AIME Practice Set 2015

David Altizio

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Abstract

These problems are ones I have collected from my problem-solving over the past few years that resemble AIME level problems, except that none of them have actually appeared on the AIME. Each problem (should) has a nonnegative integer answer, and each of the four sections have ten problems roughly ordered by difficulty. I tried to make the sections of similar difficulties, but this is probably not the case (Combinatorics and Number Theory seem easier than Algebra and Geometry). Have fun with the problems!

The majority of these problems have been selected from both collections of questions (AoPS, Brilliant, various math camps) and actual contests (HMMT, iTest, Math League, Mandelbrot[1], NIMO, etc.).

[1]Several of the Mandelbrot problems that appear in this collection came from the book *Mandelbrot Morsels*. To be honest, it has some pretty awesome problems, and it's one of the few books that has a bunch of problems that haven't all been released onto the Internet yet. It's definitely worth checking out!

1 Algebra

1. Let c be the larger solution to the equation $x^2 - 20x + 13 = 0$. Compute the area of the circle with center (c, c) passing through the point $(13, 7)$.

2. Suppose a, b, and c are real numbers such that
$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{a}\right)=\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right).$$
If $abc = 13$, what is $|a+b+c|$?

3. What is the only real number $x > 1$ which satisfies the equation
$$\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x?$$

4. Suppose that P is the polynomial of least degree with integral coefficients for which $P(\sqrt{3}+\sqrt{2}) = \sqrt{3}-\sqrt{2}$. Compute $P(2)$.

5. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$,
$$a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1.$$
Find a_{513}.

6. Let $\{x_n\}_{n=1}^{150}$ be a sequence of real numbers such that $x_i \in \{\sqrt{2}+1, \sqrt{2}-1\}$ for all positive integers i with $1 \leq i \leq 150$. For how many positive integers $1 \leq S \leq 1000$ does there exist such a sequence $\{x_n\}$ with the property that
$$x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{149}x_{150} = S?$$

7. Let $c_1, c_2, c_3, \ldots, c_{2008}$ be complex numbers such that
$$|c_1| = |c_2| = |c_3| = \cdots = |c_{2008}| = 2011,$$
and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of
$$\left|\frac{S(2008, 1492)}{S(2008, 516)}\right|.$$
Find the remainder when Q is divided by 1000.

8. Let S be the sum of all x such that $1 \leq x \leq 99$ and
$$\{x^2\} = \{x\}^2.$$
Find the number formed by the first three digits of $\lfloor S \rfloor$. (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

9. Let a, b, c, and d be positive real numbers such that
$$\begin{aligned} a^2+b^2 = c^2+d^2 &= 2008, \\ ac = bd &= 1000. \end{aligned}$$
If $S = a+b+c+d$, compute the value of $\lfloor S \rfloor$.

10. Suppose a, b, c, d are integers such that
$$a+b+c+d=0 \qquad \text{and} \qquad (ab-cd)(ac-bd)(ad-bc)+528^2=0.$$
What is the maximum possible value of a?

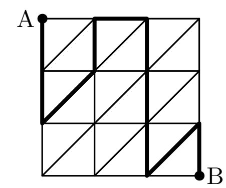
2 Combinatorics

1. A random pizza is made by flipping a fair coin to decide whether to include pepperoni, then doing the same for sausage, mushrooms, and onions. The probability that two random pizzas have at least one topping in common can be written in the form $\frac{m}{n}$ where m and n are positive integers. Find $m+n$.

2. How many permutations $(a_1, a_2, \ldots, a_{10})$ of $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ satisfy

$$|a_1 - 1| + |a_2 - 2| + \cdots + |a_{10} - 10| = 4?$$

3. How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is highlighted.



4. Consider the set $S = \{1, 2, 3, 4, 5, \ldots, 100\}$. How many subsets of this set with two or more elements satisfy:
   (i) the terms of the subset form an arithmetic sequence, and
   (ii) we cannot include another element from S with this subset to form an even longer arithmetic sequence?

5. Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the first game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the final hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n. Find the sum of all possible values of n.

6. An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point (5,5)?

7. Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:
   1. Choose a row, and subtract 1 from each entry.
   2. Chooses a column, and add 1 to each entry.

   He would like to get an array where all integers are divisible by 2013. Let M be the number of initial arrays for which this is possible. What is the number formed by the last three digits of M?

8. If you flip a fair coin 1000 times, let P be the expected value of the product of the number of heads and the number of tails. What are the first three digits of P?

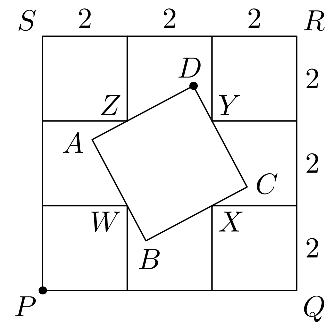
9. Suppose N is the number of ways to partition the counting numbers from 1 to 12 (inclusive) into four sets with three numbers in each set so that the product of the numbers in each set is divisible by 6. What is the number formed by the last three digits of N?

10. Seven points are spaced equally around a circle each having labeled with some number. A labeling is *clean* if for any two pairs of points a, b and c, d with a having the same label as b and c as d, but a not having the same label as c, the chords connecting ab and cd do not intersect. Additionally, two *clean* labelings are the same if the set of points that have the same label in one labeling are the same as in the other and if the points can be rotated to equal the other. How many unique *clean* labelings are there?

3 Geometry

1. Regular hexagon $ABCDEF$ is given in the plane. If the area of the triangle whose vertices are the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ is 225, what is the area of $ABCDEF$?

2. In the corners of a square $PQRS$ with side length 6 cm four smaller squares are placed with side lengths 2 cm. Let us denote their vertices by W, X, Y, Z like in the picture. A square $ABCD$ is constructed in such a way, that points W, X, Y, Z lie inside the sides AB, BC, CD, DA respectively. Find the square of the largest possible distance between points P and D.



3. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find the remainder when R^2 is divided by 1000, where R is the radius of the sphere.

4. Two circles, ω_1 and ω_2, have radii of 5 and 12 respectively, and their centers are 13 units apart. The circles intersect at two different points P and Q. A line l is drawn through P and intersects the circle ω_1 at $X \neq P$ and ω_2 at $Y \neq P$. Find the maximum value of $PX \cdot PY$.

5. Two circles in the Cartesian plane have four common tangent lines. If the slopes of these lines are 2, 3, 4, and m, in increasing order, then calculate $\lfloor 100m \rfloor$.

6. Let point O be the origin of a three-dimensional coordinate system, and let points A, B, and C be located on the positive x, y, and z axes, respectively. Suppose $OA = \sqrt[4]{75}$ and $m\angle BAC = 30^\circ$. Compute $100K$, where K is the area of $\triangle ABC$.

7. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$, and denote by ω the unique circle inscribed inside the triangle. Suppose the orthocenter of $\triangle ABC$ lies on ω. Then there exist relatively prime positive integers m and n such that $\cos \angle BAC = \frac{m}{n}$. Find $m + n$.

8. Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $m + n$.

9. Given a convex, n-sided polygon P, form a $2n$-sided polygon $\text{clip}(P)$ by cutting off each corner of P at the edges' trisection points. In other words, $\text{clip}(P)$ is the polygon whose vertices are the $2n$ edge trisection points of P, connected in order around the boundary of P. Let P_1 be an isosceles trapezoid with side lengths $13, 13, 13$, and 3, and for each $i \geq 2$, let $P_i = \text{clip}(P_{i-1})$. This iterative clipping process approaches a limiting shape $P_\infty = \lim_{i \to \infty} P_i$. If the difference of the areas of P_{10} and P_∞ is written as a fraction $\frac{x}{y}$ in lowest terms, calculate the number of positive integer factors of $x \cdot y$.

10. Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, find the sum of the numerator and denominator of $(DP/DQ)^2$ when written in lowest terms.

4 Number Theory

1. Jack chose nine different integers from 1 through 19 and found their sum. From the remaining ten integers, Jill chose nine and found their sum. If the ratio of Jack's sum to Jill's sum was $7:15$, which of the nineteen integers was chosen by *neither* Jack nor Jill?

2. There exist unique positive integers x and y such that $4^y - 615 = x^2$. What is the value of $x + y$?

3. In the binary expansion of
$$\frac{2^{2007}-1}{2^{225}-1},$$
how many of the first $10,000$ digits to the right of the radix point are 0's?

4. For positive integers $n \geq 2$, define $g(n)$ to be one more than the largest proper divisor of n. Hence $g(35) = 8$, since the proper divisors of 35 are 1, 5, and 7. For how many n in the range $2 \leq n \leq 100$ do we have $g(g(n)) = 2$?

5. A positive integer n is called a *good* number if
$$n^3 + 7n - 133 = m^3$$
for some positive integer m. What is the sum of all good numbers?

6. How many zeroes occur at the end of the number $1999^6 + 6 \cdot 1999 + 5$?

7. All the digits of the positive integer N are either 0 or 1. The remainder after dividing N by 37 is 18. What is the smallest number of times that the digit 1 can appear in N?

8. It is well-known that the n^{th} triangular number can be given by the formula $n(n+1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a, b, c) such that $a^2 + b^2 = c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a, b, c) such that $a \leq b < c$ and
$$\frac{a(a+1)}{2} + \frac{b(b+1)}{2} = \frac{c(c+1)}{2}.$$
For instance, $(3, 5, 6)$ is a PTTN $(6 + 15 = 21)$. Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

9. How many of the first 2010 rows of Pascal's Triangle (rows 0 through 2009) have exactly 256 odd entries?

10. For how many integers $1 \leq n \leq 9999$ is there a solution to the congruence
$$\phi(n) \equiv 2 \pmod{12},$$
where $\phi(n)$ is the Euler phi-function?