MATHEMATICAL ASSOCIATION OF AMERICA
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Solutions Pamphlet

American Mathematics Competitions

30th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 18, 2014

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions should be addressed to:

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Orders for prior year exam questions and solutions pamphlets should be addressed to:

American Mathematics Competitions
Attn: Publications
PO Box 471
Annapolis Junction, MD 20701

1. **Answer (A):** Harry's answer is $H = 8 - (2 + 5) = 8 - 7 = 1$. Terry's answer is $T = 8 - 2 + 5 = 6 + 5 = 11$. The difference $H - T$ is $1 - 11 = -10$.

2. **Answer (E):** To use the largest number of coins, Paul would use only 5-cent coins. Because $35 = 5 \cdot 7$, the largest number of coins Paul can use is 7. To use the smallest number of coins, Paul would use a 25-cent coin and a 10-cent coin, for a total of 2 coins. The difference between the largest and the smallest number of coins he can use is $7 - 2 = 5$.

3. **Answer (B):** The number of pages in the book is

$$3 \cdot 36 + 3 \cdot 44 + 10 = 3(36 + 44) + 10 = 3 \cdot 80 + 10 = 250.$$

4. **Answer (E):** The sum of two odd primes is an even number. Since the sum 85 is odd, one of the primes must be 2, which is the only even prime. The two primes are 2 and 83, so the product is $2 \cdot 83 = 166$.

5. **Answer (C):** For \$20, Margie can buy $\frac{20}{4} = 5$ gallons of gas. She can drive 32 miles on each gallon, for a total of $32 \cdot 5 = 160$ miles.

6. **Answer (D):** The areas of the six rectangles are 2, 8, 18, 32, 50, and 72. Adding yields 182.

OR

The sum of areas is

$$2 \cdot 1 + 2 \cdot 4 + 2 \cdot 9 + 2 \cdot 16 + 2 \cdot 25 + 2 \cdot 36 = 2(1 + 4 + 9 + 16 + 25 + 36) = 2 \cdot 91 = 182.$$

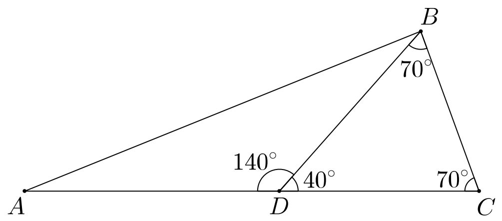
7. **Answer (B):** If there were an equal number of girls and boys, there would be 14 of each. By increasing the number of girls by 2 and decreasing the number of boys by 2, we see that there are 16 girls and 12 boys for a ratio of $16 : 12$ or $4 : 3$.

OR

If there were 4 fewer girls, then the class would be half boys and half girls. Remove 4 girls from the 28, and the other 24 students are evenly split into 12 boys and 12 girls. Add back the 4 girls to get 16 girls and 12 boys for a $16 : 12$ ratio, which simplifies to $4 : 3$.

8. **Answer (D):** The multiples of 11 between 102 and 192 are 110, 121, 132, 143, 154, 165, 176, and 187. Only 132 satisfies the condition, so $A = 3$.

9. **Answer (D):** Triangle BCD is isosceles, so $\angle BCD = \angle CBD = 70°$ and $\angle BDC = 180° - 2 \cdot 70° = 40°$. Hence $\angle ADB = 180° - 40° = 140°$.

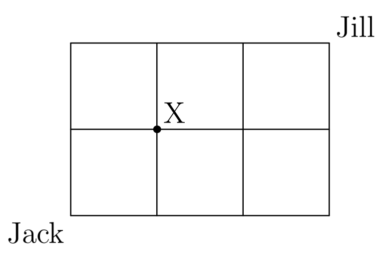


10. **Answer (A):** The seventh AMC 8 was given in 1991. So Samantha was born in $1991 - 12 = 1979$.

OR

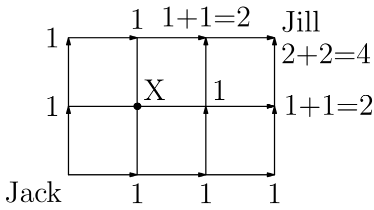
Because the seventh AMC 8 was given when Samantha was 12, the first was 6 years earlier and she was 6 that first year in 1985. She was born 6 years earlier, in 1979.

11. **Answer (A):** Let E represent traveling a block east and N represent traveling a block north. To avoid the dangerous intersection the first two blocks must be EE or NN. So there 4 possible paths: EEENN, EENEN, EENNE, and NNEEE.



OR

In the following diagram, the numbers indicate the number of ways to get to each of the intersections. In each case, the number of ways to get to any particular intersection is the sum of the numbers of ways to get to any of the intersections leading directly to it. Thus, there are four paths to Jill's house, avoiding the dangerous intersection.



12. **Answer (B):** Call the celebrities L, M, and N. There are six possible orderings: LMN, LNM, MLN, MNL, NLM, and NML. Only one of these identifies all three correctly. Therefore the probability is $\frac{1}{6}$.

13. **Answer (D):** If $n^2 + m^2$ is even, then n^2 and m^2 are either both even or both odd, which means n and m are either both even or both odd. If n and m are both even, their sum is even. If n and m are both odd, their sum is even. Because $n + m$ is never odd, (D) is the impossible choice.

14. **Answer (B):** The area of rectangle $ABCD$ is $5 \cdot 6 = 30$. The area of triangle DCE is also 30, which is half of the product $CD \cdot CE$, so that product is 60. Because $CD = AB = 5$, CE must equal $\frac{60}{5} = 12$, and by the Pythagorean Theorem, $DE = \sqrt{CD^2 + CE^2} = \sqrt{5^2 + 12^2} = \sqrt{169} = 13$.

15. **Answer (C):** Angle AOE is $\frac{4}{12}$ of $360°$ or $120°$ degrees, while $\angle GOI$ is $\frac{2}{12}$ of $360°$ or $60°$. Both triangles are isosceles, so the equal base angles are $\frac{60°}{2}$ and $\frac{120°}{2}$ respectively. The sum of angles x and y then is $(60° + 120°)/2 = 90°$.

16. **Answer (B):** Each team plays 4 non-conference games for a total of 32 games against non-conference opponents. Each team plays 7 conference games at home for a total of 56 games within the conference. The total number of games is $32 + 56 = 88$.

17. **Answer (B):** To walk 1 mile at 3 miles per hour requires $\frac{1}{3}$ of an hour, or 20 minutes. This is the amount of time George allows himself to get to school. To walk $\frac{1}{2}$ mile at 2 miles per hour requires $\frac{\frac{1}{2}}{2} = \frac{1}{4}$ of an hour, or 15 minutes, so George has only 5 minutes to cover the remaining $\frac{1}{2}$ mile. Because 5 minutes is $\frac{5}{60} = \frac{1}{12}$ of an hour, George needs to run at a speed of $\frac{1/2}{1/12} = 6$ miles per hour.

18. **Answer (D):** The 16 equally likely outcomes may be grouped as follows:

4 boys: BBBB

3 boys, 1 girl: BBBG, BBGB, BGBB, GBBB

2 boys, 2 girls: BBGG, BGBG, BGGB, GBBG, GBGB, GGBB

1 boy, 3 girls: BGGG, GBGG, GGBG, GGGB

4 girls: GGGG

There are 8 equally likely outcomes that produce 3 of one gender and 1 of the other gender, so that result is most likely.

19. **Answer (A):** The amount of white surface area is smallest when you place one white cube in the interior of the larger cube. Place each of the other 5 white cubes at the center of a face so that 1 white face and 8 red faces are visible on that face. The total surface area of the larger cube is $6 \cdot 3^2 = 54$ square inches, so the fraction of the surface area that is white is $\frac{5}{54}$.

20. **Answer (B):** The areas of the quarter-circles are $\frac{\pi}{4}$, π and $\frac{9\pi}{4}$. Their total area is $\frac{7\pi}{2}$. Using $\frac{22}{7}$ as an approximation of π, this is $\frac{7}{2} \cdot \frac{22}{7} = 11$, leaving $15-11 = 4$ for the desired area. (Using 3.14 for π yields 4.01.)

21. **Answer (A):** For $\underline{7}\,\underline{4}\,\underline{A}\,\underline{5}\,\underline{2}\,\underline{B}\,\underline{1}$ to be a multiple of 3, the sum $7+4+A+5+2+B+1 = 19+A+B$ must be a multiple of 3. Therefore $A+B$ is 1 less than a multiple of 3. The sum $3+2+6+A+B+4+C = 15+A+B+C$ must be a multiple of 3, so $A+B+C$ must be a multiple of 3. Since $A+B$ is 1 less than a multiple of 3, C must be 1 more than a multiple of 3. Only choice (A) meets this requirement.

22. **Answer (E):** Test ten consecutive numbers with unit's digits 0 through 9. For 10 through 19 we find that adding the product of the digits and the sum of the digits yields the sequence 1, 3, 5, 7, 9, 11, 13, 15, 17, and 19. In this sequence only 19 meets the desired condition. It is easy to verify that 29, 39, . . ., 99 also meet the desired condition.

OR

With a as the tens digit and b as the units digit, the number is $10a + b$. So $a + b + ab = 10a + b, a + ab = 10a, ab = 9a$ and $b = 9$.

23. **Answer (A):** The sum of any two of the girls' uniform numbers must be no greater than 31. The only possible sums of two 2-digit primes that are no greater than 31 are

$$11 + 13 = 24, \quad 11 + 17 = 28, \quad 11 + 19 = 30, \quad \text{and} \quad 13 + 17 = 30.$$

Therefore the required dates are 24, 28, and 30. Caitlin's uniform number must appear in the two smallest sums, $11 + 13 = 24$ and $11 + 17 = 28$. So Caitlin's uniform number is 11. The other two girl's uniform numbers are 13 for Ashley and 17 for Bethany.

24. **Answer (C):** Suppose the numbers of cans purchased by the 100 customers are listed in increasing order. The median is the average of the 50th and 51st numbers in the ordered list. To maximize the median, minimize the first 49 numbers by taking them all to be 1. If the 50th number is 4, then the sum of all 100 numbers would at least $49 + 51 \cdot 4 = 253$, which is too large. If instead the 50th number is 3 and the following numbers all equal 4, then the sum of the 100 numbers is $49 + 3 + 50 \cdot 4 = 252$ and the median is $(3 + 4) \div 2 = 3.5$.

OR

To maximize the median, the largest 50 should be the same and as large as possible. The lower 49 should be as small as possible. The median of the list will be the average of the 50th and 51st numbers. If every customer has 1 can of soda, there are 152 left to distribute. Giving the upper 50 three more each gives the top 50 four cans each (200 total) and the lower 50 one each (50 total). There are 2 cans left. Giving the 50th person the extra 2 means the 50th has 3 cans, and the 51st has 4 cans for a median of $(3 + 4) \div 2 = 3.5$.

25. **Answer (B):** Each semicircle moves Robert 40 feet ahead, so he would have to ride $5280 \div 40 = 132$ semicircles to cover 1 mile. Riding 132 semicircles is equal to the distance of 66 full circles. Each circle has a circumference of 40π, so Robert rides $66 \cdot 40\pi$ feet. Converting to miles, that is $\frac{66 \cdot 40\pi}{5280} = \frac{\pi}{2}$ miles. Since he is riding at 5 miles per hour, it will take him $\frac{\pi}{2} \div 5 = \frac{\pi}{10}$ hours.

OR

Each semi-circular path is $\frac{\pi}{2}$ times as long as the straight path. Since the straight path would take $\frac{1}{5}$ hour to ride, the curved path will take $\frac{1}{5} \cdot \frac{\pi}{2} = \frac{\pi}{10}$ hours to ride.

The problems and solutions in this contest were proposed by Steve Blasberg, Tom Butts, John Cocharo, Barb Currier, Steve Dunbar, Josh Frost, Joe Kennedy, Norb Kuenzi, Jeganathan Srikandarajah, and Dave Wells.

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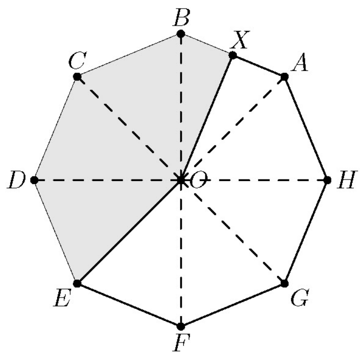
1. **Answer (A):** The floor is $\frac{12}{3} = 4$ yards long and $\frac{9}{3} = 3$ yards wide, so it will take $4 \times 3 = 12$ square yards of carpet to cover it.

OR

The area of the floor is 12×9 square feet. There are $3^2 = 9$ square feet in a square yard, so the number of square yards required is $\frac{12 \times 9}{9} = 12$.

2. **Answer (D):** The octagon can be divided into 8 congruent triangles, 3 of which are $\triangle BOC$, $\triangle COD$, and $\triangle DOE$. The area of $\triangle XOB$ is half the area of one of these, so the fraction of the area of the octagon that is shaded is $3 \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} = \frac{7}{16}$.



3. **Answer (D):** Jill takes $\frac{1}{10}$ of an hour, or 6 minutes, to get to the pool, and Jack takes $\frac{1}{4}$ of an hour, or 15 minutes, so Jill arrives $15 - 6 = 9$ minutes before Jack.

4. **Answer (E):** There are 2 ways to seat the boys, one on each end, and $3 \cdot 2 \cdot 1 = 6$ ways to seat the three girls in the middle. So there are $2 \cdot 6 = 12$ possible arrangements.

5. **Answer (A):** The range is the high score minus the low score, so the range changes from 31 to 33. The range is the only listed statistic that will increase. Because 40 is the lowest score for the season, it will cause the mean to decrease. The median value of the first 11 games is the 6th highest score, or 58. The median value of the first 12 games will be the average of the 6th highest and 7th highest scores, or $(58 + 58)/2 = 58$, so no change will occur in the median. Similarly, the score that occurs most frequently in either situation is 58, so the mode will not change. The mid-range is the average of the highest score and the lowest score. The mid-range of the first 11 games is $(73 + 42)/2 = 57.5$. The mid-range of the first 12 games is 56.5, a decrease from 57.5.

6. **Answer (B):** Let D be the midpoint of side $\overline{AC}$. Then $\overline{BD}$ is the altitude to $\overline{AC}$ and $\triangle BDC$ is a right triangle with $BC = 29$ and $DC = 21$. So $BD = \sqrt{29^2 - 21^2} = \sqrt{400} = 20$. The area of $\triangle ABC = \frac{1}{2} \cdot 20 \cdot 42 = 420$.

OR

Heron's formula gives the area of a triangle in terms of the lengths of its sides. If the side lengths are a, b and c, then let $s = \frac{a+b+c}{2}$. The area is then $\sqrt{s(s-a)(s-b)(s-c)}$. In this problem, $s = \frac{29+29+42}{2} = 50$, and the area is $\sqrt{50 \cdot 21 \cdot 21 \cdot 8} = 21\sqrt{400} = 21 \cdot 20 = 420$.

7. **Answer (E):** The nine possible equally likely outcomes are:

$$(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)$$

In five of the nine outcomes the product is even. Therefore the probability is $\frac{5}{9}$.

OR

The only way the product of the two values could be odd is if an odd number is drawn from each box. The probability that this occurs is $\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. So the probability that the product is even is $1 - \frac{4}{9} = \frac{5}{9}$.

8. **Answer (D):** Let t be the length of the third side of the triangle. By the Triangle Inequality, $t < 5 + 19 = 24$. So the perimeter $5 + 19 + t < 5 + 19 + (5 + 19) = 48$.

9. **Answer (D):** Note that Janabel has sold a total of 1 widget after 1 day, $1 + 3 = 4 = 2^2$ after 2 days and $1 + 3 + 5 = 3^2$ widgets after 3 days. It can be shown that this pattern continues so that after 20 days, Janabel has sold a total of $20^2 = 400$.

OR

The sum of the first 20 odd numbers $1, 3, 5, \ldots, 39$ is needed. The sum of the first and last is 40, the sum of the second and 19th is also 40, and in fact, there are 10 pairs of numbers that each add up to 40. Thus the required sum is 400.

10. **Answer (B):** The thousands position can be filled by the digits 1 through 9 (0 is excluded). Without repetition, the hundreds position can be filled with any of the remaining 9 digits (including 0). Similarly without repetition, the tens position and ones position can be filled with the remaining 8 and 7 digits, respectively. Thus there are $9 \cdot 9 \cdot 8 \cdot 7 = 4536$ integers between 1000 and 9999 that have distinct digits.

11. **Answer (B):** The first symbol can be any of the 5 vowels, the second can be any of the 21 consonants, the third can be any of the 20 other consonants, and the fourth can be any of the 10 digits. The total number of possible license plates is $5 \cdot 21 \cdot 20 \cdot 10 = 21{,}000$. Only one plate will read "AMC8", so the probability is $\frac{1}{21{,}000}$.

12. **Answer (C):** Each of the 12 edges is parallel to 3 other edges giving 36 possible pairs of parallel edges. But each pair of parallel edges is counted twice in this process, so there are 18 pairs of parallel edges.

OR

There are 6 pairs of parallel edges related to $\overline{AB}$ ($\overline{AB} \| \overline{EF}$, $\overline{AB} \| \overline{HG}$,$\overline{AB} \| \overline{DC}$, $\overline{EF} \| \overline{HG}$, $\overline{EF} \| \overline{DC}$, $\overline{HG} \| \overline{DC}$). Similarly there are 6 pairs of parallel edges related to $\overline{AE}$ and 6 pairs of parallel edges related to $\overline{AD}$ for a total of 18 pairs of parallel edges.

13. **Answer (D):** If the average of the remaining 9 numbers is 6, then their sum is 54. Because the sum of the numbers in the original set is 66, the sum of the two numbers removed must be 12. There are five such subsets: $\{1, 11\}$, $\{2, 10\}$, $\{3, 9\}$, $\{4, 8\}$, and $\{5, 7\}$.

14. **Answer (D):** The sum of 4 consecutive odd integers is always a multiple of 8, $(2n-3)+(2n-1)+(2n+1)+(2n+3) = 8n$. Among the given choices, only 100 is not a multiple of 8. The other four numbers can each be written as the sum of four consecutive odd numbers:

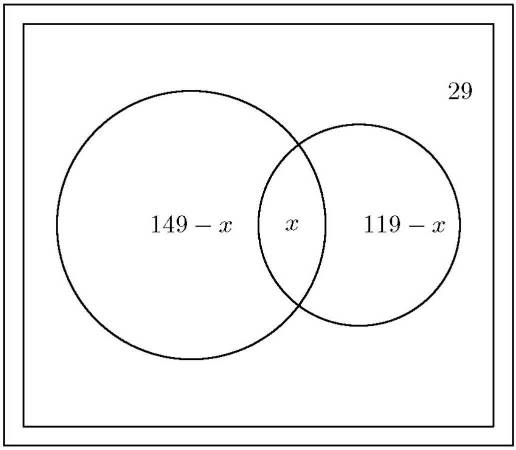
$$\begin{aligned} 16 &= 1+3+5+7 \\ 40 &= 7+9+11+13 \\ 72 &= 15+17+19+21 \\ 200 &= 47+49+51+53 \end{aligned}$$

15. **Answer (D):** The sum $149 + 119 + 29 = 297$ counts the number of students who voted for both issues twice. So the number who voted in favor of both issues is $297 - 198$ or 99.

OR

In the diagram below, the left circle represents the 149 students who voted for the first issue, and the right circle represents the 119 students who voted for the second issue. Let x be the number of students who voted for both issues. Then $149 - x$ students voted for the first issue but not the second, $119 - x$ students voted for the second issue but not the first and 29 students voted against both issues. The sum of the numbers in the diagram must be 198, so

$$\begin{aligned}(149 - x) + x + (119 - x) + 29 &= 198,\\ 297 - x &= 198,\\ x &= 99.\end{aligned}$$



16. **Answer (B):** 2 out of every 5 sixth graders are paired with a ninth grade buddy, and 2 out of every 6 ninth graders are paired with a sixth grade buddy. (The ratios are now expressed so that the number of sixth graders matches the number of ninth graders.) So 4 out of every 11 students are in the mentoring program. The fraction is $\frac{4}{11}$.

OR

Suppose that n sixth graders are paired with n ninth graders. Then the total number of sixth graders is $\frac{5}{2}n$, the total number of ninth graders is $3n$, and the total number of sixth and ninth graders is $\frac{5}{2}n + 3n = \frac{11}{2}n$. There are $2n$ students in the mentoring program, which is $\dfrac{2n}{\frac{11}{2}n} = \dfrac{4}{11}$ of the total number of students.

17. **Answer (D):** Because the new time is $\frac{12}{20} = \frac{3}{5}$ of the original time, the new speed must be $\frac{5}{3} = 1\frac{2}{3}$ of the original speed. Then the additional 18 miles per hour must be $\frac{2}{3}$ of the original speed, which is then 27 mph. In 20 minutes, Jeremy's father travels $\frac{1}{3} \cdot 27 = 9$ miles.

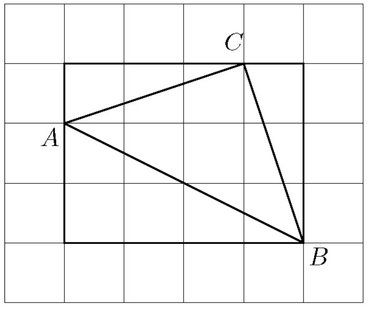
OR

Let r be Jeremy's original speed in miles per hour. Twenty minutes is $\frac{20}{60} = \frac{1}{3}$ of an hour and twelve minutes is $\frac{12}{60} = \frac{1}{5}$ of an hour. Then $\frac{1}{3}r = \frac{1}{5}(r + 18)$, so $5r = 3r + 54$, and $r = 27$. Thus the distance to the school is $\frac{1}{3}r = 9$ miles.

18. **Answer (B):** The middle number in an arithmetic sequence with 5 terms is the average of the first and last numbers. The average of 1 and 25 is 13. The average of 17 and 81 is 49. Thus, X is the average of 13 and 49, or 31. Alternatively, $X = \frac{9+53}{2} = 31$. In fact X is the average of the four corner entries.

1		13		25
9		31		53
17		49		81

19. **Answer (A):** The triangle is inscribed in a 4×3 rectangle with vertices at $(1, 1)$, $(1, 4)$, $(5, 4)$, and $(5, 1)$. Three triangular regions are inside the 4×3 rectangle but outside $\triangle ABC$. The area of the lower-left triangle is $\frac{1}{2} \cdot 4 \cdot 2 = 4$ square units. The area of the upper-left triangle is $\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$ square units. The area of the third triangle is also $\frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$ square units. So the area of $\triangle ABC$ is $12 - 4 - \frac{3}{2} - \frac{3}{2} = 5$ square units. The area of the 6×5 grid is 30 square units. Thus, the fraction covered by the triangle is $\frac{5}{30} = \frac{1}{6}$.



OR

Pick's Theorem says that the area of a polygonal region whose vertices are at lattice points (points whose coordinates are integers) is given by $A = I + \frac{1}{2}B - 1$ where I is the number of lattice points in the interior of the region and B is the number of lattice points on the boundary. Referring to the figure above, there are $B = 4$ lattice points on the boundary of $\triangle ABC$ at $(1,3)$, $(3,2)$, $(5,1)$, and $(4,4)$. There are $I = 4$ points with integer coordinates in the interior of $\triangle ABC$ at $(2,3)$, $(3,3)$, $(4,2)$, and $(4,3)$. Then the area of $\triangle ABC$ is $I + \frac{1}{2}B - 1 = 4 + 2 - 1 = 5$ square units. As before, the fraction of the rectangle covered by the triangle is $\frac{5}{30} = \frac{1}{6}$.

20. **Answer (D):** If Ralph buys 6 pairs of \$1 socks, then the other 6 pairs of socks would cost at least \$19 making the total cost more than \$24. Buying fewer than 6 pairs of \$1 socks would make Ralph's cost even higher. If he bought 8 pairs of \$1 socks, then the other 4 pairs would cost less than \$16 making the total cost less than \$24. Buying more than 8 pairs of \$1 socks would make his total cost even lower. So Ralph bought 7 pairs of \$1 socks, 3 pairs of \$3 socks, and 2 pairs of \$4 socks.

OR

Let a, b and c be the number of pairs of \$1, \$3 and \$4 socks, respectively. Then $a + b + c = 12$ and $a + 3b + 4c = 24$. Subtracting the first equation from the second gives $2b + 3c = 12$. Since 3 is a factor of both 12 and $3c$, 3 must also be a factor of $2b$. Since $c > 0$, it follows that $b = 3$, $c = 2$, and $a = 7$.

21. **Answer (C):** The area of the square $ABJI$ is 18 and $\triangle KJB$ is equilateral, so $KB = JB = \sqrt{18} = 3\sqrt{2}$. The area of the square $FEHG$ is 32, so $BC = FE = \sqrt{32} = 4\sqrt{2}$. Each interior angle of the hexagon is $120°$, so $\angle KBC = 360° - 60° - 90° - 120° = 90°$ and $\triangle KBC$ is a right triangle. Its area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot 4\sqrt{2} = 12$.

22. **Answer (C):** The number of students must be a multiple of 6 and also a multiple of 15. So the number of students must be divisible by the least common multiple of 6 and 15 which is 30. The divisors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30, so there are only 8 divisors. The divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30 and 60. So 60 has 12 divisors and 60 is the smallest possible number of students.

23. **Answer (D):**

The sum of the numbers is 35. So the 5 consecutive numbers in the cups must be 5, 6, 7, 8, and 9. It is impossible to get a sum of 5 or 7 using the slip with

3.5. Cup B needs a sum of 6, but it already has a slip with 3 on it so the slip with a 3.5 can't go there. Cup E needs a sum of 9, but with a slip with 2 in it the slip with 3.5 can't go there. The only place the slip with 3.5 on it can go is Cup D. One possibility is:

Cup A. 2, 3

Cup B. 3, 3

Cup C. 2.5, 4.5

Cup D. 2, 2.5, 3.5

Cup E. 2, 3, 4

24. **Answer (B):** The number of games played by a team is $3N + 4M = 76$. Because $M > 4$ and $N > 2M$ it follows that $N > 8$. Because 4 divides both 76 and $4M$, 4 must divide $3N$ and hence N. If $N = 12$ then $M = 10$ and the condition $N > 2M$ is not satisfied. If $N \geq 20$ then $M \leq 4$ and the condition $M > 4$ is not satisfied. So the only possibility is $N = 16$ and $M = 7$. So each team plays $3 \cdot 16 = 48$ games within its division and $4 \cdot 7 = 28$ games against the other division.

OR

The total number of games played by each team is $3N + 4M = 76$. Make a chart of possibilities with $M > 4$:

M	$4M$	$76 - 4M = 3N$	N
5	20	56	(not an integer)
6	24	52	(not an integer)
7	28	48	16
8	32	44	(not an integer)
9	36	40	(not an integer)
10	40	36	12
11	44	32	(not an integer)
12	48	28	(not an integer)
13	52	24	8

Only $M = 7$ and $N = 16$ satisfy the conditions.

The case $M = 10$ and $N = 12$ violates the condition $N > 2M$.

So each team plays $3N = 48$ divisional games and $4M = 28$ games against the other division.

This is modeled on the Pioneer Baseball League with teams in Colorado, Idaho, Montana, and Utah.

25. **Answer (C):** Let $EQ = c$ and $TQ = s$ as indicated in the figure. Triangles QUV and FEQ are similar since $\angle FQE$ and $\angle QVU$ are congruent because

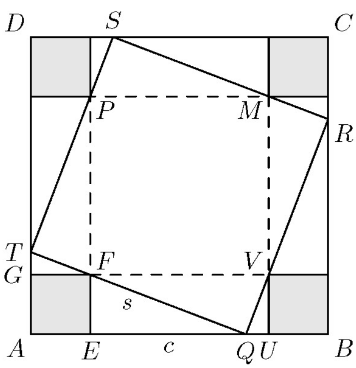
both are complementary to $\angle VQU$. So

$$\frac{QU}{UV} = \frac{FE}{EQ}$$

and thus $QU = \frac{1}{c}$. Then $AB = 1 + c + 1/c + 1 = 5$ and so $c + 1/c = 3$. Since the area of square $ABCD$ equals the sum of areas of square $QRST$, four unit squares, four $1 \times c$ triangles, and four $\frac{1}{c} \times 1$ triangles, it follows that

$$\begin{aligned} 25 &= s^2 + 4\left(1 + \frac{c}{2} + \frac{1}{2c}\right) \\ &= s^2 + 4 + 2\left(c + \frac{1}{c}\right) \\ &= s^2 + 4 + 2 \cdot 3 \end{aligned}$$

Therefore, the area of square $QRST = s^2 = 15$.



OR

Square $FVMP$ has area 9, the four triangles FQV, VRM, MSP, and PTF each have area $\frac{3}{2}$. So the area of square $STQR$ is $9 + 6 = 15$.

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MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

32nd Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 15, 2016

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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Attn: Publications
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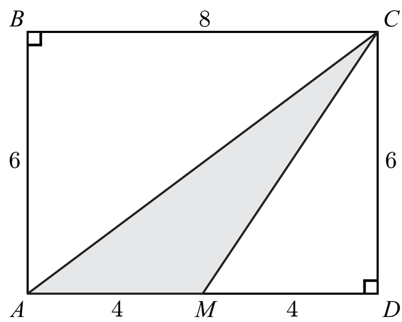
1. Answer (C):

   There are 60 minutes in 1 hour, so 11 hours plus 5 minutes is equal to $11 \cdot 60 + 5 = 665$ minutes.

2. Answer (A):

   The area of $\triangle ACD$ is $\frac{1}{2} \cdot 8 \cdot 6 = 24$. The area of $\triangle MCD$ is $\frac{1}{2} \cdot 4 \cdot 6 = 12$. So the area of $\triangle AMC$ is $24 - 12 = 12$.

OR



As seen in the diagram above, the altitude from C to the line of the base $\overline{AM}$ is $\overline{CD}$. Thus the area of the shaded $\triangle AMC$ is

$$\frac{1}{2} \cdot AM \cdot CD = \frac{1}{2} \cdot 4 \cdot 6 = 12.$$

3. Answer (A):

   The given scores 70, 80, and 90 are a total of 30 above the stated average. Thus the remaining score is 30 points below the average, and $70 - 30 = 40$.

OR

Let x be the missing score. Then the sum $70 + 80 + 90 + x = 70 \cdot 4 = 280$. So x must be 40.

4. Answer (B):

   As a boy it took Cheenu 3 hours and 30 minutes, which is 210 minutes, to go 15 miles. That is a rate of $210 \div 15 = 14$ minutes per mile. As an old man it takes him 4 hours, or 240 minutes, to travel 10 miles. That is a rate of $240 \div 10 = 24$ minutes per mile. It takes him $24 - 14 = 10$ minutes more to walk a mile as an old man.

5. Answer (E):

   The two-digit numbers that leave a remainder of 3 when divided by 10 are: 13, 23, 33, 43, 53, 63, 73, 83, 93. The two-digit numbers that leave a remainder of 1 when divided 9 are: 10, 19, 28, 37, 46, 55, 64, 73, 82, 91. Among these two sets, 73 is the only common number. When 73 is divided by 11 the remainder is 7.

6. Answer (B):

   The 19 name lengths are 3, 3, 3, 3, 3, 3, 3, 4, 4, 4, 5, 6, 6, 6, 6, 7, 7, 7, 7. The tenth value, 4, is the median.

7. Answer (B):

   The numbers 1^{2016}, 3^{2018}, 5^{2020} have even exponents and hence are squares. The number 2^{2017} is not a perfect square because it is twice a square $2(2^{1008})^2$. SInce $4^{2019} = (2^2)^{2019} = 2^{4038}$, it is also a perfect square.

   OR

   A positive integer power of a square is again a square. This eliminates choices (A) and (D). An even power of any integer is a square. This eliminates choices (C) and (E) The only remaining choice is (B), and in fact, an odd power of a non-square cannot be a square.

8. Answer (C) :

   Evaluate the expression by grouping as follows:

   $(100-98)+(96-94)+\cdots+(8-6)+(4-2) = 2+2+\cdots+2+2 = 2\cdot 25 =$ 50.

9. Answer (B):

   The prime factorization of 2016 is: $2016 = (2^5)(3^2)(7)$, so the distinct prime divisors of 2016 are 2, 3, and 7, and their sum is $2+3+7 = 12$.

10. Answer (D):

    Since $2*(5*x) = 1$, it follows that $6-(5*x) = 1$, and so $5*x = 5$. Applying the formula again, $15-x = 5$, and therefore $x = 10$.

11. Answer (B):

    Let $\underline{a}\,\underline{b}$ be the two digit number. Then $132 = (10a+b)+(10b+a) = 11(a+b)$.

    Thus $a+b = 12$. The possible numbers are: 39, 93, 48, 84, 57, 75, and 66. There are seven two-digit numbers that meet this criterion.

12. Answer (B):

    Converting the given fractions to the same denominator, we see that $\frac{9}{12}$ of the girls and $\frac{8}{12}$ of the boys went on the trip. So the ratio of the number of girls to the number of boys was $9:8$, and it follows that $\frac{9}{17}$ of the students on the trip were girls.

    OR

    The number of boys and girls must be a common multiple of 4 and 3, the denominators of the fractions given in the problem. Suppose there are 12 boys and 12 girls in Jefferson Middle School. Then 9 girls and 8 boys went on the trip, for a total of 17 students. The fraction of girls on the trip is 9/17.

13. Answer (D):

    There are $6 \cdot 5 = 30$ possible pairs of numbers. For a product to be 0, either the first factor or the second factor must be 0, so there are $1 \cdot 5 + 5 \cdot 1 = 10$ such products. The desired probability is $10/30 = 1/3$.

14. Answer (A):

    In driving 350 miles, Karl used $\frac{350}{35} = 10$ gallons of gas, so he had $14 - 10 = 4$ gallons left in his tank. After buying 8 more gallons, he had $4 + 8 = 12$ gallons. When he arrived at his destination, he had $\frac{14}{2} = 7$ gallons left, so he used an additional $12 - 7 = 5$ gallons. This let him drive an additional $5 \cdot 35 = 175$ miles, so he drove a total of $350 + 175 = 525$ miles.

    OR

    Karl used $14 + 8 - \frac{14}{2} = 15$ gallons of gas on his trip, so he drove $15 \cdot 35 = 525$ miles.

15. Answer (C):

    Factor, using a difference of two squares:

    $$\begin{aligned} 13^4 - 11^4 &= \left(13^2 + 11^2\right)(13+11)(13-11) \\ &= 290 \cdot 24 \cdot 2 \\ &= 2 \cdot 145 \cdot 8 \cdot 3 \cdot 2 \\ &= 32 \cdot 435 \end{aligned}$$

    So the largest power of 2 that is a divisor of $13^4 - 11^4$ is 32.

16. Answer (D):

    Let N be the number of laps run by Annie when she passes Bonnie for the first time. The number of laps run by Bonnie is $N-1$. Then $\frac{N}{N-1} = 1.25 = \frac{5}{4}$. So $N = 5$.

OR

For each lap Bonnie completes, Annie runs $1^1/4$ laps, thus gaining $^1/4$ of a lap on Bonnie during that time. Annie will pass Bonnie when Bonnie has run 4 laps, at which point Annie will have run 5.

17. Answer (D):

    If there were no restrictions, then 10^4 passwords would be possible. Among these, 10 passwords begin with 9 1 1, and have 10 options for the fourth digit. Thus $10^4 - 10 = 9990$ passwords satisfy the condition.

18. Answer (C):

    Divide the 216 sprinters into 36 groups of 6. Run 36 races to eliminate 180 sprinters, leaving 36 winners. Divide the 36 winners into 6 groups of 6, run 6 races to eliminate 30 sprinters, leaving 6 winners. Finally run the last race to determine the champion. The number of races run is $36+6+1 = 43$.

OR

When all the races have been run, 215 sprinters will have been eliminated. Since 5 sprinters are eliminated in each race, there are $\frac{215}{5} = 43$ races needed to determine the champion.

19. Answer (E):

    The average of the 25 even integers is $10000/25 = 400$. So 12 consecutive even integers will be larger than 400 and 12 consecutive even integers will be smaller than 400. The sum $376+378+\cdots+398+400+402+\cdots+422+424 = 10000$. The largest of these numbers is 424.

OR

The average of the 25 even integers is $\frac{10000}{25} = 400$. Since 12 of the consecutive even integers are larger than 400, the largest is $400 + 12 \cdot 2 = 424$.

20. Answer (A):

    If $b = 1$, then $a = 12$ and $c = 15$, and the least common multiple of a and c is 60. If $b > 1$, then any prime factor of b must also be a factor of both 12 and 15, and thus the only possible value is $b = 3$. In this case, a must be a multiple of 4 and a divisor of 12, so $a = 4$ or $a = 12$. Similarly, c must be a multiple of 5 and a divisor of 15, so $c = 5$ or $c = 15$. It follows that the least common multiple of a and c must be a multiple of 20. When $a = 4$, $b = 3$, and $c = 5$, the least common multiple of a and c is exactly 20.

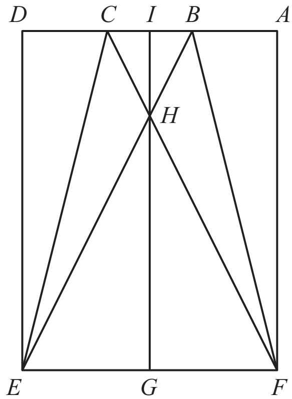
21. Answer (B):

    Consider drawing all five chips and listing the 10 possible outcomes: RRRGG, RRGRG, RGRRG, GRRRG, GGRRR, GRGRR, RGGRR, GRRGR, RGRGR, RRGGR.

    All 10 of these outcomes are equally likely. The outcomes that end in G correspond to the outcomes where the 3 reds are drawn and the outcomes that end in R correspond to the outcomes where the 2 greens are drawn. The probability that the 3 reds are drawn is $\frac{4}{10} = \frac{2}{5}$.

22. Answer (C):

    The area of $\triangle BCE$ is $\frac{1}{2}(1)(4) = 2$. Triangles $\triangle CBH$ and $\triangle EFH$ are similar. Since $CB = \frac{1}{3}EF$, it follows that $IH = \frac{1}{3}GH = \frac{1}{4}IG = 1$. The area of $\triangle CBH$ is $\frac{1}{2}$, so the area of $\triangle ECH$ is $2 - \frac{1}{2} = \frac{3}{2}$. Thus the batwing's area is 3.

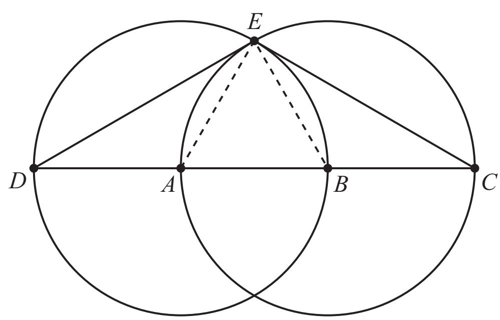


23. Answer (C):

We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measure of $\angle AEB$ is 60°. Since $\overline{DB}$ is a diameter of circle A and $\overline{AC}$ is a diameter of circle B, it follows that $\angle DEB$ and $\angle AEC$ are both right angles. Therefore the degree measure of $\angle DEC$ is $90^\circ + 90^\circ - 60^\circ = 120^\circ$.

OR

We know $\triangle AEB$ is equilateral since each of its sides is a radius of one of the congruent circles. Thus the measures of $\angle AEB$ and $\angle EAB$ are both 60°. Then the measure of $\angle DAE$ is 120°, and since $\triangle DAE$ is isosceles, the measure of $\angle DEA$ is 30°. Similarly, the measure of $\angle BEC$ is also 30°. Therefore the degree measure of $\angle DEC$ is $30^\circ + 60^\circ + 30^\circ = 120^\circ$.



24. Answer (A):

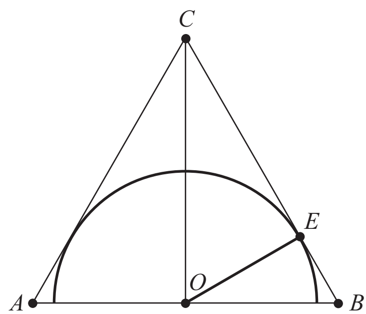
Since QRS is divisible by 5, we know that $S = 5$. Since PQR is divisible by 4, we know that QR is 12, 32, or 24. So RST will be either $25T$ or $45T$ and divisible by 3. Using the available digits, 453 is the only number that is divisible by 3. So $T = 3$, $R = 4$, and $P = 1$.

25. Answer (B):

Let O be the midpoint of base $\overline{AB}$ of $\triangle ABC$ and the center of the semicircle. Triangle $\triangle OBC$ is a right triangle with $OB = 8$ and $OC = 15$, and so, by the Pythagorean Theorem, $BC = 17$. Let E be the point where the semicircle intersects $\overline{BC}$, so radius $\overline{OE}$ is perpendicular to $\overline{BC}$. Then $\triangle OEB$ and $\triangle COB$ are similar, and therefore, $OE : CO = OB : CB$. Hence, $\frac{OE}{15} = \frac{8}{17}$ and so $OE = \frac{120}{17}$.

OR

Let O be the center of the semicircle, which is also the midpoint of base $\overline{AB}$. Since $OB = 8$ and $OC = 15$, then by the Pythagorean Theorem $BC = 17$. Let E be the point where the semicircle intersects $\overline{BC}$, so radius $\overline{OE}$ is perpendicular to $\overline{BC}$. Since the area of $\triangle OBC$ is $\frac{1}{2}(BC)(OE) = \frac{1}{2}(OB)(OC)$, then $\frac{1}{2}(17)(OE) = \frac{1}{2}(8)(15)$ and so $OE = 120/17$.



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MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

33rd Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 14, 2017

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1. **Answer (A):** The values of the expressions are, in order, 10, 8, 9, 9, and 0.

2. **Answer (E):** Observe that 36 votes made up 30% of the total number of votes. Thus 12 votes made up 10% of the total number of votes and therefore there were 120 total votes.

3. **Answer (C):** Simplifying yields

$$\sqrt{16\sqrt{8\sqrt{4}}} = \sqrt{16\sqrt{16}} = \sqrt{64} = 8$$

4. **Answer (D):** The product may be estimated as

$$\left(3 \cdot 10^{-4}\right)\left(8 \cdot 10^{6}\right) = 24 \cdot 10^{2} = 2400.$$

The exact value of the product is 2497.498.

5. **Answer (B):** The sum $1 + 2 + 3 + \cdots + 8 = 36$, so the desired quotient is

$$\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8}{36} = 4 \cdot 5 \cdot 7 \cdot 8 = 1120.$$

6. **Answer (D):** Let the degree measures of the angles of the triangle be $3x$, $3x$, and $4x$. Then $3x + 3x + 4x = 10x = 180$, and $x = 18$. So the largest angle has degree measure $4x = 4 \cdot 18 = 72$.

OR

The degree measure of the largest angle is $\frac{4}{3+3+4} = \frac{2}{5}$ of the sum of the degree measures of the angles in the triangle, so it is $\frac{2}{5} \cdot 180 = 72$ degrees.

7. **Answer (A):** Assume Z has the form $abcabc$. Then

$$Z = 1001 \cdot abc = 7 \cdot 11 \cdot 13 \cdot abc$$

So 11 must be a factor of Z.

OR

A positive integer is divisible by 11 if and only if the difference of the sums of the digits in the even and odd positions in the number is divisible by 11. For $Z = abcabc$ the sum of the digits in the even positions is equal to the sum of the digits in the odd positions, so the difference of the two sums is 0. Hence, 11 divides Z.

8. **Answer (D):** There are no two-digit even primes, so statements (1) and (2) cannot both be true. Also, no two-digit prime is divisible by 7, so statements (1) and (3) cannot both be true. Because there is only one false statement, it must be (1), so Isabella's house number is an even two-digit multiple of 7 that has a digit of 9. The number is even, so the 9 must be the tens digit. The only even multiple of 7 between 90 and 99 is 98, so the units digit is 8.

9. **Answer (D):** The total number of Marcy's marbles must be divisible by both 3 and 4 thus it must be a multiple of 12. If she has just 12 marbles, then 4 are blue and 3 are red, leaving just 5 other marbles, so she could not have 6 green marbles. If she has 24 marbles, then 8 are blue and 6 are red, leaving 10 other marbles, so she could have 6 green marbles and 4 yellow marbles. The smallest number of yellow marbles that Marcy would have is 4.

10. **Answer (C):** There are 10 possible equally likely outcomes:

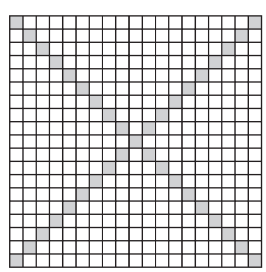
1, 2, 3	1, 2, 4	1, 2, 5	1, 3, 4	1, 3, 5
1, 4, 5	2, 3, 4	2, 3, 5	2, 4, 5	3, 4, 5

The three highlighted outcomes have 4 as the largest value selected. Hence the probability is $\frac{3}{10}$.

OR

There are 10 ways to select 3 cards without replacement from a box of 5 cards. If the largest value selected is 4, then the remaining two cards can be selected from the cards 1, 2, and 3 in 3 ways. So, the probability that 4 is the largest value selected is $\frac{3}{10}$.

11. **Answer (C):** If the total number of tiles in the two diagonals is 37, there are 19 tiles in each diagonal (with one tile appearing in both diagonals). The number of tiles on a diagonal is equal to the number of tiles on a side. Therefore, the square floor is covered by $19 \times 19 = 361$ square tiles.



12. **Answer (D):** The least common multiple of 4, 5, and 6 is 60. Numbers that leave a remainder of 1 when divided by 4, 5, and 6 are 1 more than a whole number multiple of 60. So the smallest positive number greater than 1 that leaves a remainder of 1 when divided by 4, 5, and 6 is 61.

13. **Answer (B):** Whenever a person wins a game, another person loses that game. So the total number of wins equals the total number of losses. Peter, Emma, and Kyler lost $2+3+3=8$ games altogether, and Peter and Emma won $4+3=7$ games in total. Therefore, Kyler won 1 game.

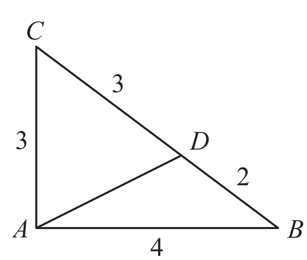
14. **Answer (C):** For simplicity, suppose that the assignment contained 100 problems. Chloe correctly solved 80% of the 50 problems she worked on alone, which was 40 problems. She had a total of 88 correct answers, so $88-40=48$ of the 50 problems that she and Zoe worked on together had correct answers. In addition Zoe correctly solved 90% of the 50 problems that she worked on alone, which was 45 problems. Her overall percentage of correct answers was $45+48=93$.

OR

Chloe's percentage of success on the half that she solved alone was 80, which is 8 points less than 88. So her percentage of success on the other half was 8 points above 88, or 96. Zoe's percentage of success was 90 on half of the problems and 96 on the other half, so her overall percentage was the average of 90 and 96, which is 93.

15. **Answer (D):** Starting at A, there are 4 choices for the M, each of which is followed by 3 choices for the C, each of which is followed by 2 choices for the 8. So all together, there are $4\cdot 3\cdot 2=24$ paths.

16. **Answer (D):** Because the perimeters of $\triangle ADC$ and $\triangle ADB$ are equal, $CD=3$ and $BD=2$.



$\triangle ADC$ and $\triangle ADB$ have the same altitude from A, so the area of $\triangle ADC$ will be 3/5 of the area of $\triangle ABC$, and $\triangle ADB$ will be $\frac{2}{5}$ of the area of $\triangle ABC$. The area of $\triangle ABC$ is $\frac{1}{2}\cdot 3\cdot 4=6$, so the area of $\triangle ADB$ is $\frac{2}{5}\cdot 6=12/5$.

17. **Answer (C):** After putting 9 coins in all but two of the chests, I could take 3 coins out of each chest to leave 6 coins in those chests. Doing this would allow me to fill the remaining 2 chests with 6 coins each, and have another 3 coins left over. So I must have removed 15 coins (3 from each of 5 chests). Thus I initially had put 9 coins into each of 5 chests, making for a total of 45 coins (and 7 chests).

OR

Let c be the number of chests. Then $9(c-2) = 6c+3$. Solving yields $c = 7$, so the number of coins is $6 \cdot 7 + 3 = 9(7-2) = 45$.

18. **Answer (B):** In right triangle BCD, $3^2 + 4^2 = 5^2$, so $BD = 5$. In $\triangle ABD$, $13^2 = 12^2 + 5^2$, so $\triangle ABD$ is a right triangle with right angle $\angle ABD$. The area of $\triangle ABD$ is $\frac{1}{2} \cdot 5 \cdot 12 = 30$. The area of $\triangle BCD$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. So the area of the quadrilateral is $30 - 6 = 24$.

19. **Answer (D):** Factoring yields $98! + 99! + 100! = 98!(1 + 99 + 100 \cdot 99) = 98!(100 + 100 \cdot 99) = 98!(100)(1+99) = 98! \cdot 100^2$. The exponent of 5 in $98!$ is $19 + 3 = 22$, one for each multiple of 5 and one more for each multiple of 25. Thus the exponent of 5 in the product is $22 + 4 = 26$ as $100^2 = 2^4 \cdot 5^4$.

20. **Answer (B):** There are 9000 integers between 1000 and 9999 inclusive. For an integer to be odd it must end in 1, 3, 5, 7, or 9. So there are 5 choices for the units digit. For a number to be between 1000 and 9999 the thousands digit must be nonzero and so there are now 8 choices for the thousands digit. For the hundreds digit there are 8 choices and for the tens digit there are 7 choices for a total number of $5 \cdot 8 \cdot 8 \cdot 7 = 2240$ choices. So the probability is $\frac{2240}{9000} = \frac{56}{225}$.

21. **Answer (A):** Since $a + b + c = 0$, then these three numbers cannot be all positive or all negative. The value of $\dfrac{X}{|X|} = 1$ for X positive and -1 for X negative.

Case I. When there are two positive numbers and one negative number,

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} = 1,$$

and $\dfrac{abc}{|abc|} = -1$, so

$$\frac{a}{|a|} + \frac{b}{|b|} + \frac{c}{|c|} + \frac{abc}{|abc|} = 0.$$

Case II. When there are two negative numbers and one positive number,

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}=-1,$$

and $\dfrac{abc}{|abc|}=1$, so

$$\frac{a}{|a|}+\frac{b}{|b|}+\frac{c}{|c|}+\frac{abc}{|abc|}=0.$$

Therefore the only possible value of $\dfrac{a}{|a|}+\dfrac{b}{|b|}+\dfrac{c}{|c|}+\dfrac{abc}{|abc|}$ is 0.

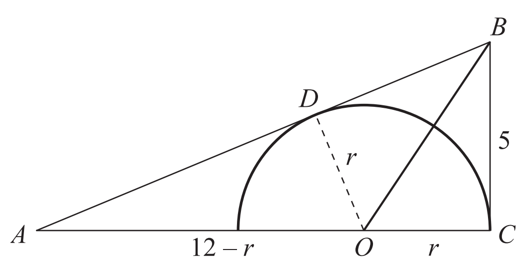
22. **Answer (D):** Let O be the center of the inscribed semicircle on $\overline{AC}$, and let D be the point at which $\overline{AB}$ is tangent to the semicircle. Because $\overline{OD}$ is a radius of the semicircle it is perpendicular to $\overline{AB}$, making $\overline{OD}$ an altitude of $\triangle AOB$. By the Pythagorean Theorem, $AB=13$. In the diagram, $\overline{OB}$ partitions $\triangle ABC$ so that

$$\text{Area}(\triangle ABC)=\text{Area}(\triangle BOC)+\text{Area}(\triangle AOB)$$

Since we know $\triangle ABC$ has area 30, we have

$$\begin{aligned}30&=\text{Area}(\triangle BOC)+\text{Area}(\triangle AOB)\\&=\frac{1}{2}(BC)r+\frac{1}{2}(AB)r=\frac{5}{2}r+\frac{13}{2}r=9r.\end{aligned}$$

Therefore $r=\frac{30}{9}=\frac{10}{3}$.



OR

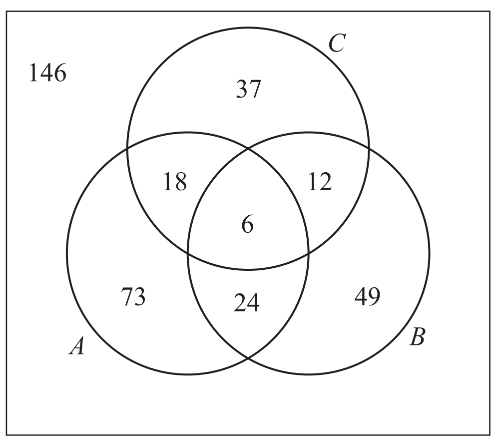
Because $\overline{OD}$ is a radius of the semicircle, it is perpendicular to $\overline{AB}$, making $\triangle ADO$ similar to $\triangle ACB$. Because $\overline{BC}$ and $\overline{BD}$ are both tangent to the semicircle, they are congruent. So $BD=5$ and $AD=8$. It follows that $\frac{r}{8}=\frac{5}{12}$ and so $r=\frac{40}{12}=\frac{10}{3}$.

23. **Answer (C):** Her time for each trip was 60 minutes. The factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. Her daily number of minutes for a mile form a sequence of four numbers where each number is 5 more than the previous number. Also, these numbers must each be a factor of 60 since the number of miles traveled must be an integer. The only such sequence from among the factors of 60 is 5, 10, 15, 20. So her rates in miles per minute for the four days were $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{15}$, $\frac{1}{20}$, and multiplying each by 60 minutes gives her distances in miles as 12, 6, 4, and 3, for a total distance of 25 miles.

24. **Answer (D):** During a 60-day cycle, there are 20 days that the first one calls, 15 days that the second one calls, and 12 days that the third one calls. The sum $20 + 15 + 12 = 47$ over-counts the number of days when more than one grandchild called. There were $60 \div 12 = 5$ days when the first and second called. There were $60 \div 15 = 4$ days when the first and third called. There were $60 \div 20 = 3$ days when the second and third called. Subtracting $5 + 4 + 3$ from 47 leaves 35 days. But the 60$^{\text{th}}$ day was added in three times and subtracted out three times, so there were 36 days in which she received at least one phone call. Thus, in each 60-day cycle, there were $60 - 36 = 24$ days without a phone call. In a year, there are six full cycles. Additionally, she receives no phone call on the 361$^{\text{st}}$ or 362$^{\text{nd}}$ day. Therefore, the total number of days that she does not receive a phone call is $6 \cdot 24 + 2 = 146$ days.

OR

In the Venn diagram below, let A be the set of days in the year in which the first grandchild calls her, let B be the set of days in which the second calls her, and let C be the set of days in which the third calls her. Then the region common to the 3 sets represents the days on which all 3 call, which are every $3 \cdot 4 \cdot 5 = 60$ days, or 6 days in the year.

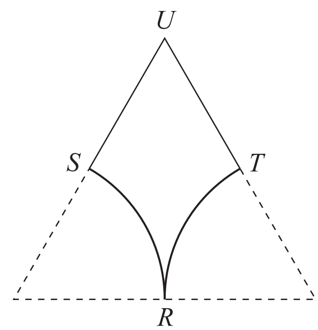


The region common to A and B represents the days on which the first and second grandchild call, which are every $3 \cdot 4 = 12$ days, or 30 days in the year. Since 6 of those days have already been counted, we label the region below the 6 with $30 - 6 = 24$. Similarly, the region common to A and C is 24, so the region to the left of the common region is $24 - 6 = 18$, and the region to the right is $18 - 6 = 12$.

Now the first grandchild calls on 121 days, of which 48 have already been counted. Thus the lower left region contains 73 days. Similarly, B contains 91 days, so the lower right region contains 49 days, and the top region has 37 days.

All the regions total 219 days, so the number of days without a call is $365 - 219 = 146$ days. Note that in leap years, the lower left region increases to 74, so the answer is still $366 - 220 = 146$ days.

25. **Answer (B):** The region shown is what remains when two one-sixth sectors of a circle of radius 2 are removed from an equilateral triangle with side length 4.



The area of an equaliateral triangle with side length s is $\dfrac{\sqrt{3}}{4}s^2$. Thus the area of the region is $4\sqrt{3} - 2\left(\dfrac{1}{6} \cdot 4\pi\right) = 4\sqrt{3} - \dfrac{4\pi}{3}$.

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MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

34th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 13, 2018

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination beyond the classroom at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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1. An amusement park has a collection of scale models, with ratio 1:20, of buildings and other sights from around the country. The height of the United States Capitol is 289 feet. What is the height in feet of its replica at this park, rounded to the nearest whole number?

   (A) 14 **(B)** 15 **(C)** 16 **(D)** 18 **(E)** 20

   Answer (A): The height of the replica is $\frac{289}{20} = 14.45$ feet, so the answer is 14 feet when rounded to the nearest whole number.

2. What is the value of the product

$$\left(1+\frac{1}{1}\right)\cdot\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\left(1+\frac{1}{6}\right)?$$

   (A) $\frac{7}{6}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{7}{2}$ **(D)** 7 **(E)** 8

   Answer (D): The product may be written as

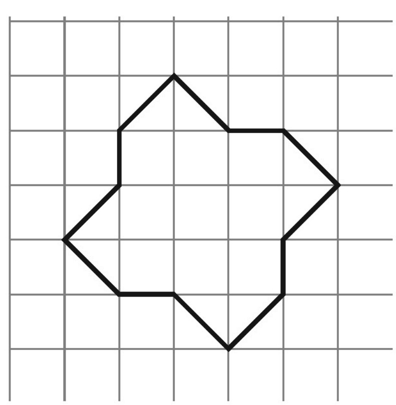
$$2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdot\frac{6}{5}\cdot\frac{7}{6}=7.$$

3. Students Arn, Bob, Cyd, Dan, Eve, and Fon are arranged in that order in a circle. They start counting: Arn first, then Bob, and so forth. When the number contains a 7 as a digit (such as 47) or is a multiple of 7 that person leaves the circle and the counting continues. Who is the last one present in the circle?

   (A) Arn **(B)** Bob **(C)** Cyd **(D)** Dan **(E)** Eve

   Answer (D): Dan is the last one present because Arn is out on 7, Cyd on 14, Fon on 17, Bob on 21 and Eve on 27.

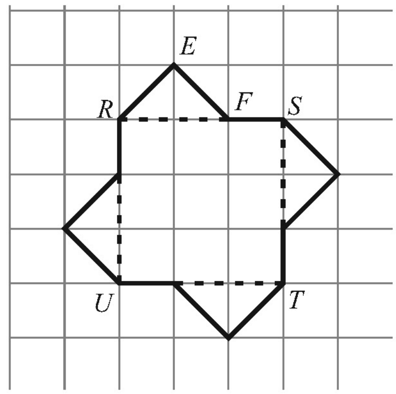
4. The twelve-sided figure shown has been drawn on 1 cm × 1 cm graph paper. What is the area of the figure in cm^2?



   (A) 12 **(B)** 12.5 **(C)** 13 **(D)** 13.5 **(E)** 14

Answer (C):

In the figure below, the square $RSTU$ has area $3 \cdot 3 = 9$, and $\triangle REF$ has area $\frac{1}{2} \cdot 2 \cdot 1 = 1$. The other three triangles are congruent to $\triangle REF$, so the total area is $9 + 4 \cdot 1 = 13$.



5. What is the value of $1 + 3 + 5 + \cdots + 2017 + 2019 - 2 - 4 - 6 - \cdots - 2016 - 2018$?

(A) -1010 **(B)** -1009 **(C)** 1008 **(D)** 1009 **(E)** 1010

Answer (E): The expression can be written as

$$1 + (3 - 2) + (5 - 4) + (7 - 6) + \cdots + (2019 - 2018) = 1 + \frac{2018}{2} = 1010.$$

6. On a trip to the beach, Anh traveled 50 miles on the highway and 10 miles on a coastal access road. He drove three times as fast on the highway as on the coastal road. If Anh spent 30 minutes driving on the coastal road, how many minutes did his entire trip take?

(A) 50 **(B)** 70 **(C)** 80 **(D)** 90 **(E)** 100

Answer (C): Anh's speed on the coastal road was 20 miles per hour because he drove 10 miles in half an hour. This implies that Anh traveled 60 miles per hour on the highway, and so it took Anh 50 minutes to drive the 50 miles. Anh's entire trip took $30 + 50 = 80$ minutes.

OR

Because Anh drives three times as fast on the highway, he needs only $\frac{30}{3} = 10$ minutes to drive 10 miles on the highway and five times 10 minutes to drive 50 miles on the highway. Thus his total time is $30 + 5(10) = 80$ minutes.

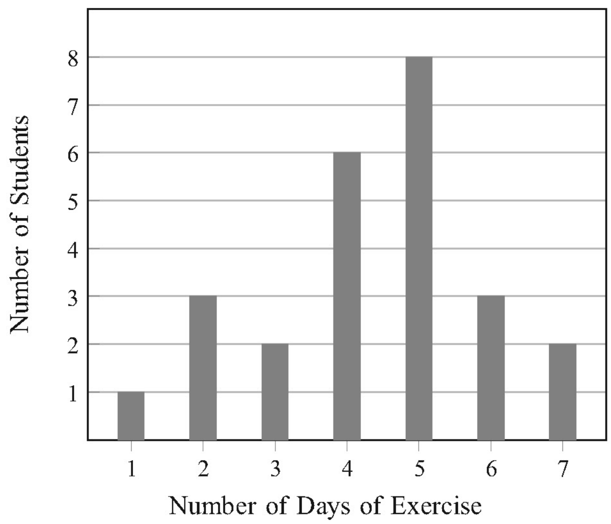
7. The 5-digit number $\underline{2}\ \underline{0}\ \underline{1}\ \underline{8}\ \underline{U}$ is divisible by 9. What is the remainder when this number is divided by 8?

   (A) 1 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 7

   Answer (B):

   To be divisible by 9, the sum of the digits must be divisible by 9, so $2+0+1+8+U = 11+U$ is divisible by 9. Thus $U = 7$, and the 5-digit number is 20187. Then because $20187 = 8 \cdot 2523+3$, the remainder is 3.

8. Mr. Garcia asked the members of his health class how many days last week they exercised for at least 30 minutes. The results are summarized in the following bar graph, where the heights of the bars represent the number of students.



   What was the mean number of days of exercise last week, rounded to the nearest hundredth, reported by the students in Mr. Garcia's class?

   (A) 3.50 **(B)** 3.57 **(C)** 4.36 **(D)** 4.50 **(E)** 5.00

   Answer (C): The number of students surveyed was $1+3+2+6+8+3+2 = 25$, and the number of days of exercise reported was $1 \cdot 1 + 3 \cdot 2 + 2 \cdot 3 + 6 \cdot 4 + 8 \cdot 5 + 3 \cdot 6 + 2 \cdot 7 = 109$. So the mean number of days of exercise was $109 \div 25 = 4.36$.

9. Tyler is tiling the floor of his 12 foot by 16 foot living room. He plans to place one-foot by one-foot square tiles to form a border along the edges of the room and to fill in the rest of the floor with two-foot by two-foot square tiles. How many tiles will he use?

   (A) 48 **(B)** 87 **(C)** 91 **(D)** 96 **(E)** 120

   Answer (B): The perimeter of the room is $2(12+16) = 56$ feet. The 4 corner tiles each occupy 2 feet of the perimeter, so Tyler needs those 4 plus another $56 - 4 \cdot 2 = 48$ one-foot square tiles to make the border. The remaining space in the room is a 10 foot by 14 foot rectangle, so Tyler needs $\frac{10}{2} \cdot \frac{14}{2} = 35$ two-foot square tiles to cover it. Therefore he needs a total of $4 + 48 + 35 = 87$ tiles.

OR

If Tyler used only one-foot square tiles, he would need a total of $12 \cdot 16 = 192$ tiles. As above, the room minus the border is a 10 foot by 14 foot rectangle to be covered with two-foot square tiles, each of which has the same area as 4 one-foot square tiles. Therefore the number of tiles needed to cover that portion of the room is not $10 \cdot 14 = 140$, but $\frac{140}{4} = 35$, and the total number of tiles needed is $(192 - 140) + 35 = 87$.

10. The *harmonic mean* of a set of non-zero numbers is the reciprocal of the average of the reciprocals of the numbers. What is the harmonic mean of 1, 2, and 4?

(A) $\dfrac{3}{7}$ **(B)** $\dfrac{7}{12}$ **(C)** $\dfrac{12}{7}$ **(D)** $\dfrac{7}{4}$ **(E)** $\dfrac{7}{3}$

Answer (C): The reciprocals of 1, 2, and 4 are 1, $\frac{1}{2}$, and $\frac{1}{4}$, and their average is $\frac{1+\frac{1}{2}+\frac{1}{4}}{3} = \frac{7}{12}$. So the harmonic mean of 1, 2, and 4 is the reciprocal of $\frac{7}{12}$, which is $\frac{12}{7}$.

11. Abby, Bridget, and four of their classmates will be seated in two rows of three for a group picture, as shown.

$$\begin{matrix} X & X & X \\ X & X & X \end{matrix}$$

If the seating positions are assigned randomly, what is the probability that Abby and Bridget are adjacent to each other in the same row or the same column?

(A) $\dfrac{1}{3}$ **(B)** $\dfrac{2}{5}$ **(C)** $\dfrac{7}{15}$ **(D)** $\dfrac{1}{2}$ **(E)** $\dfrac{2}{3}$

Answer (C): There are 6 possible positions for Abby, and this leaves 5 possible positions for Bridget. Because their order doesn't matter, the two girls can be placed in any of $\frac{6 \cdot 5}{2} = 15$ pairs of positions. There are 2 pairs of positions that are adjacent in the top row, 2 pairs that are adjacent in the bottom row, and 3 pairs that are adjacent in the same column. So the probability that they occupy adjacent positions is $\frac{2+2+3}{15} = \frac{7}{15}$.

OR

There is a $\frac{2}{3}$ chance Abby is assigned to a corner position and a $\frac{1}{3}$ chance that Abby is assigned to a middle position. If Abby is assigned to a corner position, there is a $\frac{2}{5}$ chance that Bridget is adjacent to Abby. If Abby is assigned to a middle position, there is a $\frac{3}{5}$ chance that Bridget is adjacent to Abby. Thus, the probability that Abby and Bridget are adjacent to each other in the same row or column is $\frac{2}{3} \cdot \frac{2}{5} + \frac{1}{3} \cdot \frac{3}{5} = \frac{7}{15}$.

12. The clock in Sri's car, which is not accurate, gains time at a constant rate. One day as he begins shopping he notes that his car clock and his watch (which is accurate) both say 12:00 noon. When he is done shopping, his watch says 12:30 and his car clock says 12:35. Later that day, Sri loses his watch. He looks at his car clock and it says 7:00. What is the actual time?

(A) 5:50 **(B)** 6:00 **(C)** 6:30 **(D)** 6:55 **(E)** 8:10

Answer (B): The ratio of time as measured by Sri's car clock to actual time is $\frac{35}{30} = \frac{7}{6}$. Therefore when the clock shows that 7 hours have passed since noon, only 6 hours have actually passed, so the actual time is 6:00.

OR

Gaining 5 minutes every half hour is equivalent to gaining 1 hour every 6 hours. Hence when the car clock reads 7:00 the actual time is 6:00.

13. Laila took five math tests, each worth a maximum of 100 points. Laila's score on each test was an integer between 0 and 100, inclusive. Laila received the same score on the first four tests, and she received a higher score on the last test. Her average score on the five tests was 82. How many values are possible for Laila's score on the last test?

(A) 4 **(B)** 5 **(C)** 9 **(D)** 10 **(E)** 18

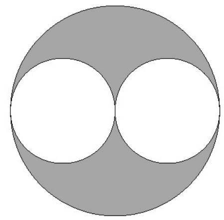
Answer (A): Because Laila's score on the last test was higher than her other test scores and all of her scores were integers, for each point below 82 that she scored on each of the first four tests, she had to score four points above 82 on the last test. Therefore, her possible scores on the last test were 86, 90, 94, and 98 (with scores on the first four tests of 81, 80, 79, and 78, respectively). Thus there are four possible values for her score on the last test.

14. Let N be the greatest five-digit number whose digits have a product of 120. What is the sum of the digits of N?

(A) 15 **(B)** 16 **(C)** 17 **(D)** 18 **(E)** 20

Answer (D): The leftmost digit of N must be the greatest single-digit factor of $120 = 2^3 \cdot 3 \cdot 5$. Note that $2^3 = 8$ is a factor of 120, but 9 is not, hence the leftmost digit of N must be 8. Because $5 \cdot 3 = 15$ cannot be a digit, it follows that 5 must be immediately to the right of the 8, and 3 must be immediately to the right of the 5. The remaining rightmost two digits of N must be 1, so that $N = 85311$. Thus the sum of the digits of N is 18.

15. In the diagram below, a diameter of each of the two smaller circles is a radius of the larger circle. If the two smaller circles have a combined area of 1 square unit, then what is the area of the shaded region, in square units?



(A) $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** $\frac{\pi}{2}$

Answer (D): If one of the smaller circles has radius r, then its area is πr^2, and the area of the larger circle is $\pi(2r)^2 = 4\pi r^2$. Thus the area of the large circle is 4 times the area of one of the small circles. So the area of the shaded region is equal to the combined area of the two smaller circles, which is 1 square unit.

OR

The ratio of the areas of two similar figures is the square of the ratio of their corresponding lengths. Because the diameter of the larger circle is twice the diameter of each smaller circle, it follows that the area of the larger circle is four times the area of each smaller circle. Therefore the shaded area is equal to the combined area of the two smaller circles.

16. Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

(A) 1440 **(B)** 2880 **(C)** 5760 **(D)** 182,440 **(E)** 362,880

Answer (C): There are two ways to arrange the two Arabic books among themselves and there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ ways to arrange the four Spanish books among themselves. If the two Arabic books are considered as a unit and the four Spanish books are considered as a unit, then, including the three German books, there are five objects to arrange on the shelf. These five objects can be arranged on the shelf in $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways. So altogether the books can be arranged in $2 \cdot 24 \cdot 120 = 5760$ ways.

17. Bella begins to walk from her house toward her friend Ella's house. At the same time, Ella begins to ride her bicycle toward Bella's house. They each maintain a constant speed, and Ella rides 5 times as fast as Bella walks. The distance between their houses is 2 miles, which is 10,560 feet, and Bella covers $2\frac{1}{2}$ feet with each step. How many steps will Bella take by the time she meets Ella?

(A) 704 **(B)** 845 **(C)** 1056 **(D)** 1760 **(E)** 3520

Answer (A): By the time that the two girls meet, Ella will ride 5 times the distance that Bella walks, so Ella will ride $\frac{5}{6}$ of the total distance, and Bella will walk $\frac{1}{6}$ of the total distance, or $\frac{1}{6}(10,560) = 1760$ feet. Each of her steps covers $2\frac{1}{2} = \frac{5}{2}$ feet, so the number of steps she will take is $1760 \div \frac{5}{2} = 1760 \cdot \frac{2}{5} = 704$.

OR

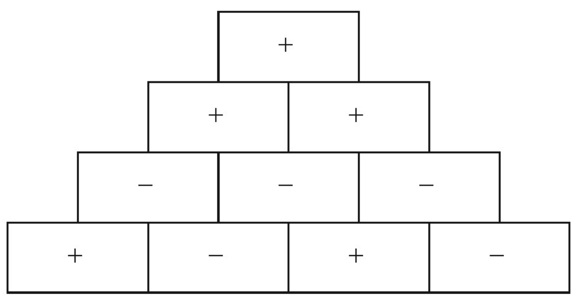
Bella's rate is 2.5 feet per step whereas Ella travels 12.5 feet for each of Bella's steps. So together they travel 15 feet for each of Bella's steps. Hence the number of steps taken by Bella when she meets Ella is $\frac{10{,}560}{15} = 704$.

18. How many positive factors does 23,232 have?

(A) 9 **(B)** 12 **(C)** 28 **(D)** 36 **(E)** 42

Answer (E): The prime factorization of 23,232 is $2^6 \cdot 3 \cdot 11^2$. Each factor of 23,232 must be of the form $2^a \cdot 3^b \cdot 11^c$, where $a = 0, 1, 2, 3, 4, 5$, or 6, $b = 0$ or 1, and $c = 0, 1$, or 2. Therefore, the number of factors of 23,232 is $7 \cdot 2 \cdot 3 = 42$.

19. In a sign pyramid a cell gets a "+" if the two cells below it have the same sign, and it gets a "–" if the two cells below it have different signs. The diagram below illustrates a sign pyramid with four levels. How many possible ways are there to fill the four cells in the bottom row to produce a "+" at the top of the pyramid?



(A) 2 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 16

Answer (C): Think of the + sign as +1, and the − sign as −1. Let a, b, c, and d denote the values of the four cells at the bottom of the pyramid, in that order. Then the cells in the second row from the bottom have values $a \cdot b$, $b \cdot c$ and $c \cdot d$, and the cells in the row above this are $a \cdot b \cdot b \cdot c = a \cdot c$ and $b \cdot c \cdot c \cdot d = b \cdot d$ (because both 1 and −1 squared are 1.) Finally, the top cell has value $a \cdot b \cdot c \cdot d$. This value is +1 if all four variables are +1 or all four are −1, giving two ways; or, if two of the variables are +1 and two are −1, giving 6 additional ways (+ + - -, + - + -, + - - +, - + + -, - + - +, and - - + +). Thus there are a total of 8 ways to fill the fourth row.

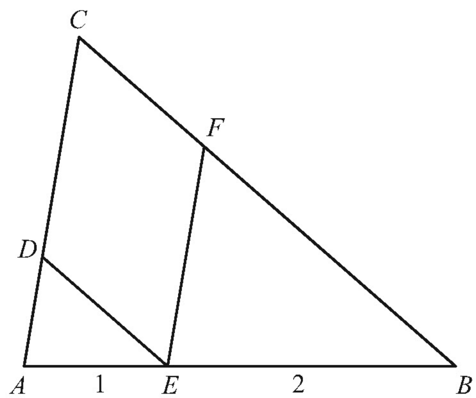
OR

In order to produce a + at the top of the pyramid, the second row must contain either + + or − −. Each leads to two possible arrangements for the third row. Consider the following cases.

Second row	Third row	Fourth row
$++$	$+++$	$++++$ or $----$
$++$	$---$	$+-+-$ or $-+-+$
$--$	$+-+$	$++--$ or $--++$
$--$	$-+-$	$+--+$ or $-++-$

Thus, there are eight possible ways to fill the fourth row.

20. In ΔABC, point E is on $\overline{AB}$ with $AE = 1$ and $EB = 2$. Point D is on $\overline{AC}$ so that $\overline{DE} \parallel \overline{BC}$ and point F is on $\overline{BC}$ so that $\overline{EF} \parallel \overline{AC}$. What is the ratio of the area of $CDEF$ to the area of ΔABC?

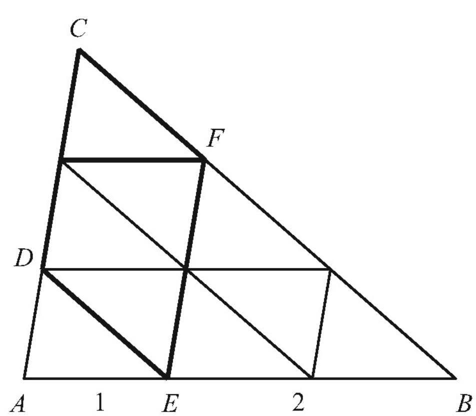


(A) $\frac{4}{9}$ **(B)** $\frac{1}{2}$ **(C)** $\frac{5}{9}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{2}{3}$

Answer (A): Triangles AED, EBF, and ABC are similar with their sides in the ratio of 1:2:3. Therefore their areas are in the ratio of 1:4:9. The combined areas of ΔAED and ΔEBF constitute $\frac{5}{9}$ of the area of ΔABC, so the area of $CDEF$ is $\frac{4}{9}$ of the area of ΔABC.

OR

Connecting corresponding points of trisection of the three sides of the triangle, as shown in the figure below, results in nine congruent triangles, four of which constitute $CDEF$.

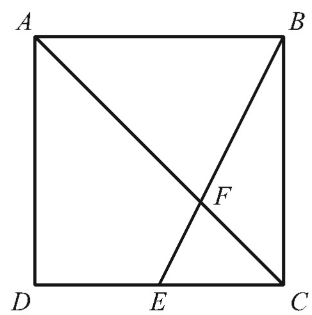


21. How many positive three-digit integers have a remainder of 2 when divided by 6, a remainder of 5 when divided by 9, and a remainder of 7 when divided by 11?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

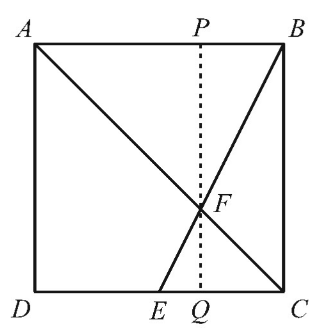
Answer (E): Let n be a positive three-digit integer that satisfies the conditions stated in the problem. Note that because n has a remainder of 2 when divided by 6, $n+4$ is divisible by 6. Similarly, $n+4$ is divisible by 9 and 11. Hence $n+4$ is divisible by the least common multiple of 6, 9, and 11, which is 198. Thus $n+4 = 198k$ where $k = 1$, 2, 3, 4, or 5. So $n = 194$, 392, 590, 788, or 986, and there are therefore five positive three-digit integers satisfying the given conditions.

22. Point E is the midpoint of side $\overline{CD}$ in square $ABCD$, and $\overline{BE}$ meets diagonal $\overline{AC}$ at F. The area of quadrilateral $AFED$ is 45. What is the area of $ABCD$?



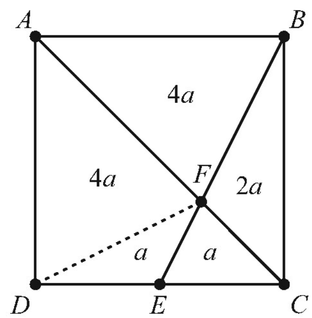
(A) 100 **(B)** 108 **(C)** 120 **(D)** 135 **(E)** 144

Answer (B): To see what fraction of the square is occupied by quadrilateral $AFED$, first note that ΔADC occupies half of the area. Next note that because $\overline{AB}$ and $\overline{CD}$ are parallel, it follows that ΔABF and ΔCEF are similar, and $CE = \frac{1}{2}AB$. Draw $\overline{PQ}$ through F such that $\overline{PF}$ and $\overline{FQ}$ are altitudes of ΔABF and ΔCEF, respectively. Then $FQ = \frac{1}{2}PF$, so $FQ = \frac{1}{3}PQ = \frac{1}{3}BC$. Therefore the area of ΔCEF is $\frac{1}{2}(CE)(FQ) = \frac{1}{2}\left(\frac{1}{2}AB\right)\left(\frac{1}{3}BC\right) = \frac{1}{12}(AB)(BC)$, which is $\frac{1}{12}$ of the area of the square. Because the area of quadrilateral $AFED$ equals the area of ΔADC minus the area of ΔCEF, the fraction of the square occupied by quadrilateral $AFED$ is $\frac{1}{2} - \frac{1}{12} = \frac{5}{12}$. Because the area of $AFED$ is 45, the area of $ABCD$ is $\left(\frac{12}{5}\right)(45) = 108$.

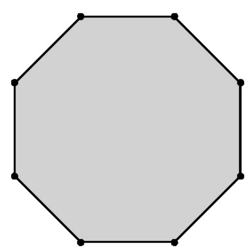


OR

Let a denote the area of ΔCEF. Note that ΔABF is similar to ΔCEF with ratio of similarity $AB : CE = 2 : 1$. Therefore the area of ΔABF is $4a$. Because ΔDEF and ΔCEF have equal bases and have a common altitude from F, the area of ΔDEF is a. Because $BF = 2FE$ and because ΔCBF and ΔCEF have a common altitude from C, it follows that the area of ΔCBF is $2a$. Similarly, ΔDAF has an area twice that of ΔDCF, so the area of ΔDAF is 4a. Finally, the area of quadrilateral $AFED$ is $45 = 5a$, so $a = 9$, and therefore the area of square $ABCD$ is $12a = 108$.



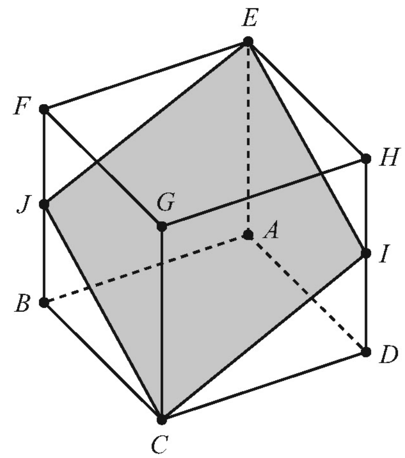
23. From a regular octagon, a triangle is formed by connecting three randomly chosen vertices of the octagon. What is the probability that at least one of the sides of the triangle is also a side of the octagon?



(A) $\dfrac{2}{7}$ **(B)** $\dfrac{5}{42}$ **(C)** $\dfrac{11}{14}$ **(D)** $\dfrac{5}{7}$ **(E)** $\dfrac{6}{7}$

Answer (D): For each side of the octagon, there are 6 triangles containing that side. Because the 8 triangles containing two adjacent sides of the octagon are counted twice, there are a total of $8 \cdot 6 - 8 = 40$ triangles sharing a side with the octagon. The total number of triangles that can be formed from the eight vertices is $\frac{8\cdot7\cdot6}{3!} = 56$, so the probability is $\frac{40}{56} = \frac{5}{7}$.

24. In the cube $ABCDEFGH$ with opposite vertices C and E, J and I are the midpoints of edges $\overline{FB}$ and $\overline{HD}$, respectively. Let R be the ratio of the area of the cross-section $EJCI$ to the area of one of the faces of the cube. What is R^2?



(A) $\frac{5}{4}$ **(B)** $\frac{4}{3}$ **(C)** $\frac{3}{2}$ **(D)** $\frac{25}{16}$ **(E)** $\frac{9}{4}$

Answer (C): Let s denote the length of an edge of the cube. Now $EJCI$ is a non-square rhombus whose area is $\frac{1}{2}EC \cdot JI$, because the area of a rhombus is half the product of the lengths of its diagonals. By the Pythagorean Theorem, $JI = FH = s\sqrt{2}$, and using the Pythagorean Theorem twice, $EC = s\sqrt{3}$. Thus $R = \frac{\frac{1}{2}EC \cdot JI}{s^2} = \frac{\frac{1}{2}(s\sqrt{3})(s\sqrt{2})}{s^2} = \frac{\sqrt{6}}{2}$ and $R^2 = \frac{6}{4} = \frac{3}{2}$.

25. How many perfect cubes lie between $2^8 + 1$ and $2^{18} + 1$, inclusive?

(A) 4 **(B)** 9 **(C)** 10 **(D)** 57 **(E)** 58

Answer (E): Note that $2^8 + 1 = 257$ and that $216 = 6^3 < 257 < 7^3 = 343$. Also note that $2^{18} = (2^6)^3 = 64^3$, so the perfect cube that is closest to and less than $2^{18} + 1$ is 64^3. Thus the numbers $7^3, 8^3, \ldots, 63^3, 64^3$ are precisely the perfect cubes that lie between the two given numbers, and so there are $64 - 6 = 58$ of these perfect cubes.

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We acknowledge the generosity of the following organizations in supporting the MAA AMC and Invitational Competitions:

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2019 AMC 8 Solutions

Problem 1

Ike and Mike go into a sandwich shop with a total of $\$30.00$ to spend. Sandwiches cost $\$4.50$ each and soft drinks cost $\$1.00$ each. Ike and Mike plan to buy as many sandwiches as they can, and use any remaining money to buy soft drinks. Counting both sandwiches and soft drinks, how many items will they buy?

$\textbf{(A) }6\qquad\textbf{(B) }7\qquad\textbf{(C) }8\qquad\textbf{(D) }9\qquad\textbf{(E) }10$

Solution 1

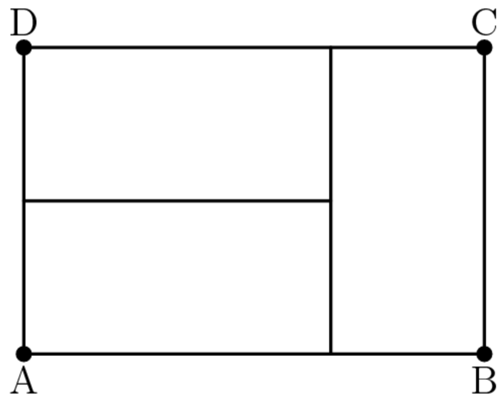
We know that the sandwiches cost 4.50 dollars. Guessing will bring us to multiplying 4.50 by 6, which gives us 27.00. Since they can spend 30.00 they have 3 dollars left. Since sodas cost 1.00 dollar each, they can buy 3 sodas, which makes them spend 30.00 Since they bought 6 sandwiches and 3 sodas, they bought a total of 9 items. Therefore, the answer is $\boxed{D = 9}$

Solution 2 (Using Algebra)

Let s be the number of sandwiches and d be the number of sodas. We have to satisfy the equation of$4.50s + d = 30$In the question, it states that Ike and Mike buys as many sandwiches as possible. So, we drop the number of sodas for a while. We have:$4.50s = 30$ $s = \frac{30}{4.5}s = 6R30$We don't want a remainder so the maximum number of sandwiches is 6. The total money spent is $6 \cdot 4.50 = 27$. The number of dollar left to spent on sodas is $30 - 27 = 3$ dollars. 3 dollars can buy 3 sodas leading us to a total of $6 + 3 = 9$ items. Hence, the answer is $\boxed{(D) = 9}$

Problem 2

Three identical rectangles are put together to form rectangle $ABCD$, as shown in the figure below. Given that the length of the shorter side of each of the smaller rectangles is 5 feet, what is the area in square feet of rectangle $ABCD$?



$\textbf{(A) } 45 \qquad \textbf{(B) } 75 \qquad \textbf{(C) } 100 \qquad \textbf{(D) } 125 \qquad \textbf{(E) } 150$

Solution 1

We can see that there are 2 rectangles lying on top of the other and that is the same as the length of one rectangle. Now we know that the shorter side is 5, if we take the information of the problem, and the bigger side is 10, if we do $5 \cdot 2 = 10$. Now we get the sides of the big rectangles being 15 and 10, so the area is $\boxed{\textbf{(E) } 150}$.

Solution 2

Using the diagram we find that the larger side of the small rectangle is 2 times the length of the smaller side. Therefore, the longer side is $5 \cdot 2 = 10$. So the area of the identical rectangles is $5 \cdot 10 = 50$. We have 3 identical rectangles that form the large rectangle. Therefore the area of the large rectangle is $50 \cdot 3 = \boxed{\textbf{(E) } 150}$.

Solution 3

We see that if the short sides are 5, the long side has to be $5 \cdot 2 = 10$ because the long side is equal to the 2 short sides and because the rectangles are congruent. If that is to be, then the long side of the BIG rectangle(rectangle $ABCD$) is $10 + 5 = 15$ because long side + short side of the small rectangle is 15. The short side of rectangle $ABCD$ is 10 because it is the long side of the short rectangle. Multiplying 15 and 10 together gets us $15 \cdot 10$ which is $\boxed{\textbf{(E) } 150}$.

Problem 3

Which of the following is the correct order of the fractions $\frac{15}{11}, \frac{19}{15},$ and $\frac{17}{13},$ from least to greatest?

$\textbf{(A)}\ \frac{15}{11}< \frac{17}{13}< \frac{19}{15} \qquad\textbf{(B)}\ \frac{15}{11}< \frac{19}{15}<\frac{17}{13} \qquad\textbf{(C)}\ \frac{17}{13}< \frac{19}{15}<\frac{15}{11} \qquad\textbf{(D)}\ \frac{19}{15}< \frac{15}{11}<\frac{17}{13} \qquad\textbf{(E)}\ \frac{19}{15}< \frac{17}{13}<\frac{15}{11}$

Solution 1

We take a common denominator:

$$\frac{15}{11},\frac{19}{15}, \frac{17}{13} = \frac{15\cdot 15 \cdot 13}{11\cdot 15 \cdot 13},\frac{19\cdot 11 \cdot 13}{15\cdot 11 \cdot 13}, \frac{17\cdot 11 \cdot 15}{13\cdot 11 \cdot 15} = \frac{2925}{2145},\frac{2717}{2145},\frac{2805}{2145}.$$

Since $2717 < 2805 < 2925$ it follows that the answer is $\boxed{\textbf{(E)}\frac{19}{15}<\frac{17}{13}<\frac{15}{11}}$.

Solution 2

When $\frac{x}{y}>1$ and $z>0$, $\frac{x+z}{y+z}<\frac{x}{y}$. Hence, the answer is $\boxed{\textbf{(E)}\frac{19}{15}<\frac{17}{13}<\frac{15}{11}}$.

This is also similar to Problem 20 on the AMC 2012.

Solution 3 (probably won't use this solution)

We use our insane mental calculator to find out that $\frac{15}{11} \approx 1.36$, $\frac{19}{15} \approx 1.27$, and $\frac{17}{13} \approx 1.31$. Thus, our answer is $\boxed{\textbf{(E)}\frac{19}{15}<\frac{17}{13}<\frac{15}{11}}$.

Solution 4

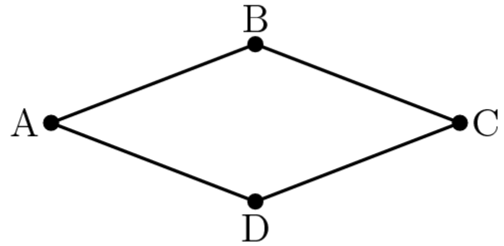
Suppose each fraction is expressed with denominator 2145: $\frac{2925}{2145}, \frac{2717}{2145}, \frac{2805}{2145}$. Clearly $2717 < 2805 < 2925$ so the answer is $\boxed{\textbf{(E)}}$.

Solution 5 -SweetMango77

We notice that each of these fraction's numerator $-$ denominator $= 4$. If we take each of the fractions, and subtract 1 from each, we get $\frac{4}{11}$, $\frac{4}{15}$, and $\frac{4}{19}$. These are easy to order because the numerators are the same, we get $\frac{4}{15}<\frac{4}{13}<\frac{4}{11}$. Because it is a subtraction by a constant, in order to order them, we keep the inequality signs to get $\boxed{\textbf{(E)}\frac{19}{15}<\frac{17}{13}<\frac{15}{11}}$.

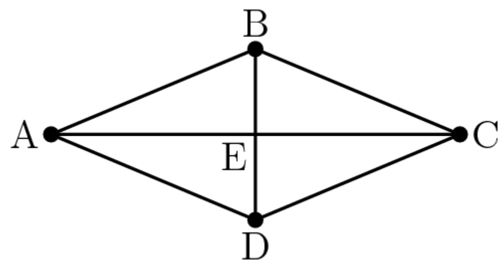
Problem 4

Quadrilateral $ABCD$ is a rhombus with perimeter 52 meters. The length of diagonal $\overline{AC}$ is 24 meters. What is the area in square meters of rhombus $ABCD$?



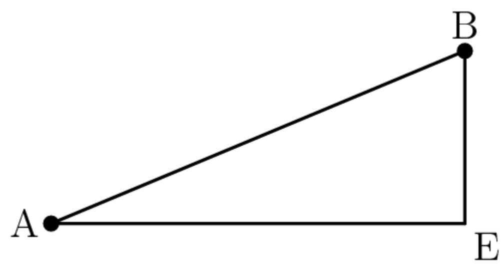
$\textbf{(A) }60\qquad\textbf{(B) }90\qquad\textbf{(C) }105\qquad\textbf{(D) }120\qquad\textbf{(E) }144$

Solution 1



A rhombus has sides of equal length. Because the perimeter of the rhombus is 52, each side is $\frac{52}{4}=13$. In a rhombus, diagonals are perpendicular and bisect each other, which means $\overline{AE}$ = 12 = $\overline{EC}$.

Consider one of the right triangles:



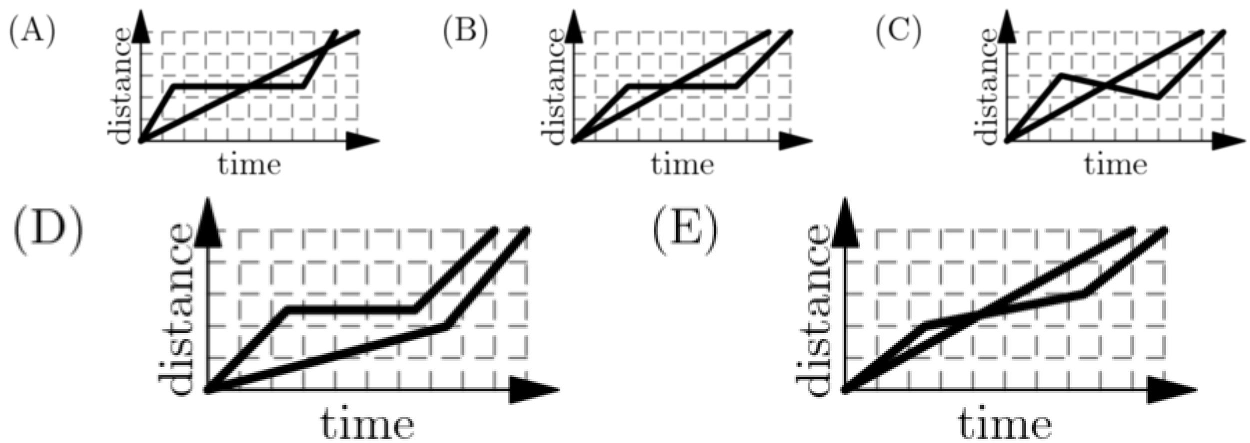
$\overline{AB}$ = 13, and $\overline{AE}$ = 12. Using Pythagorean theorem, we find that $\overline{BE}$ = 5. "You may recall the famous Pythagorean triple, (5, 12, 13), that's how I did it" - Zack2008

Thus the values of the two diagonals are $\overline{AC}$ = 24 and $\overline{BD}$ = 10. The area of a rhombus is = $\frac{d_1\cdot d_2}{2}$ = $\frac{24\cdot 10}{2}$ = 120

$\boxed{\textbf{(D) } 120}$

Problem 5

A tortoise challenges a hare to a race. The hare eagerly agrees and quickly runs ahead, leaving the slow-moving tortoise behind. Confident that he will win, the hare stops to take a nap. Meanwhile, the tortoise walks at a slow steady pace for the entire race. The hare awakes and runs to the finish line, only to find the tortoise already there. Which of the following graphs matches the description of the race, showing the distance d traveled by the two animals over time t from start to finish?



Solution 1

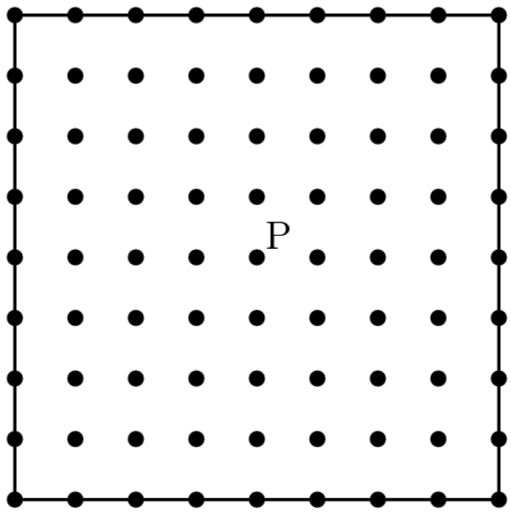
First, the tortoise walks at a constant rate, ruling out (D) Second, when the hare is resting, the distance will stay the same, ruling out (E) and (C). Third, the tortoise wins the race, meaning that the non-constant one should go off the graph last, ruling out (A). Therefore, the answer $\boxed{\textbf{(B)}}$ is the only one left.

Solution 2

First, we know that the rabbit beats the tortoise in the first half of the race. So he is going to be ahead of the tortoise. We also know, while he rested, he didn't move. The only graph portraying that is going to be $\boxed{\textbf{(B)}}$. This is our answer.

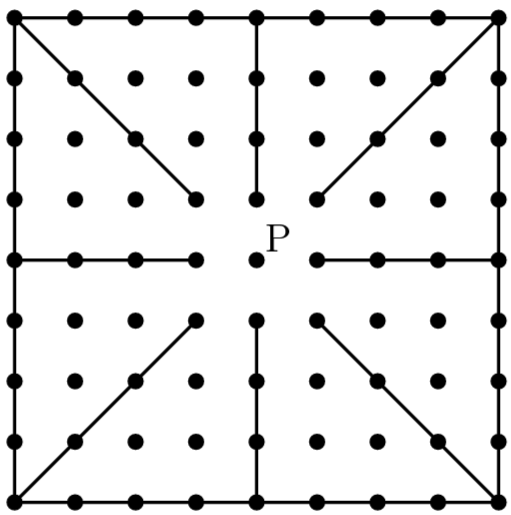
Problem 6

There are 81 grid points (uniformly spaced) in the square shown in the diagram below, including the points on the edges. Point P is in the center of the square. Given that point Q is randomly chosen among the other 80 points, what is the probability that the line PQ is a line of symmetry for the square?



(A) $\frac{1}{5}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{2}{5}$ **(D)** $\frac{9}{20}$ **(E)** $\frac{1}{2}$

Solution 1



Lines of symmetry go through point P, and there are 8 directions the lines could go, and there are 4 dots at each direction. $\frac{4\times 8}{80}=\boxed{\textbf{(C)}\frac{2}{5}}$.

Problem 7

Shauna takes five tests, each worth a maximum of 100 points. Her scores on the first three tests are 76, 94, and 87. In order to average 81 for all five tests, what is the lowest score she could earn on one of the other two tests?

(A) 48 **(B)** 52 **(C)** 66 **(D)** 70 **(E)** 74

Solution 1

We should notice that we can turn the information we are given into a linear equation and just solve for our set variables. I'll use the variables x and y for the scores on the last two tests. $\frac{76+94+87+x+y}{5}=81$,$\frac{257+x+y}{5}=81$. We can now cross multiply to get rid of the denominator.$257+x+y=405$,$x+y=148$.Now that we have this equation, we will assign y as the lowest score of the two other tests, and so:$x=100$,$y=48$.Now we know that the lowest score on the two other tests is $\boxed{48}$.

Solution 2

Right now, she scored $76, 94,$ and 87 points, for a total of 257 points. She wants her average to be 81 for her 5 tests, so she needs to score 405 points in total. This means she needs to score a total of $405-257=148$ points in her next 2 tests. Since the maximum score she can get on one of her 2 tests is 100, the least possible score she can get is $\boxed{\textbf{(A)}\ 48}$.

Note: You can verify that $\boxed{48}$ is the right answer because it is the lowest answer out of the 5. Since it is possible to get 48, we are guaranteed that that is the right answer.

Solution 3

We can compare each of the scores with the average of 81: $76 \rightarrow -5$, $94 \rightarrow +13$, $87 \rightarrow +6$, $100 \rightarrow +19$;

So the last one has to be -33 (since all the differences have to sum to 0), which corresponds to $81-33=\boxed{48}$.

Problem 8

Gilda has a bag of marbles. She gives 20% of them to her friend Pedro. Then Gilda gives 10% of what is left to another friend, Ebony. Finally, Gilda gives 25% of what is now left in the bag to her brother Jimmy. What percentage of her original bag of marbles does Gilda have left for herself?

$\textbf{(A)}\ 20 \qquad \textbf{(B)}\ 33\frac{1}{3} \qquad \textbf{(C)}\ 38 \qquad \textbf{(D)}\ 45 \qquad \textbf{(E)}\ 54$

Solution 1

After Gilda gives 20% of the marbles to Pedro, she has 80% of the marbles left. If she then gives 10% of what's left to Ebony, she has $(0.8*0.9)$ = 72% of what she had at the beginning. Finally, she gives 25% of what's left to her brother, so she has $(0.75*0.72)$ $\boxed{\textbf{(E)}\ 54}$. of what she had in the beginning left.

Solution 2

Suppose Gilda has 100 marbles.

Then she gives Pedro 20% of 100 = 20, she remains with 80 marbles.

Out of 80 marbles she gives 10% of 80 = 8 to Ebony.

Thus she remains with 72 marbles.

Then she gives 25% of 72 = 18 to Jimmy, finally leaving her with 54.

And $\frac{54}{100}$=54%=$\boxed{\textbf{(E)}\ 54}$

Solution 3

(Only if you have lots of time do it this way) Since she gave away 20% and 10% of what is left and then another 25% of what is actually left, we can do 20+10+25 or 55%. But it is actually going to be a bit more than 55% because 10% of what is left is not 10% of the total amount. So the only option that is greater than 100% - 55% is $\boxed{\textbf{(E)}\ 54}$.

Problem 9

Alex and Felicia each have cats as pets. Alex buys cat food in cylindrical cans that are 6 cm in diameter and 12 cm high. Felicia buys cat food in cylindrical cans that are 12 cm in diameter and 6 cm high. What is the ratio of the volume one of Alex's cans to the volume one of Felicia's cans?

$\textbf{(A)}\ 1:4 \qquad \textbf{(B)}\ 1:2 \qquad \textbf{(C)}\ 1:1 \qquad \textbf{(D)}\ 2:1 \qquad \textbf{(E)}\ 4:1$

Solution 1

Using the formula for the volume of a cylinder, we get Alex, $\pi 108$, and Felicia, $\pi 216$. We can quickly notice that π cancels out on both sides, and that Alex's volume is $1/2$ of Felicia's leaving $1/2 = \boxed{1:2}$ as the answer.

Solution 2

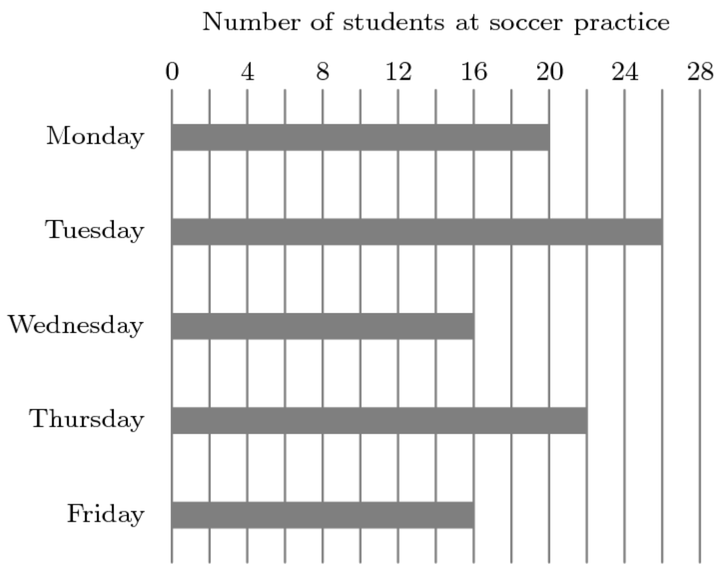
Using the formula for the volume of a cylinder, we get that the volume of Alex's can is $3^2 \cdot 12 \cdot \pi$, and that the volume of Felicia's can is $6^2 \cdot 6 \cdot \pi$. Now we divide the volume of Alex's can by the volume of Felicia's can, so we get $\frac{1}{2}$, which is $\boxed{\textbf{(B)}\ 1:2}$

Solution 3

The ratio of the numbers is $1/2$. Looking closely at the formula $r^2 * h * \pi$, we see that the $r * h * \pi$ will cancel, meaning that the ratio of them will be $\frac{1(2)}{2(2)}$ = $\boxed{\textbf{(B) } 1:2}$

Problem 10

The diagram shows the number of students at soccer practice each weekday during last week. After computing the mean and median values, Coach discovers that there were actually 21 participants on Wednesday. Which of the following statements describes the change in the mean and median after the correction is made?



$\textbf{(A)}$The mean increases by 1 and the median does not change.

$\textbf{(B)}$The mean increases by 1 and the median increases by 1.

$\textbf{(C)}$The mean increases by 1 and the median increases by 5.

$\textbf{(D)}$The mean increases by 5 and the median increases by 1.

$\textbf{(E)}$The mean increases by 5 and the median increases by 5.

Solution 1

On Monday, 20 people come. On Tuesday, 26 people come. On Wednesday, 16 people come. On Thursday, 22 people come. Finally, on Friday, 16 people come. $20+26+16+22+16=100$, so the mean is 20. The median is $(16, 16, 20, 22, 26) 20$. The coach figures out that actually 21 people come on Wednesday. The new mean is 21, while the new median is $(16, 20, 21, 22, 26) 21$. The median and mean both change, so the answer is $\boxed{\textbf{(B)}}$ Another way to compute the change in mean is to notice that the sum increased by 5 with the correction. So the average increased by $5/5 = 1$.

Problem 11

The eighth grade class at Lincoln Middle School has 93 students. Each student takes a math class or a foreign language class or both. There are 70 eighth graders taking a math class, and there are 54 eighth graders taking a foreign language class. How many eighth graders take *only* a math class and *not* a foreign language class?

$\textbf{(A) }16\qquad\textbf{(B) }23\qquad\textbf{(C) }31\qquad\textbf{(D) }39\qquad\textbf{(E) }70$

Solution 1

Let x be the number of students taking both a math and a foreign language class.

By P-I-E, we get $70 + 54 - x$ = 93.

Solving gives us $x = 31$.

But we want the number of students taking only a math class.

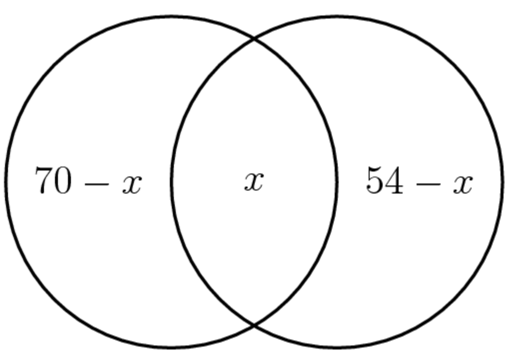
Which is $70 - 31 = 39$.

$\boxed{\textbf{(D) }39}$

Solution 2

We have $70 + 54 = 124$ people taking classes. However we over-counted the number of people who take both classes. If we subtract the original amount of people who take classes we get that 31 people took the two classes. To find the amount of people who took only math class web subtract the people who didn't take only one math class, so we get $70 - 31 = \boxed{\textbf{D}\, 39}$ -fath2012

Solution 3



We know that the sum of all three areas is 93 So, we have:$93 = 70 - x + x + 54 - x$
$93 = 70 + 54 - x 93 = 124 - x -31 = -x x = 31$

We are looking for the number of students in only math. This is $70 - x$.

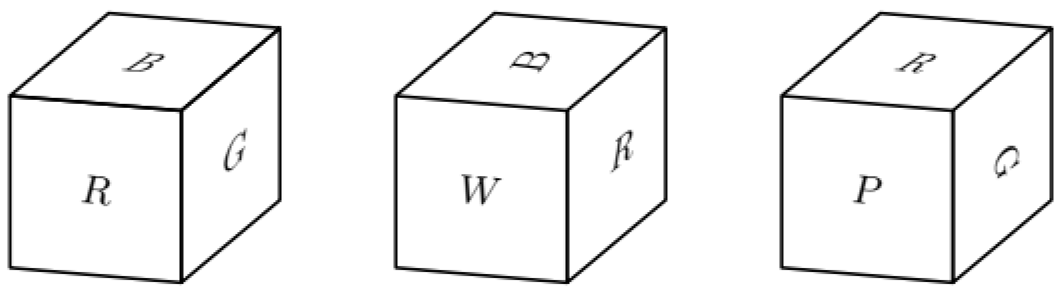
Substituting x with 31, our answer is $\boxed{39}$.

Solution 4

Associated video - https://www.youtube.com/watch?v=onPaMTO3dSA

Problem 12

The faces of a cube are painted in six different colors: red (R), white (W), green (G), brown (B), aqua (A), and purple (P). Three views of the cube are shown below. What is the color of the face opposite the aqua face?



Solution 1

B is on the top, and R is on the side, and G is on the right side. That means that (image 2) W is on the left side. From the third image, you know that P must be on the bottom since G is sideways. That leaves us with the back, so the back must be A. The front is opposite of the back, so the answer is $\boxed{\textbf{(A)}\ R}$.

Solution 2

Looking closely we can see that all faces are connected with R except for A. Thus the answer is $\boxed{\textbf{(A)}\ R}$.

It is A, just draw it out!

Solution 3

Associated video - https://www.youtube.com/watch?v=K5vaX_EzjEM

Problem 13

A *palindrome* is a number that has the same value when read from left to right or from right to left. (For example, 12321 is a palindrome.) Let N be the least three-digit integer which is not a palindrome but which is the sum of three distinct two-digit palindromes. What is the sum of the digits of N?

$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 3 \qquad \textbf{(C)}\ 4 \qquad \textbf{(D)}\ 5 \qquad \textbf{(E)}\ 6$

Solution 1

Note that the only positive 2-digit palindromes are multiples of 11, namely $11, 22, \ldots, 99$. Since N is the sum of 2-digit palindromes, N is necessarily a multiple of 11. The smallest 3-digit multiple of 11 which is not a palindrome is 110, so $N = 110$ is a candidate solution. We must check that 110 can be written as the sum of three distinct 2-digit palindromes; this suffices as $110 = 77 + 22 + 11$. Then $N = 110$, and the sum of the digits of N is $1 + 1 + 0 = \boxed{\textbf{(A)}\ 2}$.

Solution 2

Associated video - https://www.youtube.com/watch?v=bOnNFeZs7S8

Problem 14

Isabella has 6 coupons that can be redeemed for free ice cream cones at Pete's Sweet Treats. In order to make the coupons last, she decides that she will redeem one every 10 days until she has used them all. She knows that Pete's is closed on Sundays, but as she circles the 6 dates on her calendar, she realizes that no circled date falls on a Sunday. On what day of the week does Isabella redeem her first coupon?

$\textbf{(A)}\ \text{Monday} \qquad \textbf{(B)}\ \text{Tuesday} \qquad \textbf{(C)}\ \text{Wednesday} \qquad \textbf{(D)}\ \text{Thursday} \qquad \textbf{(E)}\ \text{Friday}$

Solution 1

Let $\text{Day } 1$ to $\text{Day } 2$ denote a day where one coupon is redeemed and the day when the second coupon is redeemed.

If she starts on a Monday she redeems her next coupon on Thursday.

Thursday to Sunday.

Thus $\textbf{(A)}\ \text{Monday}$ is incorrect.

If she starts on a Tuesday she redeems her next coupon on Friday.

Friday to Monday.

Monday to Thursday.

Thursday to Sunday.

Thus $\textbf{(B)}\ \text{Tuesday}$ is incorrect.

If she starts on a Wednesday she redeems her next coupon on Saturday.
Saturday to Tuesday.
Tuesday to Friday.
Friday to Monday.
Monday to Thursday.
And on Thursday she redeems her last coupon.

No Sunday occured thus $\boxed{\textbf{(C)}\ \text{Wednesday}}$ is correct.

Checking for the other options,

If she starts on a Thursday she redeems her next coupon on Sunday.
Thus $\textbf{(D)}\ \text{Thursday}$ is incorrect.

If she starts on a Friday she redeems her next coupon on Monday.
Monday to Thursday.
Thursday to Sunday.

Checking for the other options gave us negative results, thus the answer is $\boxed{\textbf{(C)}\ \text{Wednesday}}$.

Solution 2

Let
$Sunday \equiv 0 \pmod{7}$
$Monday \equiv 1 \pmod{7}$
$Tuesday \equiv 2 \pmod{7}$
$Wednesday \equiv 3 \pmod{7}$
$Thursday \equiv 4 \pmod{7}$
$Friday \equiv 5 \pmod{7}$
$Saturday \equiv 6 \pmod{7}$

$10 \equiv 3 \pmod{7}$
$20 \equiv 6 \pmod{7}$

$30 \equiv 2 \pmod 7$
$40 \equiv 5 \pmod 7$
$50 \equiv 1 \pmod 7$
$60 \equiv 4 \pmod 7$

Which clearly indicates if you start form a $x \equiv 3 \pmod 7$ you will not get a $y \equiv 0 \pmod 7$.

Any other starting value may lead to a $y \equiv 0 \pmod 7$.

Which means our answer is $\boxed{\textbf{(C)}\ Wednesday}$.

Solution 3

Like Solution 2, let the days of the week be numbers $\pmod 7$. 3 and 7 are coprime, so continuously adding 3 to a number $\pmod 7$ will cycle through all numbers from 0 to 6. If a string of 6 numbers in this cycle does not contain 0, then if you minus 3 from the first number of this cycle, it will always be 0. So, the answer is $\boxed{\textbf{(C)}\ Wednesday}$.

Solution 4

Since Sunday is the only day that has not been counted yet. We can just add the 3 days as it will become $\boxed{\textbf{(C)}\ Wednesday}$. Note: This only works when 7 and 3 are relatively prime.

Solution 5

Let Sunday be Day 0, Monday be Day 1, Tuesday be Day 2, and so forth. We see that Sundays fall on Day n, where n is a multiple of seven. If Isabella starts using her coupons on Monday (Day 1), she will fall on a Day that is a multiple of seven, a Sunday (her third coupon will be "used" on Day 21). Similarly, if she starts using her coupons on Tuesday (Day 2), Isabella will fall on a Day that is a multiple of seven (Day 42). Repeating this process, if she starts on Wednesday (Day 3), Isabella will first fall on a Day that is a multiple of seven, Day 63 (13, 23, 33, 43, 53 are not multiples of seven), but on her seventh coupon, of which she only has six. So, the answer is $\boxed{\textbf{(C)}\ \text{Wednesday}}$.

Solution 6

Associated video - https://www.youtube.com/watch?v=LktgMtgb_8E

Problem 15

On a beach 50 people are wearing sunglasses and 35 people are wearing caps. Some people are wearing both sunglasses and caps. If one of the people wearing a cap is selected at random, the probability that this person is is also wearing sunglasses is $\frac{2}{5}$. If instead, someone wearing sunglasses is selected at random, what is the probability that this person is also wearing a cap? $\textbf{(A) }\frac{14}{85}\qquad\textbf{(B) }\frac{7}{25}\qquad\textbf{(C) }\frac{2}{5}\qquad\textbf{(D) }\frac{4}{7}\qquad\textbf{(E) }\frac{7}{10}$

Video Solution

https://youtu.be/6xNkyDgIhEE?t=252

Solution 1

The number of people wearing caps and sunglasses is $\frac{2}{5}\cdot35=14$. So then 14 people out of the 50 people wearing sunglasses also have caps. $\frac{14}{50}=\boxed{\textbf{(B)}\frac{7}{25}}$

Video Solution

https://www.youtube.com/watch?v=gKlYlAiBzrs

Another video - https://www.youtube.com/watch?v=afMsUqER13c

https://youtu.be/37UWNaltvQo

Problem 16

Qiang drives 15 miles at an average speed of 30 miles per hour. How many additional miles will he have to drive at 55 miles per hour to average 50 miles per hour for the entire trip?

$\textbf{(A) }45\qquad\textbf{(B) }62\qquad\textbf{(C) }90\qquad\textbf{(D) }110\qquad\textbf{(E) }135$

Solution 1

The only option that is easily divisible by 55 is 110. Which gives 2 hours of travel. And by the formula $\frac{15}{30} + \frac{110}{55} = \frac{5}{2}$

And Average Speed = $\frac{\text{Total Distance}}{\text{Total Time}}$

Thus $\frac{125}{50} = \frac{5}{2}$

Both are equal and thus our answer is $\boxed{\textbf{(D) }110}$.

Solution 2

Note that the average speed is simply the total distance over the total time. Let the number of additional miles he has to drive be x. Therefore, the total distance is $15+x$ and the total time (in hours) is $\frac{15}{30}+\frac{x}{55}=\frac{1}{2}+\frac{x}{55}$. We can set up the following equation: $\frac{15+x}{\frac{1}{2}+\frac{x}{55}}=50.$ Simplifying the equation, we get $15+x=25+\frac{10x}{11}$. Solving the equation yields $x=110$, so our answer is $\boxed{\textbf{(D) }110}$.

Solution 3

If he travels 15 miles at a speed of 30 miles per hour, he travels for 30 min. Average rate is total distance over total time so $(15+d)/(0.5+t)=50$, where d is the distance left to travel and t is the time to travel that distance. solve for d to get $d=10+50t$. you also know that he has to travel 55 miles per hour for some time, so $d=55t$ plug that in for d to get $55t=10+50t$ and $t=2$ and since $d=55t$, $d=2\cdot 55=110$ the answer is $\boxed{\textbf{(D) }110}$.

Video Solution

Associated Video - https://www.youtube.com/watch?v=OC1KdFeZFeE

https://youtu.be/5K1AgeZ8rUQ

Problem 17

What is the value of the product $\left(\frac{1\cdot3}{2\cdot2}\right)\left(\frac{2\cdot4}{3\cdot3}\right)\left(\frac{3\cdot5}{4\cdot4}\right)\cdots\left(\frac{97\cdot99}{98\cdot98}\right)\left(\frac{98\cdot100}{99\cdot99}\right)?$

$\textbf{(A) }\frac{1}{2}\qquad\textbf{(B) }\frac{50}{99}\qquad\textbf{(C) }\frac{9800}{9801}\qquad\textbf{(D) }\frac{100}{99}\qquad\textbf{(E) }50$

Solution 1(Telescoping)

We rewrite: $\frac{1}{2}\cdot\left(\frac{3\cdot2}{2\cdot3}\right)\left(\frac{4\cdot3}{3\cdot4}\right)\cdots\left(\frac{99\cdot98}{98\cdot99}\right)\cdot\frac{100}{99}$

The middle terms cancel, leaving us with

$$\left(\frac{1\cdot100}{2\cdot99}\right)=\boxed{\textbf{(B)}\frac{50}{99}}$$

Solution 2

If you calculate the first few values of the equation, all of the values tend to $\frac{1}{2}$, but are not equal to it. The answer closest to $\frac{1}{2}$ but not equal to it is $\boxed{\textbf{(B)}\frac{50}{99}}$.~hpotter0104

Solution 3

Rewriting the numerator and the denominator, we get $\frac{\frac{100!\cdot 98!}{2}}{(99!)^2}$. We can simplify by canceling 99! on both sides, leaving us with: $\frac{100 \cdot 98!}{2 \cdot 99!}$ We rewrite $99!$ as $99 \cdot 98!$ and cancel $98!$, which gets $\boxed{\frac{50}{99}}$. Answer B.

Video Solution

Associated video - https://www.youtube.com/watch?v=yPQmvyVyvaM

https://www.youtube.com/watch?v=ffHl1dAjs7g&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=1

Problem 18

The faces of each of two fair dice are numbered 1, 2, 3, 5, 7, and 8. When the two dice are tossed, what is the probability that their sum will be an even number?

$\textbf{(A)}\ \frac{4}{9} \qquad \textbf{(B)}\ \frac{1}{2} \qquad \textbf{(C)}\ \frac{5}{9} \qquad \textbf{(D)}\ \frac{3}{5} \qquad \textbf{(E)}\ \frac{2}{3}$

Solution 1

The approach to this problem: There are two cases in which the sum can be an even number: both numbers are even and both numbers are odd. This results in only one case where the sum of the numbers are odd (one odd and one even in any order) . We can solve for how many ways the 2 numbers add up to an odd number and subtract the answer from 1.

How to solve the problem: The probability of getting an odd number first is $\frac{4}{6} = \frac{2}{3}$. In order to make the sum odd, we must select an even number next. The probability of getting an even number is $\frac{2}{6} = \frac{1}{3}$. Now we multiply the two fractions: $\frac{2}{3} \times \frac{1}{3} = 2/9$. However, this is not the answer because we could pick an even number first then an odd number. The equation is the same except backward and by the Communitive Property of Multiplication, the equations are it does not matter is the equation is backward or not. Thus we do $\frac{2}{9} \times 2 = \frac{4}{9}$. This is the probability of getting an odd number. In order to get the probability of getting an even number we do $1 - \frac{4}{9} = \boxed{(\mathbf{C}) \frac{5}{9}}$

- ViratKohli2018 (VK18)

Solution 2

We have a 2 die with 2 evens and 4 odds on both dies. For the sum to be even, the rolls must consist of 2 odds or 2 evens.

Ways to roll 2 odds (Case 1): The total number of ways to roll 2 odds is $4 * 4 = 16$, as there are 4 choices for the first odd on the first roll and 4 choices for the second odd on the second roll.

Ways to roll 2 evens (Case 2): Similarly, we have $2 * 2 = 4$ ways to roll $36 = \frac{20}{36} = \frac{5}{9}$, or $\boxed{\text{C}}$.

Solution 3 (Complementary Counting)

We count the ways to get an odd. If the sum is odd, then we must have an even and an odd. The probability of an even is $\frac{1}{3}$, and the probability of an odd is $\frac{2}{3}$. We have to multiply by $2!$ because the even and odd can be in any order. This gets us $\frac{4}{9}$, so the answer is $1 - \frac{4}{9} = \frac{5}{9} = \boxed{(\mathbf{C})}$.

Solution 4

To get an even, you must get either 2 odds or 2 evens. The probability of getting 2 odds is $\frac{4}{6} * \frac{4}{6}$. The probability of getting 2 evens is $\frac{2}{6} * \frac{2}{6}$. If you add them together, you get $\frac{16}{36} + \frac{4}{36}$ = $\boxed{(\mathbf{C}) \frac{5}{9}}$.

Solution 5 (Casework)

To get an even number, we must either have two odds or two evens. We will solve this through casework. The probability of rolling a 1 is 1/6, and the probability of rolling another odd number after this is 4/6=2/3, so the probability of getting a sum of an even number is (1/6)(2/3)=1/9. The probability of rolling a 2 is 1/6, and the probability of rolling another even number after this is 2/6=1/3, so the probability of rolling a sum of an even number is

(1/6)(1/3)=1/18. Now, notice that the probability of getting an even sum with two odd numbers is identical for all odd numbers. This is because the probability of probability of getting an even number is identical for all even numbers, so the probability of getting an even sum with only even numbers is (2)(1/18)=1/9. Adding these two up, we get our desired $\boxed{(\textbf{C})\frac{5}{9}}$.

Video Solution

Associated video - https://www.youtube.com/watch?v=EoBZy_WYWEw

https://www.youtube.com/watch?v=H52AqAl4nt4&t=2s

Problem 19

In a tournament there are six teams that play each other twice. A team earns 3 points for a win, 1 point for a draw, and 0 points for a loss. After all the games have been played it turns out that the top three teams earned the same number of total points. What is the greatest possible number of total points for each of the top three teams?

$\textbf{(A) }22\qquad\textbf{(B) }23\qquad\textbf{(C) }24\qquad\textbf{(D) }26\qquad\textbf{(E) }30$

Solution 1

After fully understanding the problem, we immediately know that the three top teams, say team A, team B, and team C, must beat the other three teams D, E, F. Therefore, A,B, C must each obtain $(3+3+3)=9$ points. However, they play against each team twice, for a total of 18 points against D, E, and F. For games between A, B, C, we have 2 cases. In both cases, there is an equality of points between A, B, and C.

Case 1: A team ties the two other teams. For a tie, we have 1 point, so we have $(1+1)*2=4$ points (they play twice). Therefore, this case brings a total of $4+18=22$ points.

Case 2: A team beats one team while losing to another. This gives equality, as each team wins once and loses once as well. For a win, we have 3 points, so a team gets $3\times 2=6$ points if they each win a game and lose a game. This case brings a total of $18+6=24$ points.

Therefore, we use Case2 since it brings the greater amount of points, or $\boxed{24}$, so the answer is $\boxed{C}$.

Note that case 2 can be easily seen to be better as follows. Let x_A be the number of points A gets, x_B be the number of points B gets, and x_C be the number of points C gets. Since $x_A=x_B=x_C$, to maximize x_A, we can just maximize $x_A+x_B+x_C$. But in each match, if one team wins then the total sum increases by 3 points, whereas if they tie, the total sum increases by 2 points. So it is best if there are the fewest ties possible.

Solution 2

(1st match(3) + 2nd match(1)) * number of teams(6) = 24, $\boxed{C}$.

Explanation: So after reading the problem we see that there are 6 teams and each team versus each other twice. This means one of the two matches has to be a win, so 3 points so far. Now if we say that the team won again and make it 6 points, that would mean that team would be dominating the leader-board and the problem says that all the top 3 people have the same score. So that means the maximum amount of points we could get is 1 so that each team gets the same amount of matches won & drawn so that adds up to 4. 4 * the number of teams(6) = 24 so the answer is $\boxed{C}$.

Solution 3

We can name the top three teams as $A, B,$ and C. We can see that $A = B = C$, because these teams have the same points. If we look at the matches that involve the top three teams, we see that there are some duplicates: $AB, BC,$ and AC come twice. In order to even out the scores and get the maximum score, we can say that in match AB, A and B each win once out of the two games that they play. We can say the same thing for AC and BC. This tells us that each team $A, B,$ and C win and lose twice. This gives each team a total of 3 + 3 + 0 + 0 = 6 points. Now, we need to include the other three teams. We can label these teams as $D, E,$ and F. We can write down every match that $A, B,$ or C plays in that we haven't counted yet: $AD, AD, AE, AE, AF, AF, BD, BD, BE, BE, BF, BF, CD, CD, CE, CE, CF,$ and CF. We can say $A, B,$ and C win each of these in order to obtain the maximum score that $A, B,$ and C can have. If $A, B,$ and C win all six of their matches, $A, B,$ and C will have a score of 18. $18 + 6$ results in a maximum score of $\boxed{24}$. This tells us that the correct answer choice is $\boxed{C}$.

Video Solutions

Associated Video - https://youtu.be/s0O3_uXZrOI

https://youtu.be/hM4sHJSMNDs

Problem 20

How many different real numbers x satisfy the equation $(x^2 - 5)^2 = 16?$

$\textbf{(A) } 0 \qquad \textbf{(B) } 1 \qquad \textbf{(C) } 2 \qquad \textbf{(D) } 4 \qquad \textbf{(E) } 8$

Solution 1

We have that $(x^2-5)^2 = 16$ if and only if $x^2-5 = \pm 4$. If $x^2-5 = 4$, then $x^2 = 9 \implies x = \pm 3$, giving 2 solutions. If $x^2-5 = -4$, then $x^2 = 1 \implies x = \pm 1$, giving 2 more solutions. All four of these solutions work, so the answer is $\boxed{\textbf{(D)} 4}$. Further, the equation is a quartic in x, so by the Fundamental Theorem of Algebra, there can be at most four real solutions.

Solution 2

We can expand $(x^2-5)^2$ to get x^4-10x^2+25, so now our equation is $x^4-10x^2+25=16$. Subtracting 16 from both sides gives us $x^4-10x^2+9=0$. Now, we can factor the left hand side to get $(x^2-9)(x^2-1)=0$. If x^2-9 and/or x^2-1 equals 0, then the whole left side will equal 0. Since the solutions can be both positive and negative, we have 4 solutions: $-3, 3, -1, 1$ (we can find these solutions by setting x^2-9 and x^2-1 equal to 0 and solving for x). So the answer is $\boxed{\textbf{(D)} 4}$.

Solution 3

Associated Video - https://www.youtube.com/watch?v=Q5yfodutpsw

https://youtu.be/0AY1klX3gBo

Solution 4

https://youtu.be/5BXh0JY4klM (Uses a difference of squares & factoring method, different from above solutions)

Solution 5 (Fast)

https://www.youtube.com/watch?v=44vrsk_CbF8&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=2

Video Solution

https://youtu.be/V3HxkJhSn08

Problem 21

What is the area of the triangle formed by the lines $y=5$, $y=1+x$, and $y=1-x$?

$\textbf{(A) } 4 \qquad \textbf{(B) } 8 \qquad \textbf{(C) } 10 \qquad \textbf{(D) } 12 \qquad \textbf{(E) } 16$

Solution 1

First we need to find the coordinates where the graphs intersect.

We want the points x and y to be the same. Thus, we set $5 = x + 1$, and get $x = 4$. Plugging this into the equation, $y = 1 - x$, $y = 5$, and $y = 1 + x$ intersect at $(4, 5)$, we call this line x.

Doing the same thing, we get $x = -4$. Thus $y = 5$ also. $y = 5$, and $y = 1 - x$ intersect at $(-4, 5)$, we call this line y.

It's apparent the only solution to $1 - x = 1 + x$ is 0. Thus, $y = 1$. $y = 1 - x$ and $y = 1 + x$ intersect at $(0, 1)$, we call this line z.

Using the Shoelace Theorem we get: $\left(\frac{(20-4)-(-20+4)}{2}\right) = \frac{32}{2} =$ So our answer is $\boxed{\textbf{(E) } 16.}$

We might also see that the lines y and z are mirror images of each other. This is because, when rewritten, their slopes can be multiplied by -1 to get the other. As the base is horizontal, this is a isosceles triangle with base 8, as the intersection points have distance 8. The height is $5 - 1 = 4$, so $\frac{4 \cdot 8}{2} = \boxed{\textbf{(E)} 16.}$

Warning: Do not use the distance formula for the base then use heron's formula. It will take you half of the time you have left!

Solution 2

Graphing the lines, using the intersection points we found in Solution 1, we can see that the height of the triangle is 4, and the base is 8. Using the formula for the area of a triangle, we get $\frac{4 \cdot 8}{2}$ which is equal to $\boxed{\textbf{(E) } 16}$.

Solution 3

$y = x + 1$ and $y = -x + 1$ have y-intercepts at $(1, 0)$ and slopes of 1 and -1, respectively. Since the product of these slopes is -1, the two lines are perpendicular. From $y = 5$, we see that $(-4, 5)$ and $(4, 5)$ are the other two intersection points, and they are 8 units apart. By symmetry, this triangle is a $45 - 45 - 90$ triangle, so the legs are $4\sqrt{2}$ each and the area is $\frac{(4\sqrt{2})^2}{2} = \boxed{\textbf{(E) } 16}$.

Video Solutions

Associated Video - https://www.youtube.com/watch?v=ie3tlSNyiaY https://www.youtube.com/watch?v=9nlX9VCisQc

https://www.youtube.com/watch?v=mz3DY1rc5ao

https://www.youtube.com/watch?v=Z27G0xy5AgA&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=3

https://youtu.be/RvtOX17DemY

Problem 22

A store increased the original price of a shirt by a certain percent and then lowered the new price by the same amount. Given that the resulting price was 84% of the original price, by what percent was the price increased and decreased?

$\textbf{(A) }16\qquad\textbf{(B) }20\qquad\textbf{(C) }28\qquad\textbf{(D) }36\qquad\textbf{(E) }40$

Solution 1

Suppose the fraction of discount is x. That means $(1-x)(1+x)=0.84$; so $1-x^2=0.84$, and $(x^2)=0.16$, obtaining $x=0.4$. Therefore, the price was increased and decreased by 40%, or $\boxed{\textbf{(E) }40}$.

Solution 2 (Answer options)

We can try out every option and see which one works out. By this method, we get $\boxed{\textbf{(E) }40}$.

Solution 3

Let x be the discount. We can also work in reverse such as (84)$(\frac{100}{100-x})$ $(\frac{100}{100+x})$ = 100.

Thus 8400 = $(100+x)(100-x)$. Solving for x gives us $x=40,-40$. But x has to be positive. Thus x = 40.

Solution 4 ~ using the answer choices

Let our original cost be $\$100$. We are looking for a result of $\$84,$ then. We try 16% and see it gets us higher than 84. We try 20% and see it gets us lower than 16 but still higher than 84. We know that the higher the percent, the less the value. We try 36, as we are not progressing much, and we are close! We try $\boxed{40\%}$, and we have the answer; it worked.

Video explaining solution

Associated video - https://www.youtube.com/watch?v=aJX27Cxvwlc

https://youtu.be/gX_l0PGsQao

https://www.youtube.com/watch?v=_TheVi-6LWE

https://www.youtube.com/watch?v=RcBDdB35Whk&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=4

https://youtu.be/h2GlK7itc2g

Problem 23

After Euclid High School's last basketball game, it was determined that $\frac{1}{4}$ of the team's points were scored by Alexa and $\frac{2}{7}$ were scored by Brittany. Chelsea scored 15 points. None of the other 7 team members scored more than 2 points. What was the total number of points scored by the other 7 team members?

$\textbf{(A) } 10 \qquad \textbf{(B) } 11 \qquad \textbf{(C) } 12 \qquad \textbf{(D) } 13 \qquad \textbf{(E) } 14$

Solution 1

Since $\frac{\text{total points}}{4}$ and $\frac{2(\text{total points})}{7}$ are integers, we have $28|\text{total points}$. We see that the number of points scored by the other team members is less than or equal to 14 and greater than or equal to 0. We let the total number of points be t and the total number of points scored by the other team members be x, which means that $\frac{t}{4}+\frac{2t}{7}+15+x=t \quad\implies\quad 0\leq\frac{13t}{28}-15=x\leq 14$, which means $15\leq\frac{13t}{28}\leq 29$. The only value of t that satisfies all conditions listed is 56, so $x=\boxed{\textbf{(B)} 11}$.

Solution 2

Starting from the above equation $\frac{t}{4}+\frac{2t}{7}+15+x=t$ where t is the total number of points scored and $x\leq 14$ is the number of points scored by the remaining 7 team members, we can simplify to obtain the Diophantine equation $x+15=\frac{13}{28}t$, or $28x+28\cdot 15=13t$. Since t is necessarily divisible by 28, let $t = 28u$ where $u \geq 0$ and divide by 28 to obtain $x + 15 = 13u$. Then it is easy to see $u = 2$ ($t = 56$) is the only candidate, giving $x = \boxed{\textbf{(B)} 11}$. -scrabbler94

Solution 3

We first start by setting the total number of points as 28, since $\text{LCM}(4,7) = 28$. However, we see that this does not work since we surpass the number of points just with the information given ($28 \cdot \frac{1}{4} + 28 \cdot \frac{2}{7} + 15 = 30$ (> 28)). Next, we can see that the total number of points scored is 56 as, if it is more than or equal to 84, at least one of the others will score more than 2 points. With this, we have that Alexa, Brittany, and Chelsea score: $56 \cdot \frac{1}{4} + 56 \cdot \frac{2}{7} + 15 = 45$, and thus, the other seven players would have scored a total of $56 - 45 = \boxed{\textbf{(B)} 11}$ (We see that this works since we could have 4 of them score 2 points, and the other 3 of them score 1 point)

Solution 4 — Modular Arithmetic

Adding together Alexa's and Brittany's fractions, we get $\frac{15}{28}$ as the fraction of the total number of points they scored together. However, this is just a ratio, so we can introduce a variable: $\frac{15x}{28x}$ where x is the common ratio. Let y and z and w be the number of people who scored 1, 2, and 0 points, respectively. Writing an equation, we have $\frac{13x}{28x} = 15 + y + 2z + 0w.$ We want all of our variables to be integers. Thus, we want $15 + y + 2z = 0 \pmod{13}.$ Simplifying, $y + 2z = 11 \pmod{13}.$ The only possible value, as this integer sum has to be less than $7 \cdot 2 + 1 = 15,$ must be 11. Therefore $y + 2z = 11,$ and the answer is $\boxed{\textbf{(B) } 11}$ - ab2024

Video explaining solution

Associated video - https://www.youtube.com/watch?v=jE-7Se7ay1c

https://www.youtube.com/watch?v=3Mae_6qFxoU&t=204s

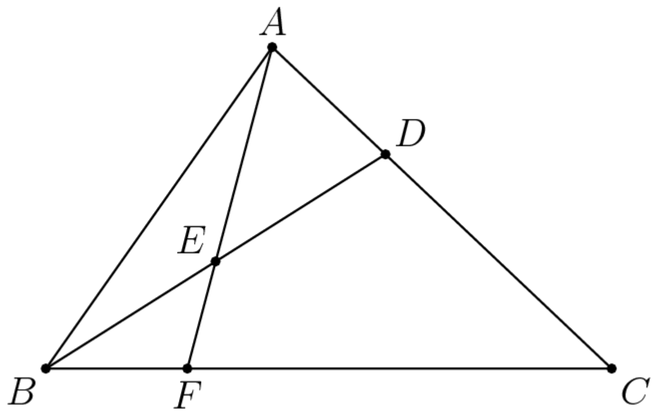
https://youtu.be/wsYCn2FqZJE

https://www.youtube.com/watch?v=fKjmw_zzCUU

https://www.youtube.com/watch?v=o2mcnLOVFBA&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=5

Problem 24

In triangle ABC, point D divides side $\overline{AC}$ so that $AD : DC = 1 : 2$. Let E be the midpoint of $\overline{BD}$ and let F be the point of intersection of line BC and line AE. Given that the area of $\triangle ABC$ is 360, what is the area of $\triangle EBF$?

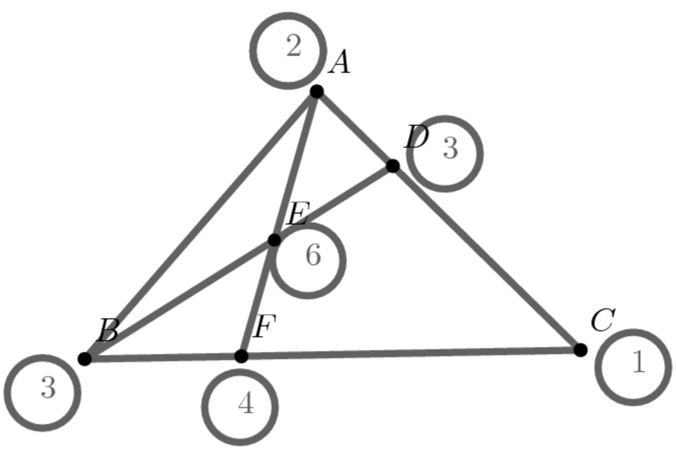


$\textbf{(A) } 24 \qquad \textbf{(B) } 30 \qquad \textbf{(C) } 32 \qquad \textbf{(D) } 36 \qquad \textbf{(E) } 40$

Solution 1

Draw X on $\overline{AF}$ such that XD is parallel to BC. That makes triangles BEF and EXD congruent since $BE = ED$. $FC = 3XD$ so $BC = 4BF$. Since $AF = 3EF$ ($XE = EF$ and $AX = \frac{1}{3}AF$, so $XE = EF = \frac{1}{3}AF$), the altitude of triangle BEF is equal to $\frac{1}{3}$ of the altitude of ABC. The area of ABC is 360, so the area of $BEF = \frac{1}{3} \cdot \frac{1}{4} \cdot 360 = \boxed{\textbf{(B) } 30}$

Solution 2 (Mass Points)



First, when we see the problem, we see ratios, and we see that this triangle basically has no special properties (right, has medians, etc.) and this screams mass points at us.

First, we assign a mass of 2 to point A. We figure out that C has a mass of 1 since $2 \times 1 = 1 \times 2$. Then, by adding $1 + 2 = 3$, we get that point D has a mass of 3. By equality, point B has a mass of 3 also.

Now, we add $3 + 3 = 6$ for point E and $3 + 1 = 4$ for point F.

Now, BF is a common base for triangles ABF and EBF, so we figure out that the ratios of the areas is the ratios of the heights which is $\frac{AE}{EF} = 2:1$. So, EBF's area is one third the area of ABF, and we know the area of ABF is $\frac{1}{4}$ the area of ABC since they have the same heights but different bases.

So we get the area of EBF as $\frac{1}{3} \times \frac{1}{4} \times 360 = \boxed{\textbf{(B) } 30}$

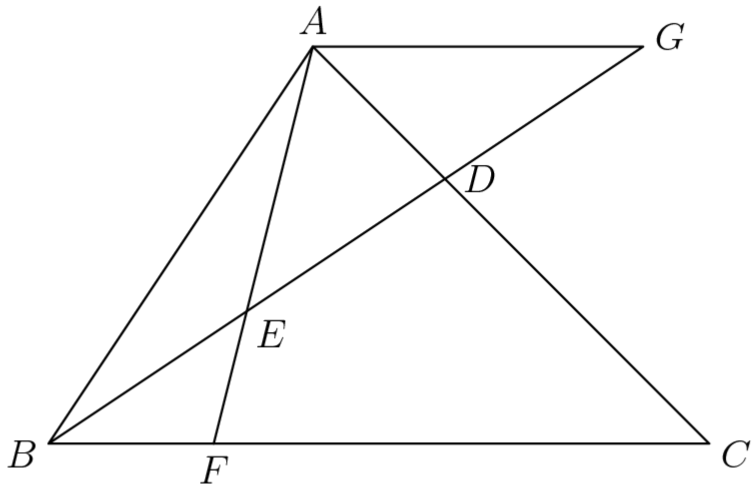
Note: We can also find the ratios of the areas using the reciprocal of the product of the mass points of EBF over the product of the mass points of ABC which is $\frac{2 \times 3 \times 1}{3 \times 6 \times 4} \times 360$ which also yields $\boxed{\textbf{(B) } 30}$

Solution 3

$\frac{BF}{FC}$ is equal to $\frac{\text{The area of triangle ABE}}{\text{The area of triangle ACE}}$. The area of triangle ABE is equal to 60 because it is equal to on half of the area of triangle ABD, which is equal to one third of the area of triangle ABC, which is 360. The area of triangle ACE is the sum of the areas of triangles AED and CED, which is respectively 60 and 120. So, $\frac{BF}{FC}$ is equal to $\frac{60}{180}$=$\frac{1}{3}$, so the area of triangle ABF is 90. That minus the area of triangle ABE is $\boxed{\textbf{(B) } 30}$.

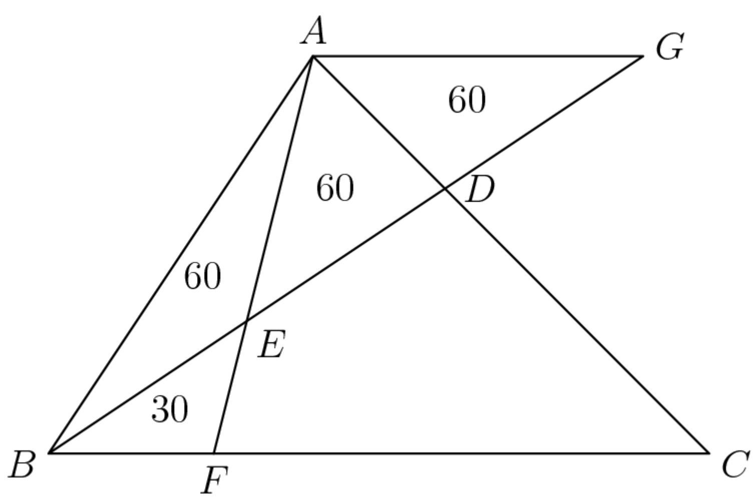
Solution 4 (Similar Triangles)

Extend $\overline{BD}$ to G such that $\overline{AG} \parallel \overline{BC}$ as shown:

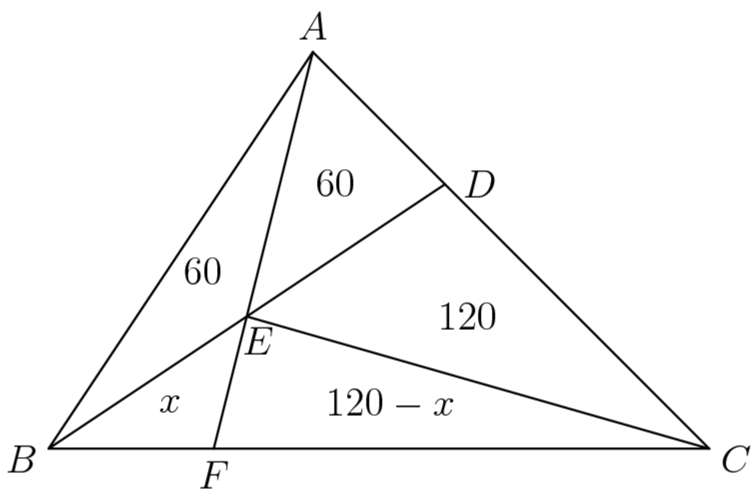


Then $\triangle ADG \sim \triangle CDB$ and $\triangle AEG \sim \triangle FEB$. Since $CD = 2AD$, triangle CDB has four times the area of triangle ADG. Since $[CDB] = 240$, we get $[ADG] = 60$.

Since $[AED]$ is also 60, we have $ED = DG$ because triangles AED and ADG have the same height and same areas and so their bases must be the congruent. Thus triangle AEG has twice the side lengths and therefore four times the area of triangle BEF, giving $[BEF] = (60+60)/4 = \boxed{\textbf{(B) } 30}$.



Solution 5 (Area Ratios)



As before we figure out the areas labeled in the diagram. Then we note that $\frac{EF}{AE} = \frac{x}{60} = \frac{120-x}{180}$. Solving gives $x = \boxed{\textbf{(B) } 30}$.

Solution 6 (Coordinate Bashing)

Let ADB be a right triangle, and $BD = CD$

Let $A = (-2\sqrt{30}, 0)$

$B = (0, 4\sqrt{30})$

$C = (4\sqrt{30}, 0)$

$D = (0, 0)$

$E = (0, 2\sqrt{30})$

$F = (\sqrt{30}, 3\sqrt{30})$

The line $\overleftrightarrow{AE}$ can be described with the equation $y = x - 2\sqrt{30}$

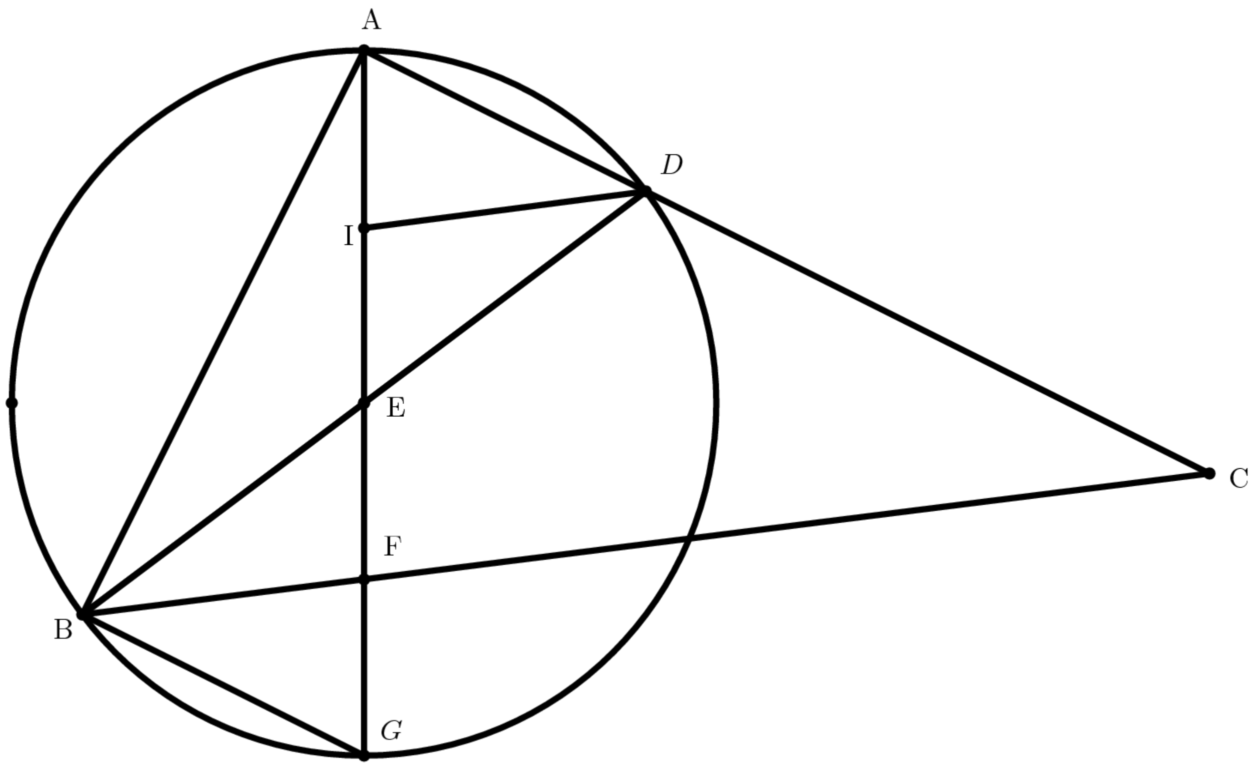
The line $\overleftrightarrow{BC}$ can be described with $x + y = 4\sqrt{30}$

Solving, we get $x = 3\sqrt{30}$ and $y = \sqrt{30}$

Now we can find $EF = BF = 2\sqrt{15}$

$[\triangle EBF] = \frac{(2\sqrt{15})^2}{2} = \boxed{\textbf{(B) } 30} \blacksquare$

Solution 7



Let $A[\Delta XYZ]$ = $\text{Area of Triangle XYZ}$

$A[\Delta ABD] : A[\Delta DBC] :: 1 : 2 :: 120 : 240$

$A[\Delta ABE] = A[\Delta AED] = 60$ (the median divides the area of the triangle into two equal parts)

Construction: Draw a circumcircle around ΔABD with BD as is diameter.
Extend AF to G such that it meets the circle at G. Draw line BG.

$A[\Delta ABD] = A[\Delta ABG] = 120$ (Since $\square ABGD$ is cyclic)

But $A[\Delta ABE]$ is common in both with an area of 60. So, $A[\Delta AED] = A[\Delta BEG]$.
\therefore $A[\Delta AED] \cong A[\Delta BEG]$ (SAS Congruency Theorem).
In ΔAED, let DI be the median of ΔAED.
Which means $A[\Delta AID] = 30 = A[\Delta EID]$

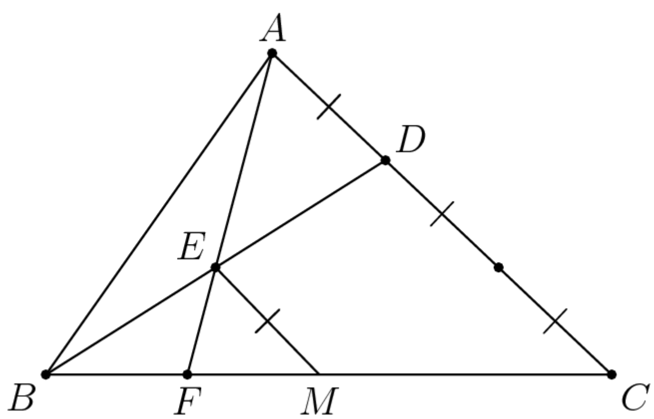
Rotate ΔDEA to meet D at B and A at G. DE will fit exactly in BE (both are radii of the circle). From the above solutions, $\dfrac{AE}{EF} = 2:1$.

AE is a radius and EF is half of it implies EF = $\frac{radius}{2}$.

Which means $A[\Delta BEF] \cong A[\Delta DEI]$

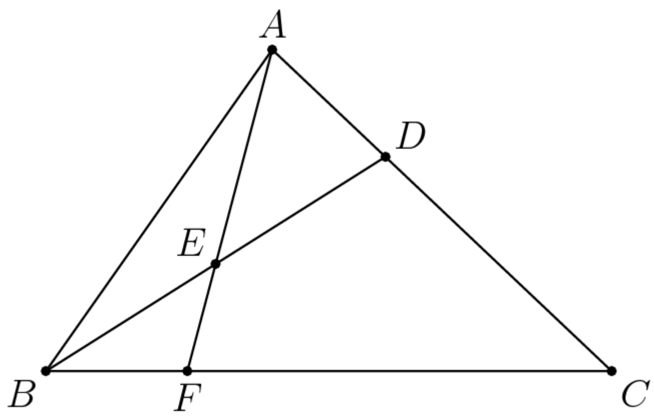
Thus $A[\Delta BEF] = \boxed{\textbf{(B) } 30}$

Solution 8



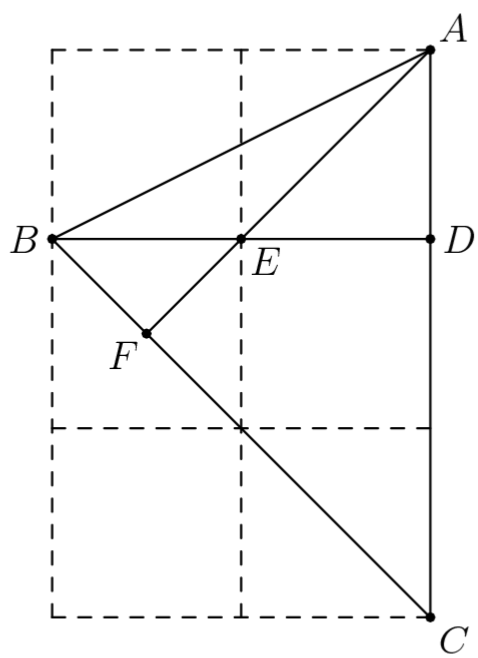
Using the ratio of $\overline{AD}$ and $\overline{CD}$, we find the area of $\triangle ADB$ is 120 and the area of $\triangle BDC$ is 240. Also using the fact that E is the midpoint of $\overline{BD}$, we know $\triangle ADE = \triangle ABE = 60$. Let M be a point such $\overline{EM}$ is parellel to $\overline{CD}$. We immediatley know that $\triangle BEM \sim BDC$ by 2. Using that we can conclude EM has ratio 1. Using $\triangle EFM \sim \triangle AFC$, we get $EF : AE = 1 : 2$. Therefore using the fact that $\triangle EBF$ is in $\triangle ABF$, the area has ratio $\triangle BEF : \triangle ABE = 1 : 2$ and we know $\triangle ABE$ has area 60 so $\triangle BEF$ is $\boxed{\textbf{(B) } 30}$.

Solution 9 (Menelaus's Theorem)



By Menelaus's Theorem on triangle BCD, we have $\frac{BF}{FC}\cdot\frac{CA}{DA}\cdot\frac{DE}{BE}=3\frac{BF}{FC}=1\implies\frac{BF}{FC}=\frac{1}{3}\implies\frac{BF}{BC}=\frac{1}{4}$. Therefore, $[EBF]=\frac{BE}{BD}\cdot\frac{BF}{BC}\cdot[BCD]=\frac{1}{2}\cdot\frac{1}{4}\cdot\left(\frac{2}{3}\cdot[ABC]\right)=\boxed{\textbf{(B) } 30}$.

Solution 10 (Graph Paper)



Note: If graph paper is unavailable, this solution can still be used by constructing a small grid on a sheet of blank paper.

As triangle ABC is loosely defined, we can arrange its points such that the diagram fits nicely on a coordinate plane. By doing so, we can construct it on graph paper and be able to visually determine the relative sizes of the triangles.

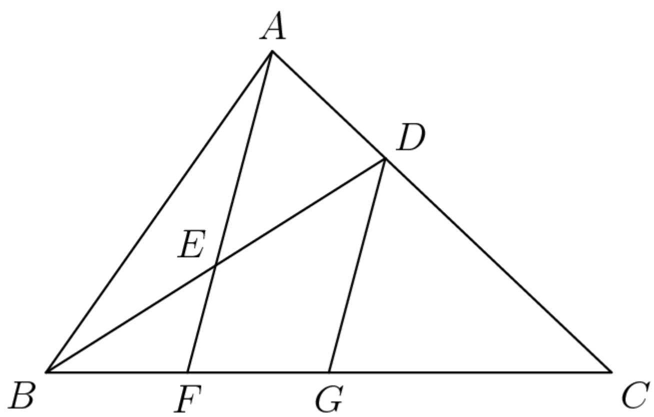
As point D splits line segment $\overline{AC}$ in a $1:2$ ratio, we draw $\overline{AC}$ as a vertical line segment 3 units long. Point D is thus 1 unit below point A and 2 units above point C. By definition, Point E splits line segment $\overline{BD}$ in a $1:1$ ratio, so we draw $\overline{BD}$ 2 units long directly left of D and draw E directly between B and D, 1 unit away from both.

We then draw line segments $\overline{AB}$ and $\overline{BC}$. We can easily tell that triangle ABC occupies 3 square units of space. Constructing line AE and drawing F at the intersection of AE and BC, we can easily see that triangle EBF forms a right triangle occupying $\frac{1}{4}$ of a square unit of space.

The ratio of the areas of triangle EBF and triangle ABC is thus $\frac{1}{4} \div 3 = \frac{1}{12}$, and since the area of triangle ABC is 360, this means that the area of triangle EBF is $\frac{1}{12} \times 360 = \boxed{\textbf{(B) } 30}$.

Additional note: There are many subtle variations of this triangle; this method is one of the more compact ones.

Solution 11



We know that $AD = \frac{1}{3}AC$, so $[ABD] = \frac{1}{3}[ABC] = 120$. Using the same method, since $BE = \frac{1}{2}BD$, $[ABE] = \frac{1}{2}[ABD] = 60$. Next, we draw G on $\overline{BC}$ such that $\overline{DG}$ is parallel to $\overline{AF}$ and create segment DG. We then observe that $\triangle AFC \sim \triangle DGC$, and since $AD : DC = 1 : 2$, $FG : GC$ is also equal to $1 : 2$. Similarly (no pun intended), $\triangle DBG \sim \triangle EBF$, and since $BE : ED = 1 : 1$, $BF : FG$ is also equal to $1 : 1$. Combining the information in these two ratios, we find that $BF : FG : GC = 1 : 1 : 2$, or equivalently, $BF = \frac{1}{4}BC$.

Thus, $[BFA] = \frac{1}{4}[BCA] = 90$. We already know that $[ABE] = 60$, so the area of $\triangle EBF$ is $[BFA] - [ABE] = \boxed{\textbf{(B) } 30}$.

Solution 12 (Fastest Solution if you have no time)

The picture is misleading. Assume that the triangle ABC is right.

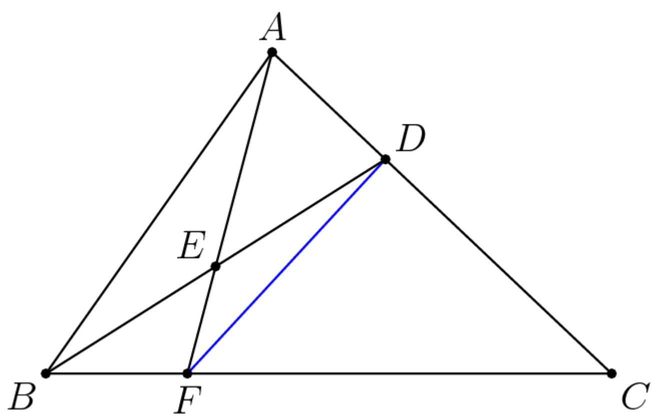
Then find two factors of 720 that are the closest together so that the picture becomes easier in your mind. Quickly searching for squares near 720 to use difference of squares, we find 24 and 30 as our numbers. Then the coordinates of D are $(10, 16)$(note, A=0,0). E is then $(5, 8)$. Then the equation of the line AE is $-16x/5 + 24 = y$. Plugging in $y = 0$, we have $x = \frac{15}{2}$. Now notice that we have both the height and the base of EBF.

Solving for the area, we have $(8)(15/2)(1/2) = 30$.

Solution 13

$AD : DC = 1 : 2$, so ADB has area 120 and CDB has area 240. $BE = ED$ so the area of ABE is equal to the area of $ADE = 60$. Draw $\overline{DG}$ parallel to $\overline{AF}$.
Set area of BEF = x. BEF is similar to BDG in ratio of 1:2
so area of BDG = $4x$, area of EFDG=$3x$, and area of CDG$= 240 - 4x$.
CDG is similar to CAF in ratio of 2:3 so area CDG = $4/9$ area CAF, and area AFDG=$5/4$ area CDG.
Thus $60 + 3x = 5/4(240 - 4x)$ and $x = 30$.

Solution 14 - Geometry & Algebra



We draw line FD so that we can define a variable x for the area of $\triangle BEF = \triangle DEF$. Knowing that $\triangle ABE$ and $\triangle ADE$ share both their height and base, we get that $ABE = ADE = 60$.

Since we have a rule where 2 triangles, ($\triangle A$ which has base a and vertex c), and ($\triangle B$ which has Base b and vertex c)who share the same vertex (which is vertex c in this case), and share a common height, their relationship is : Area of $A : B = a : b$ (the length of the two bases), we can list the equation where $\frac{\triangle ABF}{\triangle ACF} = \frac{\triangle DBF}{\triangle DCF}$. Substituting x into the equation we get:

$\dfrac{x+60}{300-x}=\dfrac{2x}{240-2x}$.$(2x)(300-x)=(60+x)(240-x)$.
$600-2x^2=14400-120x+240x-2x^2$.$480x=14400$.and we now have that $\triangle BEF=30$.

Video Solutions

Associated video - https://www.youtube.com/watch?v=DMNbExrK2oo

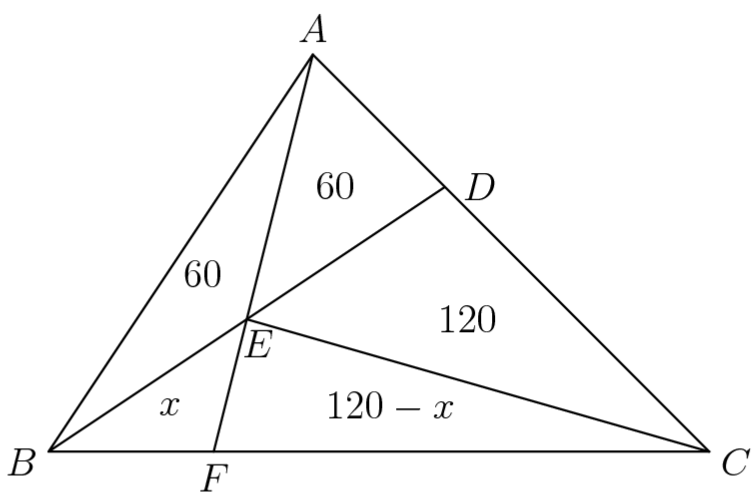
https://youtu.be/Ns34Jiq9ofc

https://www.youtube.com/watch?v=nm-Vj_fsXt4

https://www.youtube.com/watch?v=nyevg9w-CCI&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=6

https://www.youtube.com/watch?v=m04K0Q2SNXY&t=1s

Solution 16 (Straightforward Solution)



Since $AD:DC=1:2$ thus $\triangle ABD=\dfrac{1}{3}\cdot 360=120.$ Similarly, $\triangle DBC=\dfrac{2}{3}\cdot 360=240.$ Now, since E is a midpoint of BD, $\triangle ABE=\triangle AED=120\div 2=60$. We can use the fact that E is a midpoint of BD even further. Connect lines E and C so that $\triangle BEC$ and $\triangle DEC$ share 2 sides. We know that $\triangle BEC=\triangle DEC=240\div 2=120$ since E is a midpoint of BD. Let's label $\triangle BEF$ x. We know that $\triangle EFC$ is $120-x$ since $\triangle BEC=120$. Note that with this information now, we can deduct more things that are needed to finish the solution.

Note that $\dfrac{EF}{AE}=\dfrac{120-x}{180}=\dfrac{x}{60}$. because of triangles $EBF, ABE, AEC,$ and EFC. We want to find x. This is a simple equation, and solving we get $x=\boxed{\textbf{(B)}30}$.

By mathboy282, an expanded solution of Solution 5, credit to scrabbler94 for the idea.

Problem 25

Alice has 24 apples. In how many ways can she share them with Becky and Chris so that each of the three people has at least two apples? $\textbf{(A)}\ 105\qquad\textbf{(B)}\ 114\qquad\textbf{(C)}\ 190\qquad\textbf{(D)}\ 210\qquad\textbf{(E)}\ 380$

Solution 1

We use stars and bars. Let Alice get k apples, let Becky get r apples, let Chris get y apples. $\implies k + r + y = 24$We can manipulate this into an equation which can be solved using stars and bars.

All of them get at least 2 apples, so we can subtract 2 from k, 2 from r, and 2 from y. $\implies (k-2) + (r-2) + (y-2) = 18$Let $k' = k - 2$, let $r' = r - 2$, let $y' = y - 2$. $\implies k' + r' + y' = 18$We can allow either of them to equal to 0, hence this can be solved by stars and bars.

By Stars and Bars, our answer is just $$\binom{18+3-1}{3-1} = \binom{20}{2} = \boxed{\textbf{(C)}\ 190}.$$

Solution 2

First assume that Alice has 2 apples. There are 19 ways to split the rest of the apples with Becky and Chris. If Alice has 3 apples, there are 18 ways to split the rest of the apples with Becky and Chris. If Alice has 4 apples, there are 17 ways to split the rest. So the total number of ways to split 24 apples between the three friends is equal to $19 + 18 + 17 ....\ldots\ldots + 1 = 20 \times \frac{19}{2} = \boxed{\textbf{(C)}\ 190}$

Solution 3

Let's assume that the three of them have x, y, z apples. Since each of them has to have at least 2 apples, we say that $a+2=x, b+2=y$ and $c+2=z$.
Thus, $a+b+c+6=24 \implies a+b+c=18$, and so by stars and bars, the number of solutions for this is $\binom{n+k-1}{k} \implies \binom{18+3-1}{3-1} \implies \binom{20}{2} = \boxed{\textbf{(C)}\ 190}$ -
aops5234

Solution 4

Since we have to give each of the 3 friends at least 2 apples, we need to spend a total of $2+2+2=6$ apples to solve the restriction. Now we have $24-6=18$ apples left to be divided among Alice, Becky, and Chris, without any constraints. We use the Ball-and-urn technique, or sometimes known as ([Sticks and Stones]/[Stars and Bars]), to divide the apples. We now have 18 stones and 2 sticks, which have a total of $\binom{18+2}{2} = \binom{20}{2} = \frac{20 \times 19}{2} = \boxed{190}$ ways to arrange.

Solution 5

Equivalently, we split 21 apples among 3 friends with each having at least 1 apples. We put sticks between apples to split apples into three stacks. So there are 20 spaces to put 2 sticks. We have $\binom{20}{2} = 190$ different ways to arrange the two sticks. So, there are $\boxed{190}$ ways to split the apples among them.

Video Solutions

https://www.youtube.com/channel/UCFoUjg2IndpAryBtHbfOETg?view_as=subscriber

https://www.youtube.com/watch?v=wJ7uvypbB28

https://www.youtube.com/watch?v=2dBUklyUaNI

https://www.youtube.com/watch?v=EJzSOPXULBc

https://youtu.be/ZsCRGK4VgBE

https://www.youtube.com/watch?v=3qp0wTq-LI0&list=PLLCzevlMcsWNBsdpItBT4r7Pa8cZb6Viu&index=7