AMERICAN MATHEMATICS COMPETITIONS
51st Annual Mathematics Contest →12

AMC → 12
Solutions Pamphlet
TUESDAY, FEBRUARY 15, 2000

Sponsored by
Mathematical Association of America
Society of Actuaries Mu Alpha Theta
National Council of Teachers of Mathematics
Casualty Actuarial Society American Statistical Association
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
Consortium for Mathematics and its Applications
Pi Mu Epsilon National Association of Mathematicians
School Science and Mathematics Association
Clay Mathematics Institute University of Nebraska-Lincoln

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of calculus or a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC →12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, eMail, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Professor Vince Schielack, AMC →12 Chair
Department of Mathematics
Texas A & M University
College Station, TX 77843 USA

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. **Answer (E):** Factor 2001 into primes to get $2001 = 3 \cdot 23 \cdot 29$. The largest possible sum of three distinct factors whose product is the one which combines the two largest prime factors, namely $I = 23 \cdot 29 = 667$, $M = 3$, and $O = 1$, so the largest possible sum is $1 + 3 + 667 = 671$.

2. **Answer (A):** $2000(2000^{2000}) = (2000^1)(2000^{2000}) = 2000^{1+2000} = 2000^{2001}$. All the other options are greater than 2000^{2001}.

3. **Answer (B):** Since Jenny ate 20% of the jellybeans remaining each day, 80% of the jellybeans are left at the end of each day. If x is the number of jellybeans in the jar originally, then$(0.8)^2 x = 32$. Thus $x = 50$.

4. **Answer (C):** The sequence of units digits is

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots$$

   The digit 6 is the last of the ten digits to appear.

5. **Answer (C):** Since $x < 2$, it follows that $|x - 2| = 2 - x$. If $2 - x = p$, then $x = 2 - p$. Thus $x - p = 2 - 2p$.

6. **Answer (C):** There are five prime numbers between 4 and 18: 5,7,11,13, and 17. Hence the product of any two of these is odd and the sum is even. Because $xy - (x+y) = (x-1)(y-1) - 1$ increases as either x or y increases (since both x and y are bigger than 1), the answer must be an odd number that is no smaller than $23 = 5 \cdot 7 - (5 + 7)$ and no larger than $191 = 13 \cdot 17 - (13 + 17)$. The only possibility among the options is 119, and indeed $119 = 11 \cdot 13 - (11 + 13)$.

7. **Answer (E):** If $log_b 729 = n$, then $b^n = 729 = 3^6$, so n must be an integer factor of 6; that is, $n = 1$, 2, 3, or 6. Since $729 = 720^1 = 27^2 = 9^3 = 3^6$, the corresponding values of b are 3^6, 3^3, 3^2, and 3.

8. **Answer (C):** Calculating the number of squares in the first few figures uncovers a pattern. Figure 0 has $2(0)+1 = 2(0^2)+1$ squares, figure 1 has $2(1)+3 = 2(1^2)+3$ squares, figure 2 has $2(1+3)+5 = 2(2^2)+5$ squares, and figure 3 has $2(1+3+5)+7 = 2(3^2)+7$ squares. In general, the number of unit squares in figure n is

$$2(1+3+5+\cdots+(2n-1))+2n+1 = 2(n^2)+2n+1.$$

Therefore, the figure 100 has $2(100^2)+2\cdot 100+1 = 20201$.

OR

Each figure can be considered as a large square with identical small pieces deleted from each of the four corners. Figure 1 has $3^2-4(1)$ unit squares, figure 2 has $5^2-4(1+2)$ unit squares, and figure 3 has $7^2-4(1+2+3)$ unit squares. In general, figure n has

$$(2n-1)^2-4(1+2+\cdots+n) = (2n+1)^2-2n(n+1) \text{ unit squares.}$$

Thus figure 100 has $201^2-200(101) = 20201$ unit squares.

OR

The number of unit squares in figure n is the sum of the first n positive odd integers plus the sum of the first $n+1$ positive odd integers. Since the sum of the first k positive odd integers is k^2, figure n has $n^2+(n+1)^2$ unit squares. So figure 100 has $100^2+101^2 = 20201$ unit squares.

9. **Answer (C):** Note that the integer average condition means that the sum of the scores of the first n students is a multiple of n. The scores of the first two students must be both even or both odd, and the sum of the scores of the first three students must be divisible by 3. The remainders when 71, 76, 80, 82, and 91 are divided by 3 are 2, 1, 2, 1, and 1, respectively. Thus the only sum of three scores divisible by 3 is $76+82+91 = 249$, so the first two scores entered are 76 and 82 (in some order), and the third score is 91. Since 249 is 1 larger than a multiple of 4, the fourth score must be 3 larger than a multiple of 4, and the only possible is 71, leaving 80 as the score of the fifth student.

10. **Answer (E):** Reflecting the point $(1,2,3)$ in the xy-plane produces $(1,2,-3)$. A half turn about the x-axis yields $(1,-2,3)$. Finally, the translation gives $(1,3,3)$.

11. **Answer (E):** Find the common denominator and replace the ab in the numerator with $a-b$ to get

$$\begin{aligned}\frac{a}{b}+\frac{b}{a}-ab &= \frac{a^2+b^2-(ab)^2}{ab}\\ &= \frac{a^2+b^2-(a-b)^2}{ab}\\ &= \frac{a^2+b^2-(a^2-2ab+b^2)}{ab}\\ &= \frac{2ab}{ab}=2.\end{aligned}$$

OR

Note that $a=a/b-1$ and $b=1-b/a$. It follows that $\frac{a}{b}+\frac{b}{a}-ab=(a+1)+(1-b)-(a-b)=2$.

12. **Answer (E):** Note that

$$AMC+AM+MC+CA=(A+1)(M+1)(C+1)-(A+M+C)-1=pqr-13,$$

where p, q, and r are positive integers whose sum is 15. A case-by-case analysis shows that pqr is largest when $p=5$, $q=5$, and $r=5$. Thus the answer is $5\cdot5\cdot5-13=112$.

13. **Answer (C):** Suppose that the whole family drank x cups of milk and y cups of coffee. Let n denote the number of people in the family. The information given implies that $x/4+y/6=(x+y)/n$. This leads to

$$3x(n-4)=2y(6-n).$$

Since x and y are positive, the only positive integer n for which both sides have the same sign is $n=5$.

OR

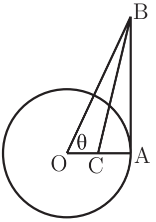
If Angela drank c cups of coffee and m cups of mile, then $0<c<1$ and $m+c=1$. The number of people in the family is $6c+4m=4+2c$, which is an integer if and only if $c=\frac{1}{2}$. Thus, there are 5 people in the family.

14. **Answer (E):** If x were less than or equal to 2, then 2 would be both the median and the mode of the list. Thus $x > 2$. Consider the two cases $2 < x < 4$, and $x \geq 4$.
Case 1: If $2 < x < 4$, then 2 is the mode, x is the median, and $\frac{25+x}{7}$ is the mean, which must equal $2-(x-2)$, $\frac{x+2}{2}$, or $x+(x-2)$, depending on the size of the mean relative to 2 and x. These give $x = \frac{3}{8}$, $x = \frac{36}{5}$, and $x = 3$, of which $x = 3$ is the only value between 2 and 4.
Case 2: If $x \geq 4$, then 4 is the median, 2 is the mode, and $\frac{25+x}{7}$ is the mean, which must be 0,3, or 6. Thus $x = -25$, -4, or 17, of which 17 is the only one of these values greater than or equal to 4.
Thus the x-value sum to $3 + 17 = 20$.

15. **Answer (B):** Let $x = 9z$. Then $f(3z) = f(9z/3) = f(3z) = (9z)^2 + 9z + 1 = 7$. Simplifying and solving the equation for z yields $81z^2 + 9z - 6 = 0$, so $3(3z+1)(9z-2) = 0$. Thus $z = -1/3$ or $z = 2/9$. The sum of these values is $-1/9$.
Note. The answer can also be obtained by using the sum-of-roots formula on $81z^2 + 9z - 6 = 0$. The sum of the roots is $-9/81 = -1/9$.

16. **Answer (D):** Suppose each square is identified by an ordered pair (m, n), where m is the row and n is the column in which it lies. In the original system, each square (m, n) has the number $17(m-1) + n$ assigned; in the renumbered system, it has the number $13(n-1) + m$ assigned to it. Equating the two expressions yields $4m - 3n = 1$, whose acceptable solutions are $(1, 1)$, $(4, 5)$, $(7, 9)$, $(10, 13)$, and $(13, 17)$. These squares are numbered 1, 56, 111, 166 and 221, respectively, and the sum is 555.

17. **Answer (D):** The fact that $OA = 1$ implies that $BA = \tan\theta$ and $BO = \sec\theta$. Since $\overline{BC}$ bisects $\angle ABP$, it follows that $\frac{OB}{BA} = \frac{OC}{CA}$, which implies $\frac{OB}{OB+BA} = \frac{OC}{OC+CA} = OC$. Substituting yields



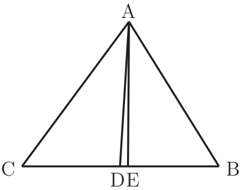
$$OC = \frac{\sec\theta}{\sec\theta + \tan\theta} = \frac{1}{1+\sin\theta}.$$

OR

Let $\alpha = \angle CBO = \angle ABC$. Using the *Law of Sines* on triangle BCO yields $\frac{\sin\theta}{BC} = \frac{\sin\alpha}{OC}$, so $OC = \frac{BC\sin\alpha}{\sin\theta}$. In right triangle ABC, $\sin\alpha = \frac{1-OC}{BC}$. Hence $OC = \frac{1-OC}{\sin\theta}$. Solving this for OC yields $OC = \frac{1}{1+\sin\theta}$.

18. **Answer (A):** Note that, if a Tuesday is d days after a Tuesday, then d is a multiple of 7. Next, we need to consider whether any of the years $N-1$, N, $N+1$ is a leap year. If N is not a leap year, the 200^{th} day of year $N+1$ is $365 - 300 + 200 = 265$ days after a Tuesday, and thus is a Monday, since 265 if 6 larger than a multiple of 7. Thus, year N is a leap year and the 200^{th} day of year $N+1$ is another Tuesday (as given), being 266 days after a Tuesday. It follows that year $N-1$ is not a leap year. Therefore, the 100^{th} day of year $N-1$ precedes the given Tuesday in year N by $365-100+300 = 565$ days, and therefore is a Thursday, since $565 = 7\cdot 80 + 5$ is 5 larger than a multiple of 7.

19. **Answer (B):** By Heron's Formula the area of triangle ABC is $\sqrt{(21)(8)(7)(6)}$, which is 84, so the altitude from vertex A is $2(84)/14 = 12$. The midpoint D divides $\overline{BC}$ into two segments of length 7, and the bisector of angle BAC divides $\overline{BC}$ into segments of length $14(13/28) = 6.5$ and $14(15/28) = 7.5$ (since the angle bisector divides the opposite side into lengths proportional to the remaining two sides). Thus the triangle ADE has base $DE = 7 - 6.5 = 0.5$ and altitude 12, so its area is 3.



20. **Answer (B):** Note that $(x+1/y)+(y+1/z)+(z+1/x) = 4+1+7/3 = 22/3$ and that

$$\begin{aligned} 28/3 = 4\cdot 1\cdot 7/3 &= (x+1/y)(y+1/z)(z+1/x) \\ &= xyz + x + y + z + 1/x + 1/y + 1/z + 1/(xyz) \\ &= xyz + 22/3 + 1/(xyz). \end{aligned}$$

It follows that $xyz + 1/(xyz) = 2$ and $(xyz-1)^2 = 0$. Hence $xyz = 1$.

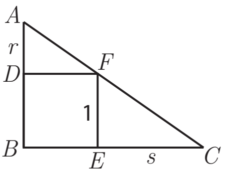
OR

By substitution,

$$4 = x + \frac{1}{y} = x + \frac{1}{1-1/z} = x + \frac{1}{1-3x/(7x-3)} = x + \frac{7x+3}{4x-3}.$$

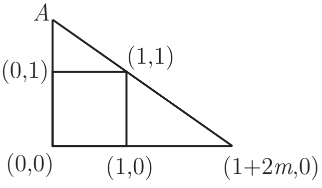
Thus $4(4x-3) = x(4x-3) + 7x - 3$, which simplifies to $(2x-3)^2 = 0$. Accordingly, $x = 3/2$, $z = 7/3 - 2/3 = 5/3$, and $y = 1 - 3/5 = 2/5$, so $xyz = (3/2)(2/5)(5/3) = 1$.

21. **Answer (D):** With out loss of generality, let the side of the square have length 1 unit and let the area of triangle ADF be m. Let $AD = r$ and $EC = s$. Because triangles ADF and FEC are similar, $s/1 = 1/r$. Since $\frac{1}{2}r = m$, the area of triangle FEC is $\frac{1}{2}s = \frac{1}{2r} = \frac{1}{4m}$.

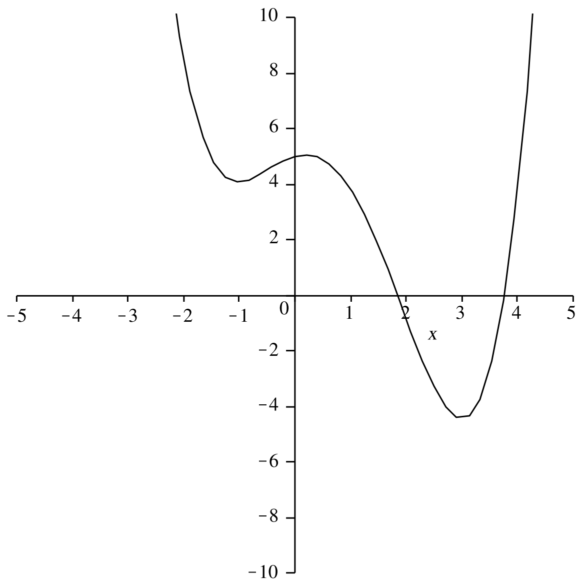


OR

Let $B = (0,0)$, $E = (1,0)$, $F = (1,1)$ and $D = (0,1)$ be the vertices of the square. Let $C = (1+2m, 0)$, and notice that the area of $BEFD$ is 1 and the area of triangle FEC is m. The slope of the line through C and F is $-\frac{1}{2m}$; thus, it intersects the y-axis at $A = \left(0, 1 + \frac{1}{2m}\right)$. The area of triangle ADF is therefore $\frac{1}{4m}$.



22. **Answer (C):** First note that the quartic polynomial can have no more real zeros than the two shown. (If it did, the quartic $P(x)-5$ would have more than four zeros.) The sum of the coefficients of P is $P(1)$, which is greater than 3. The product of all zeros of P is the constant term of the polynomial, which is the y-intercept, which is greater than 5. The sum of the real zeros of P (the sum of the x-intercepts) is greater than 4.5, and $P(-1)$ is greater than 4. However, since the product of the real zeros of P is greater than 4.5 and the product of all the zeros is less than 6, it follows that the product of all the zeros is less than 6, it follows that the product of the non-real zeros of P is less than 2, making it the smallest of the numbers.



23. **Answer (B):** In order for the sum of the logarithms of six numbers to be an integer k, the product of the numbers must be 10^k. The only prime factors of 10 are 2 and 5, so the six integers must be chosen from the list 1, 2, 4, 5, 8, 10, 16, 20, 25, 32, 40. For each of these, subtract the number of times that 5 occurs as a factor from the number of times 2 occurs as a factor. This yields the list 0, 1, 2, -1, 3, 0, 4, 1, -2, 5, 2. Because 10^k has just as many factors of 2 as it has of 5, the six chosen integers must correspond to six integers in the latter list that sum to 0. Two of the numbers must be -1 and -2, because there are only two zeros in the list, and no number greater than 2 can appear in the sum, which must therefore be $(-1)+(-1)+0+0+1+2=0$. It follows that Professor Gamble chose 25, 5, 1, 10, one number from {2,20}, and one number from {4,40}. There are four possible tickets Professor Gamble could have bought and only one is a winner, so the probability that Professor Gamble wins the lottery is 1/4.

OR

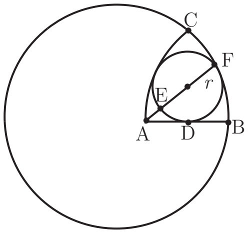
As before, the six integers must be chosen from the set $S = \{1, 2, 4, 5, 6, 10, 16, 20, 25, 32, 40\}$. The product of the smallest six numbers in S is 3,200 $> 10^3$, so the product of the numbers on the ticket must be 10^k for some $k \geq 4$. On the other hand, there are only six factors of 5 available among the numbers in S, so the product p can only be 10^4, 10^5, or 10^6.

Case 1, $p = 10^6$. There is only one way to produce 10^6, since all six factors of 5 must be used and their product is already 10^6, leaving 1 as the other number: 1, 5, 10, 20, 25, 40.

Case 2, $p = 10^5$. To produce a product of 10^5 we must use six numbers that include five factors of 5 and five factors of 2 among them. We cannot use both 20 and 40, because these numbers combine to give five factors of 2 among them and the other four numbers would have to be odd (whereas there are only three odd numbers in S). If we omit 40, we must include the other multiples of 5 $(5, 10, 20, 25)$ plus two numbers whose product is 4 (necessarily 1 and 4). If we omit 20, we must include 5, 10, 25, and 40, plus two numbers with a product of 2 (necessarily 1 and 2).

Case 3, $p = 10^4$. To produce a product of 10^4 we must use six numbers that include four factors of 5 and four factors of 2 among them. So that there are only four factors of 2, we must include 1, 5, 25, 2, and 10. These include two factors of 2 and four factors of 5, so the sixth number must contain two factors of 2 and no 5's, so must be 4.

Thus there are four lottery tickets whose numbers have base-ten logarithms with an integer sum: {1, 5, 10, 20, 25, 40}, {1, 2, 5, 10, 25, 40}, {1, 2, 4, 5, 10, 25}, and {1, 4, 5, 10, 20, 25}. Professor Gamble has a 1/4 probability of being a winner.

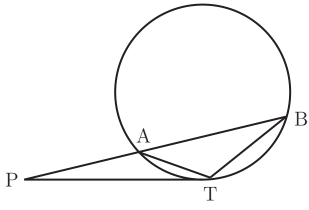


24. **Answer (D):** Construct the circle with center A and radius AB. Let F be the point of tangency of the two circles. Draw $\overline{AF}$, and let E be the point of intersection of $\overline{AF}$ and the given circle. By the *Power of a Point Theorem*, $AD^2 = AF \cdot AE$ (see Note below). Let r be the radius of the smaller circle. Since $\overline{AF}$ and $\overline{AB}$ are radii of the larger circle, $AF = AB$ and $AE = AF - EF = AB - 2r$. Because $AD = AB/2$, substitution into the first equation yields

$$(AB/2)^2 = AB \cdot (AB - 2r),$$

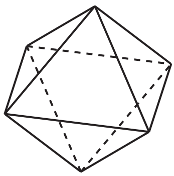
or, equivalently, $\frac{r}{AB} = \frac{3}{8}$. Points A, B, and C are equidistant from each other, so $\overset{\frown}{BC} = 60^\circ$ and thus the circumference of the larger circle is $6 \cdot (\text{length of } \overset{\frown}{BC}) = 6 \cdot 12$. Let c be the circumference of the smaller circle. Since the circumferences of the two circles are in the same ratio as their radii, $\frac{c}{72} = \frac{r}{AB} = \frac{3}{8}$. Therefore $c = \frac{3}{8}(72) = 27$.

Note. From any exterior point P, a secant PAB and a tangent PT are drawn. Consider triangles PAT and PTB. They have a common angle P. Since angles ATP and PBT intercept the same arc $\overset{\frown}{AT}$, they are congruent. Therefore triangles PAT and PTB are similar, and it follows that $PA/PT = PT/PB$ and $PA \cdot PB = PT^2$. The number PT^2 is called *the power of the point* P with respect to the circle. Intersecting secants, tangents, and chords, paired in any manner create various cases of this theorem, which is sometimes called *Crossed Chords.*



25. **Answer (E):** The octahedron has 8 congruent equilateral triangular faces that form 4 pairs of parallel faces. Choose one color for the bottom face. There are 7 choices for the color of the top face. Three of the remaining faces have an edge in common with the bottom face. There are $\binom{6}{3} = 20$ ways of choosing the colors for these faces and two ways to arrange these on the three faces (clockwise and counterclockwise). Finally, there are $3! = 6$ ways to fix the last three colors. Thus the total number of distinguishable octahedrons that can be constructed is $7 \cdot 20 \cdot 2 \cdot 6 = 1680$.

OR



Place a cube inside the octahedron so that each of its vertices touches a face of the octahedron. Then assigning colors to the faces of the octahedron is equivalent to assigning colors to the vertices of the cube. Pick one vertex and assign it a color. Then the remaining colors can be assigned in 7! ways. Since three vertices are joined by edges to the first vertex, they are interchangeable by a rotation of the cube, hence the answer is $7!/3 = 1680$.

American Mathematics Competitions

52st Annual Mathematics Contest 12

AMC 12
Solutions Pamphlet

TUESDAY, FEBRUARY 13, 2001

Sponsored by

Mathematical Association of America

University of Nebraska

American Statistical Association
Casualty Actuarial Society
Society of Actuaries
National Council of Teachers of Mathematics
American Society of Pension Actuaries
American Mathematical Society
American Mathematical Association of Two Year Colleges
Pi Mu Epsilon
Consortium for Mathematics and its Applications
Mu Alpha Theta
National Association of Mathematicians
Kappa Mu Epsilon
School Science and Mathematics Association
Clay Mathematics Institute

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of calculus or a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Professor Vince Schielack, AMC 12 Chair
Department of Mathematics
Texas A & M University
College Station, TX 77843 USA

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

A03t-200112

1. **(E)** Suppose the two numbers are a and b. Then the desired sum is

$$2(a_3) + 2(b + 3) = 2(a + b) + 12 = 2S + 12.$$

2. **(E)** Suppose $N = 10a + b$. Then $10a + b = ab + (a + b)$. It follows that $9a = ab$, which implies that $b = 9$, since $a \neq 0$.

3. **(B)** If Kristin's annual income is $x \geq 28{,}000$ dollars, then

$$\frac{p}{100} \cdot 28{,}000 + \frac{p+2}{100} \cdot (x - 28{,}000) = \frac{p + 0.25}{100} \cdot x.$$

Multiplying by 100 and expanding yields

$$28{,}000p + px + 2x - 28{,}000p - 56{,}000 = px + 0.25x.$$

So, $1.75x = \frac{7}{4}x = 56{,}000$ and $x = 32{,}000$.

4. **(D)** Since the median is 5, we can write the three numbers as x, 5, and y, where

$$\frac{1}{3}(x + 5 + y) = x + 10 \text{ and } \frac{1}{3}(x + 5 + y) + 15 = y.$$

If we add these equations, we get

$$\frac{2}{3}(x + 5 + y) + 15 = x + y + 10$$

and solving for $x+y$ gives $x+y = 25$. Hence the sum of the numbers $x+y+5 = 30$.

OR

Let m be the mean of the three numbers. Then the least of the numbers is $m - 10$ and the greatest is $m + 15$. The middle of the three numbers is the median, 5. So

$$\frac{1}{3}((m - 10) + 5 + (m + 15)) = m$$

and $m = 10$. Hence, the sum of the three numbers is $3(10) = 30$.

5. **(D)** Note that

$$1 \cdot 3 \cdots 9999 = \frac{1 \cdot 2 \cdot 3 \cdots 9999 \cdot 10000}{2 \cdot 4 \cdots 10000} = \frac{10000!}{2^{5000} \cdot 1 \cdot 2 \cdots 5000} = \frac{10000!}{2^{5000} \cdot 5000!}$$

6. **(E)** The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is 810-642-9753.

7. **(A)** Let n be the number of full-price tickets and p be the price of each in dollars. Then

$$np + (140 - n) \cdot \frac{p}{2} = 2001, \text{ so } p(n + 140) = 4002.$$

Thus $n + 140$ must be a factor of $4002 = 2 \cdot 3 \cdot 23 \cdot 29$. Since $0 \le n \le 140$, we have $140 \le n + 140 \le 280$, and the only factor of 4002 that is in the required range for $n + 140$ is $174 = 2 \cdot 3 \cdot 29$. Therefore, $n + 140 = 174$, so $n = 34$ and $p = 23$. The money raised by the full-price tickets is $34 \cdot 23 = 782$ dollars.

8. **(C)** The slant height of the cone is 10, the radius of the sector. The circmference of the base of the cone is the same as the length of the secotr's arc. This is $252/360 = 7/10$ of the circumference, 20π, of the circle from which the sector is cut. The base circumference of the cone is 14π, so its radius is 7.

9. **(C)** Note that

$$f(600) = f(500 \cdot \frac{6}{5}) = \frac{f(500)}{6/5} = \frac{3}{6/5} = \frac{5}{2}.$$

OR

For all positive x,

$$f(x) = f(1 \cdot x) = \frac{f(1)}{x},$$

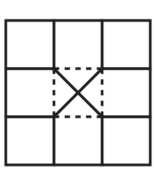
so $xf(x)$ is the constant $f(1)$. Therefore,

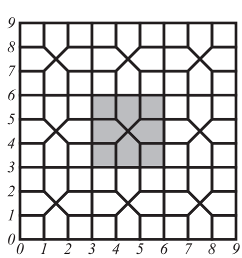
$$600f(600) = 500f(500) = 500(3) = 1500,$$

so $f(600) = \frac{1500}{600} = \frac{5}{2}$. Note. $f(x) = \frac{1500}{x}$ is the unique function satisfying the given conditions.

10. **(D)** The pattern shown at left is repeated in the plane. In fact, nine repetitions of it are shown in the statement of the problem. Note that four of the nine squres in the three-by-three square are not in the four pentagons that make up the three-by-three square. Therefore, the percentage of the plane that is enclosed by pentagons is

$$1 - \frac{4}{9} = \frac{5}{9} = 55\frac{5}{9}\%$$





11. **(D)** Think of continuing the drawing until all five chips are removed form the box. There are ten possible orderings of the colors: RRRWW, RRWRW, RWRRW, WRRRW, RRWWR, RWRWR, WRRWR, RWWRR, WRWRR, and WWRRR. The six orderings that end in R represent drawings that would have ended when the second white chip was drawn.

OR

Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability 3/5.

12. **(B)** For integers not exceeding 2001, there are $\lfloor 2001/3 \rfloor = 667$ multiples of 3 and $\lfloor 2001/4 \rfloor = 500$ multiples of 4. The total, 1167, counts the $\lfloor 2001/12 \rfloor = 166$ multiples of 12 twice, so there are $1167 - 166 = 1001$ multiples of 3 or 4. From these we exclude the $\lfloor 2001/15 \rfloor = 133$ multiples of 15 and the $\lfloor 2001/20 \rfloor = 100$ multiples of 20, since these are multiples of 5. However, this excludes the $\lfloor 2001/60 \rfloor = 33$ multiples of 60 twice, so we must re-include these. The number of integers satisfying the conditions is $1001 - 133 - 100 + 33 = 801$.

13. **(E)** The equation of the first parabola can be written in the form

$$y = a(x-h)^2 + k = ax^2 - 2axh + ah^2 + k,$$

and the equation for the second (having the same shape and vertex, but opening in the opposite direction) can be written in the form

$$y = -a(x-h)^2 + k = -ax^2 + 2axh - ah^2 + k.$$

Hence,

$$a+b+c+d+e+f = a + (-2ah) + (ah^2+k) + (-a) + (2ah) + (-ah^2+k) = 2k.$$

OR

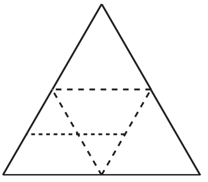
The reflection of a point (x, y) about the line $y = k$ is $(x, 2k - y)$. Thus, the equation of the reflected parabola is

$$2k - y = ax^2 + bx + c, \text{ or equivalently, } y = 2k - (ax^2 + bx + c).$$

Hence $a + b + c + d + e + f = 2k$.

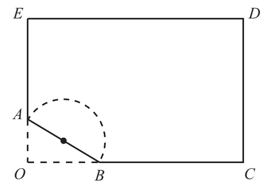
14. **(D)** Each of the $\binom{9}{2} \equiv 9C2 = 36$ pairs of vertices determines two equilateral triangles, for a total of 72 triangles. However, the three triangles $A_1A_4A_7$, $A_2A_5A_8$, and $A_3A_6A_9$ are each counted 3 times, resulting in an overcount of 6. Thus, there are 66 distinct equilateral triangles.

15. **(B)** Unfold the tetrahedron onto a plane. The two opposite-edge midpoints become the midpoints of opposite sides of a rhombus with sides of length 1, so are now 1 unit apart. Folding back to a tetrahedron does not change the distance and it remains minimal.



16. **(D)** Number the spider's legs from 1 through 8, and let a_k and b_k denote the sock and shoe that will go on leg k. A possible arrangement of the socks and shoes is a permutation of the sixteen symbols $a_1, b_1, \ldots a_8, b_8$, in which a_k precedes b_k for $1 \le k \le 8$. There are 16! permutations of the sixteen symbols, and a_1 precedes b_1 in exactly half of these, or $16!/2$ permutations. Similarly, a_2 precedes b_2 in exactly half of those, or $16!/2^2$ permutations. Continuing, we can conclude that a_k precedes b_k for $1 \le k \le 8$ in exactly $16!/2^8$ permutations.

17. **(C)** Since $\angle APB = 90°$ if and only if P lies on the semicircle with center $(2,1)$ and radius $\sqrt{5}$, the angel is obtuse if and only if the point P lies inside this semicircle. The semicircle lies entirely inside the pentagon, since the distance, 3, from $(2,1)$ to $\overline{DE}$ is greater than the radius of the circle. Thus the probability that the nagle is obtuse is the ratio of the area of the semicircle to the area of the pentagon.



Let $O = (0,0)$, $A = (0,2)$, $B = (4,0)$, $C = (2\pi+1,0)$, $D = (2\pi+1,4)$, and $E = (0,4)$. Then the area of the pentagon is

$$[ABCDE] = [OCDE] - [OAB] = 4 \cdot (2\pi + 1) - \frac{1}{2}(2 \cdot 4) = 8\pi,$$

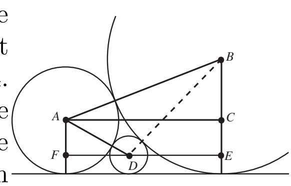
and the area of the semicircle is

$$\frac{1}{2}\pi(\sqrt{5})^2 = \frac{5}{2}\pi.$$

The probability is

$$\frac{\frac{5}{2}\pi}{8\pi} = \frac{5}{16}.$$

18. **(D)** Let C be the intersection of the horizontal line through A and the vertical line through B. In right triangle ABC, we have $BC = 3$ and $AB = 5$, so $AC = 4$. Let x be the radius of the third circle, and D be the center. Let E and F be the points of intersection of the horizontal line through D with the vertical lines through B and A, respectively, as shown.



In $\triangle BED$ we have $BD = 4 + x$ and $BE = 4 - x$, so

$$DE^2 = (4+x)^2 - (4-x)^2 = 16x,$$

and $DE = 4\sqrt{x}$. In $\triangle ADF$ we have $AD = 1 + x$ and $AF = 1 - x$, so

$$FD^2 = (1+x)^2 - (1-x)^2 = 4x,$$

and $FD = 2\sqrt{x}$. Hence,

$$4 = AC = FD + DE = 2\sqrt{x} + 4\sqrt{x} = 6\sqrt{x}$$

and $\sqrt{x} = \frac{2}{3}$, which implies $x = \frac{4}{9}$.

19. **(A)** The sum and product of the zeros of $P(x)$ are $-a$ and $-c$, respectively. Therefore,

$$-\frac{a}{3} = -c = 1 + a + b + c.$$

Since $c = P(0)$ is the y-intercept of $y = P(x)$, it follows that $c = 2$. Thus $a = 6$ and $b = -11$.

20. **(C)** Let the midpoints of sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ be denoted M,N,P,and Q, respectively. Then $M = (2, 5)$ and $N = (3, 2)$. Since $\overline{MN}$ has slope -3, the slope of $\overline{MQ}$ must be $1/3$, and $MQ = MN = \sqrt{10}$. An equation for the line containing $\overline{MQ}$ is thus $y-5 = \frac{1}{3}(x-2)$, or $y = (x+13)/3$. So Q has coordinates of the form $(a, \frac{1}{3}(a + 13))$. Since $MQ = \sqrt{10}$, we have

$$\begin{aligned}
(a-2)^2 + (\frac{a+13}{3} - 5)^2 &= 10 \\
(a-2)^2 + (\frac{a-2}{3})^2 &= 10 \\
\frac{10}{9}(a-2)^2 &= 10 \\
(a-2)^2 &= 9 \\
a - 2 &= \pm 3
\end{aligned}$$

Since Q is in the first quadrant, $a = 5$ and $Q = (5, 6)$. Since Q is the midpoint of $\overline{AD}$ and $A = (3, 9)$, we have $D = (7, 3)$, and $7 + 3 = 10$.

OR

Use translation vectors. As before, $M = (2, 5)$ and $N = (3, 2)$. So $\overrightarrow{NM} = \langle -1, 3\rangle$. The vector $\overrightarrow{MQ}$ must have the same length as $\overrightarrow{MN}$ and be perpendicular to it, so $\overrightarrow{MQ} = \langle 3, 1\rangle$. Thus, $Q = (5, 6)$. As before, $D = (7, 3)$, and the answer is 10.

OR

Each pair of opposite sides of the square are parallel to a diagonal of $ABCD$, so the diagonals of $ABCD$ are perpendicular. Similarly, each pair of opposite sides of the square has length half that of a diagonal, so the diagonals of $ABCD$ are congruent. Since the slope of $\overline{AC}$ is -3 and $\overline{AC}$ is perpendicular to $\overline{BD}$, we have

$$\frac{b-1}{a-1} = \frac{1}{3}, \text{ so } a - 1 = 3(b-1).$$

Since $AC = BD$,

$$40 = (a-1)^2 + (b-1)^2 = 9(b-1)^2 + (b-1)^2 = 10(b-1)^2,$$

and since b is positive, $b = 3$ and $a = 1 + 3(b - 1) = 7$. So the answer is 10.

21. **(D)** Note that

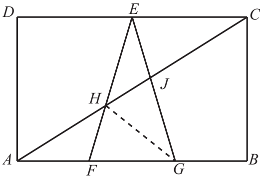
$$(a+1)(b+1) = ab+a+b+1 = 524+1 = 525 = 3 \cdot 5^2 \cdot 7,$$

and

$$(b+1)(c+1) = bc+b+c+1 = 146+1 = 147 = 3 \cdot 7^2.$$

Since $(a+1)(b+1)$ is a multiple of 25 and $(b+1)(c+1)$ is not a multiple of 5, it follows that $a+1$ must be a multiple of 25. Since $a+1$ divides 525, a is one of 24, 74, 174, or 524. Among these only 24 is a divisor of 8!, so $a = 24$. This implies that $b+1 = 21$, and $b = 20$. From this it follows that $c+1 = 7$ and $c = 6$. Finally, $(c+1)(d+1) = 105 = 3 \cdot 5 \cdot 7$, so $d+1 = 15$ and $d = 14$. Therefore, $a - d = 24 - 14 = 10$.

22. **(C)** The area of triangle EFG is $(1/6)(70) = 35/3$. Triangles AFH and CEH are similar, so $3/2 = EC/AF = EH/HF$ and $EH/EF = 3/5$. Triangles AGJ and CEJ are similar, so $3/4 = EC/AG = EJ/JG$ and $EJ/EG = 3/7$.



Since the areas of the triangles that have a common altitude are proportional to their bases, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EHG$ is $3/7$, and the ratio of the area of $\triangle EHG$ to that of $\triangle EFG$ is $3/5$. Therefore, the ratio of the area of $\triangle EHJ$ to the area of $\triangle EFG$ is $(3/5)(3/7) = 9/35$. Thus, the area of $\triangle EHJ$ is $(9/35)(35/3) = 3$.

23. **(A)** If r and s are the integer zeros, the polynomial can be written in the form

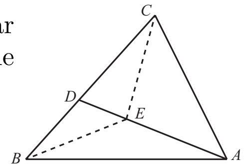
$$P(x) = (x-r)(x-s)(x^2+\alpha x+\beta).$$

The coefficient of x^3, $\alpha - (r+s)$, is an integer, so α is an integer. The coefficient of x^2, $\beta - \alpha(r+s) + rs$, is an integer, so β is also an integer. Applying the quadratic formula gives the remaining zeros as

$$\frac{1}{2}(-\alpha \pm \sqrt{\alpha^2 - 4\beta}) = -\frac{\alpha}{2} \pm i\frac{\sqrt{4\beta-\alpha^2}}{2}.$$

Answer choices (A), (B), (C), and (E) require that $\alpha = -1$, which implies that the imaginary parts of the remaining zeros have the form $\pm\sqrt{4\beta-1}/2$. This is true only for choice (A).
Note that choice (D) is not possible since this choice requires $\alpha = -2$, which produces an imaginary part of the form $\sqrt{\beta-1}$, which cannot be $\frac{1}{2}$.

24. **(D)** Let E be a point on $\overline{AD}$ such that $\overline{CE}$ is perpendicular to $\overline{AD}$, and draw $\overline{BE}$. Since $\angle ADC$ is an exterior angle of $\triangle ADB$ it follows that



$$\angle ADC = \angle DAB + \angle ABD = 15^\circ + 45^\circ = 60^\circ.$$

Thus, $\triangle CDE$ is a $30^\circ - 60^\circ - 90^\circ$ triangle and $DE = \frac{1}{2}CD = BD$. Hence, $\triangle BDE$ is isosceles and $\angle EBD = \angle BED = 30^\circ$. But $\angle ECB$ is also equal to 30° and therefore $\triangle BEC$ is isosceles with $BE = EC$. On the other hand,

$$\angle ABE = \angle ABD - \angle EBD = 45^\circ - 30^\circ = 15^\circ = \angle EAB.$$

Thus, $\triangle ABE$ is isosceles with $AE = BE$. Hence $AE = BE = EC$. The right triangle AEC is also isosceles with $\angle EAC = \angle ECA = 45^\circ$. Hence,

$$\angle ACB = \angle ECA + \angle ECD = 45^\circ + 30^\circ = 75^\circ.$$

25. **(D)** If a, b, and c are three consecutive terms of such a sequence, then $ac-1 = b$, which can be rewritten as $c = (1+b)/a$. Applying this rule recursively and simplifying yields

$$\ldots, a, b, \frac{1+b}{a}, \frac{1+a+b}{ab}, \frac{1+a}{b}, a, b, \ldots$$

This shows that at most five different terms can appear in such a sequence. Moreover, the value of a is determined once the value 2000 is assigned to b and the value 2001 is assigned to another of the first five terms. Thus, there are four such sequences that contain 2001 s a term, namely

$$2001, 2000, 1, \frac{1}{1000}, \frac{1001}{1000}, 2001, \ldots,$$
$$1, 2000, 2001, \frac{1001}{1000}, \frac{1}{1000}, 1, \ldots,$$
$$\frac{2001}{4001999}, 2000, 4001999, 2001, \frac{2002}{4001999}, \frac{2001}{4001999}, \ldots, \text{ and}$$
$$4001999, 2000, \frac{2001}{4001999}, \frac{2002}{4001999}, 2001, 4001999, \ldots,$$

respectively. The four values of x are 2001, 1, $\frac{2001}{4001999}$, and 4001999.

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

The Akamai Foundation

53rd Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 12, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(A)** Factor to get $(2x+3)(2x-10) = 0$, so the two roots are $-3/2$ and 5, which sum to $7/2$.

2. **(A)** Let x be the number she was given. Her calculations produce

$$\frac{x-9}{3} = 43,$$

so

$$x - 9 = 129 \quad \text{and} \quad x = 138.$$

The correct answer is

$$\frac{138-3}{9} = \frac{135}{9} = 15.$$

3. **(B)** No matter how the exponentiations are performed, 2^{2^2} always gives 16. Depending on which exponentiation is done last, we have

$$\left(2^{2^2}\right)^2 = 256, \quad 2^{\left(2^{2^2}\right)} = 65,536, \quad \text{or} \quad \left(2^2\right)^{\left(2^2\right)} = 256,$$

so there is one other possible value.

4. **(B)** The appropriate angle x satisfies

$$90 - x = \frac{1}{4}(180 - x), \quad \text{so} \quad 360 - 4x = 180 - x.$$

Solving for x gives $3x = 180$, so $x = 60$.

5. **(C)** The large circle has radius 3, so its area is $\pi \cdot 3^2 = 9\pi$. The seven small circles have a total area of $7\left(\pi \cdot 1^2\right) = 7\pi$. So the shaded region has area $9\pi - 7\pi = 2\pi$.

6. **(E)** When $n = 1$, the inequality becomes $m \le 1 + m$, which is satisfied by all integers m. Thus, there are infinitely many of the desired values of m.

7. **(A)** Let $C_A = 2\pi R_A$ be the circumference of circle A, let $C_B = 2\pi R_B$ be the circumference of circle B, and let L the common length of the two arcs. Then

$$\frac{45}{360} C_A = L = \frac{30}{360} C_B.$$

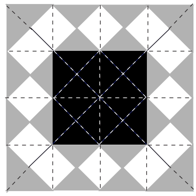
Therefore

$$\frac{C_A}{C_B} = \frac{2}{3} \quad \text{so} \quad \frac{2}{3} = \frac{2\pi R_A}{2\pi R_B} = \frac{R_A}{R_B}.$$

Thus, the ratio of the areas is

$$\frac{\text{Area of Circle } (A)}{\text{Area of Circle } (B)} = \frac{\pi R_A^2}{\pi R_B^2} = \left(\frac{R_A}{R_B}\right)^2 = \frac{4}{9}.$$

8. **(A)** Draw additional lines to cover the entire figure with congruent triangles. There are 24 triangles in the blue region, 24 in the white region, and 16 in the red region. Thus, $B = W$.



9. **(B)** First note that the amount of memory needed to store the 30 files is

$$3(0.8) + 12(0.7) + 15(0.4) = 16.8 \text{ mb},$$

so the number of disks is at least

$$\frac{16.8}{1.44} = 11 + \frac{2}{3}.$$

However, a disk that contains a 0.8-mb file can, in addition, hold only one 0.4-mb file, so on each of these disks at least 0.24 mb must remain unused. Hence, there is at least $3(0.24) = 0.72$ mb of unused memory, which is equivalent to half a disk. Since

$$\left(11 + \frac{2}{3}\right) + \frac{1}{2} > 12,$$

at least 13 disks are needed.
To see that 13 disks suffice, note that:
Six disks could be used to store the 12 files containing 0.7 mb;
Three disks could be used to store the three 0.8-mb files together with three of the 0.4-mb files;
Four disks could be used to store the remaining twelve 0.4-mb files.

10. **(D)** After the first transfer, the first cup contains two ounces of coffee, and the second cup contains two ounces of coffee and four ounces of cream. After the second transfer, the first cup contains $2 + (1/2)(2) = 3$ ounces of coffee and $(1/2)(4) = 2$ ounces of cream. Therefore, the fraction of the liquid in the first cup that is cream is $2/(2+3) = 2/5$.

11. **(B)** Let t be the number of hours Mr. Bird must travel to arrive on time. Since three minutes is the same as 0.05 hours, $40(t + 0.05) = 60(t - 0.05)$. Thus,

$$40t + 2 = 60t - 3, \quad \text{so } t = 0.25.$$

The distance from his home to work is $40(0.25 + 0.05) = 12$ miles. Therefore, his average speed should be $12/0.25 = 48$ miles per hour.

OR

Let d be the distance from Mr. Bird's house to work, and let s be the desired average speed. Then the desired driving time is d/s. Since $d/60$ is three minutes too short and $d/40$ is three minutes too long, the desired time must be the average, so

$$\frac{d}{s} = \frac{1}{2}\left(\frac{d}{60} + \frac{d}{40}\right).$$

This implies that $s = 48$.

12. **(B)** Let p and q be two primes that are roots of $x^2 - 63x + k = 0$. Then

$$x^2 - 63x + k = (x - p)(x - q) = x^2 - (p + q)x + p \cdot q,$$

so $p + q = 63$ and $p \cdot q = k$. Since 63 is odd, one of the primes must be 2 and the other 61. Thus, there is exactly one possible value for k, namely $k = p \cdot q = 2 \cdot 61 = 122$.

13. **(C)** A number x differs by one from its reciprocal if and only if $x - 1 = 1/x$ or $x + 1 = 1/x$. These equations are equivalent to $x^2 - x - 1 = 0$ and $x^2 + x - 1 = 0$. Solving these by the quadratic formula yields the positive solutions

$$\frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{-1+\sqrt{5}}{2},$$

which are reciprocals of each other. The sum of the two numbers is $\sqrt{5}$.

14. **(D)** We have

$$N = \log_{2002} 11^2 + \log_{2002} 13^2 + \log_{2002} 14^2 = \log_{2002} 11^2 \cdot 13^2 \cdot 14^2 = \log_{2002} (11 \cdot 13 \cdot 1$$

Simplifying gives

$$N = \log_{2002} (11 \cdot 13 \cdot 14)^2 = \log_{2002} 2002^2 = 2.$$

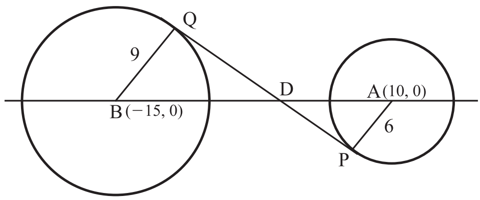
15. **(D)** The values 6, 6, 6, 8, 8, 8, 8, 14 satisfy the requirements of the problem, so the answer is at least 14. If the largest number were 15, the collection would have the ordered form 7, __, __, 8, 8, __, __, 15. But $7 + 8 + 8 + 15 = 38$, and a mean of 8 implies that the sum of all values is 64. In this case, the four missing values would sum to $64 - 38 = 26$, and their average value would be 6.5. This implies that at least one would be less than 7, which is a contradiction. Therefore, the largest integer that can be in the set is 14.

16. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is 1/10. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

$$\left(\frac{1}{10}\right)\left(1 + \frac{9}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} + 0 + 0 + 0\right) = \frac{2}{5}.$$

17. **(B)** First, observe that 4, 6, and 8 cannot be the units digit of any two-digit prime, so they must contribute at least $40 + 60 + 80 = 180$ to the sum. The remaining digits must contribute at least $1 + 2 + 3 + 5 + 7 + 9 = 27$ to the sum. Thus, the sum must be at least 207, and we can achieve this minimum only if we can construct a set of three one-digit primes and three two-digit primes. Using the facts that nine is not prime and neither two nor five can be the units digit of any two-digit prime, we can construct the sets $\{2, 3, 5, 41, 67, 89\}$, $\{2, 3, 5, 47, 61, 89\}$, or $\{2, 5, 7, 43, 61, 89\}$, each of which yields a sum of 207.

18. **(C)** The centers are at $A = (10, 0)$ and $B = (-15, 0)$, and the radii are 6 and 9, respectively. Since the internal tangent is shorter than the external tangent, $\overline{PQ}$ intersects $\overline{AB}$ at a point D that divides $\overline{AB}$ into parts proportional to the radii. The right triangles $\triangle APD$ and $\triangle BQD$ are similar with ratio of similarity $2 : 3$. Therefore, $D = (0, 0)$, $PD = 8$, and $QD = 12$. Thus $PQ = 20$.



19. **(D)** The equation $f(f(x)) = 6$ implies that $f(x) = -2$ or $f(x) = 1$. The horizontal line $y = -2$ intersects the graph of f twice, so $f(x) = -2$ has two solutions. Similarly, $f(x) = 1$ has 4 solutions, so there are 6 solutions of $f(f(x)) = 6$.

20. **(C)** Since $0.\overline{ab} = \frac{ab}{99}$, the denominator must be a factor of $99 = 3^2 \cdot 11$. The factors of 99 are 1, 3, 9, 11, 33, and 99. Since a and b are not both nine, the denominator cannot be 1. By choosing a and b appropriately, we can make fractions with each of the other denominators.

21. **(B)** Writing out more terms of the sequence yields

$$4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3,\ 4,\ 7,\ 1\ldots.$$

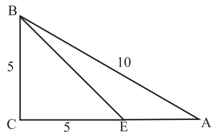
The sequence repeats itself, starting with the 13th term. Since $S_{12} = 60$, $S_{12k} = 60k$ for all positive integers k. The largest k for which $S_{12k} \leq 10,000$ is

$$k = \left\lfloor \frac{10,000}{60} \right\rfloor = 166,$$

and $S_{12\cdot 166} = 60 \cdot 166 = 9960$. To have $S_n > 10,000$, we need to add enough additional terms for their sum to exceed 40. This can be done by adding the next 7 terms of the sequence, since their sum is 42. Thus, the smallest value of n is $12 \cdot 166 + 7 = 1999$.

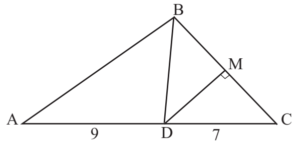
22. **(C)** Since AB is 10, we have $BC = 5$ and $AC = 5\sqrt{3}$. Choose E on $\overline{AC}$ so that $CE = 5$. Then $BE = 5\sqrt{2}$. For BD to be greater than $5\sqrt{2}$, P has to be inside $\triangle ABE$. The probability that P is inside $\triangle ABE$ is

$$\frac{\text{Area of } \triangle ABE}{\text{Area of } \triangle ABC} = \frac{\frac{1}{2}EA \cdot BC}{\frac{1}{2}CA \cdot BC} = \frac{EA}{AC} = \frac{5\sqrt{3}-5}{5\sqrt{3}} = \frac{\sqrt{3}-1}{\sqrt{3}} = \frac{3-\sqrt{3}}{3}.$$



23. **(D)** By the angle-bisector theorem, $\frac{AB}{BC} = \frac{9}{7}$. Let $AB = 9x$ and $BC = 7x$, let $m\angle ABD = m\angle CBD = \theta$, and let M be the midpoint of $\overline{BC}$. Since M is on the perpendicular bisector of $\overline{BC}$, we have $BD = DC = 7$. Then

$$\cos\theta = \frac{\frac{7x}{2}}{7} = \frac{x}{2}.$$



Applying the Law of Cosines to $\triangle ABD$ yields

$$9^2 = (9x)^2 + 7^2 - 2(9x)(7)\left(\frac{x}{2}\right),$$

from which $x = 4/3$ and $AB = 12$. Apply Heron's formula to obtain the area of triangle ABD as $\sqrt{14 \cdot 2 \cdot 5 \cdot 7} = 14\sqrt{5}$.

24. **(E)** Let $z = a + bi$, $\overline{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \overline{z}$. Note that

$$|z|^{2002} = \left|z^{2002}\right| = |\overline{z}| = |z|,$$

from which it follows that

$$|z|\left(|z|^{2001} - 1\right) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \overline{z}$, which is equivalent to $z^{2003} = \overline{z} \cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

25. **(B)** The sum of the coefficients of P and the sum of the coefficients of Q will be equal, so $P(1) = Q(1)$. The only answer choice with an intersection at $x = 1$ is (B). (The polynomials in graph B are $P(x) = 2x^4 - 3x^2 - 3x - 4$ and $Q(x) = -2x^4 - 2x^2 - 2x - 2$.)

This page left intentionally blank.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

53rd Annual American Mathematics Contest 12

AMC 12 - Contest B

Solutions Pamphlet

WEDNESDAY, FEBRUARY 27, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068
Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(A)** The number M is equal to

$$\frac{1}{9}(9+99+999+\ldots+999{,}999{,}999) = 1+11+111+\ldots+111{,}111{,}111 = 123{,}456{,}789.$$

The number M does not contain the digit 0.

2. **(D)** Since

$$\begin{aligned}(3x-2)(4x+1)-(3x-2)4x+1 &= (3x-2)(4x+1-4x)+1\\ &= (3x-2)\cdot 1+1 = 3x-1,\end{aligned}$$

when $x = 4$ we have the value $3 \cdot 4 - 1 = 11$.

3. **(B)** If $n \geq 4$, then

$$n^2 - 3n + 2 = (n-1)(n-2)$$

is the product of two integers greater than 1, and thus is not prime. For $n = 1$, 2, and 3 we have, respectively,

$$(1-1)(1-2) = 0, \quad (2-1)(2-2) = 0, \quad \text{and} \quad (3-1)(3-2) = 2.$$

Therefore, $n^2 - 3n + 2$ is prime only when $n = 3$.

4. **(E)** The number $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is greater than 0 and less than $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{1} < 2$. Hence,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n} = \frac{41}{42}+\frac{1}{n}$$

is an integer precisely when it is equal to 1. This implies that $n = 42$, so the answer is (E).

5. **(D)** A pentagon can be partitioned into three triangles, so the sum of the degree measures of the angles of the pentagon is

$$v + w + x + y + z = 540.$$

The arithmetic sequence can be expressed as $x-2d$, $x-d$, x, $x+d$, and $x+2d$, where d is the common difference, so

$$(x-2d)+(x-d)+x+(x+d)+(x+2d) = 5x = 540.$$

Thus, $x = 108$.

6. **(C)** The given conditions imply that

$$x^2 + ax + b = (x - a)(x - b) = x^2 - (a + b)x + ab,$$

so

$$a + b = -a \quad \text{and} \quad ab = b.$$

Since $b \neq 0$, the second equation implies that $a = 1$. The first equation gives $b = -2$, so $(a, b) = (1, -2)$.

7. **(B)** Let $n - 1$, n, and $n + 1$ denote the three integers. Then

$$(n - 1)n(n + 1) = 8(3n).$$

Since $n \neq 0$, we have $n^2 - 1 = 24$. It follows that $n^2 = 25$ and $n = 5$. Thus,

$$(n - 1)^2 + n^2 + (n + 1)^2 = 16 + 25 + 36 = 77.$$

8. **(D)** Since July has 31 days, Monday must be one of the last three days of July. Therefore, Thursday must be one of the first three days of August, which also has 31 days. So Thursday must occur five times in August.

9. **(C)** We have $b = a + r$, $c = a + 2r$, and $d = a + 3r$, where r is a positive real number. Also, $b^2 = ad$ yields $(a + r)^2 = a(a + 3r)$, or $r^2 = ar$. It follows that $r = a$ and $d = a + 3a = 4a$. Hence $\dfrac{a}{d} = \dfrac{1}{4}$.

10. **(A)** Each number in the given set is one more than a multiple of 3. Therefore the sum of any three such numbers is itself a multiple of 3. It is easily checked that every multiple of 3 from $1 + 4 + 7 = 12$ through $13 + 16 + 19 = 38$ is obtainable. There are 13 multiples of 3 between 12 and 48 inclusive.

11. **(E)** The numbers $A - B$ and $A + B$ are both odd or both even. However, they are also both prime, so they must both be odd. Therefore, one of A and B is odd and the other even. Because A is a prime between $A - B$ and $A + B$, A must be the odd prime. Therefore, $B = 2$, the only even prime. So $A - 2$, A, and $A + 2$ are consecutive odd primes and thus must be 3, 5, and 7. The sum of the four primes 2, 3, 5, and 7 is the prime number 17.

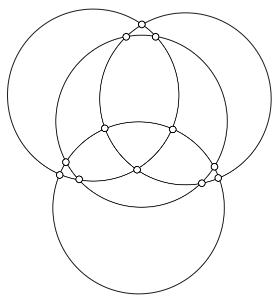
12. **(D)** If $\frac{n}{20-n} = k^2$, for some $k \geq 0$, then $n = \frac{20k^2}{k^2+1}$. Since k^2 and k^2+1 have no common factors and n is an integer, k^2+1 must be a factor of 20. This occurs only when $k = 0, 1, 2$, or 3. The corresponding values of n are $0, 10, 16$, and 18.

13. **(B)** Let $n, n+1, \ldots, n+17$ be the 18 consecutive integers. Then the sum is

$$18n + (1 + 2 + \cdots 17) = 18n + \frac{17 \cdot 18}{2} = 9(2n + 17).$$

Since 9 is a perfect square, $2n + 17$ must also be a perfect square. The smallest value of n for which this occurs is $n = 4$, so $9(2n + 17) = 9 \cdot 25 = 225$.

14. **(D)** Each pair of circles has at most two intersection points. There are $\binom{4}{2} = 6$ pairs of circles, so there are at most $6 \times 2 = 12$ points of intersection. The following configuration shows that 12 points of intersection are indeed possible:



15. **(D)** Let a denote the leftmost digit of N and let x denote the three-digit number obtained by removing a. Then $N = 1000a + x = 9x$ and it follows that $1000a = 8x$. Dividing both sides by 8 yields $125a = x$. All the values of a in the range 1 to 7 result in three-digit numbers.

16. **(C)** The product will be a multiple of 3 if and only if at least one of the two rolls is a 3 or a 6. The probability that Juan rolls 3 or 6 is $2/8 = 1/4$. The probability that Juan does not roll 3 or 6, but Amal does is $(3/4)(1/3) = 1/4$. Thus, the probability that the product of the rolls is a multiple of 3 is

$$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

17. **(B)** Let A be the number of square feet in Andy's lawn. Then $A/2$ and $A/3$ are the areas of Beth's lawn and Carlos' lawn, respectively, in square feet. Let R be the rate, in square feet per minute, that Carlos' lawn mower cuts. Then Beth's mower and Andy's mower cut at rates of $2R$ and $3R$ square feet per minute, respectively. Thus,

$$\text{Andy takes } \frac{A}{3R} \text{ minutes to mow his lawn,}$$

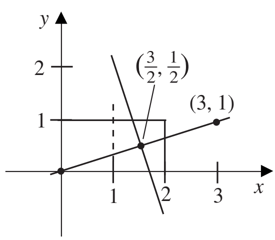
$$\text{Beth takes } \frac{A/2}{2R} = \frac{A}{4R} \text{ minutes to mow hers,}$$

and

$$\text{Carlos takes } \frac{A/3}{R} = \frac{A}{3R} \text{ minutes to mow his.}$$

Since $\dfrac{A}{4R} < \dfrac{A}{3R}$, Beth will finish first.

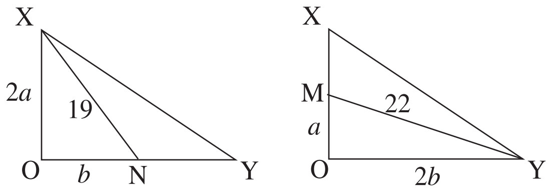
18. **(C)** The area of the rectangular region is 2. Hence the probability that P is closer to $(0,0)$ than it is to $(3,1)$ is half the area of the trapezoid bounded by the lines $y = 1$, the x- and y- axes, and the perpendicular bisector of the segment joining $(0,0)$ and $(3,1)$. The perpendicular bisector goes through the point $(3/2, 1/2)$, which is the center of the square whose vertices are $(1,0)$, $(2,0)$, $(2,1)$, and Hence, the line cuts the square into two quadrilaterals of equal area $1/2$. Thus the area of the trapezoid is $3/2$ and the probability is $3/4$.



19. **(D)** Adding the given equations gives $2(ab+bc+ca) = 484$, so $ab+bc+ca = 242$. Subtracting from this each of the given equations yields $bc = 90$, $ca = 80$, and $ab = 72$. It follows that $a^2b^2c^2 = 90 \cdot 80 \cdot 72 = 720^2$. Since $abc > 0$, we have $abc = 720$.

20. **(B)** Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$



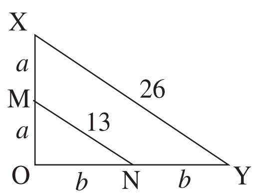
Hence

$$5(a^2 + b^2) = 19^2 + 22^2 = 845.$$

It follows that

$$MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13.$$

Since $\triangle XOY$ is similar to $\triangle MON$ and $XO = 2 \cdot MO$, we have $XY = 2 \cdot MN = 26$.



21. **(A)** Since $2002 = 11 \cdot 13 \cdot 14$, we have

$$a_n = \begin{cases} 11, & \text{if } n = 13 \cdot 14 \cdot i, \text{ where } i = 1, 2, \ldots, 10; \\ 13, & \text{if } n = 14 \cdot 11 \cdot j, \text{ where } j = 1, 2, \ldots, 12; \\ 14, & \text{if } n = 11 \cdot 13 \cdot k, \text{ where } k = 1, 2, \ldots, 13; \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\displaystyle\sum_{n=1}^{2001} a_n = 11 \cdot 10 + 13 \cdot 12 + 14 \cdot 13 = 448.$

22. **(B)** We have $a_n = \frac{1}{\log_n 2002} = \log_{2002} n$, so

$$\begin{aligned} b - c &= (\log_{2002} 2 + \log_{2002} 3 + \log_{2002} 4 + \log_{2002} 5) \\ &\quad - (\log_{2002} 10 + \log_{2002} 11 + \log_{2002} 12 + \log_{2002} 13 + \log_{2002} 14) \\ &= \log_{2002} \frac{2 \cdot 3 \cdot 4 \cdot 5}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14} = \log_{2002} \frac{1}{11 \cdot 13 \cdot 14} = \log_{2002} \frac{1}{2002} = -1. \end{aligned}$$

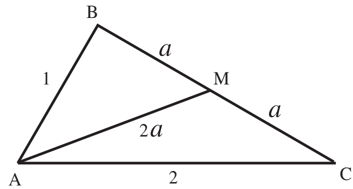
23. **(C)** Let M be the midpoint of $\overline{BC}$, let $AM = 2a$, and let $\theta = \angle AMB$. Then $\cos\angle AMC = -\cos\theta$. Applying the Law of Cosines to $\triangle ABM$ and to $\triangle AMC$ yields, respectively,

$$a^2 + 4a^2 - 4a^2\cos\theta = 1$$

and

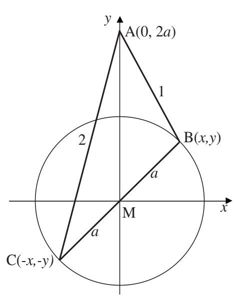
$$a^2 + 4a^2 + 4a^2\cos\theta = 4.$$

Adding, we obtain $10a^2 = 5$, so $a = \sqrt{2}/2$ and $BC = 2a = \sqrt{2}$.



OR

As above, let M be the midpoint of $\overline{BC}$ and $AM = 2a$. Put a rectangular coordinate system in the plane of the triangle with the origin at M so that A has coordinates $(0, 2a)$. If the coordinates of B are (x, y), then the point C has coordinates $(-x, -y)$,



so

$$x^2 + (2a - y)^2 = 1 \quad \text{and} \quad x^2 + (2a + y)^2 = 4.$$

Combining the last two equations gives $2\left(x^2 + y^2\right) + 8a^2 = 5$. But, $x^2 + y^2 = a^2$, so $10a^2 = 5$. Thus, $a = \sqrt{2}/2$ and $BC = \sqrt{2}$.

24. **(E)** We have

$$\text{Area }(ABCD) \le \frac{1}{2} AC \cdot BD,$$

with equality if and only if $AC \perp BD$. Since

$$\begin{aligned} 2002 = \text{Area}(ABCD) &\le \frac{1}{2} AC \cdot BD \\ &\le \frac{1}{2}(AP + PC) \cdot (BP + PD) = \frac{52 \cdot 77}{2} = 2002, \end{aligned}$$

it follows that the diagonals AC and BD are perpendicular and intersect at P. Thus, $AB = \sqrt{24^2 + 32^2} = 40$, $BC = \sqrt{28^2 + 32^2} = 4\sqrt{113}$, $CD = \sqrt{28^2 + 45^2} = 53$, and $DA = \sqrt{45^2 + 24^2} = 51$. The perimeter of $ABCD$ is therefore

$$144 + 4\sqrt{113} = 4\left(36 + \sqrt{113}\right).$$

25. **(E)** Note that

$$f(x) + f(y) = x^2 + 6x + y^2 + 6y + 2 = (x+3)^2 + (y+3)^2 - 16$$

and

$$f(x) - f(y) = x^2 - y^2 + 6(x - y) = (x - y)(x + y + 6).$$

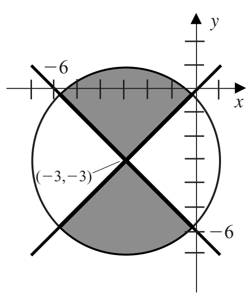
The given conditions can be written as

$$(x+3)^2 + (y+3)^2 \le 16 \quad \text{and} \quad (x - y)(x + y + 6) \le 0.$$

The first inequality describes the region on and inside the circle of radius 4 with center $(-3, -3)$. The second inequality can be rewritten as

$$(x - y \ge 0 \text{ and } x + y + 6 \le 0) \quad \text{or} \quad (x - y \le 0 \text{ and } x + y + 6 \ge 0).$$

Each of these inequalities describes a half-plane bounded by a line that passes through $(-3, -3)$ and has slope 1 or -1. Thus, the set R is the shaded region in the following diagram, and its area is half the area of the circle, which is $8\pi \approx 25.13$.



This page left intentionally blank.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

54th Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 11, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. (D) Each even counting number, beginning with 2, is one more than the preceding odd counting number. Therefore the difference is $(1)(2003) = 2003$.

2. (B) The cost for each member is the price of two pairs of socks, \$8, and two shirts, \$18, for a total of \$26. So there are $2366/26 = 91$ members.

3. (D) The total volume of the eight removed cubes is $8 \times 3^3 = 216$ cubic centimeters, and the volume of the original box is $15 \times 10 \times 8 = 1200$ cubic centimeters. Therefore the volume has been reduced by $\left(\frac{216}{1200}\right)(100\%) = 18\%$.

4. (A) Mary walks a total of 2 km in 40 minutes. Because 40 minutes is 2/3 hr, her average speed, in km/hr, is $2/(2/3) = 3$.

5. (E) Since the last two digits of $AMC10$ and $AMC12$ sum to 22, we have

$$AMC + AMC = 2(AMC) = 1234.$$

   Hence $AMC = 617$, so $A = 6$, $M = 1$, $C = 7$, and $A + M + C = 6 + 1 + 7 = 14$.

6. (C) For example, $-1 \heartsuit 0 = |-1 - 0| = 1 \neq -1$. All the other statements are true:
   (A) $x \heartsuit y = |x - y| = |-(y - x)| = |y - x| = y \heartsuit x$ for all x and y.
   (B) $2(x \heartsuit y) = 2|x - y| = |2x - 2y| = (2x) \heartsuit (2y)$ for all x and y.
   (D) $x \heartsuit x = |x - x| = 0$ for all x.
   (E) $x \heartsuit y = |x - y| > 0$ if $x \neq y$.

7. (B) The longest side cannot be greater than 3, since otherwise the remaining two sides would not be long enough to form a triangle. The only possible triangles have side lengths 1–3–3 or 2–2–3.

8. (E) The factors of 60 are

$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, \text{ and } 60.$$

   Six of the twelve factors are less than 7, so the probability is 1/2.

9. (D) The set S is symmetric about the line $y = x$ and contains $(2, 3)$, so it must also contain $(3, 2)$. Also S is symmetric about the x-axis, so it must contain $(2, -3)$ and $(3, -2)$. Finally, since S is symmetric about the y-axis, it must contain $(-2, 3)$, $(-3, 2)$, $(-2, -3)$, and $(-3, -2)$. Since the resulting set of 8 points is symmetric about both coordinate axes, it is also symmetric about the origin.

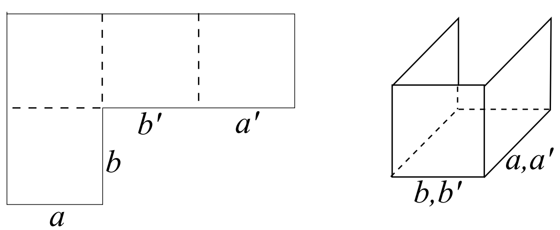
10. (D) Al, Bert, and Carl are to receive, respectively, $1/2, 1/3$, and $1/6$ of the candy. However, each believes he is the first to arrive. Therefore they leave behind, respectively, $1/2, 2/3$, and $5/6$ of the candy that was there when they arrived. The amount of unclaimed candy is $(1/2)(2/3)(5/6) = 5/18$ of the original amount, regardless of the order in which they arrive.

11. (C) Rescaling to different units does not affect the ratio of the areas, so let the perimeter be 12. Each side of the square then has length 3, and each side of the triangle has length 4. The diameter of the circle circumscribing the square is the diagonal of the square, $3\sqrt{2}$. Thus $A = \pi(3\sqrt{2}/2)^2 = 9\pi/2$. The altitude of the triangle is $2\sqrt{3}$, so the radius of the circle circumscribing the triangle is $4\sqrt{3}/3$, and $B = \pi(4\sqrt{3}/3)^2 = 16\pi/3$. Therefore

$$\frac{A}{B} = \frac{9\pi}{2}\frac{3}{16\pi} = \frac{27}{32}.$$

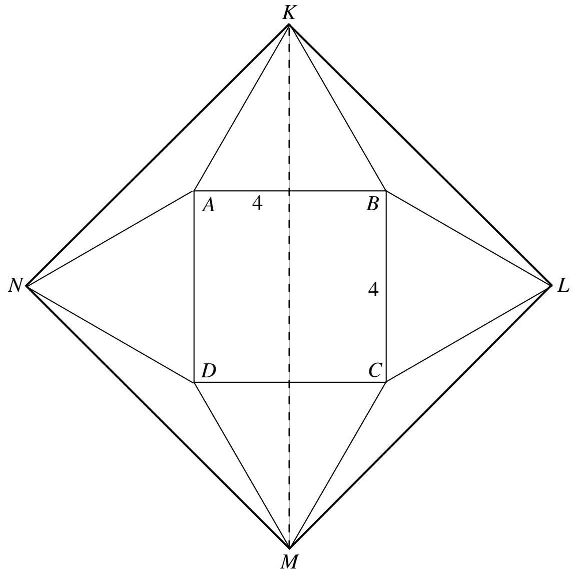
12. (E) Let R1, ..., R5 and B3, ..., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, ..., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, ..., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

13. (E) If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached at any of the six remaining edges.



14. (D) Quadrilateral $KLMN$ is a square because it has $90°$ rotational symmetry, which implies that each pair of adjacent sides is congruent and perpendicular. Since $ABCD$ has sides of length 4 and K is $2\sqrt{3}$ from side $\overline{AB}$, the length of the diagonal $\overline{KM}$ is $4 + 4\sqrt{3}$. Thus the area is

$$\frac{1}{2}(4 + 4\sqrt{3})^2 = 32 + 16\sqrt{3}.$$



OR

Note that $m(\angle NAK) = 150°$. By the Law of Cosines,

$$(NK)^2 = 4^2 + 4^2 - 2(4)(4)\left(-\frac{\sqrt{3}}{2}\right) = 32 + 16\sqrt{3}.$$

Since $KLMN$ is a square, its area is $(NK)^2 = 32 + 16\sqrt{3}$.

15. (C) First note that the area of the region determined by the triangle topped by the semicircle of diameter 1 is

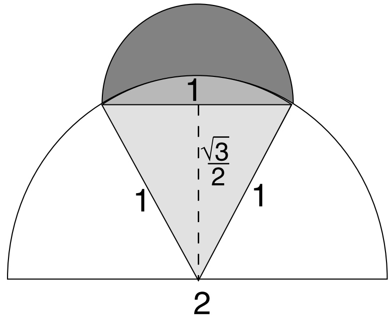
$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2 = \frac{1}{6}\pi.$$

So the area of the lune is

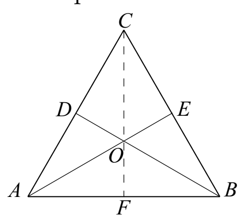
$$\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$



Note that the answer does not depend on the position of the lune on the semicircle.

16. (C) Since the three triangles ABP, ACP, and BCP have equal bases, their areas are proportional to the lengths of their altitudes.

    Let O be the centroid of $\triangle ABC$, and draw medians $\overline{AOE}$ and $\overline{BOD}$. Any point above $\overline{BOD}$ will be farther from $\overline{AB}$ than from $\overline{BC}$, and any point above $\overline{AOE}$ will be farther from $\overline{AB}$ than from $\overline{AC}$. Therefore the condition of the problem is met if and only if P is inside quadrilateral $CDOE$.

    If $\overline{CO}$ is extended to F on $\overline{AB}$, then $\triangle ABC$ is divided into six congruent triangles, of which two comprise quadrilateral $CDOE$. Thus $CDOE$ has one-third the area of $\triangle ABC$, so the required probability is 1/3.

    OR

    By symmetry, each of $\triangle ABP$, $\triangle ACP$, and $\triangle BCP$ is largest with the same probability, so the probability must be 1/3 for each.

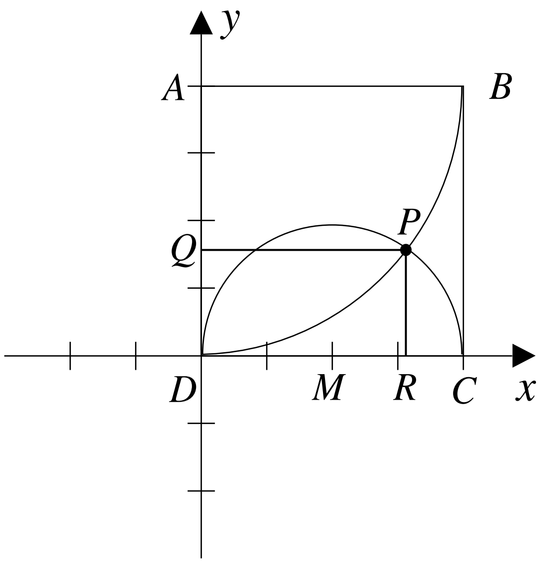
17. (B) Place an xy-coordinate system with origin at D and points C and A on the positive x- and y-axes, respectively. Then the circle centered at M has equation

    $$(x-2)^2 + y^2 = 4,$$

    and the circle centered at A has equation

    $$x^2 + (y-4)^2 = 16.$$

    Solving these equations for the coordinates of P gives $x = 16/5$ and $y = 8/5$, so the answer is 16/5.



OR

We have $AP = AD = 4$ and $PM = MD = 2$, so $\triangle ADM$ is congruent to $\triangle APM$, and $\angle APM$ is a right angle. Draw $\overline{PQ}$ and $\overline{PR}$ perpendicular to $\overline{AD}$ and $\overline{CD}$, respectively. Note that $\angle APQ$ and $\angle MPR$ are both complements of $\angle QPM$. Thus $\triangle APQ$ is similar to $\triangle MPR$, and

$$\frac{AQ}{MR} = \frac{AP}{MP} = \frac{4}{2} = 2.$$

Let $MR = x$. Then $AQ = 2x$, $PR = QD = 4 - 2x$, and $PQ = RD = x + 2$. Therefore

$$2 = \frac{AQ}{MR} = \frac{PQ}{PR} = \frac{x+2}{4-2x},$$

so $x = 6/5$ and $PQ = 6/5 + 2 = 16/5$.

OR

Let $\angle MAD = \alpha$. Then

$$PQ = (PA)\sin(\angle PAQ) = 4\sin(2\alpha) = 8\sin\alpha\cos\alpha = 8\left(\frac{2}{\sqrt{20}}\right)\left(\frac{4}{\sqrt{20}}\right) = \frac{16}{5}.$$

18. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \le n \le 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

19. (D) The original parabola has equation $y = a(x-h)^2 + k$, for some a, h, and k, with $a \neq 0$. The reflected parabola has equation $y = -a(x-h)^2 - k$. The translated parabolas have equations

$$f(x) = a(x - h \pm 5)^2 + k \quad \text{and} \quad g(x) = -a(x - h \mp 5)^2 - k,$$

so

$$(f+g)(x) = \pm 20a(x-h).$$

Since $a \neq 0$, the graph is a non-horizontal line.

20. (A) Since the first group of five letters contains no A's, it must contain k B's and $(5-k)$ C's for some integer k with $0 \le k \le 5$. Since the third group of five letters contains no C's, the remaining k C's must be in the second group, along with $(5-k)$ A's.

Similarly, the third group of five letters must contain k A's and $(5-k)$ B's. Thus each arrangement that satisfies the conditions is determined uniquely by the location of the k B's in the first group, the k C's in the second group, and the k A's in the third group.

For each k, the letters can be arranged in $\binom{5}{k}^3$ ways, so the total number of arrangements is

$$\sum_{k=0}^{5} \binom{5}{k}^3.$$

21. (D) Since $P(0) = 0$, we have $e = 0$ and $P(x) = x(x^4 + ax^3 + bx^2 + cx + d)$. Suppose that the four remaining x-intercepts are at p, q, r, and s. Then

$$x^4 + ax^3 + bx^2 + cx + d = (x-p)(x-q)(x-r)(x-s),$$

and $d = pqrs \neq 0$.
Any of the other constants could be zero. For example, consider

$$P_1(x) = x^5 - 5x^3 + 4x = x(x+2)(x+1)(x-1)(x-2)$$

and

$$P_2(x) = x^5 - 5x^4 + 20x^2 - 16x = x(x+2)(x-1)(x-2)(x-4).$$

OR

Since $P(0) = 0$, we must have $e = 0$, so

$$P(x) = x(x^4 + ax^3 + bx^2 + cx + d).$$

If $d = 0$, then

$$P(x) = x(x^4 + ax^3 + bx^2 + cx) = x^2(x^3 + ax^2 + bx + c),$$

which has a double root at $x = 0$. Hence $d \neq 0$.

OR

There is also a calculus-based solution. Since $P(x)$ has five distinct zeros and $x = 0$ is one of the zeros, it must be a zero of multiplicity one. This is equivalent to having $P(0) = 0$, but $P'(0) \neq 0$. Since

$$P'(x) = 5x^4 + 4ax^3 + 3bx^2 + 2cx + d, \quad \text{we must have} \quad 0 \neq P'(0) = d.$$

22. (C) Since there are twelve steps between $(0,0)$ and $(5,7)$, A and B can meet only after they have each moved six steps. The possible meeting places are $P_0 = (0,6)$, $P_1 = (1,5)$, $P_2 = (2,4)$, $P_3 = (3,3)$, $P_4 = (4,2)$, and $P_5 = (5,1)$. Let a_i and b_i denote the number of paths to P_i from $(0,0)$ and $(5,7)$, respectively. Since A has to take i steps to the right and B has to take $i+1$ steps down, the number of ways in which A and B can meet at P_i is

$$a_i \cdot b_i = \binom{6}{i}\binom{6}{i+1}.$$

Since A and B can each take 2^6 paths in six steps, the probability that they meet is

$$\begin{aligned}\sum_{i=0}^{5}\left(\frac{a_i}{2^6}\right)\left(\frac{b_i}{2^6}\right) &= \frac{\binom{6}{0}\binom{6}{1} + \binom{6}{1}\binom{6}{2} + \binom{6}{2}\binom{6}{3} + \binom{6}{3}\binom{6}{4} + \binom{6}{4}\binom{6}{5} + \binom{6}{5}\binom{6}{6}}{2^{12}} \\ &= \frac{99}{512} \approx 0.20.\end{aligned}$$

OR

Consider the $\binom{12}{5}$ walks that start at $(0,0)$, end at $(5,7)$, and consist of 12 steps, each one either up or to the right. There is a one-to-one correspondence between these walks and the set of (A,B)-paths where A and B meet. In particular, given one of the $\binom{12}{5}$ walks from $(0,0)$ to $(5,7)$, the path followed by A consists of the the first six steps of the walk, and the path followed by B is obtained by starting at $(5,7)$ and reversing the last six steps of the walk. There are 2^6 paths that take 6 steps from $(0,0)$ and 2^6 paths that take 6 steps from $(5,7)$, so there are 2^{12} pairs of paths that A and B can take. The probability that they meet is

$$P = \frac{1}{2^{12}}\binom{12}{5} = \frac{99}{2^9}.$$

23. (B) We have

$$\begin{aligned}1! \cdot 2! \cdot 3! \cdots 9! &= (1)(1\cdot 2)(1\cdot 2\cdot 3)\cdots(1\cdot 2\cdots 9) \\ &= 1^9 2^8 3^7 4^6 5^5 6^4 7^3 8^2 9^1 = 2^{30}3^{13}5^5 7^3.\end{aligned}$$

The perfect square divisors of that product are the numbers of the form

$$2^{2a}3^{2b}5^{2c}7^{2d}$$

with $0 \le a \le 15$, $0 \le b \le 6$, $0 \le c \le 2$, and $0 \le d \le 1$. Thus there are $(16)(7)(3)(2) = 672$ such numbers.

24. (B) We have

$$\begin{aligned}\log_a \frac{a}{b} + \log_b \frac{b}{a} =& \log_a a - \log_a b + \log_b b - \log_b a\\ =& 1 - \log_a b + 1 - \log_b a\\ =& 2 - \log_a b - \log_b a.\end{aligned}$$

Let $c = \log_a b$, and note that $c > 0$ since a and b are both greater than 1. Thus

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c} = \frac{c^2 - 2c + 1}{-c} = \frac{(c-1)^2}{-c} \le 0.$$

This expression is 0 when $c = 1$, that is, when $a = b$.

OR

As above

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - c - \frac{1}{c}$$

From the Arithmetic-Geometric Mean Inequality we have

$$\frac{c + 1/c}{2} \ge \sqrt{c \cdot \frac{1}{c}} = 1, \quad \text{so} \quad c + \frac{1}{c} \ge 2$$

and

$$\log_a \frac{a}{b} + \log_b \frac{b}{a} = 2 - \left(c + \frac{1}{c}\right) \le 0$$

with equality when $c = \frac{1}{c}$, that is, when $a = b$.

25. (C) The domain of f is $\{x \mid ax^2 + bx \ge 0\}$. If $a = 0$, then for every positive value of b, the domain and range of f are each equal to the interval $[0, \infty)$, so 0 is a possible value of a.

If $a \ne 0$, the graph of $y = ax^2 + bx$ is a parabola with x-intercepts at $x = 0$ and $x = -b/a$. If $a > 0$, the domain of f is $(-\infty, -b/a] \cup [0, \infty)$, but the range of f cannot contain negative numbers. If $a < 0$, the domain of f is $[0, -b/a]$. The maximum value of f occurs halfway between the x-intercepts, at $x = -b/2a$, and

$$f\left(-\frac{b}{2a}\right) = \sqrt{a\left(\frac{b^2}{4a^2}\right) + b\left(-\frac{b}{2a}\right)} = \frac{b}{2\sqrt{-a}}.$$

Hence, the range of f is $[0, b/2\sqrt{-a}]$. For the domain and range to be equal, we must have

$$-\frac{b}{a} = \frac{b}{2\sqrt{-a}} \quad \text{so} \quad 2\sqrt{-a} = -a.$$

The only solution is $a = -4$. Thus there are two possible values of a, and they are $a = 0$ and $a = -4$.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

54th Annual American Mathematics Contest 12

AMC 12 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 26, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

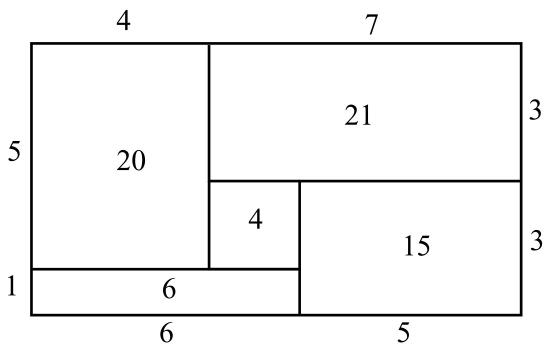
1. (C) We have

$$\frac{2-4+6-8+10-12+14}{3-6+9-12+15-18+21} = \frac{2(1-2+3-4+5-6+7)}{3(1-2+3-4+5-6+7)} = \frac{2}{3}.$$

2. (D) The cost of each day's pills is $546/14 = 39$ dollars. If x denotes the cost of one green pill, then $x + (x-1) = 39$, so $x = 20$.

3. (A) To minimize the cost, Rose should place the most expensive flowers in the smallest region, the next most expensive in the second smallest, etc. The areas of the regions are shown in the figure, so the minimal total cost, in dollars, is

$$(3)(4) + (2.5)(6) + (2)(15) + (1.5)(20) + (1)(21) = 108.$$



4. (C) The area of the lawn is

$$90 \cdot 150 = 13{,}500 \text{ ft}^2.$$

Moe cuts about two square feet for each foot he pushes the mower forward, so he cuts $2(5000) = 10{,}000$ ft^2 per hour. Therefore, it takes about $\frac{13{,}500}{10{,}000} = 1.35$ hours.

5. (D) The height, length, and diagonal are in the ratio $3 : 4 : 5$. The length of the diagonal is 27, so the horizontal length is

$$\frac{4}{5}(27) = 21.6 \text{ inches}.$$

6. (B) Let the sequence be denoted $a, ar, ar^2, ar^3, \ldots$, with $ar = 2$ and $ar^3 = 6$. Then $r^2 = 3$ and $r = \sqrt{3}$ or $r = -\sqrt{3}$. Therefore $a = \frac{2\sqrt{3}}{3}$ or $a = -\frac{2\sqrt{3}}{3}$.

7. (D) Let n represent the number of nickels in the bank, d the number of dimes, and q the number of quarters. Then $n + d + q = 100$ and $5n + 10d + 25q = 835$. Further, n, d, and q must all be nonnegative integers. Dividing the second equation by 5 yields $n + 2d + 5q = 167$. Subtracting the first equation from this gives $d + 4q = 67$. Since q cannot be negative, d is at most 67, and we check that

67 dimes and 33 nickels indeed produces \$8.35. On the other hand, d cannot be 0, 1, or 2 because then q would not be an integer. Thus the smallest d can be is 3, leaving $q = 16$. We check that 16 quarters, 3 dimes, and 81 nickels also produces \$8.35. Thus the largest d can be is 67, the smallest is 3, and the difference is 64.

8. (E) Let $y = \clubsuit(x)$. Since $x \le 99$, we have $y \le 18$. Thus if $\clubsuit(y) = 3$, then $y = 3$ or $y = 12$. The 3 values of x for which $\clubsuit(x) = 3$ are 12, 21, and 30, and the 7 values of x for which $\clubsuit(x) = 12$ are 39, 48, 57, 66, 75, 84, and 93. There are 10 values in all.

9. (D) Since f is a linear function, its slope is constant. Therefore

$$\frac{f(6) - f(2)}{6 - 2} = \frac{f(12) - f(2)}{12 - 2}, \quad \text{so} \quad \frac{12}{4} = \frac{f(12) - f(2)}{10},$$

and $f(12) - f(2) = 30$.

OR

Since f is a linear function, it has a constant rate of change, given by

$$\frac{f(6) - f(2)}{6 - 2} = \frac{12}{4} = 3.$$

Therefore $f(12) - f(2) = 3(12 - 2) = 30$.

OR

If $f(x) = mx + b$, then

$$12 = f(6) - f(2) = 6m + b - (2m + b) = 4m,$$

so $m = 3$. Hence

$$f(12) - f(2) = 12m + b - (2m + b) = 10m = 30.$$

10. (B) We may assume that one of the triangles is attached to side $\overline{AB}$. The second triangle can be attached to $\overline{BC}$ or $\overline{CD}$ to obtain two non-congruent figures. If the second triangle is attached to $\overline{AE}$ or to $\overline{DE}$, the figure can be reflected about the vertical axis of symmetry of the pentagon to obtain one of the two already counted. Thus the total is two.

11. (C) When it is 1:00 PM it is 60 minutes after noon, but Cassandra's watch has recorded only 57 minutes and 36 seconds, that is, 57.6 minutes. Thus when her watch has recorded t minutes since noon, the actual number of minutes passed is

$$\frac{60}{57.6}t = \frac{25}{24}t.$$

In particular, when her watch reads 10:00 PM it has recorded 600 minutes past noon, so the actual number of minutes past noon is

$$\frac{25}{24}(600) = 625,$$

or 10 hours and 25 minutes. Therefore the actual time is 10:25 PM.

12. (D) Among five consecutive odd numbers, at least one is divisible by 3 and exactly one is divisible by 5, so the product is always divisible by 15. The cases $n = 2$, $n = 10$, and $n = 12$ demonstrate that no larger common divisor is possible, since 15 is the greatest common divisor of $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$, $11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$, and $13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$.

13. (B) Let r be the radius of the sphere and cone, and let h be the height of the cone. Then the conditions of the problem imply that

$$\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h, \quad \text{so } h = 3r.$$

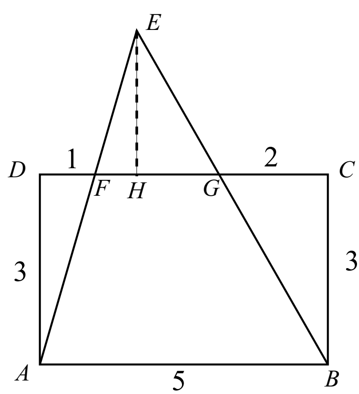
Therefore, the ratio of h to r is $3 : 1$.

14. (D) Let H be the foot of the perpendicular from E to $\overline{DC}$. Since $CD = AB = 5$, $FG = 2$, and $\triangle FEG$ is similar to $\triangle AEB$, we have

$$\frac{EH}{EH+3} = \frac{2}{5}, \quad \text{so} \quad 5EH = 2EH + 6,$$

and $EH = 2$. Hence

$$\text{Area}(\triangle AEB) = \frac{1}{2}(2+3) \cdot 5 = \frac{25}{2}.$$



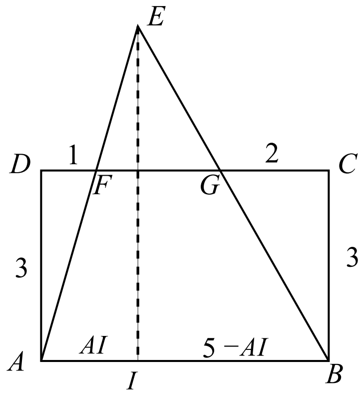
OR

Let I be the foot of the perpendicular from E to $\overline{AB}$. Since

$$\triangle EIA \text{ is similar to } \triangle ADF \quad \text{and} \quad \triangle EIB \text{ is similar to } \triangle BCG,$$

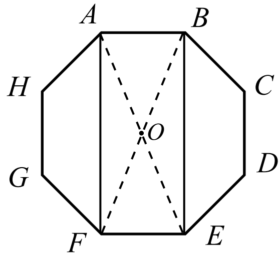
we have

$$\frac{AI}{EI} = \frac{1}{3} \quad \text{and} \quad \frac{5 - AI}{EI} = \frac{2}{3}.$$



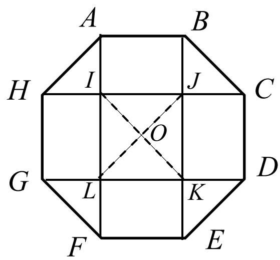
Adding gives $5/EI = 1$, so $EI = 5$. The area of the triangle is $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

15. (D) Let O be the intersection of the diagonals of $ABEF$. Since the octagon is regular, $\triangle AOB$ has area $1/8$. Since O is the midpoint of $\overline{AE}$, $\triangle OAB$ and $\triangle BOE$ have the same area. Thus $\triangle ABE$ has area $1/4$, so $ABEF$ has area $1/2$.



OR

Let O be the intersection of the diagonals of the square $IJKL$. Rectangles $ABJI$, $JCDK$, $KEFL$, and $LGHI$ are congruent. Also $IJ = AB = AH$, so the right isosceles triangles $\triangle AIH$ and $\triangle JOI$ are congruent. By symmetry, the area in the center square $IJKL$ is the sum of the areas of $\triangle AIH$, $\triangle CJB$, $\triangle EKD$, and $\triangle GLF$. Thus the area of rectangle $ABEF$ is half the area of the octagon.



16. (E) The area of the larger semicircle is

$$\frac{1}{2}\pi\,(2)^2 = 2\pi.$$

The region deleted from the larger semicircle consists of five congruent sectors and two equilateral triangles. The area of each of the sectors is

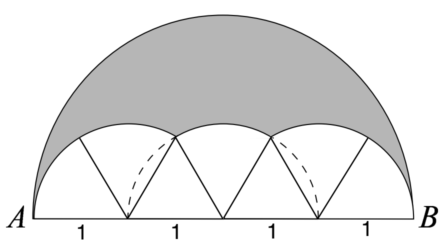
$$\frac{1}{6}\pi\,(1)^2 = \frac{\pi}{6}$$

and the area of each triangle is

$$\frac{1}{2}\cdot 1\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4},$$

so the area of the shaded region is

$$2\pi - 5\cdot\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$



17. (D) We have

$$1 = \log(xy^3) = \log x + 3\log y \quad \text{and} \quad 1 = \log(x^2y) = 2\log x + \log y.$$

Solving yields $\log x = \frac{2}{5}$ and $\log y = \frac{1}{5}$. Thus

$$\log(xy) = \log x + \log y = \frac{3}{5}.$$

OR

The given equations imply that $xy^3 = 10 = x^2y$. Thus

$$y = \frac{10}{x^2} \quad \text{and} \quad x\left(\frac{10}{x^2}\right)^3 = 10.$$

It follows that $x = 10^{2/5}$ and $y = 10^{1/5}$, so $\log(xy) = \log(10^{3/5}) = 3/5$.

OR

Since $\log(xy^3) = \log(x^2y)$, we have $xy^3 = x^2y$, so $x = y^2$. Hence

$$1 = \log(xy^3) = \log(y^5) = 5\log y, \quad \text{and} \quad \log y = \frac{1}{5}.$$

So $\log(xy) = \log(y^3) = 3/5$.

18. (B) We have
$$11y^{13} = 7x^5 = 7(a^cb^d)^5 = 7a^{5c}b^{5d}.$$
The minimum value of x is obtained when neither x nor y contains prime factors other than 7 and 11. Therefore we may assume that $a = 7$ and $b = 11$, so $x = 7^c11^d$ and $7x^5 = 7^{5c+1}11^{5d}$. Letting $y = 7^m11^n$ we obtain $11y^{13} = 7^{13m}11^{13n+1}$. Hence $7^{5c+1}11^{5d} = 7^{13m}11^{13n+1}$. The smallest positive integer solutions are $c = 5$, $d = 8$, $m = 2$, and $n = 3$. Thus $a + b + c + d = 7 + 11 + 5 + 8 = 31$.

19. (E) Since the first term is not 1, the probability that it is 2 is 1/4. If the first term is 2, then the second term cannot be 2. If the first term is not 2, there are four equally likely values, including 2, for the second term. Thus the probability that the second term is 2 is
$$\frac{1}{4}\cdot 0 + \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16},$$
so $a + b = 3 + 16 = 19$.

OR

The set S contains $(4)(4!) = 96$ permutations, since there are 4 choices for the first term, and for each of these choices there are 4! arrangements of the remaining terms. The number of permutations in S whose second term is 2 is $(3)(3!) = 18$, since there are 3 choices for the first term, and for each of these choices there are 3! arrangements of the last 3 terms. Thus the requested probability is $18/96 = 3/16$, and $a + b = 19$.

20. (B) We have
$$0 = f(-1) = -a + b - c + d \quad \text{and} \quad 0 = f(1) = a + b + c + d,$$
so $b + d = 0$. Also $d = f(0) = 2$, so $b = -2$.

OR

The polynomial is divisible by $(x+1)(x-1) = x^2 - 1$, its leading term is ax^3, and its constant term is 2, so
$$f(x) = (x^2-1)(ax-2) = ax^3 - 2x^2 - ax + 2 \quad \text{and} \quad b = -2.$$

21. (D) Let $\beta = \pi - \alpha$. Apply the Law of Cosines to $\triangle ABC$ to obtain

$$(AC)^2 = 8^2 + 5^2 - 2(8)(5)\cos\beta = 89 - 80\cos\beta.$$

Thus $AC < 7$ if and only if

$$89 - 80\cos\beta < 49, \quad \text{that is, if and only if} \quad \cos\beta > \frac{1}{2}.$$

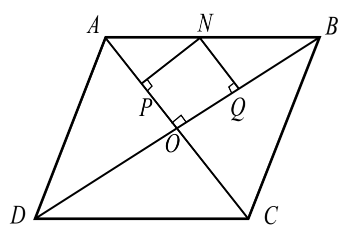
Therefore we must have $0 < \beta < \dfrac{\pi}{3}$, and the requested probability is $\dfrac{\pi/3}{\pi} = \dfrac{1}{3}$.

22. (C) Let O be the point of intersection of $\overline{AC}$ and $\overline{BD}$. Then AOB is a right triangle with legs $OA = 8$ and $OB = 15$. Quadrilateral $OPNQ$ is a rectangle because it has right angles at O, P, and Q. Thus $PQ = ON$, because the diagonals of a rectangle are of equal length. The minimum value of PQ is the minimum value of ON. This is achieved if and only if N is the foot of the altitude from O in triangle AOB. Writing the area of $\triangle AOB$ in two different ways yields

$$\frac{1}{2}AB \cdot ON = \frac{1}{2}OA \cdot OB.$$

Hence the minimum value of PQ is

$$ON = \frac{OA \cdot OB}{AB} = \frac{OA \cdot OB}{\sqrt{OA^2 + OB^2}} = \frac{8 \cdot 15}{17} = \frac{120}{17} \approx 7.06.$$



23. (A) The intercepts occur where $\sin(1/x) = 0$, that is, where $x = 1/(k\pi)$ and k is a nonzero integer. Solving

$$0.0001 < \frac{1}{k\pi} < 0.001$$

yields

$$\frac{1000}{\pi} < k < \frac{10,000}{\pi}.$$

Thus the number of x intercepts in $(0.0001, 0.001)$ is

$$\left\lfloor \frac{10,000}{\pi} \right\rfloor - \left\lfloor \frac{1000}{\pi} \right\rfloor = 3183 - 318 = 2865,$$

which is closest to 2900.

24. (C) Since the system has exactly one solution, the graphs of the two equations must intersect at exactly one point. If $x < a$, the equation $y = |x-a| + |x-b| + |x-c|$ is equivalent to $y = -3x + (a+b+c)$. By similar calculations we obtain

$$y = \begin{cases} -3x + (a+b+c), & \text{if } x < a \\ -x + (-a+b+c), & \text{if } a \le x < b \\ x + (-a-b+c), & \text{if } b \le x < c \\ 3x + (-a-b-c), & \text{if } c \le x. \end{cases}$$

Thus the graph consists of four lines with slopes -3, -1, 1, and 3, and it has corners at $(a, b+c-2a)$, $(b, c-a)$, and $(c, 2c-a-b)$.

On the other hand, the graph of $2x + y = 2003$ is a line whose slope is -2. If the graphs intersect at exactly one point, that point must be $(a, b+c-2a)$. Therefore

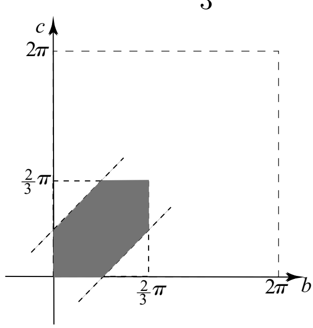
$$2003 = 2a + (b+c-2a) = b + c.$$

Since $b < c$, the minimum value of c is 1002.

25. (D) We can assume that the circle has its center at $(0,0)$ and a radius of 1. Call the three points A, B, and C, and let a, b, and c denote the length of the counterclockwise arc from $(1,0)$ to A, B, and C, respectively. Rotating the circle if necessary, we can also assume that $a = \pi/3$. Since b and c are chosen at random from $[0, 2\pi)$, the ordered pair (b,c) is chosen at random from a square with area $4\pi^2$ in the bc-plane. The condition of the problem is met if and only if

$$0 < b < \frac{2\pi}{3}, \quad 0 < c < \frac{2\pi}{3}, \quad \text{and} \quad |b-c| < \frac{\pi}{3}.$$

This last inequality is equivalent to $b - \dfrac{\pi}{3} < c < b + \frac{\pi}{3}$.



The graph of the common solution to these inequalities is the shaded region shown. The area of this region is

$$\left(\frac{6}{8}\right)\left(\frac{2\pi}{3}\right)^2 = \pi^2/3,$$

so the requested probability is

$$\frac{\pi^2/3}{4\pi^2} = \frac{1}{12}.$$

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

55th Annual American Mathematics Contest 12

AMC 12 - Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 10, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(E)** Since \$20 is 2000 cents, she pays $(0.0145)(2000) = 29$ cents per hour in local taxes.

2. **(C)** The 8 unanswered problems are worth $(2.5)(8) = 20$ points, so Charlyn must earn at least 80 additional points. The smallest multiple of 6 that is at least 80 is $(6)(14) = 84$, so Charlyn must have at least 14 correct answers.

3. **(B)** The value of $x = 100 - 2y$ is a positive integer for each positive integer y with $1 \le y \le 49$.

4. **(E)** Bertha has $30 - 6 = 24$ granddaughters, none of whom have any daughters. The granddaughters are the children of $24/6 = 4$ of Bertha's daughters, so the number of women having no daughters is $30 - 4 = 26$.

5. **(B)** The y-intercept of the line is between 0 and 1, so $0 < b < 1$. The slope is between -1 and 0, so $-1 < m < 0$. Thus $-1 < mb < 0$.

6. **(A)** Since none of V, W, X, Y, or Z exceeds 2004^{2005}, the difference $U - V = 2004^{2005}$ is the largest.

7. **(B)** After three rounds the players A, B, and C have 14, 13, and 12 tokens, respectively. Every subsequent three rounds of play reduces each player's supply of tokens by one. After 36 rounds they have 3, 2, and 1 token, respectively, and after the 37$^{\text{th}}$ round Player A has no tokens.

8. **(B)** Let x, y, and z be the areas of $\triangle ADE$, $\triangle BDC$, and $\triangle ABD$, respectively. The area of $\triangle ABE$ is $(1/2)(4)(8) = 16 = x + z$, and the area of $\triangle BAC$ is $(1/2)(4)(6) = 12 = y + z$. The requested difference is

$$x - y = (x + z) - (y + z) = 16 - 12 = 4.$$

9. **(C)** Let r, h, and V, respectively, be the radius, height, and volume of the jar that is currently being used. The new jar will have a radius of $1.25r$ and volume V. Let H be the height of the new jar. Then

$$\pi r^2 h = V = \pi(1.25r)^2 H, \quad \text{so} \quad \frac{H}{h} = \frac{1}{(1.25)^2} = 0.64.$$

   Thus H is 64% of h, so the height must be reduced by $(100 - 64)\% = 36\%$.

   OR

   Multiplying the diameter by $5/4$ multiplies the area of the base by $(5/4)^2 = 25/16$, so in order to keep the same volume, the height must be multiplied by $16/25$. Thus the height must be decreased by $9/25$, or 36%.

10. **(C)** The sum of a set of integers is the product of the mean and the number of integers, and the median of a set of consecutive integers is the same as the mean. So the median must be $7^5/49 = 7^3$.

11. **(A)** If n is the number of coins in Paula's purse, then their total value is $20n$ cents. If she had one more quarter, she would have $n+1$ coins whose total value in cents could be expressed both as $20n+25$ and as $21(n+1)$. Therefore

$$20n+25 = 21(n+1), \quad \text{so} \quad n = 4.$$

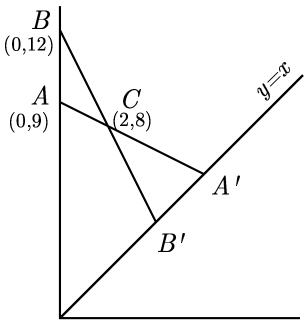
Since Paula has four coins with a total value of 80 cents, she must have three quarters and one nickel, so the number of dimes is 0.

12. **(B)** Line AC has slope $\ \frac{1}{2}$ and y-intercept (0,9), so its equation is

$$y = \quad \frac{1}{2}x + 9.$$

Since the coordinates of A' satisfy both this equation and $y = x$, it follows that $A' = (6, 6)$. Similarly, line BC has equation $y = \quad 2x + 12$, and $B' = (4, 4)$. Thus

$$A'B' = \sqrt{(6 \quad 4)^2 + (6 \quad 4)^2} = 2\sqrt{2}.$$



13. **(B)** There are $\ {9 \atop 2}\ = 36$ pairs of points in S, and each pair determines a line. However, there are three horizontal, three vertical, and two diagonal lines that pass through three points of S, and these lines are each determined by three di erent pairs of points in S. Thus the number of distinct lines is $36 \quad 2 \ 8 = 20$.

OR

There are 3 vertical lines, 3 horizontal lines, 3 each with slopes 1 and $\ 1$, and 2 each with slopes 2, $\ 2$, 1/2, and $\ 1/2$, for a total of 20.

14. **(A)** The terms of the arithmetic progression are 9, $9+d$, and $9+2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11+d)^2 = 9(29+2d) \quad \text{so} \quad d^2 + 4d \quad 140 = 0.$$

Thus $d = 10$ or $d = \quad 14$. The corresponding geometric progressions are 9, 21, 49 and 9, $\ 3$, 1, so the smallest possible value for the third term of the geometric progression is 1.

15. **(C)** When they first meet, they have run a combined distance equal to half the length of the track. Between their first and second meetings, they run a combined distance equal to the full length of the track. Because Brenda runs at a constant speed and runs 100 meters before their first meeting, she runs $2(100) = 200$ meters between their first and second meetings. Therefore the length of the track is $200 + 150 = 350$ meters.

16. **(B)** The given expression is defined if and only if

$$\log_{2003}(\log_{2002}(\log_{2001} x)) > 0,$$

that is, if and only if

$$\log_{2002}(\log_{2001} x) > 2003^0 = 1.$$

This inequality in turn is satisfied if and only if

$$\log_{2001} x > 2002,$$

that is, if and only if $x > 2001^{2002}$.

17. **(D)** Note that

$$\begin{aligned}
f(2^1) &= f(2) = f(2 \cdot 1) = 1 \cdot f(1) = 2^0 \cdot 2^0 = 2^0,\\
f(2^2) &= f(4) = f(2 \cdot 2) = 2 \cdot f(2) = 2^1 \cdot 2^0 = 2^1,\\
f(2^3) &= f(8) = f(2 \cdot 4) = 4 \cdot f(4) = 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2)},\\
f(2^4) &= f(16) = f(2 \cdot 8) = 8 \cdot f(8) = 2^3 \cdot 2^2 \cdot 2^1 \cdot 2^0 = 2^{(1+2+3)},
\end{aligned}$$

and in general

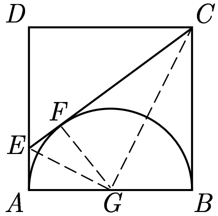
$$f(2^n) = 2^{(1+2+3+\ldots+(n-1))} = 2^{n(n-1)/2}$$

It follows that $f(2^{100}) = 2^{(100)(99)/2} = 2^{4950}$.

18. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2 - x)^2 + 2^2 = (2 + x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



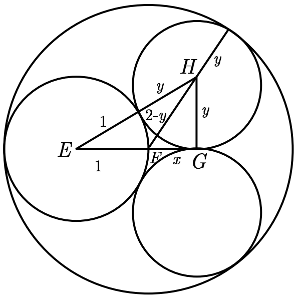
19. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1+y)^2 = (1+x)^2 + y^2 \quad \text{and} \quad (2 \quad y)^2 = x^2 + y^2,$$

from which it follows that

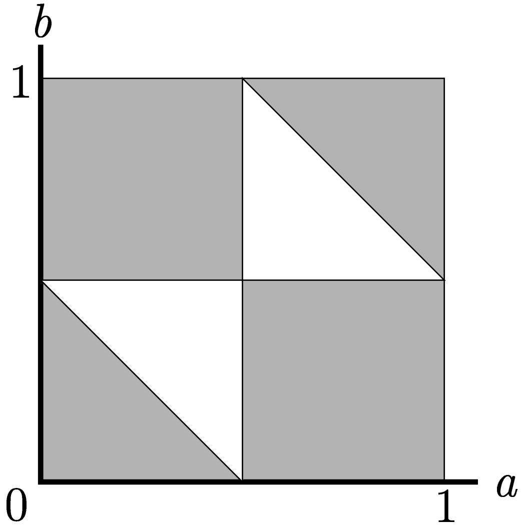
$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 \quad \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (\quad 2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.



20. **(E)** The conditions under which $A + B = C$ are as follows.
(i) If $a + b < 1/2$, then $A = B = C = 0$.
(ii) If $a \quad 1/2$ and $b < 1/2$, then $B = 0$ and $A = C = 1$.
(iii) If $a < 1/2$ and $b \quad 1/2$, then $A = 0$ and $B = C = 1$.
(iv) If $a + b \quad 3/2$, then $A = B = 1$ and $C = 2$.

These conditions correspond to the shaded regions of the graph shown. The combined area of those regions is $3/4$, and the area of the entire square is 1, so the requested probability is $3/4$.



21. **(D)** The given series is geometric with an initial term of 1 and a common ratio of $\cos^2\theta$, so its sum is

$$5 = \sum_{n=0}^{\infty} \cos^{2n}\theta = \frac{1}{1 \quad \cos^2\theta} = \frac{1}{\sin^2\theta}.$$

Therefore $\sin^2\theta = \frac{1}{5}$, and

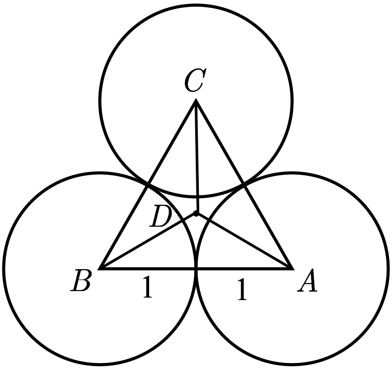
$$\cos 2\theta = 1 \quad 2\sin^2\theta = 1 \quad \frac{2}{5} = \frac{3}{5}.$$

22. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 \quad \frac{2\sqrt{3}}{3}\Bigg)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



23. **(E)** Since $z_1 = 0$, it follows that $c_0 = P(0) = 0$. The nonreal zeros of P must occur in conjugate pairs, so $\sum_{k=1}^{2004} b_k = 0$ and $\sum_{k=1}^{2004} a_k = 0$ also. The coefficient c_{2003} is the sum of the zeros of P, which is

$$\sum_{k=1}^{2004} z_k = \sum_{k=1}^{2004} a_k + i \sum_{k=1}^{2004} b_k = 0.$$

Finally, since the degree of P is even, at least one of $z_2, \ldots, z_{2004}$ must be real, so at least one of $b_2, \ldots, b_{2004}$ is 0 and consequently $b_2 \quad b_3 \quad \ldots \quad b_{2004} = 0$. Thus the quantities in **(A)**, **(B)**, **(C)**, and **(D)** must all be 0.

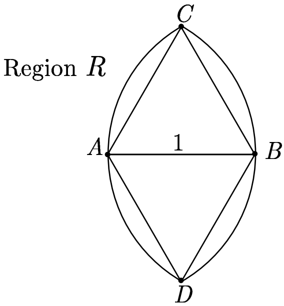
Note that the polynomial

$$P(x) = x(x \quad 2)(x \quad 3) \qquad (x \quad 2003) \quad x + \sum_{k=2}^{2003} k \Bigg)$$

satis es the given conditions, and $\sum_{k=1}^{2004} c_k = P(1) \neq 0$.

24. **(C)** The center of the disk lies in a region R, consisting of all points within 1 unit of both A and B. Let C and D be the points of intersection of the circles of radius 1 centered at A and B. Because $\triangle ABC$ and $\triangle ABD$ are equilateral, arcs CAD and CBD are each 120 . Thus the sector bounded by $\overline{BC}$, $\overline{BD}$, and arc CAD has area /3, as does the sector bounded by $\overline{AC}$, $\overline{AD}$, and arc CBD. The intersection of the two sectors, which is the union of the two triangles, has area $\sqrt{3}/2$, so the area of R is

$$\frac{2}{3} \quad \frac{\sqrt{3}}{2}.$$

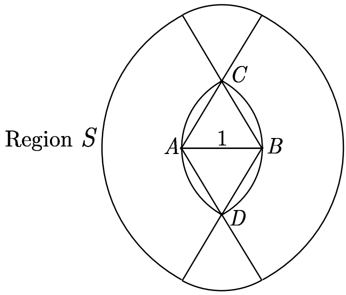


The region S consists of all points within 1 unit of R. In addition to R itself, S contains two 60 sectors of radius 1 and two 120 annuli of outer radius 2 and inner radius 1. The area of each sector is /6, and the area of each annulus is

$$\frac{\ }{3}(2^2 \quad 1^2) = \quad .$$

Therefore the area of S is

$$\frac{2}{3} \quad \frac{\sqrt{3}}{2}\Bigg) + 2 \quad \frac{\ }{6} + \quad = 3 \quad \frac{\sqrt{3}}{2}.$$



25. **(E)** Note that $n^3 a_n = 133.\overline{133}_n = a_n + n^2 + 3n + 3$, so

$$a_n = \frac{n^2+3n+3}{n^3 \quad 1} = \frac{(n+1)^3 \quad 1}{n(n^3 \quad 1)}.$$

Therefore

$$\begin{aligned} a_4 a_5 \quad a_{99} &= \frac{5^3 \quad 1}{4(4^3 \quad 1)} \quad \frac{6^3 \quad 1}{5(5^3 \quad 1)} \quad\quad \frac{100^3 \quad 1}{99(99^3 \quad 1)} \\ &= \frac{3!}{99!} \quad \frac{100^3 \quad 1}{4^3 \quad 1} = \frac{6}{99!} \quad \frac{99(100^2+100+1)}{63} \\ &= \frac{(2)(10{,}101)}{(21)(98!)} = \frac{962}{98!}. \end{aligned}$$

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Art of Problem Solving

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Institute for Operations Research and the Management Sciences

Mu Alpha Theta

National Council of Teachers of Mathematics

Pedagoguery Software Inc.

Pi Mu Epsilon

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

55th Annual American Mathematics Contest 12

AMC 12 - Contest B

Solutions Pamphlet

WEDNESDAY, FEBRUARY 25, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. David Wells,
Department of Mathematics
Penn State University,
New Kensington, PA 15068

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(A)** At Jenny's fourth practice she made $\frac{1}{2}(48) = 24$ free throws. At her third practice she made 12, at her second practice she made 6, and at her first practice she made 3.

2. **(D)** If $d \neq 0$, the value of the expression can be increased by interchanging 0 with the value of d. Therefore the maximum value must occur when $d = 0$. If $a = 1$, the value is c, which is 2 or 3. If $b = 1$, the value is $c \cdot a = 6$. If $c = 1$, the value is a^b, which is $2^3 = 8$ or $3^2 = 9$. Thus the maximum value is 9.

3. **(A)** Note that $1296 = 6^4 = 2^4 3^4$, so $x = y = 4$ and $x + y = 8$.

4. **(B)** There are 90 possible choices for x. Ten of these have a units digit of 7, and nine have a tens digit of 7. Because 77 has been counted twice, there are $10 + 9 - 1 = 18$ choices of x for which at least one digit is a 7. Therefore the probability is $\frac{18}{90} = \frac{1}{5}$.

5. **(A)** Isabella received $10d/7$ Canadian dollars at the border and spent 60 of them. Thus $10d/7 - 60 = d$, from which it follows that $d = 140$, and the sum of the digits of d is 5.

6. **(A)** Let downtown St. Paul, downtown Minneapolis, and the airport be located at S, M, and A, respectively. Then $\triangle MAS$ has a right angle at A, so by the Pythagorean Theorem,

$$MS = \sqrt{10^2 + 8^2} = \sqrt{164} \approx \sqrt{169} = 13.$$

7. **(B)** The areas of the regions enclosed by the square and the circle are $10^2 = 100$ and $\pi(10)^2 = 100\pi$, respectively. One quarter of the second region is also included in the first, so the area of the union is

$$100 + 100\pi - 25\pi = 100 + 75\pi.$$

8. **(D)** If there are n rows in the display, the bottom row contains $2n - 1$ cans. The total number of cans is therefore the sum of the arithmetic series

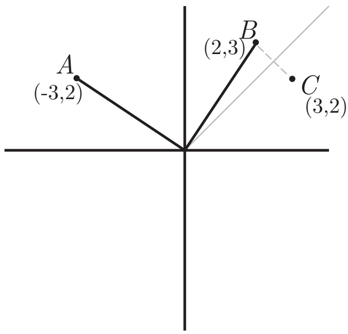
$$1 + 3 + 5 + \cdots + (2n - 1),$$

which is

$$\frac{n}{2}[(2n - 1) + 1] = n^2.$$

Thus $n^2 = 100$, so $n = 10$.

9. **(E)** The rotation takes $(-3, 2)$ into $B = (2, 3)$, and the reflection takes B into $C = (3, 2)$.



10. **(A)** The area of the annulus is the difference between the areas of the two circles, which is $\pi b^2 - \pi c^2$. Because the tangent $\overline{XZ}$ is perpendicular to the radius $\overline{OZ}$, $b^2 - c^2 = a^2$, so the area is πa^2.

11. **(D)** Each score of 100 is 24 points above the mean, so the five scores of 100 represent a total of $(5)(24) = 120$ points above the mean. Those scores must be balanced by scores totaling 120 points below the mean. Since no student scored more than $76-60 = 16$ points below the mean, the number of other students in the class must be an integer no less than $120/16$. The smallest such integer is 8, so the number of students in the class is at least 13. Note that the conditions of the problem are met if 5 students score 100 and 8 score 61.

OR

If there are k students in the class, the sum of their scores is $76k$. If the five scores of 100 are excluded, the sum of the remaining scores is $76k - 500$. Since each student scored at least 60, the sum is at least $60(k-5)$. Thus

$$76k - 500 \geq 60(k-5),$$

so $k \geq 12.5$. Since k must be an integer, $k \geq 13$.

12. **(C)** Let a_k be the k^{th} term of the sequence. For $k \geq 3$,

$$a_{k+1} = a_{k-2} + a_{k-1} - a_k, \quad \text{so} \quad a_{k+1} - a_{k-1} = -(a_k - a_{k-2}).$$

Because the sequence begins

$$2001, 2002, 2003, 2000, 2005, 1998, \ldots,$$

it follows that the odd-numbered terms and the even-numbered terms each form arithmetic progressions with common differences of 2 and -2, respectively. The 2004^{th} term of the original sequence is the 1002^{nd} term of the sequence 2002, 2000, 1998,…, and that term is $2002 + 1001(-2) = 0$.

13. **(A)** Since $f(f^{-1}(x)) = x$, it follows that $a(bx+a)+b = x$, so $ab = 1$ and $a^2 + b = 0$. Hence $a = b = -1$, so $a + b = -2$.

14. **(D)** Because $\triangle ABC$, $\triangle NBK$, and $\triangle AMJ$ are similar right triangles whose hypotenuses are in the ratio $13 : 8 : 1$, their areas are in the ratio $169 : 64 : 1$.

    The area of $\triangle ABC$ is $\frac{1}{2}(12)(5) = 30$, so the areas of $\triangle NBK$ and$\triangle AMJ$ are $\frac{64}{169}(30)$ and $\frac{1}{169}(30)$, respectively.
    Thus the area of pentagon $CMJKN$ is $(1 - \frac{64}{169} - \frac{1}{169})(30) = 240/13$.

15. **(B)** Let Jack's age be $10x + y$ and Bill's age be $10y + x$. In five years Jack will be twice as old as Bill. Therefore

$$10x + y + 5 = 2(10y + x + 5),$$

so $8x = 19y + 5$. The expression $19y + 5 = 16y + 8 + 3(y - 1)$ is a multiple of 8 if and only if $y - 1$ is a multiple of 8. Since both x and y are 9 or less, the only solution is $y = 1$ and $x = 3$. Thus Jack is 31 and Bill is 13, so the difference between their ages is 18.

16. **(C)** From the definition of f,

$$f(x + iy) = i(x - iy) = y + ix$$

for all real numbers x and y, so the numbers that satisfy $f(z) = z$ are the numbers of the form $x + ix$. The set of all such numbers is a line through the origin in the complex plane. The set of all numbers that satisfy $|z| = 5$ is a circle centered at the origin of the complex plane. The numbers satisfying both equations correspond to the points of intersection of the line and circle, of which there are two.

17. **(A)** Let r_1, r_2, and r_3 be the roots. Then

$$5 = \log_2 r_1 + \log_2 r_2 + \log_2 r_3 = \log_2 r_1 r_2 r_3,$$

so $r_1 r_2 r_3 = 2^5 = 32$. Since

$$8x^3 + 4ax^2 + 2bx + a = 8(x - r_1)(x - r_2)(x - r_3),$$

it follows that $a = -8r_1 r_2 r_3 = -256$.

18. **(E)** Let $B = (a, b)$ and $A = (-a, -b)$. Then

$$4a^2 + 7a - 1 = b \quad \text{and} \quad 4a^2 - 7a - 1 = -b.$$

Subtracting gives $b = 7a$, so $4a^2 + 7a - 1 = 7a$. Thus

$$a^2 = \frac{1}{4} \quad \text{and} \quad b^2 = (7a)^2 = \frac{49}{4},$$

so

$$AB = 2\sqrt{a^2 + b^2} = 2\sqrt{\frac{50}{4}} = 5\sqrt{2}.$$

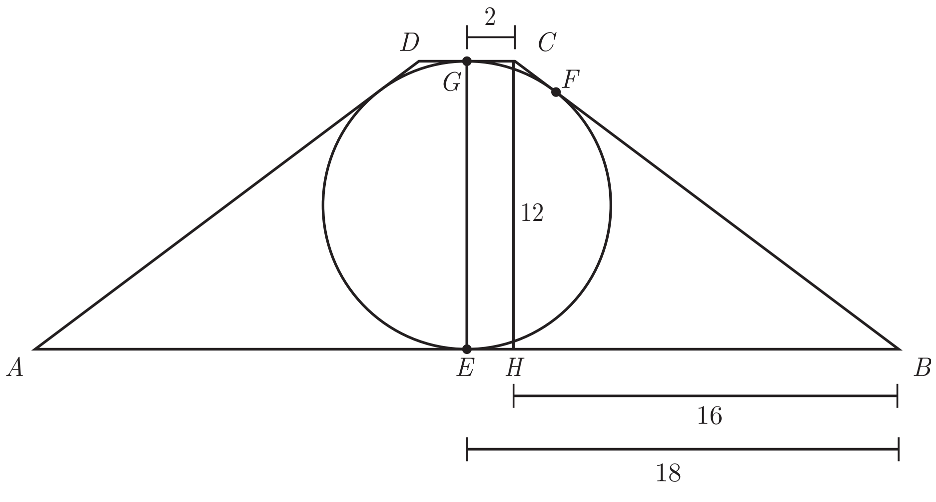
19. **(A)** Let $\overline{AB}$ and $\overline{DC}$ be parallel diameters of the bottom and top bases, respectively. A great circle of the sphere is tangent to all four sides of trapezoid $ABCD$. Let E, F, and G be the points of tangency on $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$, respectively. Then

$$FB = EB = 18 \quad \text{and} \quad FC = GC = 2,$$

so $BC = 20$. If H is on $\overline{AB}$ such that $\angle CHB$ is a right angle, then $HB = 18 - 2 = 16$. Thus

$$CH = \sqrt{20^2 - 16^2} = 12,$$

and the radius of the sphere is $(1/2)(12) = 6$.



20. **(B)** If the orientation of the cube is fixed, there are $2^6 = 64$ possible arrangements of colors on the faces. There are

$$2\binom{6}{6} = 2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5} = 12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3) = 6$ such arrangements. The total number of suitable arrangements is therefore $2 + 12 + 6 = 20$, and the probability is $20/64 = 5/16$.

21. **(C)** A line $y = mx$ intersects the ellipse in 0, 1, or 2 points. The intersection consists of exactly one point if and only if $m = a$ or $m = b$. Thus a and b are the values of m for which the system

$$2x^2 + xy + 3y^2 - 11x - 20y + 40 = 0$$
$$y = mx$$

has exactly one solution. Substituting mx for y in the first equation gives

$$2x^2 + mx^2 + 3m^2x^2 - 11x - 20mx + 40 = 0,$$

or, by rearranging the terms,

$$(3m^2 + m + 2)x^2 - (20m + 11)x + 40 = 0.$$

The discriminant of this equation is

$$(20m + 11)^2 - 4 \cdot 40 \cdot (3m^2 + m + 2) = -80m^2 + 280m - 199,$$

which must be zero if $m = a$ or $m = b$. Thus $a + b$ is the sum of the roots of the equation $-80m^2 + 280m - 199 = 0$, which is $\frac{280}{80} = \frac{7}{2}$.

22. **(C)** All the unknown entries can be expressed in terms of b. Since $100e = beh = ceg = def$, it follows that $h = 100/b$, $g = 100/c$, and $f = 100/d$. Comparing rows 1 and 3 then gives

$$50bc = 2\frac{100}{b}\frac{100}{c},$$

from which $c = 20/b$.

Comparing columns 1 and 3 gives

$$50d\frac{100}{c} = 2c\frac{100}{d},$$

from which $d = c/5 = 4/b$.

Finally, $f = 25b$, $g = 5b$, and $e = 10$. All the entries are positive integers if and only if $b = 1$, 2, or 4. The corresponding values for g are 5, 10, and 20, and their sum is 35.

23. **(C)** Let a denote the zero that is an integer. Because the coefficient of x^3 is 1, there can be no other rational zeros, so the two other zeros must be $\frac{a}{2} \pm r$ for some irrational number r. The polynomial is then

$$(x-a)\left[x-\left(\frac{a}{2}+r\right)\right]\left[x-\left(\frac{a}{2}-r\right)\right]$$
$$= x^3 - 2ax^2 + \left(\frac{5}{4}a^2 - r^2\right)x - a\left(\frac{1}{4}a^2 - r^2\right).$$

Therefore $a = 1002$ and the polynomial is

$$x^3 - 2004x^2 + (5(501)^2 - r^2)x - 1002((501)^2 - r^2).$$

All coefficients are integers if and only if r^2 is an integer, and the zeros are positive and distinct if and only if $1 \le r^2 \le 501^2 - 1 = 251{,}000$. Because r cannot be an integer, there are $251{,}000 - 500 = 250{,}500$ possible values of n.

24. **(B)** Let $\angle DBE = \alpha$ and $\angle DBC = \beta$. Then $\angle CBE = \alpha - \beta$ and $\angle ABE = \alpha + \beta$, so $\tan(\alpha-\beta)\tan(\alpha+\beta) = \tan^2\alpha$. Thus

$$\frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} \cdot \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \tan^2\alpha,$$

from which it follows that

$$\tan^2\alpha - \tan^2\beta = \tan^2\alpha(1-\tan^2\alpha\tan^2\beta).$$

Upon simplifying, $\tan^2\beta(\tan^4\alpha - 1) = 0$, so $\tan\alpha = 1$ and $\alpha = \frac{\pi}{4}$. Let $DC = a$ and $BD = b$. Then $\cot\angle DBC = \frac{b}{a}$. Because $\angle CBE = \frac{\pi}{4} - \beta$ and $\angle ABE = \frac{\pi}{4} + \beta$, it follows that $\cot\angle CBE = \tan\angle ABE = \tan\left(\frac{\pi}{4}+\beta\right) = \frac{1+\frac{a}{b}}{1-\frac{a}{b}} = \frac{b+a}{b-a}$. Thus the numbers 1, $\frac{b+a}{b-a}$, and $\frac{b}{a}$ form an arithmetic progression, so $\frac{b}{a} = \frac{b+3a}{b-a}$. Setting $b = ka$ yields $k^2 - 2k - 3 = 0$, and the only positive solution is $k = 3$. Hence $b = \frac{BE}{\sqrt{2}} = 5\sqrt{2}$, $a = \frac{5\sqrt{2}}{3}$, and the area of $\triangle ABC$ is $ab = \frac{50}{3}$.

25. **(B)** The smallest power of 2 with a given number of digits has a first digit of 1, and there are elements of S with n digits for each positive integer $n \le 603$, so there are 603 elements of S whose first digit is 1. Furthermore, if the first digit of 2^k is 1, then the first digit of 2^{k+1} is either 2 or 3, and the first digit of 2^{k+2} is either 4,5,6, or 7. Therefore there are 603 elements of S whose first digit is 2 or 3, 603 elements whose first digit is 4,5,6, or 7, and $2004 - 3(603) = 195$ whose first digit is 8 or 9. Finally, note that the first digit of 2^k is 8 or 9 if and only if the first digit of 2^{k-1} is 4, so there are 195 elements of S whose first digit is 4.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Art of Problem Solving

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Institute for Operations Research and the Management Sciences

Mu Alpha Theta

National Council of Teachers of Mathematics

Pedagoguery Software Inc.

Pi Mu Epsilon

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

56th Annual American Mathematics Contest 12

AMC 12 – Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 1, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **(D)** It is given that $0.1x = 2$ and $0.2y = 2$, so $x = 20$ and $y = 10$. Thus $x - y = 10$.

2. **(B)** Since $2x + 7 = 3$ we have $x = -2$. Hence

$$-2 = bx - 10 = -2b - 10, \quad \text{so} \quad 2b = -8, \text{ and } b = -4.$$

3. **(B)** Let w be the width of the rectangle. Then the length is $2w$, and

$$x^2 = w^2 + (2w)^2 = 5w^2.$$

The area is consequently $w(2w) = 2w^2 = \frac{2}{5}x^2$.

4. **(A)** If Dave buys seven windows separately he will purchase six and receive one free, for a cost of \$600. If Doug buys eight windows separately, he will purchase seven and receive one free, for a total cost of \$700. The total cost to Dave and Doug purchasing separately will be \$1300. If they purchase fifteen windows together, they will need to purchase only 12 windows, for a cost of \$1200, and will receive 3 free. This will result in a savings of \$100.

5. **(B)** The sum of the 50 numbers is $20 \cdot 30 + 30 \cdot 20 = 1200$. Their average is $1200/50 = 24$.

6. **(B)** Because (rate)(time) = (distance), the distance Josh rode was $(4/5)(2) = 8/5$ of the distance that Mike rode. Let m be the number of miles that Mike had ridden when they met. Then the number of miles between their houses is

$$13 = m + \frac{8}{5}m = \frac{13}{5}m.$$

Thus $m = 5$.

7. **(C)** The symmetry of the figure implies that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles. So

$$BH = CE = \sqrt{BC^2 - BE^2} = \sqrt{50 - 1} = 7,$$

and $EH = BH - BE = 7 - 1 = 6$. Hence the square $EFGH$ has area $6^2 = 36$.

OR

As in the first solution, $BH = 7$. Now note that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles, so

$$\text{Area}(EFGH) = \text{Area}(ABCD) - 4\text{Area}(\triangle ABH) = 50 - 4\left(\tfrac{1}{2} \cdot 1 \cdot 7\right) = 36.$$

8. **(D)** Since $A, M,$ and C are digits we have

$$0 \le A + M + C \le 9 + 9 + 9 = 27.$$

The prime factorization of 2005 is $2005 = 5 \cdot 401$, so

$$100A + 10M + C = 401 \quad \text{and} \quad A + M + C = 5.$$

Hence $A = 4$, $M = 0$, and $C = 1$.

9. **(A)** The quadratic formula yields

$$x = \frac{-(a+8) \pm \sqrt{(a+8)^2 - 4 \cdot 4 \cdot 9}}{2 \cdot 4}.$$

The equation has only one solution precisely when the value of the discriminant, $(a+8)^2 - 144$, is 0. This implies that $a = -20$ or $a = 4$, and the sum is -16.

OR

The equation has one solution if and only if the polynomial is the square of a binomial with linear term $\pm\sqrt{4x^2} = \pm 2x$ and constant term $\pm\sqrt{9} = \pm 3$. Because $(2x \pm 3)^2$ has a linear term $\pm 12x$, it follows that $a + 8 = \pm 12$. Thus a is either -20 or 4, and the sum of those values is -16.

10. **(B)** The unit cubes have a total of $6n^3$ faces, of which $6n^2$ are red. Therefore

$$\frac{1}{4} = \frac{6n^2}{6n^3} = \frac{1}{n}, \quad \text{so} \quad n = 4.$$

11. **(E)** The first and last digits must be both odd or both even for their average to be an integer. There are $5 \cdot 5 = 25$ odd-odd combinations for the first and last digits. There are $4 \cdot 5 = 20$ even-even combinations that do not use zero as the first digit. Hence the total is 45.

12. **(D)** The slope of the line is

$$\frac{1000 - 1}{100 - 1} = \frac{111}{11},$$

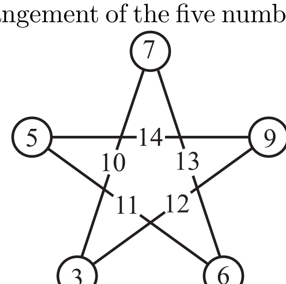
so all points on the line have the form $(1+11t, 1+111t)$. Such a point has integer coordinates if and only if t is an integer, and the point is strictly between A and B if and only if $0 < t < 9$. Thus there are 8 points with the required property.

13. **(D)** Each number appears in two sums, so the sum of the sequence is

$$2(3 + 5 + 6 + 7 + 9) = 60.$$

The middle term of a five-term arithmetic sequence is the mean of its terms, so $60/5 = 12$ is the middle term.

The figure shows an arrangement of the five numbers that meets the requirement.



14. **(D)** A standard die has a total of 21 dots. For $1 \le n \le 6$, a dot is removed from the face with n dots with probability $n/21$. Thus the face that originally has n dots is left with an odd number of dots with probability $n/21$ if n is even and $1 - n/21$ if n is odd. Each face is the top face with probability $1/6$. Therefore the top face has an odd number of dots with probability

$$\frac{1}{6}\left(\left(1-\frac{1}{21}\right)+\frac{2}{21}+\left(1-\frac{3}{21}\right)+\frac{4}{21}+\left(1-\frac{5}{21}\right)+\frac{6}{21}\right) = \frac{1}{6}\left(3+\frac{3}{21}\right) = \frac{1}{6}\cdot\frac{66}{21} = \frac{11}{21}.$$

OR

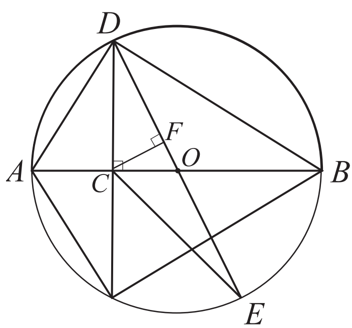
The probability that the top face is odd is $1/3$ if a dot is removed from an odd face, and the probability that the top face is odd is $2/3$ if a dot is removed from an even face. Because each dot has the probability $1/21$ of being removed, the top face is odd with probability

$$\left(\frac{1}{3}\right)\left(\frac{1+3+5}{21}\right)+\left(\frac{2}{3}\right)\left(\frac{2+4+6}{21}\right) = \frac{33}{63} = \frac{11}{21}.$$

15. **(C)** Let O be the center of the circle. Each of $\triangle DCE$ and $\triangle ABD$ has a diameter of the circle as a side. Thus the ratio of their areas is the ratio of the two altitudes to the diameters. These altitudes are $\overline{DC}$ and the altitude from C to $\overline{DO}$ in $\triangle DCE$. Let F be the foot of this second altitude. Since $\triangle CFO$ is similar to $\triangle DCO$,

$$\frac{CF}{DC} = \frac{CO}{DO} = \frac{AO - AC}{DO} = \frac{\frac{1}{2}AB - \frac{1}{3}AB}{\frac{1}{2}AB} = \frac{1}{3},$$

which is the desired ratio.



OR

Because $AC = AB/3$ and $AO = AB/2$, we have $CO = AB/6$. Triangles DCO and DAB have a common altitude to $\overline{AB}$ so the area of $\triangle DCO$ is $\frac{1}{6}$ the area of $\triangle ADB$. Triangles DCO and ECO have equal areas since they have a common base $\overline{CO}$ and their altitudes are equal. Thus the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$ is $1/3$.

16. **(D)** Consider a right triangle as shown. By the Pythagorean Theorem,

$$(r+s)^2 = (r-3s)^2 + (r-s)^2$$

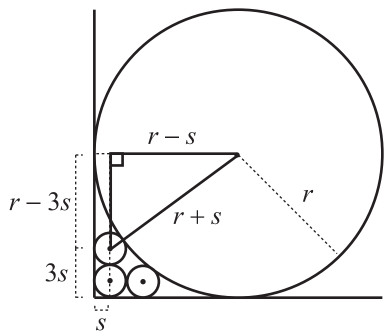
so

$$r^2 + 2rs + s^2 = r^2 - 6rs + 9s^2 + r^2 - 2rs + s^2$$

and

$$0 = r^2 - 10rs + 9s^2 = (r-9s)(r-s).$$

But $r \neq s$, so $r = 9s$ and $r/s = 9$.

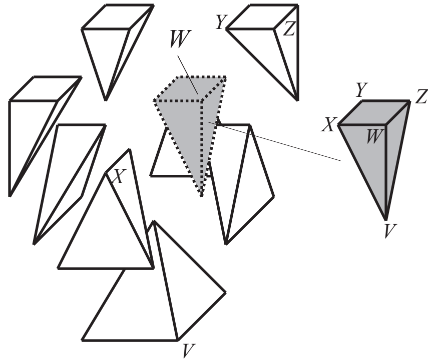


OR

Because the ratio r/s is independent of the value of s, assume that $s = 1$ and proceed as in the previous solution.

17. **(A)** The piece that contains W is shown. It is a pyramid with vertices V, W, X, Y, and Z. Its base $WXYZ$ is a square with sides of length $1/2$ and its altitude VW is 1. Hence the volume of this pyramid is

$$\frac{1}{3}\left(\frac{1}{2}\right)^2(1) = \frac{1}{12}.$$



18. **(A)** Of the numbers less than 1000, 499 of them are divisible by two, 333 are divisible by 3, and 199 are divisible by 5. There are 166 multiples of 6, 99 multiples of 10, and 66 multiples of 15. And there are 33 numbers that are divisible by 30. So by the Inclusion-Exclusion Principle there are

$$499 + 333 + 199 - 166 - 99 - 66 + 33 = 733$$

numbers that are divisible by at least one of 2, 3, or 5. Of the remaining $999 - 733 = 266$ numbers, 165 are primes other than 2, 3, or 5. Note that 1 is neither prime nor composite. This leaves exactly 100 prime-looking numbers.

19. **(B)** Because the odometer uses only 9 digits, it records mileage in base-9 numerals, except that its digits 5, 6, 7, 8, and 9 represent the base-9 digits 4, 5, 6, 7, and 8. Therefore the mileage is

$$2004_{\text{base } 9} = 2 \cdot 9^3 + 4 = 2 \cdot 729 + 4 = 1462.$$

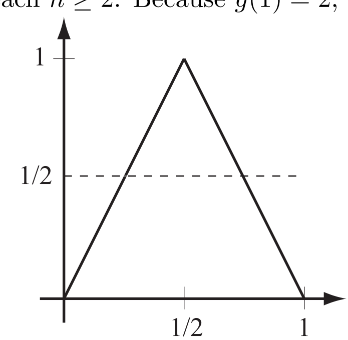
OR

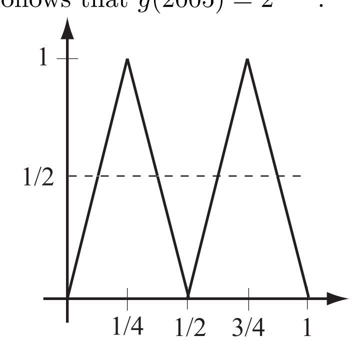
The number of miles traveled is the same as the number of integers between 1 and 2005, inclusive, that do not contain the digit 4. First consider the integers less than 2000. There are two choices for the first digit, including 0, and 9 choices for each of the other three. Because one combination of choices is 0000, there are $2 \cdot 9^3 - 1 = 1457$ positive integers less than 2000 that do not contain the digit 4. There are 5 integers between 2000 and 2005, inclusive, that do not have a 4 as a digit, so the car traveled $1457 + 5 = 1462$ miles.

20. **(E)** The graphs of $y = f(x)$ and $y = f^{[2]}(x)$ are shown below. For $n \geq 2$ we have

$$f^{[n]}(x) = f^{[n-1]}(f(x)) = \begin{cases} f^{[n-1]}(2x), & \text{if } 0 \leq x \leq \frac{1}{2}, \\ f^{[n-1]}(2-2x), & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let $g(n)$ be the number of values of x in $[0,1]$ for which $f^{[n]}(x) = 1/2$. Then $f^{[n]}(x) = 1/2$ for $g(n-1)$ values of x in $[0, 1/2]$ and $g(n-1)$ values of x in $[1/2, 1]$. Furthermore $f^{[n]}(1/2) = f^{[n-1]}(1) = 0 \neq 1/2$ for $n \geq 2$. Hence $g(n) = 2g(n-1)$ for each $n \geq 2$. Because $g(1) = 2$, it follows that $g(2005) = 2^{2005}$.





21. **(C)** The two equations are equivalent to $b = a^{(c^{2005})}$ and $c = 2005 - b - a$, so

$$c = 2005 - a^{(c^{2005})} - a.$$

If $c > 1$, then

$$b \geq 2^{(2^{2005})} > 2005 > 2005 - a - c = b,$$

which is a contradiction. For $c = 0$ and for $c = 1$, the only solutions are the ordered triples $(2004, 1, 0)$ and $(1002, 1002, 1)$, respectively. Thus the number of solutions is 2.

22. **(B)** Let the dimensions of P be x, y, and z. The sum of the lengths of the edges of P is $4(x + y + z)$, and the surface area of P is $2xy + 2yz + 2xz$, so

$$x + y + z = 28 \quad \text{and} \quad 2xy + 2yz + 2xz = 384.$$

Each internal diagonal of P is a diameter of the sphere, so

$$(2r)^2 = (x^2 + y^2 + z^2) = (x + y + z)^2 - (2xy + 2xz + 2yz) = 28^2 - 384 = 400.$$

So $2r = 20$ and $r = 10$.

Note: There are infinitely many positive solutions of the system $x+y+z = 28$, $2xy+2yz+2xz = 384$, so there are infinitely many non-congruent boxes meeting the given conditions, but each can be inscribed in a sphere of radius 10.

23. **(B)** Let $a = 2^j$ and $b = 2^k$. Then

$$\log_a b = \log_{2^j} 2^k = \frac{\log 2^k}{\log 2^j} = \frac{k \log 2}{j \log 2} = \frac{k}{j},$$

so $\log_a b$ is an integer if and only if k is an integer multiple of j. For each j, the number of integer multiples of j that are at most 25 is $\left\lfloor \frac{25}{j} \right\rfloor$. Because $j \neq k$, the number of possible values of k for each j is $\left\lfloor \frac{25}{j} \right\rfloor - 1$. Hence the total number of ordered pairs (a, b) is

$$\sum_{j=1}^{25} \left(\left\lfloor \frac{25}{j} \right\rfloor - 1 \right) = 24 + 11 + 7 + 5 + 4 + 3 + 2(2) + 4(1) = 62.$$

Since the total number of possibilities for a and b is $25 \cdot 24$, the probability that $\log_a b$ is an integer is

$$\frac{62}{25 \cdot 24} = \frac{31}{300}.$$

24. **(B)** The polynomial $P(x) \cdot R(x)$ has degree 6, so $Q(x)$ must have degree 2. Therefore Q is uniquely determined by the ordered triple $(Q(1), Q(2), Q(3))$. When $x = 1$, 2, or 3, we have $0 = P(x) \cdot R(x) = P\left(Q(x)\right)$. It follows that $(Q(1), Q(2), Q(3))$ is one of the 27 ordered triples (i, j, k), where i, j, and k can

be chosen from the set $\{1, 2, 3\}$. However, the choices $(1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$, $(1, 2, 3)$, and $(3, 2, 1)$ lead to polynomials $Q(x)$ defined by $Q(x) = 1$, 2, 3, x, and $4-x$, respectively, all of which have degree less than 2. The other 22 choices for $(Q(1), Q(2), Q(3))$ yield non-collinear points, so in each case $Q(x)$ is a quadratic polynomial.

25. **(C)** Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, and $C(x_3, y_3, z_3)$ be the vertices of such a triangle. Let

$$(\Delta x_k, \Delta y_k, \Delta z_k) = (x_{k+1} - x_k, y_{k+1} - y_k, z_{k+1} - z_k), \text{ for } 1 \le k \le 3,$$

where $(x_4, y_4, z_4) = (x_1, y_1, z_1)$. Then $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the ordered triples $(0, 0, 1)$, $(0, 0, 2)$, $(0, 1, 1)$, $(0, 1, 2)$, $(0, 2, 2)$, $(1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 2)$, or $(2, 2, 2)$. Since $\triangle ABC$ is equilateral, $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ correspond to permutations of the same ordered triple (a, b, c). Because

$$\sum_{k=1}^{3} \Delta x_k = \sum_{k=1}^{3} \Delta y_k = \sum_{k=1}^{3} \Delta z_k = 0,$$

the sums

$$\sum_{k=1}^{3} |\Delta x_k|, \quad \sum_{k=1}^{3} |\Delta y_k|, \quad \text{and} \quad \sum_{k=1}^{3} |\Delta z_k|$$

are all even. Therefore $(|\Delta x_k|, |\Delta y_k|, |\Delta z_k|)$ is a permutation of one of the triples $(0, 0, 2)$, $(0, 1, 1)$, $(0, 2, 2)$, $(1, 1, 2)$, or $(2, 2, 2)$.

If $(a, b, c) = (0, 0, 2)$, each side of $\triangle ABC$ is parallel to one of the coordinate axes, which is impossible.

If $(a, b, c) = (2, 2, 2)$, each side of $\triangle ABC$ is an interior diagonal of the $2 \times 2 \times 2$ cube that contains S, which is also impossible.

If $(a, b, c) = (0, 2, 2)$, each side of $\triangle ABC$ is a face diagonal of the $2 \times 2 \times 2$ cube that contains S. The three faces that join at any vertex determine such a triangle, so the triple $(0, 2, 2)$ produces a total of 8 triangles.

If $(a, b, c) = (0, 1, 1)$, each side of $\triangle ABC$ is a face diagonal of a unit cube within the larger cube that contains S. There are 8 such unit cubes producing a total of $8 \cdot 8 = 64$ triangles.

There are two types of line segments for which $(a, b, c) = (1, 1, 2)$. One type joins the center of the face of the $2 \times 2 \times 2$ cube to a vertex on the opposite face. The other type joins the midpoint of one edge of the cube to the midpoint of another edge. Only the second type of segment can be a side of $\triangle ABC$. The midpoint of each of the 12 edges is a vertex of two suitable triangles, so there are $12 \cdot 2/3 = 8$ such triangles.

The total number of triangles is $8 + 64 + 8 = 80$.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

The Mathematical Association of America
University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour & Company
Mu Alpha Theta
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
USA Math Talent Search
W. H. Freeman & Company

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

56th Annual American Mathematics Contest 12

AMC 12 – Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 16, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **(A)** The scouts bought $1000/5 = 200$ groups of five candy bars at a total cost of $200 \cdot 2 = 400$ dollars. They sold $1000/2 = 500$ groups of two candy bars for a total of $500 \cdot 1 = 500$ dollars. Their profit was $\$500 - \$400 = \$100$.

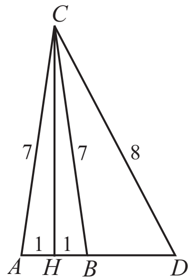
2. **(D)** We have
$$\frac{x}{100} \cdot x = 4, \quad \text{so} \quad x^2 = 400.$$
Because $x > 0$, it follows that $x = 20$.

3. **(C)** The number of CDs that Brianna will finally buy is three times the number she has already bought. The fraction of her money that will be required for all the purchases is $(3)(1/5) = 3/5$. The fraction she will have left is $1 - 3/5 = 2/5$.

4. **(B)** To earn an A on at least 80% of her quizzes, Lisa needs to receive an A on at least $(0.8)(50) = 40$ quizzes. Thus she must earn an A on at least $40 - 22 = 18$ of the remaining 20. So she can earn a grade lower than an A on at most 2 of the remaining quizzes.

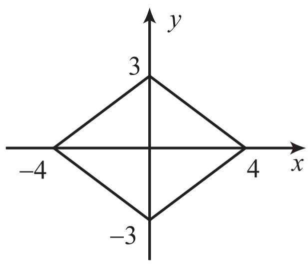
5. **(A)** The four white quarter circles in each tile have the same area as a whole circle of radius 1/2, that is, $\pi(1/2)^2 = \pi/4$ square feet. So the area of the shaded portion of each tile is $1 - \pi/4$ square feet. Since there are $8 \cdot 10 = 80$ tiles in the entire floor, the area of the total shaded region in square feet is
$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

6. **(A)** Let $\overline{CH}$ be an altitude of $\triangle ABC$. Applying the Pythagorean Theorem to $\triangle CHB$ and to $\triangle CHD$ produces
$$8^2 - (BD+1)^2 = CH^2 = 7^2 - 1^2 = 48, \quad \text{so} \quad (BD+1)^2 = 16.$$
Thus $BD = 3$.



7. **(D)** The graph is symmetric with respect to both coordinate axes, and in the first quadrant it coincides with the graph of the line $3x + 4y = 12$. Therefore the region is a rhombus, and the area is

$$\text{Area} = 4\left(\frac{1}{2}(4 \cdot 3)\right) = 24.$$



8. **(C)** The vertex of the parabola is $(0, a^2)$. The line passes through the vertex if and only if $a^2 = 0 + a$. There are two solutions to this equation, namely $a = 0$ and $a = 1$.

9. **(B)** The percentage of students getting 95 points is

$$100 - 10 - 25 - 20 - 15 = 30,$$

so the mean score on the exam is

$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$

Since fewer than half of the scores were less than 85, and fewer than half of the scores were greater than 85, the median score is 85. The difference between the mean and the median score on this exam is $86 - 85 = 1$.

10. **(E)** The sequence begins 2005, 133, 55, 250, 133, ... Thus after the initial term 2005, the sequence repeats the cycle 133, 55, 250. Because $2005 = 1 + 3 \cdot 668$, the 2005$^{\text{th}}$ term is the same as the last term of the repeating cycle, 250.

11. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

12. **(D)** Let r_1 and r_2 be the roots of $x^2+px+m=0$. Since the roots of $x^2+mx+n=0$ are $2r_1$ and $2r_2$, we have the following relationships:

$$m = r_1r_2, \quad n = 4r_1r_2, \quad p = -(r_1+r_2), \quad \text{and} \quad m = -2(r_1+r_2).$$

So

$$n = 4m, \quad p = \frac{1}{2}m, \quad \text{and} \quad \frac{n}{p} = \frac{4m}{\frac{1}{2}m} = 8.$$

OR

The roots of

$$\left(\frac{x}{2}\right)^2 + p\left(\frac{x}{2}\right) + m = 0$$

are twice those of $x^2+px+m=0$. Since the first equation is equivalent to $x^2+2px+4m=0$, we have

$$m = 2p \quad \text{and} \quad n = 4m, \quad \text{so} \quad \frac{n}{p} = 8.$$

13. **(D)** Since $4^{x_1} = 5$, $5^{x_2} = 6, \ldots,$ $127^{x_{124}} = 128$, we have

$$4^{7/2} = 128 = 127^{x_{124}} = \left(126^{x_{123}}\right)^{x_{124}} = 126^{x_{123}\cdot x_{124}} = \cdots = 4^{x_1x_2\cdots x_{124}}.$$

So $x_1x_2\cdots x_{124} = 7/2$.

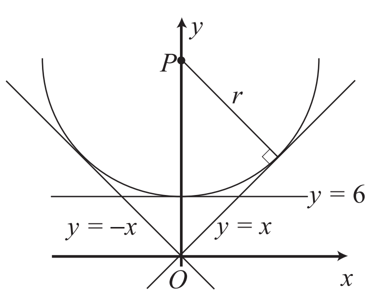
OR

We have

$$\begin{aligned} x_1x_2\cdots x_{124} &= \log_4 5 \cdot \log_5 6 \cdots \log_{127} 128 \\ &= \frac{\log 5}{\log 4}\cdot\frac{\log 6}{\log 5}\cdots\frac{\log 128}{\log 127} = \frac{\log 128}{\log 4} = \frac{\log 2^7}{\log 2^2} = \frac{7\log 2}{2\log 2} = \frac{7}{2}. \end{aligned}$$

14. **(E)** Let O denote the origin, P the center of the circle, and r the radius. A radius from the center to the point of tangency with the line $y = x$ forms a right triangle with hypotenuse $\overline{OP}$. This right triangle is isosceles since the line $y = x$ forms a 45° angle with the y-axis. So

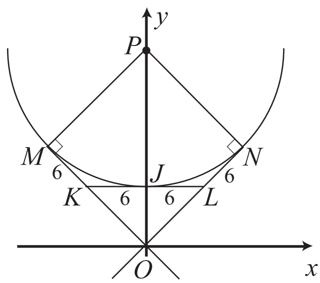
$$r\sqrt{2} = r + 6 \quad \text{and} \quad r = \frac{6}{\sqrt{2}-1} = 6\sqrt{2}+6.$$



OR

Let the line $y = -x$ intersect the circle and the line $y = 6$ at M and K, respectively, and let the line $y = x$ intersect the circle and the line $y = 6$ at N and L, respectively. Quadrilateral $PMON$ has four right angles and $MP = PN$, so $PMON$ is a square. In addition, $MK = KJ = 6$ and $KO = 6\sqrt{2}$. Hence

$$r = MO = MK + KO = 6 + 6\sqrt{2}.$$



15. **(D)** The sum of the digits 1 through 9 is 45, so the sum of the eight digits is between 36 and 44, inclusive. The sum of the four units digits is between $1+2+3+4 = 10$ and $6+7+8+9 = 30$, inclusive, and also ends in 1. Therefore the sum of the units digits is either 11 or 21. If the sum of the units digits is 11, then the sum of the tens digits is 21, so the sum of all eight digits is 32, an impossibility. If the sum of the units digits is 21, then the sum of the tens digits is 20, so the sum of all eight digits is 41. Thus the missing digit is $45 - 41 = 4$. Note that the numbers 13, 25, 86, and 97 sum to 221.

OR

Each of the two-digit numbers leaves the same remainder when divided by 9 as does the sum of its digits. Therefore the sum of the four two-digit numbers leaves the same remainder when divided by 9 as the sum of all eight digits. Let d be the missing digit. Because 221 when divided by 9 leaves a remainder of 5, and the sum of the digits from 1 through 9 is 45, the number $(45 - d)$ must leave a remainder of 5 when divided by 9. Thus $d = 4$.

16. **(D)** The centers of the unit spheres are at the 8 points with coordinates $(\pm1, \pm1, \pm1)$, which are at a distance

$$\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

from the origin. Hence the maximum distance from the origin to any point on the spheres is $1 + \sqrt{3}$.

17. **(B)** The given equation is equivalent to

$$\log_{10}\left(2^a \cdot 3^b \cdot 5^c \cdot 7^d\right) = 2005, \quad \text{so} \quad 2^a \cdot 3^b \cdot 5^c \cdot 7^d = 10^{2005} = 2^{2005} \cdot 5^{2005}.$$

Let M be the least common denominator of a, b, c and d. It follows that

$$2^{Ma} \cdot 3^{Mb} \cdot 5^{Mc} \cdot 7^{Md} = 2^{2005M} \cdot 5^{2005M}.$$

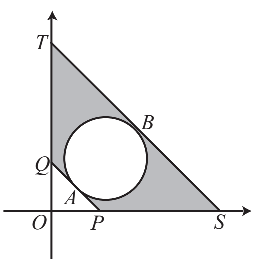
Since the exponents are all integers, the Fundamental Theorem of Arithmetic implies that

$$Ma = 2005M, \quad Mb = 0, \quad Mc = 2005M, \quad \text{and} \quad Md = 0.$$

Hence the only solution is $(a, b, c, d) = (2005, 0, 2005, 0)$.

18. **(C)** For $\triangle ABC$ to be acute, all angles must be acute. For $\angle A$ to be acute, point C must lie above the line passing through A and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, point C must lie below the line through B and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$. For $\angle C$ to be acute, point C must lie outside the circle U that has $\overline{AB}$ as a diameter. Let O denote the origin. Region R, shaded below, has area equal to

$$\begin{aligned} \text{Area}(\triangle OST) - \text{Area}(\triangle OPQ) - \text{Area}(\text{Circle } U) &= \frac{1}{2} \cdot 14^2 - \frac{1}{2} \cdot 4^2 - \pi\left(\frac{\sqrt{50}}{2}\right)^2 \\ &= 90 - \frac{25}{2}\pi \approx 51. \end{aligned}$$



19. **(E)** By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since $99(a^2 - b^2)$ must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2,$$

for some positive integer k. Because a and b are distinct digits, we have $a - b \leq 9 - 1 = 8$ and $a + b \leq 9 + 8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and k is either 1 or 2.

If $k = 2$, then $(a,b) = (15/2, 7/2)$, which is impossible. Thus $k = 1$ and $(a,b) = (6,5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

20. **(C)** Note that the sum of the elements in the set is 8. Let $x = a+b+c+d$, so $e+f+g+h = 8-x$. Then

$$\begin{aligned}(a+b+c+d)^2 + (e+f+g+h)^2 &= x^2 + (8-x)^2\\ &= 2x^2 - 16x + 64 = 2(x-4)^2 + 32 \geq 32.\end{aligned}$$

The value of 32 can be attained if and only if $x = 4$. However, it may be assumed without loss of generality that $a = 13$, and no choice of b, c, and d gives a total of 4 for x. Thus $(x-4)^2 \geq 1$, and

$$(a+b+c+d)^2 + (e+f+g+h)^2 = 2(x-4)^2 + 32 \geq 34.$$

A total of 34 can be attained by letting a, b, c, and d be distinct elements in the set $\{-7, -5, 2, 13\}$.

21. **(C)** Let $n = 7^k Q$, where Q is the product of primes, none of which is 7. Let d be the number of divisors of Q. Then n has $(k+1)d$ divisors. Also $7n = 7^{k+1}Q$, so $7n$ has $(k+2)d$ divisors. Thus

$$\frac{(k+2)d}{(k+1)d} = \frac{80}{60} = \frac{4}{3} \quad \text{and} \quad 3(k+2) = 4(k+1).$$

Hence $k = 2$. Note that $n = 2^{19}7^2$ meets the conditions of the problem.

22. **(E)** Note that

$$z_{n+1} = \frac{iz_n}{\overline{z_n}} = \frac{iz_n^2}{z_n\overline{z_n}} = \frac{iz_n^2}{|z_n|^2}.$$

Since $|z_0| = 1$, the sequence satisfies

$$z_1 = iz_0^2,\ z_2 = iz_1^2 = i\left(iz_0^2\right)^2 = -iz_0^4,$$

and, in general, when $k \geq 2$,

$$z_k = -iz_0^{2^k}.$$

Hence z_0 satisfies the equation $1 = -iz_0^{(2^{2005})}$, so $z_0^{(2^{2005})} = i$. Because every nonzero complex number has n distinct nth roots, this equation has 2^{2005} solutions. So there are 2^{2005} possible values for z_0.

OR

Define

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta.$$

Then if $z_n = r\,\text{cis}\,\theta$ we have

$$z_{n+1} = \frac{\text{cis}\,(\theta + 90^\circ)}{\text{cis}\,(-\theta)} = \text{cis}\,(2\theta + 90^\circ).$$

The first terms of the sequence are $z_0 = \text{cis}\,\alpha$, $z_1 = \text{cis}\,(2\alpha + 90^\circ) = iz_0^2$, $z_2 = \text{cis}\,(4\alpha + 270^\circ) = \text{cis}\,(4\alpha - 90^\circ) = \frac{z_0^4}{i}$, $z_3 = \text{cis}\,(8\alpha - 90^\circ) = \frac{z_0^8}{i}$, and, in general,

$$z_n = \frac{z_0^{(2^n)}}{i} \quad \text{for } n \geq 2.$$

So

$$z_{2005} = \frac{z_0^{(2^{2005})}}{i} = 1 \quad \text{and} \quad z_0^{(2^{2005})} = i.$$

As before, there are 2^{2005} possible solutions for z_0.

23. **(B)** From the given conditions it follows that

$$x + y = 10^z, \quad x^2 + y^2 = 10 \cdot 10^z \quad \text{and} \quad 10^{2z} = (x+y)^2 = x^2 + 2xy + y^2.$$

Thus

$$xy = \frac{1}{2}(10^{2z} - 10 \cdot 10^z).$$

Also

$$(x+y)^3 = 10^{3z} \quad \text{and} \quad x^3 + y^3 = (x+y)^3 - 3xy(x+y),$$

which yields

$$\begin{aligned} x^3 + y^3 &= 10^{3z} - \frac{3}{2}(10^{2z} - 10 \cdot 10^z)(10^z) \\ &= 10^{3z} - \frac{3}{2}(10^{3z} - 10 \cdot 10^{2z}) = -\frac{1}{2}10^{3z} + 15 \cdot 10^{2z}, \end{aligned}$$

and $a + b = -\frac{1}{2} + 15 = 29/2$.

No other value of $a + b$ is possible for all members of S, because the triple $\left(\frac{1}{2}(1+\sqrt{19}), \frac{1}{2}(1-\sqrt{19}), 0\right)$ is in S, and for this ordered triple, the equation $x^3 + y^3 = a \cdot 10^{3z} + b \cdot 10^{2z}$ reduces to $a + b = 29/2$.

24. **(A)** Suppose that the triangle has vertices $A(a, a^2)$, $B(b, b^2)$ and $C(c, c^2)$. The slope of line segment $\overline{AB}$ is

$$\frac{b^2 - a^2}{b - a} = b + a,$$

so the slopes of the three sides of the triangle have a sum

$$(b + a) + (c + b) + (a + c) = 2 \cdot \frac{m}{n}.$$

The slope of one side is $2 = \tan\theta$, for some angle θ, and the two remaining sides have slopes

$$\tan\left(\theta \pm \frac{\pi}{3}\right) = \frac{\tan\theta \pm \tan(\pi/3)}{1 \mp \tan\theta\tan(\pi/3)} = \frac{2 \pm \sqrt{3}}{1 \mp 2\sqrt{3}} = -\frac{8 \pm 5\sqrt{3}}{11}.$$

Therefore

$$\frac{m}{n} = \frac{1}{2}\left(2 - \frac{8+5\sqrt{3}}{11} - \frac{8-5\sqrt{3}}{11}\right) = \frac{3}{11},$$

and $m + n = 14$.

Such a triangle exists. The x-coordinates of its vertices are $(11 \pm 5\sqrt{3})/11$ and $-19/11$.

OR

Define the vertices as in the first solution, with the added stipulations that $a < b$ and $\overline{AB}$ has slope 2. Then

$$2 = \frac{b^2 - a^2}{b - a} = b + a, \quad \text{so} \quad a = 1 - k \text{ and } b = 1 + k,$$

for some $k > 0$. If D is the midpoint of $\overline{AB}$, then

$$D = \left(1, \frac{(1-k)^2 + (1+k)^2}{2}\right) = (1, 1 + k^2).$$

The slope of the altitude $\overline{CD}$ is $-1/2$, so

$$1 - c = 2(c^2 - 1 - k^2).$$

Therefore

$$CD^2 = (1-c)^2 + \left(c^2 - 1 - k^2\right)^2 = \frac{5}{4}(1-c)^2.$$

Because $\triangle ABC$ is equilateral, we also have

$$CD^2 = \frac{3}{4}AB^2 = \frac{3}{4}\left((2k)^2 + (4k)^2\right) = 15k^2.$$

Hence

$$\frac{5}{4}(1-c)^2 = 15k^2, \quad \text{so} \quad k^2 = \frac{(1-c)^2}{12}.$$

Substitution into the equation $1 - c = 2(c^2 - 1 - k^2)$ yields $c = 1$ or $c = -19/11$. Because $c < 1$, it follows that

$$a + b + c = 2 - \frac{19}{11} = \frac{3}{11} = \frac{m}{n}, \quad \text{so} \quad m + n = 14.$$

25. **(A)** Because each ant can move from its vertex to any of four adjacent vertices, there are 4^6 possible combinations of moves. In the following, consider only those combinations in which no two ants arrive at the same vertex. Label the vertices as A, B, C, A', B' and C', where A', B' and C' are opposite A, B and C, respectively. Let f be the function that maps each ant's starting vertex onto its final vertex. Then neither of $f(A)$ nor $f(A')$ can be either A or A', and similar statements hold for the other pairs of opposite vertices. Thus there are $4 \cdot 3 = 12$ ordered pairs of values for $f(A)$ and $f(A')$. The vertices $f(A)$ and $f(A')$ are opposite each other in four cases and adjacent to each other in eight.

Suppose that $f(A)$ and $f(A')$ are opposite vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = B'$. Then $f(C)$ must be either A or A' and $f(C')$ must be the other. Similarly, $f(B)$ must be either C or C' and $f(B')$ must be the other. Therefore there are $4 \cdot 2 \cdot 2 = 16$ combinations of moves in which $f(A)$ and $f(A')$ are opposite each other.

Suppose now that $f(A)$ and $f(A')$ are adjacent vertices, and, without loss of generality, that $f(A) = B$ and $f(A') = C$. Then one of $f(B)$ and $f(B')$ must be C' and the other cannot be B'. So there are four possible ordered pairs of values for $f(B)$ and $f(B')$. For each of those there are two possible ordered pairs of values for $f(C)$ and $f(C')$. Therefore there are $8 \cdot 4 \cdot 2 = 64$ combinations of moves in which $f(A)$ and $f(A')$ are adjacent to each other.

Hence the probability that no two ants arrive at the same vertex is

$$\frac{16 + 64}{4^6} = \frac{5 \cdot 2^4}{2^{12}} = \frac{5}{256}.$$

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

The Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour & Company
Mu Alpha Theta
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
USA Math Talent Search
W. H. Freeman & Company

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

57th Annual American Mathematics Contest 12

AMC 12 – Contest A

Solutions Pamphlet

Tuesday, JANUARY 31, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **(A)** Five sandwiches cost $5 \cdot 3 = 15$ dollars and eight sodas cost $8 \cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

2. **(C)** By the definition we have

$$h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

3. **(B)** Mary is $(3/5)(30) = 18$ years old.

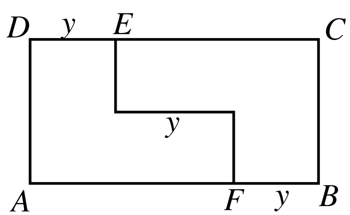
4. **(E)** The largest possible sum of the two digits representing the minutes is $5+9 = 14$, occurring at 59 minutes past each hour. The largest possible single digit that can represent the hour is 9. This exceeds the largest possible sum of two digits that can represent the hour, which is $1 + 2 = 3$. Therefore, the largest possible sum of all the digits is $14 + 9 = 23$, occurring at 9:59.

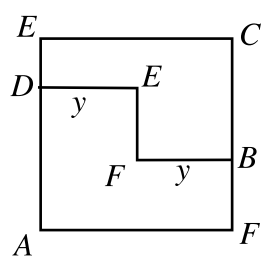
5. **(D)** Each slice of plain pizza cost \$1. Dave paid \$2 for the anchovies in addition to \$5 for the five slices of pizza that he ate, for a total of \$7. Doug paid only \$3 for the three slices of pizza that he ate. Hence Dave paid $7 - 3 = 4$ dollars more than Doug.

6. **(A)** Let E represent the end of the cut on $\overline{DC}$, and let F represent the end of the cut on $\overline{AB}$. For a square to be formed, we must have

$$DE = y = FB \quad \text{and} \quad DE + y + FB = 18, \quad \text{so} \quad y = 6.$$

The rectangle that is formed by this cut is indeed a square, since the original rectangle has area $8 \cdot 18 = 144$, and the rectangle that is formed by this cut has a side of length $2 \cdot 6 = 12 = \sqrt{144}$.





7. **(B)** Let Danielle be x years old. Sally is 40% younger, so she is $0.6x$ years old. Mary is 20% older than Sally, so Mary is $1.2(0.6x) = 0.72x$ years old. The sum of their ages is $23.2 = x + 0.6x + 0.72x = 2.32x$ years, so $x = 10$. Therefore Mary's age is $0.72x = 7.2$ years, and she will be 8 on her next birthday.

8. **(C)** First note that, in general, the sum of n consecutive integers is n times their median. If the sum is 15, we have the following cases:

   if $n = 2$, then the median is 7.5 and the two integers are 7 and 8;

   if $n = 3$, then the median is 5 and the three integers are 4, 5, and 6;

   if $n = 5$, then the median is 3 and the five integers are 1, 2, 3, 4, and 5.

   Because the sum of four consecutive integers is even, 15 cannot be written in such a manner. Also, the sum of more than five consecutive integers must be more than $1+2+3+4+5 = 15$. Hence there are 3 sets satisfying the condition.

   Note: It can be shown that the number of sets of two or more consecutive positive integers having a sum of k is equal to the number of odd positive divisors of k, excluding 1.

9. **(A)** Let p be the cost (in cents) of a pencil, and let s be the cost (in cents) of a set of one pencil and one eraser. Because Oscar buys 3 sets and 10 extra pencils for \$1.00, we have

   $$3s + 10p = 100.$$

   Thus $3s$ is a multiple of 10 that is less than 100, so s is 10, 20, or 30. The corresponding values of p are 7, 4, and 1. Since the cost of a pencil is more than half the cost of the set, the only possibility is $s = 10$.

10. **(E)** Suppose that $k = \sqrt{120 - \sqrt{x}}$ is an integer. Then $0 \le k \le \sqrt{120}$, and because k is an integer, we have $0 \le k \le 10$. Thus there are 11 possible integer values of k. For each such k, the corresponding value of x is $\left(120 - k^2\right)^2$. Because $\left(120 - k^2\right)^2$ is positive and decreasing for $0 \le k \le 10$, the 11 values of x are distinct.

11. **(C)** The equation $(x+y)^2 = x^2 + y^2$ is equivalent to $x^2 + 2xy + y^2 = x^2 + y^2$, which reduces to $xy = 0$. Thus the graph of the equation consists of the two lines that are the coordinate axes.

12. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are $16\,\text{cm}, 15\,\text{cm}, \ldots, 1\,\text{cm}$. Thus the total distance is

    $$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

    OR

    The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

    $$(3+4+5+\cdots+20) - 2 \cdot 17 = (1+2+3+4+5+\cdots+20) - 3 - 34 = \frac{20 \cdot 21}{2} - 37 = 173\,\text{cm}.$$

13. **(E)** Let r, s, and t be the radii of the circles centered at A, B, and C, respectively. Then $r+s=3$, $r+t=4$, and $s+t=5$, from which $r=1$, $s=2$, and $t=3$. Thus the sum of the areas of the circles is

$$\pi(1^2+2^2+3^2)=14\pi.$$

14. **(C)** If a debt of D dollars can be resolved in this way, then integers p and g must exist with

$$D=300p+210g=30(10p+7g).$$

As a consequence, D must be a multiple of 30, so no positive debt of less than \$30 can be resolved. A debt of \$30 can be resolved since

$$30=300(-2)+210(3).$$

This is done by giving 3 goats and receiving 2 pigs.

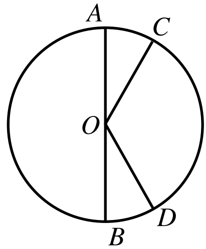
15. **(A)** Because $\cos x=0$ and $\cos(x+z)=1/2$, it follows that $x=m\pi/2$ for some odd integer m and $x+z=2n\pi\pm\pi/3$ for some integer n. Therefore

$$z=2n\pi-\frac{m\pi}{2}\pm\frac{\pi}{3}=k\pi+\frac{\pi}{2}\pm\frac{\pi}{3}$$

for some integer k. The smallest value of k that yields a positive value for z is 0, and the smallest positive value of z is $\pi/2-\pi/3=\pi/6$.

OR

Let O denote the center of the unit circle. Because $\cos x=0$, the terminal side of an angle of measure x, measured counterclockwise from the positive x-axis, intersects the circle at $A=(0,1)$ or $B=(0,-1)$.



Because $\cos(x+z)=1/2$, the terminal side of an angle of measure $x+z$ intersects the circle at $C=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ or $D=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$. Thus all angles of positive measure $z=(x+z)-x$ can be measured counterclockwise from either $\overline{OA}$ or $\overline{OB}$ to either $\overline{OC}$ or $\overline{OD}$. The smallest such angle is $\angle BOD$, which has measure $\pi/6$ and is attained, for example, when $x=-\pi/2$ and $x+z=-\pi/3$.

16. **(B)** Radii $\overline{AC}$ and $\overline{BD}$ are each perpendicular to $\overline{CD}$. By the Pythagorean Theorem,

$$CE = \sqrt{5^2 - 3^2} = 4.$$

Because $\triangle ACE$ and $\triangle BDE$ are similar,

$$\frac{DE}{CE} = \frac{BD}{AC}, \quad \text{so} \quad DE = CE \cdot \frac{BD}{AC} = 4 \cdot \frac{8}{3} = \frac{32}{3}.$$

Therefore

$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

17. **(B)** Let $B = (0,0)$, $C = (s,0)$, $A = (0,s)$, $D = (s,s)$, and $E = \left(s + \frac{r}{\sqrt{2}}, s + \frac{r}{\sqrt{2}}\right)$. Apply the Pythagorean Theorem to $\triangle AFE$ to obtain

$$r^2 + \left(9 + 5\sqrt{2}\right) = \left(s + \frac{r}{\sqrt{2}}\right)^2 + \left(\frac{r}{\sqrt{2}}\right)^2,$$

from which $9 + 5\sqrt{2} = s^2 + rs\sqrt{2}$. Because r and s are rational, it follows that $s^2 = 9$ and $rs = 5$, so $r/s = 5/9$.

OR

Extend $\overline{AD}$ past D to meet the circle at $G \neq D$. Because E is collinear with B and D, $\triangle EDG$ is an isosceles right triangle. Thus $DG = r\sqrt{2}$. By the Power of a Point Theorem,

$$9 + 5\sqrt{2} = AF^2 = AD \cdot AG = AD \cdot (AD + DG) = s\left(s + r\sqrt{2}\right) = s^2 + rs\sqrt{2}.$$

As in the first solution, conclude that $r/s = 5/9$.

18. **(E)** The conditions on f imply that both

$$x = f(x) + f\left(\frac{1}{x}\right) \quad \text{and} \quad \frac{1}{x} = f\left(\frac{1}{x}\right) + f\left(\frac{1}{1/x}\right) = f\left(\frac{1}{x}\right) + f(x).$$

Thus if x is in the domain of f, then $x = 1/x$, so $x = \pm 1$.

The conditions are satisfied if and only if $f(1) = 1/2$ and $f(-1) = -1/2$.

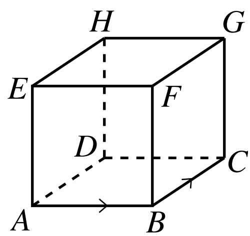
19. **(E)** The slope of the line l containing the centers of the circles is $5/12 = \tan\theta$, where θ is the acute angle between the x-axis and line l. The equation of line l is $y - 4 = (5/12)(x - 2)$. This line and the two common external tangents are concurrent. Because one of these tangents is the x-axis, the point of intersection is the x-intercept of line l, which is $(-38/5, 0)$. The acute angle between the x-axis and the other tangent is 2θ, so the slope of that tangent is

$$\tan 2\theta = 2 \cdot \frac{5/12}{1 - (5/12)^2} = \frac{120}{119}.$$

Thus the equation of that tangent is $y = (120/119)\,(x + (38/5))$, and

$$b = \frac{120}{119} \cdot \frac{38}{5} = \frac{912}{119}.$$

20. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as $A, B,$ and C, in that order. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, and D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug can travel from C to D and then from D to H. Each of these occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow\ E \to F \to G \\ \searrow\ G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3} + \frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$.

21. **(E)** For $j = 1$ and 2, the given inequality is equivalent to

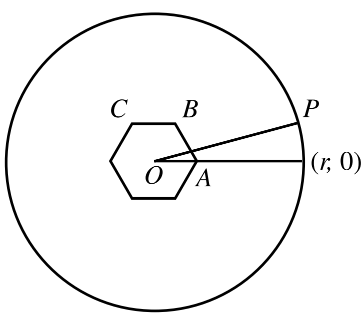
$$j + x^2 + y^2 \le 10^j(x + y),$$

or to

$$\left(x - \frac{10^j}{2}\right)^2 + \left(y - \frac{10^j}{2}\right)^2 \le \frac{10^{2j}}{2} - j,$$

provided that $x + y > 0$. These inequalities define regions bounded by circles. For $j = 1$ the circle has center $(5, 5)$ and radius 7. For $j = 2$ the circle has center $(50, 50)$ and radius $\sqrt{4998}$. In each case the center is on the line $y = x$ in the first quadrant, and the radius is less than the distance from the center to the origin. Thus $x + y > 0$ at each interior point of each circle, as was required to ensure the equivalence of the inequalities. The squares of the radii of the circles are 49 and 4998 for $j = 1$ and 2, respectively. Therefore the ratio of the area of S_2 to that of S_1 is $(4998\pi)/(49\pi) = 102$.

22. **(D)** Place the hexagon in a coordinate plane with center at the origin O and vertex A at $(2, 0)$. Let B, C, D, E, and F be the other vertices in counterclockwise order.



Corresponding to each vertex of the hexagon, there is an arc on the circle from which only the two sides meeting at that vertex are visible. The given probability condition implies that those arcs have a combined degree measure of 180°, so by symmetry each is 30°. One such arc is centered at $(r, 0)$. Let P be the endpoint of this arc in the upper half-plane. Then $\angle POA = 15°$. Side $\overline{BC}$ is visible from points immediately above P, so P is collinear with B and C. Because the perpendicular distance from O to $\overline{BC}$ is $\sqrt{3}$, we have

$$\sqrt{3} = r \sin 15° = r \sin(45° - 30°) = r\left(\sin 45° \cos 30° - \sin 30° \cos 45°\right).$$

So

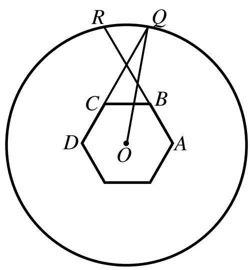
$$\sqrt{3} = r \cdot \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) = r \cdot \frac{\sqrt{6}-\sqrt{2}}{4}.$$

Therefore

$$r = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} = \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \cdot \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} = \sqrt{18}+\sqrt{6} = 3\sqrt{2}+\sqrt{6}.$$

OR

Call the hexagon $ABCDEF$. Side $\overline{AB}$ is visible from point X if and only if X lies in the half-plane that is in the exterior of the hexagon and that is determined by the line AB. The region from which the three sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are visible is the intersection of three such half-planes.



Let rays AB and DC intersect the circle at R and Q, respectively. Then QR is one of the six arcs of the circle from which three sides are visible. Symmetry implies that the six arcs are congruent, and because the given probability is $1/2$, the measure of each arc is 30°. Let O be the center of the hexagon and the circle. Then $\angle QOR = 30^\circ$, so

$$\angle QOC = \angle QOR + \frac{1}{2}(\angle BOC - \angle QOR) = 30^\circ + \frac{1}{2}(60^\circ - 30^\circ) = 45^\circ.$$

Thus

$$\angle OQC = 180^\circ - \angle QOC - \angle OCQ = 15^\circ.$$

Apply the Law of Sines in $\triangle OQC$ to obtain

$$\frac{r}{\sin 120^\circ} = \frac{2}{\sin 15^\circ}, \quad \text{and then} \quad r = \frac{\sqrt{3}}{\sin 15^\circ}.$$

Then proceed as in the first solution.

23. **(B)** For every sequence $S = (a_1, a_2, \ldots, a_n)$ of at least three terms,

$$A^2(S) = \left(\frac{a_1+2a_2+a_3}{4}, \frac{a_2+2a_3+a_4}{4}, \ldots, \frac{a_{n-2}+2a_{n-1}+a_n}{4}\right).$$

Thus for $m = 1$ and 2, the coefficients of the terms in the numerator of $A^m(S)$ are the binomial coefficients $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$, and the denominator is 2^m. Because $\binom{m}{r} + \binom{m}{r+1} = \binom{m+1}{r+1}$ for all integers $r \geq 0$, the coefficients of the terms in the numerators of $A^{m+1}(S)$ are $\binom{m+1}{0}, \binom{m+1}{1}, \ldots, \binom{m+1}{m+1}$ for $2 \leq m \leq n-2$. The definition implies that the denominator of each term in $A^{m+1}(S)$ is 2^{m+1}. For the given sequence, the sole term in $A^{100}(S)$ is

$$\frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} a_{m+1} = \frac{1}{2^{100}} \sum_{m=0}^{100} \binom{100}{m} x^m = \frac{1}{2^{100}} (x+1)^{100}.$$

Therefore

$$\left(\frac{1}{2^{50}}\right) = A^{100}(S) = \left(\frac{(1+x)^{100}}{2^{100}}\right),$$

so $(1+x)^{100} = 2^{50}$, and because $x > 0$, we have $x = \sqrt{2} - 1$.

24. **(D)** There is exactly one term in the simplified expression for every monomial of the form $x^a y^b z^c$, where a, b, and c are non-negative integers, a is even, and $a + b + c = 2006$. There are 1004 even values of a with $0 \leq a \leq 2006$. For each such value, b can assume any of the $2007 - a$ integer values between 0 and $2006 - a$, inclusive, and the value of c is then uniquely determined as $2006 - a - b$. Thus the number of terms in the simplified expression is

$$(2007 - 0) + (2007 - 2) + \cdots + (2007 - 2006) = 2007 + 2005 + \cdots + 1.$$

This is the sum of the first 1004 odd positive integers, which is $1004^2 = 1{,}008{,}016$.

OR

The given expression is equal to

$$\sum \frac{2006!}{a!b!c!} \left(x^a y^b z^c + x^a (-y)^b (-z)^c\right),$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c = 2006$. Because the number of non-negative integer solutions of $a + b + c = k$ is $\binom{k+2}{2}$, the sum is taken over $\binom{2008}{2}$ terms, but those for which b and c have opposite parity have a sum of zero. If b is odd and c is even, then a is odd, so $a = 2A+1, b = 2B+1,$ and $c = 2C$ for some non-negative integers $A, B,$ and C. Therefore $2A + 1 + 2B + 1 + 2C = 2006$, so $A + B + C = 1002$. Because the last equation has $\binom{1004}{2}$ non-negative integer solutions, there are $\binom{1004}{2}$ terms for which b is odd and c is even. The number of terms for which b is even and c is odd is the same. Thus the number of terms in the simplified expression is

$$\binom{2008}{2} - 2\binom{1004}{2} = 1004 \cdot 2007 - 1004 \cdot 1003 = 1004^2 = 1{,}008{,}016.$$

25. **(E)** For $1 \le k \le 15$, the k-element sets with properties (1) and (2) are the k-element subsets of $U_k = \{k, k+1, \ldots, 15\}$ that contain no two consecutive integers. If $\{a_1, a_2, \ldots, a_k\}$ is such a set, with its elements listed in increasing order, then $\{a_1 + k - 1, a_2 + k - 2, \ldots, a_{k-1} + 1, a_k\}$ is a k-element subset of U_{2k-1}. Conversely, if $\{b_1, b_2, \ldots, b_k\}$ is a k-element subset of U_{2k-1}, with its elements listed in increasing order, then $\{b_1 - k + 1, b_2 - k + 2, \ldots, b_{k-1} - 1, b_k\}$ is a set with properties (1) and (2). Thus for each k, the number of k-element sets with properties (1) and (2) is equal to the number of k-element subsets of the $(17 - 2k)$-element set U_{2k-1}. Because $k \le 17 - 2k$ only if $k \le 5$, the total number of such sets is

$$\sum_{k=1}^{5} \binom{17-2k}{k} = \binom{15}{1} + \binom{13}{2} + \binom{11}{3} + \binom{9}{4} + \binom{7}{5} = 15+78+165+126+21 = 405.$$

The

AMERICAN MATHEMATICS COMPETITIONS

are Sponsored by

The Mathematical Association of America
University of Nebraska-Lincoln
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Council of Teachers of Mathematics
National Assessment & Testing
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

57th Annual American Mathematics Contest 12

AMC 12 – Contest B

Solutions Pamphlet

Wednesday, February 15, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 12 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **(C)** Because

$$(-1)^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \text{ is odd,} \end{cases}$$

the sum can be written as

$$(-1+1)+(-1+1)+\cdots+(-1+1) = 0+0+\cdots+0 = 0.$$

2. **(A)** Because $4 \spadesuit 5 = (4+5)(4-5) = -9$, it follows that

$$3 \spadesuit (4 \spadesuit 5) = 3 \spadesuit (-9) = (3+(-9))(3-(-9)) = (-6)(12) = -72.$$

3. **(A)** Let c and p represent the number of points scored by the Cougars and the Panthers, respectively. The two teams scored a total of 34 points, so $c+p=34$. The Cougars won by 14 points, so $c-p=14$. The solution is $c=24$ and $p=10$, so the Panthers scored 10 points.

4. **(A)** The five items cost approximately $8+5+3+2+1=19$ dollars, so Mary's change is about \$1.00, which is 5 percent of her \$20.00.

5. **(A)** In order to catch up to John, Bob must walk 1 mile farther in the same amount of time. Because Bob's speed exceeds John's speed by $5-3=2$ miles per hour, the time required for Bob to catch up to John is $1/2$ hour, or 30 minutes.

6. **(B)** Francesca's 600 grams of lemonade contains $25+386=411$ calories, so 200 grams of her lemonade contains $411/3=137$ calories.

7. **(B)** There are only two possible occupants for the driver's seat. After the driver is chosen, any of the remaining three people can sit in the front, and there are two arrangements for the other two people in the back. Thus there are $2\cdot3\cdot2=12$ possible seating arrangements.

8. **(E)** Substituting $x = 1$ and $y = 2$ into the equations gives

$$1 = \frac{2}{4} + a \quad \text{and} \quad 2 = \frac{1}{4} + b.$$

It follows that

$$a + b = \left(1 - \frac{2}{4}\right) + \left(2 - \frac{1}{4}\right) = 3 - \frac{3}{4} = \frac{9}{4}.$$

OR

Because

$$a = x - \frac{y}{4} \quad \text{and} \quad b = y - \frac{x}{4}, \quad \text{we have} \quad a + b = \frac{3}{4}(x + y).$$

Since $x = 1$ when $y = 2$, this implies that $a + b = \frac{3}{4}(1 + 2) = \frac{9}{4}$.

9. **(B)** Let the integer have digits a, b, and c, read left to right. Because $1 \le a < b < c$, none of the digits can be zero and c cannot be 2. If $c = 4$, then a and b must each be chosen from the digits 1, 2, and 3. Therefore there are $\binom{3}{2} = 3$ choices for a and b, and for each choice there is one acceptable order. Similarly, for $c = 6$ and $c = 8$ there are, respectively, $\binom{5}{2} = 10$ and $\binom{7}{2} = 21$ choices for a and b. Thus there are altogether $3 + 10 + 21 = 34$ such integers.

10. **(A)** The sides of the triangle are x, $3x$, and 15 for some positive integer x. By the Triangle Inequality, these three numbers are the sides of a triangle if and only if $x + 3x > 15$ and $x + 15 > 3x$. Because x is an integer, the first inequality is equivalent to $x \ge 4$, and the second inequality is equivalent to $x \le 7$. Thus the greatest possible perimeter is $7 + 21 + 15 = 43$.

11. **(E)** Joe has 2 ounces of cream in his cup. JoAnn has drunk 2 ounces of the 14 ounces of coffee-cream mixture in her cup, so she has only $12/14 = 6/7$ of her 2 ounces of cream in her cup. Therefore the ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee is

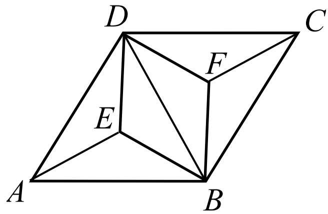
$$\frac{2}{\frac{6}{7} \cdot 2} = \frac{7}{6}.$$

12. **(D)** A parabola with the given equation and with vertex (p, p) must have equation $y = a(x - p)^2 + p$. Because the y-intercept is $(0, -p)$ and $p \ne 0$, it follows that $a = -2/p$. Thus

$$y = -\frac{2}{p}(x^2 - 2px + p^2) + p = -\frac{2}{p}x^2 + 4x - p,$$

so $b = 4$.

13. **(C)** Since $\angle BAD = 60^\circ$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

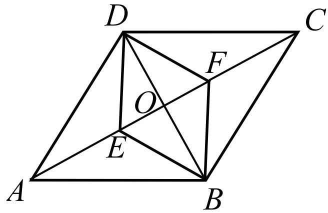


OR

Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90^\circ$ triangle. Therefore $AO = \sqrt{3} \cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

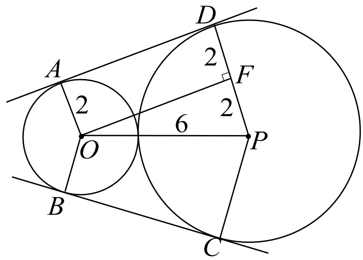
$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



14. **(D)** The total cost of the peanut butter and jam is $N(4B + 5J) = 253$ cents, so N and $4B + 5J$ are factors of $253 = 11 \cdot 23$. Because $N > 1$, the possible values of N are 11, 23, and 253. If $N = 253$, then $4B + 5J = 1$, which is impossible since B and J are positive integers. If $N = 23$, then $4B + 5J = 11$, which also has no solutions in positive integers. Hence $N = 11$ and $4B + 5J = 23$, which has the unique positive integer solution $B = 2$ and $J = 3$. So the cost of the jam is $11(3)(5¢) = \$1.65$.

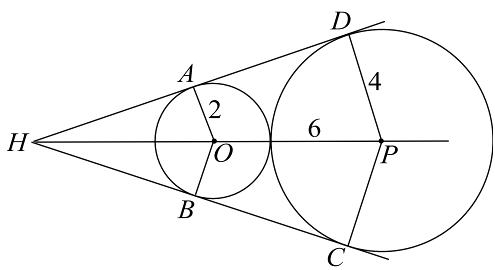
15. **(B)** Through O draw a line parallel to $\overline{AD}$ intersecting $\overline{PD}$ at F.



Then $AOFD$ is a rectangle and OPF is a right triangle. Thus $DF = 2$, $FP = 2$, and $OF = 4\sqrt{2}$. The area of trapezoid $AOPD$ is $12\sqrt{2}$, and the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

OR

Lines $AD, BC,$ and OP intersect at a common point H.



Because $\angle PDH = \angle OAH = 90°$, triangles PDH and OAH are similar with ratio of similarity 2. Thus $2HO = HP = HO + OP = HO + 6$, so $HO = 6$ and $AH = \sqrt{HO^2 - OA^2} = 4\sqrt{2}$. Hence the area of $\triangle OAH$ is $(1/2)(2)(4\sqrt{2}) = 4\sqrt{2}$, and the area of $\triangle PDH$ is $(2^2)(4\sqrt{2}) = 16\sqrt{2}$. The area of the hexagon is twice the area of $\triangle PDH$ minus twice the area of $\triangle OAH$, so it is $24\sqrt{2}$.

16. **(C)** Diagonals $\overline{AC}$, $\overline{CE}$, $\overline{EA}$, $\overline{AD}$, $\overline{CF}$, and $\overline{EB}$ divide the hexagon into twelve congruent $30-60-90°$ triangles, six of which make up equilateral $\triangle ACE$. Because $AC = \sqrt{7^2 + 1^2} = \sqrt{50}$, the area of $\triangle ACE$ is $\frac{\sqrt{3}}{4}\left(\sqrt{50}\right)^2 = \frac{25}{2}\sqrt{3}$. The area of hexagon $ABCDEF$ is $2\left(\frac{25}{2}\sqrt{3}\right) = 25\sqrt{3}$.

OR

Let O be the center of the hexagon. Then triangles $ABC, CDE,$ and EFA are congruent to triangles $AOC, COE,$ and EOA, respectively. Thus the area of the hexagon is twice the area of equilateral $\triangle ACE$. Then proceed as in the first solution.

17. **(C)** On each die the probability of rolling k, for $1 \le k \le 6$, is

$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1\cdot 6+2\cdot 5+3\cdot 4+4\cdot 3+5\cdot 2+6\cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

18. **(B)** Each step changes either the x-coordinate or the y-coordinate of the object by 1. Thus if the object's final point is (a,b), then $a+b$ is even and $|a|+|b| \le 10$. Conversely, suppose that (a,b) is a lattice point with $|a|+|b| = 2k \le 10$. One ten-step path that ends at (a,b) begins with $|a|$ horizontal steps, to the right if $a \ge 0$ and to the left if $a < 0$. It continues with $|b|$ vertical steps, up if $b \ge 0$ and down if $b < 0$. It has then reached (a,b) in $2k$ steps, so it can finish with $5-k$ steps up and $5-k$ steps down. Thus the possible final points are the lattice points that have even coordinate sums and lie on or inside the square with vertices $(\pm 10, 0)$ and $(0, \pm 10)$. There are 11 such points on each of the 11 lines $x+y = 2k$, $-5 \le k \le 5$, for a total of 121 different points.

19. **(B)** The 4-digit number on the license plate has the form *aabb* or *abab* or *baab*, where a and b are distinct integers from 0 to 9. Because Mr. Jones has a child of age 9, the number on the license plate is divisible by 9. Hence the sum of the digits, $2(a+b)$, is also divisible by 9. Because of the restriction on the digits a and b, this implies that $a+b=9$. Moreover, since Mr. Jones must have either a 4-year-old or an 8-year-old, the license plate number is divisible by 4. These conditions narrow the possibilities for the number to 1188, 2772, 3636, 5544, 6336, 7272, and 9900. The last two digits of 9900 could not yield Mr. Jones's age, and none of the others is divisible by 5, so he does not have a 5-year-old.

Note that 5544 is divisible by each of the other eight non-zero digits.

20. **(C)** The given condition is equivalent to $\lfloor \log_{10} x \rfloor = \lfloor \log_{10} 4x \rfloor$. Thus the condition holds if and only if

$$n \le \log_{10} x < \log_{10} 4x < n+1$$

for some negative integer n. Equivalently,

$$10^n \le x < 4x < 10^{n+1}.$$

This inequality is true if and only if

$$10^n \le x < \frac{10^{n+1}}{4}.$$

Hence in each interval $[10^n, 10^{n+1})$, the given condition holds with probability

$$\frac{(10^{n+1}/4) - 10^n}{10^{n+1} - 10^n} = \frac{10^n((10/4) - 1)}{10^n(10 - 1)} = \frac{1}{6}.$$

Because each number in $(0, 1)$ belongs to a unique interval $[10^n, 10^{n+1})$ and the probability is the same on each interval, the required probability is also $1/6$.

21. **(C)** Let $2a$ and $2b$, respectively, be the lengths of the major and minor axes of the ellipse, and let the dimensions of the rectangle be x and y. Then $x + y$ is the sum of the distances from the foci to point A on the ellipse, which is $2a$. The length of a diagonal of the rectangle is the distance between the foci of the ellipse, which is $2\sqrt{a^2 - b^2}$. Thus $x + y = 2a$ and $x^2 + y^2 = 4a^2 - 4b^2$. The area of the rectangle is

$$2006 = xy = \frac{1}{2}\left[(x + y)^2 - (x^2 + y^2)\right] = \frac{1}{2}\left[(2a)^2 - (4a^2 - 4b^2)\right] = 2b^2,$$

so $b = \sqrt{1003}$. Thus the area of the ellipse is

$$2006\pi = \pi ab = \pi a\sqrt{1003},$$

so $a = 2\sqrt{1003}$, and the perimeter of the rectangle is $2(x + y) = 4a = 8\sqrt{1003}$.

22. **(B)** Note that n is the number of factors of 5 in the product $a!b!c!$, and $2006 < 5^5$. Thus

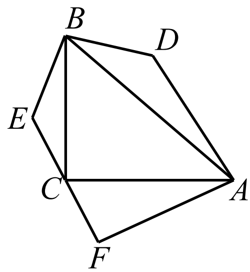
$$n = \sum_{k=1}^{4} \left(\lfloor a/5^k \rfloor + \lfloor b/5^k \rfloor + \lfloor c/5^k \rfloor\right).$$

Because $\lfloor x \rfloor + \lfloor y \rfloor + \lfloor z \rfloor \geq \lfloor x + y + z \rfloor - 2$ for all real numbers x, y, and z, it follows that

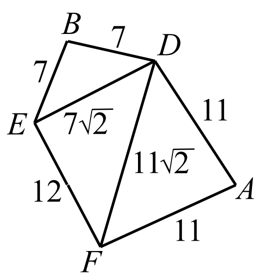
$$\begin{aligned} n &\geq \sum_{k=1}^{4} \left(\lfloor (a + b + c)/5^k \rfloor - 2\right) \\ &= \sum_{k=1}^{4} \left(\lfloor 2006/5^k \rfloor - 2\right) \\ &= 401 + 80 + 16 + 3 - 4 \cdot 2 = 492. \end{aligned}$$

The minimum value of 492 is achieved, for example, when $a = b = 624$ and $c = 758$.

23. **(E)** Let D, E, and F be the reflections of P about $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Then $\angle FAD = \angle DBE = 90^\circ$, and $\angle ECF = 180^\circ$. Thus the area of pentagon $ADBEF$ is twice that of $\triangle ABC$, so it is s^2.



Observe that $DE = 7\sqrt{2}$, $EF = 12$, and $FD = 11\sqrt{2}$. Furthermore, $(7\sqrt{2})^2 + 12^2 = 98 + 144 = 242 = (11\sqrt{2})^2$, so $\triangle DEF$ is a right triangle. Thus the pentagon can be tiled with three right triangles, two of which are isosceles, as shown.



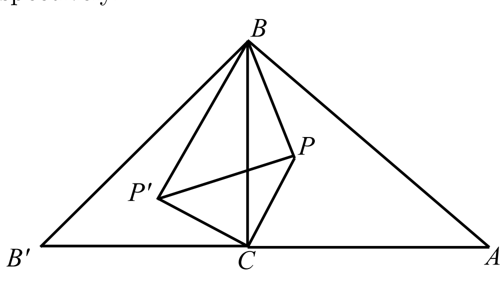
It follows that

$$s^2 = \frac{1}{2} \cdot (7^2 + 11^2) + \frac{1}{2} \cdot 12 \cdot 7\sqrt{2} = 85 + 42\sqrt{2},$$

so $a + b = 127$.

OR

Rotate $\triangle ABC$ 90° counterclockwise about C, and let B' and P' be the images of B and P, respectively.



Then $CP' = CP = 6$, and $\angle PCP' = 90^\circ$, so $\triangle PCP'$ is an isosceles right triangle. Thus $PP' = 6\sqrt{2}$, and $BP' = AP = 11$. Because $\left(6\sqrt{2}\right)^2 + 7^2 = 11^2$, the converse of the Pythagorean Theorem implies that $\angle BPP' = 90^\circ$. Hence $\angle BPC = 135^\circ$. Applying the Law of Cosines in $\triangle BPC$ gives

$$BC^2 = 6^2 + 7^2 - 2 \cdot 6 \cdot 7 \cos 135^\circ = 85 + 42\sqrt{2},$$

and $a + b = 127$.

24. **(C)** For a fixed value of y, the values of $\sin x$ for which $\sin^2 x - \sin x \sin y + \sin^2 y = \frac{3}{4}$ can be determined by the quadratic formula. Namely,

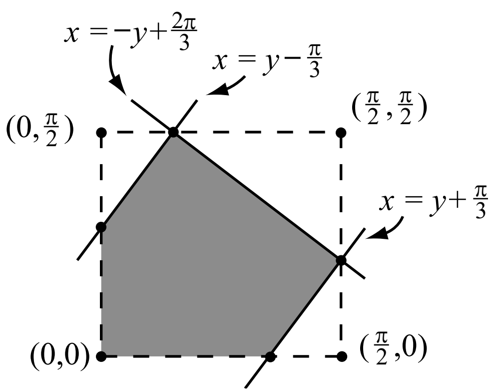
$$\sin x = \frac{\sin y \pm \sqrt{\sin^2 y - 4(\sin^2 y - \frac{3}{4})}}{2} = \frac{1}{2}\sin y \pm \frac{\sqrt{3}}{2}\cos y.$$

Because $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, this implies that

$$\sin x = \cos\left(\frac{\pi}{3}\right)\sin y \pm \sin\left(\frac{\pi}{3}\right)\cos y = \sin\left(y \pm \frac{\pi}{3}\right).$$

Within S, $\sin x = \sin(y - \frac{\pi}{3})$ implies $x = y - \frac{\pi}{3}$. However, the case $\sin x = \sin(y + \frac{\pi}{3})$ implies $x = y + \frac{\pi}{3}$ when $y \le \frac{\pi}{6}$, and $x = -y + \frac{2\pi}{3}$ when $y \ge \frac{\pi}{6}$. Those three lines divide the region S into four subregions, within each of which the truth value of the inequality is constant. Testing the points $(0, 0)$, $(\frac{\pi}{2}, 0)$, $(0, \frac{\pi}{2})$, and $(\frac{\pi}{2}, \frac{\pi}{2})$ shows that the inequality is true only in the shaded subregion. The area of this subregion is

$$\left(\frac{\pi}{2}\right)^2 - \frac{1}{2}\cdot\left(\frac{\pi}{3}\right)^2 - 2\cdot\frac{1}{2}\cdot\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{6}.$$



25. **(B)** The condition $a_{n+2} = |a_{n+1} - a_n|$ implies that a_n and a_{n+3} have the same parity for all $n \ge 1$. Because a_{2006} is odd, a_2 is also odd. Because $a_{2006} = 1$ and a_n is a multiple of $\gcd(a_1, a_2)$ for all n, it follows that $1 = \gcd(a_1, a_2) = \gcd(3^3 \cdot 37, a_2)$. There are 499 odd integers in the interval $[1, 998]$, of which 166 are multiples of 3, 13 are multiples of 37, and 4 are multiples of $3 \cdot 37 = 111$. By the Inclusion-Exclusion Principle, the number of possible values of a_2 cannot exceed $499 - 166 - 13 + 4 = 324$.

To see that there are actually 324 possibilities, note that for $n \ge 3$, $a_n < \max(a_{n-2}, a_{n-1})$ whenever a_{n-2} and a_{n-1} are both positive. Thus $a_N = 0$ for some $N \le 1999$. If $\gcd(a_1, a_2) = 1$, then $a_{N-2} = a_{N-1} = 1$, and for $n > N$ the sequence cycles through the values 1, 1, 0. If in addition a_2 is odd, then a_{3k+2} is odd for $k \ge 1$, so $a_{2006} = 1$.

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

58th Annual American Mathematics Contest 12

AMC 12
CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 6, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **Answer (C):** Susan pays $(4)(0.75)(20) = 60$ dollars. Pam pays $(5)(0.70)(20) = 70$ dollars, so she pays $70 - 60 = 10$ more dollars than Susan.

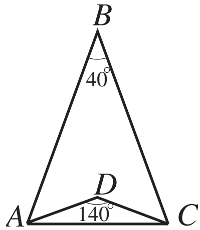
2. **Answer (D):** The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have
$$8000 = 4000h \quad \text{and} \quad h = 2 \text{ centimeters}$$

3. **Answer (A):** Let the smaller of the integers be x. Then the larger is $x + 2$. So $x + 2 = 3x$, from which $x = 1$. Thus the two integers are 1 and 3, and their sum is 4.

4. **Answer (A):** Kate rode for 30 minutes $= 1/2$ hour at 16 mph, so she rode 8 miles. She walked for 90 minutes $= 3/2$ hours at 4 mph, so she walked 6 miles. Therefore she covered a total of 14 miles in 2 hours, so her average speed was 7 mph.

5. **Answer (D):** After paying the federal taxes, Mr. Public had 80% of his inheritance money left. He paid 10% of that, or 8% of his inheritance, in state taxes. Hence his total tax bill was 28% of his inheritance, and his inheritance was \$10,500/0.28 = \$37,500.

6. **Answer (D):** Because $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = 70^\circ$.



Similarly,
$$\angle DAC = \frac{1}{2}(180^\circ - \angle ADC) = 20^\circ.$$
Thus $\angle BAD = \angle BAC - \angle DAC = 50^\circ$.

OR

Because $\triangle ABC$ and $\triangle ADC$ are isosceles triangles, applying the Exterior Angle Theorem to $\triangle ABD$ gives $\angle BAD = 70^\circ - 20^\circ = 50^\circ$.

7. **Answer (C):** Let D be the difference between consecutive terms of the sequence. Then $a = c - 2D$, $b = c - D$, $d = c + D$, and $e = c + 2D$, so

$$a + b + c + d + e = (c - 2D) + (c - D) + c + (c + D) + (c + 2D) = 5c.$$

Thus $5c = 30$, so $c = 6$.

To see that the values of the other terms cannot be found, note that the sequences $4, 5, 6, 7, 8$ and $10, 8, 6, 4, 2$ both satisfy the given conditions.

8. **Answer (C):** Consider the two chords with an endpoint at 5. The arc subtended by the angle determined by these chords extends from 10 to 12, so the degree measure of the arc is $(2/12)(360) = 60$. By the Central Angle Theorem, the degree measure of this angle is $(1/2)(60) = 30$. By symmetry, the degree measure of the angle at each vertex is 30.

9. **Answer (B):** Let w be Yan's walking speed, and let x and y be the distances from Yan to his home and to the stadium, respectively. The time required for Yan to walk to the stadium is y/w, and the time required for him to walk home is x/w. Because he rides his bicycle at a speed of $7w$, the time required for him to ride his bicycle from his home to the stadium is $(x + y)/(7w)$. Thus

$$\frac{y}{w} = \frac{x}{w} + \frac{x + y}{7w} = \frac{8x + y}{7w}.$$

As a consequence, $7y = 8x + y$, so $8x = 6y$. The required ratio is $x/y = 6/8 = 3/4$.

OR

Because we are interested only in the ratio of the distances, we may assume that the distance from Yan's home to the stadium is 1 mile. Let x be his present distance from his home. Imagine that Yan has a twin, Nay. While Yan walks to the stadium, Nay walks to their home and continues $1/7$ of a mile past their home. Because walking $1/7$ of a mile requires the same amount of time as riding 1 mile, Yan and Nay will complete their trips at the same time. Yan has walked $1 - x$ miles while Nay has walked $x + \frac{1}{7}$ miles, so $1 - x = x + \frac{1}{7}$. Thus $x = 3/7$, $1 - x = 4/7$, and the required ratio is $x/(1 - x) = 3/4$.

10. **Answer (A):** Let the sides of the triangle have lengths $3x$, $4x$, and $5x$. The triangle is a right triangle, so its hypotenuse is a diameter of the circle. Thus $5x = 2 \cdot 3 = 6$, so $x = 6/5$. The area of the triangle is

$$\frac{1}{2} \cdot 3x \cdot 4x = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

OR

A right triangle with side lengths 3, 4, and 5 has area $(1/2)(3)(4) = 6$. Because the given right triangle is inscribed in a circle with diameter 6, the hypotenuse of this triangle has length 6. Thus the sides of the given triangle are $6/5$ as long as those of a $3-4-5$ triangle, and its area is $(6/5)^2$ times that of a $3-4-5$ triangle. The area of the given triangle is

$$\left(\frac{6}{5}\right)^2 (6) = \frac{216}{25} = 8.64.$$

11. **Answer (D):** A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S = 111k = 3 \cdot 37k$, so S must be divisible by 37. To see that S need not be divisible by any larger prime, note that the sequence 123, 231, 312 gives $S = 666 = 2 \cdot 3^2 \cdot 37$.

12. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2) \cdot (1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{5}{8}.$$

13. **Answer (B):** The point (a, b) is the foot of the perpendicular from $(12, 10)$ to the line $y = -5x + 18$. The perpendicular has slope $\frac{1}{5}$, so its equation is

$$y = 10 + \frac{1}{5}(x - 12) = \frac{1}{5}x + \frac{38}{5}.$$

The x-coordinate at the foot of the perpendicular satisfies the equation

$$\frac{1}{5}x + \frac{38}{5} = -5x + 18,$$

so $x = 2$ and $y = -5 \cdot 2 + 18 = 8$. Thus $(a, b) = (2, 8)$, and $a + b = 10$.

OR

If the mouse is at $(x, y) = (x, 18 - 5x)$, then the square of the distance from the mouse to the cheese is

$$(x - 12)^2 + (8 - 5x)^2 = 26\left(x^2 - 4x + 8\right) = 26\left((x - 2)^2 + 4\right).$$

The value of this expression is smallest when $x = 2$, so the mouse is closest to the cheese at the point $(2, 8)$, and $a + b = 2 + 8 = 10$.

14. **Answer (C):** If 45 is expressed as a product of five distinct integer factors, the absolute value of the product of any four is at least $|(-3)(-1)(1)(3)| = 9$, so no factor can have an absolute value greater than 5. Thus the factors of the given expression are five of the integers $\pm 1, \pm 3$, and ± 5. The product of all six of these is $-225 = (-5)(45)$, so the factors are -3, -1, 1, 3, and 5. The corresponding values of a, b, c, d, and e are 9, 7, 5, 3, and 1, and their sum is 25.

15. **Answer (E):** The mean of the augmented set is $(28+n)/5$. If $n < 6$, the median of that set is 6, so $28 + n = 5 \cdot 6$, and $n = 2$. If $6 < n < 9$, the median is n, so $28 + n = 5n$, and $n = 7$. If $n > 9$, the median is 9, so $28 + n = 5 \cdot 9$, and $n = 17$. Thus the sum of all possible values of n is $2 + 7 + 17 = 26$.

16. **Answer (C):** The set of the three digits of such a number can be arranged to form an increasing arithmetic sequence. There are 8 possible sequences with a common difference of 1, since the first term can be any of the digits 0 through 7. There are 6 possible sequences with a common difference of 2, 4 with a common difference of 3, and 2 with a common difference of 4. Hence there are 20 possible arithmetic sequences. Each of the 4 sets that contain 0 can be arranged to form $2 \cdot 2! = 4$ different numbers, and the 16 sets that do not contain 0 can be arranged to form $3! = 6$ different numbers. Thus there are a total of $4 \cdot 4 + 16 \cdot 6 = 112$ numbers with the required properties.

17. **Answer (B):** Square both sides of both given equations to obtain

$$\sin^2 a + 2\sin a \sin b + \sin^2 b = 5/3 \quad \text{and} \quad \cos^2 a + 2\cos a \cos b + \cos^2 b = 1.$$

Then add corresponding sides of the resulting equations to obtain

$$\left(\sin^2 a + \cos^2 a\right) + \left(\sin^2 b + \cos^2 b\right) + 2(\sin a \sin b + \cos a \cos b) = \frac{8}{3}.$$

Because $\sin^2 a + \cos^2 a = \sin^2 b + \cos^2 b = 1$, it follows that

$$\cos(a-b) = \sin a \sin b + \cos a \cos b = \frac{1}{3}.$$

One ordered pair (a, b) that satisfies the given condition is approximately $(0.296, 1.527)$.

18. **Answer (D):** Because $f(x)$ has real coefficients and $2i$ and $2+i$ are zeros, so are their conjugates $-2i$ and $2-i$. Therefore

$$\begin{aligned} f(x) = (x+2i)(x-2i)(x-(2+i))(x-(2-i)) &= (x^2+4)(x^2-4x+5) \\ &= x^4 - 4x^3 + 9x^2 - 16x + 20. \end{aligned}$$

Hence $a + b + c + d = -4 + 9 - 16 + 20 = 9$.

OR

As in the first solution,

$$f(x) = (x + 2i)(x - 2i)(x - (2 + i))(x - (2 - i)),$$

so

$$a+b+c+d = f(1)-1 = (1+2i)(1-2i)(-1-i)(-1+i)-1 = (1+4)(1+1)-1 = 9.$$

19. **Answer (E):** Let h be the length of the altitude from A in $\triangle ABC$. Then

$$2007 = \frac{1}{2} \cdot BC \cdot h = \frac{1}{2} \cdot 223 \cdot h,$$

so $h = 18$. Thus A is on one of the lines $y = 18$ or $y = -18$. Line DE has equation $x - y - 300 = 0$. Let A have coordinates (a, b). By the formula for the distance from a point to a line, the distance from A to line DE is $|a-b-300|/\sqrt{2}$. The area of $\triangle ADE$ is

$$7002 = \frac{1}{2} \cdot \frac{|a - b - 300|}{\sqrt{2}} \cdot DE = \frac{1}{2} \cdot \frac{|a \pm 18 - 300|}{\sqrt{2}} \cdot 9\sqrt{2}.$$

Thus $a = \pm 18 \pm 1556 + 300$, and the sum of the four possible values of a is $4 \cdot 300 = 1200$.

OR

As above, conclude that A is on one of the lines $y = \pm 18$. By similar reasoning, A is on one of two particular lines l_1 and l_2 parallel to $\overline{DE}$. Therefore there are four possible positions for A, determined by the intersections of the lines $y = 18$ and $y = -18$ with each of l_1 and l_2. Let the line $y = 18$ intersect l_1 and l_2 in points (x_1, y_1) and (x_2, y_2), and let the line $y = -18$ intersect l_1 and l_2 in points (x_3, y_3) and (x_4, y_4). The four points of intersection are the vertices of a parallelogram, and the center of the parallelogram has x-coordinate $(1/4)(x_1 + x_2 + x_3 + x_4)$. The center is the intersection of the line $y = 0$ and line DE. Because line DE has equation $y = x - 300$, the center of the parallelogram is $(300, 0)$. Thus the sum of all possible x-coordinates of A is $4 \cdot 300 = 1200$.

20. **Answer (B):** Removing the corners removes two segments of equal length from each edge of the cube. Call that length x. Then each octagon has side length $\sqrt{2}x$, and the cube has edge length $1 = \left(2 + \sqrt{2}\right) x$, so

$$x = \frac{1}{2 + \sqrt{2}} = \frac{2 - \sqrt{2}}{2}.$$

Each removed corner is a tetrahedron whose altitude is x and whose base is an isosceles right triangle with leg length x. Thus the total volume of the eight tetrahedra is

$$8 \cdot \frac{1}{3} \cdot x \cdot \frac{1}{2}x^2 = \frac{1}{6}\left(2 - \sqrt{2}\right)^3 = \frac{10 - 7\sqrt{2}}{3}.$$

21. **Answer (A):** The product of the zeros of f is c/a, and the sum of the zeros is $-b/a$. Because these two numbers are equal, $c = -b$, and the sum of the coefficients is $a + b + c = a$, which is the coefficient of x^2. To see that none of the other choices is correct, let $f(x) = -2x^2 - 4x + 4$. The zeros of f are $-1 \pm \sqrt{3}$, so the sum of the zeros, the product of the zeros, and the sum of the coefficients are all -2. However, the coefficient of x is -4, the y-intercept is 4, the x-intercepts are $-1 \pm \sqrt{3}$, and the mean of the x-intercepts is -1.

22. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy
$$n \ge 2007 - 28 - 10 = 1969.$$
In addition, $n, S(n),$ and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n),$ and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

23. **Answer (A):** Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Then $AB = 6 = |p - q|$. Because $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$. Thus $|q^2 - q| = 6$, and the only positive solution is $q = 3$. Note that $C = (q, 3\log_a q)$, so $BC = 6 = \log_a q$, from which $a^6 = q = 3$ and $a = \sqrt[6]{3}$.

24. **Answer (D):** Note that $F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$. For each n, $\sin nx \ge 0$ on each interval $[(2k-2)\pi/n, (2k-1)\pi/n]$ where k is a positive integer and $2k - 1 \le n$. The number of such intervals is $n/2$ if n is even and $(n+1)/2$ if n is odd. The graphs intersect twice on each interval unless $\sin x = 1 = \sin nx$ at some point in the interval, in which case the graphs intersect once. This last equation is satisfied if and only if $n \equiv 1 \pmod 4$ and the interval contains $\pi/2$. If n is even, this count does not include the point of intersection at $(\pi, 0)$. Therefore $F(n) = 2(n/2) + 1 = n + 1$ if n is even, $F(n) = 2(n+1)/2 = n + 1$ if $n \equiv 3 \pmod 4$, and $F(n) = n$ if $n \equiv 1 \pmod 4$. Hence
$$\sum_{n=2}^{2007} F(n) = \left(\sum_{n=2}^{2007} (n+1)\right) - \left\lfloor \frac{2007-1}{4} \right\rfloor = \frac{(2006)(3+2008)}{2} - 501 = 2{,}016{,}532.$$

25. **Answer (E):** For each positive integer n, let $S_n = \{k : 1 \le k \le n\}$, and let c_n be the number of spacy subsets of S_n. Then $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. For $n \ge 4$, the spacy subsets of S_n can be partitioned into two types: those that contain n and those that do not. Those that do not contain n are precisely the spacy subsets of S_{n-1}. Those that contain n do not contain either $n-1$ or $n-2$ and are therefore in one-to-one correspondence with the spacy subsets of S_{n-3}. It follows that $c_n = c_{n-3} + c_{n-1}$. Thus the first twelve terms in the sequence (c_n) are 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, and there are $c_{12} = 129$ spacy subsets of S_{12}.

OR

Note that each spacy subset of S_{12} contains at most 4 elements. For each such subset $a_1, a_2, \ldots, a_k$, let $b_1 = a_1 - 1$, $b_j = a_j - a_{j-1} - 3$ for $2 \le j \le k$, and $b_{k+1} = 12 - a_k$. Then $b_j \ge 0$ for $1 \le j \le k+1$, and

$$b_1 + b_2 + \cdots + b_{k+1} = 12 - 1 - 3(k-1) = 14 - 3k.$$

The number of solutions for $(b_1, b_2, \ldots, b_{k+1})$ is $\binom{14-2k}{k}$ for $0 \le k \le 4$, so the number of spacy subsets of S_{12} is

$$\binom{14}{0} + \binom{12}{1} + \binom{10}{2} + \binom{8}{3} + \binom{6}{4} = 1 + 12 + 45 + 56 + 15 = 129.$$

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

58th Annual American Mathematics Contest 12

AMC 12
CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 21, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **Answer (E):** The perimeter of each bedroom is $2(12 + 10) = 44$ feet, so the surface to be painted in each bedroom has an area of $44 \cdot 8 - 60 = 292$ square feet. Since there are 3 bedrooms, Isabella must paint $3 \cdot 292 = 876$ square feet.

2. **Answer (B):** The student used $120/30 = 4$ gallons on the trip home and $120/20 = 6$ gallons on the trip back to school. So the average gas mileage for the round trip was

$$\frac{240 \text{ miles}}{10 \text{ gallons}} = 24 \text{ miles per gallon.}$$

3. **Answer (D):** Since $OA = OB = OC$, triangles AOB, BOC, and COA are all isosceles. Hence

$$\angle ABC = \angle ABO + \angle OBC = \frac{180^\circ - 140^\circ}{2} + \frac{180^\circ - 120^\circ}{2} = 50^\circ.$$

OR

Since

$$\angle AOC = 360^\circ - 140^\circ - 120^\circ = 100^\circ,$$

the Central Angle Theorem implies that

$$\angle ABC = \frac{1}{2}\angle AOC = 50^\circ.$$

4. **Answer (B):** Because 3 bananas cost as much as 2 apples, 18 bananas cost as much as 12 apples. Because 6 apples cost as much as 4 oranges, 12 apples cost as much as 8 oranges. Therefore 18 bananas cost as much as 8 oranges.

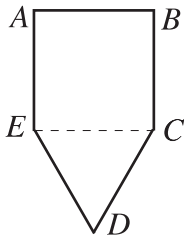
5. **Answer (D):** Sarah will receive 4.5 points for the three questions she leaves unanswered, so she must earn at least $100 - 4.5 = 95.5$ points on the first 22 problems. Because

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 of the first 22 problems correctly. This would give her a score of 100.5.

6. **Answer (D):** The perimeter of the triangle is $5 + 6 + 7 = 18$, so the distance that each bug crawls is 9. Therefore $AB + BD = 9$, and $BD = 4$.

7. **Answer (E):** Because $AB = BC = EA$ and $\angle A = \angle B = 90°$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90°$.



Also $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60°$. Therefore

$$\angle E = \angle AEC + \angle CED = 90° + 60° = 150°.$$

8. **Answer (D):** Tom's age N years ago was $T - N$. The sum of his three children's ages at that time was $T - 3N$. Therefore $T - N = 2(T - 3N)$, so $5N = T$ and $T/N = 5$. The conditions of the problem can be met, for example, if Tom's age is 30 and the ages of his children are 9, 10, and 11. In that case $T = 30$ and $N = 6$.

9. **Answer (A):** Let $u = 3x - 1$. Then $x = (u+1)/3$, and

$$f(u) = \left(\frac{u+1}{3}\right)^2 + \frac{u+1}{3} + 1 = \frac{u^2+2u+1}{9} + \frac{u+1}{3} + 1 = \frac{u^2+5u+13}{9}.$$

In particular,

$$f(5) = \frac{5^2 + 5 \cdot 5 + 13}{9} = \frac{63}{9} = 7.$$

OR

The value of $3x - 1$ is 5 when $x = 2$. Thus

$$f(5) = f(3 \cdot 2 - 1) = 2^2 + 2 + 1 = 7.$$

10. **Answer (C):** Let g be the number of girls and b the number of boys initially in the group. Then $g = 0.4(g+b)$. After two girls leave and two boys arrive, the size of the entire group is unchanged, so $g - 2 = 0.3(g+b)$. The solution of the system of equations

$$g = 0.4(g+b) \quad \text{and} \quad g - 2 = 0.3(g+b)$$

is $g = 8$ and $b = 12$, so there were initially 8 girls.

OR

After two girls leave and two boys arrive, the size of the group is unchanged. So the two girls who left represent $40\% - 30\% = 10\%$ of the group. Thus the size of the group is 20, and the original number of girls was 40% of 20, or 8.

11. **Answer (D):** Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

12. **Answer (C):** Let N be the number of students in the class. Then there are $0.1N$ juniors and $0.9N$ seniors. Let s be the score of each junior. The scores totaled $84N = 83(0.9N) + s(0.1N)$, so

$$s = \frac{84N - 83(0.9N)}{0.1N} = 93.$$

Note: In this problem, we could assume that the class has one junior and nine seniors. Then

$$9 \cdot 83 + s = 10 \cdot 84 = 9 \cdot 84 + 84 \quad \text{and} \quad s = 9(84 - 83) + 84 = 93.$$

13. **Answer (D):** The light completes a cycle every 63 seconds. Leah sees the color change if and only if she begins to look within three seconds before the change from green to yellow, from yellow to red, or from red to green. Thus she sees the color change with probability $(3 + 3 + 3)/63 = 1/7$.

14. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

15. **Answer (E):** The terms involving odd powers of r form the geometric series $ar + ar^3 + ar^5 + \cdots$. Thus

$$7 = a + ar + ar^2 + \cdots = \frac{a}{1-r},$$

and

$$3 = ar + ar^3 + ar^5 + \cdots = \frac{ar}{1-r^2} = \frac{a}{1-r} \cdot \frac{r}{1+r} = \frac{7r}{1+r}.$$

Therefore $r = 3/4$. It follows that $a/(1/4) = 7$, so $a = 7/4$ and

$$a + r = \frac{7}{4} + \frac{3}{4} = \frac{5}{2}.$$

OR

The sum of the terms involving even powers of r is $7 - 3 = 4$. Therefore

$$3 = ar + ar^3 + ar^5 + \cdots = r(a + ar^2 + ar^4 + \cdots) = 4r,$$

so $r = 3/4$. As in the first solution, $a = 7/4$ and $a + r = 5/2$.

16. **Answer (A):** Let r, w, and b be the number of red, white, and blue faces, respectively. Then (r, w, b) is one of 15 possible ordered triples, namely one of the three permutations of $(4, 0, 0)$, $(2, 2, 0)$, or $(2, 1, 1)$, or one of the six permutations of $(3, 1, 0)$. The number of distinguishable colorings for each of these ordered triples is the same as for any of its permutations. If $(r, w, b) = (4, 0, 0)$, then exactly one coloring is possible. If $(r, w, b) = (3, 1, 0)$, the tetrahedron can be placed with the white face down. If $(r, w, b) = (2, 2, 0)$, the tetrahedron can be placed with one white face down and the other facing forward. If $(r, w, b) = (2, 1, 1)$, the tetrahedron can be placed with the white face down and the blue face forward. Therefore there is only one coloring for each ordered triple, and the total number of distinguishable colorings is 15.

17. **Answer (D):** Because $b < 10^b$ for all $b > 0$, it follows that $\log_{10} b < b$. If $b \geq 1$, then $0 < (\log_{10} b)/b^2 < 1$, so a cannot be an integer. Therefore $0 < b < 1$, so $\log_{10} b < 0$ and $a = (\log_{10} b)/b^2 < 0$. Thus $a < 0 < b < 1 < 1/b$, and the median of the set is b.

    Note that the conditions of the problem can be met with $b = 0.1$ and $a = -100$.

18. **Answer (C):** Let N^2 be the smaller of the two squares. Then the difference between the two squares is $(N+1)^2 - N^2 = 2N + 1$. The given conditions state that

$$100a + 10b + c = N^2 + \frac{2N+1}{3} \quad \text{and} \quad 100a + 10c + b = N^2 + \frac{2(2N+1)}{3}.$$

Subtraction yields $9(c - b) = (2N + 1)/3$, from which $27(c - b) = 2N + 1$. If $c - b = 0$ or 2, then N is not an integer. If $c - b \geq 3$, then $N \geq 40$, so N^2 is not a three-digit integer. If $c - b = 1$, then $N = 13$. The numbers that are, respectively, one third of the way and two thirds of the way from 13^2 to 14^2 are 178 and 187, so $a + b + c = 1 + 7 + 8 = 16$.

19. **Answer (A):** Let $\theta = \angle ABC$. The base of the cylinder is a circle with circumference 6, so the radius of the base is $6/(2\pi) = 3/\pi$. The height of the cylinder is the altitude of the rhombus, which is $6 \sin\theta$. Thus the volume of the cylinder is

$$6 = \pi \left(\frac{3}{\pi}\right)^2 (6 \sin\theta) = \frac{54}{\pi} \sin\theta,$$

so $\sin\theta = \pi/9$.

20. **Answer (D):** Two vertices of the first parallelogram are at $(0, c)$ and $(0, d)$. The x-coordinates of the other two vertices satisfy $ax+c = bx+d$ and $ax+d = bx + c$, so the x-coordinates are $\pm(c - d)/(b - a)$. Thus the parallelogram is composed of two triangles, each of which has area

$$9 = \frac{1}{2} \cdot |c - d| \cdot \left|\frac{c - d}{b - a}\right|.$$

It follows that $(c - d)^2 = 18|b - a|$. By a similar argument using the second parallelogram, $(c + d)^2 = 72|b - a|$. Subtracting the first equation from the second yields $4cd = 54|b - a|$, so $2cd = 27|b - a|$. Thus $|b - a|$ is even, and $a + b$ is minimized when $\{a, b\} = \{1, 3\}$. Also, cd is a multiple of 27, and $c + d$ is minimized when $\{c, d\} = \{3, 9\}$. Hence the smallest possible value of $a+b+c+d$ is $1+3+3+9 = 16$. Note that the required conditions are satisfied when $(a, b, c, d) = (1, 3, 3, 9)$.

21. **Answer (A):** Because $3^6 = 729 < 2007 < 2187 = 3^7$, it is convenient to begin by counting the number of base-3 palindromes with at most 7 digits. There are two palindromes of length 1, namely 1 and 2. There are also two palindromes of length 2, namely 11 and 22. For $n \geq 1$, each palindrome of length $2n + 1$ is obtained by inserting one of the digits 0, 1, or 2 immediately after the nth digit in a palindrome of length $2n$. Each palindrome of length $2n+2$ is obtained by similarly inserting one of the strings 00, 11, or 22. Therefore there are 6 palindromes of each of the lengths 3 and 4, 18 of each of the lengths 5 and 6, and 54 of length 7. Because the base-3 representation of 2007 is 2202100, that integer is less than each of the palindromes 2210122, 2211122, 2212122, 2220222, 2221222, and 2222222. Thus the required total is $2+2+6+6+18+18+54-6 = 100$.

22. **Answer (A):** Imagine a third particle that moves in such a way that it is always halfway between the first two. Let D, E, and F denote the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively, and let X, Y, and Z denote the midpoints of $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively. When the first particle is at A, the second is at D and the third is at X. When the first particle is at F, the second is at C and the third is at Z. Between those two instants, both coordinates of the first two particles are linear functions of time. Because the average of two linear functions is linear, the third particle traverses $\overline{XZ}$. Similarly, the third particle traverses $\overline{ZY}$ as the first traverses $\overline{FB}$ and the second traverses $\overline{CE}$. Finally, as the first particle traverses $\overline{BD}$ and the second traverses $\overline{EA}$, the third traverses $\overline{YX}$. As the first two particles return to A and D, respectively, the third makes a second circuit of $\triangle XYZ$.

If O is the center of $\triangle ABC$, then by symmetry O is also the center of equilateral $\triangle XYZ$. Note that

$$OZ = OC - ZC = \frac{2}{3}CF - \frac{1}{2}CF = \frac{1}{6}CF,$$

so the ratio of the area of $\triangle XYZ$ to that of $\triangle ABC$ is

$$\left(\frac{OZ}{OC}\right)^2 = \left(\frac{\frac{1}{6}CF}{\frac{2}{3}CF}\right)^2 = \frac{1}{16}.$$

23. **Answer (A):** Let the triangle have leg lengths a and b, with $a \leq b$. The given condition implies that

$$\frac{1}{2}ab = 3\left(a + b + \sqrt{a^2 + b^2}\right),$$

so

$$ab - 6a - 6b = 6\sqrt{a^2 + b^2}.$$

Squaring both sides and simplifying yields

$$ab\,(ab - 12a - 12b + 72) = 0,$$

from which

$$(a - 12)(b - 12) = 72.$$

The positive integer solutions of the last equation are $(a, b) = (3, 4)$, $(13, 84)$, $(14, 48)$, $(15, 36)$, $(16, 30)$, $(18, 24)$, and $(20, 21)$. However, the solution $(3, 4)$ is extraneous, and there are six right triangles with the required property.

Query: Why do the given conditions imply that the hypotenuse also has integer length?

24. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation

$$\frac{a}{b}+\frac{14b}{9a}=k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a}=bk-a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b}=9ak-14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a,b) is a solution, then $b=1$, 3, or 9, and $a=1$, 2, 7, or 14. This gives a total of twelve possible solutions (a,b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b}+\frac{14b}{9a}$$

is an integer are when (a,b) is $(1,3)$, $(2,3)$, $(7,3)$, or $(14,3)$.

25. **Answer (C):** Introduce a coordinate system in which $D=(-1,0,0)$, $E=(1,0,0)$, and $\triangle ABC$ lies in a plane $z=k>0$. Because $\angle CDE$ and $\angle DEA$ are right angles, A and C are located on circles of radius 2 centered at E and D in the planes $x=1$ and $x=-1$, respectively. Thus $A=(1,y_1,k)$ and $C=(-1,y_2,k)$, where $y_j=\pm\sqrt{4-k^2}$ for $j=1$ and 2. Because $AC=2\sqrt{2}$, it follows that $(1-(-1))^2+(y_1-y_2)^2=\left(2\sqrt{2}\right)^2$. If $y_1=y_2$, there is no solution, so $y_1=-y_2$. It may be assumed without loss of generality that $y_1>0$, in which case $y_1=1$ and $y_2=-1$. It follows that $k=\sqrt{3}$, so $A=(1,1,\sqrt{3})$, $C=\left(-1,-1,\sqrt{3}\right)$, and B is one of the points $\left(1,-1,\sqrt{3}\right)$ or $\left(-1,1,\sqrt{3}\right)$. In the first case, $BE=2$ and $\overline{BE}\perp\overline{DE}$. In the second case, $BD=2$ and $\overline{BD}\perp\overline{DE}$. In either case, the area of $\triangle BDE$ is $(1/2)(2)(2)=2$.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

59th Annual American Mathematics Contest 12

AMC 12
CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 12, 2008

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **Answer (D):** The machine worked for 2 hours and 40 minutes, or 160 minutes, to complete one third of the job, so the entire job will take $3 \cdot 160 = 480$ minutes, or 8 hours. Hence the doughnut machine will complete the job at 4:30 PM.

2. **Answer (A):** Note that

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} = \frac{7}{6}.$$

The reciprocal of $\frac{7}{6}$ is $\frac{6}{7}$.

3. **Answer (C):** Note that $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas, which are worth as much as 8 oranges. So one banana is worth as much as $8 \cdot \frac{3}{20} = \frac{6}{5}$ oranges. Therefore $\frac{1}{2}$ of 5 bananas are worth as much as $\frac{5}{2} \cdot \frac{6}{5} = 3$ oranges.

4. **Answer (B):** Because each denominator except the first can be canceled with the previous numerator, the product is $\frac{2008}{4} = 502$.

5. **Answer (B):** Because

$$\frac{2x}{3} - \frac{x}{6} = \frac{x}{2}$$

is an integer, x must be even. The case $x = 4$ shows that x is not necessarily a multiple of 3 and that none of the other statements must be true.

6. **Answer (A):** Let x denote the sticker price, in dollars. Heather pays $0.85x - 90$ dollars at store A and would have paid $0.75x$ dollars at store B. Thus the sticker price x satisfies $0.85x - 90 = 0.75x - 15$, so $x = 750$.

7. **Answer (D):** At the rate of 4 miles per hour, Steve can row a mile in 15 minutes. During that time $15 \cdot 10 = 150$ gallons of water will enter the boat. LeRoy must bail $150 - 30 = 120$ gallons of water during that time. So he must bail at the rate of at least $\frac{120}{15} = 8$ gallons per minute.

OR

Steve must row for 15 minutes to reach the shore, so the amount of water in the boat can increase by at most $\frac{30}{15} = 2$ gallons per minute. Therefore LeRoy must bail out at least $10 - 2 = 8$ gallons per minute.

8. **Answer (C):** Let x be the side length of the larger cube. The larger cube has surface area $6x^2$, and the smaller cube has surface area 6. So $6x^2 = 2 \cdot 6 = 12$, and $x = \sqrt{2}$. The volume of the larger cube is $x^3 = \left(\sqrt{2}\right)^3 = 2\sqrt{2}$.

9. **Answer (D):** Let h and w be the height and width of the screen, respectively, in inches. By the Pythagorean Theorem, $h{:}w{:}27 = 3{:}4{:}5$, so

$$h = \frac{3}{5}\cdot 27 = 16.2 \quad \text{and} \quad w = \frac{4}{5}\cdot 27 = 21.6.$$

The height of the non-darkened portion of the screen is half the width, or 10.8 inches. Therefore the height of each darkened strip is

$$\frac{1}{2}(16.2 - 10.8) = 2.7 \quad \text{inches.}$$

OR

The screen has dimensions $4a \times 3a$ for some a. The portion of the screen not covered by the darkened strips has aspect ratio 2:1, so it has dimensions $4a \times 2a$. Thus the darkened strips each have height $\frac{a}{2}$. By the Pythagorean Theorem, the diagonal of the screen is $5a = 27$ inches. Hence the height of each darkened strip is $\frac{27}{10} = 2.7$ inches.

10. **Answer (D):** In one hour Doug can paint $\frac{1}{5}$ of the room, and Dave can paint $\frac{1}{7}$ of the room. Working together, they can paint $\frac{1}{5}+\frac{1}{7}$ of the room in one hour. It takes them t hours to do the job, but because they take an hour for lunch, they work for only $t-1$ hours. The fraction of the room that they paint in this time is

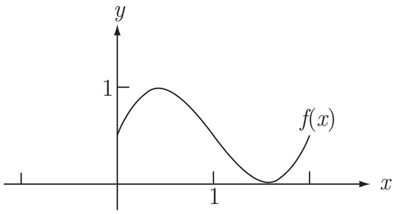
$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1),$$

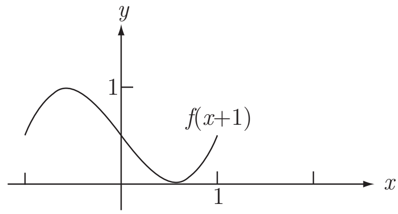
which must be equal to 1. It may be checked that the solution, $t = \frac{47}{12}$, does not satisfy the equation in any of the other answer choices.

11. **Answer (C):** The sum of the six numbers on each cube is $1+2+4+8+16+32 = 63$. The three pairs of opposite faces have numbers with sums $1+32 = 33$, $2+16 = 18$, and $4+8 = 12$. On the two lower cubes, the numbers on the four visible faces have the greatest sum when the 4 and the 8 are hidden. On the top cube, the numbers on the five visible faces have the greatest sum when the 1 is hidden. Thus the greatest possible sum is $3\cdot 63 - 2\cdot(4+8) - 1 = 164$.

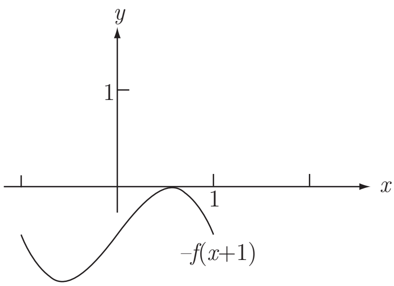
12. **Answer (B):** Because the domain of f is $[0,2]$, $f(x+1)$ is defined for $0 \le x+1 \le 2$, or $-1 \le x \le 1$. Thus $g(x)$ is also defined for $-1 \le x \le 1$, so its domain is $[-1,1]$. Because the range of f is $[0,1]$, the values of $f(x+1)$ are all the numbers between 0 and 1, inclusive. Thus the values of $g(x)$ are all the numbers between $1-0=1$ and $1-1=0$, inclusive, so the range of g is $[0,1]$.

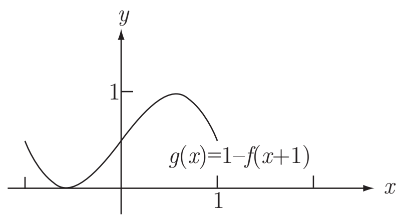
OR



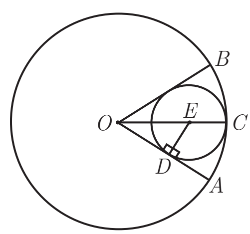


The graph of $y = f(x+1)$ is obtained by shifting the graph of $y = f(x)$ one unit to the left. The graph of $y = -f(x+1)$ is obtained by reflecting the graph of $y = f(x+1)$ across the x-axis. The graph of $y = g(x) = 1 - f(x+1)$ is obtained by shifting the graph of $y = -f(x+1)$ up one unit. As the figures illustrate, the domain and range of g are $[-1, 1]$ and $[0, 1]$, respectively.





13. **Answer (B):** Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90°$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.



14. **Answer (A):** The boundaries of the region are the two pairs of parallel lines

$$(3x - 18) + (2y + 7) = \pm 3 \quad \text{and} \quad (3x - 18) - (2y + 7) = \pm 3.$$

These lines intersect at $(6, -2)$, $(6, -5)$, $(5, -\frac{7}{2})$, and $(7, -\frac{7}{2})$. Thus the region is a rhombus whose diagonals have lengths 2 and 3. The area of the rhombus is half the product of the diagonal lengths, which is 3.

15. **Answer (D):** The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

16. **Answer (D):** The first three terms of the sequence can be written as $3\log a + 7\log b$, $5\log a + 12\log b$, and $8\log a + 15\log b$. The difference between consecutive terms can be written either as

$$(5\log a + 12\log b) - (3\log a + 7\log b) = 2\log a + 5\log b$$

or as

$$(8\log a + 15\log b) - (5\log a + 12\log b) = 3\log a + 3\log b.$$

Thus $\log a = 2\log b$, so the first term of the sequence is $13\log b$, and the difference between consecutive terms is $9\log b$. Hence the 12^{th} term is

$$(13 + (12-1)\cdot 9)\log b = 112\log b = \log(b^{112}).$$

17. **Answer (D):** If a_1 is even, then $a_2 = (a_1/2) < a_1$, so the required condition is not met. If $a_1 \equiv 1 \pmod 4$, then $a_2 = 3a_1 + 1$ is a multiple of 4, so $a_3 = (3a_1+1)/2$, and $a_4 = (3a_1+1)/4 \le a_1$. Hence the required condition is also not met in this case. If $a_1 \equiv 3 \pmod 4$, then a_2 is even but not a multiple of 4. It follows that $a_3 = (3a_1+1)/2 > a_1$, and a_3 is odd, so $a_4 = 3a_3 + 1 > a_3 > a_1$. Because 2008 is a multiple of 4, a total of $\frac{2008}{4} = 502$ possible values of a_1 are congruent to $3 \pmod 4$. These 502 values of a_1 meet the required condition.

Note: It is a famous unsolved problem to show whether or not the number 1 must be a term of this sequence for every choice of a_1.

18. **Answer (C):** It may be assumed that $A = (a,0,0)$, $B = (0,b,0)$, $C = (0,0,c)$, $AB = 5$, $BC = 6$, and $CA = 7$. Then

$$a^2 + b^2 = 5^2, \quad b^2 + c^2 = 6^2, \quad \text{and} \quad a^2 + c^2 = 7^2,$$

from which

$$a^2 + b^2 + c^2 = \frac{1}{2}\left(5^2 + 6^2 + 7^2\right) = 55.$$

It follows that $a = \sqrt{55 - 6^2} = \sqrt{19}$, $b = \sqrt{55 - 7^2} = \sqrt{6}$, $c = \sqrt{55 - 5^2} = \sqrt{30}$, and the volume of tetrahedron $OABC$ can be expressed as

$$\frac{1}{3}\cdot OC \cdot \text{Area}(\triangle OAB) = \frac{1}{6}\sqrt{6\cdot 19\cdot 30} = \sqrt{95}.$$

19. **Answer (C):** Each term in the expansion has the form x^{a+b+c}, where $0 \le a \le 27$, $0 \le b \le 14$, and $0 \le c \le 14$. There are $(14+1)^2 = 225$ possible combinations of values for b and c, and for every combination except $(b, c) = (0, 0)$, there is a unique a with $a + b + c = 28$. Thus the coefficient of x^{28} is 224.

OR

Let $P(x) = (1 + x + x^2 + \cdots + x^{14})^2 = 1 + r_1x + r_2x^2 + \cdots + r_{28}x^{28}$ and $Q(x) = 1 + x + x^2 + \cdots + x^{27}$. The coefficient of x^{28} in the product $P(x)Q(x)$ is $r_1 + r_2 + \cdots + r_{28} = P(1) - 1 = 15^2 - 1 = 224$.

20. **Answer (E):** By the Angle Bisector Theorem,

$$AD = 5 \cdot \frac{3}{3+4} = \frac{15}{7} \quad \text{and} \quad BD = 5 \cdot \frac{4}{3+4} = \frac{20}{7}.$$

To determine CD, start with the relation Area($\triangle ADC$) + Area($\triangle BCD$) = Area($\triangle ABC$) to get

$$\frac{3 \cdot CD}{2\sqrt{2}} + \frac{4 \cdot CD}{2\sqrt{2}} = \frac{3 \cdot 4}{2}.$$

This gives $CD = \frac{12\sqrt{2}}{7}$. Now use the fact that the area of a triangle is given by rs, where r is the radius of the inscribed circle and s is half the perimeter of the triangle. The ratio of the area of $\triangle ADC$ to the area of $\triangle BCD$ is the ratio of the altitudes to their common base $\overline{CD}$, which is $\frac{AD}{BD} = \frac{3}{4}$. Hence

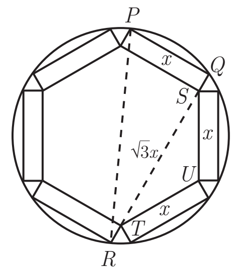
$$\frac{3}{4} = \frac{\text{Area}(\triangle ADC)}{\text{Area}(\triangle BCD)} = \frac{r_a(3 + \frac{15}{7} + \frac{12\sqrt{2}}{7})}{r_b(4 + \frac{20}{7} + \frac{12\sqrt{2}}{7})}.$$

which yields

$$\frac{r_a}{r_b} = \frac{3(4+\sqrt{2})}{4(3+\sqrt{2})} = \frac{3}{28}(10 - \sqrt{2}).$$

21. **Answer (D):** Call a permutation <u>balanced</u> if $a_1 + a_2 = a_4 + a_5$, and consider the number of balanced permutations. The sum of all five entries is odd, so in a balanced permutation, a_3 must be 1, 3, or 5. For each choice of a_3, there is a unique way to group the remaining four numbers into two sets whose elements have equal sums. For example, if $a_3 = 1$, the two sets must be $\{2, 5\}$ and $\{3, 4\}$. Any one of the four numbers can be a_1, and the value of a_2 is then determined. Either of the two remaining numbers can be a_4, and the value of a_5 is then determined. Thus there are $3 \cdot 2 \cdot 4 = 24$ balanced permutations of $(1, 2, 3, 4, 5)$, and $5! - 24 = 96$ permutations that are not balanced. Call a permutation <u>heavy-headed</u> if $a_1 + a_2 > a_4 + a_5$. Reversing the entries in a heavy-headed permutation yields a unique heavy-tailed permutation, and vice versa, so there are exactly as many heavy-headed permutations as heavy-tailed ones. Therefore the number of heavy-tailed permutations is $\frac{1}{2} \cdot 96 = 48$.

22. **Answer (C):** Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle $120°$. It follows that $ST = \sqrt{3}x$, so $QR = QS+ST+TR = \sqrt{3}x+2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x+2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain



$$x = \frac{3\sqrt{7}-\sqrt{3}}{2}.$$

23. **Answer (D):** Adding $1+i$ to each side of the given equation gives

$$1+i = (z^4+4z^3i-6z^2-4zi-i)+1+i = z^4+4z^3i-6z^2-4zi+1 = (z+i)^4.$$

Let $w = z+i = r(\cos\theta + i\sin\theta)$. Since

$$i+1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

the solutions of $w^4 = 1+i$ satisfy

$$r^4 = \sqrt{2} \quad \text{and} \quad \theta = \frac{1}{4}\left(\frac{\pi}{4}+2k\pi\right) = \frac{\pi}{16}+\frac{\pi}{2}k,$$

for $k = 0$, 1, 2, or 3. Thus

$$w_k = 2^{1/8}\left(\cos\left(\frac{\pi}{16}+\frac{\pi}{2}k\right) + i\sin\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)\right) \quad \text{for } k = 0,1,2, \text{ or } 3,$$

and the four solutions for $z = w - i$ are

$$z_k = 2^{1/8}\left(\cos\left(\frac{\pi}{16}+\frac{\pi}{2}k\right) + i\sin\left(\frac{\pi}{16}+\frac{\pi}{2}k\right)\right) - i \quad \text{for } k = 0,1,2, \text{ or } 3.$$

Note that w_0, w_1, w_2, and w_3 are equally spaced around the circle of radius $2^{1/8}$ centered at $(0,0)$, so z_0, z_1, z_2, and z_3 are equally spaced around the circle of radius $2^{1/8}$ centered at $(0,-1)$. Therefore z_0, z_1, z_2, and z_3 are vertices of a square with side length $2^{1/8}\sqrt{2} = 2^{5/8}$ and area $(2^{5/8})^2 = 2^{5/4}$.

OR

The Binomial Theorem gives

$$(z+i)^4 = z^4+4z^3i-6z^2-4zi+1 = (z^4+4z^3i-6z^2-4zi-i)+1+i = 1+i.$$

Let a satisfy $a^4 = 1+i$, and let $w = z+i$. Then $w^4 = a^4$, so the possible values for w are a, ia, $-a$, and $-ia$, which are the vertices of a square with diagonal $2|a| = 2\sqrt[8]{2}$. The transformation $w = z+i$ is a translation, so it preserves area. Hence the area of the original polygon is $(2\sqrt[8]{2})^2/2 = 2\sqrt[4]{2} = 2^{5/4}$.

24. **Answer (D):** Let $C = (0,0)$, $B = (2, 2\sqrt{3})$, and $A = (x,0)$ with $x > 0$. Then $D = (1, \sqrt{3})$. Let P be on the positive x-axis to the right of A. Then $\angle BAD = \angle PAD - \angle PAB$. Provided $\angle PAD$ and $\angle PAB$ are not right angles, it follows that

$$\begin{aligned}\tan(\angle BAD) = \tan(\angle PAD - \angle PAB) &= \frac{\tan(\angle PAD) - \tan(\angle PAB)}{1+\tan(\angle PAD)\tan(\angle PAB)} \\ &= \frac{m_{AD} - m_{AB}}{1 + m_{AD}m_{AB}} = \frac{\frac{\sqrt{3}}{1-x} - \frac{2\sqrt{3}}{2-x}}{1 + \frac{\sqrt{3}}{1-x}\cdot\frac{2\sqrt{3}}{2-x}} = \frac{\sqrt{3}x}{x^2-3x+8} \\ &= \frac{\sqrt{3}}{\left(\sqrt{x} - \frac{2\sqrt{2}}{\sqrt{x}}\right)^2 + (4\sqrt{2}-3)} \le \frac{\sqrt{3}}{4\sqrt{2}-3},\end{aligned}$$

with equality when $x = 2\sqrt{2}$. If $\angle PAD = 90°$, then

$$\tan(\angle BAD) = \cot(\angle PAB) = \frac{1}{2\sqrt{3}} < \frac{\sqrt{3}}{4\sqrt{2}-3}.$$

If $\angle PAB = 90°$, then

$$\tan(\angle BAD) = -\cot(\angle PAD) = \frac{1}{\sqrt{3}} < \frac{\sqrt{3}}{4\sqrt{2}-3}.$$

Therefore the largest possible value of $\tan(\angle BAD)$ is $\sqrt{3}/(4\sqrt{2}-3)$.

OR

Because the circle with diameter $\overline{BD}$ does not intersect the line AC, it follows that $\angle BAD < 90°$. Thus the value of $\tan(\angle BAD)$ is greatest when $\angle BAD$ is greatest. This occurs when A is placed to minimize the size of the circle passing through A, B, and D, so the maximum is attained when that circle is tangent to $\overline{AC}$ at A. For this location of A, the Power of a Point Theorem implies that

$$AC^2 = CB \cdot CD = 4 \cdot 2 = 8, \text{ and } AC = \sqrt{8} = 2\sqrt{2}.$$

Because $\frac{CA}{CB} = \frac{CD}{CA}$, it follows that $\triangle CAD$ is similar to $\triangle CBA$. Thus $AB = \sqrt{2}AD$. The Law of Cosines, applied to $\triangle ADC$, gives

$$AD^2 = CD^2 + CA^2 - 2CD \cdot CA \cdot \cos 60° = 12 - 4\sqrt{2}.$$

Let O be the center of the circle passing through A, B, and D. The Extended Law of Sines, applied to $\triangle ABD$ and $\triangle ADC$, gives

$$2OB = \frac{AB}{\sin(\angle BDA)} = \frac{AB}{\sin(\angle ADC)}$$

$$= \frac{AB \cdot AD}{AC \cdot \sin 60^\circ} = \frac{2AB \cdot AD}{\sqrt{3}AC}$$

$$= \frac{2\sqrt{2}AD^2}{2\sqrt{2}\sqrt{3}} = \frac{AD^2}{\sqrt{3}}.$$

Let M be the midpoint of BD. Because $\angle BAD = \frac{1}{2}\angle BOD = \angle BOM$, it follows that

$$\begin{aligned}\tan(\angle BAD) = \tan(\angle BOM) &= \frac{MB}{OM}\\ &= \frac{1}{\sqrt{OB^2-1}} = \frac{1}{\sqrt{\frac{AD^4}{12}-1}}\\ &= \frac{\sqrt{3}}{\sqrt{(6-2\sqrt{2})^2-3}} = \frac{\sqrt{3}}{4\sqrt{2}-3}.\end{aligned}$$

25. **Answer (D):** Let $z_n = a_n + b_n i$. Then

$$\begin{aligned}z_{n+1} &= (\sqrt{3}a_n - b_n) + (\sqrt{3}b_n + a_n)i = (a_n + b_n i)(\sqrt{3}+i)\\ &= z_n(\sqrt{3}+i) = z_1(\sqrt{3}+i)^n.\end{aligned}$$

Noting that $\sqrt{3}+i = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and applying DeMoivre's formula gives

$$\begin{aligned}2+4i = z_{100} &= z_1\left(2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right)^{99}\\ &= z_1 \cdot 2^{99}\left(\cos\left(\frac{99\pi}{6}\right) + i\sin\left(\frac{99\pi}{6}\right)\right)\\ &= (a_1 + b_1 i)\cdot 2^{99}\cdot i = -2^{99}b_1 + 2^{99}a_1 i.\end{aligned}$$

So $2 = -2^{99}b_1$, $4 = 2^{99}a_1$, and

$$a_1 + b_1 = \frac{4}{2^{99}} - \frac{2}{2^{99}} = \frac{1}{2^{98}}.$$

OR

Note that

$$\begin{aligned}
(a_{n+2}, b_{n+2}) &= \Big(\sqrt{3}(\sqrt{3}a_n - b_n) - (\sqrt{3}b_n + a_n), \\
&\qquad \sqrt{3}(\sqrt{3}b_n + a_n) + (\sqrt{3}a_n - b_n)\Big) \\
&= \left(-2\sqrt{3}b_n + 2a_n, 2\sqrt{3}a_n + 2b_n\right), \\
(a_{n+3}, b_{n+3}) &= \Big(\sqrt{3}(-2\sqrt{3}b_n + 2a_n) - (2\sqrt{3}a_n + 2b_n), \\
&\qquad \sqrt{3}(2\sqrt{3}a_n + 2b_n) + (-2\sqrt{3}b_n + 2a_n)\Big) \\
&= 8(-b_n, a_n),
\end{aligned}$$

and $(a_{n+6}, b_{n+6}) = 8(-b_{n+3}, a_{n+3}) = -64(a_n, b_n)$. Because $97 = 1 + 16 \cdot 6$, we have

$$(a_{97}, b_{97}) = (-64)^{16}(a_1, b_1) = 2^{96}(a_1, b_1)$$

and

$$(2, 4) = (a_{100}, b_{100}) = 2^3\,(-b_{97}, a_{97}) = 2^{99}(-b_1, a_1).$$

The conclusion follows as in the first solution.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

59th Annual American Mathematics Contest 12

AMC 12 CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 27, 2008

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. David Wells, Department of Mathematics
Penn State University, New Kensington, PA 15068

1. **Answer (E):** The number of points could be any integer between $5 \cdot 2 = 10$ and $5 \cdot 3 = 15$, inclusive. The number of possibilities is $15 - 10 + 1 = 6$.

2. **Answer (B):** The two sums are $1 + 10 + 17 + 22 = 50$ and $4+9+16+25 = 54$, so the positive difference between the sums is $54 - 50 = 4$.

1	2	3	4
11	10	9	8
15	16	17	18
25	24	23	22

Query: If a different 4×4 block of dates had been chosen, the answer would be unchanged. Why?

3. **Answer (C):** A single player can receive the largest possible salary only when the other 20 players on the team are each receiving the minimum salary of \$15,000. Thus the maximum salary for any player is \$700,000 − 20 · \$15,000 = \$400,000.

4. **Answer (D):** The measure of $\angle COD$ is $180° - 30° - 45° = 105°$. Therefore the ratio of the area of the sector to the area of the circle is $\frac{105}{360} = \frac{7}{24}$.

5. **Answer (C):** The total cost of the carnations must be an even number of dollars. The total number of dollars spent is the even number 50, so the number of roses purchased must also be even. In addition, the number of roses purchased cannot exceed $\frac{50}{3}$. Therefore the number of roses purchased must be one of the even integers between 0 and 16, inclusive. This gives 9 possibilities for the number of roses purchased, and consequently 9 possibilities for the number of bouquets.

6. **Answer (A):** During the year Pete takes

$$44 \times 10^5 + 5 \times 10^4 = 44.5 \times 10^5$$

steps. At 1800 steps per mile, the number of miles Pete walks is

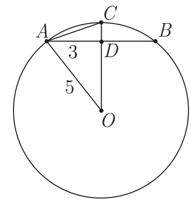
$$\frac{44.5 \times 10^5}{18 \times 10^2} = \frac{44.5}{18} \times 10^3 \approx 2.5 \times 10^3 = 2500.$$

7. **Answer (A):** Note that $(y - x)^2 = (x - y)^2$, so

$$(x-y)^2\$(y-x)^2 = (x-y)^2\$(x-y)^2 = \left((x-y)^2 - (x-y)^2\right)^2 = 0^2 = 0.$$

8. **Answer (C):** Because $AB + BD = AD$ and $AB = 4BD$, it follows that $BD = \frac{1}{5} \cdot AD$. By similar reasoning, $CD = \frac{1}{10} \cdot AD$. Thus

$$BC = BD - CD = \frac{1}{5} \cdot AD - \frac{1}{10} \cdot AD = \frac{1}{10} \cdot AD.$$



9. **Answer (A):** Let O be the center of the circle, and let D be the intersection of $\overline{OC}$ and $\overline{AB}$. Because $\overline{OC}$ bisects minor arc AB, $\overline{OD}$ is a perpendicular bisector of chord $\overline{AB}$. Hence $AD = 3$, and applying the Pythagorean Theorem to $\triangle ADO$ yields $OD = \sqrt{5^2 - 3^3} = 4$. Therefore $DC = 1$, and applying the Pythagorean Theorem to $\triangle ADC$ yields $AC = \sqrt{3^2 + 1^2} = \sqrt{10}$.

10. **Answer (B):** Let n be the number of bricks in the chimney. Then the number of bricks per hour Brenda and Brandon can lay working alone is $\frac{n}{9}$ and $\frac{n}{10}$, respectively. Working together they can lay $(\frac{n}{9} + \frac{n}{10} - 10)$ bricks in an hour, or

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right)$$

bricks in 5 hours to complete the chimney. Thus

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right) = n,$$

and the number of bricks in the chimney is $n = 900$.

OR

Suppose that Brenda can lay x bricks in an hour and Brandon can lay y bricks in an hour. Then the number of bricks in the chimney can be expressed as $9x$, $10y$, or $5(x + y - 10)$. The equality of these expressions leads to the system of equations

$$4x - 5y = -50$$
$$-5x + 5y = -50.$$

It follows that $x = 100$, so the number of bricks in the chimney is $9x = 900$.

11. **Answer (A):** The portion of the mountain that is above the water forms a cone that is similar to the entire mountain. The ratio of the volumes of the cones is the cube of the ratio of their heights. Let d be the depth of the ocean, in feet. Then the height of the mountain above the water is $8000 - d$ feet, and

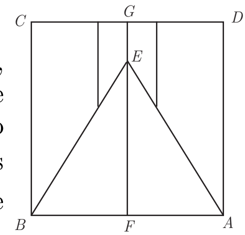
$$\frac{(8000 - d)^3}{8000^3} = \frac{1}{8}.$$

Taking cube roots on both sides gives

$$\frac{8000 - d}{8000} = \frac{1}{2},$$

from which $16{,}000 - 2d = 8000$, and $d = 4000$.

12. **Answer (B):** Because the mean of the first n terms is n, their sum is n^2. Therefore the nth term is $n^2 - (n-1)^2 = 2n - 1$, and the 2008th term is $2 \cdot 2008 - 1 = 4015$.



13. **Answer (B):** Draw a line parallel to $\overline{AD}$ through point E, intersecting $\overline{AB}$ at F and intersecting $\overline{CD}$ at G. Triangle AEF is a $30-60-90°$ triangle with hypotenuse $AE = 1$, so $EF = \frac{\sqrt{3}}{2}$. Region R consists of two congruent trapezoids of height $\frac{1}{6}$, shorter base $EG = 1-\frac{\sqrt{3}}{2}$, and longer base the weighted average

$$\frac{2}{3}EG + \frac{1}{3}AD = \frac{2}{3}\left(1-\frac{\sqrt{3}}{2}\right) + \frac{1}{3}\cdot 1 = 1 - \frac{\sqrt{3}}{3}.$$

Therefore the area of R is

$$2\cdot\frac{1}{6}\cdot\frac{1}{2}\left(\left(1-\frac{\sqrt{3}}{2}\right)+\left(1-\frac{\sqrt{3}}{3}\right)\right) = \frac{1}{6}\left(2-\frac{5\sqrt{3}}{6}\right) = \frac{12-5\sqrt{3}}{36}$$

OR

Place $ABCD$ in a coordinate plane with $B = (0,0)$, $A = (1,0)$, and $C = (0,1)$. Then the equation of the line BE is $y = \sqrt{3}x$, so $E = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, and the point of R closest to B is $(\frac{1}{3}, \frac{\sqrt{3}}{3})$. Thus the region R consists of two congruent trapezoids with height $\frac{1}{6}$ and bases $1-\frac{\sqrt{3}}{2}$ and $1-\frac{\sqrt{3}}{3}$. Then proceed as in the first solution.

14. **Answer (C):** The given information implies that $2\pi\log_{10}(a^2) = \log_{10}(b^4)$ or, equivalently, that $4\pi\log_{10} a = 4\log_{10} b$. Thus

$$\log_a b = \frac{\log_{10} b}{\log_{10} a} = \pi.$$

15. **Answer (C):** The region inside S but outside R consists of four triangles, each of which has two sides of length 1. The angle between those two sides is $360° - 90° - 4\cdot 60° = 30°$. Thus the area of each triangle is

$$\frac{1}{2}\cdot 1\cdot 1\cdot\sin 30° = \frac{1}{4},$$

so the required area is $4\cdot\frac{1}{4} = 1$.

16. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a-2$ by $b-2$ feet. Hence $ab = 2(a-2)(b-2)$, from which $0 = ab - 4a - 4b + 8$. Add 8 to each side of the equation to produce

$$8 = ab - 4a - 4b + 16 = (a-4)(b-4).$$

Because the only integer factorizations of 8 are

$$8 = 1 \cdot 8 = 2 \cdot 4 = (-4) \cdot (-2) = (-8) \cdot (-1),$$

and because $b > a > 0$, the only possible ordered pairs satisfying this equation for $(a-4, b-4)$ are $(1,8)$ and $(2,4)$. Hence (a,b) must be one of the two ordered pairs $(5,12)$, or $(6,8)$.

17. **Answer (C):**

Let $A = (a, a^2)$ and $C = (c, c^2)$. Then $B = (-a, a^2)$. If either $\angle A$ or $\angle B$ is $90°$, then $c = \pm a$, but this is impossible because A, B, and C must have distinct x-coordinates. Thus $\angle C = 90°$, so $AC \perp BC$. Consequently

$$\frac{c^2 - a^2}{c - a} \cdot \frac{c^2 - a^2}{c + a} = -1,$$

from which $1 = a^2 - c^2$, which is the length of the altitude from C to $\overline{AB}$. Because $\triangle ABC$ has area 2008, it follows that $AB = 4016$, $|a| = 2008$ and $a^2 = 2008^2 = 4032064$. Therefore $c^2 = a^2 - 1 = 4032063$ and the sum of the digits of c^2 is 18.

18. **Answer (E):** Square $ABCD$ has side length 14. Let F and G be the feet of the altitudes from E in $\triangle ABE$ and $\triangle CDE$, respectively. Then $FG = 14$, $EF = 2 \cdot \frac{105}{14} = 15$ and $EG = 2 \cdot \frac{91}{14} = 13$. Because $\triangle EFG$ is perpendicular to the plane of $ABCD$, the altitude to $\overline{FG}$ is the altitude of the pyramid. By Heron's Formula, the area of $\triangle EFG$ is $\sqrt{(21)(6)(7)(8)} = 84$, so the altitude to $\overline{FG}$ is $2 \cdot \frac{84}{14} = 12$. Therefore the volume of the pyramid is $(\frac{1}{3})(196)(12) = 784$.

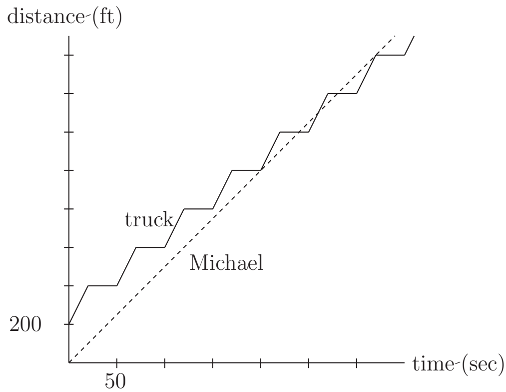
19. **Answer (B):** Let $\alpha = a + bi$ and $\gamma = c + di$, where a, b, c, and d are real numbers. Then $f(1) = (4 + a + c) + (1 + b + d)i$, and $f(i) = (-4 - b + c) + (-1 + a + d)i$. Because both $f(1)$ and $f(i)$ are real, it follows that $a = 1 - d$ and $b = -1 - d$. Thus

$$\begin{aligned}
|\alpha| + |\gamma| &= \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2} \\
&= \sqrt{(1-d)^2 + (-1-d)^2} + \sqrt{c^2 + d^2} \\
&= \sqrt{2 + 2d^2} + \sqrt{c^2 + d^2}.
\end{aligned}$$

The minimum value of $|\alpha| + |\gamma|$ is consequently $\sqrt{2}$, which is achieved when $c = d = 0$. In this case we also have $a = 1$ and $b = -1$.

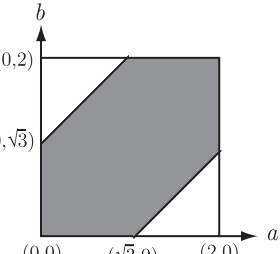
20. **Answer (B):** Number the pails consecutively so that Michael is presently at pail 0 and the garbage truck is at pail 1. Michael takes $200/5 = 40$ seconds to walk between pails, so for $n \geq 0$ he passes pail n after $40n$ seconds. The truck takes 20 seconds to travel between pails and stops for 30 seconds at each pail. Thus for $n \geq 1$ it leaves pail n after $50(n-1)$ seconds, and for $n \geq 2$ it arrives at pail n after $50(n-1) - 30$ seconds. Michael will meet the truck at pail n if and only if

$$50(n-1) - 30 \leq 40n \leq 50(n-1) \quad \text{or, equivalently, } 5 \leq n \leq 8.$$



Hence Michael first meets the truck at pail 5 after 200 seconds, just as the truck leaves the pail. He passes the truck at pail 6 after 240 seconds and at pail 7 after 280 seconds. Finally, Michael meets the truck just as it arrives at pail 8 after 320 seconds. These conditions imply that the truck is ahead of Michael between pails 5 and 6 and that Michael is ahead of the truck between pails 7 and 8. However, the truck must pass Michael at some point between pails 6 and 7, so they meet a total of five times.

21. **Answer (E):** Circles A and B both have radius 1, so they intersect if and only if the distance between their centers is no greater than 2. Let the centers of the circles be $(a, 0)$ and $(b, 1)$. The distance between these points is $\sqrt{(a-b)^2 + 1}$, so the circles intersect if and only if $\sqrt{(a-b)^2 + 1} \leq 2$. This condition is equivalent to $(a-b)^2 \leq 3$, or $-\sqrt{3} \leq a - b \leq \sqrt{3}$. Points in the square correspond to ordered pairs (a, b) with $0 \leq a \leq 2$ and $0 \leq b \leq 2$. The shaded region corresponds to the points that satisfy $-\sqrt{3} \leq a - b \leq \sqrt{3}$. Its area is $4 - \left(2 - \sqrt{3}\right)^2$. The requested probability is the area of the shaded region divided by the area of the square, which is



$$\frac{4 - \left(2 - \sqrt{3}\right)^2}{4} = \frac{4\sqrt{3} - 3}{4}.$$

22. **Answer (E):** The four vacant spaces can be located in any of $\binom{16}{4} = 1820$ combinations of positions. The arrangements in which Auntie Em is unable to park may be divided into two cases. If the rightmost space is occupied, then every vacant space is immediately to the left of an occupied space. Let X denote the union of a vacant space and the occupied space immediately to its right, and let Y denote a single occupied space not immediately to the right of a vacant space. The arrangement of cars and spaces can be represented by a sequence of four X's and eight Y's in some order, and there are $\binom{12}{4} = 495$ possible orders. If the rightmost space is vacant, the arrangement in the remaining 15 spaces can be represented by a sequence of three X's and nine Y's in some order, and there are $\binom{12}{3} = 220$ possible orders. Therefore there are $1820 - 495 - 220 = 1105$ arrangements in which Auntie Em can park, and the requested probability is $\frac{1105}{1820} = \frac{17}{28}$.

OR

Let O denote an occupied space, and let V denote a vacant space. The problem is equivalent to finding the probability p that in a string of 12 O's and 4 V's, there are at least two consecutive V's. Then $1-p$ is the probability that no two V's are consecutive. In a string of 12 O's, there are 13 spaces in which to insert 4 V's to create a string in which no two V's are consecutive. Thus

$$p = 1 - \frac{\binom{13}{4}}{\binom{16}{4}} = \frac{17}{28}.$$

23. **Answer (A):** Because the prime factorization of 10 is $2 \cdot 5$, the positive divisors of 10^n are the numbers $2^a \cdot 5^b$ with $0 \le a \le n$ and $0 \le b \le n$. Thus

$$\begin{aligned}
792 = \sum_{a=0}^{n}\sum_{b=0}^{n} \log_{10}\left(2^a 5^b\right) &= \sum_{a=0}^{n}\sum_{b=0}^{n} (a\log_{10} 2 + b\log_{10} 5) \\
&= \sum_{b=0}^{n}\sum_{a=0}^{n} (a\log_{10} 2) + \sum_{a=0}^{n}\sum_{b=0}^{n} (b\log_{10} 5) \\
&= (n+1)(\log_{10} 2)\sum_{a=0}^{n} a + (n+1)(\log_{10} 5)\sum_{b=0}^{n} b \\
&= (n+1)(\log_{10} 2 + \log_{10} 5)\left(\frac{1}{2}n(n+1)\right) \\
&= \frac{1}{2}n(n+1)^2(\log_{10} 10) = \frac{1}{2}n(n+1)^2.
\end{aligned}$$

Hence $n(n+1)^2 = 2 \cdot 792 = 2 \cdot 11 \cdot 72 = 11 \cdot 12^2$, so $n = 11$.

OR

Let $d(M)$ denote the number of divisors of a positive integer M. The sum of the logs of the divisors of M is equal to the log of the product of its divisors.

If M is not a square, its divisors can be arranged in pairs, each with a product of M. Thus the product of the divisors is $M^{d(M)/2}$. A similar argument shows that this result is also true if M is a square. Therefore

$$792 = \log\left((10^n)^{d(10^n)/2}\right) = \frac{1}{2}d(10^n)\cdot n = \frac{1}{2}d(2^n\cdot 5^n)\cdot n = \frac{1}{2}(n+1)^2\cdot n,$$

and the conclusion follows as in the first solution.

24. **Answer (C):** For $n \ge 0$, let $A_n = (a_n, 0)$, and let $c_{n+1} = a_{n+1} - a_n$. Let $B_0 = A_0$, and let $c_0 = 0$. Then for $n \ge 0$,

$$B_{n+1} = \left(a_n + \frac{c_{n+1}}{2}, \frac{\sqrt{3}c_{n+1}}{2}\right),$$

so

$$\left(\frac{\sqrt{3}c_{n+1}}{2}\right)^2 = a_n + \frac{c_{n+1}}{2},$$

from which $3c_{n+1}^2 - 2c_{n+1} - 4a_n = 0$. For $n \ge 1$,

$$B_n = \left(a_n - \frac{c_n}{2}, \frac{\sqrt{3}c_n}{2}\right),$$

so

$$\left(\frac{\sqrt{3}c_n}{2}\right)^2 = a_n - \frac{c_n}{2},$$

from which $3c_n^2 + 2c_n - 4a_n = 0$. Hence $3c_{n+1}^2 - 2c_{n+1} = 4a_n = 3c_n^2 + 2c_n$, and

$$2(c_{n+1} + c_n) = 3(c_{n+1}^2 - c_n^2) = 3(c_{n+1} + c_n)(c_{n-1} - c_n).$$

Thus $c_{n+1} = c_n + \frac{2}{3}$ for $n \ge 0$. It follows that

$$a_n = \frac{2}{3} + \frac{4}{3} + \frac{6}{3} + \cdots + \frac{2n}{3} = \frac{2}{3}\cdot\frac{n(n+1)}{2} = \frac{n(n+1)}{3}.$$

Solving $n(n+1)/3 \ge 100$ gives $n \ge 17$.

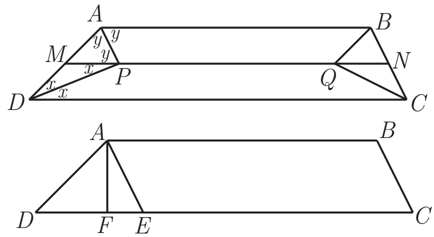
25. **Answer (B):** Let M and N be the midpoints of sides AD and BC. Set $\angle BAD = 2y$ and $\angle ADC = 2x$. We have $x + y = 90°$, from which it follows that $\angle APD = 90°$. Hence in triangle APD, MP is the median to the hypotenuse AD, so $AM = MD = MP$ and $\angle MPA = \angle MAP = \angle PAB$. Thus, $\overline{MP} \parallel \overline{AB}$. Likewise, $\overline{QN} \parallel \overline{AB}$. It follows that M, P, Q, and N are collinear, and

$$PQ = MN - MP - QN = \frac{AB + CD - AD - BC}{2} = 9.$$

The area of $ABQCDP$ is equal to the sum of the areas of two trapezoids $ABQP$ and $CDPQ$. Let F be the foot of the perpendicular from A to $\overline{CD}$. Then the area of $ABQCDP$ is equal to

$$\frac{AB+PQ}{2}\cdot\frac{AF}{2}+\frac{CD+PQ}{2}\cdot\frac{AF}{2}=12AF.$$

Let E lie on $\overline{DC}$ so that $\overline{AE}\parallel\overline{BC}$. Then $AE=BC=5$ and $DE=CD-CE=CD-AB=8$. We have $AD^2-DF^2=AF^2=AE^2-EF^2=AE^2-(DE-DF)^2$, or $49-DF^2=25-(8-DF)^2$. Solving the last equation gives $DF=\frac{11}{2}$. Thus $AF=\frac{5\sqrt{3}}{2}$ and the area of $ABQCDP$ is $12AF=30\sqrt{3}$.



OR

As in the first solution, conclude that $AE=5$ and $DE=8$. Apply the Law of Cosines to $\triangle ADE$ to obtain

$$\cos(\angle AED)=\frac{8^2+5^2-7^2}{2\cdot 8\cdot 5}=\frac{1}{2}.$$

Therefore $\angle AED=60^\circ$, so $AF=5\sqrt{3}/2$, and the area of $ABQCDP$ is $30\sqrt{3}$.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

The MATHEMATICAL ASSOCIATION of AMERICA

American Mathematics Competitions

60th Annual American Mathematics Contest 12

AMC 12 CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 10, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo Abrego, Dept. of Mathematics,
California State University-Northridge, Northridge, CA 91330-8313

1. **Answer (A):** There are $60-34=26$ minutes from 10:34 AM to 11:00 AM, there are 2 hours from 11:00 AM to 1:00 PM, and there are 18 minutes from 1:00 PM to 1:18 PM. Thus the flight lasted 2 hours and $26+18=44$ minutes. Hence $h+m=2+44=46$.

2. **Answer (C):** Simplifying the expression,

$$1+\frac{1}{1+\frac{1}{1+1}}=1+\frac{1}{1+\frac{1}{2}}=1+\frac{1}{\frac{3}{2}}=1+\frac{2}{3}=\frac{5}{3}.$$

3. **Answer (B):** The number is

$$\frac{1}{4}+\frac{1}{3}\left(\frac{3}{4}-\frac{1}{4}\right)=\frac{1}{4}+\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{4}+\frac{1}{6}=\frac{5}{12}.$$

4. **Answer (A):** The value of any combination of four coins that includes pennies cannot be a multiple of 5 cents, and the value of any combination of four coins that does not include pennies must exceed 15 cents. Therefore the total value cannot be 15 cents. The other four amounts can be made with, respectively, one dime and three nickels; three dimes and one nickel; one quarter, one dime and two nickels; and one quarter and three dimes.

5. **Answer (D):** Let x be the side length of the cube. Then the volume of the cube was x^3, and the volume of the new solid is $x(x+1)(x-1)=x^3-x$. Therefore $x^3-x=x^3-5$, from which $x=5$, and the volume of the cube was $5^3=125$.

6. **Answer (E):** Note that

$$12^{mn}=(2^2\cdot 3)^{mn}=2^{2mn}\cdot 3^{mn}=(2^m)^{2n}\cdot(3^n)^m=P^{2n}Q^m.$$

Remark: The pair of integers $(2,1)$ shows that the other choices are not possible.

7. **Answer (B):** Because the difference between consecutive terms is constant,

$$(5x-11)-(2x-3)=(3x+1)-(5x-11).$$

Therefore $x=4$, and the first three terms are 5, 9, and 13. Thus the difference between consecutive terms is 4. The nth term is $2009=5+(n-1)\cdot 4$, and it follows that $n=502$.

8. **Answer (A):** Let the lengths of the shorter and longer side of each rectangle be x and y, respectively. The outer and inner squares have side lengths $y + x$ and $y - x$, respectively, and the ratio of their side lengths is $\sqrt{4} = 2$. Therefore $y + x = 2(y - x)$, from which $y = 3x$.

9. **Answer (D):** Expanding, we have $f(x+3) = a(x^2+6x+9)+b(x+3)+c = ax^2+(6a+b)x+(9a+3b+c)$. Equating coefficients implies that $a = 3$, $6\cdot 3+b = 7$, whence $b = -11$, and then $9 \cdot 3 + 3 \cdot (-11) + c = 4$, and so $c = 10$. Therefore $a + b + c = 3 - 11 + 10 = 2$.

OR

Note that

$$\begin{aligned} f(x) = f((x-3)+3) &= 3(x-3)^2 + 7(x-3) + 4 \\ &= 3(x^2 - 6x + 9) + 7x - 21 + 4 \\ &= 3x^2 - 11x + 10. \end{aligned}$$

Therefore $a = 3$, $b = -11$, and $c = 10$, giving $a + b + c = 2$.

OR

The sum $a + b + c$ is $f(1) = f(-2+3) = 3(-2)^2 + 7(-2) + 4 = 2$.

10. **Answer (C):** Let x be the length of $\overline{BD}$. By the triangle inequality on $\triangle BCD$, $5 + x > 17$, so $x > 12$. By the triangle inequality on $\triangle ABD$, $5 + 9 > x$, so $x < 14$. Since x must be an integer, $x = 13$.

11. **Answer (E):** The outside square for F_n has 4 more diamonds on its boundary than the outside square for F_{n-1}. Because the outside square of F_2 has 4 diamonds, the outside square of F_n has $4(n-2)+4 = 4(n-1)$ diamonds. Hence the number of diamonds in figure F_n is the number of diamonds in F_{n-1} plus $4(n-1)$, or

$$\begin{aligned} &1 + 4 + 8 + 12 + \cdots + 4(n-2) + 4(n-1) \\ =\,&1 + 4(1 + 2 + 3 + \cdots + (n-2) + (n-1)) \\ =\,&1 + 4\frac{(n-1)n}{2} \\ =\,&1 + 2(n-1)n. \end{aligned}$$

Therefore figure F_{20} has $1 + 2 \cdot 19 \cdot 20 = 761$ diamonds.

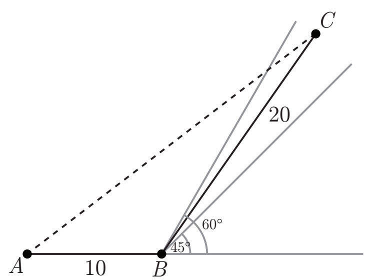
12. **Answer (B):** The only such number is 54. A single-digit number would have to satisfy $6u = u$, implying $u = 0$, which is impossible. A two-digit number would have to satisfy $10t + u = 6(t + u)$, so $4t = 5u$ and then necessarily $t = 5$ and $u = 4$; hence the number is 54. A three-digit number would have to satisfy $100h+10t+u = 6(h+t+u)$ or $94h+4t = 5u$. But the left side of the expression is at least 94 while the right side of the expression is at most 45, so no solution is possible.

13. **Answer (D):** By the Law of Cosines,

$$AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos \angle ABC = 500 - 400 \cos \angle ABC.$$

Because $\cos \angle ABC$ is between $\cos 120° = -\frac{1}{2}$ and $\cos 135° = -\frac{\sqrt{2}}{2}$, it follows that

$$700 = 500 + 200 \le AC^2 \le 500 + 200\sqrt{2} < 800.$$



14. **Answer (B):** The line must contain the midpoint of the segment joining $(1, 1)$ and $(6m, 0)$, which is $\left(\frac{6m+1}{2}, \frac{1}{2}\right)$. Thus

$$m = \frac{\frac{1}{2}}{\frac{6m+1}{2}} = \frac{1}{6m+1},$$

from which $0 = 6m^2 + m - 1 = (3m - 1)(2m + 1)$. The two possible values of m are $-\frac{1}{2}$ and $\frac{1}{3}$, and their sum is $-\frac{1}{6}$.

If $m = -\frac{1}{2}$ then the triangle with vertices $(0, 0)$, $(1, 1)$, and $(-3, 0)$ is bisected by the line passing through the origin and $(-1, \frac{1}{2})$. Similarly, when $m = \frac{1}{3}$ the triangle with vertices $(0, 0)$, $(1, 1)$, and $(2, 0)$ is bisected by the line passing through the origin and $(\frac{3}{2}, \frac{1}{2})$.

15. **Answer (D):** Let k be a multiple of 4. For $k \geq 0$,

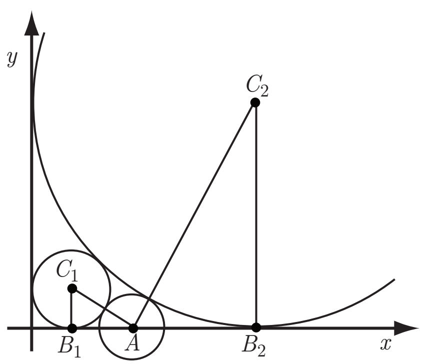
$$(k+1)\,i^{k+1} + (k+2)\,i^{k+2} + (k+3)\,i^{k+3} + (k+4)\,i^{k+4} =$$
$$(k+1)\,i + (k+2)\,(-1) + (k+3)\,(-i) + (k+4) = 2 - 2i.$$

Thus when $n = 4 \cdot 24 = 96$, we have $i + 2i^2 + \cdots + ni^n = 24\,(2 - 2i) = 48 - 48i$. Adding the term $97i^{97} = 97i$ gives $(48 - 48i) + 97i = 48 + 49i$ when $n = 97$.

16. **Answer (D):** Let r be the radius of a circle with center C, $A = (3, 0)$, and $B = (r, 0)$. Then, $AC = 1 + r$ and $CB = r$. Applying the Pythagorean Theorem to $\triangle ABC$ gives

$$AB^2 = (1+r)^2 - r^2 = 1 + 2r.$$

Also, $AB = |3 - r|$, so $1 + 2r = (3 - r)^2$, which simplifies to $r^2 - 8r + 8 = 0$. Thus $r = 4 \pm 2\sqrt{2}$, both of which are positive, and the sum of all possible values of r is 8.



17. **Answer (C):** The sum of the first series is

$$\frac{a}{1 - r_1} = r_1,$$

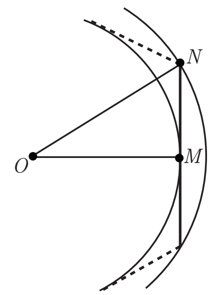
from which $r_1^2 - r_1 + a = 0$, and $r_1 = \frac{1}{2}(1 \pm \sqrt{1 - 4a})$. Similarly, $r_2 = \frac{1}{2}(1 \pm \sqrt{1 - 4a})$. Because r_1 and r_2 must be different, $r_1 + r_2 = 1$. Such series exist as long as $0 < a < \frac{1}{4}$.

18. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by 2^6, so $N(k) < 6$. For $k > 4$, the first term is divisible by 2^7, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6 + 1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6 + 1) = ((5^2)^3 + 1^3) = (5^2 + 1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2+1=26$ is divisible by 2 but not 4, and the second factor is odd, so 5^6+1 contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

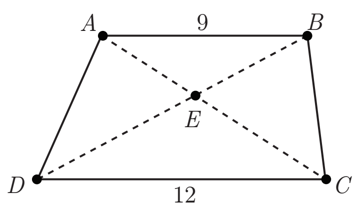
19. **Answer (C):** Consider a regular n-gon with side length 2. Let the radii of its inscribed and circumscribed circles be r and R, respectively. Let O be the common center of the circles, let M be the midpoint of one side of the polygon, and let N be one endpoint of that side. Then $\triangle OMN$ has a right angle at M, $MN=1$, $OM=r$, and $ON=R$. By the Pythagorean Theorem, $R^2-r^2=1$. Thus the area of the annulus between the circles is $\pi(R^2-r^2)=\pi$ for all $n\geq 3$. Hence $A=B$.



20. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from A and B to $\overline{CD}$ have the same length. Thus $\overline{AB}\parallel\overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

$$\frac{AE}{EC}=\frac{AB}{CD}=\frac{9}{12}=\frac{3}{4}.$$

Let $AE=3x$ and $EC=4x$. Then $7x=AE+EC=AC=14$, so $x=2$, and $AE=3x=6$.

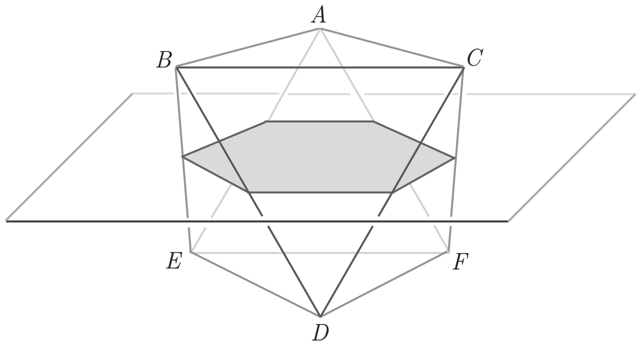


21. **Answer (C):** Because $x^{12}+ax^8+bx^4+c=p(x^4)$, the value of this polynomial is 0 if and only if

$$x^4=2009+9002\pi i \qquad\text{or}\qquad x^4=2009 \qquad\text{or}\qquad x^4=9002.$$

The first of these three equations has four distinct nonreal solutions, and the second and third each have two distinct nonreal solutions. Thus $p(x^4)=x^{12}+ax^8+bx^4+c$ has 8 distinct nonreal zeros.

22. **Answer (E):** Let $\triangle ABC$ and $\triangle DEF$ be the two faces of the octahedron parallel to the cutting plane. The plane passes through the midpoints of the six edges of the octahedron that are not sides of either of those triangles. Hence the intersection of the plane with the octahedron is an equilateral hexagon with side length $\frac{1}{2}$. Then by symmetry the hexagon is also equiangular and hence regular. The area of the hexagon is 6 times that of an equilateral triangle with side length $\frac{1}{2}$, so the area is $6\left(\frac{1}{2}\right)^2 \frac{\sqrt{3}}{4} = \frac{3\sqrt{3}}{8}$. Therefore $a+b+c = 3+3+8 = 14$.



23. **Answer (D):** Let (h, k) be the vertex of the graph of f. Because the graph of f intersects the x-axis twice, we can assume that $f(x) = a(x-h)^2 + k$ with $\frac{-k}{a} > 0$. Let $s = \sqrt{\frac{-k}{a}}$; then the x-intercepts of the graph of f are $h \pm s$. Because $g(x) = -f(100-x) = -a(100-x-h)^2 - k$, it follows that the x-intercepts of the graph of g are $100 - h \pm s$.

The graph of g contains the point (h, k); thus

$$k = f(h) = g(h) = -a(100-2h)^2 - k,$$

from which $h = 50 \pm \frac{\sqrt{2}}{2}s$. Regardless of the sign in the expression for h, the four x-intercepts in order are

$$50 - s\left(1 + \frac{\sqrt{2}}{2}\right) < 50 - s\left(1 - \frac{\sqrt{2}}{2}\right) < 50 + s\left(1 - \frac{\sqrt{2}}{2}\right) < 50 + s\left(1 + \frac{\sqrt{2}}{2}\right).$$

Because $x_3 - x_2 = 150$, it follows that $150 = s(2-\sqrt{2})$, that is $s = 150\left(1 + \frac{\sqrt{2}}{2}\right)$. Therefore $x_4 - x_1 = s(2+\sqrt{2}) = 450 + 300\sqrt{2}$, and then $m+n+p = 450+300+2 = 752$.

OR

The graphs of f and g intersect the x-axis twice each. By symmetry, and because the graph of g contains the vertex of f, we can assume x_1 and x_3 are the roots of f, and x_2 and x_4 are the roots of g. A point (p, q) is on the graph of f if and only if $(100-p, -q)$ is on the graph of g, so the two graphs are reflections of each other with respect to the point $(50, 0)$. Thus $x_2+x_3 = x_1+x_4 = 100$, and since $x_3 - x_2 = 150$, it follows that $x_2 = -25$ and $x_3 = 125$. The average of x_1 and $x_3 = 125$ is h. It follows that $x_1 = 2h-125$, from which $x_4 = 100-x_1 = 225-2h$, and $x_4 - x_1 = 350 - 4h$.

Moreover, $f(x) = a(x - x_1)(x - x_3) = a(x + 125 - 2h)(x - 125)$ and $g(x) = -f(100 - x) = -a(x + 25)(x + 2h - 225)$. The vertex of the graph of f lies on the graph of g; thus

$$1 = \frac{f(h)}{g(h)} = \frac{(125 - h)(h - 125)}{-(h + 25)(3h - 225)},$$

from which $h = -25 \pm 75\sqrt{2}$. However, $h < x_2 < 0$; thus $h = -25 - 75\sqrt{2}$. Therefore $x_4 - x_1 = 450 + 300\sqrt{2}$ and then $m + n + p = 450 + 300 + 2 = 752$.

24. **Answer (E):** Define the *k-iterated logarithm* as follows: $\log_2^1 x = \log_2 x$ and $\log_2^{k+1} x = \log_2(\log_2^k x)$ for $k \geq 1$. Because $\log_2 T(n + 1) = T(n)$ for $n \geq 1$, it follows that $\log_2 A = T(2009) \log_2 T(2009) = T(2009)T(2008)$ and $\log_2 B = A \log_2 T(2009) = A \cdot T(2008)$. Then $\log_2^2 B = \log_2 A + \log_2 T(2008) = T(2009)T(2008) + T(2007)$. Now,

$$\log_2^3 B > \log_2(T(2009)T(2008)) > \log_2 T(2009) = T(2008),$$

and recursively for $k \geq 1$,

$$\log_2^{k+3} B > T(2008 - k).$$

In particular $\log_2^{2010} B > T(1) = 2$, and then $\log_2^{2012} B > 0$. Thus $\log_2^{2013} B$ is defined.

On the other hand, because $T(2007) < T(2008)T(2009)$ and $1 + T(2007) < T(2008)$, it follows that

$$\log_2^3 B < \log_2 (2T(2008)T(2009)) = 1 + T(2007) + T(2008) < 2T(2008) \text{ and}$$
$$\log_2^4 B < \log_2 (2T(2008)) = 1 + T(2007) < T(2008).$$

Applying $\log_2$ recursively for $k \geq 1$ we get

$$\log_2^{4+k} B < T(2008 - k).$$

In particular $\log_2^{2011} B < T(1) = 2$, and then $\log_2^{2013} B < 0$. Thus $\log_2^{2014} B$ is undefined.

25. **Answer (A):** Recognize the similarity between the recursion formula given and the trigonometric identity

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}.$$

Also note that the first two terms of the sequence are tangents of familiar angles, namely $\frac{\pi}{4}$ and $\frac{\pi}{6}$. Let $c_1 = 3$, $c_2 = 2$, and $c_{n+2} = (c_n + c_{n+1}) \bmod 12$. We claim that the sequence $\{a_n\}$ satisfies $a_n = \tan\left(\frac{\pi c_n}{12}\right)$. Note that

$$a_1 = 1 = \tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{\pi c_1}{12}\right) \text{ and}$$
$$a_2 = \frac{1}{\sqrt{3}} = \tan\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi c_2}{12}\right).$$

By induction on n, the formula for the tangent of the sum of two angles, and the fact that the period of $\tan x$ is π,

$$\begin{aligned} a_{n+2} = \frac{a_n + a_{n+1}}{1 - a_n a_{n+1}} &= \frac{\tan\left(\frac{\pi c_n}{12}\right) + \tan\left(\frac{\pi c_{n+1}}{12}\right)}{1 - \tan\left(\frac{\pi c_n}{12}\right)\tan\left(\frac{\pi c_{n+1}}{12}\right)} \\ &= \tan\left(\frac{\pi(c_n + c_{n+1})}{12}\right) = \tan\left(\frac{\pi c_{n+2}}{12}\right). \end{aligned}$$

The first few terms of the sequence $\{c_n\}$ are:

$$3, 2, 5, 7, 0, 7, 7, 2, 9, 11, 8, 7, 3, 10, 1, 11, 0, 11, 11, 10, 9, 7, 4, 11, 3, 2.$$

So the sequence c_n is periodic with period 24. Because $2009 = 24 \cdot 83 + 17$, it follows that $c_{2009} = c_{17} = 0$. Thus $|a_{2009}| = |\tan\left(\frac{\pi c_{17}}{12}\right)| = 0$.

The problems and solutions in this contest were proposed by Bernardo Ábrego, Betsy Bennett, Thomas Butts, George Brauer, Steve Dunbar, Douglas Faires, Sister Josanne Furey, Gregory Galperin, John Haverhals, Elgin Johnston, Joe Kennedy, David Wells, LeRoy Wenstrom, and Woody Wenstrom.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Math Zoom Academy
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

60th Annual American Mathematics Contest 12

AMC 12 CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 25, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo Abrego, Dept. of Mathematics
California State University-Northridge, Northridge, CA 91330-8313

1. **Answer (B):** Make a table for the cost of the muffins and bagels:

Cost of Muffins	Cost of Bagels	Total Cost
$0 \cdot 0.50 = 0.00$	$5 \cdot 0.75 = 3.75$	3.75
$1 \cdot 0.50 = 0.50$	$4 \cdot 0.75 = 3.00$	3.50
$2 \cdot 0.50 = 1.00$	$3 \cdot 0.75 = 2.25$	3.25
$3 \cdot 0.50 = 1.50$	$2 \cdot 0.75 = 1.50$	3.00
$4 \cdot 0.50 = 2.00$	$1 \cdot 0.75 = 0.75$	2.75
$5 \cdot 0.50 = 2.50$	$0 \cdot 0.75 = 0.00$	2.50

The only combination which is a whole number of dollars is the cost of 3 muffins and 2 bagels.

2. **Answer (C):** The loss of 3 cans of paint resulted in 5 fewer rooms being painted, so the ratio of cans of paint to rooms painted is $3 : 5$. Hence for 25 rooms she would require $\frac{3}{5} \cdot 25 = 15$ cans of paint.

OR

If she used x cans of paint for 25 rooms, then $\frac{x+3}{30} = \frac{x}{25}$. Hence $25x + 75 = 30x$, and $x = 15$.

3. **Answer (D):** Twenty percent less than 60 is $\frac{4}{5} \cdot 60 = 48$. One-third more than a number n is $\frac{4}{3}n$. Therefore $\frac{4}{3}n = 48$, and the number is 36.

4. **Answer (C):** Each triangle has leg length $\frac{1}{2} \cdot (25 - 15) = 5$ meters and area $\frac{1}{2} \cdot 5^2 = \frac{25}{2}$ square meters. Thus the flower beds have a total area of 25 square meters. The entire yard has length 25 and width 5, so its area is 125. The fraction of the yard occupied by the flower beds is $\frac{25}{125} = \frac{1}{5}$.

5. **Answer (D):** The age of each person is a factor of $128 = 2^7$. So the twins could be $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$ years of age and, consequently, Kiana could be $\frac{128}{1^2} = 128$, $\frac{128}{2^2} = 32$, $\frac{128}{4^2} = 8$, or $\frac{128}{8^2} = 2$ years old, respectively. Because Kiana is younger than her brothers, she must be 2 years old. The sum of their ages is $2 + 8 + 8 = 18$.

6. **Answer (C):** The three operations can be performed in any of $3! = 6$ orders. However, if the addition is performed either first or last, then multiplying in either order produces the same result. Thus at most four distinct values can be obtained. It is easily checked that the values of the four expressions

$$(2 \times 3) + (4 \times 5) = 26,$$

$$((2 \times 3 + 4) \times 5) = 50,$$
$$2 \times (3 + (4 \times 5)) = 46,$$
$$2 \times (3 + 4) \times 5 = 70$$

are in fact all distinct.

7. **Answer (B):** Let p denote the price at the beginning of January. The price at the end of March was $(1.2)(0.8)(1.25)p = 1.2p$. Because the price at the end of April was p, the price decreased by $0.2p$ during April, and the percent decrease was
$$x = 100 \cdot \frac{0.2p}{1.2p} = \frac{100}{6} \approx 16.7\,.$$
To the nearest integer, x is 17.

8. **Answer (E):** Let x be the weight of the bucket and let y be the weight of the water in a full bucket. Then we are given that $x + \frac{2}{3}y = a$ and $x + \frac{1}{2}y = b$. Hence $\frac{1}{6}y = a - b$, so $y = 6a - 6b$. Thus $x = b - \frac{1}{2}(6a - 6b) = -3a + 4b$. Finally, $x + y = 3a - 2b$.

OR

The difference between a kg and b kg is the weight of water that would fill $\frac{1}{6}$ of a bucket. So the weight of water that would fill $\frac{1}{2}$ of a bucket is $3(a - b)$. Therefore the weight of a bucket filled with water is $b + 3(a - b) = 3a - 2b$.

9. **Answer (A):** Because the line $x + y = 7$ is parallel to $\overline{AB}$, the area of $\triangle ABC$ is independent of the location of C on the line. Therefore it may be assumed that $C = (7, 0)$. In that case the triangle has base $AC = 4$ and altitude 3, so its area is $\frac{1}{2} \cdot 4 \cdot 3 = 6$.

OR

The base of the triangle is $AB = \sqrt{3^2 + 3^2} = 3\sqrt{2}$. Its altitude is the distance between the point A and the parallel line $x + y = 7$, which is
$$\frac{|3 + 0 - 7|}{\sqrt{1^2 + 1^2}} = 2\sqrt{2}.$$
Therefore its area is $\frac{1}{2} \cdot 3\sqrt{2} \cdot 2\sqrt{2} = 6$.

10. **Answer (A):** The clock will display the incorrect time for the entire hours of 1, 10, 11, and 12. So the correct hour is displayed correctly $\frac{2}{3}$ of the time.

The minutes will not display correctly whenever either the tens digit or the ones digit is a 1, so the minutes that will not display correctly are 10, 11, 12, ..., 19, and 01, 21, 31, 41, and 51. This is 15 of the 60 possible minutes for a given hour. Hence the fraction of the day that the clock shows the correct time is $\frac{2}{3} \cdot (1 - \frac{15}{60}) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$.

11. **Answer (D):** On Monday, day 1, the birds find $\frac{1}{4}$ quart of millet in the feeder. On Tuesday they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4}$$

quarts of millet. On Wednesday, day 3, they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

quarts of millet. The number of quarts of millet they find on day n is

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \cdots + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{(\frac{1}{4})(1 - (\frac{3}{4})^n)}{1 - \frac{3}{4}} = 1 - \left(\frac{3}{4}\right)^n.$$

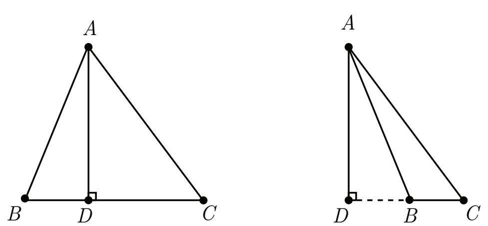
The birds always find $\frac{3}{4}$ quart of other seeds, so more than half the seeds are millet if $1 - (\frac{3}{4})^n > \frac{3}{4}$, that is, when $(\frac{3}{4})^n < \frac{1}{4}$. Because $(\frac{3}{4})^4 = \frac{81}{256} > \frac{1}{4}$ and $(\frac{3}{4})^5 = \frac{243}{1024} < \frac{1}{4}$, this will first occur on day 5 which is Friday.

12. **Answer (E):** Let the nth term of the series be ar^{n-1}. Because

$$\frac{8!}{7!} = \frac{ar^7}{ar^4} = r^3 = 8,$$

it follows that $r = 2$ and the first term is $a = \frac{7!}{r^4} = \frac{7!}{16} = 315$.

13. **Answer (D):** Let D be the foot of the altitude to $\overline{BC}$. Then $BD = \sqrt{13^2 - 12^2} = 5$ and $DC = \sqrt{15^2 - 12^2} = 9$. Thus $BC = BD + DC = 5 + 9 = 14$ or $BC = DC - BD = 9 - 5 = 4$. The sum of the two possible values is $14 + 4 = 18$.



14. **Answer (C):** The area of the entire region is 5. The shaded region consists of a triangle with base $3-a$ and altitude 3, with one unit square removed. Therefore

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2}.$$

Solving this equation yields $a = \frac{2}{3}$.

15. **Answer (B):**

Rearrange the equations to the form

$$x = \frac{\log(\frac{7}{3})}{\log(1 + f(r))}.$$

Because $f(r)$ is positive, for each answer choice, x will be largest when $f(r)$ is the smallest. Because $r > 0$, we have $\frac{r}{10} < r < 2r$. Because $r^2 < 9 < 10$ we have $\frac{r}{10} < \frac{1}{r}$. Finally, $\sqrt{r} < 10$, so $\frac{r}{10} < \sqrt{r}$.

16. **Answer (B):** Extend $\overline{AB}$ and $\overline{DC}$ to meet at E. Then

$$\begin{aligned}\angle BED &= 180^\circ - \angle EDB - \angle DBE \\ &= 180^\circ - 134^\circ - 23^\circ = 23^\circ.\end{aligned}$$

Thus $\triangle BDE$ is isosceles with $DE = BD$. Because $\overline{AD} \parallel \overline{BC}$, it follows that the triangles BCD and ADE are similar. Therefore

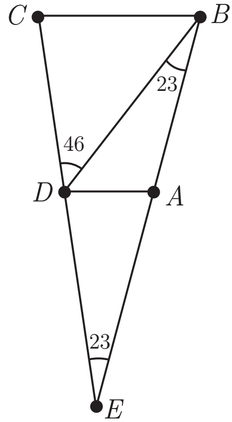
$$\frac{9}{5} = \frac{BC}{AD} = \frac{CD + DE}{DE} = \frac{CD}{BD} + 1 = CD + 1,$$

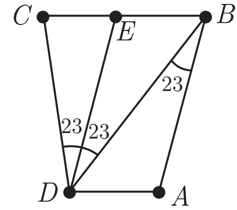
so $CD = \frac{4}{5}$.

OR

Let E be the intersection of $\overline{BC}$ and the line through D parallel to $\overline{AB}$. By construction $BE = AD$ and $\angle BDE = 23^\circ$; it follows that DE is the bisector of the angle BDC. By the Bisector Theorem we get

$$CD = \frac{CD}{BD} = \frac{EC}{BE} = \frac{BC - BE}{BE} = \frac{BC}{AD} - 1 = \frac{9}{5} - 1 = \frac{4}{5}.$$





17. **Answer (B):** The stripe on each face of the cube will be oriented in one of two possible directions, so there are $2^6 = 64$ possible stripe combinations on the cube. There are 3 pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in 2 different ways. Thus a total of $3 \cdot 2 \cdot 2 = 12$ stripe combinations on the cube result in a continuous stripe around the cube, and the requested probability is $\frac{12}{64} = \frac{3}{16}$.

OR

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom, and back faces are painted to complete an encircling stripe with the top face's stripe, or the front, right, back, and left faces are painted to form an encircling stripe. The probability of the first cases is $(\frac{1}{2})^3 = \frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4 = \frac{1}{16}$, so the answer is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$.

OR

There are three possible orientations of an encircling stripe. For any one of these to appear, the four faces through which the stripe is to pass must be properly aligned. The probability of one such stripe alignment is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Because there are 3 such possibilities, and these events are disjoint, the total probability is $3\left(\frac{1}{16}\right) = \frac{3}{16}$.

18. **Answer (C):** After 10 min. = 600 sec., Rachel will have completed 6 laps and

be 30 seconds from the finish line. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture taking region between

$$30 - \frac{22.5}{2} = 18.75 \quad \text{and} \qquad 30 + \frac{22.5}{2} = 41.25$$

seconds of the 10th minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds from the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture taking region between 30 and 50 seconds of the 10th minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the 10th minute. The probability that the picture is snapped during this time is

$$\frac{41.25 - 30}{60} = \frac{3}{16}.$$

19. **Answer (E):** Note that $f(n) = n^4 + 40n^2 + 400 - 400n^2 = (n^2+20)^2 - (20n)^2 = (n^2 + 20n + 20)(n^2 - 20n + 20)$. Because the first factor is greater than 1, the product cannot be prime unless the second factor is 1. The solutions of the equation $n^2 - 20n + 20 = 1$ are 1 and 19. The values of $f(1) = 1^2 + 20 \cdot 1 + 20 = 41$ and $f(19) = 19^2 + 20 \cdot 19 + 20 = 761$ are prime, and the requested sum is $41 + 761 = 802$.

20. **Answer (C):** Each edge of Q is cut by two planes, so R has 200 vertices. Three edges of R meet at each vertex, so R has $\frac{1}{2} \cdot 3 \cdot 200 = 300$ edges.

OR

At each vertex, as many new edges are created by this process as there are original edges meeting that vertex. Thus the total number of new edges is the total number of endpoints of the original edges, which is 200. A middle portion of each original edge is also present in R, so R has $100 + 200 = 300$ edges.

OR

Euler's Polyhedron Formula applied to Q gives $n - 100 + F = 2$, where F is the number of faces of Q. Each edge of Q is cut by two planes, so R has 200 vertices. Each cut by a plane P_k creates an additional face on R, so Euler's Polyhedron Formula applied to R gives $200 - E + (F + n) = 2$, where E is the number of edges of R. Subtracting the first equation from the second gives $300 - E = 0$, so $E = 300$.

21. **Answer (A):** Let S_n denote the number of ways that n women in n seats can be reseated so that each woman reseats herself in the seat she occupied before

or a seat next to it. It is easy to see that $S_1 = 1$ and $S_2 = 2$. Now consider the case with $n \geq 3$ women, and focus on the woman at the right end of the line. If this woman sits again in this end seat, then the remaining $n-1$ women can reseat themselves in S_{n-1} ways. If this end woman sits in the seat next to hers, then the former occupant of this new seat must sit on the end. Then the remaining $n-2$ women can seat themselves in S_{n-2} ways. Thus for $n \geq 3$, $S_n = S_{n-1} + S_{n-2}$. Therefore $(S_1, S_2, \ldots, S_{10}) = (1, 2, 3, 5, 8, 13, 21, 34, 55, 89)$, which are some of the first few terms of the Fibonacci Sequence. Thus $S_{10} = 89$.

22. **Answer (C):** Let $B = (b, b)$ and $D = (d, kd)$, so $C = (b+d, b+kd)$. Let $E = (b+d, 0)$ and $F = (0, b+kd)$. Rectangle $AECF$ is the disjoint union of parallelogram $ABCD$, two rectangles with length d and height b, two isosceles right triangles with leg length b, and two right triangles with leg lengths d and kd. It follows that the area of $ABCD$ is

$$(b+d)(b+kd) - 2bd - b^2 - kd^2 = (k-1)bd.$$

Therefore each parallelogram with the required properties determines, and is determined by, an ordered triple $(k-1, b, d)$ of positive integers whose product is $1{,}000{,}000 = 2^6 5^6$. The number of ways to distribute the six factors of 2 among the three integers $k-1$, b, and d is $\binom{6+3-1}{3-1} = \binom{8}{2} = 28$. The six factors of 5 can also be distributed in 28 ways, so there are $28^2 = 784$ parallelograms with the required property.

OR

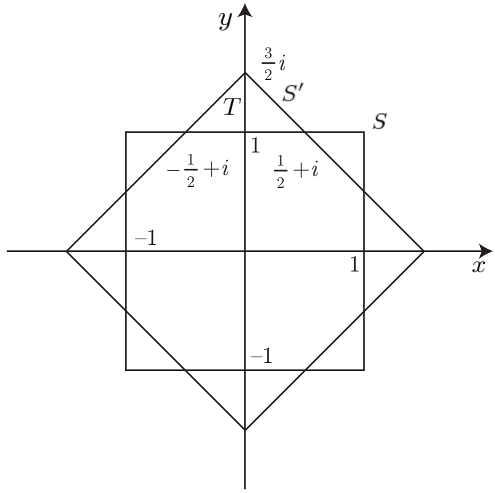
The area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

Thus the area of $\triangle ABD$ is $\frac{1}{2}(k-1)bd$ and the area of $\triangle CBD$ is the same. Then proceed as in the first solution.

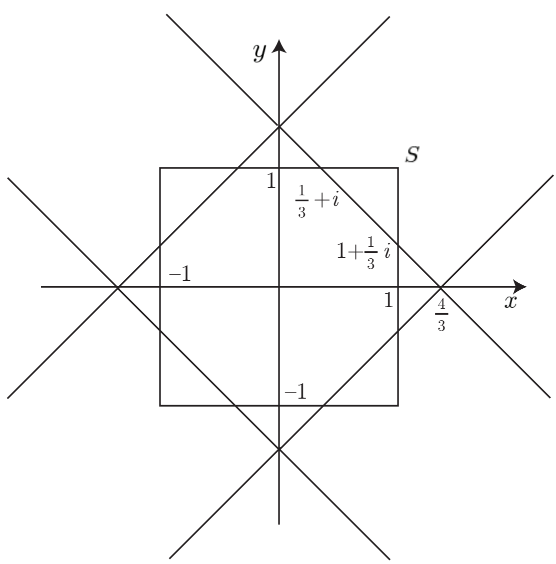
23. **Answer (D):** Let $f(z) = (\frac{3}{4} + \frac{3}{4}i)z$. The effect of multiplying z by $(\frac{3}{4} + \frac{3}{4}i)$ is to rotate z an angle equal to $\arg(\frac{3}{4} + \frac{3}{4}i) = \frac{\pi}{4}$ from the origin, and to magnify by a factor of $\left|\frac{3}{4} + \frac{3}{4}i\right| = \frac{3}{4}\sqrt{2}$. Thus the image S' of S under f is a square region with vertices $\pm\frac{3}{2}$ and $\pm\frac{3}{2}i$. The area of S' is $(\frac{3}{4}\sqrt{2} \cdot 2)^2$. The intersection of S and S' is an octagonal region obtained from S' by removing four congruent triangular regions. The topmost of these triangles T has vertices $\frac{1}{2} + i$, $\frac{3}{2}i$, and $-\frac{1}{2} + i$, so its area equals $\frac{1}{4}$. Then the requested probability is

$$\frac{\left(\frac{3}{4}\sqrt{2} \cdot 2\right)^2 - 4 \cdot \frac{1}{4}}{\left(\frac{3}{4}\sqrt{2} \cdot 2\right)^2} = \frac{7}{9}.$$



OR

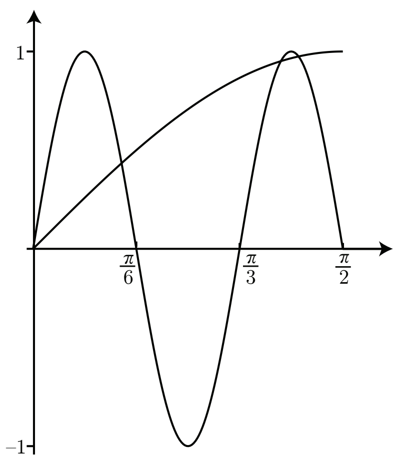
The product is $(\frac{3}{4}+\frac{3}{4}i)(x+iy) = (\frac{3}{4}x - \frac{3}{4}y) + (\frac{3}{4}x + \frac{3}{4}y)i$. The point $x+iy$ will be in S if and only if $-1 \le \frac{3}{4}x - \frac{3}{4}y \le 1$ and $-1 \le \frac{3}{4}x + \frac{3}{4}y \le 1$, which are equivalent to $-\frac{4}{3} \le x - y \le \frac{4}{3}$ and $-\frac{4}{3} \le x + y \le \frac{4}{3}$. Thus $x+yi$ must be inside the square with vertices $\pm\frac{4}{3}$ and $\pm\frac{4}{3}i$. By symmetry we can look at just the first quadrant. Because the portion of S in the first quadrant has area 1, the desired probability is the area of the portion of the interior of this square within S. The squares intersect at $1+\frac{1}{3}i$ and $\frac{1}{3}+i$, so the desired probability is $1 - \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{7}{9}$.



24. **Answer (B):** Let $f(x) = \sin^{-1}(\sin 6x)$ and $g(x) = \cos^{-1}(\cos x)$. If $0 \le x \le \pi$, then $g(x) = x$. If $0 \le x \le \pi/12$, then $f(x) = 6x$. Note also that $\sin\left(6\left(\frac{\pi}{6} - x\right)\right) = \sin 6x$, $\sin\left(6\left(\frac{\pi}{3} - x\right)\right) = -\sin 6x$, and $\sin\left(6\left(\frac{\pi}{3} + x\right)\right) = \sin 6x$, from which it follows that $f(\frac{\pi}{6} - x) = f(x)$, $f(\frac{\pi}{3} - x) = -f(x)$, and $f(\frac{\pi}{3} + x) = f(x)$. Thus the graph of $y = f(x)$ has period $\frac{\pi}{3}$ and consists of line segments with slopes of 6 or -6 and endpoints at $((4k+1)\frac{\pi}{12}, \frac{\pi}{2})$ and $((4k+3)\frac{\pi}{12}, -\frac{\pi}{2})$ for integer values of k. The graphs of f and g intersect twice in the interval $[0, \frac{\pi}{6}]$ and twice more in the interval $[\frac{\pi}{3}, \frac{\pi}{2}]$. If $\frac{\pi}{2} < x \le \pi$, then $g(x) = x > \frac{\pi}{2}$, so the graphs of f and g do not intersect.

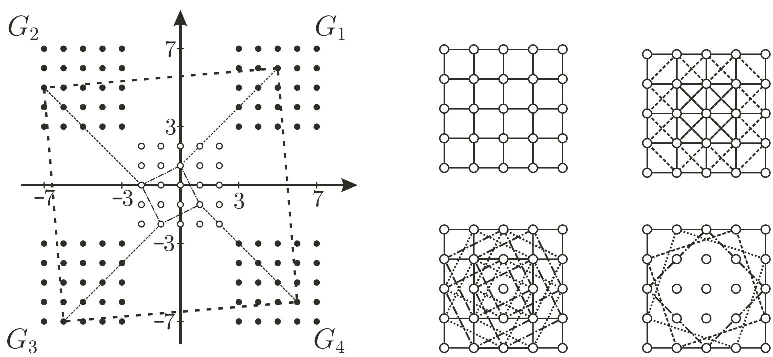
OR

In the range $[0, \pi]$, we have $\cos^{-1}(\cos x) = x$. Since the range of $\sin^{-1} x$ is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, it suffices to solve the equation $\sin^{-1}(\sin(6x)) = x$ on the interval $[0, \frac{\pi}{2}]$. Since $\sin x$ is one-to-one in $[0, \frac{\pi}{2}]$, we can consider the equivalent equation $\sin(\sin^{-1}(\sin(6x))) = \sin x$, or $\sin(6x) = \sin x$. Let $f(x) = \sin(6x)$ and $g(x) = \sin x$. Note that $f(0) = 0$, $f(\frac{\pi}{12}) = 1$, $f(\frac{\pi}{4}) = -1$, $f(\frac{5\pi}{12}) = 1$, and $f(\frac{\pi}{2}) = 0$. Moreover $f(x)$ is increasing on $(0, \frac{\pi}{12})$ and $(\frac{\pi}{4}, \frac{5\pi}{12})$, and decreasing on $(\frac{\pi}{12}, \frac{\pi}{4})$ and $(\frac{5\pi}{12}, \frac{\pi}{2})$. Similarly $g(0) = 0$, $g(\frac{\pi}{2}) = 1$, and $g(x)$ is increasing on $[0, \frac{\pi}{2}]$. Thus the graphs of $y = f(x)$ and $y = g(x)$ intersect at $x = 0$, once in the interval $[\frac{\pi}{12}, \frac{\pi}{4}]$, once in the interval $[\frac{\pi}{4}, \frac{5\pi}{12}]$, and once more in the interval $[\frac{5\pi}{12}, \frac{\pi}{2}]$. Therefore there are 4 solutions to the given equation.



25. **Answer (E):** Let G_i be the subset of G contained in the ith quadrant, $1 \le i \le 4$. For a fixed i, the maximum distance among points in G_i is $4\sqrt{2} < 6$, also the distance from a point in G_i to a point in $G_j \ne G_i$ is at least 6. Thus the required squares are exactly the squares in G with exactly one vertex in each of the G_i. Let $S = p_1p_2p_3p_4$ be a square with vertices $p_i \in G_i$. Let $p_1' = p_1 + (-5, -5)$,

$p_2' = p_2 + (5, -5)$, $p_3' = p_3 + (5, 5)$, and $p_4' = p_4 + (-5, 5)$. Observe that p_1', p_2', p_3', and p_4' are all lattice points inside the square region G' defined by the points (x, y) with $|x|$, $|y| \leq 2$; moreover, by symmetry, $S' = p_1'p_2'p_3'p_4'$ is either a square or $p_1' = p_2' = p_3' = p_4'$. Reciprocally, if $S' = p_1'p_2'p_3'p_4'$ is a square in G', then the points $p_1 = p_1' + (5, 5)$, $p_2 = p_2' + (-5, 5)$, $p_3 = p_3' + (-5, -5)$, and $p_4 = p_4' + (5, -5)$ satisfy that $p_i \in G_i$ and $S = p_1p_2p_3p_4$ is a square. The same conclusion holds if $p_1' = p_2' = p_3' = p_4'$. Therefore the required count consists of the number of points in G' plus four times the number of squares with vertices in G'.



There are 5^2 points in G' and the following number of squares with vertices in G': 4^2 of side 1, 3^2 of side 2, 3^2 of side $\sqrt{2}$ (each inscribed in a unique square of side 2), 2^2 of side 3, $2 \cdot 2^2$ of side $\sqrt{5}$ (exactly two inscribed in every square of side 3), 1^2 of side 4, 1^2 of side $2\sqrt{2}$, and $2 \cdot 1^2$ of side $\sqrt{10}$ (exactly two inscribed in the square of side 4). Thus the answer is

$$5^2 + 4 \cdot (4^2 + 2 \cdot 3^2 + 3 \cdot 2^2 + 4 \cdot 1^2) = 25 + 4 \cdot 50 = 225.$$

The problems and solutions in this contest were proposed by Bernardo Ábrego, Betsy Bennett, Thomas Butts, Gerald Bergum, Steve Dunbar, Sister Josanne Furey, Gregory Galperin, John Haverhals, Jerrold Grossman, Elgin Johnston, Joe Kennedy, David Wells, and LeRoy Wenstrom.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Math Zoom Academy
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

61st Annual

AMC 12 A

American Mathematics Contest 12A

Solutions Pamphlet

Tuesday, February 9, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 And AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego
bernardo.abrego@csun.edu

1. **Answer (C):** Distributing the negative signs gives

$$\begin{aligned}&(20-(2010-201))+(2010-(201-20))\\&=20-2010+201+2010-201+20\\&=40.\end{aligned}$$

2. **Answer (A):** The ferry boat makes 6 trips to the island. The number of tourists shuttled was

$$\begin{aligned}100+(100-1)+(100-2)+(100-3)+(100-4)+(100-5)\\=6\cdot 100-(1+2+3+4+5)\\=600-15\\=585.\end{aligned}$$

3. **Answer (E):** Let s equal the side length of the square. Because half of the area of the rectangle is in the square, $\frac{1}{2}AB = s$. Because one fifth of the square's area is in the shaded region, $s = 5 \cdot AD$. Therefore $\frac{1}{2}AB = 5 \cdot AD$, and $\frac{AB}{AD} = 10$.

4. **Answer (D):** Choice **(D)** may be written as $-\frac{1}{x}$. If x is negative, choice **(D)** is positive. To see that the other choices need not be positive, let $x=-1$ and then

$$\begin{aligned}&\textbf{(A)}\frac{-1}{|-1|}=-1,\\&\textbf{(B)}-(-1)^2=-1,\\&\textbf{(C)}-2^{-1}=-\frac{1}{2},\\&\textbf{(E)}\sqrt[3]{-1}=-1.\end{aligned}$$

5. **Answer (C):** The second place archer could score a maximum of $50 \cdot 10 = 500$ points with the remaining shots. Therefore Chelsea needs to score more than $500 - 50 = 450$ points to guarantee victory. If Chelsea's next n shots will score $10n$ points, her remaining $50-n$ shots will score at least $4(50-n)$ points. To guarantee victory,

$$10\cdot n+4\cdot(50-n)>450$$

$$6n + 200 > 450$$
$$n > 41\frac{2}{3}.$$

Therefore Chelsea needs at least 42 bullseyes to guarantee victory.

OR

If Chelsea does not make a bullseye, the maximum number of points her opponents could gain per shot would be $10 - 4 = 6$. Chelsea must make enough bullseyes to prevent her opponents from gaining 50 points. Because $8 \cdot 6 < 50 < 9 \cdot 6$, the most non-bullseyes she can afford to score is 8, leaving $50 - 8 = 42$ bullseyes needed to guarantee her victory.

6. **Answer (E):** Let $x + 32$ be written in the form $CDDC$. Because x has three digits, $1000 < x + 32 < 1032$, and so $C = 1$ and $D = 0$. Hence $x = 1001 - 32 = 969$, and the sum of the digits of x is $9 + 6 + 9 = 24$.

7. **Answer (C):** The volume scale for Logan's model is $0.1 : 100{,}000 = 1 : 1{,}000{,}000$. Therefore the linear scale is $1 : \sqrt[3]{1{,}000{,}000}$, which is $1 : 100$. Logan's water tower should stand $\frac{40}{100} = 0.4$ meters tall.

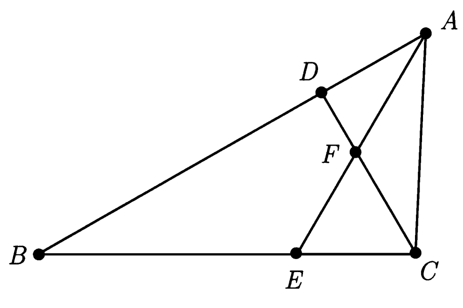
8. **Answer (C):** Let $\alpha = \angle BAE = \angle ACD = \angle ACF$. Because $\triangle CFE$ is equilateral, it follows that $\angle CFA = 120°$ and then

$$\angle FAC = 180° - 120° - \angle ACF = 60° - \alpha.$$

Therefore

$$\angle BAC = \angle BAE + \angle FAC = \alpha + (60° - \alpha) = 60°.$$

Because $AB = 2 \cdot AC$, it follows that $\triangle BAC$ is a $30-60-90°$ triangle, and thus $\angle ACB = 90°$.



9. **Answer (A):** The volume of the solid cube is 27 in^3. The first hole to be cut removes $2 \times 2 \times 3 = 12$ in^3 from the volume. The other holes remove $2 \times 2 \times 0.5 = 2$ in^3 from each of the four remaining faces. The volume of the remaining solid is $27 - 12 - 4(2) = 7$ in^3.

10. **Answer (A):** Consecutive terms in an arithmetic sequence have a common difference d. Thus $(3p+q)-(3p-q) = 2q = d$. Further, the second term is equal to $p+d$, so $p+d = 9$, and the third term is equal to $p+2d$, so $p+2d = 3p-q$. These three equations form a system that can be solved to yield $p = 5$, $q = 2$, and $d = 4$. Therefore the 2010^{th} term of the sequence is $p + 2009d = 5 + 2009 \cdot 4 = 8041$.

11. **Answer (C):** If $x = \log_b 7^7$, then $b^x = 7^7$. Thus

$$(7b)^x = 7^x \cdot b^x = 7^{x+7} = 8^x.$$

Because $x > 0$, it follows that $7b = 8$ and so $b = \frac{8}{7}$.

OR

Taking the logarithm of both sides gives us $(x+7)\log 7 = x \log 8$. Solving, we have $\frac{x+7}{x} = \frac{\log 8}{\log 7}$, $x \log 8 = x \log 7 + 7 \log 7$, $x(\log 8 - \log 7) = 7 \log 7$, and we have $x = \dfrac{\log 7^7}{\log \frac{8}{7}}$. Using the change of base rule for logarithms, $b = \frac{8}{7}$.

12. **Answer (D):** LeRoy and Chris cannot both be frogs, because their statements would be true and frogs lie. Also LeRoy and Chris cannot both be toads, because then their statements would be false, and toads tell the truth. Hence between LeRoy and Chris, exactly one must be a toad.

If Brian is a toad, then Mike must be a frog, but this is a contradiction as Mike's statement would then be true. Hence Brian is a frog, so Brian's statement must be false, and Mike must be a frog. Altogether there are 3 frogs: Brian, Mike, and either LeRoy or Chris.

13. **Answer (C):** When $k = 0$, the graphs of $x^2 + y^2 = 0$ and $xy = 0$ consist of the single point $\{(0,0)\}$ and the union of the two lines $x = 0$ and $y = 0$, respectively; so the two graphs intersect. When $k \neq 0$, the graph of $x^2 + y^2 = k^2$ is a circle of radius k centered at the origin and the graph of $xy = k$ is an equilateral hyperbola centered at the origin. The vertices of the hyperbola, located at

$(\pm\sqrt{k}, \pm\sqrt{k})$ if $k > 0$ or at $(\pm\sqrt{-k}, \mp\sqrt{-k})$ if $k < 0$, are the closest points on the graph to the origin. If $|k| \geq 2$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2\,|k| \leq k^2,$$

thus the graphs intersect. If $|k| = 1$, then

$$(\sqrt{|k|})^2 + (\sqrt{|k|})^2 = 2 > 1 = k^2,$$

and thus the graphs do not intersect. Thus the graphs do not intersect for $k = 1$ or $k = -1$.

14. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

15. **Answer (D):** Let p be the requested probability. If the coin is flipped four times, the probability of heads and tails appearing twice is $\binom{4}{2}p^2(1-p)^2 = \frac{1}{6}$, and because $0 \leq p \leq 1$ it follows that $p(1-p) = \frac{1}{6}$. Solving for p yields $p = \frac{1}{6}(3 \pm \sqrt{3})$ and because $p < 1/2$, the answer is $p = \frac{1}{6}(3 - \sqrt{3})$.

16. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability

$$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3} + \frac{2}{3} \cdot \frac{1}{2}(1 - \frac{1}{56}) = \frac{111}{168} = \frac{37}{56}.$$

17. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it

follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2}+1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2 + r + 1).$$

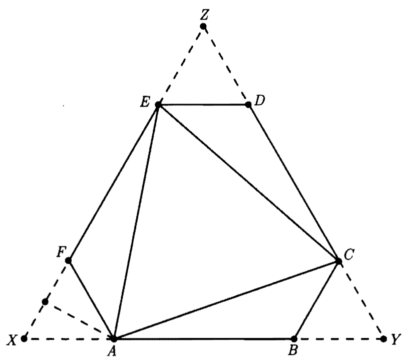
The area of hexagon $ABCDEF$ is equal to

$$[XYZ] - [XAF] - [YCB] - [ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2 - 3r^2\right) = \frac{\sqrt{3}}{4}(r^2 + 4r + 1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

$$r^2 + r + 1 = \frac{7}{10}(r^2 + 4r + 1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



18. **Answer (D):** Each such path intersects the line $y = -x$ at exactly one of the points $(\pm 4, \mp 4)$, $(\pm 3, \mp 3)$, or $(\pm 2, \mp 2)$. For $j = 0$, 1, and 2, the number of paths from $(-4, 4)$ to either of $(\pm(4-j), \mp(4-j))$ is $\binom{8}{j}$, and the number of paths to $(4, 4)$ from either of $(\pm(4-j), \mp(4-j))$ is the same. Therefore the number of paths that meet the requirement is $2\left(\binom{8}{0}^2 + \binom{8}{1}^2 + \binom{8}{2}^2\right) = 2(1^2 + 8^2 + 28^2) = 1698$.

19. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \geq 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2+n-2010 > 0$, from which $n > \frac{1}{2}(-1+\sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

20. **Answer (C):** Because $a_n = 1+(n-1)d_1$ and $b_n = 1+(n-1)d_2$ for some integers d_1 and d_2, it follows that $n-1$ is a factor of $\gcd(a_n-1, b_n-1)$. The ordered pair (a_n, b_n) must be one of $(2, 1005), (3, 670), (5, 402), (6, 335), (10, 201), (15, 134)$, or $(30, 67)$. For every pair except the sixth pair, the numbers $a_n - 1$ and $b_n - 1$ are relatively prime, so $n = 2$. In the exceptional case, $\gcd(15-1, 134-1) = 7$. The sequences defined by $a_n = 2n-1$ and $b_n = 19n-18$ satisfy the conditions, so $n = 8$.

21. **Answer (A):** Let the three points of intersections have x-coordinates p, q, and r, and let $f(x) = x^6 - 10x^5 + 29x^4 - 4x^3 + ax^2 - bx - c$. Then $f(p) = f(q) = f(r) = 0$, and $f(x) \geq 0$ for all x, so $f(x) = ((x-p)(x-q)(x-r))^2 = (x^3 - Ax^2 + Bx - C)^2$, where $A = p+q+r$, $B = pq+qr+rp$, and $C = pqr$. The coefficient of x^5 is $-10 = -2A$, so $A = 5$. The coefficient of x^4 is $29 = A^2+2B = 25+2B$, so $B = 2$. The coefficient of x^3 is $-4 = -2C - 2AB = -2C - 20$, so $C = -8$. Thus $f(x) = (x^3 - 5x^2 + 2x + 8)^2$. Because the sums of the coefficients of the even and odd powers of x are equal, -1 is a zero of $f(x)$. Factoring gives $f(x) = ((x+1)(x^2-6x+8))^2 = ((x+1)(x-2)(x-4))^2$, and the largest of the three zeros is 4.

22. **Answer (A):** Note that

$$f(x) = \begin{cases} -(x-1)-(2x-1)-\cdots-(119x-1), & \text{if } x \leq \frac{1}{119}; \\ -(x-1)-(2x-1)-\cdots-((m-1)x-1) & \\ \quad +(mx-1)+\cdots+(119x-1), & \text{if } \frac{1}{m} \leq x \leq \frac{1}{m-1};\ 2 \leq m \leq \\ (x-1)+(2x-1)+\cdots+(119x-1), & \text{if } x \geq 1. \end{cases}$$

The graph of $f(x)$ consists of a negatively sloped ray for $x \leq \frac{1}{119}$, a positively sloped ray for $x \geq 1$, and for $\frac{1}{119} \leq x \leq 1$ a sequence of line segments whose slopes increase as x increases. The minimum value of $f(x)$ occurs at the right endpoint of the rightmost interval in which the graph has a non-positive slope. The slope on the interval $[\frac{1}{m}, \frac{1}{m-1}]$ is

$$\sum_{k=m}^{119} k - \sum_{k=1}^{m-1} k = \sum_{k=1}^{119} k - 2\sum_{k=1}^{m-1} k = 7140 - (m-1)(m).$$

The inequality $7140 + m - m^2 \leq 0$ is satisfied in the interval $[-84, 85]$ with equality at the endpoints. Therefore on the interval $\left[\frac{1}{85}, \frac{1}{84}\right]$ the graph of $f(x)$ has a slope of 0 and a constant value of $(84)(1) + (119 - 84)(-1) = 49$.

23. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \leq 99$, then n is a multiple of 4.

Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k+j) \equiv 5k(1+2+3+4) + 1 \cdot 2 \cdot 3 \cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1 \cdot 2 \cdot 3 \cdot 4) \cdot (6 \cdot 7 \cdot 8 \cdot 9) \cdot \cdots \cdot (86 \cdot 87 \cdot 88 \cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

Similarly,

$$B = (5 \cdot 10 \cdot 15 \cdot 20) \cdot (30 \cdot 35 \cdot 40 \cdot 45) \cdot (55 \cdot 60 \cdot 65 \cdot 70) \cdot (80 \cdot 85 \cdot 90) \cdot (25 \cdot 50 \cdot 75),$$

thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1 \cdot 2 \cdot 3 \cdot 4) \cdot (6 \cdot 7 \cdot 8 \cdot 9) \cdot (11 \cdot 12 \cdot 13 \cdot 14) \cdot (16 \cdot 17 \cdot 18) \cdot (1 \cdot 2 \cdot 3) \\ &\equiv 24^3 \cdot (-9) \cdot (-8) \cdot (-7) \cdot 6 \equiv (-1)^3 \cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

Finally, $2^{21} = 2 \cdot (2^{10})^2 = 2 \cdot (1024)^2 \equiv 2 \cdot (-1)^2 \equiv 2 \pmod{25}$, so $13 \cdot 2^{21} \equiv 13 \cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13 \cdot 2^{21})N = 13 \cdot \frac{90!}{5^{21}} = 13 \cdot A \cdot \frac{B}{5^{21}} \equiv 13 \cdot 1 \cdot (-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

24. **Answer (B):**

Let $g(x) = \sin(\pi x) \cdot \sin(2\pi x) \cdot \sin(3\pi x) \cdots \sin(8\pi x)$. The domain of $f(x)$ is the union of all intervals on which $g(x) > 0$. Note that $\sin(n\pi(1x)) = (-1)^{k+1}\sin(n\pi x)$, so $g(1x) = g(x)$. Because $g(1/2) = 0$, it suffices to consider the subintervals of $(0, 1/2)$ on which $g(x) > 0$. In this interval the distinct solutions of the equation $g(x) = 0$ are the numbers k/n, where $2 \le n \le 8$, $1 \le k < n/2$, and k and n are relatively prime. For $n = 2, 3, 4, 5, 6, 7$, and 8 there are, respectively, 0, 1, 1, 2, 1, 3, and 2 values of k. Thus there are $1 + 1 + 2 + 1 + 3 + 2 = 10$ solutions of $g(x) = 0$ in the interval $(0, 1/2)$. The sign of $g(x)$ changes at k/n unless an even number of factors of $g(x)$ are zero at k/n, that is unless there are an even number of ways to represent k/n as a rational number with a positive denominator not exceeding 8. Thus the sign of $g(n)$ changes except at $1/4 = 2/8$ and $1/3 = 2/6$.

Let the solutions of $g(x) = 0$ in the interval $(0, 1/2)$ be $x_1, x_2, \ldots, x_{10}$ in increasing order, and let $x_0 = 0$ and $x_{11} = 1/2$. It is easily verified that $x_5 = 1/4$ and $x_7 = 1/3$, so for $0 \le j \le 10$, the sign of $g(x)$ changes at x_j except for $j = 5$ and 7. Because 5 and 7 have the same parity and $g(x) > 0$ in (x_0, x_1), the solution of $g(x) > 0$ in $(0, 1/2)$ consists of 6 disjoint open intervals. The solution of $g(x) > 0$ in $(1/2, 1)$ also consists of 6 disjoint open intervals, so the requested number of intervals is 12.

25. **Answer (C):** Suppose that a quadrilateral with sides $a \ge b \ge c \ge d$ and with perimeter 32 exists. By the triangle inequality $a < b+c+d = 32-a$, so $a \le 15$. Reciprocally, if (a, b, c, d) is a quadruple of positive integers whose sum equals 32, and whose maximum entry is $a \le 15$, then $b+c+d = 32-a \ge 17 > a$, so the triangle inequality is satisfied. This is the only condition required to guarantee the existence of a convex quadrilateral with given side lengths. Moreover, if the cyclic order of the sides is specified, then there is exactly one such cyclic quadrilateral.

The problem reduces to counting all the quadruples (a, b, c, d) of positive integers with $a + b + c + d = 32$, $\max(a, b, c, d) \le 15$, and where two quadruples are considered the same if they generate the same quadrilateral, that is if one is a cyclic permutation of the other one. For example $(12, 4, 5, 11)$ and $(5, 11, 12, 4)$ generate the same quadrilateral.

The number of quadruples (a, b, c, d) with $a + b + c + d = 32$ can be counted as follows: consider 31 spots on a line to be filled with 28 ones and 3 plus signs. There are $\binom{31}{3}$ ways to choose the locations of the plus signs, and every such assignment is in one-to-one correspondence to the quadruple (a', b', c', d'), where each entry indicates the number of ones between consecutive plus signs. Setting $(a, b, c, d) = (1, 1, 1, 1) + (a', b', c', d')$ gives precisely all quadruples where $a, b, c, d \ge 1$ and $a + b + c + d = 32$. To count those where the maximum entry is 16 or more, consider 13 ones and 3 plus signs. There are $\binom{16}{3}$ quadruples (a', b', c', d') where $a', b', c', d' \ge 0$ and $a' + b' + c' + d' = 13$, there are 4 ways to choose one of the coordinates, say a', to be the maximum. Then the quadruple $(a, b, c, d) = (16, 1, 1, 1) + (a', b', c', d')$ satisfies our requirements. Thus there are exactly $4\binom{16}{3}$ quadruples (a, b, c, d) where $a, b, c, d \ge 1$, $a + b + c + d = 32$, and

$\max(a,b,c,d) \geq 16$; consequently, there are

$$\binom{31}{3} - 4\binom{16}{3} \tag{1}$$

quadruples (a,b,c,d) where $a,b,c,d \geq 1$, $a+b+c+d = 32$, and $\max(a,b,c,d) \leq 15$.

If (a,b,c,d) consists of distinct entries, then it has exactly 4 cyclic permutations. The same occurs if only two entries are equal to each other, or three entries are equal to each other and the remaining entry is not. If (a,b,c,d) has two pairs of entries equal to each other ordered (a,a,b,b), then it has 4 cyclic permutations, but if they are ordered (a,b,a,b) then it has only 2 cyclic permutations. Finally, if all entries are equal then there is only one cyclic permutation.

There are exactly $2 \cdot 7 = 14$ quadruples of the form (a,b,a,b) with $a \neq b$ and $a+b = 16$ and there is only one quadruple $(a,a,a,a) = (8,8,8,8)$ with four equal entries. Adding to (1) the number of quadruples of the form (a,b,a,b) and 3 times the number of quadruples of the form (a,a,a,a), guarantees that all classes of equivalence under cyclic permutations are counted exactly 4 times. Therefore the required number of cyclic quadrilaterals is

$$\begin{aligned} \frac{1}{4}\left(\binom{31}{3} - 4\binom{16}{3} + 14 + 3\right) &= \frac{1}{4}(31 \cdot 5 \cdot 29 - 32 \cdot 5 \cdot 14 + 17) \\ &= \frac{1}{4}(5 \cdot 451 + 17) = 568. \end{aligned}$$

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Blasberg, Steven Davis, Steven Dunbar, Douglas Faires, Silvia Fernandez, David Grabiner, Jerrold Grossman, Brian Hartwig, Dan Kennedy, Glen Marr, John Morrison, Raymond Scacalossi, David Wells, LeRoy Wenstrom, and Woody Wenstrom.

MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

61st Annual

AMC 12 B

American Mathematics Contest 12B

Solutions Pamphlet

Wednesday, February 24, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

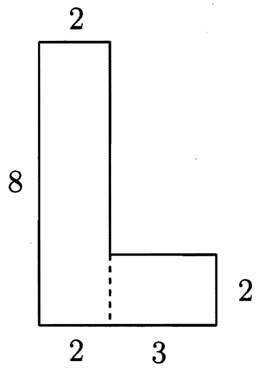
American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 12 And AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego
[illegible]

1. **Answer (C):** Makayla spent $45 + 2 \cdot 45 = 135$ minutes, or $\frac{135}{60} = \frac{9}{4}$ hours in meetings. Hence she spent $100 \cdot \frac{9/4}{9} = 25$ percent of her time in meetings.

2. **Answer (A):** The region consists of two rectangles: an 8-by-2 rectangle, and a 3-by-2 rectangle. The desired area is $8 \cdot 2 + 3 \cdot 2 = 22$.



3. **Answer (E):** The cost of an individual ticket must divide 48 and 64. The common factors of 48 and 64 are $1, 2, 4, 8$, and 16. Each of these may be the cost of one ticket, so there are 5 possible values for x.

4. **Answer (B):** A month with 31 days has 3 successive days of the week appearing five times and 4 successive days of the week appearing four times. If Monday and Wednesday appear five times then Monday must be the first day of the month. If Monday and Wednesday appear only four times then either Thursday or Friday must be the first day of the month. Hence there are 3 days of the week that could be the first day of the month.

5. **Answer (D):** The correct answer was $1-(2-(3-(4+e))) = 1-2+3-4-e = -2-e$. Larry's answer was $1-2-3-4+e = -8+e$. Therefore $-2-e = -8+e$, so $e = 3$.

6. **Answer (D):** Assume there are 100 students in Mr. Wells' class. Then at least $70 - 50 = 20$ students answered "No" at the beginning of the school year and "Yes" at the end, so $x \geq 20$. Because only 30 students answered "No" at the end of the school year, at least $50-30 = 20$ students who answered "Yes"' at the

beginning of the year gave the same answer at the end, so $x \leq 80$. The difference between the maximum and minimum possible values of x is $80 - 20 = 60$. The minimum $x = 20$ is achieved if exactly 20 students answered "No" at the beginning and "Yes" at the end of the school year. The maximum $x = 80$ is achieved if exactly 20 students answered "Yes" at the beginning and the end.

7. **Answer (C):** Let t be the number of minutes Shelby spent driving in the rain. Then she traveled $20\frac{t}{60}$ miles in the rain, and $30\frac{40-t}{60}$ miles in the sun. Solving $20\frac{t}{60} + 30\frac{40-t}{60} = 16$ results in $t = 24$ minutes.

8. **Answer (B):** If there are n schools in the city, then there are $3n$ contestants, so $3n \geq 64$, and $n \geq 22$. Because Andrea received the median score and each student received a different score, n is odd, so $n \geq 23$. Andrea's position is $\frac{3n+1}{2}$, and Andrea finished ahead of Beth, so $\frac{3n+1}{2} < 37$, and $3n < 73$. Because n is an odd integer, $n \leq 23$. Therefore $n = 23$.

9. **Answer (E):** Because n is divisible by 20, $n = 2^{2+a} \cdot 5^{1+b} \cdot k$, where a and b are nonnegative integers and k is a positive integer not divisible by 2 or 5. Because $n^2 = 2^{2(2+a)} \cdot 5^{2(1+b)} \cdot k^2$ is a perfect cube, 3 divides $2(2+a)$ and 3 divides $2(1+b)$. Because $n^3 = 2^{3(2+a)} \cdot 5^{3(1+b)} \cdot k^3$ is a perfect square, 2 divides $3(2+a)$ and 2 divides $3(1+b)$. Therefore 6 divides $2+a$ and 6 divides $1+b$. The smallest possible choices for a, b, and k, are $a = 4$, $b = 5$, and $k = 1$. In this case $n = 2^6 \cdot 5^6 = 1{,}000{,}000$, and n has 7 digits.

OR

The only prime factors of 20 are 2 and 5, so n has the form $2^a \cdot 5^b$ for integers $a \geq 2$ and $b \geq 1$. Because n^2 is a perfect cube, $2a$ and $2b$ are both multiples of 3, so a and b are also both multiples of 3. Similarly, because n^3 is a perfect square, a and b are both multiples of 2. Therefore both a and b are multiples of 6. Note that $n = 2^6 \cdot 5^6 = 1{,}000{,}000$ satisfies the given conditions, and n has 7 digits.

"No"

10. **Answer (B):** The average of the numbers is

$$\frac{1+2+\cdots+99+x}{100} = \frac{\frac{99\cdot 100}{2}+x}{100} = \frac{99\cdot 50+x}{100} = 100x.$$

This equation is equivalent to $9999x = (99 \cdot 101)x = 99 \cdot 50$, so $x = \frac{50}{101}$.

11. **Answer (E):** Each four-digit palindrome has digit representation $abba$ with $1 \le a \le 9$ and $0 \le b \le 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

12. **Answer (D):** Rewriting each logarithm in base 2 gives

$$\frac{\frac{1}{2}\log_2 x}{\frac{1}{2}} + \log_2 x + \frac{2\log_2 x}{2} + \frac{3\log_2 x}{3} + \frac{4\log_2 x}{4} = 40.$$

Therefore $5\log_2 x = 40$, so $\log_2 x = 8$, and $x = 256$.

OR

For $a \ne 0$ the expression $\log_{2^a}(x^a) = y$ if and only if $2^{ay} = x^a$. Thus $2^y = x$ and $y = \log_2 x$. Therefore the given equation is equivalent to $5\log_2 x = 40$, so $\log_2 x = 8$ and $x = 256$.

13. **Answer (C):** The maximum value for $\cos x$ and $\sin x$ is 1; hence $\cos(2A-B) = 1$ and $\sin(A+B) = 1$. Therefore $2A - B = 0°$ and $A + B = 90°$, and solving gives $A = 30°$ and $B = 60°$. Hence $\triangle ABC$ is a $30-60-90°$ right triangle and $BC = 2$.

14. **Answer (B):** Note that $3M > (a+b) + c + (d+e) = 2010$, so $M > 670$. Because M is an integer $M \ge 671$. The value of 671 is achieved if $(a,b,c,d,e) = (669,1,670,1,669)$.

15. **Answer (D):** There are three cases to consider.

First, suppose that $i^x = (1+i)^y \ne z$. Note that $|i^x| = 1$ for all x, and $|(1+i)^y| \ge |1+i| = \sqrt{2} > 1$ for $y \ge 1$. If $y = 0$, then $(1+i)^y = 1 = i^x$ if x is a multiple of 4. The ordered triples that satisfy this condition are $(4k, 0, z)$ for $0 \le k \le 4$ and $0 \le z \le 19$, $z \ne 1$. There are $5 \cdot 19 = 95$ such triples.

Next, suppose that $i^x = z \ne (1+i)^y$. The only nonnegative integer value of i^x is 1, which is assumed when $x = 4k$ for $0 \le k \le 4$. In this case $i^x = 1$ and $y \ne 0$. The ordered triples that satisfy this condition are $(4k, y, 1)$ for $0 \le k \le 4$ and $1 \le y \le 19$. There are $5 \cdot 19 = 95$ such triples.

Finally, suppose that $(1+i)^y = z \ne i^x$. Note that $(1+i)^2 = 2i$, so $(1+i)^y$ is a positive integer only when y is a multiple of 8. Because $(1+i)^0 = 1$, $(1+i)^8 = (2i)^4 = 16$, and $(1+i)^{16} = 16^2 = 256$, the only possible ordered triples are $(x, 0, 1)$ with $x \neq 4k$ for $0 \leq k \leq 4$ and $(x, 8, 16)$ for any x. There are $15 + 20 = 35$ such triples.

The total number of ordered triples that satisfy the given conditions is $95+95+35 = 225$.

16. **Answer (E):** Let $N = abc+ab+a = a(bc+b+1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1. Then

$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.
Therefore the requested probability is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$.

17. **Answer (D):** Let a_{ij} denote the entry in row i and column j. he given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5,$ or 6. If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

18. **Answer (C):** Let A denote the frog's starting point, and let P, Q, and B denote its positions after the first, second, and third jumps, respectively. Introduce a coordinate system with P at $(0, 0)$, Q at $(1, 0)$, A at $(\cos\alpha, \sin\alpha)$, and B at $(1 + \cos\beta, \sin\beta)$. It may be assumed that $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq 2\pi$. For $\alpha = 0$, the required condition is met for all values of β. For $\alpha = \pi$, the required condition is met only if $\beta = \pi$. For $0 < \alpha < \pi$, $AB = 1$ if and only if $\beta = \alpha$ or $\beta = \pi$, and the required condition is met if and only if $\alpha \leq \beta \leq \pi$. In

the $\alpha\beta$-plane, the rectangle $0 \le \alpha \le \pi$, $0 \le \beta \le 2\pi$ has area $2\pi^2$. The triangle $0 \le \alpha \le \pi$, $\alpha \le \beta \le \pi$ has area $\frac{\pi^2}{2}$, so the requested probability is $\frac{1}{4}$.

19. **Answer (E):** The Raiders' score was $a(1+r+r^2+r^3)$, where a is a positive integer and $r > 1$. Because ar is also an integer, $r = m/n$ for relatively prime positive integers m and n with $m > n$. Moreover $ar^3 = a \cdot \frac{m^3}{n^3}$ is an integer, so n^3 divides a. Let $a = n^3 A$. Then the Raiders' score was $R = A(n^3 + mn^2 + m^2 n + m^3)$, and the Wildcats' score was $R-1 = a+(a+d)+(a+2d)+(a+3d) = 4a+6d$ for some positive integer d. Because $A \ge 1$, the condition $R \le 100$ implies that $n \le 2$ and $m \le 4$. The only possibilities are $(m,n) = (4,1), (3,2), (3,1)$, or $(2,1)$. The corresponding values of R are, respectively, $85A$, $65A$, $40A$, and $15A$. In the first two cases $A = 1$, and the corresponding values of $R-1$ are, respectively, $64 = 32+6d$ and $84 = 4+6d$. In neither case is d an integer. In the third case $40A = 40a = 4a+6d+1$ which is impossible in integers. In the last case $15a = 4a+6d+1$, from which $11a = 6d+1$. The only solution in positive integers for which $4a+6d \le 100$ is $(a,d) = (5,9)$. Thus $R = 5+10+20+40 = 75$, $R-1 = 5+14+23+32 = 74$, and the number of points scored in the first half was $5+10+5+14 = 34$.

20. **Answer (E):** The ratio between consecutive terms of the sequence is

$$r = \frac{a_2}{a_1} = \cot x,$$

so $a_4 = (\tan x)(\cot x) = 1$, and r is also equal to

$$\sqrt{\frac{a_4}{a_2}} = \frac{1}{\sqrt{\cos x}}.$$

Therefore x satisfies the equation $\cos^3 x = \sin^2 x = 1 - \cos^2 x$, which can be written as $(\cos^2 x)(1+\cos x) = 1$. The given conditions imply that $\cos x \neq 0$, so this equation is equivalent to

$$1 + \cos x = \frac{1}{\cos^2 x} = r^4.$$

Thus $1 + \cos x = 1 \cdot r^4 = a_4 \cdot r^4 = a_8$.

21. **Answer (B):** Because $1, 3, 5$, and 7 are roots of the polynomial $P(x) - a$, it follows that

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2, 4, 6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \operatorname{lcm}(15, 9, 105)$ divides a, that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A + (x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A, the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

22. **Answer (D):** Let R be the circumradius of $ABCD$ and let $a = AB$, $b = BC$, $c = CD$, $d = DA$, and $k = bc = ad$. Because the areas of $\triangle ABC$, $\triangle CDA$, $\triangle BCD$, and $\triangle ABD$ are

$$\frac{ab \cdot AC}{4R}, \quad \frac{cd \cdot AC}{4R}, \quad \frac{bc \cdot BD}{4R}, \quad \text{and} \quad \frac{ad \cdot BD}{4R},$$

respectively, and Area($\triangle ABC$)+Area($\triangle CDA$) = Area($\triangle BCD$)+Area($\triangle ABD$), it follows that

$$\frac{AC}{4R}(ab+cd) = \frac{BD}{4R}(bc+ad) = \frac{BD}{4R}(2k);$$

that is, $(ab+cd) \cdot AC = 2k \cdot BD$. By Ptolemy's Theorem $ac + bd = AC \cdot BD$. Solving for AC and substituting into the previous equation gives

$$BD^2 = \frac{1}{2k}(ac+bd)(ab+cd) = \frac{1}{2k}(a^2k+c^2k+b^2k+d^2k) = \frac{1}{2}(a^2+b^2+c^2+d^2).$$

None of the sides can be equal to 11 or 13 because by assumption a, b, c, and d are pairwise distinct and less than 15, and so it is impossible to have a factor of 11 or 13 on each side of the equation $bc = ad$. If the largest side length is 12 or less, then $2BD^2 \le 12^2 + 10^2 + 9^2 + 8^2 = 389$, and so $BD \le \sqrt{\frac{389}{2}}$. If the largest side is 14 and the other sides are $s_1 > s_2 > s_3$, then $14s_3 = s_1 s_2$. Thus 7 divides $s_1 s_2$ and because $0 < s_2 < s_1 < 14$, it follows that either $s_1 = 7$ or $s_2 = 7$. If $s_1 = 7$, then $2BD^2 < 14^2 + 7^2 + 6^2 + 5^2 = 306$. If $s_2 = 7$, then $2s_3 = s_1$, and it follows that $2BD^2 \le 14^2 + 7^2 + 12^2 + 6^2 = 425$. Therefore $BD \le \sqrt{\frac{425}{2}}$ with equality for a cyclic quadrilateral with $a = 14$, $b = 12$, $c = 7$, and $d = 6$.

23. **Answer (A):** Because both $P(Q(x))$ and $Q(P(x))$ have four distinct real zeros, both $P(x)$ and $Q(x)$ must have two distinct real zeros, so there are real numbers h_1, k_1, h_2, and k_2 such that $P(x) = (x-h_1)^2 - k_1^2$ and $Q(x) = (x-h_2)^2 - k_2^2$. The zeros of $P(Q(x))$ occur when $Q(x) = h_1 \pm k_1$. The solutions of each equation are equidistant from h_2, so $h_2 = -19$. It follows that $Q(-15) - Q(-17) = (16 - k_2^2) - (4 - k_2^2) = 12$, and also $Q(-15) - Q(-17) = 2k_1$, so $k_1 = 6$. Similarly $h_1 = -54$, so $2k_2 = P(-49) - P(-51) = (25 - k_1^2) - (9 - k_1^2) = 16$, and $k_2 = 8$. Thus the sum of the minimum values of $P(x)$ and $Q(x)$ is $-k_1^2 - k_2^2 = -100$.

24. **Answer (C):** Let $f(x) = \frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011}$. Note that

$$f(x) - f(y) = (y-x)\left(\frac{1}{(x-2009)(y-2009)} + \frac{1}{(x-2010)(y-2010)} + \frac{1}{(x-2011)(y-2011)}\right).$$

If $x < y < 2009$, then $y - x > 0$,

$$\frac{1}{(x-2009)(y-2009)} > 0, \quad \frac{1}{(x-2010)(y-2010)} > 0,$$
$$\text{and } \frac{1}{(x-2011)(y-2011)} > 0.$$

Thus f is decreasing on the interval $x < 2009$, and because $f(x) < 0$ for $x < 0$, it follows that no values $x < 2009$ satisfy $f(x) \geq 1$.

If $2009 < x < y < 2010$, then $f(x) - f(y) > 0$ as before. Thus f is decreasing in the interval $2009 < x < 2010$. Moreover, $f(2009 + \frac{1}{10}) = 10 - \frac{10}{9} - \frac{10}{19} > 1$ and $f(2010 - \frac{1}{10}) = \frac{10}{9} - 10 - \frac{10}{11} < 1$. Thus there is a number $2009 < x_1 < 2010$ such that $f(x) \geq 1$ for $2009 < x \leq x_1$ and $f(x) < 1$ for $x_1 < x < 2010$.

Similarly, f is decreasing on the interval $2010 < x < 2011$, $f(2010 + \frac{1}{10}) > 1$, and $f(2011 - \frac{1}{10}) < 1$. Thus there is a number $2010 < x_2 < 2011$ such that $f(x) \geq 1$ for $2010 < x \leq x_2$ and $f(x) < 1$ for $x_2 < x < 2011$.

Finally, f is decreasing on the interval $x > 2011$, $f(2011 + \frac{1}{10}) > 1$, and $f(2014) = \frac{1}{5} + \frac{1}{4} + \frac{1}{3} < 1$. Thus there is a number $x_3 > 2011$ such that $f(x) \geq 1$ for $2011 < x \leq x_3$ and $f(x) < 1$ for $x > x_3$.

The required sum of the lengths of these three intervals is

$$x_1 - 2009 + x_2 - 2010 + x_3 - 2011 = x_1 + x_2 + x_3 - 6020.$$

Multiplying both sides of the equation

$$\frac{1}{x-2009} + \frac{1}{x-2010} + \frac{1}{x-2011} = 1$$

by $(x-2009)(x-2010)(x-2011)$ and collecting terms on one side of the equation gives

$$x^3 - x^2(2009 + 2010 + 2011 + 1 + 1 + 1) + ax + b = 0$$

where a and b are real numbers. The three roots of this equation are x_1, x_2, and x_3. Thus $x_1 + x_2 + x_3 = 6020 + 3$, and consequently the required sum equals 3.

25. **Answer (D):** Observe that $2010 = 2 \cdot 3 \cdot 5 \cdot 67$. Let $P = \prod_{n=2}^{5300} \text{pow}(n) = 2^a \cdot 3^b \cdot 5^c \cdot 67^d \cdot Q$ where Q is relatively prime with $2, 3, 5$, and 67. The largest power of 2010 that divides P is equal to 2010^m where $m = \min(a, b, c, d)$.

By definition $\operatorname{pow}(n) = 2^k$ if and only if $n = 2^k$. Because $2^{12} = 4096 < 5300 < 8192 = 2^{13}$, it follows that

$$a = 1 + 2 + \cdots + 12 = \frac{12 \cdot 13}{2} = 78.$$

Similarly, $\operatorname{pow}(n) = 67$ if and only if $n = 67N$ and the largest prime dividing N is smaller than 67. Because $5300 = 79 \cdot 67 + 7$ and 71, 73, and 79 are the only primes p in the range $67 < p \le 79$; it follows that for $n \le 5300$, $\operatorname{pow}(n) = 67$ if and only if

$$n \in \{67k : 1 \le k \le 79\} \setminus \{67^2, 67 \cdot 71, 67 \cdot 73, 67 \cdot 79\}.$$

Because $67^2 < 5300 < 2 \cdot 67^2$, the only $n \le 5300$ for which $\operatorname{pow}(n) = 67^k$ with $k \ge 2$, is $n = 67^2$. Therefore

$$d = 79 - 4 + 2 = 77.$$

If $n = 2^j \cdot 3^k$ for $j \ge 0$ and $k \ge 1$, then $\operatorname{pow}(n) = 3^k$. Moreover, if $0 \le j \le 2$ and $1 \le k \le 6$, or if $0 \le j \le 1$ and $k = 7$; then $n = 2^j \cdot 3^k \le 2 \cdot 3^7 = 4374 < 5300$. Thus

$$b \ge 3\,(1 + 2 + \cdots + 6) + 7 + 7 = 3 \cdot 21 + 14 = 77.$$

If $n = 2^i \cdot 3^j \cdot 5^k$ for $i, j \ge 0$ and $k \ge 1$, then $\operatorname{pow}(n) = 5^k$. Moreover, If $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3\}$ and $1 \le k \le 3$, or if $2^i \cdot 3^j \in \{1, 2, 3, 2^2, 2 \cdot 3, 2^3\}$ and $k = 4$, or if $2^i \cdot 3^j = 1$ and $k = 5$; then $n = 2^i \cdot 3^j \cdot 5^k \le 8 \cdot 5^4 = 5000 < 5300$. Thus

$$c \ge 8\,(1 + 2 + 3) + 6 \cdot 4 + 5 = 77.$$

Therefore $m = d = 77$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsty Bennett, Steven Blasberg, Tom Butts, Steven Davis, Sundeep Desai, Steven Dunbar, Sylvia Fernandez, Jerrold Grossman, Joe Kennedy, Leon La Spina, David Wells, LeRoy Wenstrom and Woody Wenstrom.

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

62nd Annual

AMC 12 A

American Mathematics Contest 12 A

Tuesday, February 8, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

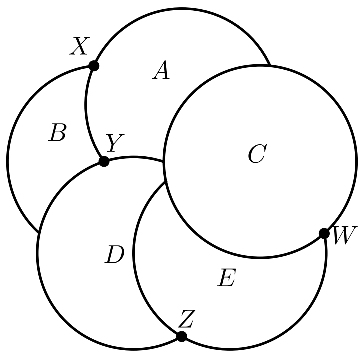
American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego
bernardo.abrego@csun.edu

1. **Answer (D):** The text messages cost $\$0.05 \cdot 100 = \$5.00$, and the 30 minutes of excess chatting cost $\$0.10 \cdot 30 = \$3.00$. Therefore the total bill came to $\$5 + \$3 + \$20 = \$28$.

2. **Answer (E):** The circumference of coin B in the figure does not continue past the point X where it intersects the circumference of coin A. Thus coin A is above coin B. Similarly, the points Y, Z, and W in the figure show that coin D is above coin A, coin E is above coin D, and coin C is above coin E, respectively. Thus the order of the coins from top to bottom is (C, E, D, A, B).



3. **Answer (E):** Because $14 \cdot 35 = 490 < 500$ and $15 \cdot 35 = 525 \geq 500$, the minimum number of bottles that she needs to buy is 15.

4. **Answer (C):** Let N equal the number of fifth graders. Then there are $2N$ fourth graders and $4N$ third graders. The total number of minutes run per day by the students is $4N \cdot 12 + 2N \cdot 15 + N \cdot 10 = 88N$. There are a total of $4N + 2N + N = 7N$ students, so the average number of minutes run by the students per day is $\frac{88N}{7N} = \frac{88}{7}$.

5. **Answer (C):** Because 75% of the birds were not swans and 30% of the birds were geese, it follows that $\frac{30}{75} \cdot 100\% = 40\%$ of the birds that were not swans were geese.

6. **Answer (A):** Let x, y, and z be the number of successful three-point shots, two-point shots, and free throws, respectively. Then the given conditions imply

$$3x + 2y + z = 61,$$

$$2y = 3x, \text{ and}$$
$$y + 1 = z.$$

Solving results in $x = 8$, $y = 12$, and $z = 13$. Hence the team made 13 free throws.

7. **Answer (B):** Let C be the cost of a pencil in cents, N be the number of pencils each student bought, and S be the number of students who bought pencils. Then $C \cdot N \cdot S = 1771 = 7 \cdot 11 \cdot 23$, and $C > N > 1$. Because a majority of the students bought pencils, $30 \geq S > \frac{30}{2} = 15$. Therefore $S = 23, N = 7$, and $C = 11$.

8. **Answer (C):** Note that for any four consecutive terms, the first and last terms must be equal. For example, consider B, C, D, and E; because

$$B + C + D = 30 = C + D + E\,,$$

we must have $B = E$. Hence $A = D = G$, and $C = F = 5$. The required sum $A + H = G + (30 - G - F) = 30 - 5 = 25$.

OR

Note that

$$\begin{aligned} A + C + H &= (A + B + C) - (B + C + D) + (C + D + E) \\ &\quad - (E + F + G) + (F + G + H) \\ &= 3 \cdot 30 - 2 \cdot 30 = 30. \end{aligned}$$

Hence $A + H = 30 - C = 25$.

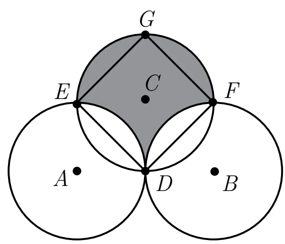
9. **Answer (B):** Each of the 18 twins shook hands with 16 twins and 9 triplets, giving a total of $18 \cdot 25$ handshakes. Similarly, each of the 18 triplets shook hands with 15 triplets and 9 twins, giving a total of $18 \cdot 24$ handshakes. This tally counts every handshake twice, so the number of handshakes is $\frac{1}{2}(18 \cdot 25 + 18 \cdot 24) = 9 \cdot 49 = 441$.

10. **Answer (B):** Let d be the sum of the numbers rolled. The conditions are satisfied if and only if $\pi \left(\frac{d}{2}\right)^2 < \pi d$, that is, $d < 4$. Of the 36 equally likely outcomes for the roll of the two dice, one has a sum of 2 and two have sums of 3. Thus the desired probability is $\frac{1+2}{36} = \frac{1}{12}$.

11. **Answer (C):** Let D be the midpoint of $\overline{AB}$, and let circle C intersect circles A and B at E and F, respectively, distinct from D. The shaded portion of unit square $ADCE$ has area $1-\frac{\pi}{4}$, as does the shaded portion of unit square $BDCF$. The portion of the shaded region which is outside these squares is a semicircle of radius 1 and has area $\frac{\pi}{2}$. The total shaded area is $2\left(1-\frac{\pi}{4}\right)+\frac{\pi}{2}=2$.

OR

Let D, E, and F be defined as in the first solution, and let G be diametrically opposite D on circle C. The shaded area is equal to the area of square $DFGE$, which has diagonal length 2. Its side length is $\sqrt{2}$, and its area is $\left(\sqrt{2}\right)^2=2$.



12. **Answer (D):** Assume the power boat and raft met at point O on the river. Let x be the speed of the boat and y be the speed of the raft and the river current. Then $x+y$ is the speed of the power boat downstream and $x-y$ is the speed of the power boat upstream. Let the distance AB between the docks be S, so that $AO=9y$ and $OB=S-9y$. Then because time is equal to distance divided by rate,

$$\frac{S}{x+y}+\frac{S-9y}{x-y}=9\,.$$

Rearrange to find that $S=\frac{9}{2}(x+y)$. Then the time it took the power boat to go from A to B is

$$\frac{S}{x+y}=\frac{9(x+y)}{2(x+y)}=4.5\,.$$

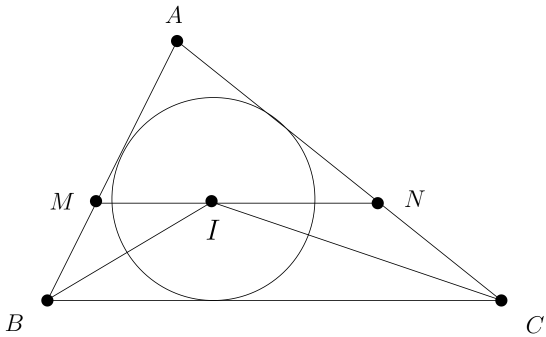
OR

In the reference frame of the raft, the boat simply went away, turned around, and came back, all at the same speed. Because the trip took 9 hours, the boat must have turned around after 4.5 hours.

13. **Answer (B):** Let I be the incenter of $\triangle ABC$. Because I is the intersection of the angle bisectors of the triangle and $\overline{MN}$ is parallel to $\overline{BC}$, it follows that $\angle IBM=\angle CBI=\angle MIB$ and $\angle NCI=\angle ICB=\angle CIN$. Hence $\triangle BMI$ and

$\triangle CNI$ are isosceles with $MB = MI$ and $CN = IN$. Thus the perimeter of $\triangle AMN$ is

$$\begin{aligned} AM + MN + NA &= AM + MI + IN + NA \\ &= AM + MB + CN + NA \\ &= AB + AC = 12 + 18 = 30. \end{aligned}$$



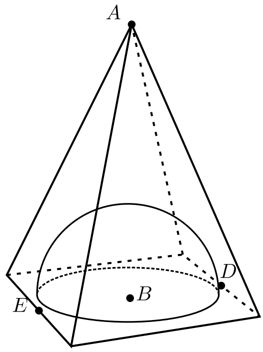
14. **Answer (E):** The point (a, b) is above the parabola if and only if $b > a^3 - ab$. Because a is positive, this is equivalent to $b > \frac{a^3}{a+1}$. If $a = 1$, then b can be any digit from 1 to 9 inclusive. If $a = 2$, then b can be any digit between 3 and 9 inclusive. If $a = 3$, then b can be any digit between 7 and 9 inclusive. If $a > 3$, there is no b that satisfies $b > \frac{a^3}{a+1}$. Therefore there are $9 + 7 + 3 = 19$ pairs satisfying the condition, out of a total of $9 \cdot 9 = 81$ pairs. The requested probability is $\frac{19}{81}$.

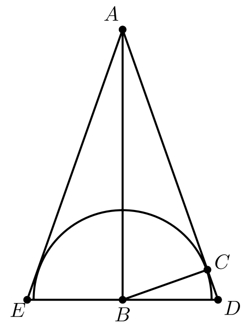
15. **Answer (A):** A cross section of the figure is shown, where A is the apex, B is the center of the base, D and E are the midpoints of opposite sides of the base, and the hemisphere meets $\overline{AD}$ at C.

Right triangle ABC has $AB = 6$ and $BC = 2$, so $AC = 4\sqrt{2}$. Because $\triangle ABC \sim \triangle ADB$, it follows that

$$BD = \frac{BC \cdot AB}{AC} = \frac{2 \cdot 6}{4\sqrt{2}} = \frac{3\sqrt{2}}{2}.$$

Hence the length of the base edge is $DE = 2 \cdot BD = 3\sqrt{2}$.





16. **Answer (C):** If five distinct colors are used, then there are $\binom{6}{5} = 6$ different color choices possible. They may be arranged in $5! = 120$ ways on the pentagon, resulting in $120 \cdot 6 = 720$ colorings.

    If four distinct colors are used, then there is one duplicated color, so there are $\binom{6}{4}\binom{4}{1} = 60$ different color choices possible. The duplicated color must appear on neighboring vertices. There are 5 neighbor choices and $3! = 6$ ways to color the remaining three vertices, resulting in a total of $60 \cdot 5 \cdot 6 = 1800$ colorings.

    If three distinct colors are used, then there must be two duplicated colors, so there are $\binom{6}{3}\binom{3}{2} = 60$ different color choices possible. The non-duplicated color may appear in 5 locations. As before, a duplicated color must appear on neighboring vertices, so there are 2 ways left to color the remaining vertices. In this case there are $60 \cdot 5 \cdot 2 = 600$ colorings possible.

    There are no colorings with two or fewer colors. The total number of colorings is $720 + 1800 + 600 = 3120$.

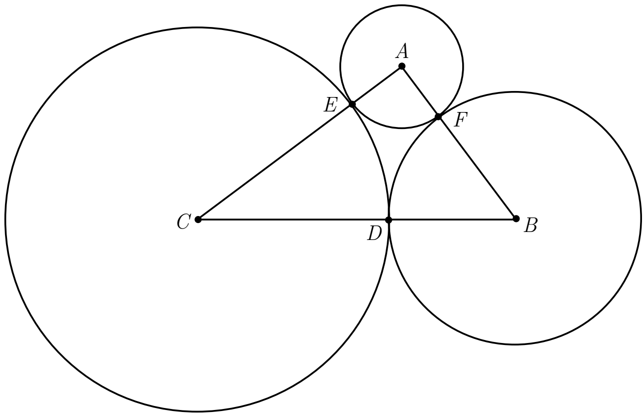
17. **Answer (D):** Let A, B, and C be the centers of the circles with radii 1, 2, and 3, respectively. Let D, E, and F be the points of tangency, where D is on the circles B and C, E is on the circles A and C, and F is on the circles A and B. Because $AB = AF + FB = 1 + 2 = 3$, $BC = BD + DC = 2 + 3 = 5$, and $CA = CE + EA = 3 + 1 = 4$, it follows that $\triangle ABC$ is a 3-4-5 right triangle. Therefore

$$[ABC] = \frac{1}{2}AB \cdot AC = 6, \quad [AEF] = \frac{1}{2}AE \cdot AF = \frac{1}{2},$$
$$[BFD] = \frac{1}{2}BD \cdot BF \cdot \sin(\angle FBD) = \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{4}{5} = \frac{8}{5}, \text{ and}$$
$$[CDE] = \frac{1}{2}CD \cdot CE \cdot \sin(\angle DCE) = \frac{1}{2} \cdot 3 \cdot 3 \cdot \frac{3}{5} = \frac{27}{10}.$$

Hence

$$[DEF] = [ABC] - [AEF] - [BFD] - [CDE]$$

$$= 6 - \frac{1}{2} - \frac{8}{5} - \frac{27}{10} = \frac{6}{5}.$$



18. **Answer (D):** The graph of the equation $|x+y|+|x-y| = 2$ is a square formed by the lines $x = \pm 1$ and $y = \pm 1$. For $c > -9$, the equation $c = x^2 - 6x + y^2 = (x-3)^2 + y^2 - 9$ is the equation of a circle with center $(3, 0)$ and radius $\sqrt{c+9}$. Among all such circles that intersect the square, the largest one contains the points $(-1, \pm 1)$ and has radius $\sqrt{4^2 + 1^2} = \sqrt{17}$. It follows that the maximum value of c is $17 - 9 = 8$.

19. **Answer (C):** The given conditions imply that $N \geq 19$ and $1+\lfloor \log_2(N-1) \rfloor = \log_2(N+19)$. Because $1 + \lfloor \log_2(N-1) \rfloor$ is a positive integer, so is $\log_2(N+19)$; thus $2^k = N + 19 \geq 38$ for some integer k. It follows that $k \geq 6$ and the two smallest values of N are $2^6 - 19 = 45$ and $2^7 - 19 = 109$, whose sum is 154.

   Note: This formula for the number of elite status players provides a method used to determine the number of first-round byes in a single-elimination tournament.

20. **Answer (C):** Note that $f(1) = a+b+c = 0$, so $f(7) = 49a+7b+c = 48a+6b = 6(8a+b)$. Thus $f(7)$ is an integer multiple of 6 strictly between 50 and 60, so $f(7) = 54$ and $8a+b = 9$. Similarly, $f(8) = 64a+8b+c = 63a+7b = 7(9a+b)$. Thus $f(8)$ is an integer multiple of 7 strictly between 70 and 80, so $f(8) = 77$ and $9a + b = 11$. It follows that $a = 2$, $b = -7$, and $c = 5$. Therefore $f(100) = 2 \cdot 100^2 - 7 \cdot 100 + 5 = 19{,}305$, and thus $k = 3$.

21. **Answer (A):** Because $f_2(x) = \sqrt{1 - \sqrt{4-x}}$, $f_2(x)$ is defined if and only if $0 \leq \sqrt{4-x} \leq 1$, so the domain of f_2 is the interval $[3, 4]$. Similarly, the

domain of f_3 is the solution set of the inequality $3 \le \sqrt{9-x} \le 4$, which is the interval $[-7, 0]$, and the domain of f_4 is the solution set of the inequality $-7 \le \sqrt{16-x} \le 0$, which is $\{16\}$. The domain of f_5 is the solution set of the equation $\sqrt{25-x} = 16$, which is $\{-231\}$, and because the equation $\sqrt{36-x} = -231$ has no real solutions, the domain of f_6 is empty. Therefore $N + c = 5 + (-231) = -226$.

22. **Answer (C):** Assume without loss of generality that R is bounded by the square with vertices $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, and $D = (0,1)$, and let $X = (x,y)$ be n-ray partitional. Because the n rays partition R into triangles, they must include the rays from X to A, B, C, and D. Let the number of rays intersecting the interiors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ be n_1, n_2, n_3, and n_4, respectively. Because $\triangle ABX \cup \triangle CDX$ has the same area as $\triangle BCX \cup \triangle DAX$, it follows that $n_1 + n_3 = n_2 + n_4 = \frac{n}{2} - 2$, so n is even. Furthermore, the $n_1 + 1$ triangles with one side on $\overline{AB}$ have equal area, so each has area $\frac{1}{2} \cdot \frac{1}{n_1+1} \cdot y$. Similarly, the triangles with sides on $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have areas $\frac{1}{2} \cdot \frac{1}{n_2+1} \cdot (1-x)$, $\frac{1}{2} \cdot \frac{1}{n_3+1} \cdot (1-y)$, and $\frac{1}{2} \cdot \frac{1}{n_4+1} \cdot x$, respectively. Setting these expressions equal to each other gives

$$x = \frac{n_4+1}{n_2+n_4+2} = \frac{2(n_4+1)}{n} \text{ and } y = \frac{n_1+1}{n_1+n_3+2} = \frac{2(n_1+1)}{n}.$$

Thus an n-ray partitional point must have the form $X = \left(\frac{2a}{n}, \frac{2b}{n}\right)$ with $1 \le a < \frac{n}{2}$ and $1 \le b < \frac{n}{2}$. Conversely, if X has this form, R is partitioned into n triangles of equal area by the rays from X that partition $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ into b, $\frac{n}{2} - a$, $\frac{n}{2} - b$, and a congruent segments, respectively.

Assume X is 100-ray partitional. If X is also 60-ray partitional, then $X = \left(\frac{a}{50}, \frac{b}{50}\right) = \left(\frac{c}{30}, \frac{d}{30}\right)$ for some integers $1 \le a, b \le 49$ and $1 \le c, d \le 29$. Thus $3a = 5c$ and $3b = 5d$; that is, both a and b are multiples of 5. Conversely, if a and b are multiples of 5, then

$$X = \left(\frac{a}{50}, \frac{b}{50}\right) = \left(\frac{\frac{3a}{5}}{30}, \frac{\frac{3b}{5}}{30}\right)$$

is 60-ray partitional. Because there are exactly 9 multiples of 5 between 1 and 49, the required number of points X is equal to $49^2 - 9^2 = 40 \cdot 58 = 2320$.

23. **Answer (C):** Note that

$$g(z) = \frac{\frac{z+a}{z+b} + a}{\frac{z+a}{z+b} + b} = \frac{(1+a)z + a(1+b)}{(1+b)z + (a+b^2)} = \frac{Az+B}{Cz+D},$$

where $A = 1 + a$, $B = a(1+b)$, $C = 1 + b$, and $D = a + b^2$. Then

$$g(g(z)) = \frac{Ag(z) + B}{Cg(z) + D}.$$

Setting $g(g(z)) = z$ and solving for $g(z)$ gives

$$g(z) = \frac{-Dz + B}{Cz - A}.$$

Equating the two expressions for $g(z)$ gives

$$(Az + B)(Cz - A) = (-Dz + B)(Cz + D),$$

that is,

$$(A + D)(z^2C + z(D - A) - B) = 0.$$

Therefore either $B = C = 0$ (and $A = D$) or $A + D = 0$. In the former case $b = -1$, $f(z) = \frac{z+a}{z-1}$, and $g(z) = \frac{(1+a)z}{1+a} = z$, as required, unless $a = -1$. (Note that $a = -1$ in this case would imply $f(z) = 1$, which contradicts $g(g(z)) = z$.) In the latter case $1 + 2a + b^2 = 0$, so $|b|^2 = |2a + 1|$. Because $|a| = 1$, the triangle inequality yields

$$1 = |2|a| - 1| \le |2a + 1| \le 2|a| + 1 = 3,$$

so $1 \le |b| \le \sqrt{3}$. The minimum $|b| = 1$ is attained when $a = -1$ and $b = 1$ (or as above, when $b = -1$). The maximum $|b| = \sqrt{3}$ is attained when $a = 1$ and $b = \pm\sqrt{3}i$. The required difference is $\sqrt{3} - 1$.

Note: The conditions imply that a lies on the unit circle in the complex plane, so $2a+1$ lies on a circle of radius 2 centered at 1. The steps above are reversible, so if $b^2 = -1 - 2a$, then $g(g(z)) = z$ (unless $a = b = -1$). Therefore b^2 can be anywhere on the circle of radius 2 centered at -1, and $|b|$ can take on any value between 1 and $\sqrt{3}$.

24. **Answer (C):** Because $AB + CD = 21 = BC + DA$, it follows that $ABCD$ always has an inscribed circle tangent to its four sides. Let r be the radius of the inscribed circle. Note that $[ABCD] = \frac{1}{2}r\,(AB + BC + CD + DA) = 21r$. Thus the radius is maximum when the area is maximized. Note that $[ABC] = \frac{1}{2} \cdot 14 \cdot 9 \sin B = 63 \sin B$ and $[ACD] = \frac{1}{2} \cdot 12 \cdot 7 \sin D = 42 \sin D$. On the one hand,

$$\begin{aligned}[ABCD]^2 &= ([ABC] + [ACD])^2 \\ &= 63^2 \sin^2 B + 42^2 \sin^2 D + 2 \cdot 42 \cdot 63 \sin B \sin D.\end{aligned}$$

On the other hand, by the Law of Cosines,

$$AC^2 = 12^2 + 7^2 - 2 \cdot 7 \cdot 12 \cos D = 14^2 + 9^2 - 2 \cdot 9 \cdot 14 \cos B.$$

Thus

$$21^2 = \left(\frac{2 \cdot 26 + 2 \cdot 16}{4}\right)^2 = \left(\frac{14^2 - 12^2 + 9^2 - 7^2}{4}\right)^2 = (63 \cos B - 42 \cos D)^2$$

$$= 63^2 \cos^2 B + 42^2 \cos^2 D - 2 \cdot 42 \cdot 63 \cos B \cos D.$$

Adding these two identities yields

$$[ABCD]^2 + 21^2 = 63^2 + 42^2 - 2 \cdot 42 \cdot 63 \cos(B + D)$$
$$\leq 63^2 + 42^2 + 2 \cdot 42 \cdot 63 = (63 + 42)^2 = 105^2,$$

with equality if and only if $B + D = \pi$ (that is $ABCD$ is cyclic). Therefore $[ABCD]^2 \leq 105^2 - 21^2 = 21^2(5^2 - 1) = 42^2 \cdot 6$, and the required maximum $r = \frac{1}{21}[ABCD] = 2\sqrt{6}$.

OR

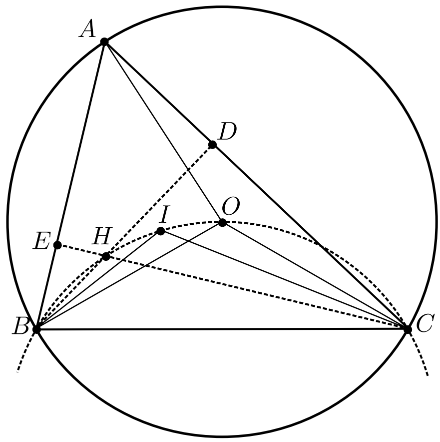
Establish as in the first solution that r is maximized when the area is maximized. Bretschneider's formula, which generalizes Brahmagupta's formula, states that the area of an arbitrary quadrilateral with side lengths a, b, c, and d, is given by

$$\sqrt{(s-a)(s-b)(s-c)(s-d) - abcd \cos^2 \theta},$$

where $s = \frac{1}{2}(a+b+c+d)$ and θ is half the sum of either pair of opposite angles. For a, b, c, and d fixed, the area is maximized when $\cos\theta = 0$. Thus the area is maximized when $\theta = \frac{1}{2}\pi$, that is, when the quadrilateral is cyclic. In this case, the area equals $\sqrt{7 \cdot 12 \cdot 14 \cdot 9} = 42\sqrt{6}$, and the required maximum radius $r = \frac{1}{21} \cdot 42\sqrt{6} = 2\sqrt{6}$.

25. **Answer (D):** By the Inscribed Angle Theorem, $\angle BOC = 2\angle BAC = 120°$. Let D and E be the feet of the altitudes of $\triangle ABC$ from B and C, respectively. Because $\overline{CE}$ and $\overline{BD}$ intersect at H,

$$\angle BHC = \angle DHE = 360° - \angle HEA - \angle ADH - \angle EAD$$
$$= 360° - 90° - 90° - 60° = 120°.$$



Because the lines BI and CI are bisectors of $\angle CBA$ and $\angle ACB$, respectively, it follows that

$$\begin{aligned}\angle BIC = 180^\circ - \angle ICB - \angle CBI &= 180^\circ - \frac{1}{2}(\angle ACB + \angle CBA)\\ &= 180^\circ - \frac{1}{2}(180^\circ - \angle BAC) = 120^\circ.\end{aligned}$$

Thus the points B, C, O, I, and H are all on a circle. Further,

$$\begin{aligned}\angle OCI = \angle ACI - \angle ACO &= \frac{1}{2}\angle ACB - \left(90^\circ - \frac{1}{2}\angle COA\right)\\ &= \frac{1}{2}\angle ACB - (90^\circ - \angle CBA)\\ &= \frac{1}{2}\angle ACB - 90^\circ + (120^\circ - \angle ACB) = 30^\circ - \frac{1}{2}\angle ACB,\end{aligned}$$

and $\angle ICH = \angle ACH - \angle ACI = (90^\circ - \angle EAC) - \frac{1}{2}\angle ACB = 30^\circ - \frac{1}{2}\angle ACB$. Thus $OI = IH$. Because $[BCOIH] = [BCO] + [BOIH]$ and BCO is an isosceles triangle with $BC = 1$ and $OB = OC = \frac{1}{\sqrt{3}}$, it is sufficient to maximize the area of quadrilateral $BOIH$. If P_1, P_2 are two points in an arc of circle BO with $BP_1 < BP_2$, then the maximum area of BOP_1P_2 occurs when $BP_1 = P_1P_2 = P_2O$. Indeed, if $BP_1 \neq P_1P_2$, then replacing P_1 by the point P_1' located halfway in the arc of circle BP_2 yields a triangle $BP_1'P_2$ with larger area than $\triangle BP_1P_2$, and the area of $\triangle BOP_2$ remains the same. Similarly, if $P_1P_2 \neq P_2O$, then replacing P_2 by the midpoint P_2' of the arc P_1O causes the area of $\triangle P_1P_2'O$ to increase and the area of $\triangle BP_1O$ to remain the same.

Therefore the maximum is achieved when $OI = IH = HB$, that is, when $\angle OCI = \angle ICH = \angle HCB = \frac{1}{3}\angle OCB = 10^\circ$. Thus $30^\circ - \frac{1}{2}\angle ACB = 10^\circ$, so $\angle ACB = 40^\circ$ and $\angle CBA = 80^\circ$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America

The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Institute for Operations Research and the Management Sciences
Jane Street
MathPath
Math Zoom Academy
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

62nd Annual

AMC 12 B

American Mathematics Contest 12 B

Wednesday, February 23, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego
bernardo.abrego@csun.edu

1. **Answer (C):** The given expression is equal to
$$\frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}.$$

2. **Answer (E):** The sum of her first 5 test scores is 385, yielding an average of 77. To raise her average to 80, her 6^{th} test score must be the difference between $6 \cdot 80 = 480$ and 385, which is 95.

3. **Answer (C):** Bernardo has paid $B - A$ dollars more than LeRoy. If LeRoy gives Bernardo half of that difference, $\frac{B-A}{2}$, then each will have paid the same amount.

4. **Answer (E):** Because $161 = 23 \cdot 7$, the only two digit factor of 161 is 23. The correct product must have been $32 \cdot 7 = 224$.

5. **Answer (A):**

   Because N is divisible by 3, 4, and 5, the prime factorization of N must contain one 3, two 2s, and one 5. Furthermore $2^2 \cdot 3 \cdot 5 = 60$ is divisible by every integer less than 7. Therefore the numbers with this property are precisely the positive multiples of 60. The second smallest positive multiple of 60 is 120, and the sum of its digits is 3.

6. **Answer (C):** Let O be the center of the circle, and let the degree measures of the minor and major arcs be $2x$ and $3x$, respectively. Because $2x + 3x = 360°$, it follows that $x = 72°$ and $\angle BOC = 2x = 144°$. In quadrilateral $ABOC$, the segments AB and AC are tangent to the circle, thus $\angle ABO = \angle ACO = 90°$ and $\angle BAC = 360° - (144° + 90° + 90°) = 36°$.

7. **Answer (B):** Because $x \le 99$ and $\frac{1}{2}(x+y) = 60$, it follows that $y = 120 - x \ge 120 - 99 = 21$. Thus the maximum value of $\frac{x}{y}$ is $\frac{99}{21} = \frac{33}{7}$.

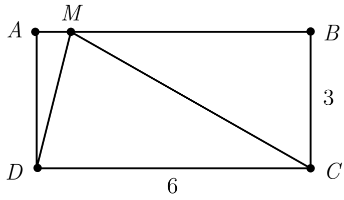
8. **Answer (A):** The only parts of the track that are longer walking on the outside edge rather than the inside edge are the two semicircular portions. If the radius of the inner semicircle is r, then the difference in the lengths of the two paths is $2\pi(r+6) - 2\pi r = 12\pi$. Let x be her speed in meters per second. Then $36x = 12\pi$, and $x = \frac{\pi}{3}$.

9. **Answer (D):** Consider all ordered pairs (a, b) with each of the numbers a and b in the closed interval $[-20, 10]$. These pairs fill a 30×30 square in the coordinate plane, with an area of 900 square units. Ordered pairs in the first and third quadrants have the desired property, namely $a \cdot b > 0$. The areas of the portions of the 30×30 square in the first and third quadrants are $10^2 = 100$ and $20^2 = 400$, respectively. Therefore the probability of a positive product is $\frac{100+400}{900} = \frac{5}{9}$.

OR

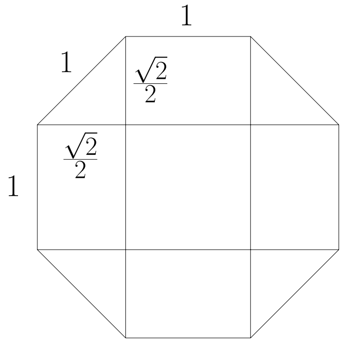
Each of the numbers is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$. Their product is positive if and only if both numbers are positive or both are negative, so the requested probability is $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

10. **Answer (E):** Sides $\overline{AB}$ and $\overline{CD}$ are parallel, so $\angle CDM = \angle AMD$. Because $\angle AMD = \angle CMD$, it follows that $\triangle CMD$ is isosceles and $CD = CM = 6$. Therefore $\triangle MCB$ is a $30-60-90°$ right triangle with $\angle BMC = 30°$. Finally, $2 \cdot \angle AMD + 30° = \angle AMD + \angle CMD + 30° = 180°$, so $\angle AMD = 75°$.



11. **Answer (B):** Because $AB = 1$, the smallest number of jumps is at least 2. The perpendicular bisector of $\overline{AB}$ is the line with equation $x = \frac{1}{2}$, which has no points with integer coordinates, so 2 jumps are not possible. A sequence of 3 jumps is possible; one such sequence is $(0, 0)$ to $(3, 4)$ to $(6, 0)$ to $(1, 0)$.

12. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$.



13. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

$$9 = w - z = (w - n) + (n - z).$$

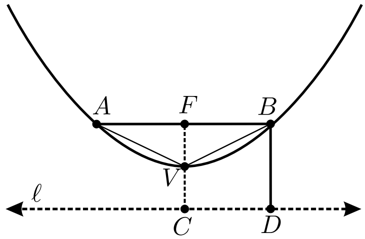
Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w-3, w-4, w-9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

14. **Answer (D):** Let ℓ be the directrix of the parabola, and let C and D be the projections of F and B onto ℓ, respectively. For any point in the parabola, its distance to F and to ℓ are the same. Because V and B are on the parabola, it follows that $p = FV = VC$ and $2p = FC = BD = FB$. By the Pythagorean Theorem, $VB = \sqrt{FV^2 + FB^2} = \sqrt{5}p$, and thus $\cos(\angle FVB) = \frac{FV}{VB} = \frac{p}{\sqrt{5}p} = \frac{\sqrt{5}}{5}$. Because $\angle AVB = 2(\angle FVB)$, it follows that

$$\cos(\angle AVB) = 2\cos^2(\angle FVB) - 1 = 2\left(\frac{\sqrt{5}}{5}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}.$$



OR

Establish as in the first solution that $FV = p$, $FB = 2p$, and $VB = \sqrt{5}p$. Then $AB = 2 \cdot FB = 4p$, and by the Law of Cosines applied to $\triangle ABV$,

$$\cos \angle AVB = \frac{VA^2 + VB^2 - AB^2}{2(VA)(VB)} = \frac{5p^2 + 5p^2 - 16p^2}{2(5p^2)} = -\frac{3}{5}.$$

Note: The segment AB is called the *latus rectum*.

15. **Answer (D):** Factoring results in the following product of primes:

$$\begin{aligned} 2^{24} - 1 &= (2^{12} - 1)(2^{12} + 1) = (2^6 - 1)(2^6 + 1)(2^4 + 1)(2^8 - 2^4 + 1) \\ &= 63 \cdot 65 \cdot 17 \cdot 241 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241. \end{aligned}$$

The two-digit integers that can be formed from these prime factors are:

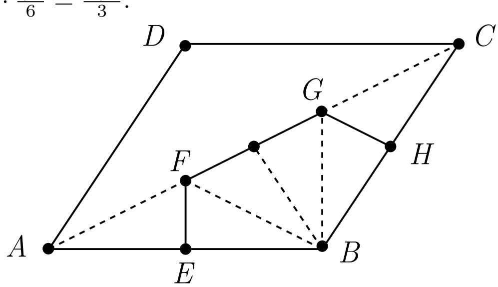
$$\begin{array}{r} 17, \quad 3 \cdot 17 = 51, \quad 5 \cdot 17 = 85, \\ 13, \quad 3 \cdot 13 = 39, \quad 5 \cdot 13 = 65, \quad 7 \cdot 13 = 91, \\ 3 \cdot 7 = 21, \quad 5 \cdot 7 = 35, \quad 3 \cdot 3 \cdot 7 = 63, \\ 3 \cdot 5 = 15, \quad \text{and} \quad 3 \cdot 3 \cdot 5 = 45. \end{array}$$

Thus there are 12 positive two-digit factors.

16. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from H. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90^\circ$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120^\circ - \angle FBE - \angle GBH =$

$60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



17. **Answer (B):** Note that

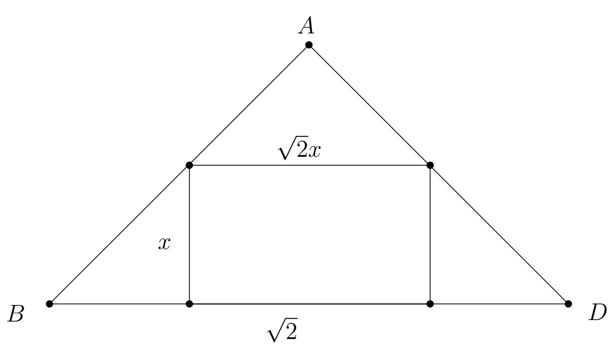
$$h_1(x) = \log_{10}\left(\frac{10^{10x}}{10}\right) = \log_{10}\left(10^{10x-1}\right) = 10x - 1.$$

Therefore $h_2(x) = 10^2x-(1+10)$, $h_3(x) = 10^3x-(1+10+10^2)$, and in general,

$$h_n(x) = 10^n x - \sum_{k=0}^{n-1} 10^k.$$

Hence $h_n(1)$ is an n-digit integer whose units digit is 9 and whose other digits are all 8's. The sum of the digits of $h_{2011}(1)$ is $8 \cdot 2010 + 9 = 16{,}089$.

18. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

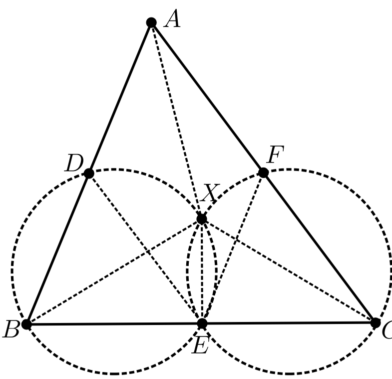
$$(\sqrt{2}-1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = (\frac{s}{2}, s)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

19. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0, 2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

20. **Answer (C):** Because $\overline{DE}$ is parallel to $\overline{AC}$ and $\overline{EF}$ is parallel to $\overline{AB}$ it follows that $\angle BDE = \angle BAC = \angle EFC$. By the Inscribed Angle Theorem, $\angle BDE = \angle BXE$ and $\angle EFC = \angle EXC$. Therefore $\angle BXE = \angle EXC$. Furthermore $BE = EC$, so by the Angle Bisector Theorem $XB = XC$. Note that $\angle BXC = 2\angle BXE = 2\angle BDE = 2\angle BAC$, and by the Inscribed Angle Theorem, it follows that X is the circumcenter of $\triangle ABC$, so $XA = XB = XC = R$ the circumradius of $\triangle ABC$.



Let $a = BC$, $b = AC$, and $c = AB$. The area of $\triangle ABC$ equals $\frac{1}{4R}(abc)$, and by Heron's Formula it also equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Thus

$$R = \frac{abc}{4\sqrt{s\,(s-a)\,(s-b)\,(s-c)}} = \frac{13\cdot 14\cdot 15}{4\sqrt{21\cdot 8\cdot 7\cdot 6}} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}$.

21. **Answer (D):** Let the arithmetic and geometric means of x and y be $10a + b$ and $10b + a$, respectively. Then

$$\frac{x+y}{2} = 10a + b \Rightarrow (x+y)^2 = 400a^2 + 80ab + 4b^2$$

and

$$\sqrt{xy} = 10b + a \Rightarrow xy = 100b^2 + 20ab + a^2,$$

so

$$(x-y)^2 = (x+y)^2 - 4xy = 396(a^2 - b^2) = 11\cdot 6^2\cdot (a+b)(a-b)$$

Because x and y are distinct, a and b are distinct digits, and the last expression is a perfect square if and only if $a + b = 11$ and $a - b$ is a perfect square. The cases $a - b = 1$, 4, and 9 give solutions $(a, b) = (6, 5)$, $(7.5, 3.5)$, and $(10, 1)$, respectively. Because a and b are digits only the first solution is valid. Thus $(x-y)^2 = 11\cdot 6^2\cdot 11 = 66^2$ and $|x-y| = 66$. Note that the given conditions are satisfied if $\{x, y\} = \{32, 98\}$.

22. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a+c-b)$. Similarly $AD = \frac{1}{2}(b+c-a)$ and $CF = \frac{1}{2}(a+b-c)$. In the given $\triangle ABC$, suppose that $AB = x+1$, $BC = x-1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}\left((x-1) + (x+1) - x\right) = \frac{1}{2}x, \\ AD &= \frac{1}{2}\left(x + (x+1) - (x-1)\right) = \frac{1}{2}x + 1, \text{ and} \\ CF &= \frac{1}{2}\left((x-1) + x - (x+1)\right) = \frac{1}{2}x - 1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of

T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011+2012+2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

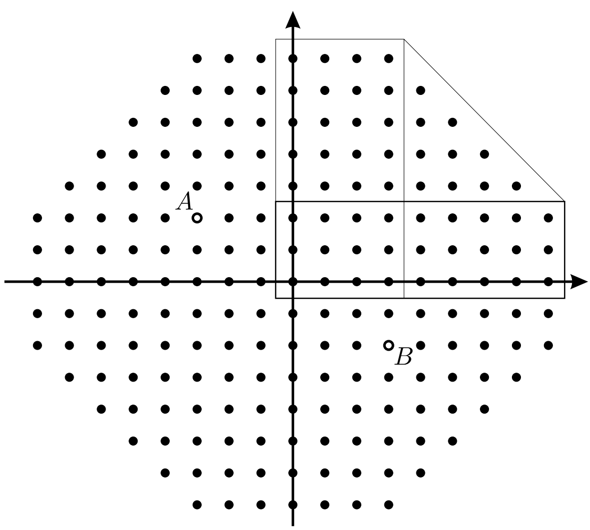
$$-1 + \frac{2012}{2^N} + \frac{2012}{2^N} \le 1 + \frac{2012}{2^N}.$$

Equivalently, $2012 \le 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

23. **Answer (C):** Let $X = (x, y)$. The distance traveled by the bug from A to X is at least $|x+3| + |y-2|$. Similarly, the distance traveled by the bug from X to B is at least $|x-3| + |y+2|$. It follows that X belongs to a path from A to B traveled by the bug if and only if

$$d = |x-3| + |x+3| + |y-2| + |y+2| \le 20.$$

The expression for d is invariant if x is replaced by $-x$ or y is replaced by $-y$. By symmetry, it is enough to count the number of points X with $x \ge 0$ and $y \ge 0$, multiply by 4, and subtract the points that were overcounted, that is those in the x-axis or in the y-axis. Consider four cases:



Case 1. $0 \le x \le 3$ and $0 \le y \le 2$. In this case $|x-3| + |x+3| = 6$ and $|y-2| + |y+2| = 4$. Thus $d = 10 < 20$ and there are $4 \cdot 3 = 12$ points X in this case. This includes the origin and 5 other points for which $xy = 0$.

Case 2. $0 \le x \le 3$ and $y \ge 3$. In this case $|x-3|+|x+3| = 6$ and $|y-2|+|y+2| = 2y$. Thus $d = 6 + 2y \le 20$ if and only if $y \le 7$. There are $4 \cdot 5 = 20$ points X in this case. This includes 5 points for which $xy = 0$.

Case 3. $x \ge 4$ and $0 \le y \le 2$. In this case $|x-3| + |x+3| = 2x$ and $|y-2| + |y+2| = 4$. Thus $d = 4 + 2x \le 20$ if and only if $x \le 8$. There are $5 \cdot 3 = 15$ points X in this case. This includes 5 points for which $xy = 0$.

Case 4. $x \ge 4$ and $y \ge 3$. In this case $|x-3|+|x+3| = 2x$ and $|y-2|+|y+2| = 2y$. Thus $d = 2x + 2y \le 20$ if and only if $x + y \le 10$. The number of points X in this case is equal to

$$\sum_{x=4}^{7} \sum_{y=3}^{10-x} 1 = \sum_{x=4}^{7} (10 - x - 2) = \sum_{x=4}^{7} (8 - x) = 4 + 3 + 2 + 1 = 10,$$

and there are no points with $xy = 0$.

By symmetry the required total is $4(12 + 20 + 15 + 10) - 2(5 + 5 + 5) - 3 = 4 \cdot 57 - 2 \cdot 15 - 3 = 195$.

24. **Answer (B):** Factoring or using the quadratic formula with z^4 as the variable yields $P(z) = (z^4 - 1)(z^4 + (4\sqrt{3} + 7))$. Moreover, $4\sqrt{3} + 7 = (\sqrt{3} + 2)^2$ and $2(\sqrt{3}+2) = 2\sqrt{3}+4 = (\sqrt{3}+1)^2$; thus $4\sqrt{3}+7 = (\frac{1}{2}(\sqrt{6}+\sqrt{2}))^4$. If $w = \frac{1}{2}(\sqrt{3}+1)$, then the eight zeros of $P(z)$ are $1, -1, i, -i, w(1+i), w(-1+i), w(-1-i)$, and $w(1-i)$.

The distances from 1 to the other zeros are

$$\begin{aligned} |1-(-1)| = 2, |1 \pm i| = \sqrt{2}, |1 - w(1 \pm i)| &= \sqrt{(1-w)^2 + w^2} = \sqrt{2}, \text{ and} \\ |1 - w(-1 \pm i)| &= \sqrt{(1+w)^2 + w^2} = \sqrt{2\sqrt{3} + 4} = \sqrt{3} + 1. \end{aligned}$$

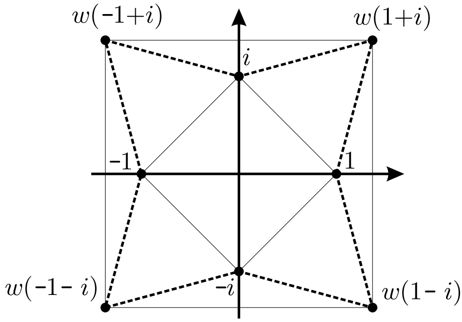
Similarly, the distances from $w(1+i)$ to the other zeros are

$$\begin{aligned} |w(1+i) - w(1-i)| = |w(1+i) - w(-1+i)| &= 2w = \sqrt{3} + 1, \\ |w(1+i) - w(-1-i)| &= 2\sqrt{2}w = \sqrt{6} + \sqrt{2}, \end{aligned}$$

and by symmetry,

$$\begin{aligned} |w(1+i) - 1| = |w(1+i) - i| &= \sqrt{2}, \text{ and} \\ |w(1+i) + 1| = |w(1+i) + i| &= \sqrt{3} + 1. \end{aligned}$$

Because the set of zeros is 4-fold symmetric with respect to the origin, it follows that every line segment joining two of the zeros has length at least $\sqrt{2}$. This shows that any polygon with vertices at the zeros has perimeter at least $8\sqrt{2}$. Finally, note that the polygon with consecutive vertices $1, w(1+i)$, i, $w(-1+i)$, -1, $w(-1-i)$, $-i$, and $w(1-i)$ has perimeter $8\sqrt{2}$.



25. **Answer (D):** Let

$$100 = qk + r, \text{ with } q, r \in \mathbb{Z} \text{ and } |r| \leq \frac{k-1}{2}, \text{ and}$$

$$n = q_1 k + r_1, \text{ with } q_1, r_1 \in \mathbb{Z} \text{ and } |r_1| \leq \frac{k-1}{2},$$

so that $\left[\frac{100}{k}\right] = q$ and $\left[\frac{n}{k}\right] = q_1$. Note that $\left[\frac{n+mk}{k}\right] = \left[\frac{n}{k}\right] + m$ for every integer m. Thus n satisfies the required identity if and only if $n + mk$ satisfies the identity for all integers m. Thus all members of a residue class mod k either satisfy the required equality or not; moreover, k divides 99! for every $1 \leq k \leq 99$, so every residue class mod k in the interval $1 \leq n \leq 99!$ has the same number of elements. Suppose $r \geq 0$. If $r_1 \geq r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1)\,k + (r - r_1)\,,$$

where $0 \leq r - r_1 \leq \frac{k-1}{2}$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 = \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. Similarly, if $r_1 < r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1 + 1)\,k + (r - r_1 - k)\,,$$

where $-\frac{k-1}{2} \leq r - r_1 - k \leq -1$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 + 1 > \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. It follows that the only residue classes r_1 that satisfy the identity are those in the interval $r - \frac{k-1}{2} \leq r_1 \leq \frac{k-1}{2}$. Thus for $r \geq 0$,

$$P(k) = \frac{1}{k}\left(\frac{k-1}{2} + 1 - \left(r - \frac{k-1}{2}\right)\right) = \frac{k-r}{k} = 1 - \frac{|r|}{k}.$$

Similarly, if $r < 0$ then the identity is satisfied only by the residue classes r_1 in the interval $-\frac{k-1}{2} \leq r_1 \leq r + \frac{k-1}{2}$. Thus for $r < 0$,

$$P(k) = \frac{1}{k}\left(r + \frac{k-1}{2} + 1 - \left(-\frac{k-1}{2}\right)\right) = \frac{k+r}{k} = 1 - \frac{|r|}{k}.$$

To minimize $P(k)$ in the range $1 \leq k \leq 99$, where k is odd, first suppose that $r = \frac{k-1}{2}$. Note that $P(k) = \frac{1}{2} + \frac{1}{2k}$, $100 = qk + \frac{k-1}{2}$, and so $201 = k\,(2q+1)$.

The minimum of $P(k)$ in this case is achieved by the largest possible k under this restriction. Because $201 = 3 \cdot 67$, it follows that the largest factor k of 201 in the given range is $k = 67$. In this case $P(67) = \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67}$. Second, suppose $r = \frac{1-k}{2}$. In this case $P(k) = \frac{1}{2} + \frac{1}{2k}$ and $199 = k(2q-1)$. Because 199 is prime, it follows that $k = 1$ and $P(k) = 1 > \frac{34}{67}$. Finally, if $|r| \le \frac{k-3}{2}$, then

$$\begin{aligned} P(k) &= 1 - \frac{|r|}{k} > 1 - \frac{k-3}{2k} = \frac{1}{2} + \frac{3}{2k} \\ &\ge \frac{1}{2} + \frac{3}{2 \cdot 99} > \frac{1}{2} + \frac{1}{2 \cdot 67} = \frac{34}{67}. \end{aligned}$$

Therefore the minimum value of $P(k)$ in the required range is $\frac{34}{67}$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Doug Faires, Sister Josanna Furey, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Joe Kennedy, Joachim Rebholz, Eugene Veklerov, David Wells, and LeRoy Wenstrom

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America

The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Institute for Operations Research and the Management Sciences
Jane Street
MathPath
Math Zoom Academy
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company