100 Geometry Problems: Bridging the Gap from AIME to USAMO

David Altizio

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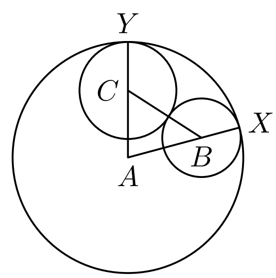
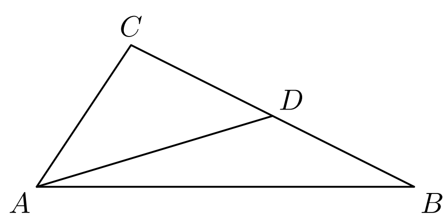
Abstract

This is a collection of one-hundred geometry problems from all around the globe designed for bridging the gap between computational geometry and proof geometry. Problems start middle-AMC level and go all the way to early IMO Shortlist level. As there are computational and proof problems mixed in with each other, relative difficulties may not be exact, so feel free to skip around. Enjoy![1]

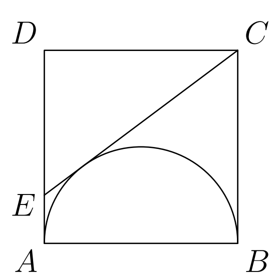
1. [MAΘ ????] In the figure shown below, circle B is tangent to circle A at X, circle C is tangent to circle A at Y, and circles B and C are tangent to each other. If $AB = 6$, $AC = 5$, and $BC = 9$, what is AX?



2. [AHSME ????] In triangle ABC, $AC = CD$ and $\angle CAB - \angle ABC = 30°$. What is the measure of $\angle BAD$?



3. [AMC 10A 2004] Square $ABCD$ has side length 2. A semicircle with diameter AB is constructed inside the square, and the tangent to the semicircle from C intersects side AD at E. What is the length of CE?



4. [AMC 10B 2011] Rectangle $ABCD$ has $AB = 6$ and $BC = 3$. Point M is chosen on side AB so that $\angle AMD = \angle CMD$. What is the degree measure of $\angle AMD$?

[1] This is the second version of the PDF. It fixes a few typoes and inaccuracies found in the first version.

5. [AIME 2011] On square $ABCD$, point E lies on side AD and point F lies on side BC, so that $BE = EF = FD = 30$. Find the area of the square.

6. Points A, B, and C are situated in the plane such that $\angle ABC = 90^\circ$. Let D be an arbitrary point on $\overline{AB}$, and let E be the foot of the perpendicular from D to $\overline{AC}$. Prove that $\angle DBE = \angle DCE$.

7. [AMC 10B 2012] Four distinct points are arranged in a plane so that the segments connecting them have lengths $a, a, a, a, 2a$, and b. What is the ratio of b to a?

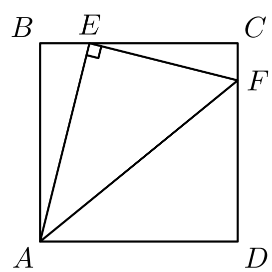
8. [Britain 2010] Let ABC be a triangle with $\angle CAB$ a right angle. The point L lies on the side BC between B and C. The circle BAL meets the line AC again at M and the circle CAL meets the line AB again at N. Prove that L, M, and N lie on a straight line.

9. [OMO 2014] Let ABC be a triangle with incenter I and $AB = 1400$, $AC = 1800$, $BC = 2014$. The circle centered at I passing through A intersects line BC at two points X and Y. Compute the length XY.

10. [India RMO 2014] Let ABC be an isosceles triangle with $AB = AC$ and let Γ denote its circumcircle. A point D is on arc AB of Γ not containing C. A point E is on arc AC of Γ not containing B. If $AD = CE$ prove that BE is parallel to AD.

11. A closed planar shape is said to be *equiable* if the numerical values of its perimeter and area are the same. For example, a square with side length 4 is equiable since its perimeter and area are both 16. Show that any closed shape in the plane can be dilated to become equiable. (A dilation is an affine transformation in which a shape is stretched or shrunk. In other words, if $\mathcal{A}$ is a dilated version of $\mathcal{B}$ then $\mathcal{A}$ is similar to $\mathcal{B}$.)

12. [David Altizio] Triangle AEF is a right triangle with $AE = 4$ and $EF = 3$. The triangle is inscribed inside square $ABCD$ as shown. What is the area of the square?



13. Points A and B are located on circle Γ, and point C is an arbitrary point in the interior of Γ. Extend AC and BC past C so that they hit Γ at M and N respectively. Let X denote the foot of the perpendicular from M to BN, and let Y denote the foot of the perpendicular from N to AM. Prove that $AB \parallel XY$.

14. [AIME 2007] Square $ABCD$ has side length 13, and points E and F are exterior to the square such that $BE = DF = 5$ and $AE = CF = 12$. Find EF^2.

15. Let Γ be the circumcircle of $\triangle ABC$, and let D, E, F be the midpoints of arcs AB, BC, CA respectively. Prove that $DF \perp AE$.

16. [AIME 1984] In tetrahedron $ABCD$, edge AB has length 3 cm. The area of face ABC is 15 cm^2 and the area of face ABD is 12 cm^2. These two faces meet each other at a 30° angle. Find the volume of the tetrahedron in cm^3.

17. Let $P_1P_2P_3P_4$ be a quadrilateral inscribed in a circle with diameter of length D, and let X be the intersection of its diagonals. If $P_1P_3 \perp P_2P_4$ prove that

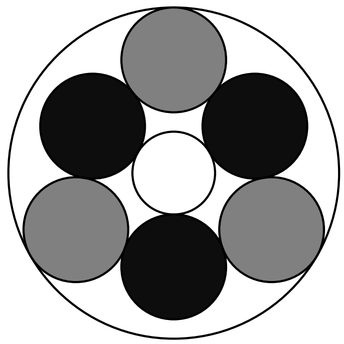
$$D^2 = XP_1^2 + XP_2^2 + XP_3^2 + XP_4^2.$$

18. [iTest 2008] Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find R^2, where R is the radius of the sphere.

19. [AIME 2008] In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37^\circ$, $\angle D = 53^\circ$, and M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length MN.

20. [Sharygin 2014] Let ABC be an isosceles triangle with base AB. Line ℓ touches its circumcircle at point B. Let CD be a perpendicular from C to ℓ, and AE, BF be the altitudes of ABC. Prove that D, E, F are collinear.

21. [Purple Comet 2013] Two concentric circles have radii 1 and 4. Six congruent circles form a ring where each of the six circles is tangent to the two circles adjacent to it as shown. The three lightly shaded circles are internally tangent to the circle with radius 4 while the three darkly shaded circles are externally tangent to the circle with radius 1. The radius of the six congruent circles can be written $\frac{k+\sqrt{m}}{n}$, where k, m, and n are integers with k and n relatively prime. Find $k + m + n$.



22. Let A, B, C, and D be points in the plane such that $\angle BAC = \angle CBD$. Prove that the circumcircle of $\triangle ABC$ is tangent to BD.

23. [Britain 1995] Triangle ABC has a right angle at C. The internal bisectors of angles BAC and ABC meet BC and CA at P and Q respectively. The points M and N are the feet of the perpendiculars from P and Q to AB. Find angle MCN.

24. Let $ABCD$ be a parallelogram with $\angle A$ obtuse, and let M and N be the feet of the perpendiculars from A to sides BC and CD. Prove that $\triangle MAN \sim \triangle ABC$.

25. For a given triangle $\triangle ABC$, let H denote its orthocenter and O its circumcenter.

    (a) Prove that $\angle HAB = \angle OAC$.[2]

    (b) Prove that $\angle HAO = |\angle B - \angle C|$.

26. Suppose P, A, B, C, and D are points in the plane such that $\triangle PAB \sim \triangle PCD$. Prove that $\triangle PAC \sim \triangle PBD$.

27. [AMC 12A 2012] Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

28. Let $ABCD$ be a cyclic quadrilateral with no two sides parallel. Lines AD and BC (extended) meet at K, and AB and CD (extended) meet at M. The angle bisector of $\angle DKC$ intersects CD and AB at points E and F, respectively; the angle bisector of $\angle CMB$ intersects BC and AD at points G and H, respectively. Prove that quadrilateral $EGFH$ is a rhombus.

29. [David Altizio] In $\triangle ABC$, $AB = 13$, $AC = 14$, and $BC = 15$. Let M denote the midpoint of $\overline{AC}$. Point P is placed on line segment $\overline{BM}$ such that $\overline{AP} \perp \overline{PC}$. Suppose that p, q, and r are positive integers with p and r relatively prime and q squarefree such that the area of $\triangle APC$ can be written in the form $\frac{p\sqrt{q}}{r}$. What is $p + q + r$?

30. [All-Russian MO 2013] Acute-angled triangle ABC is inscribed into circle Ω. Lines tangent to Ω at B and C intersect at P. Points D and E are on AB and AC such that PD and PE are perpendicular to AB and AC respectively. Prove that the orthocenter of triangle ADE is the midpoint of BC.

[2] As a result of this equality condition, lines AH and AO are said to be *isogonal conjugates*, i.e. reflections across the A-angle bisector.

31. For an acute triangle $\triangle ABC$ with orthocenter H, let H_A be the foot of the altitude from A to BC, and define H_B and H_C similarly. Show that H is the incenter of $\triangle H_A H_B H_C$.

32. [AMC 10A 2013] In $\triangle ABC$, $AB = 86$, and $AC = 97$. A circle with center A and radius AB intersects $\overline{BC}$ at points B and X. Moreover $\overline{BX}$ and $\overline{CX}$ have integer lengths. What is BC?

33. [APMO 2010] Let ABC be a triangle with $\angle BAC \neq 90^\circ$. Let O be the circumcenter of the triangle ABC and Γ be the circumcircle of the triangle BOC. Suppose that Γ intersects the line segment AB at P different from B, and the line segment AC at Q different from C. Let ON be the diameter of the circle Γ. Prove that the quadrilateral $APNQ$ is a parallelogram.

34. [AMC 10A 2013] A unit square is rotated 45° about its center. What is the area of the region swept out by the interior of the square?

35. [Canada 1986] A chord ST of constant length slides around a semicircle with diameter AB. M is the midpoint of ST and P is the foot of the perpendicular from S to AB. Prove that angle SPM is constant for all positions of ST.

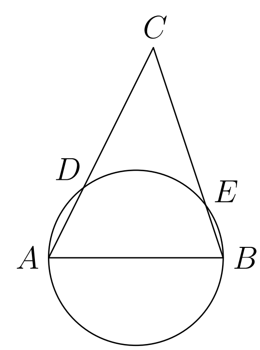
36. [Sharygin 2012] On side AC of triangle ABC an arbitrary point is selected D. The tangent in D to the circumcircle of triangle BDC meets AB in point C_1; point A_1 is defined similarly. Prove that $A_1C_1 \parallel AC$.

37. [AMC 10B 2013] In triangle ABC, $AB = 13$, $BC = 14$, and $CA = 15$. Distinct points D, E, and F lie on segments $\overline{BC}$, $\overline{CA}$, and $\overline{DE}$, respectively, such that $\overline{AD} \perp \overline{BC}$, $\overline{DE} \perp \overline{AC}$, and $\overline{AF} \perp \overline{BF}$. The length of segment $\overline{DF}$ can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

38. [Mandelbrot] In triangle ABC, $AB = 5$, $AC = 6$, and $BC = 7$. If point X is chosen on BC so that the sum of the areas of the circumcircles of triangles AXB and AXC is minimized, then determine BX.

39. [Sharygin 2014] Given a rectangle $ABCD$. Two perpendicular lines pass through point B. One of them meets segment AD at point K, and the second one meets the extension of side CD at point L. Let F be the common point of KL and AC. Prove that $BF \perp KL$.

40. [AIME Unused] In the figure, ABC is a triangle and $AB = 30$ is a diameter of the circle. If $AD = AC/3$ and $BE = BC/4$, then what is the area of the triangle?



41. [MOSP 1995] An interior point P is chosen in the rectangle $ABCD$ such that $\angle APD + \angle BPC = 180^\circ$. Find the sum of the angles $\angle DAP$ and $\angle BAP$.

42. Let ABC be a triangle and P, Q, R points on the sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ respectively. Prove that the circumcircles of $\triangle AQR$, $\triangle BRP$, and $\triangle CPQ$ intersect in a common point. This point is named the *Miquel point* of the configuration.

43. [AIME 2013] Let $\triangle PQR$ be a triangle with $\angle P = 75^\circ$ and $\angle Q = 60^\circ$. A regular hexagon $ABCDEF$ with side length 1 is drawn inside $\triangle PQR$ so that side $\overline{AB}$ lies on $\overline{PQ}$, side $\overline{CD}$ lies on $\overline{QR}$, and one of the remaining vertices lies on $\overline{RP}$. There are positive integers a, b, c, and d such that the area of $\triangle PQR$ can be expressed in the form $\frac{a+b\sqrt{c}}{d}$, where a and d are relatively prime and c is not divisible by the square of any prime. Find $a + b + c + d$.

44. ["Fact 5"] Let Γ be the circumcircle of an arbitrary triangle $\triangle ABC$. Furthermore, denote I its incenter and M the midpoint of minor arc $\widehat{BC}$. Prove that M is the circumcenter of $\triangle BIC$.

45. [AIME 2001] In triangle ABC, angles A and B measure 60 degrees and 45 degrees, respectively. The bisector of angle A intersects $\overline{BC}$ at T, and $AT = 24$. The area of triangle ABC can be written in the form $a + b\sqrt{c}$, where a, b, and c are positive integers, and c is not divisible by the square of any prime. Find $a + b + c$.

46. Let O be the circumcenter of a triangle ABC with $AB > AC$. Define M as the midpoint of $\overline{BC}$, D the foot of the altitude from A, and E the point on line AO such that $BE \perp AO$. Prove that $MD = ME$.

47. [India RMO 2008] Let ABC be an acute angled triangle; let D, F be the midpoints of BC, AB respectively. Let the perpendicular from F to AC and the perpendicular from B to BC meet at N. Prove that ND is equal in length to the circumradius of $\triangle ABC$.

48. [Sharygin 2012] Let ABC be a triangle, and let M be the midpoint of side BC. Point P is the foot of the altitude from B to the perpendicular bisector of segment AC. Suppose that lines PM and AB intersect at point Q. Prove that triangle QPB is isosceles.

49. [ELMO SL 2013] Let ABC be a triangle with incenter I. Let U, V and W be the intersections of the angle bisectors of angles A, B, and C with the incircle, so that V lies between B and I, and similarly with U and W. Let X, Y, and Z be the points of tangency of the incircle of triangle ABC with BC, AC, and AB, respectively. Let triangle UVW be the *David Yang triangle* of ABC and let XYZ be the *Scott Wu triangle* of ABC. Prove that the David Yang and Scott Wu triangles of a triangle are congruent if and only if ABC is equilateral.

50. [AIME 2001] Triangle ABC has $AB = 21$, $AC = 22$, and $BC = 20$. Points D and E are located on $\overline{AB}$ and $\overline{AC}$, respectively, such that $\overline{DE}$ is parallel to $\overline{BC}$ and contains the center of the inscribed circle of triangle ABC. Then $DE = m/n$, where m and n are relatively prime positive integers. Find $m + n$.

51. Inscribe equilateral triangle ABC inside a circle. Pick a point P on arc BC, and let AP intersect BC at Q. Prove that
$$\frac{1}{PQ} = \frac{1}{PB} + \frac{1}{PC}.$$

52. [Sharygin 2012] Let BM be the median of right-angled triangle $ABC(\angle B = 90^\circ)$. The incircle of triangle ABM touches sides AB, AM in points A_1, A_2; points C_1, C_2 are defined similarly. Prove that lines A_1A_2 and C_1C_2 meet on the bisector of angle ABC.

53. [IMSA] Let ω be a circle centered at point O. Lines AB and AC are tangent to ω at points B and C respectively. On line segment BC a point X is chosen, and ℓ is the line that passes through X perpendicular to XO. Let ℓ intersect AB and BC (or their extensions) at points K and L respectively. Prove that X is the midpoint of segment KL.

54. [Sharygin 2008] Quadrilateral $ABCD$ is circumscribed around a circle with center I. Prove that the projections of points B and D to the lines IA and IC lie on a single circle.

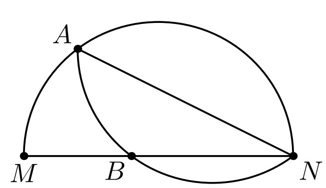
55. [HMMT] Let $ABCD$ be an isosceles trapezoid such that $AB = 10$, $BC = 15$, $CD = 28$, and $DA = 15$. There is a point E such that $\triangle AED$ and $\triangle AEB$ have the same area and such that EC is minimal. Find EC.

56. [Canada 2008] $ABCD$ is a convex quadrilateral for which AB is the longest side. Points M and N are located on sides AB and BC respectively, so that each of the segments AN and CM divides the quadrilateral into two parts of equal area. Prove that the segment MN bisects the diagonal BD.

57. [India RMO 2011] Let ABC be an acute angled scalene triangle with circumcentre O and orthocentre H. If M is the midpoint of BC, then show that AO and HM intersect on the circumcircle of ABC.

58. [Sharygin 2009] Let ABC be a triangle. Points M, N are the projections of B and C to the bisectors of angles C and B respectively. Prove that line MN intersects sides AC and AB in their points of contact with the incircle of ABC.

59. [PUMaC 2010] In the following diagram, a semicircle is folded along a chord AN and intersects its diameter MN at B. Suppose $MB : BN = 2 : 3$ and $MN = 10$. If $AN = x$, find x^2.



60. [BAMO 2001] Let $JHIZ$ be a rectangle, and let A and C be points on sides ZI and ZJ, respectively. The perpendicular from A to CH intersects line HI in X, and the perpendicular from C to AH intersects line HJ in Y. Prove that X, Y, and Z are collinear.

61. Let ABC be a triangle, and let D be a point on BC. Suppose O_1 and O_2 are the centers of the circles that circumscribe $\triangle ABD$ and $\triangle ACD$ respectively. Prove that $\triangle AO_1O_2 \sim \triangle ABC$.

62. [Ray Li] In triangle ABC, $AB = 36, BC = 40, CA = 44$. The bisector of angle A meet BC at D and the circumcircle at E different from A. Calculate the value of DE^2.

63. [APMO 2007] Let ABC be an acute angled triangle with $\angle BAC = 60^\circ$ and $AB > AC$. Let I be the incenter and H the orthocenter of the triangle ABC. Prove that $2\angle AHI = 3\angle ABC$.

64. [Brazil 2008] Let $ABCD$ be a cyclic quadrilateral and r and s the lines obtained reflecting AB with respect to the internal bisectors of $\angle CAD$ and $\angle CBD$, respectively. If P is the intersection of r and s and O is the center of the circumscribed circle of $ABCD$, prove that OP is perpendicular to CD.

65. [AIME 1986] In $\triangle ABC$, $AB = 425$, $BC = 450$, and $AC = 510$. An interior point P is then drawn, and segments are drawn through P parallel to the sides of the triangle. If these three segments are of an equal length d, find d.

66. Let $ABCD$ be a convex quadrilateral, and define P_1, P_2, P_3, P_4, P_5, and P_6 to be the midpoints of line segments AB, BC, CD, DA, AC, and BD respectively. Prove that lines P_1P_3, P_2P_4, and P_5P_6 all intersect in a single point.

67. [PUMaC 2013] An equilateral triangle is given. A point lies on the incircle of this triangle. If the smallest two distances from the point to the sides of the triangle is 1 and 4, the sidelength of this equilateral triangle can be expressed as $\frac{a\sqrt{b}}{c}$ where $(a, c) = 1$ and b is not divisible by the square of an integer greater than 1. Find $a + b + c$.

68. [IberoAmerican 2012] Let ABC be a triangle, P and Q the intersections of the parallel line to BC that passes through A with the external angle bisectors of angles B and C, respectively. The perpendicular to BP at P and the perpendicular to CQ at Q meet at R. Let I be the incenter of ABC. Show that $AI = AR$.

69. [Mexico 2012] Let $\mathcal{C}_1$ be a circumference with center O, P a point on it and ℓ the line tangent to $\mathcal{C}_1$ at P. Consider a point Q on ℓ different from P, and let $\mathcal{C}_2$ be the circumference passing through O, P and Q. Segment OQ cuts $\mathcal{C}_1$ at S and line PS cuts $\mathcal{C}_2$ at a point R diffferent from P. If r_1 and r_2 are the radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively, Prove
$$\frac{PS}{SR} = \frac{r_1}{r_2}.$$

70. [AMC 12B 2008] Let $ABCD$ be a trapezoid with $AB \parallel CD$, $AB = 11$, $BC = 5$, $CD = 19$, and $DA = 7$. Bisectors of $\angle A$ and $\angle D$ meet at P, and bisectors of $\angle B$ and $\angle C$ meet at Q. What is the area of hexagon $ABQCDP$?

71. [Sharygin 2010] Suppose X and Y are the common points of two circles ω_1 and ω_2. The third circle ω is internally tangent to ω_1 and ω_2 in P and Q respectively. Segment XY intersects ω in points M and N. Rays PM and PN intersect ω_1 in points A and D; rays QM and QN intersect ω_2 in points B and C respectively. Prove that $AB = CD$.

72. [Italy TST 2001] The diagonals AC and BD of a convex quadrilateral $ABCD$ intersect at point M. The bisector of $\angle ACD$ meets the ray BA at K. Given that
$$MA \cdot MC + MA \cdot CD = MB \cdot MD,$$
prove that $\angle BKC = \angle CDB$.

73. [Sharygin 2012] In acute triangle ABC inscribed in circle ω, let A' be the projection of A onto BC and B', C' the projections of A' onto AC, AB respectively. Line $B'C'$ intersects ω at X and Y and line AA' intersects ω for the second time at D. Prove that A' is the incenter of triangle XYD.

74. Let P, Q, R be arbitary points on the sides BC, CA, AB respectively of triangle ABC. Prove that the circumcenters of triangles AQR, BRP, CPQ form a triangle similar to triangle ABC.

75. [Mandelbrot 2008] Triangle ABC has sides of length $AB = \sqrt{41}$, $AC = 5$, and $BC = 8$. Let O be the center of the circumcircle of $\triangle ABC$, and let A' be the point diametrically opposite A with respect to circle O. Determine the area of $\triangle A'BC$.

76. [AIME 2008] In triangle ABC, $AB = AC = 100$, and $BC = 56$. Circle P has radius 16 and is tangent to $\overline{AC}$ and $\overline{BC}$. Circle Q is externally tangent to P and is tangent to $\overline{AB}$ and $\overline{BC}$. No point of circle Q lies outside of $\triangle ABC$. The radius of circle Q can be expressed in the form $m - n\sqrt{k}$, where m, n, and k are positive integers and k is the product of distinct primes. Find $m + nk$.

77. Let P, A, B, C, D be points in the plane such that $\triangle PAB \sim \triangle PCD$, and let M and N be the midpoints of $\overline{AC}$ and $\overline{BD}$ respectively. Show that $\triangle PAB \sim \triangle PMN \sim \triangle PCD$.

78. [AIME 2002] In triangle ABC the medians $\overline{AD}$ and $\overline{CE}$ have lengths 18 and 27, respectively, and $AB = 24$. Extend $\overline{CE}$ to intersect the circumcircle of ABC at F. The area of triangle AFB is $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Find $m + n$.

79. [USAMO 1999] Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

80. [IMO 2000] Two circles G_1 and G_2 intersect at two points M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through the point M, with C on G_1 and D on G_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

81. [AIME 2008] Let $\overline{AB}$ be a diameter of circle ω. Extend AB through A to C. Point T lies on ω so that line CT is tangent to ω. Point P is the foot of the perpendicular from A to line CT. Suppose $AB = 18$, and let m denote the maximum possible length of segment BP. Find m^2.

82. [IberoAmerican 2003] Let C and D be two points on the semicircle with diameter AB such that B and C are on distinct sides of the line AD. Denote by M, N and P the midpoints of AC, BD and CD respectively. Let O_A and O_B the circumcentres of the triangles ACP and BDP. Show that the lines O_AO_B and MN are parallel.

83. [AIME 2009] In triangle ABC, $AB = 10$, $BC = 14$, and $CA = 16$. Let D be a point in the interior of $\overline{BC}$. Let I_B and I_C denote the incenters of triangles ABD and ACD, respectively. The circumcircles of triangles BI_BD and CI_CD meet at distinct points P and D. The maximum possible area of $\triangle BPC$ can be expressed in the form $a - b\sqrt{c}$, where a, b, and c are positive integers and c is not divisible by the square of any prime. Find $a + b + c$.

84. [IMO 2014] Let P and Q be on segment BC of an acute triangle ABC such that $\angle PAB = \angle BCA$ and $\angle CAQ = \angle ABC$. Let M and N be the points on AP and AQ, respectively, such that P is the midpoint of AM and Q is the midpoint of AN. Prove that the intersection of BM and CN is on the circumcircle of triangle ABC.

85. [AIME 2005] Triangle ABC has $BC = 20$. The incircle of the triangle evenly trisects the median AD. If the area of the triangle is $m\sqrt{n}$ where m and n are integers and n is not divisible by the square of a prime, find $m + n$.

86. [Japanese Theorem] Let A_1, A_2, A_3, A_4 be arbitrary points on circle ω in that order. For each positive integer $1 \le k \le 4$, define I_k to be the incenter of $\triangle A_kA_{k+1}A_{k+2}$, where indeces are taken modulo 4 (so that $A_5 = A_1$, etc.). Show that $I_1I_2I_3I_4$ is a rectangle.

87. [Iran 2007] Two circles C, D are exterior tangent to each other at point P. Point A is in the circle C. We draw 2 tangents AM, AN from A to the circle D (M,N are the tangency points). The second meet points of AM, AN with C are E, F, respectively. Prove that $\frac{PE}{PF} = \frac{ME}{NF}$.

88. [Sharygin 2009] Let CL be a bisector of triangle ABC. Points A_1 and B_1 are the reflections of A and B in CL, points A_2 and B_2 are the reflections of A and B in L. Let O_1 and O_2 be the circumcenters of triangles AB_1B_2 and BA_1A_2 respectively. Prove that angles O_1CA and O_2CB are equal.

89. [IMO 1990] Chords $\overline{AB}$ and $\overline{CD}$ of a circle intersect at a point E inside the circle. Let M be an interior point of the segment $\overline{EB}$. The tangent line at E to the circle through D, E, and M intersects the lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ at F and G, respectively. If $AM/AB = t$, find EG/EF in terms of t.

90. [All-Russian MO 2001] Let the circle ω_1 be internally tangent to another circle ω_2 at N. Take a point K on ω_1 and draw a tangent AB which intersects ω_2 at A and B. Let M be the midpoint of the arc AB which is on the opposite side of N. Prove that the circumradius of $\triangle KBM$ does not depend on the choice of K.

91. [USAJMO 2011] Points A, B, C, D, E lie on a circle ω and point P lies outside the circle. The given points are such that (i) lines PB and PD are tangent to ω, (ii) P, A, C are collinear, and (iii) $\overline{DE} \parallel \overline{AC}$. Prove that $\overline{BE}$ bisects $\overline{AC}$.

92. [Iran 2011] Let ABC be a triangle and denote its circumcircle centered at O by ω. Points M and N lie on sides AB and AC respectively. The circumcircle of triangle AMN intersects ω for the second time at Q. Let P be the intersection point of MN and BC. Prove that PQ is tangent to ω if and only if $OM = ON$.

93. [ISL 2007] Denote by M midpoint of side BC in an isosceles triangle $\triangle ABC$ with $AC = AB$. Take a point X on a smaller arc $\widehat{MA}$ of the circumcircle of triangle $\triangle ABM$. Denote by T point inside of angle BMA such that $\angle TMX = 90$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on the choice of X.

94. [Italy TST 2005] The circle Γ and the line ℓ have no common points. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \neq A$, B is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Let H be the reflection of G in AB. Show that F, C, and H are collinear.

95. Let Ω be the circumcircle of a triangle ABC. A circle ω with center O passes through B and C and meets the segments $\overline{AC}$ and $\overline{AB}$ again at D and E respectively. Let $P \neq A$ be the point at which the circumcircle of $\triangle ADE$ meets Ω. Prove that $AP \perp PO$.

96. [All-Russian MO 2008] A circle ω with center O is tangent to the rays of an angle BAC at B and C. Point Q is taken inside the angle BAC. Assume that point P on the segment AQ is such that $AQ \perp OP$. The line OP intersects the circumcircles ω_1 and ω_2 of triangles BPQ and CPQ again at points M and N. Prove that $OM = ON$.

97. [OMO 2014] Let $AXYBZ$ be a convex pentagon inscribed in a circle with diameter $\overline{AB}$. The tangent to the circle at Y intersects lines BX and BZ at L and K, respectively. Suppose that $\overline{AY}$ bisects $\angle LAZ$ and $AY = YZ$. If the minimum possible value of

$$\frac{AK}{AX} + \left(\frac{AL}{AB}\right)^2$$

can be written as $\frac{m}{n} + \sqrt{k}$, where m, n and k are positive integers with $\gcd(m, n) = 1$, compute $m + 10n + 100k$.

98. [ISL 2006] Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE, \qquad \angle ABC = \angle ACD = \angle ADE.$$

Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

99. [ISL 2011] Let $A_1A_2A_3A_4$ be a non-cyclic quadrilateral. Let O_1 and r_1 be the circumcentre and the circumradius of the triangle $A_2A_3A_4$. Define O_2, O_3, O_4 and r_2, r_3, r_4 in a similar way. Prove that

$$\frac{1}{O_1A_1^2 - r_1^2} + \frac{1}{O_2A_2^2 - r_2^2} + \frac{1}{O_3A_3^2 - r_3^2} + \frac{1}{O_4A_4^2 - r_4^2} = 0.$$

100. [USAMO 2008] Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$ intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle.