MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

American Mathematics Competitions

63rd Annual

# AMC 12 A

**American Mathematics Contest 12 A**

**Tuesday, February 7, 2012**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*



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*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

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1. **Answer (E):** The distance from $-2$ to $-6$ is $|(-6) - (-2)| = 4$ units. The distance from $-6$ to $5$ is $|5 - (-6)| = 11$ units. Altogether the bug crawls $4 + 11 = 15$ units.

2. **Answer (D):** Because 20 seconds is $\frac{1}{3}$ of a minute, Cagney can frost $5 \div \frac{1}{3} = 15$ cupcakes in five minutes. Because 30 seconds is $\frac{1}{2}$ of a minute, Lacey can frost $5 \div \frac{1}{2} = 10$ cupcakes in five minutes. Altogether they can frost $15 + 10 = 25$ cupcakes in five minutes.

3. **Answer (D):** The volume of the second box is $2 \cdot 3 = 6$ times the volume of the first box. Hence it can hold $6 \cdot 40 = 240$ grams of clay.

4. **Answer (C):** The ratio of blue marbles to red marbles is $3 : 2$. If the number of red marbles is doubled, the ratio will be $3 : 4$, and the fraction of marbles that are red will be $\frac{4}{3+4} = \frac{4}{7}$.

5. **Answer (D):**

   For each blueberry in the fruit salad there are 2 raspberries, 8 cherries, and 24 grapes. Thus there are $1 + 2 + 8 + 24 = 35$ pieces of fruit for each blueberry. Because $280 = 35 \cdot 8$, it follows that there are a total of 8 blueberries, $8 \cdot 2 = 16$ raspberries, $8 \cdot 8 = 64$ cherries, and $8 \cdot 24 = 192$ grapes in the fruit salad. Thus there are 64 cherries.

6. **Answer (D):** Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a + b, a + c, b + c) = (12, 17, 19)$. Thus $2(a + b + c) = (a + b) + (a + c) + (b + c) = 12 + 17 + 19 = 48$, and $a + b + c = 24$. If follows that $(a, b, c) = (24 - 19, 24 - 17, 24 - 12) = (5, 7, 12)$. Therefore the middle number is 7.

7. **Answer (C):** Let $a$ be the initial term and $d$ the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is
$$a + (a + d) + \cdots + (a + 11d) = 12a + 66d = 360,$$
   so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for $a$, namely $a = 8$.

8. **Answer (C):** If the numbers are arranged in the order $a$, $b$, $c$, $d$, $e$, then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a+b+2c+4d+8e}{16}.$$

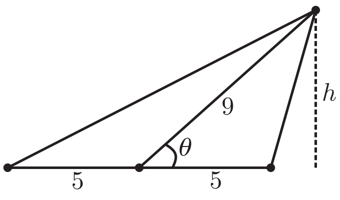
The largest value is obtained by letting $(a, b, c, d, e) = (1, 2, 3, 4, 5)$ or $(2, 1, 3, 4, 5)$, and the smallest value is obtained by letting $(a, b, c, d, e) = (5, 4, 3, 2, 1)$ or $(4, 5, 3, 2, 1)$. In the former case the iterative average is $65/16$, and in the latter case the iterative average is $31/16$, so the desired difference is

$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

9. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

10. **Answer (D):**

The area of a triangle equals one half the product of two sides and the sine of the included angle. Because the median divides the base in half, it partitions the triangle in two triangles with equal areas. Thus $\frac{1}{2} \cdot 5 \cdot 9 \sin\theta = 15$, and $\sin\theta = \frac{2\cdot 15}{5\cdot 9} = \frac{2}{3}$.




OR

The altitude $h$ to the base forms a right triangle with the median as its hypotenuse, and thus $h = 9\sin\theta$. Hence the area of the original triangle is $\frac{1}{2} \cdot 10h = \frac{1}{2} \cdot 10 \cdot 9 \sin\theta = 30$, so $\sin\theta = \frac{2\cdot 30}{10\cdot 9} = \frac{2}{3}$.

11. **Answer (B):**

If Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round, then the game's outcomes will be a permutation of AAAMMC, where the $i^{th}$ letter represents the initial of the winner of the $i^{th}$ round. There are

$$\frac{6!}{3!2!1!} = 60$$

such permutations.

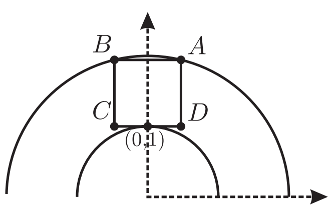
Because each round has only one winner, it follows that $P(\text{M}) + P(\text{C}) = 1 - P(\text{A}) = \frac{1}{2}$. Also $P(\text{M}) = 2P(\text{C})$ and so $P(\text{M}) = \frac{1}{3}$ and $P(\text{C}) = \frac{1}{6}$.

The probability that Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round is therefore

$$\frac{6!}{3!2!1!}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right) = \frac{60}{2^3 \cdot 3^2 \cdot 6} = \frac{5}{36}.$$

12. **Answer (D):**

Suppose by symmetry that $A = (a, b)$ with $a > 0$. Because $ABCD$ is tangent to the circle with equation $x^2 + y^2 = 1$ at $(0, 1)$ and both $A$ and $B$ are on the concentric circle with equation $x^2 + y^2 = 4$, it follows that $B = (-a, b)$. Then the horizontal length of the square is $2a$ and its vertical height is $b-1$. Therefore $2a = b - 1$, or $b = 2a + 1$. Substituting this into the equation $a^2 + b^2 = 4$ leads to the equation $5a^2 + 4a - 3 = 0$. By the quadratic formula, the positive root is $\frac{1}{5}(\sqrt{19} - 2)$, and so the side length $2a$ is $\frac{1}{5}(2\sqrt{19} - 4)$.




13. **Answer (D):** Let the length of the lunch break be $m$ minutes. Then the three painters each worked $480-m$ minutes on Monday, the two helpers worked $372-m$ minutes on Tuesday, and Paula worked $672-m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$\begin{aligned}(p + h)(480 - m) &= 50,\\ h(372 - m) &= 24, \text{and}\end{aligned}$$

$$p(672 - m) = 26.$$

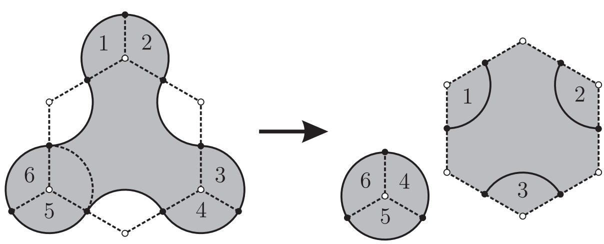
Adding the last two equations gives $672p+372h-mp-mh = 50$, and subtracting this equation from the first one gives $108h - 192p = 0$, so $h = \frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\frac{25p}{9}(480 - m) = 50,$$
$$p(672 - m) = 26.$$

The solution of this system is $p = \frac{1}{24}$ and $m = 48$. Note that $h = \frac{2}{27}$.

14. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4} \cdot 2^2\right) = \pi + 6\sqrt{3}\,.$$




15. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of $BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

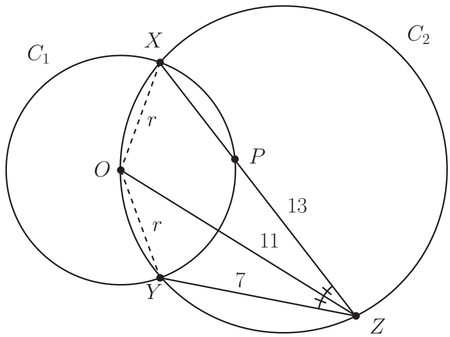
16. **Answer (E):** Let $r$ be the radius of $C_1$. Because $OX = OY = r$, it follows that $\angle OZY = \angle XZO$. Applying the Law of Cosines to triangles $XZO$ and $OZY$ gives

$$\frac{11^2 + 13^2 - r^2}{2 \cdot 11 \cdot 13} = \cos\angle XZO = \cos\angle OZY = \frac{7^2 + 11^2 - r^2}{2 \cdot 7 \cdot 11}.$$

Solving for $r^2$ gives $r^2 = 30$ and so $r = \sqrt{30}$.

OR

Let $P$ be the point on $\overline{XZ}$ such that $ZP = ZY = 7$. Because $\overline{OZ}$ is the bisector of $\angle XZY$, it follows that $\triangle OPZ \cong \triangle OYZ$. Therefore $OP = OY = r$ and thus $P$ is on $C_1$. By the Power of a Point Theorem, $13 \cdot 7 = ZX \cdot ZP = OZ^2 - r^2 = 11^2 - r^2$. Solving for $r^2$ gives $r^2 = 30$ and so $r = \sqrt{30}$.




17. **Answer (B):** For $1 \le j \le 5$, let $S_j = \{5n + j : 0 \le n \le 5\}$. Because no pair of elements in $S$ can have a sum that is divisible by 5, at least one of the sets $S \cap S_1$ and $S \cap S_4$ must be empty. Similarly, at least one of $S \cap S_2$ and $S \cap S_3$ must be empty, and $S \cap S_5$ can contain at most one element. Thus $S$ can contain at most $30 - 6 - 6 - 5 = 13$ elements. An example of a set that meets the requirements is $S = \{1, 2, 6, 7, 11, 12, 16, 17, 21, 22, 26, 27, 30\}$.

OR

The set $S$ from the previous solution shows that size 13 is possible. Consider the following partition of $\{1, 2, \ldots, 30\}$:

$$\{5,10,15,20,25,30\}, \{1,4\}, \{2,3\}, \{6,9\}, \{7,8\}, \{11,14\},$$
$$\{12,13\}, \{16,19\}, \{17,18\}, \{21,24\}, \{22,23\}, \{26,29\}, \{27,28\}.$$

There are 13 sets in this partition, and the sum of any pair of elements in the same part is a multiple of 5. Thus by the pigeon-hole principle any set $S$ with at least 14 elements has at least two elements whose sum is divisible by 5. Therefore 13 is the largest possible size of $S$.

18. **Answer (A):**

Let $a = BC$, $b = AC$, and $c = AB$. Let $D$, $E$, and $F$ be the feet of the perpendiculars from $I$ to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Because $\overline{BF}$ and $\overline{BD}$ are common tangent segments to the incircle of $\triangle ABC$, it follows that $BF = BD$. Similarly, $CD = CE$ and $AE = AF$. Thus

$$\begin{aligned} 2 \cdot BD = BD + BF &= (BC - CD) + (AB - AF) = BC + AB - (CE + AE) \\ &= a + c - b = 25 + 27 - 26 = 26, \end{aligned}$$

so $BD = 13$.

Let $s = \frac{1}{2}(a + b + c) = 39$ be the semiperimeter of $\triangle ABC$ and $r = DI$ the inradius of $\triangle ABC$. The area of $\triangle ABC$ is equal to $rs$ and also equal to $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. Thus

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} = \frac{14 \cdot 13 \cdot 12}{39} = 56.$$

Finally, by the Pythagorean Theorem applied to the right triangle $BDI$, it follows that

$$BI^2 = DI^2 + BD^2 = r^2 + BD^2 = 56 + 13^2 = 56 + 169 = 225,$$

so $BI = 15$.

19. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let $n$ be the number of friends each person has; then $1 \le n \le 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

20. **Answer (B):**

A factor in the product defining $P(x)$ has degree 2012 if and only if the sum of the exponents in $x$ is equal to 2012. Because there is only one way to write 2012 as a sum of distinct powers of 2, namely the one corresponding to its binary expansion $2012 = 11111011100_2$, it follows that the coefficient of $x^{2012}$ is equal to $2^0 \cdot 2^1 \cdot 2^5 = 2^6$.

Note: In general, if $0 \le n \le 2047$ and $n = \sum_{j\in A} 2^j$ for $A \subseteq \{0, 1, 2, \ldots, 10\}$, then the coefficient of $x^n$ is equal to $2^a$ where $a = \binom{11}{2} - \sum_{j\in A} j$.

21. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a - b$, $b - c$, and $c - a$ are integers with their sum equal to 0 and $a \ge b \ge c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c + 3, c + 1, c)$ or $(a, b, c) = (c + 3, c + 2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a - c)(a + c) + (a - b)b = 3(2c + 3) + 2(c + 1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

22. **Answer (C):** Label the vertices of $Q$ as in the figure. Let $m_{xy}$ denote the midpoint of $\overline{v_x v_y}$. Call a segment *long* if it joins midpoints of opposite edges of a face and *short* if it joins midpoints of adjacent edges.

Let $p$ be one of the $k$ planes. Assume $p$ intersects the face $v_1v_2v_3v_4$. First suppose $p$ intersects $v_1v_2v_3v_4$ by a long segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{12}m_{34}}$. Because $p$ intersects the interior of $Q$, it follows that $p$ intersects the face $v_3v_4v_8v_7$. By symmetry there are two cases: 1.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$ and 1.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

In Case 1.1 the plane $p$ is the plane determined by the square $m_{12}m_{34}m_{78}m_{56}$. Note that $p$ contains 4 long segments and by symmetry there are 3 planes like $p$, one for every pair of opposite faces of $Q$.

In Case 1.2 the plane $p$ is determined by the rectangle $m_{12}m_{34}m_{48}m_{15}$. Note that $p$ contains 2 long segments and 2 short segments, and by symmetry there are 12 planes like $p$, one for every edge of $Q$.

Second, suppose $p$ intersects $v_1v_2v_3v_4$ by a short segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{23}m_{34}}$. Again $p$ must intersect the face $v_3v_4v_8v_7$. There are three cases: 2.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{37}}$, 2.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$, and 2.3 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

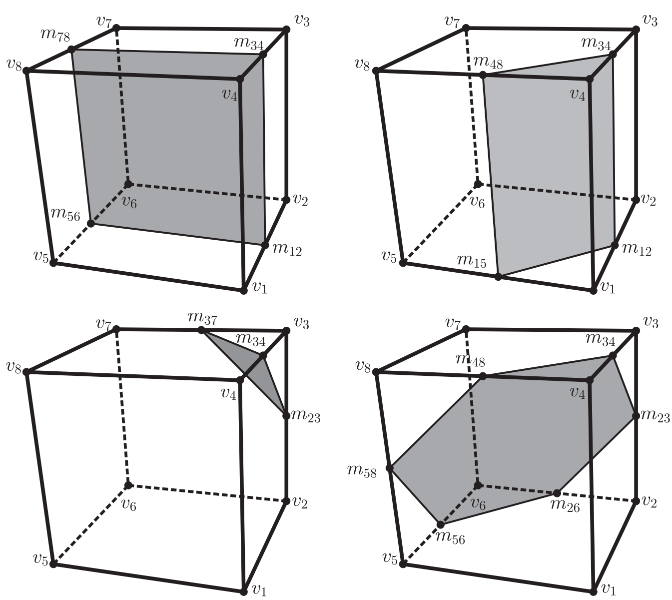
In Case 2.1 the plane $p$ is the plane determined by the triangle $m_{23}m_{34}m_{37}$. Note that $p$ contains 3 short segments and by symmetry there are 8 planes like $p$, one for every vertex of $Q$.

Case 2.2 duplicates Case 1.2.

In Case 2.3 the plane $p$ is determined by the hexagon $m_{23}m_{34}m_{48}m_{58}m_{56}m_{26}$. Note that $p$ contains 6 short segments, and by symmetry there are 4 planes like $p$, one for every pair of opposite vertices of $Q$.

Therefore the maximum possible value of $k$ is $3 + 12 + 8 + 4 = 27$, obtained by considering all possible planes classified so far.

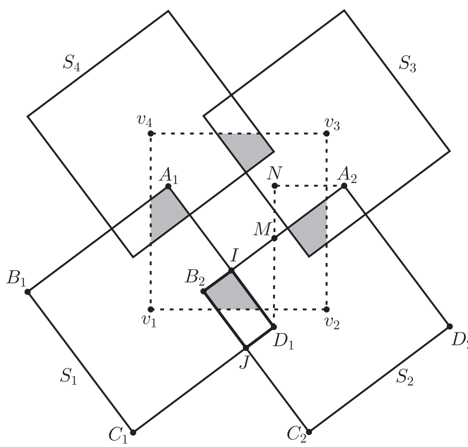
To find the minimum, note that $P \cap S$ consists of 24 short segments and 12 long segments. Every plane $p \in P$ can contain at most 6 short segments; moreover, the union of the 4 planes obtained from Case 2.3 contains all 24 short segments. Similarly, every plane $p \in P$ can contain at most 4 long segments; moreover, the union of the 3 planes obtained from Case 1.1 contains all 12 long segments. Thus the minimum possible value of $k$ is $4 + 3 = 7$, and the required difference is $27 - 7 = 20$.




23. **Answer (C):** Consider the unit square $U$ with vertices $v_1 = (0,0), v_2 = (1,0), v_3 = (1,1)$, and $v_4 = (0,1)$, and the squares $S_i = T(v_i)$ with $i = 1,2,3,4$. Note that $T(v)$ contains $v_i$ if and only if $v \in S_i$. First choose a point $v = (x,y)$ uniformly at random over all pairs of real numbers $(x,y)$ such that $0 \le x \le 1$ and $0 \le y \le 1$. In this case, the probability that $T(v)$ contains $v_i$ and $v_j$ is the

area of the intersection of the squares $U$, $S_i$, and $S_j$. This intersection is empty when $v_iv_j$ is a diagonal of $U$ and it is equal to Area$(U \cap S_i \cap S_j)$ when $v_iv_j$ is a side of $U$. By symmetry, the probability that $T(v)$ contains two vertices of $U$ is $4 \cdot \text{Area}(U \cap S_1 \cap S_2) = 2 \cdot \text{Area}(S_1 \cap S_2)$. By periodicity, this probability is the same as when the point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers $(x, y)$ such that $0 \le x \le 2012$ and $0 \le y \le 2012$.

For $i = 1$ and 2, let $A_i, B_i, C_i$, and $D_i$ be the vertices of $S_i$ in counterclockwise order, where $A_1 = (0.1, 0.7)$ and $A_2 = (1.1, 0.7)$. Then $B_2 = (0.3, 0.1)$ and $D_1 = (0.7, -0.1)$. Let $M = (0.7, 0.4)$ be the midpoint of $A_2B_2$ and $N = (0.7, 0.7)$. Let $I \in \overline{A_2B_2}$ and $J \in \overline{C_1D_1}$ be the points of intersection of the boundaries of $S_1$ and $S_2$. Then $S_1 \cap S_2$ is the rectangle $IB_2JD_1$. Because $D_1, M$, and $N$ are collinear and $D_1M = MA_2 = 0.5$, the right triangles $A_2NM$ and $D_1IM$ are congruent. Hence $ID_1 = NA_2 = 1.1 - 0.7 = 0.4$ and $IB_2 = MB_2 - MI = MB_2 - MN = 0.5 - 0.3 = 0.2$. Therefore Area$(S_1 \cap S_2) = \text{Area}(IB_2JD_1) = 0.2 \cdot 0.4 = 0.08$, and thus the required probability is 0.16.




24. **Answer (C):**

Because $y = a^x$ is decreasing for $0 < a < 1$ and $y = x^b$ is increasing on the interval $[0, \infty)$ for $b > 0$, it follows that

$$1 > a_2 = (0.2011)^{a_1} > (0.201)^{a_1} > (0.201)^1 = a_1,$$
$$a_3 = (0.20101)^{a_2} < (0.2011)^{a_2} < (0.2011)^{a_1} = a_2,$$

and

$$a_3 = (0.20101)^{a_2} > (0.201)^{a_2} > (0.201)^1 = a_1.$$

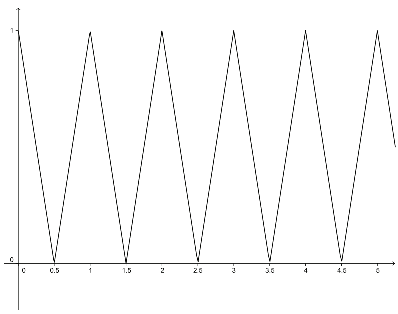
Therefore $1 > a_2 > a_3 > a_1 > 0$. More generally, it can be shown by induction that

$$1 > b_1 = a_2 > b_2 = a_4 > \cdots > b_{1005} = a_{2010}$$

$$> b_{1006} = a_{2011} > b_{1007} = a_{2009} > \cdots > b_{2011} = a_1 > 0.$$

Hence $a_k = b_k$ if and only if $2(k - 1006) = 2011 - k$, so $k = 1341$.

25. **Answer (C):** Because $-1 \le 2\{x\} - 1 \le 1$ it follows that $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$. Thus $0 \le nf(xf(x)) \le n$, and therefore all real solutions $x$ of the required equation are in the interval $[0, n]$. Also $f(x)$ is periodic with period 1, $f(x) = 1 - 2x$ if $0 \le x \le \frac{1}{2}$, and $f(x) = 2x - 1$ if $\frac{1}{2} \le x \le 1$. Thus the graph of $y = f(x)$ for $x \ge 0$ consists of line segments joining the points with coordinates $(k, 1)$, $(k + \frac{1}{2}, 0)$, $(k + 1, 1)$ for integers $k \ge 0$ as shown.




Let $a$ be an integer such that $0 \le a \le n - 1$. Consider the interval $[a, a + \frac{1}{2})$. If $x \in [a, a + \frac{1}{2})$, then $f(x) = |2\{x\} - 1| = |2(x - a) - 1| = 1 + 2a - 2x$ and thus $g(x) := xf(x) = x(1 + 2a - 2x)$. Suppose $a \ge 1$ and $a \le x < y < a + \frac{1}{2}$. Then $2x + 2y - 2a - 1 > 2a - 1 \ge 1$ and so $(y - x)(2x + 2y - 2a - 1) > 0$, which is equivalent to $g(x) = x(1 + 2a - 2x) > y(1 + 2a - 2y) = g(y)$. Thus $g$ is strictly decreasing on $[a, a + \frac{1}{2})$ and so it maps $[a, a + \frac{1}{2})$ bijectively to $(0, a]$ . Thus the graph of the function $y = f(g(x))$ on the interval $[a, a + \frac{1}{2})$ oscillates from 1 to 0 as many times as the graph of the function $y = f(x)$ on the interval $(0, a]$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a, a + \frac{1}{2})$ exactly $2a$ times.

If $a = 0$ and $x \in [a, a + \frac{1}{2})$, then $g(x) = x(1 - 2x)$ satisfies $0 \le g(x) \le \frac{1}{8}$, so $f(g(x)) = 1 - 2g(x) = 4x^2 - 2x + 1$. If $x \in [0, \frac{1}{2})$ and $n \ge 1$, then $0 \le \frac{x}{n} < \frac{1}{2n} \le \frac{1}{2}$. Because $\frac{1}{2} \le 1 - 2g(x) \le 1$, it follows that the parabola $y = f(g(x))$ does not intersect any of the lines with equation $y = \frac{x}{n}$ on the interval $[0, \frac{1}{2})$.

Similarly, if $x \in [a + \frac{1}{2}, a + 1)$, then $f(x) = |2\{x\} - 1| = |2(x - a) - 1| = 2x - 2a - 1$ and $g(x) := xf(x) = x(2x - 2a - 1)$. This time if $a + \frac{1}{2} \le x < y < a + 1$, then $2x + 2y - 2a + 1 \ge 2a + 1 \ge 1$ and so $(x - y)(2x + 2y - 2a + 1) < 0$, which is equivalent to $g(x) < g(y)$. Thus $g$ is strictly increasing on $[a + \frac{1}{2}, a + 1)$ and so it maps $[a + \frac{1}{2}, a + 1)$ bijectively to $[0, a + 1)$. Thus the graph of the function $y = f(g(x))$ on the interval $[a + \frac{1}{2}, a + 1)$ oscillates as many times as the graph of $y = f(x)$ on the interval $[0, a + 1)$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a + \frac{1}{2}, a + 1)$ exactly $2(a + 1)$

times. Therefore the total number of intersections of the line $y = \frac{x}{n}$ and the graph of $y = f(g(x))$ is equal to

$$\sum_{a=0}^{n-1}(2a + 2(a + 1)) = 2\sum_{a=0}^{n-1}(2a + 1) = 2n^2.$$

Finally the smallest $n$ such that $2n^2 \geq 2012$ is $n = 32$ because $2 \cdot 31^2 = 1922$ and $2 \cdot 32^2 = 2048$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Davis, Zuming Feng, Silvia Fernández, Sister Josanne Furey, Peter Gilchrist, Jerrold Grossman, Leon LaSpina, Kevin Wang, David Wells, and LeRoy Wenstrom.

MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

American Mathematics Competitions

63rd Annual

# AMC 12 B

**American Mathematics Contest 12 B**

**Wednesday, February 22, 2012**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*



*Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:*

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

Prof. Bernardo M. Abrego

1. **Answer (C):** There are $18-2=16$ more students than rabbits per classroom. Altogether there are $4\cdot 16=64$ more students than rabbits.

2. **Answer (E):** The width of the rectangle is the diameter of the circle, so the width is $2\cdot 5=10$. The length of the rectangle is $2\cdot 10=20$. Therefore the area of the rectangle is $10\cdot 20=200$.

3. **Answer (D):** Let $h$ be the number of holes dug by the chipmunk. Then the chipmunk hid $3h$ acorns, while the squirrel hid $4(h-4)$ acorns. Since they hid the same number of acorns, $3h=4(h-4)$. Solving gives $h=16$. Thus the chipmunk hid $3\cdot 16=48$ acorns.

4. **Answer (B):** Diana's money is worth 500 dollars and Étienne's money is worth $400\cdot 1.3=520$ dollars. Hence the value of Étienne's money is greater than the value of Diana's money by
$$\frac{520-500}{500}\cdot 100\%=4\%.$$

5. **Answer (A):** The sum of two integers is even if they are both even or both odd. The sum of two integers is odd if one is even and one is odd. Only the middle two integers have an odd sum, namely $41-26=15$. Hence at least one integer must be even. A list satisfying the given conditions in which there is only one even integer is 1, 25, 1, 14, 1, 15.

6. **Answer (A):** Consider $x$ and $y$ as points on the real number line, with $x$ necessarily to the right of $y$. Then $x-y$ is the distance between $x$ and $y$. Xiaoli's rounding moved $x$ to the right and moved $y$ to the left. Therefore the distance between them increased, and her estimate is larger than $x-y$.

   To see that the other answer choices are not correct, let $x=2.9$ and $y=2.1$, and round each by $0.1$. Then $x-y=0.8$ and Xiaoli's estimated difference is $(2.9+0.1)-(2.1-0.1)=1.0$.

7. **Answer (E):** Consider consecutive red, red, green, green, green lights as a unit. There are $5\cdot 6\cdot \frac{1}{12}=2.5$ feet between corresponding lights in successive units. The 3rd red light begins the 2nd unit, and the 21st red light begins the 11th unit. Therefore the distance between the desired lights is $(11-2)\cdot 2.5=22.5$ feet.

8. **Answer (A):** There are 3 choices for Saturday (anything except cake) and for the same reason 3 choices for Thursday. Similarly there are 3 choices for Wednesday, Tuesday, Monday, and Sunday (anything except what was to be served the following day). Therefore there are $3^6 = 729$ possible dessert menus.

OR

If any dessert could be served on Friday, there would be 4 choices for Sunday and 3 for each of the other six days. There would be a total of $4 \cdot 3^6$ dessert menus for the week, and each dessert would be served on Friday with equal frequency. Because cake is the dessert for Friday, this total is too large by a factor of 4. The actual total is $3^6 = 729$.

9. **Answer (B):** Let $x$ be Clea's rate of walking and $r$ be the rate of the moving escalator. Because the distance is constant, $24(x+r) = 60x$. Solving for $r$ yields $r = \frac{3}{2}x$. Let $t$ be the time required for Clea to make the escalator trip while just standing on it. Then $rt = 60x$, so $\frac{3}{2}xt = 60x$. Therefore $t = 40$ seconds.

10. **Answer (B):** Solve the first equation for $y^2$ and substitute into the second equation to get $x^2+x-20 = 0$, so $x = 4$ or $x = -5$. This leads to the intersection points $(-5, 0)$, $(4, 3)$, and $(4, -3)$. The vertical side of the triangle with these three vertices has length $3 - (-3) = 6$, and the horizontal height to that side has length $4 - (-5) = 9$, so its area is $\frac{1}{2} \cdot 6 \cdot 9 = 27$.

11. **Answer (C):** First assume $B = A - 1$. By the definition of number bases,

$$A^2 + 3A + 2 + 4(A - 1) + 3 = 6(A + A - 1) + 9.$$

Simplifying yields $A^2 - 5A - 2 = 0$, which has no integer solutions.

Next assume $B = A + 1$. In this case

$$A^2 + 3A + 2 + 4(A + 1) + 3 = 6(A + A + 1) + 9,$$

which simplifies to $A^2 - 5A - 6 = (A - 6)(A + 1) = 0$. The only positive solution is $A = 6$. Letting $A = 6$ and $B = 7$ in the original equation produces $132_6 + 43_7 = 69_{13}$, or $56 + 31 = 87$, which is true. The required sum is $A + B = 13$.

12. **Answer (E):** By symmetry, half of all such sequences end in zero. Of those, exactly one consists entirely of zeros. Each of the others contains a single subsequence of one or more consecutive ones beginning at position $j$ and ending at position $k$ with $1 \le j \le k \le 19$. Thus the number of sequences that meet the requirements is

$$2\left(1+\sum_{k=1}^{19}\sum_{j=1}^{k}1\right) = 2(1+(1+2+3+\cdots+19)) = 2\left(1+\frac{19\cdot 20}{2}\right) = 382.$$

OR

Let $A$ be the set of zero-one sequences of length 20 where all the zeros appear together, and let $B$ be the equivalent set of sequences where all the ones appear together. Set $A$ contains one sequence with no zeros and 20 sequences with exactly one zero. Each sequence of $A$ with more than one zero has a position where the first zero appears and a position where the last zero appears, so there are $\binom{20}{2} = 190$ such sequences, and thus $|A| = 1+20+190 = 211$. By symmetry $|B| = 211$. A sequence in $A \cap B$ begins with zero and contains from 1 to 20 zeros, or it begins with one and contains from 1 to 20 ones; thus $|A \cap B| = 40$. Therefore the required number of sequences equals

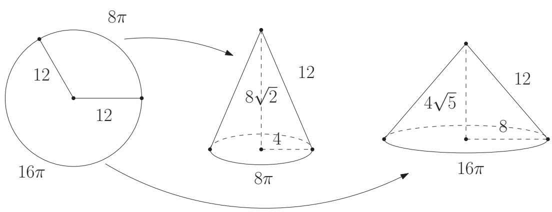
$$|A \cup B| = |A| + |B| - |A \cap B| = 211 + 211 - 40 = 382.$$

13. **Answer (D):** The parabolas have no points in common if and only if the equation $x^2+ax+b = x^2+cx+d$ has no solution. This is true if and only if the lines with equations $y = ax+b$ and $y = cx+d$ are parallel, which happens if and only if $a = c$ and $b \ne d$. The probability that $a = c$ is $\frac{1}{6}$ and the probability that $b \ne d$ is $\frac{5}{6}$, so the probability that the two parabolas have a point in common is $1 - \frac{1}{6}\cdot\frac{5}{6} = \frac{31}{36}$.

14. **Answer (A):** The smallest initial number for which Bernardo wins after one round is the smallest integer solution of $2n + 50 \ge 1000$, which is 475. The smallest initial number for which he wins after two rounds is the smallest integer solution of $2n + 50 \ge 475$, which is 213. Similarly, the smallest initial numbers for which he wins after three and four rounds are 82 and 16, respectively. There is no initial number for which Bernardo wins after more than four rounds. Thus $N = 16$, and the sum of the digits of $N$ is 7.

15. **Answer (C):** Each sector forms a cone with slant height 12. The circumference of the base of the smaller cone is $\frac{120}{360} \cdot 2 \cdot 12 \cdot \pi = 8\pi$. Hence the radius of the base of the smaller cone is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}$. Similarly, the circumference of the base of the larger cone is $16\pi$. Hence the radius of the base of the larger cone is 8 and its height is $4\sqrt{5}$. The ratio of the volume of the smaller cone to the volume of larger cone is

$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$



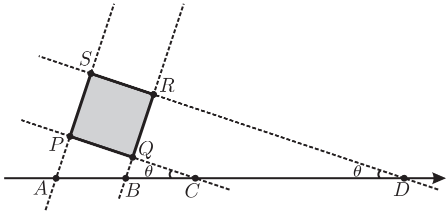

16. **Answer (B):** There are two cases to consider.

Case 1
Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This case can occur in $3 \cdot 6 \cdot 2 = 36$ ways.

Case 2
Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. There are $4! = 24$ ways to assign the songs to these four categories, and the last song can be liked by Amy, Beth, Jo, or no one. This case can occur in $24 \cdot 4 = 96$ ways.

The total number of possibilities is $96 + 36 = 132$.

17. **Answer (C):** Let $A = (3, 0)$, $B = (5, 0)$, $C = (7, 0)$, $D = (13, 0)$, and $\theta$ be the acute angle formed by the line $PQ$ and the $x$-axis. Then $SR = PQ = AB\cos\theta = 2\cos\theta$, and $SP = QR = CD\sin\theta = 6\sin\theta$. Because $PQRS$ is a square, it follows that $2\cos\theta = 6\sin\theta$ and $\tan\theta = \frac{1}{3}$. Therefore lines $SP$ and $RQ$ have slope 3, and lines $SR$ and $PQ$ have slope $-\frac{1}{3}$. Let the points $M = (4, 0)$ and $N = (10, 0)$ be the respective midpoints of segments $AB$ and $CD$. Let $\ell_1$ be the line through $M$ parallel to line $SP$. Let $\ell_2$ be the line through $N$ parallel to line $SR$. Lines $\ell_1$ and $\ell_2$ intersect at the center of the square $PQRS$. Line $\ell_1$ satisfies the equation $y = 3(x - 4)$, and line $\ell_2$ satisfies the equation $y = -\frac{1}{3}(x - 10)$. Thus the lines $\ell_1$ and $\ell_2$ intersect at the point $(4.6, 1.8)$, and the required sum of coordinates is $6.4$.




18. **Answer (B):** If $a_1 = 1$, then the list must be an increasing sequence. Otherwise let $k = a_1$. Then the numbers 1 through $k-1$ must appear in increasing order from right to left, and the numbers from $k$ through 10 must appear in increasing order from left to right. For $2 \le k \le 10$ there are $\binom{9}{k-1}$ ways to choose positions in the list for the numbers from 1 through $k-1$, and the positions of the remaining numbers are then determined. The number of lists is therefore

$$1 + \sum_{k=2}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

19. **Answer (A):** Let $s$ be the length of the octahedron's side, and let $Q_i$ and $Q_i'$ be the vertices of the octahedron on $\overline{P_1P_i}$ and $\overline{P_1'P_i'}$, respectively. If $Q_2$ and $Q_3$ were opposite vertices of the octahedron, then the midpoint $M$ of $\overline{Q_2Q_3}$ would be the center of the octahedron. Because $M$ lies on the plane $P_1P_2P_3$, the vertex of the octahedron opposite $Q_4$ would be outside the cube. Therefore $Q_2$, $Q_3$, and $Q_4$ are all adjacent vertices of the octahedron, and by symmetry so are $Q_2'$, $Q_3'$, and $Q_4'$. For $2 \le i < j \le 4$, the Pythagorean Theorem applied to $\triangle P_1Q_iQ_j$ gives

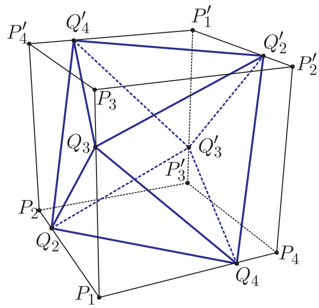
$$s^2 = (Q_iQ_j)^2 = (P_1Q_i)^2 + (P_1Q_j)^2.$$

It follows that $P_1Q_2 = P_1Q_3 = P_1Q_4 = \frac{\sqrt{2}}{2}s$, and by symmetry, $P_1'Q_2' = P_1'Q_3' = P_1'Q_4' = \frac{\sqrt{2}}{2}s$. Consequently $Q_i$ and $Q_i'$ are opposite vertices of the octahedron. The Pythagorean Theorem on $\triangle Q_2P_2P_3'$ and $\triangle Q_3'P_3'Q_2$ gives

$$(Q_2P_3')^2 = (P_2P_3')^2 + (Q_2P_2)^2 = 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2 \quad \text{and}$$

$$s^2 = (Q_2Q_3')^2 = (P_3'Q_3')^2 + (Q_2P_3')^2 = \left(1 - \frac{\sqrt{2}}{2}s\right)^2 + 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2.$$

Solving for $s$ gives $s = \frac{3\sqrt{2}}{4}$.




20. **Answer (D):** Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$ and $AB < CD$. Let $E$ be the point on $\overline{CD}$ such that $CE = AB$. Then $ABCE$ is a parallelogram. Set $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Then the side lengths of $\triangle ADE$ are $b$, $d$, and $c - a$. If one of $b$ or $d$ is equal to 11, say $b = 11$ by symmetry, then $d + (c - a) \leq 7 + (5 - 3) < 11 = d$, which contradicts the triangle inequality. Thus $c = 11$. There are three cases to consider, namely, $a = 3$, $a = 5$, and $a = 7$.

If $a = 3$, then $\triangle ADE$ has side lengths 5, 7, and 8 and by Heron's formula its area is
$$\frac{1}{4}\sqrt{(5+7+8)(7+8-5)(8+5-7)(5+7-8)} = 10\sqrt{3}.$$
The area of $\triangle AEC$ is $\frac{3}{8}$ of the area of $\triangle ADE$, and triangles $ABC$ and $AEC$ have the same area. It follows that the area of the trapezoid is $\frac{1}{2}(35\sqrt{3})$.

If $a = 5$, then $\triangle ADE$ has side lengths 3, 6, and 7, and area
$$\frac{1}{4}\sqrt{(3+6+7)(6+7-3)(7+3-6)(3+6-7)} = 4\sqrt{5}.$$
The area of $\triangle AEC$ is $\frac{5}{6}$ of the area of $\triangle ADE$, and triangles $ABC$ and $AEC$ have the same area. It follows that the area of the trapezoid is $\frac{1}{3}(32\sqrt{5})$.

If $a = 7$, then $\triangle ADE$ has side lengths 3, 4, and 5. Hence this is a right trapezoid with height 3 and base lengths 7 and 11. This trapezoid has area $\frac{1}{2}(3(7+11)) = 27$.

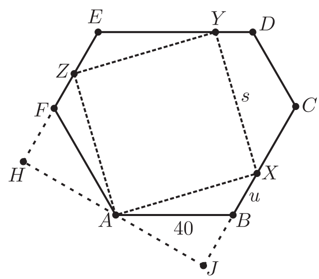
The sum of the three possible areas is $\frac{35}{2}\sqrt{3}+\frac{32}{3}\sqrt{5}+27$. Hence $r_1 = \frac{35}{2}$, $r_2 = \frac{32}{3}$, $r_3 = 27$, $n_1 = 3$, $n_2 = 5$, and $r_1+r_2+r_3+n_1+n_2 = \frac{35}{2}+\frac{32}{3}+27+3+5 = 63+\frac{1}{6}$. Thus the required integer is 63.

21. **Answer (A):** Extend $\overline{EF}$ to $H$ and extend $\overline{CB}$ to $J$ so that $\overline{HJ}$ contains $A$ and is perpendicular to lines $EF$ and $CB$. Let $s$ be the side length of the square and let $u = BX$. Because $\angle ABJ = 60^\circ$, it follows that $BJ = 20$ and $AJ = 20\sqrt{3}$. Then by the Pythagorean Theorem
$$AX^2 = s^2 = (20+u)^2 + (20\sqrt{3})^2.$$

Because $ABCDEF$ is equiangular, it follows that $\overline{ED} \parallel \overline{AB}$ and so $\overline{EY} \parallel \overline{AB}$. Also $\overline{ZY} \parallel \overline{AX}$ and thus it follows that $\angle EYZ = \angle BAX$ and so $\triangle EYZ \cong \triangle BAX$. Thus $EZ = u$. Also, $\angle HZA = 90^\circ - \angle YZE = 90^\circ - \angle AXJ = \angle JAX$; thus $\triangle AXJ \cong \triangle ZAH$ and so $ZH = 20\sqrt{3}$ and $HA = 20 + u$. Moreover, $\angle HFA = 60^\circ$ and so $FH = \frac{HA}{\sqrt{3}} = \frac{1}{\sqrt{3}}(20+u)$. But $EZ + ZH = EF + FH$, and so

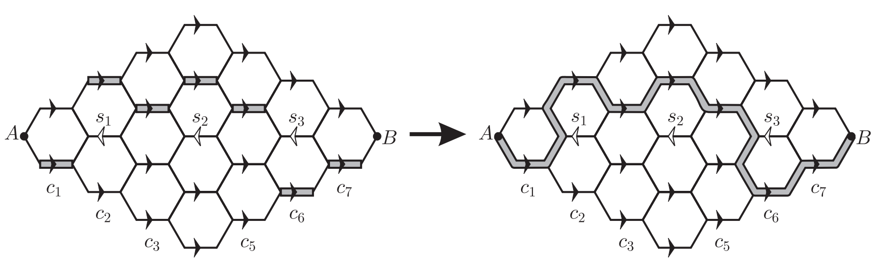
$$u + 20\sqrt{3} = 41(\sqrt{3}-1) + \frac{20+u}{\sqrt{3}}.$$

Solving for $u$ yields $u = 21\sqrt{3} - 20$. Then $s^2 = (21\sqrt{3})^2 + (20\sqrt{3})^2 = 3 \cdot 29^2$ and therefore $s = 29\sqrt{3}$.

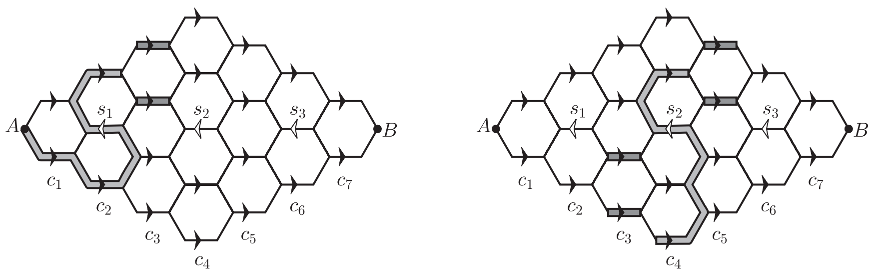



22. **Answer (E):** Label the columns having arrows as $c_1, c_2, c_3, \ldots, c_7$ according to the figure. Call those segments that can be traveled only from left to right *forward segments*. Call the segments $s_1$, $s_2$, and $s_3$, in columns $c_2$, $c_4$, and $c_6$, respectively, which can be traveled only from right to left, *back segments*. Denote $S$ as the set of back segments traveled for a path.

First suppose that $S = \emptyset$. Because it is not possible to travel a segment more than once, it follows that the path is uniquely determined by choosing one forward segment in each of the columns $c_j$. There are 2, 2, 4, 4, 4, 2, and 2 choices for the forward segment in columns $c_1$, $c_2$, $c_3$, $c_4$, $c_5$, $c_6$, and $c_7$, respectively. This gives a total of $2^{10}$ total paths in this case.




Next suppose that $S = \{s_1\}$. The two forward segments in $c_2$, together with $s_1$, need to be part of the path, and once the forward segment from $c_1$ is chosen, the order in which the segments of $c_2$ are traveled is determined. Moreover, there are only 2 choices for possible segments in $c_3$ depending on the last segment traveled in $c_2$, either the bottom 2 or the top 2. For the rest of the columns, the path is determined by choosing any forward segment. Thus the total number of paths in this case is $2 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 2 \cdot 2 = 2^8$, and by symmetry this is also the total for the number of paths when $S = \{s_3\}$. A similar argument gives $2 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 2 = 2^6$ trips for the case when $S = \{s_1, s_3\}$.




Suppose $S = \{s_2\}$. Because $s_2$ is traveled, it follows that 2 forward segments in $c_4$ need to belong to the path, one of them above $s_2$ (2 choices) and the other below it (2 choices). Once these are determined, there are 2 possible choices for the order in which these segments are traveled: the bottom forward segment first, then $s_2$, then the top forward segment, or vice versa. Next, there are only 2 possible forward segments that can be selected in $c_3$ and also only 2 possible forward segments that can be selected in $c_5$. The forward segments in $c_1$, $c_2$, $c_6$, and $c_7$ can be freely selected (2 choices each). This gives a total of $(2^3 \cdot 2 \cdot 2) \cdot 2^4 = 2^9$ paths.

If $S = \{s_1, s_2\}$, then the analysis is similar, except for the last step, where the forward segments of $c_1$ and $c_2$ are determined by the previous choices. Thus there are $(2^3 \cdot 2 \cdot 2) \cdot 2^2 = 2^7$ possibilities, and by symmetry the same number when $S = \{s_2, s_3\}$.

Finally, if $S = \{s_1, s_2, s_3\}$, then in the last step, all forward segments of $c_1$, $c_2$, $c_6$, and $c_7$ are determined by the previous choices and hence there are $2^3 \cdot 2 \cdot 2 = 2^5$ possible paths. Altogether the total number of paths is $2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400$.

23. **Answer (B):** If $z_0^k$ is equal to a positive real $r$, then $1 = |z_0|^k = |z_0^k| = |r| = r$, so $z_0^k = 1$. Suppose that $z_0^k = 1$. If $k = 1$, then $z_0 = 1$, but $P(1) = 4 + a + b + c + d \geq 4$ so $z_0 = 1$ is not a zero of the polynomial. If $k = 2$, then $z_0 = \pm 1$. If $z_0 = -1$, then $0 = P(-1) = (4 - a) + (b - c) + d$ and by assumption $4 \geq a$, $b \geq c$, and $d \geq 0$. Thus $a = 4$, $b = c$, and $d = 0$. Conversely, if $a = 4$, $b = c$, and $d = 0$, then $P(z) = 4z^4 + 4z^3 + bz^2 + bz = z(z + 1)(4z^2 + b)$ satisfies the required conditions. If $k = 3$, then $z_0 = 1$ or $z_0 = \gamma$ where $\gamma$ is any of the roots of $\gamma^2 + \gamma + 1 = 0$. If $z_0 = \gamma$, then $0 = P(\gamma) = 4\gamma + a + b(-1 - \gamma) + c\gamma + d =$

$(a-b)+d+\gamma((4-b)+c)$ and by assumption $a \geq b$, $d \geq 0$, $4 \geq b$, and $c \geq 0$. Thus $a=b$, $d=0$, $b=4$, and $c=0$. Conversely, if $a=b=4$ and $c=d=0$, then $P(z)=4z^4+4z^3+4z^2=4z^2(z^2+z+1)$ satisfies the given conditions because $z_0=\cos(2\pi/3)+i\sin(2\pi/3)$ is a zero of this polynomial. If $k=4$, then $z_0=\pm 1$ or $z_0=\pm i$. If $z_0=\pm i$, then $0=P(\pm i)=4\mp ia-b\pm ic+d=(4-b)+d\mp i(a-c)$ and by assumption $4\geq b$, $d\geq 0$, and $4\geq a\geq b\geq c$. Thus $b=4$, $d=0$, and $a=c=4$. Conversely, if $a=b=c=4$ and $d=0$, then $P(z)=4z^4+4z^3+4z^2+4z=4z(z+1)(z^2+1)$ satisfies the given conditions, but it was already considered in the case when $z_0=-1$. The remaining case is that $z_0^k$ is not a positive real number for $1\leq k\leq 4$. In this case,

$$4z^5-(z-1)P(z)=z^4(4-a)+z^3(a-b)+z^2(b-c)+z(c-d)+d.$$

If $z=z_0$, then the triangle inequality yields

$$\begin{aligned}
4 &= |z_0^4(4-a)+z_0^3(a-b)+z_0^2(b-c)+z_0(c-d)+d| \\
&\leq \left|z_0^4(4-a)\right|+\left|z_0^3(a-b)\right|+\left|z_0^2(b-c)\right|+|z_0(c-d)|+|d| \\
&= |z_0|^4(4-a)+|z_0|^3(a-b)+|z_0|^2(b-c)+|z_0|(c-d)+d \\
&= 4-a+a-b+b-c+c-d+d=4.
\end{aligned}$$

Thus equality must occur throughout. This means that the vectors $v_4=z_0^4(4-a)$, $v_3=z_0^3(a-b)$, $v_2=z_0^2(b-c)$, $v_1=z_0(c-d)$, and $v_0=d$ are parallel and they belong to the same quadrant. If two of these vectors are nonzero, then the quotient must be a positive real number; but dividing the vector with the largest exponent of $z_0$ by the other would yield a positive rational number times $z_0^k$ for some $1\leq k\leq 4$. Because not all of the $v_j$ can be zero, it follows that there is exactly one of them that is nonzero. If $v_0=d\neq 0$ and $v_1=v_2=v_3=v_4=0$, then $4=a=b=c=d$, and $P(z)=4z^4+4z^3+4z^2+4z+4$ satisfies the given conditions because $z_0=\cos(2\pi/5)+i\sin(2\pi/5)$ is a zero of this polynomial. Finally, if $v_j\neq 0$ for some $1\leq j\leq 4$ and the rest are zero, then $4z_0^5=v_j=z_0^j n$ for some positive integer $n$, and so $z_0^{5-j}=\frac{1}{4}n$ is a positive real.

Therefore the complete list of polynomials is: $4z^4+4z^3+4z^2+4z+4$, $4z^4+4z^3+4z^2$, and $4z^4+4z^3+bz^2+bz$ with $0\leq b\leq 4$. The required sum is $20+12+\sum_{b=0}^{4}(8+2b)=32+40+(2+4+6+8)=92$.

24. **Answer (D):** Let $S_N=(f_1(N),f_2(N),f_3(N),\ldots)$. If $N_1$ divides $N_2$, then $f_1(N_1)$ divides $f_1(N_2)$. Thus $S_{N_2}$ is unbounded if $S_{N_1}$ is unbounded. Call $N$ *essential* if $S_N$ is unbounded and $N\leq 400$ is not divisible by any smaller number $n$ such that $S_n$ is unbounded. Assume $N=p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is essential. If $e_j=1$ for some $j$, then $f_1(N)=f_1(\frac{N}{p_j})$. Let $n=\frac{N}{p_j}$ and note that $S_N$ and $S_n$ coincide after the first term and consequently $S_n$ is unbounded. This contradicts the fact that $N$ is essential. Thus $e_j\geq 2$ for all $1\leq j\leq k$. Moreover, $(p_1p_2\cdots p_k)^2\leq p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}=N\leq 400$; thus $p_1p_2\cdots p_k\leq\sqrt{400}=20$. Because $2\cdot 3\cdot 5>20$ it follows that $k\leq 2$.

First analyze the case when $n = 2^a \cdot 3^b$. In that case $f_2(n) = f_1(2^{2b-2} \cdot 3^{a-1}) = 2^{2a-4} \cdot 3^{2b-3}$; thus $S_n$ is unbounded if and only if $a \ge 5$ or $b \ge 4$, and $n$ is essential if and only if $n = 2^5$ or $n = 3^4$.

If $k = 1$, then $N = p^e$ for some prime $p \le 19$. The cases $p = 2$ or $p = 3$ have been considered before. If $p = 5$, then $f_1(5^a) = 2^{a-1} \cdot 3^{a-1}$ and because $a \le 3$, no power of 5 in the given range is essential. If $p = 7$, then $f_1(7^a) = 2^{3a-3}$, and thus $N = 7^3$ is essential. If $p \ge 11$, then $p^3 > 400$. Because $f_1(11^2) = 2^2 \cdot 3$, $f_2(13^2) = f_1(2 \cdot 7) = 1$, $f_1(17^2) = 2 \cdot 3^2$, and $f_2(19^2) = f_1(2^2 \cdot 5) = 3$, no powers of 11, 13, 17, or 19 are essential.

If $k = 2$, then the only possible pairs of primes $(p_1, p_2)$ are $(2, 3)$, $(2, 5)$, $(2, 7)$, and $(3, 5)$. The pair $(2, 3)$ was analyzed before and it yields no essential $N$. If $N = 2^a \cdot 5^b \le 400$ is essential, then $2 \le a \le 4$ and $b = 2$. Moreover $f_1(N) = 2 \cdot 3^a$, so $a = 4$ and thus only $N = 2^4 \cdot 5^2$ is essential in this case. If $(p_1, p_2) = (2, 7)$ or $(3, 5)$ and $N = p_1^{e_1} p_2^{e_2} \le 400$ is essential, then $N \in \{2^2 \cdot 7^2, 2^3 \cdot 7^2, 3^2 \cdot 5^2\}$. Because $f_1(2^2 \cdot 7^2) = 2^3 \cdot 3$, $f_1(2^3 \cdot 7^2) = 2^3 \cdot 3^2$, and $f_1(3^2 \cdot 5^2) = 2^3 \cdot 3$, it follows that there are no essential $N$ in this case.

Therefore the only essential values of $N$ are $2^5 = 32$, $3^4 = 81$, $7^3 = 343$, and $2^4 \cdot 5^2 = 400$. These values have $\lfloor \frac{400}{32} \rfloor = 12$, $\lfloor \frac{400}{81} \rfloor = 4$, $\lfloor \frac{400}{343} \rfloor = 1$, and $\lfloor \frac{400}{400} \rfloor = 1$ multiples, respectively, in the range $1 \le N \le 400$. Because there are no common multiples, the required answer is $12 + 4 + 1 + 1 = 18$.

25. **Answer (B):** First note that the isosceles right triangles $t$ can be excluded from the product because $f(t) = 1$ for these triangles. All triangles mentioned from now on are scalene right triangles. Let $O = (0, 0)$. First consider all triangles $t = \triangle ABC$ with vertices in $S \cup \{O\}$. Let $R_1$ be the reflection with respect to the line with equation $x = 2$. Let $A_1 = R_1(A)$, $B_1 = R_1(B)$, $C_1 = R_1(C)$, and $t_1 = \triangle A_1 B_1 C_1$. Note that $\triangle ABC \cong \triangle A_1 B_1 C_1$ with right angles at $A$ and $A_1$, but the counterclockwise order of the vertices of $t_1$ is $A_1$, $C_1$, and $B_1$. Thus $f(t_1) = \tan(\angle A_1 C_1 B_1) = \tan(\angle ACB)$ and

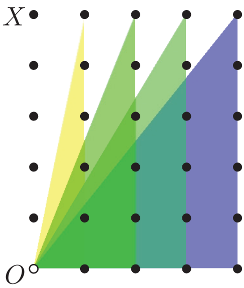
$$f(t)f(t_1) = \tan(\angle CBA)\tan(\angle ACB) = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1.$$

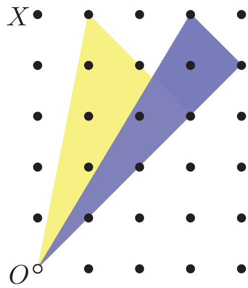
The reflection $R_1$ is a bijection of $S \cup \{O\}$ and it induces a partition of the triangles in pairs $(t, t_1)$ such that $f(t)f(t_1) = 1$. Thus the product over all triangles in $S \cup \{O\}$ is equal to 1, and thus the required product is equal to the reciprocal of $\prod_{t \in T_1} f(t)$, where $T_1$ is the set of triangles with vertices in $S \cup \{O\}$ having $O$ as one vertex.

Let $S_1 = \{(x, y) : x \in \{0, 1, 2, 3, 4\}, \text{ and } y \in \{0, 1, 2, 3, 4\}\}$ and let $R_2$ be the reflection with respect to the line with equation $x = y$. For every right triangle $t = \triangle OBC$ with vertices $B$ and $C$ in $S_1$, let $B_2 = R_2(B)$, $C_2 = R_2(C)$, and $t_2 = \triangle OB_2C_2$. Similarly as before, $R_2$ is a bijection of $S_1$ and it induces a partition of the triangles in pairs $(t, t_2)$ such that $f(t)f(t_2) = 1$. Thus

$\prod_{t\in T_1} f(t) = \prod_{t\in T_2} f(t)$, where $T_2$ is the set of triangles with vertices in $S \cup \{O\}$ with $O$ as one vertex, and another vertex with $y$ coordinate equal to 5.

Next, consider the reflection $R_3$ with respect to the line with equation $y = \frac{5}{2}$. Let $X = (0,5)$. For every right triangle $t = \triangle OXC$ with $C$ in $S$, let $C_3 = R_3(C)$, and $t_3 = \triangle OXC_3$. As before $R_3$ induces a partition of these triangles in pairs $(t, t_3)$ such that $f(t)f(t_3) = 1$. Therefore to calculate $\prod_{t\in T_2} f(t)$, the only triangles left to consider are the triangles of the form $t = \triangle OYZ$ where $Y \in \{(x,5) : x \in \{1,2,3,4\}\}$ and $Z \in S \setminus \{X\}$.




The following argument shows that there are six such triangles. Because the $y$ coordinate of $Y$ is greater than zero, the right angle of $t$ is not at $O$. The slope of the line $OY$ has the form $\frac{5}{x}$ with $1 \le x \le 4$, so if the right angle were at $Y$, then the vertex $Z$ would need to be at least 5 horizontal units away from $Y$, which is impossible. Therefore the right angle is at $Z$. There are 4 such triangles with $Z$ on the $x$-axis, with vertices $O, Z = (x,0)$, and $Y = (x,5)$ for $1 \le x \le 4$. There are two more triangles: with vertices $O$, $Z = (3,3)$, and $Y = (1,5)$, and with vertices $O$, $Z = (4,4)$, and $Y = (3,5)$. The product of the values $f(t)$ over these six triangles is equal to

$$\frac{1}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{4}{5}\cdot\frac{3\sqrt{2}}{2\sqrt{2}}\cdot\frac{4\sqrt{2}}{\sqrt{2}} = \frac{144}{625}.$$

Thus the required product equals

$$\prod_{t\in T} f(t) = \left(\prod_{t\in T_1} f(t)\right)^{-1} = \left(\prod_{t\in T_2} f(t)\right)^{-1} = \left(\frac{144}{625}\right)^{-1} = \frac{625}{144}.$$

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Barb Currier, Steve Davis, Zuming Feng, Silvia Fernández, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Wells, LeRoy Wenstrom.

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1. **Answer (E):** The legs of $\triangle ABE$ have lengths $AB = 10$ and $BE$. Therefore $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $BE = 8$.

2. **Answer (C):** The softball team could only have scored twice as many runs as their opponent when they scored an even number of runs. In those games their opponents scored

$$\frac{2}{2} + \frac{4}{2} + \frac{6}{2} + \frac{8}{2} + \frac{10}{2} = 15 \text{ runs.}$$

In the games the softball team lost, their opponents scored

$$(1+1) + (3+1) + (5+1) + (7+1) + (9+1) = 30 \text{ runs.}$$

The total number of runs scored by their opponents was $15 + 30 = 45$ runs.

3. **Answer (E):** Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

4. **Answer (C):** Factoring $2^{2012}$ from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}.$$

5. **Answer (B):** The total shared expenses were $105+125+175 = 405$ dollars, so each traveler's fair share was $\frac{1}{3} \cdot 405 = 135$ dollars. Therefore $t = 135 - 105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

OR

Because Dorothy paid 20 dollars more than Tom, Sammy must receive 20 more dollars from Tom than from Dorothy.

6. **Answer (B):** If Shenille attempted $x$ three-point shots and $30 - x$ two-point shots, then she scored a total of $\frac{20}{100} \cdot 3 \cdot x + \frac{30}{100} \cdot 2 \cdot (30 - x) = 18$ points.

**Remark:** The given information does not allow the value of $x$ to be determined.

7. **Answer (C):**

   Note that $110 = S_9 = S_7 + S_8 = 42 + S_8$, so $S_8 = 110 - 42 = 68$. Thus $68 = S_8 = S_6 + S_7 = S_6 + 42$, so $S_6 = 68 - 42 = 26$. Similarly, $S_5 = 42 - 26 = 16$, and $S_4 = 26 - 16 = 10$.

8. **Answer (D):**

   Multiplying the given equation by $xy \neq 0$ yields $x^2y + 2y = xy^2 + 2x$. Thus

   $$x^2y - 2x - xy^2 + 2y = x(xy - 2) - y(xy - 2) = (x - y)(xy - 2) = 0.$$

   Because $x - y \neq 0$, it follows that $xy = 2$.

9. **Answer (C):** Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

10. **Answer (D):** If $n$ satisfies the equation $\frac{1}{n} = 0.\overline{ab}$, then $\frac{100}{n} = ab.\overline{ab}$ and subtracting gives $\frac{99}{n} = ab$. The positive factors of 99 are 1, 3, 9, 11, 33, and 99. Only $n = 11$, 33, and 99 give a number $\frac{99}{n}$ consisting of two different digits, namely 09, 03, and 01, respectively. Thus the requested sum is $11+33+99 = 143$.

11. **Answer (C):**

    Let $x = DE$ and $y = FG$. Then the perimeter of $ADE$ is $x + x + x = 3x$, the perimeter of $DFGE$ is $x + (y - x) + y + (y - x) = 3y - x$, and the perimeter of $FBCG$ is $y + (1 - y) + 1 + (1 - y) = 3 - y$. Because the perimeters are equal, it follows that $3x = 3y - x = 3 - y$. Solving this system yields $x = \frac{9}{13}$ and $y = \frac{12}{13}$. Thus $DE + FG = x + y = \frac{21}{13}$.

12. **Answer (A):** Let the angles of the triangle be $\alpha - \delta$, $\alpha$, and $\alpha + \delta$. Then $3\alpha = \alpha - \delta + \alpha + \alpha + \delta = 180^\circ$, so $\alpha = 60^\circ$. There are three cases depending on which side is opposite to the $60^\circ$ angle. Suppose that the triangle is $ABC$ with $\angle BAC = 60^\circ$. Let $D$ be the foot of the altitude from $C$. The triangle $CAD$ is a 30-60-90° triangle, so $AD = \frac{1}{2}AC$ and $CD = \frac{\sqrt{3}}{2}AC$. There are three cases to consider. In each case the Pythagorean Theorem can be used to solve for the unknown side.

    If $AB = 5$, $AC = 4$, and $BC = x$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = 3$. It follows that $x^2 = BC^2 = CD^2 + BD^2 = 21$, so $x = \sqrt{21}$.

If $AB = x$, $AC = 4$, and $BC = 5$, then $AD = 2$, $CD = 2\sqrt{3}$, and $BD = |AB - AD| = |x - 2|$. It follows that $25 = BC^2 = CD^2 + BD^2 = 12 + (x-2)^2$, and the positive solution is $x = 2 + \sqrt{13}$.

If $AB = x$, $AC = 5$, and $BC = 4$, then $AD = \frac{5}{2}$, $CD = \frac{5\sqrt{3}}{2}$, and $BD = |AB - AD| = |x - \frac{5}{2}|$. It follows that $16 = BC^2 = CD^2 + BD^2 = \frac{75}{4} + (x - \frac{5}{2})^2$, which has no solution because $\frac{75}{4} > 16$.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

OR

As in the first solution, there are three cases depending on which side is opposite to the $60^\circ$ angle. In each case, the Law of Cosines can be used to solve for the unknown side. If the unknown side is opposite to the $60^\circ$ angle, then

$$x^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot \cos(60^\circ) = 21,$$

so $x = \sqrt{21}$.

If the side of length 5 is opposite to the $60^\circ$ angle, then

$$5^2 = x^2 + 4^2 - 2 \cdot 4 \cdot x \cdot \cos(60^\circ) = x^2 - 4x + 16,$$

and the positive solution is $2 + \sqrt{13}$.

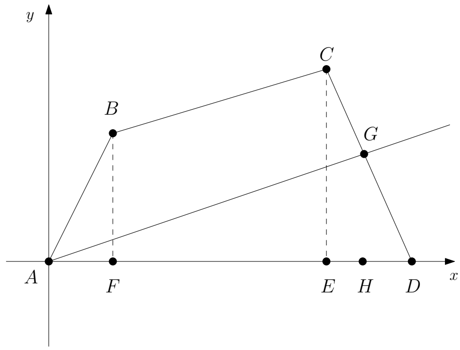
If the side of length 4 is opposite to the $60^\circ$ angle, then

$$4^2 = x^2 + 5^2 - 2 \cdot x \cdot 5 \cdot \cos(60^\circ) = x^2 - 5x + 25,$$

which has no real solutions.

The sum of all possible side lengths is $2 + \sqrt{13} + \sqrt{21}$. The requested sum is $2 + 13 + 21 = 36$.

13. **Answer (B):** Let line $AG$ be the required line, with $G$ on $\overline{CD}$. Divide $ABCD$ into triangle $ABF$, trapezoid $BCEF$, and triangle $CDE$, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p + q + r + s = 58$.




14. **Answer (B):**

Because the terms form an arithmetic sequence,

$$\begin{aligned}\log_{12} y &= \frac{1}{2}(\log_{12} 162 + \log_{12} 1250) = \frac{1}{2}\log_{12}(162 \cdot 1250)\\ &= \frac{1}{2}\log_{12}(2^2 3^4 5^4) = \log_{12}(2 \cdot 3^2 5^2).\end{aligned}$$

Then

$$\begin{aligned}\log_{12} x &= \frac{1}{2}(\log_{12} 162 + \log_{12} y) = \frac{1}{2}\left(\log_{12}(2 \cdot 3^4) + \log_{12}(2 \cdot 3^2 5^2)\right)\\ &= \frac{1}{2}\log_{12}(2^2 3^6 5^2) = \log_{12}(2 \cdot 3^3 5) = \log_{12} 270.\end{aligned}$$

Therefore $x = 270$.

OR

If $(B_k) = (\log_{12} A_k)$ is an arithmetic sequence with common difference $d$, then $(A_k)$ is a geometric sequence with common ratio $r = 12^d$. Therefore $162, x, y, z, 1250$ is a geometric sequence. Let $r$ be their common ratio. Then $1250 = 162r^4$ and $r = \frac{5}{3}$. Thus $x = 162r = 162 \cdot \frac{5}{3} = 270$.

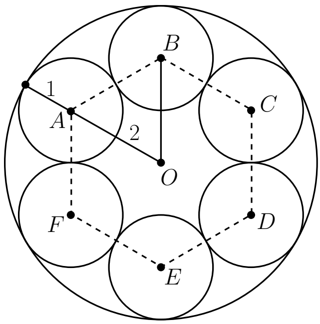
15. **Answer (D):** There are two cases. If Peter and Pauline are given to the same pet store, then there are 4 ways to choose that store. Each of the children must then be assigned to one of the other three stores, and this can be done in $3^3 = 27$ ways. Therefore there are $4 \cdot 27 = 108$ possible assignments in this case. If Peter and Pauline are given to different stores, then there are $4 \cdot 3 = 12$ ways to choose those stores. In this case, each of the children must be assigned to one of the other two stores, and this can be done in $2^3 = 8$ ways. Therefore there are $12 \cdot 8 = 96$ possible assignments in this case. The total number of assignments is $108 + 96 = 204$.

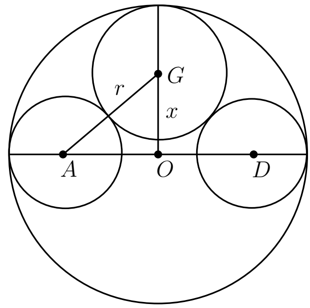
16. **Answer (E):** Let $a$, $b$, and $c$ be the number of rocks in piles $A$, $B$, and $C$, respectively. Then
$$\frac{40a+50b}{a+b}=43 \text{ and } 7b=3a.$$
Because 7 and 3 are relatively prime, there is a positive integer $k$ such that $a=7k$ and $b=3k$. Let $\mu_C$ equal the mean weight in pounds of the rocks in $C$ and $\mu_{BC}$ equal the mean weight in pounds of the rocks in $B$ and $C$. Then
$$\frac{40\cdot 7k+\mu_C\cdot c}{7k+c}=44, \text{ so } \mu_C=\frac{28k+44c}{c},$$
and
$$\mu_{BC}=\frac{50\cdot 3k+(28k+44c)}{3k+c}=\frac{178k+44c}{3k+c}.$$
Clearing the denominator and rearranging yields $(\mu_{BC}-44)c=(178-3\mu_{BC})k$. Because the mean weight of the rocks in the combined piles $A$ and $C$ is 44 pounds, and the mean weight of the rocks in $B$ is greater than the mean weight of the rocks in $A$, it follows that the mean weight of the rocks in $B$ and $C$ must be greater than 44 pounds. Thus $(\mu_{BC}-44)c>0$ and therefore $178-3\mu_{BC}$ must be greater than zero. This implies that $\mu_{BC}<\frac{178}{3}=59\frac{1}{3}$. If $k=15c$ and $\mu_C=464$, then $\mu_{BC}=59$. Thus the greatest possible integer value for the weight in pounds of the combined piles $B$ and $C$ is 59.

17. **Answer (D):** For $1\le k\le 11$, the number of coins remaining in the chest before the $k^{\text{th}}$ pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are $n$ coins left for the $12^{\text{th}}$ pirate to take, the number of coins originally in the chest is
$$\frac{12^{11}\cdot n}{11!}=\frac{2^{22}\cdot 3^{11}\cdot n}{2^8\cdot 3^4\cdot 5^2\cdot 7\cdot 11}=\frac{2^{14}\cdot 3^7\cdot n}{5^2\cdot 7\cdot 11}.$$
The smallest value of $n$ for which this is a positive integer is $5^2\cdot 7\cdot 11=1925$. In this case there are
$$2^{14}\cdot 3^7\cdot\frac{11!}{(12-k)!\cdot 12^{k-1}}$$
coins left for the $k^{\text{th}}$ pirate to take, and note that this amount is an integer for each $k$. Hence the $12^{\text{th}}$ pirate receives 1925 coins.

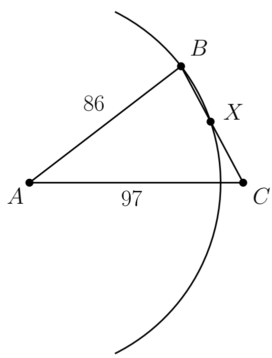
18. **Answer (B):** Let the vertices of the regular hexagon be labeled in order $A$, $B$, $C$, $D$, $E$, and $F$. Let $O$ be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center $G$ and radius $r$. Because the centers of the six small spheres are each a distance 2 from $O$ and the small spheres have radius 1, the radius of the largest sphere is 3. Because $G$

is equidistant from $A$ and $D$, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let $x$ be the distance from $G$ to $O$. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at $O$.




19. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and $CX$ are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.




20. **Answer (B):** Consider the elements of $S$ as integers modulo 19. Assume $a \succ b$. If $a > b$, then $a - b \le 9$. If $a < b$, then $b - a > 9$; that is $b - a \ge 10$ and so $(a + 19) - b \le 9$. Thus $a \succ b$ if and only if $0 < (a - b) \pmod{19} \le 9$.

Suppose that $(x, y, z)$ is a triple in $S \times S \times S$ such that $x \succ y$, $y \succ z$, and $z \succ x$. There are 19 possibilities for the first entry $x$. Once $x$ is chosen, $y$ can equal

$x+i$ for any $i$, $1 \le i \le 9$. Then $z$ is at most $x+9+i$ and at least $x+10$, so once $y$ is chosen, there are $i$ possibilities for the third element $z$.

The number of required triples is equal to $19(1+2+\cdots+9) = 19 \cdot \frac{1}{2} \cdot 9 \cdot 10 = 19 \cdot 45 = 855$.

21. **Answer (A):**

Let $A_n = \log(n + \log((n-1) + \log(\cdots + \log(3 + \log 2) \cdots)))$. Note that $0 < \log 2 = A_2 < 1$. If $0 < A_{k-1} < 1$, then $k < k + A_{k-1} < k+1$. Hence $0 < \log k < \log(k + A_{k-1}) = A_k < \log(k+1) \le 1$, as long as $\log k > 0$ and $\log(k+1) \le 1$, which occurs when $2 \le k \le 9$. Thus $0 < A_n < 1$ for $2 \le n \le 9$.

Because $0 < A_9 < 1$, it follows that $10 < 10 + A_9 < 11$, and so $1 = \log(10) < \log(10 + A_9) = A_{10} < \log(11) < 2$. If $1 < A_{k-1} < 2$, then $k+1 < k + A_{k-1} < k+2$. Hence $1 < \log(k+1) < \log(k + A_{k-1}) = A_k < \log(k+2) \le 2$, as long as $\log(k+1) > 1$ and $\log(k+2) \le 2$, which occurs when $10 \le k \le 98$. Thus $1 < A_n < 2$ for $10 \le n \le 98$.

In a similar way, it can be proved that $2 < A_n < 3$ for $99 \le n \le 997$, and $3 < A_n < 4$ for $998 \le n \le 9996$.

For $n = 2012$, it follows that $3 < A_{2012} < 4$, so $2016 < 2013 + A_{2012} < 2017$ and $\log 2016 < A_{2013} < \log 2017$.

22. **Answer (E):** Let $n$ be a 6-digit palindrome, $m = \frac{n}{11}$, and suppose $m$ is a palindrome as well. First, if $m$ is a 4-digit number, then $n = 11m < 11 \cdot 10^5 = 10^6 + 10^5$. Thus the first and last digit of $n$ is 1. Thus the last digit of $m$ is 1 and then the first digit of $m$ must be 1 as well. Then $m \le 1991 < 2000$ and $n = 11m < 11 \cdot 2000 = 22\,000$, which is a contradiction. Therefore $m$ is a 5-digit number $abcba$. If $a+b \le 9$ and $b+c \le 9$, then there are no carries in the sum $n = 11m = abcba0 + abcba$; thus the digits of $n$ in order are $a$, $a+b$, $b+c$, $b+c$, $a+b$, and $a$. Conversely, if $a+b \ge 10$, then the first digit of $n$ is $a+1$ and the last digit $a$; and if $a+b \le 9$ but $b+c \ge 10$, then the second digit of $n$ is $a+b+1$ if $a+b < 9$, or 0 if $a+b = 9$, and the previous to last digit is $a+b$. In any case $n$ is not a palindrome. Therefore $n = 11m$ is a palindrome if and only if $a+b \le 9$ and $b+c \le 9$.

Thus the number of pairs $(m,n)$ is equal to

$$\sum_{b=0}^{9} \sum_{c=0}^{9-b} \sum_{a=1}^{9-b} 1 = \sum_{b=0}^{9} (10-b)(9-b).$$

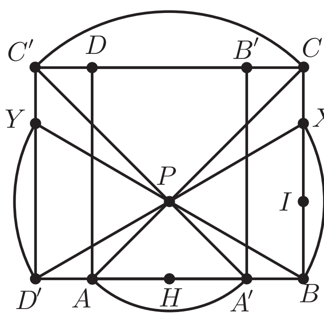
Letting $j = 10 - b$ gives

$$\sum_{j=1}^{10} j(j-1) = \frac{10 \cdot 11 \cdot 21}{6} - \frac{10 \cdot 11}{2} = 330.$$

The total number of 6-digit palindromes *abccba* is determined by 10 choices for each of $b$ and $c$, and 9 choices for $a$, for a total of $9 \cdot 10^2 = 900$. Thus the required probability is $\frac{330}{900} = \frac{11}{30}$.

23. **Answer (C):**

Assume that the vertices of $ABCD$ are labeled in counterclockwise order. Let $A'$, $B'$, $C'$, and $D'$ be the images of $A$, $B$, $C$, and $D$, respectively, under the rotation. Because $\triangle A'PA$ and $\triangle C'PC$ are isosceles right triangles, points $A'$ and $C'$ are on lines $AB$ and $CD$, respectively. Moreover, because $AP = \sqrt{2}$ and $PC = AC - AP = \sqrt{2}(\sqrt{3}+1) - \sqrt{2} = \sqrt{6}$, it follows that $AA' = \sqrt{2}AP = 2$ and $CC' = \sqrt{2}CP = 2\sqrt{3}$. By symmetry, points $B'$ and $D'$ are on lines $CD$ and $AB$, respectively. Let $X \neq B$ and $Y \neq D'$ be the intersections of $\overline{BC}$ and $\overline{C'D'}$, respectively, with the circle centered at $P$ with radius $PB$. Note that $PD' = PD = PB$, so this circle also contains $D'$. Therefore the required region consists of sectors $APA'$, $BPX$, $CPC'$, and $YPD'$, and triangles $BPA'$, $CPX$, $YPC'$, and $APD'$.




Sector $APA'$ has area $\frac{1}{4} \cdot (\sqrt{2})^2\pi = \frac{\pi}{2}$, and sector $CPC'$ has area $\frac{1}{4} \cdot (\sqrt{6})^2\pi = \frac{3\pi}{2}$. Let $H$ and $I$ be the midpoints of $\overline{AA'}$ and $\overline{BX}$, respectively. Then $PH = AH = \frac{\sqrt{2}}{2}AP = 1$, and $PI = HB = AB - AH = \sqrt{3}$. Thus $\triangle BPH$ is a 30-60-90° triangle, implying that $PB = 2$ and $\triangle XPB$ is equilateral. Therefore congruent sectors $BPX$ and $YPD'$ each have area $\frac{1}{6} \cdot 2^2\pi = \frac{2\pi}{3}$.

Congruent triangles $BPA'$ and $D'PA$ each have altitude $PH = 1$ and base $A'B = AB - AH - HA' = \sqrt{3} - 1$, so each has area $\frac{1}{2}(\sqrt{3}-1)$. Congruent triangles $CPX$ and $C'PY$ each have altitude $PI = \sqrt{3}$ and base $XC = BC - BX = \sqrt{3} - 1$, so each has area $\frac{1}{2}(3 - \sqrt{3})$.

The area of the entire region is

$$\frac{\pi}{2} + \frac{3\pi}{2} + 2 \cdot \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}-1}{2}\right) + 2\left(\frac{3-\sqrt{3}}{2}\right) = \frac{10\pi + 6}{3},$$

and $a + b + c = 10 + 6 + 3 = 19$.

24. **Answer (E):** Assume without loss of generality that the regular 12-gon is inscribed in a circle of radius 1. Every segment with endpoints in the 12-gon subtends an angle of $\frac{360}{12}k = 30k$ degrees for some $1 \le k \le 6$. Let $d_k$ be the length of those segments that subtend an angle of $30k$ degrees. There are 12 such segments of length $d_k$ for every $1 \le k \le 5$ and 6 segments of length $d_6$. Because $d_k = 2\sin(15k^\circ)$, it follows that $d_2 = 2\sin(30^\circ) = 1$, $d_3 = 2\sin(45^\circ) = \sqrt{2}$, $d_4 = 2\sin(60^\circ) = \sqrt{3}$, $d_6 = 2\sin(90^\circ) = 2$,

$$\begin{aligned} d_1 &= 2\sin(15^\circ) = 2\sin(45^\circ - 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) - 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}-\sqrt{2}}{2}, \text{ and} \\ d_5 &= 2\sin(75^\circ) = 2\sin(45^\circ + 30^\circ) \\ &= 2\sin(45^\circ)\cos(30^\circ) + 2\sin(30^\circ)\cos(45^\circ) = \frac{\sqrt{6}+\sqrt{2}}{2}. \end{aligned}$$

If $a \le b \le c$, then $d_a \le d_b \le d_c$ and the segments with lengths $d_a$, $d_b$, and $d_c$ do not form a triangle with positive area if and only if $d_c \ge d_a + d_b$. Because $d_2 = 1 < \sqrt{6} - \sqrt{2} = 2d_1 < \sqrt{2} = d_3$, it follows that for $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5),(1,1,6)\}$, the segments of lengths $d_a$, $d_b$, $d_c$ do not form a triangle with positive area. Similarly,

$$\begin{aligned} d_3 &= \sqrt{2} < \frac{\sqrt{6}-\sqrt{2}}{2} + 1 = d_1 + d_2 < \sqrt{3} = d_4, \\ d_4 &< d_5 = \frac{\sqrt{6}+\sqrt{2}}{2} = \frac{\sqrt{6}-\sqrt{2}}{2} + \sqrt{2} = d_1 + d_3, \text{ and} \\ d_5 &< d_6 = 2 = 1 + 1 = 2d_2, \end{aligned}$$

so for $(a,b,c) \in \{(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$, the segments of lengths $d_a$, $d_b$, $d_c$ do not form a triangle with positive area. Finally, if $a \ge 2$ and $b \ge 3$, then $d_a + d_b \ge d_2 + d_3 = 1 + \sqrt{2} > 2 \ge d_c$, and also if $a \ge 3$, then $d_a + d_b \ge 2d_3 = 2\sqrt{2} > 2 = d_c$. Therefore the complete list of forbidden triples $(d_a, d_b, d_c)$ is given by $(a,b,c) \in \{(1,1,3), (1,1,4), (1,1,5),(1,1,6),(1,2,4),(1,2,5),(1,2,6),(1,3,5),(1,3,6),(2,2,6)\}$.

For each $(a,b,c) \in \{(1,1,3),(1,1,4),(1,1,5)\}$, there are $\binom{12}{2}$ pairs of segments of length $d_a$ and 12 segments of length $d_c$. For each $(a,b,c) \in \{(1,1,6),(2,2,6\}\}$, there are $\binom{12}{2}$ pairs of segments of length $d_a$ and 6 segments of length $d_c$. For each $(a,b,c) \in \{(1,2,4),(1,2,5),(1,3,5)\}$, there are $12^3$ triples of segments with lengths $d_a$, $d_b$, and $d_c$. Finally, for each $(a,b,c) \in \{(1,2,6),(1,3,6)\}$, there are $12^2$ pairs of segments with lengths $d_a$ and $d_b$, and 6 segments of length $d_c$. Because the total number of triples of segments equals $\binom{\binom{12}{2}}{3} = \binom{66}{3}$, the required probability equals

$$\begin{aligned} &1 - \frac{3 \cdot 12 \cdot \binom{12}{2} + 2 \cdot 6 \cdot \binom{12}{2} + 3 \cdot 12^3 + 2 \cdot 12^2 \cdot 6}{\binom{66}{3}} \\ &= 1 - \frac{63}{286} = \frac{223}{286}. \end{aligned}$$

25. **Answer (A):** Let $H = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. If $z_1, z_2 \in H$ and $f(z_1) = f(z_2)$, then $z_1^2 - z_2^2 + i(z_1 - z_2) = (z_1 - z_2)(z_1 + z_2 + i) = 0$. Because $\operatorname{Im}(z_1) > 0$ and $\operatorname{Im}(z_2) > 0$, it follows that $z_1 + z_2 + i \neq 0$. Thus $z_1 = z_2$; that is, the function $f$ is one-to-one on $H$. Let $r$ be a positive real number. Note that $f(r) = r^2+1+ir$ describes the top part of the parabola $x = y^2+1$. Similarly, $f(-r) = r^2+1-ir$ describes the bottom part of the parabola $x = y^2 + 1$. Because $f(i) = -1$, it follows that the image set $f(H)$ equals $\{w \in \mathbb{C} : \operatorname{Re}(w) < (\operatorname{Im}(w))^2 + 1\}$. Thus the set of complex numbers $w \in f(H)$ with integer real and imaginary parts of absolute value at most 10 is equal to

$$S = \{w = a + ib \in \mathbb{C} : a, b \in \mathbb{Z},\ |a| \le 10,\ |b| \le 10, \text{ and } a < b^2 + 1\}.$$

Because $f$ is one-to-one, the required answer is $|f^{-1}(S)| = |S|$ and

$$\begin{aligned} |S| &= 21^2 - \sum_{b=-3}^{3} \sum_{a=b^2+1}^{10} 1 = 441 - \sum_{b=-3}^{3} (10 - b^2) \\ &= 441 - (1 + 6 + 9 + 10 + 9 + 6 + 1) = 399. \end{aligned}$$

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Tom Butts, Steve Davis, Doug Faires, Michelle Ghrist, Jerry Grossman, Elign Johnston, Dan Kennedy, Joe Kennedy, Cap Khoury, Patrick Vennebush, Kevin Wang, Dave Wells, and LeRoy Wenstrom.

1. **Answer (C):** The difference between the high and low temperatures was 16 degrees, so the difference between each of these and the average temperature was 8 degrees. The low temperature was 8 degrees less than the average, so it was $3° - 8° = -5°$.

2. **Answer (A):** The garden is $2 \cdot 15 = 30$ feet wide and $2 \cdot 20 = 40$ feet long. Hence Mr. Green expects $\frac{1}{2} \cdot 30 \cdot 40 = 600$ pounds of potatoes.

3. **Answer (D):** The number 201 is the $1^{\text{st}}$ number counted when proceeding backwards from 201 to 3. In turn, 200 is the $2^{\text{nd}}$ number, 199 is the $3^{\text{rd}}$ number, and $x$ is the $(202-x)^{\text{th}}$ number. Therefore 53 is the $(202-53)^{\text{th}}$ number, which is the $149^{\text{th}}$ number.

4. **Answer (B):** Let $D$ equal the distance traveled by each car. Then Ray's car uses $\frac{D}{40}$ gallons of gasoline and Tom's car uses $\frac{D}{10}$ gallons of gasoline. The cars combined miles per gallon of gasoline is
$$\frac{2D}{\left(\frac{D}{40} + \frac{D}{10}\right)} = 16.$$

5. **Answer (C):** The sum of all the ages is $55 \cdot 33 + 33 \cdot 11 = 33 \cdot 66$, so the average of all the ages is
$$\frac{33 \cdot 66}{55 + 33} = \frac{33 \cdot 66}{88} = \frac{33 \cdot 3}{4} = 24.75\,.$$

6. **Answer (B):** By completing the square the equation can be rewritten as follows:
$$x^2 + y^2 = 10x - 6y - 34,$$
$$x^2 - 10x + 25 + y^2 + 6y + 9 = 0,$$
$$(x-5)^2 + (y+3)^2 = 0.$$
Therefore $x = 5$ and $y = -3$, so $x + y = 2\,$.

7. **Answer (E):** Note that Jo starts by saying 1 number, and this is followed by Blair saying 2 numbers, then Jo saying 3 numbers, and so on. After someone completes her turn after saying the number $n$, then $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

numbers have been said. If $n = 9$ then 45 numbers have been said. Therefore there are $53 - 45 = 8$ more numbers that need to be said. The $53^{\text{rd}}$ number said is 8.

8. **Answer (B):** The solution to the system of equations $3x - 2y = 1$ and $y = 1$ is $B = (x, y) = (1, 1)$. The perpendicular distance from $A$ to $\overline{BC}$ is 3. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot BC = 3$, so $BC = 2$. Thus point $C$ is 2 units to the right or to the left of $B = (1, 1)$. If $C = (-1, 1)$ then the line $AC$ is vertical and the slope is undefined. If $C = (3, 1)$, then the line $AC$ has slope $\frac{1-(-2)}{3-(-1)} = \frac{3}{4}$.

9. **Answer (C):** Because $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, the largest perfect square that divides $12!$ is $2^{10} \cdot 3^4 \cdot 5^2$ which has square root $2^5 \cdot 3^2 \cdot 5$. The sum of the exponents is $5 + 2 + 1 = 8$.

10. **Answer (E):** After Alex makes $m$ exchanges at the first booth and $n$ exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m - n \geq 74$ and $3n - m \geq 73$. Equality can be achieved when $(m, n) = (59, 44)$, and Alex will have $59 + 44 = 103$ silver tokens.

    Note that the following exchanges produce 103 silver tokens:

| | Red Tokens | Blue Tokens | Silver Tokens |
|---|---|---|---|
| Exchange 75 blue tokens | 100 | 0 | 25 |
| Exchange 100 red tokens | 0 | 50 | 75 |
| Exchange 48 blue tokens | 16 | 2 | 91 |
| Exchange 16 red tokens | 0 | 10 | 99 |
| Exchange 9 blue tokens | 3 | 1 | 102 |
| Exchange 2 red tokens | 1 | 2 | 103 |

11. **Answer (A):** Suppose that the two bees start at the origin and that the positive directions of the $x$, $y$, and $z$ coordinate axes correspond to the directions east, north, and up, respectively. Note that the bees are always getting farther apart from each other. After bee $A$ has traveled 7 feet it will have gone 3 feet north, 2 feet east, and 2 feet up. Its position would be the point $(2, 3, 2)$. In the same time bee $B$ will have gone 4 feet south and 3 feet west, and its position would be the point $(-3, -4, 0)$. This puts them at a distance

$$\sqrt{(2 - (-3))^2 + (3 - (-4))^2 + 2^2} = \sqrt{78} < 10$$

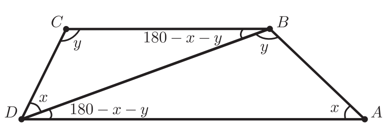
After this moment, bee A will travel east to the point $(3, 3, 2)$ and bee B will travel west to the point $(-4, -4, 0)$. Their distance after traveling one foot will be

$$\sqrt{(3-(-4))^2 + (3-(-4))^2 + 2^2} = \sqrt{102} > 10.$$

Hence bee $A$ is traveling east and bee $B$ is traveling west when they are exactly 10 feet away from each other.

12. **Answer (D):** Cities $C$ and $E$ and the roads leading in and out of them can be replaced by a second $A$–$D$ road and a second $B$–$D$ road, respectively. If routes are designated by the list of cities they visit in order, then there are 4 types of routes: $ABDADB$, $ADABDB$, $ADBADB$, and $ADBDAB$. Each type of route represents 4 actual routes, because the trip between $A$ and $D$ can include the detour through $E$ either the first or the second time, and a similar choice applies for the trip between $B$ and $D$. Therefore there are $4 \cdot 4 = 16$ different routes.

13. **Answer (D):** Let the degree measures of the angles be as shown in the figure. The angles of a triangle form an arithmetic progression if and only if the median angle is $60°$, so one of $x$, $y$, or $180 - x - y$ must be equal to 60. By symmetry of the role of the triangles $ABD$ and $DCB$, assume that $x \le y$. Because $x \le y < 180 - x$ and $x < 180 - y \le 180 - x$, it follows that the arithmetic progression of the angles in $ABCD$ from smallest to largest must be either $x, y, 180-y, 180-x$ or $x, 180-y, y, 180-x$. Thus either $x+180-y = 2y$, in which case $3y = x + 180$; or $x + y = 2(180 - y)$, in which case $3y = 360 - x$. Neither of these is compatible with $y = 60$ (the former forces $x = 0$ and the latter forces $x = 180$), so either $x = 60$ or $x + y = 120$.




First suppose that $x = 60$. If $3y = x + 180$, then $y = 80$, and the sequence of angles in $ABCD$ is $(x, y, 180 - y, 180 - x) = (60, 80, 100, 120)$. If $3y = 360 - x$, then $y = 100$, and the sequence of angles in $ABCD$ is $(x, 180 - y, y, 180 - x) = (60, 80, 100, 120)$. Finally, suppose that $x+y = 120$. If $3y = x+180$, then $y = 75$, and the sequence of angles in $ABCD$ is $(x, y, 180-y, 180-x) = (45, 75, 105, 135)$. If $3y = 360 - x$, then $y = 120$ and $x = 0$, which is impossible.

Therefore, the sum in degrees of the two largest possible angles is $105 + 135 = 240$.

14. **Answer (C):** Let the two sequences be $(a_n)$ and $(b_n)$, and assume without loss of generality that $a_1 < b_1$. The definitions of the sequences imply that

$a_7 = 5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Because 5 and 8 are relatively prime, 8 divides $b_1 - a_1$ and 5 divides $a_2 - b_2$. It follows that $a_1 \le b_1 - 8 \le b_2 - 8 \le a_2 - 13$. The minimum value of $N$ results from choosing $a_1 = 0$, $b_1 = b_2 = 8$, and $a_2 = 13$, in which case $N = 104$.

15. **Answer (B):** The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. There must be a factor of 61 in the numerator, so $a_1 \ge 61$. Since $a_1!$ will have a factor of 59 and 2013 does not, there must be a factor of 59 in the denominator, and $b_1 \ge 59$. Thus $a_1 + b_1 \ge 120$, and this minimum value can be achieved only if $a_1 = 61$ and $b_1 = 59$. Furthermore, this minimum value is attainable because

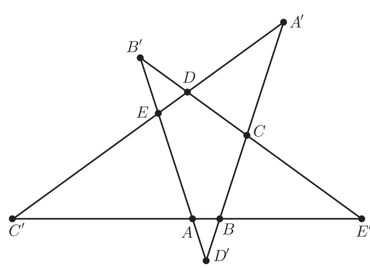
$$2013 = \frac{(61!)(11!)(3!)}{(59!)(10!)(5!)}.$$

Thus $|a_1 - b_1| = a_1 - b_1 = 61 - 59 = 2$.

16. **Answer (A):** The sum of the internal angles of the pentagon $ABCDE$ is $3 \cdot 180° = 540°$ and by assumption all internal angles are equal, so they are all equal to $\frac{1}{5}(540°) = 108°$. Therefore the supplementary angles at each of the vertices are all equal to $180° - 108° = 72°$. It follows that all the triangles making up the points of the star are isosceles triangles with angles measuring $72°$, $72°$, and $36°$. Label the rest of the vertices of the star as in the figure. By the above argument, there is a constant $c$ such that $A'C = A'D = c \cdot CD$ and similar expressions for the other four points of the star. Therefore the required perimeter equals

$$A'C + A'D + B'D + B'E + C'A + C'E + D'A + D'B + E'B + E'C = \\ 2c(CD + DE + EA + AB + BC) = 2c,$$

and therefore the maximum and minimum values are the same and their difference is 0.




Note: The constant $c$ equals $\frac{1}{2}\csc(\frac{\pi}{10}) = \frac{1}{2}(\sqrt{5} + 1)$.

17. **Answer (D):**

From the equations, $a+b=2-c$ and $a^2+b^2=12-c^2$. Let $x$ be an arbitrary real number, then $(x-a)^2+(x-b)^2 \geq 0$; that is, $2x^2-2(a+b)x+(a^2+b^2) \geq 0$. Thus

$$2x^2-2(2-c)x+(12-c^2) \geq 0$$

for all real values $x$. That means the discriminant $4(2-c)^2-4\cdot 2(12-c^2) \leq 0$. Simplifying and factoring gives $(3c-10)(c+2) \leq 0$. So the range of values of $c$ is $-2 \leq c \leq \frac{10}{3}$. Both maximum and minimum are attainable by letting $(a,b,c)=(2,2,-2)$ and $(a,b,c)=(-\frac{2}{3},-\frac{2}{3},\frac{10}{3})$. Therefore the difference between the maximum and minimum possible values of $c$ is $\frac{10}{3}-(-2)=\frac{16}{3}$.

18. **Answer (B):**

If the game starts with 2013 coins and Jenna starts, then she picks 3 coins, and then no matter how many Barbara chooses, Jenna responds by keeping the number of remaining coins congruent to 0 (mod 5). That is, she picks 3 if Barbara picks 2, and she picks 1 if Barbara picks 4. This ensures that on her last turn Jenna will leave 0 coins and thus she will win. Similarly, if Barbara starts, then Jenna can reply as before so that the number of remaining coins is congruent to 3 (mod 5). On her last turn Barbara will have 3 coins available. She is forced to remove 2 and thus Jenna will win by taking the last coin.

If the game starts with 2014 coins and Jenna starts, then she picks 1 coin and reduces the game to the previous case of 2013 coins where she wins. If Barbara starts, she selects 4 coins. Then no matter what Jenna chooses, Barbara responds by keeping the number of remaining coins congruent to 0 (mod 5). This ensures that on her last turn Barbara will leave 0 coins and win the game. Thus whoever goes first will win the game with 2014 coins.

19. **Answer (B):** The Pythagorean Theorem applied to right triangles $ABD$ and $ACD$ gives $AB^2-BD^2=AD^2=AC^2-CD^2$; that is, $13^2-BD^2=15^2-(14-BD)^2$, from which it follows that $BD=5$, $CD=9$, and $AD=12$. Because triangles $AED$ and $ADC$ are similar,

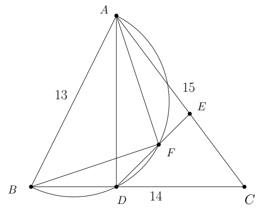
$$\frac{AE}{12}=\frac{DE}{9}=\frac{12}{15},$$

implying that $ED=\frac{36}{5}$ and $AE=\frac{48}{5}$.

Because $\angle AFB=\angle ADB=90^\circ$, it follows that $ABDF$ is cyclic. Thus $\angle ABD+\angle AFD=180^\circ$ from which $\angle ABD=\angle AFE$. Therefore right triangles $ABD$ and $AFE$ are similar. Hence

$$\frac{FE}{5}=\frac{\frac{48}{5}}{12},$$

from which it follows that $FE=4$. Consequently $DF=DE-FE=\frac{36}{5}-4=\frac{16}{5}$.




20. **Answer (A):** Because $135^\circ < x < 180^\circ$, it follows that $\cos x < 0 < \sin x$ and $|\sin x| < |\cos x|$. Thus $\tan x < 0$, $\cot x < 0$, and

$$|\tan x| = \frac{|\sin x|}{|\cos x|} < 1 < \frac{|\cos x|}{|\sin x|} = |\cot x|.$$

Therefore $\cot x < \tan x$. Moreover, $\cot x = \frac{\cos x}{\sin x} < \cos x$. Thus the four vertices $P, Q, R$, and $S$ are located on the parabola $y = x^2$ and $P$ and $S$ are in between $Q$ and $R$. If $\overline{AB}$ and $\overline{CD}$ are chords on the parabola $y = x^2$ such that the $x$-coordinates of $A$ and $B$ are less than the $x$-coordinates of $C$ and $D$, then the slope of $\overline{AB}$ is less than the slope of $\overline{CD}$. It follows that the two parallel sides of the trapezoid must be $\overline{QR}$ and $\overline{PS}$. Thus the slope of $\overline{QR}$ is equal to the slope of $\overline{PS}$. Thus,

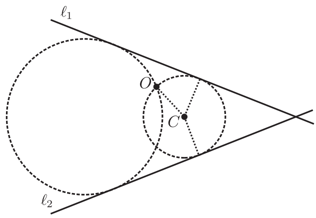
$$\cot x + \sin x = \tan x + \cos x.$$

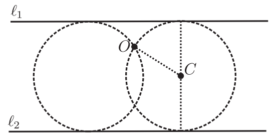
Multiplying by $\sin x \cos x \neq 0$ and rearranging gives the equivalent identity

$$(\cos x - \sin x)(\cos x + \sin x - \sin x \cos x) = 0.$$

Because $\cos x - \sin x \neq 0$ in the required range, it follows that $\cos x + \sin x - \sin x \cos x = 0$. Squaring and using the fact that $2 \sin x \cos x = \sin(2x)$ gives $1 + \sin(2x) = \frac{1}{4}\sin^2(2x)$. Solving this quadratic equation in the variable $\sin(2x)$ gives $\sin(2x) = 2 \pm 2\sqrt{2}$. Because $-1 < \sin 2x < 1$, the only solution is $\sin(2x) = 2 - 2\sqrt{2}$. There is indeed such a trapezoid for $x = 180^\circ + \frac{1}{2}\arcsin(2 - 2\sqrt{2}) \approx 152.031^\circ$.

21. **Answer (C):** If the directrices of two parabolas with the same focus intersect, then the corresponding parabolas intersect in exactly two points. The same conclusion holds if the directrices are parallel and the focus is between the two lines. Moreover, if the directrices are parallel and the focus is not between the two lines, then the corresponding parabolas do not intersect. Indeed, a point $C$ belongs to the intersection of the parabolas with focus $O$ and directrices $\ell_1$ and $\ell_2$, if and only if, $d(C, \ell_1) = OC = d(C, \ell_2)$. That is, the circle with center $C$ and radius $OC$ is tangent to both $\ell_1$ and $\ell_2$. If $\ell_1$ and $\ell_2$ are parallel and $O$ is not between them, then clearly such circle does not exist. If $\ell_1$ and $\ell_2$ intersect and $O$ is not on them, then there are are exactly two circles tangent to both $\ell_1$ and $\ell_2$ that go through $O$. The same is true if $\ell_1$ and $\ell_2$ are parallel and $O$ is between them.

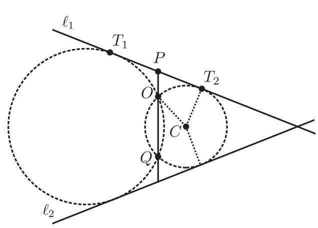



Thus there are $\binom{30}{2}$ pairs of parabolas and the pairs that do not intersect are exactly those whose directrices have the same slope and whose $y$-intercepts have the same sign. There are 5 different slopes and $2 \cdot \binom{3}{2} = 6$ pairs of $y$-intercepts with the same sign taken from $\{-3, -2, -1, 1, 2, 3\}$. Because the pairs of parabolas that intersect do so at exactly two points and no point is in three parabolas, it follows that the total number of intersection points is

$$2\left(\binom{30}{2} - 5 \cdot 6\right) = 810.$$

Note: It is possible to construct the two circles through $O$ and tangent to the lines $\ell_1$ and $\ell_2$ as follows: Let $\ell'$ be the bisector of the angle determined by the angular sector spanned by $\ell_1$ and $\ell_2$ that contains $O$ (or the midline of $\ell_1$ and $\ell_2$ if these lines are parallel and $O$ is between them). Let $Q$ be the symmetric point of $O$ with respect to $\ell'$ and let $P$ be the intersection of $\ell_1$ and the line $OQ$ (if $O = Q$ then let $P$ be the intersection of $\ell_1$ and a perpendicular line to $\ell'$ by $O$). If $C$ is one of the desired circles, then $C$ passes through $O$ and $Q$ and is tangent to $\ell_1$. Let $T$ be the point of tangency of $C$ and $\ell_1$. By the Power of a Point Theorem, $PT^2 = PO \cdot PQ$. The circle with center $P$ and radius $\sqrt{PO \cdot PQ}$ intersects $\ell_1$ in two points $T_1$ and $T_2$. The circumcircles of $OQT_1$ and $OQT_2$ are the desired circles.




22. **Answer (A):**

Using the change of base identity gives $\log n \cdot \log_n x = \log x$ and $\log m \cdot \log_m x = \log x$. The equivalent equation is

$$(\log x)^2 - \frac{1}{8}(7 \log m + 6 \log n) \log x - \frac{2013}{8} \log m \cdot \log n = 0.$$

As a quadratic equation in $\log x$, the sum of the two solutions $\log x_1$ and $\log x_2$ is equal to the negative of the linear coefficient. It follows that

$$\log(x_1x_2) = \log x_1 + \log x_2 = \frac{1}{8}(7\log m + 6\log n) = \log\left(\left(m^7n^6\right)^{1/8}\right).$$

Let $N = x_1x_2$ be the product of the solutions. Suppose $p$ is a prime dividing $m$. Let $p^a$ and $p^b$ be the largest powers of $p$ that divide $m$ and $n$ respectively. Then $p^{7a+6b}$ is the largest power of $p$ that divides $m^7n^6 = N^8$. It follows that $7a+6b \equiv 0 \pmod 8$. If $a$ is odd, then there is no solution to $7a+6b \equiv 0 \pmod 8$ because $7a$ is not divisible by $\gcd(6,8) = 2$. If $a \equiv 0 \pmod 8$, then because $a > 0$, it follows that $N^8 = m^7n^6 \geq (p^8)^7 = p^{56} \geq 2^{56}$, so $N \geq 2^7 = 128$. If $a \equiv 2 \pmod 8$ then $14 + 6b \equiv 0 \pmod 8$ is equivalent to $3b + 3 \equiv 3b + 7 \equiv 0 \pmod 4$. Thus $b \equiv 3 \pmod 4$ and then $N^8 = m^7n^6 \geq (p^2)^7(p^3)^6 = p^{32} \geq 2^{32}$, so $N \geq 2^4 = 16$ with equality for $m = 2^2$ and $n = 2^3$. Finally, if $a \geq 4$ and $a$ is not a multiple of 8, then $b \geq 1$ and thus $N^8 = m^7n^6 \geq (p^4)^7(p^1)^6 = p^{34} \geq 2^{34}$, so $N \geq 2^{17/4} > 2^4 = 16$. Therefore the minimum product is $N = 16$ obtained uniquely when $m = 2^2$ and $n = 2^3$. The requested sum is $m + n = 4 + 8 = 12$.

23. **Answer (E):** Expand the set of three-digit positive integers to include integers $N$, $0 \leq N \leq 99$, with leading zeros appended. Because $\operatorname{lcm}(5^2, 6^2, 10^2) = 900$, such an integer $N$ meets the required condition if and only if $N + 900$ does. Therefore $N$ can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of $N$ are $a_1$ and $a_0$. Similarly, suppose that the last two digits of the base-6 representation of $N$ are $b_1$ and $b_0$. By assumption, $2N \equiv a_0 + b_0 \pmod{10}$, but $N \equiv a_0 \pmod 5$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \leq a_0 \leq 4$ and $0 \leq b_0 \leq 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod 5$, it follows that there is an integer $N_1$ such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod 6$ implies that $5N_1 + a_0 \equiv a_0 \pmod 6$ and so $N_1 \equiv 0 \pmod 6$. It follows that $N_1 = 6N_2$ for some integer $N_2$ and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2 + a_0 \equiv 5a_1 + a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod 5$. It follows that $N_2 = 5N_3 + a_1$ for some integer $N_3$ and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1 + a_0 \pmod{36}$ implies that $6N_3 - 6a_1 + a_0 \equiv 150N_3 + 30a_1 + a_0 \equiv 6b_1 + a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod 6$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer $N_4$ and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod 5$ and $a_1 \equiv b_1 \pmod 2$. Conversely, if $N = 900N_4 + 180a_1 + 150b_1 + a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs $(a_1, b_1)$, namely $(0,0)$, $(2,0)$, $(4,0)$, $(1,5)$, and $(3,5)$. Each of these can be combined with 5 different values of $a_0$ $(0 \le a_0 \le 4)$, to determine exactly 25 different numbers $N$ with the required property.

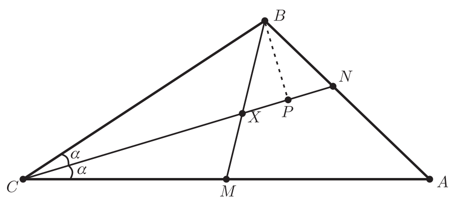
24. **Answer (A):** Let $\alpha = \angle ACN = \angle NCB$ and $x = BN$. Because $\triangle BXN$ is equilateral it follows that $\angle BXC = \angle CNA = 120°$, $\angle CBX = \angle BAC = 60° - \alpha$, and $\angle CBA = \angle BMC = 120° - \alpha$. Thus $\triangle ABC \sim \triangle BMC$ and $\triangle ANC \sim \triangle BXC$. Then

$$\frac{BC}{2} = \frac{BC}{AC} = \frac{MC}{BC} = \frac{1}{BC},$$

so $BC = \sqrt{2}$; and

$$\frac{CX + x}{2} = \frac{CN}{AC} = \frac{CX}{BC} = \frac{CX}{\sqrt{2}},$$

so $CX = (\sqrt{2} + 1)x$.




Let $P$ be the midpoint of $\overline{XN}$. Because $\triangle BXN$ is equilateral, the triangle $BPC$ is a right triangle with $\angle BPC = 90°$. Then by the Pythagorean Theorem,

$$\begin{aligned} 2 = BC^2 = CP^2 + PB^2 &= (CX + XP)^2 + PB^2 \\ &= \left(CX + \frac{1}{2}BN\right)^2 + \left(\frac{\sqrt{3}}{2}BN\right)^2 \\ &= \left(\sqrt{2} + \frac{3}{2}\right)^2 x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 x^2 = (5 + 3\sqrt{2})x^2. \end{aligned}$$

Therefore

$$x^2 = \frac{2}{5 + 3\sqrt{2}} = \frac{10 - 6\sqrt{2}}{7}.$$

OR

Establish as in the first solution that $CX = (\sqrt{2}+1)x$. Then the Law of Cosines applied to $\triangle BCX$ gives

$$2 = BC^2 = BX^2 + CX^2 - 2BX \cdot CX \cdot \cos(120°)$$

$$= x^2 + (1+\sqrt{2})^2 x^2 + (1+\sqrt{2})x^2$$
$$= (5+3\sqrt{2})x^2,$$

and solving for $x^2$ gives the requested answer.

25. **Answer (B):** Let $P(z)$ be a polynomial in $G$. Because the coefficients of $P(z)$ are real, it follows that the nonreal roots of $P(z)$ must be paired by conjugates; that is, if $a+ib$ is a root, then $a-ib$ is a root as well. In particular, $P(z)$ can be factored into the product of pairwise different linear polynomials of the form $(z-c)$ with $c \in \mathbb{Z}$ and quadratic polynomials of the form $(z-(a+ib))(z-(a-ib)) = z^2-2az+(a^2+b^2)$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Moreover, the product of the independent terms of these polynomials must be equal to 50, so each of $a^2+b^2$ or $c$ must be a factor of 50. Call these linear or quadratic polynomials *basic* and for every $d \in \{1, 2, 5, 10, 25, 50\}$, let $B_d$ be the set of basic polynomials with independent term equal to $\pm d$.

The equation $a^2+b^2=1$ has a pair of conjugate solutions in integers with $b \neq 0$, namely $(a,b)=(0,\pm 1)$. Thus there is only 1 basic quadratic polynomial with independent term of magnitude 1: $(z-i)(z+i) = z^2+1$. Similarly, the equation $a^2+b^2=2$ has 2 pairs of conjugate solutions with $b \neq 0$, $(a,b)=(1,\pm 1)$ and $(-1,\pm 1)$. These give the following 2 basic polynomials with independent term $\pm 2$: $(z-1-i)(z-1+i) = z^2-2z+2$ and $(z+1+i)(z+1-i) = z^2+2z-2$. In the same way the equations $a^2+b^2=5$, $a^2+b^2=10$, $a^2+b^2=25$, and $a^2+b^2=50$ have 4, 4, 5, and 6 respective pairs of conjugate solutions $(a,b)$. These are $(2,\pm 1)$, $(-2,\pm 1)$, $(1,\pm 2)$, and $(-1,\pm 2)$; $(3,\pm 1)$, $(-3,\pm 1)$, $(1,\pm 3)$, and $(-1,\pm 3)$; $(3,\pm 4)$, $(-3,\pm 4)$, $(4,\pm 3)$, $(-4,\pm 3)$, and $(0,\pm 5)$; and $(7,\pm 1)$, $(-7,\pm 1)$, $(1,\pm 7)$, $(-1,\pm 7)$, $(5,\pm 5)$, and $(-5,\pm 5)$. These generate all possible basic quadratic polynomials with nonreal roots and independent term that divides 50. The basic linear polynomials with real roots are $z-c$ where $c \in \{\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50\}$. Thus the linear basic polynomials contribute 2 to $|B_d|$. It follows that $|B_1|=3$, $|B_2|=4$, $|B_5|=6$, $|B_{10}|=6$, $|B_{25}|=7$ , and $|B_{50}|=8$.

Because $P$ has independent term 50, there are either 8 choices for a polynomial in $B_{50}$, or $7 \cdot 4$ choices for a product of two polynomials, one in $B_{25}$ and the other in $B_2$, or $6 \cdot 6$ choices for a product of two polynomials, one in $B_{10}$ and the other in $B_5$, or $4 \cdot \binom{6}{2}$ choices for a product of three polynomials, one in $B_2$ and the other two in $B_5$. Finally, each of the polynomials $z+1$ and $z^2+1$ in $B_1$ can appear or not in the product, but the presence of the polynomial $z-1$ is determined by the rest: if the product of the remaining independent terms is $-50$, then it has to be present, and if the product is 50, then it must not be in the product. Thus, the grand total is

$$2^2\left(8+7\cdot 4+6\cdot 6+4\cdot\binom{6}{2}\right) = 2^2(8+28+36+60) = 4\cdot 132 = 528.$$

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Davis, Zuming Feng, Sister Josanne Furey, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Cap Khoury, Patrick Vennebush, Kevin Wang, and Dave Wells.

MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

American Mathematics Competitions

65th Annual

# AMC 12 A

**American Mathematics Contest 12 A**

**Tuesday, February 4, 2014**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*



*Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:*

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

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1. **Answer (C):** Note that
$$10 \cdot \left(\frac{1}{2}+\frac{1}{5}+\frac{1}{10}\right)^{-1} = 10 \cdot \left(\frac{8}{10}\right)^{-1} = \frac{25}{2}.$$

2. **Answer (B):** Because child tickets are half the price of adult tickets, the price of 5 adult tickets and 4 child tickets is the same as the price of $5 + \frac{1}{2} \cdot 4 = 7$ adult tickets. In the same way, the price of 8 adult tickets and 6 child tickets is the same as the price of $8 + \frac{1}{2} \cdot 6 = 11$ adult tickets, which is equal to $11 \cdot \frac{1}{7} \cdot 24.50 = 38.50$ dollars.

3. **Answer (B):** If Ralph passed the orange house first, then because the blue and yellow houses are not neighbors, the house color ordering must be orange, blue, red, yellow. If Ralph passed the blue house first, then there are 2 possible placements for the yellow house, and each choice determines the placement of the orange and red houses. These 2 house color orderings are blue, orange, yellow, red, and blue, orange, red, yellow. There are 3 possible orderings for the colored houses.

4. **Answer (A):** One cow gives $\frac{b}{a}$ gallons in $c$ days, so one cow gives $\frac{b}{ac}$ gallons in 1 day. Thus $d$ cows will give $\frac{bd}{ac}$ gallons in 1 day. In $e$ days $d$ cows will give $\frac{bde}{ac}$ gallons of milk.

5. **Answer (C):** Because over 50% of the students scored 90 or lower, and over 50% of the students scored 90 or higher, the median score is 90. The mean score is
$$\frac{10}{100} \cdot 70 + \frac{35}{100} \cdot 80 + \frac{30}{100} \cdot 90 + \frac{25}{100} \cdot 100 = 87,$$
for a difference of $90 - 87 = 3$.

6. **Answer (D):** Let $10a + b$ be the larger number. Then $10a + b - (10b + a) = 5(a + b)$, which simplifies to $2a = 7b$. The only nonzero digits that satisfy this equation are $a = 7$ and $b = 2$. Therefore the larger number is 72, and the required sum is $72 + 27 = 99$.

7. **Answer (A):** Each term in a geometric progression is $r$ times the preceding term. The ratio is
$$r = \frac{3^{\frac{1}{3}}}{3^{\frac{1}{2}}} = 3^{\frac{1}{3}-\frac{1}{2}} = 3^{-\frac{1}{6}}.$$

Thus the third term is correctly given as $r \cdot 3^{\frac{1}{3}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{3}} = 3^{\frac{1}{6}}$, and the fourth term is $r \cdot 3^{\frac{1}{6}} = 3^{-\frac{1}{6}} \cdot 3^{\frac{1}{6}} = 3^0 = 1$.

8. **Answer (C):** Let $P > 100$ be the listed price. Then the price reductions in dollars are as follows:

   Coupon 1: $\frac{P}{10}$
   Coupon 2: 20
   Coupon 3: $\frac{18}{100}(P - 100)$

   Coupon 1 gives a greater price reduction than coupon 2 when $\frac{P}{10} > 20$, that is, $P > 200$. Coupon 1 gives a greater price reduction than coupon 3 when $\frac{P}{10} > \frac{18}{100}(P - 100)$, that is, $P < 225$. The only choice that satisfies these inequalities is \$219.95.

9. **Answer (B):** The five consecutive integers starting with $a$ are $a$, $a+1$, $a+2$, $a+3$, and $a+4$. Their average is $a+2 = b$. The average of five consecutive integers starting with $b$ is $b+2 = a+4$.

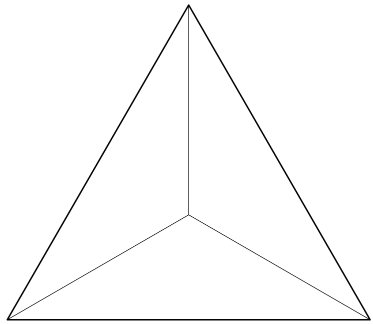
10. **Answer (B):**

    Let $h$ represent the altitude of each of the isosceles triangles from the base on the equilateral triangle. Then the area of each of the congruent isosceles triangles is $\frac{1}{2} \cdot 1 \cdot h = \frac{1}{2}h$. The sum of the areas of the three isosceles triangles is the same as the area of the equilateral triangle, so $3 \cdot \frac{1}{2}h = \frac{1}{4}\sqrt{3}$ and $h = \frac{1}{6}\sqrt{3}$. As a consequence, the Pythagorean Theorem implies that the side length of the isosceles triangles is

    $$\sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{6}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}.$$

    **OR**

    Suppose that the isosceles triangles are constructed internally with respect to the equilateral triangle. Because the sum of their areas is equal to the area of the equilateral triangle, it follows that the center of the equilateral triangle is a vertex common to all three isosceles triangles. The distance from the center of the equilateral triangle to any of its vertices is two thirds of its height. Thus the required side length is equal to $\frac{2}{3} \cdot \frac{1}{2}\sqrt{3} = \frac{1}{3}\sqrt{3}$.



11. **Answer (C):** Let $d$ be the remaining distance after one hour of driving, and let $t$ be the remaining time until his flight. Then $d = 35(t+1)$, and $d = 50(t-0.5)$. Solving gives $t = 4$ and $d = 175$. The total distance from home to the airport is $175 + 35 = 210$ miles.

**OR**

Let $d$ be the distance between David's home and the airport. The time required to drive the entire distance at 35 MPH is $\frac{d}{35}$ hours. The time required to drive at 35 MPH for the first 35 miles and 50 MPH for the remaining $d - 35$ miles is $1 + \frac{d-35}{50}$. The second trip is 1.5 hours quicker than the first, so

$$\frac{d}{35} - \left(1 + \frac{d-35}{50}\right) = 1.5.$$

Solving yields $d = 210$ miles.

12. **Answer (D):**

Let the larger and smaller circles have radii $R$ and $r$, respectively. Then the length of chord $\overline{AB}$ can be expressed as both $r$ and $2R\sin 15^\circ$. The ratio of the areas of the circles is

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{4\sin^2 15^\circ} = \frac{1}{2(1-\cos 30^\circ)} = \frac{1}{2-\sqrt{3}} = 2+\sqrt{3}.$$

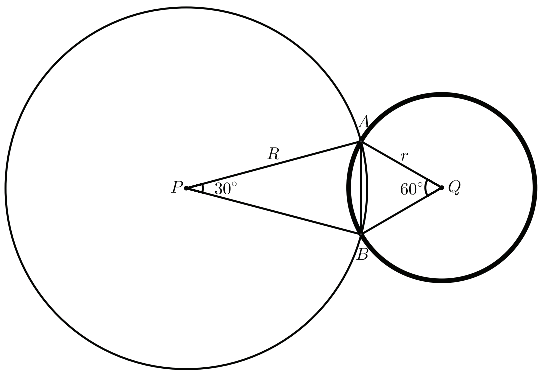
**OR**

Let the larger and smaller circles have radii $R$ and $r$, and centers $P$ and $Q$, respectively. Because $\triangle QAB$ is equilateral, it follows that $r = AB$. The Law of Cosines applied to $\triangle PBA$ gives

$$\begin{aligned} r^2 = AB^2 &= PA^2 + PB^2 - 2PA \cdot PB \cos 30^\circ \\ &= 2R^2 - 2R^2 \cos 30^\circ = R^2(2 - \sqrt{3}). \end{aligned}$$

Thus

$$\frac{\pi R^2}{\pi r^2} = \frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}.$$




13. **Answer (B):** If each friend rooms alone, then there are $5! = 120$ ways to assign the guests to the rooms. If one pair of friends room together and the others room alone, then there are $\binom{5}{2} = 10$ ways to choose the roommates and then $5 \cdot 4 \cdot 3 \cdot 2 = 120$ ways to assign the rooms to the 4 sets of occupants, for a total of $10 \cdot 120 = 1200$ possible arrangements. The only other possibility is to have two sets of roommates. In this case the roommates can be chosen in $5 \cdot \frac{1}{2}\binom{4}{2} = 15$ ways (choose the solo lodger first), and then there are $5 \cdot 4 \cdot 3 = 60$ ways to assign the rooms, for a total of $15 \cdot 60 = 900$ possibilities. Therefore the answer is $120 + 1200 + 900 = 2220$.

14. **Answer (C):** Let $d = b - a$ be the common difference of the arithmetic progression. Then $b = a + d$, $c = a + 2d$, and because $a, c, b$ is a geometric progression,

$$\frac{a + 2d}{a} = \frac{a + d}{a + 2d}.$$

Thus $(a + 2d)(a + 2d) = a(a + d)$, which simplifies to $3ad + 4d^2 = 0$. Because $d > 0$, it follows that $3a + 4d = 0$ and therefore $a = -4k$ and $d = 3k$ for some positive integer $k$. Thus $c = (-4k) + 2(3k) = 2k$, and the smallest value of $c$ is $2 \cdot 1 = 2$.

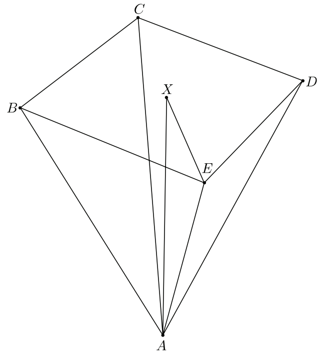
15. **Answer (B):** Note that $abcba = a000a + b0b0 + c00$. Because $1+2+\cdots+9 = \frac{1}{2}(9\cdot 10) = 45$, the sum of all integers of the form $a000a$ is 450 045. For each value of $a$ there are $10 \cdot 10 = 100$ choices for $b$ and $c$. Similarly, the sum of all integers of the form $b0b0$ is 45 450. For each value of $b$ there are $9 \cdot 10 = 90$ choices for $a$ and $c$. The sum of all integers of the form $c00$ is 4500, and for each $c$ there are $9 \cdot 10 = 90$ choices for $a$ and $b$. Thus $S = 100 \cdot 450\,045 + 90 \cdot 45\,450 + 90 \cdot 4500 = 45(1\,000\,100 + 90\,900 + 9000) = 45(1\,100\,000) = 49\,500\,000$. The sum of the digits is 18.

16. **Answer (D):** By direct multiplication, $8 \cdot 888\ldots 8 = 7111\ldots 104$, where the product has 2 fewer ones than the number of digits in $888\ldots 8$. Because $7 + 4 = 11$, the product must have $1000 - 11 = 989$ ones, so $k - 2 = 989$ and $k = 991$.

17. **Answer (A):** Connect the centers of the large sphere and the four small spheres at the top to form an inverted square pyramid as shown. Since $BCDE$ is a square of side 2, $EX = \sqrt{2}$. Also, $AE = 3$ and $\triangle AXE$ is a right triangle, so

$$AX = \sqrt{3^2 - \left(\sqrt{2}\right)^2} = \sqrt{7}.$$

The distance from the plane containing $BCDE$ to the top of the box is 1. Therefore the total height of the box is $2(1 + AX) = 2 + 2\sqrt{7}$.




18. **Answer (C):** For every $a > 0$, $a \neq 1$, the domain of $\log_a x$ is the set $\{x : x > 0\}$. Moreover, for $0 < a < 1$, $\log_a x$ is a decreasing function on its domain, and for $a > 1$, $\log_a x$ is an increasing function on its domain. Thus the function $f(x)$ is defined if and only if $\log_4(\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}}(x)))) > 0$, and this inequality is

equivalent to each of the following:

$$\log_{\frac{1}{4}}(\log_{16}(\log_{\frac{1}{16}}(x))) > 1, \quad 0 < \log_{16}(\log_{\frac{1}{16}} x) < \frac{1}{4},$$
$$1 < \log_{\frac{1}{16}} x < 2, \text{ and } \quad \frac{1}{256} < x < \frac{1}{16}.$$

Thus $\frac{m}{n} = \frac{1}{16} - \frac{1}{256} = \frac{15}{256}$, and $m + n = 271$.

19. **Answer (E):** Solve the equation for $k$ to obtain $k = -\frac{12}{x} - 5x$. For each integer value of $x$ except $x = 0$, there is a corresponding rational value for $k$. As a function of $x$, $|k| = \frac{12}{x} + 5x$ is increasing for $x \geq 2$. Thus by inspection, the integer values of $x$ that ensure $|k| < 200$ satisfy the inequality $-39 \leq x \leq 39$. There are 78 such values. Assume that $a$ and $b$ are two different integer values of $x$ that produce the same $k$. Then $k = -\frac{12}{a} - 5a = -\frac{12}{b} - 5b$, which simplifies to $(5ab - 12)(a - b) = 0$. Because $a \neq b$, it follows that $5ab = 12$, but there are no integers satisfying this equation. Thus the values of $k$ corresponding to the 78 values of $x$ are all distinct, and the answer is therefore 78.

20. **Answer (D):** Let $B'$ be the reflection of point $B$ across $\overline{AC}$, and let $C'$ be the reflection of point $C$ across $\overline{AB}$. Then $AB' = AB = 10$, $AC' = AC = 6$, $BE = B'E$, $CD = C'D$, and $\angle B'AC' = 120^\circ$. By the Law of Cosines, $B'C'^2 = 10^2 + 6^2 - 2 \cdot 10 \cdot 6 \cos 120^\circ = 196$; thus $B'C' = 14$. Furthermore, $B'C' \leq B'E + DE + C'D = BE + DE + CD$. Therefore the answer is 14.

21. **Answer (A):** If $x = n + r$, where $n$ is an integer, $1 \leq n \leq 2013$, and $0 \leq r < 1$, then $f(x) = n(2014^r - 1)$. The condition $f(x) \leq 1$ is equivalent to $2014^r \leq 1 + \frac{1}{n}$, or $0 \leq r \leq \log_{2014}\left(\frac{n+1}{n}\right)$. Thus the required sum is

$$\log_{2014}\frac{2}{1} + \log_{2014}\frac{3}{2} + \log_{2014}\frac{4}{3} + \cdots + \log_{2014}\frac{2014}{2013}$$
$$= \log_{2014}\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2014}{2013}\right) = \log_{2014}(2014) = 1.$$

22. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for each $n$ there is at most one $m$ satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let $d$ (respectively, $t$) be the number of nonnegative integers $n$ less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between

$5^n$ and $5^{n+1}$. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.

23. **Answer (B):** Note that

$$\frac{10^n}{99^2} = \frac{10^n}{9801} = b_{n-1}b_{n-2}\ldots b_2b_1b_0.\overline{b_{n-1}b_{n-2}\ldots b_2b_1b_0}.$$

Subtracting the original equation gives

$$\frac{10^n - 1}{99^2} = b_{n-1}b_{n-2}\ldots b_2b_1b_0.$$

Thus $10^n - 1 = 99^2 \cdot b_{n-1}b_{n-2}\ldots b_2b_1b_0$. It follows that $10^n - 1$ is divisible by 11 and thus $n$ is even, say $n = 2N$. For $0 \le j \le N-1$, let $a_j = 10b_{2j+1} + b_{2j}$. Note that $0 \le a_j \le 99$, and because

$$\frac{10^{2N} - 1}{10^2 - 1} = 1 + 10^2 + 10^4 + \cdots + 10^{2(N-1)},$$

it follows that

$$\sum_{k=0}^{N-1} 10^{2k} = (10^2 - 1)\sum_{k=0}^{N-1} a_k 10^{2k},$$

and so

$$\sum_{k=0}^{N-1} 10^{2k} + \sum_{k=0}^{N-1} a_k 10^{2k} = \sum_{k=1}^{N} a_{k-1}10^{2k}.$$

Considering each side of the equation as numbers written in base 100, it follows that $1 + a_0 \equiv 0 \pmod{100}$, so $a_0 = 99$ and there is a carry for the $10^2$ digit in the sum on the left side. Thus $1 + (1 + a_1) \equiv a_0 = 99 \pmod{100}$ and so $a_1 = 97$, and there is no carry for the $10^4$ digit. Next, $1 + a_2 \equiv a_1 = 97 \pmod{100}$, and so $a_2 = 96$ with no carry for the $10^6$ digit. In the same way $a_j = 98 - j$ for $1 \le j \le 98$. Then $1 + a_{99} \equiv a_{98} = 0 \pmod{100}$ would yield $a_{99} = 99$, and then the period would start again. Therefore $N = 99$ and $b_{n-1}b_{n-2}\ldots b_2b_1b_0 = 0001020304\ldots969799$. By momentarily including 9 and 8 as two extra digits, the sum would be $(0 + 1 + 2 + \cdots + 9) \cdot 20 = 900$, so the required sum is $900 - 9 - 8 = 883$.

24. **Answer (C):** For integers $n \ge 1$ and $k \ge 0$, if $f_{n-1}(x) = \pm k$, then $f_n(x) = k - 1$. Thus if $f_0(x) = \pm k$, then $f_k(x) = 0$. Furthermore, if $f_n(x) = 0$, then $f_{n+1}(x) = -1$ and $f_{n+2}(x) = 0$. It follows that the zeros of $f_{100}$ are the solutions of $f_0(x) = 2k$ for $-50 \le k \le 50$. To count these solutions, note that

$$f_0(x) = \begin{cases} x + 200 & \text{if } x < -100, \\ -x & \text{if } -100 \le x < 100, \text{ and} \\ x - 200 & \text{if } x \ge 100. \end{cases}$$

The graph of $f_0(x)$ is piecewise linear with turning points at $(-100, 100)$ and $(100, -100)$. The line $y = 2k$ crosses the graph three times for $-49 \le k \le 49$ and twice for $k = \pm 50$. Therefore the number of zeros of $f_{100}(x)$ is $99 \cdot 3 + 2 \cdot 2 = 301$.

25. **Answer (B):** Let $O = (0, 0)$, $A = (4, 3)$, and $B = (-4, -3)$. Because $A, B \in P$ and $O$ is the midpoint of $\overline{AB}$, it follows that $\overline{AB}$ is the latus rectum of the parabola $P$. Thus the directrix is parallel to $\overline{AB}$. Let $T$ be the foot of the perpendicular from $O$ to the directrix of $P$. Because $OT = OA = OB = 5$ and $\overline{OT}$ is perpendicular to $\overline{AB}$, it follows that $T = (3, -4)$. Thus the equation of the directrix is $y+4 = \frac{3}{4}(x-3)$, and in general form the equation is $4y-3x+25 = 0$.

Using the formula for the distance from a point to a line, as well as the definition of $P$ as the locus of points equidistant from $O$ and the directrix, the equation of $P$ is

$$\sqrt{x^2 + y^2} = \frac{|4y - 3x + 25|}{\sqrt{4^2 + 3^2}}.$$

After squaring and rearranging, this is equivalent to

$$\begin{aligned} 25x^2 + 25y^2 &= 25(x^2 + y^2) = (4y - 3x + 25)^2 \\ &= 16y^2 + 9x^2 - 24xy + 25^2 + 50(4y - 3x), \end{aligned}$$

and

$$(4x + 3y)^2 = 25(25 + 2(4y - 3x)). \tag{1}$$

Assume $x$ and $y$ are integers. Then $4x + 3y$ is divisible by 5. If $4x + 3y = 5s$ for $s \in \mathbb{Z}$, then $2s^2 = 50+16y-12x = 50+16y-3(5s-3y) = 50+25y-15s$. Thus $s$ is divisible by 5. If $s = 5t$ for $t \in \mathbb{Z}$, then $2t^2 = 2+y-3t$, and so $y = 2t^2+3t-2$. In addition $4x = 5s - 3y = 25t - 3y = 25t - 3(2t^2 + 3t - 2) = -6t^2 + 16t + 6$, and thus $t$ is odd. If $t = 2u + 1$ for $u \in \mathbb{Z}$, then

$$x = -6u^2 + 2u + 4 \text{ and } y = 8u^2 + 14u + 3. \tag{2}$$

Conversely, if $x$ and $y$ are defined as in (2) for $u \in \mathbb{Z}$, then $x$ and $y$ are integers and they satisfy (1), which is the equation of $P$. Lastly, with $u \in \mathbb{Z}$,

$$\begin{aligned} |4x + 3y| &= \left|-24u^2 + 8u + 16 + 24u^2 + 42u + 9\right| \\ &= |50u + 25| \le 1000 \end{aligned}$$

if and only if $u$ is an integer such that $|2u + 1| \le 39$. That is, $-20 \le u \le 19$, and so the required answer is $19 - (-21) = 40$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.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MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

American Mathematics Competitions

65th Annual

# AMC 12 B

**American Mathematics Contest 12 B**

**Wednesday, February 19, 2014**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*



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*The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:*

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1. **Answer (C):** Leah has 7 pennies and 6 nickels, which are worth 37 cents.

2. **Answer (C):** The special allows Orvin to purchase balloons at $\frac{1+\frac{2}{3}}{2} = \frac{5}{6}$ times the regular price. Because Orvin had just enough money to purchase 30 balloons at the regular price, he may now purchase $30 \cdot \frac{6}{5} = 36$ balloons.

3. **Answer (E):** The fraction of Randy's trip driven on pavement was $1 - \frac{1}{3} - \frac{1}{5} = \frac{7}{15}$. Therefore the entire trip was $20 \div \frac{7}{15} = \frac{300}{7}$ miles.

4. **Answer (B):** Let a muffin cost $m$ dollars and a banana cost $b$ dollars. Then $2(4m + 3b) = 2m + 16b$, and simplifying gives $m = \frac{5}{3}b$.

5. **Answer (A):** Denote the height of a pane by $5x$ and the width by $2x$. Then the square window has height $2 \cdot 5x + 6$ inches and width $4 \cdot 2x + 10$ inches. Solving $2 \cdot 5x + 6 = 4 \cdot 2x + 10$ gives $x = 2$. The side length of the square window is 26 inches.

6. **Answer (D):** Let $a$ be the amount in a regular lemonade. Then a large lemonade holds $\frac{3}{2}a$, and Ann had $\frac{1}{4} \cdot \frac{3}{2}a = \frac{3}{8}a$ lemonade left right before she gave Ed part of her drink. She gave him $\frac{1}{3} \cdot \frac{3}{8}a + 2 = \frac{1}{8}a + 2$ ounces. Because Ann and Ed drank the same amount of lemonade, it follows that $a + (\frac{1}{8}a + 2) = \frac{3}{2}a - (\frac{1}{8}a + 2)$, and $4 = \frac{1}{4}a$. Thus $a = 16$ ounces, $\frac{3}{2}a = 24$ ounces, and together they drank $16 + 24 = 40$ ounces.

7. **Answer (D):** Let $x = \frac{n}{30-n}$ so that $n = \frac{30x}{x+1}$. Because $x$ and $x+1$ are relatively prime, it follows that $x + 1$ must be a factor of 30. Because $n$ is positive and less than 30 it follows that $x + 1 \geq 2$. Thus $x + 1$ equals 2, 3, 5, 6, 10, 15, or 30. Hence there are 7 possible values for $n$, namely 15, 20, 24, 25, 27, 28, and 29.

8. **Answer (C):** As indicated by the leftmost column $A + B \leq 9$. Then both the second and fourth columns show that $C = 0$. Because $A$, $B$, and $C$ are distinct digits, $D$ must be at least 3. The following values for $(A, B, C, D)$ show that $D$ may be any of the 7 digits that are at least 3: $(1, 2, 0, 3)$, $(1, 3, 0, 4)$, $(2, 3, 0, 5)$, $(2, 4, 0, 6)$, $(2, 5, 0, 7)$, $(2, 6, 0, 8)$, $(2, 7, 0, 9)$.

9. **Answer (B):** By the Pythagorean Theorem, $AC = 5$. Because $5^2 + 12^2 = 13^2$, the converse of the Pythagorean Theorem applied to $\triangle DAC$ implies that $\angle DAC = 90^\circ$. The area of $\triangle ABC$ is 6 and the area of $\triangle DAC$ is 30. Thus the area of the quadrilateral is $6 + 30 = 36$.

10. **Answer (D):** Let $m$ be the total mileage of the trip. Then $m$ must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, it is a multiple of 9. Therefore $m$ is a multiple of 495. Because $m$ is at most a 3-digit number and $a$ is not equal to 0, $m = 495$. Therefore $c - a = 5$. Because $a + b + c \leq 7$, the only possible $abc$ is 106, so $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

**OR**

Let $m$ be the total mileage of the trip. Then $m$ must be a multiple of 55. Also, because $m = cba - abc = 99(c - a)$, $c - a$ is a multiple of 5. Because $a \geq 1$ and $a+b+c \leq 7$, it follows that $c = 6$ and $a = 1$. Therefore $b = 0$, so $a^2+b^2+c^2 = 37$.

11. **Answer (E):** The numbers in the list have a sum of $11 \cdot 10 = 110$. The value of the 11th number is maximized when the sum of the first ten numbers is minimized subject to the following conditions.

- If the numbers are arranged in nondecreasing order, the sixth number is 9.
- The number 8 occurs either 2, 3, 4, or 5 times, and all other numbers occur fewer times.

If 8 occurs 5 times, the smallest possible sum of the first 10 numbers is

$$8 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 9 + 10 = 86.$$

If 8 occurs 4 times, the smallest possible sum of the first 10 numbers is

$$1 + 8 + 8 + 8 + 8 + 9 + 9 + 9 + 10 + 10 = 80.$$

If 8 occurs 3 times, the smallest possible sum of the first 10 numbers is

$$1 + 1 + 8 + 8 + 8 + 9 + 9 + 10 + 10 + 11 = 75.$$

If 8 occurs 2 times, the smallest possible sum of the first 10 numbers is

$$1 + 2 + 3 + 8 + 8 + 9 + 10 + 11 + 12 + 13 = 77.$$

Thus the largest possible value of the 11th number is $110 - 75 = 35$.

12. **Answer (B):** Denote a triangle by the string of its side lengths written in nonincreasing order. Then $S$ has at most one equilateral triangle and at most one of the two triangles 442 and 221. The other possible elements of $S$ are 443, 441, 433, 432, 332, 331, and 322. All other strings are excluded by the triangle inequality. Therefore $S$ has at most 9 elements.

13. **Answer (C):** There is a triangle with side lengths 1, $a$, and $b$ if and only if $a > b - 1$. There is a triangle with side lengths $\frac{1}{b}$, $\frac{1}{a}$, and 1 if and only if $\frac{1}{a} > 1 - \frac{1}{b}$, that is, $a < \frac{b}{b-1}$. Therefore there are no such triangles if and only if $b - 1 \geq a \geq \frac{b}{b-1}$. The smallest possible value of $b$ satisfies $b - 1 = \frac{b}{b-1}$, or $b^2 - 3b + 1 = 0$. The solution with $b > 1$ is $\frac{1}{2}(3 + \sqrt{5})$. The corresponding value of $a$ is $\frac{1}{2}(1 + \sqrt{5})$.

14. **Answer (D):** Denote the edge lengths by $x$, $y$, and $z$. The surface area is $2(xy+yz+zx) = 94$ and the sum of the lengths of the edges is $4(x+y+z) = 48$. Therefore $144 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = x^2+y^2+z^2+94$, so $x^2 + y^2 + z^2 = 50$. By the Pythagorean Theorem applied twice, each of the 4 internal diagonals has length $\sqrt{50}$, and their total length is $4\sqrt{50} = 20\sqrt{2}$. A right rectangular prism with edge lengths 3, 4, and 5 satisfies the conditions of the problem.

15. **Answer (C):** Because $k \ln k = \ln(k^k)$ and the log of a product is the sum of the logs, $p = \ln \prod_{k=1}^{6} k^k$. Therefore $e^p$ is the integer $1^1 \cdot 2^2 \cdot 3^3 \cdot 4^4 \cdot 5^5 \cdot 6^6 = 2^{16} \cdot 3^9 \cdot 5^5$, and the largest power of 2 dividing $e^p$ is $2^{16}$.

16. **Answer (E):** Because $P(0) = k$, it follows that $P(x) = ax^3 + bx^2 + cx + k$. Thus $P(1) = a+b+c+k = 2k$ and $P(-1) = -a+b-c+k = 3k$. Adding these equations gives $2b = 3k$. Hence

$$\begin{aligned} P(2) + P(-2) &= (8a + 4b + 2c + k) + (-8a + 4b - 2c + k) \\ &= 8b + 2k = 12k + 2k = 14k. \end{aligned}$$

**OR**

Let $(P(-2), P(-1), P(0), P(1), P(2)) = (r, 3k, k, 2k, s)$. The sequence of first differences of consecutive values is $(3k-r, -2k, k, s-2k)$, the sequence of second differences is $(r - 5k, 3k, s - 3k)$, and the sequence of third differences is $(8k - r, s - 6k)$. Because $P$ is a cubic polynomial, the third differences are equal, so $P(-2) + P(2) = r + s = 14k$.

17. **Answer (E):** The line passing through point $Q = (20, 14)$ with slope $m$ has equation $y - 14 = m(x - 20)$. The requested values for $m$ are those for which the system

$$\begin{cases} y - 14 = m(x - 20) \\ y = x^2 \end{cases}$$

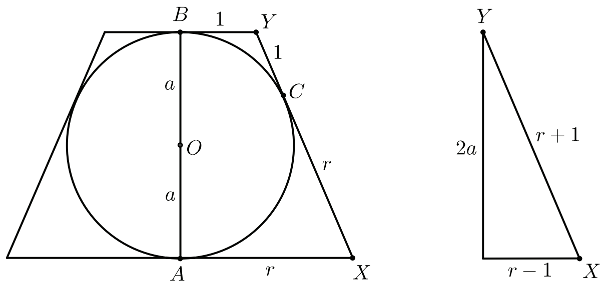
has no solutions. Solving for $y$ in the first equation and substituting into the second yields $m(x-20)+14 = x^2$, which reduces to $x^2 - mx + (20m - 14) = 0$. This equation has no solution for $x$ when the discriminant is negative, that is, when $m^2 - 4 \cdot (20m - 14) = m^2 - 80m + 56 < 0$. This quadratic in $m$ is negative between its two roots $40 \pm \sqrt{40^2 - 56}$, which are the required values of $r$ and $s$. The requested sum is $r + s = 2 \cdot 40 = 80$.

18. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum $n$ can be achieved, then the sum $15 - n$ can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is $1bc5e$. Furthermore, $\{b, c\}$ cannot equal $\{2, 3\}$ because $1 + 2 + 3 = 6$; similarly $\{b, c\}$ cannot equal $\{2, 4\}$. It follows that $e = 2$, which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality the arrangement is $2bc5e$. Reasoning as before, $\{b, c\}$ cannot equal $\{3, 4\}$ or $\{1, 4\}$, so $e = 4$, and then $b = 3$ and $c = 1$, to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

19. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by $r$ and the radius of the sphere by $a$. The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields $r = a^2$. The volume of the frustum is

$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere, $\frac{4}{3}\pi a^3$, and simplifying gives $a^4 - 3a^2 + 1 = 0$, or $r^2 - 3r + 1 = 0$. Therefore $r = \frac{3+\sqrt{5}}{2}$.




20. **Answer (B):** The domain of $\log_{10}(x-40)+\log_{10}(60-x)$ is $40 < x < 60$. Within this domain, the inequality $\log_{10}(x-40)+\log_{10}(60-x) < 2$ is equivalent to each of the following: $\log_{10}((x-40)(60-x)) < 2$, $(x-40)(60-x) < 10^2 = 100$, $x^2 - 100x + 2500 > 0$, and $(x-50)^2 > 0$. The last inequality is true for all $x \neq 50$. Thus the integer solutions to the original inequality are $41, 42, \ldots, 49, 51, 52, \ldots, 59$, and their number is 18.

21. **Answer (C):** Let $x = BE = GH = CF$, and let $\theta = \angle DHG = \angle AGJ = \angle FKH$. Note that $AD = GJ = HK = 1$. In right triangle $GDH$, $x \sin\theta = DG = 1 - AG = 1 - \cos\theta$, so $x = \frac{1-\cos\theta}{\sin\theta}$. Then $1 = CD = CF + FH + HD = x + \sin\theta + x\cos\theta$. Substituting for $x$ gives

$$\begin{aligned} 1 &= \frac{1-\cos\theta}{\sin\theta} + \sin\theta + \frac{1-\cos\theta}{\sin\theta}\cdot\cos\theta \\ &= \frac{(1-\cos\theta)(1+\cos\theta)}{\sin\theta} + \sin\theta \\ &= \frac{\sin^2\theta}{\sin\theta} + \sin\theta = 2\sin\theta. \end{aligned}$$

It follows that $\sin\theta = \frac{1}{2}$, so $\theta = 30^\circ$, and

$$x = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2 - \sqrt{3}.$$

**OR**

Let $a = EK$, $b = EJ$, and $c = JK = BE$. Then triangles $KEJ$, $GDH$, and $JAG$ are similar right triangles and it follows that $a^2 + b^2 = c^2$, $\frac{a}{c} = 1 - b - c$, and $\frac{b}{c} = 1 - a$. The first equation is equivalent to $a^2 = (c+b)(c-b)$, and the last equation is equivalent to $ac = c - b$. Multiplying by $c + b$ and equating to the first equation gives $ac(c+b) = (c+b)(c-b) = a^2$. Because $a > 0$, it follows that $a = c(c+b)$. Plugging into the second equation gives $c(1-b-c) = c(c+b)$. Because $c > 0$, it follows that $c + b = \frac{1}{2}$. Thus $a = \frac{c}{2}$ and

$$c^2 = a^2 + b^2 = \frac{c^2}{4} + \left(\frac{1}{2} - c\right)^2.$$

Solving for $c$ gives $c = 2 \pm \sqrt{3}$. If $c = 2 + \sqrt{3}$, then $b = \frac{1}{2} - c = -\frac{3}{2} - \sqrt{3} < 0$. Thus $BE = c = 2 - \sqrt{3}$.

22. **Answer (C):** First note that once the frog is on pad 5, it has probability $\frac{1}{2}$ of eventually being eaten by the snake, and a probability $\frac{1}{2}$ of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability $\frac{9}{10} \cdot \frac{8}{10} = \frac{72}{100}$, will be back at pad 1 with probability $\frac{9}{10} \cdot \frac{2}{10} = \frac{18}{100}$, and will retreat to pad 0 and be eaten with probability $\frac{1}{10}$. Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is $\frac{72}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{36}{41}$, and the probability that it ultimately makes it to pad 0 is $\frac{10}{100} \div \left(\frac{72}{100} + \frac{10}{100}\right) = \frac{5}{41}$.

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability $\frac{7}{10} \cdot \frac{6}{10} = \frac{42}{100}$, will be back at pad 3 with probability $\frac{7}{10} \cdot \frac{4}{10} + \frac{3}{10} \cdot \frac{8}{10} = \frac{52}{100}$, and will retreat to pad 1 with probability $\frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100}$. Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is $\frac{42}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{7}{8}$, and the probability that it ultimately makes it to pad 1 is $\frac{6}{100} \div \left(\frac{42}{100} + \frac{6}{100}\right) = \frac{1}{8}$.

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 5, 1 \to 3 \to 1 \to 3 \to 1 \to 3 \to 5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned}
&\frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{1}{8} \cdot \frac{36}{41} \cdot \frac{7}{8} + \cdots \\
&= \frac{63}{82}\left(1 + \frac{9}{82} + \left(\frac{9}{82}\right)^2 + \cdots\right) \\
&= \frac{63}{82} \div \left(1 - \frac{9}{82}\right)
\end{aligned}$$

$$= \frac{63}{73}.$$

Therefore the requested probability is $\frac{1}{2} \cdot \frac{63}{73} = \frac{63}{146}$.

**OR**

For $1 \le j \le 5$, let $p_j$ be the probability that the frog eventually reaches pad 10 starting at pad $j$. By symmetry $p_5 = \frac{1}{2}$. For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability $\frac{2}{5}$ or to pad 5 with probability $\frac{3}{5}$, and then continues on a successful sequence from either of these pads. Thus $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$. Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability $\frac{3}{10}$ or to pad 4 with probability $\frac{7}{10}$. Thus $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$, and substituting from the previous equation for $p_4$ gives $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$. In the same way, $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$ and after substituting for $p_3$ gives $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$. Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability $\frac{9}{10}$, and then escape starting from pad 2. Thus $p_1 = \frac{9}{10}p_2 = \frac{9}{10}(\frac{3}{10}p_1 + \frac{7}{20})$, and solving the equation gives $p_1 = \frac{63}{146}$.

**Note:** This type of random process is called a Markov process.

23. **Answer (C):** Let $n = \binom{2014}{k}$. Note that $2016 \cdot 2015 \equiv (-1)(-2) = 2 \pmod{2017}$ and $2016 \cdot 2015 \cdots (2015-k) \equiv (-1)(-2)\cdots(-(k+2)) = (-1)^k(k+2)! \pmod{2017}$. Because $n \cdot k! \cdot (2014-k)! = 2014!$, it follows that

$$n \cdot k! \cdot (2014-k)! \cdot ((2015-k)\cdots 2015 \cdot 2016) \cdot 2 \equiv$$
$$2014! \cdot 2015 \cdot 2016 \cdot (-1)^k (k+2)! \pmod{2017}.$$

Thus

$$2n \cdot k! \cdot 2016! \equiv (-1)^k (k+2)! \cdot 2016! \pmod{2017}.$$

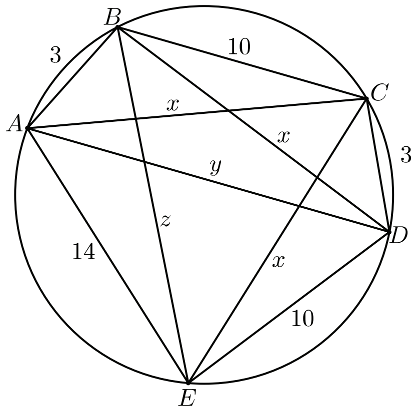
Dividing by $2016! \cdot k!$, which is relatively prime to 2017, gives

$$2n \equiv (-1)^k (k+2)(k+1) \pmod{2017}.$$

Thus $n \equiv (-1)^k \binom{k+2}{2} \pmod{2017}$. It follows that

$$S \equiv \sum_{k=0}^{62} (-1)^k \binom{k+2}{2} = 1 + \sum_{k=1}^{31} \left( \binom{2k+2}{2} - \binom{2k+1}{2} \right)$$
$$= 1 + \sum_{k=1}^{31} (2k+1) = 32^2 = 1024 \pmod{2017}.$$

24. **Answer (D):** Let $x = AC$, $y = AD$, and $z = BE$. Because the arcs $ABC$, $BCD$, and $CDE$ are congruent, it follows that $AC = BD = CE = x$.




By Ptolemy's Theorem applied to the quadrilaterals $ABCD$, $ABDE$, and $BCDE$, it follows that

$$10y + 9 = x^2,\ 30 + 14x = yz, \text{ and } 100 + 3z = x^2.$$

Solving for $y$ and $z$ in the first and third equations and substituting in the second equation gives

$$30 + 14x = \left(\frac{x^2 - 9}{10}\right)\left(\frac{x^2 - 100}{3}\right) = \frac{x^4 - 109x^2 + 900}{30},$$

which implies that

$$900 + 420x = x^4 - 109x^2 + 900.$$

Thus $x^3 - 109x - 420 = 0$. This equation factors as $(x-12)(x+5)(x+7) = 0$. Because $x > 0$ it follows that $x = 12$, $y = \frac{1}{10}(x^2 - 9) = \frac{135}{10} = \frac{27}{2}$, and $z = \frac{1}{3}(x^2 - 100) = \frac{44}{3}$. The required sum of diagonals equals $3x + y + z = \frac{385}{6}$, so $m + n = 385 + 6 = 391$.

25. **Answer (D):** If $x = \frac{1}{2}\pi y$, then the given equation is equivalent to

$$2\cos(\pi y)\left(\cos(\pi y) - \cos\left(\frac{4028\pi}{y}\right)\right) = \cos(2\pi y) - 1.$$

Dividing both sides by 2 and using the identity $\frac{1}{2}(1 - \cos(2\pi y)) = \sin^2(\pi y)$ yields

$$\cos^2(\pi y) - \cos(\pi y)\cos\left(\frac{4028\pi}{y}\right) = \frac{1}{2}\left(\cos(2\pi y) - 1\right) = -\sin^2(\pi y).$$

This is equivalent to

$$1 = \cos(\pi y) \cos\left(\frac{4028\pi}{y}\right).$$

Thus either $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = 1$ or $\cos(\pi y) = \cos(\frac{4028\pi}{y}) = -1$. It follows that $y$ and $\frac{4028}{y}$ are both integers having the same parity. Therefore $y$ cannot be odd or a multiple of 4. Finally, let $y = 2a$ with $a$ a positive odd divisor of $4028 = 2^2 \cdot 19 \cdot 53$, that is $a \in \{1, 19, 53, 19 \cdot 53\}$. Then $\cos(\pi y) = \cos(2a\pi) = 1$ and $\cos(\frac{4028\pi}{y}) = \cos(\frac{2014\pi}{a}) = 1$. Therefore the sum of all solutions $x$ is $\pi(1 + 19 + 53 + 19 \cdot 53) = \pi(19 + 1)(53 + 1) = 1080\pi$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

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# Solutions Pamphlet

American Mathematics Competitions

66th Annual

# AMC 12 A

**American Mathematics Contest 12A**

**Tuesday, February 3, 2015**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*



Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 11 Subcommittee Chair:

Jerrold W. Grossman

1. **Answer (C):**
$$(1-1+25+0)^{-1}\times 5=\frac{1}{25}\times 5=\frac{1}{5}$$

2. **Answer (E):** By the Triangle Inequality the third side must be greater than $20-15=5$ and less than $20+15=35$. Therefore the perimeter must be greater than $5+20+15=40$ and less than $35+20+15=70$. Among the choices only 72 cannot be the perimeter.

3. **Answer (E):** The sum of the 14 test scores was $14\cdot 80=1120$. The sum of all 15 test scores was $15\cdot 81=1215$. Therefore Payton's score was $1215-1120=95$.

**OR**

To bring the average up to 81, the total must include 1 more point for each of the 14 students, in addition to 81 points for Payton. Therefore Payton's score was $81+14=95$.

4. **Answer (B):** Let $x$ and $y$ be the two positive numbers, with $x>y$ . Then $x+y=5(x-y)$. Thus $4x=6y$, so $\frac{x}{y}=\frac{3}{2}$.

5. **Answer (D):** As long as $x$ and $y$ and their rounded values are positive, rounding the dividend $x$ up in a division problem $\frac{x}{y}$ makes the answer larger, and rounding $x$ down makes the answer smaller. Similarly, rounding the divisor $y$ up makes the answer smaller, and rounding $y$ down makes the answer larger. In a subtraction problem $x-y$, rounding $x$ up or rounding $y$ down increases the answer, and rounding $x$ down or rounding $y$ up decreases it. Only in choice (D) do all the roundings contribute to increasing the answer. In the other situations, the estimate may be larger or smaller than the exact value, depending on the the amount by which each number is rounded and their values. In particular, the rounding may make the answer smaller.
For (A), $\frac{999\,999}{900}-490>\frac{1\,000\,000}{1\,000}-500$.
For (B), $\frac{999\,999}{900}-510>\frac{1\,000\,000}{1\,000}-500$.
For (C), $\frac{999\,999}{1\,001}-490>\frac{1\,000\,000}{1\,000}-500$.
For (E), $\frac{1\,000\,001}{1\,001}-490>\frac{1\,000\,000}{1\,000}-500$.

6. **Answer (B):** Let $p$ be Pete's present age, and let $c$ be Claire's age. Then $p-2=3(c-2)$ and $p-4=4(c-4)$. Solving these equations gives $p=20$ and

$c = 8$. Thus Pete is 12 years older than Claire, so the ratio of their ages will be $2 : 1$ when Claire is 12 years old. That will occur $12 - 8 = 4$ years from now.

7. **Answer (D):** Let $r, h, R, H$ be the radii and heights of the first and second cylinders, respectively. The volumes are equal, so $\pi r^2 h = \pi R^2 H$. Also $R = r + 0.1r = 1.1r$. Thus $\pi r^2 h = \pi(1.1r)^2 H = \pi(1.21r^2)H$. Dividing by $\pi r^2$ yields $h = 1.21H = H + 0.21H$. Thus the first height is 21% more than the second height.

8. **Answer (C):** Let the sides of the rectangle have lengths $3a$ and $4a$. By the Pythagorean Theorem, the diagonal has length $5a$. Because $5a = d$, the side lengths are $\frac{3}{5}d$ and $\frac{4}{5}d$. Therefore the area is $\frac{3}{5}d \cdot \frac{4}{5}d = \frac{12}{25}d^2$, so $k = \frac{12}{25}$.

9. **Answer (C):** Because the marbles left for Cheryl are determined at random, the second of Cheryl's marbles is equally likely to be any of the 5 marbles other than her first marble. One of those 5 marbles matches her first marble in color. Therefore the probability is $\frac{1}{5}$.

**OR**

Because all the choices are made at random, Cheryl is equally likely to take any of the $\binom{6}{2} = 15$ possible pairs of marbles. Exactly 3 of these are pairs of same-colored marbles. Therefore the requested probability is $\frac{3}{15} = \frac{1}{5}$.

10. **Answer (E):** Adding 1 to both sides of the equation and factoring yields $(x + 1)(y + 1) = 81 = 3^4$. Because $x$ and $y$ are distinct positive integers and $x > y$, the only possibility is that $x+1 = 3^3 = 27$ and $y+1 = 3^1 = 3$. Therefore $x = 26$.

11. **Answer (D):** If the smaller circle is in the interior of the larger circle, there are no common tangent lines. If the smaller circle is internally tangent to the larger circle, there is exactly one common tangent line. If the circles intersect at two points, there are exactly two common tangent lines. If the circles are externally tangent, there are exactly three tangent lines. Finally, if the circles do not intersect, there are exactly four tangent lines. Therefore, $k$ can be any of the numbers 0, 1, 2, 3, or 4, which gives 5 possibilities.

12. **Answer (B):** The $y$-intercepts of the two parabolas are $-2$ and 4, respectively,

and in order to intersect the $x$-axis, the first must open upward and the second downward. Because the area of the kite is 12, the $x$-intercepts of both parabolas must be $-2$ and 2. Thus $4a-2=0$ so $a=\frac{1}{2}$, and $4-4b=0$ so $b=1$. Therefore $a+b=1.5$.

13. **Answer (E):** Note that each of the 12 teams plays 11 games, so $\frac{12\cdot 11}{2}=66$ games are played in all. If every game ends in a draw, then each team will have a score of 11, so statement (E) is not true. Each of the other statements is true. Each of the games generates 2 points in the score list, regardless of its outcome, so the sum of the scores must be $66\cdot 2=132$; thus (D) is true. Because the sum of an odd number of odd numbers plus any number of even numbers is odd, and 132 is even, there must be an even number of odd scores; thus (A) is true. Because there are 12 scores in all, there must also be an even number of even scores; thus (B) is true. Two teams cannot both have a score of 0 because the game between them must result in 1 point for each of them or 2 points for one of them; thus (C) is true.

14. **Answer (D):** By the change of base formula, $\dfrac{1}{\log_m n}=\log_n m$. Thus

$$1=\frac{1}{\log_2 a}+\frac{1}{\log_3 a}+\frac{1}{\log_4 a}=\log_a 2+\log_a 3+\log_a 4=\log_a 24.$$

It follows that $a=24$.

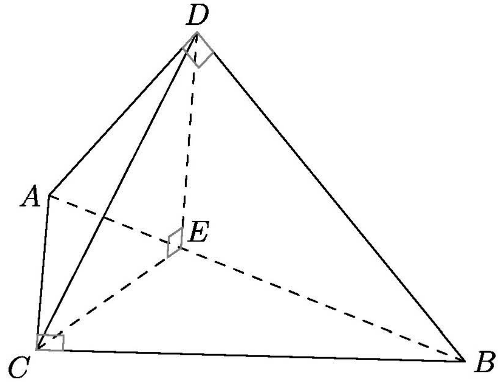
15. **Answer (C):** The numerator and denominator of this fraction have no common factors. To express the fraction as a decimal requires rewriting it with a power of 10 as the denominator. The smallest denominator that permits this is $10^{26}$:

$$\frac{123\,456\,789}{2^{26}\cdot 5^4}=\frac{123\,456\,789\cdot 5^{22}}{2^{26}\cdot 5^4\cdot 5^{22}}=\frac{123\,456\,789\cdot 5^{22}}{10^{26}},$$

so the numeral will have 26 places after the decimal point. In fact

$$\frac{123\,456\,789}{2^{26}\cdot 5^4}=0.00294\,34392\,21382\,14111\,32812\,5.$$

16 **Answer (C):** Triangles $ABC$ and $ABD$ are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE=\frac{12}{5}$. Likewise in $\triangle ABD$, $DE=\frac{12}{5}$. Triangle $CDE$ has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle $CED$. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base $ABC$. The tetrahedron's volume is $\frac{1}{3}\cdot 6\cdot\frac{12}{5}=\frac{24}{5}$.




17. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1+8+(28-8)+(56-8-32)+2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

18. **Answer (C):** The zeros of $f$ are integers and their sum is $a$, so $a$ is an integer. If $r$ is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of $r$ are 6, 4, 3, 1, 0, and $-2$. The possible values of $a$ are 9, 8, 0, and $-1$, all of which yield integer zeros of $f$, and their sum is 16.

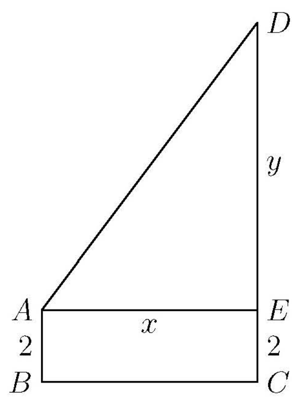
**OR**

As above, $a$ must be an integer. The function $f$ has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer $w$. Solving for $a$ in terms of $w$ gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for $w$ are 0 and $\pm 3$, from which it follows that the values of $a$ are 0, 8, 9, and $-1$, all of which yield integer values of $x$. The requested sum is 16.

19. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let $E$ be the foot of the perpendicular from $A$ to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2 + y$. By the Pythagorean Theorem, $x^2 + y^2 = (2 + y)^2$, or $x^2 = 4 + 4y$. Therefore $x$ is even, say $x = 2z$, and $z^2 = 1 + y$. The perimeter of the quadrilateral is $x + 2y + 6 = 2z^2 + 2z + 4$. Increasing positive integer values of $z$ give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.




20. **Answer (A):** Let $g$ and $h$ be the lengths of the altitudes of $T$ and $T'$ from the sides with lengths 8 and $b$, respectively. The Pythagorean Theorem implies that $g = \sqrt{5^2 - 4^2} = 3$, and so the area of $T$ is $\frac{1}{2} \cdot 8 \cdot 3 = 12$, and the perimeter is $5 + 5 + 8 = 18$. The Pythagorean Theorem implies that $h = \frac{1}{2}\sqrt{4a^2 - b^2}$. Thus $18 = 2a + b$ and

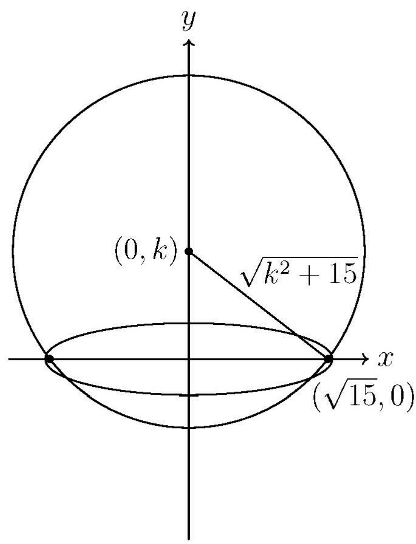
$$12 = \frac{1}{2}b \cdot \frac{1}{2}\sqrt{4a^2 - b^2} = \frac{1}{4}b\sqrt{4a^2 - b^2}.$$

Solving for $a$ and substituting in the square of the second equation yields

$$\begin{aligned} 12^2 &= \frac{b^2}{16}\left(4a^2 - b^2\right) = \frac{b^2}{16}\left((18 - b)^2 - b^2\right) \\ &= \frac{b^2}{16} \cdot 18 \cdot (18 - 2b) = \frac{9}{4}b^2\,(9 - b)\,. \end{aligned}$$

Thus $64 - b^2(9-b) = b^3 - 9b^2 + 64 = (b-8)(b^2-b-8) = 0$. Because $T$ and $T'$ are not congruent, it follows that $b \neq 8$. Hence $b^2 - b - 8 = 0$ and the positive solution of this equation is $\frac{1}{2}(\sqrt{33}+1)$. Because $25 < 33 < 36$, the solution is between $\frac{1}{2}(5+1) = 3$ and $\frac{1}{2}(6+1) = 3.5$, so the closest integer is 3.

21. **Answer (D):** The ellipse with equation $x^2 + 16y^2 = 16$ is centered at the origin, with a major axis of length 8 and a minor axis of length 2. If the foci have coordinates $(\pm c, 0)$, then $c^2 + 1^2 = 4^2$. Thus $c = \pm\sqrt{15}$. Any circle passing through both foci must have its center on the $y$-axis; thus $r$ is at least as large as the distance from the foci to the $y$-axis. That is, $r \geq \sqrt{15}$. For any $k \geq 0$, the circle of radius $\sqrt{k^2+15}$ and center $(0,k)$ passes through both foci (in the interior of the ellipse) and the points $(0, k \pm \sqrt{k^2+15})$. The point $(0, k + \sqrt{k^2+15})$ is in the exterior of the ellipse since $k + \sqrt{k^2+15} > \sqrt{15} > 1$. The point $(0, k - \sqrt{k^2+15})$ is in the exterior of the ellipse if and only if $k - \sqrt{k^2+15} < -1$, that is, if and only if $k < 7$. Thus, for $k \geq 0$, the circle with center $(0,k)$ intersects the ellipse in four points if and only if $0 \leq k < 7$. As $k$ increases, the radius $r = \sqrt{k^2+15}$ increases as well, so the set of possible radii is the interval $[\sqrt{15}, \sqrt{7^2+15}) = [\sqrt{15}, 8)$. The requested answer is $\sqrt{15} + 8$.




22. **Answer (D):** Note that $S(1) = 2$, $S(2) = 4$, and $S(3) = 8$. Call a sequence with $A$ and $B$ entries valid if it does not contain 4 or more consecutive symbols that are the same. For $n \geq 4$, every valid sequence of length $n-1$ can be extended to a valid sequence of length $n$ by appending a symbol different from its last symbol. Similarly, valid sequences of length $n-2$ or $n-3$ can be extended

to valid sequences of length $n$ by appending either two or three equal symbols different from its last symbol. Note that all of these sequences are pairwise distinct. Conversely, every valid sequence of length $n$ ends with either one, two, or three equal consecutive symbols. Removal of these equal symbols at the end produces every valid sequence of length $n-1$, $n-2$, or $n-3$, respectively. Thus $S(n) = S(n-1) + S(n-2) + S(n-3)$. This recursive formula implies that the remainders modulo 3 of the sequence $S(n)$ for $1 \le n \le 16$ are

$$2, 1, 2, 2, 2, 0, 1, 0, 1, 2, 0, 0, 2, 2, 1, 2.$$

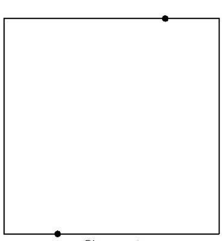
Thus the sequence is periodic with period-length 13. Because $2015 = 13 \cdot 155$, it follows that $S(2015) \equiv S(13) \equiv 2 \pmod 3$. Similarly, the remainders modulo 4 of the sequence $S(n)$ for $1 \le n \le 7$ are $2, 0, 0, 2, 2, 0, 0$. Thus the sequence is periodic with period-length 4. Because $2015 = 4 \cdot 503 + 3$, it follows that $S(2015) \equiv S(3) \equiv 0 \pmod 4$. Therefore $S(2015) = 4k$ for some integer $k$, and $4k \equiv 2 \pmod 3$. Hence $k \equiv 2 \pmod 3$ and $S(2015) = 4k \equiv 8 \pmod{12}$.

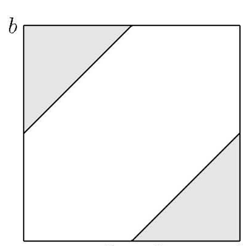
23. **Answer (A):** Let the square have vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

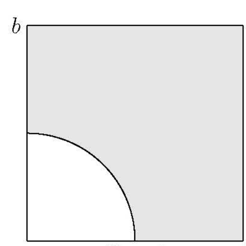
Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a, 0)$ and $(b, 0)$. The pairs of points in the $ab$-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a, 0)$ and $(0, b)$. The pairs of points in the $ab$-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2 + b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1 - \frac{1}{4}\pi(\frac{1}{2})^2 = 1 - \frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.


Case 1



Case 2



Case 3

Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\left(1-\frac{\pi}{16}\right)=\frac{26-\pi}{32},$$

and $a+b+c=59$.

24. **Answer (D):** There are 20 possible values for each of $a$ and $b$, namely those in the set

$$S=\left\{0,1,\frac{1}{2},\frac{3}{2},\frac{1}{3},\frac{2}{3},\frac{4}{3},\frac{5}{3},\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4},\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{6}{5},\frac{7}{5},\frac{8}{5},\frac{9}{5}\right\}.$$

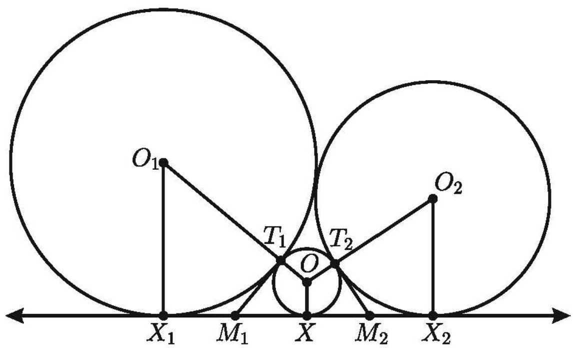
If $x$ and $y$ are real numbers, then $(x+iy)^2=x^2-y^2+i(2xy)$ is real if and only if $xy=0$, that is, $x=0$ or $y=0$. Therefore $(x+iy)^4$ is real if and only if $x^2-y^2=0$ or $xy=0$, that is, $x=0$, $y=0$, or $x=\pm y$. Thus $((\cos(a\pi)+i\sin(b\pi))^4$ is a real number if and only if $\cos(a\pi)=0$, $\sin(b\pi)=0$, or $\cos(a\pi)=\pm\sin(b\pi)$. If $\cos(a\pi)=0$ and $a\in S$, then $a=\frac{1}{2}$ or $a=\frac{3}{2}$ and $b$ has no restrictions, so there are 40 pairs $(a,b)$ that satisfy the condition. If $\sin(b\pi)=0$ and $b\in S$, then $b=0$ or $b=1$ and $a$ has no restrictions, so there are 40 pairs $(a,b)$ that satisfy the condition, but there are 4 pairs that have been counted already, namely $(\frac{1}{2},0)$, $(\frac{1}{2},1)$, $(\frac{3}{2},0)$, and $(\frac{3}{2},1)$. Thus the total so far is $40+40-4=76$.

Note that $\cos(a\pi)=\sin(b\pi)$ implies that $\cos(a\pi)=\cos(\pi(\frac{1}{2}-b))$ and thus $a\equiv\frac{1}{2}-b \pmod 2$ or $a\equiv-\frac{1}{2}+b \pmod 2$. If the denominator of $b\in S$ is 3 or 5, then the denominator of $a$ in simplified form would be 6 or 10, and so $a\notin S$. If $b=\frac{1}{2}$ or $b=\frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a=0$ and $a=1$, respectively. For every $b\in\left\{\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$, there is exactly one solution $a\in S$ to each of the previous congruences. None of the solutions are equal to each other because if $\frac{1}{2}-b\equiv-\frac{1}{2}+b \pmod 2$, then $2b\equiv 1 \pmod 2$; that is, $b=\frac{1}{2}$ or $b=\frac{3}{2}$. Similarly, $\cos(a\pi)=-\sin(b\pi)=\sin(-b\pi)$ implies that $\cos(a\pi)=\cos(\pi(\frac{1}{2}+b))$ and thus $a\equiv\frac{1}{2}+b \pmod 2$ or $a\equiv-\frac{1}{2}-b \pmod 2$. If the denominator of $b\in S$ is 3 or 5, then the denominator of $a$ would be 6 or 10, and so $a\notin S$. If $b=\frac{1}{2}$ or $b=\frac{3}{2}$, then there is a unique solution to either of the two congruences, namely $a=1$ and $a=0$, respectively. For every $b\in\left\{\frac{1}{4},\frac{3}{4},\frac{5}{4},\frac{7}{4}\right\}$, there is exactly one solution $a\in S$ to each of the previous congruences, and, as before, none of these solutions are equal to each other. Thus there are a total of $2+8+2+8=20$ pairs $(a,b)\in S^2$ such that $\cos(a\pi)=\pm\sin(b\pi)$. The requested probability is $\frac{76+20}{400}=\frac{96}{400}=\frac{6}{25}$.

**Note:** By de Moivre's Theorem the fourth power of the complex number $x+iy$ is real if and only if it lies on one of the four lines $x=0$, $y=0$, $x=y$, or $x=-y$. Then the counting of $(a,b)$ pairs proceeds as above.

25. **Answer (D):** Suppose that circles $C_1$ and $C_2$ in the upper half-plane have

centers $O_1$ and $O_2$ and radii $r_1$ and $r_2$, respectively. Assume that $C_1$ and $C_2$ are externally tangent and tangent to the $x$-axis at $X_1$ and $X_2$, respectively. Let $C$ with center $O$ and radius $r$ be the circle externally tangent to $C_1$ and $C_2$ and tangent to the $x$-axis. Let $X$ be the point of tangency of $C$ with the $x$-axis, and let $T_1$ and $T_2$ be the points of tangency of $C$ with $C_1$ and $C_2$, respectively. Let $M_1$ and $M_2$ be the points on the $x$-axis such that $\overline{M_1T_1} \perp \overline{O_1T_1}$ and $\overline{M_2T_2} \perp \overline{O_2T_2}$.




Because $\overline{M_1X_1}$ and $\overline{M_1T_1}$ are both tangent to $C_1$, it follows that $X_1M_1 = M_1T_1$. Similarly, $\overline{M_1T_1}$ and $\overline{M_1X}$ are both tangent to $C$, and thus $M_1T_1 = M_1X$. Because $\angle OT_1M_1$, $\angle M_1X_1O_1$, $\angle M_1T_1O$, and $\angle OXM_1$ are all right angles and $\angle T_1M_1X = \pi - \angle X_1M_1T_1$, it follows that quadrilaterals $O_1X_1M_1T_1$ and $M_1XOT_1$ are similar. Thus

$$\frac{r_1}{X_1M_1} = \frac{O_1X_1}{X_1M_1} = \frac{M_1X}{XO} = \frac{X_1M_1}{r}.$$

Therefore $X_1M_1 = \sqrt{rr_1}$, and similarly $M_2X_2 = \sqrt{rr_2}$. By the distance formula,

$$(r_1 + r_2)^2 = (O_1O_2)^2 = (X_1X_2)^2 + (r_1 - r_2)^2.$$

Thus

$$\begin{aligned} 2\sqrt{r_1r_2} &= X_1X_2 = X_1M_1 + M_1X + XM_2 + M_2X_2 \\ &= 2(X_1M_1 + M_2X_2) = 2\sqrt{r}(\sqrt{r_1} + \sqrt{r_2}); \end{aligned}$$

that is,

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}. \tag{1}$$

It follows that

$$\sum_{C \in L_k} \frac{1}{\sqrt{r(C)}} = \sum_{C \in L_k} \left( \frac{1}{\sqrt{r(C_1)}} + \frac{1}{\sqrt{r(C_2)}} \right),$$

where $C_1$ and $C_2$ are the consecutive circles in $\bigcup_{j=0}^{k-1} L_j$ that are tangent to $C$. Note that every circle in $\bigcup_{j=0}^{k-1} L_j$ appears twice in the sum on the right-hand side, except for the two circles in $L_0$, which appear only once. Thus

$$\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = 2\sum_{j=1}^{k-1}\sum_{C\in L_j} \frac{1}{\sqrt{r(C)}} + \sum_{C\in L_0} \frac{1}{\sqrt{r(C)}}.$$

In particular, if $k = 1$, then

$$\sum_{C\in L_1} \frac{1}{\sqrt{r(C)}} = \sum_{C\in L_0} \frac{1}{\sqrt{r(C)}} = \frac{1}{70} + \frac{1}{73}.$$

For simplicity let $x = \frac{1}{70} + \frac{1}{73}$. Let $k \geq 2$, and suppose by induction that for $1 \leq j \leq k-1$,

$$\sum_{C\in L_j} \frac{1}{\sqrt{r(C)}} = 3^{j-1}x.$$

It follows that

$$\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = 2\left(\sum_{j=1}^{k-1} 3^{j-1}x\right) + x = 2x\left(\frac{3^{k-1}-1}{2}\right) + x = x3^{k-1}.$$

Therefore

$$\sum_{C\in S} \frac{1}{\sqrt{r(C)}} = \sum_{k=0}^{6}\sum_{C\in L_k} \frac{1}{\sqrt{r(C)}} = x + \sum_{k=1}^{6} x3^{k-1} = x\left(1 + \frac{3^6-1}{2}\right)$$
$$= x\left(\frac{3^6+1}{2}\right) = \frac{143}{70\cdot 73}\left(\frac{730}{2}\right) = \frac{143}{14}.$$

**Note:** Equation (1) is a special case of the Kissing Circles Theorem.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Barb Currier, Steven Davis, Steve Dunbar, Silvia Fernandez, Charles Garner, Peter Gilchrist, Jerry Grossman, Jon Kane, Dan Kennedy, Joe Kennedy, Michael Khoury, Roger Waggoner, Dave Wells, Ronald Yannone, and Carl Yerger.

*The*

**MAA American Mathematics Competitions**

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1. **Answer (C):**
$$2-(-2)^{-2}=2-\frac{1}{(-2)^2}=2-\frac{1}{4}=\frac{7}{4}$$

2. **Answer (B):** The first two tasks together took 100 minutes—from 1:00 to 2:40. Therefore each task took 50 minutes. Marie began the third task at 2:40 and finished 50 minutes later, at 3:30 PM.

3. **Answer (A):** Let $x$ be the integer Isaac wrote two times, and let $y$ be the integer Isaac wrote three times. Then $2x+3y=100$. If $x=28$, then $3y=100-2\cdot 28=44$, and $y$ cannot be an integer. Therefore $y=28$ and $2x=100-3\cdot 28=16$, so $x=8$.

4. **Answer (B):** Marta finished 6th, so Jack finished 5th. Therefore Todd finished 3rd and Rand finished 2nd. Because Hikmet was 6 places behind Rand, it was Hikmet who finished 8th. (David finished 10th.)

5. **Answer (B):** If the Sharks win the next $N$ games, then they win $\frac{1+N}{3+N}\cdot 100\%$ of the games. Therefore $\frac{1+N}{3+N}\geq\frac{95}{100}=\frac{19}{20}$, so $20+20N\geq 57+19N$. Therefore $N\geq 37$.

**OR**

If the Tigers win no more games, then their 2 wins should be no more than 5%, or $\frac{1}{20}$, of the games played. So the minimum number of games played must be at least 40, and $N\geq 37$.

6. **Answer (A):** There are $13\cdot 13=169$ entries in the body of the table. An entry is odd if and only if both its row factor and its column factor are odd. There are 6 odd whole numbers between 0 and 12, so there are $6\cdot 6=36$ odd entries in the body of the table. The required fraction is $\frac{36}{169}=0.213\ldots\approx 0.21$.

7. **Answer (D):** The lines of symmetry are the 15 lines joining a vertex to the midpoint of the opposite side, so $L=15$. There is rotational symmetry around the center of the 15-gon, and the smallest positive angle of rotation that will transform the 15-gon onto itself is $\frac{360}{15}=24$ degrees; therefore $R=24$. The sum is $15+24=39$.

8. **Answer (D):**

$$\left(625^{\log_5 2015}\right)^{\frac{1}{4}} = \left((5^4)^{\log_5 2015}\right)^{\frac{1}{4}} = \left(5^{4\log_5 2015}\right)^{\frac{1}{4}} = \left(5^{\log_5 2015}\right)^{4\cdot\frac{1}{4}} = 2015$$

9. **Answer (C):** Let $x$ be the probability that Larry wins the game. Then $x = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot x$. To see this, note that Larry can win by knocking the bottle off the ledge on his first throw; if he and Julius both miss, then it is as if they started the game all over. Thus $x = \frac{1}{2} + \frac{1}{4}x$, so $\frac{3}{4}x = \frac{1}{2}$ or $x = \frac{2}{3}$.

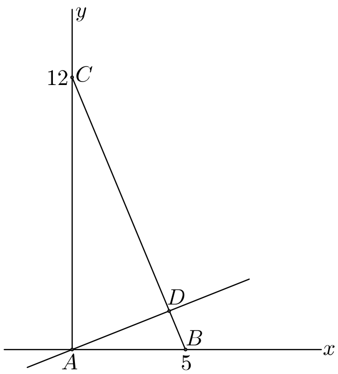
**OR**

For Larry to win on his $n$th throw, there must be $2n-2$ misses—$n-1$ by Larry and $n-1$ by Julius—followed by a hit by Larry. Because the probability of each of these independent events is $\frac{1}{2}$, the probability that Larry wins on his $n$th throw is $\left(\frac{1}{2}\right)^{2n-1}$. Therefore the probability that Larry wins the game is given by a geometric series:

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{2n-1} &= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots \\ &= \frac{1}{2}\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right) \\ &= \frac{1}{2} \cdot \frac{1}{1-\frac{1}{4}} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}.\end{aligned}$$

10. **Answer (C):** Let the side lengths be $a < b < c$. By the Triangle Inequality $a + b > c$; it follows that perimeter $P = a + b + c > 2c$. Then $2c < P < 15$, $2c < 14$, and $c < 7$. The only triangles (denoted by three-digit numbers with decreasing digits) that are not equilateral or isosceles are 653, 652, 643, 543, 542, and 432. Of these, only 543 is a right triangle, so the answer is 5.

11. **Answer (E):** Label the vertices of the triangle $A = (0, 0)$, $B = (5, 0)$, and $C = (0, 12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2} \cdot AD \cdot BC = 30$. Therefore $AD = \frac{2\cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.




12. **Answer (D):** If $(x-a)(x-b)+(x-b)(x-c)=0$, then $(x-b)(2x-(a+c))=0$, so the two roots are $b$ and $\frac{a+c}{2}$. The maximum value of their sum is $9+\frac{8+7}{2}=16.5$.

13. **Answer (B):** Because $\angle BAC$ and $\angle BDC$ intercept the same arc, $\angle BDC = 70^\circ$. Then $\angle ADC = 110^\circ$ and $\angle ABC = 180^\circ - \angle ADC = 70^\circ$. Thus $\triangle ABC$ is isosceles, and therefore $AC = BC = 6$.

14. **Answer (D):** Let $x$ equal the area of the circle, $y$ the area of the triangle, and $z$ the area of the overlapped sector. The answer is $(x-z)-(y-z)=x-y$. The area of the circle is $4\pi$ and the area of the triangle is $\frac{\sqrt{3}}{4}\cdot 4^2 = 4\sqrt{3}$, so the result is $4(\pi-\sqrt{3})$.

15. **Answer (D):** Rachelle needs a total of at least 14 points to get a 3.5 or higher GPA, so she needs a total of at least 6 points in English and History. The probability of a C in English is $1-\frac{1}{6}-\frac{1}{4}=\frac{7}{12}$, and the probability of a C in History is $1-\frac{1}{4}-\frac{1}{3}=\frac{5}{12}$. The probability that Rachelle earns exactly 6, 7, or 8 total points is computed as follows:

    6 points: $\frac{1}{6}\cdot\frac{5}{12}+\frac{1}{4}\cdot\frac{1}{3}+\frac{7}{12}\cdot\frac{1}{4}=\frac{43}{144}$

    7 points: $\frac{1}{6}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{4}=\frac{17}{144}$

    8 points: $\frac{1}{6}\cdot\frac{1}{4}=\frac{6}{144}$

    The probability that Rachelle will get at least a 3.5 GPA is

$$\frac{43}{144}+\frac{17}{144}+\frac{6}{144}=\frac{66}{144}=\frac{11}{24}.$$

16. **Answer (C):** The distance from a vertex of the hexagon to its center is 6. The height of the pyramid can be calculated by the Pythagorean Theorem using the right triangle with other leg 6 and hypotenuse 8; it is $\sqrt{8^2 - 6^2} = 2\sqrt{7}$. The volume is then

$$\frac{1}{3}Bh = \frac{1}{3} \cdot 6\left(6^2 \cdot \frac{\sqrt{3}}{4}\right) \cdot 2\sqrt{7} = 36\sqrt{21}.$$

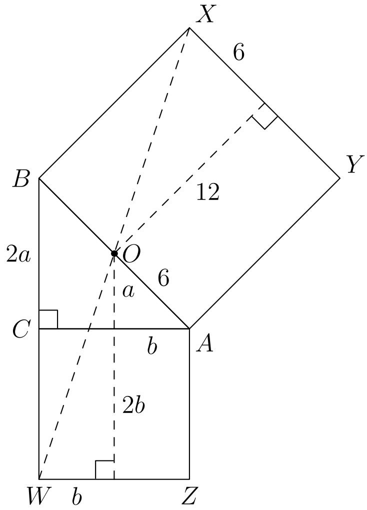
17. **Answer (D):** The probability of exactly two heads is $\binom{n}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^{n-2}$, and this must equal the probability of three heads, $\binom{n}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^{n-3}$. This results in the equation

$$\frac{n(n-1)}{2} \cdot \frac{3}{4} = \frac{n(n-1)(n-2)}{6} \cdot \frac{1}{4} \qquad \text{or} \qquad \frac{3}{8} = \frac{n-2}{24}.$$

Therefore $n = 11$.

18. **Answer (D):** To be composite, a number must have at least two prime factors, and the smallest prime number is 2. Therefore the smallest element in the range of $r$ is $2 + 2 = 4$. To see that all integers greater than 3 are in the range, note that $r(2^n) = 2n$ for all $n \geq 2$, and $r(2^n \cdot 3) = 2n + 3$ for all $n \geq 1$.

19. **Answer (C):** Let $O$ be the center of the circle on which $X$, $Y$, $Z$, and $W$ lie. Then $O$ lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that $O$ lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, $O$ is the circumcenter of $\triangle ABC$. Because $\angle C = 90°$, $O$ is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12+2a+2b = 12 + 12\sqrt{2}$.




20. **Answer (B):** Computing from the definition leads to the following values of $f(i,j)$ for $i = 0,1,2,3,4,5,6$ (the horizontal coordinate in the table) and $j = 0,1,2,3,4$ (the vertical coordinate).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 4 | 0 | 1 | 1 | 0 | 3 | 1 | 1 |
| 3 | 4 | 0 | 4 | 1 | 1 | 1 | 1 |
| 2 | 3 | 4 | 2 | 4 | 3 | 1 | 1 |
| 1 | 2 | 3 | 0 | 3 | 1 | 1 | 1 |
| 0 | 1 | 2 | 3 | 0 | 3 | 1 | 1 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 |

If follows that $f(i,2) = 1$ for all $i \geq 5$.

21. **Answer (D):** Assume that there are $t$ steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of $t$ are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d+20$ jumps, and $t = 2d+39$ or $t = 2d+40$. There are 10 possible combinations but only 3 of them lead to integer values of $d$: $t = 5d+3 = 2d+39$, or $t = 5d+1 = 2d+40$, or $t = 5d+4 = 2d+40$. The possible values of $t$ are 63, 66, and 64, and $s = 63+66+64 = 193$. The answer is $1+9+3 = 13$.

22. **Answer (D):** To make the analysis easier, suppose first that everyone gets up and moves to the chair directly across the table. The reseating rule now is that each person must sit in the same chair or in an adjacent chair. There must be either 0, 2, 4, or 6 people who choose the same chair; otherwise there would be an odd-sized gap, which would not permit all the people in that gap to sit in an adjacent chair. If no people choose the same chair, then either everyone moves left, which can be done in 1 way, or everyone moves right, which can be done in 1 way, or people swap with a neighbor, which can be done in 2 ways, for a total of 4 possibilities. If two people choose the same chair, then they must be either directly opposite each other or next to each other; there are $3 + 6 = 9$ such pairs. The remaining four people must swap in pairs, and that can be done in just 1 way in each case. If four people choose the same chair, there are 6 ways to choose those people and the other two people swap. Finally, there is 1 way for everyone to choose the same chair. Therefore there are $4 + 9 + 6 + 1 = 20$ ways in which the reseating can be done.

23. **Answer (B):** Because the volume and surface area are numerically equal, $abc = 2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6) = 0$ shows that $a \leq 6$. The original equation can also be written as $(a-2)bc-2ab-2ac = 0$. Note that if $a = 2$, this becomes $b + c = 0$, and there are no solutions. Otherwise, multiplying both sides by $a - 2$ and adding $4a^2$ to both sides gives $[(a - 2)b - 2a][(a - 2)c - 2a] = 4a^2$. Consider the possible values of $a$.

   $a = 1$: $(b + 2)(c + 2) = 4$
   There are no solutions in positive integers.

   $a = 3$: $(b - 6)(c - 6) = 36$
   The 5 solutions for $(b, c)$ are $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, and $(12, 12)$.

   $a = 4$: $(b - 4)(c - 4) = 16$
   The 3 solutions for $(b, c)$ are $(5, 20)$, $(6, 12)$, and $(8, 8)$.

   $a = 5$: $(3b - 10)(3c - 10) = 100$
   Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2, 50)$ and $(5, 20)$. The solutions for $(b, c)$ are $(4, 20)$ and $(5, 10)$, but only $(5, 10)$ has $a \leq b$.

   $a = 6$: $(b - 3)(c - 3) = 9$
   The solutions for $(b, c)$ are $(4, 12)$ and $(6, 6)$, but only $(6, 6)$ has $a \leq b$.

   Thus in all there are 10 ordered triples $(a, b, c)$: $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, and $(6, 6, 6)$.

24. **Answer (D):** Points $A$, $B$, $C$, $D$, and $R$ all lie on the perpendicular bisector of $\overline{PQ}$. Assume $R$ lies between $A$ and $B$. Let $y = AR$ and $x = \frac{AP}{5}$. Then $BR = 39 - y$ and $BP = 8x$, so $y^2 + 24^2 = 25x^2$ and $(39 - y)^2 + 24^2 = 64x^2$. Subtracting the two equations gives $x^2 = 39-2y$, from which $y^2+50y-399 = 0$, and the only positive solution is $y = 7$. Thus $AR = 7$, and $BR = 32$.

Note that circles $A$ and $B$ are determined by the assumption that $R$ lies between $A$ and $B$. Thus because the four circles are noncongruent, $R$ does not lie between $C$ and $D$. Let $w = CR$ and $z = \frac{CP}{5}$. Then $DR = 39 + w$ and $DP = 8z$, so $w^2 + 24^2 = 25z^2$ and $(39 + w)^2 + 24^2 = 64z^2$. Subtracting the two equations gives $z^2 = 39 + 2w$, from which $w^2 - 50w - 399 = 0$, and the only positive solution is $w = 57$. Thus $CR = 57$ and $DR = 96$. Again, the uniqueness of the solution implies that $R$ must indeed lie between $A$ and $B$.

The requested sum is $7 + 32 + 57 + 96 = 192$.

25. **Answer (B):** Modeling the bee's path with complex numbers, set $P_0 = 0$ and $z = e^{\pi i/6}$. It follows that for $j \geq 1$,

$$P_j = \sum_{k=1}^{j} kz^{k-1}.$$

Thus

$$P_{2015} = \sum_{k=0}^{2015} kz^{k-1} = \sum_{k=0}^{2014} (k+1)\, z^k = \sum_{k=0}^{2014} \sum_{j=0}^{k} z^k.$$

Interchanging the order of summation and summing the geometric series gives

$$\begin{aligned}
P_{2015} &= \sum_{j=0}^{2014} \sum_{k=j}^{2014} z^k = \sum_{j=0}^{2014} z^j \sum_{k=0}^{2014-j} z^k \\
&= \sum_{j=0}^{2014} \frac{z^j\left(z^{2015-j} - 1\right)}{z-1} = \sum_{j=0}^{2014} \frac{z^{2015} - z^j}{z-1} = \frac{1}{z-1} \sum_{j=0}^{2014} (z^{2015} - z^j) \\
&= \frac{1}{z-1}\left(2015z^{2015} - \sum_{j=0}^{2014} z^j\right) = \frac{1}{z-1}\left(2015z^{2015} - \frac{z^{2015} - 1}{z-1}\right) \\
&= \frac{1}{(z-1)^2}\left(2015z^{2015}(z-1) - z^{2015} + 1\right) \\
&= \frac{1}{(z-1)^2}\left(2015z^{2016} - 2016z^{2015} + 1\right).
\end{aligned}$$

Note that $z^{12} = 1$ and thus $z^{2016} = (z^{12})^{168} = 1$ and $z^{2015} = \frac{1}{z}$. It follows that

$$P_{2015} = \frac{2016}{(z-1)^2}\left(1 - \frac{1}{z}\right) = \frac{2016}{z(z-1)}.$$

Finally,

$$|z-1|^2 = \left|\cos\left(\frac{\pi}{6}\right) - 1 + i\sin\left(\frac{\pi}{6}\right)\right|^2 = \left|\frac{\sqrt{3}}{2} - 1 + \frac{i}{2}\right|^2 = 2 - \sqrt{3} = \frac{(\sqrt{3}-1)^2}{2},$$

and thus

$$\begin{aligned}|P_{2015}| &= \left|\frac{2016}{z(z-1)}\right| = \frac{2016}{|z-1|} = \frac{2016\sqrt{2}}{\sqrt{3}-1} = 1008\sqrt{2}(\sqrt{3}+1)\\ &= 1008\sqrt{6} + 1008\sqrt{2}.\end{aligned}$$

The requested sum is $1008 + 6 + 1008 + 2 = 2024$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Steve Blasberg, Tom Butts, Barbara Currier, Steven Davis, Steve Dunbar, Zuming Feng, Silvia Fernandez, Charles Garner, Richard Gibbs, Jerry Grossman, Joe Kennedy, Cap Khoury, Steve Miller, David Wells, and Carl Yerger.

MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

# Solutions Pamphlet

American Mathematics Competitions

67th Annual

# AMC 12A

**American Mathematics Contest 12A**

**Tuesday, February 2, 2016**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (B):**

$$\frac{11!-10!}{9!}=\frac{10!\cdot(11-1)}{9!}=\frac{10\cdot 9!\cdot 10}{9!}=100$$

2. **Answer (C):** The equation can be written $10^x\cdot(10^2)^{2x}=(10^3)^5$ or $10^x\cdot 10^{4x}=10^{15}$. Thus $10^{5x}=10^{15}$, so $5x=15$ and $x=3$.

3. **Answer (B):**

$$\frac{3}{8}-\left(-\frac{2}{5}\right)\left\lfloor\frac{\frac{3}{8}}{-\frac{2}{5}}\right\rfloor=\frac{3}{8}+\frac{2}{5}\left\lfloor-\frac{15}{16}\right\rfloor=\frac{3}{8}+\frac{2}{5}(-1)=-\frac{1}{40}$$

4. **Answer (D):** The mean of the data values is

$$\frac{60+100+x+40+50+200+90}{7}=\frac{x+540}{7}=x.$$

Solving this equation for $x$ gives $x=90$. Thus the data in nondecreasing order are $40,50,60,90,90,100,200$, so the median is 90 and the mode is 90, as required.

5. **Answer (E):** A counterexample must satisfy the hypothesis of being an even integer greater than 2 but fail to satisfy the conclusion that it can be written as the sum of two prime numbers.

6. **Answer (D):** There are

$$1+2+\cdots+N=\frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1)=2\cdot 2016=4032$. Because $N(N+1)\approx N^2$, it follows that $N\approx\sqrt{4032}\approx\sqrt{2^{12}}=2^6=64$. Indeed, $63\cdot 64=4032$, so $N=63$ and the sum of the digits of $N$ is 9.

7. **Answer (D):** The given equation is equivalent to $(x^2-y^2)(x+y+1)=0$, which is in turn equivalent to $(x+y)(x-y)(x+y+1)=0$. A product is 0 if and only if one of the factors is 0, so the graph is the union of the graphs of $x+y=0$, $x-y=0$, and $x+y+1=0$. These are three straight lines, two of which intersect at the origin and the third of which does not pass through the origin. Therefore the graph consists of three lines that do not all pass through a common point.

8. **Answer (D):** The diagonal of the rectangle from upper left to lower right divides the shaded region into four triangles. Two of them have a 1-unit horizontal base and altitude $\frac{1}{2} \cdot 5 = 2\frac{1}{2}$, and the other two have a 1-unit vertical base and altitude $\frac{1}{2} \cdot 8 = 4$. Therefore the total area is $2 \cdot \frac{1}{2} \cdot 1 \cdot 2\frac{1}{2} + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = 6\frac{1}{2}$.

9. **Answer (E):** Let $x$ be the common side length. Draw a diagonal between opposite corners of the unit square. The length of this diagonal is $\sqrt{2}$. The diagonal consists of two small-square diagonals and one small-square side length. Combining the previous two observations yields

$$2x\sqrt{2} + x = \sqrt{2}.$$

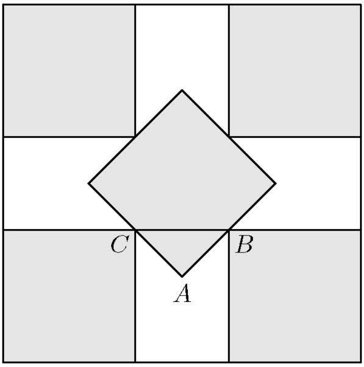
Solving this equation for $x$ gives $x = \frac{4-\sqrt{2}}{7}$. The requested sum is $4 + 7 = 11$.

**OR**

Again let $x$ be the common side length. Triangle $ABC$ in the figure shown is a right triangle with sides $\frac{x}{2}$, $\frac{x}{2}$, and $1 - 2x$. By the Pythagorean Theorem,

$$\left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^2 = (1 - 2x)^2.$$

Solving this equation and noting that $x < \frac{1}{2}$ yields $x = \frac{4-\sqrt{2}}{7}$, as above.




10. **Answer (B):** The total number of seats moved to the right among the five friends must equal the total number of seats moved to the left. One of Dee and Edie moved some number of seats to the right, and the other moved the same number of seats to the left. Because Bea moved two seats to the right and Ceci moved one seat to the left, Ada must also move one seat to the left upon her

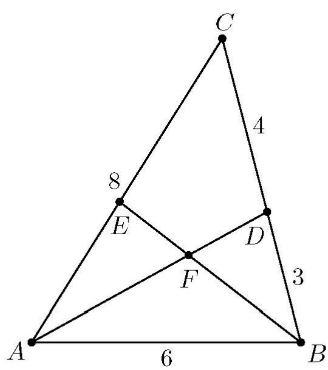
return. Because her new seat is an end seat and its number cannot be 5, it must be seat 1. Therefore Ada occupied seat 2 before she got up. The order before moving was Bea-Ada-Ceci-Dee-Edie (or Bea-Ada-Ceci-Edie-Dee), and the order after moving was Ada-Ceci-Bea-Edie-Dee (or Ada-Ceci-Bea-Dee-Edie).

11. **Answer (E):** Because 42 students cannot sing, $100 - 42 = 58$ can sing. Similarly, $100 - 65 = 35$ can dance, and $100 - 29 = 71$ can act. This gives a total of $58 + 35 + 71 = 164$. However, the students with two talents have been counted twice in this sum. Because there are 100 students in all, $164 - 100 = 64$ students must have been counted twice.

**OR**

Consider the three sets referred to in the problem: those who cannot sing, those who cannot dance, and those who cannot act. Students with one talent are in two of those sets, whereas students with two talents are in only one. Thus the total $42 + 65 + 29 = 136$ counts all students twice except those with two talents. The number of students with two talents is therefore $2 \cdot 100 - 136 = 64$.

12. **Answer (C):** Applying the Angle Bisector Theorem to $\triangle BAC$ gives $BD : DC = 6 : 8$, so $BD = \frac{6}{6+8} \cdot 7 = 3$. Then applying the Angle Bisector Theorem to $\triangle ABD$ gives $AF : FD = 6 : 3 = 2 : 1$.




**Note:** More generally the ratio $AF : FD$ is $(AB+CA) : BC$, which equals $2 : 1$ whenever $AB, BC, CA$ forms an arithmetic progression.

13. **Answer (A):** Let $N = 5k$, where $k$ is a positive integer. There are $5k + 1$ equally likely possible positions for the red ball in the line of balls. Number

these $0, 1, 2, 3, \ldots, 5k-1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k+1, 2k+2, \ldots, 3k-1$. This includes $(3k-1)-2k = k-1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for $k$ yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of $k$ corresponding to the required least value of $N$ is therefore 96, so $N = 480$. The sum of the digits of $N$ is 12.

14. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6} \cdot 3 \cdot (1+2+\cdots+8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, and $\{1, 4, 6, 7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can belong to at most two faces, the vertices of one face must be labeled with the numbers $1, 4, 6, 7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

15. **Answer (D):** Let $X$ be the foot of the perpendicular from $P$ to $\overline{QQ'}$, and let $Y$ be the foot of the perpendicular from $Q$ to $\overline{RR'}$. By the Pythagorean Theorem,

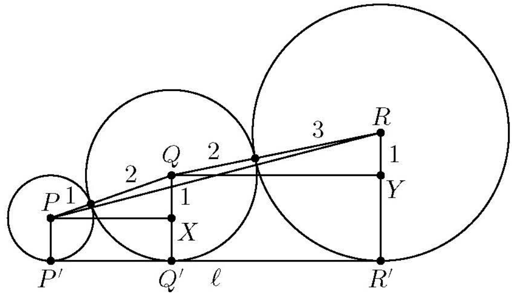
$$P'Q' = PX = \sqrt{(2+1)^2 - (2-1)^2} = \sqrt{8}$$

and

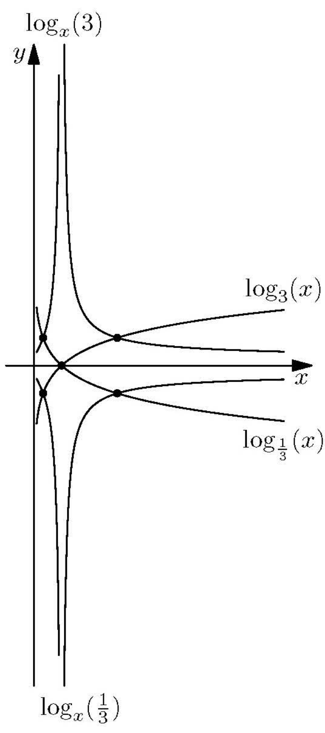
$$Q'R' = QY = \sqrt{(3+2)^2 - (3-2)^2} = \sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8} + \frac{2+3}{2}\sqrt{24} - \frac{1+3}{2}\left(\sqrt{8}+\sqrt{24}\right) = \sqrt{6} - \sqrt{2}.$$




16. **Answer (D):** Let $u = \log_3 x$. Then $\log_x 3 = \frac{1}{u}$, $\log_{\frac{1}{3}} x = -u$, and $\log_x \frac{1}{3} = -\frac{1}{u}$. Thus each point at which two of the graphs of the given functions intersect in the $(x, y)$-plane corresponds to a point at which two of the graphs of $y = u$, $y = \frac{1}{u}$, $y = -u$, and $y = -\frac{1}{u}$ intersect in the $(u, y)$-plane. There are 5 such points $(u, y)$, namely $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$, and $(-1, -1)$. The corresponding points of intersection on the graphs of the given functions are $(1, 0)$, $(3, 1)$, $(\frac{1}{3}, 1)$, $(3, -1)$, and $(\frac{1}{3}, -1)$.



17. **Answer (B):** Without loss of generality, let the square and equilateral triangles have side length 6. Then the height of the equilateral triangles is $3\sqrt{3}$, and the distance of each of the triangle centers, $E$, $F$, $G$, and $H$, to the square $ABCD$ is $\sqrt{3}$. It follows that the diagonal of square $ABCD$ has length $6\sqrt{2}$, and the diagonal of square $EFGH$ has length equal to the side length of square $ABCD$ plus twice the distance from the center of an equilateral triangle to square $ABCD$ or $6+2\sqrt{3}$. The required ratio of the areas of the two squares is equal to the square of the ratio of the lengths of the diagonals of the two squares, or

$$\left(\frac{6+2\sqrt{3}}{6\sqrt{2}}\right)^2 = \left(\frac{3+\sqrt{3}}{3\sqrt{2}}\right)^2 = \frac{12+6\sqrt{3}}{18} = \frac{2+\sqrt{3}}{3}.$$

**OR**

Without loss of generality, place the square in the Cartesian plane with coordinates $A(-6,0)$, $B(0,0)$, $C(0,-6)$, and $D(-6,-6)$. The center of each equilateral triangle is the point at which the medians intersect, and this point is one third of the way from the midpoint of a side of the triangle to the opposite vertex. The height of an equilateral triangle with side 6 is $3\sqrt{3}$, so the centers are $\sqrt{3}$ units from the sides of the square. Therefore the coordinates are $E(-3,\sqrt{3})$, $F(\sqrt{3},-3)$, $G(-3,-6-\sqrt{3})$, and $H(-6-\sqrt{3},-3)$. The area of square $EFGH$ is half the product of the lengths of its diagonals, or $\frac{1}{2}(6+2\sqrt{3})^2 = 24+12\sqrt{3}$. Square $ABCD$ has area 36, so the desired ratio is $\frac{2+\sqrt{3}}{3}$.

18. **Answer (D):** Let $110n^3 = p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where the $p_j$ are distinct primes and the $r_j$ are positive integers. Then $\tau(110n^3)$, the number of positive integer divisors of $110n^3$, is given by

$$\tau(110n^3) = (r_1+1)(r_2+1)\cdots(r_k+1) = 110.$$

Because $110 = 2\cdot 5\cdot 11$, it follows that $k=3$, $\{p_1, p_2, p_3\} = \{2, 5, 11\}$, and, without loss of generality, $r_1 = 1$, $r_2 = 4$, and $r_3 = 10$. Therefore

$$n^3 = \frac{p_1\cdot p_2^4\cdot p_3^{10}}{110} = p_2^3\cdot p_3^9, \quad \text{so} \quad n = p_2\cdot p_3^3.$$

It follows that $81n^4 = 3^4\cdot p_2^4\cdot p_3^{12}$, and because 3, $p_2$, and $p_3$ are distinct primes, $\tau(81n^4) = 5\cdot 5\cdot 13 = 325$.

19. **Answer (B):** Jerry arrives at 4 for the first time after an even number of tosses. Because Jerry tosses 8 coins, he arrives at 4 for the first time after either

4, 6, or 8 tosses. If Jerry arrives at 4 for the first time after 4 tosses, then he must have tossed HHHH. The probability of this occurring is $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 6 tosses, he must have tossed 5 heads and 1 tail among the 6 tosses, and the 1 tail must have come among the first 4 tosses. Thus, there are 4 possible sequences of valid tosses, each with probability $\frac{1}{64}$, for a total of $\frac{1}{16}$. If Jerry arrives at 4 for the first time after 8 tosses, then he must have tossed 6 heads and 2 tails among the 8 tosses. Both tails must occur among the first 6 tosses; otherwise Jerry would have already reached 4 before the $8^{\text{th}}$ toss. Further, at least 1 tail must occur in the first 4 tosses; otherwise Jerry would have already reached 4 after the $4^{\text{th}}$ toss. Therefore there are $\binom{6}{2} - 1 = 14$ sequences for which Jerry first arrives at 4 after 8 tosses, each with probability $\frac{1}{256}$, for a total of $\frac{14}{256} = \frac{7}{128}$. Thus the probability that Jerry reaches 4 at some time during the process is $\frac{1}{16} + \frac{1}{16} + \frac{7}{128} = \frac{23}{128}$. The requested sum is $23 + 128 = 151$.

**OR**

Count the sequences of 8 heads or tails that result in Jerry arriving at 4. Any sequence with T appearing fewer than 3 times results in Jerry reaching 4. There are $\binom{8}{0} + \binom{8}{1} + \binom{8}{2} = 1+8+28 = 37$ such sequences. If Jerry's sequence contains exactly 3 Ts, then he reaches 4 only if he does so before getting his second T. As a result, Jerry can get at most one T in his first 5 tosses. This happens if the first 4 tosses are H and there is exactly one H in the last 4 tosses, or there is one T within the first 4 tosses followed by the remaining 5 Hs, accounting for $4+4 = 8$ ways for Jerry to get to 4 with exactly 3 Ts. Finally, the only way for Jerry to get to 4 by tossing exactly 4 Ts is HHHHTTTT. Jerry cannot get to 4 by tossing fewer than 4 Hs. Thus there are $37 + 8 + 1 = 46$ sequences where he reaches 4, out of $2^8 = 256$ equally likely ways to toss the coin 8 times. The required probability is $\frac{46}{256} = \frac{23}{128}$, and the requested sum is $23 + 128 = 151$.

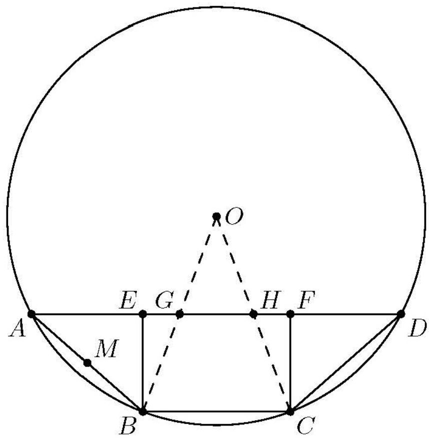
20. **Answer (A):** From the given properties, $a \diamondsuit 1 = a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a = 1 \cdot a = a$ for all nonzero $a$. Then for nonzero $a$ and $b$, $a = a \diamondsuit 1 = a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b$. It follows that $a \diamondsuit b = \frac{a}{b}$. Thus

$$100 = 2016 \diamondsuit (6 \diamondsuit x) = 2016 \diamondsuit \frac{6}{x} = \frac{2016}{\frac{6}{x}} = 336x,$$

so $x = \frac{100}{336} = \frac{25}{84}$. The requested sum is $25 + 84 = 109$.

21. **Answer (E):** Let $ABCD$ be the given cyclic quadrilateral with $AB = BC = CD = 200$, and let $E$ and $F$ be the feet of the perpendicular segments from $B$ and $C$, respectively, to $\overline{AD}$, as shown in the figure. Let the center of the circle be $O$, and let $\angle AOB = \angle BOC = \angle COD = \theta$. Because inscribed $\angle BAD$ is half the

size of central $\angle BOD = 2\theta$, it follows that $\angle BAD = \theta$. Let $M$ be the midpoint of $\overline{AB}$. Then $\sin\left(\frac{\theta}{2}\right) = \frac{AM}{AO} = \frac{100}{200\sqrt{2}} = \frac{1}{2\sqrt{2}}$. Then $\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right) = \frac{3}{4}$. Hence $AE = AB\cos\theta = 200 \cdot \frac{3}{4} = 150$, and $FD = 150$ as well. Because $EF = BC = 200$, the remaining side $AD = AE + EF + FD = 150 + 200 + 150 = 500$.




**OR**

Label the quadrilateral $ABCD$ and the center of the circle as in the first solution. Because the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are shorter than the radius, each of $\angle AOB$, $\angle BOC$, and $\angle COD$ is less than $60^\circ$, so $O$ is outside the quadrilateral $ABCD$. Let $G$ and $H$ be the intersections of $\overline{AD}$ with $\overline{OB}$ and $\overline{OC}$, respectively. Because $\overline{AD}$ and $\overline{BC}$ are parallel, and $\triangle OAB$ and $\triangle OBC$ are congruent and isosceles, it follows that $\angle ABO = \angle OBC = \angle OGH = \angle AGB$. Thus $\triangle ABG$, $\triangle OGH$, and $\triangle OBC$ are similar and isosceles with $\frac{AB}{BG} = \frac{OG}{GH} = \frac{OB}{BC} = \frac{200\sqrt{2}}{200} = \sqrt{2}$. Then $AG = AB = 200$, $BG = \frac{AB}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$, and $GH = \frac{OG}{\sqrt{2}} = \frac{BO-BG}{\sqrt{2}} = \frac{200\sqrt{2}-100\sqrt{2}}{\sqrt{2}} = 100$. Therefore $AD = AG + GH + HD = 200 + 100 + 200 = 500$.

**OR**

Let $\theta$ be the central angle that subtends the side of length 200. Then by the Law of Cosines, $\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos\theta = 200^2$, which gives $\cos\theta = \frac{3}{4}$. The Law of Cosines also gives the square of the fourth side of the quadrilateral as

$$\left(200\sqrt{2}\right)^2 + \left(200\sqrt{2}\right)^2 - 2\left(200\sqrt{2}\right)^2\cos(3\theta)$$

$$= 160{,}000 - 160{,}000(4\cos^3\theta - 3\cos\theta) = 250{,}000.$$

Thus the fourth side has length $\sqrt{250{,}000} = 500$.

22. **Answer (A):** Because $\operatorname{lcm}(x, y) = 2^3 \cdot 3^2$ and $\operatorname{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, it follows that $5^2$ divides $z$, but neither $x$ nor $y$ is divisible by 5. Furthermore, $y$ is divisible by $3^2$, and neither $x$ nor $z$ is divisible by $3^2$, but at least one of $x$ or $z$ is divisible by 3. Finally, because $\operatorname{lcm}(y, z) = 2^2 \cdot 3^2 \cdot 5^2$, at least one of $y$ or $z$ is divisible by $2^2$, but neither is divisible by $2^3$. However, $x$ must be divisible by $2^3$. Thus $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, where $\max(j, n) = 1$ and $\max(k, m) = 2$. There are 3 choices for $(j, n)$ and 5 choices for $(k, m)$, so there are 15 possible ordered triples $(x, y, z)$.

23. **Answer (C):** Let the chosen numbers be $x$, $y$, and $z$. The set of possible ordered triples $(x, y, z)$ forms a solid unit cube, two of whose vertices are $(0, 0, 0)$ and $(1, 1, 1)$. The numbers fail to be the side lengths of a triangle with positive area if and only if one of the numbers is at least as great as the sum of the other two. The ordered triples that satisfy $z \geq x + y$ lie in the region on and above the plane $z = x+y$. The intersection of this region with the solid cube is a solid tetrahedron with vertices $(0, 0, 0)$, $(0, 0, 1)$, $(0, 1, 1)$, and $(1, 0, 1)$. The volume of this tetrahedron is $\frac{1}{6}$. The intersections of the solid cube with the regions defined by the inequalities $y \geq x+z$ and $x \geq y+z$ are solid tetrahedra with the same volume. Because at most one of the inequalities $z > x + y$, $y > x + z$, and $x > y + z$ can be true for any choice of $x$, $y$, and $z$, the three tetrahedra have disjoint interiors. Thus the required probability is $1 - 3 \cdot \frac{1}{6} = \frac{1}{2}$.

**OR**

As in the first solution, the set of possible ordered triples $(x, y, z)$ forms a solid unit cube. First consider only the points for which $x > y$ and $x > z$. These points form a square pyramid whose vertex is $(0, 0, 0)$ and whose base has vertices at $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$, and $(0, 1, 1)$. Such an ordered triple corresponds to the side lengths of a triangle if and only if $z < x + y$. The plane $z = x + y$ passes through the vertex of the pyramid and bisects its base, so it bisects the volume of the pyramid. The probability of forming a triangle is the same as the probability of not forming a triangle. The same argument applies when $y$ or $z$ is the largest element in the triple. The probability of any two of $x$, $y$, and $z$ being equal is 0, so this case can be ignored. Thus this event and its complement are equally likely; the probability is $\frac{1}{2}$.

24. **Answer (B):** Because $a$ and $b$ are positive, all the roots must be positive. Let the roots be $r$, $s$, and $t$. Then

$$x^3-ax^2+bx-a=(x-r)(x-s)(x-t)=x^3-(r+s+t)x^2+(rs+st+tr)x-rst.$$

Therefore $r+s+t=a=rst$. The Arithmetic Mean–Geometric Mean Inequality implies that $27rst \le (r+s+t)^3=(rst)^3$, from which $a=rst\ge 3\sqrt{3}$. Furthermore, equality is achieved if and only if $r=s=t=\sqrt{3}$. In this case $b=rs+st+tr=9$.

25. **Answer (E):** Assume that $k=2j\ge 2$ is even. The smallest perfect square with $k+1$ digits is $10^k=(10^j)^2$. Thus the sequence of numbers written on the board after Silvia erases the last $k$ digits of each number is the sequence

$$1=\left\lfloor\frac{\left(10^j\right)^2}{10^k}\right\rfloor,\left\lfloor\frac{\left(10^j+1\right)^2}{10^k}\right\rfloor,\ldots,\left\lfloor\frac{n^2}{10^k}\right\rfloor,\ldots.$$

The sequence ends the first time that

$$\left\lfloor\frac{(n+1)^2}{10^k}\right\rfloor-\left\lfloor\frac{n^2}{10^k}\right\rfloor\ge 2;$$

before that, every two consecutive terms are either equal or they differ by 1. Suppose that

$$\left\lfloor\frac{n^2}{10^k}\right\rfloor=a \quad\text{and}\quad \left\lfloor\frac{(n+1)^2}{10^k}\right\rfloor\ge a+2.$$

Then $n^2<(a+1)10^k$ and $(a+2)10^k\le(n+1)^2$. Thus

$$10^k=(a+2)10^k-(a+1)10^k<(n+1)^2-n^2=2n+1.$$

It follows that $n=\frac{10^k}{2}+m$ for some positive integer $m$. Note that

$$\frac{n^2}{10^k}=\frac{1}{10^k}\left(\frac{10^k}{2}+m\right)^2=\frac{1}{10^k}\left(\frac{10^{2k}}{4}+m\cdot 10^k+m^2\right)=\frac{10^k}{4}+m+\frac{m^2}{10^k}.$$

Because $k\ge 2$, it follows that $10^k$ is divisible by 4, and so

$$\left\lfloor\frac{n^2}{10^k}\right\rfloor=\frac{10^k}{4}+m+\left\lfloor\frac{m^2}{10^k}\right\rfloor \quad\text{and}\quad \left\lfloor\frac{(n+1)^2}{10^k}\right\rfloor=\frac{10^k}{4}+m+1+\left\lfloor\frac{(m+1)^2}{10^k}\right\rfloor.$$

The difference will be at least 2 for the first time when

$$\left\lfloor\frac{m^2}{10^k}\right\rfloor=0 \quad\text{and}\quad \left\lfloor\frac{(m+1)^2}{10^k}\right\rfloor\ge 1,$$

that is, for $m$ such that $m^2 < 10^k \le (m+1)^2$, equivalently, $m < 10^j \le m+1$. Thus $m = 10^j - 1$ and then

$$f(k) = f(2j) = a + 1 = \left\lfloor \frac{n^2}{10^k} \right\rfloor + 1 = \frac{10^k}{4} + m + 1 = \frac{10^{2j}}{4} + 10^j.$$

Therefore

$$\begin{aligned}\sum_{j=1}^{1008} f(2j) &= \sum_{j=1}^{1008} \left( \frac{10^{2j}}{4} + 10^j \right) = 25 \sum_{j=0}^{1007} 10^{2j} + 10 \sum_{j=0}^{1007} 10^j \\ &= \underbrace{252525\ldots25}_{2016 \text{ digits}} + \underbrace{111\ldots10}_{1009 \text{ digits}}.\end{aligned}$$

Because there are no carries in the sum, the required sum of digits equals $1008 \cdot (2+5) + 1008 \cdot 1 = 1008 \cdot 8 = 8064$.

Problems and solutions were contributed by Bernardo Abrego, Sam Baethge, Tom Butts, Steve Dunbar, Marta Eso, Jacek Fabrykowski, Chuck Garner, Peter Gilchrist, Jerry Grossman, Elgin Johnston, Joe Kennedy, Michael Khoury, Krassimir Penev, and David Wells.

*The*

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# Solutions Pamphlet

MAA American Mathematics Competitions

67th Annual

# AMC 12B

**American Mathematics Contest 12B**

**Wednesday, February 17, 2016**

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (D):**

$$\frac{2(\frac{1}{2})^{-1} + \frac{(\frac{1}{2})^{-1}}{2}}{\frac{1}{2}} = \left(2 \cdot 2 + \frac{2}{2}\right) \cdot 2 = 10$$

2. **Answer (A):** The harmonic mean of 1 and 2016 is

$$\frac{2 \cdot 1 \cdot 2016}{1 + 2016} = 2 \cdot \frac{2016}{2017} \approx 2 \cdot 1 = 2.$$

3. **Answer (D):**

$$\Bigg|\Big|\,|-2016|-(-2016)\,\Big| - |-2016|\Bigg| - (-2016)$$
$$= \Bigg|\Big|\,2016 + 2016\,\Big| - 2016\Bigg| + 2016 = 2016 + 2016 = 4032$$

4. **Answer (C):** Let $\alpha$ and $\beta$ be the measures of the angles, with $\alpha < \beta$. Then $\frac{\beta}{\alpha} = \frac{5}{4}$. Because $\alpha < \beta$, it follows that $90^\circ - \beta < 90^\circ - \alpha$, so $90^\circ - \alpha = 2(90^\circ - \beta)$. This leads to the system of linear equations $4\beta - 5\alpha = 0$ and $2\beta - \alpha = 90^\circ$. Solving the system gives $\alpha = 60^\circ$, $\beta = 75^\circ$. The requested sum is $\alpha + \beta = 135^\circ$.

5. **Answer (B):** Because $919 = 7 \cdot 131 + 2$, the war lasted 131 full weeks plus 2 days. Therefore it ended 2 days beyond Thursday, which is Saturday.

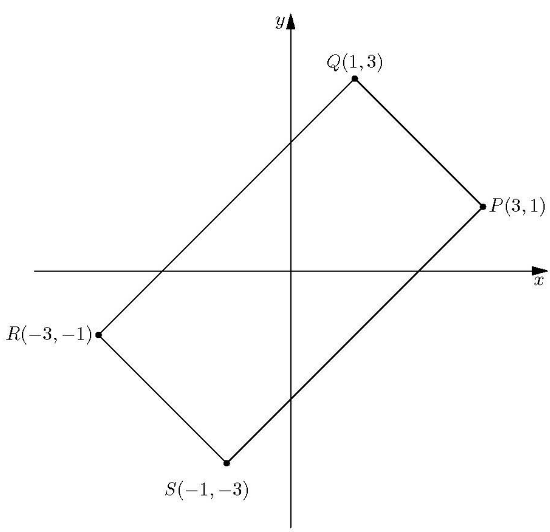
6. **Answer (C):** Let the vertex of the triangle that lies in the first quadrant be $(x, x^2)$. Then the base of the triangle is $2x$ and the height is $x^2$, so $\frac{1}{2} \cdot 2x \cdot x^2 = 64$. Thus $x^3 = 64$, $x = 4$, and $BC = 2x = 8$.

7. **Answer (D):** In the first pass Josh marks out the odd numbers $1, 3, 5, 7, \ldots, 99$, leaving the multiples of 2: $2, 4, 6, 8, \ldots, 100$. In the second pass Josh marks out $2, 6, 10, \ldots, 98$, leaving the multiples of 4: $4, 8, 12, \ldots, 100$. Similarly, in the $n^{\text{th}}$ pass Josh marks out the numbers that are not multiples of $2^n$, leaving the numbers that are multiples of $2^n$. It follows that in the $6^{\text{th}}$ pass Josh marks out the numbers that are multiples of $2^5$ but not multiples of $2^6$, namely 32 and 92. This leaves 64, the only number in his original list that is a multiple of $2^6$. Thus the last number remaining is 64.

8. **Answer (D):** The weight of an object of uniform density is proportional to its volume. The volume of the triangular piece of wood of uniform thickness is proportional to the area of the triangle. The side length of the second piece is $\frac{5}{3}$ times the side length of the first piece, so the area of the second piece is $\left(\frac{5}{3}\right)^2$ times the area of the first piece. Therefore the weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

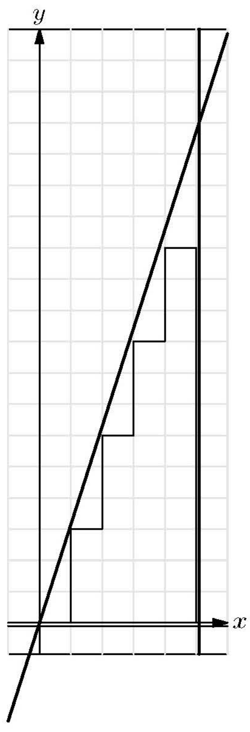
9. **Answer (B):** Let $x$ be the number of posts along the shorter side; then there are $2x$ posts along the longer side. When counting the number of posts on all the sides of the garden, each corner post is counted twice, so $2x + 2(2x) = 20 + 4$. Solving this equation gives $x = 4$. Thus the dimensions of the rectangle are $(4-1)\cdot 4 = 12$ yards by $(8-1)\cdot 4 = 28$ yards. The requested area is given by the product of these dimensions, $12 \cdot 28 = 336$ square yards.

10. **Answer (A):** The slopes of $\overline{PQ}$ and $\overline{RS}$ are $-1$, and the slopes of $\overline{QR}$ and $\overline{PS}$ are 1, so the figure is a rectangle. The side lengths are $PQ = (a-b)\sqrt{2}$ and $PS = (a+b)\sqrt{2}$, so the area is $2(a-b)(a+b) = 2(a^2-b^2) = 16$. Therefore $a^2 - b^2 = 8$. The only perfect squares whose difference is 8 are 9 and 1, so $a = 3$, $b = 1$, and $a + b = 4$.

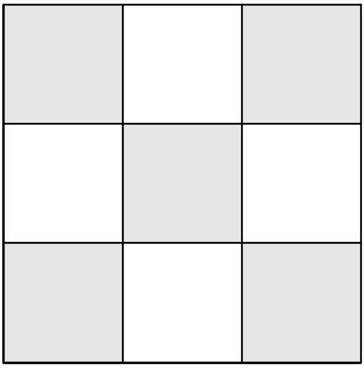



11. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in

the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3 + 6 + 9 + 12 + 2 + 5 + 8 + 1 + 4 = 50$ squares of the desired type within the given region.




12. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1 + 3 + 5 + 7 + 9 = 25$. Because all but the center add up to $18 = 25 - 7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.



| | | |
|---|---|---|
| 3 | 4 | 5 |
| 2 | 7 | 6 |
| 1 | 8 | 9 |

13. **Answer (E):** Let Alice, Bob, and the airplane be located at points $A$, $B$, and $C$, respectively. Let $D$ be the point on the ground directly beneath the airplane, and let $h$ be the airplane's altitude, in miles. Then $\triangle ACD$ and $\triangle BCD$ are $30-60-90^\circ$ right triangles with right angles at $D$, so $AD = \sqrt{3}h$ and $BD = \frac{h}{\sqrt{3}}$. Then by the Pythagorean Theorem applied to the right triangle on the ground,

$$100 = AB^2 = AD^2 + BD^2 = \left(\sqrt{3}h\right)^2 + \left(\frac{h}{\sqrt{3}}\right)^2 = \frac{10h^2}{3}.$$

Thus $h = \sqrt{30}$, and the closest of the given choices is 5.5.

14. **Answer (E):** Let $r$ be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

Because $S > 0$, the smallest value of $S$ occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of $S$ is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

15. **Answer (D):** Suppose that one pair of opposite faces of the cube are assigned the numbers $a$ and $b$, a second pair of opposite faces are assigned the numbers $c$ and $d$, and the remaining pair of opposite faces are assigned the numbers $e$ and $f$. Then the needed sum of products is $ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f)$. The sum of these three factors is $2+3+4+5+6+7 = 27$. A product of positive numbers whose sum is fixed is maximized when the factors are all equal. Thus the greatest possible value occurs when $a+b = c+d = e+f = 9$, as in $(a,b,c,d,e,f) = (2,7,3,6,4,5)$. This results in the value $9^3 = 729$.

16. **Answer (E):** A sum of consecutive integers is equal to the number of integers in the sum multiplied by their median. Note that $345 = 3 \cdot 5 \cdot 23$. If there are an odd number of integers in the sum, then the median and the number of integers must be complementary factors of 345. The only possibilities are 3 integers with median $5 \cdot 23 = 115$, 5 integers with median $3 \cdot 23 = 69$, $3 \cdot 5 = 15$ integers with median 23, and 23 integers with median $3 \cdot 5 = 15$. Having more integers in the sum would force some of the integers to be negative. If there are an even number of integers in the sum, say $2k$, then the median will be $\frac{j}{2}$, where $k$ and $j$ are complementary factors of 345. The possibilities are 2 integers with median $\frac{345}{2}$, 6 integers with median $\frac{115}{2}$, and 10 integers with median $\frac{69}{2}$. Again, having more integers in the sum would force some of the integers to be negative. This gives a total of 7 solutions.

17. **Answer (D):** Let $x = BH$. Then $CH = 8 - x$ and $AH^2 = 7^2 - x^2 = 9^2 - (8-x)^2$, so $x = 2$ and $AH = \sqrt{45}$. By the Angle Bisector Theorem in $\triangle ACH$, $\frac{AP}{PH} = \frac{CA}{CH} = \frac{9}{6}$, so $AP = \frac{3}{5}AH$. Similarly, by the Angle Bisector Theorem in $\triangle ABH$, $\frac{AQ}{QH} = \frac{BA}{BH} = \frac{7}{2}$, so $AQ = \frac{7}{9}AH$. Then $PQ = AQ - AP = (\frac{7}{9} - \frac{3}{5})AH = \frac{8}{45}\sqrt{45} = \frac{8}{15}\sqrt{5}$.

18. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2 + y^2 - x - y = 0$. Completing the square gives $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2}, \frac{1}{2})$ and contains the points $A(1,0)$ and $B(0,1)$. Because $C$ is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle $AOB$, where $O(0,0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2} \cdot \pi\left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2} + \frac{\pi}{4}) = \pi + 2$.

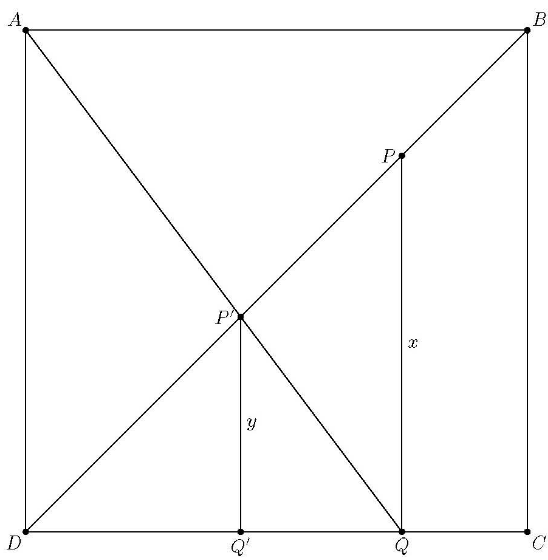
19. **Answer (B):** The probability that a flipper obtains his first head on the $n^{\text{th}}$ flip is $(\frac{1}{2})^n$, because the sequence of outcomes must be exactly TT ... TH, with $n-1$ Ts. Therefore the probability that all of them obtain their first heads on the $n^{\text{th}}$ flip is $\left((\frac{1}{2})^n\right)^3 = (\frac{1}{8})^n$. The probability that all three flip their coins the same number of times is computed by summing an infinite geometric series:

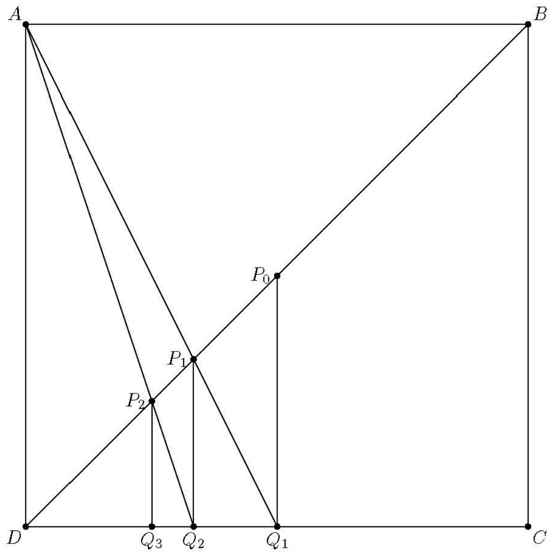
$$\left(\frac{1}{8}\right)^1 + \left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^3 + \cdots = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

20. **Answer (A):** There must have been $10 + 10 + 1 = 21$ teams, and therefore there were $\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$ subsets $\{A, B, C\}$ of three teams. If such a

subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2} = 45$ choices for the other two teams in the subset. This gives $21 \cdot 45 = 945$ such subsets. The required answer is $1330 - 945 = 385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.

21. **Answer (B):** For any point $P$ between $B$ and $D$, let $Q$ be the foot of the perpendicular from $P$ to $\overline{CD}$, let $P'$ be the intersection of $\overline{AQ}$ and $\overline{BD}$, and let $Q'$ be the foot of the perpendicular from $P'$ to $\overline{CD}$. Let $x = PQ$ and $y = P'Q'$. Because $\triangle PQD$ and $\triangle P'Q'D$ are isosceles right triangles, $DQ = x$ and $DQ' = y$. Because $\triangle ADQ$ is similar to $\triangle P'Q'Q$, $\frac{1}{x} = \frac{y}{x-y}$. Solving for $y$ gives $y = \frac{x}{1+x}$.






Now let $P_0$ be the midpoint of $\overline{BD}$. Then $P_0Q_1 = DQ_1 = \frac{1}{2}$. It follows from the analysis above that $P_1Q_2 = DQ_2 = \frac{1}{3}$, $P_2Q_3 = DQ_3 = \frac{1}{4}$, and in general $P_iQ_{i+1} = DQ_{i+1} = \frac{1}{i+2}$. The area of $\triangle DQ_iP_i$ is

$$\frac{1}{2} \cdot DQ_i \cdot P_iQ_{i+1} = \frac{1}{2} \cdot \frac{1}{i+1} \cdot \frac{1}{i+2} = \frac{1}{2}\left(\frac{1}{i+1} - \frac{1}{i+2}\right).$$

The requested infinite sum telescopes:

$$\sum_{i=1}^{\infty} \text{Area of } \triangle DQ_iP_i = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots\right).$$

Its value is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

22. **Answer (B):** Because $\frac{1}{n} = \frac{abcdef}{999999}$, it follows that $n$ is a divisor of $10^6 - 1 = (10^3-1)(10^3+1) = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$. Because $\frac{1}{n+6} = \frac{wxyz}{9999}$, it follows that $n+6$ divides $10^4 - 1 = 3^2 \cdot 11 \cdot 101$. However, $n+6$ does not divide $10^2 - 1 = 3^2 \cdot 11$, because otherwise the decimal representation of $\frac{1}{n+6}$ would have period 1 or 2. Thus $n = 101k - 6$, where $k = 1, 3, 9, 11, 33$, or $99$. Because $n < 1000$, the only possible values of $k$ are 1, 3, and 9, and the corresponding values of $n$ are 95, 297, and 903. Of these, only $297 = 3^3 \cdot 11$ divides $10^6 - 1$. Thus $n \in [201, 400]$. It may be checked that $\frac{1}{297} = 0.\overline{003367}$ and $\frac{1}{303} = 0.\overline{0033}$.

23. **Answer (A):** In the first octant, the first inequality reduces to $x + y + z \le 1$, and the inequality defines the region under a plane that intersects the coordinate

axes at $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. By symmetry, the first inequality defines the region inside a regular octahedron centered at the origin and having internal diagonals of length 2. The upper half of this octahedron is a pyramid with altitude 1 and a square base of side length $\sqrt{2}$, so the volume of the octahedron is $2 \cdot \frac{1}{3} \cdot \left(\sqrt{2}\right)^2 \cdot 1 = \frac{4}{3}$. The second inequality defines the region obtained by translating the first region up 1 unit. The intersection of the two regions is bounded by another regular octahedron with internal diagonals of length 1. Because the linear dimensions of the third octahedron are half those of the first, its volume is $\frac{1}{8}$ that of the first, or $\frac{1}{6}$.

24. **Answer (D):** Note that $\gcd(a,b,c,d) = 77$ and $\operatorname{lcm}(a,b,c,d) = n$ if and only if $\gcd(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = 1$ and $\operatorname{lcm}(\frac{a}{77}, \frac{b}{77}, \frac{c}{77}, \frac{d}{77}) = \frac{n}{77}$. Thus there are 77,000 ordered quadruples $(a,b,c,d)$ such that $\gcd(a,b,c,d) = 1$ and $\operatorname{lcm}(a,b,c,d) = \frac{n}{77}$. Let $m = \frac{n}{77}$ and suppose that $p$ is a prime that divides $m$. Let $A = A(p)$, $B = B(p)$, $C = C(p)$, $D = D(p)$, and $M = M(p) \geq 1$ be the exponents of $p$ such that $p^A$, $p^B$, $p^C$, $p^D$, and $p^M$ are the largest powers of $p$ that divide $a$, $b$, $c$, $d$, and $m$, respectively. The gcd and lcm requirements are equivalent to $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$. For a fixed value of $M$, there are $(M+1)^4$ quadruples $(A,B,C,D)$ with each entry in $\{0,1,\ldots,M\}$. There are $M^4$ of them for which $\min(A,B,C,D) \geq 1$, and also $M^4$ of them such that $\max(A,B,C,D) \leq M-1$. Finally, there are $(M-1)^4$ quadruples $(A,B,C,D)$ such that $\min(A,B,C,D) \geq 1$ and $\max(A,B,C,D) \leq M-1$. Thus the number of quadruples such that $\min(A,B,C,D) = 0$ and $\max(A,B,C,D) = M$ is equal to $(M+1)^4 - 2M^4 + (M-1)^4 = 12M^2 + 2 = 2(6M^2+1)$. Multiplying these quantities over all primes that divide $m$ yields the total number of quadruples $(a,b,c,d)$ with the required properties. Thus

$$77{,}000 = 2^3 \cdot 5^3 \cdot 7 \cdot 11 = \prod_{p|m} 2(6(M(p))^2 + 1).$$

Note that $6(M(p))^2 + 1$ is odd and this product must contain three factors of 2, so there must be exactly three primes that divide $m$. Let $p_1$, $p_2$, and $p_3$ be these primes. Note that $6 \cdot 1^2 + 1 = 7$, $6 \cdot 2^2 + 1 = 5^2$, and $6 \cdot 3^2 + 1 = 5 \cdot 11$. None of these could appear as a factor more than once because 77,000 is not divisible by $7^2$, $5^4$, or $11^2$. Moreover, the product of these three is equal to $5^3 \cdot 7 \cdot 11$. All other factors of the form $6M^2 + 1$ are greater than these three, so without loss of generality the only solution is $M(p_1) = 1$, $M(p_2) = 2$, and $M(p_3) = 3$. It follows that $m = p_1^1 p_2^2 p_3^3$, and the smallest value of $m$ occurs when $p_1 = 5$, $p_2 = 3$, and $p_3 = 2$. Therefore the smallest possible values of $m$ and $n$ are $5 \cdot 3^2 \cdot 2^3 = 360$ and $77(5 \cdot 3^2 \cdot 2^3) = 27{,}720$, respectively.

25. **Answer (A):** Express each term of the sequence $(a_n)$ as $2^{\frac{b_n}{19}}$. (Equivalently, let $b_n$ be the logarithm of $a_n$ to the base $\sqrt[19]{2}$.) The recursive definition of the

sequence $(a_n)$ translates into $b_0 = 0$, $b_1 = 1$, and $b_n = b_{n-1} + 2b_{n-2}$ for $n \geq 2$. Then the product $a_1 a_2 \cdots a_k$ is an integer if and only if $\sum_{i=1}^{k} b_i$ is divisible by 19. Let $c_n = b_n \bmod 19$. It follows that $a_1 a_2 \cdots a_k$ is an integer if and only if $p_k = \sum_{i=1}^{k} c_i$ is divisible by 19. Let $q_k = p_k \bmod 19$. Because the largest answer choice is 21, it suffices to compute $c_k$ and $q_k$ successively for $k$ from 1 up to at most 21, until $q_k$ first equals 0. The modular computations are straightforward from the definitions.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $c_k$ | 1 | 1 | 3 | 5 | 11 | 2 | 5 | 9 | 0 | 18 | 18 | 16 | 14 | 8 | 17 | 14 | 10 |
| $q_k$ | 1 | 2 | 5 | 10 | 2 | 4 | 9 | 18 | 18 | 17 | 16 | 13 | 8 | 16 | 14 | 9 | 0 |

Thus the requested answer is 17.

**OR**

Using standard techniques, the recurrence relation for $b_n$ can be solved to get $b_n = \frac{1}{3}(2^n - (-1)^n)$. Let $S_k = b_1 + b_2 + \cdots + b_k$. Then it is straightforward to show that $S_k = \frac{1}{3}(2^{k+1} - 1)$ for $k$ odd, and $S_k = \frac{2}{3}(2^k - 1)$ for $k$ even. Let $P_k = a_1 a_2 \cdots a_k$. It follows that, for $k$ odd, $P_k$ is an integer if and only if 19 divides $2^{k+1} - 1$; and, for $k$ even, $P_k$ is an integer if and only if 19 divides $2^k - 1$. A little computation shows that this first occurs at $k = 17$, when $2^{18} - 1 = 2^{18} - 1 = (2^9 - 1)(2^9 + 1) = 511 \cdot 513 = 511 \cdot 19 \cdot 27$. (In fact, one can show that $P_k$ is an integer if and only if $k$ is congruent to 0 or $-1$ mod 18.)

Problems and solutions were contributed by Bernardo Abrego, Sam Baethge, Barb Currier, Marta Eso, Chuck Garner, Jerry Grossman, Joe Kennedy, Michael Khoury, Matthew McMullen, Harold Reiter, and David Wells.

*The*

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MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

MAA American Mathematics Competitions

68th Annual

# AMC 12 A

**American Mathematics Competition 12A**

**Tuesday, February 7, 2017**

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

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The problems and solutions for this AMC 12 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (D):** The cheapest popsicles cost $\$3.00 \div 5 = \$0.60$ each. Because $14 \cdot \$0.60 = \$8.40$ and Pablo has just \$8, he could not pay for 14 popsicles even if he were allowed to buy partial boxes. The best he can hope for is 13 popsicles, and he can achieve that by buying two 5-popsicle boxes (for \$6) and one 3-popsicle box (for \$2).

**OR**

If Pablo buys two single popsicles for \$1 each, he could have bought a 3-popsicle box for the same amount of money. Similarly, if Pablo buys three single popsicles or both one 3-popsicle box and one single popsicle, he could have bought a 5-popsicle box for the same amount of money. If Pablo buys two 3-popsicle boxes, he could have bought a 5-popsicle box and a single popsicle for the same amount of money. The previous statements imply that a maximum number of popsicles for a given amount of money can be obtained by buying either at most one single popsicle and the rest 5-popsicle boxes, or a single 3-popsicle box and the rest 5-popsicle boxes. When Pablo has \$8, he can obtain the maximum number of popsicles by buying two 5-popsicle boxes and one 3-popsicle box. This gives a total of $2 \cdot 5 + 1 \cdot 3 = 13$ popsicles.

2. **Answer (C):** Let the two numbers be $x$ and $y$. Then $x + y = 4xy$. Dividing this equation by $xy$ gives $\frac{1}{y} + \frac{1}{x} = 4$. One such pair of numbers is $x = \frac{1}{3}$, $y = 1$.

3. **Answer (B):** The given statement is logically equivalent to its contrapositive: If a student did not receive an A on the exam, then the student did not get all the multiple choice questions right, which means that he got at least one of them wrong. None of the other statements follows logically from the given implication; the teacher made no promises concerning students who did not get all the multiple choice questions right. In particular, a statement does not imply its inverse or its converse; and the negation of the statement that Lewis got all the questions right is not the statement that he got all the questions wrong.

4. **Answer (A):** If the square had side length $x$, then Jerry's path had length $2x$, and Silvia's path along the diagonal, by the Pythagorean Theorem, had length $\sqrt{2}x$. Therefore Silvia's trip was shorter by

$2x - \sqrt{2}x$, and the required percentage is

$$\frac{2x - \sqrt{2}x}{2x} = 1 - \frac{\sqrt{2}}{2} \approx 1 - 0.707 = 0.293 = 29.3\%.$$

The closest of the answer choices is 30%.

5. **Answer (B):** Each of the 20 people who know each other shakes hands with 10 people. Each of the 10 people who know no one shakes hands with 29 people. Because each handshake involves two people, the number of handshakes is $\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) = 245$.

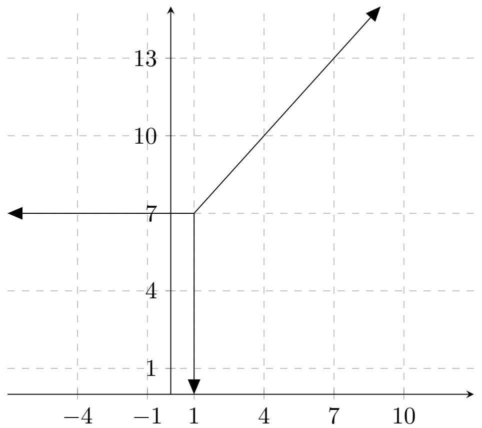
6. **Answer (B):** Four rods can form a quadrilateral with positive area if and only if the length of the longest rod is less than the sum of the lengths of the other three. Therefore if the fourth rod has length $n$ cm, then $n$ must satisfy the inequalities $15 < 3+7+n$ and $n < 3+7+15$, that is, $5 < n < 25$. Because $n$ is an integer, it must be one of the 19 integers from 6 to 24, inclusive. However, the rods of lengths 7 cm and 15 cm have already been chosen, so the number of rods that Joy can choose is $19 - 2 = 17$.

7. **Answer (B):** It is clear after listing the first few values, $f(1) = 2$, $f(2) = f(1) + 1 = 3$, $f(3) = f(1) + 2 = 4$, $f(4) = f(3) + 1 = 5$, and so on, that $f(n) = n + 1$ for all positive integers $n$. Indeed, the function is uniquely determined by the recursive description, and the function defined by $f(n) = n + 1$ fits the description. Therefore $f(2017) = 2018$.

8. **Answer (D):** Let $h = AB$. The region consists of a solid circular cylinder of radius 3 and height $h$, together with two solid hemispheres of radius 3 centered at $A$ and $B$. The volume of the cylinder is $\pi \cdot 3^2 \cdot h = 9\pi h$, and the two hemispheres have a combined volume of $\frac{4}{3}\pi \cdot 3^3 = 36\pi$. Therefore $9\pi h + 36\pi = 216\pi$, and $h = 20$.

9. **Answer (E):** Suppose that the two larger quantities are the first and the second. Then $3 = x + 2 \geq y - 4$. This is equivalent to $x = 1$ and $y \leq 7$, and its graph is the downward-pointing ray with endpoint $(1, 7)$. Similarly, if the two larger quantities are the first and third, then $3 = y - 4 \geq x + 2$. This is equivalent to $y = 7$ and $x \leq 1$, and its graph is the leftward-pointing ray with endpoint $(1, 7)$. Finally, if the

two larger quantities are the second and third, then $x+2 = y-4 \geq 3$. This is equivalent to $y = x+6$ and $x \geq 1$, and its graph is the ray with endpoint $(1,7)$ that points upward and to the right. Thus the graph consists of three rays with common endpoint $(1,7)$.




**Note:** This problem is related to a relatively new area of mathematics called tropical geometry.

10. **Answer (C):** Half of the time Laurent will pick a number between 2017 and 4034, in which case the probability that his number will be greater than Chloé's number is 1. The other half of the time, he will pick a number between 0 and 2017, and by symmetry his number will be the larger one in half of those cases. Therefore the requested probability is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

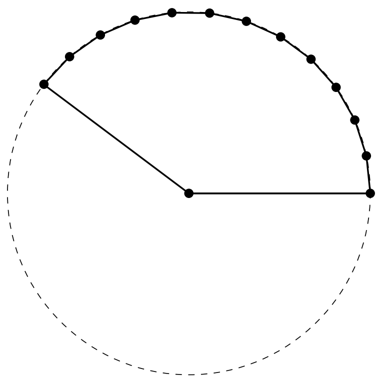
**OR**

The choices of numbers can be represented in the coordinate plane by points in the rectangle with vertices at $(0,0)$, $(2017,0)$, $(2017,4034)$, and $(0,4034)$. The portion of the rectangle representing the event that Laurent's number is greater than Chloé's number is the portion above the line segment with endpoints $(0,0)$ and $(2017,2017)$. This area is $\frac{3}{4}$ of the area of the entire rectangle, so the requested probability is $\frac{3}{4}$.

11. **Answer (D):** If the polygon has $n$ sides and the degree measure of the forgotten angle is $\alpha$, then $(n-2)180 = 2017 + \alpha$. Because $0 < \alpha < 180$,

$$2017 < (n-2)180 < 2197,$$

which implies that $n = 14$, the angle sum is 2160, and $\alpha = 143$. To see that such a polygon exists, draw a circle and a central angle of measure $143°$, and divide the minor arc spanned by the angle into 12 small arcs. The polygon is then formed by the two radii and 12 small chords, as illustrated.



12. **Answer (B):** Horse $k$ will again be at the starting point after $t$ minutes if and only if $k$ is a divisor of $t$. Let $I(t)$ be the number of integers $k$ with $1 \le k \le 10$ that divide $t$. Then $I(1) = 1$, $I(2) = 2$, $I(3) = 2$, $I(4) = 3$, $I(5) = 2$, $I(6) = 4$, $I(7) = 2$, $I(8) = 4$, $I(9) = 3$, $I(10) = 4$, $I(11) = 1$, and $I(12) = 5$. Thus $T = 12$ and the requested sum of digits is $1 + 2 = 3$.

13. **Answer (B):** Let $d$ be the requested distance in miles, and suppose that Sharon usually drives at speed $r$ in miles per hour. Then $\frac{d}{r} = 3$. The total time in hours for Sharon's trip with the snowstorm is then $\frac{\frac{1}{3}d}{r} + \frac{\frac{2}{3}d}{r-20} = \frac{23}{5}$. Because $\frac{d}{r} = 3$, this reduces to

$$1 + \frac{\frac{2}{3}}{\frac{r}{d} - \frac{20}{d}} = 1 + \frac{\frac{2}{3}}{\frac{1}{3} - \frac{20}{d}} = \frac{23}{5}.$$

Solving for $d$ gives $d = 135$.

**OR**

The last $\frac{2}{3}$ of the drive takes $276 - \frac{1}{3} \cdot 180 = 216$ minutes, which is $\frac{216}{60}$ hours. If $r$ is the original speed in miles per hour, then $\frac{2}{3}$ of the distance is both $2r$ and $\frac{216}{60} \cdot (r - 20)$. Setting these expressions equal and solving yields $r = 45$. Therefore the original speed is 45 miles per hour, and the requested distance is $3 \cdot 45 = 135$ miles.

14. **Answer (C):** Let $X$ be the set of ways to seat the five people in which Alice sits next to Bob. Let $Y$ be the set of ways to seat the five people in which Alice sits next to Carla. Let $Z$ be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is $5! - |X \cup Y \cup Z|$. The Inclusion–Exclusion Principle gives

$$|X \cup Y \cup Z| = \big(|X|+|Y|+|Z|\big) - \big(|X \cap Y|+|X \cap Z|+|Y \cap Z|\big) + |X \cap Y \cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are $2 \cdot 4! = 48$ elements of $X$. Similarly there are 48 elements of $Y$ and 48 elements of $Z$. Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that $|X \cap Y| = 2 \cdot 3! = 12$. Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$. Similarly $|Y \cap Z| = 24$. Finally, there are $2 \cdot 2 \cdot 2! = 8$ elements of $X \cap Y \cap Z$. Therefore $|X \cup Y \cup Z| = (48+48+48)-(12+24+24)+8 = 92$, and the answer is $120 - 92 = 28$.

**OR**

There are three cases based on where Alice is seated.

• If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of $2^4 = 16$ arrangements.

• If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of $2^3 = 8$ arrangements.

• If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in $2^2 = 4$ arrangements.

Thus there are $16 + 8 + 4 = 28$ possible arrangements in total.

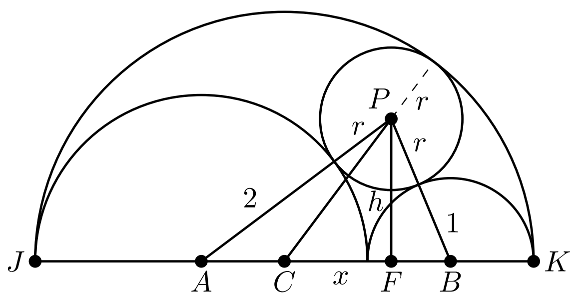
15. **Answer (D):** For $0 < x < \frac{\pi}{2}$ all three terms are positive, and $f(x)$ is undefined when $x = \frac{\pi}{2}$. For $\frac{\pi}{2} < x < \frac{3\pi}{4}$, the term $3\tan x$ is less than $-3$ and dominates the other two terms, so $f(x) < 0$ there. For $\frac{3\pi}{4} \le x < \pi$, $|\cos(x)| \ge |\sin(x)|$ and $\cos x$ and $\tan x$ are negative, so $\sin x + 2\cos x + 3\tan x < 0$. Therefore there is no positive solution of $f(x) = 0$ for $x < \pi$. Because the range of $f$ includes all values between $f(\pi) = -2 < 0$ and $f(\frac{5\pi}{4}) = -\frac{3}{2}\sqrt{2} + 3 > -1.5 \cdot 1.5 + 3 > 0$ on the interval $[\pi, \frac{5\pi}{4}]$, the smallest positive solution of $f(x) = 0$ lies

between $\pi$ and $\frac{5\pi}{4}$. Because $\pi > 3$ and $\frac{5\pi}{4} < 4$, the requested interval is $(3, 4)$.

16. **Answer (B):** Let $C$ be the center of the largest semicircle, and let $r$ denote the radius of the circle centered at $P$. Note that $PA = 2+r$, $PC = 3 - r$, $PB = 1 + r$, $AC = 1$, $BC = 2$, and $AB = 3$. Let $F$ be the foot of the perpendicular from $P$ to $\overline{AB}$, let $h = PF$, and let $x = CF$. The Pythagorean Theorem in $\triangle PAF$, $\triangle PCF$, and $\triangle PBF$ gives

$$h^2 = (2 + r)^2 - (1 + x)^2 = (3 - r)^2 - x^2 = (1 + r)^2 - (2 - x)^2.$$

This reduces to two linear equations in $r$ and $x$, whose solution is $r = \frac{6}{7}$, $x = \frac{9}{7}$.




**OR**

With the notation and observations above, apply Heron's Formula to $\triangle PCB$ and $\triangle PAC$, noting that the former has twice the area of the latter and each has semiperimeter 3. Thus

$$\sqrt{3 \cdot 1 \cdot r \cdot (r - 2)} = 2\sqrt{3 \cdot 2 \cdot r \cdot (r - 1)},$$

from which it follows that $r = \frac{6}{7}$.

**OR**

With the notation and observations above, let $\theta = \angle PAB$. The Law of Cosines applied to $\triangle PAC$ gives

$$(3 - r)^2 = (2 + r)^2 + 1^2 - 2 \cdot (2 + r) \cdot 1 \cdot \cos\theta,$$

and simplifying yields $(2+r)\cos\theta = 5r - 2$. Applying the Law of Cosines to $\triangle PAB$ gives

$$(1+r)^2 = (2+r)^2 + 3^2 - 2\cdot(2+r)\cdot 3\cdot\cos\theta,$$

and simplifying yields $3(2+r)\cos\theta = r+6$. Hence $r+6 = 3(5r-2)$, so $r = \frac{6}{7}$.

17. **Answer (D):** The complex numbers $z$ such that $z^{24} = 1$ are the roots of $z^{24} - 1 = (z^6-1)(z^6+1)((z^6)^2+1)$. The factors can have at most 6, 6, and 12, roots, respectively. Because $z^{24} - 1$ has 24 distinct roots, the factors do actually have 6, 6, and 12 distinct roots, respectively. The six roots of the first factor satisfy $z^6 = 1$, and the six roots of the second factor satisfy $z^6 = -1$. The twelve roots of the third factor satisfy $(z^6)^2 = -1$, so $z^6$ is never real in this case. There are $6+6 = 12$ roots such that $z^6$ is real.

**OR**

The complex values of $z$ such that $z^{24} = 1$ are the 24th roots of unity. These values can be written in the form $e^{\frac{1}{12}k\pi i}$, where $k$ is an integer between 0 and 23, inclusive. By Euler's Theorem,

$$z^6 = e^{\frac{1}{2}k\pi i} = \cos\left(\frac{1}{2}k\pi\right) + i\sin\left(\frac{1}{2}k\pi\right).$$

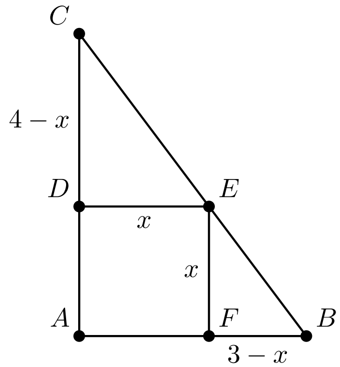
This quantity is a real number if and only if $\sin\left(\frac{1}{2}k\pi\right) = 0$, which occurs if and only if $k$ is even. There are therefore 12 complex values of $z$ such that $z^6$ is real.

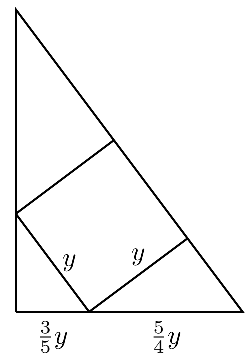
18. **Answer (D):** Note that $S(n+1) = S(n) + 1$ unless the numeral for $n$ ends with a 9. Moreover, if the numeral for $n$ ends with exactly $k$ 9s, then $S(n+1) = S(n) + 1 - 9k$. Thus the possible values of $S(n+1)$ when $S(n) = 1274$ are all of the form $1275 - 9k$, where $k \in \{0, 1, 2, 3, \ldots, 141\}$. Of the choices, only 1239 can be formed in this manner, and $S(n+1)$ will equal 1239 if, for example, $n$ consists of 4 consecutive 9s preceded by 1238 1s.

**OR**

The value of a positive integer is congruent to the sum of its digits modulo 9. Therefore $n \equiv S(n) = 1274 \equiv 5 \pmod 9$, so $S(n+1) \equiv n+1 \equiv 6 \pmod 9$. Of the given choices, only 1239 meets this requirement.

19. **Answer (D):** In the first figure $\triangle FEB \sim \triangle DCE$, so $\frac{x}{3-x} = \frac{4-x}{x}$ and $x = \frac{12}{7}$. In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving $\frac{3}{5}y + \frac{5}{4}y = 3$ yields $y = \frac{60}{37}$. Thus $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.




20. **Answer (E):** Let $u = \log_b a$. Because $u^{2017} = 2017u$, either $u = 0$ or $u = \pm\sqrt[2016]{2017}$. If $u = 0$, then $a = 1$ and $b$ can be any integer from 2 to 200. If $u = \pm\sqrt[2016]{2017}$, then $a = b^{\pm\sqrt[2016]{2017}}$, where again $b$ can be any integer from 2 to 200. Therefore there are $3 \cdot 199 = 597$ such ordered pairs.

21. **Answer (D):** Because $-1$ is a root of $10x + 10$, $-1$ is added to $S$. Then 1 is also added to $S$, because it is a root of $(-1)x^{10} + (-1)x^9 + \cdots + (-1)x + 10$. At this point $-10$, a root of $1 \cdot x + 10$, can be added to $S$. Because 2 is a root of $1 \cdot x^3 + 0 \cdot x^2 + 1 \cdot x + (-10)$, and $-2$ is a root of $1 \cdot x + 2$, both 2 and $-2$ can be added to $S$. The polynomials $2x + (-10)$ and $2x + 10$ allow 5 and $-5$ into $S$. At this point $S = \{0, \pm1, \pm2, \pm5, \pm10\}$. No more elements can be added to $S$, because by the Rational Root Theorem, any integer root of a polynomial with integer coefficients whose constant term is a factor of 10 must be a factor of 10. Therefore $S$ contains 9 elements.

   **Note:** It is not true that in general if $S$ starts with $\{0, c\}$ then all factors of $c$ can be added to $S$. For example, applying the procedure to $\{0, 35\}$ gives only $\{0, \pm1, \pm35\}$, although of course it takes some argument to rule out $\pm5$ and $\pm7$.

22. **Answer (E):** Let $A = \{(1,0), (0,1), (-1,0), (0,-1)\}$, let $C = \{(0,0)\}$, and let $I = \{(1,1), (-1,1), (-1,-1), (1,-1)\}$. A particle

in $A$ will move to $A$ with probability $\frac{2}{8}$, to $C$ with probability $\frac{1}{8}$, to $I$ with probability $\frac{2}{8}$, and to an interior point of a side of the square with probability $\frac{3}{8}$. Similarly, a particle in $C$ will move to $A$ with probability $\frac{4}{8}$ and to $I$ with probability $\frac{4}{8}$; and a particle in $I$ will move to $A$ with probability $\frac{2}{8}$, to $C$ with probability $\frac{1}{8}$, to a corner of the square with probability $\frac{1}{8}$, and to an interior point of a side of the square with probability $\frac{4}{8}$. Let $a$, $c$, and $i$ be the probabilities that the particle will first hit the square at a corner, given that it is currently in $A$, $C$, and $I$, respectively. The transition probabilities noted above lead to the following system of equations.

$$\begin{aligned} a &= \frac{2}{8}a + \frac{1}{8}c + \frac{2}{8}i \\ c &= \frac{4}{8}a + \frac{4}{8}i \\ i &= \frac{2}{8}a + \frac{1}{8}c + \frac{1}{8} \end{aligned}$$

This system can be solved by elimination to yield $a = \frac{1}{14}$, $c = \frac{4}{35}$, and $i = \frac{11}{70}$. The required fraction is $c$, whose numerator and denominator sum to 39.

23. **Answer (C):** Let $q$ be the additional root of $f(x)$. Then

$$\begin{aligned} f(x) &= (x-q)(x^3 + ax^2 + x + 10) \\ &= x^4 + (a-q)x^3 + (1-qa)x^2 + (10-q)x - 10q. \end{aligned}$$

Thus $100 = 10 - q$, so $q = -90$ and $c = -10q = 900$. Also $1 = a - q = a + 90$, so $a = -89$. It follows, using the factored form of $f$ shown above, that $f(1) = \big(1-(-90)\big)\cdot(1-89+1+10) = 91\cdot(-77) = -7007$.

24. **Answer (A):** Because $\overline{YE}$ and $\overline{EF}$ are parallel to $\overline{AD}$ and $\overline{AC}$, respectively, $\triangle XEY \sim \triangle XAD$ and $\triangle XEF \sim \triangle XAC$. Therefore

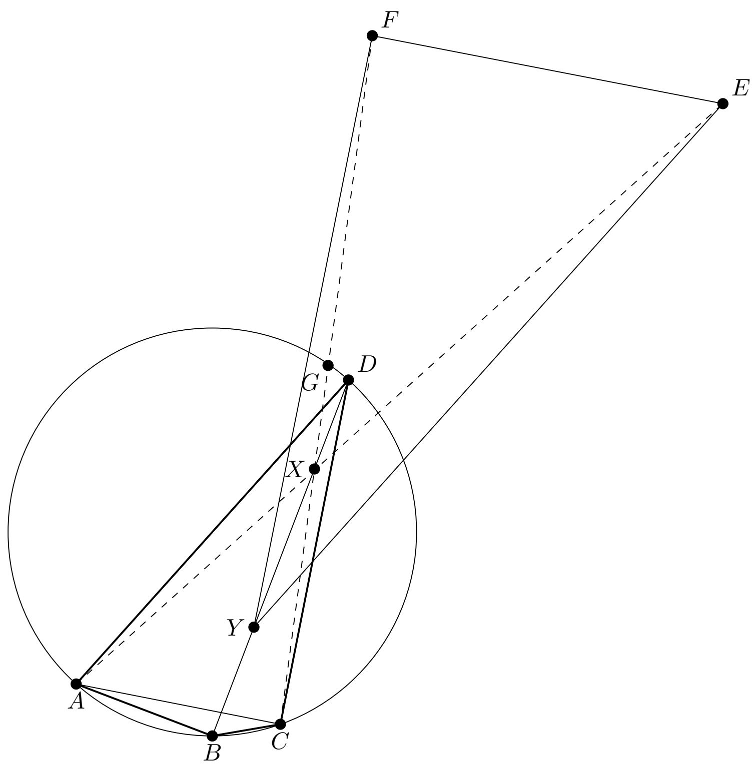
$$\frac{XY}{XE} = \frac{XD}{XA} \quad \text{and} \quad \frac{XF}{XE} = \frac{XC}{XA}.$$

It follows that

$$\frac{XC}{XD} = \frac{XF}{XY}.$$

The Power of a Point Theorem applied to circle $O$ and point $X$ implies that $XC \cdot XG = XD \cdot XB$. Together with the previous equation this implies that $XF \cdot XG = XB \cdot XY$. Let $d = BD$; then $DX = \frac{1}{4}d$ and $BY = \frac{11}{36}d$. It follows that

$$\begin{aligned} XF \cdot XG = XB \cdot XY &= (BD - DX) \cdot (BD - DX - BY) \\ &= \left(d - \frac{1}{4}d\right)\left(d - \frac{1}{4}d - \frac{11}{36}d\right) \\ &= \frac{3}{4} \cdot \frac{4}{9}d^2 = \frac{d^2}{3}. \end{aligned}$$




To determine $d$, note that because $ABCD$ is a cyclic quadrilateral it follows that $\alpha = \angle BAD = \pi - \angle DCB$. Applying the Law of Cosines to $\triangle ABD$ and $\triangle BCD$ yields

$$\cos\alpha = \frac{AB^2 + AD^2 - BD^2}{2 \cdot AB \cdot AD} = \frac{3^2 + 8^2 - d^2}{2 \cdot 3 \cdot 8} = \frac{73 - d^2}{48},$$

and

$$-\cos\alpha = \cos(\pi - \alpha) = \frac{CB^2 + CD^2 - BD^2}{2 \cdot CB \cdot CD} = \frac{2^2 + 6^2 - d^2}{2 \cdot 2 \cdot 6} = \frac{40 - d^2}{24}.$$

Therefore

$$\frac{73-d^2}{48}=\frac{d^2-40}{24},$$

and solving for $d^2$ gives $d^2=51$. Hence $XF\cdot XG=\frac{1}{3}d^2=17$.

25. **Answer (E):** If $z_j$ is an element of the set $A=\{\sqrt{2}i,-\sqrt{2}i\}$, then $|z_j|=\sqrt{2}$. Otherwise $z_j$ is an element of

$$B=V\setminus A=\left\{\frac{1}{\sqrt{8}}(1+i),\frac{1}{\sqrt{8}}(-1+i),\frac{1}{\sqrt{8}}(1-i),\frac{1}{\sqrt{8}}(-1-i)\right\}$$

and $|z_j|=\frac{1}{2}$. It follows that $|P|=\prod_{j=1}^{12}|z_j|=1$ exactly when 8 of the 12 factors $z_j$ are in $A$ and 4 of the factors are in $B$. The product of 8 complex numbers each of which is in $A$ is a real number, either 16 or $-16$. The product of 4 numbers each of which is in $B$ is one of $\frac{1}{16}$, $\frac{1}{16}i$, $-\frac{1}{16}$, or $-\frac{1}{16}i$. Thus a product $P=\prod_{j=1}^{12}z_j$ is $-1$ exactly when 8 of the $z_j$ are from $A$, 4 of the $z_j$ are from $B$, and the last of the 4 elements from $B$ is chosen so that the product is $-1$ rather than $i$, $-i$, or 1. Because the probability is $\frac{1}{3}$ that a particular factor $z_j$ is from $A$, the probability is $\frac{2}{3}$ that a particular factor $z_j$ is from $B$, and the probability is $\frac{1}{6}$ that a particular factor $z_j$ is a specific element of $V$, the probability that the product $P$ will be $-1$ is given by

$$\binom{12}{4}\left(\frac{1}{3}\right)^8\left(\frac{2}{3}\right)^3\left(\frac{1}{6}\right)=\frac{12\cdot 11\cdot 10\cdot 9}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{1}{3^8}\cdot\frac{2^3}{3^3}\cdot\frac{1}{6}=\frac{2^2\cdot 5\cdot 11}{3^{10}}.$$

Problems and solutions were contributed by Bernardo Abrego, Stephen Adams, Paul Baird-Smith, Thomas Butts, Barb Currier, Steven Davis, Marta Eso, Silvia Fernandez, Peter Gilchrist, Jerrold Grossman, Jonathan Kane, Joe Kennedy, Michael Khoury, Matthew McMullen, Steven Miller, Hugh Montgomery, Joachim Rebholz, Mark Saul, Roger Waggoner, Dave Wells, and Carl Yerger.

*The*

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MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

MAA American Mathematics Competitions

68th Annual

# AMC 12B

**American Mathematics Competition 12B**

**Wednesday, February 15, 2017**

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:


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The problems and solutions for this AMC 12 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (E):** After $m$ months, Kymbrea's collection will have $30+2m$ comic books and LaShawn's collection will have $10+6m$ comic books. Solving $10+6m = 2(30+2m)$ yields $m = 25$, so LaShawn's collection will have twice as many comic books as Kymbrea's after 25 months.

2. **Answer (E):** Adding the inequalities $y > -1$ and $z > 1$ yields $y+z > 0$. The other four choices give negative values if, for example, $x = \frac{1}{8}$, $y = -\frac{1}{4}$, and $z = \frac{3}{2}$.

3. **Answer (D):** The given equation implies that $3x+y = -2(x-3y)$, which is equivalent to $x = y$. Therefore
$$\frac{x+3y}{3x-y} = \frac{4y}{2y} = 2.$$

4. **Answer (C):** Let $2d$ be the distance in kilometers to the friend's house. Then Samia bicycled distance $d$ at rate 17 and walked distance $d$ at rate 5, for a total time of
$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}$$
hours. Solving this equation yields $d = \frac{17}{6} = 2.833\ldots$. Therefore Samia walked about 2.8 kilometers.

5. **Answer (B):** Because 1.5 times the interquartile range for this data set is $1.5\cdot(43-33) = 15$, outliers are data values less than $33-15 = 18$ or greater than $43+15 = 58$. Only the value 6 meets this condition, so there is 1 outlier.

6. **Answer (D):** The center of the circle is the midpoint of the diameter, which is $(4,3)$, and the radius is $\sqrt{4^2+3^2} = 5$. Therefore the equation of the circle is $(x-4)^2 + (y-3)^2 = 25$. If $y = 0$, then $(x-4)^2 = 16$, so $x = 0$ or $x = 8$. The circle intersects the $x$-axis at $(8,0)$.

**OR**

Any diameter of a circle is a line of symmetry. Because the line $x = 4$ goes through the center of the circle, $(4,3)$, it contains a diameter. The reflection of $(0,0)$ in this line is $(8,0)$. Alternatively, $(8,6)$ can be reflected in the line $y = 3$, resulting in the same point.

7. **Answer (B):** Because $\cos(\sin(x+\pi)) = \cos(-\sin(x)) = \cos(\sin(x))$, the function is periodic with period $\pi$. Furthermore, $\cos(\sin(x)) = 1$ if and only if $\sin(x) = 0$, which occurs if and only if $x$ is a multiple of $\pi$, so the period cannot be less than $\pi$. Therefore the function $\cos(\sin(x))$ has least period $\pi$.

8. **Answer (C):** Let $x$ be the length of the short side of the rectangle, and let $y$ be the length of the long side. Then the length of the diagonal is $\sqrt{x^2+y^2}$, and

$$\frac{x^2}{y^2} = \frac{y^2}{x^2+y^2}, \qquad \text{so} \qquad \frac{y^2}{x^2} = \frac{x^2+y^2}{y^2} = \frac{x^2}{y^2} + 1.$$

Let $r = \frac{x^2}{y^2}$ be the requested squared ratio. Then $\frac{1}{r} = r + 1$, so $r^2 + r - 1 = 0$. By the quadratic formula, the positive solution is $r = \frac{\sqrt{5}-1}{2}$.

9. **Answer (A):** The first circle has equation $(x+10)^2+(y+4)^2 = 169$, and the second circle has equation $(x-3)^2+(y-9)^2 = 65$. Expanding these two equations, subtracting, and simplifying yields $x + y = 3$. Because the points of intersection of the two circles must satisfy this new equation, it must be the required equation of the line through those points, so $c = 3$. In fact, the circles intersect at $(2,1)$ and $(-5,8)$.

10. **Answer (D):** The students who like dancing but say they dislike it constitute $60\% \cdot (100\% - 80\%) = 12\%$ of the students. Similarly, the students who dislike dancing and say they dislike it constitute $(100\% - 60\%) \cdot 90\% = 36\%$ of the students. Therefore the requested fraction is $\frac{12}{12+36} = \frac{1}{4} = 25\%$.

11. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1,2,3,4,5,6,7,8,9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence with the subsets of $\{0,1,2,3,4,5,6,7,8,9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10} - 2 = 1022$. The single-digit numbers are included in both sets, so there are $511 + 1022 - 9 = 1524$ monotonous positive integers.

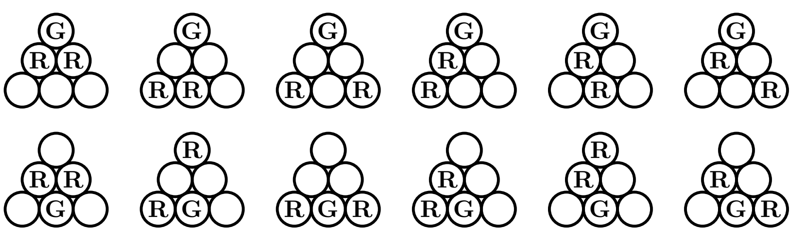
12. **Answer (D):** The principal root of the equation $z^{12} = 64$ is

$$z = 64^{\frac{1}{12}} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right).$$

The 12 roots lie in the complex plane on the circle of radius $\sqrt{2}$ centered at the origin. The roots with positive real part make angles of $0$, $\pm\frac{\pi}{6}$, and $\pm\frac{\pi}{3}$ with the positive real axis. When these five numbers are added, the imaginary parts cancel out and the sum is

$$\sqrt{2} + 2\sqrt{2} \cdot \cos\frac{\pi}{6} + 2\sqrt{2} \cdot \cos\frac{\pi}{3} = \sqrt{2} \cdot (1 + \sqrt{3} + 1) = 2\sqrt{2} + \sqrt{6}.$$

13. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1 + 2 + 3 + 1 + 2 + 3 = 12$.



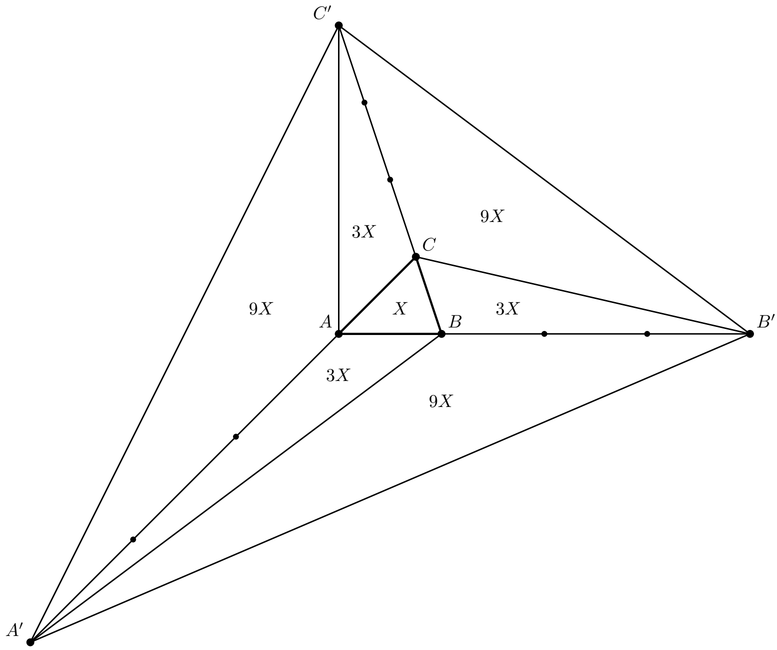
14. **Answer (E):** A frustum is constructed by removing a right circular cone from a larger right circular cone. The volume of the given frustum is the volume of a right circular cone with a 4-inch-diameter base and a height of 8 inches, minus the volume of a right circular cone with a 2-inch-diameter base and a height of 4 inches. (The stated heights come from considering similar right triangles.) Because the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, the volume of the frustum is

$$\frac{1}{3}\pi \cdot 2^2 \cdot 8 - \frac{1}{3}\pi \cdot 1^2 \cdot 4 = \frac{28}{3}\pi.$$

The volume of the top cone of the novelty is $\frac{1}{3}\pi \cdot 2^2 \cdot 4 = \frac{16}{3}\pi$. The requested volume of ice cream is the sum of the volume of each part of the novelty, namely $\frac{28}{3}\pi + \frac{16}{3}\pi = \frac{44}{3}\pi$.

**Note:** In general, the volume of a frustum with height $h$ and base radii $R$ and $r$ is $\frac{1}{3}\pi h(r^2 + rR + R^2)$.

15. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let $X$ be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X + 3(3X) + 3(9X) = 37X$, and the requested ratio is $37 : 1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.




**OR**

Let $s = AB$. Applying the Law of Cosines to $\triangle B'BC'$ gives

$$\begin{aligned}(B'C')^2 &= (3s)^2 + (4s)^2 - 2 \cdot 3s \cdot 4s \cdot \cos 120^\circ \\ &= s^2 \left(25 - 24\left(-\frac{1}{2}\right)\right) = 37s^2.\end{aligned}$$

By symmetry, $\triangle A'B'C'$ is also equilateral and therefore is similar to

$\triangle ABC$ with similarity ratio $\sqrt{37}$. Hence the ratio of their areas is $37:1$.

**OR**

Let $s = AB$. The areas of $\triangle B'BC'$, $\triangle C'CA'$, and $\triangle A'AB'$ are all

$$\frac{1}{2}(3s)(4s)\sin 120^\circ = 3\sqrt{3}s^2.$$

Therefore the requested ratio is

$$\frac{3\left(3\sqrt{3}s^2\right) + \frac{1}{4}\sqrt{3}s^2}{\frac{1}{4}\sqrt{3}s^2} = \frac{37}{1}.$$

16. **Answer (B):** There are $\left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where $k$ is odd. A divisor of 21! must be of the form $2^i b$ where $0 \le i \le 18$ and $b$ is a divisor of $k$. For each choice of $b$, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so it has $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60{,}800$ positive integer divisors, of which $10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3{,}200$ are odd.

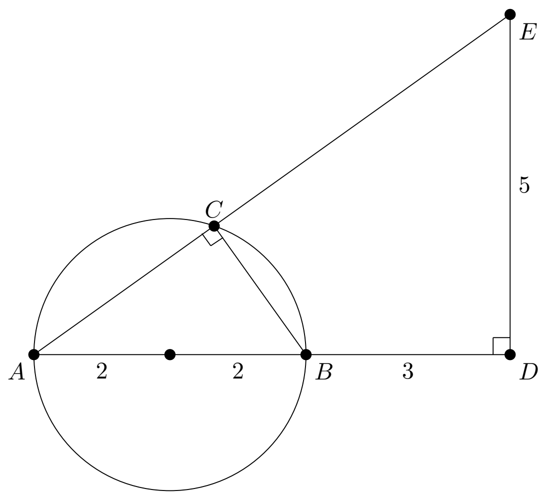
17. **Answer (D):** Let $p$ be the probability of heads. To win Game A requires that all three tosses be heads, which occurs with probability $p^3$, or all three tosses be tails, which occurs with probability $(1-p)^3$. To win Game B requires that the first two tosses be the same, the probability of which is $p^2+(1-p)^2$, and that the last two tosses be the same, which occurs with the same probability. Therefore the probability of winning Game A minus the probability of winning Game B is
$$\left(p^3 + (1-p)^3\right) - \left(p^2 + (1-p)^2\right)^2.$$
As $p = \frac{2}{3}$, this gives
$$\left(\left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3\right) - \left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right)^2 = \frac{1}{3} - \frac{25}{81} = \frac{2}{81}.$$
Thus the probability of winning Game A is $\frac{2}{81}$ greater than the probability of winning Game B.

**Note:** Expanding and then factoring the general expression above for the probability of winning Game A minus the probability of winning Game B yields $p(1-p)(2p-1)^2$. This value is always nonnegative, so the player should never choose Game B. It equals 0 if and only if $p = 0$, $\frac{1}{2}$, or 1. It is maximized when $p = \frac{2\pm\sqrt{2}}{4}$, which is about 85% or 15%, and in this case winning Game A is 6.25 percentage points more likely than winning Game B.

18. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as $AB^2$ is to $AE^2$. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.

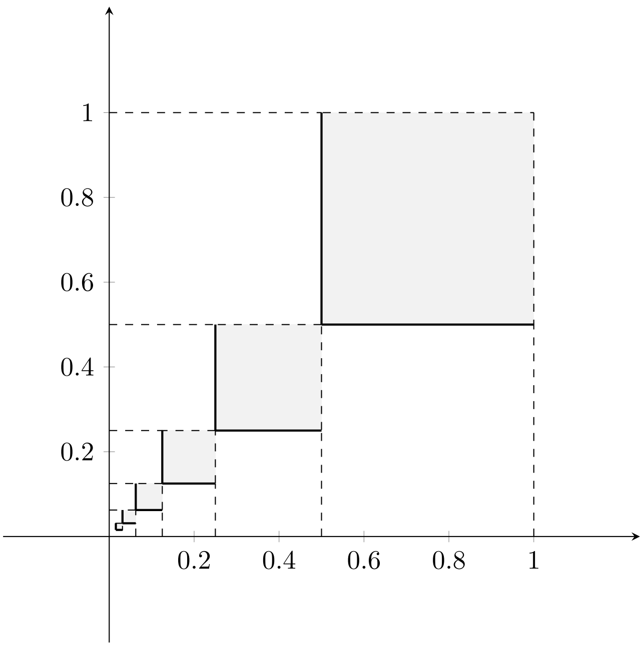



19. **Answer (C):** The remainder when $N$ is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of $N$ is $4(0+1+2+\cdots+9)+10\cdot 1+10\cdot 2+10\cdot 3+(4+0)+(4+1)+(4+2)+(4+3)+(4+4) = 270$, so $N$ must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and $N$ leaves a remainder of 9 when divided by 45.

20. **Answer (D):** The set of all possible ordered pairs $(x, y)$ is bounded by the unit square in the coordinate plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. For each positive integer $n$, $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ if and only if $\frac{1}{2^n} \le x < \frac{1}{2^{n-1}}$ and $\frac{1}{2^n} \le y < \frac{1}{2^{n-1}}$. Thus the set of ordered pairs $(x, y)$ such that $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor = -n$ is bounded by a square with side length $\frac{1}{2^n}$ and therefore area $\frac{1}{4^n}$. The union of these squares over all positive integers $n$ has area

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3},$$

and therefore the requested probability is $\frac{1}{3}$. (It is also clear from the diagram that one third of the square is shaded.)




**OR**

The problem can be modeled with Xerxes and Yolanda each repeatedly flipping a fair coin to determine the binary (base two "decimal") expansions of $x$ and $y$, respectively. If Xerxes flips a head, he writes down a 0 as the next binary digit; if he flips a tail, he writes down a 1. Yolanda does the same. Then $\lfloor \log_2 x \rfloor = \lfloor \log_2 y \rfloor$ if and only if the first time that either of them flips a tail, so does the other. There are three equally likely outcomes: tail-tail, tail-head, and head-tail. Therefore the requested probability is $\frac{1}{3}$.

21. **Answer (E):** Let $S$ be the sum of Isabella's 7 scores. Then $S$ is a multiple of 7, and

$$658 = 91 + 92 + 93 + \cdots + 97 \leq S \leq 94 + 95 + 96 + \cdots + 100 = 679,$$

so $S$ is one of 658, 665, 672, or 679. Because $S - 95$ is a multiple of 6, it follows that $S = 665$. Thus the sum of Isabella's first 6 scores was $665 - 95 = 570$, which is a multiple of 5, and the sum of her first 5 scores was also a multiple of 5. Therefore her sixth score must have been a multiple of 5. Because her seventh score was 95 and her scores were all different, her sixth score was 100. One possible sequence of scores is $91, 93, 92, 96, 98, 100, 95$.

22. **Answer (B):** There are $4 \cdot 3 = 12$ outcomes for each set of draws and therefore $12^4$ outcomes in all. To count the number of outcomes in which each player will end up with four coins, note that this can happen in four ways:

• For some permutation $(w, x, y, z)$ of $\{\text{Abby}, \text{Bernardo}, \text{Carl}, \text{Debra}\}$, the outcomes of the four draws are that $w$ gives a coin to $x$, $x$ gives a coin to $y$, $y$ gives a coin to $z$, and $z$ gives a coin to $w$, in one of $4! = 24$ orders. There are 3 ways to choose whom Abby gives her coin to and 2 ways to choose whom that person gives his or her coin to, which makes 6 ways to choose the givers and receivers for these transaction. Therefore there are $24 \cdot 6 = 144$ ways for this to happen.

• One pair of the players exchange coins, and the other two players also exchange coins, in one of $4! = 24$ orders. There are 3 ways to choose the pairings. Therefore there are $24 \cdot 3 = 72$ ways for this to happen.

• Two of the players exchange coins twice. There are $\binom{4}{2} = 6$ ways to choose those players and $\binom{4}{2} = 6$ ways to choose the orders of the exchanges, for a total of $6 \cdot 6 = 36$ ways for this to happen.

• One of the players is involved in all four transactions, giving and receiving a coin from each of two others. There are 4 ways to choose this player, 3 ways to choose the other two players, and $4! = 24$ ways to choose the order in which the transactions will take place. Therefore there are $4 \cdot 3 \cdot 24 = 288$ ways for this to happen.

In all, there are $144 + 72 + 36 + 288 = 540$ outcomes that will result in each player having four coins. The requested probability is $\frac{540}{12^4} = \frac{5}{192}$.

23. **Answer (D):** Let $g(x) = f(x) - x^2$. Then $g(2) = g(3) = g(4) = 0$, so for some constant $a \neq 0$, $g(x) = a(x-2)(x-3)(x-4)$. Thus the coefficients of $x^3$ and $x^2$ in $f(x)$ are $a$ and $1-9a$, respectively, so the sum of the roots of $f(x)$ is $9 - \frac{1}{a}$. If $L(x)$ is any linear function, then the roots of $f(x) - L(x)$ have the same sum. The given information implies that the sets of roots for three such functions are $\{2, 3, x_1\}$, $\{2, 4, x_2\}$, and $\{3, 4, x_3\}$, where

$$24 = x_1 + x_2 + x_3 = 3\left(9 - \frac{1}{a}\right) - 2(2+3+4) = 9 - \frac{3}{a},$$

so $a = -\frac{1}{5}$. Therefore $f(x) = x^2 - \frac{1}{5}(x-2)(x-3)(x-4)$, and $f(0) = \frac{24}{5}$. (In fact, $D = (9, 39)$, $E = (8, 40)$, $F = (7, 37)$, and the roots of $f$ are 12, $1+i$, and $1-i$.)

24. **Answer (D):** Let $F$ lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Because $BCDF$ is a rectangle, $\angle FCB \cong \angle DBC \cong \angle CAB \cong \angle BCE$, so $E$ lies on $\overline{CF}$ and it is the foot of the altitude to the hypotenuse in $\triangle CBF$. Therefore $\triangle BEF \sim \triangle CBF \cong \triangle BCD \sim \triangle ABC$. Because

$$\overline{DF} \perp \overline{AB}, \quad \overline{FE} \perp \overline{EB}, \quad \text{and} \quad \frac{AB}{DF} = \frac{AB}{BC} = \frac{BE}{FE},$$

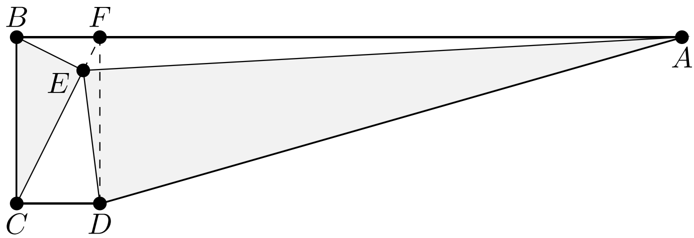
it follows that $\triangle ABE \sim \triangle DFE$. Thus $\angle DEA = \angle DEF - \angle AEF = \angle AEB - \angle AEF = \angle FEB = 90^\circ$. Furthermore,

$$\frac{AE}{ED} = \frac{BE}{EF} = \frac{AB}{BC},$$

so $\triangle AED \sim \triangle ABC$. Assume without loss of generality that $BC = 1$, and let $AB = r > 1$. Because $\frac{AB}{BC} = \frac{BC}{CD}$, it follows that $BF = CD = \frac{1}{r}$. Then

$$17 = \frac{\text{Area}(\triangle AED)}{\text{Area}(\triangle CEB)} = AD^2 = FD^2 + AF^2 = 1 + \left(r - \frac{1}{r}\right)^2,$$

and because $r > 1$ this yields $r^2 - 4r - 1 = 0$, with positive solution $r = 2 + \sqrt{5}$.



**OR**

Without loss of generality, assume that $BC = 1$. The given conditions imply that the quadrilateral can be placed in the coordinate plane with $C = (0,0)$, $B = (0,1)$, $A = (r,1)$, and $D = (\frac{1}{r}, 0)$. Let $E$ have positive coordinates $(x,y)$. Because $\triangle ABC \sim \triangle CEB$, these coordinates must satisfy

$$\frac{x}{y} = \tan(\angle ECB) = \tan(\angle BAC) = \frac{1}{r}$$

and

$$\sqrt{x^2+y^2} = \frac{CE}{1} = \frac{r}{\sqrt{1+r^2}}.$$

Solving this system of equations gives

$$x = \frac{r}{1+r^2} \quad \text{and} \quad y = \frac{r^2}{1+r^2}.$$

The area of $\triangle CEB$ is $\frac{x}{2}$. The area of $\triangle AED$ can be computed using the fact that the area of a polygon with vertices $(x_1,y_1)$, $(x_2,y_2)$, $\ldots$, $(x_n,y_n)$ in counterclockwise order is

$$\frac{1}{2}\big((x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) \\ - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1)\big).$$

In this case,

$$\text{Area}(\triangle AED) = \frac{1}{2}\left(y \cdot r + \frac{1}{r} - x - \frac{y}{r}\right).$$

Substituting in the expressions for $x$ and $y$ in terms of $r$, setting $\text{Area}(\triangle AED) = 17 \cdot \text{Area}(\triangle CEB)$, and simplifying yields the equation $r^4 - 18r^2 + 1 = 0$. Applying the quadratic formula, and noting that $r > 1$, gives $r^2 = 9 + 4\sqrt{5} = (2+\sqrt{5})^2$, so $r = 2 + \sqrt{5}$.

**OR**

Let $\theta = \angle ACB$, and without loss of generality assume $BC = 1$. Let $F$ lie on $\overline{AB}$ so that $\overline{DF} \perp \overline{AB}$. Then the requested fraction is $AB = \tan\theta$. Because $\triangle ABC \sim \triangle BCD \sim \triangle CEB \sim \triangle BEF$, it follows that $CD = \cot\theta$, $BE = \cos\theta$, and $CE = \sin\theta$. Then the area of quadrilateral $ABCD$ is $[ABCD] = \frac{1}{2}(\tan\theta + \cot\theta) = \frac{1}{2\sin\theta\cos\theta}$; and the areas of three of the four triangles into which that area can be decomposed are $[ABE] = \frac{1}{2}\tan\theta\cos^2\theta = \frac{1}{2}\sin\theta\cos\theta$, $[BCE] = \frac{1}{2}\sin\theta\cos\theta$, and $[CDE] = \frac{1}{2}\sin^2\theta\cot\theta = \frac{1}{2}\sin\theta\cos\theta$. (Interestingly, the three triangles all have the same area.) Then

$$[AED] = \frac{1}{2\sin\theta\cos\theta} - \frac{3}{2}\sin\theta\cos\theta = 17\cdot\frac{1}{2}\sin\theta\cos\theta.$$

This last equation simplifies to $20\sin^2\theta\cos^2\theta = 1$, so $(2\sin\theta\cos\theta)^2 = \frac{1}{5}$. Then $\sin(2\theta) = \frac{1}{\sqrt{5}}$, $\cos(2\theta) = \frac{-2}{\sqrt{5}}$ (because $AB > BC$ implies $\frac{\pi}{4} < \theta < \frac{\pi}{2}$), and

$$\tan\theta = \frac{\sin(2\theta)}{\cos(2\theta)+1} = \frac{1}{-2+\sqrt{5}} = 2+\sqrt{5}.$$

25. **Answer (D):** Let $T$ be the number of teams participating in the tournament, and let $P$ be the set of participants. For every $A \subseteq P$ let $f(A)$ be the number of teams whose 5 players are in $A$. According to the described property,

$$\left(\frac{1}{\binom{n}{9}}\sum_{\substack{A\subseteq P\\|A|=9}} f(A)\right)\cdot\left(\frac{1}{\binom{n}{8}}\sum_{\substack{A\subseteq P\\|A|=8}} f(A)\right) = 1.$$

Note that each of the $T$ teams is counted exactly $\binom{n-5}{4}$ times in the sum $\sum_{\substack{A\subseteq P\\|A|=9}} f(A)$. Indeed, once a particular team is fixed, there are exactly $\binom{n-5}{4}$ ways of choosing the remaining 4 persons to determine a set $A$ of size 9. Thus the sum in the first factor is equal to $\binom{n-5}{4}T$; similarly, the sum in the second factor is equal to $\binom{n-5}{3}T$. The described property is now equivalent to

$$\frac{\binom{n-5}{4}\binom{n-5}{3}T^2}{\binom{n}{9}\binom{n}{8}} = 1.$$

Therefore

$$T^2 = \frac{(n!)^2 4!3!}{((n-5)!)^2 9!8!} = \frac{n^2(n-1)^2(n-2)^2(n-3)^2(n-4)^2}{9\cdot 8^2\cdot 7^2\cdot 6^2\cdot 5^2\cdot 4},$$

so

$$T = \frac{n(n-1)(n-2)(n-3)(n-4)}{8\cdot 7\cdot 6\cdot 5\cdot 3\cdot 2} = \frac{n(n-1)(n-2)(n-3)(n-4)}{2^5\cdot 3^2\cdot 5\cdot 7}.$$

Thus a number $n$ has the required property if and only if $T$ is an integer and $n \geq 9$. Let $N = n(n-1)(n-2)(n-3)(n-4)$; because

$N$ consists of the product of five consecutive integers, it is always a multiple of 5. Similarly, $N \equiv 0 \pmod 7$ if and only if $n \equiv 0, 1, 2, 3, 4 \pmod 7$, $N \equiv 0 \pmod 9$ if and only if $n \equiv 0, 1, 2, 3, 4, 6, 7 \pmod 9$, and $N \equiv 0 \pmod{32}$ if and only if $n \equiv 0, 1, 2, 3, 4, 8, 10, 12 \pmod{16}$. Therefore by the Chinese Reminder Theorem there are exactly $5 \cdot 7 \cdot 8 = 280$ residue-class solutions mod $16 \cdot 9 \cdot 7 = 1008$. Thus there are $2 \cdot 280 = 560$ values of $n$ with the desired property in the interval $1 \le n \le 2 \cdot 1008 = 2016$. The numbers 1, 2, 3, and 4 are among them, and 5, 6, 7, and 8 are not. In addition, $2017 \equiv 1 \pmod{1008}$; thus 2017 is also a valid value of $n$. Therefore there are $560 - 4 + 1 = 557$ possible values of $n$ in the required range.

Problems and solutions were contributed by Bernardo Abrego, Thomas Butts, Barb Currier, Steven Davis, Marta Eso, Silvia Fernandez, Devin Gardella, Jerrold Grossman, Jonathan Kane, Joe Kennedy, Michael Khoury, Pamela Mishkin, Hugh Montgomery, Joachim Rebholz, Mark Saul, Gabriel Staton, Roger Waggoner, Dave Wells, Barry Weng, and Carl Yerger.

*The*

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MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

MAA American Mathematics Competitions

69th Annual

# AMC 12A

**American Mathematics Competition 12A**

**Wednesday, February 7, 2018**

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:


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The problems and solutions for this AMC 12 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (D):** There are currently 36 red balls in the urn. In order for the 36 red balls to represent 72% of the balls in the urn after some blue balls are removed, there must be $36 \div 0.72 = 50$ balls left in the urn. This requires that $100 - 50 = 50$ blue balls be removed.

2. **Answer (C):** The 5-pound rocks have a value of $\$14 \div 5 = \$2.80$ per pound; the 4-pound rocks have a value of $\$11 \div 4 = \$2.75$ per pound; the 1-pound rocks have a value of \$2 per pound. It is not to Carl's advantage to take 1-pound rocks when he can take the larger rocks. Therefore the only issue is how many of the more valuable 5-pound rocks to take, including as many 4-pound rocks as possible in each case. The viable choices are displayed in the following table.

| 5-pound rocks (\$14 each) | 4-pound rocks (\$11 each) | 1-pound rocks (\$2 each) | value |
|---|---|---|---|
| 3 | 0 | 3 | \$48 |
| 2 | 2 | 0 | \$50 |
| 1 | 3 | 1 | \$49 |
| 0 | 4 | 2 | \$48 |

The maximum possible value is \$50.

**Note:** Although the 5-pound rocks are the most valuable per pound, it was not to Carl's advantage to take as many of them as possible. This situation is an example of the classic knapsack problem for which the so-called "greedy algorithm" is not optimal.

3. **Answer (E):** There are 4 choices for the periods in which the mathematics courses can be taken: periods 1, 3, 5; periods 1, 3, 6; periods 1, 4, 6; and periods 2, 4, 6. Each choice of periods allows $3! = 6$ ways to order the 3 mathematics courses. Therefore there are $4 \cdot 6 = 24$ ways of arranging a schedule.

4. **Answer (D):** Because the statements of Alice, Bob, and Charlie are all incorrect, the actual distance $d$ satisfies $d < 6$, $d > 5$, and $d > 4$. Hence the actual distance lies in the interval $(5, 6)$.

5. **Answer (E):** Factoring $x^2 - 3x + 2$ as $(x - 1)(x - 2)$ shows that its roots are 1 and 2. If 1 is a root of $x^2 - 5x + k$, then $1^2 - 5 \cdot 1 + k = 0$ and $k = 4$. If 2 is a root of $x^2 - 5x + k$, then $2^2 - 5 \cdot 2 + k = 0$ and $k = 6$. The sum of all possible values of $k$ is $4 + 6 = 10$.

6. **Answer (B):** Note that the given conditions imply that the 6 values are listed in increasing order. Because the median of the these 6 values is $n$, the mean of the middle two values must be $n$, so

$$\frac{(m+10)+(n+1)}{2}=n,$$

which implies $m=n-11$. Because the mean of the set is also $n$,

$$\frac{(n-11)+(n-7)+(n-1)+(n+1)+(n+2)+2n}{6}=n,$$

so $7n-16=6n$ and $n=16$. Then $m=16-11=5$, and the requested sum is $5+16=21$.

7. **Answer (E):** Because $4000=2^5\cdot 5^3$,

$$4000\cdot\left(\frac{2}{5}\right)^n=2^{5+n}\cdot 5^{3-n}.$$

This product will be an integer if and only if both of the factors $2^{5+n}$ and $5^{3-n}$ are integers, which happens if and only if both exponents are nonnegative. Therefore the given expression is an integer if and only if $5+n\geq 0$ and $3-n\geq 0$. The solutions are exactly the integers satisfying $-5\leq n\leq 3$. There are $3-(-5)+1=9$ such values.

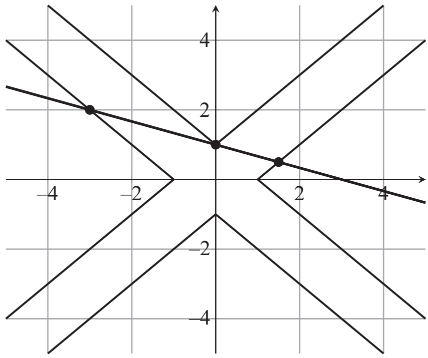
8. **Answer (E):** The length of the base $\overline{DE}$ of $\triangle ADE$ is 4 times the length of the base of a small triangle, so the area of $\triangle ADE$ is $4^2\cdot 1=16$. Therefore the area of $DBCE$ is the area of $\triangle ABC$ minus the area of $\triangle ADE$, which is $40-16=24$.

9. **Answer (E):** If $0\leq x\leq\pi$ and $0\leq y\leq\pi$, then $\sin(x)\geq 0$, $\sin(y)\geq 0$, $\cos(x)\leq 1$, and $\cos(y)\leq 1$. Therefore

$$\sin(x+y)=\sin(x)\cdot\cos(y)+\cos(x)\cdot\sin(y)\leq\sin(x)+\sin(y).$$

The given inequality holds for all $y$ such that $0\leq y\leq\pi$.

10. **Answer (C):** The graph of the system is shown below.




The graph of the first equation is a line with $x$-intercept $(3, 0)$ and $y$-intercept $(0, 1)$. To draw the graph of the second equation, consider the equation quadrant by quadrant. In the first quadrant $x > 0$ and $y > 0$, and thus the second equation is equivalent to $|x - y| = 1$, which in turn is equivalent to $y = x \pm 1$. Its graph consists of the rays with endpoints $(0, 1)$ and $(1, 0)$, as shown. In the second quadrant $x < 0$ and $y > 0$. The corresponding graph is the reflection of the first quadrant graph across the $y$-axis. The rest of the graph can be sketched by further reflections of the first-quadrant graph across the coordinate axes, resulting in the figure shown. There are 3 intersection points: $(-3, 2)$, $(0, 1)$, and $\left(\frac{3}{2}, \frac{1}{2}\right)$, as shown.

**OR**

The second equation implies that $x = y \pm 1$ or $x = -y \pm 1$. There are four cases:

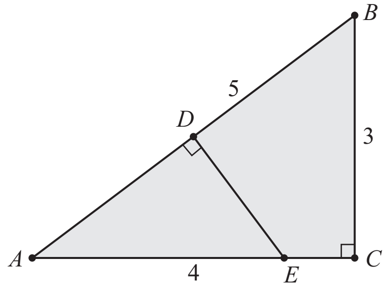
- If $x = y + 1$, then $(y + 1) + 3y = 3$, so $(x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$.
- If $x = y - 1$, then $(y - 1) + 3y = 3$, so $(x, y) = (0, 1)$.
- If $x = -y + 1$, then $(-y + 1) + 3y = 3$, so again $(x, y) = (0, 1)$.
- If $x = -y - 1$, then $(-y - 1) + 3y = 3$, so $(x, y) = (-3, 2)$.

It may be checked that each of these ordered pairs actually satisfies the given equations, so the total number of solutions is 3.

11. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle $ACB$, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC > BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let $D$ be the midpoint of $\overline{AB}$, and let $E$ be the intersection of $\overline{AC}$ and the line through $D$ perpendicular to $\overline{AB}$. Then the crease in the paper

is $\overline{DE}$. Because $\triangle ADE \sim \triangle ACB$, it follows that $\frac{DE}{AD} = \frac{CB}{AC} = \frac{3}{4}$. Thus

$$DE = AD \cdot \frac{CB}{AC} = \frac{5}{2} \cdot \frac{3}{4} = \frac{15}{8}.$$




12. **Answer (C):** If $1 \in S$, then $S$ can have only 1 element, not 6 elements. If 2 is the least element of $S$, then 2, 3, 5, 7, 9, and 11 are available to be in $S$, but 3 and 9 cannot both be in $S$, so the largest possible size of $S$ is 5. If 3 is the least element, then 3, 4, 5, 7, 8, 10, and 11 are available, but at most one of 4 and 8 can be in $S$ and at most one of 5 and 10 can be in $S$, so again $S$ has size at most 5. The set $S = \{4, 6, 7, 9, 10, 11\}$ has the required property, so 4 is the least possible element of $S$.

**OR**

At most one integer can be selected for $S$ from each of the following 6 groups: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. Because $S$ consists of 6 distinct integers, exactly one integer must be selected from each of these 6 groups. Therefore 7, 9, and 11 must be in $S$. Because 9 is in $S$, 3 must not be in $S$. This implies that either 6 or 12 must be selected from the second group, so neither 1 nor 2 can be selected from the first group. If 4 is selected from the first group, the collection of integers $\{4, 5, 6, 7, 9, 11\}$ is one possibility for the set $S$. Therefore 4 is the least possible element of $S$.

**Note:** The two collections given in the solutions are the only ones with least element 4 that have the given property. This problem is a special case of the following result of Paul Erdős from the 1930s: Given $n$ integers $a_1, a_2, \ldots, a_n$, no one of them dividing any other, with $a_1 < a_2 < \cdots < a_n \leq 2n$, let the integer $k$ be determined by the inequalities $3^k < 2n < 3^{k+1}$. Then $a_1 \geq 2^k$, and this bound is sharp.

13. **Answer (D):** Let $S$ be the set of integers, both negative and nonnegative, having the given form. Increasing the value of $a_i$ by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between $S$ and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so $S$ contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so $S$ contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

**OR**

First note that if an integer $N$ can be written in this form, then $N - 1$ can also be written in this form as long as not all the $a_i$ in the representation of $N$ are equal to $-1$. A procedure to alter the representation of $N$ so that it will represent $N - 1$ instead is to find the least value of $i$ such that $a_i \neq -1$, reduce the value of that $a_i$ by 1, and set $a_i = 1$ for all lower values of $i$. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from $-3280$ through $-1$ can also be written in this way.)

**OR**

Think of the indicated sum as an expansion in base 3 using "digits" $-1$, 0, and 1. Note that the leftmost digit $a_k$ of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining $k$ digits to the right of $a_k$, resulting in $3^k$ positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

14. **Answer (D):** By the change-of-base formula, the given equation is equivalent to

$$\begin{aligned}
\frac{\log 4}{\log 3x} &= \frac{\log 8}{\log 2x} \\
\frac{2\log 2}{\log 3 + \log x} &= \frac{3\log 2}{\log 2 + \log x} \\
2\log 2 + 2\log x &= 3\log 3 + 3\log x \\
\log x &= 2\log 2 - 3\log 3 \\
\log x &= \log \frac{4}{27}.
\end{aligned}$$

Therefore $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

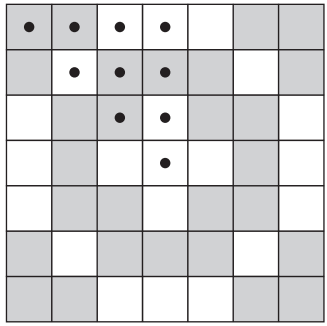
**OR**

Changing to base-2 logarithms transforms the given equation into

$$\begin{aligned}
\frac{2}{\log_2 3x} &= \frac{3}{\log_2 2x} \\
2\log_2 2x &= 3\log_2 3x \\
\log_2 (2x)^2 &= \log_2 (3x)^3 \\
(2x)^2 &= (3x)^3,
\end{aligned}$$

so $x = \frac{4}{27}$, and the requested sum is $4 + 27 = 31$.

15. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of "all black" and "all white". On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



**OR**

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

| | | | | | | |
|---|---|---|---|---|---|---|
| *A* | *B* | *C* | *D* | *C* | *B* | *A* |
| *B* | *E* | *F* | *G* | *F* | *E* | *B* |
| *C* | *F* | *H* | *I* | *H* | *F* | *C* |
| *D* | *G* | *I* | *J* | *I* | *G* | *D* |
| *C* | *F* | *H* | *I* | *H* | *F* | *C* |
| *B* | *E* | *F* | *G* | *F* | *E* | *B* |
| *A* | *B* | *C* | *D* | *C* | *B* | *A* |

16. **Answer (E):** Solving the second equation for $x^2$ gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in $y$ can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for $x$ are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

**OR**

Substituting the value for $y$ from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a-1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \le \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

17. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4,0)$, $(0,3)$, and $(0,0)$, with $[0,s]\times[0,s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12=0$, so the distance from $(s,s)$, the corner of $S$ closest to the hypotenuse, to this line is given by

$$\frac{|3s+4s-12|}{\sqrt{3^2+4^2}}.$$

Setting this equal to 2 and solving for $s$ gives $s = \frac{22}{7}$ and $s = \frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$, and the requested fraction is

$$1 - \frac{\frac{4}{49}}{\frac{1}{2}\cdot 4\cdot 3} = \frac{145}{147}.$$

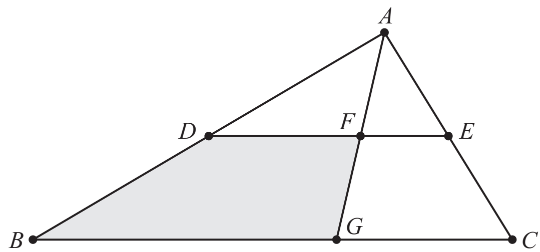
**OR**

Let the given triangle be described as $\triangle ABC$ with the right angle at $B$ and $AB = 3$. Let $D$ be the vertex of the square that is in the interior of the triangle, and let $s$ be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side $s$, a right triangle with legs $s$ and $3-s$, and a right triangle with legs $s$ and $4-s$. The sum of the areas of these four regions is

$$\frac{1}{2}\cdot 5\cdot 2 + s^2 + \frac{1}{2}s(3-s) + \frac{1}{2}s(4-s) = 5 + \frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5 + \frac{7}{2}s = 6$ for $s$ gives $s = \frac{2}{7}$, and the solution concludes as above.

18. **Answer (D):** Because $AB$ is $\frac{5}{6}$ of $AB + AC$, it follows from the Angle Bisector Theorem that $DF$ is $\frac{5}{6}$ of $DE$, and $BG$ is $\frac{5}{6}$ of $BC$. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120-30 = 90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6} \cdot 90 = 75$.




**Note:** The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC \approx 59.0$ and $\angle BAC \approx 151°$. It is also possible for $\angle ACB$ to be obtuse, with $BC \approx 41.5$ and $\angle BAC \approx 29°$. These values can be calculated using the Law of Cosines and the sine formula for area.

19. **Answer (C):** Elements of set $A$ are of the form $2^i \cdot 3^j \cdot 5^k$ for nonnegative integers $i$, $j$, and $k$. Note that the product

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\cdots\right)$$

will produce the desired sum. By the formula for infinite geometric series, this product evaluates to

$$\frac{1}{1-\frac{1}{2}} \cdot \frac{1}{1-\frac{1}{3}} \cdot \frac{1}{1-\frac{1}{5}} = 2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4}.$$

The requested sum is $15 + 4 = 19$.

20. **Answer (D):** It follows from the Pythagorean Theorem that $CM = MB = \frac{3}{2}\sqrt{2}$. Because quadrilateral $AIME$ is cyclic, opposite angles are supplementary and thus $\angle IME$ is a right angle. Let $x = CI$ and $y = BE$; then $AI = 3 - x$ and $AE = 3 - y$. By the Law of Cosines in $\triangle MCI$,

$$IM^2 = x^2 + \left(\frac{3}{2}\sqrt{2}\right)^2 - 2 \cdot x \cdot \frac{3}{2}\sqrt{2} \cdot \cos 45° = x^2 - 3x + \frac{9}{2}.$$

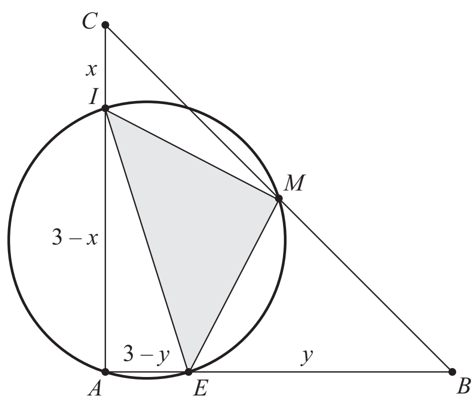
Similarly, $ME^2 = y^2 - 3y + \frac{9}{2}$. By the Pythagorean Theorem in right triangles $EMI$ and $IAE$,

$$\left(x^2 - 3x + \frac{9}{2}\right) + \left(y^2 - 3y + \frac{9}{2}\right) = (3 - x)^2 + (3 - y)^2,$$

which simplifies to $x + y = 3$. Because the area of $\triangle EMI$ is 2, it follows that $IM^2 \cdot ME^2 = 16$. Therefore

$$\left(x^2 - 3x + \frac{9}{2}\right)\left((3 - x)^2 - 3(3 - x) + \frac{9}{2}\right) = 16,$$

which simplifies to $\left(x^2 - 3x + \frac{9}{2}\right)^2 = 16$. Because $y > x$, the only real solution is $x = \frac{3-\sqrt{7}}{2}$. The requested sum is $3 + 7 + 2 = 12$.




**OR**

Place the figure in the coordinate plane with $A$ at $(0, 0)$, $B$ at $(3, 0)$, and $C$ at $(0, 3)$. Then $M$ is at $\left(\frac{3}{2}, \frac{3}{2}\right)$ . Let $s = AE$ and $t = CI$. Then the coordinates of $E$ are $(s, 0)$, and the coordinates of $I$ are $(0, 3 - t)$. Because $AIME$ is a cyclic quadrilateral and $\angle EAI$ is a right angle, $\angle IME$ is a right angle. Therefore $\overline{MI}$ and $\overline{ME}$ are perpendicular, so the product of their slopes is

$$\frac{\frac{3}{2}}{\frac{3}{2} - s} \cdot \frac{t - \frac{3}{2}}{\frac{3}{2}} = -1;$$

this equation simplifies to $s = t$. Then, with brackets indicating area,

$$\begin{aligned}[ABC] &= [CIM] + [BME] + [AEI] + [IME]\\ \frac{9}{2} &= \frac{1}{2} \cdot \frac{3}{2} \cdot t + \frac{1}{2} \cdot \frac{3}{2} \cdot (3 - t) + \frac{1}{2} \cdot t \cdot (3 - t) + 2,\end{aligned}$$

which simplifiles to $2t^2 - 6t + 1 = 0$. Therefore $t = \frac{3\pm\sqrt{7}}{2}$, and because $AI > AE$, the length of $CI$ is $\frac{3-\sqrt{7}}{2}$ and the requested sum is $3 + 7 + 2 = 12$.

21. **Answer (B):** By Descartes' Rule of Signs, none of these polynomials has a positive root, and each one has exactly one negative root. Because each polynomial is positive at $x = 0$ and negative at $x = -1$, it follows that each has exactly one root between $-1$ and $0$. Note also that each polynomial is increasing throughout the interval $(-1, 0)$. Because $x^{19} > x^{17}$ for all $x$ in the interval $(-1, 0)$, it follows that the polynomial in choice **A** is greater than the polynomial in choice **B** on that interval, which implies that the root of the polynomial in choice **A** is less than the root of the polynomial in choice **B**. Because $x^{13} > x^{11}$ for all $x$ in the interval $(-1, 0)$, it follows that the polynomial in choice **C** is greater than the polynomial in choice **A** on that interval, which implies that the root of the polynomial in choice **C** is less than the root of the polynomial in choice **A** and therefore less than the root of the polynomial in choice **B**. The same reasoning shows that the root of the polynomial in choice **D** is less than the root of the polynomial in choice **B**.

   Furthermore, $2018 > 2018x^6$ on the interval $(-1, 0)$, so $x^6 + 2018 > 2019x^6$, from which it follows that $x^{11}(x^6 + 2018) < 2019x^{17}$. Therefore the polynomial in choice **B** is less than $2019x^{17} + 1$ on the interval $(-1, 0)$. The polynomial in choice **E** has root $-\left(1 - \frac{1}{2019}\right)$. Bernoulli's Inequality shows that $(1 + x)^{17} > 1 + 17x$ for all $x > -1$, which implies that

$$-2019\left(1 - \frac{1}{2019}\right)^{17} + 1 < -2019\left(1 - \frac{17}{2019}\right) + 1 = -2001 < 0,$$

   so the polynomial in choice **B** is negative at the root of the polynomial in choice **E**. This shows that the root of the polynomial in choice **B** is greater than the root in choice **E**.

   Because the unique real root of the polynomial in choice **B** is greater than the unique root of the polynomial in each of the other choices, that polynomial has the greatest real root.

22. **Answer (A):** Let $z = a+bi$ be a solution of the first equation, where $a$ and $b$ are real numbers. Then $(a + bi)^2 = 4 + 4\sqrt{15}i$. Expanding the left-hand side and equating real and imaginary parts yields

$$a^2 - b^2 = 4 \quad \text{and} \quad 2ab = 4\sqrt{15}.$$

From the second equation, $b = \frac{2\sqrt{15}}{a}$, and substituting this into the first equation and simplifying gives $(a^2)^2 - 4a^2 - 60 = 0$, which factors as $(a^2-10)(a^2+6) = 0$. Because $a$ is real, it follows that $a = \pm\sqrt{10}$, from which it then follows that $b = \pm\sqrt{6}$. Thus two vertices of the parallelogram are $\sqrt{10}+\sqrt{6}i$ and $-\sqrt{10}-\sqrt{6}i$. A similar calculation with the other given equation shows that the other two vertices of the parallelogram are $\sqrt{3}+i$ and $-\sqrt{3}-i$. The area of this parallogram can be computed using the shoelace formula, which gives the area of a polygon in terms of the coordinates of its vertices $(x_1, y_1)$, $(x_2, y_2)$, $\ldots$, $(x_n, y_n)$ in clockwise or counter-clockwise order:

$$\frac{1}{2} \cdot \Big| (x_1y_2 + x_2y_3 + \cdots + x_{n-1}y_n + x_ny_1) \\ - (y_1x_2 + y_2x_3 + \cdots + y_{n-1}x_n + y_nx_1) \Big|.$$

In this case $x_1 = \sqrt{10}$, $y_1 = \sqrt{6}$, $x_2 = \sqrt{3}$, $y_2 = 1$, $x_3 = -\sqrt{10}$, $y_3 = -\sqrt{6}$, $x_4 = -\sqrt{3}$, and $y_4 = -1$. The area is $6\sqrt{2} - 2\sqrt{10}$, and the requested sum of the four positive integers in this expression is 20.

**OR**

The solutions of $z^2 = 4 + 4\sqrt{15}i = 16 \operatorname{cis} 2\theta_1$ are $z_1 = 4 \operatorname{cis} \theta_1$ and its opposite, with $0 < \theta_1 < \frac{\pi}{4}$ and $\tan 2\theta_1 = \sqrt{15}$. Then $\cos 2\theta_1 = \frac{1}{4}$, and by the half-angle identities, $\cos\theta_1 = \frac{\sqrt{10}}{4}$ and $\sin\theta_1 = \frac{\sqrt{6}}{4}$. Similarly, the solutions of $z^2 = 2 + 2\sqrt{3}i = 4 \operatorname{cis} \theta_2$ are $z_2 = 2 \operatorname{cis} \theta_2$ and its opposite, with $0 < \theta_2 < \frac{\pi}{4}$ and $\tan 2\theta_2 = \sqrt{3}$. Then $\cos\theta_2 = \frac{\sqrt{3}}{2}$ and $\sin\theta_2 = \frac{1}{2}$.

The area of the parallelogram in the complex plane with vertices $z_1$, $z_2$, and their opposites is 4 times the area of the triangle with vertices 0, $z_1$, and $z_2$, and because the area of a triangle is one-half the product of the lengths of two of its sides and the sine of their included angle, it follows that the area of the parallelogram is

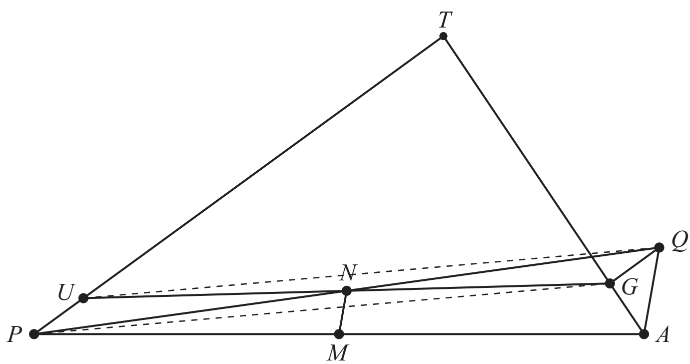
$$\begin{aligned} 4\left(\frac{1}{2} \cdot 4 \cdot 2 \cdot \sin(\theta_1 - \theta_2)\right) &= 16(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2) \\ &= 16\left(\frac{\sqrt{6}}{4} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{10}}{4} \cdot \frac{1}{2}\right) \\ &= 6\sqrt{2} - 2\sqrt{10}. \end{aligned}$$

Therefore, $p + q + r + s = 6 + 2 + 2 + 10 = 20$.

23. **Answer (E):** Extend $\overline{PN}$ through $N$ to $Q$ so that $PN = NQ$. Segments $\overline{UG}$ and $\overline{PQ}$ bisect each other, implying that $UPGQ$ is a parallelogram. Therefore $\overline{GQ} \parallel \overline{PT}$, so $\angle QGA = 180^\circ - \angle T = \angle TPA + \angle TAP = 36^\circ + 56^\circ = 92^\circ$. Furthermore $GQ = PU = AG$, so $\triangle QGA$ is isosceles, and $\angle QAG = \frac{1}{2}(180^\circ - 92^\circ) = 44^\circ$. Because $\overline{MN}$ is a midline of $\triangle QPA$, it follows that $\overline{MN} \parallel \overline{AQ}$ and

$$\angle NMP = \angle QAP = \angle QAG + \angle GAP = 44^\circ + 56^\circ = 100^\circ,$$

so acute $\angle NMA = 80^\circ$. (Note that the value of the common length $PU = AG$ is immaterial.)




**OR**

Place the figure in the coordinate plane with $P = (-5, 0)$, $M = (0, 0)$, $A = (5, 0)$, and $T$ in the first quadrant. Then

$$U = (-5 + \cos 36^\circ, \sin 36^\circ) \quad \text{and} \quad G = (5 - \cos 56^\circ, \sin 56^\circ),$$

and the midpoint $N$ of $\overline{UG}$ is

$$\left(\frac{1}{2}(\cos 36^\circ - \cos 56^\circ), \frac{1}{2}(\sin 36^\circ + \sin 56^\circ)\right).$$

The tangent of $\angle NMA$ is the slope of line $MN$, which is calculated as follows using the sum-to-product trigonometric identites:

$$\begin{aligned}\tan(\angle NMA) &= \frac{\sin 36^\circ + \sin 56^\circ}{\cos 36^\circ - \cos 56^\circ} \\ &= \frac{2 \sin \frac{36^\circ + 56^\circ}{2} \cos \frac{36^\circ - 56^\circ}{2}}{-2 \sin \frac{36^\circ + 56^\circ}{2} \sin \frac{36^\circ - 56^\circ}{2}} \\ &= \frac{\cos 10^\circ}{\sin 10^\circ} = \cot 10^\circ = \tan 80^\circ,\end{aligned}$$

and it follows that $\angle NMA = 80^\circ$.

24. **Answer (B):** Because Alice and Bob are choosing their numbers uniformly at random, the cases in which two or three of the chosen numbers are equal have probability 0 and can be ignored. Suppose Carol chooses the number $c$. She will win if her number is greater than Alice's number and less than Bob's, and she will win if her number is less than Alice's number and greater than Bob's. There are three cases.

  • If $c \le \frac{1}{2}$, then Carol's number is automatically less than Bob's, so her chance of winning is the probability that Alice's number is less than $c$, which is just $c$. The best that Carol can do in this case is to choose $c = \frac{1}{2}$, in which case her chance of winning is $\frac{1}{2}$.

  • If $c \ge \frac{2}{3}$, then Carol's number is automatically greater than Bob's, so her chance of winning is the probability that Alice's number is greater than $c$, which is just $1-c$. The best that Carol can do in this case is to choose $c = \frac{2}{3}$, in which case her chance of winning is $\frac{1}{3}$.

  • Finally suppose that $\frac{1}{2} < c < \frac{2}{3}$. The probability that Carol's number is less than Bob's is

$$\frac{\frac{2}{3} - c}{\frac{2}{3} - \frac{1}{2}} = 4 - 6c,$$

  so the probability that her number is greater than Alice's and less than Bob's is $c(4-6c)$. Similarly, the probability that her number is less than Alice's and greater than Bob's is $(1-c)(6c-3)$. Carol's probability of winning in this case is therefore

$$c(4-6c) + (1-c)(6c-3) = -12c^2 + 13c - 3.$$

  The value of a quadratic polynomial with a negative coefficient on its quadratic term is maximized at $\frac{-b}{2a}$, where $a$ is the coefficient on its quadratic term and $b$ is the coefficient on its linear term; here that is when $c = \frac{13}{24}$, which is indeed between $\frac{1}{2}$ and $\frac{2}{3}$. Her probability of winning is then

$$-12 \cdot \left(\frac{13}{24}\right)^2 + 13 \cdot \frac{13}{24} - 3 = \frac{25}{48} > \frac{24}{48} = \frac{1}{2}.$$

  Because the probability of winning in the third case exceeds the probabilities obtained in the first two cases, Carol should choose $\frac{13}{24}$.

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9}\right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values $n_1$ and $n_2$, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit $a$ must be 3, 6, or 9, with corresponding values 1, 4, or 9 for $c$, and 2, 8, or 18 for $b$. The case $b = 18$ is invalid, so there are just two triples of possible values for $a$, $b$, and $c$, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers $n$; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.

Problems and solutions were contributed by Risto Atanasov, George Bauer, Chris Bolognese, Ivan Borsenco, Thomas Butts, Barbara Currier, Steven Davis, Steve Dunbar, Marta Eso, Zuming Feng, Zachary Franco, Peter Gilchrist, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Joe Kennedy, Michael Khoury, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, David Wells, Barry Weng, and Carl Yerger.

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MATHEMATICAL ASSOCIATION OF AMERICA

# Solutions Pamphlet

MAA American Mathematics Competitions

69th Annual

# AMC 12B

**American Mathematics Competition 12B**

**Thursday, February 15, 2018**

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:


MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
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The problems and solutions for this AMC 12 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (A):** The total area of cornbread is $20 \cdot 18 = 360$ in$^2$. Because each piece of cornbread has area $2 \cdot 2 = 4$ in$^2$, the pan contains $360 \div 4 = 90$ pieces of cornbread.

**OR**

When cut, there are $20 \div 2 = 10$ pieces of cornbread along a long side of the pan and $18 \div 2 = 9$ pieces along a short side, so there are $10 \cdot 9 = 90$ pieces.

2. **Answer (D):** Sam covered $\frac{1}{2} \cdot 60 = 30$ miles during the first 30 minutes and $\frac{1}{2} \cdot 65 = 32.5$ miles during the second 30 minutes, so he needed to cover $96 - 30 - 32.5 = 33.5$ miles during the last 30 minutes. Thus his average speed during the last 30 minutes was

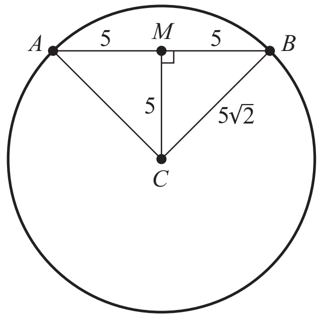
$$\frac{33.5 \text{ miles}}{\frac{1}{2} \text{ hour}} = 67 \text{ mph}.$$

3. **Answer (B):** The line with slope 2 containing the point $(40, 30)$ has the equation $y - 30 = 2(x - 40)$. Similarly, the line with slope 6 containing the point $(40, 30)$ has the equation $y - 30 = 6(x - 40)$. To find the $x$-intercepts of these two lines, let $y = 0$ and solve for $x$ separately in each of these two equations. With the first equation the $x$-intercept is 25, and with the second equation the $x$-intercept is 35. Thus the distance between the two $x$-intercepts is $|25 - 35| = 10$.

**OR**

As the line with slope 2 rises from $y = 0$ to $y = 30$, $x$ increases by 15. As the line with slope 6 rises from $y = 0$ to $y = 30$, $x$ increases by 5. Thus the distance between the $x$-intercepts is $|15 - 5| = 10$.

4. **Answer (B):** Let the chord have endpoints $A$ and $B$, and let $C$ be the center of the circle. The segment from $C$ to the midpoint $M$ of $\overline{AB}$ is perpendicular to $\overline{AB}$ and has length 5. This creates the $45-45-90^\circ$ triangle $CMB$, whose sides are 5, 5, and $CB = 5\sqrt{2}$. Therefore the radius of the circle is $5\sqrt{2}$, and the area of the circle is $\pi \cdot \left(5\sqrt{2}\right)^2 = 50\pi$.




5. **Answer (D):** The number of qualifying subsets equals the difference between the total number of subsets of $\{2,3,4,5,6,7,8,9\}$ and the number of such subsets containing no prime numbers, which is the number of subsets of $\{4,6,8,9\}$. A set with $n$ elements has $2^n$ subsets, so the requested number is $2^8 - 2^4 = 256 - 16 = 240$.

**OR**

A subset meeting the condition must be the union of a nonempty subset of $\{2,3,5,7\}$ and a subset of $\{4,6,8,9\}$. There are $2^4 - 1 = 15$ of the former and $2^4 = 16$ of the latter, which gives $15 \cdot 16 = 240$ choices in all.

6. **Answer (B):** The cost of 1 can is $\frac{Q}{S}$ quarters, which is $\frac{Q}{4S}$ dollars. Hence the number of cans that can be purchased with $D$ dollars is

$$\frac{D}{\left(\frac{Q}{4S}\right)} = \frac{4DS}{Q}.$$

7. **Answer (C):** The change of base formula states that $\log_a b = \frac{\log b}{\log a}$. Thus the product telescopes:

$$\begin{aligned}\frac{\log 7}{\log 3} \cdot \frac{\log 9}{\log 5} \cdot \frac{\log 11}{\log 7} \cdot \frac{\log 13}{\log 9} \cdots \frac{\log 25}{\log 21} \cdot \frac{\log 27}{\log 23} &= \frac{\log 25}{\log 3} \cdot \frac{\log 27}{\log 5} \\ &= \frac{\log 5^2}{\log 3} \cdot \frac{\log 3^3}{\log 5} \\ &= \frac{2\log 5}{\log 3} \cdot \frac{3\log 3}{\log 5}. \\ &= 6.\end{aligned}$$

**OR**

Let

$$a = \log_3 7 \cdot \log_7 11 \cdot \log_{11} 15 \cdot \log_{15} 19 \cdot \log_{19} 23 \cdot \log_{23} 27$$

and

$$b = \log_5 9 \cdot \log_9 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25.$$

The required product is $ab$. Now

$$\begin{aligned}
b &= \log_5 9 \cdot \log_9 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 9^{\log_9 13} \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 13 \cdot \log_{13} 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 13^{\log_{13} 17} \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 17 \cdot \log_{17} 21 \cdot \log_{21} 25 \\
&= \log_5 17^{\log_{17} 21} \cdot \log_{21} 25 \\
&= \log_5 21 \cdot \log_{21} 25 \\
&= \log_5 21^{\log_{21} 25} \\
&= \log_5 25 \\
&= 2.
\end{aligned}$$

Similarly, $a = \log_3 27 = 3$, so $ab = 2 \cdot 3 = 6$.

8. **Answer (C):** Let $O$ be the center of the circle. Triangle $ABC$ is a right triangle, and $O$ is the midpoint of the hypotenuse $\overline{AB}$. Then $\overline{OC}$ is a radius, and it is also one of the medians of the triangle. The centroid is located one third of the way along the median from $O$ to $C$, so the centroid traces out a circle with center $O$ and radius $\frac{1}{3} \cdot 12 = 4$ (except for the two missing points corresponding to $C = A$ or $C = B$). The area of this smaller circle is then $\pi \cdot 4^2 = 16\pi \approx 16 \cdot 3.14 \approx 50$.

9. **Answer (E):** Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$\begin{aligned}
\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\sum_{j=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j \\
&= \sum_{j=1}^{100}\sum_{i=1}^{100} i + \sum_{i=1}^{100}\sum_{j=1}^{100} j
\end{aligned}$$

$$
\begin{aligned}
&= 100\sum_{i=1}^{100} i + 100\sum_{j=1}^{100} j \\
&= 100(5050 + 5050) \\
&= 1{,}010{,}000.
\end{aligned}
$$

**OR**

Note that the sum of the first 100 positive integers is $\frac{1}{2} \cdot 100 \cdot 101 = 5050$. Then

$$
\begin{aligned}
\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j) &= \sum_{i=1}^{100}\big((i+1)+(i+2)+\cdots+(i+100)\big) \\
&= \sum_{i=1}^{100}(100i + 5050) \\
&= 100 \cdot 5050 + 100 \cdot 5050 \\
&= 1{,}010{,}000.
\end{aligned}
$$

**OR**

The sum contains 10,000 terms, and the average value of both $i$ and $j$ is $\frac{101}{2}$, so the sum is equal to

$$10{,}000\left(\frac{101}{2} + \frac{101}{2}\right) = 1{,}010{,}000.$$

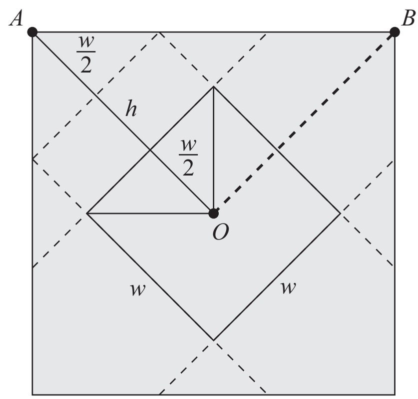
10. **Answer (D):** The list has $2018 - 10 = 2008$ entries that are not equal to the mode. Because the mode is unique, each of these 2008 entries can occur at most 9 times. There must be at least $\left\lceil \frac{2008}{9} \right\rceil = 224$ distinct values in the list that are different from the mode, because if there were fewer than this many such values, then the size of the list would be at most $9 \cdot \left(\left\lceil \frac{2008}{9} \right\rceil - 1\right) + 10 = 2017 < 2018$. (The ceiling function notation $\lceil x \rceil$ represents the least integer greater than or equal to $x$.) Therefore the least possible number of distinct values that can occur in the list is 225. One list satisfying the conditions of the problem contains 9 instances of each of the numbers 1 through 223, 10 instances of the number 224, and one instance of 225.

11. **Answer (A):** The figure shows that the distance $AO$ from a corner of the wrapping paper to the center is

$$\frac{w}{2} + h + \frac{w}{2} = w + h.$$

The side of the wrapping paper, $\overline{AB}$ in the figure, is the hypotenuse of a $45-45-90^\circ$ right triangle, so its length is $\sqrt{2} \cdot AO = \sqrt{2}(w+h)$. Therefore the area of the wrapping paper is

$$\left(\sqrt{2}(w+h)\right)^2 = 2(w+h)^2.$$




**OR**

The area of the wrapping paper, excluding the four small triangles indicated by the dashed lines, is equal to the surface area of the box, which is $2w^2 + 4wh$. The four triangles are isosceles right triangles with leg length $h$, so their combined area is $4 \cdot \frac{1}{2}h^2 = 2h^2$. Thus the total area of the wrapping paper is $2w^2 + 4wh + 2h^2 = 2(w+h)^2$.

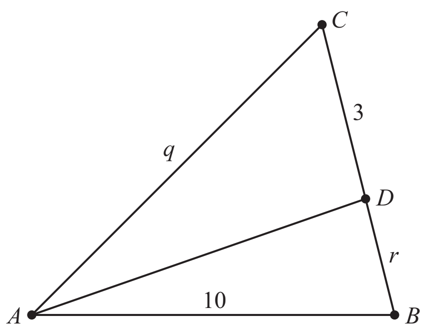
12. **Answer (C):** Let $q = AC$ and $r = BD$. By the Angle Bisector Theorem,

$$\frac{AC}{CD} = \frac{AB}{BD}, \quad \text{which means} \quad \frac{q}{3} = \frac{10}{r}, \quad \text{so} \quad r = \frac{30}{q}.$$

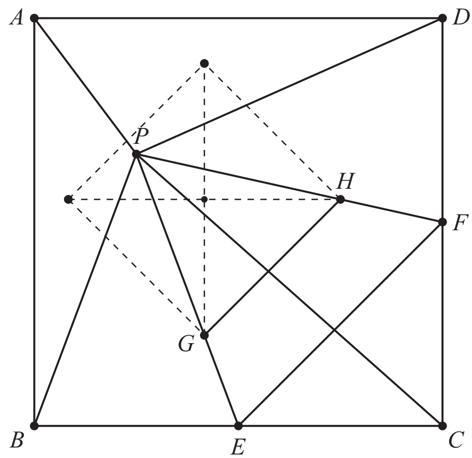
The possible values of $AC$ can be determined by considering the three Triangle Inequalities in $\triangle ABC$.

- $AC + BC > AB$, which means $q + 3 + r > 10$. Substituting for $r$ and simplifying gives $q^2 - 7q + 30 > 0$, which always holds because $q^2 - 7q + 30 = \left(q - \frac{7}{2}\right)^2 + \frac{71}{4}$.
- $BC + AB > AC$, which means $3 + r + 10 > q$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q - 15)(q + 2) < 0$, which holds if and only if $-2 < q < 15$.
- $AB + AC > BC$, which means $10 + q > 3 + r$. Substituting $r = \frac{30}{q}$, simplifying, and factoring gives $(q + 10)(q - 3) > 0$, which holds if and only if $q > 3$ or $q < -10$.

Combining these inequalities shows that the set of possible values of $q$ is the open interval $(3, 15)$, and the requested sum of the endpoints of the interval is $3 + 15 = 18$.




13. **Answer (C):** Let $E$ and $F$ be the midpoints of sides $\overline{BC}$ and $\overline{CD}$, respectively. Let $G$ and $H$ be the centroids of $\triangle BCP$ and $\triangle CDP$, respectively. Then $G$ is on $\overline{PE}$, a median of $\triangle BCP$, a distance $\frac{2}{3}$ of the way from $P$ to $E$. Similarly, $H$ is on $\overline{PF}$ a distance $\frac{2}{3}$ of the way from $P$ to $F$. Thus $\overline{GH}$ is parallel to $\overline{EF}$ and $\frac{2}{3}$ the length of $\overline{EF}$. Because $BC = 30$, it follows that $EC = 15$, $EF = 15\sqrt{2}$, and $GH = 10\sqrt{2}$. The midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ form a square, so the centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ also form a square, and that square has side length $10\sqrt{2}$. The requested area is $\left(10\sqrt{2}\right)^2 = 200$.




**OR**

Place the figure in the coordinate plane with $A = (0, 30)$, $B = (0, 0)$, $C = (30, 0)$, $D = (30, 30)$, and $P = (3x, 3y)$. Then the coordinates of the centroids of the four triangles are found by averaging the coordinates of the vertices: $(x, y + 10)$, $(x + 10, y)$, $(x + 20, y + 10)$, and $(x + 10, y + 20)$. It can be seen that the quadrilateral formed by the centroids is a square with center $(x + 10, y + 10)$ and vertices aligned vertically and horizontally. Its area is half the product of the lengths of its diagonals, $\frac{1}{2} \cdot 20 \cdot 20 = 200$.

**Note:** As the solutions demonstrate, the inner quadrilateral is always a square, and its size is independent of the location of point $P$. The location of the square within square $ABCD$ does depend on the location of $P$.

14. **Answer (E):** Let Chloe be $n$ years old today, so she is $n - 1$ years older than Zoe. For integers $y \geq 0$, Chloe's age will be a multiple of Zoe's age $y$ years from now if and only if

$$\frac{n + y}{1 + y} = 1 + \frac{n - 1}{1 + y}$$

is an integer, that is, $1 + y$ is a divisor of $n - 1$. Thus $n - 1$ has exactly 9 positive integer divisors, so the prime factorization of $n - 1$ has one of the two forms $p^2q^2$ or $p^8$. There are no two-digit integers of the form $p^8$, and the only one of the form $p^2q^2$ is $2^2 \cdot 3^2 = 36$. Therefore Chloe is 37 years old today, and Joey is 38. His age will be a multiple of Zoe's age in $y$ years if and only if $1 + y$ is a divisor of

$38 - 1 = 37$. The nonnegative integer solutions for $y$ are 0 and 36, so the only other time Joey's age will be a multiple of Zoe's age will be when he is $38 + 36 = 74$ years old. The requested sum is $7 + 4 = 11$.

15. **Answer (A):** Let $\underline{a}\,\underline{b}\,\underline{c}$ be a 3-digit positive odd multiple of 3 that does not include the digit 3. There are 8 possible values for $a$, namely 1, 2, 4, 5, 6, 7, 8, and 9, and 4 possible values for $c$, namely 1, 5, 7, and 9. The possible values of $b$ can be put into three groups of the same size: $\{0, 6, 9\}$, $\{1, 4, 7\}$, and $\{2, 5, 8\}$. Recall that an integer is divisible by 3 if and only if the sum of its digits is divisible by 3. Thus for every possible pair of digits $(a, c)$, the choices for $b$ such that $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 3 constitute one of those groups. Hence the answer is $8 \cdot 4 \cdot 3 = 96$.

**OR**

There are $\frac{1}{2} \cdot \frac{1}{3} \cdot 900 = 150$ odd 3-digit multiples of 3. Those including the digit 3 have the form $\underline{a}\,\underline{b}\,\underline{3}$, $\underline{a}\,\underline{3}\,\underline{b}$, or $\underline{3}\,\underline{a}\,\underline{b}$. There are 30 of the first type, where the number $\underline{a}\,\underline{b}$ is one of 12, 15, 18, ..., 99. There are 15 of the second type, where the number $\underline{a}\,\underline{b}$ is one of 15, 21, 27, ..., 99. There are 17 of the third type, where the number $\underline{a}\,\underline{b}$ is one of 03, 09, 15, ..., 99. The numbers 303, 339, 363, 393, 633, and 933 are each counted twice, and 333 is counted 3 times. By the Inclusion–Exclusion Principle there are $150 - (30 + 15 + 17) + (1 \cdot 6 + 2 \cdot 1) = 96$ such numbers.

16. **Answer (B):** The answer would be the same if the equation were $z^8 = 81$, resulting from a horizontal translation of 6 units. The solutions to this equation are the 8 eighth roots of 81, each of which is $\sqrt[8]{3^4} = \sqrt{3}$ units from the origin. These 8 points form a regular octagon. The triangle of minimum area occurs when the vertices of the triangle are consecutive vertices of the octagon, so without loss of generality they have coordinates $A\left(\frac{1}{2}\sqrt{6}, \frac{1}{2}\sqrt{6}\right)$, $B\left(\sqrt{3}, 0\right)$, and $C\left(\frac{1}{2}\sqrt{6}, -\frac{1}{2}\sqrt{6}\right)$. This triangle has base $AC = \sqrt{6}$ and height $\sqrt{3} - \frac{1}{2}\sqrt{6}$, so its area is

$$\frac{1}{2} \cdot \sqrt{6} \cdot \left(\sqrt{3} - \frac{1}{2}\sqrt{6}\right) = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

**OR**

The complex solutions form a regular octagon centered at $z = -6$. The distance from the center to any one of the vertices is $\sqrt[8]{81} =$

$\sqrt[8]{3^4} = \sqrt{3}$. By the Law of Cosines, the side length $s$ of the octagon satisfies

$$s^2 = \left(\sqrt{3}\right)^2 + \left(\sqrt{3}\right)^2 - 2 \cdot \sqrt{3} \cdot \sqrt{3} \cdot \cos 45^\circ = 6 - 6 \cdot \frac{\sqrt{2}}{2} = 6 - 3\sqrt{2}.$$

The least possible area of $\triangle ABC$ occurs when two of the sides of $\triangle ABC$ are adjacent sides of the octagon; the angle between these two sides is $135^\circ$. The sine formula for area gives

$$\frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \sin 135^\circ = \frac{1}{2} \cdot \left(6 - 3\sqrt{2}\right) \cdot \frac{\sqrt{2}}{2} = \frac{3}{2}\sqrt{2} - \frac{3}{2}.$$

17. **Answer (A):** The first inequality is equivalent to $9p > 5q$, and because both sides are integers, it follows that $9p - 5q \geq 1$. Similarly, $4q - 7p \geq 1$. Now

$$\begin{aligned}\frac{1}{63} = \frac{4}{7} - \frac{5}{9} &= \left(\frac{p}{q} - \frac{5}{9}\right) + \left(\frac{4}{7} - \frac{p}{q}\right) \\ &= \frac{9p - 5q}{9q} + \frac{4q - 7p}{7q} \\ &\geq \frac{1}{9q} + \frac{1}{7q} \\ &= \frac{16}{63q}.\end{aligned}$$

Thus $q \geq 16$. Because

$$\frac{8}{16} < \frac{5}{9} < \frac{9}{16} < \frac{4}{7} < \frac{10}{16},$$

the fraction $\frac{9}{16}$ lies in the required interval, but $\frac{8}{16}$ and $\frac{10}{16}$ do not. Therefore when $q$ is as small as possible, $q = 16$ and $p = 9$, and the requested difference is $16 - 9 = 7$.

**Note:** A theorem in the study of Farey fractions states that if $\frac{a}{p} < \frac{b}{q}$ and $bp - aq = 1$, then the rational number with least denominator between $\frac{a}{p}$ and $\frac{b}{q}$ is $\frac{a+b}{p+q}$.

18. **Answer (B):** Applying the recursion for several steps leads to the conjecture that

$$f(n) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 6, \\ n & \text{if } n \equiv 1 \pmod 6, \\ n-1 & \text{if } n \equiv 2 \pmod 6, \\ n & \text{if } n \equiv 3 \pmod 6, \\ n+2 & \text{if } n \equiv 4 \pmod 6, \\ n+3 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The conjecture can be verified using the strong form of mathematical induction with two base cases and six inductive steps. For example, if $n \equiv 2 \pmod 6$, then $n = 6k+2$ for some nonnegative integer $k$ and

$$\begin{aligned} f(n) &= f(6k+2) \\ &= f(6k+1) - f(6k) + 6k + 2 \\ &= (6k+1) - (6k+2) + 6k + 2 \\ &= 6k + 1 \\ &= n - 1. \end{aligned}$$

Therefore $f(2018) = f(6 \cdot 336 + 2) = 2018 - 1 = 2017$.

**OR**

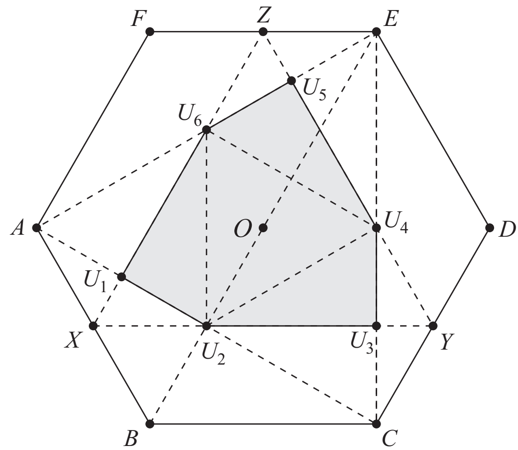
Note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + (n-1)] - f(n-2) + n \\ &= -[f(n-4) - f(n-5) + (n-3)] + 2n - 1 \\ &= -[f(n-5) - f(n-6) + (n-4)] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that $f(2018) = f(2) + 2016 = 2017$.

19. **Answer (C):** Let $d$ be the next divisor of $n$ after 323. Then $\gcd(d, 323) \neq 1$, because otherwise $n \geq 323d > 323^2 > 100^2 = 10000$, contrary to the fact that $n$ is a 4-digit number. Therefore $d - 323 \geq \gcd(d, 323) > 1$. The prime factorization of 323 is $17 \cdot 19$. Thus the next divisor of $n$ is at least $323 + 17 = 340 = 17 \cdot 20$. Indeed, 340 will be the next number in Mary's list when $n = 17 \cdot 19 \cdot 20 = 6460$.

20. **Answer (C):** Let $O$ be the center of the regular hexagon. Points $B$, $O$, $E$ are collinear and $BE = BO + OE = 2$. Trapezoid $FABE$ is isosceles, and $\overline{XZ}$ is its midline. Hence $XZ = \frac{3}{2}$ and analogously $XY = ZY = \frac{3}{2}$.




Denote by $U_1$ the intersection of $\overline{AC}$ and $\overline{XZ}$ and by $U_2$ the intersection of $\overline{AC}$ and $\overline{XY}$. It is easy to see that $\triangle AXU_1$ and $\triangle U_2XU_1$ are congruent $30-60-90°$ right triangles.

By symmetry the area of the convex hexagon enclosed by the intersection of $\triangle ACE$ and $\triangle XYZ$, shaded in the figure, is equal to the area of $\triangle XYZ$ minus 3 times the area of $\triangle U_2XU_1$. The hypotenuse of $\triangle U_2XU_1$ is $XU_2 = AX = \frac{1}{2}$, so the area of $\triangle U_2XU_1$ is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{32}\sqrt{3}.$$

The area of the equilateral triangle $XYZ$ with side length $\frac{3}{2}$ is equal to $\frac{1}{4}\sqrt{3} \cdot \left(\frac{3}{2}\right)^2 = \frac{9}{16}\sqrt{3}$. Hence the area of the shaded hexagon is

$$\frac{9}{16}\sqrt{3} - 3 \cdot \frac{1}{32}\sqrt{3} = 3\sqrt{3}\left(\frac{3}{16} - \frac{1}{32}\right) = \frac{15}{32}\sqrt{3}.$$

**OR**

Let $U_1$ and $U_2$ be as above, and continue labeling the vertices of the shaded hexagon counterclockwise with $U_3$, $U_4$, $U_5$, and $U_6$ as shown. The area of $\triangle ACE$ is half the area of hexagon $ABCDEF$. Triangle $U_2U_4U_6$ is the midpoint triangle of $\triangle ACE$, so its area is $\frac{1}{4}$ of the area of $\triangle ACE$, and thus $\frac{1}{8}$ of the area of $ABCDEF$. Each of $\triangle U_2U_3U_4$, $\triangle U_4U_5U_6$, and $\triangle U_6U_1U_2$ is congruent to half of $\triangle U_2U_4U_6$, so the total shaded area is $\frac{5}{2}$ times the area of $\triangle U_2U_4U_6$ and therefore $\frac{5}{2}\cdot\frac{1}{8} = \frac{5}{16}$ of the area of $ABCDEF$. The area of $ABCDEF$ is $6\cdot\frac{\sqrt{3}}{4}\cdot 1^2$, so the requested area is $\frac{15}{32}\sqrt{3}$.

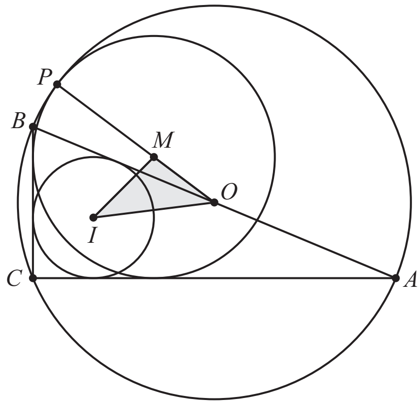
21. **Answer (E):** Place the figure on coordinate axes with coordinates $A(12,0)$, $B(0,5)$, and $C(0,0)$. The center of the circumscribed circle is the midpoint of the hypotenuse of right triangle $ABC$, so the coordinates of $O$ are $\left(6,\frac{5}{2}\right)$. The radius $r$ of the inscribed circle equals the area of the triangle divided by its semiperimeter, which here is $30 \div 15 = 2$, so the center of the inscribed circle is $I(2,2)$. Because the circle with center $M$ is tangent to both coordinate axes, its center has coordinates $(\rho,\rho)$, where $\rho$ is its radius. Let $P$ be the point of tangency of this circle and the circumscribed circle. Then $M$, $P$, and $O$ are collinear because $\overline{MP}$ and $\overline{OP}$ are perpendicular to the common tangent line at $P$. Thus $MO = OP - MP = \frac{13}{2} - \rho$. By the distance formula, $MO = \sqrt{(\rho-6)^2 + (\rho-\frac{5}{2})^2}$. Equating these expressions and solving for $\rho$ shows that $\rho = 4$. The area of $\triangle MOI$ can now be computed using the shoelace formula:

$$\left|\frac{4\cdot\frac{5}{2} + 6\cdot 2 + 2\cdot 4 - (4\cdot 6 + \frac{5}{2}\cdot 2 + 2\cdot 4)}{2}\right| = \frac{7}{2}.$$

Alternatively, the area can be computed as $\frac{1}{2}$ times $MI$, which by the distance formula is $\sqrt{(4-2)^2 + (4-2)^2} = 2\sqrt{2}$, times the distance from point $O$ to the line $MI$, whose equation is $x - y + 0 = 0$. This last value is

$$\frac{\left|1\cdot 6 + (-1)\cdot\frac{5}{2} + 0\right|}{\sqrt{1^2 + (-1)^2}} = \frac{7}{4}\sqrt{2},$$

so the area is $\frac{1}{2}\cdot(2\sqrt{2})\cdot\frac{7}{4}\sqrt{2} = \frac{7}{2}$.




22. **Answer (D):** Let $P(x) = ax^3 + bx^2 + cx + d$, where $a$, $b$, $c$, and $d$ are integers between 0 and 9, inclusive. The condition $P(-1) = -9$ is equivalent to $-a + b - c + d = -9$. Adding 18 to both sides gives $(9 - a) + b + (9 - c) + d = 9$ where $0 \le 9 - a, b, 9 - c, d \le 9$. By the stars and bars argument, there are $\binom{9+4-1}{4-1} = \binom{12}{3} = 220$ nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 = 9$. Each of these give rise to one of the desired polynomials.

**OR**

With the notation above, note that $(a+c) - (b+d) = 9$ can occur in several ways: $b+d = k$, $a+c = 9+k$ where $k = 0, 1, 2, \ldots, 9$. There are $k+1$ solutions to $b+d = k$ and $10-k$ solutions to $a+c = 9+k$ under the restrictions on $a$, $b$, $c$, and $d$, yielding $\sum_{k=0}^{9}(k+1)(10-k) = 220$ solutions in all.

23. **Answer (C):** To travel from $A$ to $B$, one could circle $135^\circ$ east along the equator and then $45^\circ$ north. Construct an $x$-$y$-$z$ coordinate system with origin at Earth's center $C$, the positive $x$-axis running through $A$, the positive $y$-axis running through the equator at $160^\circ$ west longitude, and the positive $z$-axis running through the North Pole. Set Earth's radius to be 1. The coordinates of $A$ are $(1, 0, 0)$. Let $b$ be the $y$-coordinate of $B$; note that $b > 0$. Then the $x$-coordinate of $B$ will be $-b$, and the $z$-coordinate will be $\sqrt{2}b$. Because the distance from the center of Earth is 1,

$$\sqrt{(-b)^2 + b^2 + \left(\sqrt{2}b\right)^2} = 1,$$

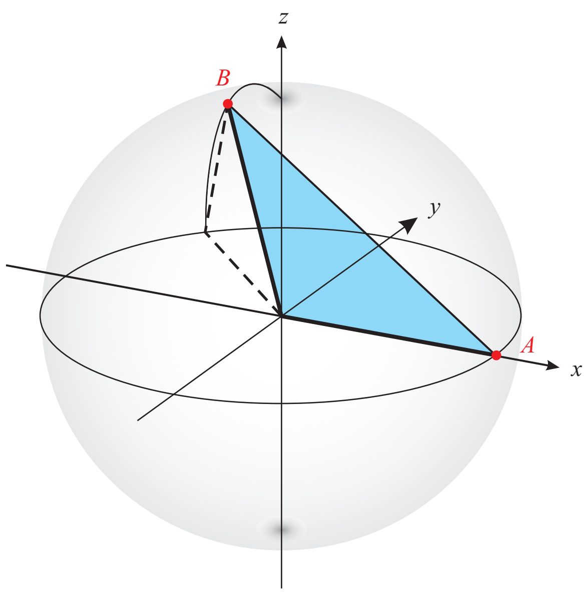
so $b = \frac{1}{2}$, and the coordinates are $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$. The distance $AB$ is therefore

$$\sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{3}.$$

Applying the Law of Cosines to $\triangle ACB$ gives

$$3 = 1 + 1 - 2 \cdot 1 \cdot 1 \cdot \cos\angle ACB,$$

so $\cos\angle ACB = -\frac{1}{2}$ and $\angle ACB = 120^\circ$. An alternative to using the Law of Cosines to find $\cos\angle ACB$ is to compute the dot product of the unit vectors $(1, 0, 0)$ and $\left(-\frac{1}{2}, \frac{1}{2}, \frac{\sqrt{2}}{2}\right)$.




24. **Answer (C):** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of $x$. Then $0 \le \{x\} < 1$. The given equation is equivalent to $x^2 = 10{,}000\{x\}$, that is,

$$\frac{x^2}{10{,}000} = \{x\}.$$

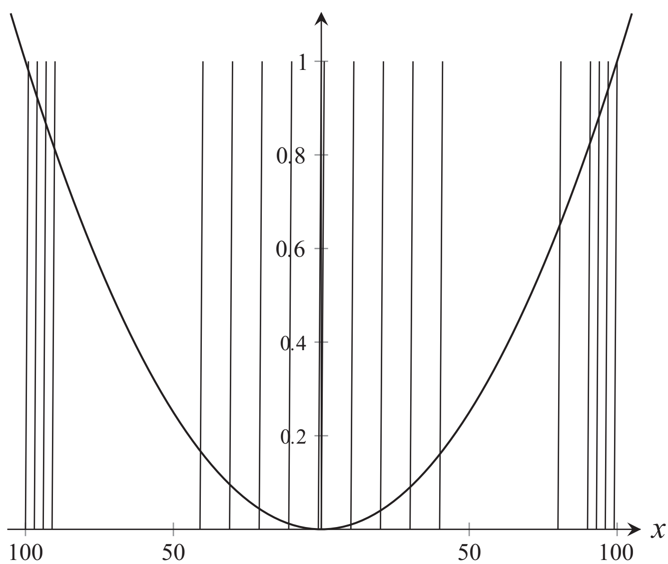
Therefore if $x$ satisfies the equation, then

$$0 \le \frac{x^2}{10{,}000} < 1.$$

This implies that $x^2 < 10{,}000$, so $-100 < x < 100$. The figure shows a sketch of the graphs of

$$f(x) = \frac{x^2}{10{,}000} \qquad \text{and} \qquad g(x) = \{x\}$$

for $-100 < x < 100$ on the same coordinate axes. The graph of $g$ consists of the 200 half-open line segments with slope 1 connecting the points $(k, 0)$ and $(k + 1, 1)$ for $k = -100, -99, \ldots, 98, 99$. (The endpoints of these intervals that lie on the $x$-axis are part of the graph, but the endpoints with $y$-coordinate 1 are not.) It is clear that there is one intersection point for $x$ lying in each of the intervals $[-100, -99)$, $[-99, -98)$, $[-98, -97)$, $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $\ldots$, $[97, 98)$, $[98, 99)$ but no others. Thus the equation has 199 solutions.




**OR**

The solutions to the equation correspond to points of intersection of the graphs $y = 10000\lfloor x \rfloor$ and $y = 10000x - x^2$. There will be a point of intersection any time the parabola intersects the half-open horizontal segment from the point $(a, 10000a)$ to the point $(a+1, 10000a)$, where $a$ is an integer. This occurs for every integer value of $a$ for which

$$10000a - a^2 \leq 10000a < 10000(a + 1) - (a + 1)^2.$$

This is equivalent to $(a+1)^2 < 10000$, which occurs if and only if $-101 < a < 99$. Thus points of intersection occur on the intervals $[a, a+1)$ for $a = -100, -99, -98, \ldots, -1, 0, 1, \ldots, 97, 98$, resulting in 199 points of intersection.

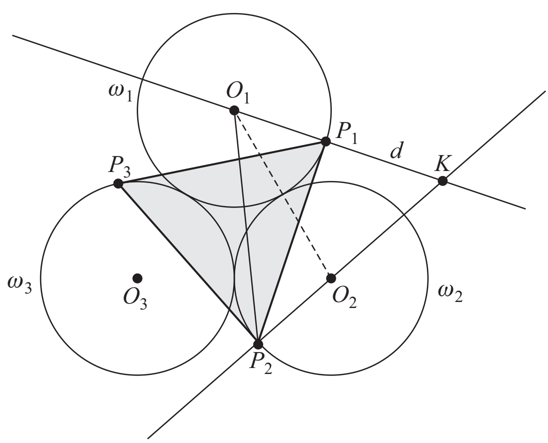
25. **Answer (D):** Let $O_i$ be the center of circle $\omega_i$ for $i = 1, 2, 3$, and let $K$ be the intersection of lines $O_1P_1$ and $O_2P_2$. Because $\angle P_1P_2P_3 = 60^\circ$, it follows that $\triangle P_2KP_1$ is a $30-60-90^\circ$ triangle. Let $d = P_1K$; then $P_2K = 2d$ and $P_1P_2 = \sqrt{3}d$. The Law of Cosines in $\triangle O_1KO_2$ gives

$$8^2 = (d+4)^2 + (2d-4)^2 - 2(d+4)(2d-4)\cos 60^\circ,$$

which simplifies to $3d^2 - 12d - 16 = 0$. The positive solution is $d = 2 + \frac{2}{3}\sqrt{21}$. Then $P_1P_2 = \sqrt{3}d = 2\sqrt{3} + 2\sqrt{7}$, and the required area is

$$\frac{\sqrt{3}}{4} \cdot \left(2\sqrt{3} + 2\sqrt{7}\right)^2 = 10\sqrt{3} + 6\sqrt{7} = \sqrt{300} + \sqrt{252}.$$

The requested sum is $300 + 252 = 552$.




Problems and solutions were contributed by David Altizio, Risto Atanasov, Chris Bolognese, Ivan Borsenco, Thomas Butts, Barbara Currier, Steven Davis, Steve Dunbar, Marta Eso, Zuming Feng, Zachary Franco, Peter Gilchrist, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Michael Khoury, Hugh Montgomery, Mohamed Omar, Albert Otto, David Wells, and Carl Yerger.

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