AMERICAN MATHEMATICS COMPETITIONS

15th ANNUAL

AMC → 8

(formerly AJHSME)

Solutions Pamphlet

TUESDAY, NOVEMBER 16, 1999

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This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. *However, the publication, reproduction, or communication of the problems or solutions of the AMC → 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication at any time via copier, telephone, eMail, World Wide Web or media of any type is a violation of the copyright law.

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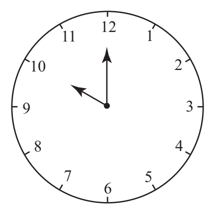
1. **Answer (A):**

$$
\begin{aligned}
&(6\ ?\ 3) + 4 - (2 - 1) = 5 & \\
&(6\ ?\ 3) + 4 - 1 = 5 & \text{(subtract:}2 - 1 = 1) \\
&(6\ ?\ 3) + 3 = 5 & \text{(subtract:}4 - 1 = 3) \\
&(6\ ?\ 3) = 2 & \text{(subtract 3 from both sides)} \\
&(6 \div 3) = 2 &
\end{aligned}
$$

The other operations produce the following result:

$$
\begin{aligned}
(6 + 3) + 4 - (2 - 1) &= 9 + 4 - 1 = 12 \\
(6 - 3) + 4 - (2 - 1) &= 3 + 4 - 1 = 6 \\
(6 \times 3) + 4 - (2 - 1) &= 18 + 4 - 1 = 21
\end{aligned}
$$

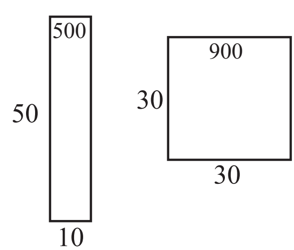
2. **Answer (C):** There are 360°(degrees) in a circle and twelve spaces on a clock. This means that each space measures 30°. At 10 o'clock the hands point to 10 and 12. They are two spaces or 60° apart.



3. **Answer (D):** $1.1 + (-2.1) + 1.0 = 0$. The other triplets add to 1 .

4. **Answer (A):** Four hours after starting, Alberto has gone about 60 miles and Bjorn has gone about 45 miles. Therefore, Alberto has biked about 15 more miles.

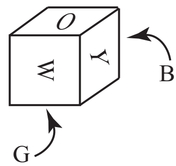
5. **Answer (D):** The area of the garden was 500 square feet(50×10) and its perimeter was 120 feet, $2 \times (50 + 10)$. The square garden is also enclosed by 120 feet of fence so its sides are each 30 feet long. The square garden's area is 900 square feet (30×30). and this has increased the garden area by 400 square feet.



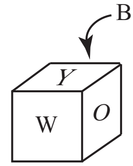
6. **Answer (E):** From the second sentence, Flo has more than someone so she can't have the least. From the third sentence both Bo and Coe have more than someone so that eliminates them. And, from the fourth sentence, Jo has more than someone, so that leaves only poor Moe!

7. **Answer (E):** There are $160 - 40 = 120$ miles between the third and tenth exits, so the service center is at milepost $40 + (3/4)120 = 40 + 90 = 130$.

8. **Answer (A):** When G is arranged to be the base, B is the back face and W is the front face. Thus, B is opposite W.

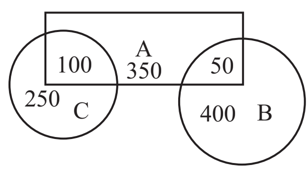


OR



Let Y be the top and fold G, O, and W down. Then B will fold to become the back face and be opposite W.

9. **Answer (C):** Bed A has 350 plants it doesn't share with B or C. Bed B has 400 plants it doesn't share with A or C. And C has 250 it doesn't share with A or B. The total is $350 + 400 + 250 + 50 + 100 = 1150$ plants.



OR

Plants shared by two beds have been counted twice, so the total is $500 + 450 + 350 - 50 - 100 = 1150$.

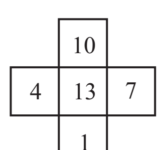
10. **Answer (E):**

$$\frac{\text{time not green}}{\text{total time}} = \frac{\mathrm{R} + \mathrm{Y}}{\mathrm{R} + \mathrm{Y} + \mathrm{G}} = \frac{35}{60} = \frac{7}{12}.$$

OR

The probability of green is $\frac{25}{60} = \frac{5}{12}$. so the probability of not green is $1 - \frac{5}{12} = \frac{7}{12}$.

11. **Answer (D):** The largest sum occurs when 13 is placed in the center. This sum is $13 + 10 + 1 = 13 + 7 + 4 = 24$. Note: Two other common sums, 18 and 21, are possible.



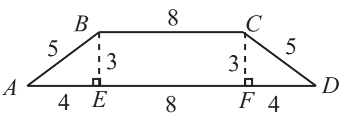
OR

Since the horizontal sum equals the vertical sum, twice this sum will be the sum of the five numbers plus the number in the center. When the center number is 13, the sum is the largest, $[10 + 4 + 1 + 7 + 2(13)]/2 = 48/2 = 24$. The other four numbers are divided into two pairs with equal sums.

12. **Answer (B):** The Won/Lost ratio is 11/4 so, for some number N, the team won $11N$ games and lost $4N$ games. Thus, the team played $15N$ games and the fraction of games lost is $\dfrac{4N}{15N} = \dfrac{4}{15} \approx 0.27 = 27\%$.

13. **Answer (C):** The sum of all ages is $40 \times 17 = 680$. The sum of the girls' ages is $20 \times 15 = 300$ and the sum of the boys' ages is $15 \times 16 = 240$. The sum of the five adults' ages is $680 - 300 - 240 = 140$. Therefore, their average is $\frac{140}{5} = 28$.

14. **Answer (D):** When the figure is divided, as shown the unknown sides are the hypotenuses of right triangles with legs of 3 and 4. Using the Pythagorean Theorem yields $AB = CD = 5$. The total perimeter is $16 + 5 + 8 + 5 = 34$.



15. **Answer (D):** Before new letters were added, five different letters could have been chosen for the first position, three for the second, and four for the third. This means that $5 \cdot 3 \cdot 4 = 60$ plates could have been made.
If two letters are added to the second set, then $5 \cdot 5 \cdot 4 = 100$ plates can be made. If one letter is added to each of the second and third sets, then $5 \cdot 4 \cdot 5 = 100$ plates can be made. None of the other four ways to place the two letters will create as many plates. So, $100 - 60 = 40$ ADDITIONAL plates can be made.

Note: Optimum results can usually be obtained in such problems by making the factors as nearly equal as possible.

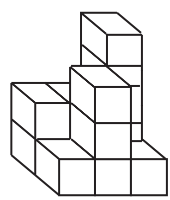
16. **Answer (B):** Since $70\%(10) + 40\%(30) + 60\%(35) = 7 + 12 + 21 = 40$, she answered 40 questions correctly. She needed $60\%(75) = 45$ to pass, so she needed 5 more correct answers.

17. **Answer (C):** One recipe makes 15 cookies, so $216 \div 15 = 14.4$ recipes are needed, but this must be rounded up to 15 recipes to make enough cookies. Each recipe requires 2 eggs. So 30 eggs are needed. This is 5 half-dozens.

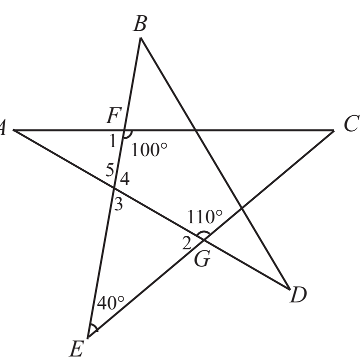
18. **Answer (E):** The $108(0.75) = 81$ students need 2 cookies each so 162 cookies are to be baked. Since $162 \div 15 = 10.8$, Walter and Gretel must bake 11 recipes. A few leftovers are a good thing!

19. **Answer (B):** Since $216 \div 15 = 14.4$, they will have to bake 15 recipes. This requires $15 \times 3 = 45$ tablespoons of butter. So, $45 \div 8 = 5.625$, and 6 sticks are needed.

20. **Answer (B):** The front view shows the larger of the numbers of cubes in the front or back stack in each column. Therefore the desired front view will have, from left to right, 2, 3, and 4 cubes. This is choice B.



21. **Answer (B):** Since $\angle 1$ forms a straight line with angle $100°$, $\angle 1 = 80°$. Since $\angle 2$ forms a straight line with angle $110°$, $\angle 2 = 70°$. Angle 3 is the third angle in a triangle with $\angle E = 40°$ and $\angle 2 = 70°$, so $\angle 3 = 180° - 40° - 70° = 70°$.Angle $4 = 110°$ since it forms a straight angle with $\angle 3$. Then $\angle 5$ forms a straight angle with $\angle 4$, so $\angle 5 = 70°$. (Or $\angle 3 = \angle 5$ because they are vertical angles.) Therefore, $\angle A = 180° - \angle 1 - \angle 5 = 180° - 80° - 70° = 30°$.



OR

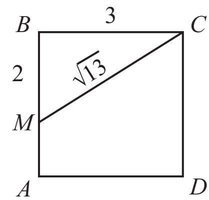
The angle sum in $\triangle CEF$ is $180°$, so $\angle C = 180° - 40° - 100° = 40°$. In $\triangle ACG$, $\angle G = 110°$ and $\angle C = 40°$, so $\angle A = 180° - 110° - 40° = 30°$.

22. **Answer (D):** One fish is worth $\frac{2}{3}$ of a loaf of bread and $\frac{2}{3}$ of a loaf of bread is worth $\frac{2}{3} \cdot 4 = \frac{8}{3} = 2\frac{2}{3}$ bags of rice.

OR

$$\begin{gathered} 3F = 2B \\ \tfrac{3}{2}F = B = 4R \\ (\tfrac{2}{3})(\tfrac{3}{2}) = \tfrac{2}{3}(4R) \\ F = \tfrac{8}{3}R = 2\tfrac{2}{3}R. \end{gathered}$$

23. **Answer (C):** One-third of the square's area is 3, so triangle MBC has area $3 = \frac{1}{2}(MB)(BC)$. Since side BC is 3, side MB must be 2. The hypotenuse CM of this right triangle is $\sqrt{2^2 + 3^2} = \sqrt{13}$.



24. **Answer (D):** Since any positive integer(expressed in base ten) is some multiple of 5 plus its last digit, its remainder when divided by 5 can be obtained by knowing its last digit.

Note that 1999^1 ends in 9, 1999^2 ends in 1, 1999^3 ends in 9, 1999^4 ends in 1, and this alternation of 9 and 1 endings continues with all even powers ending in 1. Therefore, the remainder when 1999^{2000} is divided by 5 is 1.

25. **Answer (A):** At each stage the area of the shaded triangle is one-third of the trapezoidal region not containing the smaller triangle being divided in the next step. Thus, the total area of the shaded triangles comes closer and closer to one-third of the area of the triangular region ACG and this is $\frac{1}{3} \cdot \frac{1}{2} \cdot 6 \cdot 6 = 6$. The shaded areas for the first six stages are: 4.5, 5.625, 5.906, 5.976, 5.994, and 5.998.

These are the calculations for the first three steps.

$\frac{1}{2} \cdot \frac{6}{2} \cdot \frac{6}{2} = 4.5$

$\frac{1}{2} \cdot \frac{6}{2} \cdot \frac{6}{2} + \frac{1}{2} \cdot \frac{6}{4} \cdot \frac{6}{4} = 4.5 + 1.125 = 5.625$

$\frac{1}{2} \cdot \frac{6}{2} \cdot \frac{6}{2} + \frac{1}{2} \cdot \frac{6}{4} \cdot \frac{6}{4} + \frac{1}{2} \cdot \frac{6}{8} \cdot \frac{6}{8} = 5.625 + 0.281 = 5.906$

American Mathematics Competitions

16th Annual

AMC 8

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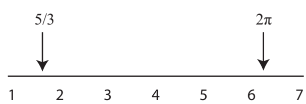
Lincoln, NE 68501-1606

1. **Answer (B):** Brianna is half as old as Aunt Anna, so Brianna is 21 years old. Caitlin is 5 years younger than Brianna, so Caitlin is 16 years old.

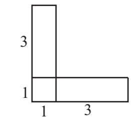
2. **Answer (A):** The number 0 has no reciprocal, and 1 and -1 are their own reciprocals. This leaves only 2 and -2. The reciprocal of 2 is $\frac{1}{2}$, but 2 is not less than $\frac{1}{2}$. The reciprocal of -2 is $-\frac{1}{2}$, and -2 is less than $-\frac{1}{2}$.

3. **Answer (D):** The smallest whole number in the interval is 2 because $\frac{5}{3}$ is more than 1 but less than 2. The largest whole number in the interval is 6 because 2π is more than 6 but less than 7. There are five whole numbers in the interval. They are 2, 3, 4, 5, and 6.



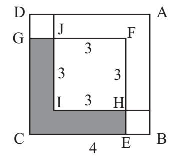
4. **Answer (E):** The data are 1960(5%), 1970(8%), 1980(15%), and 1990(30%). Only graph (E) has these entries.

5. **Answer (C):** If the first year of the 8-year period was the final year of a principal's term, then in the next six years two more principals would serve, and the last year of the period would be the first year of the fourth principal's term. Therefore, the maximum number of principals who can serve during an 8-year period is 4.

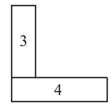


6. **Answer (A):** The L-shaped region is made up of two rectangles with area $3 \times 1 = 3$ plus the corner square with area $1 \times 1 = 1$, so the area of the L-shaped figure is $2 \times 3 + 1 = 7$.

OR



Square $FECG-$ square $FHIJ = 4 \times 4 - 3 \times 3 = 16 - 9 = 7$.



OR

The L-shaped region can be decomposed into a 4×1 rectangle and a 3×1 rectangle. So the total area is 7.

7. **Answer (B):** The only way to get a negative product using three numbers is to multiply one negative number and two positives or three negatives. Only two reasonable choices exist: $(-8) \times (-6) \times (-4) = (-8) \times (24) = -192$ and $(-8) \times 5 \times 7 = (-8) \times 35 = -280$. The latter is smaller.

8. **Answer (D):** The numbers on one die total $1 + 2 + 3 + 4 + 5 + 6 = 21$, so the numbers on the three dice total 63. Numbers 1, 1, 2, 3, 4, 5, 6 are visible, and these total 22. This leaves $63 - 22 = 41$ not seen.

9. **Answer (D):** The 3-digit powers of 5 are 125 and 625, so space 2 is filled with a 2. The only 3-digit power of 2 beginning with 2 is 256, so the outlined block is filled with a 6.

10. **Answer (E):** Shea is 60 inches tall. This is 1.2 times the common starting height, so the starting height was $\frac{60}{1.2} = 50$ inches. Shea has grown $60 - 50 = 10$ inches. Therefore, Ara grew 5 inches and is now 55 inches tall.

11. **Answer (C):** Twelve numbers ending with 1, 2, or 5 have this property. They are 11, 12, 15, 21, 22, 25, 31, 32, 35, 41, 42, and 35. In addition, we have 33, 24, 44, 36, and 48, for a total of 17. (Note that 20, 30, and 40 are not divisible by 0, since division by 0 is not defined.)

12. **Answer (D):** If the vertical joins were not staggered, the wall could be build with $\frac{1}{2}(100 \times 7) = 350$ of the two-foot blocks. To stagger the joins, we need only to replace, in every other row, one of the longer blocks by two shorter ones, placing one at each end. To minimize the number of blocks this should be done in rows 2, 4, and 6. This adds 3 blocks to the 350, making a total of 353.

13. **Answer (C):** Since $\angle ACT = \angle ATC$ and $\angle CAT = 36°$, we have $2(\angle ATC) = 180° - 36° = 144°$ and $\angle ATC = \angle ACT = 72°$. Because $\overline{TR}$ bisects $\angle ATC$, $\angle CTR = \frac{1}{2}(72°) = 36°$. In triangle CRT, $\angle CRT = 180° - 36° - 72° = 72°$. Note that some texts use $\angle ACT$ to define the angle and $m\angle ACT$ to indicate its measure.

14. **Answer (D):** The units digit of a power of an integer is determined by the units digit of the integer; that is, the tens digit, hundreds digit, etc... of the integer have no effect on the units digit of the result. In this problem, the units digit of 19^{19} is the units digit of 9^{19}. Note that $9^1 = 9$ ends in 9, $9^2 = 81$ ends in 1, $9^3 = 729$ ends in 9, and, in general, the units digit of odd powers of 9 is 9, whereas the units digit of even powers of 9 is 1. Since both exponents are odd, the sum of their units digits is $9 + 9 = 18$, the units digit of which is 8.

15. **Answer (C):** We have

$$AB + BC + CD + DE + EF + FG + GA = \\ 4 + 4 + 2 + 2 + 1 + 1 + 1 = 15$$

16. **Answer (C):** The perimeter is $1000 \div 10 = 100$, and this is two lengths and two widths. The length of the backyard is $1000 \div 25 = 40$. Since two lengths total 80, the two widths total 20, and the width is 10. The area is $10 \times 40 = 400$.

17. **Answer (A):** We have

$$(1 \otimes 2) \otimes 3 = \frac{1^2}{2} \otimes 3 = \frac{1}{2} \otimes 3 = \frac{(\frac{1}{2})^2}{3} = \frac{\frac{1}{4}}{3} = \frac{1}{12},$$

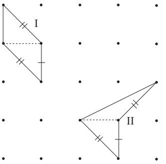
and

$$1 \otimes (2 \otimes 3) = 1 \otimes (\frac{2^2}{3}) = 1 \otimes \frac{4}{3} = \frac{1^2}{\frac{4}{3}} = \frac{3}{4}.$$

Therefore,

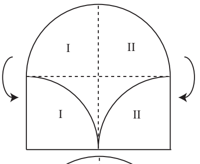
$$(1 \otimes 2) \otimes 3 - 1 \otimes (2 \otimes 3) = \frac{1}{12} - \frac{3}{4} = \frac{1}{12} - \frac{9}{12} = -\frac{8}{12} = -\frac{2}{3}.$$

18. **Answer (E):** Divide each quadrilateral as shown. The resulting triangles each have base 1, altitude 1, and area $\frac{1}{2}$, so the quadrilaterals each have area 1.



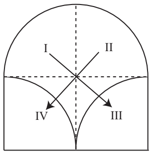
Three sides of quadrilateral I match those of quadrilateral II as indicated by matching marks. The fourth side of quadrilateral I is less than the fourth side of quadrilateral II, hence its perimeter is less, and choice (E) is correct.

19. **Answer (C):** Divide the semicircle in half and rotate each half down to fill the space below the quarter-circles. The figure formed is a rectangle with dimensions 5 and 10. The area is 50.



OR

Slide I into III and II into IV as indicated by the arrows to create the 5×10 rectangle.



20. **Answer (A):** Since the total value is \$1.02, you must have either 2 or 7 pennies. It is impossible to have 7 pennies, since the two remaining coins cannot have a value of 95 cents. With 2 pennies the remaining 7 coins have a value of \$1.00. Either 2 or 3 of these must be quarters. If you have 2 quarters, the other 5 coins would be dimes, and you would have no nickels. The only possible solution is 3 quarters, 1 dime, 3 nickels and 2 pennies.

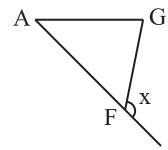
21. **Answer (B):** Make a complete list of equally likely outcomes:

Keiko	Ephraim	Same Number of Heads?
H	HH	No
H	HT	Yes
H	TH	Yes
H	TT	No
T	HH	No
T	HT	No
T	TH	No
T	TT	Yes

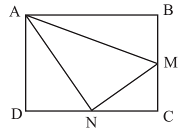
The probability that they have the same number of heads is $\frac{3}{8}$.

22. **Answer (C):** The area of each face of the larger cube is $2^2 = 4$. There are six faces of the cube, so its surface area is $6(4) = 24$. When we add the smaller cube, we decrease the original surface area by 1, but we add $5(1^2) = 5$ units of area (1 unit for each of the five unglued faces of the smaller cube), This is a net increase of 4 from the original surface area, and 4 is $\frac{4}{24} = \frac{1}{6} \approx 16.7\%$ of 24. The closest value given is 17.

23. **Answer (B):** Since the average of all seven numbers is $6\frac{4}{7} = \frac{46}{7}$, the sum of the seven numbers is $7 \times \frac{46}{7} = 46$. The sum of the first four numbers is $4 \times 5 = 20$ and the sum of the last four numbers is $4 \times 8 = 32$. Since the fourth number is used in each of these two sums, the fourth number must be $(20 + 32) - 46 = 6$.



24. **Answer (D):** Since $\angle AFG = \angle AGF$ and $\angle GAF + \angle AFG + \angle AGF = 180°$, we have $20° + 2(\angle AFG) = 180°$. So $\angle AFG = 80°$. Also, $\angle AFG + \angle BFD = 190°$, so $\angle BFD = 100°$. The sum of the angles of $\triangle BFD$ is $180°$, so $\angle B + \angle D = 80°$.
Note: In $\triangle AFG, \angle AFG = \angle B + \angle D$. In general, an exterior angle of a triangle equals the sum of its remote interior angles. For example, in $\triangle GAF, \angle x = \angle GAF + \angle AGF$.
Note that, as in Problem 13, some texts use different symbols to represent an angle and its degree measure.



25. **Answer (B):** Three right triangles lie outside $\triangle AMN$. Their areas are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{8}$ for a total of $\frac{5}{8}$ of the rectangle. The area of $\triangle AMN$ is $\frac{3}{8}(72) = 27$.

OR

Let the rectangle have sides of $2a$ and $2b$ so that $4ab = 72$ and $ab = 18$.Three right triangles lie outside triangle AMN, and their areas are $\frac{1}{2}(2a)(b)$, $\frac{1}{2}(2b)(a)$, $\frac{1}{2}(a)(b)$, for a total of $\frac{5}{2}(ab) = \frac{5}{2}(18) = 45$. The area of triangle AMN is $72 - 45 = 27$.

MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions
PRESENTED BY THE AKAMAI FOUNDATION

17th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 13, 2001

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. (D) At 2 seconds per dimple, it takes $300 \times 2 = 600$ seconds to paint them. Since there are 60 seconds in a minute, he will need $600 \div 60 = 10$ minutes.

2. (D) Since their sum is to be 11, only positive factors need to be considered. Number pairs whose product is 24 are $(1, 24)$, $(2, 12)$, $(3, 8)$ and $(4, 6)$. The sum of the third pair is 11, so the numbers are 3 and 8. The larger one is 8.

3. (E) Anjou has one-third as much money as Granny Smith, so Anjou has \$21. Elberta has \$2 more than Anjou, and $\$21 + \$2 = \$23$.

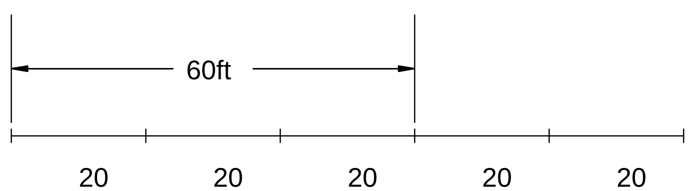
4. (E) To make the number as small as possible, the smaller digits are placed in the higher-value positions. To make the number even, the larger even digit 4 must be the units digit. The smallest possible even number is 12394 and 9 is in the tens place.

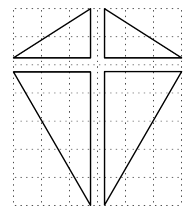
5. (C) Use the formula $d = rt$ (distance equals rate times time): 1088 feet per second $\times 10$ seconds $= 10880$ feet, which is just 320 feet more than two miles. Therefore, Snoopy is just about two miles from the flash of lightning.

   OR

   Since this is an estimate, round the speed of sound down to 1000 feet per second and the length of a mile down to 5000 feet. Then $5000 \div 1000 = 5$ seconds per mile, so in 10 seconds the sound will travel about 2 miles.

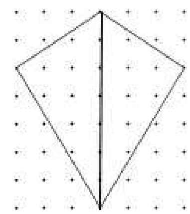
6. (B) There are three spaces between the first tree and the fourth tree, so the distance between adjacent trees is 20 feet. There are 6 trees with five of these 20-foot spaces, so the distance between the first and last trees is 100 feet.



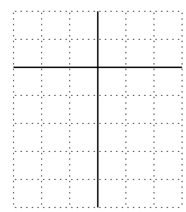


7. (A) The area is made up of two pairs of congruent triangles. The top two triangles can be arranged to form a 2×3 rectangle. The bottom two triangles can be arranged to form a 5×3 rectangle. The kite's area is $6 + 15 = 21$ square inches.

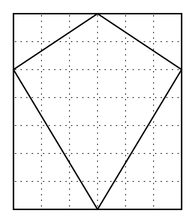
OR



The kite can be divided into two triangles, each with base 7 and altitude 3. Each area is $(1/2)(7)(3) = 10.5$, so the total area is $2(10.5) = 21$ square inches.



8. (E) The small kite is 6 inches wide and 7 inches high, so the larger kite is 18 inches wide and 21 inches high. The amount of bracing needed is $18 + 21 = 39$ inches.



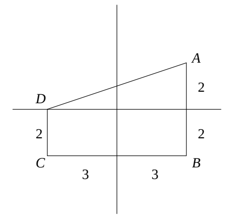
9. (D) The upper corners can be arranged to form a 6×9 rectangle and the lower corners can be arranged to form a 15×9 rectangle. The total area is $54 + 135 = 189$ square inches. (Note that the kite's area is also 189 square inches.)

OR

The area cut off equals the area of the kite. If each dimension is tripled, the area is $3 \times 3 = 9$ times as large as the original area and $21 \times 9 = 189$ square inches. In general, if one dimension is multiplied by a number x and the other by a number y, the area is multiplied by $x \times y$.

10. (A) $2000\% = 20.00$, so the quarters are worth 20 times their face value. That makes the total value $20(4)(\$0.25) = \$20$.

11. (C) The lower part is a 6×2 rectangle with area 12. The upper part is a triangle with base 6 and altitude 2 with area 6. The total area is $12 + 6 = 18$.



OR

Trapezoid $ABCD$ has bases 2 and 4 with altitude 6. Using the formula:
$$A = \frac{h(b_1 + b_2)}{2}, \text{ the area is } \frac{6(2+4)}{2} = 18.$$

12. (A) $6 \otimes 4 = \dfrac{6+4}{6-4} = \dfrac{10}{2} = 5$, and $5 \otimes 3 = \dfrac{5+3}{5-3} = \dfrac{8}{2} = 4$.

Note: $(6 \otimes 4) \otimes 3 \neq 6 \otimes (4 \otimes 3)$. Does $(6 \otimes 4) \otimes 3 = 3 \otimes (6 \otimes 4)$?

13. (D) Since $12 + 8 + 6 = 26$, there are $36 - 26 = 10$ children who prefer cherry or lemon pie. These ten are divided into equal parts of 5 each.
$$\frac{5}{36} \times 360^\circ = 5 \times 10^\circ = 50^\circ.$$

14. (C) There are 3 choices for the meat and 4 for dessert.

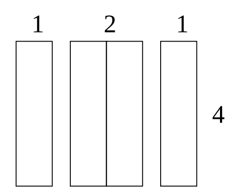
There are 6 ways to choose the two vegetables. The first vegetable may be chosen in 4 ways and the second in 3 ways. This would seem to make 12 ways, but since the order is not important the 12 must be divided by 2. Otherwise, for example, both tomatoes/corn and corn/tomatoes would be included. The 6 choices are beans/corn, beans/potatoes, beans/tomatoes, corn/potatoes, corn/tomatoes and potatoes/tomatoes.

The answer is 3(4)(6)=72.

15. (A) After 4 minutes Homer had peeled 12 potatoes. When Christen joined him, the combined rate of peeling was 8 potatoes per minute, so the remaining 32 potatoes required 4 minutes to peel. In these 4 minutes Christen peeled 20 potatoes.

OR

	minute	Homer	Christen	running total
	1	3		3
	2	3		6
	3	3		9
	4	3		12
	5	3	5	20
	6	3	5	28
	7	3	5	36
	8	3	5	44
Totals		24	20	



16. (E) The dimensions of the new rectangles are shown. The perimeter of a small rectangle is $4+1+4+1=10$ inches and for the large one it is $4+2+4+2=12$ inches. The ratio is $10/12=5/6$.

17. (B) The percent increase from a to b is given by

$$\frac{b-a}{a}(100\%)$$

For example, the percent increase for the first two questions is

$$\frac{200-100}{100}(100\%)=100\%$$

Each time the amount doubles there is a 100% increase. The only exceptions in this game are 2 to 3 (50%), 3 to 4 ($66\frac{2}{3}\%$) and 11 to 12 (about 95%). The answer is (B).

OR

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Value	100	200	300	500	1K	2K	4K	8K	16K	32K	64K	125K	250K	500K	1000K
% Increase		100	50	66.7	100	100	100	100	100	100	100	95	100	100	100

18. (D) There would be $6 \times 6 = 36$ entries in the table if it were complete, but only the 11 entries that are multiples of 5 are shown. The probability of getting a multiple of 5 is $11/36$.

$\times$	1	2	3	4	5	6
1					5	
2					10	
3					15	
4					20	
5	5	10	15	20	25	30
6					30	

OR

Probability questions are sometimes answered by calculating the ways the event will NOT happen, then subtracting. In this problem the 1, 2, 3, 4 and 6 faces are paired to create $5 \times 5 = 25$ number pairs whose product is NOT multiples of 5. This leaves $36 - 25 = 11$ ways to get a multiple of 5, so the probability is $11/36$.

19. (D) The second car travels the same distance at twice the speed; therefore, it needs half the time required for the first car. Graph D shows this relationship.

20. (A) Quay indicates that she has the same score as Kaleana. Marty's statement indicates that her score is higher than Kaleana's, and Shana's statement indicates that her score is lower than Kaleana's. The sequence S,Q,M is the correct one.

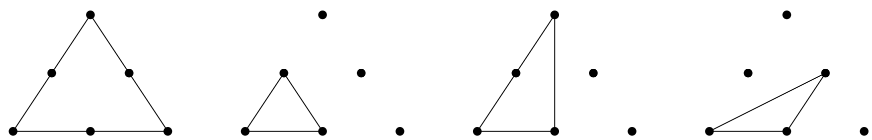
21. (D) The sum of all five numbers is 5(15)=75. Let the numbers be $W, X, 18, Y$ and Z in increasing order. For Z to be as large as possible, make W, X and Y as small as possible. The smallest possible values are $W = 1, X = 2$ and $Y = 19$. Then the sum of $W, X, 18$ and Y is 40, and the difference, $75 - 40 = 35$, is the largest possible value of Z.

22. (E) To get a score in the 90s, a student must get 18 or 19 correct answers. If the number is 18, then the other two questions are worth 0+0, 0+1, 1+0 or 1+1, producing total scores of 90, 91 or 92. If the number correct is 19, then the total is 95+0 or 95+1. Therefore, the only possible scores in the 90s are 90, 91, 92, 95 and 96. This leaves 97 as an impossible score.

OR

The highest possible score is 100 for 20 correct answers. For 19 correct the total is 95+0 or 95+1. This shows that 97 is not possible. As above, 90, 91 and 92 are possible.

23. (D) There are four noncongruent triangles.



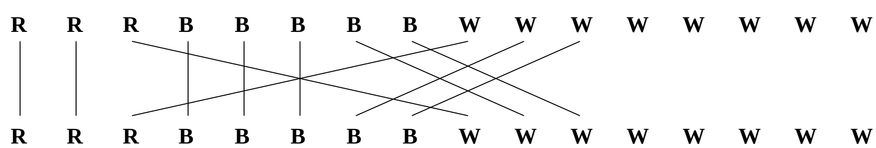
OR

The seventeen possible triangles may be divided into four congruence classes: {RST}; {RXY, XTZ, YZS, XYZ}; {RXS, TXS, RZS, RZT, TYR, TYS}; {RXZ, RYZ, TXY, TZY, XYS, XZS}

24. (B) All six red triangles are accounted for, so the two unmatched upper blue triangles must coincide with lower white triangles. Since one lower white triangle is matched with a red triangle and two are matched with blue triangles, there are five left and these must match with upper white triangles.

OR

It may be helpful to construct a diagram such as this:



25. (D) Six of the 24 numbers are in the 2000s, six in the 4000s, six in the 5000s and six in the 7000s. Doubling and tripling numbers in the 2000s produce possible solutions, but any multiple of those in the other sets is larger than 8000.

Units digits of the numbers are 2, 4, 5 and 7, so their doubles will end in 4, 8, 0 and 4, respectively. Choice (A) 5724 ends in 4 but $5724/2 = 2862$, not one of the 24 numbers. Likewise, choice (C) 7254 produces $7254/2 = 3627$, also not one of the numbers. When the units digits are tripled the resulting units digits are 6, 2, 5 and 1 and choices (B) 7245, (D) 7425 and (E) 7542 are possibilities. Division by 3 yields 2415, 2475 and 2514 respectively. Only the second of these numbers is one of the 24 given numbers. Choice (D) is correct.

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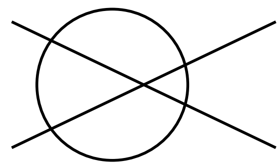
1. **(D)** Two distinct lines can intersect in one point whereas a line can intersect a circle in two points. The maximum number 5 can be achieved if the lines and circle are arranged as shown. Note that the lines could also meet outside the circle for the same result. (Other arrangements of the lines and circle can produce 0, 1, 2, 3, or 4 points of intersection.)



2. **(A)** Since the total \$17 is odd, there must be an odd number of \$5 bills. One \$5 bill plus six \$2 bills is a solution, as is three \$5 bills plus one \$2 bill. Five \$5 bills exceeds \$17, so these are the only two combinations that work.

3. **(C)** The smallest average will occur when the numbers are as small as possible. The four smallest distinct positive even integers are 2, 4, 6, and 8 and their average is 5.

   Note: These numbers form an arithmetic sequence. The average of the numbers in any arithmetic sequence is the average of the first and last terms.

4. **(B)** The next palindrome is 2112. The product of its digits is $2 \cdot 1 \cdot 1 \cdot 2 = 4$.

5. **(C)** Since 706 days is 700 plus 6 days, it is 100 weeks plus 6 days. Friday is 6 days after Saturday.

6. **(A)** Initially, volume increases with time as shown by graphs A, C, and E. But once the birdbath is full, the volume remains constant as the birdbath overflows. Only graph A shows both features.

7. **(E)** There are $6 + 8 + 4 + 2 + 5 = 25$ students. Of the 25 students 5 prefer candy E and $\frac{5}{25} = \frac{20}{100} = 20\%$.

8. **(D)** There are 15 French stamps and 9 Spanish stamps issued in the '80s. So there are $15 + 9 = 24$ European stamps listed in the table in the '80s.

9. **(B)** His South American stamps issued before the '70s include $4 + 7 = 11$ from Brazil that cost $11 \times \$0.06 = \$0.66$ and $6 + 4 = 10$ from Peru that cost $10 \times \$0.04 = \$0.40$. Their total cost is $\$0.66 + \$0.40 = \$1.06$.

10. **(E)** The '70s stamps cost: Brazil, $12(\$0.06) = \$0.72$; Peru, $6(\$0.04) = \$0.24$; France, $12(\$0.06) = \$0.72$; Spain, $13(\$0.05) = \$0.65$. The total is \$2.33 for the 43 stamps and the average price is $\frac{\$2.33}{43} \approx \$0.054 \approx 5.5$¢.

11. **(C)** To build the second square from the first, add 3 tiles. To build the third from the second, add 5 tiles. The pattern of adding an odd number of tiles continues. For the fourth square, add 7; for the fifth, add 9; for the sixth, add 11 and for the seventh, add 13.

OR

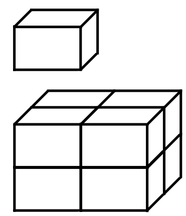
The number of additional tiles needed is $7^2 - 6^2 = 49 - 36 = 13$.

OR

To build the second square from the first, add $2+1=3$ tiles. To build the third from the second, add $3+2=5$ tiles. The pattern continues and $7+6=13$.

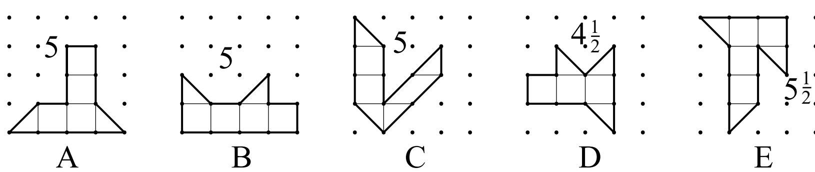
12. **(B)** Since the sum of the three probabilities is 1, the probability of stopping on region C is $1-\frac{1}{3}-\frac{1}{2}=\frac{6}{6}-\frac{2}{6}-\frac{3}{6}=\frac{1}{6}$.

13. **(E)** Since the exact dimensions of Bert's box do not matter, assume the box is $1\times2\times3$. Its volume is 6. Carrie's box is $2\times4\times6$, so its volume is 48 or 8 times the volume of Bert's box. Carrie has approximately $8(125)=1000$ jellybeans.
Note: Other examples may help to see that the ratio is always 8 to 1.



14. **(B)** The first discount means that the customer will pay 70% of the original price. The second discount means a selling price of 80% of the discounted price. Because $0.80(0.70)=0.56=56\%$, the customer pays 56% of the original price and thus receives a 44% discount.

15. **(E)** Areas may be found by dividing each polygon into triangles and squares as shown.



Note: Pick's Theorem may be used to find areas of geoboard polygons. If I is the number of dots inside the figure, B is the number of dots on the boundary and A is the area, then $A=I+\frac{B}{2}-1$. Geoboard figures in this problem have no interior points, so the formula simplifies to $A=\frac{B}{2}-1$. For example, in polygon D the number of boundary points is 11 and $\frac{11}{2}-1=4\frac{1}{2}$.

16. **(E)** The areas are $W = \frac{1}{2}(3)(4) = 6$, $X = \frac{1}{2}(3)(3) = 4\frac{1}{2}$, $Y = \frac{1}{2}(4)(4) = 8$ and $Z = \frac{1}{2}(5)(5) = 12\frac{1}{2}$. Therefore, (E) is correct. $X + Y = 4\frac{1}{2} + 8 = 12\frac{1}{2} = Z$. The other choices are incorrect.

OR

By the Pythagorean Theorem, if squares are constructed on each side of any right triangle, the sum of the areas of the squares on the legs equal the area of the square on the hypotenuse. So $2X + 2Y = 2Z$, and $X + Y = Z$.

17. **(C)** Olivia solved at least 6 correctly to have scored over 25. Her score for six correct would be $6(+5) + 4(-2) = 22$, which is too low. If she answered 7 correctly, her score would be $7(+5) + 3(-2) = 29$, and this was her score. The correct choice is (C).

OR

Make a table:

Right	+ score	Wrong	- score	Total
9	45	1	-2	43
8	40	2	-4	36
7	35	3	-6	29

OR

If Olivia answers x problems correctly, then she answered $10 - x$ incorrectly and her score was

$$\begin{aligned} 5x - 2(10 - x) &= 29 \\ 5x - 20 + 2x &= 29 \\ 7x &= 49 \\ x &= 7 \end{aligned}$$

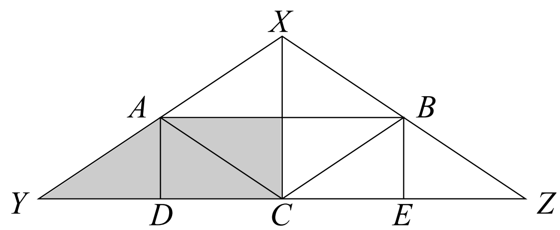
18. **(E)** In 5 days, Gage skated for $5 \times 75 = 375$ minutes, and in 3 days he skated for $3 \times 90 = 270$ minutes. So, in 8 days he skated for $375 + 270 = 645$ minutes. To average 85 minutes per day for 9 days he must skate $9 \times 85 = 765$ minutes, so he must skate $765 - 645 = 120$ minutes $= 2$ hours the ninth day.

OR

For 5 days Gage skated 10 minutes under his desired average, and for 3 days he skated 5 minutes over his desired average. So, on the ninth day he needs to make up $5 \times 10 - 3 \times 5 = 50 - 15 = 35$ minutes. To do this, on the ninth day he must skate for $85 + 35 = 120$ minutes $= 2$ hours.

19. **(D)** Numbers with exactly one zero have the form _0_ or _ _0, where the blanks are not zeros. There are $(9 \cdot 1 \cdot 9) + (9 \cdot 9 \cdot 1) = 81 + 81 = 162$ such numbers.

20. **(D)** Segments $\overline{AD}$ and $\overline{BE}$ are drawn perpendicular to $\overline{YZ}$. Segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ divide $\triangle XYZ$ into four congruent triangles. Vertical line segments $\overline{AD}$, $\overline{XC}$ and $\overline{BE}$ divide each of these in half. Three of the eight small triangles are shaded, or $\frac{3}{8}$ of $\triangle XYZ$. The shaded area is $\frac{3}{8}(8) = 3$.



OR

Segments $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ divide $\triangle XYZ$ into four congruent triangles, so the area of $\triangle XAB$ is one-fourth the area of $\triangle XYZ$. That makes the area of trapezoid $ABZY$ three-fourths the area of $\triangle XYZ$. The shaded area is one-half the area of trapezoid $ABZY$, or three-eighths the area of $\triangle XYZ$, and $\frac{3}{8}(8) = 3$.

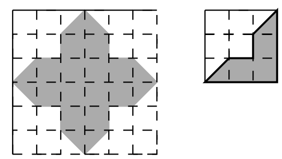
21. **(E)** There are 16 possible outcomes: $HHHH$, $HHHT$, $HHTH$, $HTHH$, $THHH$, $HHTT$, $HTHT$, $HTTH$, $THTH$, $THHT$, $TTHH$ and $HTTT$, $THTT$, $TTHT$, $TTTH$, $TTTT$. The first eleven have at least as many heads as tails. The probability is $\frac{11}{16}$.

22. **(C)** When viewed from the top and bottom, there are 4 faces exposed; from the left and right sides, there are 4 faces exposed and from the front and back, there are 5 faces exposed. The total is $4 + 4 + 4 + 4 + 5 + 5 = 26$ exposed faces.

OR

Before the cubes were glued together, there were $6 \times 6 = 36$ faces exposed. Five pairs of faces were glued together, so $5 \times 2 = 10$ faces were no longer exposed. This leaves $36 - 10 = 26$ exposed faces.

23. **(B)** The 6×6 square in the upper left-hand region is tessellated, so finding the proportion of darker tiles in this region will answer the question. The top left-hand corner of this region is a 3×3 square that has $3 + 2\left(\frac{1}{2}\right) = 4$ darker tiles. So $\frac{4}{9}$ of the total area will be made of darker tiles.



24. **(B)** Use 6 pears to make 16 oz of pear juice and 6 oranges to make 24 oz of orange juice for a total of 40 oz of juice. The percent of pear juice is $\frac{16}{40} = \frac{4}{10} = 40\%$.

OR

Miki can make 8 oz of orange juice with 2 oranges, so she can make 4 oz of orange juice with 1 orange. She can make 8 oz of pear juice from 3 pears, so she can make $\frac{8}{3}$ oz of pear juice from 1 pear. With 1 orange and 1 pear, she can make $4 + \frac{8}{3} = \frac{20}{3}$ oz of the blend, of which $\frac{8}{3}$ oz is pear juice. As a percent, $\frac{\frac{8}{3}}{\frac{20}{3}} = \frac{8}{20} = \frac{4}{10} = 40\%$ of the blend is pear juice.

25. **(B)** Only the fraction of each friend's money is important, so we can assume any convenient amount is given to Ott. Suppose that each friend gave Ott \$1. If this is so, then Moe had \$5 orignally and now has \$4, Loki had \$ 4 and now has \$3, and Nick had \$3, and now has \$2. The four friends now have $\$4 + \$3 + \$2 + \$3 = \$12$, so Ott has $\frac{3}{12} = \frac{1}{4}$ of the group's money. This same reasoning applies to any amount of money.

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Solutions Pamphlet

Tuesday, NOVEMBER 18, 2003

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the copyright law.

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1. **(E)** A cube has 12 edges, 8 corners and 6 faces. The sum is 26.

2. **(C)** The smallest prime is 2, which is a factor of every even number. Because 58 is the only even number, it has the smallest prime factor.

3. **(D)** Since 30 of the 120 grams are filler, $\frac{30}{120} = 25\%$ of the burger is filler. So $100\% - 25\% = 75\%$ of the burger is not filler.

   OR

   There are $120 - 30 = 90$ grams that are not filler. So $\frac{90}{120} = 75\%$ is not filler.

4. **(C)** The following chart shows that the answer must be 5 tricycles.

Bicycles	Tricycles	Wheels
0	7	21
1	6	20
2	5	19
3	4	18

   OR

   Let b equal the number of bicycles and t equal the number of tricycles. Then the number of vehicles is $b + t = 7$, and the number of wheels is $2b + 3t = 19$. Because $b = 7 - t$, it follows that

   $$\begin{aligned} 2(7-t) + 3t &= 19 \\ 14 - 2t + 3t &= 19 \\ 14 + t &= 19 \\ t &= 5. \end{aligned}$$

   OR

   If each child had a bicycle, there would be 14 wheels. Since there are 19 wheels, 5 of the vehicles must be tricycles.

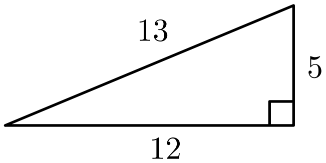
5. **(B)** If 20% of the number is 12, the number must be 60. Then 30% of 60 is $0.30 \times 60 = 18$.

   OR

   Since 20% of the number is 12, it follows that 10% of the number is 6. So 30% of the number is 18.

6. **(B)**

$$A = \frac{1}{2}(\sqrt{144})(\sqrt{25})$$
$$A = \frac{1}{2} \cdot 12 \cdot 5$$
$$A = 30 \text{ square units}$$



7. **(A)** Blake scored a total of $4 \times 78 = 312$ points on the four tests. Jenny scored $10 - 10 + 20 + 20 = 40$ more points than Blake, so her average was $\frac{352}{4} = 88$, or 10 points higher than Blake's.

OR

The total point difference between Jenny's and Blake's tests was $10 - 10 + 20 + 20 = 40$ points. The average difference is $\frac{40}{4} = 10$ points.

8. **(A)** Because all of the cookies have the same thickness, only the surface area of their shapes needs to be considered. The surface area of each of Art's trapezoid cookies is $\frac{1}{2} \cdot 3 \cdot 8 = 12$ in^2. Since he makes 12 cookies, the surface area of the dough is $12 \times 12 = 144$ in^2.

   Roger's rectangle cookies each have surface area $2 \cdot 4 = 8$ in^2; therefore, he makes $144 \div 8 = 18$ cookies.

   Paul's parallelogram cookies each have surface area $2 \cdot 3 = 6$ in^2. He makes $144 \div 6 = 24$ cookies.

   Trisha's triangle cookies each have surface area $\frac{1}{2} \cdot 4 \cdot 3 = 6$ in^2. She makes $144 \div 6 = 24$ cookies.

   So Art makes the fewest cookies.

9. **(C)** Art's 12 cookies sell for $12 \times \$0.60 = \$7.20$. Roger's 18 cookies should cost $\$7.20 \div 18 = \$.40$ each.

OR

The trapezoid's area is 12 in^2 and the rectangle's area is 8 in^2. So the cost of a rectangle cookie should be $\left(\frac{8}{12}\right) 60¢ = 40¢$.

10. **(E)** The triangle's area is 6 in^2, or half that of the trapezoid. So Trisha will make twice as many cookies as Art, or 24.

11. **(B)** Thursday's price of \$40 is increased 10% or \$4, so on Friday the shoes are marked \$44. Then 10% of \$44 or \$4.40 is taken off, so the price on Monday is \$44 – \$4.40 = \$39.60.

OR

Marking a price 10% higher multiplies the original price by 1.1, and reducing a price by 10% multiplies the price by 0.9. So the price of a pair of shoes that was originally \$40 will be $\$40 \times 1.1 \times 0.9 = \$39.60$.

12. **(E)** If 6 is one of the visible faces, the product will be divisible by 6. If 6 is not visible, the product of the visible faces will be $1 \times 2 \times 3 \times 4 \times 5 = 120$, which is also divisible by 6. Because the product is always divisible by 6, the probability is 1.

13. **(B)** A cube has four red faces if it is attached to exactly two other cubes. The four top cubes are each attached to only one other cube, so they have five red faces. The four bottom corner cubes are each attached to three others, so they have three red faces. The remaining six each have four red faces.

14. **(D)** As given, $T = 7$. This implies that $F = 1$ and that O equals either 4 or 5. Since O is even, $O = 4$. Therefore, $R = 8$. Replacing letters with numerals gives

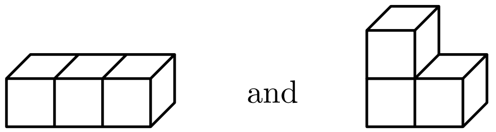
$$\begin{array}{cccc} & 7 & W & 4 \\ + & 7 & W & 4 \\ \hline 1 & 4 & U & 8 \end{array}$$

$W + W$ must be less than 10; otherwise, a 1 would be carried to the next column, and O would be 5. So $W < 5$. $W \neq 0$ because $W \neq U$, $W \neq 1$ because $F = 1$, $W \neq 2$ because if $W = 2$ then $U = 4 = O$, and $W \neq 4$ because $O = 4$. So $W = 3$.

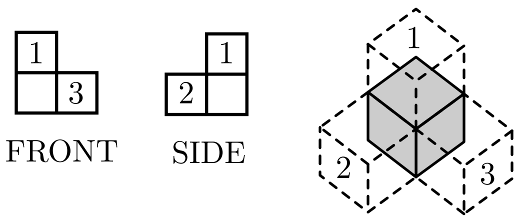
The addition problem is

$$\begin{array}{cccc} & 7 & 3 & 4 \\ + & 7 & 3 & 4 \\ \hline 1 & 4 & 6 & 8 \end{array}$$

15. **(B)** There are only two ways to construct a solid from three cubes so that each cube shares a face with at least one other:



Neither of these configurations has both the front and side views shown. The four-cube configuration has the required front and side views. Thus at least four cubes are necessary.

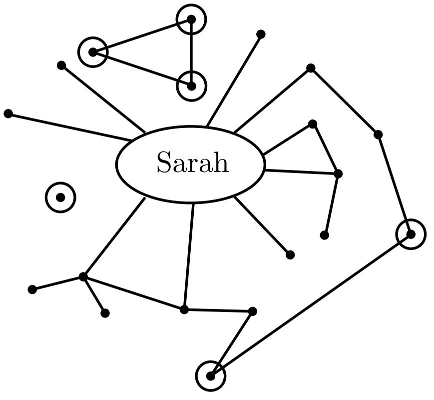


Question: Is it possible to construct a five-cube configuration with these front and side views?

16. **(D)** There are 2 choices for the driver. The other three can seat themselves in $3 \times 2 \times 1 = 6$ different ways. So the number of seating arrangements is $2 \times 6 = 12$.

17. **(E)** Because Jim has brown eyes and blond hair, none of his siblings can have both blue eyes and black hair. Therefore, neither Benjamin nor Tevyn can be Jim's sibling. Consequently, there are only three possible pairs for Jim's siblings – Nadeen and Austin, Nadeen and Sue, or Austin and Sue. Since Nadeen has different hair color and eye color from both Austin and Sue, neither can be Nadeen's sibling. So Austin and Sue are Jim's siblings. Benjamin, Nadeen and Tevyn are siblings in the other family.

18. **(D)** In the graph below, the six classmates who are not friends with Sarah or with one of Sarah's friends are circled. Consequently, six classmates will not be invited to the party.



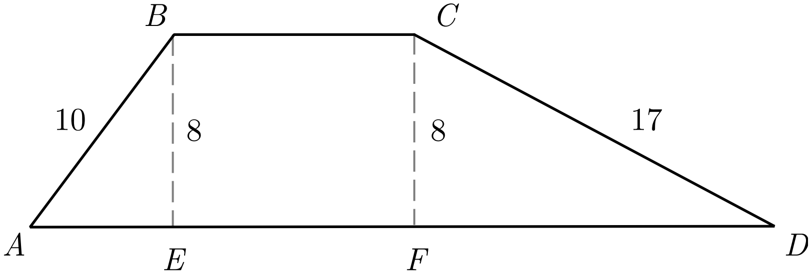
19. **(C)** A number with 15, 20 and 25 as factors must be divisible by their least common multiple (LCM). Because $15 = 3 \times 5$, $20 = 2^2 \times 5$, and $25 = 5^2$, the LCM of 15, 20 and 25 is $2^2 \times 3 \times 5^2 = 300$. There are three multiples of 300 between 1000 and 2000: 1200, 1500 and 1800.

20. **(D)** Note that in the same period of time, the hour hand moves $\frac{1}{12}$ as far as the minute hand. At 4:00 a.m., the minute hand is at 12 and the hour hand is at 4. By 4:20 a.m., the minute hand has moved $\frac{1}{3}$ of way around the clock to 4, and the hour hand has moved $\frac{1}{12} \times \frac{1}{3} = \frac{1}{36}$ of the way around the clock from 4. Therefore, the angle formed by the hands at 4:20 a.m. is $\frac{1}{36} \cdot 360° = 10°$.

OR

As the minute hand moves $\frac{1}{3}$ of the way around the clock face from 12 to 4, the hour hand will move $\frac{1}{3}$ of the way from 4 to 5. So the hour hand will move $\frac{1}{3}$ of $\frac{1}{12}$ of $360°$, or $10°$.

21. **(B)** Label the feet of the altitudes from B and C as E and F respectively. Considering right triangles AEB and DFC, $AE = \sqrt{10^2 - 8^2} = \sqrt{36} = 6$ cm, and $FD = \sqrt{17^2 - 8^2} = \sqrt{225} = 15$ cm. So the area of $\triangle AEB$ is $\frac{1}{2}(6)(8) = 24$ cm^2, and the area of $\triangle DFC$ is $\left(\frac{1}{2}\right)(15)(8) = 60$ cm^2. Rectangle $BCFE$ has area $164 - (24 + 60) = 80$ cm^2. Because $BE = CF = 8$ cm, it follows that $BC = 10$ cm.



OR

Let $BC = EF = x$. From the first solution we know that $AE = 6$ and $FD = 15$. Therefore, $AD = x + 21$, and the area of the trapezoid $ABCD$ is $(8)\left\{\frac{1}{2}\left[x + (x + 21)\right]\right\} = 164$. So

$$\begin{aligned} 4(2x + 21) &= 164, \\ 2x + 21 &= 41, \\ 2x &= 20, \end{aligned}$$

and $x = 10$.

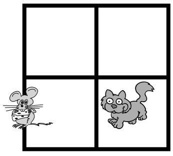
22. **(C)** For Figure A, the area of the square is $2^2 = 4$ cm^2. The diameter of the circle is 2 cm, so the radius is 1 cm and the area of the circle is π cm^2. So the area of the shaded region is $4 - \pi$ cm^2.

For Figure B, the area of the square is also 4 cm^2. The radius of each of the four circles is $\frac{1}{2}$ cm, and the area of each circle is $\left(\frac{1}{2}\right)^2 \pi = \frac{1}{4}\pi$ cm^2. The combined area of all four circles is π cm^2. So the shaded regions in A and B have the same area.

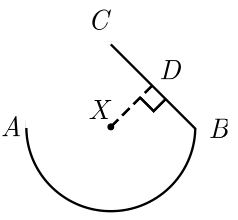
For Figure C, the radius of the circle is 1 cm, so the area of the circle is π cm^2. Because the diagonal of the inscribed square is the hypotenuse of a right triangle with legs of equal lengths, use the Pythagorean Theorem to determine the length s of one side of the inscribed square. That is, $s^2 + s^2 = 2^2 = 4$. So $s^2 = 2$ cm^2, the area of the square. Therefore, the area of the shaded region is $\pi - 2$ cm^2. Because $\pi - 2 > 1$ and $4 - \pi < 1$, the shaded region in Figure C has the largest area.

Note that the second figure consists of four small copies of the first figure. Because each of the four small squares has sides half the length of the sides of the big square, the area of each of the four small figures is $\frac{1}{4}$ the area of Figure A. Because there are four such small figures in Figure B, the shaded regions in A and B have the same area.

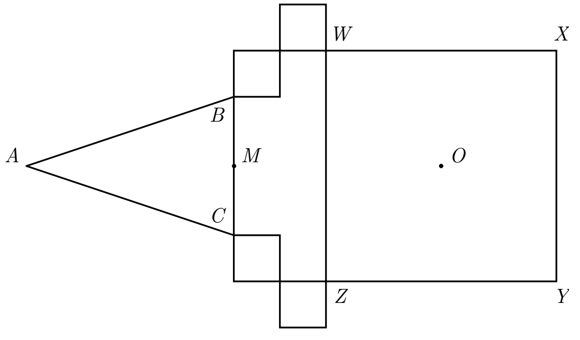
23. **(A)** There are four different positions for the cat in the 2×2 array, so after every fourth move, the cat will be in the same location. Because $247 = 4 \times 61 + 3$, the cat will be in the 3rd position clockwise from the first, or the lower right quadrant. There are eight possible positions for the mouse. Because $247 = 8 \times 30 + 7$, the mouse will be in the 7th position counterclockwise from the first, or the left-hand side of the lower left quadrant.



24. **(B)** All points along the semicircular part of the course are the same distance from X, so the first part of the graph is a horizontal line. As the ship moves from B to D, its distance from X decreases, then it increases as the ship moves from D to C. Only graph B has these features.



25. **(C)** Let M be the midpoint of $\overline{BC}$. Since $\triangle ABC$ is isosceles, $\overline{AM}$ is an altitude to base $\overline{BC}$. Because A coincides with O when $\triangle ABC$ is folded along $\overline{BC}$, it follows that $AM = MO = \frac{5}{2} + 1 + 1 = \frac{9}{2}$ cm. Also, $BC = 5 - 1 - 1 = 3$ cm, so the area of $\triangle ABC$ is $\frac{1}{2} \cdot BC \cdot AM = \frac{1}{2} \cdot 3 \cdot \frac{9}{2} = \frac{27}{4}$ cm^2.



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1. **(B)** If 12 centimeters represents 72 kilometers, then 1 centimeter represents 6 kilometers. So 17 centimeters represents $17 \times 6 = 102$ kilometers.

2. **(B)** To form a four-digit number using 2, 0, 0 and 4, the digit in the thousands place must be 2 or 4. There are three places available for the remaining nonzero digit, whether it is 4 or 2. So the final answer is 6.

   OR

   Make a list: 2004, 2040, 2400, 4002, 4020 and 4200. So 6 numbers are possible.

3. **(A)** If 12 people order $\frac{18}{12} = 1\frac{1}{2}$ times too much food, they should have ordered $\frac{12}{\frac{3}{2}} = \frac{2}{3} \times 12 = 8$ meals.

   OR

   Let x be the number of meals they should have ordered. Then,
   $$\frac{12}{18} = \frac{x}{12},$$
   so
   $$x = 8.$$

4. **(B)** When three players start, one is the alternate. Because any of the four players might be the alternate, there are four ways to select a starting team: Lance-Sally-Joy, Lance-Sally-Fred, Lance-Joy-Fred and Sally-Joy-Fred.

5. **(D)** It takes 15 games to eliminate 15 teams.

6. **(C)** If Sally makes 55% of her 20 shots, she makes $0.55 \times 20 = 11$ shots. If Sally makes 56% of her 25 shots, she makes $0.56 \times 25 = 14$ shots. So she makes $14 - 11 = 3$ of the last 5 shots.

7. **(B)** A 26-year-old's target heart rate is $0.8(220 - 26) = 155.2$ beats per minute. The nearest whole number is 155.

8. **(B)** There are 7 two-digit numbers whose digits sum to 7: 16, 61, 25, 52, 34, 43 and 70.

9. **(D)** The sum of all five numbers is $5 \times 54 = 270$. The sum of the first two numbers is $2 \times 48 = 96$, so the sum of the last three numbers is $270 - 96 = 174$. The average of the last three numbers is $\frac{174}{3} = 58$.

10. **(E)** Aaron worked 75 minutes on Monday, 50 on Tuesday, 145 on Wednesday and 30 on Friday, for a total of 300 minutes or 5 hours. He earned $5 \times \$3 = \$15$.

11. **(C)** The largest, smallest and median occupy the three middle places, so the other two numbers, 9 and 4, are in the first and last places. The average of 9 and 4 is $\frac{9+4}{2} = 6.5$.

12. **(B)** The phone has been used for 60 minutes, or 1 hour, to talk, during which time it has used $\frac{1}{3}$ of the battery. In addition, the phone has been on for 8 hours without talking, which used an additional $\frac{8}{24}$ or $\frac{1}{3}$ of the battery. Consequently, $\frac{1}{3}+\frac{1}{3}=\frac{2}{3}$ of the battery has been used, meaning that $\frac{1}{3}$ of the battery, or $\frac{1}{3}\times 24 = 8$ hours remain if Niki does not talk on her phone.

OR

Niki's battery has 24 hours of potential battery life. By talking for one hour, she uses $\frac{1}{3}\times 24 = 8$ hours of battery life. In addition, the phone is left on and unused for 8 hours, using an additional 8 hours. This leaves $24-8-8=8$ hours of battery life if the phone is on and unused.

13. **(E)** Bill is not the oldest, because if he were, the first two statements would be true. Celine is not the oldest, because if she were, the last two statements would be true. Therefore, Amy is the oldest. So the first two statements are false. The last statement must be true. This means that Celine is not the youngest, so Bill is the youngest. The correct order from oldest to youngest is Amy, Celine, Bill.

OR

The possible cases for the three statements are

	I	II	III
1	T	F	F
2	F	T	F
3	F	F	T

Case 1 leads to a contradiction of statements I and II.

Case 2 means Celine is youngest and neither Bill nor Amy is oldest, but one must be oldest.

Only case 3 is possible. The correct order from oldest to youngest is Amy, Celine, Bill.

14. **(C)** To find the area, subtract the areas of regions A, B, C, D and E from that of the surrounding square.

Square: $10\times 10 = 100$

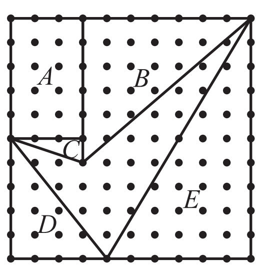
Region A: $3\times 5 = 15$

Region B: $\frac{1}{2}\times 6\times 7 = 21$

Region C: $\frac{1}{2}\times 1\times 3 = 1\frac{1}{2}$

Region D: $\frac{1}{2}\times 4\times 5 = 10$

Region E: $\frac{1}{2}\times 6\times 10 = 30$



The area is $100-\left(15+21+1\frac{1}{2}+10+30\right) = 100-77\frac{1}{2} = 22\frac{1}{2}$ square units.

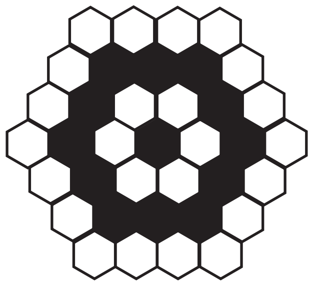
OR

By Pick's Theorem, the area of a polygon with vertices in a lattice is

$$(\text{number of points inside}) + \frac{\text{number of points on boundary}}{2} - 1.$$

In this case, the area is $21 + \frac{5}{2} - 1 = 22\frac{1}{2}$.

15. **(C)** The next border requires an additional $6 \times 3 = 18$ white tiles. A total of 24 white and 13 black tiles will be used, so the difference is $24 - 13 = 11$.



16. **(C)** Because the first pitcher was $\frac{1}{3}$ full of orange juice, after filling with water it contains 200 ml of juice and 400 ml of water. Because the second pitcher was $\frac{2}{5}$ full of orange juice, after filling it contains 240 ml of orange juice and 360 ml of water. In all, the amount of orange juice is 440 ml out of a total of 1200 ml or $\frac{440}{1200} = \frac{11}{30}$ of the mixture.

17. **(D)** The largest number of pencils that any friend can have is four. There are 3 ways that this can happen: $(4,1,1)$, $(1,4,1)$ and $(1,1,4)$. There are 6 ways one person can have 3 pencils: $(3,2,1)$, $(3,1,2)$, $(2,3,1)$, $(2,1,3)$, $(1,2,3)$ and $(1,3,2)$. There is only one way all three can have two pencils each: $(2,2,2)$. The total number of possibilities is $3 + 6 + 1 = 10$.

OR

The possible distributions of 6 pencils among 3 friends are the following:

1	1	4
1	2	3
1	3	2
1	4	1
2	1	3
2	2	2
2	3	1
3	1	2
3	2	1
4	1	1

The number of possible distributions is 10.

18. **(A)** Ben must hit 1 and 3. This means Cindy must hit 5 and 2, because she scores 7 using two different numbers, neither of which is 1 or 3. By similar reasoning, Alice hits 10 and 6, Dave hits 7 and 4, and Ellen hits 9 and 8. Alice hits the 6.

OR

Ellen's score can be obtained by either $10 + 7$ or $9 + 8$. In the first case, it is impossible for Alice to score 16. So Ellen's 17 is obtained by scoring 9 and 8, and Alice's total of 16 is the result of her hitting 10 and 6. The others scored $11 = 7 + 4$, $7 = 5 + 2$ and $4 = 3 + 1$.

19. **(B)** The numbers that leave a remainder of 2 when divided by 4 and 5 are 22, 42, 62 and so on. Checking these numbers for a remainder of 2 when divided by both 3 and 6 yields 62 as the smallest.

OR

The smallest whole number that is evenly divided by each of 3, 4, 5 and 6 is $\mathsf{LCM}\{3, 4, 5, 6\} = 2^2 \times 3 \times 5 = 60$. So the smallest whole number greater than 2 that leaves a remainder of 2 when divided by each of 3, 4, 5 and 6 is 62.

20. **(D)** Because the 6 empty chairs are $\frac{1}{4}$ of the chairs in the room, there are $6 \times 4 = 24$ chairs in all. The number of seated people is $\left(\frac{3}{4}\right)24 = 18$, and this is $\frac{2}{3}$ of the people present. It follows that

$$\frac{18}{\text{people present}} = \frac{2}{3}.$$

So there are 27 people in the room.

21. **(D)** In eight of the twelve outcomes the product is even: 1×2, 2×1, 2×2, 2×3, 3×2, 4×1, 4×2, 4×3. In four of the twelve, the product is odd: 1×1, 1×3, 3×1, 3×3. So the probability that the product is even is $\frac{8}{12}$ or $\frac{2}{3}$.

OR

To get an odd product, the result of both spins must be odd. The probability of odd is $\frac{1}{2}$ on Spinner A and $\frac{2}{3}$ on Spinner B. So the probability of an odd product is $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = \frac{1}{3}$. The probability of an even product, then, is $1 - \frac{1}{3} = \frac{2}{3}$.

22. **(B)** Because $\frac{2}{5}$ of all the women in the room are single, there are two single women for every three married women in the room. There are also two single women for every three married men in the room. So out of every $2 + 3 + 3 = 8$ people, 3 are men. The fraction of the people who are married men is $\frac{3}{8}$.

23. **(D)** The distance increases as Tess moves from J to K, and continues at perhaps a different rate as she moves from K to L. The greatest distance from home will occur at L. The distance decreases as she runs from L to M and continues at perhaps a different rate as she moves from M to J. Graph D shows these changes.

24. **(C)** By the Pythagorean Theorem, $HE = 5$. Rectangle $ABCD$ has area $10 \times 8 = 80$, and the corner triangles have areas $\frac{1}{2} \times 3 \times 4 = 6$ and $\frac{1}{2} \times 6 \times 5 = 15$. So the area of $EFGH$ is $80 - (2)(6) - (2)(15) = 38$. Because the area of $EFGH$ is $EH \times d$ and $EH = 5$, $38 = 5 \times d$, so $d = 7.6$.

25. **(D)** The overlap of the two squares is a smaller square with side length 2, so the area of the region covered by the squares is $2(4 \times 4) - (2 \times 2) = 32 - 4 = 28$. The diameter of the circle has length $\sqrt{2^2 + 2^2} = \sqrt{8}$, the length of the diagonal of the smaller square. The shaded area created by removing the circle from the squares is $28 - \pi\left(\frac{\sqrt{8}}{2}\right)^2 = 28 - 2\pi$.

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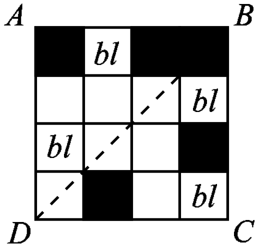
1. **(B)** If multiplying a number by 2 results in 60, then the number must be 30. If 30 is divided by 2, the correct answer is 15.

2. **(C)** Karl spent $5 \times \$2.50 = \$12.50$ on the folders. If he had purchased the folders a day later, he would have saved 20% of this total, or $0.20 \times \$12.50 = \$2.50$.

OR

Karl could have bought five folders for the price of four in the 20%-off sale, so he could have saved \$2.50.

3. **(D)** For diagonal BD to lie on a line of symmetry in square $ABCD$, the four small squares labeled bl must be colored black.

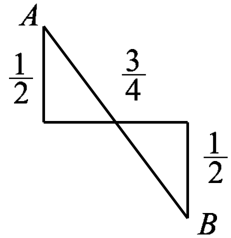


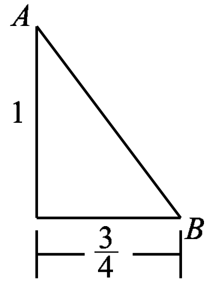
4. **(C)** The perimeter of the triangle is $6.1 + 8.2 + 9.7 = 24$ cm. The perimeter of the square is also 24 cm. Each side of the square is $24 \div 4 = 6$ cm. The area of the square is $6^2 = 36$ square centimeters.

5. **(B)** To get the minimum number of packs, purchase as many 24-packs as possible: three 24-packs contain 72 cans, which leaves $90 - 72 = 18$ cans. To get the remaining 18 cans, purchase one 12-pack and one 6-pack. The minimum number of packs is 5.

6. **(C)** The number $2.00d5$ is greater than 2.005 for $d = 5, 6, 7, 8$ and 9. Therefore, there are five digits satisfying the inequality.

7. **(B)** The diagram on the left shows the path of Bill's walk. As the diagram on the right illustrates, he could also have walked from A to B by first walking 1 mile south then $\frac{3}{4}$ mile east.





By the Pythagorean Theorem,

$$(AB)^2 = 1^2 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16},$$

so $AB = \frac{5}{4} = 1\frac{1}{4}$.

8. **(E)**

To check the possible answers, choose the easiest odd numbers for m and n. If $m = n = 1$, then

$$m + 3n = 4, \quad 3m - n = 2, \quad 3m^2 + 3n^2 = 6, \quad (mn + 3)^2 = 16 \text{ and } 3mn = 3.$$

This shows that (A), (B), (C) and (D) can be even when m and n are odd. On the other hand, because the product of odd integers is always odd, $3mn$ is always odd if m and n are odd.

Questions: Which of the expressions are <u>always</u> even if m and n are odd? What are the possibilities if m and n are both even? If one is even and the other odd?

9. **(D)** Triangle ACD is an isosceles triangle with a $60°$ angle, so it is also equilateral. Therefore, the length of $\overline{AC}$ is 17.

10. **(D)** Covering the same distance three times as fast takes one-third the time. So Joe ran for 2 minutes. His total time was $6 + 2 = 8$ minutes.

11. **(C)** To add 6% sales tax to an item, multiply the price by 1.06. To calculate a 20% discount, multiply the price by 0.8. Because both actions require only multiplication, and because multiplication is commutative, the order of operations doesn't matter. Jack and Jill will get the same total.

Note: Jack's final computation is $0.80(1.06 \times \$90.00)$ and Jill's is $1.06(0.80 \times \$90.00)$. Both yield the same product, \$76.32.

12. **(D)** You can solve this problem by guessing and checking. If Big Al had eaten 10 bananas on May 1, then he would have eaten $10 + 16 + 22 + 28 + 34 = 110$ bananas. This is 10 bananas too many, so he actually ate 2 fewer bananas each day. Thus, Big Al ate 8 bananas on May 1 and 32 bananas on May 5.

OR

The average number of bananas eaten per day was $\frac{100}{5} = 20$. Because the number of bananas eaten on consecutive days differs by 6 and there are an odd number of days, the average is also the median. Therefore, the average number of bananas he ate per day, 20, is equal to the number of bananas he ate on May 3. So on May 5 Big Al ate $20 + 12 = 32$ bananas.

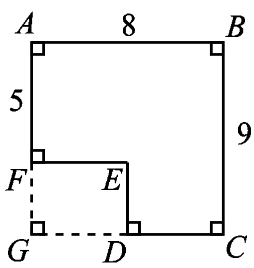
OR

Let x be the number of bananas that Big Al ate on May 5. The following chart documents his banana intake for the five days.

May 5	May 4	May 3	May 2	May 1
x	$x - 6$	$x - 12$	$x - 18$	$x - 24$

The total number of bananas Big Al ate was $5x - 60$, which must be 100. So Big Al ate $x = \frac{160}{5} = 32$ bananas on May 5.

13. **(C)**



Rectangle $ABCG$ has area $8 \times 9 = 72$, so rectangle $FEDG$ has area $72 - 52 = 20$. The length of $\overline{FG}$ equals $DE = 9 - 5 = 4$, so the length of $\overline{EF}$ is $\frac{20}{4} = 5$. Therefore, $DE + EF = 4 + 5 = 9$.

14. **(B)** Each team plays 10 games in its own division and 6 games against teams in the other division. So each of the 12 teams plays 16 conference games. Because each game involves two teams, there are $\frac{12 \times 16}{2} = 96$ games scheduled.

15. **(C)** Because the perimeter of such a triangle is 23, and the sum of the two equal side lengths is even, the length of the base is odd. Also, the length of the base is less than the sum of the other two side lengths, so it is less than half of 23. Thus the six possible triangles have side lengths 1, 11, 11; 3, 10, 10; 5, 9, 9; 7, 8, 8; 9, 7, 7 and 11, 6, 6.

16. **(D)** It is possible for the Martian to pull out at most 4 red, 4 white and 4 blue socks without having a matched set. The next sock it pulls out must be red, white or blue, which gives a matched set. So the Martian must select $4 \times 3 + 1 = 13$ socks to be guaranteed a matched set of five socks.

17. **(E)** Evelyn covered more distance in less time than Briana, Debra and Angela, so her average speed is greater than any of their average speeds. Evelyn went almost as far as Carla in less than half the time that it took Carla, so Evelyn's average speed is also greater than Carla's.

OR

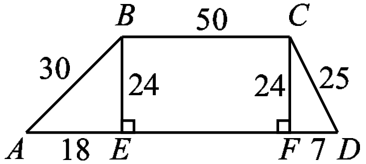
The ratio of distance to time, or average speed, is indicated by the slope of the line from the origin to each runner's point in the graph. Therefore, the line from the origin with the greatest slope will correspond to the runner with the greatest average speed. Because Evelyn's line has the greatest slope, she has the greatest average speed.

18. **(C)** The smallest three-digit number divisible by 13 is $13 \times 8 = 104$, so there are seven two-digit multiples of 13. The greatest three-digit number divisible by 13 is $13 \times 76 = 988$. Therefore, there are $76 - 7 = 69$ three-digit numbers divisible by 13.

OR

Because the integer part of $\frac{999}{13}$ is 76, there are 76 multiples of 13 less than or equal to 999. Because the integer part of $\frac{99}{13}$ is 7, there are 7 multiples of 13 less than or equal to 99. That means there are $76 - 7 = 69$ multiples of 13 between 100 and 999.

19. **(A)**



By the Pythagorean Theorem, $AE = \sqrt{30^2 - 24^2} = \sqrt{324} = 18$. (Or note that triangle AEB is similar to a 3-4-5 right triangle, so $AE = 3 \times 6 = 18$.)

Also $CF = 24$ and $FD = \sqrt{25^2 - 24^2} = \sqrt{49} = 7$. The perimeter of the trapezoid is $50 + 30 + 18 + 50 + 7 + 25 = 180$.

20. **(A)** Write the points where Alice and Bob will stop after each move and compare points.

Move	0	1	2	3	4	5	6
Alice:	12	5	10	3	8	1	6
Bob:	12	3	6	9	12	3	6

So Alice and Bob will be together again after six moves.

OR

If Bob does not move and Alice moves $9 + 5 = 14$ points or 2 points each time, they will still be in the same relative position from each other after each turn. If Bob does not move, they will be on the same point when Alice first stops on point 12, where she started. So Alice will have to move 2 steps 6 times to stop at her starting point.

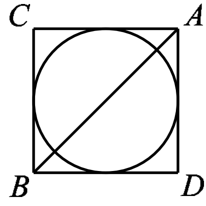
21. **(C)** To make a triangle, select as vertices two dots from one row and one from the other row. To select two dots in the top row, decide which dot is not used. This can be done in three ways. There are also three ways to choose one dot to use from the bottom row. So there are $3 \times 3 = 9$ triangles with two vertices in the top row and one in the bottom. Similarly, there are nine triangles with one vertex in the top row and two in the bottom. This gives a total of $9 + 9 = 18$ triangles.
Note: Can you find the four noncongruent triangles?

22. **(E)** Neither the units of size nor the cost are important in this problem. So for convenience, suppose the small size costs \$1 and weighs 10 ounces. To determine the relative value, we compare the cost per unit weight.

$$\text{S}: \frac{\$1.00}{10} = 10\,¢ \text{ per oz.}$$
$$\text{M}: \frac{\$1.50}{0.8 \times 20} = 9.375\,¢ \text{ per oz.}$$
$$\text{L}: \frac{1.3 \times \$1.50}{20} = 9.75\,¢ \text{ per oz.}$$

So the value, or buy, from best to worst is medium, large and small, that is MLS.

23. **(B)** Reflect the triangle and the semicircle across the hypotenuse $\overline{AB}$ to obtain a circle inscribed in a square. The circle has area 4π. The radius of a circle with area 4π is 2. The side length of the square is 4 and the area of the square is 16. So the area of the triangle is 8.



24. **(B)** One way to solve the problem is to work backward, either dividing by 2 if the number is even or subtracting 1 if the number is odd.

$$200/2 \to 100/2 \to 50/2 \to 25-1 \to 24/2 \to 12/2 \to 6/2 \to 3-1 \to 2/2 \to 1$$

So if you press [×2] [+1] [×2] [×2] [×2] [+1] [×2] [×2] [×2] or 9 keystrokes, you can reach "200" from "1."

To see that no sequence of eight keystrokes works, begin by noting that of the four possible sequences of two keystrokes, [×2] [×2] produces the maximum result. Furthermore, [+1] [×2] produces a result larger than either [×2] [+1] or [+1] [+1]. So the largest possible result of a sequence of eight keystrokes is "256," produced by either

[×2] [×2] [×2] [×2] [×2] [×2] [×2] [×2]

or

[+1] [×2] [×2] [×2] [×2] [×2] [×2] [×2].

The second largest result is "192," produced by

[×2] [+1] [×2] [×2] [×2] [×2] [×2] [×2].

Thus no sequence of eight keystrokes produces a result of "200."

25. **(A)** Because the circle and square share the same interior region and the area of the two exterior regions indicated are equal, the square and the circle must have equal area. The area of the square is 2^2 or 4. Because the area of both the circle and the square is 4, $4 = \pi r^2$. Solving for r, the radius of the circle, yields $r^2 = \frac{4}{\pi}$, so $r = \sqrt{\frac{4}{\pi}} = \frac{2}{\sqrt{\pi}}$.

Note: It is not necessary that the circle and square have the same center.

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American Mathematics Competitions

22nd Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 14, 2006

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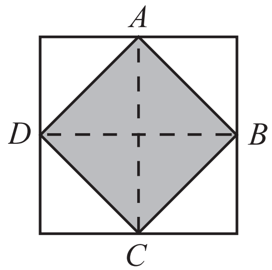
1. **(D)** Mindy's total was approximately $2 + 5 + 10 = \$17$.

2. **(C)** On the AMC 8 a student's score is the number of problems answered correctly. So Billy's score is 13. Because there is no penalty for guessing, if he wants to increase his score, he probably should fill in the last five answers.

3. **(A)** When Elisa started, she completed a lap in $\frac{25}{10} = 2.5$ minutes. Now she can complete a lap in $\frac{24}{12} = 2$ minutes. She has improved her lap time by $2.5-2 = 0.5$ or $\frac{1}{2}$ minute.

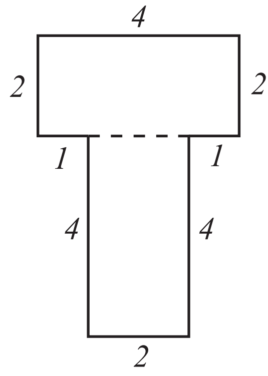
4. **(B)** Ignore the number of complete revolutions because they do not affect direction. One-fourth of the distance around the circle clockwise from west is north. Three-fourths of the distance counterclockwise around the circle from north is east. Chenille's spinner points east.

5. **(D)** Divide the larger square into 8 congruent triangles, as shown, 4 of which make up the smaller square.



The area of the smaller square is $\frac{4}{8}$ or $\frac{1}{2}$ of the area of the larger square, so the area of the smaller square is equal to 30.

6. **(C)**



The perimeter is $4 + 2 + 1 + 4 + 2 + 4 + 1 + 2 = 20$ inches.

OR

Each rectangle has perimeter $= 2l + 2w = 2(4) + 2(2) = 8 + 4 = 12$ inches. When the two rectangles are positioned to form the T, a two-inch segment of each rectangle is inside the T and is not on the perimeter of the T. So the perimeter of the T is $2(12) - 2(2) = 24 - 4 = 20$ inches.

7. **(B)** Because circumference $C = 2\pi r$ and circle Y has circumference 8π, its radius is $\frac{8\pi}{2\pi} = 4$. Because area $A = \pi r^2$ and circle Z has area 9π, its radius is $\sqrt{9} = 3$. Ordering the radii gives $3 < \pi < 4$, so the circles in ascending order of radii length are Z, X and Y.

8. **(E)** Because $200 - 96 = 104$ of those surveyed were male, $104 - 26 = 78$ of those surveyed are male listeners.

	Listen	Don't Listen	Total
Male	78	26	104
Female	58	38	96
Total	136	64	200

The percentage of males surveyed who listen to KAMC is $\frac{78}{104} \times 100\% = 75\%$.

9. **(C)** Note that in each fraction, the numerator is the same as the denominator in the next fraction, so they divide. The product of $\frac{\cancel{3}}{2} \times \frac{\cancel{4}}{\cancel{3}} \times \frac{\cancel{5}}{\cancel{4}} \times \ldots \times \frac{2006}{\cancel{2005}} = \frac{2006}{2} = 1003$.

10. **(A)** When the area of a rectangle is 12 square units and the sides are integers, the factors of 12 are the possible lengths of the sides. In point form, the side lengths could be $(1, 12), (2, 6), (3, 4), (4, 3), (6, 2)$ and $(12, 1)$. Only graph A fits these points.

11. **(C)** The sum of the digits of a two-digit number is at most $9 + 9 = 18$. This means the only possible perfect square sums are $1, 4, 9$ and 16. Each square has the following two-digit possibilities:
$1 : 10$
$4 : 40, 31, 22, 13$
$9 : 90, 81, 72, 63, 54, 45, 36, 27, 18$
$16 : 97, 88, 79$
There are 17 two-digit numbers in all.

12. **(D)** Note that 70% of 10 is 7, 80% of 20 is 16 and 90% of 30 is 27. Antonette answers $7 + 16 + 27 = 50$ problems correctly out of 60 problems in all. Her overall score is $\frac{50}{60}$ or $83.\overline{3}\%$.

13. **(D)** Between 8:30 and 9:00 AM Cassie travels 6 miles. At 9:00 Cassie and Brian are only 56 miles apart. After 9:00, because they are both biking towards each other, the distance between them decreases at the rate of $12 + 16 = 28$ miles per hour. At that rate, it will take them $\frac{56}{28} = 2$ hours to meet. So they will meet at 11:00 AM.

14. **(B)** Bob takes $45 - 30 = 15$ more seconds per page than Chandra. So the difference in their total reading times is $760 \cdot 15 =$ 11,400 seconds. Bob will spend 11,400 more seconds reading than Chandra.

15. **(C)** The ratio of time it takes Bob to read a page to the time it takes Chandra to read a page is 45:30 or 3:2, so Bob should read $\frac{2}{3}$ of the number of pages that Chandra reads. Divide the book into 5 parts, each with $\frac{760}{5} = 152$ pages. Chandra will read the first $3 \cdot 152 = 456$ pages, while Bob reads the last $2 \cdot 152 = 304$ pages.

OR

If Chandra reads x pages, she will read for $30x$ seconds. Bob has to read $760-x$ pages, and this takes him $45(760-x)$ seconds. Because Chandra and Bob read the same amount of time, $30x = 45(760-x)$.
Solving for x,

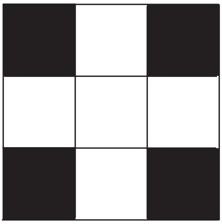
$$\begin{aligned} 30x &= 45 \cdot 760 - 45x, \\ 75x &= 45 \cdot 760, \\ x &= \tfrac{45 \cdot 760}{75} = 456. \end{aligned}$$

So Chandra will read the first 456 pages.

16. **(E)** The least common multiple of 20, 45 and 30 is $2^2 \cdot 3^2 \cdot 5 = 180$. Using the LCM, in 180 seconds Alice reads $\frac{180}{20} = 9$ pages, Chandra reads $\frac{180}{30} = 6$ pages and Bob reads $\frac{180}{45} = 4$ pages. Together they read a total of 19 pages in 180 seconds. The total number of seconds each reads is $\frac{760}{19} \cdot 180 = 7200$.

17. **(B)** Because the sum of a number from spinner Q and a number from spinner R is always odd, the sum of the numbers on the three spinners will be odd exactly when the number from spinner P is even. Because 2 is the only even number on spinner P, the probability of getting an odd sum is $\frac{1}{3}$.

18. **(D)** Four black and five white squares are visible on each of the six faces of the cube. So $\frac{5}{9}$ of the surface will be white.



19. **(D)** Because triangles ABD and ECD are congruent and triangle ABC is isosceles, $EC = AB = BC = 11$. That means $BD = \frac{11}{2}$ or 5.5.

20. **(C)** Each of the six players played 5 games, and each game involved two players. So there were $\frac{6 \cdot 5}{2} = 15$ games. Helen, Ines, Janet, Kendra and Lara won a total of $4 + 3 + 2 + 2 + 2 = 13$ games, so Monica won $15 - 13 = 2$ games.

21. **(A)** Using the volume formula $lwh = V$, the volume of water in the aquarium is $100 \times 40 \times 37 = 148,000\,\text{cm}^3$. When the rock is put in, the water and the rock will occupy a box-shaped region with volume $148,000 + 1000 = 149,000\,\text{cm}^3$. The volume of the water and the rock is $100 \times 40 \times h$, where h is the new height of the water. The new volume $= 4000h = 149,000\,\text{cm}^3$, so the new height is
$$h = \frac{149000}{4000} = 37.25\,\text{cm}.$$
After adding the rock, the water rises $37.25 - 37 = 0.25\,\text{cm}$.

OR

Because the shape of the rock is irrelevant, we may assume that the rock is shaped like a rectangular box with base measuring $100\,\text{cm} \times 40\,\text{cm}$ and height h cm. Using the volume formula, $100 \times 40 \times h = 1000$, so $h = \frac{1000}{100 \times 40} = 0.25\,\text{cm}$. When the rock is put into the aquarium, the water level will rise by $0.25\,\text{cm}$.

22. **(D)** If the lower cells contain A, B and C, then the second row will contain $A + B$ and $B + C$, and the top cell will contain $A + 2B + C$. To obtain the smallest sum, place 1 in the center cell and 2 and 3 in the outer ones. The top number will be 7. For the largest sum, place 9 in the center cell and 7 and 8 in the outer ones. This top number will be 33. The difference is $33 - 7 = 26$.

23. **(A)** The counting numbers that leave a remainder of 4 when divided by 6 are $4, 10, 16, 22, 28, 34, \ldots$. The counting numbers that leave a remainder of 3 when divided by 5 are $3, 8, 13, 18, 23, 28, 33, \ldots$. So 28 is the smallest possible number of coins that meets both conditions. Because $4 \times 7 = 28$, there are no coins left when they are divided among seven people.

OR

If there were two more coins in the box, the number of coins would be divisible by both 6 and 5. The smallest number that is divisible by 6 and 5 is 30, so the smallest possible number of coins in the box is 28.

24. **(A)** We can decompose $CDCD$ into $CD \times 100 + CD = CD(101)$. That means that $A = 1$ and $B = 0$. The sum is $1 + 0 = 1$.

25. **(B)** There are one odd and two even numbers showing. Because all primes other than 2 are odd and the sum of an even number and an odd number is odd, the common sum must be odd. That means 2 must be opposite 59 and the common sum is $2 + 59 = 61$. The other two hidden numbers are $61 - 44 = 17$ and $61 - 38 = 23$. The average of 2, 17 and 23 is $\frac{2+17+23}{3} = \frac{42}{3} = 14$.

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23rd Annual

AMC 8

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1. **(D)** The first 5 weeks Theresa works a total of $8+11+7+12+10=48$ hours. She has promised to work $6\times 10=60$ hours. She must work $60-48=12$ hours during the final week.

2. **(E)** The ratio of the number of students who preferred spaghetti to the number of students who preferred manicotti is $\dfrac{250}{100}=\dfrac{5}{2}$.

3. **(C)** The prime factorization of 250 is $2\cdot 5\cdot 5\cdot 5$. The sum of 2 and 5 is 7.

4. **(D)** Georgie has 6 choices for the window in which to enter. After entering, Georgie has 5 choices for the window from which to exit. So altogether there are $6\times 5=30$ different ways for Georgie to enter one window and exit another.

5. **(B)** For his birthday, Chandler gets $50+35+15=100$ dollars. Therefore, he needs $500-100=400$ dollars more. It will take Chandler $400\div 16=25$ weeks to earn 400 dollars, so he can buy his bike after 25 weeks.

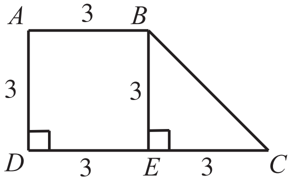
6. **(E)** The difference in the cost of a long-distance call per minute from 1985 to 2005 was $41-7=34$ cents. The percent decrease is $100\times\dfrac{34}{41}\approx 100\times\dfrac{32}{40}=100\times\dfrac{8}{10}=80\%$.

7. **(D)** Originally the sum of the ages of the people in the room is $5\times 30=150$. After the 18-year-old leaves, the sum of the ages of the remaining people is $150-18=132$. So the average age of the four remaining people is $\frac{132}{4}=33$ years.

   OR

   The 18-year-old is 12 years younger than 30, so the four remaining people are an average of $\frac{12}{4}=3$ years older than 30.

8. **(B)** Note that $ABED$ is a square with side 3. Subtract DE from DC, to find that $\overline{EC}$, the base of $\triangle BEC$, has length 3. The area of $\triangle BEC$ is $\dfrac{1}{2}\cdot 3\cdot 3=\dfrac{9}{2}=4.5$.



   OR

   The area of the $\triangle BEC$ is the area of the trapezoid $ABCD$ minus the area of the square $ABED$. The area of $\triangle BEC$ is $\frac{1}{2}(3+6)3-3^2=13.5-9=4.5$.

9. **(B)** The number in the last column of the second row must be 1 because there are already a 2 and a 3 in the second row and a 4 in the last column. By similar reasoning, the number above the 1 must be 3. So the number in the lower right-hand square must be 2. This is not the only way to find the solution.

1		2	3
2	3		1
			4
			2

The completed square is

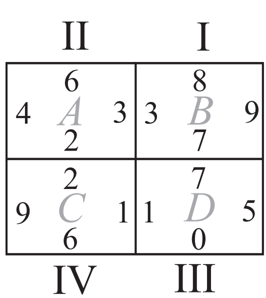
1	4	2	3
2	3	4	1
3	2	1	4
4	1	3	2

10. **(D)** First calculate $\boxed{11} = 1 + 11 = 12$. So

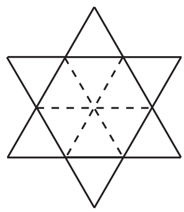
$$\boxed{\boxed{11}} = \boxed{12} = 1 + 2 + 3 + 4 + 6 + 12 = 28$$

.

11. **(D)** Because Tile III has a 0 on the bottom edge and there is no 0 on any other tile, Tile III must be placed on C or D. Because Tile III has a 5 on the right edge and there is no 5 on any other tile, Tile III must be placed on the right, on D. Because Tile III has a 1 on the left edge and only Tile IV has a 1 on the right edge, Tile IV must be placed to the left of Tile III, that is, on C .



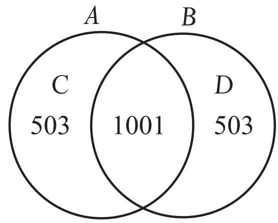
12. **(A)** Use diagonals to cut the hexagon into 6 congruent triangles. Because each exterior triangle is also equilateral and shares an edge with an internal triangle, each exterior triangle is congruent to each interior triangle. Therefore, the ratio of the area of the extensions to the area of the hexagon is 1:1.



13. **(C)** Let C denote the set of elements that are in A but not in B. Let D denote the set of elements that are in B but not in A. Because sets A and B have the same number of elements, the number of elements in C is the same as the number of elements in D. This number is half the number of elements in the union of A and B minus the intersection of A and B. That is, the number of elements in each of C and D is

$$\frac{1}{2}(2007 - 1001) = \frac{1}{2} \cdot 1006 = 503.$$

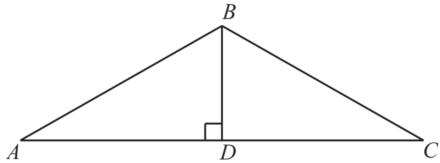
Adding the number of elements in A and B to the number in A but not in B gives $1001 + 503 = 1504$ elements in A.



OR

Let x be the number of elements each in A and B. Then $2x - 1001 = 2007$, $2x = 3008$ and $x = 1504$.

14. **(C)** Let $\overline{BD}$ be the altitude from B to $\overline{AC}$ in $\triangle ABC$.



Then $60 =$ the area of $\triangle ABC = \frac{1}{2} \cdot 24 \cdot BD$, so $BD = 5$. Because $\triangle ABC$ is isosceles, $\triangle ABD$ and $\triangle CBD$ are congruent right triangles. This means that $AD = DC = \dfrac{24}{2} = 12$. Applying the Pythagorean Theorem to $\triangle ABD$ gives

$$AB^2 = 5^2 + 12^2 = 169 = 13^2, \text{ so } AB = 13.$$

15. **(A)** Because $b < c$ and $0 < a$, adding corresponding sides of the inequalities gives $b < a + c$, so (A) is impossible. To see that the other choices are possible, consider the following choices for a, b, and c:

(B) and (C): $a = 1$, $b = 2$, and $c = 4$;

(D): $a = \dfrac{1}{3}$, $b = 1$, and $c = 2$;

(E): $a = \dfrac{1}{2}$, $b = 1$, and $c = 2$.

16. **(A)** The circumferences of circles with radii 1 through 5 are 2π, 4π, 6π, 8π and 10π, respectively. Their areas are, respectively, $\pi, 4\pi$, 9π, 16π and 25π. The points $(2\pi, \pi)$, $(4\pi, 4\pi)$, $(6\pi, 9\pi)$, $(8\pi, 16\pi)$ and $(10\pi, 25\pi)$ are graphed in **(A)**. It is the only graph of an increasing quadratic function, called a parabola.

17. **(C)** There are $0.30(30) = 9$ liters of yellow tint in the original 30-liter mixture. After adding 5 liters of yellow tint, 14 of the 35 liters of the new mixture are yellow tint. The percent of yellow tint in the new mixture is $100 \times \dfrac{14}{35} = 100 \times \dfrac{2}{5}$ or 40%.

18. **(D)** To find A and B, it is sufficient to consider only $303 \cdot 505$, because 0 is in the thousands place in both factors.

$$\begin{array}{rr} & \cdots 303 \\ \times & \cdots 505 \\ \hline & \cdots 1515 \\ & \cdots 1500 \\ \hline & \cdots 3015 \end{array}$$

So $A = 3$ and $B = 5$, and the sum is $A + B = 3 + 5 = 8$.

19. **(C)** One of the squares of two consecutive integers is odd and the other is even, so their difference must be odd. This eliminates A, B and D. The largest consecutive integers that have a sum less than 100 are 49 and 50, whose squares are 2401 and 2500, with a difference of 99. Because the difference of the squares of consecutive positive integers increases as the integers increase, the difference cannot be 131. The difference between the squares of 40 and 39 is 79.

OR

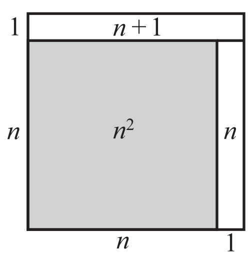
Let the consecutive integers be n and $n + 1$, with $n \leq 49$. Then

$$(n + 1)^2 - n^2 = (n^2 + 2n + 1) - n^2 = 2n + 1 = n + (n + 1).$$

That means the difference of the squares is an odd number. Therefore, the difference is an odd number less than or equal to $49 + (49 + 1) = 99$, and choice C is the only possible answer. The sum of $n = 39$ and $n + 1 = 40$ is 79.

Note: The difference of the squares of any two consecutive integers is not only odd but also the sum of the two consecutive integers. Every positive odd integer greater than 1 and less than 100 could be the answer.

Seen in geometric terms, $(n + 1)^2 - n^2$ looks like



20. **(A)** Because 45% is the same as the simplified fraction $\frac{9}{20}$, the Unicorns won 9 games for each 20 games they played. This means that the Unicorns must have played some multiple of 20 games before district play. The table shows the possibilities that satisfy the conditions in the problem.

Before District Play			**After District Play**		
Games Played	Games Won	Games Lost	Games Played	Games Won	Games Lost
20	9	11	28	15	13
40	**18**	**22**	**48**	**24**	**24**
60	27	33	68	33	35
80	36	44	88	42	46
...	...	...	...	...	...

Only when the Unicorns played 40 games before district play do they finish winning half of their games. So the Unicorns played $24 + 24 = 48$ games.

OR

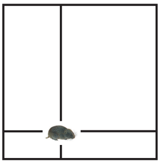
Let n be the number of Unicorn games before district play. Then $0.45n + 6 = 0.5(n + 8)$. Solving for n yields

$$\begin{aligned} 0.45n + 6 &= 0.5n + 4, \\ 2 &= 0.05n, \\ 40 &= n. \end{aligned}$$

So the total number of games is $40 + 8 = 48$.

21. **(D)** After the first card is dealt, there are seven left. The three cards with the same color as the initial card are winners and so is the card with the same letter but a different color. That means four of the remaining seven cards form winning pairs with the first card, so the probability of winning is $\frac{4}{7}$.

22. **(C)** Wherever the lemming is inside the square, the sum of the distances to the two horizontal sides is 10 meters and the sum of the distances to the two vertical sides is 10 meters. Therefore the sum of all four distances is 20 meters, and the average of the four distances is $\dfrac{20}{4} = 5$ meters.



23. **(B)** Find the area of the unshaded portion of the 5×5 grid, then subtract the unshaded area from the total area of the grid. The unshaded triangle in the middle of the top of the 5×5 grid has a base of 3 and an altitude of $\dfrac{5}{2}$. The four unshaded triangles have a total area of $4 \times \dfrac{1}{2} \times 3 \times \dfrac{5}{2} = 15$ square units. The four corner squares are also unshaded, so the shaded pinwheel has an area of $25 - 15 - 4 = 6$ square units.

24. **(C)** A number is a multiple of three when the sum of its digits is a multiple of 3. If the number has three distinct digits drawn from the set $\{1, 2, 3, 4\}$, then the sum of the digits will be a multiple of three when the digits are $\{1, 2, 3\}$ or $\{2, 3, 4\}$. That means the number formed is a multiple of three when, after the

three draws, the number remaining in the bag is 1 or 4. The probability of this occurring is $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

25. **(B)** The outer circle has area 36π and the inner circle has area 9π, making the area of the outer ring $36\pi - 9\pi = 27\pi$. So each region in the outer ring has area $\frac{27\pi}{3} = 9\pi$, and each region in the inner circle has area $\frac{9\pi}{3} = 3\pi$. The probability of hitting a given region in the inner circle is $\frac{3\pi}{36\pi} = \frac{1}{12}$, and the probability of hitting a given region in the outer ring is $\frac{9\pi}{36\pi} = \frac{1}{4}$. For the score to be odd, one of the numbers must be 1 and the other number must be 2. The probability of hitting a 1 is

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{12} = \frac{7}{12},$$

and the probability of hitting a 2 is

$$1 - \frac{7}{12} = \frac{5}{12}.$$

Therefore, the probability of hitting a 1 and a 2 in either order is

$$\frac{7}{12} \cdot \frac{5}{12} + \frac{5}{12} \cdot \frac{7}{12} = \frac{70}{144} = \frac{35}{72}.$$

The Mathematical Association of America
American Mathematics Competitions

24th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 18, 2008

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We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, e-mail, World Wide Web or media of any type is a violation of the competition rules.*

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1. **Answer (B):** Susan spent $2 \times 12 = \$24$ on rides, so she had $50 - 12 - 24 = \$14$ to spend.

2. **Answer (A):** Because the key to the code starts with zero, all the letters represent numbers that are one less than their position. Using the key, C is $9 - 1 = 8$, and similarly L is 6, U is 7, and E is 1.

BEST	OF	LUCK
0 1 2 3	4 5	6 7 8 9

   CLUE = 8671

3. **Answer (A):** A week before the 13^{th} is the 6^{th}, which is the first Friday of the month. Counting back from that, the 5^{th} is a Thursday, the 4^{th} is a Wednesday, the 3^{rd} is a Tuesday, the 2^{nd} is a Monday, and the 1^{st} is a Sunday.

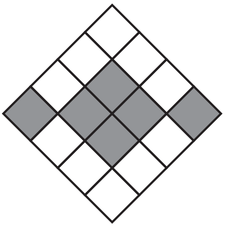
   OR

   Counting forward by sevens, February 1 occurs on the same day of the week as February 8 and February 15. Because February 13 is a Friday, February 15 is a Sunday, and so is February 1.

4. **Answer (C):** The area of the outer triangle with the inner triangle removed is $16 - 1 = 15$, the total area of the three congruent trapezoids. Each trapezoid has area $\frac{15}{3} = 5$.

5. **Answer (E):** Barney rides $1661 - 1441 = 220$ miles in 10 hours, so his average speed is $\frac{220}{10} = 22$ miles per hour.

6. **Answer (D):** After subdividing the central gray square as shown, 6 of the 16 congruent squares are gray and 10 are white. Therefore, the ratio of the area of the gray squares to the area of the white squares is $6 : 10$ or $3 : 5$.



7. **Answer (E):** Note that $\frac{M}{45} = \frac{3}{5} = \frac{3 \cdot 9}{5 \cdot 9} = \frac{27}{45}$, so $M = 27$. Similarly, $\frac{60}{N} = \frac{3}{5} = \frac{3 \cdot 20}{5 \cdot 20} = \frac{60}{100}$, so $N = 100$. The sum $M + N = 27 + 100 = 127$.

OR

Note that $\frac{M}{45} = \frac{3}{5}$, so $M = \frac{3}{5} \cdot 45 = 27$. Also $\frac{60}{N} = \frac{3}{5}$, so $\frac{N}{60} = \frac{5}{3}$, and $N = \frac{5}{3} \cdot 60 = 100$. The sum $M + N = 27 + 100 = 127$.

8. **Answer (D):** The sales in the 4 months were \$100, \$60, \$40 and \$120. The average sales were $\dfrac{100 + 60 + 40 + 120}{4} = \dfrac{320}{4} = \$80$.

OR

In terms of the \$20 intervals, the sales were 5, 3, 2 and 6 on the chart. Their sum is $5 + 3 + 2 + 6 = 16$ and the average is $\frac{16}{4} = 4$. The average sales were $4 \cdot \$20 = \$80$.

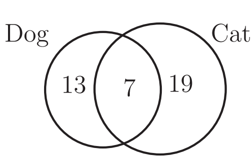
9. **Answer (D):** At the end of the first year, Tammy's investment was 85% of the original amount, or \$85. At the end of the second year, she had 120% of her first year's final amount, or 120% of $\$85 = 1.2(\$85) = \$102$. Over the two-year period, Tammy's investment changed from \$100 to \$102, so she gained 2%.

10. **Answer (D):** The sum of the ages of the 6 people in Room A is $6 \times 40 = 240$. The sum of the ages of the 4 people in Room B is $4 \times 25 = 100$. The sum of the ages of the 10 people in the combined group is $100 + 240 = 340$, so the average age of all the people is $\frac{340}{10} = 34$.

11. **Answer (A):** The number of cat owners plus the number of dog owners is $20 + 26 = 46$. Because there are only 39 students in the class, there are $46 - 39 = 7$ students who have both.

OR

Because each student has at least a cat or a dog, there are $39 - 20 = 19$ students with a cat but no dog, and $39 - 26 = 13$ students with a dog but no cat. So there are $39 - 13 - 19 = 7$ students with both a cat and a dog.



12. **Answer (C):** The table gives the height of each bounce.

Bounce	1	2	3	4	5
Height in Meters	2	$\frac{2}{3} \cdot 2 = \frac{4}{3}$	$\frac{2}{3} \cdot \frac{4}{3} = \frac{8}{9}$	$\frac{2}{3} \cdot \frac{8}{9} = \frac{16}{27}$	$\frac{2}{3} \cdot \frac{16}{27} = \frac{32}{81}$

Because $\frac{16}{27} > \frac{16}{32} = \frac{1}{2}$ and $\frac{32}{81} < \frac{32}{64} = \frac{1}{2}$, the ball first rises to less than 0.5 meters on the fifth bounce.
Note: Because all the fractions have odd denominators, it is easier to double the numerators than to halve the denominators. So compare $\frac{16}{27}$ and $\frac{32}{81}$ to their numerators' fractional equivalents of $\frac{1}{2}$, $\frac{16}{32}$ and $\frac{32}{64}$.

13. **Answer (C):** Because each box is weighed two times, once with each of the other two boxes, the total $122 + 125 + 127 = 374$ pounds is twice the combined weight of the three boxes. The combined weight is $\frac{374}{2} = 187$ pounds.

14. **Answer (C):** There are only two possible spaces for the B in row 1 and only two possible spaces for the A in row 2. Once these are placed, the entries in the remaining spaces are determined.

The four arrangements are:

A	B	C
B	C	A
C	A	B

A	B	C
C	A	B
B	C	A

A	C	B
C	B	A
B	A	C

A	C	B
B	A	C
C	B	A

OR

The As can be placed either

A		
	A	
		A

or

A		
		A
	A	

In each case, the letter next to the top A can be B or C. At that point the rest of the grid is completely determined. So there are $2+2 = 4$ possible arrangements.

15. **Answer (B):** The sum of the points Theresa scored in the first 8 games is 37. After the ninth game, her point total must be a multiple of 9 between 37 and $37 + 9 = 46$, inclusive. The only such point total is $45 = 37 + 8$, so in the ninth game she scored 8 points. Similarly, the next point total must be a multiple of 10 between 45 and $45 + 9 = 54$. The only such point total is $50 = 45 + 5$, so in the tenth game she scored 5 points. The product of the number of points scored in Theresa's ninth and tenth games is $8 \cdot 5 = 40$.

16. **Answer (D):** The volume is $7 \times 1 = 7$ cubic units. Six of the cubes have 5 square faces exposed. The middle cube has no face exposed. So the total surface area of the figure is $5 \times 6 = 30$ square units. The ratio of the volume to the surface area is $7 : 30$.

OR

The volume is $7 \times 1 = 7$ cubic units. There are five unit squares facing each of six directions: front, back, top, bottom, left and right, for a total of 30 square units of surface area. The ratio of the volume to the surface area is $7 : 30$.

17. **Answer (D):** The formula for the perimeter of a rectangle is $2l + 2w$, so $2l + 2w = 50$, and $l + w = 25$. Make a chart of the possible widths, lengths, and areas, assuming all the widths are shorter than all the lengths.

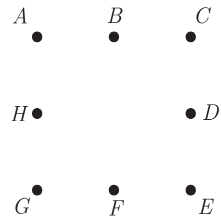
Width	1	2	3	4	5	6	7	8	9	10	11	12
Length	24	23	22	21	20	19	18	17	16	15	14	13
Area	24	46	66	84	100	114	126	136	144	150	154	156

The largest possible area is $13 \times 12 = 156$ and the smallest is $1 \times 24 = 24$, for a difference of $156 - 24 = 132$ square units.
Note: The product of two numbers with a fixed sum increases as the numbers get closer together. That means, given the same perimeter, the square has a larger area than any rectangle, and a rectangle with a shape closest to a square will have a larger area than other rectangles with equal perimeters.

18. **Answer (E):** The length of first leg of the aardvark's trip is $\frac{1}{4}(2\pi \times 20) = 10\pi$ meters. The third and fifth legs are each $\frac{1}{4}(2\pi \times 10) = 5\pi$ meters long. The second and sixth legs are each 10 meters long, and the length of the fourth leg is 20 meters. The length of the total trip is $10\pi + 5\pi + 5\pi + 10 + 10 + 20 = 20\pi + 40$ meters.

19. **Answer (B):** Choose two points. Any of the 8 points can be the first choice, and any of the 7 other points can be the second choice. So there are $8 \times 7 = 56$

ways of choosing the points in order. But each pair of points is counted twice, so there are $\frac{56}{2} = 28$ possible pairs.



Label the eight points as shown. Only segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$, $\overline{GH}$ and $\overline{HA}$ are 1 unit long. So 8 of the 28 possible segments are 1 unit long, and the probability that the points are one unit apart is $\frac{8}{28} = \frac{2}{7}$.

OR

Pick the two points, one at a time. No matter how the first point is chosen, exactly 2 of the remaining 7 points are 1 unit from this point. So the probability of the second point being 1 unit from the first is $\frac{2}{7}$.

20. **Answer (B):** Because $\frac{2}{3}$ of the boys passed, the number of boys in the class is a multiple of 3. Because $\frac{3}{4}$ of the girls passed, the number of girls in the class is a multiple of 4. Set up a chart and compare the number of boys who passed with the number of girls who passed to find when they are equal.

Total boys	Boys passed
3	2
6	4
9	6

Total girls	Girls passed
4	3
8	6

The first time the number of boys who passed equals the number of girls who passed is when they are both 6. The minimum possible number of students is $9 + 8 = 17$.

OR

Because $\frac{2}{3}$ of the boys passed, the number of boys who passed must be a multiple of 2. Because $\frac{3}{4}$ of the girls passed, the number of girls who passed must be a multiple of 3. Because the same number of boys and girls passed, the smallest possible number is 6, the least common multiple of 2 and 3. If 6 of 9 boys and 6 of 8 girls passed, there are 17 students in the class, and that is the minimum number possible.

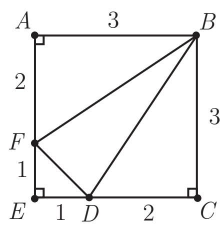
OR

Let $G =$ the number of girls and $B =$ the number of boys. Then $\frac{2}{3}B = \frac{3}{4}G$, so $8B = 9G$. Because 8 and 9 are relatively prime, the minimum number of boys and girls is 9 boys and 8 girls, for a total of $9 + 8 = 17$ students.

21. **Answer (C):** Using the formula for the volume of a cylinder, the bologna has volume $\pi r^2 h = \pi \times 4^2 \times 6 = 96\pi$. The cut divides the bologna in half. The half-cylinder will have volume $\frac{96\pi}{2} = 48\pi \approx 151$ cm^3.
Note: The value of π is slightly greater than 3, so to estimate the volume multiply $48(3) = 144$ cm^3. The product is slightly less than and closer to answer C than any other answer.

22. **Answer (A):** Because $\frac{n}{3}$ is at least 100 and is an integer, n is at least 300 and is a multiple of 3. Because $3n$ is at most 999, n is at most 333. The possible values of n are 300, 303, 306, $\ldots$, $333 = 3 \cdot 100, 3 \cdot 101, 3 \cdot 102, \ldots, 3 \cdot 111$, so the number of possible values is $111 - 100 + 1 = 12$.

23. **Answer (C):** Because the answer is a ratio, it does not depend on the side length of the square. Let $AF = 2$ and $FE = 1$. That means square $ABCE$ has side length 3 and area $3^2 = 9$ square units. The area of $\triangle BAF$ is equal to the area of $\triangle BCD = \frac{1}{2} \cdot 3 \cdot 2 = 3$ square units. Triangle DEF is an isosceles right triangle with leg lengths $DE = FE = 1$. The area of $\triangle DEF$ is $\frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$ square units. The area of $\triangle BFD$ is equal to the area of the square minus the areas of the three right triangles: $9 - (3 + 3 + \frac{1}{2}) = \frac{5}{2}$. So the ratio of the area of $\triangle BFD$ to the area of square $ABCE$ is $\dfrac{\frac{5}{2}}{9} = \dfrac{5}{18}$.



24. **Answer (C):** There are $10 \times 6 = 60$ possible pairs. The squares less than 60 are 1, 4, 9, 16, 25, 36 and 49. The possible pairs with products equal to the given squares are $(1,1)$, $(2,2)$, $(1,4)$, $(4,1)$, $(3,3)$, $(9,1)$, $(4,4)$, $(8,2)$, $(5,5)$, $(6,6)$ and $(9,4)$. So the probability is $\frac{11}{60}$.

25. **Answer (A):**

Circle #	Radius	Area
1	2	4π
2	4	16π
3	6	36π
4	8	64π
5	10	100π
6	12	144π

The total black area is $4\pi + (36 - 16)\pi + (100 - 64)\pi = 60\pi$ in^2.
So the percent of the design that is black is $100 \times \frac{60\pi}{144\pi} = 100 \times \frac{5}{12}$ or about 42%.

The Mathematical Association of America

American Mathematics Competitions

25th Annual

AMC 8

(American Mathematics Contest 8)

Solutions Pamphlet

Tuesday, NOVEMBER 17, 2009

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American Mathematics Competitions

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1. **Answer (E):** Work backwards. Bridget had 7 apples before she gave Cassie 3 apples. These 7 apples were half of Bridget's 14 original apples.

OR

Let B = Bridget's original number of apples.

$$\frac{B}{2} - 3 = 4$$
$$\frac{B}{2} = 7$$
$$B = 14$$

So Bridget originally had 14 apples.

2. **Answer (D):** Let s = number of sedans. Set up a proportion: $\frac{4}{7} = \frac{28}{s} = \frac{4(7)}{7(7)} = \frac{28}{49}$. So the dealership expects to sell 49 sedans.

OR

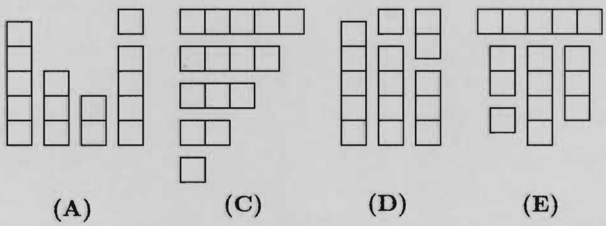
Because selling 4 sports cars corresponds to selling 7 sedans, 28 sports cars = 7(4 sports cars) corresponds to 7(7 sedans) = 49 sedans.

3. **Answer (C):** Suzanna rides at a constant rate of five minutes per mile. In 30 minutes there are six 5-minute intervals, so she travels six miles.

OR

Suzanna rides 3 miles in 15 minutes, so she will ride 6 miles in 30 minutes.

4. **Answer (B):** Figure B does not contain any 5-square-long piece. One solution is given for each of the other four figures. There are other solutions.



5. **Answer (D):** Make the list:

Position	1	2	3	4	5	6	7	8
Number	1	2	3	$1+2+3$ $=6$	$2+3+6$ $=11$	$3+6+11$ $=20$	$6+11+20$ $=37$	$11+20+37$ $=\mathbf{68}$

So the eighth number in the sequence is 68.

6. **Answer (A):** Together the hoses supply 10 gallons per minute to the pool. The pool holds 24,000 gallons, so it will take a total of $\dfrac{24{,}000 \text{ gallons}}{10 \text{ gallons/minute}} =$ 2400 minutes. Because 2400 minutes equals 40 hours, it takes 40 hours to fill Steve's pool.

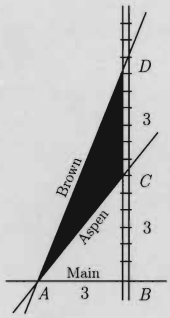
OR

The hoses supply (10 gallons/minute)(60 minutes/hour) = 600 gallons/hour. So it will take $\dfrac{24{,}000 \text{ gallons}}{600 \text{ gallons/hour}} = 40$ hours to fill Steve's pool.

7. **Answer (C):** The area of $\triangle ABC$ is $\frac{1}{2}(3)(3) = \frac{9}{2}$ square miles. The area of $\triangle ABD = \frac{1}{2}(3)(6) = 9$ square miles. The shaded area is the area of $\triangle ABD$ minus the area of $\triangle ABC$, which is $9 - \frac{9}{2} = \frac{9}{2} = 4.5$ square miles.

OR

The base $\overline{CD}$ of $\triangle ACD$ is 3 miles. The altitude $\overline{AB}$ of $\triangle ACD$ is 3 miles. The area of $\triangle ACD$ is $\frac{1}{2} \cdot 3 \cdot 3 = \frac{9}{2} = 4.5$ squares miles.



8. **Answer (B):** A rectangle with length L and width W has area LW. The new rectangle has area $(1.1)L \times (0.9)W = 0.99LW$. The new area $0.99LW$ is 99% of the old area.

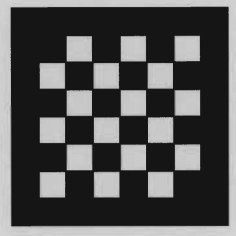
9. **Answer (B):** One side of the triangle and one side of the octagon will each touch one other polygon. Two sides of the other polygons will touch other polygons. Make a table and add the appropriate number of sides.

Number of sides	3	4	5	6	7	8
Number of sides that touch other polygons	1	2	2	2	2	1
Number of sides that don't	2	2	3	4	5	7

The resulting polygon has $2+2+3+4+5+7 = 23$ sides.

10. **Answer (D):** The checkerboard has 64 unit squares. There are $2 \cdot 8 + 2 \cdot 6 = 28$ unit squares on the outer edge, and $64 - 28 = 36$ unit squares in the interior. Therefore the probability of choosing a unit square that does not touch the outer edge is $\frac{36}{64} = \frac{18}{32} = \frac{9}{16}$.

OR



There are $(8-2)^2 = 36$ unit squares in the interior. Therefore, the probability of choosing a unit square that is does not touch the outer edge is $\frac{36}{64} = \frac{18}{32} = \frac{9}{16}$.

11. **Answer (D):** The number of sixth graders who bought a pencil is 195 divided by the cost of a pencil. Similarly the number of seventh graders who bought a pencil is 143 divided by the cost of a pencil. That means both 195 and 143 are multiples of the price of the pencil. Factor $195 = 1 \cdot 3 \cdot 5 \cdot 13$ and $143 = 1 \cdot 11 \cdot 13$. The only common divisors are 1 and 13. If a pencil cost 1 cent, then 195 sixth graders bought a pencil. However, there are only 30 sixth graders, so a pencil must cost 13 cents. Using that fact, $\frac{195}{13} - \frac{143}{13} = 15 - 11 = 4$ more sixth graders than seventh graders bought pencils.

12. **Answer (D):** Make a table.

	1	3	5
2	$1+2=3$	$3+2=5$	$5+2=7$
4	$1+4=5$	$3+4=7$	$5+4=9$
6	$1+6=7$	$3+6=9$	$5+6=11$

The table shows that seven of the nine equally likely events have prime numbers for their outcomes. So the probability of a prime outcome is $\frac{7}{9}$.

13. **Answer (B):** There are 6 three-digit numbers possible using the digits 1, 3 and 5 once each: 135, 153, 315, 351, 513 and 531. Because the numbers divisible by 5 end in 0 or 5, only 135 and 315 are divisible by 5. The probability that the three-digit number is divisible by 5 is $\frac{2}{6} = \frac{1}{3}$.

OR

The number is equally likely to end in 1, 3 or 5. The number is divisible by 5 only if it ends in 5, so the probability is $\frac{1}{3}$.

14. **Answer (B):** Find the time traveling to Temple by dividing the distance, 50 miles, by the rate, 60 miles per hour: $\frac{50}{60} = \frac{5}{6}$ hours. Find the time returning by dividing the distance, 50 miles, by the rate, 40 miles per hour: $\frac{50}{40} = \frac{5}{4}$ hours. Find the average speed for the round trip by dividing the total distance, $2 \cdot 50 = 100$ miles, by the total time, $\frac{5}{6} + \frac{5}{4} = \frac{10}{12} + \frac{15}{12} = \frac{25}{12}$ hours. The average speed is $\dfrac{100}{\frac{25}{12}} = 100(\frac{12}{25}) = 48$ miles per hour.

NOTE: The harmonic mean h of 2 numbers a and b is found using the formula $h = \frac{2ab}{a+b}$. The harmonic mean is the average rate if the same distance is traveled at two different rates.

If $a = 60$ and $b = 40$, then $h = \frac{2 \cdot 60 \cdot 40}{60+40} = \frac{4800}{100} = 48$ miles per hour.

15. **Answer (D):** Jordan has 5 squares of chocolate, which is $2\frac{1}{2}$ times the amount the recipe calls for. She has $2 \div \frac{1}{4} = 8$ times the amount of sugar and $\frac{7}{4} = 1\frac{3}{4}$ times the amount of milk necessary to make the recipe. So the amount of milk limits the number of servings. Jordan cannot make more than $5\left(1\frac{3}{4}\right) = 5\left(\frac{7}{4}\right) = \frac{35}{4} = 8\frac{3}{4}$ servings of hot chocolate.

16. **Answer (D):** The possible ways of expressing 24 as a product of 3 digits are $(1 \cdot 3 \cdot 8)$, $(1 \cdot 4 \cdot 6)$, $(2 \cdot 3 \cdot 4)$ and $(2 \cdot 2 \cdot 6)$. From the first product, the six integers 138, 183, 318, 381, 813 and 831 can be formed. Similarly, six integers can be formed from each of the products $(1 \cdot 4 \cdot 6)$ and $(2 \cdot 3 \cdot 4)$. From the product $(2 \cdot 2 \cdot 6)$, the three integers 226, 262 and 622 can be formed. The total number of integers whose digits have a product of 24 is $6 + 6 + 6 + 3 = 21$.

17. **Answer (B):** Factor 360 into $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5$. First increase the number of each factor as little as possible to form a square: $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 5 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5)(2 \cdot 5) = (360)(10)$, so x is 10. Then increase the number of each factor as little as possible to form a cube: $2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 5 = (2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5)(3 \cdot 5 \cdot 5) = (360)(75)$, so y is 75. The sum of x and y is $10 + 75 = 85$.

18. **Answer (C):** To maintain the pattern, white squares will always occupy the corners, and every edge of the square pattern will have an odd number of tiles. Create a table, starting with a white square in the corner of the pattern, and increase the sides by 2 tiles.

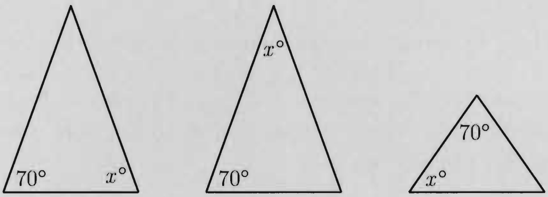
Floor area	# of white squares	Pattern
1×1	1	1^2
3×3	4	2^2
5×5	9	3^2
7×7	16	4^2
9×9	25	5^2

Following the pattern, an 11×11 area has 36 squares, a 13×13 area has 49, and a 15×15 has 64.

OR

There will be 8 rows that each contain 8 white tiles, so the total is $8(8) = 64$.

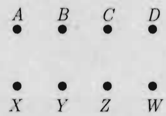
19. **Answer (D):** The two angles measuring $70°$ and $x°$, in an isosceles triangle, could be positioned in three ways, as shown.



If $70°$ and $x°$ are the degree measures of the congruent angles, then $x = 70$. If x is the degree measure of the vertex, then x is $180 - 70 - 70 = 40$. If x is the degree measure of one of the base angles, but not 70, then x is $\frac{1}{2}(180-70) = 55$.

The possible values of x are 70, 40 and 55. The sum of these values is $70+40+55 = 165$.

20. **Answer (D):** With the points labeled as shown, one set of non-congruent triangles is AXY, AXZ, AXW, AYZ, AYW, AZW, BXZ and BXW.



Every other possible triangle is congruent to one of the 8 listed triangles.

CHALLENGE: Find the 48 distinct triangles possible and group them into sets of congruent triangles.

21. **Answer (D):** There are 40 rows, so the sum of the 40 row sums is $40A$. This number is also the sum of all of the numbers in the array because each number in the array is added to obtain one of the row sums. Similarly, there are 75 columns, so the sum of the 75 column sums is $75B$, and this, too, is the sum of all of the numbers in the array. So $40A = 75B$, and $\frac{A}{B} = \frac{75}{40} = \frac{15}{8}$.

22. **Answer (D):** There are 8 one-digit positive integers, excluding 1. There are $8 \cdot 9 = 72$ two-digit integers that do not contain the digit 1. There are $8 \cdot 9 \cdot 9 = 648$ three-digit integers that do not contain the digit 1. There are $8 + 72 + 648 = 728$ integers between 1 and 1000 that do not contain the digit 1.

OR

Think of each number between 1 and 1000 as a three-digit number. For example, think of 2 as 002 and 27 as 027. There are $9^3 = 729$ three-digit numbers that do not use the digit 1. Because 000 does not represent a whole number between 1 and 1000, the total is 728.

23. **Answer (B):** Mrs. Wonderful gave $400 - 6 = 394$ jelly beans to the class. Make a table, starting with a small, reasonable number of girls and boys.

Girls	Boys	Number of jelly beans
9	11	$(9 \times 9) + (11 \times 11) = 202$
10	12	$(10 \times 10) + (12 \times 12) = 244$
11	13	$(11 \times 11) + (13 \times 13) = 290$
12	14	$(12 \times 12) + (14 \times 14) = 340$
13	15	$(13 \times 13) + (15 \times 15) = 394$

The number of students is $13 + 15 = 28$.

24. **Answer (E):** Because $A+B=A$, $B=0$. Subtracting 10 from A and adding 10 to B, $10-A=A$, so A must be 5, and AB is 50. Because $50-C5=5$, $C=4$ and $D=5+4=9$.

25. **Answer (E):** Looking from either end, the visible area totals $\frac{1}{2}$ square foot because piece A measures $\frac{1}{2}\times 1=\frac{1}{2}$ ft^2, and the other pieces decrease in height from that piece. The two side views each show four blocks that can stack to a unit cube. So the area as seen from each side is 1 ft^2. Finally, the top and bottom views each show four unit squares. So the top and bottom view each contribute 4 ft^2 to the area. Summing, the total surface area is

$$\frac{1}{2}+\frac{1}{2}+1+1+4+4=11 \text{ square feet.}$$

CHALLENGE: Suppose the cuts are $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$. Does this change the solution?

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

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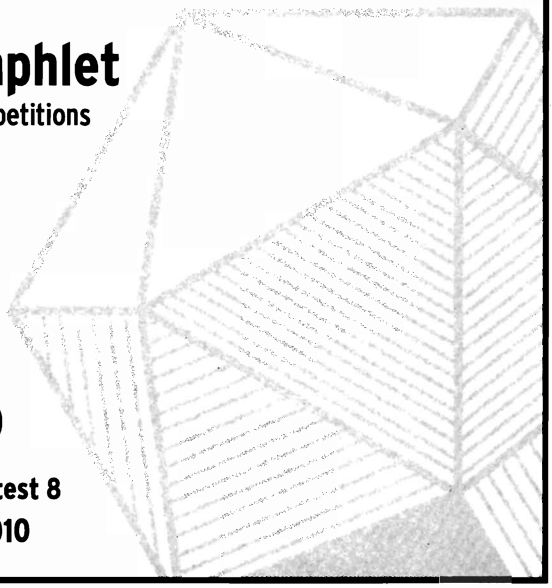

26th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 16, 2010

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1. **Answer (C):** The total number of students taking the test is $11 + 8 + 9 = 28$.

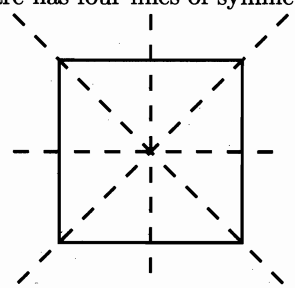
2. **Answer (D):** $5 * 10 = \frac{5\times10}{5+10} = \frac{50}{15} = \frac{5\times10}{5\times3} = \frac{10}{3}$.

3. **Answer (C):** The highest price in January was \$17 and the lowest in March was \$10. The \$17 price was \$7 more than the \$10 price, and 7 is 70% of 10.

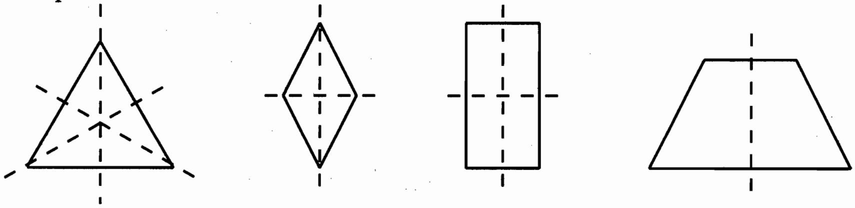
4. **Answer (C):** Arrange the numbers in increasing order: $0, 0, 1, 2, 3, 3, 3, 4$. The mean is the sum divided by 8, or $\frac{16}{8} = 2$. The median is halfway between 2 and 3, or 2.5. The mode is 3, because there are more 3's than any other number. The sum of the mean, median, and mode is $2 + 2.5 + 3 = 7.5$.

5. **Answer (B):** The ceiling is $2.4 - 1.5 = 0.9$ meters $= 90$ centimeters above Alice's head. She can reach 46 centimeters above the top of her head, and the light bulb is 10 centimeters below the ceiling, so the stool is $90 - 46 - 10 = 34$ centimeters high.

6. **Answer (E):** A square has four lines of symmetry.



The number of lines of symmetry for the other figures are:
equilateral triangle 3, non-square rhombus 2, non-square rectangle 2, isosceles trapezoid 1.



7. **Answer (B):** Four pennies are needed to make small change. Adding two nickels means that change up to fourteen cents can be made as efficiently as possible. Adding a dime extends the efficiency up to 24 cents. Adding three quarters permits any amount of change up to \$0.99.

$$4 \text{ pennies} + 2 \text{ nickels} + 1 \text{ dime} + 3 \text{ quarters} = 10 \text{ coins worth } \$0.99$$

Note that 4 pennies, 1 nickel, 2 dimes, and 3 quarters also satisfies the requirements. For 10 to be the smallest number of coins, one must prove that 9 coins will not work. Freddie will still need at least 4 pennies and 1 nickel to make any amount up to 9. Freddie will still need at least 1 more nickel and 1 dime to make any amount up to 24. If Freddie chose only 2 quarters, then he would need at least 9 other coins (4 dimes, 1 nickel, and 4 pennies) to reach 99 cents. He would need at least 11 coins. Thus 10 is the smallest number of coins as shown above.

8. **Answer (D):** Emily gains on Emerson at the rate of $12 - 8 = 4$ miles per hour. To get from $\frac{1}{2}$ mile behind Emerson to $\frac{1}{2}$ mile in front of him, she must gain 1 mile on him. This takes $\frac{1}{4}$ hour, which is 15 minutes.

OR

Make a chart that shows the position of Emily and Emerson every 5 minutes. Note that for every 5 minutes Emily rides $\frac{12}{60}$ miles per minute $\times$ 5 minutes $= 1$ mile, and Emerson skates $\frac{8}{12}$ mile. Eventually, Emily will be $\frac{1}{2}$ mile ahead of Emerson. Notice that happens after 15 minutes.

Time (minutes)	0	5	10	15
Emily's Distance (miles)	0	1	2	3
Emerson's Distance (miles)	$\frac{1}{2}$	$1\frac{1}{6}$	$1\frac{5}{6}$	$2\frac{1}{2}$

9. **Answer (D):** The three tests contain a total of 75 problems. Ryan received 80% of $25 = 20$, 90% of $40 = 36$, and 70% of $10 = 7$. Ryan correctly answered $20 + 36 + 7$ problems, for a total of 63 problems. The percent of problems Ryan answered correctly was:

$$\frac{63}{75} = \frac{21}{25} = \frac{21 \cdot 4}{25 \cdot 4} = \frac{84}{100} = 84\%$$

10. **Answer (B):** If six pepperonis fit across the diameter, then each pepperoni circle has a diameter of 2 inches and a radius of 1 inch. The area of each

pepperoni is $\pi(1)^2 = \pi$ square inches. The 24 pepperoni circles cover 24π square inches of the pizza. The area of the pizza is $\pi(6)^2 = 36\pi$ square inches. The fraction of the pizza covered by pepperoni is $\frac{24\pi}{36\pi} = \frac{2}{3}$.

11. **Answer (B):** The sum of the heights of the two trees can be divided into 7 parts where one part is 16 feet. The taller tree has 4 parts so its height is $4 \times 16 = 64$ feet.

12. **Answer (D):** The number of blue balls in the bag is 20% of 500, which is $(0.20)(500) = 100$. After some red balls are removed, the 100 blue balls must be 25%, or $\frac{1}{4}$, of the number in the bag. There must be $(4)(100) = 400$ balls in the bag, so $500 - 400 = 100$ red balls must be removed.

OR

Let x represent the number of red balls removed. Remember that $0.80(500) = 400$ is the number of red balls that were originally in the bag. Setup and solve the proportion:

$$\begin{aligned} 75\% = \frac{3}{4} &= \frac{400 - x}{500 - x} \\ 3(500 - x) &= 4(400 - x) \\ 1500 - 3x &= 1600 - 4x \end{aligned}$$

The number of red balls removed is $x = 100$.

13. **Answer (E):** One strategy is to try the choices:

$$\begin{array}{ll} 5 + 6 + 7 = 18; & 5 \neq 30\% \text{ of } 18 \\ 6 + 7 + 8 = 21; & 6 \neq 30\% \text{ of } 21 \\ 7 + 8 + 9 = 24; & 7 \neq 30\% \text{ of } 24 \\ 8 + 9 + 10 = 27; & 8 \neq 30\% \text{ of } 27 \\ 9 + 10 + 11 = 30; & 9 = 30\% \text{ of } 30 \end{array}$$

If the shortest side is 9, then the longest side is 11.

OR

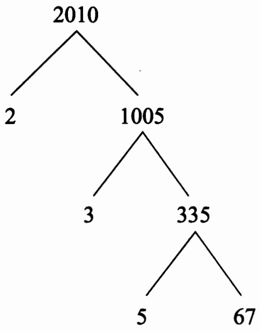
Let the three consecutive integers be side lengths $x, x - 1$, and $x - 2$.

$$\begin{aligned} x - 2 &= 0.3(x + x - 1 + x - 2) \\ x - 2 &= 0.3(3x - 3) \\ x - 2 &= 0.9x - 0.9 \\ 0.1x &= 1.1 \end{aligned}$$

$$x = 11$$

The longest side is 11.

14. **Answer (C):** The prime factors of 2010 are: 2, 3, 5, 67



The sum of the prime factors is $2 + 3 + 5 + 67 = 77$.

15. **Answer (C):** The 30 green gum drops are $100\% - (30+20+15+10)\% = 25\%$ of the total gum drops, so there are 120 gum drops in the jar. The number of blue gum drops is 30% of 120, which is 36, and the number of brown gum drops is 20% of 120, which is 24. After half the blue gum drops are replaced by brown ones, the number of brown gum drops is $24 + \frac{1}{2}(36) = 42$.

16. **Answer (B):** Let the radius of the circle be 1. Then the area of the circle is $\pi(1)^2 = \pi$. The area of the square is π, so its side length $\sqrt{\pi}$. The ratio of the side length of the square to the radius of the circle is $\frac{\sqrt{\pi}}{1} = \sqrt{\pi}$.

    Note: The "squaring of a circle" is a classical problem. In the latter part of the 19th century it was proven that a square having an area equal to that of a given circle cannot be constructed with the standard tools of straightedge and compasss because it is impossible to construct a transcendental number, e. g. $\sqrt{\pi}$.

17. **Answer (D):** The area below $\overline{PQ}$ is

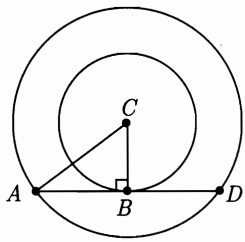
$$\begin{aligned} 1 + \frac{1}{2} \cdot 5 \cdot (1 + QY) &= 5 \\ \frac{5}{2} \cdot (1 + QY) &= 4 \\ 1 + QY &= \frac{8}{5} \end{aligned}$$

$$QY = \frac{3}{5}$$

Then $XQ = 1 - QY = 1 - \frac{3}{5} = \frac{2}{5}$, so $\frac{XQ}{QY} = \frac{2/5}{3/5} = \frac{2}{3}$.

18. **Answer (C):** The given ratio implies that $AD = \dfrac{3}{2}AB = \dfrac{3}{2} \cdot 30 = 45$ inches, so the area of the rectangle is $AB \cdot AD = 30 \cdot 45$ square inches. The 2 semicircles make 1 circle with radius $= 15$ inches. The area of the circle is $15^2\pi$ square inches. The ratio of the areas is $\dfrac{30 \cdot 45}{15 \cdot 15\pi} = \dfrac{6}{\pi}$.

19. **Answer (C):** In the right triangle ABC, AB is 8. By the Pythagorean Theorem, $8^2+BC^2 = 10^2$ so $BC = 6$. The area of the outer circle is $10^2\pi = 100\pi$ and the area of the inner circle is $6^2\pi = 36\pi$. The area between the circles is $100\pi - 36\pi = 64\pi$.



20. **Answer (A):** Because $\frac{2}{5}$ and $\frac{3}{4}$ of the people in the room are whole numbers, the number of people in the room is a multiple of both 5 and 4. The least common multiple of 4 and 5 is 20, so the minimum number of people in the room is 20. If $\frac{2}{5}$ of 20 people are wearing gloves, then 8 people are wearing gloves. If $\frac{3}{4}$ of 20 people are wearing hats, then 15 are wearing hats. The minimum number wearing gloves and hats occurs if the 5 not wearing hats are each wearing gloves. This leaves $8 - 5 = 3$ people wearing both gloves and hats.

OR

If 8 are wearing gloves and 15 are wearing hats, then $8 + 15$ are wearing gloves and/or hats. There is a minimum of 20 people in the room, so $23 - 20 = 3$ people are wearing both a hat and gloves.

21. **Answer (C):** Reason backward as follows. On the third day, Hui reads $\frac{1}{3}$ of the remaining pages plus 18 more, leaving her with 62 pages left to read. This means that $62+18 = 80$ is $\frac{2}{3}$ of the number of pages remaining at the end of the

second day. She had $\frac{3}{2} \times 80 = 120$ pages left to read at the end of the second day. On the second day she read $\frac{1}{4}$ of the remaining pages plus 15 more, so $120 + 15 = 135$ is $\frac{3}{4}$ of the number of pages remaining at the end of the first day. She had $\frac{4}{3} \times 135 = 180$ pages left to read at the end of the first day. On the first day she read $\frac{1}{5}$ of the pages plus 12 more. So $180 + 12 = 192$ is $\frac{4}{5}$ of the number of pages in the book. The total number of pages is $\frac{5}{4} \times 192 = 240$.

OR

Setup a chart working backwards through the days, to show the number of pages that Hui has left to read each day.

Day	Extra Pages Read	Fraction Read	Fraction Left	Calculation	Pages to be Read
4	0				62
3	18	1/3	2/3	$(62+18)/(2/3)$	120
2	15	1/4	3/4	$(120+15)/(3/4)$	180
1	12	1/5	4/5	$(180+12)/(4/5)$	240

22. **Answer (E):** Take 452 as the original number, and subtract 254. The difference is 198 and the units digit is 8. The same result will be obtained with any number that meets the criteria.

OR

The original number is $100a + 10b + c$, with $a - c = 2$. The reversed number will be $100c + 10b + a$. The difference is $99a - 99c = 99(a - c) = 99(2) = 198$. So the units digit is 8.

23. **Answer (B):** By the Pythagorean Theorem, the radius OQ of circle O is $\sqrt{2}$. Given the coordinates $P, Q, R,$ and S, the diameters $\overline{PQ}$ and $\overline{RS}$ of the semicircles have length 2. So the areas of the two semicircles will equal the area of a circle of radius 1. Thus the desired ratio is $\dfrac{\pi \cdot 1^2}{\pi(\sqrt{2})^2} = \dfrac{1}{2}$.

24. **Answer (A):** $2^{24} = (2^8) \cdot (2^{16}) = (2^8) \cdot (4^8) < (2^8) \cdot (5^8) = 10^8 = (4^4) \cdot (5^8) < (5^4) \cdot (5^8) = 5^{12}$.

25. **Answer (E):** Jo can climb 1 stair in 1 way, 2 stairs in 2 ways: $1+1$ or 2, and 3 stairs in 4 ways: $1+1+1$, $1+2$, $2+1$ or 3. Jo can start a flight of 4 stairs with 1 stair, leaving 3 stairs to go and 4 ways to climb them, or with 2 stairs, leaving 2 stairs to go and 2 ways to climb them or with 3 stairs, leaving 1 to

go and 1 way to climb it. This means there are $1 + 2 + 4 = 7$ ways to climb 4 stairs. By the same argument, there must be $2 + 4 + 7 = 13$ ways to climb 5 stairs and $4 + 7 + 13 = 24$ ways to climb 6 stairs.

OR

Making a systematic list is another approach.

Number of stair moves	Stair move sequence	Count
6	1-1-1-1-1-1	1
5	1-1-1-1-2; 1-1-1-2-1; 1-1-2-1-1; 1-2-1-1-1; and 2-1-1-1-1	5
4	1-1-1-3; 1-1-3-1; 1-3-1-1; 3-1-1-1; 1-1-2-2; 1-2-1-2; 1-2-2-1; 2-1-1-2; 2-1-2-1; and 2-2-1-1	10
3	1-2-3; 1-3-2; 2-1-3; 2-3-1; 3-1-2; 3-2-1; and 2-2-2	7
2	3-3	1

The number of ways Jo can climb the stairs is $1 + 5 + 10 + 7 + 1 = 24$.

The problems and solutions for this contest were proposed by Steve Blasberg, Thomas Butts, John Cocharo, Steve Davis, Melissa Desjarlais, Steven Dunbar, Margie Raub Hunt, Joe Kennedy, Norbert Kuenzi, Sister Josanne Furey, Jeganathan Sriskandarajah, David Wells, LeRoy Wenstrom and Ron Yannone.

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Solutions Pamphlet

American Mathematics Competitions

27th Annual

AMC 8

American Mathematics Contest 8

Tuesday, November 15, 2011

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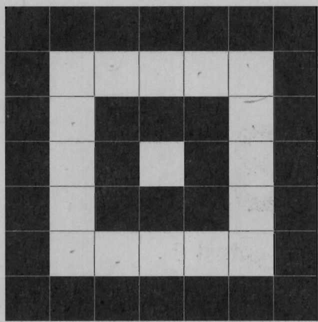
Lincoln, NE 68501-1606

1. **Answer (E):** The cost of the apples was $3 \times \$0.50 = \$1.50$. Her change was $\$5.00 - \$1.50 = \$3.50$.

2. **Answer (E):** Karl's garden is $20 \times 45 = 900$ square feet. Makenna's garden is $25 \times 40 = 1000$ square feet. Makenna's garden is larger by $1000 - 900 = 100$ square feet.

3. **Answer (D):** There are 32 black tiles and 17 white tiles in the extended pattern. So the ratio is $32 : 17$.



4. **Answer (C):** The ordered list is 0, 0, 1, 1, 2, 2, 3, 3, 3. The mean is $\frac{15}{9} = \frac{5}{3}$, the median is 2, and the mode is 3. Because $\frac{5}{3} < 2 < 3$, the correct order is mean < median < mode.

5. **Answer (D):** Convert 2011 minutes to 33 hours and 31 minutes. January 1 uses 24 hours. January 2 gets the remainder of the 9 hours and 31 minutes. The time at the end of 2011 minutes was 9:31AM on January 2.

6. **Answer (D):** In a population of 351 people, 45 people own a motorcycle. Therefore there are $351 - 45 = 306$ car owners who do not own a motorcycle.

7. **Answer (C):** The upper left and the lower right squares are each one-fourth shaded, for a total of one-half square. The shaded portions of the upper right and lower left squares make up one-half square. So the total shaded area is one full square, which is 25% of the total area.

8. **Answer (B):** Make a table of possibilities.

+	1	3	5
2	3	5	7
4	5	7	9
6	7	9	11

The possible sums are 3, 5, 7, 9, and 11, for a total of 5 possibilities.

9. **Answer (E):** Carmen covers 35 miles in 7 hours, making her average speed $\frac{35}{7} = 5$ mph.

10. **Answer (C):** Including a \$2 tip, a 0.5 mile ride would cost \$4.40. The remaining \$5.60 would take you an additional $0.1 \times \frac{5.60}{0.2} = 2.80$ miles, so the total distance is $0.5 + 2.8 = 3.3$ miles.

11. **Answer (A):** Asha's study time totals $60+90+100+80+70 = 400$ minutes, for an average of $\frac{400}{5} = 80$ minutes per day. Sasha's total is $70+80+120+110+50 = 430$ minutes, for an average of $\frac{430}{5} = 86$ minutes per day, so Sasha averages 6 minutes more per day than Asha.

OR

The daily differences between Sasha and Asha are $+10$, -10, $+20$, $+30$, and -20 minutes for a total of $+30$ minutes. The average difference is $\frac{30}{5} = 6$ minutes per day.

12. **Answer (B):** Proceeding clockwise from Angie, the seating could be: BCD, BDC, CBD, CDB, DBC, or DCB. In 2 of these 6 possibilities Carlos is opposite Angie, so the probability is $\frac{2}{6} = \frac{1}{3}$.

OR

If Angie sits down first, there are three equally likely places for Carlos to sit. Only one of these is opposite Angie. Thus the probability is $\frac{1}{3}$.

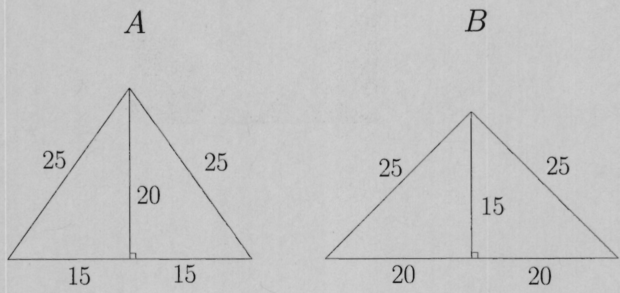
13. **Answer (C):** The shaded rectangle $PBCS$ has height $BC = 15$ and length $SC = DC + SR - DR = 15 + 15 - 25 = 5$.

Rectangle $AQRD$ has the same height and length 25. The portion of rectangle $AQRD$ that is shaded is $\frac{15\times5}{15\times25} = \frac{5}{25}$, which is 20%.

14. **Answer (C):** The number of girls at the dance is $\frac{4}{9}(270)+\frac{5}{9}(180) = 120+100 = 220$. So the fraction of the students that are girls is $\frac{220}{450} = \frac{22}{45}$.

15. **Answer (D):** The product $4^5 \cdot 5^{10} = 2^{10} \cdot 5^{10} = 10^{10}$ is a number with a 1 followed by 10 zeros for a total of 11 digits.

16. **Answer (C):**



The altitude shown divides each triangle into two congruent right triangles. The hypotenuse of each right triangle is 25. In $\triangle A$ the horizontal leg of each right triangle is 15, so the vertical leg is $\sqrt{25^2 - 15^2} = 20$. In $\triangle B$ the horizontal leg of each right triangle is 20, so the vertical leg is 15. The area of $\triangle A$ is $\frac{1}{2}(30)(20) = 300$, and the area of $\triangle B$ is $\frac{1}{2}(40)(15) = 300$, so the two areas are equal.

17. **Answer (A):** Factor 588 into $2^2 \cdot 3^1 \cdot 5^0 \cdot 7^2$. Thus $w = 2, x = 1, y = 0$, and $z = 2$, and $2w + 3x + 5y + 7z = 21$.

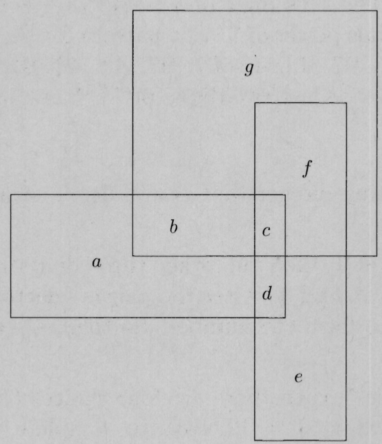
18. **Answer (D):** Make a table of 36 possible equally-likely outcomes. The first number is greater than or equal to the second in the 21 cases indicated by the asterisks, so the probability is $\frac{21}{36} = \frac{7}{12}$.

	1	2	3	4	5	6
1	1,1*	1,2	1,3	1,4	1,5	1,6
2	2,1*	2,2*	2,3	2,4	2,5	2,6
3	3,1*	3,2*	3,3*	3,4	3,5	3,6
4	4,1*	4,2*	4,3*	4,4*	4,5	4,6
5	5,1*	5,2*	5,3*	5,4*	5,5*	5,6
6	6,1*	6,2*	6,3*	6,4*	6,5*	6,6*

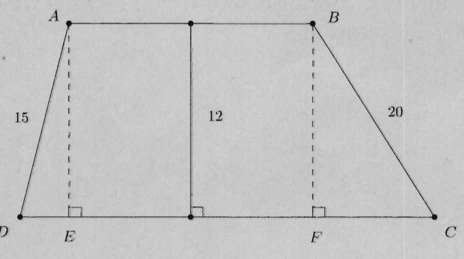
OR

In 6 of the 36 possible outcomes the two numbers are equal. The first number is greater than the second in half of the remaining 30 outcomes, so the first number is greater than or equal to the second in $6 + 15 = 21$ outcomes. The probability is $\frac{21}{36} = \frac{7}{12}$.

19. **Answer (D):** Partition the figure into non-overlapping regions as shown. The list of rectangles are b, c, d, ab, bc, cd, cf, de, $abcd$, $bcfg$, and $cdef$.



20. **Answer (D):** Let E and F be the feet of the perpendiculars from A and B to $\overline{DC}$. In right $\triangle AED$, $DE^2 = 15^2 - 12^2 = 225 - 144 = 81$, so $DE = 9$. In right $\triangle BFC$, $FC^2 = 20^2 - 12^2 = 400 - 144 = 256$, so $FC = 16$.



Right $\triangle AED$ has area $\frac{1}{2} \cdot 9 \cdot 12 = 54$, right $\triangle BFC$ has area $\frac{1}{2} \cdot 16 \cdot 12 = 96$, and rectangle $ABFE$ has area $50 \cdot 12 = 600$. The trapezoid $ABCD$ has area $54 + 96 + 600 = 750$.

OR

Begin as in the first solution and note that $DC = DE+EF+FC = 9+50+16 = 75$. Then the area of trapezoid is $\frac{1}{2}(AB+DC)\cdot AE = \frac{1}{2}(50+75)\cdot 12 = 125\cdot 6 = 750$.

21. **Answer (C):** Because half are too low, Norb is at least 37. Because two are off by one, he must be between 36 and 38 or between 47 and 49. Because 37 is prime and 48 is not, Norb is 37.

22. **Answer (D):** The tens digit of a power of 7 is determined by the last two digits of the previous power of 7. The pattern for the last two digits of successive powers of 7 is **01**, **07**, **49**, **43**, **01**, **07**, **49**, **43**, **01**, **07**, **49**, **43**, **01**,.... Since $2011 = 4\cdot 502 + 3$, the last two digits of 7^{2011} are 43 and the tens digit is 4.

23. **Answer (D):** Any integer that is a multiple of 5 must have a 0 or 5 as the units digit.

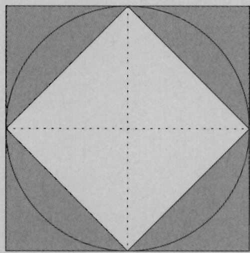
    If the units digit is 0, then the other three digits must be a 5 and two digits selected from 1, 2, 3, and 4. After the pair is selected, there are 6 possible ways to arrange them to form the number. So there are $6\cdot 6 = 36$ possible numbers with units digit 0.

    If the units digit is 5, then there are four choices (1, 2, 3, 4) for the thousands digit and there are $4\cdot 3 = 12$ ways to complete the number. So there are $4\cdot 12 = 48$ possible numbers with units digit 5.

    Together there are $36 + 48 = 84$ possible numbers.

24. **Answer (A):** If the sum of two numbers is odd, one number must be even and the other number must be odd. Because all primes except 2 are odd, 2 must be one of the summands. Because $10{,}001 = 2 + 9999$, and $9999 = 9\cdot 1111$ is not prime, there are no solutions.

25. **Answer (A):** The area of a circle of radius 1 is $\pi(1)^2 = \pi$. The side length of the big square is the diameter of the circle, which is 2, so its area is $2^2 = 4$. The big square can be divided into 8 congruent triangles, and the shaded area is made up of 4 of those triangles. The shaded area is half the area of the big square, which is 2. The requested ratio of the two shaded areas is $\frac{\pi-2}{2} \approx \frac{3.14-2}{2} \approx \frac{1}{2}$.



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The problems and solutions in this contest were proposed by Bernardo Abrego, Sam Baethge, Betsy Bennett, Bruce Brombacher, Thomas Butts, John Cocharo, Barbara Currier, Steven Davis, Melissa Desjarlais, Steve Dunbar, Sister Josanne Furey, Michele Ghrist, Peter Gilchrist, Dick Gibbs, Jerrold Grossman, Joel Haack, Margie Raub Hunt, J. Alfredo Jimenez, Elgin Johnston, Dan Kennedy, Joe Kennedy, Geneivieve M. Knight, Gerald A. Krause, Sheila Krilov, Norb Kuenzi, Yo Ju Kuo, Sylvia Lazarnick, Bonnie Leitch, Victor Levine, Yung-Way Liu, Jeff Misener, Ellen Panofsky, Harry Sedinger, J. Sriskandarajah, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

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1. **Answer (E):** Rachelle needs $\frac{24}{8} = 3$ times the amount of meat for the picnic than she would use for her family. So she needs $3 \times 3 = 9$ pounds of meat.

OR

Set up a proportion to compare the two ratios of pounds of meat to number of hamburgers.

$$\frac{3}{8} = \frac{x}{24}$$

Solving for x, $8x = 72$, so $x = 9$ pounds of meat.

2. **Answer (B):** The net growth per day in East Westmore is 3 births − 1 death = 2 people. There are typically 365 days in a year, so the population grows by about $2 \times 365 = 730$, or close to 700 people a year.

3. **Answer (B):** From 6:57 AM to 12:00 PM (noon) is 5 hours and 3 minutes. Since the length of daylight is 10 hours and 24 minutes, there must be another 5 hours and 21 minutes until sunset. The correct sunset time is 5:21 PM.

4. **Answer (C):** The whole slice that Peter ate was $\frac{1}{12}$ of the pizza. His half of the second slice was half of $\frac{1}{12}$, or $\frac{1}{24}$, of the pizza. The fraction of the pizza that Peter ate was

$$\frac{1}{12} + \frac{1}{24} = \frac{2}{24} + \frac{1}{24} = \frac{3}{24} = \frac{1}{8}.$$

5. **Answer (E):** The vertical sides on the left add up to $5 + X$ while the vertical sides on the right add up to 10. Therefore $X = 5$.

6. **Answer (E):** The width of the frame is $10 + 2 + 2 = 14$ inches, and its height is $8 + 2 + 2 = 12$ inches. It encloses an area of $14 \times 12 = 168$ square inches. The photograph occupies $10 \times 8 = 80$ square inches of that area, so the area of the border itself is $168 - 80 = 88$ square inches.

7. **Answer (B):** To achieve an average grade of 95 on the four tests, Isabella must score a total of $4 \times 95 = 380$ points. She scored a total of $97 + 91 = 188$ points on her first two tests, so she must score a total of at least $380 - 188 = 192$ points on her last two tests. Because she can score at most 100 on her fourth test, she must have scored at least 92 on her third test.

OR

Isabella's first score was $95 + 2$, her second score was $95 - 4$, and her fourth score can be at most $95 + 5$. Because she can still achieve an average score of 95, her third score must have been at least $95 - 2 + 4 - 5 = 92$.

8. **Answer (D):** The price of an item costing $\$d$ after both discounts are applied is $.80(.50d) = .40d$, a discount of 60% off the original price.

9. **Answer (C):** All 200 heads belonged to animals with at least two legs, accounting for 400 of the 522 legs. The additional 122 legs belonged to four-legged mammals, each of which had two additional legs. So Margie saw $\frac{122}{2} = 61$ four-legged mammals and $200 - 61 = 139$ birds.

10. **Answer (D):** To form a number greater than 1000, the first digit must be 1 or 2. If the first digit is a 1, the remaining numbers could be 022, 202, or 220. If the first digit is a 2, the remaining numbers could be 012, 021, 102, 120, 201, or 210. So there are $3 + 6 = 9$ ways to form a number greater than 1000.

11. **Answer (D):** Because the mode is unique, it must be 6, so the mean must also be 6. The sum of the seven numbers is $31 + x$, which must be equal to $7 \times 6 = 42$. Therefore $x = 11$. The median of the numbers 3, 4, 5, 6, 6, 7, 11 is also 6.

12. **Answer (A):** To determine the units digit of 13^{2012}, looking for patterns is a good approach to finding the solution. The units digit of each power of 13 depends only on the units digit of the previous power, as follows:
For 13^1, the units digit is 3.
For 13^2, $3 \times 3 = 9$, so the units digit is 9.
For 13^3, $9 \times 3 = 27$, so the units digits is 7.
For 13^4, $7 \times 3 = 21$, so the units digit is 1.
For 13^5, $1 \times 3 = 3$, so the units digit is 3.
The units digits of successive powers of 13 follow the pattern 3, 9, 7, 1, 3, 9, 7, 1, ...
Since $2012 = 4 \times 503$, the units digit of 13^{2012} is 1.

13. **Answer (C):** Since $143 = 11 \times 13$ and $187 = 11 \times 17$, the pencils cost 11 cents each. That means Sharona bought $17 - 13 = 4$ pencils more than Jamar.

14. **Answer (B):** If there were 2 teams in the conference, then there would be only 1 game played. Each additional team plays a game against each previously listed team.

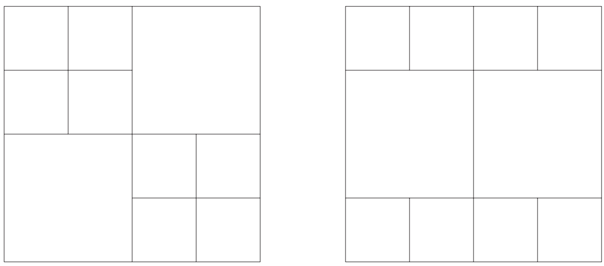
Number of teams	Number of games
2	1
3	$1 + 2 = 3$
4	$3 + 3 = 6$
5	$6 + 4 = 10$
6	$10 + 5 = 15$
7	$15 + 6 = 21$

If 21 conference games were played, then there were 7 teams in the conference.

15. **Answer (D):** Two less than the number must be divisible by 3, 4, 5, and 6. The least common multiple of these numbers is 60, so 62 is the smallest number greater than 2 to leave a remainder of 2 when divided by 3, 4, 5, and 6. Therefore the number lies between 61 and 65.

16. **Answer (C):** The sum will be as large as possible when the largest digits are placed in the most significant places. The 8 and 9 should be in the ten-thousands place, the 6 and 7 in the thousands place, the 4 and 5 in the hundreds place, the 2 and 3 in the tens place, and the 0 and 1 in the units place. The only choice that fits that description is 87431. (In this case the other number is 96520, giving the largest sum of 183951.)

17. **Answer (B):** The area of the original square is a square number that is more than 8, so 16 is the least possible value for the area of the original square. Its side has length 4. Two possible ways of cutting the square are shown below:



18. **Answer (A):** Since the integer is neither prime nor square, it is divisible either by two distinct primes or by the cube of a prime. The smallest prime numbers not less than 50 are 53 and 59. Since $53 \times 59 = 3127 < 53^3$, the smallest number satisfying this description is 3127.

19. **Answer (C):** There are 4 non-blue marbles. That is, there are altogether 4 red and green marbles. There are also 6 non-red marbles and 8 non-green marbles, so there are two more red marbles than green marbles. Therefore there are 3 red marbles and 1 green marble. Because 3 of the 8 non-green marbles are red, the other 5 must be blue. The total number of marbles is $1 + 3 + 5 = 9$.

OR

Let g, b, and r be the number of green, blue, and red marbles respectively. Then

$$g + b = 6$$

$$r + b = 8$$
$$r + g = 4$$

Adding all three equations together, $2g + 2b + 2r = 18$, so $g + b + r = 9$.

20. **Answer (B):** Using a common denominator, $\frac{5}{19} = \frac{105}{399}$ and $\frac{7}{21} = \frac{133}{399}$, so $\frac{5}{19} < \frac{7}{21}$.
Also $\frac{7}{21} = \frac{161}{483}$ and $\frac{9}{23} = \frac{189}{483}$, so $\frac{7}{21} < \frac{9}{23}$.

OR

Comparing each fraction, $\frac{7}{21} = \frac{1}{3}$, $\frac{5}{19} < \frac{5}{15} = \frac{1}{3}$, and $\frac{9}{23} > \frac{9}{27} = \frac{1}{3}$, so the correct increasing order is $\frac{5}{19} < \frac{7}{21} < \frac{9}{23}$.

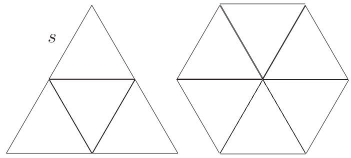
21. **Answer (D):** The surface area of the cube is $6 \times 10^2 = 600$ square feet. The green paint covers 300 square feet, so the total area of the white squares is $600 - 300 = 300$ square feet. There are 6 white squares, so each has area $\frac{300}{6} = 50$ square feet.

22. **Answer (D):** If nine distinct integers are written in increasing order, the median, which is the middle number, would be in the fifth place. If all of the remaining integers are greater than 9, then the median is 9. If all of the remaining integers are less than 3, then the median is 3. All the integers from 3 to 9 are possible medians, as shown in the following table.

Include in R	Median of R
-1, 0, 1	3
0, 1, 5	4
1, 5, 7	5
5, 7, 8	6
7, 8, 10	7
8, 10, 11	8
10, 11, 12	9

So, 3, 4, 5, 6, 7, 8, and 9 are the possible medians, and their count is 7.

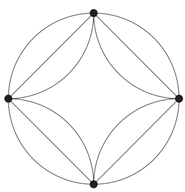
23. **Answer (C):** The equilateral triangle can be divided into 4 smaller congruent equilateral triangles, each with area of 1 and side length s, as shown. Then the original equilateral triangle has perimeter $6s$. A hexagon with perimeter $6s$ can be divided into 6 congruent equilateral triangles, each with area of 1. Therefore the area of the hexagon is 6.



24. **Answer (A):** Translate the star into the circle so that the points of the star coincide with the points on the circle. Construct four segments connecting the consecutive points of the circle and the star, creating a square concentric to the circle.

The area of the circle is $\pi(2)^2 = 4\pi$. The square is made up of four congruent right triangles with area $\frac{1}{2}(2 \times 2) = 2$, so the area of the square is $4 \times 2 = 8$. The area inside the circle but outside the square is $4\pi - 8$.

This is also the area inside the square but outside the star. So, the area of the star is $8 - (4\pi - 8) = 16 - 4\pi$. The ratio of the area of the star figure to the area of the original circle is $\frac{16-4\pi}{4\pi} = \frac{4-\pi}{\pi}$.



25. **Answer (C):** The area of the region inside the larger square and outside the smaller square has total area $5-4 = 1$ and is equal to the area of four congruent right triangles, each with one side of length a and the other of length b. The area of each triangle is $\frac{1}{4}$. If $\frac{1}{2}ab = \frac{1}{4}$, then $ab = \frac{1}{2}$.

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American Mathematics Contest 8

Tuesday, November 19, 2013

This Solutions Pamphlet gives at least one solution for each problem on this year's exam and shows that all the problems can be solved using material normally associated with the mathematics curriculum for students in eighth grade or below. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students. However, the publication, reproduction, or communication of the problems or solutions of the AMC 8 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. *Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

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1. **Answer (A):** The smallest multiple of 6 that is at least 23 is 24, so Danica must buy 1 additional car.

2. **Answer (D):** A half-pound package costs \$3 at the sale price, so it would cost \$6 at the regular price. A whole pound would cost \$12 at the regular price.

3. **Answer (E):** Inside the parentheses are 500 pairs of numbers, each with a sum of 1. Therefore the expression equals $4 \cdot 500 = 2000$.

4. **Answer (C):** Judi's share of the bill was $7(\$2.50) = \$17.50$, so the total bill was $8(\$17.50) = \$140$.

5. **Answer (E):** The total weight is 130 pounds, so the average is 26 pounds. The median is 6 pounds, so the average is greater by 20 pounds.

6. **Answer (C):** The product of the two numbers in the second row is 600, so the missing number in that row is $\frac{600}{30} = 20$. The product of 5 with the missing number in the top row is 20, so the missing number in the top row is $\frac{20}{5} = 4$.

7. **Answer (C):** Because Trey counted 6 cars in 10 seconds, close to $6 \cdot 6 = 36$ cars passed in $10 \cdot 6 = 60$ seconds or 1 minute. In 2 minutes 45 seconds, about $2 \cdot 36 + \frac{45}{60}(36) = 72 + \frac{3}{4}(36) = 72 + 27 = 99$ cars passed, so there were approximately 100 cars in the train.

   OR

   Two minutes and 45 seconds is $2(60) + 45 = 165$ seconds. Let N be the number of cars, and set up a proportion:

   $$\frac{6}{10} = \frac{N}{165}$$

   Solving gives $10N = 6(165) = 990$, so $N = 99$. Approximately 100 train cars passed.

8. **Answer (C):** List the 8 possible equally likely outcomes: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. Only HHH, HHT, THH have at least 2 consecutive heads, so the probability of at least 2 consecutive heads is $\frac{3}{8}$.

9. **Answer (C):**

Jump	Distance (meters)
1	$1 = 2^0$
2	$2 = 2^1$
3	$4 = 2^2$
$\vdots$	$\vdots$
10	$512 = 2^9$
11	$1024 = 2^{10}$

Because 1024 meters is greater than 1 kilometer, he first exceeds 1 kilometer on the 11^{th} jump.

10. **Answer (C):** Because the prime factorizations of 180 and 594 are $2^2 \cdot 3^2 \cdot 5$ and $2 \cdot 3^3 \cdot 11$, respectively, the least common multiple of 180 and 594 is $2^2 \cdot 3^3 \cdot 5 \cdot 11$, and their greatest common factor is $2 \cdot 3^2$. The ratio of their least common multiple to their greatest common factor is $\frac{2^2 \cdot 3^3 \cdot 5 \cdot 11}{2 \cdot 3^2} = 2 \cdot 3 \cdot 5 \cdot 11 = 330$.

11. **Answer (D):** Because time equals distance divided by rate, Grandfather was on the treadmill for $\frac{2}{5}$ hours or 24 minutes on Monday. Similarly, he walked for $\frac{2}{3}$ hours, or 40 minutes, on Wednesday and $\frac{2}{4}$ hours, or 30 minutes, on Friday. The total time Grandfather spent on the treadmill was $24 + 40 + 30 = 94$ minutes. If he had walked the entire 6 miles at 4 miles per hour, he would have spent $\frac{6}{4}$ hours, or 90 minutes, on the treadmill, so he would have saved 4 minutes.

12. **Answer (B):** Javier spent \$50 on the first pair, \$30 on the second pair, and \$25 on the third pair, for a total of \$105. This is a savings of \$45 off the \$150 list price, which is 30% off.

13. **Answer (A):** Switching a digit from the units column to the tens column increases the sum by 9 times the value of that digit. For example, switching a 7 from the units column to the tens column increases the sum by $70-7 = 63 = 9 \cdot 7$. Similarly, switching a digit from the tens column to the units column decreases the sum by 9 times the value of that digit. Therefore reversing two digits changes the sum by an amount that must be a multiple of 9. Among the given choices, only 45 is a possible difference.

    (Note: Other multiples of 9 are also possible.)

14. **Answer (C):** Denote Abe's jelly beans by g and r. Denote Bea's jelly beans by G, Y, R_1, and R_2. There are 8 equally likely pairings: (g, G), (g, Y), (g, R_1), (g, R_2), (r, G), (r, Y), (r, R_1), and (r, R_2). Only (g, G), (r, R_1), and (r, R_2) match, so the probability that the colors match is $\frac{3}{8}$.

15. **Answer (B):** From the first equation, $3^p + 81 = 90$, so $3^p = 9$, and $p = 2$. From the second equation, $2^r = 32$, so $r = 5$. From the third equation, $6^s + 125 = 1421$, so $6^s = 1296$, and $s = 4$. The product of p, r, and s is $2 \cdot 5 \cdot 4 = 40$.

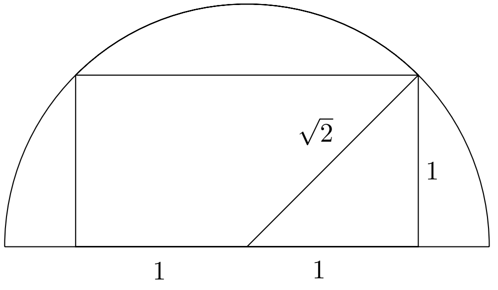
16. **Answer (E):** The number of 8^{th}-graders must be a multiple of both 5 and 8, so it must be at least 40. If there are 40 8^{th}-graders, then there are $\frac{3}{5}(40) = 24$ 6^{th}-graders and $\frac{5}{8}(40) = 25$ 7^{th}-graders, for a total of $40 + 24 + 25 = 89$ students.

17. **Answer (B):** The average of the six integers is $\frac{2013}{6} = 335.5$, so $2013 = 333 + 334 + 335 + 336 + 337 + 338$. The largest of the six integers is 338.

18. **Answer (B):** The fort, including the inside, occupies a volume of $12 \times 10 \times 5 = 600$ cubic feet. The inside of the fort is $12 - 2 = 10$ feet long, $10 - 2 = 8$ feet wide, and $5 - 1 = 4$ feet high, so it occupies a volume of $10 \times 8 \times 4 = 320$ cubic feet. Therefore the walls and floor occupy $600 - 320 = 280$ cubic feet, so the fort contains 280 blocks.

19. **Answer (D):** Cassie says, "I didn't get the lowest score," so her score is higher than Hannah's score. Bridget says, "I didn't get the highest score," so her score is lower than Hannah's score. Therefore the order, from highest to lowest, must be Cassie, Hannah, Bridget.

20. **Answer (C):**

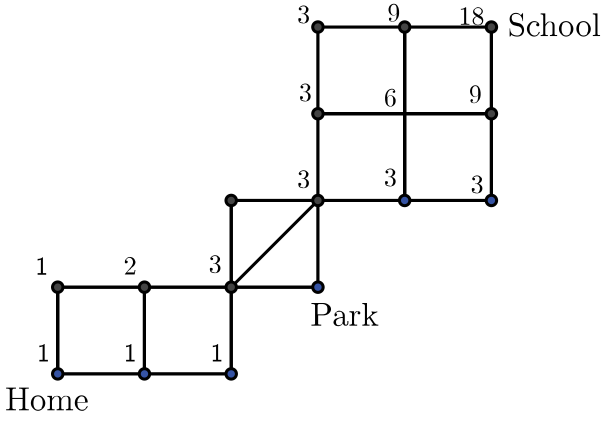


By the Pythagorean Theorem, the radius of the semicircle is $\sqrt{2}$, so its area is $\frac{\pi(\sqrt{2})^2}{2} = \pi$.

21. **Answer (E):** There are 3 ways she can bike from home to the southwest corner of the park, EEN, ENE, or NEE. There are 6 ways to bike from the northeast corner of the park to school, EENN, ENEN, ENNE, NEEN, NENE, or NNEE. So there are $6 \cdot 3 = 18$ routes.

OR

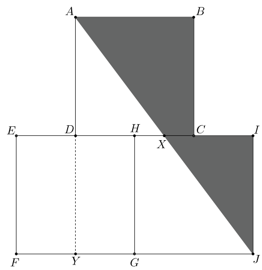
Using a Pascal's Triangle approach starting from the house to the school, count the routes to each intermediate point with the following diagram, moving only north or east at each corner.



22. **Answer (E):** A grid 60 toothpicks long and 32 toothpicks high needs 61 columns of 32 toothpicks and 33 rows of 60 toothpicks. Therefore a total of $(61 \times 32) + (33 \times 60) = 3932$ toothpicks are needed.

23. **Answer (B):** The circle with diameter $\overline{AB}$ has twice the area of the corresponding semicircle; thus the area of the circle is 16π and its radius is 4. Consequently $AB = 8$. The circle with diameter $\overline{AC}$ has circumference 17π, so $AC = 17$. $\overline{AC}$ is the hypotenuse of the right triangle. By the Pythagorean Theorem, $17^2 = 8^2 + (BC)^2$. Therefore $BC = 15$, and the radius is 7.5.

24. **Answer (C):**



Let the length of the side of each square be 1 and extend side AD to Y as shown. The total area of the three squares is 3. The unshaded area is area $(EDYF)$ + area $(AYJ) = 1(\frac{1}{2}) + \frac{1}{2} \cdot \frac{3}{2} \cdot 2 = \frac{1}{2} + \frac{3}{2} = 2$, so the shaded area is 1 and the desired ratio is $\frac{1}{3}$.

OR

Label point X as shown. Intuitively, rotating $\triangle XIJ$ $180°$ about X takes it to $\triangle XDA$ so the shaded area is the same as the area of square $ABCD$ and the desired ratio is $\frac{1}{3}$. More precisely, segments AX, XD, and DA are parallel to segments JX, XI, and IJ, respectively. Also, $DA = IJ$, so $\triangle ADX$ is congruent to $\triangle JIX$.

25. **Answer (A):** The diameter of the ball is 4 inches, so its radius is 2 inches. The center of the ball rolls through semicircles of radii $R_1 - 2 = 100 - 2 = 98$ inches, $R_2 + 2 = 60 + 2 = 62$ inches, and $R_3 - 2 = 80 - 2 = 78$ inches, respectively. The length of the path is then $\pi(98 + 62 + 78) = 238\pi$ inches.