AIME Solution Set 2015

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Abstract

These problems are ones I have collected from my problem-solving over the past few years that resemble AIME level problems, except that none of them have actually appeared on the AIME. Each problem (should) has a nonnegative integer answer, and each of the four sections have ten problems roughly ordered by difficulty. I tried to make the sections of similar difficulties, but this is probably not the case (Combinatorics and Number Theory seem easier than Algebra and Geometry). Have fun with the problems!

The majority of these problems have been selected from both collections of questions (AoPS, Brilliant, various math camps) and actual contests (HMMT, iTest, Math League, Mandelbrot[1], NIMO, etc.).

[1] Several of the Mandelbrot problems that appear in this collection came from the book *Mandelbrot Morsels*. To be honest, it has some pretty awesome problems, and it's one of the few books that has a bunch of problems that haven't all been released onto the Internet yet. It's definitely worth checking out!

1 Algebra

1. Let c be the larger solution to the equation $x^2 - 20x + 13 = 0$. Compute the area of the circle with center (c, c) passing through the point $(13, 7)$.

   Solution. It suffices to find the distance between the two points (c, c) and $(13, 7)$ - this value will be the radius r of the circle and the desired answer is πr^2. We manipulate carefully and hope for the best:

$$\begin{aligned} r &= \sqrt{(c-13)^2 + (c-7)^2} \\ &= \sqrt{(c^2 - 26c + 169) + (c^2 - 14c + 49)} \\ &= \sqrt{2c^2 - 40c + 218} \\ &= \sqrt{2(c^2 - 20c + 13) + 192} = \sqrt{192}. \end{aligned}$$

   Success! As per our reasoning earlier the area of the circle is therefore $\boxed{192\pi}$.

   Source. Mandelbrot 2013-2014

2. Suppose a, b, and c are real numbers such that

$$\left(a + \frac{1}{b}\right)\left(b + \frac{1}{c}\right)\left(c + \frac{1}{a}\right) = \left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right).$$

   If $abc = 13$, what is $|a + b + c|$?

   Solution. Let $s = a + b + c$. Expanding both sides of the equality and manipulating yields

$$\begin{aligned} abc + a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{abc} &= 1 + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc} + \frac{1}{abc} \\ \implies abc + a + b + c &= 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \\ &= 1 + \frac{a+b+c}{abc} \\ \implies 13 + s &= 1 + \frac{s}{13}. \end{aligned}$$

   Solving this equation yields $|s| = \boxed{013}$.

   Source. AoPS, otherwise unknown

3. What is the only real number $x > 1$ which satisfies the equation

$$\log_2 x \log_4 x \log_6 x = \log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x?$$

   Solution. This equation looks scary, but in fact it has a simple solution. Divide both sides by $\log_2 x \log_4 x \log_6 x$ to get

$$\begin{aligned} 1 &= \frac{\log_2 x \log_4 x + \log_2 x \log_6 x + \log_4 x \log_6 x}{\log_2 x \log_4 x \log_6 x} \\ &= \frac{1}{\log_6 x} + \frac{1}{\log_4 x} + \frac{1}{\log_2 x} \\ &= \frac{1}{\log x / \log 6} + \frac{1}{\log x / \log 4} + \frac{1}{\log x / \log 2} \\ &= \frac{\log 6}{\log x} + \frac{\log 4}{\log x} + \frac{\log 2}{\log x} = \frac{\log 48}{\log x}. \end{aligned}$$

   Therefore $\log x = \log 48 \implies x = \boxed{48}$.

   Source. Math League 1988-1989

4. Suppose that P is the polynomial of least degree with integral coefficients for which $P(\sqrt{3}+\sqrt{2}) = \sqrt{3}-\sqrt{2}$. Compute $P(2)$.

Solution. Let $t = \sqrt{3}+\sqrt{2}$. Note that $\sqrt{3}-\sqrt{2} = \frac{1}{t}$, so we wish to find a polynomial $P(x)$ such that $P(t) = \frac{1}{t}$. Squaring gives

$$t^2 = (\sqrt{3}+\sqrt{2})^2 = 5+\sqrt{24} \implies t^2 - 5 = \sqrt{24}.$$

Squaring again gives $(t^2-5)^2 = t^4 - 10t^2 + 25 = 24$, so $t^4 - 10t^2 = -1$ and $10t - t^3 = \frac{1}{t}$. Hence $P(x) = 10x - x^3$ and the requested answer is $P(2) = 20 - 8 = \boxed{012}$.

Source. Math League 1997-1998/2009-2010

5. Let $a_1, a_2, \ldots$ be a sequence defined by $a_1 = 1$ and for $n \geq 1$,

$$a_{n+1} = \sqrt{a_n^2 - 2a_n + 3} + 1.$$

Find a_{513}.

Solution. Note that the given recursion can be rewritten as

$$a_{n+1} - 1 = \sqrt{a_n^2 - 2a_n + 3} = \sqrt{(a_n-1)^2+2} \implies (a_{n+1}-1)^2 = (a_n-1)^2 + 2.$$

Let $b_n = (a_n - 1)^2$. Then $b_{n+1} = b_n + 2$ and $b_1 = 0$, so $a_n = 2(n-1)$ for all positive integers n. (This is easily provable by induction or anything else.) Hence $b_{513} = 1024$, so

$$(a_{513}-1)^2 = 1024 \implies a_{513} - 1 = 32 \implies a_{513} = \boxed{033}.$$

Source. OMO Fall 2012, Ray Li

6. Let $\{x_n\}_{n=1}^{150}$ be a sequence of real numbers such that $x_i \in \{\sqrt{2}+1, \sqrt{2}-1\}$ for all positive integers i with $1 \leq i \leq 150$. For how many positive integers $1 \leq S \leq 1000$ does there exist such a sequence $\{x_n\}$ with the property that

$$x_1x_2 + x_3x_4 + x_5x_6 + \cdots + x_{149}x_{150} = S?$$

Solution. Let $a_i = x_{2i-1}x_{2i}$ for all positive integers i. Note that each a_i can take one of three possible values: 1, $3+\sqrt{8}$, and $3-\sqrt{8}$. Let a, b, and c be the number of the a_i that are each of these values respectively. Then by considering both the total number of terms and the value of each term we can set up the system of equations

$$\begin{cases} a+b+c & = 75, \\ a + (3+\sqrt{8})b + (3-\sqrt{8})c & = S. \end{cases}$$

Note that the LHS of the second equation can be written in the form $m + n\sqrt{2}$ where m and n are integers. However, we want it to be an integer, so $n = 0 \implies b = c$. Therefore we can substitute b into every place a c exists and simplify to get a new simplified system

$$\begin{cases} a + 2b & = 75, \\ a + 6b & = S. \end{cases}$$

This implies there exists a bijection between the number of possible values of S and the number of solutions in positive integers to the Diophantine equation $a + 2b = 75$. To compute this new value, we simply realize that a can take on every odd integer between 1 and 75, and that each of these values of a gives a unique b. Therefore the number of possible values of S is $\frac{75+1}{2} = \boxed{038}$.

Source. Adapted from Argentina TST 2005

7. Let $c_1, c_2, c_3, \ldots, c_{2008}$ be complex numbers such that

$$|c_1| = |c_2| = |c_3| = \cdots = |c_{2008}| = 2011,$$

and let $S(2008, t)$ be the sum of all products of these 2008 complex numbers taken t at a time. Let Q be the maximum possible value of

$$\left|\frac{S(2008,1492)}{S(2008,516)}\right|.$$

Find the remainder when Q is divided by 1000.

Solution (Official). Let $c_k = 2011z_k$ for $1 \le k \le 2008$ such that $z_k = \operatorname{cis}\theta_k$ for some angle θ_k. Noting that $1492 + 516 = 2008$, we recognize that there are $\binom{2008}{1492} = \binom{2008}{516}$ terms in both the numerator and denominator. Letting $p_1, p_2, \ldots, p_{\binom{2008}{1492}}$ be the products of the z_k taken 1492 at a time, and

$$P = z_1 z_2 \ldots z_{2008},$$

we have

$$\begin{aligned}
\left|\frac{S(2008,1492)}{S(2008,516)}\right| &= \left|\frac{2011^{1492}\left(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}}\right)}{2011^{516}P\left(\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_{\binom{2008}{1492}}}\right)}\right| \\
&= \left|\frac{2011^{976}\left(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}}\right)}{P\left(\overline{p_1} + \overline{p_2} + \cdots + \overline{p_{\binom{2008}{1492}}}\right)}\right| \\
&= 2011^{976} \cdot \left|\frac{1}{P}\right| \cdot \left|\frac{\left(p_1 + p_2 + \cdots + p_{\binom{2008}{1492}}\right)}{\left(\overline{p_1 + p_2 + \cdots + p_{\binom{2008}{1492}}}\right)}\right| \\
&= 2011^{976} \cdot 1 \cdot 1 = 2011^{976}.
\end{aligned}$$

Finally, to compute the last three digits of this integer, we note that

$$2011^{976} \equiv 11^{976} \equiv (10+1)^{976} \equiv 10^2\binom{976}{2} + 10\binom{976}{1} + 1 \equiv \boxed{761} \pmod{1000}.$$

Source. Modified from iTest Tournament of Champions 2008

8. Let S be the sum of all x such that $1 \le x \le 99$ and

$$\{x^2\} = \{x\}^2.$$

Find the number formed by the first three digits of $\lfloor S \rfloor$. (Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x.)

Solution. Using the definition $\{x\} = x - \lfloor x \rfloor$ gives

$$x^2 - \lfloor x^2 \rfloor = (x - \lfloor x \rfloor)^2 \implies \lfloor x^2 \rfloor = 2x\lfloor x \rfloor - \lfloor x \rfloor^2.$$

Let $\lfloor x \rfloor = k$. Note that we must have $2kx - k^2$ to be integral, so $x = k + \dfrac{a}{2k}$ for some positive integer $0 \le a < 2k$. I claim that all of these work. Indeed, note that

$$\{x^2\} = \left\{\left(k + \frac{a}{2k}\right)\right\}^2 = \left\{k^2 + a + \left(\frac{a}{2k}\right)^2\right\} = \left(\frac{a}{2k}\right)^2 = \{x\}^2.$$

Now it remains to find their sum. This is calculation; if we are careful, we should get that the sum equals

$$\begin{aligned}
&99+\sum_{k=1}^{98}\left[k+\left(k+\frac{1}{2k}\right)+\left(k+\frac{2}{2k}\right)+\cdots+\left(k+\frac{2k-1}{2k}\right)\right] \\
&=99+\sum_{k=1}^{98}\left[k(2k)+\frac{1}{2k}\sum_{a=0}^{2k-1}a\right] \\
&=99+\sum_{k=1}^{98}\left[2k^2+k-\frac{1}{2}\right] \\
&=99+2\left[\frac{98(99)(197)}{6}\right]+\frac{98(99)}{2}-48=641999.
\end{aligned}$$

Finding $\lfloor S\rfloor$ as opposed to S is unnecessary and the resulting answer is $\boxed{641}$.

Source. iTest 2007

9. Let a, b, c, and d be positive real numbers such that

$$\begin{aligned}
a^2+b^2 &= c^2+d^2 = 2008, \\
ac &= bd = 1000.
\end{aligned}$$

If $S=a+b+c+d$, compute the value of $\lfloor S\rfloor$.

Solution. Note that since $a^2+b^2=c^2+d^2=2008$, we have $(a^2+b^2)(c^2+d^2)=2008^2$. Multiplying this out gives

$$(ac)^2+(ad)^2+(bc)^2+(bd)^2=2008^2.$$

Next note that if x and y are two equal numbers then $x^2+y^2=2xy$. In this case, letting $x=ac$ and $y=bd$ gives $(ac)^2+(bd)^2=2abcd$. Therefore

$$(ac)^2+(ad)^2+(bc)^2+(bd)^2=(ad)^2+2abcd+(bc)^2=(ad+bc)^2=2008^2,$$

so $ad+bc=2008$.

Now note that since $a^2+b^2=c^2+d^2=ad+bc$, we have

$$a^2+b^2+c^2+d^2-2(ad+bc)=0 \implies (a-d)^2+(b-c)^2=0,$$

implying that $a=d$ and $b=c$. Finally, remark that

$$(a+b)(c+d)=ac+bd+(ad+bc)=4008.$$

From the equalities established above, we have $a+b=c+d=\sqrt{4008}$, so $\lfloor S\rfloor=\lfloor 2\sqrt{2008}\rfloor=\boxed{126}$.

Source. iTest 2008

10. Suppose a,b,c,d are integers such that

$$a+b+c+d=0 \qquad \text{and} \qquad (ab-cd)(ac-bd)(ad-bc)+528^2=0.$$

What is the maximum possible value of a?

Solution. Note that using the condition $a+b+c+d=0$ we can see that

$$\begin{aligned}
ab-cd &= ab+ac-ac-cd=a(b+c)-c(a+d) \\
&= a(b+c)+c(b+c)=(a+c)(b+c).
\end{aligned}$$

Similarly, $ac-bd=(a+d)(c+d)$ and $ad-bc=(a+b)(d+b)$. Therefore, the second equation becomes

$$(a+b)(a+c)(a+d)(b+c)(b+d)(c+d)=-528^2.$$

Noting once again that $a+b=-(c+d)$, $a+c=-(b+d)$, and $a+d=-(b+c)$, we can simplify this product to

$$-(a+b)^2(a+c)^2(a+d)^2=-528^2 \implies (a+b)(a+c)(a+d)=528.$$

(It is easy to see why the negative solution should be thrown out here.) It remains to maximize the sum of the three factors on the left hand side, as $(a+b)+(a+c)+(a+d)=3a+b+c+d=2a$, and maximizing this sum will in turn maximize a.

Lemma 1. *Suppose x, y, and z are positive integers such that $xyz=c$. Then $x+y+z\le c+2$.*

Proof. Note that since $x\ge 1$ and $y\ge 1$, we have

$$(x-1)(y-1)\ge 0 \implies x+y\le xy+1.$$

By applying this repeatedly we get

$$x+y+z\le xy+z+1\le xyz+2=c+2$$

as desired. Equality holds when $(x,y,z)=(c,1,1)$ or permutations. □

Applying this lemma to our original problem, we have

$$2a\le 528+1+1=530 \implies a\le \boxed{265}.$$

Source. Brilliant.org

2 Combinatorics

1. A random pizza is made by flipping a fair coin to decide whether to include pepperoni, then doing the same for sausage, mushrooms, and onions. The probability that two random pizzas have at least one topping in common can be written in the form $\frac{m}{n}$ where m and n are positive integers. Find $m+n$.

   Solution. We first compute the probability that the two random pizzas have no toppings in common. Note that for any one topping, there is a $\frac{3}{4}$ probability that it is not common to both pizzas (since the probability of this is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$). Since there are four toppings, and each of the coin flips is independent, the probability that two random pizzas have no toppings in common is $(\frac{3}{4})^4=\frac{81}{256}$. Hence the probability that the two pizzas have at least one topping in common is $1-\frac{81}{256}=\frac{175}{256}$ and the requested answer is $\boxed{431}$.

   Source. Mandelbrot 2010-2011

2. How many permutations $(a_1,a_2,\ldots,a_{10})$ of $1,2,3,4,5,6,7,8,9,10$ satisfy

$$|a_1-1|+|a_2-2|+\cdots+|a_{10}-10|=4?$$

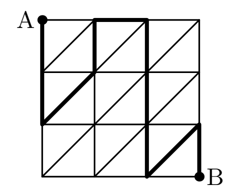
   Solution (Zimbalono). Any transposition alters the sum by an even amount. All permutations can be constructed out of transpositions, so the sum must be even. No element can move more than two steps away from its original position without making the sum greater than 4. So to create a sum of 4, you need

   - two transpositions of pairs of consecutive numbers, or
   - one transposition of numbers differing by two, or
   - one cycle of length three of consecutive numbers (forward or back).

   In the first case, you have $7+6+5+4+3+2+1=28$, in the second case, you have 8, and in the third case, you have $8\cdot 2=16$. Total of $\boxed{52}$.

   Source. Turkey First Round 2012

3. How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is highlighted.



   Solution. The important part of this problem is that the path can not move to the left in any way, shape, or form. Once it leaves the first column, it can not traverse back there again. Thus, it is advantageous to consider each column independently.

   Note that there are seven ways to travel from the first column to the second column: meander through either the three diagonal sides or the four horizontal ones. Once this first column is left, it can be ignored, and the problem is reduced to determining the number of ways to travel from the second column to the third one. But note that this is the exact same problem as before! No matter where the path enters the second column, there are seven ways the path can go to the third column. Similarly, there are seven ways the path can go from the third to the fourth column, at which point the path drops down to B.

   This is thus an algorithm to construct any of the $7^3=\boxed{343}$ possible paths.

   Source. Mandelbrot 2013-2014

4. Consider the set $S=\{1,2,3,4,5,\ldots,100\}$. How many subsets of this set with two or more elements satisfy:

   (i) the terms of the subset form an arithmetic sequence, and

   (ii) we cannot include another element from S with this subset to form an even longer arithmetic sequence?

Solution. The key idea in this problem is that each subset of S we desire is determined uniquely by the arithmetic sequence it contains. To prove this, suppose S_1 and S_2 are two distinct subsets with $S_1 \subset S_2$. Then we can add elements in S_2 to the front or back of the arithmetic sequence to obtain S_1, but then S_1 and S_2 are not distinct, contradiction!

Let a and d be the first term and common difference of the arithmetic sequence in S. We now do casework based on the value of d.

CASE 1: $d \leq 50$. Here, remark that a can take on any of the values $1, 2, \ldots, d$. This is because if $a = d+1$, then a 1 can be appended to the sequence to get a subset we have already counted. As a result, for every d there are d possible values of a.

CASE 2: $d > 50$. In this case, we have to be careful, since there can be only at most two elements in the subset we desire. Hence S must be one of the sets

$$\{1, d+1\}, \quad \{2, d+2\}, \quad \ldots, \quad \{100-d, 100\},$$

so for every d in this case there are $100 - d$ possible values of a.

In conclusion, the total number of subsets S that satisfy the problem condition is

$$1 + 2 + \cdots + 49 + 50 + 49 + \cdots + 2 + 1 = 50 + 2(1 + 2 + \cdots + 49) = 50 + 2\left(\frac{49 \cdot 50}{2}\right) = \boxed{2500}.$$

Source. AoPS Intermediate Counting and Probability

5. Simon and Garfunkle play in a round-robin golf tournament. Each player is awarded one point for a victory, a half point for a tie, and no points for a loss. Simon beat Garfunkle in the first game by a record margin as Garfunkle sent a shot over the bridge and into troubled waters on the final hole. Garfunkle went on to score 8 total victories, but no ties at all. Meanwhile, Simon wound up with exactly 8 points, including the point for a victory over Garfunkle. Amazingly, every other player at the tournament scored exactly n. Find the sum of all possible values of n.

Solution. Consider the complete graph with k vertices of all the games played at the competition. There are two ways to count the total number of points awarded to all players in the tourney. First is the obvious way: counting with respect to the vertices. Simon and Garfunkle both score 8 points (since from the information given Garfunkle has 8 wins and some number of losses but no ties for 8 points total), while the other $k-2$ competitors get n points for a total of $16 + n(k-2)$. However, there's another way to count the number of points total: counting with respect to the edges. For every match played, the total number of points goes up by 1 (since each player is awarded half a point in a tie). For k players, the number of total matches (and hence total points) is $\binom{k}{2}$. Setting these equal to each other gives $16 + n(k-2) = \binom{k}{2}$, or

$$n = \frac{\binom{k}{2} - 16}{k-2} = \frac{k^2 - k - 32}{2(k-2)} = \frac{1}{2}\left[k + 1 - \frac{30}{k-2}\right].$$

The only integer k greater than nine that make the RHS an integer are 12, 17, and 32. This gives $n = 5, 8, 16$ and the requested sum is $\boxed{29}$.

Source. iTest Tournament of Champions 2008

6. An ant starts at the origin of a coordinate plane. Each minute, it either walks one unit to the right or one unit up, but it will never move in the same direction more than twice in the row. In how many different ways can it get to the point (5,5)?

Solution (Official). We can change the ant's sequence of moves to a sequence a_1, a_2, …, a_{10}, with $a_i = 0$ if the i-th step is up, and $a_i = 1$ if the i-th step is right. We define a subsequence of moves a_i, a_{i+1}, …, a_j, $(i \leq j)$ as an *up run* if all terms of the subsequence are equal to 0, and a_{i-1} and a_{j+1} either do not exist or are not equal to 0, and define a *right run* similarly. In a sequence of moves, up runs and right runs alternate, so the number of up rights can differ from the number of right runs by at most one.

Now let $f(n)$ denote the number of sequences $a_1, a_2, \ldots, a_n$ where $a_i \in \{1,2\}$ for $1 \le i \le n$, and $a_1 + a_2 + \cdots + a_n = 5$. (In essence, we are splitting the possible 5 up moves into up runs, and we are doing the same with the right moves.) We can easily compute that $f(3) = 3$, $f(4) = 4$, $f(5) = 1$, and $f(n) = 0$ otherwise.

For each possible pair of numbers of up runs and right runs, we have two choices of which type of run is first. Our answer is then

$$2\left(f(3)^2 + f(3)f(4) + f(4)^2 + f(4)f(5) + f(5)^2\right) = 2\,(9 + 12 + 16 + 4 + 1) = \boxed{84}.$$

Source. HMMT November 2010

7. Thaddeus is given a 2013×2013 array of integers each between 1 and 2013, inclusive. He is allowed two operations:

    1. Choose a row, and subtract 1 from each entry.
    2. Chooses a column, and add 1 to each entry.

    He would like to get an array where all integers are divisible by 2013. Let M be the number of initial arrays for which this is possible. What is the number formed by the last three digits of M?

    Solution (Official). We claim that the set of grids on which it is possible to obtain an array of all zeroes $\pmod{2013}$ is indexed by ordered 4025-tuples of residues $\pmod{2013}$, corresponding to the starting entries in the first row and the first column of the grid, giving the answer of $\boxed{2013^{4025}}$. To do this, we show that given after fixing all of the entries in the first row and column, there is a unique starting grid which can become an array of all zeroes after applying the appropriate operations.

    Let $a_{i,j}$ be the entry in the i-th row and j-th column. Suppose there is a sequence of operations giving all zeroes in the array; let r_i be the number of times we operate on row i, and let c_j be the number of times we operate on column j. It is enough to take all of these values to be residues modulo 2013. Clearly, $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j. In particular, $r_1 + c_1 \equiv a_{1,1}$. Now for each i and j we have

    $$\begin{aligned} a_{i,j} &\equiv -r_i - c_j \\ &\equiv (a_{i,1} + c_1) + (a_{1,j} + r_1) \\ &\equiv a_{i,1} + a_{1,j} - a_{1,1}, \end{aligned}$$

    which is fixed. Thus the rest of the entries in the grid are forced.

    Conversely, if we set $a_{i,j}$ to be the appropriate representative of the residue class of $a_{i,1} + a_{1,j} - a_{1,1}$ modulo 2013, we may take $r_i \equiv -a_{i,1} \pmod{2013}$, and $c_j \equiv a_{1,1} - a_{1,j} \pmod{2013}$ for each i and j. It is clear that $a_{i,j} + r_i + c_j \equiv 0 \pmod{2013}$ for each i, j, so we're done.

    Source. HMMT February 2013

8. If you flip a fair coin 1000 times, let P be the expected value of the product of the number of heads and the number of tails. What are the first three digits of P?

    Solution (yimingz89). Let x denote expected number of heads and n denote the total number of coin flips (in this case, $n = 1000$). We're looking for $\mathbb{E}(n(n-x))$. By linearity of expectation, $\mathbb{E}(n(n-x)) = n\mathbb{E}(x) - \mathbb{E}(x^2)$. Recall that $\mathrm{Var}(x) = \mathbb{E}(x^2) - \mathbb{E}(x)^2$. Note that the variance for each coin flip is just $\frac{1}{4}$, so $\mathrm{Var}(x) = \frac{n}{4}$. Plugging into the varience equation and solving for the expected value of x^2, we get $\mathbb{E}(x^2) = \frac{n}{4} + \frac{n^2}{4}$. Therefore,

    $$\mathbb{E}(n(n-x)) = n \times \frac{n}{2} - \left(\frac{n}{4} + \frac{n^2}{4}\right) = \frac{n^2 - n}{4}.$$

    Plugging in $n = 1000$, we get $\boxed{249}750$.

OR

Solution (Naysh). Take

$$f(x,y)=\left(\frac{x+y}{2}\right)^{1000}.$$

Then, the problem is just asking for us to compute

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}f(x,y)\right)\bigg|_{(x,y)=(1,1)}.$$

Differentiating $f(x,y)$ partially first with respect to y and then with respect to x yields the new function

$$\frac{1000\cdot 999}{4}\cdot\left(\frac{x+y}{2}\right)^{998}.$$

So, the expected value of the product of the number of heads and tails is just $999\cdot 250$, whose first three digits are just $\boxed{249}$. ■

OR

Solution (bobthesmartypants). The probability of getting k heads and $1000-k$ tails is $\frac{1}{2^{1000}}\binom{1000}{k}$. The product that this gives is $k(1000-k)$. Thus, the expected value for this case is

$$k(1000-k)\frac{\binom{1000}{k}}{2^{1000}}$$

and the expected value overall is

$$\sum_{k=0}^{1000}k(1000-k)\frac{\binom{1000}{k}}{2^{1000}}.$$

Now we do some clever algebraic manipulation to get

$$\begin{aligned}\sum_{k=0}^{1000}k(1000-k)\frac{\binom{1000}{k}}{2^{1000}}&=\frac{1}{2^{1000}}\sum_{k=0}^{1000}k(1000-k)\frac{1000!}{k!(1000-k)!}\\&=\frac{1}{2^{1000}}\sum_{k=1}^{999}\frac{1000!}{(k-1)!(1000-k-1)!}\\&=\frac{999000}{2^{1000}}\sum_{k=0}^{998}\frac{998!}{k!(998-k)!}\\&=\frac{999000}{2^{1000}}\sum_{k=0}^{998}\binom{998}{k}\\&=\frac{999000}{2^{1000}}\cdot 2^{998}=\boxed{249}750.\end{aligned}$$

Source. HMMT November 2014

9. Suppose N is the number of ways to partition the counting numbers from 1 to 12 (inclusive) into four sets with three numbers in each set so that the product of the numbers in each set is divisible by 6. What is the number formed by the last three digits of N?

Note: it is possible that this solution below contains a mistake, since although I think the answer I get differs from the official one, I have no way anymore of checking the actual solution right now. In the event that this solution is actually incorrect, do not hesitate to contact me.

Solution. First remark that there are exactly four numbers divisible by 3 to choose from - 3, 6, 9, and 12. Thus, it must be the case that each one of these goes into its own set, so the sets must be of the form

$$\{3,\ ,\ \},\quad \{6,\ ,\ \},\quad \{9,\ ,\ \},\quad \{12,\ ,\ \}.$$

Now in order to fufill the condition, the sets containing 3 and 9 must have an even number as well. There are four left to choose from - 2, 4, 8, and 10. It thus suffices to find the number of ways one can place the remaining eight numbers into the eight slots above such that the sets containing 3 and 9 have at least one even number.

It turns out it is slightly easier to count the number of configurations which *do not* satisfy this property - as we shall see later, this is because the cases are more symmetric. While a classic PIE strategy suffices here, in reality we can be a bit more clever. Consider the distribution of the even numbers among the four sets. Note that if the product of the elements of some set is not divisible by 6, then it must not be the case that an even number appers in all four subsets. This leaves only two possible distributions of even numbers left.

- First suppose two of the sets have two even numbers and the other two have none. There are 5 ways to choose which two sets get the two even numbers, $\binom{4}{2} = 6$ ways to split the even numbers between the two sets, and $\binom{4}{2} = 6$ ways to split the remaining four numbers between the two remaining sets. This gives $5 \cdot 6^2 = 180$ possibilities total.
- Now suppose one set gets two even numbers while two other sets get one number each. There are 6 ways to choose the sets which get chosen for the even numbers. Then there are $4 \cdot 3 = 12$ ways to distribute the even numbers among the given sets, and $4 \cdot 3 = 12$ ways to distribute the remaining four numbers among the remaining four positions. This yields $6 \cdot 12^2 = 864$ possibilities.

Finally, since the total number of distributions of the eight numbers is $\binom{8}{2,2,2,2} = 2520$, our answer is

$$2520 - 180 - 864 = 1476 \equiv \boxed{476} \pmod{1000}.$$

Source. Mandelbrot 2002-2003

10. Seven points are spaced equally around a circle each having labeled with some number. A labeling is *clean* if for any two pairs of points a, b and c, d with a having the same label as b and c as d, but a not having the same label as c, the chords connecting ab and cd do not intersect. Additionally, two *clean* labelings are the same if the set of points that have the same label in one labeling are the same as in the other and if the points can be rotated to equal the other. How many unique *clean* labelings are there?

Solution. I actually don't know how to do this one....

Source. Unknown

3 Geometry

1. Regular hexagon $ABCDEF$ is given in the plane. If the area of the triangle whose vertices are the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ is 225, what is the area of $ABCDEF$?

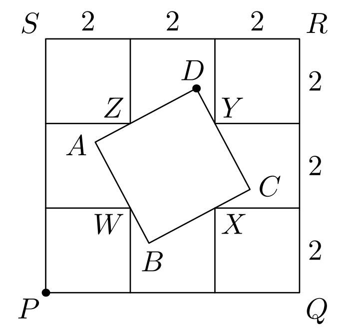
   Solution. Suppose s is the side length of the hexagon, and let M, N, and P denote the midpoints of $\overline{AB}$, $\overline{CD}$, and $\overline{EF}$ respectively. Then MN is a midline of trapezoid $ABCD$, and so $MN = \frac{3}{2}s$. Thus

$$\frac{[ABCDEF]}{[MNP]} = \frac{6 \cdot s^2\sqrt{3}/4}{(3s/2)^2\sqrt{3}/4} = \frac{8}{3},$$

   and so the requested answer is $\frac{8}{3} \cdot 225 = \boxed{600}$.

   Source. MATHCOUNTS

2. In the corners of a square $PQRS$ with side length 6 cm four smaller squares are placed with side lengths 2 cm. Let us denote their vertices by W, X, Y, Z like in the picture. A square $ABCD$ is constructed in such a way, that points W, X, Y, Z lie inside the sides AB, BC, CD, DA respectively. Find the square of the largest possible distance between points P and D.



   Solution. Let M denote the midpoint of $\overline{YZ}$. Then $PM = 5$ by Pythagorean Theorem and $MD = \frac{1}{2} \cdot YZ = 1$, and so Triangle Inequality dictates that $PD^2 \leq 6^2 = \boxed{36}$, with equality achieved when P, M, and D are collinear.

   Source. Cayley 2007

3. Two perpendicular planes intersect a sphere in two circles. These circles intersect in two points, A and B, such that $AB = 42$. If the radii of the two circles are 54 and 66, find the remainder when R^2 is divided by 1000, where R is the radius of the sphere.

   Solution. Let O, O_1, and O_2 be the centers of the sphere, the circle with radius 54, and the circle with radius 66 respectfully. In addition, let M be the midpoint of $\overline{AB}$. Note that OO_1 is perpendicular to the plane containing circle O_1, so $OO_1 \perp O_1A$. Additionally, since $O_1M \perp O_2M$, we have that $OO_1 = MO_2$. Therefore,

$$\begin{aligned} R^2 = AO^2 = AO_1^2 + OO_1^2 = AO_1^2 + MO_2^2 = AO_1^2 + (AO_2^2 - AM^2) \\ = 54^2 + 66^2 - 21^2 = \boxed{6831}. \end{aligned}$$

   Source. iTest 2008

4. Two circles, ω_1 and ω_2, have radii of 5 and 12 respectively, and their centers are 13 units apart. The circles intersect at two different points P and Q. A line l is drawn through P and intersects the circle ω_1 at $X \neq P$ and ω_2 at $Y \neq P$. Find the maximum value of $PX \cdot PY$.

   Solution. Let the centers of the circles ω_1 and ω_2 be O_1 and O_2 respectively, and let the projections of O_1 and O_2 onto XY be M and N, once again respectively. Let $\theta = \angle XPO_1$. Since $\triangle O_1PM$ is right, we have $PM = O_1P\cos\theta \implies PX = 2O_1P\cos\theta = 10\cos\theta$. Furthermore, since $O_1O_2 = 13$, $\triangle O_1PO_2$ is a right triangle, which implies that $\angle O_1PM$ and $\angle O_2PN$ are complementary. Thus, $NP = O_2P\cos\left(\frac{\pi}{2} - \theta\right) \implies YP = 2O_2P\sin\theta = 24\sin\theta$. Therefore

$$PX \cdot PY = (10\cos\theta) \cdot (24\sin\theta) = 240\sin\theta\cos\theta = 120\sin 2\theta \leq \boxed{120}.$$

   Equality holds when $\theta = \frac{\pi}{4}$.

   Source. West Windsor Plainsboro Math Tournament 2013

5. Two circles in the Cartesian plane have four common tangent lines. If the slopes of these lines are 2, 3, 4, and m, in increasing order, then calculate $\lfloor 100m \rfloor$.

Solution (ahaanomegas). First, notice that the two internal tangents are the ones with slopes 2 and m and the two external tangents are the ones with slopes 3 and 4. The internal tangents and the external tangents are symmetric about the line through the centers of the circles. Say that line has slope a. Then we have the system of equations

$$\begin{cases} \arctan(a) - \arctan(3) &= \arctan(4) - \arctan(a), \\ \arctan(a) - \arctan(2) &= \arctan(m) - \arctan(a). \end{cases}$$

Using trig identities, we get $\frac{a-3}{1+3a} = \frac{4-a}{1+4a}$ and $\frac{a-2}{1+2a} = \frac{m-a}{1+ma}$. Solving gives $m = \frac{29}{3}$. This slope is indeed the largest of the four, so we are done and our answer is $\lfloor 100 \cdot \frac{29}{3} \rfloor = \boxed{966}$.

Source. Mandelbrot 2004-2005

6. Let point O be the origin of a three-dimensional coordinate system, and let points A, B, and C be located on the positive x, y, and z axes, respectively. Suppose $OA = \sqrt[4]{75}$ and $m\angle BAC = 30^\circ$. Compute $100K$, where K is the area of $\triangle ABC$.

Solution (brandbest1). The problem reduces to finding $\frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{|\overrightarrow{AB}||\overrightarrow{AC}| \sin 30^\circ}{2}$. Let $A = (\sqrt[4]{75}, 0, 0), B = (0, b, 0), C = (0, 0, c)$. We know that

$$\overrightarrow{AB} = \begin{pmatrix} -\sqrt[4]{75} & b & 0 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{AC} = \begin{pmatrix} -\sqrt[4]{75} & 0 & c \end{pmatrix}.$$

Note that $\overrightarrow{AB} \cdot \overrightarrow{AC} = |\overrightarrow{AB}||\overrightarrow{AC}| \cos 30^\circ = \sqrt{75}$. This makes $|\overrightarrow{AB}||\overrightarrow{AC}| = 10$. Substituting this back into our equation, we get

$$K = \frac{|\overrightarrow{AB}||\overrightarrow{AC}| \sin 30^\circ}{2} = \frac{10}{4} = \frac{5}{2}$$

Therefore, our answer, $100K$, is $\boxed{250}$.

Source. Mandelbrot 2004-2005

7. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$, and denote by ω the unique circle inscribed inside the triangle. Suppose the orthocenter of $\triangle ABC$ lies on ω. Then there exist relatively prime positive integers m and n such that $\cos \angle BAC = \frac{m}{n}$. Find $m + n$.

Solution. Let the vertices of the triangle be A, B, and C, with A the vertex of the isosceles triangle. Let H be the orthocenter and I the incenter of the triangle. Additionally, let D denote the intersection of the incircle and BC, and let F be the foot of the perpendicular from B to AC.

Suppose that $\angle ABC = x$. Then $\angle FBC = 90^\circ - \angle ACB = 90^\circ - x$. Additionally, since I is the intersection of the angle bisectors of the triangle, $\angle IBD = \dfrac{x}{2}$. Next note that $\tan \angle IBD = \frac{ID}{DB}$ and $\tan \angle HBD = \frac{HD}{DB}$. Plugging in our expressions for x, this gives

$$\tan(90^\circ - x) = \frac{HD}{DB} = 2\left(\frac{ID}{DB}\right) = 2 \tan \frac{x}{2} \implies \cot x = 2 \tan \frac{x}{2}.$$

Rewriting in terms of sines and cosines (and in addition using a tangent half-angle identity) gives

$$\frac{\cos x}{\sin x} = 2\left(\frac{1 - \cos x}{\sin x}\right) \implies \cos x = 2(1 - \cos x) \implies \cos x = \frac{2}{3}.$$

Finally, we have

$$\cos \angle BAC = \cos(180^\circ - 2x) = -\cos(2x) = 1 - 2\cos^2 x = 1 - 2\left(\frac{4}{9}\right) = \boxed{\frac{1}{9}}.$$

Source. ARML 1983

8. Let $\triangle ABC$ have $AB = 6$, $BC = 7$, and $CA = 8$, and denote by ω its circumcircle. Let N be a point on ω such that AN is a diameter of ω. Furthermore, let the tangent to ω at A intersect BC at T, and let the second intersection point of NT with ω be X. The length of $\overline{AX}$ can be written in the form $\frac{m}{\sqrt{n}}$ for positive integers m and n, where n is not divisible by the square of any prime. Find $m + n$.

Solution. Note that $AT \perp AN$ and $AX \perp TN$, so it suffices to first compute AT and AN. AN is easier: by Law of Cosines $\cos A = \frac{17}{32} \implies \sin A = \frac{7\sqrt{15}}{32}$. Hence $AN = 2R = \frac{BC}{\sin\angle A} = \frac{32}{\sqrt{15}}$. AT, however, is a bit harder; we shall solve for its length for general a, b, c with $b > c$. Let $AT = x$ and $TC = y$. From $\triangle ABT \sim \triangle CAT$ we obtain two equations for similarity, $x^2 = y(y - a)$ and $cy = bx$. Multiplying both sides of the first equation by b^2 gives

$$\begin{aligned}(bx)^2 = (cy)^2 &= b^2(y^2 - ay)\\ c^2y &= b^2y - b^2a\\ y &= \frac{b^2a}{b^2 - c^2}\\ \implies x &= \frac{c}{b}\left(\frac{b^2a}{b^2 - c^2}\right) = \frac{abc}{b^2 - c^2}.\end{aligned}$$

Plugging in the specific numbers gives $AT = \frac{6\cdot 7\cdot 8}{8^2 - 6^2} = 12$.

Now we compute AX. To do this, scale computations down by a factor of 4, so that $AT = 3$ and $AN = \frac{8}{\sqrt{15}}$. By Pythagorean Theorem,

$$NT^2 = AT^2 + AN^2 = 9 + \frac{64}{15} = \frac{199}{15} \implies NT = \frac{\sqrt{199}}{\sqrt{15}}.$$

Therefore

$$AT \cdot AN = NT \cdot AX \implies AX = \frac{AT \cdot AN}{NT} = \frac{3 \cdot \frac{8}{\sqrt{15}}}{\frac{\sqrt{199}}{\sqrt{15}}} = \frac{24}{\sqrt{199}}.$$

Scaling back up gives a final value of $AX = \frac{96}{\sqrt{199}}$, and the requested answer is $96 + 199 = \boxed{295}$.

Source. NIMO 16, Own

9. Given a convex, n-sided polygon P, form a $2n$-sided polygon clip(P) by cutting off each corner of P at the edges' trisection points. In other words, clip(P) is the polygon whose vertices are the $2n$ edge trisection points of P, connected in order around the boundary of P. Let P_1 be an isosceles trapezoid with side lengths $13, 13, 13$, and 3, and for each $i \geq 2$, let $P_i = \text{clip}(P_{i-1})$. This iterative clipping process approaches a limiting shape $P_\infty = \lim_{i\to\infty} P_i$. If the difference of the areas of P_{10} and P_∞ is written as a fraction $\frac{x}{y}$ in lowest terms, calculate the number of positive integer factors of $x \cdot y$.

Solution (fedja). Let D_n be the difference in the areas between P_n and P_{n+1}. Let our trapezoid be $P_1 = ABCD$ (and $[ABCD] = \frac{12(3+13)}{2} = 96$); then without loss of generality construct diagonal BD.

Let A_1, A_2 be the trisection points on $\overline{AB}, \overline{AD}$, respectively, that are closest to A. Then the operation clip(P) deletes $\triangle A_1AA_2$. Since $A_1A/AB = 1/3, A_2A/AD = 1/3$, and $\triangle A_1AA_2, \triangle BAD$ share common $\angle A$, we have $\triangle A_1AA_2 \sim \triangle BAD$ by side ratio $1/3$. Their areas are in the ratio $(1/3)^2 = 1/9$.

Similarly, $[C_1CC_2] = \frac{1}{9}[BCD]$, and $[A_1AA_2] + [C_1CC_2] = \frac{1}{9}[ABCD]$. Cutting along diagonal AC, we get the same result, so $D_1 = \frac{2}{9}P_1$.

We now consider the effects of the second clipping. Without loss of generality consider what happens along the vertex A_1 of P_2. Let A_{11} be the trisection point along $\overline{AB}$ (again closest to A_1), and A_{12} be the trisection point along $\overline{A_1A_2}$. Now $\frac{A_1A_{11}}{AA_1} = \frac{(AB/3)/3}{AB/3} = \frac{1}{3}$ and $\frac{A_1A_{12}}{A_1A_2} = \frac{1}{3}$, and

$$\angle AA_1A_2 = 180 - \angle A_{11}A_1A_{12} \implies \sin(\angle AA_1A_2) = \sin(\angle A_{11}A_1A_{12}).$$

Using the $\frac{1}{2}ab\sin C$ definition of the area of a triangle, we see that $[A_1A_{11}A_{12}] = \frac{1}{9}[AA_1A_2]$. A similar clipping about A_2 gives $[A_2A_{21}A_{22}] = \frac{1}{9}[AA_1A_2]$; around each clipped region in D_1, we clip a new area $2/9D_1$. Generalizing, we have the recursion $D_n = \frac{2}{9}\cdot D_{n-1}$.

As a result,

$$P_n = P_1 - D_1 - D_2 - \cdots - D_{n-1} = 96 - 96\left(\left(\frac{2}{9}\right) + \left(\frac{2}{9}\right)^2 + \cdots + \left(\frac{2}{9}\right)^{n-1}\right).$$

Hence

$$P_{10} - P_\infty = 96\left(\left(\frac{2}{9}\right)^{10} + \left(\frac{2}{9}\right)^{11} + \cdots\right) = \left(\frac{2}{9}\right)^{10}\left(\frac{1}{1-2/9}\right) = \frac{2^{15}}{3^{17}\cdot 7}$$

and so $xy = 2^{15}\cdot 3^{17}\cdot 7$ has $16\cdot 18\cdot 2 = \boxed{576}$ factors.

Source. iTest 2008

10. Let ABC be a triangle, and I its incenter. Let the incircle of ABC touch side BC at D, and let lines BI and CI meet the circle with diameter AI at points P and Q, respectively. Given $BI = 6, CI = 5, DI = 3$, find the sum of the numerator and denominator of $(DP/DQ)^2$ when written in lowest terms.

Solution (Official). Let the incircle touch sides AC and AB at E and F respectively. Noe that E and F both lie on the circle with diameter AI since $\angle AEI = \angle AFI = 90^\circ$. The key observation is that D, E, and P are collinear. To prove this, suppose that P lies outside the triangle (the other case is analogous), then

$$\angle PEA = \angle PIA = \angle IBA + \angle IAB = \frac{1}{2}(\angle B + \angle A) = 90^\circ - \frac{1}{2}\angle C = \angle DEC,$$

which implies that D, E, and P are collinear. Similarly, D, F, and Q are collinear. Then, by Power of a Point, $DE\cdot DP = DF\cdot DQ$, so $DP/DQ = DF/DE$.

Now we compute DF/DE. Note that

$$DF = 2DB\sin\angle DBI = 2\sqrt{6^2-3^2}\left(\frac{3}{6}\right) = 3\sqrt{3}$$

and that

$$DE = 2DC\sin\angle DCI = 2\sqrt{5^2-3^2}\left(\frac{3}{5}\right) = \frac{24}{5}.$$

Therefore, $(DF/DE)^2 = 75/64$ and the requested answer is $\boxed{139}$.

Source. HMMT February 2008

4 Number Theory

1. Jack chose nine different integers from 1 through 19 and found their sum. From the remaining ten integers, Jill chose nine and found their sum. If the ratio of Jack's sum to Jill's sum was $7 : 15$, which of the nineteen integers was chosen by *neither* Jack nor Jill?

   Solution. Let x be a positive integer such that Jack's integers sum to $7x$ while Jill's integers sum to $15x$, and let n be the integer not chosen by either person. Then adding all nineteen integers together gives

   $$7x + 15x + n = 1 + 2 + 3 + \cdots + 19 \qquad \Longrightarrow \qquad 22x + n = 190.$$

   Taking $\pmod{22}$ of both sides (i.e. considering the remainders of various parts of the equation upon division by 22) gives $n \equiv 190 \equiv 14 \pmod{22}$. Therefore, since $n \leq 19$, we must have $n = \boxed{14}$. (Indeed, this scenario can occur if, for example, Jack picks 2, 3, 4, 5, 6, 7, 8, 9, and 12 while Jill picks 1, 10, 11, 13, 15, 16, 17, 18, and 19.)

   Source. Math League

2. There exist unique positive integers x and y such that $4^y - 615 = x^2$. What is the value of $x + y$?

   Solution. Note that the equation rearranges to

   $$615 = 4^y - x^2 = (2^y - x)(2^y + x).$$

   Now it suffices to find two numbers which multiply to 615 and which average to a power of two. Experimentation yields that the only possibility is $615 = 5 \cdot 123$, which yields $y = 6$, $x = 59$, and $x + y = \boxed{65}$.

   Source. Math League 2005-2006

3. In the binary expansion of

   $$\frac{2^{2007} - 1}{2^{225} - 1},$$

   how many of the first $10,000$ digits to the right of the radix point are 0's?

   Solution. First, I claim that the answer is the same as that of the binary expansion of $\frac{2^{207}-1}{2^{225}-1}$. To prove this, remark that $225 | 1800$, so $2^{225} | 2^{1800} - 1$. Thus $2^{1800} \equiv 1 \pmod{2^{225} - 1}$, so $2^{2007} - 1 \equiv 2^{207} - 1 \pmod{2^{225} - 1}$ as desired. Now rewrite the number as

   $$\frac{\frac{2^{207}-1}{2^{225}}}{1 - \frac{1}{2^{225}}}.$$

   In binary, $2^{207} - 1$ corresponds to the integer with 207 consecutive ones. Also, division by 2^{225} shifts the digits over 225 places. This value is the first term of a geometric series with common ratio $\frac{1}{2^{225}}$. Thus this pattern repeats, and we get that the fractional part of $\frac{2^{207}-1}{2^{225}-1}$ is

   $$0.\underbrace{00\cdots0}_{18\,0s}\,\underbrace{11\cdots1}_{207\,1s}\,\underbrace{00\cdots0}_{18\,0s}\,\underbrace{11\cdots1}_{207\,1s}\cdots.$$

   Each block of zeroes and ones takes up 225 digits, so $\lfloor\frac{10000}{225}\rfloor = 44$ complete blocks are present within the first 10000 digits of the binary expansion. In the 45^{th} block, since $10000 \equiv 100 \pmod{225}$, the entire set of 0's is located within the first 10000 digits as well. Thus the number of zeroes is $18 \cdot 45 = \boxed{810}$.

   Source. iTest 2007

4. For positive integers $n \geq 2$, define $g(n)$ to be one more than the largest proper divisor of n. Hence $g(35) = 8$, since the proper divisors of 35 are 1, 5, and 7. For how many n in the range $2 \leq n \leq 100$ do we have $g(g(n)) = 2$?

   Solution (Official). We observe that $g(n) = 2$ if and only if n is a prime, since composite numbers all have at least one proper divisor between 1 and n, making $g(n)$ larger than 2. Therefore $g(g(n)) = 2$ exactly when $g(n)$ is a prime. We now count how many times this occurs for $2 \leq n \leq 100$. Evidently we could have

$g(n) = 2$ or $g(n) = p$ for an odd prime $p > 2$. In the first case n itself must be prime (as we have just seen), and there are 25 primes from 2 to 100. But there are also cases such as $n = 44$ whose largest proper divisor 22 is one less than a prime, making $g(44) = 23$. These cases occur whenever n has the form $n = 2(p-1)$ for p an odd prime. There are 14 such values of n, from $2(3-1) = 4$ through $2(47-1) = 92$, for a grand total of $25 + 14 = \boxed{39}$ values of n.

Source. Mandelbrot 2013-2014

5. A positive integer n is called a *good* number if

$$n^3 + 7n - 133 = m^3$$

for some positive integer m. What is the sum of all good numbers?

Solution (tastymath75025). Consider when $n < m$. Then

$$m^3 - n^3 = 7n - 133 \geq 3n^2 + 3n + 1$$

since the minimum value of $m^3 - n^3$ is when $m = n+1$. Thus we reduce the inequality to $3n^2 - 4n + 134 \leq 0$, impossible by checking the discriminant.

If $n = m$, we see they are both 19.

Now consider when $n > m$. Clearly

$$n^3 - m^3 = 133 - 7n \geq 3n^2 - 3n + 1$$

since the minimum of $n^3 - m^3$ is $n^3 - (n-1)^3$. Thus, by simplifying the inequality, we find $3n^2 + 4n - 132 \leq 0$. Clearly this restricts n to $1, 2, 3, 4, 5, 6$ and we may check them manually to see that $(6, 5)$ and $(5, 3)$ are the only solutions.

So to conclude, our solutions are $\boxed{(5,3), (6,5), \text{ and } (19,19)}$ for (n, m).

Source. Unknown

6. How many zeroes occur at the end of the number $1999^6 + 6 \cdot 1999 + 5$?

Solution. Write the number in question as $n^6 + 6n + 5$ for $n = 1999$. Now remark that this factors as

$$(n+1)(n^5 - n^4 + n^3 - n^2 + n + 5) = (n+1)^2(n^4 - 2n^3 + 3n^2 - 4n + 5).$$

The $(n+1)^2 = 2000^2$ term contributes six factors of 5. With respect to the other term, note that

$$n^4 - 2n^3 + 3n^2 - 4n + 5 \equiv (-1)^4 - 2(-1)^3 + 3(-1)^2 - 4(-1) + 5 \equiv 15 \pmod{25},$$

meaning that it is divisible by 5 but not 25. Thus this term adds exactly one extra factor of 2, meaning that the desired answer is $\boxed{7}$.

Source. Mandelbrot 2003-2004

7. All the digits of the positive integer N are either 0 or 1. The remainder after dividing N by 37 is 18. What is the smallest number of times that the digit 1 can appear in N?

Solution. The key is to note that $37 \times 3 = 111$, so $10^3 \equiv 999 + 1 \equiv 1 \pmod{37}$. Hence we can break the number into blocks of 3 when computing the remainder modulo 37. As an example, the numbers 12345678 and $12 + 345 + 678$ are congruent mod 37.

Let a, b, c be the total number of ones that are ones, tens, and hundreds digits in these blocks. (For example, if $N = 110101$, then $(a, b, c) = (1, 1, 2)$.) Then since each "units digit" contributes one to the remainder, each "tens digit" 10, and each "hundreds digit" $100 \equiv -11$, we have

$$N \equiv a + 10b - 11c \equiv (a + b + c) + 3(3b - 4c) \equiv 18 \pmod{37}.$$

We first wish to find the minimum possible value of $a + b + c$; this will in turn limit our options and leave only a finite number of cases to check.

Now it just becomes trial and error. Obviously $a+b+c=1,2,3$ do not work. If $a+b+c=4$, then

$$3(3b-4c)\equiv 14 \pmod{37} \implies 3b-4c\equiv 17 \pmod{37}.$$

Simple checking yields no solutions with $b,c\le 4$. On the other hand, if $a+b+c=5$, then

$$3(3b-4c)\equiv 13 \pmod{37} \implies 3b-4c\equiv -8 \pmod{37},$$

and here it's easy to see that $(a,b,c)=(3,0,2)$ works! Hence the smallest number of possible ones is $\boxed{5}$, and it is easy to see that $1,101,101$ is the smallest possible construction under this case.

Source. AMC (Australian Mathematics Competition) 2013

8. It is well-known that the n^{th} triangular number can be given by the formula $n(n+1)/2$. A Pythagorean triple of *square numbers* is an ordered triple (a,b,c) such that $a^2+b^2=c^2$. Let a Pythagorean triple of *triangular numbers* (a PTTN) be an ordered triple of positive integers (a,b,c) such that $a\le b<c$ and

$$\frac{a(a+1)}{2}+\frac{b(b+1)}{2}=\frac{c(c+1)}{2}.$$

For instance, $(3,5,6)$ is a PTTN $(6+15=21)$. Here we call both a and b *legs* of the PTTN. Find the smallest natural number n such that n is a leg of *at least* six distinct PTTNs.

Solution. Multiplying everything by 8 and adding two to both sides gives

$$(2a+1)^2+(2b+1)^2=1+(2c+1)^2.$$

Making the substitution $x=2a+1$, etc, we now come across the simplified equation $x^2+y^2=z^2+1$. The one caveat is that all three variables are odd.

We're trying to find the smallest positive integer that appears in at least six different triples (x,y,z) that satisfy this equation (as only x or y, NOT z). Let this integer be x. Then rewrite as $x^2-1=z^2-y^2=(z-y)(z+y)$. Since both y and z are odd, $z-y$ and $z+y$ are both even and additionally have different residues $\pmod 4$. Therefore one of these two quantities must have exactly one factor of 2 while the other one must have at least two. For all odd x, $8|x^2-1$ since $a^2\equiv 0,1,4 \pmod 8$ for integer a, so we'll never have fewer than three factors of 2 at any point in time. Hence in order to get our set of solutions, we find all solutions in integers to the equation $w=(z-y)(z+y)$ (where w is the number that results after all powers of two are removed from the prime factorization of x^2-1), add these powers of two back, and convert to a PTTN.

Now we have to figure out what to do with all those odd factors. At first glance, it seems like the condition that x be contained in at least six PTTN's implies that x^2-1 must have at least six positive odd divisors. However, we need to be more strict than this. The pair $(y,z)=(1,x)$ always works as a solution, but $y=1$ translates to $b=0$, which is not a positive integer! Hence the number of odd positive divisors of x^2-1 actually has to be at least *eight*. (It can't be seven because then $x^2-1=(p^3)^2$ for some prime p, but no two perfect squares differ by 1.)

At this point, it suffices to bash out numbers. (I tried some less-bashy methods to find one solution, but then ended up grinding through the whole list anyway to confirm that no smaller solutions existed.)

$$\begin{aligned}
3^2 - 1 &= 8 = 2^3, \\
5^2 - 1 &= 24 = 2^3 \times 3, \\
7^2 - 1 &= 48 = 2^4 \times 3, \\
9^2 - 1 &= 80 = 2^4 \times 5, \\
11^2 - 1 &= 120 = 2^3 \times 3 \times 5, \\
13^2 - 1 &= 168 = 2^3 \times 3 \times 7, \\
15^2 - 1 &= 224 = 2^5 \times 7, \\
17^2 - 1 &= 288 = 2^3 \times 3^2, \\
19^2 - 1 &= 18 \times 20 = 2^3 \times 3^2 \times 5, \\
21^2 - 1 &= 20 \times 22 = 2^3 \times 5 \times 11, \\
23^2 - 1 &= 22 \times 24 = 2^4 \times 3 \times 11, \\
25^2 - 1 &= 24 \times 26 = 2^4 \times 3 \times 13, \\
27^2 - 1 &= 26 \times 28 = 2^3 \times 7 \times 13, \\
29^2 - 1 &= 28 \times 30 = 2^3 \times 3 \times 5 \times 7.
\end{aligned}$$

Success! Now for each of the 8 pairs of integers that satisfy $z^2 - y^2 = 3 \times 5 \times 7 = 105$, we can partition powers of two uniquely as mentioned previously to get our set of solutions. Indeed, when $x = 29 \implies a = 14$, there are at least six PTTN's that work (namely the seven $(14, 104, 105)$, $(14, 33, 36)$, $(14, 18, 23)$, $(11, 14, 18)$, $(5, 14, 15)$, $(14, 14, 20)$, and $(14, 51, 53)$). Hence the smallest natural numbers n that works is $\boxed{14}$.

Source. iTest Tournament of Champions 2008

9. How many of the first 2010 rows of Pascal's Triangle (rows 0 through 2009) have exactly 256 odd entries?

Solution. To solve this problem easily, we use the following lemma.

Lemma 2 (Lucas)**.** *Let m and n be positive integers and p a prime. Write $m = \overline{m_k \cdots m_1 m_0}_p$ and $n = \overline{n_k \cdots n_1 n_0}_p$ as the base-p representations of m and n respectively. Then*

$$\binom{m}{n} \equiv \binom{m_k}{n_k} \cdots \binom{m_1}{n_1}\binom{m_0}{n_0} \pmod{p}.$$

Proof. Write

$$\begin{aligned}
(1+x)^m &= (1+x)^{m_0 + m_1 p + \cdots + m_k p^k} \\
&= (1+x)^{m_0}(1+x)^{m_1 p} \cdots (1+x)^{m_k p^k} \equiv_p (1+x)^{m_0}(1+x^p)^{m_1} \cdots (1+x^{p^k})^{m_k}.
\end{aligned}$$

Now consider the coefficient of x^n of both sides. The LHS is trivially $\binom{m}{n}$. To get the right side, write $n = n_0 + n_1 p + \cdots + n_k p^k$. Then it is not hard to see from the fact that $0 \le m_i \le p-1$ for all i that the coefficient of x^n on the right hand side is

$$\begin{aligned}
&\left([x^{n_0}](1+x)^{m_0}\right)\left([x^{n_1 p}](1+x^p)^{m_1}\right) \cdots \left([x^{n_k p^k}](1+x^{p^k})^{m_k}\right) \\
\equiv_p &\binom{m_0}{n_0}\binom{m_1}{n_1} \cdots \binom{m_k}{n_k}
\end{aligned}$$

as desired. □

Now fix some $0 \le n \le 2009$, and let $0 \le k \le n$ vary. Write, as before,

$$\binom{n}{k} \equiv \binom{n_\ell}{k_\ell} \cdots \binom{n_1}{k_1}\binom{n_0}{k_0} \pmod{2}.$$

Now note that $\binom{0}{0} = \binom{1}{0} = \binom{1}{1} = 1$ and $\binom{0}{1} = 0$. Thus, $\binom{n}{k}$ is even if and only if there exists a j such that the j^{th} digit in the binary expansion of n is zero while the j^{th} digit in the binary expansion of k is one. It follows that the total number of entries in the n^{th} row which are odd is $2^{f(n)}$, where $f(n)$ denotes the number of ones in the binary expansion of n (since for each of these digits we have a choice of 0 or 1 for the corresponding digit in k, while all other digits are forced).

The condition that row n has 256 odd entries reduces down to n having eight ones in its binary representation. The number of such integers can be computed (e.g. by complementary counting) to be $\boxed{153}$.

Source. iTest 2008

10. For how many integers $1 \le n \le 9999$ is there a solution to the congruence

$$\phi(n) \equiv 2 \pmod{12},$$

where $\phi(n)$ is the Euler phi-function?

Solution (f). Suppose $n = p^k$ for some positive integer k and $p > 3$ where p is a prime number. Thus, $\phi(n) = p^{k-1}(p-1) \equiv 2 \pmod{12}$. Since $\gcd(p, 12) = 1$, we can only have $p \equiv 1, 5, 7, 11 \pmod{12}$. If $p \equiv 1$, we'd have $0 \equiv 2 \pmod{12}$, no solution. If $p \equiv 5$, then $5^{k-1}(4) \equiv 2 \pmod{12}$, no solution, because the RHS isn't divisible by 4. If $p \equiv 7 \pmod{12}$, then the LHS is divisible by 6, but not the RHS. Hence, we must have $p \equiv 11$. This would give us $11^{k-1}(10) \equiv 2 \pmod{12}$, which reduces to $(-1)^{k-1} \equiv -1 \pmod{6}$, which implies that k is even.

We must have $p < 100$ because $p^2 \le 9999 < 10000$. The primes congruent to 11 modulo 12 are 11, 23, 47, 59, 71, and 83, so 11^2, 23^2, 47^2, 59^2, 71^2, and 83^2 are solutions. But $11^4 > 10^4 = 10000 > 9999$, so we can't have $k > 2$.

Now let us consider when $p = 2$. We have $\phi(2^k) = 2^{k-1}$ so we have $2^{k-1} \equiv 2 \pmod{12}$, implying that $k = 2$ is the only solution (otherwise 4 divides the LHS and not the RHS). So we have $2^2 = 4$ is a solution.

For $p = 3$: we have $\phi(3^k) = 3^{k-1} \cdot 2$ so $3^{k-1} \equiv 1 \pmod{6}$, implying that $k = 1$ is the only solution, giving us 3 as a solution.

So now what if n is divisible by multiple primes? Say $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then, the totient of this would be

$$\phi(n) = N(p_1 - 1)(p_2 - 1)(p_3 - 1) \cdots$$

for some $N \in \mathbb{N}$. If at least two of these primes are odd, then $4 \mid \phi(n)$, and so no solution exists here. Thus n must be the product of exactly two primes, with one of them being 2.

In conclusion, our solutions are

$$3,\ 4,\ 11^2,\ 23^2,\ 47^2,\ 59^2,\ 71^2,\ 83^2,\ 2 \cdot 3,\ 2 \cdot 11^2,\ 2 \cdot 23^2,\ 2 \cdot 47^2,\ 2 \cdot 59^2.$$

There are $\boxed{13}$ in total.

Source. iTest Tournament of Champions 2008