. What is this digit?

9. A quadrilateral with positive area has four integer side lengths, with shortest side 1 and longest side 9. How many possible perimeters can this quadrilateral have?

10. Define $s(n)$ to be the sum of the digits of $n$ when expressed in base 10, and let $\gamma(n)$ be the sum of $s(d)$ over all natural number divisors $d$ of $n$. For instance, $n = 11$ has two divisors, 1 and 11, so $\gamma(11) = s(1) + s(11) = 1 + (1 + 1) = 3$. Find the value of $\gamma(2019)$.

11. How many five-digit positive integers are divisible by 9 and have 3 as the tens digit?

12. Adam owns a large rectangular block of cheese, that has a square base of side length 15 inches, and a height of 4 inches. He wants to remove a cylindrical cheese chunk of height 4, by making a circular hole that goes through the top and bottom faces, but he wants the surface area of the leftover cheese block to be the same as before. What should the diameter of his hole be, in inches?

    *Addendum on 1/26/19: the hole must have non-zero diameter.*

13. Find the smallest prime that does not divide $20! + 19! + 2019!$.

14. Convex pentagon $ABCDE$ has angles $\angle ABC = \angle BCD = \angle DEA = \angle EAB$ and angle $\angle CDE = 60°$. Given that $BC = 3$, $CD = 4$, and $DE = 5$, find $EA$.

    *Addendum on 1/26/19: $ABCDE$ is specified to be convex.*

15. Sophia has 3 pairs of red socks, 4 pairs of blue socks, and 5 pairs of green socks. She picks out two individual socks at random; what is the probability she gets a pair with matching color?

16. How many real roots does the function $f(x) = 2019^x - 2019x - 2019$ have?

17. A $30-60-90$ triangle is placed on a coordinate plane with its short leg of length 6 along the $x$-axis, and its long leg along the $y$-axis. It is then rotated 90 degrees counterclockwise, so that the short leg now lies along the $y$-axis and long leg along the $x$-axis. What is the total area swept out by the triangle during this rotation?

18. Find the number of ways to color the unit cells of a $2 \times 4$ grid in four colors such that all four colors are used and every cell shares an edge with another cell of the same color.

19. Triangle $\triangle ABC$ has centroid $G$, and $X, Y, Z$ are the centroids of triangles $\triangle BCG$, $\triangle ACG$, and $\triangle ABG$, respectively. Furthermore, for some points $D, E, F$, vertices $A, B, C$ are themselves the centroids of triangles $\triangle DBC$, $\triangle ECA$, and $\triangle FAB$, respectively. If the area of $\triangle XYZ = 7$, what is the area of $\triangle DEF$?

20. Fhomas orders three 2-gallon jugs of milk from EMCCBay for his breakfast omelette. However, every jug is actually shipped with a random amount of milk (not necessarily an integer), uniformly distributed between 0 and 2 gallons. If Fhomas needs 2 gallons of milk for his breakfast omelette, what is the probability he will receive enough milk?

## 1.2 Accuracy Test

*There are 10 problems, worth 9 points each, to be solved in 45 minutes.*

1. A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. Two hexos are considered the same if one can be rotated/reflected to match the other. Find the number of different hexos.

2. The sequence $1, 2, 3, 3, 3, 4, 5, 5, 5, 5, 5, 6, \ldots$ consists of numbers written in increasing order, where every even number $2n$ is written once, and every odd number $2n + 1$ is written $2n + 1$ times. What is the $2019^{\text{th}}$ term of this sequence?

3. On planet EMCCarth, months can only have lengths of 35, 36, or 42 days, and there is at least one month of each length. Victor knows that an EMCCarth year has $n$ days, but realizes that he cannot figure out how many months there are in an EMCCarth year. What is the least possible value of $n$?

4. In triangle $ABC$, $AB = 5$ and $AC = 9$. If a circle centered at $A$ passing through $B$ intersects $BC$ again at $D$ and $CD = 7$, what is $BC$?

5. How many nonempty subsets $S$ of the set $\{1, 2, 3, \ldots, 11, 12\}$ are there such that the greatest common factor of all elements in $S$ is greater than 1?

6. Jasmine rolls a fair 6-sided die, with faces labeled from 1 to 6, and a fair 20-sided die, with faces labeled from 1 to 20. What is the probability that the product of these two rolls, added to the sum of these two rolls, is a multiple of 3?

7. Let $\{a_n\}$ be a sequence such that $a_n$ is either $2a_{n-1}$ or $a_{n-1} - 1$. Given that $a_1 = 1$ and $a_{12} = 120$, how many possible sequences $a_1, a_2, \ldots, a_{12}$ are there?

8. A tetrahedron has two opposite edges of length 2 and the remaining edges have length 10. What is the volume of this tetrahedron?

9. In the garden of EMCCden, there is a tree planted at every lattice point $-10 \leq x, y \leq 10$ except the origin. We say that a tree is *visible* to an observer if the line between the tree and the observer does not intersect any other tree (assume that all trees have negligible thickness). What fraction of all the trees in the garden of EMCCden are visible to an observer standing at the origin?

10. Point $P$ lies inside regular pentagon $\zeta$, which lies entirely within regular hexagon $\eta$. A point $Q$ on the boundary of pentagon $\zeta$ is called *projective* if there exists a point $R$ on the boundary of hexagon $\eta$ such that $P, Q, R$ are collinear and $2019 \cdot \overline{PQ} = \overline{QR}$. Given that no two sides of $\zeta$ and $\eta$ are parallel, what is the maximum possible number of projective points on $\zeta$?

## 1.3 Team Test

*There are 15 problems, worth 20 points each, to be solved in 60 minutes.*

1. Three positive integers sum to 16. What is the least possible value of the sum of their squares?

2. Ben is thinking of an odd positive integer less than 1000. Ben subtracts 1 from his number and divides by 2, resulting in another number. If his number is still odd, Ben repeats this procedure until he gets an even number. Given that the number he ends on is 2, how many possible values are there for Ben's original number?

3. Triangle $ABC$ is isosceles, with $AB = BC = 18$ and has circumcircle $\omega$. Tangents to $\omega$ at $A$ and $B$ intersect at point $D$. If $AD = 27$, what is the length of $AC$?

4. How many non-decreasing sequences of five natural numbers have first term 1, last term 11, and have no three terms equal?

5. Adam is bored, and has written the string "EMCC" on a piece of paper. For fun, he decides to erase every letter "C", and replace it with another instance of "EMCC". For example, after one step, he will have the string "EMEMCCEMCC". How long will his string be after 8 of these steps?

6. Eric has two coins, which land heads 40% and 60% of the time respectively. He chooses a coin randomly and flips it four times. Given that the first three flips contained two heads and one tail, what is the probability that the last flip was heads?

7. In a five person rock-paper-scissors tournament, each player plays against every other player exactly once, with each game continuing until one player wins. After each game, the winner gets 1 point, while the loser gets no points. Given that each player has a 50% chance of defeating any other player, what is the probability that no two players end up with the same amount of points?

8. Let $\triangle ABC$ have $\angle A = \angle B = 75^\circ$. Points $D$, $E$, and $F$ are on sides $BC$, $CA$, and $AB$, respectively, so that $EF$ is parallel to $BC$, $EF \perp DE$, and $DE = EF$. Find the ratio of $\triangle DEF$'s area to $\triangle ABC$'s area.

9. Suppose $a$, $b$, $c$ are positive integers such that $a+b = \sqrt{c^2+336}$ and $a-b = \sqrt{c^2-336}$. Find $a+b+c$.

10. How many times on a 12-hour analog clock are there, such that when the minute and hour hands are swapped, the result is still a valid time? (Note that the minute and hour hands move continuously, and don't always necessarily point to exact minute/hour marks.)

11. Adam owns a square $S$ with side length 42. First, he places rectangle $A$, which is 6 times as long as it is wide, inside the square, so that all four vertices of $A$ lie on sides of $S$, but none of the sides of $A$ are parallel to any side of $S$. He then places another rectangle $B$, which is 7 times as long as it is wide, inside rectangle $A$, so that all four vertices of $B$ lie on sides of $A$, and again none of the sides of $B$ are parallel to any side of $A$. Find the length of the shortest side of rectangle $B$.

12. Find the value of $\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$, where the exponents are the odd natural numbers, in increasing order.

13. Jamesu and Fhomas challenge each other to a game of Square Dance, played on a $9 \times 9$ square grid. On Jamesu's turn, he colors in a $2 \times 2$ square of uncolored cells pink. On Fhomas's turn, he colors in a $1 \times 1$ square of uncolored cells purple. Once Jamesu can no longer make a move, Fhomas gets to color

in the rest of the cells purple. If Jamesu goes first, what the maximum number of cells that Fhomas can color purple, assuming both players play optimally in trying to maximize the number of squares of their color?

14. Triangle $ABC$ is inscribed in circle $\omega$. The tangents to $\omega$ from $B$ and $C$ meet at $D$, and segments $AD$ and $BC$ intersect at $E$. If $\angle BAC = 60^\circ$ and the area of $\triangle BDE$ is twice the area of $\triangle CDE$, what is $\frac{AB}{AC}$?
*Addendum on 1/26/19: $\angle A$ changed to $\angle BAC$.*

15. Fhomas and Jamesu are now having a number duel. First, Fhomas chooses a natural number $n$. Then, starting with Jamesu, each of them take turns making the following moves: if $n$ is composite, the player can pick any prime divisor $p$ of $n$, and replace $n$ by $n - p$; if $n$ is prime, the player can replace $n$ by $n - 1$. The player who is faced with 1, and hence unable to make a move, loses. How many different numbers $2 \le n \le 2019$ can Fhomas choose such that he has a winning strategy, assuming Jamesu plays optimally?

## 1.4 Guts Test

*There are 24 problems, with varying point values (indicated in square brackets), to be solved in 75 minutes.*

### 1.4.1 Round 1

1. [6] What is the smallest number equal to its cube?

2. [6] Fhomas has 5 red spaghetti and 5 blue spaghetti, where spaghetti are indistinguishable except for color. In how many different ways can Fhomas eat 6 spaghetti, one after the other? (Two ways are considered the same if the sequence of colors are identical)

3. [6] Jocelyn labels the three corners of a triangle with three consecutive natural numbers. She then labels each edge with the sum of the two numbers on the vertices it touches, and labels the center with the sum of all three edges. If the total sum of all labels on her triangle is 120, what is the value of the smallest label?

### 1.4.2 Round 2

4. [7] Adam cooks a pie in the shape of a regular hexagon with side length 12, and wants to cut it into right triangular pieces with angles 30°, 60°, and 90°, each with shortest side 3. What is the maximum number of such pieces he can make?

5. [7] If $f(x) = \frac{1}{2-x}$ and $g(x) = 1 - \frac{1}{x}$, what is the value of $f(g(f(g(\cdots f(g(f(2019)))\cdots))))$, where there are 2019 functions total, counting both $f$ and $g$?

6. [7] Fhomas is buying spaghetti again, which is only sold in two types of boxes: a 200 gram box and a 500 gram box, each with a fixed price. If Fhomas wants to buy exactly 800 grams, he must spend \$8.80, but if he wants to buy exactly 900 grams, he only needs to spend \$7.90! In dollars, how much more does the 500 gram box cost than the 200 gram box?

### 1.4.3 Round 3

7. [9] Given that

$$\begin{cases} a + 5b + 9c = 1, \\ 4a + 2b + 3c = 2, \\ 7a + 8b + 6c = 9, \end{cases}$$

what is $741a + 825b + 639c$?

8. [9] Hexagon $JAMESU$ has line of symmetry $MU$ (i.e., quadrilaterals $JAMU$ and $SEMU$ are reflections of each other), and $JA = AM = ME = ES = 1$. If all angles of $JAMESU$ are 135 degrees except for right angles at $A$ and $E$, find the length of side $US$.

9. [9] Max is parked at the 11 mile mark on a highway, when his pet cheetah, Min, leaps out of the car and starts running up the highway at its maximum speed. At the same time, Max starts his car and starts driving down the highway at $\frac{1}{2}$ his maximum speed, driving all the way to the 10 mile mark before realizing that his cheetah is gone! Max then immediately reverses directions and starts driving back up the highway at his maximum speed, finally catching up to Min at the 20 mile mark. What is the ratio between Max's max speed and Min's max speed?

### 1.4.4 Round 4

10. [11] Kevin owns three non-adjacent square plots of land, each with side length an integer number of meters, whose total area is 2019m$^2$. What is the minimum sum of the perimeters of his three plots, in meters?

11. [11] Given a $5 \times 5$ array of lattice points, how many squares are there with vertices all lying on these points?

12. [11] Let right triangle $ABC$ have $\angle A = 90°$, $AB = 6$, and $AC = 8$. Let points $D, E$ be on side $AC$ such that $AD = EC = 2$, and let points $F, G$ be on side $BC$ such that $BF = FG = 3$. Find the area of quadrilateral $FGED$.

### 1.4.5 Round 5

13. [13] Given a (not necessarily simplified) fraction $\frac{m}{n}$, where $m, n > 6$ are positive integers, when 6 is subtracted from both the numerator and denominator, the resulting fraction is equal to $\frac{4}{5}$ of the original fraction. How many possible ordered pairs $(m, n)$ are there?

14. [13] Jamesu's favorite anime show has 3 seasons, with 12 episodes each. For 8 days, Jamesu does the following: on the $n^{\text{th}}$ day, he chooses $n$ consecutive episodes of exactly one season, and watches them in order. How many ways are there for Jamesu to finish all 3 seasons by the end of these 8 days? (For example, on the first day, he could watch episode 5 of the first season; on the second day, he could watch episodes 11 and 12 of the third season, etc.)

15. [13] Let $O$ be the center of regular octagon $ABCDEFGH$ with side length 6. Let the altitude from $O$ meet side $AB$ at $M$, and let $BH$ meet $OM$ at $K$. Find the value of $BH \cdot BK$.

### 1.4.6 Round 6

16. [15] Fhomas writes the ordered pair $(2, 4)$ on a chalkboard. Every minute, he erases the two numbers $(a, b)$, and replaces them with the pair $(a^2 + b^2, 2ab)$. What is the largest number on the board after 10 minutes have passed?

17. [15] Triangle $BAC$ has a right angle at $A$. Point $M$ is the midpoint of $BC$, and $P$ is the midpoint of $BM$. Point $D$ is the point where the angle bisector of $\angle BAC$ meets $BC$. If $\angle BPA = 90^\circ$, what is $\frac{PD}{DM}$?

18. [15] A square is called *legendary* if there exist two different positive integers $a, b$ such that the square can be tiled by an equal number of non-overlapping $a$ by $a$ squares and $b$ by $b$ squares. What is the smallest positive integer $n$ such that an $n$ by $n$ square is *legendary*?

### 1.4.7 Round 7

19. [18] Let $S(n)$ be the sum of the digits of a positive integer $n$. Let $a_1 = 2019!$, and $a_n = S(a_{n-1})$. Given that $a_3$ is even, find the smallest integer $n \geq 2$ such that $a_n = a_{n-1}$.

20. [18] The local EMCC bakery sells one cookie for $p$ dollars ($p$ is not necessarily an integer), but has a special offer, where any non-zero purchase of cookies will come with one additional free cookie. With \$27.50, Max is able to buy a whole number of cookies (including the free cookie) with a single purchase and no change leftover. If the price of each cookie were 3 dollars lower, however, he would be able to buy double the number of cookies as before in a single purchase (again counting the free cookie) with no change leftover. What is the value of $p$?

21. [18] Let circle $\omega$ be inscribed in rhombus $ABCD$, with $\angle ABC < 90^\circ$. Let the midpoint of side $AB$ be labeled $M$, and let $\omega$ be tangent to side $AB$ at $E$. Let the line tangent to $\omega$ passing through $M$ other than line $AB$ intersect segment $BC$ at $F$. If $AE = 3$ and $BE = 12$, what is the area of $\triangle MFB$?

### 1.4.8 Round 8

22. [21] Find the remainder when $1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1!$ is divided by 2019.

23. [21] Two circles $\omega_1$ and $\omega_2$ have radii 1 and 2, respectively and are externally tangent to one another. Circle $\omega_3$ is externally tangent to both $\omega_1$ and $\omega_2$. Let $M$ be the common external tangent of $\omega_1$ and $\omega_3$ that doesn't intersect $\omega_2$. Similarly, let $N$ be the common external tangent of $\omega_2$ and $\omega_3$ that doesn't intersect $\omega_1$. Given that $M$ and $N$ are parallel, find the radius of $\omega_3$.

24. [21] Mana is standing in the plane at $(0, 0)$, and wants to go to the EMCCiffel Tower at $(6, 6)$. At any point in time, Mana can attempt to move 1 unit to an adjacent lattice point, or to make a knight's move, moving diagonally to a lattice point $\sqrt{5}$ units away. However, Mana is deathly afraid of negative numbers, so she will make sure never to decrease her $x$ or $y$ values. How many distinct paths can Mana take to her destination?

## 1.5 Practice Sets

*We have arranged this year's 69 EMCC problems into 12 sets, in approximately increasing order of difficulty. This sorting is based on solve rate during the contest, among other measures. The first two sets consist of 10 problems each, and the others consist of 5 problems each (except for Set 12). The goal of these Practice Sets is to provide a repository that coaches can use for training at various skill levels.*

Coming soon!

# Chapter 2

# EMC$^2$ 2019 Solutions

## 2.1 Speed Test Solutions

1. Given that $a + 19b = 3$ and $a + 1019b = 5$, what is $a + 2019b$?

   **Solution.** The answer is $\boxed{7}$.

   Note that the numbers $a+19b$, $a+1019b$, and $a+2019b$ are in arithmetic sequence, because they differ by $1000b$ each. Since we know the first two terms are 3 and 5, the third term must be 7.

2. How many multiples of 3 are there between 2019 and 2119, inclusive?

   **Solution.** The answer is $\boxed{34}$.

   Since 2019 is a multiple of 3, and our range has width 100, our multiples of 3 can be written as $2019, 2019 + 3, \ldots, 2019 + 99$, so there are a total of $\frac{99}{3} + 1 = 34$ multiples of 3.

3. What is the maximum number of quadrilaterals a 12-sided regular polygon can be *quadrangulated* into? Here *quadrangulate* means to cut the polygon along lines from vertex to vertex, none of which intersect inside the polygon, to form pieces which all have exactly 4 sides.

   **Solution.** The answer is $\boxed{5}$.

   When we cut a quadrilateral off of our polygon, three of the quadrilateral's sides originally belonged to the polygon, while the fourth side was created by our cut. The polygon itself therefore loses three sides to the quadrilateral, and gains one from the cut made, losing two sides overall. We can continue cutting off quadrilaterals until the original polygon only has 4 sides left, hence becoming a quadrilateral itself, at which point we are done. Starting from 12 sides, we can therefore cut 4 times before the original polygon is left with $12 - 2 \cdot 4 = 4$ sides. These 4 additional quadrilaterals, plus the remnant of the original polygon, give 5 total quadrilaterals.

4. What is the value of $|2\pi - 7| + |2\pi - 6|$, rounded to the nearest hundredth?

   **Solution.** The answer is $\boxed{1.00}$.

   Since $6 < 2\pi < 7$, we have $|2\pi - 7| + |2\pi - 6| = (7 - 2\pi) + (2\pi - 6) = 1$.

5. In the town of EMCCxeter, there is a 30% chance that it will snow on Saturday, and independently, a 40% chance that it will snow on Sunday. What is the probability that it snows exactly once that weekend, as a percentage?

   **Solution.** The answer is $\boxed{46\%}$.

   The probability that it snows on Saturday but not Sunday is $0.3 \cdot 0.6 = 0.18$, and the probability that it snows on Sunday but not Saturday is $0.4 \cdot 0.7 = 0.28$, so the total probability is $18\% + 28\% = 46\%$.

6. Define $n!$ to be the product of all integers between 1 and $n$ inclusive. Compute $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$.

   **Solution.** The answer is $\boxed{\frac{2019}{2017}}$.

   Notice that the product $\frac{2019!}{2017!} \times \frac{2016!}{2018!}$ can be rewritten as $\frac{2019!}{2018!} \times \frac{2016!}{2017!}$. Now, we see that the first fraction is equal to 2019 while the second is $\frac{1}{2017}$, making our final answer is $\frac{2019}{2017}$.

7. There are 2019 people standing in a row, and they are given positions $1, 2, 3, \ldots 2019$ from left to right. Next, everyone in an odd position simultaneously leaves the row, and the remaining people are assigned new positions from 1 to 1009, again from left to right. This process is then repeated until one person remains. What was this person's original position?

   **Solution.** The answer is $\boxed{1024}$.

   This process is equivalent to repeatedly removing people with odd positions, and dividing the positions of the remaining people by 2. The last person remaining, therefore, must have started with the highest number of 2's in their prime factorization, meaning their position was $2^{10} = 1024$.

8. The product $1234 \times 4321$ contains exactly one digit not in the set $\{1, 2, 3, 4\}$. What is this digit?

   **Solution.** The answer is $\boxed{5}$.

   By approximation, we can quickly check that both $13 \times 44$ and $12 \times 43$ begin with the digit 5. Therefore, the first digit of our product (which is simply a scaled up version of $12.34 \times 43.21$) must also be 5. Because this is not in the set $\{1, 2, 3, 4\}$, it is our answer. (The actual product is 5332114.)

9. A quadrilateral with positive area has four integer side lengths, with shortest side 1 and longest side 9. How many possible perimeters can this quadrilateral have?

   **Solution.** The answer is $\boxed{10}$.

   We can consider an analog of the triangle inequality, but for quadrilaterals: clearly no one side can be longer than, or equal to (if we want positive area), the sum of the other 3 sides. If the sum of our two unknown sides is $s$, this gives us $1 + s > 9$, so $s \geq 9$. Since both unknown sides are at most 9, however, we also know that $s \leq 18$, giving us 10 possible perimeters, $1 + 9 + s$ for each $9 \leq s \leq 18$.

10. Define $s(n)$ to be the sum of the digits of $n$ when expressed in base 10, and let $\gamma(n)$ be the sum of $s(d)$ over all natural number divisors $d$ of $n$. For instance, $n = 11$ has two divisors, 1 and 11, so $\gamma(11) = s(1) + s(11) = 1 + (1 + 1) = 3$. Find the value of $\gamma(2019)$.

    **Solution.** The answer is $\boxed{32}$.

    Since $2019 = 3 \cdot 673$, we compute

    $$\gamma(2019) = s(2019) + s(673) + s(3) + s(1) = 12 + 16 + 3 + 1 = 32.$$

11. How many five-digit positive integers are divisible by 9 and have 3 as the tens digit?

    **Solution.** The answer is $\boxed{1000}$.

    If we remove the tens digit of 3, we have that the resulting four-digit integer must leave a remainder of 6 when divided by 9 (since the remainder of the integer when divided by 9 is equal to the remainder of the sum of the digits when divided by 9). Since there are 9000 four-digit integers, there must be $\frac{9000}{9} = 1000$ integers with remainder 6 when divided by 9, and hence 1000 five-digit integers that satisfy the original condition.

12. Adam owns a large rectangular block of cheese, that has a square base of side length 15 inches, and a height of 4 inches. He wants to remove a cylindrical cheese chunk of height 4, by making a circular hole that goes through the top and bottom faces, but he wants the surface area of the leftover cheese block to be the same as before. What should the diameter of his hole be, in inches?

    *Addendum on 1/26/19: the hole must have non-zero diameter.*

    **Solution.** The answer is $\boxed{8}$.

    Suppose that the radius of the hole is $r$ inches. Then, the surface area lost from the top and bottom faces is $2 \cdot \pi r^2$ square inches, and the surface area gained from the wall of the cylindrical hole is $4 \cdot 2\pi r = 8\pi r$. Setting these equal gives $2\pi r^2 = 8\pi r$, or $r = 4$ (or $r = 0$, but that would mean that the hole is non-existent), so the diameter is $2r = 8$.

13. Find the smallest prime that does not divide $20! + 19! + 2019!$.

    **Solution.** The answer is $\boxed{23}$.

    Any prime $p \leq 19$ must divide $20! + 19! + 2019$, because $p$ divides each of $20!$, $19!$, and $2019!$. Hence the smallest prime not dividing $20! + 19! + 2019!$ must be at least 23. Now, note that 23 divides $2019!$, but $20! + 19! = 19! \times (20 + 1) = 19! \times 21$ is not a multiple of 23, so 23 does not divide the expression, and is the answer.

14. Convex pentagon $ABCDE$ has angles $\angle ABC = \angle BCD = \angle DEA = \angle EAB$ and angle $\angle CDE = 60°$. Given that $BC = 3$, $CD = 4$, and $DE = 5$, find $EA$.

    *Addendum on 1/26/19: $ABCDE$ is specified to be convex.*

    **Solution.** The answer is $\boxed{2}$.

    Since the sum of the five interior angles is $180° \cdot (5 - 2) = 540°$, the four equal angles must have measure $\frac{540-60}{4} = 120$ degrees. If we extend $BA$ and $DE$ to intersect at $F$, then $AEF$ is an equilateral triangle, and moreover $BCDF$ is an isosceles trapezoid. Also, it is not difficult to see that $BCDF$ can be cut into an equilateral triangle with side length 4 and a parallelogram with two sides 3 and 4, so $DF = 3 + 4 = 7$, and thus $AE = EF = DF - DE = 7 - 5 = 2$.

15. Sophia has 3 pairs of red socks, 4 pairs of blue socks, and 5 pairs of green socks. She picks out two individual socks at random; what is the probability she gets a pair with matching color?

    **Solution.** The answer is $\boxed{\frac{22}{69}}$.

    There are $2(3 + 4 + 5) = 24$ socks in total, so there are $\binom{24}{2} = 276$ possible pairs. Out of these pairs, there are $\binom{6}{2} + \binom{8}{2} + \binom{10}{2} = 88$ pairs that have matching colors, making the probability $\frac{88}{276} = \frac{22}{69}$.

16. How many real roots does the function $f(x) = 2019^x - 2019x - 2019$ have?

    **Solution.** The answer is $\boxed{2}$.

    We note that $f(-1) = \frac{1}{2019}, f(0) = -2018, f(1) = -2019, f(2) = 2019 \cdot 2016$, so by continuity there must be a root between $-1$ and 0 and a root between 1 and 2. We now show that there are at most two solutions. This is because the function $2019^x$ is *convex*, meaning that it increases at an increasing rate, and a line can intersect a convex function at most twice.

17. A 30−60−90 triangle is placed on a coordinate plane with its short leg of length 6 along the $x$-axis, and its long leg along the $y$-axis. It is then rotated 90 degrees counterclockwise, so that the short leg now lies along the $y$-axis and long leg along the $x$-axis. What is the total area swept out by the triangle during this rotation?

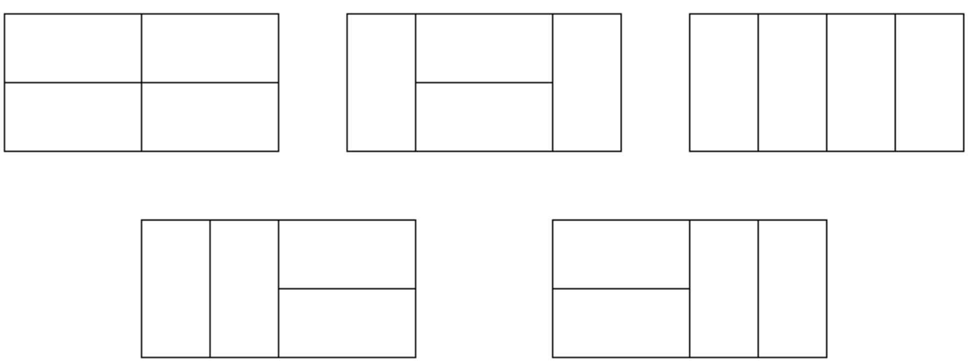
**Solution.** The answer is $\boxed{33\pi + 9\sqrt{3}}$.

Let $O$ be the origin of the coordinate plane, $OA = 6$ be the shorter leg, and $OB = 6\sqrt{3}$ be the longer leg. It is not difficult to see that the rotation must be centered around the origin. If we assume that the triangle starts in the first quadrant, then, during the rotation, everything swept in the second quadrant will be swept by $OB$, and hence the area swept in the second quadrant is a quarter circle with radius $6\sqrt{3}$, and area $\frac{\pi(6\sqrt{3})^2}{4} = 27\pi$. In the first quadrant, the area swept out is covered by both a quarter circle with radius 6, and the original triangle $OAB$. We can find the combined area by noticing that the triangle and quarter circle must intersect at $C$, the midpoint of $AB$, which means that the area consists of the $\frac{1}{6}$ circular sector $OAC$ with area $\frac{\pi(6)^2}{6} = 6\pi$, and the triangle $OBC$ with area equal to half the original area of $OAB$, or $\frac{1}{2} \cdot \frac{6 \cdot 6\sqrt{3}}{2} = 9\sqrt{3}$. Adding all three parts up, we get the total area is $27\pi + 6\pi + 9\sqrt{3} = 33\pi + 9\sqrt{3}$.

18. Find the number of ways to color the unit cells of a $2 \times 4$ grid in four colors such that all four colors are used and every cell shares an edge with another cell of the same color.

**Solution.** The answer is $\boxed{120}$.

Since each cell shares an edge with another cell of the same color, there must be at least two cells of each color, but since there are only 8 cells in total and all colors are used, there must be *exactly* two cells of each color, making a "domino" shape. There are 5 ways to partition the grid into four dominoes:



and there are $4! = 24$ ways to color the dominoes, so there are $5 \cdot 24 = 120$ colorings in total.

19. Triangle $\triangle ABC$ has centroid $G$, and $X, Y, Z$ are the centroids of triangles $\triangle BCG$, $\triangle ACG$, and $\triangle ABG$, respectively. Furthermore, for some points $D, E, F$, vertices $A, B, C$ are themselves the centroids of triangles $\triangle DBC$, $\triangle ECA$, and $\triangle FAB$, respectively. If the area of $\triangle XYZ = 7$, what is the area of $\triangle DEF$?

**Solution.** The answer is $\boxed{1008}$.

It is a well known fact that, in triangle $ABC$, a given vertex $A$ is 3 times as far from base $BC$ as the centroid $G$ is. We will utilize this fact to find our answer. Let $M$ be the midpoint of $BC$, and suppose that $AM = d$. Then, $GM$ must be $\frac{d}{3}$, by our fact, and likewise $XM$ must be $\frac{d}{9}$, so $XG = GM - XM = \frac{2d}{9}$. We also find that $DM = 3d$, so $DG = DM - GM = \frac{8}{3}d$. We also know that the centroid of a triangle lies along its median, i.e $A$ lies on $DM$, $G$ lies on $AM$, and $X$ lies on $GM$, thus $D, G, X$ are collinear. This, combined with the fact that $DG = \frac{8}{3}d = 12 \cdot \frac{2}{9}d = 12XG$ (and indeed $EG = 12YG$, $FG = 12ZG$, since we could have picked any midpoint to start our calculation), means that $\triangle DEF$ is $\triangle XYZ$ scaled around $G$ by a factor of $-12$ (here the negative refers to the fact that each points ends up on the opposite side of $G$ from where it started), so the area of $\triangle DEF$ is $12^2 \cdot 7 = 1008$.

20. Fhomas orders three 2-gallon jugs of milk from EMCCBay for his breakfast omelette. However, every jug is actually shipped with a random amount of milk (not necessarily an integer), uniformly distributed between 0 and 2 gallons. If Fhomas needs 2 gallons of milk for his breakfast omelette, what is the probability he will receive enough milk?

**Solution.** The answer is $\boxed{\frac{5}{6}}$.

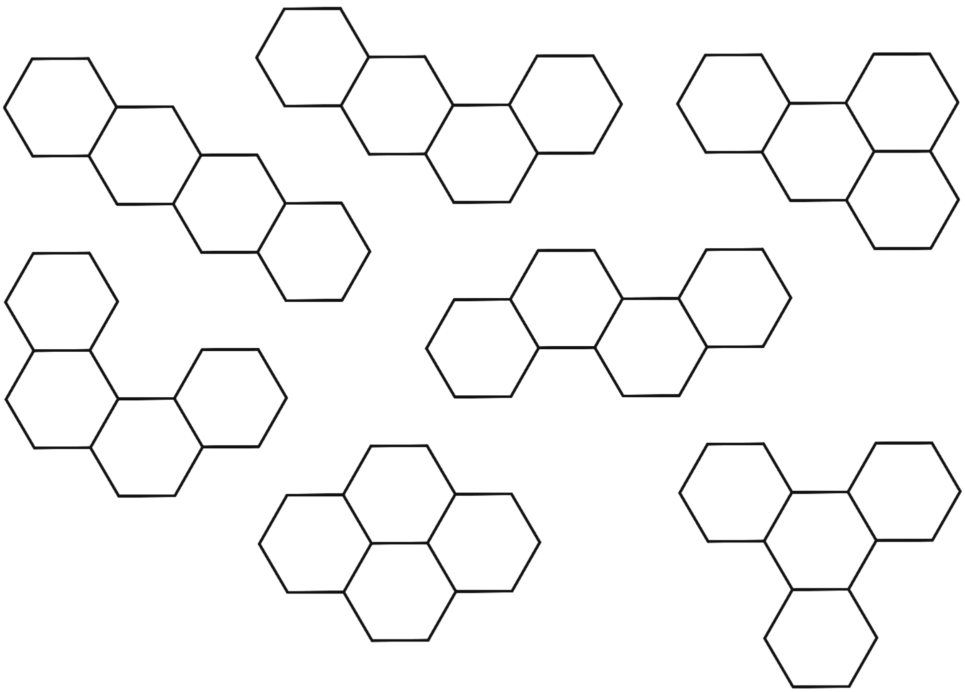
We need to compute the probability of $x + y + z \geq 2$ when $x, y, z$ are real numbers uniformly chosen between 0 and 2. If we visualize this in a cube, then we are picking a random point in a cube with volume $2^3 = 8$, and avoiding points in the region $x+y+z < 2$. Note that the plane $x+y+z = 2$ slices through the three vertices of the cube next to the origin, so the volume that it slices off is the volume of a tetrahedron, which is $\frac{1}{6} \cdot 2 \cdot 2 \cdot 2 = \frac{4}{3}$. Therefore, the probability that $x + y + z \geq 2$ is $\frac{8-\frac{4}{3}}{8} = \frac{5}{6}$.

## 2.2 Accuracy Test Solutions

1. A shape made by joining four identical regular hexagons side-to-side is called a *hexo*. Two hexos are considered the same if one can be rotated/reflected to match the other. Find the number of different hexos.

   **Solution.** The answer is $\boxed{7}$.

   The seven possible ways are shown below:



2. The sequence $1, 2, 3, 3, 3, 4, 5, 5, 5, 5, 5, 6, \ldots$ consists of numbers written in increasing order, where every even number $2n$ is written once, and every odd number $2n+1$ is written $2n+1$ times. What is the $2019^{\text{th}}$ term of this sequence?

   **Solution.** The answer is $\boxed{89}$.

   Note that numbers $2n-1$ and $2n$ are written $(2n-1)+1 = 2n$ times in total, so the first $2n$ numbers are written $2+4+6+\cdots+2n = \frac{n(2n+2)}{2} = n(n+1)$ times in total, and the largest $n$ such that $n(n+1) \leq 2019$ is $n = 44$. Therefore, because the first 88 numbers are written $44 \cdot 45 = 1980$ times in total, and 90 will be written as the $45 \cdot 46 = 2070$th number, the 2019th term must be 89.

3. On planet EMCCarth, months can only have lengths of 35, 36, or 42 days, and there is at least one month of each length. Victor knows that an EMCCarth year has $n$ days, but realizes that he cannot figure out how many months there are in an EMCCarth year. What is the least possible value of $n$?

**Solution.** The answer is $\boxed{323}$.

In order for Victor to be unable to figure out the number of months, there must be two combinations having a different number of total months, but the same number of days. Now, we can consider two cases: either these combinations have a different number of 35 day months, or they have the same number. In the second case, we can only swap between 36 and 42 day months, so the least number we need is $\text{lcm}(36, 42) = 252$, and when added to $35 + 36 + 42 = 113$ (because we need at least one of each month) gives us a total of 365 days. In the first case, however, note that both of the other month lengths are multiples of 6, and therefore if one combination uses 35 day months, it must use at least 6 at a time; the least such swap is $6 \cdot 35 = 5 \cdot 42$, giving us $210 + 113 = 323$ days. Since $323 < 365$, the first case is the optimal one, giving us our answer of 323 days.

4. In triangle $ABC$, $AB = 5$ and $AC = 9$. If a circle centered at $A$ passing through $B$ intersects $BC$ again at $D$ and $CD = 7$, what is $BC$?

   **Solution.** The answer is $\boxed{8}$.

   Let $M$ be the midpoint of $BD$, then since $ABD$ is an isosceles triangle, $AM$ is perpendicular to $BD$. Suppose that $DM = x$ and $AM = h$, then using Pythagorean theorem on $AMD$ and $AMC$ we have $x^2 + h^2 = 5^2$ and $(x + 7)^2 + h^2 = 9^2$. Subtracting the first equation from the second gives $14x + 49 = 81 - 25 = 56 \Rightarrow x = \frac{1}{2}$, so $BC = CD + BD = 7 + 2x = 8$.

   Alternatively, it is well known that the power of a point is magnitude of the difference between the square of its distance from the center of the circle and the square of the radius of the circle. Therefore, the power of $C$ is $AC^2 - AB^2 = 9^2 - 5^2 = 56$. Since the power of $C$ is also equal to $CD \cdot CB = 7CB$, $CB$ must be 8.

5. How many nonempty subsets $S$ of the set $\{1, 2, 3, \ldots, 11, 12\}$ are there such that the greatest common factor of all elements in $S$ is greater than 1?

   **Solution.** The answer is $\boxed{79}$.

   We use the Principle of Inclusion-Exclusion:

   (a) If all elements are divisible by 2, then $S$ is a subset of $\{2, 4, 6, 8, 10, 12\}$, which gives $2^6 - 1 = 63$ possibilities.
   (b) If all elements are divisible by 3, then $S$ is a subset of $\{3, 6, 9, 12\}$, which gives $2^4 - 1 = 15$ possibilities.
   (c) If all elements are divisible by 5, then $S$ is a subset of $\{5, 10\}$, which gives $2^2 - 1 = 3$ possibilities.
   (d) If all elements are divisible by 7 or 11, then $S = \{7\}$ or $S = \{11\}$, which gives 2 possibilities.
   (e) If all elements are divisible by 6, then $S$ is a subset of $\{6, 12\}$, which gives $2^2 - 1 = 3$ possibilities. *These are double-counted by cases (a) and (b).*
   (f) If all elements are divisible by 10, then $S = \{10\}$, which gives 1 possibility. *These are also double-counted by cases (a) and (c).*

   Hence, in total there are $63 + 15 + 3 + 2 - 3 - 1 = 79$ possible subsets.

6. Jasmine rolls a fair 6-sided die, with faces labeled from 1 to 6, and a fair 20-sided die, with faces labeled from 1 to 20. What is the probability that the product of these two rolls, added to the sum of these two rolls, is a multiple of 3?

**Solution.** The answer is $\boxed{\frac{13}{60}}$.

Suppose that the 6-sided die rolls $a$ and the 20-sided die rolls $b$, then we need $ab + a + b$ to be a multiple of 3, or $ab + a + b + 1 = (a + 1)(b + 1)$ to be one more than a multiple of 3. This happens in two ways:

(a) $a + 1$ and $b + 1$ are both one more than a multiple of 3, so $a \in \{3, 6\}$ and $b \in \{3, 6, 9, 12, 15, 18\}$, which gives $2 \cdot 6 = 12$ ways.

(b) $a+1$ and $b+1$ are both two more than a multiple of 3, so $a \in \{1, 4\}$ and $b \in \{1, 4, 7, 10, 13, 16, 19\}$, which gives $2 \cdot 7 = 14$ ways.

Since there are $6 \cdot 20 = 120$ possible outcomes, the probability is $\frac{12+14}{120} = \frac{13}{60}$.

7. Let $\{a_n\}$ be a sequence such that $a_n$ is either $2a_{n-1}$ or $a_{n-1} - 1$. Given that $a_1 = 1$ and $a_{12} = 120$, how many possible sequences $a_1, a_2, \ldots, a_{12}$ are there?

**Solution.** The answer is $\boxed{4}$.

We will consider all such sequences constructed via $-1$ and *doubling* moves, i.e. following $a_n$ by $a_n - 1$ or $2a_n$, respectively. An important observation to make is that, if at some point there is a $-1$ move, and there are a total of $k$ doubling moves afterwards, this $-1$ will end up having a net effect of subtracting $2^k$. In other words, if among our 11 moves from $a_1$ to $a_{12}$, there are $k$ of the $-1$ moves, and $11 - k$ of the doubling moves, then our final result will be $k$ powers of 2 subtracted off of $2^{11-k}$. As our final result must be 120, we need $2^{11-k} \geq 128$, so $k$ is at most 4.

Now, we perform some brief case checking: If $k = 4$, we need to subtract four powers of 2 from $2^{11-4} = 128$ to get 120, or in other words, we want four powers of 2 that sum to 8. Note that the ordering of our subtracted powers of 2 is necessarily from greatest to least, as we move from $a_1$ to $a_{12}$, since it is impossible for an earlier $-1$ move to be followed by fewer doubling moves than a $-1$ move occurring later in the sequence. Thus there are two distinct possibilities in this case: $(2^2, 2^1, 2^0, 2^0)$ and $(2^1, 2^1, 2^1, 2^1)$.

If $k = 3$, we need three powers of 2 summing to $2^{11-3} - 120 = 136$. Here again we can find two solutions: $(2^6, 2^6, 2^3)$ and $(2^7, 2^2, 2^2)$.

If $k = 2$, we will need two powers of 2 summing to $2^{11-2} - 120 = 392$; if we write out $392 = 2^8 + 2^7 + 2^3$ in binary, however, it is clear we cannot accomplish this with only two powers of 2. Likewise, $k = 1, 0$ are similarly impossible, due to the lack of sufficient powers of 2 to subtract off.

In conclusion, we found two cases for $k = 4$ and two for $k = 3$, giving 4 sequences in total. If anyone is curious, these sequences are:

$$1, 2, 4, 8, 16, 32, 31, 62, 61, 122, 121, 120$$

$$1, 2, 4, 8, 16, 32, 64, 63, 62, 61, 60, 120$$

$$1, 2, 4, 3, 2, 4, 8, 16, 15, 30, 60, 120$$

$$1, 2, 1, 2, 4, 8, 16, 32, 31, 30, 60, 120$$

8. A tetrahedron has two opposite edges of length 2 and the remaining edges have length 10. What is the volume of this tetrahedron?

**Solution.** The answer is $\boxed{\frac{14\sqrt{2}}{3}}$.

Let $ABCD$ be the tetrahedron, with $AB = 2$. Let $M$ be the midpoint of side $AB$, and $N$ be the midpoint of $CD$. Then, we can split tetrahedron $ABCD$ into triangular pyramids $ACDM$ and $BCDM$, each of which have height 1, and base triangle $MCD$. Since $AMC$ is a right triangle, we have $MC = \sqrt{AC^2 - AM^2} = \sqrt{99}$. Furthermore, $MCN$ is a right triangle, so $MN = \sqrt{MC^2 - CN^2} = \sqrt{98} = 7\sqrt{2}$. The area of triangle $MCD$ is therefore $\frac{1}{2}MN \cdot CD = 7\sqrt{2}$. Therefore, the volumes of $ACDM$ and $BCDM$ are each $\frac{1}{3} \cdot 7\sqrt{2} \cdot 1$, so the total volume is $\frac{14\sqrt{2}}{3}$.

9. In the garden of EMCCden, there is a tree planted at every lattice point $-10 \leq x, y \leq 10$ except the origin. We say that a tree is *visible* to an observer if the line between the tree and the observer does not intersect any other tree (assume that all trees have negligible thickness). What fraction of all the trees in the garden of EMCCden are visible to an observer standing at the origin?

**Solution.** The answer is $\boxed{\frac{32}{55}}$.

There are $21^2 - 1 = 440$ trees in the garden. We divide the garden by the four lines $x = 0, y = 0, x = y, x = -y$ into eight symmetric sections. The observer can see one tree on each of the eight directions along the four lines, and it suffices to count the number of trees visible in each section. We consider the section $\{(x, y) \mid 1 \leq x < y \leq 10\}$. It is not difficult to see that a tree is visible if and only if $\gcd(x, y) = 1$, so we count row by row in increasing order of $x$:

- $x = 1$: $y = 2, 3, \ldots, 10$, for 9 trees.
- $x = 2$: $y = 3, 5, 7, 9$, for 4 trees.
- $x = 3$: $y = 4, 5, 7, 8, 10$, for 5 trees.
- $x = 4$: $y = 5, 7, 9$, for 3 trees.
- $x = 5$: $y = 6, 7, 8, 9$, for 4 trees.
- $x = 6$: $y = 7$, for 1 tree.
- $x = 7$: $y = 8, 9, 10$, for 3 trees.
- $x = 8$: $y = 9$, for 1 tree.
- $x = 9$: $y = 10$, for 1 tree.

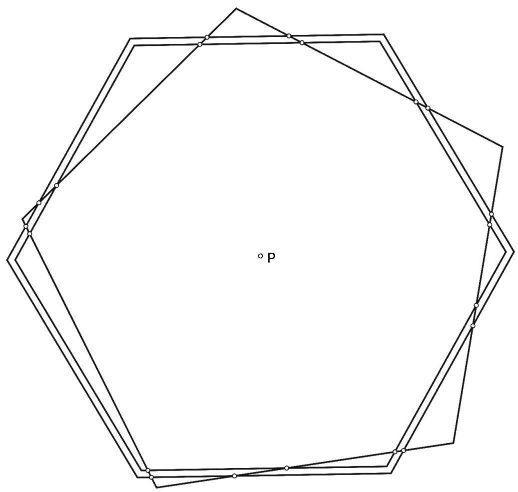
Hence one can see $9+4+5+3+4+1+3+1+1 = 31$ trees in the sector, so one can see $8 \cdot 31 + 8 = 256$ trees in total, and the fraction is therefore $\frac{256}{440} = \frac{32}{55}$.

10. Point $P$ lies inside regular pentagon $\zeta$, which lies entirely within regular hexagon $\eta$. A point $Q$ on the boundary of pentagon $\zeta$ is called *projective* if there exists a point $R$ on the boundary of hexagon $\eta$ such that $P, Q, R$ are collinear and $2019 \cdot \overline{PQ} = \overline{QR}$. Given that no two sides of $\zeta$ and $\eta$ are parallel, what is the maximum possible number of projective points on $\zeta$?

**Solution.** The answer is $\boxed{20}$.

We construct two dilations of $\eta$: $\eta'$ is $\eta$ dilated about $P$ with a ratio of $\frac{1}{2020}$, and $\eta''$ is $\eta$ dilated about $P$ with a ratio of $-\frac{1}{2018}$ ($\eta''$ will be slightly larger than $\eta'$). If $Q$ is a projective point and $R$ is on the other side of $Q$ as $P$, then we have $\overline{PR} = 2020\overline{PQ}$, so the corresponding point $R'$ on $\eta'$ coincides with $Q$. Similarly, if $R$ is on the same side as $P$, then we have $\overline{PR} = 2018\overline{PQ}$, so the corresponding point $R''$ on $\eta''$ coincides with $Q$. This means that $Q$ is projective if and only if it also lies on $\eta'$ or $\eta''$ (i.e. it is an intersection between $\zeta$ and $\eta'$ or $\eta''$).

To count the maximum number of intersections, note that each side of $\zeta$ can intersect $\eta'$ or $\eta''$ at most twice (since a line can intersect the boundary of a convex shape at most twice). Thus there can be at most $(2 \cdot 5) \cdot 2 = 20$ intersections and so at most 20 projective points. The construction of such a configuration is not difficult:



## 2.3 Team Test Solutions

1. Three positive integers sum to 16. What is the least possible value of the sum of their squares?

   **Solution.** The answer is $\boxed{86}$.

   Note that if two of the three numbers differ by two or more, say $a$ and $b$ with $b - a \geq 2$, then we will have $(a+1)^2 + (b-1)^2 = a^2 + b^2 + 2(a - b + 1) < a^2 + b^2$. Since $a > 0$, we have $b = a + 2 > 2$ so $b - 1 > 0$ is a positive integer, so taking the new triplet with $a + 1$ and $b - 1$ will give a smaller sum of squares. Hence, for the triplet minimizing the sum of squares, the three numbers must all be in an interval consisting of two consecutive integers, so the triplet must be $(5, 5, 6)$ which gives $5^2 + 5^2 + 6^2 = 86$.

2. Ben is thinking of an odd positive integer less than 1000. Ben subtracts 1 from his number and divides by 2, resulting in another number. If his number is still odd, Ben repeats this procedure until he gets an even number. Given that the number he ends on is 2, how many possible values are there for Ben's original number?

   **Solution.** The answer is $\boxed{8}$.

   Since Ben's number reduces to 2 eventually, his initial number can only be 2 with the inverse of Ben's process applied at least once. Since the inverse of Ben's process is multiplying by 2 and then adding 1, the only possible answers are $5, 11, 23, 47, 95, 191, 383, 767$, for a total of 8 numbers.

3. Triangle $ABC$ is isosceles, with $AB = BC = 18$ and has circumcircle $\omega$. Tangents to $\omega$ at $A$ and $B$ intersect at point $D$. If $AD = 27$, what is the length of $AC$?

   **Solution.** The answer is $\boxed{12}$.

   Because of the tangency condition, we know that $\angle BAC = \angle BCA = \angle BAD = \angle ABD$, so $\triangle ABC$ is similar to $\triangle ADB$. Therefore, $\frac{AD}{AB} = \frac{AB}{AC}$, so $AC = \frac{18^2}{27} = 12$.

4. How many non-decreasing sequences of five natural numbers have first term 1, last term 11, and have no three terms equal?

   **Solution.** The answer is $\boxed{255}$.

   For any non-decreasing sequence of five natural numbers that satisfy the condition, say $(a, b, c, d, e)$, consider $(a, b+1, c+2, d+3, e+4)$. Since $a \leq b \leq c \leq d \leq e$, we have $a < b+1 < c+2 < d+3 < e+4$, and $a = 1$ and $e + 4 = 15$, so there are $\binom{13}{3} = 286$ ways to pick $b+1$, $c+2$, and $d+3$ from the integers ranging from 2 to 14. Now, suppose that we have some sequence $(a, b, c, d, e)$ with three terms equal to a value $k$. If $1 < k < 11$, then we must have $b = c = d = k$ so there are 9 such sequences, one for each of value of $k$. Now, if $k = 1$, we must have $b = c = 1$ and $d$ can be any value from 1 to 11, giving 11 sequences. Similarly, if $k = 11$, we must have $c = d = 11$ and $b$ can be any value from 1 to 11, giving another 11 sequences. In total, we have $9 + 11 + 11 = 31$ sequences with three terms equal so the number of sequences satisfying the conditions is $286 - 31 = 255$.

5. Adam is bored, and has written the string "EMCC" on a piece of paper. For fun, he decides to erase every letter "C", and replace it with another instance of "EMCC". For example, after one step, he will have the string "EMEMCCEMCC". How long will his string be after 8 of these steps?

**Solution.** The answer is $\boxed{1534}$.

Replacing each "C" with "EMCC" will double the number of "C"'s in the string, so on the $i$th step, Adam will replace $2^i$ "C"'s with "EMCC". Each one of these steps increases the number of characters by 3, so the total length of the string after 8 steps will be $4+3\cdot(2^1+2^2+2^3+2^4+2^5+2^6+2^7+2^8) = 4+3\cdot(2^9-2) = 4+3\cdot 510 = 1534$.

6. Eric has two coins, which land heads 40% and 60% of the time respectively. He chooses a coin randomly and flips it four times. Given that the first three flips contained two heads and one tail, what is the probability that the last flip was heads?

   **Solution.** The answer is $\boxed{\frac{13}{25}}$.

   If the coin was the 40% one, then the probability of getting two heads and one tail would be $3\cdot(\frac{2}{5})^2\cdot(\frac{3}{5})$, as the three possibilities are $HHT$, $HTH$, and $THH$, each of which has probability $(\frac{2}{5})^2\cdot(\frac{3}{5})$. If the coin was the 60% one, the probability of getting two heads and one tail would be $3\cdot(\frac{2}{5})\cdot(\frac{3}{5})^2$ instead. Taking the ratio of these two probabilities, it is $\frac{2}{3}$ times as likely for the chosen coin to be the 40% one than the 60% one, so the coin is 40% with probability $\frac{2}{5}$ and 60% with probability $\frac{3}{5}$. Hence, the probability that the fourth flip is heads is $\frac{2}{5}\cdot\frac{2}{5}+\frac{3}{5}\cdot\frac{3}{5}=\frac{13}{25}$.

7. In a five person rock-paper-scissors tournament, each player plays against every other player exactly once, with each game continuing until one player wins. After each game, the winner gets 1 point, while the loser gets no points. Given that each player has a 50% chance of defeating any other player, what is the probability that no two players end up with the same amount of points?

   **Solution.** The answer is $\boxed{\frac{15}{128}}$.

   Since each player wins between zero and four games (inclusive), the only way for no two players to have the same amount of points is if one player wins $i$ games for every integer $0\le i\le 4$. There are 5 ways to choose who wins all 4 of their games, 4 ways to choose which of the 4 remaining players wins their remaining 3 games, 3 ways to choose which of the 3 remaining players win their remaining 2 games, 2 ways to choose which of the 2 remaining players wins their remaining game, and 1 way to choose who loses all of their games (as there is only 1 player left). In total, this gives $5\cdot4\cdot3\cdot2\cdot1=120$ ways, and each way has probability $(\frac{1}{2})^{10}$ as each of the 10 games have a $\frac{1}{2}$ chance of ending with a specific winner. Thus, the probability that no two players end up with the same amount of points is $\frac{120}{2^{10}}=\frac{120}{1024}=\frac{15}{128}$.

8. Let $\triangle ABC$ have $\angle A=\angle B=75^\circ$. Points $D$, $E$, and $F$ are on sides $BC$, $CA$, and $AB$, respectively, so that $EF$ is parallel to $BC$, $EF\perp DE$, and $DE=EF$. Find the ratio of $\triangle DEF$'s area to $\triangle ABC$'s area.

   **Solution.** The answer is $\boxed{\frac{2}{9}}$.

   Note that the conditions given tell us that $DEF$ is an isosceles right triangle with right angle at $E$. Therefore, if we drop an altitude from $D$ onto $BC$, intersecting $BC$ at $G$, we will have $DEFG$ be a square. Now, we can "glue" together triangles $BDG$ and $FEC$ by constructing a point $P$ on $BC$ such that $EP||DB$ (so $\triangle BDG\cong\triangle PEF$). This gives us two triangles, $ADE$ and $EPC$, similar to the original triangle $ABC$ by angle-angle similarity. Finally, note that $EC=2EF$, as $CFE$ is a $30-60-90$ triangle, and $EF=ED=EA$, so $CE=2EA$, meaning that $ADE$ and $EPC$ are $\frac{1}{3}$ and

$\frac{2}{3}$ as large as $ABC$, respectively. Thus these two triangles take up $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$ of the area, and the remaining $\frac{4}{9}$ is taken up by our square (after we again separate $EPC$ into $EFC$ and $BDG$), which is twice the area of $DEF$, meaning $DEF$ is $\frac{2}{9}$ the area of $ABC$.

9. Suppose $a$, $b$, $c$ are positive integers such that $a+b = \sqrt{c^2+336}$ and $a-b = \sqrt{c^2-336}$. Find $a+b+c$.

   **Solution.** The answer is $\boxed{56}$.

   Squaring both equations and subtracting, we have $4ab = 672$, so $ab = 168 = 2^3 \cdot 3 \cdot 7$. Adding the squared equations gives $2(a^2+b^2) = 2c^2$ so $a^2+b^2 = c^2$. Since $a$, $b$, and $c$ are positive integers, the only pairs $(a, b)$ such that $ab = 168$ and $a^2+b^2$ is a perfect square are $(7, 24)$ and $(24, 7)$, so $c = 25$ and $a+b+c = 7+24+25 = 56$.

10. How many times on a 12-hour analog clock are there, such that when the minute and hour hands are swapped, the result is still a valid time? (Note that the minute and hour hands move continuously, and don't always necessarily point to exact minute/hour marks.)

    **Solution.** The answer is $\boxed{143}$.

    On a normal clock, the minute hand rotates 12 times faster (in terms of angles) than the hour hand. Therefore, if a minute hand is at an angle that is 12 times the angle of the hour hand, then the hour hand, minute hand pair is a valid time. Consider a third hand that rotates 12 times faster than the minute hand (assume that they all start out pointing at 12 simultaneously). At a given time, let the hour, minute, and third hands be at positions $h$, $m$, and $t$, respectively. By definition, $(h, m)$ and $(m, t)$ are valid times. We want to count the number of times when $(m, h)$ also happens to be a valid time. Since there is only one possible valid time with a given hand at a given position, since $(m, t)$ is a valid time, $(m, h)$ can be valid if and only if $h = t$ (or differ by a whole number of revolutions, i.e. they point to the same angle).

    The third hand rotates $12 \cdot 12 = 144$ times as fast as the hour hand, and they both start facing 12, so over a 12 hour period, the hour hand makes one complete revolution while the third hand makes 144 complete revolutions. The difference of their angles is a strictly increasing function that ranges from 0 to 143 full revolutions, so there are exactly 144 times during this 12 hour interval for which the angles differ by an integer number of revolutions. However, we double count the position when all hands are facing 12, so there are 143 times for which swapping the minute and hour hands will give a valid time.

11. Adam owns a square $S$ with side length 42. First, he places rectangle $A$, which is 6 times as long as it is wide, inside the square, so that all four vertices of $A$ lie on sides of $S$, but none of the sides of $A$ are parallel to any side of $S$. He then places another rectangle $B$, which is 7 times as long as it is wide, inside rectangle $A$, so that all four vertices of $B$ lie on sides of $A$, and again none of the sides of $B$ are parallel to any side of $A$. Find the length of the shortest side of rectangle $B$.

    **Solution.** The answer is $\boxed{\frac{29}{4}}$.

    First, it is not difficult to see that the sides of $A$ are parallel to the diagonals of $S$, and cut $S$ into two congruent pairs of isosceles right triangles, and the similarity ratio between the two pairs is 6. This means that the legs of the triangles are 6 and 36 respectively, and the sides of $A$ are $6\sqrt{2}$ and $36\sqrt{2}$.

    Similarly, the sides of $B$ cut $A$ into two congruent pairs of right triangles with similarity ratio of 7. Suppose that the legs of the smaller right triangles are $x$ and $y$ (with $x < y$), then we have $7x+y = 6\sqrt{2}$

and $x+7y=36\sqrt{2}$. Solving this system of equations gives $x=\frac{1}{8}\sqrt{2}$ and $y=\frac{41}{8}\sqrt{2}$, so the length of the shorter side of $B$ is $\sqrt{x^2+y^2}=\frac{\sqrt{2(1^2+41^2)}}{8}=\frac{\sqrt{841}}{4}=\frac{29}{4}$.

12. Find the value of $\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$, where the exponents are the odd natural numbers, in increasing order.

    **Solution.** The answer is $\boxed{27}$.

    Let $x=\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}$. We would be able to calculate $x$ if we can express $x$ recursively, or in terms of itself.

    We can observe that:

$$3^2x=\sqrt{3^4*3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}=\sqrt{3^3\sqrt{3^4*3^3\sqrt{3^5\sqrt{\cdots}}}}=\sqrt{3^3\sqrt{3^5\sqrt{3^4*3^5\sqrt{\cdots}}}}=\sqrt{3^3\sqrt{3^5\sqrt{3^7\sqrt{\cdots}}}}$$

    Therefore, we can express $x$ in terms of $3^2x$, by the relation $x=\sqrt{3\sqrt{3^3\sqrt{3^5\sqrt{\cdots}}}}=\sqrt{3*(3^2x)}$, which gives $x=27$.

13. Jamesu and Fhomas challenge each other to a game of Square Dance, played on a $9\times 9$ square grid. On Jamesu's turn, he colors in a $2\times 2$ square of uncolored cells pink. On Fhomas's turn, he colors in a $1\times 1$ square of uncolored cells purple. Once Jamesu can no longer make a move, Fhomas gets to color in the rest of the cells purple. If Jamesu goes first, what the maximum number of cells that Fhomas can color purple, assuming both players play optimally in trying to maximize the number of squares of their color?

    **Solution.** The answer is $\boxed{49}$.

    Let the $9\times 9$ grid consist of squares $(x,y)$ for $1\le x,y\le 9$. Call a square special if both its $x$ and $y$ coordinates are even, so there are $4\cdot 4=16$ special squares in total. Note that every $2\times 2$ square that Jamesu can place will contain exactly one of these special squares, so if Fhomas always colors in an empty special square, Jamesu will be unable to move on his 9th move as all 16 special squares will be filled in. This strategy gives Jamesu $8\cdot 4=32$ squares, leaving Fhomas with the remaining $81-32=49$ squares. On the other hand, we can split the $9\times 9$ square into 16 $2\times 2$'s (that form an $8\times 8$ square) and each one of Fhomas's moves can cover at most one of these $2\times 2$ squares, so Jamesu is guaranteed to be able to fill in at least 8 $2\times 2$ squares. Hence, 49 squares is the maximum number of squares that Fhomas can guarantee.

14. Triangle $ABC$ is inscribed in circle $\omega$. The tangents to $\omega$ from $B$ and $C$ meet at $D$, and segments $AD$ and $BC$ intersect at $E$. If $\angle BAC=60^\circ$ and the area of $\triangle BDE$ is twice the area of $\triangle CDE$, what is $\frac{AB}{AC}$?
    *Addendum on 1/26/19: $\angle A$ changed to $\angle BAC$.*

    **Solution.** The answer is $\boxed{\sqrt{2}}$.

    Since we know that $BD$ and $CD$ are tangent to $\omega$, and $\angle BAC=60^\circ$, it follows from inscribed angles that $\angle BCD = \angle CBD = 60°$, or, in other words, $\triangle BCD$ is equilateral. Now, we can scale up $ABC$ about $A$ to draw a new triangle $AB'C'$, such that $D$ now lies on $B'C'$. This gives us the useful property that $\angle C'DC = \angle B'DB = 60°$ (since $B'C' || BC$), so by angle-angle-angle similarity, we have $\triangle BAC \sim \triangle CDC' \sim \triangle B'DB$. Now, we can see that $\frac{BE}{EC} = \frac{B'D}{DC'}$ by scaling, and $\frac{B'D}{DC'} = \frac{B'D}{BD} \cdot \frac{CD}{DC'}$ (as $BD = CD$), which in turn is equal to $\frac{BA}{CA} \cdot \frac{BA}{CA} = \frac{BA^2}{CA^2}$ by the aforementioned similarities. As we are given that $\frac{BE}{EC} = \frac{[BDE]}{[CDE]} = 2$, it follows that $\frac{BA^2}{CA^2} = 2$, so $\frac{AB}{AC} = \sqrt{2}$ as desired.

(Note that this line $AE$, constructed via tangents, is commonly known as the *symmedian* of triangle $ABC$, and indeed the general result $\frac{AB^2}{AC^2} = \frac{BE}{EC}$ is one of the symmedian's many useful properties, although this knowledge is not necessary to solve the problem.)

15. Fhomas and Jamesu are now having a number duel. First, Fhomas chooses a natural number $n$. Then, starting with Jamesu, each of them take turns making the following moves: if $n$ is composite, the player can pick any prime divisor $p$ of $n$, and replace $n$ by $n - p$; if $n$ is prime, the player can replace $n$ by $n - 1$. The player who is faced with 1, and hence unable to make a move, loses. How many different numbers $2 \le n \le 2019$ can Fhomas choose such that he has a winning strategy, assuming Jamesu plays optimally?

**Solution.** The answer is $\boxed{1015}$.

We want to find which numbers are *winning*, and which are *losing*; in other words, which numbers, when given to a player, guarantee that they will eventually win through playing optimally, and which numbers it is impossible to win from (given the opponent plays optimally). Clearly a number is losing if *every* possible move on it results in a winning number, and is winning if *at least one* possible move results in a losing number, as the optimal play will be to pick that move.

We can observe that 1 and 3 are losing numbers, while 2 is a winning number. Furthermore, note that every possible move on an odd number will necessarily result in an even number (i.e. the move is somewhat forced, as should be with a losing number). From these observations, we can begin with the assumption that all odd numbers are losing, and all even numbers are winning. Now, in order for an even number $n$ to be winning, our move must turn it into a losing, or odd, number. This is possible if $n$ has any odd prime divisor $p$, since we can make the move $n \to n - p$.

Therefore, our assumption fails if $n$ has no odd divisors, i.e. it is a power of 2 larger than 2 (note that 2 itself is still winning, since we can move to the odd number 1, which is losing). Given such a number $n = 2^k$, our only possible move is $2^k \to 2^k - 2$, so we are forced to move to a winning even number (either 2 if $k = 2$, or a non-power of 2 otherwise), so $2^k$ is losing.

Knowing this, we have to consider whether it is then possible for an odd number to be winning, through being able to make a move to $2^k$, a losing even number. However, notice that if some odd prime $p$ divides an odd $n$, then it also divides $n - p$; in other words, any move on an odd composite number cannot result in only even prime factors. Thus, the only way to move to a power of 2 from an odd number is if the number is a prime of the form $2^k + 1$. We can verify that there are four such primes in our range: $3, 5, 17, 257$; we do not count 3, however, as 2 is winning.

The final thing we must check, then, is that our assumed strategy of moving from even to odd will not accidentally land on one of these 3 winning odd primes. However, the only way to move from an even number to a prime $p$ is if our original number was in fact $2p$. Since $p = 2^k + 1$, these evens would be $2^{k+1} + 2$. In these cases, we can instead make the move $2^{k+1} + 2 \to 2^{k+1}$, since $2^{k+1}$ is a losing number. Thus it is always possible for an even non-power of two to move to a losing number, so we do not have to check any further cases.

In conclusion, a number is losing if it is odd and not one of $5, 17, 257$, or if it is a power of 2 besides 2 itself. Since Fhomas wants to give Jamesu a losing number, he can choose from among $\lfloor \frac{2019}{2} \rfloor - 3 = 1006$ odds, and 9 evens (one of $\{2^2, \cdots, 2^{10}\}$), giving 1015 total choices.

## 2.4 Guts Test Solutions

### 2.4.1 Round 1

1. [6] What is the smallest number equal to its cube?

   **Solution.** The answer is $\boxed{-1}$.

   If $x = x^3$, then either $x = 0$ or $x^2 = 1$. The latter has two solutions $x = -1$ and $x = 1$, so $x = -1$ is the smallest number equal to its cube.

2. [6] Fhomas has 5 red spaghetti and 5 blue spaghetti, where spaghetti are indistinguishable except for color. In how many different ways can Fhomas eat 6 spaghetti, one after the other? (Two ways are considered the same if the sequence of colors are identical)

   **Solution.** The answer is $\boxed{62}$.

   Each of the $2^6$ possible color sequences of length 6 work, except for $RRRRRR$ and $BBBBBB$ because there are only 5 red and 5 blue spaghetti, so there are a total of $2^6 - 2 = 62$ ways for Fhomas to eat the spaghetti.

3. [6] Jocelyn labels the three corners of a triangle with three consecutive natural numbers. She then labels each edge with the sum of the two numbers on the vertices it touches, and labels the center with the sum of all three edges. If the total sum of all labels on her triangle is 120, what is the value of the smallest label?

   **Solution.** The answer is $\boxed{7}$.

   Let the smallest label be $x$. Then, the three vertices have labels $x$, $x + 1$, and $x + 2$ so the three sides have labels $2x + 1$, $2x + 2$, and $2x + 3$. Hence, the center has label $6x + 6$ so the sum of all labels is $x + x + 1 + x + 2 + 2 \cdot (6x + 6) = 15x + 15 = 120$, so $x = 7$.

### 2.4.2 Round 2

4. [7] Adam cooks a pie in the shape of a regular hexagon with side length 12, and wants to cut it into right triangular pieces with angles 30°, 60°, and 90°, each with shortest side 3. What is the maximum number of such pieces he can make?

   **Solution.** The answer is $\boxed{48}$.

   Adam can first partition the hexagon into six equilateral triangles with side length 12, then partition each triangle into four smaller equilateral triangles with side length 6, and the partition each smaller triangle into two right triangles with the desired properties, for $6 \cdot 4 \cdot 2 = 48$ triangles in total. Because this fills the entire hexagon, it must be the greatest number of pieces.

5. [7] If $f(x) = \frac{1}{2-x}$ and $g(x) = 1 - \frac{1}{x}$, what is the value of $f(g(f(g(\cdots f(g(f(2019)))\cdots))))$, where there are 2019 functions total, counting both $f$ and $g$?

**Solution.** The answer is $\boxed{-\frac{1}{1008}}$.

Note that $g(f(x)) = 1 - \frac{1}{f(x)} = 1-(2-x) = x-1$, so $g(f(\dots f(g(f(2019)))\dots)) = 2019-1009 = 1010$ where there are 2018 functions total. Hence, the expression is equal to $f(1010) = -\frac{1}{1008}$.

6. [7] Fhomas is buying spaghetti again, which is only sold in two types of boxes: a 200 gram box and a 500 gram box, each with a fixed price. If Fhomas wants to buy exactly 800 grams, he must spend \$8.80, but if he wants to buy exactly 900 grams, he only needs to spend \$7.90! In dollars, how much more does the 500 gram box cost than the 200 gram box?

   **Solution.** The answer is $\boxed{\$1.30}$.

   Let the 200 gram box cost $a$ cents and the 500 gram box cost $b$ cents. Then, $4a = 880$ so $a = 220$, but $b + 2a = 790$ so $b = 350$. Hence, $b - a = 130$ so the 500 gram box costs \$1.30 more than the 200 gram box.

### 2.4.3 Round 3

7. [9] Given that

$$\begin{cases} a+5b+9c=1, \\ 4a+2b+3c=2, \\ 7a+8b+6c=9, \end{cases}$$

   what is $741a + 825b + 639c$?

   **Solution.** The answer is $\boxed{921}$.

   Adding 100 times the third equation, 10 times the second equation, and 1 times the first equation, we get $741a + 825b + 639c = 921$.

8. [9] Hexagon $JAMESU$ has line of symmetry $MU$ (i.e., quadrilaterals $JAMU$ and $SEMU$ are reflections of each other), and $JA = AM = ME = ES = 1$. If all angles of $JAMESU$ are 135 degrees except for right angles at $A$ and $E$, find the length of side $US$.

   **Solution.** The answer is $\boxed{2-\sqrt{2}}$.

   If we connect $MS$, we note that triangle $EMS$ is an isosceles right triangle with $MS = \sqrt{2}$. Moreover, since $\angle AME = \angle ESU = 135^\circ$, we have that $\angle UMS = \frac{135^\circ}{2} - 45^\circ = 22.5^\circ$ and $\angle MSU = 135^\circ - 45^\circ = 90^\circ$, so it suffices to find $US$ in triangle $SUM$ given these information.

   Pick $X$ on $SM$ such that $USX$ is an isosceles right triangle, then we have $\angle MUX = 22.5^\circ = \angle UMX$, which implies that $MX = UX = \sqrt{2}SX$ and thus $(1+\sqrt{2})SX = SM = \sqrt{2}$. We can then compute $US = SX = \frac{\sqrt{2}}{1+\sqrt{2}} = 2-\sqrt{2}$, as desired.

9. [9] Max is parked at the 11 mile mark on a highway, when his pet cheetah, Min, leaps out of the car and starts running up the highway at its maximum speed. At the same time, Max starts his car and

starts driving down the highway at $\frac{1}{2}$ his maximum speed, driving all the way to the 10 mile mark before realizing that his cheetah is gone! Max then immediately reverses directions and starts driving back up the highway at his maximum speed, finally catching up to Min at the 20 mile mark. What is the ratio between Max's max speed and Min's max speed?

**Solution.** The answer is $\boxed{\frac{4}{3}}$.

Assume that it takes $m$ minutes for Min to travel a mile at maximum speed and $M$ minutes for Max to travel a mile at maximum speed, then it took $9m$ minutes for Min to go from 11 miles to 20 miles, and it took $2M + 10M$ minutes for Max to go from 11 miles to 10 miles and then to 20 miles. Hence $9m = 12M$, which means that $M = \frac{3}{4}m$, so Max is $\frac{4}{3}$ times as fast as Min when at max speed.

### 2.4.4 Round 4

10. [11] Kevin owns three non-adjacent square plots of land, each with side length an integer number of meters, whose total area is $2019\text{m}^2$. What is the minimum sum of the perimeters of his three plots, in meters?

**Solution.** The answer is $\boxed{228}$.

Let the side lengths be $a, b, c$ meters with $a \geq b \geq c$. It is not difficult to see that total perimeter is minimized when $a + b + c$ is minimized, which is also minimized when the three numbers are as far apart as possible. (See solution to team round problem 1.) Note that since 2019 is congruent to 3 modulo 4, while any perfect square is congruent to 0 or 1 modulo 4, we need all three of $a, b, c$ to be odd. The largest odd $a$ with $a^2 \leq 2019$ is $a = 43$, which leaves $b^2 + c^2 = 2019 - 1849 = 170 = 13^2 + 1^2$. Hence, the sum is minimized when $a = 43, b = 13, c = 1$, for total perimeter of $4(a + b + c) = 228$.

11. [11] Given a $5 \times 5$ array of lattice points, how many squares are there with vertices all lying on these points?

**Solution.** The answer is $\boxed{50}$.

We count up by side length:

- Length 1: $4^2 = 16$ squares.
- Length $\sqrt{2}$: $3^2 = 9$ squares.
- Length 2: $3^2 = 9$ squares.
- Length $\sqrt{5}$: $2(2^2) = 8$ squares (accounting for orientation).
- Length $2\sqrt{2}$: 1 square.
- Length 3: $2^2 = 4$ squares.
- Length $\sqrt{10}$: 2 squares (accounting for orientation).
- Length 4: 1 square.

Hence, there are $16 + 9 + 9 + 8 + 1 + 4 + 2 + 1 = 50$ squares.

12. [11] Let right triangle $ABC$ have $\angle A = 90^\circ$, $AB = 6$, and $AC = 8$. Let points $D, E$ be on side $AC$ such that $AD = EC = 2$, and let points $F, G$ be on side $BC$ such that $BF = FG = 3$. Find the area of quadrilateral $FGED$.

**Solution.** The answer is $\boxed{\frac{51}{5}}$.

We have $DC = 6, EC = 2, FC = 7, GC = 4$, so we have

$$[FGED] = [FDC] - [EGC] = \left(\frac{6}{8} \cdot \frac{7}{10} - \frac{2}{8} \cdot \frac{4}{10}\right)[ABC] = \frac{17}{40} \cdot \frac{6 \cdot 8}{2} = \frac{51}{5}.$$

### 2.4.5 Round 5

13. [13] Given a (not necessarily simplified) fraction $\frac{m}{n}$, where $m, n > 6$ are positive integers, when 6 is subtracted from both the numerator and denominator, the resulting fraction is equal to $\frac{4}{5}$ of the original fraction. How many possible ordered pairs $(m, n)$ are there?

**Solution.** The answer is $\boxed{14}$.

We are given that $\frac{m-6}{n-6} = \frac{4}{5} \cdot \frac{m}{n} = \frac{4m}{5n}$. Cross-multiplying, $5mn - 30n = 4mn - 24m$, which simplifies to $mn + 24m - 30n = 0$. This can be factored as $(m - 30)(n + 24) = -720$, and $m, n$ are both positive integers so the minus sign must come from $m - 30$. Hence, $(30 - m)(n + 24) = 720$ and $n > 6$ so $n + 24 > 30$. Thus, $30 - m < 30 - 6 = 24$ is some divisor of 720 that is less than 24, so $30 - m \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 18, 20\}$. Each of these corresponds to some integer value of $m$ between 10 and 29, and we also have $n + 24 = \frac{720}{30-m} \geq 36$ so they also correspond to an integer value of $n$ greater than or equal to 12. Hence, there are 14 possible ordered pairs $(m, n)$.

14. [13] Jamesu's favorite anime show has 3 seasons, with 12 episodes each. For 8 days, Jamesu does the following: on the $n^{\text{th}}$ day, he chooses $n$ consecutive episodes of exactly one season, and watches them in order. How many ways are there for Jamesu to finish all 3 seasons by the end of these 8 days? (For example, on the first day, he could watch episode 5 of the first season; on the second day, he could watch episodes 11 and 12 of the third season, etc.)

**Solution.** The answer is $\boxed{1440}$.

It is not difficult to see that the sixth, seventh, and eighth day must be spend watching different seasons. We now separate into cases:

(a) The fifth day is the same season as the seventh day. Then we can have either $((1, 2, 3, 6), (5, 7), (4, 8))$ or $((2, 4, 6)(5, 7), (1, 3, 8))$. The former gives $3! \cdot 4! \cdot 2! \cdot 2! = 576$ ways and the latter gives $3! \cdot 3! \cdot 2! \cdot 3! = 432$ ways.

(b) The fifth day is the same season as the sixth day, then we have $((1, 5, 6), (2, 3, 7), (4, 8))$, which gives $3! \cdot 3! \cdot 3! \cdot 2! = 432$ ways.

Therefore, there are $576 + 432 + 432 = 1440$ ways in total.

15. [13] Let $O$ be the center of regular octagon $ABCDEFGH$ with side length 6. Let the altitude from $O$ meet side $AB$ at $M$, and let $BH$ meet $OM$ at $K$. Find the value of $BH \cdot BK$.

**Solution.** The answer is $\boxed{36}$.

Note that triangle $HAB$ is isosceles since $HA = AB$, and that $AKB$ is isosceles by symmetry about $OM$. Since both triangles share the $\angle HBA$, they are similar and thus we get $\frac{HB}{AB} = \frac{AB}{KB} \Rightarrow BH \cdot BK = BA^2 = 36$.

### 2.4.6 Round 6

16. [15] Fhomas writes the ordered pair $(2, 4)$ on a chalkboard. Every minute, he erases the two numbers $(a, b)$, and replaces them with the pair $(a^2 + b^2, 2ab)$. What is the largest number on the board after 10 minutes have passed?

**Solution.** The answer is $\boxed{\frac{6^{1024}+2^{1024}}{2}}$.

Let $(a_n, b_n)$ denote the two numbers on the board after $n$ minutes (so $a_0 = 2, b_0 = 4$). We also define $s_n = a_n + b_n$ and $d_n = a_n - b_n$, then we have

$$s_{n+1} = (a_n^2 + b_n^2) + (2a_nb_n) = (a_n + b_n)^2 = s_n^2 \text{ and } d_{n+1} = (a_n^2 + b_n^2) - (2a_nb_n) = (a_n - b_n)^2 = d_n^2.$$

Therefore, we have $s_{10} = s_9^2 = s_8^4 = \cdots = s_0^{1024} = 6^{1024}$ and similarly $d_{10} = d_0^{1024} = (-2)^{1024} = 2^{1024}$. Since $d_{10} > 0$, the larger of the two numbers is $a$, which is equal to $\frac{s_{10}+d_{10}}{2} = \frac{6^{1024}+2^{1024}}{2}$.

17. [15] Triangle $BAC$ has a right angle at $A$. Point $M$ is the midpoint of $BC$, and $P$ is the midpoint of $BM$. Point $D$ is the point where the angle bisector of $\angle BAC$ meets $BC$. If $\angle BPA = 90°$, what is $\frac{PD}{DM}$?

**Solution.** The answer is $\boxed{\frac{\sqrt{3}}{2}}$.

Since the altitude of $A$ onto $BM$ bisects $BM$ at $P$, we must have $AB = AM$, and since $AM = BM = CM$, we get that $ABM$ is equilateral, so $\angle B = 60°$ and $\angle C = 30°$.

Now let $AB = 1, AC = \sqrt{3}$, and $BC = 2$. We have that $BM = 1, BP = 1/2$, and $BD = \frac{2}{1+\sqrt{3}} = \sqrt{3}-1$ from angle-bisector theorem. Therefore we have

$$\frac{PD}{DM} = \frac{BD - BP}{BM - BD} = \frac{\sqrt{3} - \frac{3}{2}}{2 - \sqrt{3}} = \left(\sqrt{3} - \frac{3}{2}\right)(2 + \sqrt{3}) = \frac{\sqrt{3}}{2}.$$

18. [15] A square is called *legendary* if there exist two different positive integers $a, b$ such that the square can be tiled by an equal number of non-overlapping $a$ by $a$ squares and $b$ by $b$ squares. What is the smallest positive integer $n$ such that an $n$ by $n$ square is *legendary*?

**Solution.** The answer is $\boxed{10}$.

For ease of reference, call $n$ *legendary* if an $n$ by $n$ square is legendary. WLOG assume that $a < b$. Suppose that both types of squares are used $m$ times, then we must have $n^2 = m(a^2 + b^2)$. When $a = 1, b = 2$, we have $n$ being a multiple of 5. If $n = 5$, then we need $m = 5$, but one cannot fit five 2 by 2 squares in a 5 by 5 square. If $n = 10$, then we need $m = 20$, and it is not difficult to see that one can fit twenty 2 by 2 squares on the top 8 by 10 rectangle and twenty 1 by 1 squares in the bottom 2 by 10 rectangle. Hence $n = 10$ is legendary.

To show that no $n$ less than 10 is legendary, we first note that since obviously $m > 1$ (one cannot combine two squares into a larger square), we must have $n \geq 2b$ to fit at least two $b$ by $b$ squares, which means that $n^2 \geq 4b^2 > 2(a^2 + b^2)$ so $m \geq 3$. The case $(a, b) = (1, 2)$ has been covered above, $(a, b) = (1, 3), (2, 3), (1, 4)$ forces $n$ to be a multiple of 10, 13, or 17, which will not yield smaller legendary numbers, and $(a, b) = (2, 4)$ is pointless because we can simply cut each square into four to yield $(a, b) = (1, 2)$. These force $n^2 \geq 3 \cdot (3^2 + 4^2) = 75$, or $n \geq 9$, so it suffices to check $n = 9$. Since $m \geq 3$, we have $a^2 + b^2 \leq 81/3 = 27$, but we also have $a^2 + b^2 \geq 25$ so $m = 3$. However, it is not difficult to check (by hand or noting the remainder modulo 4) that 27 cannot be expressed as sum of two squares, so $n = 9$ is not legendary. Therefore, 10 is the smallest legendary number.

### 2.4.7 Round 7

19. [18] Let $S(n)$ be the sum of the digits of a positive integer $n$. Let $a_1 = 2019!$, and $a_n = S(a_{n-1})$. Given that $a_3$ is even, find the smallest integer $n \geq 2$ such that $a_n = a_{n-1}$.

**Solution.** The answer is $\boxed{5}$.

Note that $2019! < 10000^{2019} = 10^{8076}$ so 2019! has fewer than 8076 digits. Hence, $a_2 = S(a_1) \leq 9 \cdot 8076 < 100000$, so $a_3 = S(a_2) < 9 \cdot 5 = 45$. Since $a_3$ is even and $S(n) \equiv n \pmod 9$, we know that $a_3 \equiv a_2 \equiv a_1 \equiv 0 \pmod 9$, so $a_3 = 0$ or $a_3 \geq 18$. However, $S(n) = 0$ if and only if $n = 0$, so $a_3 \neq 0$. Thus, $18 \leq a_3 \leq 45$ and $a_3 \equiv 0 \pmod 9$ so $a_4 = 9$. It follows that $a_5 = 9$ so 5 is the smallest value of $n$ such that $a_n = a_{n-1}$.

20. [18] The local EMCC bakery sells one cookie for $p$ dollars ($p$ is not necessarily an integer), but has a special offer, where any non-zero purchase of cookies will come with one additional free cookie. With \$27.50, Max is able to buy a whole number of cookies (including the free cookie) with a single purchase and no change leftover. If the price of each cookie were 3 dollars lower, however, he would be able to buy double the number of cookies as before in a single purchase (again counting the free cookie) with no change leftover. What is the value of $p$?

**Solution.** The answer is $\boxed{\$5.50}$.

Suppose that without the special offer Max bought $x$ cookies, then after the decrease in price Max would be able to by $2(x + 1) - 1 = 2x + 1$ cookies without the special offer. Since $x = \frac{27.5}{p}$ and $2x + 1 = \frac{27.5}{p-3}$, so we have the equation

$$\frac{55}{p} + 1 = \frac{27.5}{p - 3}.$$

This simplifies to a quadratic $p^2 + 24.5p - 165 = 0$, which can be factored as $(p - 5.5)(p + 30) = 0$. Therefore the only positive solution is $p = 5.5$.

21. [18] Let circle $\omega$ be inscribed in rhombus $ABCD$, with $\angle ABC < 90°$. Let the midpoint of side $AB$ be labeled $M$, and let $\omega$ be tangent to side $AB$ at $E$. Let the line tangent to $\omega$ passing through $M$ other than line $AB$ intersect segment $BC$ at $F$. If $AE = 3$ and $BE = 12$, what is the area of $\triangle MFB$?

**Solution.** The answer is $\boxed{27}$.

Let the center of $\omega$ be $O$, then since $AOB$ is a right angle and $OE$ is perpendicular to $AB$, we have that $AE/OE = OE/BE$, implying the radius $OE = \sqrt{AE \cdot BE} = 6$.

If we drop a perpendicular from $A$ to $BC$ intersecting at $G$, then note that both $M$ and $O$ lie on the perpendicular bisector of $AF$, so by symmetry since $AM$ is tangent to $\omega$, so is $GM$. This means that $F = G$; in other words, $AFB$ is a right triangle. Since $AF$ is equal to the diameter (which is 12), we have $BF = \sqrt{AB^2 - AF^2} = 9$, so $[MFB] = [AFB]/2 = \frac{9 \cdot 12}{4} = 27$.

### 2.4.8 Round 8

22. [21] Find the remainder when $1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1!$ is divided by 2019.

**Solution.** The answer is $\boxed{1}$.

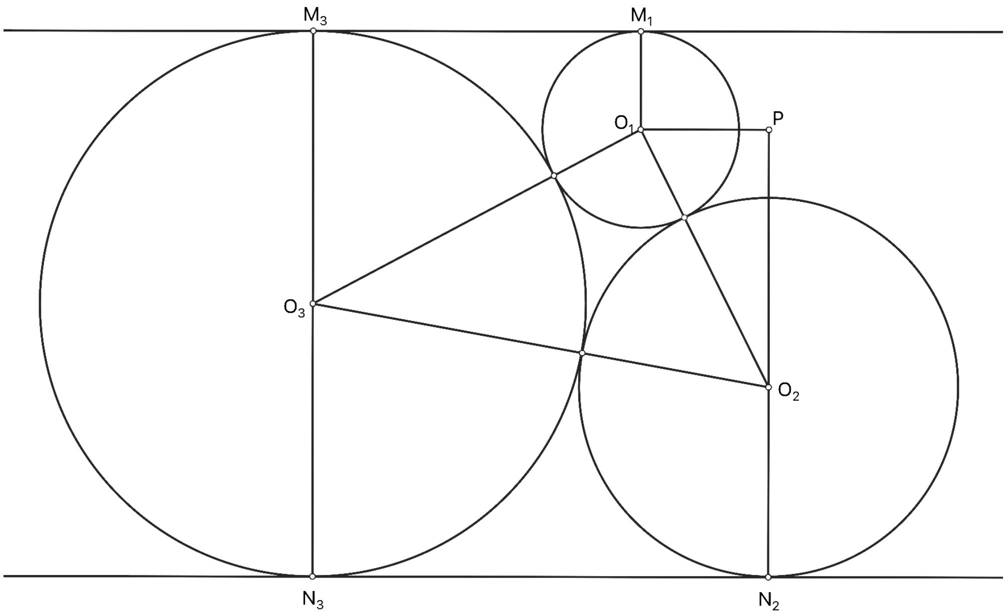
Observe that $2019 = 3 \cdot 673$, so $n!$ is a multiple of 2019 for all $n \geq 673$. Therefore we have

$$\begin{aligned}
1010 \cdot 1009! + 1011 \cdot 1008! + \cdots + 2018 \cdot 1! &\equiv -(1009 \cdot 1009! + 1008 \cdot 1008! + \cdots + 1 \cdot 1!) \\
&= -((1010! - 1009!) + (1009! - 1008!) + \cdots + (2! - 1!)) \\
&= -(1010! - 1!) \equiv -(0 - 1) \equiv 1 \pmod{2019}
\end{aligned}$$

23. [21] Two circles $\omega_1$ and $\omega_2$ have radii 1 and 2, respectively and are externally tangent to one another. Circle $\omega_3$ is externally tangent to both $\omega_1$ and $\omega_2$. Let $M$ be the common external tangent of $\omega_1$ and $\omega_3$ that doesn't intersect $\omega_2$. Similarly, let $N$ be the common external tangent of $\omega_2$ and $\omega_3$ that doesn't intersect $\omega_1$. Given that $M$ and $N$ are parallel, find the radius of $\omega_3$.

**Solution.** The answer is $\boxed{2\sqrt{2}}$.

After setting up our drawing, we can see that the condition requires that the distance between lines $M$ and $N$ equal the diameter of $\omega_3$, which we will call $2r$. Now, label the center of each circle $\omega_n$ by $O_n$. Additionally, label the tangency points of $\omega_2, \omega_3$ to $N$ as $N_2$ and $N_3$ respectively, and define $M_1$ and $M_3$ similarly. Finally, to make comparing lengths easier, we will construct a right triangle $O_1PO_2$ with right angle at $P$, such that $O_1P || M, N$ and $O_2P || M_3N_3$.




From here, we can see that $M_3N_3 = M_1O_1 + PO_2 + O_2N_2$, giving $PO_2 = M_3N_3 - M_1O_1 - O_2N_2 = 2r - 1 - 2 = 2r - 3$. Additionally, by the Pythagorean theorem,

$$N_3N_2 = \sqrt{O_3O_2^2 - (O_3N_3 - O_2N_2)^2} = \sqrt{(r+2)^2 - (r-2)^2} = 2\sqrt{2}\sqrt{r}$$

and similarly $M_3M_1 = \sqrt{(r+1)^2 - (r-1)^2} = 2\sqrt{r}$. Now, we can equate horizontal lengths to get $N_3N_2 = M_3M_1 + O_1P$, giving $O_1P = (2\sqrt{2} - 2)\sqrt{r}$.

Finally, we use the Pythagorean theorem on triangle $O_1PO_2$, giving

$$O_1O_2^2 = (2r-3)^2 + ((2\sqrt{2}-2)\sqrt{r})^2 = 4r^2 - 8\sqrt{2}r + 9$$

after expansion. However, we already know $O_1O_2 = 1+2 = 3$, so we can simplify to get $4r^2 - 8\sqrt{2}r = 0$, or $4r(r - 2\sqrt{2}) = 0$. Since $r > 0$, we must have $r = 2\sqrt{2}$.

24. [21] Mana is standing in the plane at $(0,0)$, and wants to go to the EMCCiffel Tower at $(6,6)$. At any point in time, Mana can attempt to move 1 unit to an adjacent lattice point, or to make a knight's move, moving diagonally to a lattice point $\sqrt{5}$ units away. However, Mana is deathly afraid of negative numbers, so she will make sure never to decrease her $x$ or $y$ values. How many distinct paths can Mana take to her destination?

**Solution.** The answer is $\boxed{5810}$.

We can observe that making a knights move is equivalent to first choosing to move either up or right, and then adding an extra "diagonal" movement (1 unit both up and right). We can therefore do casework on the number of knights moves. Since a knights move will add 3 to the sum of Mana's $x$ and $y$ values, and she needs to reach a destination with $x + y = 12$, we can make at most 4 knights moves.

If we do make 4 knights moves, we can ignore the extra diagonal movements, reducing our problem to moving from $(0, 0)$ to $(2, 2)$ via up/right movements. Since we have 4 moves, and must assign 2 of them the "up" direction, there are $\binom{4}{2} = 6$ ways to accomplish this.

If we make 3 knights moves, ignoring the diagonal movements again reduces our journey to moving from $(0, 0)$ to $(3, 3)$ via up/right moves. Again there are $\binom{6}{3}$ ways to choose these moves. However, we now have two different types of moves: knights moves that were *reduced* to up/right moves by subtracting off an extra diagonal movement, and moves that were already 1 unit up/right to begin with, in our original path. Out of our 6 moves, we must therefore identify which 3 are original moves, and which are reduced knights moves; there are again $\binom{6}{3}$ ways to do this, bringing the total in this case to $\binom{6}{3} \cdot \binom{6}{3} = 400$ paths.

For 2 knights moves, we again reduce to a $4 \times 4$ square, giving $\binom{8}{4}$ possible move choices, and we identify our two reduced moves in $\binom{8}{2}$ ways, giving $\binom{8}{4} \cdot \binom{8}{2} = 70 \cdot 28 = 1960$ paths.

For 1 knight move, we reduce to a $5 \times 5$ square, and assign one move as reduced, giving $\binom{10}{5} \cdot \binom{10}{1} = 2520$ paths.

Finally, for 0 knights moves, we simply have $\binom{12}{6}$ ways to assign 6 out of our 12 moves as being upwards, giving 924 paths. So in total, Mana has $6 + 400 + 1960 + 2520 + 924 = 5810$ possible paths to choose from.